MATHEMATICAL ASPECTS OF TWISTOR THEORY: NULL DECOMPOSITION OF CONFORMAL ALGEBRAS AND SELF-DUAL METRICS ON INTERSECTIONS OF QUADRICS

by

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PREFACE

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ABSTRACT

MATHEMATICAL ASPECTS OF TWISTOR THEORY: NULL DECOMPOSITION OF CONFORMAL ALGEBRAS AND SELF-DUAL METRICS ON INTERSECTIONS OF QUADRICS

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The conformal algebra of an n-dimensional affine space with a metric of arbitrary signature (p, q) with $p + q = n$ is considered. The case of broken conformal invariance is studied, by considering the subalgebra of the enveloping algebra of the conformal algebra that commutes with the squared-mass operator. This algebra, denoted $\mathcal{R}$, is generated by the generators of the Poincaré Lie algebra and an additional vector operator R which preserves the relevant information when the conformal invariance is broken. Due to the nonlinearity of the algebra, finding the Casimir invariants becomes extremely difficult.

The $\mathcal{R}$-algebra is constructed for arbitrary dimensions, but the Casimir invariants are only determined for $n \leq 5$.

The second part of this thesis describes the geometric properties of metrics on the twistor space on intersections of quadrics. Consider a generic pencil of quadrics in a complex projective space $\mathbb{CP}^5$. The base locus of this pencil is considered as a three-dimensional projective twistor space, such that each point of the associated space-time is a projective two-plane lying inside one quadric of the pencil. The time coordinate can be described as a hyperelliptic curve of genus two, over which the space time is fibered. The metrics arising on the associated twistor space of the completely null two-planes are studied. It emerges that for pencils generated by simultaneously diagonalizable quadrics, these metrics are always self-dual and, in certain cases, conformal to vacuum.

1.0 INTRODUCTION

Twistor theory was developed by Sir Roger Penrose in 1967 as a new way of describing the geometry of space-time [25], [26]. It is one of the most elegant and profound theories present these days, combining methods of algebraic, complex and differential geometry with physical theories such as general relativity, and quantum physics.

Space-time and matter have been viewed as interconnected since Einstein developed his General Relativity Theory, by means of the equation:

$$G_{\mu\nu} = kT_{\mu\nu}.$$

$G_{\mu\nu}$ is called the *Einstein tensor* and it depends entirely on the geometry of the space-time; $T_{\mu\nu}$ is known as the *energy-matter tensor*, and is purely material. As a consequence, the matter will influence the geometry of the space-time, and the geometry will influence how the matter moves. One would expect that the same fundamental structure can be used to describe both sides of this equation.

It is well-established that matter has a discrete structure. For a long time the standard description of the space-time was continuous, with points being considered the building blocks of the space-time. Although simple and intuitive, the continuous approach has not been most forthcoming with the attempts of quantizing the gravity. The four known fundamental interactions in nature, based on local invariance principles, are: electromagnetic, (nuclear) weak, (nuclear) strong and gravity. The first three are believed to be manifestations of a unified interaction, described by the Standard Model. The carriers for these three types of interactions are the photons, the W and Z bosons, and the gluons, respectively [30], [41]. One has yet to exhibit a particle carrier (graviton) for the fourth interaction.

The goal of quantum gravity is to unify all four interactions into a single one. One of the main problems arising in this endeavour is that the theory presents singularities, due in part to the lack of dimensionality of the space-time points.

String theory avoids this problem by introducing the concept of *string*, an elementary geometric object of nonzero length. The frequencies of vibration of the strings give rise to all elementary particles. Strings can split and combine, this being interpreted as interactions between particles. Some aspects of string theory make it seem contrived: for example, a consequence of this theory is that the space-time is 10, 11 or 26 dimensional, depending on the approach (e.g., bosonic theories are 26-dimensional), with the extra dimensions being explained away by compactification. Additionally, there is no freedom in choosing any of the parameters of the theory, meaning that there is a **unique** string theory [30].

Twistor theory offers another alternative to the space-time continuum, considering that the basic objects describing the geometry of the space-time are four-dimensional complex vectors, called *twistors* [25]. In this approach the points are obtained from intersections of twistors, becoming secondary objects. Twistor theory attempts to reformulate basic physics in twistor language. Similar to strings, twistors are basic objects with a dual character. They are used to replace the points as the basic geometric objects, but can also be used to describe elementary particles. Interactions between particles are explained by means of twistor diagrams [27]. One of the many advantages of twistor theory is that it has a natural complex character, which is needed in working with quantum mechanics.

Recently, there have been many developments that establish connections between twistor theory and other fundamental theories. In 2003, Witten showed that perturbative scattering amplitudes in Yang-Mills theory can be Fourier transformed from momentum space to twistor space. This observation was the starting point of what is known today as *twistor string theory* [38]. Another connection was made with the Quantum Hall Effect in four-dimensions, which was introduced by S.C. Zhang and J. Hu in 2001 [40]. G. Sparling noticed that the quantum liquid defined in [40] seems to find a natural place in twistor space, being interpreted as "more primitive than space-time itself" [23], [32], [33], [34].

The structure of this thesis is as follows: chapter 2 introduces the basic concepts and techniques used in spinor and twistor theory. This is necessary in order to understand why we are interested in the topics discussed in this thesis.

Section 2.1 presents some basic spinor theory, focusing on the properties used here.

One of the main features of twistor theory is that it is conformal. In section 2.2 we see how the conformal group arises naturally in the spinorial setting.

This chapter ends with the presentation in section 2.3 of some important concepts and results in twistor theory, ending with the representation of points as intersections of twistors.

The first original results are presented in chapter 3. We study the null decomposition of the conformal algebras of the Lie algebras of $\mathbb{SO}(p+1, q+1)$, and their Casimir invariants. These invariants are generators of the centers of the universal enveloping algebras of the algebras considered, and can be used to label irreducible representations of Lie algebras, as well as decomposing generic representations into irreducible ones [1], [6]. (The *enveloping* algebra of a Lie algebra $\mathfrak{g}$ is an associative algebra $\mathfrak{U}$ which contains $\mathfrak{g}$ as a subspace, such that the commutator on this subspace, inherited from $\mathfrak{U}$, reproduces the Lie bracket in $\mathfrak{g}$.)

The study of invariants is of great interest in many areas in mathematics and physics, in particular in situations involving symmetry breaking. In this process, the invariants of the whole symmetry group can be related to the invariants of the subgroup preserving the original symmetry [3]. The type of symmetry breaking we study here is that of the conformal invariance when the mass of the system acted on is fixed. The Poincaré group (a subgroup of the conformal group to be described in chapter 2) is known to commute with the operator representing the square of the mass. It seems natural then to study the subalgebra of the enveloping algebra of the conformal algebra that commutes with the mass.

In chapter 3 we consider the conformal algebra of an n-dimensional affine space with a metric of arbitrary signature (p, q), with $p + q = n$. The algebra of this space is the Lie algebra of $\mathbb{SO}(p + 1, q + 1)$. We obtain that the corresponding subalgebra, denoted $\mathcal{R}$, in addition to the generators of the Poincaré Lie algebra, will depend on a vector operator R_a which is shown to commute with the mass of the system and with the translations p_a.

The operator R_a preserves the relevant information when the conformal invariance is broken. The generators of the conformal group are the translations p_a, the Lorentz transformations M_{ab}, the dilation operator D and the special conformal transformations q_a. The operator R_a will be built from all the generators of the conformal group, but in the process of symmetry breaking the dilation operator D is removed. To obtain again the full conformal algebra, one only needs to add back the generator D with appropriate commutation relations.

We should remark here that the $\mathcal{R}$-algebra, although finitely generated, is not a finite-dimensional Lie-algebra, as the commutator of the vector operator R_a with itself is non-linear (cubic) in the generators [22], [31].

In chapter 3 we construct the $\mathcal{R}$-algebras and their Casimir invariants for spaces of various dimensions. The four-dimensional case has been thoroughly studied in [31] for the case of $\mathbb{SO}(2,4)$; its origins lie in the study of the twistor model of hadrons in which a hadron is described by using systems of three twistors [11], [29]. These types of representations (for $n=4$) have also been used in the AdS/CFT correspondence (a relationship between string theory and the $n=4$ supersymmetric Yang-Mills theory) [15], [16], [18].

The $\mathcal{R}$-algebra we construct here is valid in all dimensions, but a complete description of the Casimir invariants has only been obtained in low dimensions, by exploiting the characteristics of each dimensionality.

In section 3.1 we present the basic commutation relations that accompany the null decomposition of the conformal algebra of an n-dimensional affine space equipped with a metric of signature (p,q).

In section 3.2 we generalize the four-dimensional theory developed in [31] to arbitrarily many dimensions and signatures. The $\mathcal{R}$-algebra is completely determined, but due to the complexity of the expressions involved, only one of the Casimir operators is obtained.

In section 3.3, we use the general case to construct the five-dimensional conformal algebra and its Casimir operators. The $\mathcal{R}$-algebra follows directly from 3.2; in order to determine all the Casimir operators, we used a spinorial approach. Results have been written in tensor form as well [21].

Section 3.4 gives a brief outlook of the four-dimensional case studied in [31], including the statement of the main theorem and the expressions of the Casimir invariants. This case uses the Pauli-Lubanski spin-vector which can be defined only in four-dimensions.

In section 3.5 we independently develop the three-dimensional theory and verify the consistency of the results with the general theory from section 3.2. This section uses the properties of the completely skew three-dimensional tensor ε_{abc} and the properties of the cross product.

Chapter 3 ends with the two-dimensional case, where any skew tensor with two indices can be written as a multiple of the completely skew two-dimensional tensor ε_{ab}.

There are many intermediate formulas that were necessary to determine the commutation relations of the $\mathcal{R}$-algebra and the Casimir operators. Due to the fact that calculations are extremely lengthy, we listed most of the intermediate formulas used in the appendices.

This theory a priori only applies to the (non-compact) pseudo-orthogonal groups. Many other (non-compact) Lie groups such as the pseudo-unitary groups are subgroups of the pseudo-orthogonal group in a natural way. As a consequence, with appropriate modifications, this theory can be applied to these groups also.

As a particular example, note that the de Sitter algebra of n-dimensional de-Sitter space, $\mathbb{SO}(1, n)$, and the anti-de-Sitter algebra of n-dimensional anti-de-Sitter space, $\mathbb{SO}(2, n-1)$, both fall within the scope of our theory [18].

The second part of this thesis deals with the construction of a class of metrics on the twistor space on intersections of quadrics.

The standard arena for twistors is the complexified compactified Minkowski space. In chapter 4 we describe how the Minkowski space can be regarded as a four-dimensional quadric embedded in $\mathbb{CP}^5$, via the Klein representation. We consider here a linear pencil of quadrics

$$\lambda Q + \mu R = 0,$$

with $(\lambda, \mu) \in \mathbb{C}^2 \setminus (0, 0)$. This pair of coefficients can be represented by a $\mathbb{CP}^1$ and is interpreted as the time coordinate. The quadrics Q and R intersect in a three-dimensional projective space $\mathbb{CP}^3$, which corresponds to the projective twistor space.

If at least one of the quadrics is not singular, then the pencil is not singular. In this case the determinant of the pencil of quadrics, $\det(\lambda Q + \mu R)$, is a polynomial of degree six.

There are two distinct situations to be considered: when the quadrics are simultaneously diagonalizable, and when they have nontrivial Jordan Canonical Form. We studied some geometric properties of these metrics in the simultaneously diagonalizable case: flatness, Ricci flatness, conformal flatness, conformal to vacuum, and self- (or anti-self-) duality.

In section 4.1 we present the Klein representation which regards the Minkowski space-time as a quadric embedded in $\mathbb{CP}^5$.

Section 4.2 presents some important results on quadrics and pencils of quadrics.

Section 4.3 describes the construction of the metrics on the intersection of the quadrics of a linear pencil generated by two simultaneously diagonalizable quadrics, and the tensors and equations we will use to characterize these metrics.

Section 4.4 describes the geometric properties we are interested in studying.

Section 4.5 presents the results obtained: it emerges that these metrics are self dual and in most cases conformal to vacuum. The twistor space is compact, but the metrics are not defined everywhere: they are singular at the zeros of $\det(\lambda Q + \mu R)$. It has yet to be determined, in general, whether the singularities are genuine or they are coordinate singularities.

Section 4.6 presents some detailed aspects of the geometric properties for two nontrivial cases.

In the generic case, the "time" coordinate is not described by a $\mathbb{CP}^1$ anymore, but by a hyperelliptic curve of genus two. Studying this case will make the object of future research. Future plans also involve studying the case when the quadrics are not simultaneously diagonalizable, in which case the matrices involved are made of Jordan blocks of various sizes.

2.0 SPINOR TECHNIQUES AND TWISTOR THEORY

The most common approach to the study of space-time is by using a four-dimensional real continuum, with the points as basic constituents. This approach presents many problems due to the lack of dimensionality of points, in particular in the attempts of obtaining a unified theory of quantum physics and gravity. Physical laws break down at scales lower than the Planck length ($l_P = 10^{-33}$ cm), losing their power of prediction.

Twistor theory offers an alternate approach to the use of space-time continuum, where twistors are the main objects, whereas space-time points are considered to be derived objects, hence of secondary importance. A *twistor* can be thought of as a pair of spinors related by means of a differential equation (called "twistor equation"). Points in space-time are interpreted as intersections of twistors.

In the case of a four-dimensional Lorentzian metric, spinor calculus can be viewed as an alternative to the more popular (world)tensor calculus. The advantage of the latter lies in the fact that it can be used in arbitrary dimensions, but it can become very cumbersome due to the possibly large number of indices involved. On the other hand, spinors, which are two-complex-dimensional vectors, can be used only in a four-dimensional setting. The advantage of using spinors is that calculations become simpler due to working in two dimensions; moreover, one can make use of the properties of complex geometry.

In section 3.3, spinor calculus will be used to facilitate calculations in the case of a five-dimensional conformal algebra.

In chapter 4, we will describe the space-time as a four-dimensional quadric embedded in $\mathbb{CP}^5$, which will then become the arena for the study of metrics on the twistor space arising on intersections of quadrics.

2.1 BACKGROUND ON SPINORS

One of the simplest presentations of spinors and their properties can be found in [12], and this section will follow it fairly closely. For a more comprehensive description, see [28].

As mentioned earlier, our world can be described as a smooth, flat four-dimensional space, endowed with a bilinear symmetric nondegenerate pseudometric, called Minkowski space-time.

2.1.1 Minkowski Space-Time and Lorentz Transformations

A *Minkowski space-time* $\mathbb{M}$ is a four-dimensional real manifold $\mathbb{R}^4$ with line element given by the following expression:

$$ds^2 = \eta_{ab} dx^a dx^b = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{2.1}$$

where $\eta_{ab} = diag(+1, -1, -1, -1)$ is the Minkowski metric. Here $x^0 = ct$ denotes the temporal coordinate, with c the speed of light; the remaining coordinates (x^1, x^2, x^3) represent spatial coordinates. The indices $a,\ b$ assume the values $0, 1, 2,$ and 3 in this formula. Throughout this thesis, we will use Einstein's convention where summation is assumed on repeated indices.

Each point in Minkowski space-time can be characterized by four coordinates with respect to an arbitrary origin, (x^0, x^1, x^2, x^3). Such a point is called an *event* in space-time.

To each event we can associate a corresponding *light-cone* given by the vanishing of the form ds^2 in (2.1). This surface determines three regions of interest in space-time:

- the *interior* of the cone, characterized by $ds^2 > 0$. This inequality implies that the interior of the cone is **causal**, since the speed of propagation is less than c. Vectors joining the event E with points in the interior of the light-cone are called *time-like* vectors. The upper half of the cone is called *future* light-cone, and the lower half is called *past* light-cone.

- the *surface* of the cone, characterized by $ds^2 = 0$, where the speed of propagation is equal to c. Vectors joining the event E with points on the surface of the cone are called *null* vectors, of length equal to zero.
- the *exterior* of the cone, characterized by $ds^2 < 0$. This inequality implies that the exterior of the cone is **acausal**, due to the speed of propagation being greater than c. Vectors joining E with points outside of the cone are called *space-like* vectors.

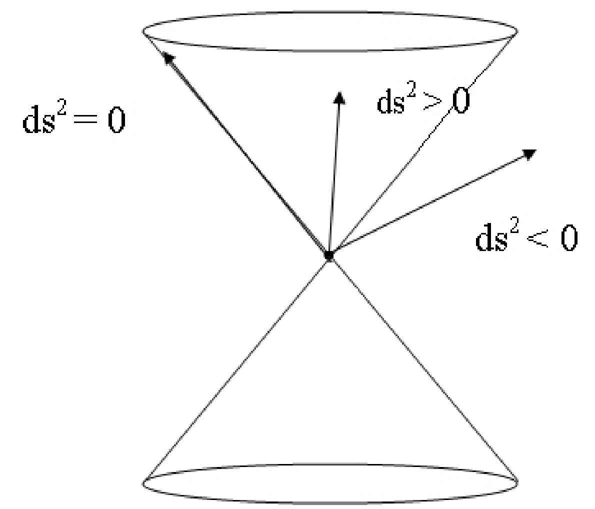

Figure 2.1: The light-cone associated to an event E in Minkowski space-time.

We should mention that the meaning of the inequalities defining these regions depends on the signature chosen. Here we will work with a signature $(+ - --)$. In a signature $(- + ++)$, space-like vectors are characterized by $ds^2 > 0$.

Next, we introduce the Lorentz transformations. A *Lorentz transformation* $\Lambda^a{}_b$ is a linear transformation of $\mathbb{M}$ that preserves the metric η_{ab}:

$$\Lambda^a{}_c \Lambda^b{}_d \eta_{ab} = \eta_{cd}, \tag{2.2}$$

or, in matrix notation:

$$\Lambda^T \eta \Lambda = \eta. \tag{2.3}$$

The *Lorentz group* $L = \mathbb{O}(1,3)$ is the group of all such linear transformations. Note that from (2.3) we have $(\det \Lambda)^2 = 1$ or $\det \Lambda = \pm 1$.

The Lorentz group is not connected, having four components. We are particularly interested in the one that contains the identity and preserves the time orientation, denoted $L_+^\uparrow$. Here $+$ denotes the sign of the determinant preserving the overall orientation, and $\uparrow$ means that ${\Lambda^0}_0 > 0$, which preserves the time orientation. $L_+^\uparrow$ is doubly covered by $\mathbb{SO}(1,3)$.

2.1.2 The Spin Space

In the Minkowski space $\mathbb{M}$, consider a vector $V^a = (V^0, V^1, V^2, V^3)$ (in some orthonormal frame). We use here the abstract index notation introduced by Penrose [28], where the index a merely indicates the type of quantity (vector, form, etc.) rather than assuming numerical values.

To each such vector one can associate by a one-to-one correspondence a Hermitian matrix as follows [12]:

$$f : \mathbb{M} \rightarrow \mathcal{M}_2(\mathbb{C}),$$

$$f(V^a) = V^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}, \tag{2.4}$$

where the matrix $V^{AA'}$ can be written also as:

$$V^{AA'} = \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix}. \tag{2.5}$$

The spinor indices $A,\ A'$ take the values $0,\ 1$, and $0',\ 1'$ respectively, and the prime stands for complex conjugation.

The determinant of the matrix $f(V^a)$ is half the length of the vector V^a:

$$\det f(V^a) = \frac{1}{2}\left[\left(V^0\right)^2 - \left(V^1\right)^2 - \left(V^2\right)^2 - \left(V^3\right)^2\right] = \frac{1}{2}\eta_{ab}V^aV^b. \tag{2.6}$$

Let $M^A_{\ B}$ be an element of $\mathbb{SL}(2,\mathbb{C})$, and $\overline{M}^{A'}_{\ \ B'}$ its Hermitian conjugate. We can define a linear transformation of the vector V^a by

$$V^a \longmapsto V^{AA'} \longmapsto M^A_{\ B} V^{BB'} \overline{M}^{A'}_{\ \ B'}.$$

Note that the result is another Hermitian matrix with the same determinant. This is in fact a Lorentz transformation.

If the vector V^a is null and future-pointing, the rank of $f\,(V^a)$ becomes equal to one. In this case, $V^{AA'}$ can be factored as [12]:

$$V^{AA'} = \alpha^A \overline{\alpha}^{A'}, \tag{2.7}$$

where α^A is a complex two-dimensional vector, and $\overline{\alpha}^{A'}$ is its complex conjugate:

$$\alpha^A = \begin{bmatrix} \alpha^0 \\ \alpha^1 \end{bmatrix}, \quad \text{and} \quad \overline{\alpha}^{A'} = \left[\overline{\alpha}^{0'}, \overline{\alpha}^{1'}\right]. \tag{2.8}$$

The vectors α^A determine a complex two-dimensional vector space S on which $\mathbb{SL}(2,\mathbb{C})$ acts, called *spin space*. The following spaces can also be defined:

- $\overline{S} = S'$: the *complex conjugate* spin space with elements $\beta^{A'}$,
- S^*: the *dual* spin space with elements γ_A,
- S'^*: the *dual of the complex conjugate* spin space, with elements $\delta_{A'}$.

The next section will present some spinor properties which will be used in this thesis.

2.1.3 Spinor Properties

1. Note that the spinors in (2.8) have valence one. Higher valence spinors can be obtained by considering tensor products of the spin spaces defined above, S, S', S^* and S'^*:

$$\Phi^{\underbrace{A...B}_{k_1}\underbrace{A'...C'}_{k_2}}{}_{\underbrace{E...F}_{k_3}\ \underbrace{E'...F'}_{k_4}} \in \left(\bigotimes_{k_1} S\right) \otimes \left(\bigotimes_{k_2} S'\right) \otimes \left(\bigotimes_{k_3} S^*\right) \otimes \left(\bigotimes_{k_4} S'^*\right), \tag{2.9}$$

   where we used the notation $\bigotimes_{k_1} S$ to mean $\underbrace{S \otimes ... \otimes S}_{k_1}$.

2. In our discussion of the five-dimensional conformal algebra we will use the concepts and properties of symmetric and antisymmetric spinors. For a spinor S of valence n we have:

$$S^{(A...B)} = \frac{1}{n!}\sum_{\sigma} S^{\sigma(A)...\sigma(B)}, \tag{2.10}$$

   and

$$S^{[A...B]} = \frac{1}{n!}\sum_{\sigma} sign(\sigma) S^{\sigma(A)...\sigma(B)}, \tag{2.11}$$

   where the sum is on all permutations σ and $sign(\sigma) = \pm 1$, depending on whether σ is an odd or an even permutation. These results hold for both primed and unprimed indices.

3. Symmetric spinors factorize into outer products of spinors of valence one:

$$S_{(A...B)} = \alpha_A ... \beta_B. \tag{2.12}$$

   The spinors $\alpha_A, ..., \beta_B$ are called the *principal null directions of the spinor* S (p.n.d.s.) This is a significant simplification of spinor calculus. We will see shortly that antisymmetric spinors simplify as well.

4. In a two-dimensional space, any completely skew quantity with more than two indices is identically equal to zero. There is thus a unique completely skew two index-spinor (up to complex multiples), denoted ϵ_{AB}. This spinor is preserved by $\mathbb{SL}(2, \mathbb{C})$, much in the way the metric η_{ab} is preserved by the Lorentz transformations in (2.2):

$$M_A{}^B M_C{}^D \epsilon_{BD} = \epsilon_{AC}, \tag{2.13}$$

   for any $M_A{}^B \in \mathbb{SL}(2, \mathbb{C})$. It follows that each spin space has such a spinor attached, and whether we mention it explicitly or not, by S we will generally mean the pair (S, ϵ_{AB}).

5. The spaces (S, ϵ_{AB}) and $(S', \epsilon_{A'B'})$ are related by an anti-isomorphism called *complex conjugation.* It is usually denoted by an overbar:

$$
\begin{aligned}
\alpha^A &\in S \Longrightarrow \overline{\alpha^A} = \overline{\alpha}^{A'} \in S', \\
\alpha^{A'} &\in S' \Longrightarrow \overline{\alpha^{A'}} = \overline{\alpha}^{A} \in S.
\end{aligned} \tag{2.14}
$$

   This extends to higher valence spinors as well, for example:

$$
\overline{\alpha^{ABC'D}} = \overline{\alpha}^{A'B'CD'}. \tag{2.15}
$$

6. We should remark here that if ϵ_{AB} is chosen such that $\epsilon_{01} = 1$ in some basis of S, we can write:

$$
\epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \overline{\epsilon}_{A'B'} = \overline{\epsilon}^{A'B'}. \tag{2.16}
$$

7. By convention, primed and unprimed indices can be commuted:

$$
T_{A'B'C} = T_{CA'B'} = T_{A'CB'}. \tag{2.17}
$$

   In general, the order among primed (unprimed) indices matters:

$$
T_{A'B'C} \neq T_{B'A'C}. \tag{2.18}
$$

8. Similar to the use of the metric η_{ab} to raise and lower indices in Minkowski space, the spinor ϵ_{AB} provides an isomorphism between the spin-space S and its dual S^* by raising and lowering indices of spinors. Since ϵ_{AB} is skew, one must be very careful when performing these operations; the adjacent indices must be descending to the right in order to avoid introducing a sign change. For example:

$$
\begin{aligned}
\epsilon^{AB}\alpha_B &= \alpha^A, \\
\beta^B \epsilon_{AB} &= -\beta^B \epsilon_{BA} = -\beta_A.
\end{aligned} \tag{2.19}
$$

   Likewise, $\epsilon_{A'B'}$ and $\epsilon^{A'B'}$ raise and lower indices in the complex conjugate space S' and its dual S'^*:

$$
\begin{aligned}
\epsilon^{A'B'}\gamma_{B'} &= \gamma^{A'} \in S', \\
\rho^{B'} \epsilon_{A'B'} &= -\rho^{B'} \epsilon_{B'A'} = -\rho_{A'} \in S'^*.
\end{aligned} \tag{2.20}
$$

9. Some important identities satisfied by the ϵ_{AB} spinor are:

$$\epsilon^{AB}\epsilon_{CB} = \delta_C^A \quad \text{and} \quad \epsilon_{AB}\epsilon^{CB} = \delta_A^C, \tag{2.21}$$

where δ_A^C is the spinor Kronecker delta, satisfying:

$$\delta_A^B = \epsilon_A{}^B = -\epsilon^B{}_A. \tag{2.22}$$

We also have:

$$\epsilon_{A[B}\epsilon_{CD]} = 0, \tag{2.23}$$

and

$$\epsilon_{AB}\epsilon^{CD} = \delta_A{}^C\delta_B{}^D - \delta_A{}^D\delta_B{}^C. \tag{2.24}$$

These relations lead to

$$\epsilon_A{}^A = 2. \tag{2.25}$$

10. All spinors α^A are *null* with respect to ϵ_{AB}, in the sense that

$$\epsilon_{AB}\alpha^A\alpha^B = \alpha_B\alpha^B = 0. \tag{2.26}$$

The complex conjugate relation holds as well.

11. A *Hermitian* spinor is a spinor with equal number of primed and unprimed indices such that the spinor and its complex conjugate are the same:

$$\overline{\alpha_{ABC'D'}} = \overline{\alpha}_{A'B'CD} = \alpha_{A'B'CD}. \tag{2.27}$$

Note that the skew spinor ϵ_{AB} is Hermitian. The Hermitian spinor $\epsilon_{AB}\epsilon_{A'B'}$ corresponds in fact to the metric η_{ab}:

$$\eta_{ab} = \epsilon_{AB}\epsilon_{A'B'}. \tag{2.28}$$

12. The correspondence between Hermitian spinors and tensors can be made rigorous by means of the *Infeld-van der Waerden symbols*, which establish a one-to-one correspondence between Hermitian spinors with n primed and n unprimed indices, and tensors of valence n; in this process each tensor index a is replaced by a pair of spinor indices AA'. For example, the correspondence between a vector V^a and a spinor $V^{AA'}$ is given by

$$\begin{aligned} V^{AA'} &\equiv V^a \sigma_a{}^{AA'}, \\ V^a &\equiv V^{AA'} \sigma^a{}_{AA'}. \end{aligned} \tag{2.29}$$

For more properties of the mixed spinor-tensor symbols $\sigma_a{}^{AA'}$ see [5], [28]. For simplicity, we will omit writing these symbols for the remaining of this thesis.

13. We mentioned in property (3) that antisymmetric spinors simplify. They do so with the help of the skew tensor ϵ_{AB}, as follows: a skew pair of indices can be removed as an ϵ spinor with a contraction on the removed indices:

$$S_{\ldots[AB]\ldots} = \frac{1}{2}\epsilon_{AB} S_{\ldots C}{}^{C}{}_{\ldots} \tag{2.30}$$

From this point of view, any spinor can be reduced to a combination of ϵ spinors and symmetric spinors. The same property holds for complex conjugate spinors as well. This, together with property (3), and the fact that spinor indices only take two values, shows that spinor calculus is much simpler than tensor calculus.

14. An example of interest that will be used in section 3.3 is a valence two skew tensor, S_{ab}. Such a tensor can be written as:

$$S_{ab} = S_{AA'BB'} = S_{ABA'B'} = S_{AB}\epsilon_{A'B'} + \overline{S}_{A'B'}\epsilon_{AB}, \tag{2.31}$$

where S_{AB} and $\overline{S}_{A'B'}$ are symmetric spinors, called the *anti-self-dual* (a.s.d.) and *self-dual* (s.d.) parts of S_{ab}, respectively, satisfying [12]:

$${}^*T_{ab} = -iT_{ab}, \text{ if } T_{ab} = S_{AB}\epsilon_{A'B'}, \tag{2.32}$$

and

$${}^*T_{ab} = iT_{ab}, \text{ if } T_{ab} = \overline{S}_{A'B'}\epsilon_{AB}. \tag{2.33}$$

In a Lorentzian space-time, S_{AB} and $\overline{S}_{A'B'}$ are related by the complex conjugation anti-isomorphism. In general, a complex space-time and a four complex-dimensional Riemannian manifold cannot be distinguished, which allows the following property to be valid in both types of spaces.

The arena for twistors, as it will be shown soon, is a complexified compactified Minkowski space-time. One can define an operation of complex conjugation in complexified space-times, but this map is not invariant under general holomorphic coordinate transformations in a complex space [28]. In this case, a real quantity is replaced by its complex conjugate, but a pair of complex conjugate quantities $(\rho, \overline{\rho})$ is replaced by independent complex quantities $(\rho, \widetilde{\rho})$.

15. The dual of a skew two-tensor S_{ab} is given by:

$$ {}^*S_{ab} = \frac{1}{2}\varepsilon_{ab}{}^{cd}S_{cd}, \tag{2.34} $$

where ε_{abcd} is a completely skew four-tensor. The spinor version of ε_{abcd} is:

$$ \varepsilon_{abcd} = \epsilon_{AA'BB'CC'DD'} = \epsilon_{ABCDA'B'C'D'}, \tag{2.35} $$

which can be simplified by using property (13) as:

$$ \varepsilon_{abcd} = i\left(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}\right). \tag{2.36} $$

Raising the last two indices, we obtain:

$$ \varepsilon_{ab}{}^{cd} = i\left(\delta_A^{\ C}\delta_B^{\ D}\delta_{A'}^{\ D'}\delta_{B'}^{\ C'} - \delta_A^{\ D}\delta_B^{\ C}\delta_{A'}^{\ C'}\delta_{B'}^{\ D'}\right), \tag{2.37} $$

which, used in (2.34), leads to:

$$ {}^*S_{ab} = -iS_{AB}\epsilon_{A'B'} + iS_{A'B'}\epsilon_{AB}. \tag{2.38} $$

16. We end this section by introducing a brief description of the spinor connection. A spinor field α^A defines a null a.s.d. skew vector (with a sign ambiguity) [12]:

$$F_{ab} = F_{AB}\epsilon_{A'B'} + \overline{F}_{A'B'}\epsilon_{AB}, \tag{2.39}$$

where F_{AB}, and $\overline{F}_{A'B'}$ are symmetric. By using property (3) we can factorize both spinors and write:

$$F_{ab} = \alpha_A\alpha_B\epsilon_{A'B'} + \overline{\alpha}_{A'}\overline{\alpha}_{B'}\epsilon_{AB}. \tag{2.40}$$

The Levi-Civita connection ∇_a of the Minkowski space $\mathbb{M}$ extends uniquely for null a.s.d. skew two-vectors to define a connection $\nabla_{AA'}$ on the spin bundles, provided:

$$\nabla_{AA'}\epsilon_{BC} = 0 = \nabla_{AA'}\epsilon_{B'C'}. \tag{2.41}$$

All these properties seem to point to the fact that spinor calculus is indeed much simpler than tensor calculus. Nonetheless, in section 3.3 we will want to write our final results in tensor language, for which physical intuition is much better developed.

2.2 THE CONFORMAL GROUP $C(1,3)$

One of the main features of twistor theory is that it is a conformal theory. This section shows that the conformal character arises naturally in spinor calculus, and consequently, becomes a natural part of twistor theory.

We shall define first what is meant by a *conformal map*: a map of the Minkowski space-time $\mathbb{M}$ to itself which preserves its conformal structure, that is sends the metric g_{ab} to

$$\widetilde{g}_{ab} = \Omega^2 g_{ab} \tag{2.42}$$

for some nowhere zero smooth function Ω.

We should mention here that $(\mathbb{M}, g_{ab})$ and $(\mathbb{M}, \widetilde{g}_{ab})$ have identical causal structures if and only if g_{ab} and $\widetilde{g}_{ab}$ are related by a conformal transformation [35]. The conformal structure of a space-time is in fact the null cone structure of that space-time.

In addition to all the spinor quantities defined in the previous section, one natural step in constructing the spinor calculus is to find the analogue of the Lie derivative from tensor calculus, that is to find an expression for the Lie derivative of a spinor α^A in the direction of a vector field X^a.

It can be shown that this is possible only for *conformal Killing vectors* X which satisfy:

$$\mathcal{L}_X g_{ab} = k g_{ab}, \tag{2.43}$$

for constant k, and indices $a, b = 0, 1, 2, 3$. Here $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector X.

(2.43) can be written as [12]:

$$\nabla_{(a} X_{b)} = \frac{1}{2} k g_{ab}, \tag{2.44}$$

with general solution of the form:

$$X_a = p_a - M_{ab} x^b + D x_a + [2(q \cdot x) x_a - q_a (x \cdot x)], \tag{2.45}$$

where p_a, $M_{ab} = -M_{ba}$, D and q_a are constants of integration.

The Killing vectors generate the *conformal group* $C(1, 3)$. From (2.45) we can see that $C(1, 3)$ is fifteen-dimensional, depending on the following parameters:

- ten of them, p_a and M_{ab}, generate the *Poincaré group* which is given by the semidirect sum of the translations p_a and the Lorentz transformations M_{ab}:

$$x_a \longmapsto M_{ba} x^b + p_a = -M_{ab} x^b + p_a. \tag{2.46}$$

As seen in section 2.2, the Lorentz transformations preserve the metric g_{ab}; the translations p_a act on x_a as:

$$x_a \longmapsto x_a + \xi_a,$$

where ξ_a is a constant. The commutation relations of the generators of the Poincaré group will be presented in detail in Chapter 3. As the full symmetry group of relativistic field theories, the representations of the Poincaré group describe all elementary particles and is therefore of major importance.

- D defines a dilation, sending

$$x_a \mapsto \rho x_a, \tag{2.47}$$

  for $\rho > 0$;

- four of them, q_a, define the *special conformal transformations*.

If the meaning of p_a, M_{ab} and D is obvious, that is not the case with the special conformal transformations. To determine their significance, set all the parameters equal to zero, except q_a, in (2.45). We obtain the equation:

$$X_a = \frac{\partial x_a}{\partial s} = 2(q \cdot x)x_a - q_a(x \cdot x), \tag{2.48}$$

with solutions [12]:

$$x_a(s) = \frac{x_a(0) - sq_a\Delta(0)}{1 - 2s(q \cdot x(0)) + s^2(q \cdot q)\Delta(0)}, \tag{2.49}$$

where $\Delta = x_a x^a = x \cdot x$.

Note that we obtain infinite values of X_a at the zeroes of the quadratic denominator. This suggests introducing some points at infinity in Minkowski space, thus compactifying it. The role of the special conformal transformations is to interchange the points at infinity with finite points of $\mathbb{M}$.

To describe the points at infinity, one considers first a six-dimensional real manifold with a flat metric of signature $(2, 4)$ which in coordinates (t, v, w, x, y, z) has the form:

$$ds^2 = dt^2 + dv^2 - dw^2 - dx^2 - dy^2 - dz^2. \tag{2.50}$$

The $\mathbb{O}(2, 4)$ null cone is then given by:

$$t^2 + v^2 - w^2 - x^2 - y^2 - z^2 = 0. \tag{2.51}$$

The group $\mathbb{O}(2, 4)$ preserves the form (2.50) and is 2-1 isomorphic to the conformal group $C(1, 3)$.

The compactified Minkowski space $\mathbb{M}^c$ consists of $\mathbb{M}$ with a null cone at infinity, and the special conformal transformations interchange this cone with the null cone of the origin [12].

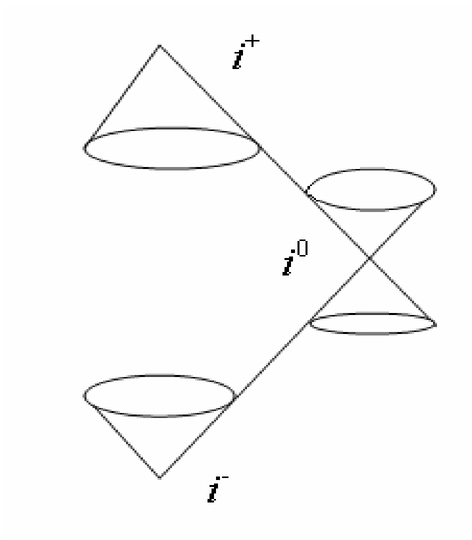

Figure 2.2: The null cones of the
origin and infinity.

Although we started this section with the apparent goal of defining a spinor Lie derivative, the real purpose was to show that the conformal group arises naturally in spinor theory. Like the Lorentz group, the conformal group is not connected either. The component of interest is the one that contains the identity, denoted by $C_+^\uparrow(1,3)$, doubly covered by $\mathbb{SO}(2,4)$.

2.3 ELEMENTS OF TWISTOR THEORY

The programme of twistor theory is to provide a complete non-local description of the geometry of space-time by using twistors rather than points, and be able to recreate basic physics in this new language [27]. Twistor theory seems to be a perfect fit in this context, as it naturally brings in complex numbers required by quantum physics.

The spinor algebra presented in section 2.1 could not accomplish this goal by itself, twistor algebra needed to be introduced.

It is not entirely clear how one should proceed to describe the concept of a twistor, for there are many ways to do so [11]:

- Geometrically, a (null) twistor can be described as an entire light ray (the "life" of a photon: its past, present, and future). A space-time event E will then be thought of as the family of light rays that pass through E, with an S^2 topology. This family of light rays is called a *celestial sphere*.
- Twistors can also be defined in terms of physical quantities characterizing the classical system of zero-rest-mass, such as (null) momentum p^a, and angular momentum M_{ab}. In this approach, twistors transform in a natural way under the group $\mathbb{SU}(2,2)$, and in particular under the Poincaré group. Twistors can also be defined as elements of the natural representation space $\mathbb{C}^4$ for $\mathbb{SU}(2,2)$, via the following covering maps:

$$\mathbb{SU}(2,2) \xrightarrow{2:1} \mathbb{SO}(2,4) \xrightarrow{2:1} C_{+}^{\uparrow}(1,3). \tag{2.52}$$

- Twistors can be viewed as solutions to a differential equation, called *twistor equation*.
- From another geometric point of view, the locations of twistors can be described in terms of the geometry of a three-dimensional complex projective space, as totally null 2-surfaces, called *α-planes*.

2.3.1 Complexified Minkowski Space-Time

For a complete description of twistors we will need an upgrade of the Minkowski space time, namely the *complexified compactified Minkowski space*, $\mathbb{CM}^c$. We discussed briefly the compactification of $\mathbb{M}$, denoted $\mathbb{M}^c$, in section (2.2).

$\mathbb{CM}$ is a four-dimensional complex manifold, $\mathbb{C}^4$, endowed with a non-degenerate complex bilinear form η, such that:

$$\eta\left(z,w\right) \equiv z^0 w^0 - z^1 w^1 - z^2 w^2 - z^3 w^3 = z_a w^a, \tag{2.53}$$

where $z = (z^0, z^1, z^2, z^3)$ and $w = (w^0, w^1, w^2, w^3)$ are arbitrary four-complex dimensional vectors.

As in the real case, to each vector z^a in $\mathbb{C}^4$ we can attach a matrix $z^{AA'}$:

$$z^a \longmapsto z^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} z^0 + z^3 & z^1 + iz^2 \\ z^1 - iz^2 & z^0 - z^3 \end{pmatrix},$$

but this matrix is not Hermitian any longer.

2.3.2 The Twistor Equation

One way in which twistors arise naturally in $\mathbb{CM}$ is as solutions of a differential equation, called *twistor equation*:

$$\nabla^{A'(A}\zeta^{B)} = 0. \tag{2.54}$$

Here $\nabla^{A'A}$ denotes the spinor covariant derivative from equation (2.41).

Twistor theory is a conformal theory. This is derived from the fact that (2.54) is invariant under a conformal rescaling of the metric tensor, and of the epsilon spinor [12]:

$$\widetilde{g}_{ab} = \Omega^2 g_{ab} \quad \text{and} \quad \widetilde{\epsilon}_{AB} = \Omega\epsilon_{AB}.$$

The general solution of (2.54), depending on the point $x \in \mathbb{CM}$, has the form:

$$\zeta^A(x) = \omega^A - ix^{AA'}\pi_{A'}, \tag{2.55}$$

where ω^A is a constant of integration, and $\pi_{A'}$ is a constant associated with this specific solution. $x^{AA'}$ is the spinor version of the position vector x^a with respect to some origin.

Note that the solutions ζ^A are completely determined by the four complex components of ω^A and $\pi_{A'}$ in a spin-frame at the origin.

The pair of spinor fields $(\omega^A, \pi_{A'})$ is called a *twistor* and is usually denoted by Z^α, with

$$Z^\alpha = (Z^0, Z^1, Z^2, Z^3) = (\omega^0, \omega^1, \pi_{0'}, \pi_{1'}). \tag{2.56}$$

The collection of all twistors determines a four-dimensional complex vector space, called *twistor space*, and denoted by $\mathbb{T}$.

The four complex components of Z^α completely determine the solutions $\zeta^A(x)$. ζ^A is called the *spinor field associated with the twistor* Z^α.

We can think of a twistor as a pair of spinors related by a differential equation, or as a nonzero four-dimensional complex vector.

Geometrically, the location of the twistor Z^α is given by the vanishing of the associated spinor ζ^A. This gives the equation:

$$\zeta^A(x) = 0 \Longrightarrow \omega^A = ix^{AA'}\pi_{A'}. \tag{2.57}$$

Since in spinor theory each equation is accompanied by its complex conjugate, we can also define a *complex conjugate twistor equation*:

$$\nabla^{A(A'}\varphi^{B')} = 0, \tag{2.58}$$

with solution

$$\varphi^{A'}(x) = \varsigma^{A'} - ix^{A'A}\nu_A. \tag{2.59}$$

In this case, the pair of spinors $(\nu_A, \varsigma^{A'})$ determines a *dual twistor*, W_α, and the collection of all dual twistors is called the *dual twistor space*, $\mathbb{T}^*$.

2.3.3 Twistor Pseudonorm

The twistor algebra has many interesting properties, but here we will only focus on aspects that will be used in this work. One such aspect is related to defining the norm of a twistor:

$$Z^\alpha \overline{Z}_\alpha = \omega^A \overline{\pi}_A + \pi_{A'}\overline{\omega}^{A'} = \omega^0\overline{\pi}_0 + \omega^1\overline{\pi}_1 + \pi_{0'}\overline{\omega}^{0'} + \pi_{1'}\overline{\omega}^{1'}, \tag{2.60}$$

where we used that the conjugate of $Z^\alpha = (\omega^A, \pi_{A'})$ is given by $\overline{Z}_\alpha = \left(\overline{\pi}_A, \overline{\omega}^{A'}\right)$.

By introducing new variables $(w, x, y, z) \in \mathbb{C}^4$ via the relations [11]:

$$\omega^0 = w + y,\ \omega^1 = x + z,\ \pi_{0'} = w - y,\ \pi_{1'} = x - z, \tag{2.61}$$

(2.60) becomes:

$$\frac{1}{2}Z^\alpha \overline{Z}_\alpha = w\overline{w} + x\overline{x} - y\overline{y} - z\overline{z}. \tag{2.62}$$

$Z^\alpha \overline{Z}_\alpha$ is called the *(pseudo)norm* of the twistor Z^α.

The following quantity is called the *helicity* of the twistor Z^α:

$$\Sigma = \frac{1}{2}Z^\alpha \overline{Z}_\alpha. \tag{2.63}$$

It can be shown that the twistor pseudonorm is conformal invariant and is a point independent property of the twistor space [28].

From (2.62), the helicity can be seen to be a quadratic Hermitian form of signature $(2,2)$, and so it is preserved by the group $\mathbb{U}(2,2)$. There is another group of interest, in fact a subgroup of $\mathbb{U}(2,2)$, which, in addition to preserving the helicity Σ, also preserves the twistor epsilon tensor $\epsilon^{\alpha\beta\gamma\delta}$. This subgroup is $\mathbb{SU}(2,2)$, and for $U^{\alpha}_{\ \beta} \in \mathbb{SU}(2,2)$ we have :

$$U^{\alpha}_{\ \rho}U^{\beta}_{\ \sigma}U^{\gamma}_{\ \mu}U^{\delta}_{\ \nu}\epsilon^{\rho\sigma\mu\nu} = \epsilon^{\alpha\beta\gamma\delta}. \tag{2.64}$$

This condition is equivalent to $\det(U^{\alpha}_{\ \beta}) = \pm 1$. As mentioned in (2.52), $\mathbb{SU}(2,2)$ is very important as it is locally isomorphic with the conformal group $C(1,3)$, and 4-1 homomorphic with $C^{\uparrow}_{+}(1,3)$. It turns out that twistors form a 4-1 representation space for $C^{\uparrow}_{+}(1,3)$.

Based on the sign of the helicity, twistors can be classified as:

- *null*, if $\Sigma = 0$. This defines the space of *null twistors*, $\mathbb{N}$.
- *right-handed*, if $\Sigma > 0$. This defines the top half $\mathbb{T}^{+}$ of the twistor space $\mathbb{T}$.
- *left-handed*, if $\Sigma < 0$. This defines the bottom half $\mathbb{T}^{-}$ of $\mathbb{T}$.

The case when the helicity is equal to zero is of particular interest. For a fixed twistor, ω^{A} and $\pi_{A'}$ are constant spinors; equation (2.57) can then be regarded as an equation for $x^{AA'}$. The solution of this equation is in general complex, and is given by

$$\gamma_{Z} : x^{AA'} = x^{AA'}(0) + \lambda^{A}\pi^{A'}, \tag{2.65}$$

where λ^{A} is an arbitrary spinor and $x^{AA'}(0)$ is a particular solution. Since the Minkowski space is an affine space, we can adjust the origin such that the particular solution is in fact the solution at the origin.

If real solutions exist, then $x^{AA'} = \overline{x}^{AA'}$, and we obtain that:

$$Z^{\alpha}\overline{Z}_{\alpha} = \omega^{A}\overline{\pi}_{A} + \pi_{A'}\overline{\omega}^{A'} = ix^{AA'}\pi_{A'}\overline{\pi}_{A} - i\overline{x}^{A'A}\overline{\pi}_{A}\pi_{A'} = 0. \tag{2.66}$$

We see that real points can only exist in the region of the twistor space of zero helicity.

It can be shown that if (2.66) holds and $\pi_{A'} \neq 0$, the solution space of (2.57) in $\mathbb{M}$ is a null geodesic for real values of r [12]:

$$x^{AA'} = x^{AA'}(0) + r\overline{\pi}^A \pi^{A'}. \tag{2.67}$$

If $\pi_{A'} = 0$, the twistor $(\omega^A, 0)$ can be regarded as a twistor at infinity, lying in the compactification of the Minkowski space. This twistor is denoted by $I_{\alpha\beta}$ and is represented by the matrix [28]:

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix}. \tag{2.68}$$

Its dual (and twistor complex conjugate) is:

$$I^{\alpha\beta} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.69}$$

This is one other way of obtaining the compactification of the complexified Minkowski space, by adding a twistor at infinity.

The infinity twistors are objects which break the conformal invariance: the conformal group $\mathbb{SU}(2,2)$ acts on the twistor space $\mathbb{T} \approx C^4 \setminus \{0\}$, but only the Poincaré group (which is a subspace of $\mathbb{SU}(2,2)$) preserves $I^{\alpha\beta}$ [13].

2.3.4 α-planes and β-planes

The locus of a twistor Z^α in $\mathbb{CM}$ is given by the region in which its associated spinor field ζ^A vanishes [11], leading to the equation:

$$\omega^A = ix^{AA'}\pi_{A'}.$$

The solution of this equation is described in (2.65). Since λ^A varies, we obtain a family of vectors $x^{AA'}$ passing through $x^{AA'}(0)$. Their endpoints determine a complex two-plane with tangent vectors of the form

$$v^a = \lambda^A \pi^{A'}, \tag{2.70}$$

for fixed $\pi^{A'}$ and varying λ^A.

One can easily show that these vectors are null:

$$v_a v^a = \left(\lambda^A \lambda_A\right) \left(\pi^{A'} \pi_{A'}\right) = 0,$$

and mutually orthogonal:

$$v^a w_a = (\lambda^A \mu_A)(\pi^{A'} \pi_{A'}) = 0.$$

This last relation also tells us that the metric η this complex two-plane inherits from the Minkowski space is null, since:

$$\eta\,(v, w) = \eta_{ab} v^a w_b = v^a w_a = 0. \tag{2.71}$$

It follows that the locus of the twistor Z^α is a null two-plane in complexified Minkowski space. Such a plane consists of all the endpoints of the complex vectors $\lambda^A \pi^{A'}$ originating from the point $x^{AA'}(0)$, and is called an *α-plane*. α-planes are totally null two-planes that are self-dual, in the sense that the two form that can be associated to any two vectors in the plane satisfies (2.34) [28].

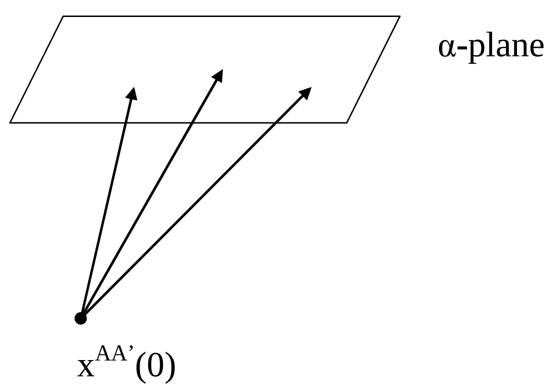

Figure 2.3: The α-plane is determined by the endpoints of the vectors corresponding to the solutions of the null twistor equation.

Similarly, the location of a dual twistor W_α in $\mathbb{CM}$ is a null two-plane, called a *β-plane*, which has the property of being anti-self-dual. By setting $\varphi^{A'}(x)$ equal to zero, we obtain the following equation for $x^{A'A}$:

$$\varsigma^{A'} = ix^{A'A}\nu_A, \tag{2.72}$$

with solution

$$x^{A'A} = x_0^{A'A} + \rho^{A'}\nu^A, \tag{2.73}$$

where $\rho^{A'}$ varies and ν^A is fixed.

It is very important to note that in complex Minkowski space, there are two distinct families of totally null two-planes: the α-planes corresponding to Z^α twistors, and the β-planes corresponding to dual twistors W_α. This will be of interest when we discuss the interpretation of the twistor space as a quadric in $\mathbb{CP}^5$.

In the case when $\pi_{A'} = 0$, there is no finite locus of the twistor Z^α. If, additionally, ω^A is nonzero, then the locus of the twistor Z^α can be interpreted as a generator of the null cone at infinity [28].

2.3.5 Projective Twistor Space

We saw from equation (2.65) that a twistor $Z^\alpha = (\omega^A, \pi_{A'})$ determines an α-plane; it is obvious that a multiple of Z^α will determine the same α-plane. Viceversa, an α-plane determines a twistor, but not uniquely, only up to a scale factor λ:

$$(\omega^A, \pi_{A'}) \sim (\lambda\omega^A, \lambda\pi_{A'}), \tag{2.74}$$

for $\lambda \in \mathbb{C} \setminus \{0\}$. This freedom is not a shortcoming of twistor theory, in fact it is of interest when one brings in quantum physics.

Equation (2.74) states that an α-plane is an equivalence class of twistors $[Z^\alpha]$, called *projective twistor*. The set of all such equivalence classes (α-planes) determine the *projective twistor space*, $\mathbb{PT}$, in which the α-planes are represented by points.

The extra information contained in the twistor space $\mathbb{T}$ compared to $\mathbb{PT}$ is the choice of scale for the spinor $\pi_{A'}$ associated to a particular α-plane.

Since the twistors Z^α are defined in $\mathbb{C}^4$ and obey the equivalence relation (2.74), it follows that the projective twistor space $\mathbb{PT}$ can be represented by a three-dimensional complex projective space.

In general, we will use the notation Z^α even if we refer to the equivalence class $[Z^\alpha]$, but in that case the components of Z^α in (2.56) will be written between square brackets and referred to as "homogeneous coordinates" of the corresponding point in $\mathbb{PT}$.

Similarly, β-planes correspond to points in a *dual projective twistor space*, denoted $\mathbb{PT}^*$, also represented by a $\mathbb{CP}^3$.

In the projective twistor space, the norm of a twistor is not well-defined any longer, but the sign of the norm can still be used to divide the projective twistor space into three regions, $\mathbb{PT}^+$, $\mathbb{PN}$, and $\mathbb{PT}^-$, corresponding to $\Sigma > 0$, $\Sigma = 0$, and $\Sigma < 0$, respectively.

2.3.6 Geometric Correspondences

We saw that points in $\mathbb{PT}$ correspond to α-planes, and from (2.67) we have that points in $\mathbb{PN}$ correspond to null geodesics. If an α-plane contains a real point, then it will contain the whole null geodesic given in (2.67).

Figure 2.4 describes some of the geometric correspondences mentioned in this section: for X and Y null twistors, their corresponding null geodesics, γ_X and γ_Y, meet at the point p. The points p and q are said to be null separated if there is a null geodesic γ joining them. Each point will be represented in $\mathbb{PT}$ by a projective line (L_p and L_q), and the null geodesic γ joining p and q in $\mathbb{CM}$, becomes the intersection point of L_p and L_q in $\mathbb{PT}$. Each null twistor is represented by a point in $\mathbb{PN}$, and the point at the intersection of the null geodesics γ_X and γ_Y is represented by a line passing through the points corresponding to the two null twistors X and Y.

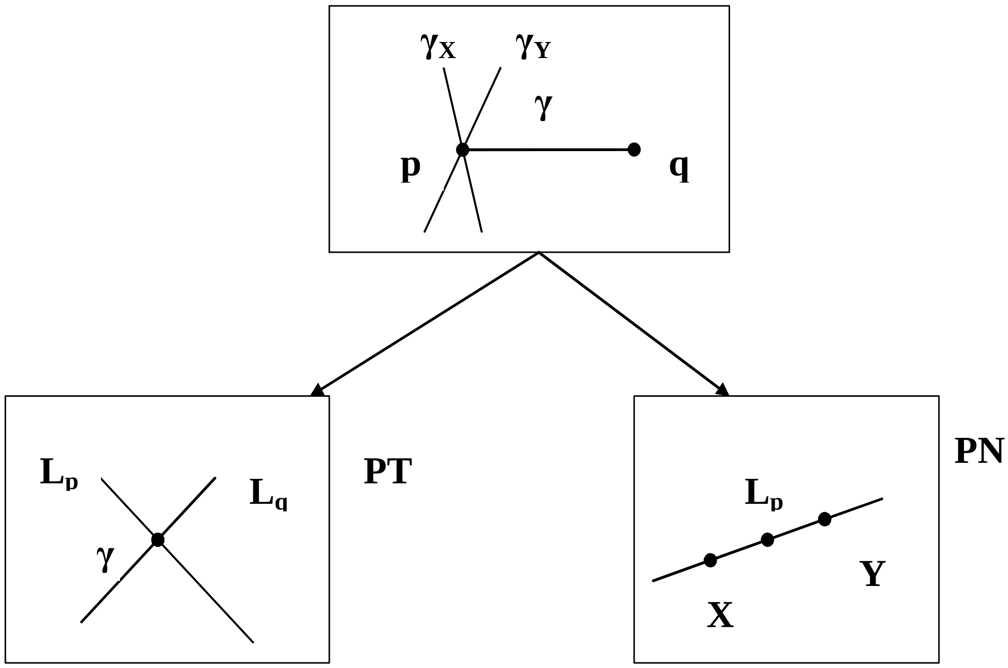

Figure 2.4: Geometric correspondences in the complexified Minkowski space, $\mathbb{PT}$ and $\mathbb{PN}$.

Other geometric correspondences can be made as follows: if we interpret (2.57) as an equation with $x^{AA'}$ fixed and solve for $(\omega^A, \pi_{A'})$, we obtain that

$$\omega^A = ix^{AA'}\pi_{A'},$$

with $\pi_{A'}$ arbitrary, which defines a complex two-plane.

Factorization by the equivalence relation (2.74) leads to a $\mathbb{CP}^1$, with the two-sphere topology. The fixed space-time point x determines a Riemann sphere in $\mathbb{PT}$. If x is real, this sphere lies entirely in $\mathbb{PN}$.

We obtain that a *complex* space-time point corresponds to a sphere in $\mathbb{PT}$, and a *real* space-time point corresponds to a sphere in $\mathbb{PN}$.

2.3.7 Space-Time Points as Intersection of Twistors

Consider two null twistors Z_1^α and Z_2^α with their respective null geodesics, γ_{Z_1} and γ_{Z_2} defined as in (2.67). Since Z_1^α and Z_2^α are null, they satisfy

$$Z_1^\alpha \overline{Z}_{1\alpha} = 0 = Z_2^\alpha \overline{Z}_{2\alpha}. \tag{2.75}$$

The condition for these geodesics to meet at a point $P \in \mathbb{M}$ is [12]:

$$Z_1^\alpha \overline{Z}_{2\alpha} = 0. \tag{2.76}$$

This is called *incidence of twistors* condition.

Since real points can only exist in $\mathbb{N}$, we may define a point in the real Minkowski space $\mathbb{M}$ by the intersection of two null geodesics. From (2.75) and (2.76) it follows that any nontrivial linear combination of the null twistors Z_1^α and Z_2^α:

$$Z^\alpha = \lambda Z_1^\alpha + \mu Z_1^\alpha, \tag{2.77}$$

for $(\lambda, \mu) \in \mathbb{C}^2 \setminus (0,0)$, will also be null and will define a null geodesic, γ_Z, which intersects the other two geodesics at the same intersection point, $P \in \mathbb{M}$. Since λ and μ are arbitrary, (2.77) defines a family of null geodesics intersecting at P, that is it defines the null cone of the point P.

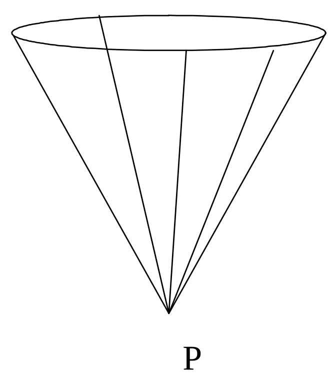

Figure 2.5: Points are represented by intersections of null twistors.

This null cone is a two-dimensional subspace of the twistor space $\mathbb{T}$, lying entirely in $\mathbb{N}$, or can be thought of as a projective line L_P lying in $\mathbb{PN}$.

The family of null geodesics corresponding to the null twistor Z^α in (2.77), intersecting at the point P, can be interpreted as actually representing the point P.

In general, any two-dimensional subspace of $\mathbb{T}$ can be interpreted as a point in Minkowski space, but the point is not real unless Z_1^α and Z_2^α are null and orthogonal [28].

Consider now the lines in $\mathbb{PT}$ which do not lie entirely in $\mathbb{PN}$. An arbitrary line passing through the two points Z_1^α and Z_2^α is given by:

$$P^{\alpha\beta} = Z_1^\alpha Z_2^\beta - Z_2^\alpha Z_1^\beta. \tag{2.78}$$

The point P corresponds thus (up to proportionality) to a simple skew 2-index twistor $P^{\alpha\beta}$, satisfying:

$$P^{\alpha\beta} = P^{[\alpha\beta]}, \text{ and } \; P^{[\alpha\beta}P^{\gamma]\delta} = 0. \tag{2.79}$$

Finally, for $P^{\alpha\beta}$ to represent a *finite* point of $\mathbb{M}$, it is also required that

$$P^{\alpha\beta} I_{\alpha\beta} \neq 0, \tag{2.80}$$

where $I_{\alpha\beta}$ is one of the infinity twistors defined in (2.68).

It has been shown thus that twistor geometry can be used to replace entirely the pointwise approach to the structure of space-time [28].

This concludes our presentation of the basic properties of spinors and twistors.

3.0 NULL DECOMPOSITION OF CONFORMAL ALGEBRAS

In chapter two we focused on a more mathematical description of twistors. In order to provide a better understanding of the motivation for studying conformal algebras, we start by describing how a twistor can be represented in terms of physical quantities as a classical zero-rest-mass (z.r.m.) system. Such a system is described by the total momentum p_a, which is a future pointing null vector field, and by the skew angular momentum $M^{ab} = -M^{ba}$ with respect to some choice of origin in $\mathbb{CM}$.

The following proposition establishes the connection between twistor quantities and the physical quantities p_a and M^{ab} [26].

Proposition 1. *A pair $\{p_a, M^{ab}\}$ represents a z.r.m. system if and only if there exists a pair of spinors $(\omega^A, \pi_{A'})$ such that*

$$p_a = \overline{\pi}_A \pi_{A'}, \tag{3.1}$$

and

$$M^{ab} = i\omega^{(A}\overline{\pi}^{B)}\epsilon^{A'B'} - i\overline{\omega}^{(A'}\pi^{B')}\epsilon^{AB}, \tag{3.2}$$

where $\overline{\pi}_A$ is the complex conjugate of $\pi_{A'}$, and $\overline{\omega}_{A'}$ is the complex conjugate of ω_A.

The position dependence of these tensors with respect to some origin is described by [28]:

$$\begin{aligned} p_a &= p_a(0), \\ M^{ab} &= M^{ab}(0) - x^a p^b + x^b p^a. \end{aligned} \tag{3.3}$$

Same quantities appeared as the generators of the conformal group, where we interpreted p_a as the translations, and M^{ab} as the Lorentz transformations.

Single twistors can be used to construct massive systems; in [11], [29], a twistor model of hadrons has been developed. It becomes clear that twistors can be viewed to be more primordial than points and particles; they represent dual objects which can be used to describe both matter *and* the geometry of space-time.

In addition to the operators associated to the momentum p_a and angular momentum M_{ab}, there is one other natural operator which arises in twistor theory, namely the squared-mass operator, p^2 [11].

We saw in chapter two that twistor theory is a conformal theory. The action of the conformal group as a whole on a quantum mechanical system changes the mass of that system, but its Poincaré subgroup commutes with the squared-mass operator. This seems to suggest that we should be interested in the Poincaré subalgebra of the system, instead of the full conformal one.

In 1939 [39], Wigner constructed a maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a particle invariant. This subgroup is called *Wigner's little group*.

Wigner used the little groups of the Poincaré group to discuss space-time symmetries of relativistic particles, and established connections between particle theory and representations of Lie groups and Lie algebras. The mass $m = \sqrt{p^2}$ is a Casimir invariant of the Poincaré group; irreducible representations of the Poincaré Lie algebra can then be classified according to whether the mass is zero or positive. The disadvantage of the Poincaré group is that it is not semisimple, and many techniques are developed for semisimple algebras.

An extension of the Poincaré group, the *conformal group*, is semisimple, but, unfortunately, it changes the mass of the quantum system it acts on.

The study of the null decomposition of conformal algebras is done in the context of broken conformal invariance, by fixing the squared-mass operator p^2. We consider here the enveloping algebra of the conformal algebra of $\mathbb{SO}(p+1, q+1)$ of an n-dimensional space-time with $p + q = n$. The associative property of the enveloping algebra is used in working with quantum operators.

To this enveloping algebra, we adjoin the element p^{-2}, and then extract the subalgebra which commutes with p^2. This subalgebra, denoted $\mathcal{R}$, in addition to the generators of the Poincaré Lie algebra, will depend on a vector operator R_a, which is shown to commute with the mass of the system and with the translations p_a.

The operator R_a preserves the relevant information when the conformal invariance is broken; it is constructed from the generators of the conformal group (the translations p_a, the Lorentz transformations M_{ab}, the dilation operator D, and the special conformal transformations q_a), but in the process of symmetry breaking the operator D is removed.

The construction of the R_a allows reducing the analysis of the representations of the conformal algebras to a discussion of the representations of a "little algebra" in analogy to Wigner's little group. The difference is that the algebra obtained is not a finite-dimensional Lie algebra, as the commutator of the R_a operator with itself contains a cubic term.

One other important aspect of this chapter is the construction of the Casimir invariants of these conformal algebras. These operators generate the center of the universal enveloping algebra, commuting with all the generators of the algebra. Constructing them for various dimensionalities has proved to be a challenging process. In the general case, due to the complexity of the expressions involved, we could construct only one of them [22]; for five dimensions and lower, we had to exploit the particularities of each dimension to obtain all the Casimir invariants, as follows:

- In five dimensions, we use spinor properties and techniques, described in Section 2.1 of this thesis [21].
- In four dimensions [31] the Pauli-Lubanski tensor has been used; at this moment we anticipate that the same spinorial approach from the five-dimensional case can be used here as well, but this will make the object of future research.
- In three dimensions we use a standard quantum mechanical model, and the properties of the three-dimensional completely antisymmetric tensor ε_{abc}.
- In two dimensions we use the fact that any skew tensor can be written as a multiple of the skew tensor ε_{ab}.

3.1 BASIC COMMUTATION RELATIONS

We start by considering the general case of the conformal algebra of an n-dimensional affine space equipped with a metric of signature (p, q), where p and q are non-negative integers such that $p + q = n$. The algebra of this space is the Lie algebra of $\mathbb{SO}(p+1, q+1)$, with generators satisfying the commutation relation [11]:

$$[M_{AB}, M_{CD}] = g_{AC}M_{BD} - g_{BC}M_{AD} - g_{AD}M_{BC} + g_{BD}M_{AC}. \tag{3.4}$$

Here M_{AB} are skew, and the upper case Latin indices run from 0 to $n + 1$. The metric $g_{AB} = g_{BA}$ is a real symmetric metric of signature $(p + 1, q + 1)$.

We use a null decomposition of the metric g_{AB}, describing it by the $(n + 2) \times (n + 2)$ symmetric matrix:

$$g_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Throughout this chapter, the lower case Latin indices run from 1 to n. Explicitly, the entries of the matrix of the metric are given by:

$$g_{00} = g_{n+1n+1} = 0,\ g_{0a} = g_{a0} = g_{n+1a} = g_{an+1} = 0,\ g_{0n+1} = g_{n+10} = 1. \tag{3.6}$$

The $n \times n$-symmetric matrix $g_{ab} = g_{ba}$ is the metric of the n-dimensional space time, of signature (p, q).

By requiring that

$$M_{0a} = p_a,\ M_{n+1a} = q_a,\ M_{0n+1} = D, \text{and } M_{ab} = -M_{ba}, \tag{3.7}$$

the tensor $M_{AB} = -M_{BA}$ can be described by an $(n + 2) \times (n + 2)$ skew matrix:

$$M_{AB} = \begin{pmatrix} 0 & p_a & D \\ -p_b & M_{ab} & -q_b \\ -D & q_a & 0 \end{pmatrix}. \tag{3.8}$$

M_{ab} is itself an $n\times n$ skew matrix and represents the analogue of the angular momentum in n dimensions, p_a represents the translations, q_a the special conformal transformations, and D is the dilation operator.

The generators of the full conformal algebra are thus: p_a, q_a, M_{ab} and D.

Remark 2. *By a notation and language abuse we will refer to the conformal algebra described by the $(n+2)\times(n+2)$ skew matrix of the generators M_{AB} as an n-dimensional conformal algebra. In this approach it is the dimensionality of the space-time to which we associate this conformal algebra that is relevant, and for the general case the space-time is n-dimensional.*

From equations (3.4)-(3.8), we obtain the following basic commutation relations of the conformal algebra as:

$$\begin{aligned}
[p_a,p_b] &= 0, \quad [q_a,q_b]=0, \quad [p_a,q_b]=M_{ab}+g_{ab}D, \\
[M_{ab},p_c] &= g_{ac}p_b-g_{bc}p_a, \quad [M_{ab},q_c]=g_{ac}q_b-g_{bc}q_a, \\
[M_{ab},M_{cd}] &= g_{ac}M_{bd}-g_{bc}M_{ad}-g_{ad}M_{bc}+g_{bd}M_{ac}, \\
[D,p_a] &= -p_a, \quad [D,q_a]=q_a, \quad [D,M_{ab}]=0.
\end{aligned} \tag{3.9}$$

We are interested in the following subalgebras, satisfying the commutation relations (3.9):

- the full conformal algebra $\mathcal{C}$, spanned by the operators p_a, M_{ab}, D, and q_a, of real dimension $(n+1)(n+2)/2$;
- the subalgebra $\mathcal{D}$ of $\mathcal{C}$, spanned by p_a, M_{ab}, and D, of real dimension $(n^2+n+2)/2$; this is called the *Weyl algebra*;
- the subalgebra $\mathcal{P}$ of $\mathcal{C}$, spanned by p_a and M_{ab}, of real dimension $n(n+1)/2$; this is called the *Poincaré algebra*, or the *Euclidean algebra*, depending on the signature of the metric ($(1,3)$ for the Poincaré algebra, and $(4,0)$ for the Euclidean case).

Following the notation introduced in [31], we denote by $\mathcal{C}_e,\ \mathcal{D}_e,\ \mathcal{P}_e$ their corresponding enveloping algebras.

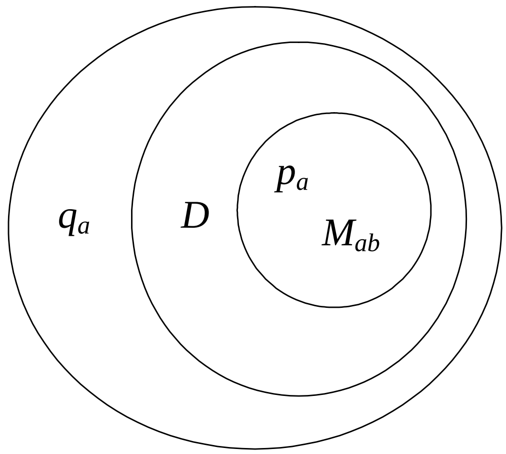

Figure 3.1: The conformal algebra,
Weyl algebra, and Poincare algebra.

Throughout this chapter we will use p^2 or $p \cdot p$ to denote $p_a p^a$, and also $p^{-2} = (p^2)^{-1} = (p \cdot p)^{-1} = (p_a p^a)^{-1}$. Adjoin to each of the above algebras the element p^{-2} and its powers $(p^{-2})^r$, where r is a positive integer, such that:

$$p^{-2}p^2 = p^2 p^{-2} = 1, \tag{3.10}$$

and

$$[p^{-2}, X] = -p^{-2}[p^2, X]p^{-2}, \tag{3.11}$$

for any X in $\mathcal{C}_e$. Note that p^{-2} is the inverse of p^2, where p^2 is the squared-mass operator of the system. The resulting algebras will be denoted by $\mathcal{C}_e\left(p^{-2}\right),\ \mathcal{D}_e\left(p^{-2}\right),\ \mathcal{P}_e\left(p^{-2}\right)$, so implicitly the representations that this work applies to are those of non-zero mass only. As mentioned earlier, the squared-mass operator p^2 is a Casimir invariant for $\mathcal{P}_e$.

3.2 THE N-DIMENSIONAL CASE

3.2.1 The $\mathcal{R}$-algebra

Recall that one of the main goals is to construct the operator R_a that will commute with the operator p^2.

Note that the commutation relation of p with the special conformal transformations q is determined by the full conformal algebra in (3.9), and this relation leads to the commutator:

$$[p^2, q_a] = p_a(2D - 1) - (M_{ab}p^b + p^b M_{ab}). \tag{3.12}$$

We want to construct an operator Q, built from the subalgebra $\mathcal{D}_e\left(p^{-2}\right)$, that will obey the same commutation relation with p as q does:

$$[p_a, Q_b] = M_{ab} + g_{ab}D.$$

This implies that the commutation relation with p^2 will also be the same:

$$[p^2, Q_a] = p_a(2D - 1) - (M_{ab}p^b + p^b M_{ab}) = [p^2, q_a].$$

Then by taking the difference $q_a - Q_a$ we should obtain the vector operator R_a which will commute with p^2, as desired.

The construction of the operator Q_a requires a few steps, described in the following. The first step consists in the introduction of a Hermitian operator constructed in $\mathcal{P}_e\left(p^{-2}\right)$ that behaves like a position operator:

$$2p^2 x_a = M_{ab}p^b + p^b M_{ab} = 2x_a p^2, \tag{3.13}$$

satisfying the relation:

$$x \cdot p = \frac{n-1}{2}. \tag{3.14}$$

From this relation we can see that x_a is not a full position operator.

Introduce the notation:

$$k = \frac{n-1}{2}. \tag{3.15}$$

By using the commutation relations (3.9), we can also write x_a as:

$$p^2 x_a = M_{ab} p^b + k p_a. \tag{3.16}$$

The operators x_a and p^2 commute since x_a lies in $\mathcal{P}_e\,(p^{-2})$.

By using (3.13) in (3.12) we obtain a new expression for the commutator of p^2 with q_a:

$$[p^2, q_a] = p_a(2D-1) - 2p^2 x_a. \tag{3.17}$$

Define also the analogue of the orbital angular momentum in n dimensions:

$$L_{ab} = x_a p_b - x_b p_a = -L_{ba}, \tag{3.18}$$

and the intrinsic spin:

$$S_{ab} = M_{ab} - L_{ab} = -S_{ba}. \tag{3.19}$$

It is now straightforward to verify that S_{ab} is orthogonal to p_a:

$$S_{ab} p^b = p^b S_{ab} = 0. \tag{3.20}$$

Finally, define a projected metric tensor:

$$h_{ab} = g_{ab} - p^{-2} p_a p_b. \tag{3.21}$$

Note that $p^a h_{ab} = h_{ab} p^a = 0$ and $h_a{}^a = n-1$. The commutation relations satisfied by these new operators can be found in appendix A.

The algebra $\mathcal{P}_e(p^{-2})$ can be written now in terms of the operators $x_a,\ p_a,\ S_{ab}$ (with $n-1,\ n$, and $(n-1)(n-2)/2$ degrees of freedom, respectively) as follows:

$$\begin{aligned}
[p_a, p_b] &= 0, \quad [x_a, p_b] = h_{ab}, \\
p^2[x_a, x_b] &= -S_{ab} - x_a p_b + x_b p_a, \\
[S_{ab}, p_c] &= 0, \quad\ \ [p^2, S_{ab}] = 0, \\
p^2[S_{ab}, x_c] &= S_{ac} p_b - S_{bc} p_a, \\
[S_{ab}, S_{cd}] &= h_{ac} S_{bd} - h_{bc} S_{ad} - h_{ad} S_{bc} + h_{bd} S_{ac}.
\end{aligned} \tag{3.22}$$

The next step is to pass to the Weyl algebra $\mathcal{D}_e(p^{-2})$, which allows us to define a full position operator:

$$p^2 y_a = p^2 x_a - p_a(D - l), \tag{3.23}$$

such that $[y_a, p_b] = g_{ab}$. Here l is a pure number to be determined, and D is the dilation operator with commutation relations described in (3.9).

This new operator, y_a, can be regarded as unconstrained if we think of D as being defined in terms of y_a by means of the relations $y \cdot p = 1 - D + k + l$ (see equations (3.27) and (3.28) below). Then y_a obeys the commutation relations:

$$[p^2, y_a] = -2p_a, \quad p^2[y_a, y_b] = -S_{ab}. \tag{3.24}$$

(Additional commutation relations can be found in the appendix A.)

Finally, consider the full conformal algebra $\mathcal{C}_e(p^{-2})$. We are ready to define the operator Q_a, from $\mathcal{D}_e(p^{-2})$, as follows:

$$Q_a = y^b S_{ab} + \alpha\,(y \cdot y)\,p_a + \gamma\,(y \cdot p)\,y_a. \tag{3.25}$$

We see that this new operator is made from M_{ab}, D and p_a only, with α and γ constants. When commuted with the p operator, we obtain that:

$$[p_a, Q_b] = S_{ab} - 2\alpha y_a p_b - \gamma y_b p_a + \gamma g_{ab}(1 - y \cdot p). \tag{3.26}$$

Requiring that this commutation relation is the same as $[p_a, q_b]$, we obtain the following values for the constants α, γ and l:

$$\alpha = -\frac{1}{2}, \ \gamma = 1, \ l = -k. \tag{3.27}$$

In addition to these, we also have:

$$y \cdot p = 1 - D. \tag{3.28}$$

Written in terms of the operator x_a, we have now that:

$$Q_a = x^b S_{ab} - \frac{1}{2}\,(x \cdot x)\,p_a - x_a D + \frac{1}{2} p^{-2} p_a \left(D^2 + D - k^2\right). \tag{3.29}$$

The newly defined operator Q satisfies:

$$\begin{aligned} [p_a, Q_b] &= M_{ab} + Dg_{ab}, \\ [p^2, Q_a] &= p_a(2D-1) - 2p^2 x_a, \\ 2p^2[Q_a, Q_b] &= -(n-3)(n-2)S_{ab} + 2S_{c[a}S^{cd}S_{b]d}. \end{aligned} \tag{3.30}$$

Note that (3.30ii) is the same as (3.17), as desired.

The final step is to define the operator R_a in $\mathcal{C}_e(p^{-2})$ to be:

$$R_a = q_a - Q_a, \tag{3.31}$$

with the key commutation relations

$$[p_a, R_b] = 0, \text{ and } [p^2, R_a] = 0.$$

This is the operator that preserves the information about the system when the conformal invariance is broken.

We obtain in this way the algebra which we will call the $\mathcal{R}$-*algebra*. It is generated by the operators R_a, p_a, and S_{ab}, satisfying the commutation relations:

$$\begin{aligned} [p_a, p_b] &= 0, \quad [R_a, p_b] = 0, \quad [S_{ab}, p_c] = 0, \\ p^2[S_{ab}, R_c] &= p^2 h_{ac} R_b - p^2 h_{bc} R_a - g_{ac}(R \cdot p)p_b + g_{bc}(R \cdot p)p_a, \\ [S_{ab}, S_{cd}] &= h_{ac}S_{bd} - h_{bc}S_{ad} - h_{ad}S_{bc} + h_{bd}S_{ac}, \\ 2p^2[R_a, R_b] &= [(n-3)(n-2) - 4(R \cdot p)]\, S_{ab} - 2S_{c[a}S^{cd}S_{b]d} \\ &\quad + (R^c S_{ac} + S_{ac}R^c)\, p_b - (R^c S_{bc} + S_{bc}R^c)\, p_a. \end{aligned} \tag{3.32}$$

The last relation is derived by a very lengthy calculation. In fact, most formulas in this chapter require significant work, and wherever the details of their derivation were not particularly important, we chose to present only final formulas.

The following theorem summarizes the results obtained in this section in the style of [31].

Theorem 3. *Define in $\mathcal{C}_e(p^{-2})$ the operator R_a by the formula:*

$$p^2 R_a = p^2 q_a - p^2 x^b S_{ab} + \frac{1}{2} p^2 (x \cdot x) p_a + p^2 x_a D - \frac{1}{2} p_a \left(D^2 + D - k^2\right).$$

Then R_a is translationally invariant:

$$[R_a, p_b] = 0.$$

Also, $p^4(R_a - q_a) \in \mathcal{D}_e$, and R_a has the following commutation relations:

$$\begin{cases} [D, R_a] = R_a, \\ p^2[S_{ab}, R_c] = p^2 h_{ac} R_b - p^2 h_{bc} R_a - g_{ac}(R \cdot p) p_b + g_{bc}(R \cdot p) p_a, \\ 2p^2[R_a, R_b] = \left[(n-3)(n-2) - 4(R \cdot p)\right] S_{ab} - 2S_{c[a} S^{cd} S_{b]d} \\ \qquad\qquad + \left(R^c S_{ac} + S_{ac} R^c\right) p_b - \left(R^c S_{bc} + S_{bc} R^c\right) p_a. \end{cases} \tag{3.33}$$

The freedom in choosing $R_a \in \mathcal{C}_e(p^{-2})$, obeying

$$[p^2, R_a] = 0, \quad p^4(R_a - q_a) \in \mathcal{D}_e,$$

is $R_a \to R_a + p^4 g_a$ where $g_a \in \mathcal{P}_e$, and $[D, g_a] = 5 g_a$.

Also, $C_e(p^{-2})$ is generated by D and the $\mathcal{R}$-algebra.

3.2.2 Casimir Invariants

Now that we constructed the $\mathcal{R}$-algebra, we will focus on finding its Casimir invariants. Recall that these are operators which commute with the generators of the algebra, hence with all the elements of the algebra. As shown in the previous section, the $\mathcal{R}$-algebra is generated by the operators R_a, p_a, and S_{ab}, such that all these generators commute with the squared-mass operator. In order to simplify the expressions, we can introduce two new operators:

$$S_a = R_b h^b{}_a, \tag{3.34}$$

with h_{ab} as in (3.21), and

$$S = R \cdot p = R_a p^a. \tag{3.35}$$

Note that $p^a S_a = S_a p^a = 0$ and $p^a S_{ab} = S_{ab} p^a = 0$. From now on, the vector operator R_a will be replaced by S_a, and the projected metric tensor h_{ab} will be used to raise and lower the indices of this new operator.

Although we replaced the operator R_a by S_a, we will still refer to the algebra constructed as the $\mathcal{R}$-algebra, since it satisfies the conditions we required, namely that all of its generators commute with p^2.

The commutation relations become:

$$\begin{aligned}
[S_{ab}, S] &= 0, \quad 2[S_a, S] = 2S^b S_{ab} - (n-2) S_a, \\
[S_{ab}, S_c] &= h_{ac} S_b - h_{bc} S_a, \\
[S_{ab}, S_{cd}] &= h_{ac} S_{bd} - h_{bc} S_{ad} - h_{ad} S_{bc} + h_{bd} S_{ac}, \\
p^2 [S_a, S_b] &= K S_{ab} - S_{c[a} S^{cd} S_{b]d},
\end{aligned} \tag{3.36}$$

where in the last relation we used the notation:

$$K = \frac{(n-3)(n-2)}{2} - 2S. \tag{3.37}$$

Note that by introducing these operators, S_a transforms like a vector with respect to the intrinsic spin S_{ab}, and the commutation relations are significantly simpler. The generators of the $\mathcal{R}$-algebra are now the operators p_a, S_{ab}, S_a, and S.

The following corollaries summarize our only general results on the Casimir invariants.

Corollary 4. *The operator:*

$$C_1 = S + \frac{1}{4} S_{ab} S^{ab}, \tag{3.38}$$

is a Casimir invariant of the algebra $C_e(p^{-2})$.

Corollary 5. C_1 *and* p^2 *are Casimir invariants for the* $\mathcal{R}$*-algebra.*

C_1 can be used to reduce the number of generators just to the operators p_a, S_a and S_{ab}, together with C_1 itself.

It is known that $\mathbb{SO}(p+1, q+1)$ with $p+q=n$ has rank $1+[\frac{n}{2}]$, this being the number of Casimir invariants as well [1], [6]. In principle, these invariants are given by the scalar operators formed from products of M_{AB} with itself, but in practice this observation is not very helpful in finding the other Casimir invariants, due to the complexity of the expressions obtained. Our attempts involved introducing yet another operator:

$$U = S_a S^a. \tag{3.39}$$

The commutation relations involving the operator U are:

$$[U, S_{ab}] = 0, \quad [U, S] = 0, \tag{3.40}$$

and

$$p^2 [U, S_a] = \left(\lambda \delta_a{}^b + \mu S_a{}^b + \nu S_a{}^c S_c{}^b + \rho S_a{}^d S_d{}^c S_b{}^c \right) S_b, \tag{3.41}$$

where

$$\begin{aligned} \lambda &= (n-2)\left(n-1+2C_1 - \tfrac{1}{2} S_{ab} S^{ab}\right), \\ \mu &= n(n-1) + 4C_1 - S_{ab} S^{ab}, \\ \nu &= 3n-4, \\ \rho &= 2. \end{aligned} \tag{3.42}$$

From (3.40) we see that U is a scalar with respect to S_{ab} and S, in the sense that it commutes with them. It is expected that U will be part of a Casimir invariant.

To produce such an invariant, the building blocks must be combinations of the operators U, C_1, and generators of the general form $S^a F_{ab}(S_{cd}) S^b$ and $G\left(S_{cd}\right)$. Here $F_{ab}(S_{cd})$ is a function of S_{cd} with a and b free indices (for example $S_{ad} S_c{}^d S_e{}^c S_b{}^e$), and $G\left(S_{cd}\right)$ is a scalar function of S_{cd} (for example $S_c{}^a S_b{}^c S_e{}^b S_a{}^e$).

If necessary, one can assume that $F_{ab}(S_{cd})$ is symmetric in the indices a and b, since the skew part allows $S^a F_{ab} S^b$ to be replaced by a commutator, which then can be assimilated into the G term. However, except for finite dimensions $n = p + q \leq 5$, we have not yet been able to construct the remaining Casimirs. The basic problem is the complexity of the relation (3.41).

3.3 THE 5-DIMENSIONAL CASE

We start by noting that the $\mathcal{R}$-algebra depending on the generators $S_{ab},\ S_a,\ S$ and p_a is valid in all dimensions. The one Casimir invariant we obtained for the general case can be used to eliminate S.

For $n = 5$, the Lie algebra of $\mathbb{SO}(p+1, q+1)$ has rank 3. This algebra will have three Casimir invariants, C_1 being one of them. In order to find the remaining ones we use the special characteristics of the dimension $n = 5$. This case is relevant for the groups $\mathbb{SO}(1,6)$, $\mathbb{SO}(2,5)$, and $\mathbb{SO}(3,4)$.

The space-time *is* five-dimensional, but since both S_a and S_{ab} are orthogonal to p^a, the problem is effectively reduced to a four-dimensional one. The representation of the space-time in this case is $\mathbb{SO}(4)$.

It is known that the Lie algebra of $\mathbb{SO}(4)$ can be written as:

$$\mathfrak{so}(4) \approx \mathfrak{su}(2) \times \mathfrak{su}(2),$$

and the irreducible representations of $\mathfrak{su}(2)$ are easy to describe by using spinors.

3.3.1 Commutation Relations

We start by rewriting all commutation relations from the previous section in terms of spinors. Recall that property (14) of spinors states that any skew tensor S_{ab} can be written as:

$$S_{ab} = \epsilon_{AB} S_{A'B'} + \epsilon_{A'B'} S_{AB}, \tag{3.43}$$

where S_{AB} and $S_{A'B'}$ are in this case independent symmetric spinors satisfying

$$[S_{AB}, S_{A'B'}] = 0. \tag{3.44}$$

We also have:

$$S_a = S_{AA'}. \tag{3.45}$$

These two formulas provide the spinor version of the generators of the $\mathcal{R}$-algebra. C_1 will be thought of abstractly, as neither tensor nor spinor, with the only relevant property that it commutes with all the elements of the algebra. p^2 can be thought of as either $p_a p^a$ or $p_{AA'} p^{AA'}$.

From (3.43) we have:

$$S_{A'B'} = \frac{1}{2} \epsilon^{AB} S_{ab} \;\text{ and }\; S_{AB} = \frac{1}{2} \epsilon^{A'B'} S_{ab}. \tag{3.46}$$

Equations (3.46) and (3.36iii) yield now:

$$[S_{AB}, S_{CD}] = 2\epsilon_{(A(C} S_{D)B)} \;\text{ and }\; [S_{A'B'}, S_{C'D'}] = 2\epsilon_{(A'(C'} S_{D')B')}. \tag{3.47}$$

Likewise, (3.46) and (3.36ii) give:

$$[S_{AB}, S_{CC'}] = -\epsilon_{C(A} S_{B)C'} \quad\text{and}\quad [S_{A'B'}, S_{CC'}] = -\epsilon_{C'(A'} S_{B')C}. \tag{3.48}$$

It is assumed that if a relation holds, then its primed analogue holds as well. For brevity, we will not keep repeating the primed analogues of unprimed equations.

Since S has been eliminated, there is one more commutation relation to be translated in spinor language, namely (3.36iv). The commutator $p^2[S_a, S_b]$ is skew in a and b, hence it can be written as:

$$p^2[S_a, S_b] = \epsilon_{AB} y_{A'B'} + \epsilon_{A'B'} y_{AB}, \tag{3.49}$$

with $y_{AB},\ y_{A'B'}$ symmetric spinors.

From (3.36iv) we have:

$$p^2[S_a, S_b] = KS_{ab} + S_{[b|d|} S^{cd} S_{a]c}, \tag{3.50}$$

with $K = \frac{(n-3)(n-2)}{2} - 2S$. For $n = 5$, we obtain $K = 3 - 2S$.

By analogy with (3.46):

$$\begin{aligned} y_{A'B'} &= \frac{1}{2}\epsilon^{AB} p^2[S_a, S_b] \\ &= \frac{1}{2}\epsilon^{AB}(KS_{ab} + S_{bd}S^{cd}S_{ac}) = KS_{A'B'} + \frac{1}{2}\epsilon^{AB} S_{bd} S^{cd} S_{ac}, \end{aligned} \tag{3.51}$$

where we dropped the antisymmetrization on a, b since $y_{A'B'}$ is symmetric.

The following notation is one of the many we will introduce in this section for the purpose of simplification; let

$$S_{BD}S^{CD} = \alpha \delta_B^{\ C} + \beta S_B^{\ C}, \tag{3.52}$$

where α and β are scalars. By choosing $B = C$ in (3.52), and using that $S_A^{\ A} = 0$ we obtain

$$2\alpha = S_{CD}S^{CD}. \tag{3.53}$$

Remark 6. *Raising and lowering one dummy index in spinor language introduces a negative sign* $S_{CD}S^{CD} = -S_C^{\ D} S^C_{\ D}$.

With these notations and by substituting (3.43) into (3.51), we obtain after simplification the following expression for $y_{A'B'}$:

$$y_{A'B'} = \left(3 - 2S + \alpha' + 3\alpha - \beta'^2\right) S_{A'B'}. \tag{3.54}$$

We eliminate now S in favor of the Casimir operator C_1 obtained in (3.38):

$$S = C_1 - \frac{1}{4} S_{ab} S^{ab}. \tag{3.55}$$

By using the notations (3.53) we can write:

$$S_{ab} S^{ab} = 4(\alpha + \alpha'), \tag{3.56}$$

and so

$$S = C_1 - (\alpha + \alpha'). \tag{3.57}$$

Equation (3.54) becomes now:

$$y_{A'B'} = \left(3 - 2C_1 + 3\alpha' + 5\alpha - \beta'^2\right) S_{A'B'}. \tag{3.58}$$

We can in fact determine the value of the scalar β by expanding equation (3.47ii):

$$2\left[S_{A'B'}, S_{C'D'}\right] = \epsilon_{A'C'} S_{B'D'} + \epsilon_{A'D'} S_{B'C'} + \epsilon_{B'C'} S_{A'D'} + \epsilon_{B'D'} S_{A'C'}, \tag{3.59}$$

which becomes:

$$2\left[S_{A'D'}, S_{C'}{}^{D'}\right] = 4 S_{A'C'}. \tag{3.60}$$

By using (3.52ii), the above equation can be written as:

$$\left[S_{A'D'}, S_{C'}{}^{D'}\right] = \alpha' \epsilon_{A'C'} + \beta' S_{A'C'} + \alpha' \epsilon_{C'A'} + \beta' S_{C'A'} = 2 S_{A'C'}, \tag{3.61}$$

which finally leads to:

$$\beta' = \beta = 1. \tag{3.62}$$

With the help of (3.57) and (3.62), equation (3.54) becomes:

$$y_{A'B'} = \left(2 - 2C_1 + 3\alpha' + 5\alpha\right) S_{A'B'}, \tag{3.63}$$

and we can finally write the last commutation relation as:

$$p^2 [S_{AA'}, S_{BB'}] = \Lambda \epsilon_{AB} S_{A'B'} + \Lambda' \epsilon_{A'B'} S_{AB}, \tag{3.64}$$

where we used the notations:

$$\Lambda = 2 - 2C_1 + 3\alpha' + 5\alpha, \qquad \Lambda' = 2 - 2C_1 + 3\alpha + 5\alpha'. \tag{3.65}$$

3.3.2 Casimir Invariants

We construct here the Casimir invariants of the five-dimensional conformal algebra. Recall that one of the Casimirs is valid in all dimensions:

$$C_1 = S + \frac{1}{4} S_{ab} S^{ab}. \tag{3.66}$$

To find the remaining invariants, consider scalars that can be formed by using the operators S_a, S_{ab} and U; additionally, we can use other scalars introduced here, such as α and α'.

3.3.2.1 Second Casimir Invariant We start by considering the following operator:

$$T = \lambda U + S^{AA'} S_{AB} S_{A'B'} S^{BB'}. \tag{3.67}$$

Choose λ of the form $a\alpha + a'\alpha'$, with a and a' constants.

As a candidate for a Casimir operator, T will be required to commute with the generators S_a and S_{ab} given in (3.43) and (3.45).

For now, we will only consider the commutation relation with S_a; once we obtain a final expression for our potential Casimir, we will show that it commutes with S_{ab} as well.

The first step is to calculate the commutator $[T, S_a]$. According to (3.67):

$$p^2[T, S_a] = p^2\lambda[U, S_a] + p^2[\lambda, S_a]U + [S^{CC'} S_{BC} S_{B'C'} S^{BB'}, S_{AA'}]. \tag{3.68}$$

From (3.41) and (3.42) with $n = 5$, we obtain after fairly lengthy calculations:

$$\begin{aligned} p^2[U, S_a] &= S_a \left(-12 - 6C_1 + \frac{3}{2} S_{bc} S^{bc}\right) + S^b S_{ab} \left(20 + 4C_1 - S_{cd} S^{cd}\right) && (3.69) \\ &\quad +11 S_b S^{bc} S_{ac} - 2S^b S_{bd} S^{cd} S_{ac}, && (3.70) \end{aligned}$$

or, in terms of spinor quantities:

$$\begin{aligned} p^2[U, S_{AA'}] &= S_{AA'} \left[\frac{21}{2} - 6C_1 + 15(\alpha + \alpha')\right] + 2S_{A'}{}^{B} S_{AB} \Lambda' \\ &\quad +2S_{A}{}^{B'} S_{A'B'} \Lambda - 10 S_{AB} S_{A'B'} S^{BB'}, \end{aligned} \tag{3.71}$$

where Λ and Λ' are as in (3.65).

For consistency, we will write formulas such that $\alpha,\ \alpha'$ and U are to the right. In order to do so, we will need some more commutation relations:

$$[\alpha, S_{AA'}] = -\frac{3}{4}S_{AA'} - S_{A'}^{\ B}S_{AB}, \tag{3.72}$$

and

$$[\Lambda, S_{AA'}] = -6S_{AA'} - 5S_{A'}^{\ B}S_{AB} - 3S_{A}^{\ B'}S_{A'B'}. \tag{3.73}$$

The primed analogues of these two relations hold as well.

Note that there are four distinct terms appearing in all these expressions: $S_{AA'}$, $S_{A'}^{\ B}S_{AB}$, $S_{A}^{\ B'}S_{A'B'}$ and $S_{AB}S_{A'B'}S^{BB'}$. Since these terms will be present throughout all calculations needed in finding the Casimir invariants, we will need the commutators of $U,\ \alpha,\ \alpha'$ (and hence of $\Lambda,\ \Lambda'$) with $S_{A'}^{\ B},\ S_{A}^{\ B'}$, and $S^{BB'}$, respectively.

Before we find these commutators though, note that from (3.53) and by means of equation (3.44), we have:

$$[\alpha, S_{A'B'}] = 0 = [\alpha', S_{AB}]\,. \tag{3.74}$$

Also,

$$[\alpha + \alpha', S_{AB}] = \left[\frac{1}{4}S_{cd}S^{cd}, \frac{1}{2}\epsilon^{A'B'}S_{ab}\right] = 0, \tag{3.75}$$

where we used (A.58). We now have:

$$[\alpha, S_{AB}] = [\alpha + \alpha', S_{AB}] - [\alpha', S_{AB}] = 0. \tag{3.76}$$

Its primed analogue is given by:

$$[\alpha', S_{A'B'}] = 0. \tag{3.77}$$

One other important relation follows from (3.40ii):

$$[U, S_{AB}] = 0 = [U, S_{A'B'}]. \tag{3.78}$$

The commutators of $U,\ \alpha,\ \alpha',\ \Lambda,\ \Lambda'$ with $S_{A'}^{\ B},\ S_{A}^{\ B'}$, and $S^{BB'}$can be obtained from (3.71) - (3.73) by noticing that we can write:

$$S_{A'}^{\ B} = \epsilon^{BA}S_{AA'},\ S_{A}^{\ B'} = \epsilon^{B'A'}S_{AA'},\ S^{BB'} = \epsilon^{BA}\epsilon^{B'A'}S_{AA'}. \tag{3.79}$$

These commutators will be listed in appendix B, but their expressions are not sufficiently enlightening to be presented here. We must mention that due to their length it is impossible to present all the calculations leading to the formulas listed in the appendix or to the intermediate commutators, without adding tens of pages to the present work.

One other simplification that we use is by introducing the following notations. Define:

$$S_{A'B}S_{B'}^{\ B} = u\epsilon_{A'B'} + U_{A'B'}, \tag{3.80}$$

with $U_{A'B'}$ symmetric. As before, we have:

$$u = \frac{1}{2}\epsilon^{A'B'}S_{A'B}S_{B'}^{\ B} = \frac{1}{2}U, \tag{3.81}$$

and so

$$S_{A'B}S_{B'}^{\ B} = \frac{1}{2}U\epsilon_{A'B'} + U_{A'B'}. \tag{3.82}$$

We also have from (3.64) that

$$p^2[S_{AA'}, S_{BB'}] = \Lambda\epsilon_{AB}S_{A'B'} + \Lambda'\epsilon_{A'B'}S_{AB}, \tag{3.83}$$

and by contracting over the A and B indices, we obtain:

$$p^2U_{A'B'} = \Lambda S_{A'B'}, \tag{3.84}$$

therefore

$$p^2S_{A'B}S_{B'}^{\ B} = \Lambda S_{A'B'} + \frac{1}{2}p^2U\epsilon_{A'B'}, \tag{3.85}$$

and

$$p^2S_{AB'}S_{B}^{\ B'} = \Lambda' S_{AB} + \frac{1}{2}p^2U\epsilon_{AB.} \tag{3.86}$$

We are now ready to check if T is a good choice for a Casimir invariant. Recall that we want to obtain an operator whose commutator with $S_{AA'}$ vanishes. We will try to make all the terms in the commutator $p^2[T, S_{AA'}]$ vanish, either by choosing constants appropriately, or by adding more terms to it.

In order to eliminate the U terms in $p^2[T, S_{AA'}]$, one needs to choose the following values for the constants a and a':

$$a = a' = \frac{1}{2}. \tag{3.87}$$

With this, we obtain:

$$\begin{aligned} p^2[T, S_{AA'}] &= S_{AA'}\left[\left(\frac{471}{8} - 15C_1\right) + \left(\frac{235}{4} - C_1\right)(\alpha + \alpha') + \frac{9}{2}\left(\alpha^2 + \alpha'^2\right) - 5\alpha\alpha'\right] \\ &+ S_{A'}{}^{B} S_{AB}[(26 - 11C_1) + \alpha\left(\frac{43}{2} - 2C_1\right) + \alpha'\left(\frac{85}{2} + 2C_1\right) \\ &\qquad -\alpha'^2 + 3\alpha^2 - 2\alpha\alpha'] + p.a. \\ &+ S_{AB}S_{A'B'}S^{BB'}\left[\left(-\frac{87}{2} - 2C_1\right) + 2(\alpha + \alpha')\right]. \end{aligned} \tag{3.88}$$

Here p.a. stands for the primed analogue of the second term, $S_{A'}{}^{B} S_{AB}$ and its coefficient.

Note that although it is not zero and is not particularly simple, this commutator does not contain a U term anymore. In order to compensate for the remaining terms, we claim that the second Casimir operator will have the form:

$$C_2 = p^2 T + f(\alpha, \alpha', C_1), \tag{3.89}$$

where $f(\alpha, \alpha', C_1)$ must be of order three in α and α'. Let

$$\begin{aligned} f(\alpha, \alpha', C_1) &= B(\alpha + \alpha') + G\left(\alpha^2 + \alpha'^2\right) + H\alpha\alpha' + E\left(\alpha^3 + \alpha'^3\right) \\ &+ F\left(\alpha^2\alpha' + \alpha'^2\alpha\right). \end{aligned} \tag{3.90}$$

Remark 7. *Although the commutator $[f(\alpha, \alpha', C_1), S_{AA'}]$ has initially been calculated by hand, we soon noticed that we can automatize the process by writing a computer program (using Maple) which can find most of the commutators needed. The main idea is described in the following paragraphs.*

In order to simplify calculations, note that when computing various commutators, we will need to move certain quantities past $S_{AA'}$, $S_{A'}^{\ B}$, $S_{A}^{\ B'}$, and $S^{BB'}$. For example, if we want to write the terms of the commutator:

$$\begin{aligned}
\left[\alpha^2, S_{AA'}\right] &= \left[\alpha\alpha, S_{AA'}\right] = \alpha\left[\alpha, S_{AA'}\right] + \left[\alpha, S_{AA'}\right]\alpha \\
&= \alpha\left(-\frac{3}{4}S_{AA'} - S_{A'}^{\ B}S_{AB}\right) + \left(-\frac{3}{4}S_{AA'} - S_{A'}^{\ B}S_{AB}\right)\alpha,
\end{aligned}$$

such that α and α' are to the right, it is necessary that we move α past $S_{AA'}$ and $S_{A'}^{\ B}$.

In general, we will need to move an arbitrary expression f_1 past the basic terms $S_{AA'}$, $S_{A'}^{\ B}S_{AB}$, $S_{A}^{\ B'}S_{A'B'}$, and $S_{AB}S_{A'B'}S^{BB'}$. We want thus a general formula for

$$f_1\left(S_{AA'}v_1 + S_{A'}^{\ B}S_{AB}v_2 + S_{A}^{\ B'}S_{A'B'}v_3 + S_{AB}S_{A'B'}S^{BB'}v_4\right), \tag{3.91}$$

knowing the commutator of f_1 with $S_{AA'}$:

$$[f_1, S_{AA'}] = S_{AA'}w_1 + S_{A'}^{\ B}S_{AB}w_2 + S_{A}^{\ B'}S_{A'B'}w_3 + S_{AB}S_{A'B'}S^{BB'}w_4. \tag{3.92}$$

This last equation implies that

$$f_1S_{AA'} = S_{AA'}(f_1 + w_1) + S_{A'}^{\ B}S_{AB}w_2 + S_{A}^{\ B'}S_{A'B'}w_3 + S_{AB}S_{A'B'}S^{BB'}w_4, \tag{3.93}$$

and so we have

$$\begin{aligned}
& f_1\left(S_{AA'}v_1 + S_{A'}^{\ B}S_{AB}v_2 + S_{A}^{\ B'}S_{A'B'}v_3 + S_{AB}S_{A'B'}S^{BB'}v_4\right) \qquad (3.94)\\
=\ & \left(f_1S_{AA'}\right)v_1 + \left(f_1S_{A'}^{\ B}\right)S_{AB}v_2 + \left(f_1S_{A}^{\ B'}\right)S_{A'B'}v_3 + S_{AB}S_{A'B'}\left(f_1S^{BB'}\right)v_4.
\end{aligned}$$

Knowing what $f_1S_{AA'}$ looks like, we can obtain the remaining terms by using:

$$f_1S_{A'}^{\ B} = \epsilon^{BA}\left(f_1S_{AA'}\right), \tag{3.95}$$

$$f_1S_{A}^{\ B'} = \epsilon^{B'A'}\left(f_1S_{AA'}\right), \tag{3.96}$$

and

$$f_1S^{BB'} = \epsilon^{BA}\epsilon^{B'A'}\left(f_1S_{AA'}\right). \tag{3.97}$$

After reordering terms accordingly, we obtain:

$$\begin{aligned} & f_1\left(S_{AA'}v_1 + S_{A'}{}^{B}S_{AB}v_2 + S_{A}{}^{B'}S_{A'B'}v_3 + S_{AB}S_{A'B'}S^{BB'}v_4\right) \\ = \; & S_{AA'}G_1 + S_{A'}{}^{B}S_{AB}G_2 + S_{A}{}^{B'}S_{A'B'}G_3 + S_{AB}S_{A'B'}S^{BB'}G_4, \end{aligned} \tag{3.98}$$

where:

$$\begin{aligned} G_1 = \; & (f_1 + w_1)\,v_1 - (\alpha w_2 v_2 + \alpha' w_3 v_3) - \frac{9}{4}\,(w_3 v_2 + w_2 v_3) \\ & -\frac{3}{2}\left[\left(\frac{3}{4}+\alpha\right)(w_4 v_2 + w_2 v_4) + \left(\frac{3}{4}+\alpha'\right)(w_4 v_3 + w_3 v_4)\right] \\ & + \left[-\frac{63}{16} - \frac{9}{4}\,(\alpha + \alpha') + \alpha\alpha'\right] w_4 v_4, \end{aligned} \tag{3.99}$$

$$\begin{aligned} G_2 = \; & w_2 v_1 + (f_1 + w_1)\,v_2 - w_2 v_2 - \frac{3}{2}\,(w_3 v_2 + w_2 v_3) \\ & - \left(\frac{3}{4}+\alpha'\right)(w_4 v_3 + w_3 v_4 + 2 w_4 v_4)\,, \end{aligned} \tag{3.100}$$

$$\begin{aligned} G_3 = \; & w_3 v_1 + (f_1 + w_1)\,v_3 - w_3 v_3 - \frac{3}{2}\,(w_3 v_2 + w_2 v_3) \\ & - \left(\frac{3}{4}+\alpha\right)(w_4 v_2 + w_2 v_4 + 2 w_4 v_4)\,, \end{aligned} \tag{3.101}$$

$$\begin{aligned} G_4 = \; & w_4 v_1 + (f_1 + w_1)\,v_4 + (w_3 v_2 + w_2 v_3) + \frac{1}{2} w_4\,(v_2 + v_3) \\ & + \left[\frac{1}{2}\,(w_2 + w_3) + 4 w_4\right] v_4. \end{aligned} \tag{3.102}$$

By using these formulas, and requiring that $[C_2, S_{AA'}] = 0$, from (3.90) we obtain that

$$\begin{aligned} B &= -\frac{1}{2} - 13 C_1, \\ G &= \frac{51}{4} - C_1, \\ H &= \frac{85}{2} + 2 C_1, \\ E &= 1, \\ F &= -1. \end{aligned} \tag{3.103}$$

Our candidate for the second Casimir invariant is thus given by:

$$\begin{aligned} C_2 \;=\; & \frac{1}{2}p^2\left(\alpha+\alpha'\right)U + p^2 S^{AA'}S_{AB}S_{A'B'}S^{BB'} - \left(\frac{1}{2}+13C_1\right)\left(\alpha+\alpha'\right) \\ & + \left(\frac{51}{4}-C_1\right)\left(\alpha^2+\alpha'^2\right) + \left(\frac{85}{2}+2C_1\right)\alpha\alpha' + \left(\alpha^3+\alpha'^3\right) \\ & - \left(\alpha^2\alpha'+\alpha'^2\alpha\right), \end{aligned} \tag{3.104}$$

satisfying

$$[C_2, S_{AA'}] = [C_2, S_a] = 0. \tag{3.105}$$

We stated that once we obtain this expression, we will show that it is indeed the desired Casimir invariant by showing that its commutator with S_{ab} is equal to zero as well. From (3.43), it is enough to show that C_2 commutes with both S_{AB} and $S_{A'B'}$. Also note that from (3.104) it is obvious that all the terms commute with both S_{AB} and $S_{A'B'}$, except, possibly, for the second term.

Consider now the commutator

$$\begin{aligned} \left[S^{AA'}S_{AB}S_{A'B'}S^{BB'}, S_{CD}\right] &= S^{AA'}S_{AB}S_{A'B'}\left[S^{BB'}, S_{CD}\right] \\ +S^{AA'}\left[S_{AB}, S_{CD}\right]S_{A'B'}S^{BB'} &+ \left[S_{AA'}, S_{CD}\right]S_{AB}S_{A'B'}S^{BB'}, \end{aligned}$$

which can be shown to be equal to zero by using the commutation relations (3.47), and (3.48). We have thus that

$$[C_2, S_{AB}] = 0, \tag{3.106}$$

with its primed analogue

$$[C_2, S_{A'B'}] = 0. \tag{3.107}$$

From (3.43), (3.106) and (3.107) we obtain

$$[C_2, S_{ab}] = 0, \tag{3.108}$$

and so C_2 in (3.104) is indeed a Casimir operator.

3.3.2.2 Third Casimir Invariant There is one more Casimir operator to find. Since C_2 depended on the first power of U, we considered the possibility of a U^2 dependency as being very likely.

Consider the following operator:

$$\begin{aligned} C_3' &= c_1 p^4 U^2 + c_2 p^2 \left(\alpha + \alpha'\right) S^{AA'} S_{AB} S_{A'B'} S^{BB'} \\ &\quad + p^2 \left[c_3 \left(\alpha^2 + \alpha'^2\right) + c_4 \left(\alpha + \alpha'\right) + c_5 \alpha\alpha'\right] U. \end{aligned} \tag{3.109}$$

Same as before, we will first try to obtain an operator that commutes with $S_{AA'}$. The commutators of the individual terms in (3.109) with $S_{AA'}$ are listed in appendix B; most of them have been obtained by using Maple, and due to their considerable length, we have not presented many of the intermediate results.

By requiring that $[C_3', S_{AA'}]$ has no U terms, we obtain the following expressions for the constants in the definition of C_3':

$$\begin{aligned} c_1 &= c_1, \\ c_2 &= -12c_1 + 2c_3, \\ c_3 &= c_3, \\ c_4 &= -4\left(2C_1 + 1\right) c_1, \\ c_5 &= 8c_1 + 2c_3. \end{aligned} \tag{3.110}$$

From (3.109) we have then:

$$\begin{aligned} C_3' &= c_1 p^4 U^2 + (-12c_1 + 2c_3) p^2 \left(\alpha + \alpha'\right) S^{AA'} S_{AB} S_{A'B'} S^{BB'} \\ &\quad + p^2 c_3 \left(\alpha^2 + \alpha'^2\right) U - 4p^2 \left(2C_1 + 1\right) c_1 \left(\alpha + \alpha'\right) \\ &\quad + (8c_1 + 2c_3) p^2 (\alpha\alpha') U. \end{aligned} \tag{3.111}$$

In order to cancel the remaining terms, add to C_3' a function of order four in α and α':

$$C_3 = C_3' + k_1\left(\alpha^4 + \alpha'^4\right) + k_2\left(\alpha^3\alpha' + \alpha'^3\alpha\right) + k_3\left(\alpha^2\alpha'^2\right) \tag{3.112}$$

$$+k_4\left(\alpha^3 + \alpha'^3\right) + k_5\left(\alpha^2\alpha' + \alpha'^2\alpha\right)$$

$$+k_6\left(\alpha^2 + \alpha'^2\right) + k_7\left(\alpha\alpha'\right) + k_8\left(\alpha + \alpha'\right). \tag{3.113}$$

Requiring that $[C_3, S_{AA'}] = 0$, and redenoting $c_1 = A,\ c_3 = B$ so that there is no confusion between these constants and the Casimir operators, we obtain the following expressions:

$$\begin{aligned}
k_1 &= -3A + 2B, \\
k_2 &= 60A, \\
k_3 &= 2\left(71A - 2B\right), \\
k_4 &= -3A\left(4C_1 + 55\right) + B\left(\tfrac{51}{2} - 2C_1\right), \\
k_5 &= -A\left(116C_1 + 715\right) + B\left(\tfrac{221}{2} + 2C_1\right), \\
k_6 &= A\left(10 + 172C_1 + 16{C_1}^2\right) - B\left(1 + 26C_1\right), \\
k_7 &= 2k_6, \\
k_8 &= 2C_2\left(6A - B\right).
\end{aligned} \tag{3.114}$$

Substituting these expressions into (3.113), and also using (3.104) to rewrite the term $p^2 S^{AA'} S_{AB} S_{A'B'} S^{BB'}$ as:

$$\begin{aligned}
p^2 S^{AA'} S_{AB} S_{A'B'} S^{BB'} \;=\;& C_2 - \frac{1}{2}p^2\left(\alpha + \alpha'\right)U + \left(\frac{1}{2} + 13C_1\right)\left(\alpha + \alpha'\right) \\
& - \left(\frac{51}{4} - C_1\right)\left(\alpha^2 + \alpha'^2\right) - \left(\frac{85}{2} + 2C_1\right)\alpha\alpha' \\
& - \left(\alpha^3 + \alpha'^3\right) + \left(\alpha^2\alpha' + \alpha'^2\alpha\right),
\end{aligned}$$

we obtain after lengthy calculations that:

$$\begin{aligned}
C_3/A \;=\;& p^4U^2 + 2p^2\left[3\left(\alpha^2 + \alpha'^2\right) + 4\left(\alpha\alpha'\right) - 2\left(\alpha + \alpha'\right)\left(1 + 2C_1\right)\right]U \\
& + 9\left(\alpha + \alpha'\right)^4 + 16\left(\alpha^2\alpha'^2\right) + 24\left(\alpha\alpha'\right)\left(\alpha + \alpha'\right)^2 - 12(1 + 2C_1)\left(\alpha + \alpha'\right)^3 \\
& - 16\left(\alpha\alpha'\right)\left(\alpha + \alpha'\right)^2\left(1 + 2C_1\right) + 4(1 + 2C_1)^2\left(\alpha + \alpha'\right)^2.
\end{aligned} \tag{3.115}$$

This allows us to rescale by A if $A \neq 0$. Substituting now $\alpha + \alpha' = C_1 - S$ from (3.57) and simplifying, we finally obtain:

$$\begin{aligned}
C_3 \;=\;& p^4U^2 + 2p^2\left[4\left(\alpha\alpha'\right) - \left(C_1 + 3S + 2\right)\left(C_1 - S\right)\right]U \\
& + \left[4\left(\alpha\alpha'\right) - \left(C_1 + 3S + 2\right)\left(C_1 - S\right)\right]^2.
\end{aligned} \tag{3.116}$$

It is obvious from (3.116) that C_3 commutes also with S_{AB} and $S_{A'B'}$, hence with S_{ab}, which shows that it is a valid Casimir invariant.

It is only after obtaining this expression that we noticed the fact that C_3 in (3.116) can be factored:

$$C_3 = \left[p^2 U + 4\alpha\alpha' - (C_1 + 3S + 2)(C_1 - S)\right]^2 .$$

We can easily check that this new expression is, in fact, the simplest Casimir invariant that can be constructed with these terms:

$$C_3 = p^2 U + 4\alpha\alpha' - (C_1 + 3S + 2)(C_1 - S) . \tag{3.117}$$

We have obtained thus all the Casimir invariants of the five-dimensional conformal algebra, which we now redenote in order of their complexity as:

$$\begin{aligned}
C_1 &= S + (\alpha + \alpha'), \\
C_2 &= p^2 U + 4\alpha\alpha' - (C_1 + 3S + 2)(C_1 - S), \\
C_3 &= \frac{1}{2} p^2 (\alpha + \alpha') U + p^2 S^{AA'} S_{AB} S_{A'B'} S^{BB'} - \left(\frac{1}{2} + 13 C_1\right)(\alpha + \alpha') \\
&\quad + \left(\frac{51}{4} - C_1\right)\left(\alpha^2 + \alpha'^2\right) + \left(\frac{85}{2} + 2C_1\right)\alpha\alpha' + \left(\alpha^3 + \alpha'^3\right) \\
&\quad - \left(\alpha^2\alpha' + \alpha'^2\alpha\right) .
\end{aligned} \tag{3.118}$$

3.3.3 Casimir Operators in Tensor Language

Although we have obtained all the Casimir invariants, it is of interest to write them also in tensor language. We hope that this will be of help in future research to derive general expressions for the Casimir invariants of the Lie algebra of $\mathbb{SO}(p+1, q+1)$ for arbitrary values of p and q.

C_1 has been originally derived in tensor notation as

$$C_1 = S + \frac{1}{4} S_{ab} S^{ab} .$$

3.3.3.1 Second Casimir Invariant From (3.118) we observe that the quantity $\alpha\alpha'$ appears in both C_2 and C_3. Our first goal will be to rewrite $\alpha\alpha'$ in tensor language.

Recall that

$$S_{ab} = \epsilon_{AB} S_{A'B'} + \epsilon_{A'B'} S_{AB},$$

and its dual is given by:

$$^*S_{ab} = i\epsilon_{AB} S_{A'B'} - i\epsilon_{A'B'} S_{AB}$$

or

$$^*S_{ab} = \frac{1}{2}\varepsilon_{abcd} S^{cd},$$

as in property (15) from section 2.1.3.

We can write then:

$$S_{ab}(^*S_{ab}) = \frac{1}{2}\varepsilon_{abcd} S^{ab} S^{cd}. \tag{3.119}$$

Introduce the following notations:

$$A = S_{ab}S^{ab} = 4\left(\alpha + \alpha'\right), \text{ and } B = (S_{ab})(^*S^{ab}) = 4i(\alpha' - \alpha), \tag{3.120}$$

where we used (3.43) and (2.34). From (3.120) we obtain that:

$$64\alpha\alpha' = A^2 + B^2. \tag{3.121}$$

We also have:

$$A^2 = \left(S_{ab}S^{ab}\right)\left(S_{cd}S^{cd}\right), \tag{3.122}$$

and

$$B^2 = (S_{ab})(^*S^{ab})(S_{cd})(^*S^{cd}) = \frac{1}{4}\varepsilon_{abcd}\varepsilon^{pqrs} S^{ab} S^{cd} S_{pq} S_{rs}. \tag{3.123}$$

Knowing that

$$\varepsilon_{abcd}\varepsilon^{pqrs} = -4!\,\delta_{[a}{}^{p}\delta_{b}{}^{q}\delta_{c}{}^{r}\delta_{d]}{}^{s}, \tag{3.124}$$

we obtain:

$$64\alpha\alpha' = \left(S_{ab}S^{ab}\right)\left(S_{cd}S^{cd}\right) - 6 S^{ab} S^{cd} S_{[ab} S_{cd]}.$$

This can be written as:

$$
\begin{aligned}
64\alpha\alpha' &= \left(S_{ab}S^{ab}\right)\left(S_{cd}S^{cd}\right) && (3.125)\\
&+S^{ab}S^{cd}(S_{ab}S_{cd} - S_{ad}S_{bc} - S_{bc}S_{ad} - S_{cd}S_{ab} + S_{ac}S_{bd} + S_{bd}S_{ac}).
\end{aligned}
$$

After reordering terms, we finally obtain:

$$
64\alpha\alpha' = -\left(S_{ab}S^{ab}\right)\left(S_{cd}S^{cd}\right) - 8S_{ab}S^{ab} + 12S_a{}^bS_b{}^cS_c{}^a + 4S_a{}^bS_b{}^cS_c{}^dS_d{}^a. \quad (3.126)
$$

This allows us to rewrite the second Casimir invariant as:

$$
\begin{aligned}
C_2 &= p^2S_aS^a - \frac{1}{16}\left(S_{ab}S^{ab}\right)\left(S_{cd}S^{cd}\right) - \frac{1}{2}S_{ab}S^{ab} && (3.127)\\
&+\frac{3}{4}S_a{}^bS_b{}^cS_c{}^a + \frac{1}{4}S_a{}^bS_b{}^cS_c{}^dS_d{}^a - \left(C_1 + 3S + 2\right)\left(C_1 - S\right).
\end{aligned}
$$

3.3.3.2 Third Casimir Invariant In order to write C_3 in terms of tensor quantities, we will have first to rewrite $S^{AA'}S_{AB}S_{A'B'}S^{BB'}$. By using (3.43), we can write

$$
2S_{AB}S_{A'B'} = \frac{1}{4}g_{ab}S_{cd}S^{cd} + S_{c(a}S_{b)}{}^c. \quad (3.128)
$$

Using now that $S^{AA'} = S^a$ and $S^{BB'} = S^b$, and the commutation relation for $n = 5$:

$$
\left[S_{cd}S^{cd}, S^b\right] = 4S^b{}_cS^c + 6S^b,
$$

we obtain:

$$
\begin{aligned}
2S^{AA'}S_{AB}S_{A'B'}S^{BB'} &= S^aS_{c(a}S_{b)}{}^cS^b + \frac{1}{4}S^ag_{ab}S_{cd}S^{cd}S^b && (3.129)\\
&= S^aS_{c(a}S_{b)}{}^cS^b + \left(C_1 - S + \frac{3}{2}\right)U + S^aS_{ab}S^b.
\end{aligned}
$$

We can also modify some of the terms involved in C_3, by writing:

$$
\left(\alpha^2 + \alpha'^2\right) = \left(\alpha + \alpha'\right)^2 - 2\left(\alpha\alpha'\right),
$$

and

$$
\left(\alpha^3 + \alpha'^3\right) - \left(\alpha^2\alpha' + \alpha'^2\alpha\right) = \left(\alpha + \alpha'\right)^2 - 4\left(\alpha + \alpha'\right)\left(\alpha\alpha'\right).
$$

Using $\alpha + \alpha' = C_1 - S$, and simplifying we obtain:

$$\begin{aligned} C_3 &= p^2 S_a S^a \left(C_1 - S + \frac{3}{4} \right) + \frac{1}{2} p^2 S^a S_{c(a} S_{b)}{}^c S^b + \frac{1}{2} p^2 S^a S_{ab} S^b \\ &\quad + (17 + 4S)(\alpha\alpha') + \frac{1}{4}(C_1 - S)\left[4S^2 - 2 - C_1 - 4S(17 + 4C_1)\right]. \end{aligned} \tag{3.130}$$

Note that the only term still written in terms of spinors is $\alpha\alpha'$ which can be gotten rid of by using (3.126). We obtain, after many simplifications, the final expression for C_3 in terms of the other two Casimir invariants:

$$\begin{aligned} C_3 &= p^2 (C_1 - 2S - 6) S_a S^a + \frac{1}{2} p^2 S^a S_{c(a} S_{b)}{}^c S^b + \frac{1}{4}(5 - 2S) S_{ab} S^{ab} \\ &\quad + \frac{1}{16}\left(S_{ab} S^{ab}\right)\left(S_{cd} S^{cd}\right) - 3 S_a{}^b S_b{}^c S_c{}^a + \frac{21 + 4S}{4} C_2 \\ &\quad + (C_1 - S)\left(5C_1 + 4S^2 + 5S + 10\right). \end{aligned} \tag{3.131}$$

We will summarize now the results obtained as follows:

Corollary 8. *The following operators are the Casimir invariants of the five-dimensional conformal algebra $C_e(p^{-2})$:*

$$C_1 = S + \frac{1}{4} S_{cd} S^{cd},$$

$$\begin{aligned} C_2 &= p^2 S_a S^a - \frac{1}{16}\left(S_{ab} S^{ab}\right)\left(S_{cd} S^{cd}\right) - \frac{1}{2} S_{ab} S^{ab} \\ &\quad + \frac{3}{4} S_a{}^b S_b{}^c S_c{}^a + \frac{1}{4} S_a{}^b S_b{}^c S_c{}^d S_d{}^a - (C_1 + 3S + 2)(C_1 - S), \end{aligned}$$

and

$$\begin{aligned} C_3 &= p^2 (C_1 - 2S - 6) S_a S^a + \frac{1}{2} p^2 S^a S_{c(a} S_{b)}{}^c S^b + \frac{1}{4}(5 - 2S) S_{ab} S^{ab} \\ &\quad + \frac{1}{16}\left(S_{ab} S^{ab}\right)\left(S_{cd} S^{cd}\right) - 3 S_a{}^b S_b{}^c S_c{}^a + \frac{21 + 4S}{4} C_2 \\ &\quad + (C_1 - S)\left(5C_1 + 4S^2 + 5S + 10\right). \end{aligned}$$

Corollary 9. *C_1, C_2, C_3, p^2 are Casimir invariants for the $\mathcal{R}$-algebra in five dimensions.*

This concludes our analysis of the five-dimensional conformal algebra.

3.4 THE 4-DIMENSIONAL CASE

This section will only present some of the results obtained in [31] for the four-dimensional case, this being the work from which the present research originates. The results apply to the conformal algebra of $\mathbb{SO}(2,4)$.

For a four-dimensional space-time, as discussed in section 2.3, the infinitesimal generators of the conformal group of space-time determine a fifteen-dimensional algebra, $\mathcal{C}$, spanned by the translations p_a, the Lorentz transformations M_{ab}, the dilation D and the special conformal transformations q_a. These generators determine the same subalgebras as the ones described in section 3.1. Their commutation relations given in [31] contain the factor i which is more appropriate in discussing Hermitian operators:

$$\begin{aligned}
[p_a, p_b] &= 0, \quad [q_a, q_b] = 0, \quad [p_a, q_b] = -i\hbar(M_{ab} - \hbar g_{ab} D), \\
[M_{ab}, p_c] &= g_{ac} p_b - g_{bc} p_a, \; [M_{ab}, q_c] = -i\hbar(g_{ac} q_b - g_{bc} q_a), \\
[M_{ab}, M_{cd}] &= -i\hbar\left(g_{ac} M_{bd} - g_{bc} M_{ad} - g_{ad} M_{bc} + g_{bd} M_{ac}\right), \\
[D, p_a] &= -i p_a, \quad [D, q_a] = i q_a, \quad [D, M_{ab}] = 0.
\end{aligned} \tag{3.132}$$

The following theorem summarizes the results obtained in [31] regarding the $\mathcal{R}$-algebra.

Theorem 10. *Define in $C_e\left(p^{-2}\right)$ the operator R_a by the formula:*

$$\begin{aligned}
p^2 R_a \;\equiv\; & p^2 q_a - \frac{1}{4} M_{bc} p_a M^{bc} - M^{c(a} p_b M^{b)}{}_c - \frac{1}{2}\hbar\left(D p_b M_a{}^b + M_a{}^b p_b D\right) \\
& -\frac{1}{2}\hbar^2 D p_a + 2\hbar^2 p_a.
\end{aligned}$$

This operator satisfies the following commutation relations:

$$\begin{aligned}
[R_a, p_b] &= 0, \\
[D, R_a] &= i R_a, \\
[M_{bc}, R_a] &= i\hbar\left(g_{ac} R_b - g_{ba} R_c\right), \\
p^2\left[R_a, R_b\right] &= 2i\hbar\left[R^c p_c M_{ab} + R^c M_{c[a} p_{b]} + p_{[a} M_{b]c} R^c\right].
\end{aligned} \tag{3.133}$$

In order to define the Casimir invariants for this case, one needs first to introduce the Pauli-Lubanski spin vector (in $\mathcal{P}_e$):

$$S_a = M^{*\ b}_{\ a} p_b = \frac{1}{2}\varepsilon_a^{\ bcd} M_{bc} p_d. \tag{3.134}$$

This operator should not be mistaken for the vector operator S_a in the five-dimensional case. It is merely an abuse of notation. Define also the relativistic spin squared:

$$p^2 \overrightarrow{J}^2 = -S^a S_a. \tag{3.135}$$

Corollary 11. *The conformal algebra of a four-dimensional space-time has three Casimir operators, given by*

$$\begin{aligned} C_1 &= R_a p^a + \overrightarrow{J}^2, \\ C_2 &= R_a S^a, \\ C_3 &= p^2 R_a R^a + 2\hbar^2 \overrightarrow{J}^2. \end{aligned} \tag{3.136}$$

We expect that the spinor approach we used in the five-dimensional case will work in this case as well, which will provide a general description of $\mathbb{SO}(p+1, q+1)$ with $p+q=4$, and not only of $\mathbb{SO}(2,4)$ as in [31].

3.5 THE 3-DIMENSIONAL CASE

This case has been studied prior to the general n-dimensional case, and so the construction of the $\mathcal{R}$-algebra is independent of the construction in section 3.2. Nonetheless, it will follow the same principles, and it uses the characteristics of a three-dimensional space. At the end of this section we will establish the connection with the n-dimensional case.

This case applies to $\mathbb{SO}(1,4)$ and $\mathbb{SO}(2,3)$ which are the standard de Sitter and anti-de Sitter spaces.

Consider the infinitesimal generators of the conformal group of a three-dimensional space: the translations p, the special conformal transformations q, the angular momentum J, and the dilation operator D.

These generators form a ten-dimensional algebra and satisfy the following commutation relations:

$$\begin{aligned}
&[D,p]=-p,\;\;[D,J]=0,\;\;[D,q]=q,\\
&[J\cdot a, p\cdot b]=-(a\times b)\cdot p,\\
&[J\cdot a, q\cdot b]=-(a\times b)\cdot q,\\
&[J\cdot a, J\cdot b]=-(a\times b)\cdot J,\\
&[p\cdot a, p\cdot b]=0,\;\;[q\cdot a, q\cdot b]=0,\\
&[p\cdot a, q\cdot b]=(a\cdot b)D-(a\times b)\cdot J.
\end{aligned} \tag{3.137}$$

In this section we will use the notation $J\cdot a$ instead of J_a and $(a\times b)\cdot J$ instead of $\varepsilon_{abc}J^c$ in order to preserve some of the characteristics of a three-dimensional space.

The commutation relations (3.137) ensure that the Jacobi identity is satisfied:

$$0=[p\cdot a,[q\cdot b,q\cdot c]]+[q\cdot c,[p\cdot a,q\cdot b]]+[q\cdot b,[q\cdot c,p\cdot a]]. \tag{3.138}$$

3.5.1 Three-Dimensional Model

We can construct a model in $\mathbb{R}^3$ that will satisfy the commutation relation of the conformal algebra. This model has roots in quantum mechanics, where p is the momentum operator, x is the position operator, J is the angular momentum and D is the dilation operator:

$$x_a = x_a,\ p_a = \frac{\partial}{\partial x^a},\quad J = x \times p,\ D = x \cdot p, \tag{3.139}$$

with the well-known commutation relations:

$$\begin{aligned} [J \cdot a, p \cdot b] &= -(a \times b) \cdot p, \\ [J \cdot a, x \cdot b] &= -(a \times b) \cdot x, \\ [p \cdot a, x \cdot b] &= \delta_{ab}. \end{aligned} \tag{3.140}$$

Define now an operator q that will satisfy the key relation of the conformal algebra, equation (3.137vi):

$$q = \frac{1}{2}(x \times J + xD). \tag{3.141}$$

Note that in this model we assumed that $J = x \times p,\ D = x \cdot p$, which leads to the relation $J \cdot p = p \cdot J = 0$. This is only a particularity of the model and will not be true in general. Once we obtain an expression for the operator q in terms of the generators of the algebra, we will drop all assumptions on any particular model.

With these formulas, we obtain that the defining relation of the conformal algebra is satisfied:

$$[p \cdot a, q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \tag{3.142}$$

and we also obtain a candidate for the special conformal transformations q:

$$2(p \cdot p)q = (p \cdot J)J - p(J \cdot J) + pD(D+3) + 2(p \times J)(D+1), \tag{3.143}$$

which can be easily seen not to commute with $p \cdot p$.

Recall that the construction of the $\mathcal{R}$-algebra involves defining an operator R that will commute with p^2 (or $p \cdot p$ in this case). The next step is to construct an operator Q that will satisfy the same commutation relation with p as q does.

If we modify q by a $p \cdot J$ term, we can define this new operator Q by:

$$2(p \cdot p)Q = 2(p \cdot J)J - p(J \cdot J) + pD(D + 3) + 2(p \times J)(D + 1). \tag{3.144}$$

It can be shown that this choice of the operator Q satisfies the desired commutation relation:

$$[p \cdot a, Q \cdot b] = (a \cdot b)D - (a \times b) \cdot J = [p \cdot a, q \cdot b], \tag{3.145}$$

and

$$[p \cdot p, Q] = p(2D + 1) + 2(p \times J) = [p \cdot p, q]. \tag{3.146}$$

The last step in constructing the conformal algebra is to define the vector $R = q - Q$, with commutators:

$$[p \cdot a, R \cdot b] = 0, \tag{3.147}$$

and

$$[p \cdot p, R] = 0. \tag{3.148}$$

R is a vector operator, so it will satisfy the following the commutation relation:

$$[J \cdot a, R \cdot b] = -(a \times b) \cdot R. \tag{3.149}$$

Moreover, R has the same dimension as q does, so we also have:

$$[D, R] = R. \tag{3.150}$$

The generators of the $\mathcal{R}$-algebra are now p, J and R. There is one last commutator needed to determine the $\mathcal{R}$-algebra, namely $[R \cdot a, R \cdot b]$, since we already have all the other commutators given in (3.147), (3.149), and (3.137ii, iv, v).

Obtaining this commutator requires a few steps. We first have:

$$[(p \cdot p)\,(Q \cdot a), (p \cdot p)\,(Q \cdot b)] = (p \cdot p)^2\,[Q \cdot a, Q \cdot b] \\ + (p \cdot p)\,[p \cdot p, Q \cdot b](Q \cdot a) - (p \cdot p)\,[p \cdot p, Q \cdot a](Q \cdot b), \tag{3.151}$$

and by using (3.144), (3.146) and many intermediate steps and formulas (some of these formulas are listed in appendix C), we obtain:

$$(p \cdot p)^2 (Q \times Q) = 2(p \cdot J)p, \tag{3.152}$$

or, as a commutator:

$$(p \cdot p)^2\,[Q \cdot a, Q \cdot b] = 2(p \cdot J)[(a \times b) \cdot p]. \tag{3.153}$$

To obtain the last commutation relation of the algebra, note first that we can write:

$$R \times R = (q - Q) \times (q - Q) = Q \times Q - (q \times Q + Q \times q) \tag{3.154}$$

or

$$\begin{aligned} (p \cdot p)^2\,[R \cdot a, R \cdot b] \;&=\; (p \cdot p)^2\,[Q \cdot a, Q \cdot b] \\ &\quad + (p \cdot p)^2\,([Q \cdot b, q \cdot a] - [Q \cdot a, q \cdot b])\,, \end{aligned} \tag{3.155}$$

where we took into account that $q \times q = 0$.

We already have the first commutator, given in (3.153). After calculating the remaining two commutators and simplifying, we obtain:

$$(p \cdot p)^2 (R \times R) = 2(p \cdot p)(p \cdot J)R - (p \cdot p)(p \times R) - 2(p \cdot J)p \tag{3.156}$$

or

$$\begin{aligned} (p \cdot p)^2 [R \cdot a, R \cdot b] \;&=\; 2(p \cdot p)(p \cdot J)[(a \times b) \cdot R] \\ &\quad - (p \cdot p)(a \times b) \cdot (p \times R) - 2(p \cdot J)[(a \times b) \cdot p]. \end{aligned} \tag{3.157}$$

The $\mathcal{R}$-algebra is now completely determined.

3.5.2 Casimir Invariants

We have constructed the subalgebra with generators p, J, and R, and obtained the complete commutation relations. As desired, the squared-mass operator is one of the Casimir operators of the algebra.

For the remaining of this section, we are trying to construct the Casimir invariants of the algebra. Since $n = 3$, there are $1 + \left[\frac{3}{2}\right]$ Casimir invariants.

The following commutation relations have been established:

$$\begin{aligned}
[R \cdot p, p] &= 0, \quad [R \cdot p, J] = 0, \\
(p \cdot p)[R \cdot p, R] &= -2(p \cdot J)(p \times R) + (R \cdot p)p - (p \cdot p)R, \\
[R \cdot R, p] &= 0, \quad [R \cdot R, J] = 0, \\
(p \cdot p)^2[R \cdot R, R] &= -2\,(p \cdot J)\,(p \times R) - (p \cdot p)R + (R \cdot p)p.
\end{aligned} \tag{3.158}$$

We see that $R \cdot p$ and $R \cdot R$ seem to be promising as parts of the Casimir invariants. To obtain the full Casimir operators, introduce:

$$S = R - \lambda \frac{p}{p \cdot p}, \tag{3.159}$$

with λ a scalar. We obtain that:

$$S \cdot p = R \cdot p - \lambda, \tag{3.160}$$

and

$$(p \cdot p)\,(S \cdot S) = (p \cdot p)\,(R \cdot R) - 2\lambda(R \cdot p) + \lambda^2. \tag{3.161}$$

By commuting it with the new generators of the algebra, namely J, S, and p, we obtain that $S \cdot S$ is a Casimir operator if $\lambda = 1/2$. This gives the following expression for S:

$$(p \cdot p)S = (p \cdot p)R - \frac{1}{2}p. \tag{3.162}$$

With λ fixed at $1/2$, we can show that the two Casimir operators of the algebra $\mathcal{C}_e$ have the simple expressions:

$$\begin{aligned} C_1 &= S \cdot S, \\ C_2 &= S \cdot p + \tfrac{1}{p \cdot p}(p \cdot J)^2. \end{aligned} \tag{3.163}$$

The Casimir invariants of the $\mathcal{R}$-algebra are $C_1,\ C_2$ and $p \cdot p$.

3.5.3 Applying the n-d Theory to the 3-d Theory

As mentioned earlier, the two theories have been developed independently. In this section we show that the two approaches are consistent with each other. The main test is showing that the operators Q and R satisfy the same relations in both theories, and that the Casimir invariant in the n-dimensional theory when applied to $n = 3$ matches the one derived in the original three-dimensional theory.

We consider the following relation that will make the connection between the two cases:

$$M_{ab} = -\epsilon_{abc} J^c \quad \text{or} \quad J_c = -\frac{1}{2} \varepsilon_{abc} M^{ab}. \tag{3.164}$$

One can show fairly easily that the commutation relations (3.137) are satisfied by these expressions.

We also have:

$$\begin{aligned} p^2 x &= -(p \times J) - p, \qquad (3.165) \\ p^2 L_{ab} &= \varepsilon_{abc}(p \cdot J) p^c - p^2 \varepsilon_{abc} J^c = (p \cdot J)\,(a \times b) \cdot p - (p \cdot p)(a \times b) \cdot J, \\ p^2 S_{ab} &= -\varepsilon_{abc}(p \cdot J) p^c = -(p \cdot J)(a \times b) \cdot p, \end{aligned}$$

which yields:

$$S_{ab} S^{ab} = 2p^{-2}(p \cdot J)^2. \tag{3.166}$$

Using these formulas in equation (3.144), we obtain the following expression for the Q operator:

$$\begin{aligned} 2(p\cdot p)Q &= 2(p\cdot J)J - p(J\cdot J) + pD(D+3) \\ &\quad +2(p\times J)(D+1) + 2p - (p\cdot p)^{-1}(p\cdot J)^2 p. \end{aligned} \tag{3.167}$$

Note that the difference between equations (3.144) and (3.167) is given by the last two terms, $2p - (p\cdot p)^{-1}(p\cdot J)^2 p$; as long as they commute with p (and p^2), these terms will not change what we require from the R operator (that is to commute with p^2). We should mention that the choice of the q and Q operators (and, therefore, the R operator and the Casimir invariants) is not unique, only up to quantities which commute with p.

In the n-dimensional case, we obtained that one of the Casimir operators was:

$$C = R\cdot p + \frac{1}{4}S_{ab}S^{ab}.$$

The corresponding Casimir operator in three-dimensions has been obtained from this general one and has the form:

$$C = R\cdot p + \frac{1}{2}S_{ab}S^{ab}. \tag{3.168}$$

Although these two relations seem slightly different, they can be shown to be the same. If we prime the quantities arising from letting $n = 3$ in the general theory, we have that:

$$2p^2 R' = 2p^2 R + p^{-2}(p\cdot J)^2 p - 2p, \tag{3.169}$$

and thus:

$$2p^2(R'\cdot p) = 2p^2(R\cdot p) + (p\cdot J)^2 - 2p^2 \tag{3.170}$$

or

$$\begin{aligned} R'\cdot p &= R\cdot p + \frac{1}{2}(p\cdot J)^2 - 1 \\ &= R\cdot p + \frac{1}{4}S_{ab}S^{ab} - 1, \end{aligned} \tag{3.171}$$

which means that

$$R'\cdot p + \frac{1}{4}S_{ab}S^{ab} = R\cdot p + \frac{1}{2}S_{ab}S^{ab}, \tag{3.172}$$

(where we ignored the constant -1).

This shows that the two Casimirs operators are the same, hence the two theories agree.

3.6 THE 2-DIMENSIONAL CASE

Consider the case, $n = p + q = 2$. This situation is relevant for the Lorentz group $\mathbb{SO}(1,3)$ (or $\mathbb{SO}(3,1)$) and for the ultra-hyperbolic group $\mathbb{SO}(2,2)$ which is used in studying solitons and integrable systems [19], [20]. The results are obtained by directly applying the general theory to $n = 2$; as a consequence, we only describe how the quantities considered there change and their new commutation relations.

In two dimensions, any skew quantity with two indices is a multiple of the completely antisymmetric tensor ε_{ab}. We have thus:

$$M_{ab} = J\varepsilon_{ab}. \tag{3.173}$$

The ε_{ab} tensor satisfies the following identities:

$$\begin{aligned} \varepsilon_{ab}\varepsilon^{cd} &= \sigma\left(\delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c\right), \\ \varepsilon_{ac}\varepsilon^{bc} &= \sigma\delta_a{}^b, \\ \varepsilon_{ab}\varepsilon^{ab} &= 2\sigma, \end{aligned} \tag{3.174}$$

with $\sigma = -1$ for $(p,q) = (1,1)$, and $\sigma = 1$ for $(p,q) = (0,2)$ or $(2,0)$. This is the only case where the signature of the metric affects the commutation relations. We will see shortly that although some of the commutation relations do change, the Casimir operators will be the same in both signatures.

3.6.1 The Case $\sigma = 1$

For $\sigma = 1$, relations (3.174) become:

$$\begin{aligned} \varepsilon_{ab}\varepsilon^{cd} &= \sigma\delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c, \\ \varepsilon_{ac}\varepsilon^{bc} &= \delta_a{}^b, \\ \varepsilon_{ab}\varepsilon^{ab} &= 2. \end{aligned} \tag{3.175}$$

The only commutation relations that will be different from (3.9) are the ones involving M_{ab} and the operators p and q.

By requiring that $[M_{ab}, p_c] = \varepsilon_{ab} [J, p_c] = g_{ac} p_b - g_{bc} p_a$, we obtain:

$$[J, p_a] = \varepsilon_a{}^b p_b.$$

Similarly we can show that:

$$[J, q_a] = \varepsilon_a{}^b q_b.$$

The basic commutation relations for the $\mathcal{C}_e$ algebra are given by:

$$\begin{aligned}
&[p_a, p_b] = 0, \ [q_a, q_b] = 0, \\
&[J, p_a] = \varepsilon_a{}^b p_b, \quad [J, q_a] = \varepsilon_a{}^b q_b, \\
&[D, p_a] = -p_a, \quad [D, q_a] = q_a, \\
&[J, J] = 0, \quad [D, D] = 0, \ [D, J] = 0, \\
&[p_a, q_b] = \varepsilon_{ab} J + g_{ab} D.
\end{aligned} \tag{3.176}$$

The new operators corresponding to (3.16), (3.18), and (3.19) are now:

$$\begin{aligned}
&p^2 x_a = \varepsilon_{ab} J p^b + \tfrac{1}{2} p_a, \\
&L_{ab} = J \varepsilon_{ab} = M_{ab}, \\
&S_{ab} = 0.
\end{aligned} \tag{3.177}$$

In the process of obtaining these relations we used repeatedly that:

$$x \cdot p = k = \frac{n-1}{2} = \frac{1}{2}, \tag{3.178}$$

and

$$\varepsilon_{[a|c|} p_{b]} = \frac{1}{2} \varepsilon_{ab} p_c. \tag{3.179}$$

One also obtains:

$$[x_a, p_b] = \delta_{ab} - p^{-2} p_a p_b,$$

and

$$p^2 [x_a, x_b] = -M_{ab}.$$

Note that these new expressions are in agreement with the commutation relations in (3.22).

The operator Q_a defined in (3.144) becomes in this case:

$$2p^2 Q_a = -J^2 p_a + p_a D^2 - 2\varepsilon_{ab} J p^b D, \tag{3.180}$$

which can be shown to satisfy:

$$[p_a, Q_b] = \varepsilon_{ab} J + \delta_{ab} D = [p_a, q_b], \tag{3.181}$$

as desired, and

$$[J, Q_a] = \varepsilon_a{}^b Q_b. \tag{3.182}$$

We also obtain that:

$$\left[p^2, q_a\right] = 2p_a \left(D - 1\right) - 2\varepsilon_{ab} J p^b = \left[p^2, Q_a\right]. \tag{3.183}$$

As in the previous cases, define $R_a = q_a - Q_a$. Relations (3.181) and (3.183) insure that

$$[p_a, R_b] = 0 \quad \text{and} \quad [p^2, R_a] = 0, \tag{3.184}$$

and (3.176) and (3.182) readily give that:

$$[J, R_a] = \varepsilon_a{}^b R_b. \tag{3.185}$$

We also have:

$$[D, R_a] = R_a. \tag{3.186}$$

There is one more commutator to be determined, namely $[R_a, R_b]$. We have:

$$[R_a, R_b] = -[Q_a, Q_b] + [Q_b, R_a] - [Q_a, R_b]. \tag{3.187}$$

The first commutator in (3.187) is given by:

$$\begin{aligned} p^4[Q_a, Q_b] &= \left[p^2 Q_a, p^2 Q_b\right] + 2p^2 \left(p_a Q_b - p_b Q_a\right) D \\ &\quad - \left[\varepsilon_{ac} J p^c \left(2p^2 Q_b\right) - \varepsilon_{bc} J p^c \left(2p^2 Q_a\right)\right]. \end{aligned} \tag{3.188}$$

After fairly lengthy calculations, we obtain that:

$$\left[p^2 Q_a, p^2 Q_b\right] = p^2 \varepsilon_{ab} J \left(1 - J^2 - D^2\right). \tag{3.189}$$

By finding the remaining commutators in (3.188), we have:

$$[Q_a, Q_b] = 0. \tag{3.190}$$

Note that this relation is consistent with (3.30iii)

$$2p^2[Q_a, Q_b] = -(n-3)(n-2)S_{ab} - 2S_{[b|d|}S^{cd}S_{a]c},$$

since $S_{ab} = 0$.

The second commutator in (3.187) can be written as:

$$\begin{aligned}
2p^2[Q_b, R_a] &= -[J^2 p_b, R_a] - 2\varepsilon_{bc}\,[Jp^c D, R_a] + p_b\left[D^2, R_a\right] \\
&= 2R_a p_b\,(D+1) - R_b p_a D + 2\varepsilon_a{}^c J R_c p_b + \varepsilon_{bc} J R_a p^c.
\end{aligned}$$

By using this expression, it follows easily from (3.187) and (3.190) that:

$$[R_a, R_b] = 0. \tag{3.191}$$

The $\mathcal{R}$-algebra will be now generated by the operators p_a, R_a and J satisfying the following commutation relations:

$$\begin{aligned}
&[R_a, p_b] = 0, \\
&[R_a, R_b] = 0, \\
&[J, R_a] = \varepsilon_a{}^b R_b, \\
&[p_a, p_b] = 0, \\
&[J, p_a] = \varepsilon_a{}^b p_b.
\end{aligned} \tag{3.192}$$

Following the n-dimensional case, one of the Casimir operators should be

$$C_1 = R \cdot p, \tag{3.193}$$

since in this case $S_{ab} = 0$. It is easy to show that the second Casimir invariant is:

$$C_2 = R \cdot R. \tag{3.194}$$

Corollary 12. *The operators $C_1 = R \cdot p$ and $C_2 = R \cdot R$ are the Casimir invariants of the $\mathcal{C}_e(p^2)$ algebra.*

Corollary 13. *The Casimir invariants of the $\mathcal{R}$-algebra are C_1, C_2 and p^2.*

In the language of the general n-dimensional theory,

$$\begin{aligned} C_1 &= S, \\ C_2 &= S_a S^a + p^{-2} C_1^2. \end{aligned} \tag{3.195}$$

3.6.2 The Case $\sigma = -1$

In this case, the identities satisfied by the ε tensor become:

$$\begin{aligned} \varepsilon_{ab}\varepsilon^{cd} &= -\left(\delta_a{}^c \delta_b{}^d - \delta_a{}^d \delta_b{}^c\right), \\ \varepsilon_{ac}\varepsilon^{bc} &= -\delta_a{}^b, \\ \varepsilon_{ab}\varepsilon^{ab} &= -2. \end{aligned} \tag{3.196}$$

Some of the formulas derived in the previous section are affected by this change of sign. We will only list some of the important changes, since the derivation of the results is identical.

(3.196ii) modifies the way the operator J acts on vector operators:

$$[J, p_a] = -\varepsilon_a{}^b p_b, \text{ and } [J, q_a] = -\varepsilon_a{}^b q_b. \tag{3.197}$$

Although the position operator x_a is still defined by

$$p^2 x_a = \varepsilon_{ab} J p^b + \frac{1}{2} p_a,$$

we have now that:

$$p^2 (x \cdot x) = -J^2 - \frac{1}{4},$$

which leads to the following expression of the operator Q_a:

$$2p^2 Q_a = J^2 p_a + p_a D^2 - 2\varepsilon_{ab} J p^b D. \tag{3.198}$$

(3.198) still yields the same commutation relations:

$$[p_a, Q_b] = \delta_{ab} D + M_{ab} = [p_a, q_b],$$

and

$$[p^2, Q_a] = 2p_a (D - 1) - 2\varepsilon_{ab} J p^b = [p^2, q_a].$$

The commutator $[p^2 Q_a, p^2 Q_b]$ becomes in this case:

$$\left[p^2 Q_a, p^2 Q_b\right] = p^2 \varepsilon_{ab} J \left(1 + J^2 - D^2\right), \tag{3.199}$$

but we still obtain:

$$[Q_a, Q_b] = 0.$$

One can show that the commutation relations of the $\mathcal{R}$-algebra are now given by:

$$\begin{aligned}
&[R_a, p_b] = 0, \\
&[R_a, R_b] = 0, \\
&[J, R_a] = -\varepsilon_a{}^b R_b, \\
&[p_a, p_b] = 0, \\
&[J, p_a] = -\varepsilon_a{}^b p_b.
\end{aligned} \tag{3.200}$$

The Casimir invariants are not affected by the change of signature, they maintain the same expressions as in (3.193) and (3.194).

This concludes the study of the null decomposition of conformal algebras.

4.0 SELF-DUAL METRICS ON INTERSECTIONS OF QUADRICS

> "A *self-dual metric* or *conformal structure* on a four-manifold $\mathbb{M}$ is a Riemannian metric g or conformal class $[g]$ for which the Weyl conformal curvature tensor C is self-dual" [14].

$\mathbb{M}$ is then called a *self-dual manifold*. Self-dual metrics have been constructed in various ways, among which as solution of nonlinear equations (Yang-Mills for example), or by solving linear equations [14], [17].

Penrose introduced the concept of nonlinear graviton [28] via the twistor correspondence, based on the idea that self-dual conformal metrics on four-manifolds arise generally from holomorphic families of $\mathbb{CP}^1$'s in complex 3-manifolds.

For metrics on complex manifolds the signature of the metric is not important. For this reason, many self-dual metrics are constructed on Riemannian manifolds. Explicitly anti-self-dual metrics are of large interest as well [4].

When studying metric manifolds, there are some tensors that are essential. Among them, the metric tensor g_{ab}, the Riemann curvature tensor R_{abcd}, the Weyl curvature tensor C_{abcd}, the Ricci tensor R_{ab}, and the scalar curvature (Ricci scalar) R.

We had mentioned earlier that twistor theory is a conformal theory. The Weyl tensor is the part of the Riemann curvature tensor that remains invariant under conformal transformations and hence is paramount in twistor theory.

Following the example of property 14 in section 2.1, we can write the Weyl tensor in spinor language as:

$$C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}, \tag{4.1}$$

where Ψ_{ABCD} and $\overline{\Psi}_{A'B'C'D'}$ are totally symmetric spinors. Here Ψ_{ABCD} is called the anti-self-dual part of the Weyl tensor, and $\overline{\Psi}_{A'B'C'D'}$ the self-dual part.

Generally, Ψ_{ABCD} and $\overline{\Psi}_{A'B'C'D'}$ are complex conjugates of each other. In complex space-times, Riemannian space-times, or real space-times of Riemannian or ultra-hyperbolic signature, the two spinors are unrelated. In this case it is possible for one of them to vanish without affecting the other. If the anti-self-dual part vanishes, the Weyl curvature tensor is said to be *self-dual*. This is related to the fact that the group $\mathbb{SO}(4)$ is not simple.

Twistor theory can be extended to curved space-times under certain circumstances; a natural extension exists for space-times whose Weyl tensor is either self-dual or anti-self-dual [36], where the dual of the Weyl tensor is given by:

$$^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{ef}C_{efcd}. \tag{4.2}$$

The complexified compactified Minkowski space-time $\mathbb{CM}^c$ can be interpreted as a complex four-dimensional quadric embedded in $\mathbb{CP}^5$, via the Klein representation. This analogy brings with it the whole richness of methods of algebraic geometry. The twistor space on this quadric is described by $\mathbb{CP}^3$, as well as the dual twistor space. On this quadric we obtain both types of metrics, self-dual and anti-self-dual. We consider here a linear pencil of quadrics, intersecting in a three-dimensional space which is interpreted as the projective twistor space. We obtain a variety of metrics on this space, depending on the choice of the parameters defining the quadrics, as it will be seen in section 4.3.

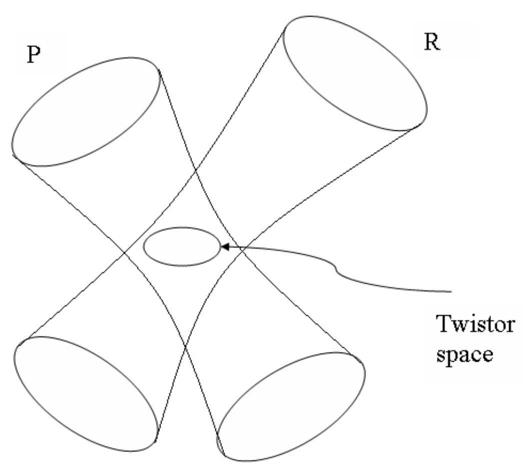

Figure 4.1: The twistor space on the intersection of quadrics.

4.1 THE KLEIN REPRESENTATION

In section 2.3.5 we have seen that (projective) twistors can be described in terms of the geometry of a complex projective space $\mathbb{CP}^3$. As we will see shortly, the complex lines in a $\mathbb{CP}^3$ will correspond to points in complex Minkowski space, that is to points on a quadric embedded in $\mathbb{CP}^5$.

Recall that in section 2.2 we briefly described one way of compactifying the Minkowski space: one starts with a six-dimensional pseudo-Euclidian space, with a flat metric of signature $(2,4)$ leading to the line element:

$$ds^2 = dt^2 + dv^2 - dw^2 - dx^2 - dy^2 - dz^2. \tag{4.3}$$

Manifolds of dimension six are particularly important due to the fact that twistors are the spinors for the group $\mathbb{O}(6,\mathbb{C})$.

The null cone N of the origin of $\mathbb{E}^6$ is then described by:

$$t^2 + v^2 - w^2 - x^2 - y^2 - z^2 = 0. \tag{4.4}$$

Recall from section 2.2 that the conformal structure of a space-time is given by the topology of its null cones. Consider $\mathcal{N}$, the space of generators of N; these generators are the set of points for which the ratio $t : v : w : x : y : z$ is constant and which satisfy (4.4). To investigate the topology of $\mathcal{N}$, intersect it with a five-dimensional sphere S^5 [12]:

$$t^2 + v^2 + w^2 + x^2 + y^2 + z^2 = 2. \tag{4.5}$$

It is obvious that this sphere will intersect each of the generators of $\mathcal{N}$ twice. From (4.4) and (4.5) we obtain the following equations:

$$\begin{aligned} w^2 + x^2 + y^2 + z^2 &= 1, \\ t^2 + v^2 = 1. \end{aligned} \tag{4.6}$$

This tells us that the intersection of S^5 with $\mathcal{N}$ has the topology of $S^1 \times S^3$.

Since the sphere S^5 intersects each generator twice, the intersection will be in fact a double cover of the space of generators $\mathcal{N}$. In order to obtain $\mathcal{N}$, we must identify antipodal points of S^1 and S^3 simultaneously, which gives again a $S^1 \times S^3$ topology [28].

$\mathcal{N}$ has a conformal metric, due to the fact that any two cuts of N have metrics which are conformally related. The group $\mathbb{O}(2,4)$ preserves (4.3) and so it maps $\mathcal{N}$ to itself, preserving its conformal metric.

Let $x^a = (t, x, y, z)$. Note that the map:

$$x^a \longmapsto \left(t, \frac{1}{2}\left(1-\Delta\right), -\frac{1}{2}\left(1+\Delta\right), x, y, z\right) \tag{4.7}$$

embeds the Minkowski space in the null cone N via $v = \frac{1}{2}\left(1-\Delta\right)$, $w = -\frac{1}{2}\left(1+\Delta\right)$, where Δ is given by $x_a x^a$, as in 2.3.

From (4.7) we see that

$$v - w = 1, \tag{4.8}$$

so the image of the Minkowski space via the map in (4.7) is the intersection of N with the hyperplane in (4.8).

This relation tells us that the embedding is in fact isometric:

$$dv = dw \Longrightarrow ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

All these observations lead to the graphical representation in Figure 4.2 [28]. We see there that the Minkowski space has the form of a "paraboloid". From (4.8) we obtain that $v = w + 1$ and so the equation satisfied by the generators of the null cone, $\mathcal{N}$, becomes:

$$v = w + 1 = \frac{1}{2}\left(t^2 - x^2 - y^2 - z^2\right).$$

On any generator of N with $v - w \neq 0$ we can find a point satisfying (4.8), hence a point of $\mathbb{M}$. (Note that the plane $v - w = 0$ intersects the null cone at the origin of $\mathbb{E}^6$.) It follows that $\mathbb{M}$ is identified with a subset of $\mathcal{N}$, and the conformal metric on $\mathbb{M}$ agrees with the conformal metric on $\mathcal{N}$.

The generators of N which lie in $v - w = 0$ correspond to points at infinity for $\mathbb{M}$. Lines through the origin in $\mathbb{E}^6$ correspond to points in a projective five-space $\mathbb{P}^5$ determined by the ratios $t : v : w : x : y : z$.

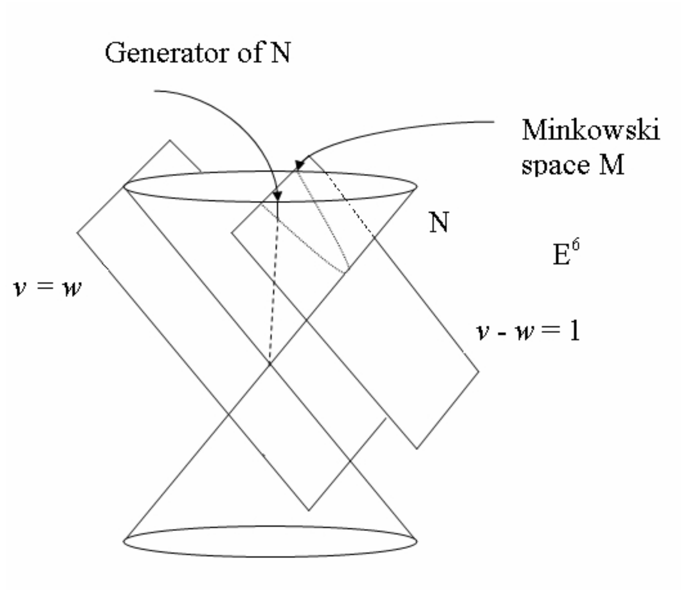

Figure 4.2: The Klein representation.

The generators of N define the points of a quadric in $\mathbb{P}^5$ whose equation is (4.4) and which will be identified with $\mathbb{M}^c$. The points of this quadric not lying in $v - w = 0$ correspond precisely to the points of $\mathbb{M}$, and the points which lie on $v - w = 0$ correspond to the points at infinity, providing thus the compactification of $\mathbb{M}$.

$\mathbb{M}^c$ has a well-defined conformal structure everywhere. As a subspace of $\mathbb{P}^5$, $\mathbb{M}^c$ has a canonically defined conformal structure of signature $(1, 3)$.

The pseudo-orthogonal group $\mathbb{O}(2, 4)$ of linear transformations in $\mathbb{E}^6$ that preserve (4.4), preserves N, and so induces a transformation on $\mathbb{P}^5$ that sends $\mathbb{M}^c$ to itself. Since it preserves linearity in $\mathbb{E}^6$, it sends projective straight lines in $\mathbb{M}^c$ to other such lines, i.e. light rays, hence it preserves the conformal structure of $\mathbb{M}^c$.

We saw in chapter 2 that $\mathbb{O}(2,4)$ is a double cover of the conformal group $C(1,3)$, and the Poincaré group is a subgroup of $C(1,3)$, characterized now by leaving invariant the hyperplane $v-w=1$, as well as the null cone N. If we add the dilations to the Poincaré group, then the plane $v-w=1$ is not invariant, but the family of hyperplanes $v-w=$ constant is transformed into itself with the only invariant $v-w=0$.

Twistor theory is a complex theory, so we will need to consider the complexification of all these spaces described above. Recall that the projective twistor space $\mathbb{PT}$ is described as a $\mathbb{CP}^3$ by means of the independent complex ratios $Z^0 : Z^1 : Z^2 : Z^3$ corresponding to the twistor $Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$. In twistor space $\mathbb{T}$ complex linear two-spaces represent points in $\mathbb{CM}$. These two-spaces can be represented as equivalence classes of proportional (nonzero) simple skew twistors $P^{\alpha\beta}$ as described in (2.78), section 2.3.6. In terms of $\mathbb{PT}$, these two-spaces are represented as complex projective lines, each by a $\mathbb{CP}^1$. These lines in $\mathbb{PT}$ represent points of $\mathbb{CM}$.

If we want to obtain a finite point of $\mathbb{CM}$, we must require that its representative line in $\mathbb{PT}$ does not meet the line I given by $P^{\alpha\beta} = I^{\alpha\beta}$, where $I^{\alpha\beta}$ is the twistor at infinity from 2.3.3. This line I represents the system of twistors Z^α with vanishing projection part $\pi_{A'}$. Such a line though, being represented by a simple skew twistor $P^{\alpha\beta}$, must correspond to a point in $\mathbb{CM}^c$, and it corresponds in fact to the vertex of $C\mathfrak{I}$, the complex null cone at infinity. We have thus that the points of $\mathbb{CM}^c$ represent the lines of $\mathbb{PT}$.

It is known that any simple skew vector with two indices F_{ab} satisfies the equation:

$$\varepsilon_{abcd} F^{ab} F^{cd} = 0.$$

It follows that the lines of $\mathbb{PT}$ are solutions of the equation:

$$\epsilon_{\alpha\beta\gamma\delta} P^{\alpha\beta} P^{\gamma\delta} = 0. \tag{4.9}$$

This is the equation of a quadric in $\mathbb{CP}^5$.

$P^{\alpha\beta}$, being skew-symmetric, has six independent components which can be organized in the following skew matrix:

$$\begin{pmatrix} 0 & p & -q & x \\ -p & 0 & z & -y \\ q & -z & 0 & r \\ -x & y & -r & 0 \end{pmatrix}.$$

Then the quadric in (4.9) can be written as:

$$\mathbf{x} \cdot \mathbf{y} = 0,$$

with

$$\mathbf{x} = (x, y, z)\,, \;\; \mathbf{y} = (p, q, r)\,.$$

Such a correspondence, in which the lines of a $\mathbb{CP}^3$ are represented as the points of a quadric in $\mathbb{CP}^5$, is known as the *Klein representation*. This representation is basic to the whole of twistor geometry and it leads to the following fundamental picture in Figure 4.3.

One important aspect is that of incidence of the various quantities considered. As seen in section 2.3, the incidence between a twistor Z^α and a dual twistor W_α is

$$Z^\alpha W_\alpha = 0,$$

and in $\mathbb{PT}$ this translates into the point Z lying on the plane W, or in $\mathbb{PT}^*$ by the point W lying on the plane Z.

Similarly, the incidence between a point X in $\mathbb{CM}^c$ and the α-plane Z, which can be written as

$$P^{[\alpha\beta} Z^{\gamma]} = 0 \text{ or } P_{\alpha\beta} Z^\beta = 0, \tag{4.10}$$

can be represented by the point Z lying on the line X in $\mathbb{PT}$.

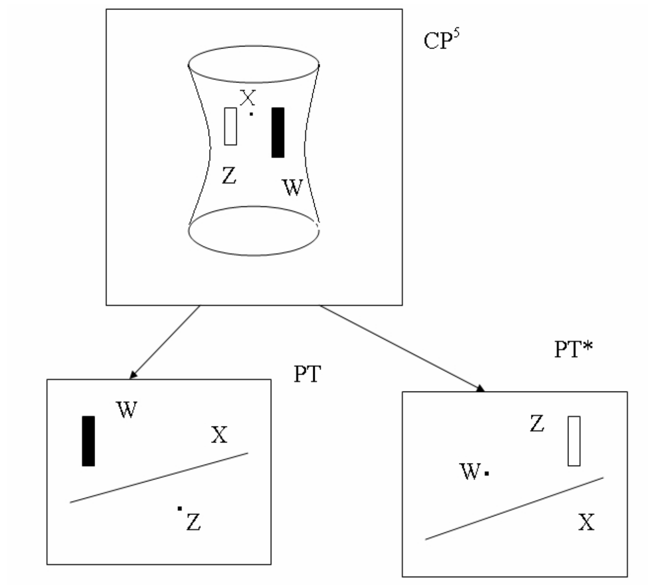

Figure 4.3: Various geometric correspondences.

One other important result is described in the following proposition [28]:

Proposition 14. *Through every null geodesic in $\mathbb{CM}^c$ there passes a unique α-plane and a unique β-plane; when an α-plane and a β-plane intersect, they always intersect in a null geodesic.*

Two distinct α-planes meet in a unique point of $\mathbb{CM}^c$ (two distinct points in $\mathbb{PT}$ can always be joined by a unique line). Same property holds for β-planes as well.

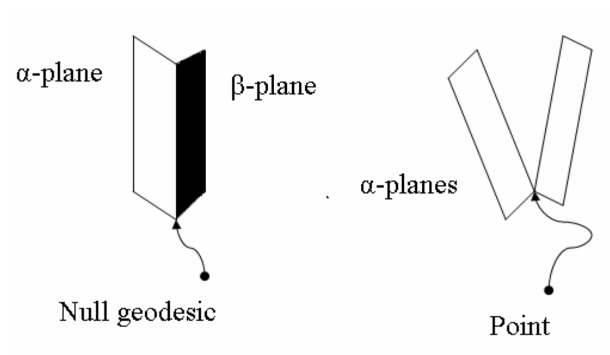

Figure 4.4: Intersections of α-planes and β-planes.

4.2 SOME BASICS ON QUADRICS

Following [8], a *quadric hypersurface* $Q \subset \mathbb{CP}^n$ may be represented as the locus of the quadratic form:

$$Q(X, X) = \sum_{i,j=0}^{n} q_{ij} X_i X_j, \tag{4.11}$$

where the $(n+1) \times (n+1)$ matrix $Q = (q_{ij})$ is symmetric. Here, by an abuse of notation we use the letter Q to denote both the quadric and the matrix of its coefficients.

The *rank* of the quadric Q is defined to be the rank of the matrix Q; since the only invariant of a symmetric form over $\mathbb{C}$ is its rank, two quadrics Q_1, $Q_2 \subset \mathbb{CP}^n$ will be projectively isomorphic iff they have the same rank.

Some useful results about quadrics are [8]:

A quadric $Q \subset \mathbb{CP}^n$ is smooth if and only if it has maximal rank $n+1$.

A quadric $Q \subset \mathbb{CP}^n$ of rank $n-k$ is singular along a k-plane $\Lambda \subset Q \subset \mathbb{CP}^n$.

The following proposition makes an important connection between twistor theory and algebraic geometry [8], [36]:

Proposition 15. *A smooth quadric Q of dimension m contains no linear subspaces of dimension greater than $m/2$; on the other hand:*

a) If $m = 2n+1$ is odd, then F contains an irreducible $(n+1)(n+2)/2$-dimensional family of n-planes; while

b) If $m = 2n$ is even, then F contains two irreducible $n(n+1)/2$-dimensional families of n-planes and moreover for any two distinct n-planes, Λ, $\Lambda' \subset F$

$$\dim(\Lambda \cap \Lambda') \equiv \begin{cases} 0, \text{ if } n \text{ is even} \\ 1, \text{ if } n \text{ is odd} \end{cases}$$

if and only if Λ and Λ' belong to the same family.

We have seen that the Minkowski space can be interpreted as a four-dimensional quadric embedded in $\mathbb{CP}^5$. If the quadric Q in the above proposition has dimension four, then according to the second part of the proposition, Q contains two irreducible 3-dimensional families of two-planes. These are the α-planes and the β-planes described in 2.3.4.

We see that if Λ and Λ' are two α-planes, the dimension of their intersection is congruent to $2 \bmod(2) = 0$, as we know they intersect in a point; same remark hold for two β-planes as well. On the other hand, if Λ is an α-plane and Λ' is a β-plane, then $\dim(\Lambda \cap \Lambda') \not\equiv n \bmod 2$, in fact it is equal to one, since they intersect in a line.

4.2.1 Pencil of Quadrics

In this thesis, we will consider intersection of quadrics, namely we will consider linear systems of quadrics, called *pencils of quadrics*.

Let P and R be two quadrics in $\mathbb{CP}^5$ given by:

$$P = \sum P_{\alpha\beta} Z^\alpha Z^\beta; \quad R = \sum R_{\alpha\beta} Z^\alpha Z^\beta, \tag{4.12}$$

with $P_{\alpha\beta}$, $R_{\alpha\beta}$ the 6×6 symmetric matrices of the coefficients. In the following we will refer to $P_{\alpha\beta}$, $R_{\alpha\beta}$ as P, and R, respectively.

Consider the generic pencil $L = \{F_t\}$ generated by these two quadrics, where the quadrics F_t are given by:

$$F_t = \lambda P + \mu R, \text{ with } (\lambda, \mu) \in \mathbb{C}^2 \setminus (0,0) \text{ and } t = \lambda : \mu. \tag{4.13}$$

Each pair (λ, μ) determines a quadric F_t, and since they are not both zero, they also determine a $\mathbb{CP}^1$. Without loss of generality, assume $\mu \neq 0$ and denote $t = \lambda/\mu$. The quadrics F_t are thus given by the locus of the following expression:

$$F_t = tP + R = 0. \tag{4.14}$$

Let X be the intersection of the quadrics P and R:

$$X = P \cap R = \bigcap_{t \in \mathbb{CP}^1} F_t. \tag{4.15}$$

Since the pencil is assumed general, the quadrics P and R will intersect transversely in X, which is a smooth variety of degree 4 and dimension 3. We regard this intersection as a projective twistor space represented by a $\mathbb{CP}^3$. In this case we have [9]:

$$L = \{F : X \subset F\}\,. \tag{4.16}$$

Proposition 16. *The pencil L spanned by the quadrics P and R contains exactly six singular elements if and only if P and R intersect transversely.*

We will say that the pencil L is *simple* if the equivalent conditions of the proposition above are satisfied. Also [9]

Lemma 17. *(Normal Form for Simple Pencils) Let L be a simple pencil of quadrics in $\mathbb{CP}^5$. Then there exists a set of homogeneous coordinates on $\mathbb{CP}^5$ in terms of which all $F \in L$ are diagonal; i.e., L is generated by quadrics Q and R where:*

$$P\left(X\right) = \sum_{i=1}^{6} X_i^2, \tag{4.17}$$

and

$$R(X) = \sum_{i=1}^{6} \lambda_i X_i^2, \tag{4.18}$$

with all λ_i distinct.

F_t will be singular exactly when the determinant $\det(tP + R)$ vanishes; since this determinant is a sextic polynomial in t, this will occur for all $t = -\lambda_i$, where $1 \leq i \leq 6$. If at least one of the quadrics generating the pencil is nonsingular, then $\det(tP + R) \not\equiv 0$.

The ratio $\lambda : \mu$ will be interpreted as the time coordinate. By varying (λ, μ) we obtain different quadrics.

For the generic case, the "time" coordinate is not described by a $\mathbb{CP}^1$ anymore, but by a hyperelliptic curve of genus two.

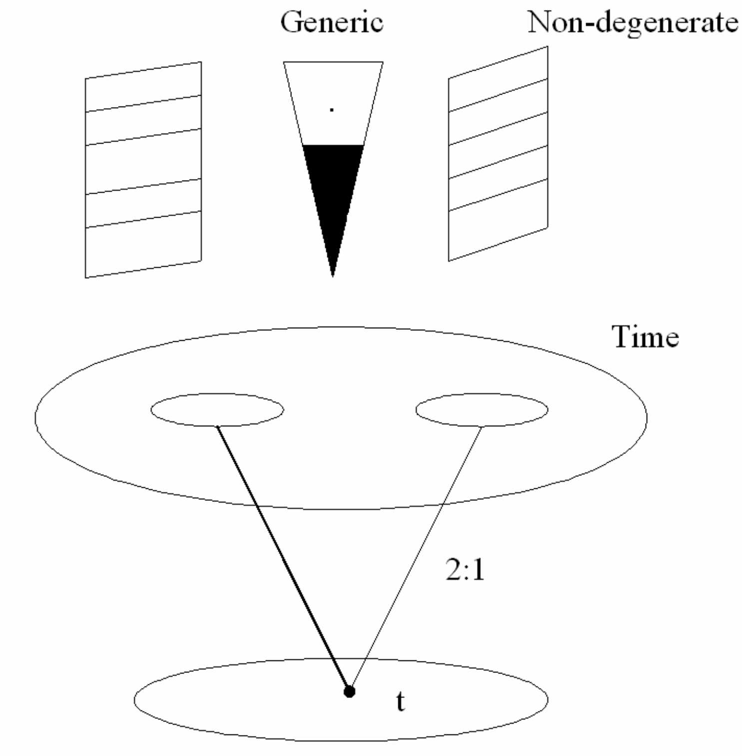

Figure 4.5: The time coordinate as a hyperelliptic curve in the generic case.

In this picture the fibers over the time curve are represented by $\mathbb{CP}^3$, and the full space-time is a $\mathbb{CP}^3$ over a hyperelliptic curve $y^2 = \det(tP + R)$. In both cases, generic and non-degenerate, the space-time manifold is compact, although at certain points the metric can be singular.

We had mentioned that $\det(tP+R)$ is a polynomial of degree six, which means that over $\mathbb{C}$ it will have six roots. The following possibilities arise:

- The roots are all distinct; in this case $\det(tP+R)$ factors into linear factors and the two quadrics P and R are simultaneously diagonalizable.
- The roots are not all distinct, but the minimal polynomial of RP^{-1} factors into distinct linear factors; in this case the quadrics P and R are again simultaneously diagonalizable.

- The roots are not all distinct, and the minimal polynomial of RP^{-1} does not factor into distinct linear factors. In this case P and R are not simultaneously diagonalizable any longer, but can be written in Jordan Canonical Form.

In this chapter we study the case of simultaneously diagonalizable quadrics. The third case can be approached by using Jordan canonical blocks for the matrices P and R, and will make the topic of future research.

We should also mention that the present approach to the study of these metrics is mostly computational; future plans involve supporting these results by a theoretical frame.

4.3 CONSTRUCTION OF THE METRICS

In this section we will describe the construction of the metrics on the twistor space on the intersection of the quadrics P and R. If the matrices are simultaneously diagonalizable, then without loss of generality we can assume that one of them is the identity matrix. Since we work over the complex numbers $\mathbb{C}$ we can choose the quadrics to have the following form:

$$\begin{aligned} P &= x^2 + y^2 + z^2 - s^2 - u^2 - v^2, \\ R &= ax^2 + by^2 + cz^2 - ps^2 - qu^2 - rv^2. \end{aligned} \tag{4.19}$$

We want to study the three-dimensional families of two-planes/subspaces inside the pencil $tP + R = 0$. In order to obtain these families on the intersection of the quadrics P and R for a fixed t value, we can write:

$$\begin{pmatrix} s \\ u \\ v \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longleftrightarrow \mathbf{y} = M\mathbf{x}, \quad \text{where } \mathbf{y} = \begin{pmatrix} s \\ u \\ v \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4.20}$$

Equation (4.19) can be written now as:

$$\begin{aligned} P &= \mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y}, \\ R &= \mathbf{x}^T A\mathbf{x} - \mathbf{y}^T B\mathbf{y}. \end{aligned} \tag{4.21}$$

Here A and B are diagonal matrices: $A = diag(a, b, c)$, $B = diag(p, q, r)$. Being diagonal, we also have $A^T = A$, and $B = B^T$.

From the equation of the pencil of quadrics (4.13), we obtain now:

$$t(\mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y}) + \mathbf{x}^T A\mathbf{x} - \mathbf{y}^T B\mathbf{y} = \mathbf{0} \tag{4.22}$$

or, by using $\mathbf{y} = M\mathbf{x}$:

$$t\left(\mathbf{x}^T\mathbf{x} - \mathbf{x}^T M^T M\mathbf{x}\right) + \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T A\mathbf{x} - \mathbf{x}^T M^T BM\mathbf{x} = \mathbf{0} \qquad \text{for all } \mathbf{x}. \tag{4.23}$$

This can be written as:

$$\begin{aligned} \mathbf{0} &= \mathbf{x}^T\left(tI - tM^T M + A - M^T BM\right)\mathbf{x} && (4.24)\\ &= \mathbf{x}^T[t(I - M^T M) + (A - M^T BM)]\mathbf{x} \quad \text{for all } \mathbf{x}. && (4.25)\end{aligned}$$

Since $\mathbf{x}^T P\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$, this is equivalent to $P = 0$ (when P is symmetric). Therefore:

$$t(I - M^T M) + (A - M^T BM) = \mathbf{0} \tag{4.26}$$

or

$$(tI + A) = M^T(tI + B)M. \tag{4.27}$$

Note that the matrices $I - M^T M$ and $A - M^T BM$ in (4.26) are symmetric, which allows three degrees of freedom for M. Since we work over complex numbers, and A and B are diagonal matrices, we can write

$$tI + A = C^2 \text{ and } tI + B = D^2 \tag{4.28}$$

for some diagonal matrices C and D.

With this notation, equation (4.27) becomes:

$$C^2 = M^T D^2 M, \tag{4.29}$$

which leads to the following possibilities for the determinant of the matrix C:

$$(\det C)^2 = (\det M)^2(\det D)^2 \Longrightarrow \det C = \pm \det M \cdot \det D. \tag{4.30}$$

From (4.29) we also have that

$$CIC = M^T DDM. \tag{4.31}$$

By multiplying on the left and on the right by C^{-1}, respectively:

$$I = (C^{-1} M^T D)(DMC^{-1}). \tag{4.32}$$

Introduce the following notation:

$$R = DMC^{-1} \tag{4.33}$$

with the transpose

$$R^T = C^{-1} M^T D, \tag{4.34}$$

since C and D are diagonal matrices. (4.32) yields then:

$$I = R^T R. \tag{4.35}$$

We obtain thus that R is orthogonal, hence it generates $\mathbb{SO}(3)$.

4.3.1 Parameterizing the rotation group $\mathbb{SO}(3)$

Note that R has three degrees of freedom. The standard way of parameterizing the rotations is by using the Euler angles. Using this method though will lead to very complicated calculations, and for this reason we will use a different method. We had mentioned earlier that the study of these metrics is mostly computational. From this point of view, one of our main objectives has been to work with the simplest possible expressions.

Define R by the following formula:

$$R = (I - X)(I + X)^{-1}, \tag{4.36}$$

where X is a 3×3 skew matrix defined by:

$$X = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}. \tag{4.37}$$

By substituting X into (4.36), we obtain the following expression for the rotation matrix:

$$R = \frac{1}{1+x^2+y^2+z^2}\begin{bmatrix} 1+x^2-y^2-z^2 & 2\,(xy-z) & 2\,(xz+y) \\ 2\,(xy+z) & 1+y^2-x^2-z^2 & 2\,(yz-x) \\ 2\,(xz-y) & 2\,(yz+x) & 1+z^2-x^2-y^2 \end{bmatrix}. \quad (4.38)$$

From (4.28) we have that

$$C = diag(\sqrt{t+a}, \sqrt{t+b}, \sqrt{t+c}), \quad (4.39)$$

and

$$D = diag(\sqrt{t+p}, \sqrt{t+q}, \sqrt{t+r}). \quad (4.40)$$

Also, from the parameterization $R = DMC^{-1}$ in (4.28), we obtain:

$$M = D^{-1}RC. \quad (4.41)$$

D being diagonal implies that dD is also diagonal, so then we have the relation:

$$d(D^{-1}) = -D^{-1}(dD)D^{-1} = -D^{-2}d(D), \quad (4.42)$$

since in this case D and dD commute.

Other useful relations are:

$$d(C^2) = 2CdC \Longrightarrow dC = \frac{1}{2}C^{-1}d(C^2). \quad (4.43)$$

Likewise

$$dD = \frac{1}{2}D^{-1}d(D^2). \quad (4.44)$$

By differentiating (4.41) now, and using (4.42)-(4.44), we obtain that:

$$\begin{aligned} dM &= d(D^{-1})RC + D^{-1}(dR)C + D^{-1}R(dC) \quad (4.45) \\ &= -D^{-2}(dD)RC + D^{-1}(dR)C + D^{-1}R(dC) \\ &= -D^{-2}\frac{1}{2}D^{-1}d(D^2)RC + D^{-1}(dR)C + D^{-1}R\frac{1}{2}C^{-1}d(C^2). \end{aligned}$$

Multiply on the left and right by $2D^3$ and C, respectively:

$$2D^3(dM)C = -d(D^2)RC^2 + 2D^2(dR)C^2 + D^2R[d(C^2)]. \tag{4.46}$$

From here we obtain:

$$\begin{aligned}\det(dM) &= \frac{\det(2D^3(dM)C)}{\det(2D^3C)} \\ &= \frac{1}{\det(2D^3C)}\det[-d(D^2)RC^2 + 2D^2(dR)C^2 + D^2Rd(C^2)],\end{aligned}$$

and by denoting:

$$G = \det(dM), \tag{4.47}$$

we also have:

$$G = \lambda \det(-d(D^2)RC^2 + 2D^2(dR)C^2 + D^2R[d(C^2)]), \tag{4.48}$$

where λ is the conformal factor

$$\lambda = \frac{1}{8\det(D^3C)}. \tag{4.49}$$

Let now:

$$G' = \det(-d(D^2)RC + 2D^2(dR)C^2 + D^2R[d(C^2)]), \tag{4.50}$$

which allows us to rewrite (4.48) as:

$$G = \lambda G'. \tag{4.51}$$

G' has a factor of dt, and so we will write $G' = gdt$ where g is the metric tensor.

4.3.2 Simplifying the Calculations

As mentioned earlier our desire for simplicity stems from the difficulty of using Maple to study the properties of the various metrics g depending on our choice of the quadrics. Calculating all the necessary tensors by hand is nearly impossible.

One way of simplifying the use of Maple is the following: in (4.39) and (4.40) we showed that the matrices C and D must be diagonal matrices with entries of the form $\sqrt{t+a_i}$ where a_i are the coefficients of the two quadrics P and R generating the pencil. We can redefine these matrices as follows.

From (4.27) we obtain that:

$$(M^T)^{-1}(tI + A) = (tI + B)M \tag{4.52}$$

or, after some simplifications:

$$(tI + B)^{-1} = M(tI + A)^{-1}M^T, \tag{4.53}$$

where

$$(tI + A)^{-1} = diag\left(\frac{1}{\sqrt{t+a}}, \frac{1}{\sqrt{t+b}}, \frac{1}{\sqrt{t+c}}\right). \tag{4.54}$$

This is true because $tI + A$ is a diagonal matrix. We define now

$$C = 1/\sqrt{tI + B} \quad \text{and} \quad D = 1/\sqrt{tI + A}. \tag{4.55}$$

The results presented here have been obtained in the general case, where the six poles take arbitrary values.

4.4 GEOMETRIC PROPERTIES OF METRICS

The problem is reduced now to studying the properties of the metrics that arise on the twistor space of the intersection of the quadrics P and R for various choices of the poles of the matrices C and D, namely for various choices of the constants a, b, c, p, q and r.

In order to describe these properties, let us first start with briefly defining the relevant tensors in the study of the properties of these metrics:

- The *Riemann curvature* tensor R_{abcd}. In terms of the metric tensor g_{ab}, the Riemann tensor is given by the following formula:

$$R^a{}_{bcd} = \frac{\partial \Gamma^a_{bd}}{\partial x^c} - \frac{\partial \Gamma^a_{bc}}{\partial x^d} + \Gamma^\rho_{bd}\Gamma^a_{\rho c} - \Gamma^\rho_{bc}\Gamma^a_{\rho d}, \tag{4.56}$$

where Γ^a_{bc} is the Christoffel symbol of the second kind, depending on the first derivatives of the metric tensor in some coordinate frame:

$$\Gamma^a_{bc} = \frac{1}{2} g^{a\rho} \left\{ \frac{\partial g_{\rho c}}{\partial x^b} + \frac{\partial g_{\rho b}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^\rho} \right\}. \tag{4.57}$$

$R^a{}_{bcd}$ is a measure of the noncommutativity of the covariant derivative [35]:

$$R_{abc}{}^d V^c = \left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) V^d. \tag{4.58}$$

- The symmetry properties of the Riemann tensor are listed below:

$$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc}, \tag{4.59}$$

and

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{4.60}$$

The number of independent components of the Riemann tensor for an n-dimensional space is

$$\frac{n^2\left(n^2-1\right)}{12}. \tag{4.61}$$

If all the components of the Riemann tensor vanish, the metric is said to be *flat*.

- The *Ricci tensor* R_{ab}: this tensor is the contraction of the Riemann tensor on its first and third index (or alternatively on its second and fourth index):

$$R^a{}_{bac} = R_{bc}. \tag{4.62}$$

The Ricci tensor is symmetric:

$$R_{ab} = R_{ba}, \tag{4.63}$$

and has $n(n+1)/2$ independent components in an n-dimensional space.

If the Ricci tensor has all components equal to zero, the space-time is said to be *Ricci flat*, and the metric g_{ab} is called the *vacuum metric* [36].

- The *Ricci scalar* R: it is obtained from the Ricci tensor by contraction on its two indices:

$$R = R^a{}_a. \tag{4.64}$$

If R is zero, the metric is said to be *scalar flat*.

- One other important tensor, especially in twistor theory, due to its conformal character, is the *Weyl* tensor: this is the trace-free part of the Riemann tensor and is the only one who is conformal invariant [35]:

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b}. \tag{4.65}$$

In an n-dimensional space-time the number of independent components of the Weyl tensor is $\frac{n(n-3)(n+1)(n+2)}{12}$ for $n > 3$, and 0 for $n \leq 3$ [24].

The Weyl tensor is trace-free on all its indices and satisfies the same symmetry properties as the Riemann tensor does. Although $C_{abc}{}^d$ is the conformally invariant quantity, due to the symmetry in the pairs of indices (ab) and (cd) we prefer working with $C_{ab}{}^{cd}$.

If this tensor vanishes, the metric is called *conformally flat* [36].

- In *conformal vacuum*, the Weyl tensor satisfies:

$$g^{ab}\left(\nabla_a - \lambda_a\right)C_{bcde} = 0, \tag{4.66}$$

where ∇_a is the covariant derivative, and λ_a is the gradient of the logarithm of the conformal factor. By solving these equations, we can obtain the conformal factor λ.

- If the Ricci tensor is a multiple of the metric tensor, the space-time is called *Einstein*:

$$R_{ab} = \frac{R}{n} g_{ab}. \tag{4.67}$$

- As defined in (4.2), in a four-dimensional space, if the dual of the Weyl tensor is the same as the original tensor, the space is called *self-dual*:

$$^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{ef} C_{efcd} = C_{abcd}. \tag{4.68}$$

4.4.1 Singularities of the metric

We mentioned earlier that although this metric is compact and self-dual, it is not defined everywhere. One way of deciding whether the singularities of the metric are removable or not is to use the characteristic polynomial for the matrix of the Weyl tensor.

The metric g has weight two:

$$g \ \sim \ w^2 \Longrightarrow g^{-1} \ \sim \ w^{-2}, \tag{4.69}$$

and so the Weyl tensor varies like w^{-2}:

$$C_{ab}{}^{cd} \ \sim \ w^{-2}. \tag{4.70}$$

The components of the Weyl tensor can be organized in 6×6 matrix, with characteristic polynomial given by:

$$\det(\mu C_{ab}{}^{cd} - \delta_{[a}{}^{[c}\delta_{b]}{}^{d]}) = 0, \tag{4.71}$$

where $\mu = \frac{1}{\lambda}$. In order to have a homogeneous equation we need

$$\mu \sim w^2 \Longrightarrow \lambda \sim w^{-2}, \tag{4.72}$$

and so we can write the characteristic polynomial in (4.71) as:

$$\lambda^6 + a_{-2}\lambda^5 + a_{-4}\lambda^4 + a_{-6}\lambda^3 + a_{-8}\lambda^2 + a_{-10}\lambda + a_{-12} = 0, \tag{4.73}$$

where the various indices indicate the weight of the corresponding term of the polynomial.

One can consider various ratios of the coefficients with overall weight equal to zero. One criterion for the singularities to be real is finite values of these ratios at the poles. Note that $a_{-2} = tr(C) = 0$, since the Weyl tensor is trace-free. Also, the coefficient of λ^4 is of the form $tr(C^2)/2$.

4.5 RESULTS

We have studied the following properties: *flatness* (F), *Ricci flatness* (RF), *conformal flatness* (CF), *Einstein* (E), *conformal to vacuum* (CV), and *self-duality* (SD).

Poles	g	$\det g$	F	RF	CF	E	CV	SD
$(6,0)$	$\equiv 0$	$\equiv 0$	–	–	–	–	–	–
$(5,1)$	$\not\equiv 0$	$\equiv 0$	–	–	–	–	–	–
$(4,2)$	$\not\equiv 0$	$\equiv 0$	–	–	–	–	–	–
$(4,1,1)$	$\not\equiv 0$	$\equiv 0$	–	–	–	–	–	–
$(3,3)$	$\not\equiv 0$	$\not\equiv$	no	no	yes	no	trivially	trivially
$(3,2,1)$	$\not\equiv 0$	$\not\equiv$	no	no	no	no	yes	yes
$(3,1,1,1)$	$\not\equiv 0$	$\not\equiv$	no	no	no	no	yes	yes
$(2,2,2)$	$\not\equiv 0$	$\not\equiv$	no	no	yes	no	trivially	trivially
$(2,2,1,1)$	$\not\equiv 0$	$\not\equiv$	no	no	no	no	?	yes
$(2,1,1,1,1)$	$\not\equiv 0$	$\not\equiv$	no	no	no	no	?	yes
$(1,1,1,1,1,1)$	$\not\equiv 0$	$\not\equiv$	no	no	no	no	?	yes

- In the first column, we considered all possibilities for the poles of the matrices C and D in (4.55): for example, $(6,0)$ means that all six poles are identical, $(3,2,1)$ means that three poles are identical, two poles are identical but different from the first set of three.
- Column 2 describes the results obtained on the metric tensor: if g is identically equal to zero, then none of the subsequent calculations make sense. If it is not identically equal to zero, but $\det g = 0$ (singular metric), none of the relevant tensors can be defined. The first nontrivial case starts with less than four identical poles.

- Column 8 determines whether the metric is conformal to vacuum; since this equation involves the Weyl tensor as described in (4.66), for the cases when the Weyl tensor is identically equal to zero, (4.66) is trivially satisfied.

  Also, in these cases the metrics are trivially self-dual.
- The equations that need to be solved in order to determine if the metrics are conformal to vacuum have increasing complexity as the poles become more distinct.

  For the last three cases we could not obtain any solutions due to the limitations of Maple. It is very likely that solutions exist, but we could not determine it at this point.
- All these metrics have singularities at the poles. It has yet to be determined if these singularities are coordinate singularities or genuine singularities.

4.6 EXAMPLES

Since the results presented in the previous section are only summarizing the main features of these metrics, we will present here some results regarding two specific cases. A sample of the Maple code is presented in appendix E.

4.6.1 Case (3,2,1)

Consider here the case when $a = 1$, $b = 2$, $c = 3$, $p = 1$, $q = 1$, $r = 2$.

- The metric in this case is given by:

$$\begin{aligned} g &= \frac{16}{(t+1)^2\,(t+2)}dt^2 - 32\,(t+1)\left(dx - \frac{3t+7}{t+1}dz\right)^2 - 32\,(t+3)\,dy^2 \\ &\quad + \frac{256\,(t+2)\,(t+3)}{(t+1)}dz^2. \end{aligned}$$

- By analyzing the signature of the metric at the poles -3, -2 and -1, we obtain that there is no change in signature:

	-3	-2	-1	
dt^2	$-$	$-$	$+$	$+$
dy^2	$+$	$-$	$-$	$-$
$dxdz$	$+$	$+$	$+$	$-$
dz^2	$-$	$+$	$-$	$+$

- On the other hand, if we study the characteristic polynomial of the matrix associated to the Weyl tensor ${C_{ab}}^{cd}$, as described in 4.4.1, we obtain:

$$\begin{aligned} char(A) &= r^6 - \frac{3\,(t+1)^2}{2^{10}\,(xz+y)^4\,(t+3)^4} r^4 + \frac{(t+1)^3}{2^{14}\,(xz+y)^6\,(t+3)^6} r^3 \\ &= r^6 - a_4 r^4 + a_6 r^3 \end{aligned}$$

- By calculating the ratios of the coefficients with zero weight, we obtain only one such ratio, given by:

$$\frac{(a_4)^3}{(a_6)^2} = -\frac{27}{4},$$

which can be seen to be independent of the poles of the quadrics.

- The eigenvalues of the characteristic polynomial in this case are:

$$0, 0, 0, -\frac{t+1}{16\,(t+3)^2\,(xz+y)^2}, \frac{t+1}{32\,(t+3)^2\,(xz+y)^2}, \frac{t+1}{32\,(t+3)^2\,(xz+y)^2}.$$

- By solving equation (4.66),we obtain that the metric is conformal to vacuum; the conformal factor Ω from (2.42) is given by:

$$\Omega = (xz+y)\,\sqrt[4]{(t+1)\,(t+3)}.$$

- Self-duality is checked by verifying that the following equation holds:

$$^*C_{abcd} = C_{abcd}.$$

4.6.2 Case (3,1,1,1)

Consider here the case when $a = 1$, $b = 1$, $c = 1$, $p = 2$, $q = 3$, $r = 4$.

- The metric in this case is given by:

$$g = \frac{6}{(t+2)(t+3)(t+4)}dt^2 + \frac{16(t+1)^2}{t+2}dx^2 + \frac{32(t+1)^2}{t+3}dy^2 + \frac{48(t+1)^2}{t+4}dz^2.$$

- By analyzing the signature of the metric at the poles -4, -3, -2 and -1, we obtain that there is a change in signature at -4 and -2:

		-4	-3	-2	-1
dt^2	$-$	$+$	$-$	$+$	$+$
dx^2	$-$	$-$	$-$	$+$	$+$
dy^2	$-$	$-$	$+$	$+$	$+$
dz^2	$-$	$+$	$+$	$+$	$+$

- If we study the characteristic polynomial of the matrix associated to the Weyl tensor $C_{ab}{}^{cd}$, as described in 4.4.1, we obtain:

$$char(A) = r^6 - a_4 r^4 + a_6 r^3,$$

with

$$a_4 = -\frac{4t^6 + 60t^5 + 378t^4 + 1292t^3 + 2553t^2 + 2790t + 1324}{48(1+x^2+y^2+z^2)^4\left[(t+1)(t+2)(t+3)(t+4)\right]^2},$$

and

$$a_6 = \frac{(4t^3 + 30t^2 + 75t + 64)(2t^3 + 18t^2 + 57t + 62)(t^3 + 6t^2 + 9t + 1)}{864(1+x^2+y^2+z^2)^6\left[(t+1)(t+2)(t+3)(t+4)\right]^3}.$$

- By calculating the ratios of the coefficients with zero weight, we again obtain only one such ratio, $C = \frac{(a_4)^3}{(a_6)^2}$, which depends only on t. We have:

$$C|_{t=-1} = -\frac{59319}{4900}, \quad C|_{t=-2} = C|_{t=-3} = C|_{t=-4} = -\frac{27}{4}.$$

- By solving equation (4.66), we obtain that the metric is conformal to vacuum; the conformal factor Ω from (2.42) is given by:

$$\Omega = \frac{\left(1 + x^2 + y^2 + z^2\right)(t+1)^{5/4}}{\left[(t+2)(t+3)(t+4)\right]^{1/4}}.$$

- Self-duality is checked again by verifying that ${}^{*}C_{abcd} = C_{abcd}$ holds.

4.7 SELF-DUALITY FROM A TWISTORIAL POINT OF VIEW

The result regarding the self-duality of these metrics can also be argued from a twistorial point of view as follows. In section 4.3, we obtained that the points of the three-dimensional space on the intersections of quadrics are described by (4.22):

$$t(\mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y}) + \mathbf{x}^T A\mathbf{x} - \mathbf{y}^T B\mathbf{y} = \mathbf{0}.$$

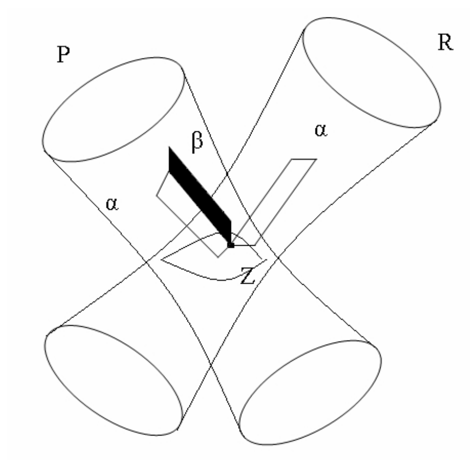

Figure 4.6: α-planes and β-planes
on intersections of quadrics.

Fix a point $Z = [\mathbf{x}_0, \mathbf{y}_0]$ on all quadrics (with $\mathbf{x}_0 \neq 0$), and consider all the planes passing through Z. All the α-planes meeting at Z determine a (projective) line in $\mathbb{PT}$ and all the β-planes meeting at Z determine a (projective) line in $\mathbb{PT}^*$.

We obtain thus that for each t there are two $\mathbb{CP}^1$'s of planes, which determine a two-surface in space-time.

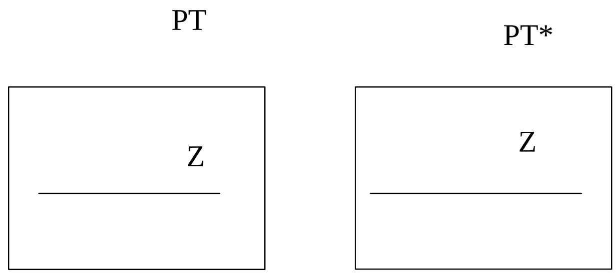

Figure 4.7: The two $\mathbb{CP}^1$'s of planes determine a two-surface in the 4-D space time.

Generically, a space-time point is given by an equation of the form:

$$\mathbf{y} = M\mathbf{x},$$

where M is fixed, and $\mathbf{x}$ varies. If the plane corresponding to the space-time point passes through $[\mathbf{x}_0, \mathbf{y}_0]$, then we have:

$$\mathbf{y}_0 = M\mathbf{x}_0.$$

As M varies, we obtain an ensemble of such space-times. For infinitesimal variations of M we have:

$$(dM)\,\mathbf{x}_0 = 0,$$

which implies that $\det(dM) = 0$ since we assumed that $\mathbf{x}_0 \neq 0$. This is the equation of the tangent vectors to the two-surfaces.

Recall now that the metric was obtained from $G = \det(dM)$ which in this case is reduced to zero. We obtain that the restriction of the metric on the two-surfaces is completely null, and it is known that three parameter families of completely null surfaces are self-dual [26]. This concludes our twistorial justification of the self-dual character of the metrics on intersections of quadrics.

5.0 CONCLUSIONS

In the first part of this thesis we have given a complete description of the $\mathcal{R}$-algebras in n-dimensions and the lower-dimensional cases: $n = 5,\ n = 3$, and $n = 2$.

In reference [1], the $\mathcal{R}$-algebra, for the case $n = 4$, was used to construct the family of unitary irreducible representations that constitute the discrete series of the conformal group. We expect that, in general, the $\mathcal{R}$-algebras constructed here can be used in the same way.

Finding the Casimir operators for these theories was not a trivial task, but it proved to be extremely difficult in the general case, preventing us from obtaining a complete result.

We plan on investigating the possibility of extending these results to super-conformal algebras [9], [10], as well as obtaining a general result for the n-dimensional case.

In the second part of this thesis we have studied a class of metrics on the twistor space arising on the intersection of quadrics for the case when the quadrics are simultaneously diagonalizable. We obtained that these metrics live on a compact space, they are not flat in most nontrivial cases, conformal to vacuum (except for three cases when we could not obtain a result), and self-dual.

One interesting interpretation arises in the generic case, when the time coordinate can be thought of as a hyperelliptic curve of genus two.

Future plans involve studying the case when the quadrics are not simultaneously diagonalizable, and the use of Jordan blocks is needed.

APPENDIX A

N-DIMENSIONAL CASE

$$[p_a, p_b] = 0, \quad [q_a, q_b] = 0, \tag{A.1}$$

$$[M_{ab}, p_c] = g_{ac}p_b - g_{bc}p_a, \tag{A.2}$$

$$[M_{ab}, q_c] = g_{ac}q_b - g_{bc}q_a, \tag{A.3}$$

$$[M_{ab}, M_{cd}] = g_{ac}M_{bd} - g_{bc}M_{ad} - g_{ad}M_{bc} + g_{bd}M_{ac}, \tag{A.4}$$

$$[D, p_a] = -p_a, \qquad [D, q_a] = q_a, \tag{A.5}$$

$$[D, D] = 0, \qquad [D, M_{ab}] = 0, \tag{A.6}$$

$$[p_a, q_b] = M_{ab} + g_{ab}D, \tag{A.7}$$

$$M_{ab}p^a p^b = 0, \tag{A.8}$$

$$p^2 x_a = \frac{1}{2}\left(M_{ab}p^b + p^b M_{ab}\right) = M_{ab}p^b + kp_a, \tag{A.9}$$

$$[p^2, q_a] = p_a(2D-1) - 2p^2 x_a, \tag{A.10}$$

$$[M_{ab}, x_c] = g_{ac}x_b - g_{bc}x_a, \tag{A.11}$$

$$p^2[x_a, x_b] = -M_{ab}, \tag{A.12}$$

$$p^4(x \cdot x) = (M_{ab}p^b)(M^a{}_c p^c) - \frac{(n-1)^2}{4}p^2, \tag{A.13}$$

$$p^2[x \cdot x, x_a] = 2x^b M_{ab} - 2kx_a, \tag{A.14}$$

$$[x \cdot x, p_a] = 2x_a, \tag{A.15}$$

$$[p^2, x_a] = 0, \tag{A.16}$$

$$[x_a, p_b] = g_{ab} - p^{-2}p_a p_b := h_{ab}, \tag{A.17}$$

$$x \cdot p = -p \cdot x := k = \frac{n-1}{2}, \tag{A.18}$$

$$L_{ab} = x_a p_b - x_b p_a, \quad L_{ab}p^a p^b = 0, \tag{A.19}$$

$$[M_{ab}, L_{cd}] = g_{ac}L_{bd} - g_{bc}L_{ad} - g_{ad}L_{bc} + g_{bd}L_{ac}, \tag{A.20}$$

$$[L_{ab}, p_c] = g_{ac}p_b - g_{bc}p_a, \tag{A.21}$$

$$S_{ab} = M_{ab} - L_{ab}, \tag{A.22}$$

$$[M_{ab}, S_{cd}] = g_{ac}S_{bd} - g_{bc}S_{ad} - g_{ad}S_{bc} + g_{bd}S_{ac}, \tag{A.23}$$

$$[L_{ab}, x_c] = g_{ac}x_b - g_{bc}x_a - p^{-2}\left(S_{ac}p_b - S_{bc}p_a\right), \tag{A.24}$$

$$[S_{ab}, p_c] = 0, \qquad S_{ab}p^b = 0, \quad [p^2, S_{ab}] = 0, \tag{A.25}$$

$$p^2[S_{ab}, x_c] = S_{ac}p_b - S_{bc}p_a, \tag{A.26}$$

$$[S_{ab}, S_{cd}] = h_{ac}S_{bd} - h_{bc}S_{ad} - h_{ad}S_{bc} + h_{bd}S_{ac}, \tag{A.27}$$

$$[S_{ab}, x^b] = 0, \tag{A.28}$$

$$p^2 y_a = p^2 x_a - p_a D, \tag{A.29}$$

$$[p^2, y_a] = -2p_a, \quad p^2[y_a, y_b] = -S_{ab}, \tag{A.30}$$

$$y_a p_b - y_b p_a = L_{ab}, \tag{A.31}$$

$$[D, M_{ab}] = 0 = [D, S_{ab}] = [D, L_{ab}], \tag{A.32}$$

$$[D, p^2] = -2p^2, \ [D, p^{-2}] = 2p^{-2}, \tag{A.33}$$

$$[D^2, p^2] = 4p^2(1 - D), \tag{A.34}$$

$$[D, p^{-2}p_a] = p^{-2}p_a, \tag{A.35}$$

$$[D^2, p_a] = p_a(-2D+1), \tag{A.36}$$

$$[D, y_a] = y_a, \qquad [D, x_a] = x_a, \quad [D, Q_a] = Q_a, \tag{A.37}$$

$$[D, x\cdot x] = 2\,(x\cdot x)\,, \tag{A.38}$$

$$f(D)x_a = x_a f(D+1),\;\; f(D)p_a = p_a f(D-1), \tag{A.39}$$

for any polynomial function f.

$$Q_a = x^b S_{ab} - \frac{1}{2}\,(x\cdot x)\,p_a - x_a D + \frac{1}{2}p^{-2}p_a\left(D^2 + D - k^2\right), \tag{A.40}$$

$$[p^2, Q_a] = [p^2, q_a], \qquad [p_a, Q_b] = [p_a, q_b], \tag{A.41}$$

$$[R_a, p_b] = 0, \tag{A.42}$$

$$[D, R_a] = 0, \tag{A.43}$$

$$[R_a, p^2] = 0, \tag{A.44}$$

$$[M_{ab}, R_c] = g_{ac}R_b - g_{bc}R_a, \tag{A.45}$$

$$p^2[x_a, R_b] = g_{ab}(R\cdot p) - R_a p_b, \tag{A.46}$$

$$p^2[S_{ab}, R_c] = p^2 h_{ac}R_b - p^2 h_{bc}R_a - g_{ac}(R\cdot p)p_b + g_{bc}(R\cdot p)p_a, \tag{A.47}$$

$$\begin{aligned} 2p^2[R_a, R_b] \;\;&=\;\; ((n-3)(n-2) - 4R\cdot p)S_{ab} - 2S_{c[a}S^{cd}S_{b]d}, \\ &\quad + \left(R^c S_{ac} + S_{ac}R^c\right)p_b - \left(R^c S_{bc} + S_{bc}R^c\right)p_a, \end{aligned} \tag{A.48}$$

$$[S_a{}^c, S_{bc}] = (n-3)S_{ab}, \tag{A.49}$$

$$[S_{ab}, x \cdot x] = 2x^c p_b S_{ac} - 2x^c p_a S_{bc} - 2S_{ab}, \tag{A.50}$$

$$R \cdot x = x \cdot R - p^{-2}(n-1)(R \cdot p), \tag{A.51}$$

$$[x_a, R \cdot p] = 0, \tag{A.52}$$

$$[x \cdot R, p_a] = R_a - p^{-2} p_a (R \cdot p), \tag{A.53}$$

$$[S_{ac}, R^c] = (n-2)\, [p^{-2}(R \cdot p) p_a - R_a], \tag{A.54}$$

$$[R \cdot p, R_a] = -\frac{1}{2}\left(R^b S_{ab} + S_{ab} R^b\right), \tag{A.55}$$

$$[S_{ab}, R \cdot p] = 0, \tag{A.56}$$

$$[S_{ac} S^{ac}, R_b] = 2\left(S_{bc} R^c + R^c S_{bc}\right), \tag{A.57}$$

$$[S_{ab} S^{ab}, S_{cd}] = 0, \tag{A.58}$$

$$S_a = R_b h^b{}_a, \quad S_a p^a = 0, \tag{A.59}$$

$$S = R \cdot p, \quad [S, S] = 0, \tag{A.60}$$

$$[S_{ab}, S] = 0, \quad [S_a, S] = S^b S_{ab} - \frac{n-2}{2} S_a, \tag{A.61}$$

$$[S_{ab}, S_c] = h_{ac}S_b - h_{bc}S_a, \tag{A.62}$$

$$p^2[S_a, S_b] = \frac{(n-3)(n-2)}{2}S_{ab} + S_{[b|d|}S^{cd}S_{a]c} - 2SS_{ab}, \tag{A.63}$$

$$U = S_a S^a, \tag{A.64}$$

$$[U, S_{ab}] = 0 = [U, S], \tag{A.65}$$

$$\begin{aligned} p^2[U, S_a] \quad = \quad & 2S_a{}^d S_d{}^c S_c{}^b S_b + \left(n(n-1) + 4C_1 - S_{cd}S^{cd}\right) S_a{}^b S_b \\ & + (3n-4)S_a{}^c S_c{}^b S_b + (n-2)\left(n - 1 + 2C_1 - \frac{1}{2}S_{cd}S^{cd}\right) S_a. \end{aligned} \tag{A.66}$$

APPENDIX B

5-DIMENSIONAL CASE

The conjugates of all the relations listed below hold (where it applies).

$$S_{ab} = \epsilon_{AB} S_{A'B'} + \epsilon_{A'B'} S_{AB}, \quad S_a = S_{AA'}, \tag{B.1}$$

$$[S_{AB}, S_{A'B'}] = 0, \tag{B.2}$$

$$[S_{AB}, S_{CD}] = \frac{1}{2} \left(\epsilon_{AC} S_{BD} + \epsilon_{AD} S_{BC} + \epsilon_{BC} S_{AD} + \epsilon_{BD} S_{AC} \right), \tag{B.3}$$

$$[S_{AB}, S_{CC'}] = \frac{1}{2} \left(\epsilon_{AC} S_{BC'} + \epsilon_{BC} S_{AC'} \right), \tag{B.4}$$

$$S_{BD} S^{CD} = \alpha \epsilon_B{}^C + S_B{}^C, \tag{B.5}$$

$$2\alpha = S_{CD} S^{CD}, \tag{B.6}$$

$$S = C_1 - (\alpha + \alpha'), \tag{B.7}$$

$$S_{ab} S^{ab} = 4 (\alpha + \alpha'), \tag{B.8}$$

$$[\alpha, S_{AB}] = [\alpha, S_{A'B'}] = 0, \tag{B.9}$$

$$[\alpha + \alpha', S_{AA'}] = -\frac{3}{2} S_{AA'} - S_A{}^{B'} S_{A'B'} - S_{A'}{}^{B} S_{AB}, \tag{B.10}$$

$$\Lambda = 2 - 2C_1 + 3\alpha' + 5\alpha, \tag{B.11}$$

$$p^2 [S_{AA'}, S_{BB'}] = \epsilon_{AB} \Lambda S_{A'B'} + \epsilon_{A'B'} \Lambda' S_{AB}, \tag{B.12}$$

$$p^2 S_{A'B} S_{B'}^{\ B} = \Lambda S_{A'B'} + \frac{1}{2} p^2 \epsilon_{A'B'} U, \tag{B.13}$$

$$p^2 [U, S_{AA'}] = S_{AA'} \left[\frac{21}{2} - 6C_1 + 15 (\alpha + \alpha') \right] + 2 \left(S_A{}^{B'} S_{A'B'} \Lambda + c.c. \right) - 10 S_{AB} S_{A'B'} S^{BB'}, \tag{B.14}$$

$$p^2 \left[U, S_A{}^{B'} \right] = S_A{}^{B'} \left[\frac{21}{2} - 6C_1 + 15 (\alpha + \alpha') \right] + 2 S_A{}^{C'} S_{C'}^{B'} \Lambda + 2 S^{CB'} S_{AC} \Lambda' - 10 S_{AC} S_{C'}^{B'} S^{CC'}, \tag{B.15}$$

$$p^2 \left[U, S^{BB'} \right] = S^{BB'} \left[\frac{21}{2} - 6C_1 + 15 (\alpha + \alpha') \right] + 2 S^{BC'} S_{C'}^{B'} \Lambda + 2 S^{B'C} S_C^{B} \Lambda' - 10 S_C^{B} S_{C'}^{B'} S^{CC'}, \tag{B.16}$$

$$\left[\alpha, S_A{}^{B'} \right] = \frac{3}{4} S_A{}^{B'} - S_{AB} S^{BB'}, \tag{B.17}$$

$$\left[\alpha, S_{A'}{}^{B} \right] = -\frac{3}{4} S_{A'}{}^{B} + S_{CA'} S^{CB}, \tag{B.18}$$

$$[\alpha, S_{AA'}] = -\frac{3}{4} S_{AA'} - S_{A'}{}^{B} S_{AB}, \tag{B.19}$$

$$\left[\alpha, S^{BB'}\right] = -\frac{3}{4}S^{BB'} + S_A^{\ B'}S^{AB}, \tag{B.20}$$

$$[\Lambda, S_{AA'}] = -6S_{AA'} - 5S_{A'}^{\ B}S_{AB} - 3S_A^{\ B'}S_{A'B'}, \tag{B.21}$$

$$\left[\Lambda, S_{A'}^{\ B}\right] = -\frac{3}{2}S_{A'}^{\ B} - 3S_{A'B'}S^{BB'} + 5S_{CA'}S^{CB}, \tag{B.22}$$

$$\left[\Lambda, S_A^{\ B'}\right] = \frac{3}{2}S_A^{\ B'} + 3S_{AC'}S^{C'B'} - 5S_{AC}S^{CB'}, \tag{B.23}$$

$$\left[\Lambda, S^{BB'}\right] = -6S^{BB'} + 3S_{A'}^{\ B}S^{A'B'} + 5S_A^{\ B'}S^{AB}, \tag{B.24}$$

$$\left[S_{AB}, S_{A'}^{\ B}\right] = S_{A'}^{\ B}S_{AB} + \frac{3}{2}S_{AA'}, \tag{B.25}$$

$$\left[S_{AB}, S^{BB'}\right] = \frac{3}{2}S_A^{\ B'}, \tag{B.26}$$

$$\left[S_{AB}S_{A'B'}, S^{BB'}\right] = \frac{3}{2}S_A^{\ B'}S_{A'B'} + \frac{3}{2}S_{A'}^{\ B}S_{AB} + \frac{9}{4}S_{AA'}, \tag{B.27}$$

$$S_{AB}S_{A'B'}S_C^{\ B'}S^{CB} = -\frac{1}{2}S_{AB}S_{A'B'}S^{BB'} + \frac{3}{2}S_{AA'}\left(\frac{3}{4} + \alpha\right) + S_A^{\ B'}S_{A'B'}\left(\frac{3}{4} + \alpha'\right), \tag{B.28}$$

$$S_{AB}S_{A'C}S^{CB} = -\frac{1}{2}S_{A'}^{\ B} + S_{AA'}\alpha, \tag{B.29}$$

$$S_{A'C}S^{CB}S_{AB} = S_{AA'}\alpha + S_{A'}^{\ B}S_{AB}, \tag{B.30}$$

$$S_{AB}\left[S_{A'B'}, S^{BB'}\right] = \frac{3}{2}S_{A'}^{\ B}S_{AB} + \frac{9}{4}S_{AA'}, \tag{B.31}$$

$$S^B_{\ C}S_{A'C'}S^{CC'}S_{AB} = -\frac{3}{2}S_{AA'}\left(\frac{3}{4} + \alpha\right) - S_A^{\ B'}S_{A'B'}\left(\alpha + \frac{3}{4}\right) + \frac{1}{2}S_{AB}S_{A'B'}S^{BB'}, \tag{B.32}$$

$$S_{AC}S_{C'}^{\ B'}S^{CC'}S_{A'B'}=-\frac{3}{2}S_{AA'}\left(\frac{3}{4}+\alpha'\right)-S_{A'}^{\ B}S_{AB}\left(\alpha'+\frac{3}{4}\right)+\frac{1}{2}S_{AB}S_{A'B'}S^{BB'}, \quad \text{(B.33)}$$

$$\begin{aligned} S_{AB}S_{A'B'}S^{BC}S^{B'C'}S_{CC'} &= S_{AA'}\left[-\frac{63}{16}-\frac{9}{4}(\alpha+\alpha')+\alpha\alpha'\right]-2S_{A'}^{\ B}S_{AB}\left(\alpha'+\frac{3}{4}\right) \\ &\quad -2S_{A}^{\ B'}S_{A'B'}\left(\alpha+\frac{3}{4}\right)+4S_{AB}S_{A'B'}S^{BB'}, \end{aligned} \quad \text{(B.34)}$$

$$\left[\alpha^2,S_{AA'}\right]=S_{AA'}\left(-\frac{5}{2}\alpha+\frac{9}{16}\right)+S_{A'}^{\ B}S_{AB}\left(\frac{1}{2}-2\alpha\right), \quad \text{(B.35)}$$

$$[\alpha\alpha',S_{AA'}]=S_{AA'}\left[-\frac{27}{16}-\frac{3}{4}(\alpha+\alpha')\right]+S_{A'}^{\ B}S_{AB}\left(-\alpha'-\frac{3}{4}\right)+p.a.+S_{AB}S_{A'B'}S^{BB'}, \quad \text{(B.36)}$$

$$[\alpha^2+\alpha'^2,S_{AA'}]=S_{AA'}\left(-\frac{5}{2}(\alpha+\alpha')+\frac{9}{8}\right)+S_{A'}^{\ B}S_{AB}\left(\frac{1}{2}-2\alpha\right)+p.a., \quad \text{(B.37)}$$

$$\begin{aligned} [\alpha^3+\alpha'^3,S_{AA'}] &= S_{AA'}\left(-\frac{21}{4}\left(\alpha^2+\alpha'^2\right)+\frac{47}{16}(\alpha+\alpha')-\frac{27}{32}\right) \quad \text{(B.38)} \\ &\quad +S_{A'}^{\ B}S_{AB}\left(-3\alpha^2+\frac{5}{2}\alpha-\frac{7}{16}\right)+p.a., \end{aligned}$$

$$\begin{aligned} [\alpha^2\alpha'+\alpha'^2\alpha,S_{AA'}] &= S_{AA'}\left[-\frac{3}{4}\left(\alpha^2+\alpha'^2\right)-\frac{33}{16}(\alpha+\alpha')-5\alpha\alpha'+\frac{45}{32}\right] \quad \text{(B.39)} \\ &\quad +S_{A'}^{\ B}S_{AB}\left(-\alpha'^2-2\alpha\alpha'-\frac{3}{2}\alpha+\frac{9}{16}\right)+p.a. \\ &\quad +S_{AB}S_{A'B'}S^{BB'}\left(2(\alpha+\alpha')-1\right), \end{aligned}$$

$$\begin{aligned} p^2\left[S^{AA'}S_{AB}S_{A'B'}S^{BB'},S_{CC'}\right] &= S_{AA'}[-\frac{3}{4}p^2U+(\alpha+\alpha')\left(-\frac{209}{4}-4C_1\right)+30\alpha\alpha' \\ &\quad +6\left(\alpha^2+\alpha'^2\right)+\left(-69+\frac{21}{2}C_1\right)] \quad \text{(B.40)} \\ &\quad +S_{A'}^{\ B}S_{AB}[-\frac{1}{2}p^2U-\frac{33}{2}\alpha+\alpha'\left(-\frac{87}{2}-4C_1\right) \\ &\quad +10\alpha\alpha'+6\alpha'^2+(-30+9C_1)]+p.a. \\ &\quad +S_{AB}S_{A'B'}S^{BB'}\left[(52+6C_1)-15(\alpha+\alpha')\right], \end{aligned}$$

$$
\begin{aligned}
p^4\left[U^2,S_{AA'}\right] &= S_{AA'}[(\alpha+\alpha')\left(266+20C_1-16C_1^2\right)+\left(\alpha^2+\alpha'^2\right)(87+4C_1) \\
&+\alpha\alpha'(378+160C_1)-220\left(\alpha^2\alpha'+\alpha'^2\alpha\right)-36\left(\alpha^3+\alpha'^3\right) \\
&-\left(\frac{351}{2}+162C_1+36C_1^2\right)+p^2U\left(21-12C_1+30\left(\alpha+\alpha'\right)\right)] \\
&+S_{A'}{}^{B}S_{AB}[\alpha(156+48C_1)+\alpha'(28-48C_1)+152\alpha\alpha'-36\alpha^2+140\alpha'^2 \\
&+p^2U(12\alpha+20\alpha'-8C_1+8)-\left(70+64C_1+16C_1^2\right)]+p.a. \\
&+S_{AB}S_{A'B'}S^{BB'}[(\alpha+\alpha')(-332-128C_1)+120\left(\alpha^2+\alpha'^2\right)+272\alpha\alpha' \\
&+\left(142+136C_1+32C_1^2\right)-20p^2U],
\end{aligned} \tag{B.41}
$$

$$
\begin{aligned}
&p^2\left[(\alpha+\alpha')S^{AA'}S_{AB}S_{A'B'}S^{BB'},S_{CC'}\right]=S_{AA'}[p^2U\left(2\left(\alpha+\alpha'\right)+\tfrac{9}{8}\right) \\
&+(\alpha+\alpha')\left(\tfrac{885}{8}-45C_1\right) \\
&+\alpha\alpha'\left(\tfrac{1201}{4}+19C_1\right)+(\alpha^2+\alpha'^2)\left(\tfrac{847}{8}+\tfrac{5}{2}C_1\right)-\tfrac{9}{2}(\alpha^3+\alpha'^3) \\
&-\tfrac{107}{2}(\alpha^2\alpha'+\alpha'^2\alpha)-\left(\tfrac{171}{2}+\tfrac{153}{4}C_1+\tfrac{3}{2}C_2\right)] \\
&+S_{A'}{}^{B}S_{AB}[p^2U\left(\alpha+\alpha'+\tfrac{1}{2}\right)+\alpha\left(\tfrac{281}{4}-20C_1\right)+\alpha'\left(\tfrac{19}{4}-28C_1\right) \\
&+\alpha^2\left(\tfrac{105}{4}-C_1\right)+\alpha'^2\left(\tfrac{285}{4}+3C_1\right)+\alpha\alpha'\left(\tfrac{245}{2}+6C_1\right) \\
&+\alpha^3-5\alpha'^3-11\alpha^2\alpha'-17\alpha\alpha'^2-\left(33+\tfrac{33}{2}C_1+C_2\right)]+p.a. \\
&+S_{AB}S_{A'B'}S^{BB'}[(\alpha+\alpha')\left(-\tfrac{299}{2}-10C_1\right)+50\alpha\alpha' \\
&+21(\alpha^2+\alpha'^2)+(70+33C_1)-p^2U,
\end{aligned} \tag{B.42}
$$

$$
\begin{aligned}
p^2\left[\left(\alpha^2+\alpha'^2\right)U,S_{AA'}\right] &= S_{AA'}[p^2U\left(\frac{9}{8}-\frac{5}{2}(\alpha+\alpha')\right)+(\alpha+\alpha')\left(-\frac{211}{8}-C_1\right)-31\alpha\alpha' \\
&+\left(\alpha^2+\alpha'^2\right)(-7-14C_1)+35\left(\alpha^2\alpha'+\alpha'^2\alpha\right)+27\left(\alpha^3+\alpha'^3\right) \\
&+\left(\frac{225}{16}+\frac{9}{4}C_1\right) \\
&+S_{A'}{}^{B}S_{AB}[p^2U\left(-2\alpha+\frac{1}{2}\right)+\alpha\left(-\frac{47}{4}+2C_1\right)+\alpha'\left(-\frac{25}{4}-2C_1\right) \\
&-14\alpha\alpha'+\alpha^2(1-4C_1)+\alpha'^2(-11-4C_1)+10\alpha^2\alpha' \\
&+6\alpha'^2\alpha+6\alpha^3+10\alpha'^3]+p.a. \\
&+S_{AB}S_{A'B'}S^{BB'}[(\alpha+\alpha')(35+8C_1)-24\alpha\alpha'-30\left(\alpha^2+\alpha'^2\right) \\
&-\left(\frac{49}{4}+4C_1\right)],
\end{aligned} \tag{B.43}
$$

$$
\begin{aligned}
p^2\left[\left(\alpha+\alpha'\right)U,S_{AA'}\right] &= S_{AA'}[-\frac{3}{2}p^2U+\left(\alpha+\alpha'\right)\left(13-10C_1\right)+21\left(\alpha^2+\alpha'^2\right) \\
&+50\alpha\alpha'-\left(\frac{81}{4}+9C_1\right)] \\
&+S_{A'}{}^{B}S_{AB}[-p^2U+\alpha\left(10-4C_1\right)+\alpha'\left(-2-4C_1\right)+6\alpha^2 \\
&+10\alpha'^2+16\alpha\alpha'-\left(8+4C_1\right)]+p.a. \\
&+S_{AB}S_{A'B'}S^{BB'}\left[17-26\left(\alpha+\alpha'\right)+8C_1\right],
\end{aligned} \tag{B.44}
$$

$$
\begin{aligned}
p^2\left[\left(\alpha\alpha'\right)U,S_{AA'}\right] &= S_{AA'}[p^2U\left(-\frac{27}{16}-\frac{3}{4}\left(\alpha+\alpha'\right)\right)+\left(\alpha+\alpha'\right)\left(-\frac{291}{16}-\frac{3}{2}C_1\right)-\frac{9}{4}\left(\alpha^2+\alpha'^2\right) \\
&+\alpha\alpha'\left(-14-14C_1\right)+31\left(\alpha^2\alpha'+\alpha'^2\alpha\right)+\left(\frac{297}{32}+\frac{45}{8}C_1\right)] \\
&+S_{A'}{}^{B}S_{AB}[p^2U\left(-\alpha'-\frac{3}{4}\right)+\alpha\left(-\frac{87}{8}-3C_1\right)+\alpha'\left(-\frac{17}{8}+3C_1\right) \\
&+\alpha\alpha'\left(-7-4C_1\right)+\frac{9}{2}\alpha^2-\frac{19}{2}\alpha'^2+\left(\frac{15}{4}+\frac{9}{4}C_1\right)]+c.c. \\
&+S_{AB}S_{A'B'}S^{BB'}[p^2U+\left(\alpha+\alpha'\right)\left(\frac{39}{2}+4C_1\right)-6\left(\alpha^2+\alpha'^2\right) \\
&-30\alpha\alpha'-\left(\frac{57}{8}+4C_1\right)].
\end{aligned} \tag{B.45}
$$

APPENDIX C

3-DIMENSIONAL CASE

$$[D, p] = -p, \quad [D, J] = 0, \quad [D, q] = q, \tag{C.1}$$

$$[J \cdot a, p \cdot b] = -(a \times b) \cdot p, \tag{C.2}$$

$$[J \cdot a, q \cdot b] = -(a \times b) \cdot q, \tag{C.3}$$

$$[J \cdot a, J \cdot b] = -(a \times b) \cdot J, \tag{C.4}$$

$$[p \cdot a, p \cdot b] = 0, \;\; [q \cdot a, q \cdot b] = 0, \tag{C.5}$$

$$[p \cdot a, q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \tag{C.6}$$

$$[p, \times q] := p \times q + q \times p, \tag{C.7}$$

$$(a \times b) \cdot (J \times p + p \times J) = -2(a \times b) \cdot p, \tag{C.8}$$

$$[J, \times p] = -2p, \;\; [J, \times q] = -2q, \tag{C.9}$$

$$J \times J = -J, \tag{C.10}$$

$$[J, p \cdot p] = 0, \tag{C.11}$$

$$[p, J \cdot J] = p \times J - J \times p = 2p \times J + 2p, \tag{C.12}$$

$$[q, J \cdot J] = q \times J - J \times q = 2q \times J + 2q, \tag{C.13}$$

$$[J, J \cdot J] = 0, \tag{C.14}$$

$$[p \cdot p, q] = pD + Dp + p \times J - J \times p = p(2D+1) + 2p \times J, \tag{C.15}$$

$$2(p \cdot p)Q = 2(p \cdot J)J - p(J \cdot J) + pD(D+3) + 2(p \times J)(D+1), \tag{C.16}$$

$$[p \cdot a, Q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \tag{C.17}$$

$$[p \cdot p, Q] = p(2D+1) + 2(p \times J) = [p \cdot p, q], \tag{C.18}$$

$$J \cdot p = p \cdot J, \;\; J \cdot q = q \cdot J, \tag{C.19}$$

$$[J \cdot p, p] = 0, [J \cdot p, J] = 0, [J \cdot p, q] = J(D-1) - q \times p, \tag{C.20}$$

$$[q, \times p] = [p, \times q] = -2J, \tag{C.21}$$

$$(p \times J) \times J = -p \times J + (p \cdot J)J - p(J \cdot J), \tag{C.22}$$

$$J \times (p \times J) = -p \times J - (p \cdot J)J + p(J \cdot J), \tag{C.23}$$

$$p \times (p \times J) = (p \cdot J)p - (p \cdot p)J, \tag{C.24}$$

$$(p \times J) \times p = -(p \cdot J)p + (p \cdot p)J, \tag{C.25}$$

$$[p \cdot J, J \cdot J] = 0, \tag{C.26}$$

$$p \cdot (p \times J) = 0, \tag{C.27}$$

$$(p \times J) \cdot p = -2p^2, \tag{C.28}$$

$$(p \times J) \cdot J = -p \cdot J, \tag{C.29}$$

$$J \cdot (p \times J) = -p \cdot J, \tag{C.30}$$

$$(p \times J) \times (p \times J) = (p \cdot p)J, \tag{C.31}$$

$$(p \times J) \cdot (p \times J) = (p \cdot p)(J \cdot J) - (p \cdot J)^2, \tag{C.32}$$

$$[p \cdot p, J \cdot J] = 0, \tag{C.33}$$

$$x_a p_b - x_b p_a = (a \times b) \cdot (x \times p), \tag{C.34}$$

$$[p_a, (p \times J)_b] = (a \cdot b)(p \cdot p) - p_a p_b, \tag{C.35}$$

$$[J \cdot q, p] = -(p \times q) - J(D+1), \tag{C.36}$$

$$(p \times J) \times q = (p \cdot q)J - (q \cdot J)p - 2(p \times q) - J(D+1), \tag{C.37}$$

$$p \times (q \times J) = (p \cdot J)q - (p \times q) - (p \cdot q)J, \tag{C.38}$$

$$(p \times q) \times J = (p \cdot J)q - (q \cdot J)p - 2(p \times q) - J(D+1), \tag{C.39}$$

$$[J \cdot q, p] = -(p \times q) - J(D+1), \tag{C.40}$$

$$(p \cdot p)^2(Q \times Q) = 2(p \cdot J)p, \tag{C.41}$$

$$R = q - Q, \tag{C.42}$$

$$[p \cdot p, R] = 0, \;\; [p \cdot a, R \cdot b] = 0, \tag{C.43}$$

$$[D, R] = R, \tag{C.44}$$

$$[J \cdot a, R \cdot b] = -(a \times b) \cdot R, \tag{C.45}$$

$$(p \cdot p)^2(R \times R) = 2(p \cdot p)(p \cdot J)R - (p \cdot p)(p \times R) - 2(p \cdot p)(p \cdot J)p, \tag{C.46}$$

$$(p \times R) \times p = (p \cdot p)R - (p \cdot R)p, \tag{C.47}$$

$$p \times (p \times R) = (p \cdot R)p - (p \cdot p)R, \tag{C.48}$$

$$[p \cdot J, R] = p \times R, \tag{C.49}$$

$$[(p \cdot J)^2, R] = 2(p \cdot J)(p \times R) + (p \cdot p)R - (R \cdot p)p, \tag{C.50}$$

$$(p \times R) \times R = R(p \cdot R) - (R \cdot R)p, \tag{C.51}$$

$$R \times (p \times R) = (R \cdot R)p - (p \cdot R)R, \tag{C.52}$$

$$[R \cdot p, p] = 0, \quad [R \cdot p, J] = 0, \tag{C.53}$$

$$(p \cdot p)[R \cdot p, R] = -2(p \cdot J)(p \times R) + (R \cdot p)p - (p \cdot p)R, \tag{C.54}$$

$$[R \cdot R, p] = 0, \quad [R \cdot R, J] = 0, \tag{C.55}$$

$$S = R - \frac{1}{2}\frac{p}{p \cdot p}, \tag{C.56}$$

$$S \cdot p = R \cdot p - \frac{1}{2}, \tag{C.57}$$

$$(p \cdot p)(S \cdot S) = (p \cdot p)(R \cdot R) - (R \cdot p) + \frac{1}{4}. \tag{C.58}$$

APPENDIX D

2-DIMENSIONAL CASE

$$M_{ab} = J\varepsilon_{ab}, \tag{D.1}$$

$$[p_a, p_b] = 0, \qquad [q_a, q_b] = 0, \tag{D.2}$$

$$[J, p_c] = \varepsilon_c{}^b p_b, \;\; [J, q_c] = \varepsilon_c{}^b q_b, \tag{D.3}$$

$$[D, p_a] = -p_a, \qquad [D, q_a] = q_a, \tag{D.4}$$

$$[J, J] = 0 = [D, D] = [D, J], \tag{D.5}$$

$$[p_a, q_b] = \varepsilon_{ab} J + g_{ab} D, \tag{D.6}$$

$$[M_{ab}, M_{cd}] = 0, \tag{D.7}$$

$$p^2 x_a = \varepsilon_{ab} J p^b + \frac{1}{2} p_a, \tag{D.8}$$

$$x \cdot p = -p \cdot x = \frac{1}{2}, \tag{D.9}$$

$$p^2(x \cdot x) = J^2 - \frac{1}{4}, \tag{D.10}$$

$$[J^2, p_a] = -p_a + 2\varepsilon_a{}^b p_b J, \tag{D.11}$$

$$L_{ab} = M_{ab}, \quad S_{ab} = 0, \tag{D.12}$$

$$p^2 Q_a = -\frac{1}{2} J^2 p_a + \frac{1}{2} p_a D^2 - \varepsilon_{ab} J p^b D, \tag{D.13}$$

$$[p_a, Q_b] = \varepsilon_{ab} J + g_{ab} D = [p_a, q_b], \tag{D.14}$$

$$p^4 [Q_a, Q_b] = 0, \tag{D.15}$$

$$R_a = q_a - Q_a, \tag{D.16}$$

$$[R_a, p_b] = 0, \quad [R_a, R_b] = 0, \quad [R_a, J] = 0, \tag{D.17}$$

$$[D, R_a] = R_a, \quad [p^2, R_a] = 0, \tag{D.18}$$

$$[R \cdot p, p_a] = 0 = [R \cdot p, J] = [R \cdot p, R_a], \tag{D.19}$$

$$[R \cdot R, p_a] = 0 = [R \cdot R, J] = [R \cdot R, R_a] = 0, \tag{D.20}$$

APPENDIX E

MAPLE CODE FOR THE STUDY OF METRICS ON INTERSECTIONS OF QUADRICS

This is the Maple code for the case (3,2,1) described in more detail in sections

```
> restart:with(linalg): with(tensor):
```

Choose the poles of the quadrics to be:

```
> a:=1:b:=2:c:=3:p:= 1: q:= 1: r:=2:
```

Define the matrix X

```
> X:= matrix([[0, z, -y], [-z, 0, x], [y, -x, 0]]):
```

and the identity matrix:

```
> J:= matrix([[1,0,0], [0,1, 0], [0, 0, 1]]):
```

Parameterize the rotation matrix

```
> R:= map(factor, evalm((J - X)&*(J + X)^(-1))):
```

Check that it is indeed a rotation:

```
> map(factor,evalm(R&*J&*transpose(R)-J));
```

Define the two matrices C and D.

```
>C:= matrix([[1/sqrt(t+a),0, 0], [0,1/sqrt(t+b) , 0], [0, 0, 1/sqrt(t + c)]]):
> D:= matrix([[1/sqrt(t + p),0, 0], [0,1/sqrt(t + q) , 0], [0, 0, 1/sqrt(t + r)]]):
> map(factor, evalm(R&*J&*transpose(R))):
> GG:= evalm(D^(-1)&*R&*C):
> M1:=map(factor,map(expand,evalm((t+a)^(1/2)*(t+b)^(1/2)*(t+c)^(1/2)
*(t+p)^(1/2)*(t+q)^(1/2)*(t+r)^(1/2)*(evalm(map(diff,evalm(GG),t)*dt
```

```
+map(diff,evalm(GG),x)*dx+map(diff,evalm(GG),y)*dy
+map(diff,evalm(GG),z)*dz))))):
```

M1 is the metric times a factor of dt: M1=dt x GGG, so GGG is going to be the metric.

```
> GGG:= collect(factor(det(evalm(M1))/(t+a)/(t+b)/(t+c)/(t + p)/(t + q)/(t + r)/dt
*8*(1 + x^2 + y^2 + z^2)^2), [dt, dx, dy, dz]);
```

Define the hessian:

```
> H:= hessian(GGG, [dt, dx, dy, dz]);
> detH:= factor(det(H));
```

Define the components of the covariant metric tensor g(i,j):

```
> unassign('coord'):unassign('coords'):unassign('g'):coord:=[t, x, y,z]:
> g_compts:=array (symmetric, sparse, 1..4, 1..4):
> g_compts[1,1]:=H[1,1]: g_compts[1,2]:=H[1,2]: g_compts[1,3]:=H[1,3]:
 g_compts[1,4]:=H[1,4]: g_compts[2,2]:=H[2,2]: g_compts[2,3]:=H[2,3]:
g_compts[2,4]:=H[2,4]: g_compts[3,3]:=H[3,3]: g_compts[3,4]:=H[3,4]:
g_compts[4,4]:=H[4,4]:
```

The contravariant metric tensor ginv:

```
> g := create( [-1,-1], eval(g_compts));ginv := invert( g, 'detg' ):
> factor(det(evalm(g_compts)));
```

Compute the first partial derivatives of the covariant metric tensor:

```
> D1g:=d1metric( g, coord ):
```

Compute the second partial derivatives of the covariant metric tensor:

```
> D2g:= d2metric( D1g, coord ):
```

Compute the Christoffel symbols of the first kind:

```
> Cf1 := Christoffel1 ( D1g ):
```

Compute the Christoffel symbols of the second kind:

```
> Cf2:= Christoffel2 (ginv, Cf1):
```

Cf2(ijk): i=upper, jk lower

Compute the covariant Riemann tensor:

```
> RMN := Riemann( ginv, D2g, Cf1 );
```

Compute the covariant Ricci tensor:

> RICCI := Ricci(ginv, RMN);

Compute the Ricci scalar:

> RS:= Ricciscalar(ginv, RICCI);

Check if it is Einstein:

> RICCI[compts][1,1]/g_compts[1,1]-RICCI[compts][2,2]/g_compts[2,2]

The covariant Weyl tensor (all components down) is:

> WEYL := Weyl(g, RMN, RICCI, RS);

> AB:=raise(ginv,WEYL,4);

The matrix of the Weyl tensor with all indices down is:

W:=subs(x=1,y=2,z=0,t=1,matrix([[WEYL[compts][1,2,1,2],WEYL[compts][1,3,1,2],
WEYL[compts][1,4,1,2],WEYL[compts][2,3,1,2],WEYL[compts][2,4,1,2],
WEYL[compts][3,4,1,2]],[WEYL[compts][1,2,1,3],
WEYL[compts][1,3,1,3],WEYL[compts][1,4,1,3],WEYL[compts][2,3,1,3],
WEYL[compts][2,4,1,3],WEYL[compts][3,4,1,3]],[WEYL[compts][1,2,1,4],
WEYL[compts][1,3,1,4],WEYL[compts][1,4,1,4],
WEYL[compts][2,3,1,4],WEYL[compts][2,4,1,4],WEYL[compts][3,4,1,4]],
[WEYL[compts][1,2,2,3],WEYL[compts][1,3,2,3],WEYL[compts][1,4,2,3],
WEYL[compts][2,3,2,3],WEYL[compts][2,4,2,3],
WEYL[compts][3,4,2,3]],[WEYL[compts][1,2,2,4],WEYL[compts][1,3,2,4],
WEYL[compts][1,4,2,4],WEYL[compts][2,3,2,4],WEYL[compts][2,4,2,4],
WEYL[compts][3,4,2,4]],[WEYL[compts][1,2,3,4],
WEYL[compts][1,3,3,4],WEYL[compts][1,4,3,4],WEYL[compts][2,3,3,4],
WEYL[compts][2,4,3,4],WEYL[compts][3,4,3,4]]]));

Calculate the Weyl tensor with two indices down and two up.

> AA:=raise(ginv,WEYL,3,4):

Write the matrix of the Weyl tensor. NOTE: the 6x6 matrix of the Weyl tensor, A, is ordered as follows: on rows: [1212,1312,1412,2312,2412,3412], [1213,1313,...], [1214,1314,...], [1223,1323,...], [1224,1324,...],[1234,1334,...]

> A:=matrix([[AA[compts][1,2,1,2],AA[compts][1,3,1,2],AA[compts][1,4,1,2],
AA[compts][2,3,1,2],AA[compts][2,4,1,2],AA[compts][3,4,1,2]],

[AA[compts][1,2,1,3],AA[compts][1,3,1,3],AA[compts][1,4,1,3],

AA[compts][2,3,1,3],AA[compts][2,4,1,3],AA[compts][3,4,1,3]],

[AA[compts][1,2,1,4],AA[compts][1,3,1,4],AA[compts][1,4,1,4],

AA[compts][2,3,1,4],AA[compts][2,4,1,4],AA[compts][3,4,1,4]],

[AA[compts][1,2,2,3],AA[compts][1,3,2,3],AA[compts][1,4,2,3],

AA[compts][2,3,2,3],AA[compts][2,4,2,3],AA[compts][3,4,2,3]],

[AA[compts][1,2,2,4],AA[compts][1,3,2,4],AA[compts][1,4,2,4],

AA[compts][2,3,2,4],AA[compts][2,4,2,4],AA[compts][3,4,2,4]],

[AA[compts][1,2,3,4],AA[compts][1,3,3,4],AA[compts][1,4,3,4],

AA[compts][2,3,3,4],AA[compts][2,4,3,4],AA[compts][3,4,3,4]]]):

> B:=subs(x=1,y=2,z=0,t=1,evalm(A));

Calculate the determinant of the matrix of the Weyl tensor:

> factor(det(A));

Find the characteristic polynomial of the matrix of the Weyl tensor:

> cp:=charpoly(A,m);

Calculate the ratios of the coefficients with overall weight equal to zero:

> a4:=-3/1024*(t^2+2*t+1)/((z*x+y)^4*(t+3)^4);

> a6:=1/16384*(t^3+3*t^2+3*t+1)/((z*x+y)^6*(t+3)^6);

> C:=simplify(a4^3/(a6^2));

Find the eigenvalues of the characteristic polynomial:

> mm:=(solve(cp=0,m));

> factor(mm[4]);factor(mm[5]);factor(mm[6]);

Next, we are checking if it is conformal to vacuum. Solve:

g_inv[a,b](cov_diff[a]-lambda[a])Weyl[b,c,d,e]=0,

where lambda[a] is the gradient of lambda.

The covariant derivative of the Weyl tensor:

> cd_Weyl:=cov_diff(WEYL,coord,Cf2):

Solve for the partial derivatives of lambda

> eq1:=ginv[compts][1,1]*cd_Weyl[compts][1,2,3,4,1]+ginv[compts][1,2]

*cd_Weyl[compts][2,2,3,4,1]+ginv[compts][1,3]*cd_Weyl[compts][3,2,3,4,1]

```
+ginv[compts][1,4]*cd_Weyl[compts][4,2,3,4,1]+ginv[compts][2,1]
*cd_Weyl[compts][1,2,3,4,2]+ginv[compts][2,2]*cd_Weyl[compts][2,2,3,4,2]
+ginv[compts][2,3]*cd_Weyl[compts][3,2,3,4,2]+ginv[compts][2,4]
*cd_Weyl[compts][4,2,3,4,2]+ginv[compts][3,1]*cd_Weyl[compts][1,2,3,4,3]
+ginv[compts][3,2]*cd_Weyl[compts][2,2,3,4,3]+ginv[compts][3,3]
*cd_Weyl[compts][3,2,3,4,3]+ginv[compts][3,4]*cd_Weyl[compts][4,2,3,4,3]
+ginv[compts][4,1]*cd_Weyl[compts][1,2,3,4,4]+ginv[compts][4,2]
*cd_Weyl[compts][2,2,3,4,4]+ginv[compts][4,3]*cd_Weyl[compts][3,2,3,4,4]
+ginv[compts][4,4]*cd_Weyl[compts][4,2,3,4,4]-(lambda1*(ginv[compts][1,1]
*WEYL[compts][1,2,3,4]+ginv[compts][1,2]*WEYL[compts][2,2,3,4]
+ginv[compts][1,3]*WEYL[compts][3,2,3,4]+ginv[compts][1,4]
*WEYL[compts][4,2,3,4])+lambda2*(ginv[compts][2,1]*WEYL[compts][1,2,3,4
+ginv[compts][2,2]*WEYL[compts][2,2,3,4]+ginv[compts][2,3]
*WEYL[compts][3,2,3,4]+ginv[compts][2,4]*WEYL[compts][4,2,3,4])
+lambda3*(ginv[compts][3,1]*WEYL[compts][1,2,3,4]+ginv[compts][3,2]
*WEYL[compts][2,2,3,4]+ginv[compts][3,3]*WEYL[compts][3,2,3,4]
+ginv[compts][3,4]*WEYL[compts][4,2,3,4])+lambda4*(ginv[compts][4,1]
*WEYL[compts][1,2,3,4]+ginv[compts][4,2]*WEYL[compts][2,2,3,4]
+ginv[compts][4,3]*WEYL[compts][3,2,3,4]+ginv[compts][4,4]
*WEYL[compts][4,2,3,4])):
> eq2:=ginv[compts][1,1]*cd_Weyl[compts][1,1,3,2,1]+ginv[compts][1,2]
*cd_Weyl[compts][2,1,3,2,1]+ginv[compts][1,3]*cd_Weyl[compts][3,1,3,2,1]
+ginv[compts][1,4]*cd_Weyl[compts][4,1,3,2,1]+ginv[compts][2,1]
*cd_Weyl[compts][1,1,3,2,2]+ginv[compts][2,2]*cd_Weyl[compts][2,1,3,2,2]
+ginv[compts][2,3]*cd_Weyl[compts][3,1,3,2,2]+ginv[compts][2,4]
*cd_Weyl[compts][4,1,3,2,2]+ginv[compts][3,1]*cd_Weyl[compts][1,1,3,2,3]
+ginv[compts][3,2]*cd_Weyl[compts][2,1,3,2,3]+ginv[compts][3,3]
*cd_Weyl[compts][3,1,3,2,3]+ginv[compts][3,4]*cd_Weyl[compts][4,1,3,2,3]
+ginv[compts][4,1]*cd_Weyl[compts][1,1,3,2,4]+ginv[compts][4,2]*
cd_Weyl[compts][2,1,3,2,4]+ginv[compts][4,3]*cd_Weyl[compts][3,1,3,2,4]
```

```
+ginv[compts][4,4]*cd_Weyl[compts][4,1,3,2,4]-(lambda1*(ginv[compts][1,1]
*WEYL[compts][1,1,3,2]+ginv[compts][1,2]*WEYL[compts][2,1,3,2]
+ginv[compts][1,3]*WEYL[compts][3,1,3,2]+ginv[compts][1,4]
*WEYL[compts][4,1,3,2])+lambda2*(ginv[compts][2,1]*WEYL[compts][1,1,3,2]
+ginv[compts][2,2]*WEYL[compts][2,1,3,2]+ginv[compts][2,3]
*WEYL[compts][3,1,3,2]+ginv[compts][2,4]*WEYL[compts][4,1,3,2])
+lambda3*(ginv[compts][3,1]*WEYL[compts][1,1,3,2]+ginv[compts][3,2]
*WEYL[compts][2,1,3,2]+ginv[compts][3,3]*WEYL[compts][3,1,3,2]
+ginv[compts][3,4]*WEYL[compts][4,1,3,2])+lambda4*(ginv[compts][4,1]
*WEYL[compts][1,1,3,2]+ginv[compts][4,2]*WEYL[compts][2,1,3,2]
+ginv[compts][4,3]*WEYL[compts][3,1,3,2]+ginv[compts][4,4]
*WEYL[compts][4,1,3,2])):
> eq3:=ginv[compts][1,1]*cd_Weyl[compts][1,1,2,4,1]+ginv[compts][1,2]
*cd_Weyl[compts][2,1,2,4,1]+ginv[compts][1,3]*cd_Weyl[compts][3,1,2,4,1]
+ginv[compts][1,4]*cd_Weyl[compts][4,1,2,4,1]+ginv[compts][2,1]
*cd_Weyl[compts][1,1,2,4,2]+ginv[compts][2,2]*cd_Weyl[compts][2,1,2,4,2]
+ginv[compts][2,3]*cd_Weyl[compts][3,1,2,4,2]+ginv[compts][2,4]
*cd_Weyl[compts][4,1,2,4,2]+ginv[compts][3,1]*cd_Weyl[compts][1,1,2,4,3]
+ginv[compts][3,2]*cd_Weyl[compts][2,1,2,4,3]+ginv[compts][3,3]
*cd_Weyl[compts][3,1,2,4,3]+ginv[compts][3,4]*cd_Weyl[compts][4,1,2,4,3]
+ginv[compts][4,1]*cd_Weyl[compts][1,1,2,4,4]+ginv[compts][4,2]
*cd_Weyl[compts][2,1,2,4,4]+ginv[compts][4,3]*cd_Weyl[compts][3,1,2,4,4]
+ginv[compts][4,4]*cd_Weyl[compts][4,1,2,4,4]-(lambda1*(ginv[compts][1,1]
*WEYL[compts][1,1,2,4]+ginv[compts][1,2]*WEYL[compts][2,1,2,4]
+ginv[compts][1,3]*WEYL[compts][3,1,2,4]+ginv[compts][1,4]
*WEYL[compts][4,1,2,4])+lambda2*(ginv[compts][2,1]*WEYL[compts][1,1,2,4]
+ginv[compts][2,2]*WEYL[compts][2,1,2,4]+ginv[compts][2,3]
*WEYL[compts][3,1,2,4]+ginv[compts][2,4]*WEYL[compts][4,1,2,4])
+lambda3*(ginv[compts][3,1]*WEYL[compts][1,1,2,4]+ginv[compts][3,2]
*WEYL[compts][2,1,2,4]+ginv[compts][3,3]*WEYL[compts][3,1,2,4]
```

```
+ginv[compts][3,4]*WEYL[compts][4,1,2,4])+lambda4*(ginv[compts][4,1]
*WEYL[compts][1,1,2,4]+ginv[compts][4,2]*WEYL[compts][2,1,2,4]
+ginv[compts][4,3]*WEYL[compts][3,1,2,4]+ginv[compts][4,4]
*WEYL[compts][4,1,2,4])):
> eq4:=ginv[compts][1,1]*cd_Weyl[compts][1,3,1,4,1]+ginv[compts][1,2]
*cd_Weyl[compts][2,3,1,4,1]+ginv[compts][1,3]*cd_Weyl[compts][3,3,1,4,1]
+ginv[compts][1,4]*cd_Weyl[compts][4,3,1,4,1]+ginv[compts][2,1]
*cd_Weyl[compts][1,3,1,4,2]+ginv[compts][2,2]*cd_Weyl[compts][2,3,1,4,2]
+ginv[compts][2,3]*cd_Weyl[compts][3,3,1,4,2]+ginv[compts][2,4]
*cd_Weyl[compts][4,3,1,4,2]+ginv[compts][3,1]*cd_Weyl[compts][1,3,1,4,3]
+ginv[compts][3,2]*cd_Weyl[compts][2,3,1,4,3]+ginv[compts][3,3]
*cd_Weyl[compts][3,3,1,4,3]+ginv[compts][3,4]*cd_Weyl[compts][4,3,1,4,3]
+ginv[compts][4,1]*cd_Weyl[compts][1,3,1,4,4]+ginv[compts][4,2]
*cd_Weyl[compts][2,3,1,4,4]+ginv[compts][4,3]*cd_Weyl[compts][3,3,1,4,4]
+ginv[compts][4,4]*cd_Weyl[compts][4,3,1,4,4]-(lambda1*(ginv[compts][1,1]
*WEYL[compts][1,3,1,4]+ginv[compts][1,2]*WEYL[compts][2,3,1,4]+
ginv[compts][1,3]*WEYL[compts][3,3,1,4]+ginv[compts][1,4]
*WEYL[compts][4,3,1,4])+lambda2*(ginv[compts][2,1]*WEYL[compts][1,3,1,4]
+ginv[compts][2,2]*WEYL[compts][2,3,1,4]+ginv[compts][2,3]
*WEYL[compts][3,3,1,4]+ginv[compts][2,4]*WEYL[compts][4,3,1,4])
+lambda3*(ginv[compts][3,1]*WEYL[compts][1,3,1,4]+ginv[compts][3,2]
*WEYL[compts][2,3,1,4]+ginv[compts][3,3]*WEYL[compts][3,3,1,4]
+ginv[compts][3,4]*WEYL[compts][4,3,1,4])+lambda4*(ginv[compts][4,1]
*WEYL[compts][1,3,1,4]+ginv[compts][4,2]*WEYL[compts][2,3,1,4]
+ginv[compts][4,3]*WEYL[compts][3,3,1,4]+ginv[compts][4,4]
*WEYL[compts][4,3,1,4])):
```

Next, solve for lambda[a].

```
> sol:=solve({eq1=0,eq2=0,eq3=0,eq4=0},{lambda1,lambda2,lambda3,lambda4});
> l4:=rhs(sol[4]);
> l2:=rhs(sol[3]);
```

```
> l1:=rhs(sol[1]);
> l3:=rhs(sol[2]);
> int(l1,t);
> unassign('C');
> L1:=1/4*ln(4*t+t^2+3)+C(x,y,z);
> diff(L1,x)=l2;
> C1:=int(z/(z*x+y),x);
> L2:=1/4*ln(4*t+t^2+3)+ln(z*x+y)+C2(y,z);
> diff(L2,y)=l3;
```

It is conformal to vacuum, with a conformal factor given by:

```
> lambda:= 1/4*ln(4*t+t^2+3)+ln(z*x+y);
```

Next, define the matrix of the epsilon tensor:

```
> E1:=subs(x=1,y=2,z=0,s12*W34+s14*W23-s13*W24+s23*W14-s24*W13
+s34*W12):
> E11:=diff(E1,s12,t12):E12:=diff(E1,s13,t12):E13:=diff(E1,s14,t12):
E14:=diff(E1,s23,t12):E15:=diff(E1,s24,t12):E16:=diff(E1,s34,t12):
> E21:=diff(E1,s12,t13):E22:=diff(E1,s13,t13):E23:=diff(E1,s14,t13):
E24:=diff(E1,s23,t13):E25:=diff(E1,s24,t13):E26:=diff(E1,s34,t13):
> E31:=diff(E1,s12,t14):E32:=diff(E1,s13,t14):E33:=diff(E1,s14,t14):
E34:=diff(E1,s23,t14):E35:=diff(E1,s24,t14):E36:=diff(E1,s34,t14):
> E41:=diff(E1,s12,t23):E42:=diff(E1,s13,t23):E43:=diff(E1,s14,t23):
E44:=diff(E1,s23,t23):E45:=diff(E1,s24,t23):E46:=diff(E1,s34,t23):
> E51:=diff(E1,s12,t24):E52:=diff(E1,s13,t24):E53:=diff(E1,s14,t24):
E54:=diff(E1,s23,t24):E55:=diff(E1,s24,t24):E56:=diff(E1,s34,t24):
> E61:=diff(E1,s12,t34):E62:=diff(E1,s13,t34):E63:=diff(E1,s14,t34):
E64:=diff(E1,s23,t34):E65:=diff(E1,s24,t34):E66:=diff(E1,s34,t34):
> E:=subs(t=1,matrix([[E11,E12,E13,E14,E15,E16],[E21,E22,E23,E24,E25,E26],
[E31,E32,E33,E34,E35,E36],[E41,E42,E43,E44,E45,E46],
[E51,E52,E53,E54,E55,E56],[E61,E62,E63,E64,E65,E66]]));
> EC:=subs(t=1,x=1,y=2,z=0,matrix([[ginv[compts][3,1]*ginv[compts][4,2]
```

```
-ginv[compts][3,2]*ginv[compts][4,1],-ginv[compts][2,1]*ginv[compts][4,2]
+ginv[compts][2,2]*ginv[compts][4,1],ginv[compts][2,1]*ginv[compts][3,2]
-ginv[compts][2,2]*ginv[compts][3,1],ginv[compts][1,1]*ginv[compts][4,2]
-ginv[compts][1,2]*ginv[compts][4,1],-ginv[compts][1,1]*ginv[compts][3,2]
+ginv[compts][1,2]*ginv[compts][3,1],ginv[compts][1,1]*ginv[compts][2,2]
-ginv[compts][1,2]*ginv[compts][2,1]],[ginv[compts][3,1]*ginv[compts][4,3]
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+ginv[compts][2,3]*ginv[compts][4,1],ginv[compts][2,1]*ginv[compts][3,3]
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-ginv[compts][1,3]*ginv[compts][4,1],-ginv[compts][1,1]*ginv[compts][3,3]
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-ginv[compts][1,3]*ginv[compts][2,1]],[ginv[compts][3,1]*ginv[compts][4,4]
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+ginv[compts][2,4]*ginv[compts][4,1],ginv[compts][2,1]*ginv[compts][3,4]
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+ginv[compts][2,3]*ginv[compts][4,2],ginv[compts][2,2]*ginv[compts][3,3]
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+ginv[compts][2,4]*ginv[compts][4,2],ginv[compts][2,2]*ginv[compts][3,4]
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+ginv[compts][1,4]*ginv[compts][3,2],ginv[compts][1,2]*ginv[compts][2,4]
-ginv[compts][1,4]*ginv[compts][2,2]],[ginv[compts][3,3]*ginv[compts][4,4]
```

```
-ginv[compts][3,4]*ginv[compts][4,3],-ginv[compts][2,3]*ginv[compts][4,4]
+ginv[compts][2,4]*ginv[compts][4,3],ginv[compts][2,3]*ginv[compts][3,4]
-ginv[compts][2,4]*ginv[compts][3,3],ginv[compts][1,3]*ginv[compts][4,4]
-ginv[compts][1,4]*ginv[compts][4,3],-ginv[compts][1,3]*ginv[compts][3,4]
+ginv[compts][1,4]*ginv[compts][3,3],ginv[compts][1,3]*ginv[compts][2,4]
-ginv[compts][1,4]*ginv[compts][2,3]]]));
> evalm(E-EC);
> subs(t=1,evalm(E^2));
> Id:=matrix([[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0],[0,0,0,1,0,0],[0,0,0,0,1,0],
[0,0,0,0,0,1]]);
> LL:=simplify(subs(x=1,y=2,z=0,t=1,map(factor,evalm(W&*(EC/2)*detgg))));
> evalm(2*LL+W);
```

This shows that the metric is self-dual.

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