MODELING MULTI-NAME DEFAULT CORRELATIONS

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This thesis focuses on the study of credit default dependence and related mathematical and computational issues.

Firstly, we derive an integral expression of the joint survival probability for the 2D first-passage-time model with default index being correlated Brownian motion and then apply it to give an alternative derivation (PDE approach) of the classical analytical formula of the default time density distribution which was first derived by Iyengar (Probabilistic approach) [1]. Furthermore, we prove that for this model both the coefficients of lower and upper tail dependence are zero.

Secondly, we create a new model, the crisis model, which is a generalization of the stress event model [2]. In the study of this model, we provide a novel identification of a set of independence conditions of defaults which enables us to derive a series expansion for the unconditional loss of a portfolio. Contrary to most bottom-up approach dynamic models, the distribution of the independence condition in the crisis model has a closed form expression which speeds up computations. We discover that by using a series expansion the loss distribution of a portfolio under the stress event model, which is a special case of the crisis model, can be computed accurately and extremely efficient. Furthermore, the computational cost for additional common factors to the stress event model is mild. This allows more flexibility for calibrations and opens up the possibility to study the multi-factor default dependence of a portfolio via a bottom-up approach. We demonstrate the effectiveness of our approach by calibrating it to investment grade CDS index tranches.

TABLE OF CONTENTS

LIST OF TABLES

PREFACE

I am heartily thankful to my thesis supervisor, Professor John Chadam, whose encouragement, guidance and support enabled me to develop an understanding of the subject. Completing this research under his supervision has been a pleasure and an exciting journey. I have benefited enormously from his incredible mathematical insight and apparently unlimited patience and generosity.

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1.0 INTRODUCTION

The world economy has been facing the worst crisis since the Great Depression. All the world's regions were experiencing enormous challenges emanating from the depression in developed countries as well as growth slowdown in developing countries. This financial and economic crisis was initiated by the credit crunch in the United States of America. One of the culprits of the crisis is the standard model, the Gaussian copula, used by the financial institutions and regulatory agencies for evaluating the risk associated with a credit portfolio. This is because the level of complexity in credit portfolios had exceeded the limitations of the model [5]. Consequently, there is an emergent need for developing new models which are computationally efficient and are able to accurately and consistently account for the risk. In fact research activities in the field of credit risk, driven by the rapid growth in the market of credit derivatives, had already been vibrant. Many of them have become valuable tools for credit risk management. Before the global financial crisis, the credit markets grew rapidly in terms of liquidity, diversity and complexity of available products. The most liquid credit derivatives were credit default swaps (CDS) and CDS index tranches. These products and their pricing methodologies are described in Appendices A and B respectively.

Bielecki and Rutkowski [6] define credit risk as "the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in the credit rating), variations of credit spreads, and the default event". Modeling of default times presents a challenging multivariate problem for researchers. Firstly, it has been observed that dependence between default times of various firms is quite significant. The sources of this dependence could be the macroeconomic or industry factors that influence firm values. While a realistic dependence structure between default times is crucial, it is often complicated by practical concerns such as parsimony of the model and computing times.

Secondly, large financial institutions typically have exposure to a very large number of credits, and many popular credit derivatives are based on underlying portfolios consisting of more than hundred of obligors. This means that the dimension of many real-world application is quite high and poses a difficult issue in computational efficiency.

The exact joint distribution of default times is not always required in credit risk applications. In many cases the distribution of the loss on a given portfolio is sufficient. Mathematically, the loss of a portfolio consisting of N credits at a given time t is

$$L_t = \sum_{i=1}^{N} \delta_i (1 - R_i) \mathbf{1}_{\{\tau_i \leq t\}}, \tag{1.1}$$

where δ_i is the notional of the i-th credit, R_i and τ_i are the corresponding recovery rate and default time. One of the major objective of this thesis is to develop a tractable and computationally efficient model for portfolio losses.

1.1 COMMON MODELING FRAMEWORKS FOR CREDIT DEFAULTS

There is a vast literature on the modeling of default times, portfolio losses and valuation of credit derivatives. A comprehensive source of cutting edge research in quantitative credit risk modeling and management is the website www.defaultrisk.com managed by Greg M. Gupton. This site is essentially a repository of published and working papers from both industry and academia.

There are three major approaches to modeling default times. One is to model their joint distribution directly, with the popular methods being copula and factor models. The second approach is to model a firm's asset value directly, and endogenously define default in terms of this value process. Models of this form are referred to as structural models and a common approach is to model default as the first time at which the value process crosses some lower barrier. The third approach is to model the default time directly. This approach typically models the default intensity of a jump process with an underlying filtration. These types of models are referred to as intensity-based models.

One key methodology in evaluation a portfolio loss is conditional independence. In general, the distribution of the number of defaults is essentially the distribution of a large number of dependent Bernoulli variables. If one can identify a set of conditions under which the underlying random variables are independent, then evaluation of the full loss distribution can be obtained.

1.1.1 Copula and Factor Models

Copulas provide a simple methodology for constructing a multivariate model when marginal distributions are given. To compute the joint distribution, one simply passes the ascribed margins through a given copula. Factor models, which are closely related to copula, allow for the indirect construction of a copula. In these models, default times are defined as functions of several underlying factors which reflect the dependence on the economic situation. A unique copula for default times is implied by the distribution of the factors and the functional transformation. Li [7] is generally credited as the first to introduce copulas to the finance community, and the Gaussian copula has become the industry standard for pricing index tranches and other credit derivatives. However, it has been well documented in the literature that the popular equicorrelated Gaussian copula provides a very poor fit to index tranche quotes (see for example [8]).

The copula approach is particularly favored by practitioners in CDO markets due to the computational efficiency enjoyed by certain simple classes of copulas. This computational efficiency is primarily due to the conditional independence enjoyed by the models, as there are several well-established techniques for evaluating the portfolio loss distribution in the conditional independence framework. See Hull and White [9], Andersen and Sidenius [10] and O'Kane [11], for good description of these techniques.

1.1.2 Structural and First-Passage Models

Structural models are widely considered to be the most intuitively appealing credit risk model. Generally speaking a structural model is one which specifies the evolution of the value of a firm's asset through time. The default time is then specified in terms of this value

process as a first-passage time through some lower barrier which represents the debt of the firm.

Merton [12] took such an approach in an effort to develop a theory for pricing bonds including the default risk. He considers a firm whose debt consists of a zero-coupon bond with fixed maturity T. The value of a firm's asset V_t was assumed to follow a geometric Brownian motion, and the firm defaults at T if V_T is below the face value of the bond. A major drawback of this model is that default can only happen at maturity. More realistic frameworks were introduced by several authors, with the simplest extension being made by Black and Cox [13], who retained the assumption of the firm's asset value process as a geometric Brownian motion, but described default as the first time at which V_t falls below a barrier of the form $Ke^{\lambda t}$. Debt is still assumed to be zero-coupon with a fixed maturity date T, and the barrier represents a safety covenant which allows bondholders to force bankruptcy and liquidation of the firm asset, should the value of the asset be too low. Other extensions include models with stochastic interest rates as well as models which determine the lower threshold endogenously as an optimal level from shareholders' perspective, typically as a function of parameters such as bankruptcy costs, tax shields on corporate interest payments and shareholder bargaining power. References to this vein of literature include Longstaff and Schwartz [14], Leland and Toft [15] and Duffie and Lando [16].

Though intuitively appealing on economic grounds, structural models are typically not able to describe actual bond yields (see Duffie and Lando [16]). In particular these models predict negligible spreads over Treasury bonds for very short maturities, which is inconsistent with observations in the bond and credit market. The negligible spread is due to the predictability of the default times of the basic structural models with respect to information flow. The predictability of default times implies that corporate defaults are never a surprise, which is certainly not true in practice. Several authors have attempted to remedy this shortcoming of the structural model. Zhou [17] introduces Poisson jumps to the Black-Cox model and finds that one can obtain significantly non-zero spreads for short maturities. Another approach, pioneered by Duffie and Lando [16], assumes imperfect information for investors. In the case of a fixed default boundary this amounts to working with a sub-filtration of that generated by the firm's continuous asset value process. They determine that defaults in this

case are not predictable with respect to the investor filtration. Several papers have investigated similar methods for modeling information reduction (for example, see Jarrow et al. [18] and references therein). Another information-based extension of Black-Cox model assumes that the default boundary is either random or unobserved. References in this direction include Schmidt [19] and Giesecke [20]. Recently, Hurd and Kuznetsov [21] consider Brownian motion time changed by an independent Lévy subordinator which effectively produces jumps in the process.

Our discussion so far has focused on structural models of single firm. Extension of the basic structural model to two firms can be found in Zhou [22] and Iyengar [1]. There are only a handful of papers which deal with dimensions greater than two of the basic structural models. Hull et al. [23] and Overbeck and Schmidt [24] investigate the direct extension of Black-Cox model to n firms. Both papers use Monte Carlo schemes for valuation and find that model prices are quite similar to those obtained from a Gaussian copula model. Instead of considering the dependence through the Brownian motions, McLeish and Metzler [25] develop a model which introduces common "systematic risk" processes that govern the trend and volatility in credit qualities.

1.1.3 Intensity-Based Models

Intensity-Based models are between the copula approaches and structural models in view of the ability to describe the dynamics of default. Basically, they directly model defaults through a jump process with an intensity λ_t which is sometimes referred to as the hazard rate. In particular the default intensity λ_t is often taken to be stochastic, and the jump process of default is usually dubbed as a "doubly-stochastic" process. Bielecki and Rutkowski [6] provide a nice introduction in this area. An advantage of the intensity-based framework with respect to copula and factor models is that they do allow for describing the dynamics of default times which is required in evaluating more complex products such as forward-starting CDOs or options on CDO tranches. Moreover, they are more tractable and computationally efficient than structural models.

Duffie et al. [2] [26] consider default intensities with a firm-specific component as well as

a common intensity process which could be interpreted as the global economic situation or the performance of a specific industry. Jumps in the common intensity process are usually needed in order to produce significant default correlation. Research papers in this vein include Mortensen [4] and Eckner [27]. Another approach to incorporate default dependence in a portfolio is through a common stochastic time process as proposed by Joshi and Stacey [28]. They call this stochastic time the business time or information arrival which measures shocks to the market.

Instead of modeling individual defaults, one can model the loss distribution L_t of a portfolio directly under the intensity framework. Basically, L_t drops abruptly if there is a jump in the underlying counting process enumerating the defaults. Brigo at el. [29] apply the dynamical Generalized-Poisson process to price the term structure of CDS index tranches and get a reasonably good fit to the market data. Longstaff and Rajan [3] propose a three-factor portfolio model which explains virtually all the time-series in an extensive data set of single maturity CDS index tranche prices.

1.2 OUTLINE OF THESIS

We introduce the concept of copula and provide an interpretation of tail dependence under the context of credit default in Chapter 2. In Chapter 3, we study the two-dimensional version of the Black-Cox first-passage time structural model. An alternative proof of the joint default time density is provided and the coefficients of the tail dependence are computed. In Chapter 4, we discuss bottom-up and top-down intensity models which motivate the creation of our model. Finally, we present our model in Chapter 5. We present a novel identification of the independence conditions of defaults which in turn suggests a series expansion of the unconditional distribution. We include examples of calibrations to CDS index tranches to demonstrate the effectiveness of our approach.

2.0 COPULA AND DEPENDENT MEASURES

After its introduction in the study of correlations among credit risky securities, the copula approach became a standard tool and language in the financial industry. The methodology of applying the Gaussian copula to credit derivatives as developed by David X. Li [7], though it has many drawbacks, is the market standard and is widely used by practitioners.

2.1 COPULA

A copula is a mathematical tool for defining associations between different random variables. Formally, it is a mechanism which links univariate marginals to a full multivariate distribution. Since the copula methodology separates the choice of marginal distributions from the choice of the dependent structure, it is especially suited to modeling credit correlation products. This is because Credit Default Swaps (CDS) prices are readily available in the market, and they provide the marginals for corresponding credit entities. Consequently, it leaves the practitioners free to choose the dependent structure (by adopting a desire copula) to capture the joint default behavior of the credits in a portfolio.

2.1.1 Motivation and Properties of a Copula

For a random vector $\tau = (\tau_1, ..., \tau_n)$, the complete information on the distribution of the random vector is given by the joint cumulative distribution function $\mathrm{P}(\tau_1 \leq t_1, ..., \tau_n \leq t_n)$. However, this function mixes information on the dependence between different components of τ with information on the marginals of the individual components. In order to separate

the mixed information, one first considers the marginal cumulative probabilities, $D_i(t) = \mathrm{P}(\tau_i \leq t)$, for $i = 1, ...n$. It is easy to see that each $D_i(\tau_i) = U_i$ is a uniformly distributed random variable on $[0, 1]$. The joint cumulative distribution of $(\tau_1, ..., \tau_n)$ is

$$\mathrm{P}(\tau_1 \leq t_1, ...\tau_n \leq t_n) = \mathrm{P}(D_1(\tau_1) \leq D_1(t_1), ...D_n(\tau_n) \leq D_n(t_n)) \tag{2.1}$$

$$= \mathrm{P}(U_1 \leq u_1, ...U_n \leq u_n), \tag{2.2}$$

and the copula function C of $\tau = (\tau_1, ...\tau_n)$ is defined as follows:

$$C(u_1, ..., u_n) = \mathrm{P}(U_1 \leq u_1, ...U_n \leq u_n) \tag{2.3}$$

$$= \mathrm{P}(D_1^{-1}(U_1) \leq D_1^{-1}(u_1), ...D_n^{-1}(U_n) \leq D_n^{-1}(u_n)) \tag{2.4}$$

$$= \mathrm{P}(\tau_1 \leq D_1^{-1}(u_1), ...\tau_n \leq D_n^{-1}(u_n)), \tag{2.5}$$

where the inverse function D_i^{-1} are assumed to exist for $i = 1, ..., n$. A copula function has the following properties:

1. $C(u_1, ..., u_n)$ is increasing in each component u_i.
2. $C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_n) = 0$
3. $C(1, ..., 1, u_k, 1, ..., 1) = u_k$
4. For all $(a_1, ..., a_n), (b_1, ..., b_n) \in [0, 1]^n$ with $a_i \leq b_i$

$$\sum_{i_1=1}^{2} \cdots \sum_{1_n=1}^{2} (-1)^{i_1+...+i_d} C(u_{1i_1}, ..., u_{di_d}) \geq 0, \tag{2.6}$$

   where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, ..., n\}$.

The first property is clearly a requirement for any multivariate cdf. The second property has to be true since no default of a single credit implies no joint defaults. The third property is the requirement of uniform marginal distribution. The last property is less obvious, it ensures the density of the distribution (if exists) is always non-negative i.e. $\partial_{u_1...u_n} C \geq 0$. It is easy to see that the dimensionality of a copula can be reduced from n to $n - 1$ simply by setting any one of the arguments to 1 and the resulting function is still a copula for $n \geq 2$. Consequently, many properties of copulas can be understood simply by studying the two-dimensional case. Although we have introduced the notion of a copula function through a multivariate cdf, it is worth noting that any function satisfying the properties stated above is also a copula function.

2.1.2 Implementation of the Copula Methodology

Let's consider an example to illustrate how copula functions can be used to price a basket of credits. Suppose we choose a Gaussian copula i.e.

$$C_\Sigma(u_1, ..., u_n) = N_\Sigma(N^{-1}(u_1), ..., N^{-1}(u_n)), \tag{2.7}$$

as the dependent structure for the credits in a basket, where N_Σ is the standard multivariate normal cumulative distribution with correlation matrix Σ and N^{-1} denotes the inverse function of the standard normal cumulative distribution. In general, a n-dimensional Gaussian copula has $n(n-1)/2$ correlation parameters. For pricing a basket that consist of a large number of credits, e.g. standard CDS index which has approximately 100 credits, a full specification of the Gaussian copula is impracticable since the number of parameters is too large. Consequently, the standard approach in the financial industry assumes a equal correlation parameter ρ for all pairs of credits in the basket. Having specified the copula, the joint distribution become uniquely determined once the marginal distributions are known. Construction of the marginal distribution is typically accomplished in practice by the term structure of associated CDS. O'Kane and Turnbull [30] discuss a common procedure which assumes piecewise constant intensity in a Poisson process. Once the intensities over the intervals are determined from the observed CDS spreads, the marginal distribution is obtained. Thus, a cumulative distribution for the default times can be constructed by specifying a copula, and then inserting into Eq.(2.5) the marginals D_i obtained from the CDS spreads. As a result, the full distribution of the default times is obtained and can be applied to price correlated products that depend only on the default times of the credits.

It is worth noting that the Gaussian copula can be constructed by a one factor model,

$$X_i = \beta_i Z + \sqrt{1-\beta_i^2} Y_i, \tag{2.8}$$

and interpreted as the asset value of firm i which is driven by a normally distributed latent common factor Z and a normally distributed independent idiosyncratic random variable Y_i. The common factor Z is a systematic component which is typically interpreted as the economic state of the market and the Y_i are interpreted as firm-specific risk factors. Default

of firm i occurs before time t if X_i falls below a time dependent threshold $C_i(t)$ which is calibrated to the term structure of survival of survival probabilities of firm i. The conditional default probabilities for a given Z are

$$\mathrm{P}(\tau_i \leq t|Z) = N\left(\frac{C_i(t) - \beta_i Z}{\sqrt{1-\beta_i^2}}\right). \tag{2.9}$$

Under the common simplifying assumption of non-random and constant recovery rate R for every firm and equal weighting for each firm in the portfolio, the percentage loss L_t takes on values of multiples of $\delta := (1-R)/N$. The loss distribution is then

$$\mathrm{P}(L_t = k\delta) = \mathrm{P}\left(\sum_{i=1}^{N} \mathbf{1}_{\{\tau_i \leq t\}} = k\right) \tag{2.10}$$

$$= \mathrm{E}\left[\mathrm{P}\left(\sum_{i=1}^{N} \mathbf{1}_{\{\tau_i \leq t\}} = k \middle| Z\right)\right]. \tag{2.11}$$

The conditional probability inside the expectation is a multi-nomial with probabilities

$$p_i(Z) = N\left(\frac{C_i(t) - \beta_i Z}{\sqrt{1-\beta_i^2}}\right) \tag{2.12}$$

and the loss distribution can therefore be obtained by integrating a multi-nomial distribution against a standard normal density for Z.

2.1.3 Limitations of the Copula Approach

The simplicity and tractability of the Gaussian copula makes it the standard tool for pricing correlated products. Brokers and investment banks usually quote tranche prices in terms of implied correlations through this standard model, analogous to the practice of quoting implied Black-Scholes volatilities in option markets. However, it is typically impossible to match the prices of all tranches at a single maturity simultaneously by calibrating a flat correlation parameter. The implied correlation as a quotation device is a parameter ρ in the Gaussian copula which is calibrated from the market price for each tranche only. If the Gaussian copula is the true dependence, the implied correlations for all tranches have to be the same. However, plotting the implied correlations for the different tranches gives a uneven curve which is the well known correlation smile.

As an extension of the Gaussian copula, Andersen and Sidenius [10] proposed a random factor loading (RFL) copula which incorporates a random regime switching mechanism in the Gaussian copula. Basically, instead of adopting a constant loading factor β_i for the latent variable Z in the one factor Gaussian model, a deterministic function $\beta_i(Z)$ depending on the random variable Z is used. In a three-parameter version of the RFL, the model shows a significant improvement in fitting different tranche prices. On the other hand, the double-t copula proposed by Hull and White [9] replaces the normal distributions of the latent common factor Z and the idiosyncratic factor Y_i in Eq.(2.8) by independent t-distributions. It turns out that the two-parameter double-t copula also matches different tranche prices reasonably well.

Although the copula approach for pricing credit correlated products is widely used in the financial industry, it is not a satisfactory modeling framework. The first drawback of the copula approach is that the parameters in the model generally lack economical meanings. As a result, it is difficult to validate the parameters obtained from calibrating the model to market data. Furthermore, even though some copula functions are able to fit the market data reasonably well, the tranche prices computed by the models are usually outside the bid-ask spreads.

2.2 TAIL DEPENDENCE

Upper and lower tail dependence are measures of pairwise dependence that depend only on the copula of a pair of random variables X_1 and X_2 with continuous marginal distributions. The motivation for looking at these coefficients is that they provide measures which capture the tendency for joint extreme movements. The coefficients of lower and upper tail dependence of X_1 and X_2 are defined as (see McNeil et al. [31])

$$\lambda_L := \lim_{u \to 0^+} \mathrm{P}(X_2 \leq D_2^{-1}(u) | X_1 \leq D_1^{-1}(u)) \tag{2.13}$$

and

$$\lambda_U := \lim_{u \to 1^-} \mathrm{P}(X_2 > D_2^{-1}(u) | X_1 > D_1^{-1}(u)), \tag{2.14}$$

where $D_i(x) = \mathrm{P}(X_i \leq x)$ are the marginal distributions. Note that the range of the coefficients of both lower and upper tail dependence is $[0, 1]$ if they exist. The random variables X_1 and X_2 are said to be asymptotically independent if both upper and lower coefficients equal zero. On the other hand, if a coefficient is greater than zero, we say that the model displays a positive tail dependence. For copula models, one can obtains simple expressions for λ_U and λ_L:

$$\lambda_L = \lim_{u \to 0^+} \frac{\mathrm{P}(X_2 \leq D_2^{-1}(u), X_1 \leq D_1^{-1}(u))}{\mathrm{P}(X_1 \leq D_1^{-1}(u))} \tag{2.15}$$

$$= \lim_{u \to 0^+} \frac{C(u, u)}{u} \tag{2.16}$$

and

$$\lambda_U = \lim_{u \to 1^-} \frac{\mathrm{P}(X_2 > D_2^{-1}(u), X_1 > D_1^{-1}(u))}{\mathrm{P}(X_1 > D_1^{-1}(u))} \tag{2.17}$$

$$= \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}. \tag{2.18}$$

Tail dependence is an important concept since it addresses the phenomenon of joint extreme values, which is one of the major concerns in financial risk management and pricing of credit derivatives. Tail dependence is usually studied in the context of returns for two different entities. It is an interesting concept in the study of the contagion of crises between

markets or countries. Large negative moves in a country or market are often found to imply large negative moves in others. These questions have been addressed by many authors, see for example Malevergne and Sornette [32] or Longin and Solnik [33]. Besides, tail dependence is also an observed feature in asset returns as shown in Mashal and Zeevi [34].

On the other hand, there is a growing awareness of tail dependence in credit risk in the financial industry (see Das and Geng [35], for example). Firstly, since it has been argued that equity return correlation is a proxy for the asset correlation in latent variable models [36], it is of interest to study models with tail dependence. Furthermore, as the one parameter Gaussian copula fails to fit the market prices of different CDO tranches simultaneously and leads to the well known implied correlation smile, some researchers (see for example Kalemanova et al. [37]) explain this phenomenon by the lack of tail dependence in the Gaussian copula and propose to use copulas with tail dependence. Furthermore, It has come to our attention through a private communication that a team from Credit Suisse headed by Christian Bluhm has performed some internal research and fitted t-copulas to empirical data which provides empirical evidence for positive tail dependence. Despite the fact that empirical studies regarding tail dependence in credit risk are rare, copulas displaying positive tail dependence are suggested for use.

2.2.1 Interpretation of the Coefficient of Lower Tail Dependence

Although many practitioners believe that copulas possessing positive tail dependence is a better approximation to the true dependence between firms, the notion of tail dependence for default times is not very clear and is seldom discussed. According to Eq.(2.17), the coefficient of upper tail dependence in the context of credit risk is

$$\lambda_U = \lim_{u \to 1^-} \frac{\mathrm{P}(\tau_2 > D_2^{-1}(u), \tau_1 > D_1^{-1}(u))}{\mathrm{P}(\tau_1 > D_1^{-1}(u))} \tag{2.19}$$

$$= \lim_{t \to \infty} \frac{\mathrm{P}(\tau_2 > t_1, \tau_1 > t_2)}{\mathrm{P}(\tau_1 > t_1)}, \tag{2.20}$$

where τ_1 and τ_2 are the default times of firm 1 and firm 2 respectively, and $t = \min\{t_1, t_2\}$. Since any real credit derivative contract has a finite maturity, the coefficient of upper tail

dependence, which is a limit when t approaches infinity, does not provide practical information of the dependence between firms. On the other hand, the coefficient of lower tail dependence is

$$\lambda_L = \lim_{u \to 0^+} \frac{\mathrm{P}(\tau_2 \leq D_2^{-1}(u), \tau_1 \leq D_1^{-1}(u))}{\mathrm{P}(\tau_1 \leq D_1^{-1}(u))} \tag{2.21}$$

$$= \lim_{t \to 0^+} \frac{\mathrm{P}(\tau_2 \leq t_2, \tau_1 \leq t_1)}{\mathrm{P}(\tau_1 \leq t_1)}, \tag{2.22}$$

where $t = \max\{t_1, t_2\}$ and $D_1(t_1) = D_2(t_2)$. This limit gives information on the default times of the firms at the beginning of a contract and is of course a relevant dependent measure that should be analyzed further. For the sake of simplicity, we first consider the case where the marginals are the same i.e. $D_1 = D_2$. It is easy to see that the conditional probability that firm 1 defaults at t given that firm 2 defaults at s can be computed by the following limit

$$\mathrm{P}(\tau_2 = t | \tau_1 = s) = \lim_{\delta \to 0} \frac{\mathrm{P}(s < \tau_1 \leq s + \delta, t < \tau_2 \leq t + \delta)}{\mathrm{P}(s < \tau_1 \leq s + \delta)}. \tag{2.23}$$

We will refer to this probability as the conditional probability of joint defaults at (s, t). In particular, in the case that $t = s = 0$,

$$\mathrm{P}(\tau_1 = 0, \tau_2 = 0 | \tau_1 = 0) = \lim_{\delta \to 0} \frac{\mathrm{P}(\tau_1 \leq \delta, \tau_2 \leq \delta)}{\mathrm{P}(\tau_1 \leq \delta)} \tag{2.24}$$

$$= \lambda_L, \tag{2.25}$$

which is just the coefficient of lower tail dependence as indicated by Eq.(2.22). This means that we can interpret the coefficient of lower tail dependence as the conditional default probability that both firms default around the beginning of the contract given that one has defaulted. However, this equivalence is correct only for the case of equal marignals. Although the values of the coefficient of lower tail dependence and the conditional probability of joint default are not necessarily the same for general marignals, they are simultaneously zero or positive, i.e. they are both zero or they are both equal to some positive numbers, under

some mild conditions on the marginals and the copula. Before we proceed to the proof of the claim, we first note that for different marginals

$$\mathrm{P}(\tau_1 = 0, \tau_2 = 0 | \tau_1 = 0) = \lim_{\delta \to 0} \frac{\mathrm{P}(\tau_1 \leq \delta, \tau_2 \leq \delta)}{\mathrm{P}(\tau_1 \leq \delta)} \tag{2.26}$$

$$= \lim_{u \to 0^+} \frac{\mathrm{P}(\tau_1 \leq D_1^{-1}(u), \tau_2 \leq D_2^{-1}(v(u)))}{\mathrm{P}(\tau_1 \leq D_1^{-1}(u))}, \tag{2.27}$$

where the function $v(u)$ is defined implicitly by the relation $\delta = D_1^{-1}(u) = D_2^{-1}(v)$ which implies that

$$v(u) = D_2 \circ D_2^{-1}(u). \tag{2.28}$$

Consequently, the conditional probability is related to the following limit in terms of the copula

$$\mathrm{P}(\tau_1 = 0, \tau_2 = 0 | \tau_1 = 0) = \lim_{u \to 0^+} \frac{C(u, v(u))}{u}. \tag{2.29}$$

Theorem 2.2.1. *Suppose the following conditions*

i. $v \in \mathcal{C}^1[0, \delta]$ *for some* $\delta > 0$,

ii. $v(0) = 0$,

iii. $v'(0) \in (0, \infty)$,

are satisfied, then

$$\lambda_L = 0 \quad \textit{iff} \quad \lim_{u \to 0^+} \frac{C(u, v(u))}{u} = 0, \tag{2.30}$$

where $C(u, v)$ *is a two-dimensional copula.*

Proof. We first prove the forward implication. If $v(u) \leq u$ then $C(u, v(u)) \leq C(u, u)$, else $C(u, v(u)) \leq C(v(u), v(u))$. Hence

$$C(u, v(u)) \leq C(u, u) + C(v(u), v(u)) \tag{2.31}$$

$$\frac{C(u, v(u))}{u} \leq \frac{C(u, u)}{u} + \frac{C(v(u), v(u))}{u} \tag{2.32}$$

$$\frac{C(u, v(u))}{u} \leq \frac{C(u, u)}{u} + \frac{v(u)}{u}\frac{C(v(u), v(u))}{v(u)} \tag{2.33}$$

$$0 \leq \lim_{u \to 0^+} \frac{C(u, v(u))}{u} \leq \lim_{u \to 0^+} \frac{C(u, u)}{u} + v'(0) \lim_{u \to 0^+} \frac{C(v(u), v(u))}{v(u)} \tag{2.34}$$

$$= (1 + v'(0))\lambda_L = 0. \tag{2.35}$$

Similarly, if $v(u) \leq u$ then $C(u, u) \leq C(v^{-1}(u), u)$, else $C(u, u) \leq C(u, v(u))$. Note that v^{-1} exists near zero as $v'(0) > 0$ and $v \in \mathcal{C}^1[0, \delta]$. Hence

$$C(u, u) \leq C(v^{-1}(u), u) + C(u, v(u)) \tag{2.36}$$

$$\frac{C(u, u)}{u} \leq \frac{C(v^{-1}(u), u)}{u} + \frac{C(u, v(u))}{u} \tag{2.37}$$

$$\frac{C(u, u)}{u} \leq \frac{C(x, v(x))}{v(x)} + \frac{C(u, v(u))}{u} \tag{2.38}$$

$$0 \leq \lim_{u \to 0^+} \frac{C(u, u)}{u} \leq \frac{1}{v'(0)} \lim_{x \to 0^+} \frac{C(x, v(x))}{x} + \lim_{u \to 0^+} \frac{C(u, v(u))}{u} \tag{2.39}$$

$$= 0. \tag{2.40}$$

This proves the backward implication. □

Theorem 2.2.1 proves our claim that the vanishing of the coefficient of lower tail dependence implies the vanishing of the conditional probability that both firm default near the beginning of the contract given that one has already defaulted and vice versa. It is easy to see that if both the marignals D_1 and D_2 are $\mathcal{C}^1$ near $t = 0$, which is always the case, then condition (i) is satisfied. For condition (ii), since $D_1(0) = D_2(0) = 0$ by definition and $v(u) = D_2 \circ D_2^{-1}(u)$, it is always true that $v(0) = 0$. Finally, if both $D_1'(0)$ and $D_2'(0)$ are greater than zero, then $v'(0) > 0$. The finiteness of the derivative of a marginal is equivalent to a nonzero default intensity for a credit. CDS spreads (even those with a very short maturity) quoted in the credit markets are always nonzero which implies finite default intensities.

As a result, condition (*iii*) is satisfied by marginals that are able to capture the empirical fact that initial default intensities are nonzero. We now move on to the proof of the second part of our claim.

Theorem 2.2.2. *In addition to the assumptions in Theorem 2.2.1, if both limits*

$$\lim_{u\to 0^+} \frac{C(u,u)}{u} \quad \textit{and} \quad \lim_{u\to 0^+} \frac{C(u,v(u))}{u} \tag{2.41}$$

exist, then

$$\lambda_L > 0 \quad \textit{iff} \quad \lim_{u\to 0^+} \frac{C(u,v(u))}{u} > 0. \tag{2.42}$$

Proof. The necessary condition in Theorem 2.2.1 is equivalent to

$$\lambda_L \neq 0 \quad \text{iff} \quad \lim_{u\to 0^+} \frac{C(u,v(u))}{u} \neq 0. \tag{2.43}$$

Since both limits exist, the result follows. □

For all common copulas, the limits in Theorem 2.2.2 always exist. In fact, it is very difficult to construct a copula such that the limits do not exist. It is worth noting that if a copula C is differentiable in $(0,\delta)\times(0,\delta)$ for some $\delta > 0$, then by L'Hopital's rule

$$\lim_{u\to 0^+} \frac{C(u,v(u))}{u} = \lim_{u\to 0^+} \left(\partial_1 C(u,v(u)) + \partial_2 C(u,v(u))v'(u)\right). \tag{2.44}$$

In particular, if $v(u) = u$ then,

$$\lambda_L = \lim_{u\to 0^+} \frac{C(u,u)}{u} = \lim_{u\to 0^+} \left(\partial_1 C(u,u) + \partial_2 C(u,u)\right). \tag{2.45}$$

This gives an easy way to evaluate the coefficient of tail dependence for differentiable copulas. It is also worth noting that if we further assume that both $\partial_1 C$ and $\partial_2 C$ are continuous in $(0,\delta)\times(0,\delta)$, the conclusions of Theorem 2.2.1 and Theorem 2.2.2 can be drawn easily. This is because

$$\lim_{u\to 0^+} \partial_i C(u,v(u)) = \lim_{u\to 0^+} \partial_i C(u,u) \tag{2.46}$$

for $i = 1, 2$ and both $\partial_1 C$ and $\partial_2 C$ are non-negative by definition. According to Eq.(2.44) and Eq.(2.45), if $\lim_{u \to 0^+} \partial_i C(u, u) = 0$, then

$$\lim_{u \to 0^+} \frac{C(u, v(u))}{u} = \lambda_L = 0. \tag{2.47}$$

On the other hand, if $\lim_{u \to 0^+} \partial_i C(u, u) > 0$, then

$$\lim_{u \to 0^+} \frac{C(u, v(u))}{u} > 0 \quad \text{and} \quad \lambda_L > 0. \tag{2.48}$$

2.2.2 Why Tail Dependence?

The reason why a copula with positive tail dependence performs better than one that has no tail dependence is not very clear. We do not intend to provide a complete and satisfactory answer to this question. Instead, we provide a partial answer by investigating the difference of the joint default probability between the Gaussian copula and copulas with positive tail dependence. This in turn provides an alternative perspective to compare the two classes of copulas. To this end, we first consider a small fixed time δ and adopt the intensity model which is the market standard for the marginal, i.e.

$$\mathrm{P}(\tau_1 \leq \delta) = 1 - e^{-\int_0^\delta \lambda_1(s) ds} \tag{2.49}$$

$$\approx \lambda_1(0)\delta, \tag{2.50}$$

where $\lambda_1(s)$ is a deterministic default intensity of firm 1 and $\lambda_1(0)$ is the initial default intensity which is always assumed finite. It is worth noting that the approximation for the marginal by Eq.(2.50) is generally true even for a stochastic $\lambda_1(s)$. Consequently, the conditional probability of joint default around the origin is

$$\frac{\mathrm{P}(\tau_1 \leq \delta, \tau_2 \leq \delta)}{\mathrm{P}(\tau_1 \leq \delta)} \approx \frac{\mathrm{P}(\tau_1 \leq \delta, \tau_2 \leq \delta)}{\lambda_1(0)\delta}. \tag{2.51}$$

The above expression for the conditional probability of joint default implies that for copulas with positive tail dependence, the joint default probability is

$$\mathrm{P}^+(\tau_1 \leq \delta, \tau_2 \leq \delta) \approx \gamma\delta, \tag{2.52}$$

where γ is a constant and the superscript '+' indicates that the probability is derived from a copula with positive tail dependence. On the other hand, for copulas with vanishing tail dependence like the Gaussian copula, the joint default probability decreases faster than δ, i.e.

$$\mathrm{P}^0(\tau_1 \leq \delta, \tau_2 \leq \delta) \leq \delta^\epsilon, \tag{2.53}$$

where $\epsilon > 1$ and the superscript '0' indicates that the probability is derived from a copula with vanishing tail dependence. As a result, for sufficiently small δ,

$$\mathrm{P}^0(\tau_1 \leq \delta, \tau_2 \leq \delta) < \mathrm{P}^+(\tau_1 \leq \delta, \tau_2 \leq \delta). \tag{2.54}$$

This explicitly shows that having a positive tail dependence can be interpreted as including more joint default probability near the beginning of a contract.

2.3 DEFAULT CORRELATION

The most commonly used measure of dependence for a pair of random variables is the linear correlation. This is a way to capture the strength of the relationship between two random variables X and Y using their linear product. It is defined as follows

$$Corr(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{(E(X^2) - E(X)^2)(E(Y^2) - E(Y)^2)}}. \tag{2.55}$$

In the context of credit risk, it is tempting to measure default correlation by employing the linear correlation of default times, i.e

$$Corr(\tau_A, \tau_B) = \frac{E(\tau_A \tau_B) - E(\tau_A)E(\tau_B)}{\sqrt{(E(\tau_A^2) - E(\tau_A)^2)(E(\tau_B^2) - E(\tau_B)^2)}}. \tag{2.56}$$

However, this quantity is not very useful in practice. This is because contracts of credit derivatives always have finite maturity while the evaluation of the linear correlation contains

irrelevant information of the distribution of large default times. It is more appropriate to investigate default correlation in terms of the default indicator defined as follows:

$$\mathbf{1}_{\{\tau \leq t\}} = \begin{cases} 1, & \text{if } \tau \leq t; \\ 0, & \text{otherwise}, \end{cases} \tag{2.57}$$

where τ is the default time of the firm and t is a fixed time. It is easy to see that the expectations of the indicators and product of indicators are:

$$\mathrm{E}(\mathbf{1}_{\{\tau_A \leq t\}}) = \mathrm{P}(\tau_A \leq t) := p_A, \tag{2.58}$$

$$\mathrm{E}(\mathbf{1}_{\{\tau_B \leq t\}}) = \mathrm{P}(\tau_B \leq t) := p_B, \tag{2.59}$$

$$\mathrm{E}(\mathbf{1}_{\{\tau_A \leq t\}}\mathbf{1}_{\{\tau_B \leq t\}}) = \mathrm{P}(\tau_A \leq t, \tau_B \leq t) := p_{AB}, \tag{2.60}$$

where p_A, p_B and p_{AB} are defined as the default probabilities of firm A, firm B and the joint default probability of them before time t respectively. Consequently, the linear correlation of the default indicators is then

$$Corr(\mathbf{1}_{\{\tau_A \leq t\}}, \mathbf{1}_{\{\tau_B \leq t\}}) := \varrho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}}. \tag{2.61}$$

In fact, ϱ_{AB} was widely used as a default correlation measure before the introduction of the copula in the credit industry. For example, Lucas [38] employs this measure in his empirical investigation of correlations of historical defaults.

In the remainder of this subsection, we establish an equivalence between the coefficients of tail dependence and the limits of ϱ_{AB} for identical marginals. We first notice that the linear correlation coefficient can be simply expressed in terms of the copula and the marginals, i.e.

$$\varrho_{AB} = \frac{C(u,v) - uv}{\sqrt{u(1-u)v(1-v)}}, \tag{2.62}$$

where C is the copula of the model, and $u = p_A(t)$ and $v = p_B(t)$ are the marginals. If the marginals are the same, Eq.(2.62) can be reduced to

$$\varrho_{AB} = \frac{C(u,u) - u^2}{u(1-u)}. \tag{2.63}$$

With this simple expression for the linear correlation coefficient, we first compute the limit of the difference:

$$\lim_{u\to 0} \frac{C(u,u)-u^2}{u(1-u)} - \frac{C(u,u)}{u}, \tag{2.64}$$

$$= \lim_{u\to 0} \frac{C(u,u)-u^2}{u(1-u)} - \frac{C(u,u)(1-u)}{u(1-u)}, \tag{2.65}$$

$$= \lim_{u\to 0} \frac{u(C(u,u)-u)}{u(1-u)}, \tag{2.66}$$

$$= 0. \tag{2.67}$$

This shows that the initial linear correlation coefficient equals λ_L when the two marginals are the same. For the case of the upper tail dependence, we compute the corresponding difference:

$$\lim_{u\to 1} \frac{C(u,u)-u^2}{u(1-u)} - \frac{1+C(u,u)-2u}{1-u}, \tag{2.68}$$

$$= \lim_{u\to 1} \frac{C(u,u)-u^2}{u(1-u)} - \frac{u+uC(u,u)-2u^2}{u(1-u)}, \tag{2.69}$$

$$= \lim_{u\to 1} \frac{C(u,u)(1-u)+u^2-u}{u(1-u)}, \tag{2.70}$$

$$= \lim_{u\to 1} \frac{C(u,u)(1-u)+u^2-u}{u(1-u)}, \tag{2.71}$$

$$= \lim_{u\to 1} \frac{C(u,u)}{u} + \frac{u(u-1)}{u(1-u)}, \tag{2.72}$$

$$= 0. \tag{2.73}$$

Thus, the linear correlation coefficient for sufficiently large t and the coefficient of λ_U are the same for identical marginals. There is no analogous equivalence for arbitrary marginals, since the linear correlation coefficient ϱ_{AB} is a function of two arbitrary marginals u and v whereas the tail dependence is defined in terms of u only.

3.0 FIRST-PASSAGE TIME MODELS

This chapter presents the model of defaults of two firms based on Black and Cox first-passage time structural framework. In this model, the dynamics of the firms' asset values follows diffusion processes and a firm defaults when its asset value first hits a default boundary. Furthermore, default correlation between two firms comes from the correlated Brownian motion. The hitting problem of correlated two-dimensional Brownian motion has been studied by several authors, including Buckholtz and Wasan [39], Iyengar [1] and Rebholz [40] and has appeared in the literature since at least 1894. Zhou [22] was the first to apply this classic model to study multiple defaults and provided an analytical formula for calculating default correlations between two firms. Overbeck and Schmidt [24] incorporated time-changes to the basic processes in such a way that exact calibration to given marginal distribution is possible.

In this chapter we begin by restating the formulation of the correlated two-dimensional Brownian motion model. An analytical formula for the joint survival probability $G(s,t) = \mathrm{P}(\tau_1 > s, \tau_2 > t)$ is then established and applied to give an alternative derivation of the analytical formula of the joint default times density distribution $h(s,t)$ which was first derived by Iyengar [1]. Furthermore, by employing the analytical formula of $G(s,t)$, we are able to prove that both the coefficients of lower and upper tail dependence of for structural models are zero.

3.1 FORMULATION OF THE MODEL

Let V_t^1 and V_t^2 denote the total asset values of firm 1 and firm 2 respectively. The dynamics for them are given by the following vector stochastic differential equation

$$\begin{pmatrix} d\ln V_t^1 \\ d\ln V_t^2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} dt + \Psi \begin{pmatrix} dZ_t^1 \\ dZ_t^2 \end{pmatrix}, \tag{3.1}$$

where μ_1 and μ_2 are constant drifts, Z_t^1 and Z_t^2 are two independent standard Brownian motions, and Ψ is a constant matrix such that

$$\Psi\Psi^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{3.2}$$

Although Ψ is not unique, it is always the case that $Corr(d\ln V_t^1, d\ln V_t^2) = \rho$. This coefficient ρ governs the correlation between the movements in the asset values of the two firms. An example of Ψ that satisfies Eq.(3.2) is

$$\Psi_0 = \begin{pmatrix} \sigma_1\sqrt{1-\rho^2} & \rho\sigma_1 \\ 0 & \sigma_2 \end{pmatrix}. \tag{3.3}$$

Assume the default of firm i is triggered when V_t^i falls to the threshold level $C_i(t)$. Following Black and Cox [13], assume that the default boundary takes an exponential form $C_i(t) = e^{\lambda_i t}K_i$. For the special case in which $\lambda_i = \mu_i$, we can transform the problem to a driftless one by defining

$$X_t^i = -\ln\left(\frac{e^{-\mu_i t}V_t^i}{V_0^i}\right), \tag{3.4}$$

then Eq.(3.1) becomes

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = -\Psi \begin{pmatrix} dZ_t^1 \\ dZ_t^2 \end{pmatrix}, \tag{3.5}$$

with the initial condition

$$\begin{pmatrix} X_0^1 \\ X_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

The default boundaries are then the vertical and horizontal lines

$$x_1 = b_1, \tag{3.7}$$

$$x_2 = b_2, \tag{3.8}$$

where

$$b_i = -\ln\left(\frac{K_i}{V_0^i}\right). \tag{3.9}$$

The two stopping times are defined as

$$\tau_i = \inf\{t \,|\, X_t^i = b_i\}, \tag{3.10}$$

which is equivalent to the first time that the asset value reaches the default boundary. Let $g(x_1, x_2, t)$ be the transition probability density of the particle in the region $\{(x_1, x_2)|x_1 < b_1,\ x_2 < b_2\}$, which is the probability density that $(X_t^1, X_t^2) = (x_1, x_2)$ given that the particle does not reach the boundary by t. The transition probability density $g(x_1, x_2, t)$ satisfies the Kolmogorov forward equation (see for example, Karatzas and Shreve [41]),

$$\frac{\partial g}{\partial t} = \frac{\sigma_1^2}{2}\frac{\partial^2 g}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 g}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 g}{\partial x_2^2}, \tag{3.11}$$

for $x_1 < b_1$ and $x_2 < b_2$, subject to the initial and boundary conditions

$$g(x_1, x_2, 0) = \delta(x_1)\delta(x_2), \tag{3.12}$$

$$g(b_1, x_2, t) = g(x_1, b_2, t) = 0, \tag{3.13}$$

$$g(-\infty, x_2, t) = g(x_1, -\infty, t) = 0, \tag{3.14}$$

where $\delta(x)$ is a Dirac delta function. Adopt the following transformation

$$x_1 = b_1 - \sigma_1(z_1\sqrt{1-\rho^2} + \rho z_2), \tag{3.15}$$

$$x_2 = b_2 - \sigma_2 z_2, \tag{3.16}$$

and write

$$f(z_1, z_2, t) = g(x_1, x_2, t)J, \tag{3.17}$$

where $J = \sigma_1\sigma_2\sqrt{1-\rho^2}$ is the jacobian of the transformation. This transformation gets rid of the mixed derivative term in the Kolmogorov forward equation and simplifies it to a heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f = \frac{1}{2}\left(\frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2}\right). \tag{3.18}$$

Furthermore, it is more convenient to express (z_1, z_2) in terms of polar coordinates (r, θ), i.e.

$$z_1 = r\cos\theta, \tag{3.19}$$

$$z_2 = r\sin\theta, \tag{3.20}$$

then Eq.(3.18) becomes

$$\frac{\partial f}{\partial t} = \frac{1}{2}\left(\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\right), \tag{3.21}$$

subject to the initial and boundary conditions

$$f(r,\theta,0) = \frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r}, \tag{3.22}$$

$$f(r,0,t) = f(r,\alpha,t) = f(\infty,\theta,t) = 0, \tag{3.23}$$

where

$$\alpha = \begin{cases} \tan^{-1}\left(-\dfrac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0, \\ \pi + \tan^{-1}\left(-\dfrac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise,} \end{cases} \tag{3.24}$$

$$\theta_0 = \begin{cases} \tan^{-1}\left(\dfrac{Y_2\sqrt{1-\rho^2}}{Y_1-\rho Y_2}\right) & \text{if } (.) > 0, \\ \pi + \tan^{-1}\left(\dfrac{Y_2\sqrt{1-\rho^2}}{Y_1-\rho Y_2}\right) & \text{otherwise,} \end{cases} \tag{3.25}$$

$$r_0 = \frac{Y_2}{\sin(\theta_0)}, \tag{3.26}$$

$$Y_i = \frac{b_i}{\sigma_i}. \tag{3.27}$$

$$\tag{3.28}$$

The transformation from (x_1, x_2) to (r, θ) is effectively a linear map which transforms the geometry of the original domain of interest from a right-angled wedge to a wedge with an angle α, followed by the moving of the intersection of the boundaries to the origin and finally representing the points in polar coordinates. It is worthwhile to notice that the boundary of $x_1 = b_1$ and $x_2 = b_2$ are then transformed to $L_1 = \{(r, \theta) | \theta = 0 \text{ or } \theta = \pi, r \geq 0\}$ and $L_2 = \{(r, \theta) | \theta = \alpha \text{ or } \theta = \alpha - \pi, r \geq 0\}$ respectively. The transition probability $f(r, \theta, t)$ which is computed by solving the PDE has an analytic formula

$$f(r, \theta, t) = \frac{2}{\alpha t} e^{-\frac{r^2 + r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right), \tag{3.29}$$

where $I_\nu(.)$ is the modified Bessel function of order ν. Integrating $f(r, \theta, t)$ over the wedge yields the survival probability by time t

$$\begin{aligned}
G(t, t) &= \mathrm{P}(\tau_1 > t, \tau_2 > t) && (3.30)\\
&= \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} g(x_1, x_2, t) dx_2 dx_1 && (3.31)\\
&= \int_0^{\alpha} \int_0^{\infty} f(r, \theta, t) r dr d\theta && (3.32)\\
&= \frac{2 r_0}{\sqrt{2\pi t}} e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right) \right], && (3.33)
\end{aligned}$$

where the infinity series converges uniformly (see Bañuelos and Smits [42]).

3.2 JOINT SURVIVAL DISTRIBUTION

The analytical formula of the survival probability $G(t, t) = \mathrm{P}(\tau_1 > t, \tau_2 > t)$ was well known for decades. An interesting extension is to consider the joint survival probability with different times, i.e. $G(s, t) = \mathrm{P}(\tau_1 > s, \tau_2 > t)$. Our aim in this section is to derive an analytical formula for $G(s, t)$ which is the probability that the particle does not hit the boundary L_1 by time s and does not hit the boundary L_2 by time t. Let $W = \{(r, \theta) | \, r \geq$

$0,\ 0 \le \theta \le \alpha\}$ and $H = \{(r,\theta)|\, r \ge 0,\ \alpha - \pi \le \theta \le \alpha\}$. Without loss of generality, suppose $s < t$. Define

$$K = \{(Z^1_\xi, Z^2_\xi) \in W^\circ,\ \forall \xi \in [0,s]\} \tag{3.34}$$

$$= \{\tau_1 > s, \tau_2 > s\}, \tag{3.35}$$

where W° is the interior of the wedge. K is the set of paths of the two dimensional Brownian motion which stay in W° in the time interval $[0,s]$. It is easy to verify the following inclusion relation

$$K \supset \{\tau_1 > s, \tau_2 > t\}. \tag{3.36}$$

We then partition K into small disjoint sets such that

$$K = \bigcup_{y \in W^\circ} K_y, \tag{3.37}$$

where

$$K_y = K \cap \{(Z^1_s, Z^2_s) \in \Delta y\} \tag{3.38}$$

$$= \{\tau_1 > s, \tau_2 > s, (Z^1_s, Z^2_s) \in \Delta y\}. \tag{3.39}$$

K_y is the set of paths that stay inside the wedge by s and land at Δy at s. By the inclusion (3.36),

$$\mathrm{P}(\tau_1 > s, \tau_2 > t) = \mathrm{P}(\tau_1 > s, \tau_2 > t, K) \tag{3.40}$$

$$= \mathrm{P}(\tau_1 > s, \tau_2 > t, \bigcup_{y \in W^\circ} K_y) \tag{3.41}$$

$$= \sum_{y \in W^\circ} \mathrm{P}(\tau_1 > s, \tau_2 > t, K_y). \tag{3.42}$$

It is easy to verify that

$$K_y \subset \{\tau_1 > s\}, \tag{3.43}$$

which implies that

$$\mathrm{P}(\tau_1 > s, \tau_2 > t, K_y) = \mathrm{P}(\tau_2 > t, K_y). \tag{3.44}$$

Substituting Eq.(3.44) into Eq.(3.42), leads to

$$\mathrm{P}(\tau_1 > s, \tau_2 > t) = \sum_{y \in W^\circ} \mathrm{P}(\tau_2 > t, K_y), \tag{3.45}$$

$$= \sum_{y \in W^\circ} \mathrm{P}(\tau_2 > t | K_y)\mathrm{P}(K_y). \tag{3.46}$$

Note that $\mathrm{P}(K_y)$ is the probability that the particle stays in the interior of W and lands at Δy at time s, whereas $\mathrm{P}(\tau_2 > t|K_y)$ is the probability that the particle starts at Δy at time s and stays in the interior of H by time t. Letting Δy tend to zero, Eq.(3.46) becomes an integral instead of a sum, i.e.

$$\mathrm{P}^x(\tau_1 > s, \tau_2 > t) = \int_{W^\circ} \mathrm{P}^y(\tau_2 > t - s)\mathrm{P}^x(\tau_1 > s, \tau_2 > s, (Z_s^1, Z_s^2) \in dy), \tag{3.47}$$

$$= \int_{W} \mathrm{P}^y(\tau_2 > t - s)\mathrm{P}^x(\tau_1 > s, \tau_2 > s, (Z_s^1, Z_s^2) \in dy), \tag{3.48}$$

where we explicitly indicate the starting positions by the superscripts. Note that

$$\mathrm{P}^y(\tau_2 > t - s) = \int_H f_H^y(z, t - s)dz, \tag{3.49}$$

where $f_H^y(z, t - s)$ is the transition density that the particle starts at y at time s, stays in the interior of H and lands at z at time t. It is worth noting that f_H^y also satisfies the heat equation (3.18) subject to the initial and boundary condition

$$f_H^y(r, \theta, 0) = \frac{\delta(r - r_y)\delta(\theta - \theta_y)}{r}, \tag{3.50}$$

$$f_H^y(r, 0, \xi) = f_H^y(r, \pi, \xi) = f_H^y(\infty, \theta, \xi) = 0, \tag{3.51}$$

where $(r_y, \theta_y) = y$ is the starting position of the particle and $\xi \in (0, t - s]$. Computing f_H^y is equivalent to solving a one dimensional PDE, and an analytical formula can be easily obtained by using the method of images. As a result, integrating f_H^y over H yields

$$\mathrm{P}^y(\tau_2 > t - s) = \left(1 - 2N\left(-\frac{b_2 - x_2}{\sigma_2\sqrt{t - s}}\right)\right) \tag{3.52}$$

$$= \mathrm{erf}\left(\frac{b_2 - x_2}{\sigma_2\sqrt{2(t - s)}}\right) \tag{3.53}$$

$$= \mathrm{erf}\left(\frac{r\sin\theta}{\sqrt{2(t - s)}}\right), \tag{3.54}$$

where $N(\cdot)$ and $\mathrm{erf}(\cdot)$ are the cumulative normal distribution and error function respectively. Note also that

$$f_W^x(y,s)dy = \mathrm{P}^x(\tau_1 > s, \tau_2 > s, (Z_s^1, Z_s^2) \in dy), \tag{3.55}$$

is the transition density that the particle starts at x, stays in W° and lands at y at time s. The analytical form of f_W^x is given by Eq.(3.29). Finally, by Eq.(3.48), Eq.(3.52) and Eq.(3.29)

$$\begin{aligned}
G(s,t) &= \mathrm{P}(\tau_1 > s, \tau_2 > t) && (3.56)\\
&= \int_{-\infty}^{b_2}\int_{-\infty}^{b_1} \mathrm{erf}\left(\frac{b_2 - x_2}{\sigma_2\sqrt{2(t-s)}}\right) g(x_1,x_2,s)dx_1dx_2 && (3.57)\\
&= \int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} \mathrm{erf}\left(\frac{r\sin\theta}{\sqrt{2(t-s)}}\right) f(r,\theta,s)rdrd\theta && (3.58)\\
&= \frac{2e^{-\frac{r_0^2}{2s}}}{\alpha s}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\int_{\theta=0}^{\alpha}\int_{r=0}^{\infty}\mathrm{erf}\left(\frac{r\sin\theta}{\sqrt{2(t-s)}}\right)e^{-\frac{r^2}{2s}}\sin\left(\frac{n\pi\theta}{\alpha}\right)I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{s}\right)rdrd\theta, && (3.59)
\end{aligned}$$

for $s < t$. The case where $t > s$ can be obtained similarly. It is worth noting that Overbeck and Schmidt [24] arrived at an equivalent expression for $G(s,t)$ where time changes of the Brownian motions were considered. For $\theta \neq 0$, it is apparent that

$$\mathrm{erf}\left(\frac{r\sin\theta}{\sqrt{2(t-s)}}\right) \to 1 \quad \text{as} \quad s \to t. \tag{3.60}$$

According to Eq.(3.32), one would expect

$$G(s,t) \to G(t,t) \quad \text{as} \quad s \to t. \tag{3.61}$$

In fact, it can be shown that $G(s,t)$ is continuous for all s and t.

3.3 JOINT DEFAULT DENSITY DISTRIBUTION

The goal of this section is to derive a formula for the joint density distribution $h(s,t)$ of the default times. Iyengar [1] was the first to derive the expression using probabilistic approach. By using the integral form of $G(s,t)$ obtained earlier, we give an alternative derivation of the analytical formula of $h(s,t)$ by a PDE approach. According to Eq. (3.48), we can write

$$G(s,t) = \int_W \int_H f_H^y(z, t-s) dz f_W^x(y,s) dy. \tag{3.62}$$

Define

$$H(s,t) = \mathrm{P}(\tau_1 < s, \tau_2 < t) \tag{3.63}$$

$$= D_1(s) + D_2(t) - 1 + G(s,t), \tag{3.64}$$

where $D_i(s) = \mathrm{P}(\tau_i < s)$ are the marginal distributions. Hence, the joint default density distribution is

$$h(s,t) = \frac{\partial^2 H}{\partial s \partial t} = \frac{\partial^2 G}{\partial s \partial t}. \tag{3.65}$$

In order to obtain an analytical expression for $h(s,t)$, we first note that

$$\frac{\partial f_H^y(z,t-s)}{\partial t} = -\frac{\partial f_H^y(z,t-s)}{\partial s} \tag{3.66}$$

and $f_H^y(z,t-s)$ satisfies the Kolmogorov backward equation, i.e.

$$-\frac{\partial f_H^y(z,t-s)}{\partial s} = \frac{1}{2}\Delta_y f_H^y(z,t-s). \tag{3.67}$$

Differentiate $G(s,t)$ with respect to t, then

$$\frac{\partial G}{\partial t} = \int_W \int_H \frac{\partial f_H^y(z,t-s)}{\partial t} dz f_W^x(y,s) dy \tag{3.68}$$

$$= \int_W \int_H -\frac{\partial f_H^y(z,t-s)}{\partial s} dz f_W^x(y,s) dy \tag{3.69}$$

$$= \int_W \int_H \frac{1}{2}\Delta_y f_H^y(z,t-s) dz f_W^x(y,s) dy \tag{3.70}$$

$$= \int_W \frac{1}{2}\Delta_y \int_H f_H^y(z,t-s) dz f_W^x(y,s) dy \tag{3.71}$$

$$= \frac{1}{2}\int_W \Delta_y \mathrm{P}^y(\tau_2 > t-s) f_W^x(y,s) dy. \tag{3.72}$$

Since $f_H^y(z, t-s)$ satisfies the Kolmogorov backward equation, it can be easily shown that $\mathrm{P}^y(\tau_2 > t-s) = \int_H f_H^y(z, t-s)dz$ also satisfies the Kolmogorov backward equation. Differentiate again with respect to s, then

$$\frac{\partial^2 G}{\partial s \partial t} = \frac{1}{2}\int_W \left(\Delta_y \frac{\partial \mathrm{P}^y}{\partial s} f_W^x(y,s) + \Delta_y \mathrm{P}^y \frac{\partial f_W^x}{\partial s}(y,s)\right) dy \tag{3.73}$$

$$= \frac{1}{2}\int_W \left(\Delta_y \frac{\partial \mathrm{P}^y}{\partial s} f_W^x(y,s) - \frac{\partial \mathrm{P}^y}{\partial s} \Delta_y f_W^x(y,s)\right) dy. \tag{3.74}$$

Recall Green's identity

$$\int_W (u\Delta v - v\Delta u)dV = \int_{\partial W} \left(\frac{\partial v}{\partial n}u - \frac{\partial u}{\partial n}v\right) dS, \tag{3.75}$$

where n is an outward pointing normal to the boundary ∂W. Thus

$$\frac{\partial^2 G}{\partial s \partial t} = \frac{1}{2}\int_{\partial W} \left(\frac{\partial^2 \mathrm{P}^y}{\partial n \partial s} f_W^x(y,s) - \frac{\partial \mathrm{P}^y}{\partial s}\frac{\partial f_W^x}{\partial n}\right) dS \tag{3.76}$$

$$= \frac{1}{2}\int_{\partial W} -\frac{\partial \mathrm{P}^y}{\partial s}\frac{\partial f_W^x}{\partial n} dS \tag{3.77}$$

$$= \int_{\partial W} \frac{\partial \mathrm{P}^y}{\partial s}\left(-\frac{1}{2}\frac{\partial f_W^x}{\partial n}\right) dS \tag{3.78}$$

The first term of the integrand on the first line vanishes since $f_W^x(y,s) = 0$ on the boundary ∂W. This line integral is exactly the same as derived by Iyengar [1] using a probabilistic approach. By Eq.(3.52), it is easy to see that

$$\frac{\partial \mathrm{P}^y(\tau_2 > t-s)}{\partial s} = \frac{r\sin\theta}{\sqrt{2\pi}(t-s)^{\frac{3}{2}}} \exp\left(-\frac{r^2\sin^2\theta}{2(t-s)}\right). \tag{3.79}$$

Furthermore,

$$\frac{\partial}{\partial n} = \begin{cases} -\dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \text{if } \theta = 0, \\[2ex] \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \text{if } \theta = \alpha. \end{cases} \tag{3.80}$$

Then, by Eq.(3.29)

$$-\frac{1}{2}\frac{\partial f_W^x}{\partial n} = \begin{cases} \frac{\pi}{\alpha^2 sr} e^{-\frac{r^2+r_0^2}{2s}} \sum_{n=1}^{\infty} n \cos\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{s}\right) & \text{if } \theta = 0, \\ -\frac{\pi}{\alpha^2 sr} e^{-\frac{r^2+r_0^2}{2s}} \sum_{n=1}^{\infty} n \cos\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{s}\right) & \text{if } \theta = \alpha. \end{cases} \tag{3.81}$$

Since $\partial W = \{\theta = 0\} \cup \{\theta = \alpha\}$,

$$\frac{\partial \mathrm{P}^y(\tau_2 > t-s)}{\partial s} = 0, \tag{3.82}$$

for $\theta = 0$, we only need to integrate along the line $\theta = \alpha$, i.e.

$$h(s,t) = \int_0^\infty \frac{-\sqrt{\pi}\sin\alpha}{\alpha^2 s\sqrt{2}(t-s)^{\frac{3}{2}}} e^{-\frac{r^2+r_0^2}{2s} - \frac{r^2\sin^2\alpha}{2(t-s)}} \sum_{n=1}^{\infty} n\cos\left(\frac{n\pi\alpha}{\alpha}\right)\sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{s}\right) dr \tag{3.83}$$

$$= \frac{e^{-\frac{r_0^2}{2s}}\sqrt{\pi}\sin\alpha}{\alpha^2 s\sqrt{2}(t-s)^{\frac{3}{2}}} \sum_{n=1}^{\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) \int_0^\infty e^{-\left(\frac{1}{2s}+\frac{\sin^2\alpha}{2(t-s)}\right)r^2} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{s}\right) dr. \tag{3.84}$$

The indefinite integral can be evaluated by using the following identity

$$\int_0^\infty e^{-c_1 r^2} I_\nu(c_2 r) dr = \frac{1}{2}\sqrt{\frac{\pi}{c_1}} e^{\frac{c_2^2}{8c_1}} I_{\frac{\nu}{2}}\left(\frac{c_2^2}{8c_1}\right), \tag{3.85}$$

which is available in Magnus et al. [43]. It is worth noting that the original identity in [43] has a typo where there is a minus sign in front of c_2^2 in the exponential term. Maple is able to give the correct identity and we also verify it by numerical integration for several different pairs of values c_1 and c_2. After applying the identity and some simplification, for $s < t$,

$$\begin{aligned} h(s,t) =& \frac{\pi\sin\alpha}{2\alpha^2\sqrt{s}(t-s)\sqrt{t-s\cos^2\alpha}} \exp\left(-\frac{r_0^2}{2s}\frac{t-s\cos 2\alpha}{(t-s)+(t-s\cos 2\alpha)}\right) \\ &\times \sum_{n=1}^{\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) I_{\frac{n\pi}{2\alpha}}\left(\frac{r_0^2}{2s}\frac{t-s}{(t-s)+(t-s\cos 2\alpha)}\right), \end{aligned} \tag{3.86}$$

The case $s > t$ can be obtained similarly. Metzler [44] also obtained the same result by following Iyengar's derivation and using the correct version of the identity for the indefinite integral. It is worth noting that when $s \to t$,

$$h(s,t) \sim (t-s)^{\frac{\pi}{2\alpha}-1}. \tag{3.87}$$

Hence, as s approaches t,

$$h(s,t) \to \begin{cases} 0 & \text{if } 0 < \alpha < \dfrac{\pi}{2}, \\ \dfrac{r_0^2 \sin 2\theta_0}{4\pi s^3} \exp\left(-\dfrac{r_0^2}{2s}\right) & \text{if } \alpha = \dfrac{\pi}{2}, \\ \infty & \text{if } \dfrac{\pi}{2} < \alpha < \pi. \end{cases} \tag{3.88}$$

Consequently, when the Brownian motion (X_t^1, X_t^2) is positively correlated which corresponds to $\alpha \in (\pi/2, \pi)$, $h(s,t)$ is discontinuous along the diagonal $s = t$.

The conditional probability of joint defaults i.e. $\mathrm{P}(\tau_2 = s|\tau_1 = s)$ is an interesting quantity one would like to compute as this value reflects the contagion of default. However, it is difficult to employ the density function $h(s,t)$ to compute the conditional probability. One possible alternative is to compute $\mathrm{P}(\tau_2 = s|\tau_1 = s)$ in terms $G(s,t)$ i.e.

$$\mathrm{P}(\tau_2 = s|\tau_1 = s) = \lim_{\delta \to 0} \frac{G(s+\delta, s+\delta) - G(s+\delta, s) - G(s, s+\delta) + G(s,s)}{D_1(s+\delta) - D_1(s)}. \tag{3.89}$$

This limit is not easy to evaluate in general. Nevertheless, we are able to calculate the limit for the special case where $s = 0$ and the computation is presented in next section.

Before we end this section, we briefly discuss the more general case where $\lambda_i \neq \mu_i$. This general case corresponds to Brownian motion with drift, thus can be analyzed by employing Girsanov theorem. Iyengar [1] provided a detailed discussion for this general situation in which expressions for both the transition probability density and the density of default times are given. However, the integral involved for the density of default times does not seem to be as tractable as the special case where $\lambda_i = \mu_i$. Surprisingly, the joint survival probability $\mathrm{P}(\tau_1 > s, \tau_2 > t)$ for the general case is still a two dimensional integral which can be computed as easily as the special case. This is because Eq.(3.48) is true even in the case that the Brownian motion is not driftless.

3.4 TAIL DEPENDENCE

We are interested in the following question: "Does correlated Brownian motion under the Black and Cox framework exhibit positive tail dependence?" This section is devoted to answer this question for the driftless case and it turns out that the answer is negative. If the financial situation requires non-zero tail dependence, then correlated Brownian motion, at least in its primitive setting, is not a good model to explain the default dependence between firms. Therefore, one should rectify the model or turn to other alternatives in the case that positive tail dependence is observed. It is worth mentioning here that the new model presented in Chapter 5 is able to exhibit positive lower tail dependence. It may be tempting to think that the computations of the coefficients of both upper and lower tail dependence are straightforward for the correlated Brownian motion since the analytical formula of the joint default density $h(s,t)$ is already known. However, it turns out that this is not really the case. The difficulty in using $h(s,t)$ for the calculations comes from the discontinuity of $h(s,t)$ along the diagonal $s = t$. It appears that integrating $h(s,t)$ to compute the probabilities in the definitions of tail dependence does not yield useful results. Thus, we turn our attention to $G(s,t)$ which is effectively an integral of h. In fact, all the probabilities involved in the definitions of tail dependence can be expressed in terms of $G(s,t)$. We first prove in the next subsection that (τ_1, τ_2) is asymptotically independent for $\rho \in (-1, 1)$ in both lower and upper tails for the special case that both marginals are the same. We then prove the asymptotical independence of (τ_1, τ_2) for the general case based on the results found in the special case.

The marginal distribution D_i that we consider in this section is that from the correlated

Brownian motion. It is easy to see that we can write D_i in terms of the error function

$$D_i(t) = \mathrm{P}(\tau_i \leq t) \tag{3.90}$$

$$= 2N\left(-\frac{b_i}{\sigma_i\sqrt{t}}\right) \tag{3.91}$$

$$= 1 + \mathrm{erf}\left(-\frac{b_i}{\sigma_i\sqrt{2t}}\right) \tag{3.92}$$

$$= 1 - \mathrm{erf}\left(\frac{b_i}{\sigma_i\sqrt{2t}}\right) \tag{3.93}$$

$$= \mathrm{erfc}\left(\frac{b_i}{\sigma_i\sqrt{2t}}\right), \tag{3.94}$$

and the inverse function of D_i is

$$D_i^{-1}(u) = \frac{b_i^2}{2\sigma_i^2(\mathrm{erf}^{-1}(1-u))^2}. \tag{3.95}$$

3.4.1 Special Case: Identical Marginals

We use $G(t,t)$ as given by Eq.(3.30) to study the special case where the marginals are the same, i.e. $D_1 = D_2 = D$. The coefficient of the lower tail dependence is then

$$\lambda_L = \lim_{u\to 0^+} \frac{H(D^{-1}(u), D^{-1}(u))}{u} \tag{3.96}$$

$$= \lim_{t\to 0^+} \frac{H(t,t)}{D(t)} \tag{3.97}$$

$$= \lim_{t\to 0^+} \frac{G(t,t) + 2D(t) - 1}{D(t)} \tag{3.98}$$

$$= 2 - \lim_{t\to 0^+} \frac{1 - G(t,t)}{D(t)}. \tag{3.99}$$

In order to evaluate the limit, we apply L'Hopital's rule. It is easy to see that

$$D'(t) = \frac{b \cdot e^{-\frac{b^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t^3}}. \tag{3.100}$$

On the other hand, the derivative of $G(t,t)$ is quite complicated. We first write each term of the infinite sum $G(t,t)$ given in Eq.(3.30) as

$$\psi_n(z) = \sqrt{z}e^{-z}(I_\nu(z) + I_{\nu-1}(z)), \tag{3.101}$$

where $z = r_0^2/4t$ and we omit the constant factor for simplicity. Thus the derivative is

$$\psi'(z) = \frac{e^{-z}}{\sqrt{z}}\left(\left(\frac{1}{2} - z\right)(I_\nu + I_{\nu-1}) + z(I'_\nu + I'_{\nu-1})\right). \tag{3.102}$$

We can simplify the expression by applying the following two recurrence relations which are available in Abramowitz and Stegun [45]

$$I'_\nu = I_{\nu-1} - \frac{\nu}{z} I_\nu, \tag{3.103}$$

$$I'_{\nu-1} = I_\nu + \frac{\nu}{z} I_{\nu-1}. \tag{3.104}$$

Then

$$\psi'_n(z) = \frac{e^{-z}}{\sqrt{z}}\left(\frac{1}{2} - \nu\right)(I_\nu - I_{\nu-1}) \tag{3.105}$$

$$= -\frac{e^{-z}}{\sqrt{z}}\frac{n\pi}{2\alpha}(I_\nu - I_{\nu-1}). \tag{3.106}$$

Putting together dz/dt and the omitted constant factor for each term, it is easy to see that

$$\frac{dG(t,t)}{dt} = \frac{r_0}{\alpha}\sqrt{\frac{\pi}{2t^3}}e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\ldots} \sin\left(\frac{n\pi\theta_0}{\alpha}\right)\left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) - I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right)\right]. \tag{3.107}$$

Since both marginals are the same, $\theta_0/\alpha = 1/2$. Hence

$$\frac{dG(t,t)}{dt} = \frac{r_0}{\alpha}\sqrt{\frac{\pi}{2t^3}}e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\ldots} \sin\left(\frac{n\pi}{2}\right)\left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) - I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right)\right] \tag{3.108}$$

$$= \frac{r_0}{\alpha}\sqrt{\frac{\pi}{2t^3}}e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\ldots} (-1)^{\frac{n-1}{2}}\left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) - I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right)\right]. \tag{3.109}$$

We first consider a more special case in which $\rho = 1$, meaning $\alpha = \pi$ and $\theta_0 = \pi/2$, and the modified Bessel functions are then of integral order. Hence,

$$\frac{dG(t,t)}{dt} = \frac{r_0}{\sqrt{2\pi t^3}}e^{-\frac{r_0^2}{4t}} \sum_{k=1}^{\infty}(-1)^{k-1}\left[I_k\left(\frac{r_0^2}{4t}\right) - I_{k-1}\left(\frac{r_0^2}{4t}\right)\right], \tag{3.110}$$

$$= \frac{r_0}{\sqrt{2\pi t^3}}e^{-\frac{r_0^2}{4t}}\left[\sum_{k=1}^{\infty}(-1)^{k-1}I_k\left(\frac{r_0^2}{4t}\right) - \sum_{k=0}^{\infty}(-1)^{k}I_k\left(\frac{r_0^2}{4t}\right)\right], \tag{3.111}$$

$$= \frac{r_0}{\sqrt{2\pi t^3}}e^{-\frac{r_0^2}{4t}}\left[\sum_{k=1}^{\infty}(-1)^{k-1}I_k\left(\frac{r_0^2}{4t}\right) + \sum_{k=0}^{\infty}(-1)^{k-1}I_k\left(\frac{r_0^2}{4t}\right)\right], \tag{3.112}$$

$$= \frac{r_0}{\sqrt{2\pi t^3}}e^{-\frac{r_0^2}{4t}}\left[-I_0\left(\frac{r_0^2}{4t}\right) + 2\sum_{k=1}^{\infty}(-1)^{k-1}I_k\left(\frac{r_0^2}{4t}\right)\right]. \tag{3.113}$$

By using the identity

$$e^{-z} = I_0(z) - 2I_1(z) + 2I_2(z) - 2I_3(z) + ..., \tag{3.114}$$

which is available in Abramowitz and Stegun [45], then

$$\frac{dG(t,t)}{dt} = \frac{r_0}{\sqrt{2\pi t^3}} e^{-\frac{r_0^2}{4t}} \left(-e^{-\frac{r_0^2}{4t}} \right), \tag{3.115}$$

$$= -\frac{r_0}{\sqrt{2\pi t^3}} e^{-\frac{r_0^2}{2t}}. \tag{3.116}$$

Since $\theta_0 = \pi/2$ implies $r_0 = b/\sigma$,

$$\frac{dG(t,t)}{dt} = -D'(t). \tag{3.117}$$

Then

$$\lambda_L = 2 - \lim_{t \to 0} \frac{-G'(t,t)}{D'(t)} = 1. \tag{3.118}$$

In fact for this special case in which $\rho = 1$, we can prove that $\lambda_L = \lambda_U = 1$ by a much simpler argument which only uses the symmetry of the problem. Since $\rho = 1$ and $b_1/\sigma_1 = b_2/\sigma_2$, the values of both firms are effectively moving exactly in the same way and must hit their boundaries at the same time if they do. Thus, both firms must either default or survive together which implies that

$$\frac{H(t,t)}{D(t)} = \frac{G(t,t)}{1 - D(t)} = 1, \tag{3.119}$$

for all $t \in (0, \infty)$ as well as the limiting cases. The real challenge, however, is to compute the coefficient of the lower tail dependence for arbitrary ρ in $(-1, 1)$. We now prove this general result by considering the following ratio

$$\frac{G'(t,t)}{D'(t)} = \frac{\sigma\sqrt{2\pi t^3}}{b \cdot e^{-\frac{b^2}{2\sigma^2 t}}} \frac{r_0}{\alpha} \sqrt{\frac{\pi}{2t^3}} e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\ldots} (-1)^{\frac{n-1}{2}} \left[I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)} \left(\frac{r_0^2}{4t} \right) - I_{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)} \left(\frac{r_0^2}{4t} \right) \right] \tag{3.120}$$

$$= \frac{\sigma\pi}{b} \frac{r_0}{\alpha} e^{-\frac{r_0^2}{4t} + \frac{b^2}{2\sigma^2 t}} \sum_{k=1}^{\infty} (-1)^{k-1} \left[I_{\frac{1}{2}\left(\frac{(2k-1)\pi}{\alpha}+1\right)} \left(\frac{r_0^2}{4t} \right) - I_{\frac{1}{2}\left(\frac{(2k-1)\pi}{\alpha}-1\right)} \left(\frac{r_0^2}{4t} \right) \right], \tag{3.121}$$

where $G'(t,t) = dG(t,t)/dt$ is given by Eq.(3.107). Since $\sin\theta_0 = b/\sigma r_0$ and $\theta_0/\alpha = 1/2$, it is easy to see that

$$-\frac{r_0^2}{4t} + \frac{b^2}{2\sigma^2 t} = -\frac{r_0^2}{4t}\left(1 - \frac{2b^2}{\sigma^2 r_0^2}\right) \tag{3.122}$$

$$= -\frac{r_0^2}{4t}\left(1 - 2\sin^2\theta_0\right) \tag{3.123}$$

$$= -\frac{r_0^2}{4t}\cos\alpha. \tag{3.124}$$

Then,

$$\frac{G'(t,t)}{D'(t)} = \frac{\pi}{\alpha\sin(\frac{\alpha}{2})} e^{-\frac{r_0^2}{4t}\cos\alpha} \sum_{k=1}^{\infty}(-1)^{k-1}\left[I_{\frac{1}{2}\left(\frac{(2k-1)\pi}{\alpha}+1\right)}\left(\frac{r_0^2}{4t}\right) - I_{\frac{1}{2}\left(\frac{(2k-1)\pi}{\alpha}-1\right)}\left(\frac{r_0^2}{4t}\right)\right]. \tag{3.125}$$

Note that when $t \to 0$, the exponential term goes to zero if $\alpha \in (0, \pi/2)$. If $\lambda_L \in [0,1]$ exists, the limit of $G'(t,t)/D'(t) \in [1,2]$. Therefore, the infinite sum must be able to cancel the effect from the exponential term if λ_L exists. The modified Bessel function has the asymptotic behavior (Abramowitz and Stegun [45])

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 + \frac{(1-4\nu^2)}{8z} + ...\right) \tag{3.126}$$

for large $z \gg |1-4\nu^2|$. Thus, it appears that the the infinite sum in $G'(t,t)/D'(t)$ with $I_\nu(z)$ written as the approximation with the factor e^z is somehow able to offset the exponential term as $t \to 0$. We would like to find an exact formula for the modified Bessel function which has the desired e^z term. The following integral expression for the modified Bessel function

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh s - \nu s}ds, \tag{3.127}$$

which is Eq.(9.6.20) in Abramowitz and Stegun [45] turns out to be the right choice. Note that the second term in the integral expression goes to zero as $z \to \infty$. Thus, we only need to consider the first term of the identity as t goes to zero. Furthermore, notice that the difference of the cosines can be computed by the compound angle formula which yields

$$\cos\left(\frac{2k-1}{\alpha}\frac{\pi}{2} + \frac{1}{2}\right)\theta - \cos\left(\frac{2k-1}{\alpha}\frac{\pi}{2} - \frac{1}{2}\right)\theta = -2\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}\right)\sin\left(\frac{\theta}{2}\right), \tag{3.128}$$

then the limit of the ratio becomes

$$\lim_{t\to 0}\frac{G'(t,t)}{D'(t)} = \lim_{t\to 0}\frac{-2}{\alpha\sin(\frac{\alpha}{2})}e^{-\frac{r_0^2}{4t}\cos\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\pi} e^{\frac{r_0^2}{4t}\cos\theta}\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)d\theta. \tag{3.129}$$

$$= \lim_{t\to 0}\frac{-2}{\alpha\sin(\frac{\alpha}{2})}e^{-\frac{r_0^2}{4t}\cos\alpha}\int_0^{\pi} e^{\frac{r_0^2}{4t}\cos\theta}\sin\left(\frac{\theta}{2}\right)\sum_{k=1}^{\infty}(-1)^{k-1}\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}\right)d\theta. \tag{3.130}$$

In order to evaluate this limit, we first compute the infinite sum,

$$\sum_{k=1}^{\infty}(-1)^{k-1}\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}\right) = \sum_{k=1}^{\infty}\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}-(k-1)\pi\right) \tag{3.131}$$

$$= \sum_{k=1}^{\infty}\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}-\left(k-\frac{1}{2}\right)\pi+\frac{\pi}{2}\right) \tag{3.132}$$

$$= \sum_{k=1}^{\infty}\cos\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}-\left(k-\frac{1}{2}\right)\pi\right) \tag{3.133}$$

$$= \sum_{k=1}^{\infty}\cos\left(\left(k-\frac{1}{2}\right)(\theta-\alpha)\frac{\pi}{\alpha}\right) \tag{3.134}$$

$$= \sum_{k=1}^{\infty}\frac{e^{i(k-\frac{1}{2})\frac{\pi}{\alpha}(\theta-\alpha)}+e^{-i(k-\frac{1}{2})\frac{\pi}{\alpha}(\theta-\alpha)}}{2} \tag{3.135}$$

$$= \frac{e^{i\frac{\pi}{2\alpha}(\theta-\alpha)}}{2}\sum_{k=1}^{\infty}e^{i(k-1)\frac{\pi}{\alpha}(\theta-\alpha)}+e^{-ik\frac{\pi}{\alpha}(\theta-\alpha)} \tag{3.136}$$

$$= \pi e^{i\frac{\pi}{2\alpha}(\theta-\alpha)}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{ik\frac{\pi}{\alpha}(\theta-\alpha)}. \tag{3.137}$$

Note that

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{ik\frac{\pi}{\alpha}(\theta-\alpha)} = \delta\left(\frac{\pi}{\alpha}(\theta-\alpha)\right) \tag{3.138}$$

is the scaled Dirac delta function (see for example Kanwal [46]). Hence

$$\sum_{k=1}^{\infty}(-1)^{k-1}\sin\left(\frac{2k-1}{\alpha}\frac{\pi\theta}{2}\right) = \pi e^{i\frac{\pi}{2\alpha}(\theta-\alpha)}\delta\left(\frac{\pi}{\alpha}(\theta-\alpha)\right) \tag{3.139}$$

$$= \pi e^{i\frac{\pi}{2\alpha}(\theta-\alpha)}\frac{\alpha}{\pi}\delta(\theta-\alpha) \tag{3.140}$$

$$= e^{i\frac{\pi}{2\alpha}(\theta-\alpha)}\alpha\delta(\theta-\alpha), \tag{3.141}$$

where we have used the scaling property of delta function $\delta(c\theta) = \delta(\theta)/|c|$. Finally, substituting the infinite sum in Eq.(3.130) by the above expression, then for $0 < \alpha < \pi$

$$\lim_{t\to 0} \frac{G'(t,t)}{D'(t)} = \lim_{t\to 0} \frac{-2}{\alpha \sin(\frac{\alpha}{2})} e^{-\frac{r_0^2}{4t}\cos\alpha} \int_0^\pi e^{\frac{r_0^2}{4t}\cos\theta} e^{i\frac{\pi}{2\alpha}(\theta-\alpha)} \alpha\delta(\theta-\alpha) \sin\left(\frac{\theta}{2}\right) d\theta \tag{3.142}$$

$$= \frac{-2}{\sin(\frac{\alpha}{2})} e^{-\frac{r_0^2}{4t}\cos\alpha} \int_0^\pi e^{\frac{r_0^2}{4t}\cos\theta} e^{i\frac{\pi}{2\alpha}(\theta-\alpha)} \delta(\theta-\alpha) \sin\left(\frac{\theta}{2}\right) d\theta \tag{3.143}$$

$$= -2. \tag{3.144}$$

Hence,

$$\lambda_L = 0, \tag{3.145}$$

for $\rho \in (-1, 1)$. It is worthwhile to note that if $\alpha = \pi$ which corresponds to the case that $\rho = 1$, integrating the Eq.(3.143) yields -1 instead of -2, and $\lambda_L = 1$ as expected. To conclude

$$\lambda_L = \begin{cases} 0 & \text{if } -1 < \rho < 1, \\ 1 & \text{if } \rho = 1. \end{cases} \tag{3.146}$$

This complete one of the most difficult calculations in this thesis.

We now move on to the derivation of the coefficient of the upper tail dependence. Under the assumption that $D_1 = D_2 = D$,

$$\lambda_U = \lim_{u\to 1^-} \frac{G(D^{-1}(u), D^{-1}(u))}{1-u} \tag{3.147}$$

$$= \lim_{t\to\infty} \frac{G(t,t)}{1-D(t)}. \tag{3.148}$$

$$\tag{3.149}$$

We first note that the asymptotic behaviors of the modified Bessel function and the default marginal are

$$I_\nu(z) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \quad \text{as } z \to 0, \tag{3.150}$$

$$D(t) \to 1 - \frac{b\sqrt{2}}{\sigma\sqrt{\pi t}} \quad \text{as } t \to \infty. \tag{3.151}$$

As a result,

$$G(t,t) \to \frac{2r_0}{\sqrt{2\pi t}} e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \left[\frac{1}{\Gamma\left(\frac{n\pi}{2\alpha}+\frac{3}{2}\right)} \left(\frac{r_0^2}{8t}\right)^{\frac{1}{2}\left(\frac{n\pi}{\alpha}+1\right)} + \frac{1}{\Gamma\left(\frac{n\pi}{2\alpha}+\frac{1}{2}\right)} \left(\frac{r_0^2}{8t}\right)^{\frac{1}{2}\left(\frac{n\pi}{\alpha}-1\right)}\right], \tag{3.152}$$

$$1 - D(t) \to \frac{b\sqrt{2}}{\sigma\sqrt{\pi t}}, \tag{3.153}$$

as $t \to \infty$. Since $\alpha \leq \pi$, we only need to consider the leading term in the infinite series in $G(t,t)$ and

$$\frac{G(t,t)}{1-D(t)} \to \frac{r_0\sigma}{b} e^{-\frac{r_0^2}{4t}} \sin\left(\frac{\pi\theta_0}{\alpha}\right) \left[\frac{1}{\Gamma\left(\frac{\pi}{2\alpha}+\frac{1}{2}\right)} \left(\frac{r_0^2}{8t}\right)^{\frac{1}{2}\left(\frac{\pi}{\alpha}-1\right)}\right], \tag{3.154}$$

as $t \to \infty$. If $\rho < 1$, then $\alpha < \pi$ and $\frac{1}{2}\left(\frac{\pi}{\alpha}-1\right) > 0$. This implies

$$\lambda_U \to 0 \quad \text{as} \quad t \to \infty. \tag{3.155}$$

On the other hand, if $\rho = 1$, then $\alpha = \pi$ and $\frac{1}{2}\left(\frac{\pi}{\alpha}-1\right) = 0$. Thus

$$\lambda_U = \lim_{t\to\infty} \frac{G(t,t)}{1-D(t)} \to \frac{r_0\sigma}{b} \sin(\theta_0) = 1, \tag{3.156}$$

which is as expected. To conclude,

$$\lambda_U = \begin{cases} 0 & \text{if } -1 < \rho < 1, \\ 1 & \text{if } \rho = 1, \end{cases} \tag{3.157}$$

which is the same as the coefficient of lower tail dependence.

The conclusions that default times for identical marginals are asymptotically independent in both lower and upper tail cases are not very surprising. In fact, the equivalence between tail dependence and asymptotic default correlations discussed in Section 2.3 sheds light on these results. The observations about the general behaviors of the default correlations of correlated Brownian motion by Zhou [22] which states that

1. default correlations are generally very small over short horizons,
2. they first increase with time, and
3. after reaching a maximum they decrease slowly,

are closely related to the conclusion we draw in this subsection. The first observation with the supporting evidence from the accompanying computational results in [22] independently corroborates that the coefficient of lower tail dependence is zero. The second observation is not related to the present discussion since it relates to finite times. Finally, our conclusion that the coefficient of upper tail dependence is zero complements the third observation by explicitly showing that default correlations decrease slowly to zero.

3.4.2 General Case: Arbitrary Marginals

We are now ready to derive the coefficients of tail dependence for the general case where the two marginals are different, i.e. $D_1 \neq D_2$. We only focus on the case that $\rho \in (-1, 1)$, the calculations of the uninteresting cases where $\rho = \pm 1$ are omitted. First notice that for any $u \in [0, 1]$, the two times

$$s = D_1^{-1}(u), \tag{3.158}$$

$$t = D_2^{-1}(u), \tag{3.159}$$

are not the same generally. The explicit inverse functions of the marginals given by Eq.(3.95), implies that

$$t = \chi s, \tag{3.160}$$

where

$$\chi = \left(\frac{b_2\sigma_1}{b_1\sigma_2}\right)^2. \tag{3.161}$$

For simplicity, write $D_1 = D$. The coefficients of lower tail dependence then becomes

$$\lambda_L = 2 - \lim_{s\to 0} \frac{1 - G(s, \chi s)}{D(s)}. \tag{3.162}$$

The trick to evaluate the limit is to rewrite the ratio into a sum of two terms,

$$\frac{1 - G(s, \chi s)}{D(s)} = \frac{1 - G(s, s)}{D(s)} + \frac{G(s, s) - G(s, \chi s)}{D(s)}. \tag{3.163}$$

As s goes to zero, the first term on the right hand side of the above equation is just the coefficient of the lower tail dependence for the special case of equal marginals which was computed in the last subsection. Thus, the task left is to evaluate the limit of the second term. Without loss of generality, suppose $\chi > 1$, i.e. $t > s$. It is easy to see that

$$1 \leq D_1(s) + D_2(\chi s) + G(s, \chi s) \tag{3.164}$$

$$1 \leq 2D(s) + G(s, \chi s) \tag{3.165}$$

$$\frac{1 - G(s, \chi s)}{D(s)} \leq 2 \tag{3.166}$$

$$\frac{G(s, s) - G(s, \chi s)}{D(s)} \leq 2 - \frac{1 - G(s, s)}{D(s)}. \tag{3.167}$$

On the other hand, since G is the survival probability and is non-increasing in both arguments, it is clear that

$$0 \leq \frac{G(s, s) - G(s, \chi s)}{D(s)}. \tag{3.168}$$

Combing the two inequalities yields

$$0 \leq \frac{G(s, s) - G(s, \chi s)}{D(s)} \leq 2 - \frac{1 - G(s, s)}{D(s)}. \tag{3.169}$$

Taking the limit that $s \to 0$, the upper bound is just the coefficient of the lower tail dependence for the same marginals, which is zero as showed before. Hence

$$\lim_{s \to 0} \frac{G(s, s) - G(s, \chi s)}{D(s)} = 0. \tag{3.170}$$

As a result, the coefficient of the lower tail dependence λ_L for different marginals is the same as we found for the special case for $\rho \in (-1, 1)$.

We now compute the coefficient of the upper tail dependence. Using the same notations as before, it is easy to see that

$$\lambda_U = \lim_{s \to \infty} \frac{G(s, \chi s)}{1 - D(s)}. \tag{3.171}$$

Since $G(s,t)$ is the joint survival probability which is always non-negative and is non-increasing in both arguments, so

$$0 \leq G(s, \chi s) \leq G(s,s). \tag{3.172}$$

As a result,

$$0 \leq \lambda_U \leq \lim_{s \to \infty} \frac{G(s,s)}{1 - D(s)} = 0, \tag{3.173}$$

where the last equality comes from the calculation in the special case. Therefore, $\lambda_U = 0$ for $\rho \in (-1, 1)$ for any marginals D_1 and D_2. Metzler [44] has also got the same result for the upper tail dependence, however the analogous analysis for the lower tail dependence was not provided.

4.0 INTENSITY-BASED MODELS

Intensity-based credit risk models for a single name assume an default event to occur at the first jump of a counting process N_t. The default intensity, sometimes referred to as the hazard rate, represents the instantaneous default probability. Suppose τ is the default time, the default intensity is defined as follows:

$$\lambda_t = \lim_{\delta \to 0} \frac{\mathrm{P}(\tau \leq t + \delta | \tau > t)}{\delta}. \tag{4.1}$$

Roughly speaking, this means that the probability of default for a small time interval $(t, t+\delta]$, given that the firm survives until time t, is proportional to δ and the proportionality constant for that particular moment is λ_t. Intensity-based credit risk models were first studied by Jarrow and Turnbull [47] using constant default intensities, in which case default is the first jump of a Poisson process. It is easy to see that

$$\lambda_t = \lim_{\delta \to 0} \frac{\mathrm{P}(t < \tau \leq t + \delta)}{\mathrm{P}(\tau > t)\delta} \tag{4.2}$$

$$= \lim_{\delta \to 0} \frac{\mathrm{P}(\tau > t) - \mathrm{P}(\tau > t + \delta)}{\mathrm{P}(\tau > t)\delta} \tag{4.3}$$

$$= \frac{1}{\mathrm{P}(\tau > t)} \frac{d\mathrm{P}(\tau > t)}{dt}. \tag{4.4}$$

Consequently, the survival probability is given by

$$\mathrm{P}(\tau > t) = e^{-\int_0^t \lambda_s ds}. \tag{4.5}$$

According to the above equation, it is clear that the full knowledge of λ_s in $[0, t)$ is equivalent to that of $\mathrm{P}(\tau > t)$ and vice versa. Lando [48] introduced stochastic intensities to model

defaults. These processes with random time-dependent λ_t are called Cox process (also known as the doubly stochastic Poisson process). Accordingly, the survival probability is

$$\mathrm{P}(\tau > t) = \mathrm{E}\left(e^{-\int_0^t \lambda_s ds}\right). \tag{4.6}$$

This expression is reminiscent of the pricing formula for a zero coupon bond. The survival probability corresponds to the bond price while the default intensity corresponds to the short rate. In fact, credit risk modeling under the intensity-based framework is mathematically equivalent to interest rate modeling. Therefore, many well known results and useful techniques from this field can be employed in the specifications of the default intensity.

4.1 CORRELATED INTENSITIES

Mutli-name correlated intensities models were pioneered by Duffie and Gârleanu [26]. In a simple form of the correlated intensity model, default intensity of the i-th firm in a portfolio consisting of N securities is chosen to be a Cox process consisting of two components as follows:

$$\lambda_i(t) = a_i Y(t) + X_i(t), \tag{4.7}$$

for $i = 1, ..., N$. The first component is a product of a Cox process $Y(t)$ which is common to every firm in the portfolio and a constant $a_i > 0$ which measures the dependence on the common process. The second term $X_i(t)$ is an idiosyncratic component pertaining to firm i and is independent of $Y(t)$. The correlation of the default times among firms comes from the common Cox process $Y(t)$.

Let's consider an extreme case to understand this correlation. Since $Y(t)$ is stochastic, it is possible that $Y(t)$ attains a very high level of intensity in some time interval. As a result, the probability of multiple defaults in the portfolio is high in that time interval, even though

defaults occur independently. A tractable choice for the Cox processes $Y(t)$ and $X_i(t)$ is the well known basic affine jump-diffusion (AJD) process, i.e.

$$dY(t) = \kappa_Y(\theta_Y - Y(t))dt + \sigma_Y\sqrt{Y(t)}dW_Y(t) + dJ_Y(t), \tag{4.8}$$

$$dX_i(t) = \kappa_i(\theta_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW_i(t) + dJ_i(t), \tag{4.9}$$

where W_Y and W_i are independent Brownian motions, and J_Y and J_i are independent pure jump processes. The jump times of these pure jump processes have intensities l_Y, l_1, ...,l_N and follow exponential distributions with mean μ_Y, μ_1, ..., μ_N. Just as for CIR processes [49], basic affine jump-diffusion process can always ensure the non-negativeness of the default intensity [50]. Furthermore, closed form solutions to a range of relevant expectations involving affine processes are available [50].

4.1.1 Loss Distribution

As mentioned earlier, the standard approach to computing the loss distribution of a portfolio relies on some form of conditional independence. Given a realization of the common Cox process $Y(t)$, defaults occur independently across entities in the portfolio. Define a common factor Z as the cumulative intensity of the common process,

$$Z(t) = \int_0^t Y(s)ds. \tag{4.10}$$

The conditional survival probability is

$$p_i(t|z) = \mathrm{P}(\tau_i > t|Z(t) = z) = e^{-a_i z}\mathrm{E}(e^{-\int_0^t X_i(s)ds}), \tag{4.11}$$

where the expectation admits closed-form solution as shown by Duffie et al. [50]. The unconditional default distribution can be written as an integral of the conditional default distribution over the common factor distribution,

$$\mathrm{P}(D(t) = j) = \int_0^\infty \mathrm{P}(D(t) = j|Z(t) = z)f_{Z(t)}(z)dz, \tag{4.12}$$

where $f_{Z(t)}$ is the density distribution of the common factor.

There are two shortcomings about the correlated intensities model. Firstly, there is no closed form expression for $f_{Z(t)}(z)$. To compute the density distribution one needs to do an inverse Fourier transform which could be quite time consuming. Secondly, including more common factors in the model would significantly reduce the efficiency of the computation, since a k-common factors model implies evaluating a k-dimensional integral of the loss distribution.

4.2 STRESS EVENT IN INTENSITY MODELS

Duffie and Singleton [2] propose an alternative approach. Instead of correlating the default intensities λ_A and λ_B, they introduce joint default events:

- $\bar{N}_A$ with $\bar{\lambda}_A$ - firm A defaults alone.
- $\bar{N}_B$ with $\bar{\lambda}_B$ - firm B defaults alone.
- N_C with λ_C - firm A defaults with probability p_A and firm B defaults with probability p_B.

In this setup, $\bar{N}_A$, $\bar{N}_B$ and N_C are independent Poisson processes and the intensities could be time dependent. The process N_C models the arrival of a stress event which may kill both firms at the same time. We call p_A and p_B the impact probabilities of the stress for firm A and B respectively. The default intensities for firms A and B are

$$\lambda_A = \bar{\lambda}_A + p_A \lambda_C, \tag{4.13}$$

$$\lambda_B = \bar{\lambda}_B + p_B \lambda_C. \tag{4.14}$$

It is easy to generalize the idea to K stress events in a portfolio of N firms and the default time of firm i could be defined as follows:

$$\tau_i = \inf \left\{ s \geq 0 : \bar{N}_i(s) + \sum_{l=1}^{L} \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t_j^l\}} X_{i,j}^l > 0 \right\}, \tag{4.15}$$

for $i = 1, .., N$, where

- All $\bar{N}_i$ and N^l are independent Poisson processes with intensities $\bar{\lambda}_i(t)$ and $\lambda^l(t)$ respectively.
- t_j^l is the j-th jump time of a Poisson process $N^l(s)$.
- $\mathbf{1}_{\{s>t_j^l\}}$ is an indicator function that equals one if $s > t_j^l$ and zero otherwise.
- $X_{i,j}^l$ are Bernoulli random variables indicating if a stress event has killed the firm or not, independent of the Poisson processes.

$\bar{N}_i$ is an idiosyncratic Poisson process associated with firm i which is driven by firm-specific factors. Once there is a jump in $\bar{N}_i$, firm i defaults immediately. Besides, if N^l has a jump at t_j^l, firm i may default with a probability $\mathrm{P}(X_{i,j}^l = 1) = p_i^l$. It is worth noting that only the first jump in $\bar{N}_i$ is relevant for default triggering of firm i and later jumps are irrelevant, whereas each jump in N^l could be the default triggering event. It is easy to show that the default intensity for each firm is simply

$$\lambda_i(t) = \bar{\lambda}_i(t) + \sum_{l=1}^{L} p_i^l \lambda^l(t). \tag{4.16}$$

4.2.1 $G(s,t)$ and the Density of (τ_A, τ_B)

The dependent structure of the default times between any two firms A and B, is governed by the default density distribution $h(s,t)$. Although there are K different kinds of stress events, we can regard them as one combined stress event with intensity

$$\lambda_C(t) = \sum_{k=1}^{K} \lambda^k(t), \tag{4.17}$$

and the probabilities of defaults of firm A and B triggered by a jump of this combined stress are then

$$p_A(t) = \frac{\sum_{k=1}^{K} p_A^k \lambda^k(t)}{\lambda_C(t)}, \tag{4.18}$$

$$p_B(t) = \frac{\sum_{k=1}^{K} p_B^k \lambda^k(t)}{\lambda_C(t)}. \tag{4.19}$$

respectively. Moreover, the default intensities are

$$\lambda_A(t) = \bar{\lambda}_A(t) + p_A(t)\lambda_C(t), \tag{4.20}$$

$$\lambda_B(t) = \bar{\lambda}_B(t) + p_B(t)\lambda_C(t). \tag{4.21}$$

Therefore, the dependent structure between firm A and B is equivalent to the simple case introduced at the beginning of this section where only a single type of stress is present.

In order to determine $h(s,t)$, we derive a formula for the joint survival probability $G(s,t)$. As usual, we first consider the situation where $s < t$:

$$\begin{aligned}
G(s,t) &= \mathrm{P}(\tau_A > s, \tau_B > t) && (4.22)\\
&= \exp\left(-\int_0^s \bar{\lambda}_A(u)du\right)\exp\left(-\int_0^t \bar{\lambda}_B(u)du\right) && (4.23)\\
&\quad\times \sum_{m=0}^{\infty} e^{-\lambda_C(s)}\frac{(\lambda_C(s))^m}{m!}\left(\int_0^s (1-p_A(u))(1-p_B(u))\frac{\lambda_C(u)du}{\lambda_C(s)}\right)^m && (4.24)\\
&\quad\times \sum_{n=0}^{\infty} e^{-(\lambda_C(t)-\lambda_C(s))}\frac{(\lambda_C(t)-\lambda_C(s))^n}{n!}\left(\int_s^t (1-p_B(v))\frac{\lambda_C(v)dv}{\lambda_C(t)-\lambda_C(s)}\right)^n && (4.25)\\
&= \exp\left(-\int_0^s \bar{\lambda}_A(u)du\right)\exp\left(-\int_0^t \bar{\lambda}_B(u)du\right) && (4.26)\\
&\quad\times \exp\left(-\int_0^s \lambda_C(u)\left(1-(1-p_A(u))(1-p_B(u))\right)du\right) && (4.27)\\
&\quad\times \exp\left(-\int_s^t \lambda_C(u)p_B(u)du\right) && (4.28)\\
&= \exp\left(-\Lambda_A(s)-\Lambda_B(t)\right)\exp\left(\int_0^s \lambda_C(u)p_A(u)p_B(u)du\right), && (4.29)
\end{aligned}$$

where the capitalized Λ is the cumulative intensity for the corresponding λ, for example

$$\Lambda_A(s) = \int_0^s \lambda_A(u)du. \tag{4.30}$$

By symmetry, we can easily derive $G(s,t)$ for $s \geq t$, consequently for any $s \geq 0$ and $t \geq 0$

$$\begin{aligned}
G(s,t) &= \exp\left(-\Lambda_A(s)-\Lambda_B(t)\right)\exp\left(\int_0^{s\wedge t} \lambda_C(u)p_A(u)p_B(u)du\right) && (4.31)\\
&= S_A(s)S_B(t)\exp\left(\int_0^{s\wedge t} \lambda_C(u)p_A(u)p_B(u)du\right), && (4.32)
\end{aligned}$$

where

$$s \wedge t = \min(s,t), \tag{4.33}$$

$$S_A(s) = \mathrm{P}(\tau_A > s) = \exp\left(-\Lambda_A(s)\right), \tag{4.34}$$

$$S_B(t) = \mathrm{P}(\tau_B > t) = \exp\left(-\Lambda_B(t)\right). \tag{4.35}$$

Marshall and Olkin computed a particular case of $G(s,t)$ where time independent intensities and impact probabilities are considered [51]. They called the model the non-fatal shock model and the corresponding $G(s,t)$ is usually referred to as the bivariate exponential distribution.

The density for (τ_A, τ_B) can then be computed easily by evaluating the mixed derivative of $G(s,t)$:

$$h(s,t) = \frac{\partial^2 G(s,t)}{\partial s \partial t} \tag{4.36}$$

$$= \begin{cases} G(s,t)\lambda_B(t)\left(\lambda_A(s) - \lambda_C(s)p_A(s)p_B(s)\right) & \text{if} \quad s < t, \\ +\infty & \text{if} \quad s = t \quad \text{and} \quad \lambda_C(s)p_A(s)p_B(s) > 0, \\ G(s,t)\lambda_A(s)\left(\lambda_B(t) - \lambda_C(t)p_A(t)p_B(t)\right) & \text{if} \quad s > t. \end{cases} \tag{4.37}$$

The discontinuity along the diagonal $s = t$ resembles the default density of the correlated Brownian motion in Chapter 3, and the condition

$$\lambda_C(s)p_A(s)p_B(s) > 0 \tag{4.38}$$

is analogous to that in the correlated Brownian motion

$$\frac{\pi}{2} < \alpha < \pi \tag{4.39}$$

where both conditions ensure positive correlations. However, the underlying reasons are different. For the stress model, the conditional probability of joint default given that firm A defaults is

$$\mathrm{P}(\tau_B = t | \tau_A = t) = \lim_{\delta \to 0} \frac{G(t,t) - G(t+\delta,t) - G(t,t+\delta) + G(t+\delta,t+\delta)}{S_A(t) - S_A(t+\delta)} \tag{4.40}$$

$$= \frac{G(t,t)}{S_A(t)} \frac{\lambda_C(t)p_A(t)p_B(t)}{\lambda_A(t)}. \tag{4.41}$$

The limit is evaluated by applying Eq.(4.32) and L'Hôpital's rule.

The main advantages of the stress model are its simplicity and the ability to attain default correlations as high as possible. Besides, numerous statistics admit close-form expressions [52] if the intensities are constant. However, there are a few disadvantages of the model as discussed by Schönbucher [53]. Firstly, the specification of the intensities for the stress events is far from trivial. If a complete specification is desired, we will have to prescribe an intensity for every subset of the firms, i.e. every subset of $\{1, ..., N\}$. The number of subsets grows exponentially. For example, a typical CDO index contains 125 firms and this means specifying 2^{125} joint intensities which is far too large and computationally unmanageable. Secondly, the more fundamental problem of this approach lies in the time resolution of defaults in this model. It is unrealistic to suppose that two or more firms would default exactly at the same time. Although clustering of default times can certainly be an important risk feature in the model, simultaneity of defaults seems a bit extreme. Furthermore, when a credit event triggers a number of defaults, one would expect an impact on other firms, yet the default intensities of them (even the ones which have nonzero finite default probabilities induced by this event) remain unchanged. This is obviously not what we observe in reality. Motivated by the intent to remedy these two drawbacks, a new model is devised which is presented in Chapter 5.

4.3 MULTI-FACTOR PORTFOLIO MODEL

In this section, we summarize the three-factor portfolio credit model proposed by Longstaff and Rajan [3] and discuss the implications from their study. Since losses on the tranches of a CDO are simple functions of the total losses on the underlying portfolio, the evolution of the distribution of total portfolio losses is sufficient for valuing tranche losses and spreads. Consequently, rather than modeling individual losses and then aggregating over the portfolio, they model the distribution of total portfolio losses directly.

Let L_t denote the total portfolio losses on a portfolio with \$1 notional amount. By

definition, $L_0 = 0$. The dynamics of L_t is assumed to satisfy the following equation

$$\frac{dL_t}{1 - L_t} = \bar{\gamma}_1 dN_{1t} + \bar{\gamma}_2 dN_{2t} + \bar{\gamma}_3 dN_{3t}, \tag{4.42}$$

where $\bar{\gamma}_i = 1 - e^{-\gamma_i}$, $i = 1, 2, 3$; γ_1, γ_2 and γ_3 are nonnegative constants defining jump sizes; and N_{1t}, N_{2t} and N_{3t} are independent Poisson processes. A closed form solution for Eq.(4.42) is given by

$$L_t = 1 - e^{-\gamma_1 N_{1t}} e^{-\gamma_2 N_{2t}} e^{-\gamma_3 N_{3t}}. \tag{4.43}$$

It is worth noting that the economic condition $0 \leq L_t \leq 1$ is satisfied for all t and the portfolio loss L_t is a non-decreasing function of time since N_{1t}, N_{2t} and N_{3t} are nondecreasing processes. The intensities of the three Poisson processes are designated λ_{1t}, λ_{2t} and λ_{3t} respectively and each of them is a Cox-Ingeroll-Ross process

$$d\lambda_{it} = (\alpha_i - \beta_i \lambda_{it})dt + \sigma_i \sqrt{\lambda_{it}} dZ_{it}, \tag{4.44}$$

for $i = 1, 2, 3$ and Z_{it} are standard independent Brownian motion. The expectation of an arbitrary function $F(L_t)$ of the portfolio losses can be calculated directly by the following expression

$$\mathrm{E}(F(L_t)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{P_{1,i}(\lambda_1, t)}{i!} \frac{P_{2,j}(\lambda_2, t)}{j!} \frac{P_{1,k}(\lambda_3, t)}{k!} F(1 - e^{-\gamma_1 i} e^{-\gamma_2 j} e^{-\gamma_3 k}), \tag{4.45}$$

where

$$P_{m,n}(\lambda_m, t) = \mathrm{E}\left[\exp\left(-\int_0^t \lambda_{ms} ds\right)\left(\int_0^t \lambda_{ms} ds\right)^n\right] \tag{4.46}$$

denotes $n!$ times the probability that $N_{mt} = n$ conditional on the initial intensity λ_{m0} for $m = 1, 2, 3$. There are two nice features about Eq.(4.45). Firstly, although it is a infinite sum, only the first few terms generally need to be evaluated since the remainder turns out to be negligibly small in practice. Secondly, $P_{m,0}$ admits a closed form from results in Cox et al. [49] and $P_{m,n}$ for $n \geq 0$ satisfies a recursive partial differential equation which has the following closed form solution

$$P_{m,i}(\lambda_m, t) = A(t) e^{-B(t)\lambda_m} \sum_{j=0}^{i} C_{i,j}(t) \lambda_m^j, \tag{4.47}$$

where $A(t)$ and $B(t)$ are simple functions and $C_{i,j}(t)$ are given as solutions to a recursive system of first order ordinary differential equations which can be solved numerically.

The major drawback of the three-factor portfolio model, like any top-down approach model, is that it disregards the information from the individual credits. In chapter 5, we provide an efficient implementation of the bottom up approach version of the three-factor portfolio model. Another issue about the three-factor portfolio model that is not very satisfactory is the computational inefficiency of Eq.(4.46) due to the recursive system of first order ordinary differential equations. In Appendices D and E, we present a much more efficient scheme for evaluating the probabilities of the form of Eq.(4.46) for intensity processes with closed form $P_{m,0}$.

5.0 CRISIS MODEL

In light of the stress model and the three-factor portfolio model discussed in Chapter 4, we develop a new model which is a generalization of the former and use the key observation from the latter to specify credit events. The generalization is straightforward. Instead of defaulting simultaneously when a stress event happens, firms affected by this event will default independently at later, random times. This approach resolves the time resolution problem arising from the stress model while maintaining the ability to achieve significant default correlation and retaining the clustering of default times when a credit event happens. Furthermore, default probabilities of firms not yet killed increase at first when an event happens and then gradually return to their ordinary levels. These are consistent with what we observe in the credit market. We call this kind of event a *crisis*. On the other hand, instead of a complete specification of joint default events as discussed in the stress model, we consider crises associated with the sectors to which each firm belongs. It is possible that a firm belongs to more than one sector. In this thesis we aim to explore the specification, which is motivated by the works of Duffie [26] and Longstaff [3], that each firm is associated with two sectors. One of the sectors is the industrial group to which a firm belongs and the other is the global market as a whole which affects every firm in the portfolio. For each sector, there is a sequence of crises that may kill the firms in it. Firms that survived previous crises may still be killed by future ones.

5.1 MODEL FORMULATION

For notational consistency, in the rest of this chapter we would reserve the subscript index i for specifying a firm and the superscript for indexing a sector. In a portfolio which consists of credit risky securities issued by N firms, the default time of firm i in the crisis model framework is defined as follows:

$$\tau_i = \inf\left\{ s \geq 0 : \bar{N}_i(s) + \sum_{k=1}^{K} \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t_j^k\}} X_{i,j}^k (s - t_j^k) > 0 \right\}, \tag{5.1}$$

for $i = 1, .., N$, where

- $\bar{N}_i$ are independent Poisson processes.
- t_j^k is the j-th jump time of a Poisson process $N^k(s)$ associated with the k-th sector for $k = 1, ..., K$ and $j = 1, 2, ...$
- $\mathbf{1}_{\{s > t_j^k\}}$ is an indicator function that equals one if $s > t_j^k$ and zero otherwise.
- $X_{i,j}^k$ are single jump processes (there is only one jump from 0 to 1) with a cumulative distribution $Z_{i,j}^k(s - t_j^k)$.
- All $\bar{N}_i$, $X_{i,j}^k$ and N^k are independent processes.

$\bar{N}_i$ is an idiosyncratic Poisson process associated with firm i which is driven by firm-specific factors. Once there is a jump in $\bar{N}_i$, firm i defaults immediately. Besides, if N^k has a jump at t_j^k, firm i may default at a later time. A jump at t_j^k in N^k turns on another jump process $X_{i,j}^k$ such that a jump of $X_{i,j}^k$ leads to a default of firm i immediately. The functional form of the cumulative distribution $Z_{i,j}^k(s - t_j^k)$ is left unspecified in order to facilitate the discussion in general. Specific forms of $Z_{i,j}^k(s - t_j^k)$ will be provided when we discuss calibrations of the model to data. The crisis model is a generalization of the default time model described by Eq.(4.15). The specification we aim to explore in this thesis is the case when $K = 2$ in Eq.(5.1), i.e.

$$\tau_i = \inf\left\{ s \geq 0 : \bar{N}_i(s) + \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t_j^{S(i)}\}} X_{i,j}^{S(i)} (s - t_j^{S(i)}) + \sum_{k=1}^{\infty} \mathbf{1}_{\{s > t_k^G\}} X_{i,k}^G (s - t_k^G) > 0 \right\} \tag{5.2}$$

where $S(i)$ denotes the index of the industrial group to which firm i belongs and G is the index of the global sector to which every firm belongs. This specification implies that the

default of a firm is caused by firm-specific factors or triggered either by a sectorally associated crisis or a globally associated crisis.

5.2 SURVIVAL PROBABILITIES

The stopping time in Eq.(5.1) is defined by using the so called multivariate delayed Poisson process. The multivariate delayed Poisson process is a type of multivariate point process, which is a point process with n types of event occurring along a time axis. Bivariate point processes were first studied by Cox and Lewis [54] and carried further by Lawrance and Lewis [55]. This section is an extension of their idea to encompass multivariate point processes with an emphasis on survival analysis. A multivariate point process is formed by subjecting the events of a main Poisson process to n independent delays. For example, N^k defined under Eq.(5.1) is a main Poisson process and the associated delayed events of the n firms in this k-th sector forms a n-dimensional multivariate delayed Poisson process. Since it is well known (for example, Cox and Lewis [56]) that a Poisson process whose events are independently and identically displaced remains a Poisson process, the events of each delayed process considered separately form n nonhomogeneous Poisson processes. They cannot, however, be independent Poisson processes because their events are associated through the events of the main Poisson process.

5.2.1 Individual Survival Probability

The univariate delayed Poisson process is constructed from a main Poisson process of intensity λ^c. We use a superscript c to indicate a generic type of crisis. Events triggered from the main process are delayed independently by random amount ζ_j with a cumulative distribution $Z_j(\cdot)$. For the sake of simplicity, we assume identical cumulative distributions for all j, i.e. $Z_j(\cdot) = Z(\cdot)$. Thus an event at time t_j^c in the main Poisson process produces delayed event at a later time $t_j^c + \zeta_j$. We assume that the random variables ζ_j are non-negative due to causality. The first time the delayed event happens, triggered by an event of the main

process occurring at an earlier time, is defined as follows:

$$\bar{\tau}^c = \inf\left\{ s \geq 0 : \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t_j^c\}} X^c(s - t_j^c) > 0 \right\} \tag{5.3}$$

$$= \min_j \{t_j^c + \zeta_j\}, \tag{5.4}$$

where X^c is a single jump process with a cumulative distribution $Z(\cdot)$. One of the key properties of Poisson processes is that given m events in $(0, t]$, they occur independently and are uniformly distributed over the interval. Hence, for constant λ^c, it is easy to see that the probability of a firm not killed by a crisis by t is

$$\mathrm{P}(\bar{\tau}^c > t) = \sum_{m=0}^{\infty} e^{-\lambda^c t} \frac{(\lambda^c t)^m}{m!} \left(\frac{1}{t} \int_0^t \mathrm{P}(\zeta > t - s) ds \right)^m \tag{5.5}$$

$$= \sum_{m=0}^{\infty} e^{-\lambda^c t} \frac{(\lambda^c t)^m}{m!} \left(\frac{1}{t} \int_0^t (1 - Z(t - s)) ds \right)^m \tag{5.6}$$

$$= e^{-\lambda^c t + \lambda^c \int_0^t (1 - Z(s)) ds} \tag{5.7}$$

$$= e^{-\int_0^t \lambda^c Z(s) ds}, \tag{5.8}$$

where ζ denotes the identically distributed random variable ζ_j. Since the delayed process is an inhomogeneous Poisson process, it can be easily verified from Eq.(5.8) that the intensity of the delayed process is

$$\lambda(s) = \lambda^c Z(s). \tag{5.9}$$

We can easily generalize the idea to the main Poisson process with nonhomogeneous intensity $\lambda^c(s)$. The probability that there are exactly m main events occurring in the interval $(0, t]$ is

$$\mathrm{P}(N(t) - N(0) = m) = e^{-\Lambda^c(t)} \frac{(\Lambda^c(t))^m}{m!}, \tag{5.10}$$

where $\Lambda^c(t)$ is the cumulative intensity, i.e. $\int_0^t \lambda^c(s)ds$. Given this condition, the crises are independent and have identical distribution $\lambda^c(s)/\Lambda^c(t)$ in the interval $(0, t]$. Thus the survival probability becomes

$$\begin{aligned}
\mathrm{P}(\bar{\tau}^c > t) &= \sum_{m=0}^{\infty} e^{-\Lambda^c(t)} \frac{(\Lambda^c(t))^m}{m!} \left(\int_0^t \mathrm{P}(\zeta > t-s) \frac{\lambda^c(s)ds}{\Lambda^c(t)} \right)^m && (5.11)\\
&= \sum_{m=0}^{\infty} e^{-\Lambda^c(t)} \frac{(\Lambda^c(t))^m}{m!} \left(\int_0^t (1 - Z(t-s)) \frac{\lambda^c(s)ds}{\Lambda^c(t)} \right)^m && (5.12)\\
&= e^{-\Lambda^c(t) + \int_0^t (1-Z(t-s))\lambda^c(s)ds} && (5.13)\\
&= e^{-\int_0^t \lambda^c(s) Z(t-s)ds}. && (5.14)
\end{aligned}$$

We further generalize to the case of K independent main Poisson processes N^k, with different $\lambda^k(s)$ and $Z^k(t)$ for each process. The first time a delayed event happens, triggered by any of the K main Poisson processes, is defined as follows:

$$\bar{\tau}^{\min} = \min\{\bar{\tau}^1, \bar{\tau}^2, ..., \bar{\tau}^K\}, \tag{5.15}$$

where each $\bar{\tau}^k$ is the default time triggered by a crisis of the k-th factor as defined by Eq.(5.3). Since the main Poisson processes N^k are independent,

$$\begin{aligned}
\mathrm{P}(\bar{\tau}^{\min} > t) &= \prod_{k=1}^{K} \mathrm{P}(\bar{\tau}^k > t) && (5.16)\\
&= \prod_{k=1}^{K} e^{-\int_0^t \lambda^k(s) Z^k(t-s)ds}, && (5.17)\\
&= \exp\left(-\sum_{k=1}^{K} \int_0^t \lambda^k(s) Z^k(t-s)ds \right). && (5.18)
\end{aligned}$$

Define

$$\bar{\tau} = \inf\left\{ s \geq 0 : \bar{N} > 0 \right\} \tag{5.19}$$

as the first jump of an idiosyncratic Poisson process $\bar{N}$. According to Eq.(5.18) and the independence of $\bar{N}$ and N^k, for the case of interest here with $K = 2$, the survival probability of a firm by time t is then

$$\mathrm{P}(\tau > t) = \mathrm{P}(\bar{\tau} > t)\mathrm{P}\left(\bar{\tau}^S > t\right)\mathrm{P}\left(\bar{\tau}^G > t\right) \tag{5.20}$$

$$= e^{-\int_0^t \bar{\lambda}(s)ds} e^{-\int_0^t \lambda^S(s)Z^S(t-s)ds} e^{-\int_0^t \lambda^G(s)Z^G(t-s)ds}, \tag{5.21}$$

where $\bar{\lambda}(s)$, $\lambda^S(s)$ and $\lambda^G(s)$ are the intensities corresponding to the Poisson processes $\bar{N}$, N^S and N^G respectively. Moreover $Z^S(\cdot)$ and $Z^G(\cdot)$ are the cumulative probability distributions of the delayed processes. The default intensity or hazard rate of individual firms of the crisis model is then

$$\lambda(s) = \lim_{h\to 0} \frac{\mathrm{P}(\tau \le s+h | \tau > s)}{h} \tag{5.22}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\mathrm{P}(s < \tau \le s+h)}{\mathrm{P}(\tau > s)} \tag{5.23}$$

$$= \frac{1}{\mathrm{P}(\tau > s)} \lim_{h\to 0} \frac{\mathrm{P}(\tau > s) - \mathrm{P}(\tau > s+h)}{h} \tag{5.24}$$

$$= -\frac{1}{\mathrm{P}(\tau > s)} \frac{d\mathrm{P}(\tau > s)}{ds} \tag{5.25}$$

$$= \bar{\lambda}(s) + \lambda^S(s)Z^S(0) + \int_0^s \lambda^S(u)\frac{dZ^S(s-u)}{ds}du$$

$$+ \lambda^G(s)Z^G(0) + \int_0^s \lambda^G(u)\frac{dZ^G(s-u)}{ds}du. \tag{5.26}$$

In the above derivation, we assume that $Z^S(\cdot)$ and $Z^G(\cdot)$ are differentiable almost everywhere. It is important to note that $\lambda(s)$ depends on the history of the paths of λ^S and λ^G, contrary to other intensity-based model that depend only on the current states of the intensities of the common Poisson processes. In particular, if $Z^S(s) = p^S$ and $Z^G(s) = p^G$, i.e. both are constant, then the crisis model reduces to the stress event model [2] with default intensity

$$\lambda(s) = \bar{\lambda}(s) + p^S\lambda^S(s) + p^G\lambda^G(s). \tag{5.27}$$

5.2.2 Joint Survival Probability

The multivariate delayed Poisson process is constructed similarly to the univariate case from a main Poisson process of intensity λ^c. Associated with each of its events is a set of independent random delays $\{\zeta^c_{i,j}\}$ and the corresponding cumulative distributions $\{Z^c_i(t)\}$ which determine the event time of the delayed process for each firm i. Thus an event at time t^c_j in the main Poisson process induces an type i event at time $t^c_j + \zeta^c_{i,j}$ for each delayed process. We also assume that each $\zeta^c_{i,j}$ is non-negative as before. Define

$$\bar{\tau}^c_i = \inf\left\{ s \geq 0 : \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t^c_j\}} X^c_i(s - t^c_j) > 0 \right\} \tag{5.28}$$

$$= \min_j \{t^c_j + \zeta^c_{i,j}\}, \tag{5.29}$$

as the time of the first occurrence of the type i delayed process triggered by a crisis of a sector. Suppose the dimension of the multivariate is n which is the number of firms in a sector. We also define

$$\bar{\tau}^c_{\min} = \min\{\bar{\tau}^c_1, \bar{\tau}^c_2, ..., \bar{\tau}^c_n\}, \tag{5.30}$$

which is the first event time of a delayed process in this sector. The probability that there is no delayed event in the sector by time t is then

$$\mathrm{P}(\bar{\tau}^c_{\min} > t) = \mathrm{P}\left(\bigcap_{i=1}^{n} \bar{\tau}^c_i > t \right) \tag{5.31}$$

$$= \sum_{m=0}^{\infty} e^{-\Lambda^c(t)} \frac{(\Lambda^c(t))^m}{m!} \left(\int_0^t \frac{\lambda^c(s)}{\Lambda^c(t)} \prod_{m=1}^{n} \mathrm{P}(\zeta^c_i > t - s) ds \right)^m \tag{5.32}$$

$$= \sum_{m=0}^{\infty} \frac{e^{-\Lambda^c(t)}}{m!} \left(\int_0^t \lambda^c(s) \prod_{i=1}^{n} (1 - Z^c_i(t-s)) ds \right)^m \tag{5.33}$$

$$= e^{-\Lambda^c(t)} e^{\int_0^t \lambda^c(s) \prod_{i=1}^{n} (1 - Z^c_i(t-s)) ds} \tag{5.34}$$

$$= e^{-\int_0^t \lambda^c(s) \left(1 - \prod_{i=1}^{n} (1 - Z^c_i(t-s))\right) ds}, \tag{5.35}$$

where ζ^c_i is a random variable which is distributed identically to $\zeta^c_{i,j}$. In the above derivation we apply both the technique we adopted in deriving Eq.(5.11) and the independence of the

delayed times $\zeta^c_{i,j}$. It can be easily verified that Eq.(5.35) gives the correct joint survival probability of the stress model by setting $Z^c_i = 1$ for all i, i.e.

$$\mathrm{P}\left(\bar{\tau}^c_{\min} > t\right) = e^{-\int_0^t \lambda^c(s)} = \mathrm{P}(t^c_1 > t), \tag{5.36}$$

where t^c_1 is the first jump of the main Poisson process.

We are now ready to derive the joint survival probability of a portfolio consisting of N securities where each firm i is affected by the corresponding sectoral intensity $\lambda^{S(i)}$ and the global intensity λ^G. To simplify the notation, we replace the superscript G with 0 and call the global market as the zero-th sector. Furthermore, $S(i)$ the index of the industrial group to which firm i belongs, is a function mapping i to $l \in \{1, .., L\}$, where L is the number of industrial groups in the portfolio. If $S(i) = l$, then we say that firm i is in the l-th sector. We also assume that the distributions of the delayed events depend only on the sectors to which each firm belongs, i.e $Z^l_i(\cdot) = Z^l(\cdot)$ for $l = 0, ..., L$. It easy to see that the default time of firm i is

$$\tau_i = \min\{\bar{\tau}_i, \bar{\tau}^{S(i)}_i, \bar{\tau}^G_i\}. \tag{5.37}$$

Define

$$\tau_{\min} = \min\{\tau_1, \tau_2, ..., \tau_N\} \tag{5.38}$$

as the first default time of a firm in the portfolio. The probability that all firms survive by time t is then

$$\begin{aligned}
\mathrm{P}\left(\tau_{\min} > t\right) &= \mathrm{P}\left(\bigcap_{i=1}^{N}(\bar{\tau}_i > t \cap \bar{\tau}_i^{S(i)} > t \cap \bar{\tau}_i^{G} > t)\right) && (5.39)\\
&= \mathrm{P}\left(\bigcap_{i=1}^{N}\bar{\tau}_i > t\right)\mathrm{P}\left(\bigcap_{i=1}^{N}\bar{\tau}_i^{S(i)} > t\right)\mathrm{P}\left(\bigcap_{i=1}^{N}\bar{\tau}_i^{G} > t)\right) && (5.40)\\
&= \mathrm{P}\left(\bigcap_{i=1}^{N}\bar{\tau}_i > t\right)\mathrm{P}\left(\bigcap_{l=1}^{L}\bar{\tau}_{\min}^{l} > t\right)\mathrm{P}\left(\bar{\tau}_{\min}^{G} > t)\right) && (5.41)\\
&= \mathrm{P}\left(\bigcap_{i=1}^{N}\bar{\tau}_i > t\right)\prod_{l=0}^{L}\mathrm{P}\left(\bar{\tau}_{\min}^{l} > t\right) && (5.42)\\
&= \prod_{i=1}^{N} e^{-\int_0^t \bar{\lambda}_i(s)ds}\prod_{l=0}^{L} e^{-\int_0^t \lambda^l(s)\left(1-(1-Z^l(t-s))^{n_l}\right)ds} && (5.43)\\
&= \exp\left(-\sum_{i=1}^{N}\int_0^t \bar{\lambda}_i(s)ds - \sum_{l=0}^{L}\int_0^t \lambda^l(s)\left(1-(1-Z^l(t-s))^{n_l}\right)ds\right), && (5.44)
\end{aligned}$$

where n_l is the number of firms in the l-th sector. The first line comes from the definition of $\tau_i = \min\{\bar{\tau}_i, \bar{\tau}_i^{S(i)}, \bar{\tau}_i^{G}\}$. The independence of $\bar{N}_i$'s, N^l's and $N_{i,j}^l$ leads to the final closed form expression. The First-to-Default (FtD) probability of the portfolio can be calculated easily as follows:

$$\mathrm{P}\left(\tau_{\min} \le t\right) = 1 - \mathrm{P}\left(\tau_{\min} > t\right) \tag{5.45}$$

$$\tag{5.46}$$

and the FtD density is

$$\begin{aligned}
\frac{d\mathrm{P}(\tau_{\min} \le t)}{dt} = \mathrm{P}\left(\tau_{\min} > t\right) \times \Bigg(&\sum_{i=1}^{N}\bar{\lambda}_i(t) + \sum_{l=0}^{L}\lambda^l(t)\left(1-(1-Z^l(0))^{n_l}\right)\\
&+ \sum_{l=0}^{L}\int_0^t \lambda^l(s)n_l(1-Z^l(t-s))^{n_l-1}\frac{dZ^l(t-s)}{dt}ds\Bigg). && (5.47)
\end{aligned}$$

We can also find the joint survival probability

$$\mathrm{P}\left(\bigcap_{i\in I}\tau_i > t\right) \tag{5.48}$$

of any sub-portfolio by the same way in deriving Eq.(5.44), where I is a subset of $\{1, ..., N\}$. Define M_t as the number of firms which still survive by time t. By standard probability argument, the distribution of M_t can be computed directly as

$$\mathrm{P}(M_t = n) = \sum_{j=n}^{N} {}_jC_n(-1)^{j-n} \sum_{|I|=j} \mathrm{P}\left(\bigcap_{i\in I} \tau_i > t\right). \tag{5.49}$$

Then the distribution of the number of defaults $D(t) = N - M_t$ can also be calculated. However, computing the complete distribution of $D(t)$ requires 2^N calculations of the joint probabilities. For example, the typical size of a CDS index has $N = 125$, i.e. 2^{125} terms must be computed, which makes Eq.(5.49) computationally unmanageable. As a result, we need to seek other computationally efficient means in order to calculate the full distribution when N is large.

5.3 LOSS DISTRIBUTION

The loss distribution of a portfolio is a dynamic process which evolves stochastically over time. A common approach in calculating the loss distribution of a credit risky portfolio for bottom-up approaches is by computing the loss under conditional independence. The unconditional default distribution is then the weighted sum of the conditional ones, i.e.

$$\mathrm{P}(D(t) = n) = \int_\Omega \mathrm{P}(D(t) = n|\omega)\mathrm{P}(d\omega), \quad n = 1, ..., N, \tag{5.50}$$

where $D(t)$ is the number of defaults by time t and ω is a condition under which defaults of firms are independent. We assume that the recovery rate of each security is a constant R and a uniform notional amount δ for all firms in the portfolio, thus

$$L_t = \sum_{i=1}^{N} \delta_i(1 - R_i)\mathbf{1}_{\tau_i \le t} = \delta(1 - R)\sum_{i=1}^{N} \mathbf{1}_{\tau_i \le t} = \delta(1 - R)D(t). \tag{5.51}$$

Therefore, modeling the loss distribution is equivalent to modeling the default distribution. The first challenge of evaluating Eq.(5.50) is to find a computationally efficient scheme to calculate the conditional loss distribution $\mathrm{P}(D(t) = n|\omega)$. There are a few ways available in

the literature. The one that we are going to describe is an efficient recursive algorithm developed by Andersen et al.[57] which is applicable to both homogeneous and inhomogeneous portfolios that can contain a large number of securities. The second challenge lies on the evaluation of $\mathrm{P}(d\omega)$. This is in fact a threefold challenge. One needs to identify conditions under which defaults are independent, choose a partition for the probability space Ω in order to enhance calculation, and evaluate the probabilities of these conditions. We present a novel identification of conditions of independence which arise naturally from the formulation of the crisis model. We also introduce a systematic way of choosing a countable partitions of Ω which automatically arranges the sizes of $\mathrm{P}(d\omega)$ in descending order. This then leads to a series expansion for the loss distribution and only the first few terms are usually needed to accurately approximate the loss distribution. Finally, we discuss the loss distribution of the special case that there is no delay if a crisis happens, i.e. a firm defaults immediately with a finite probability. This is just the stress event model with two common factors for each firm. If the mean delayed time of a crisis is short, we can regard the stress event model as an good approximation for the crisis model. It turns out that the loss distribution of the stress event model has a tractable expression and can be computed efficiently. Furthermore, the simplicity of the expression for the loss distribution in this special situation is preserved even when stochastic intensities are introduced.

5.3.1 Recursive Algorithm

This subsection summarizes the recursive algorithm introduced by Andersen, Sidenius and Basu [57] which is an efficient way to compute the conditional loss distribution of a portfolio. For a fixed time t, suppose for each $\omega \in \Omega$ the default indicators $\mathbf{1}_{\{\tau_i(\omega)\leq t\}}$ are independent. Let $D_K(t)$ denote the number of defaults by time t in the basket consisting of the first K names of the portfolio. Since defaults are conditionally independent, the conditional probability of having n defaults in the K-basket is

$$\mathrm{P}(D_K(t) = n|\omega) = \mathrm{P}(D_{K-1}(t) = n|\omega)s_K(t|\omega) + \mathrm{P}(D_{K-1}(t) = n-1|\omega)d_K(t|\omega), \qquad (5.52)$$

for $n = 1, ..., K$, where

$$d_K(t|\omega) = \mathrm{E}[\mathbf{1}_{\{\tau_K(\omega)\leq t\}}] \tag{5.53}$$

$$= \mathrm{P}(\tau_K(\omega) \leq t) \tag{5.54}$$

is the default probability of entity K by time t under the condition ω and

$$s_K(t|\omega) = \mathrm{E}[1 - \mathbf{1}_{\{\tau_K(\omega)\leq t\}}] \tag{5.55}$$

$$= 1 - \mathrm{P}(\tau_K(\omega) \leq t) \tag{5.56}$$

$$= \mathrm{P}(\tau_K(\omega) > t) \tag{5.57}$$

is the corresponding conditional survival probability. It is worth mentioning that both $d_K(t|\omega)$ and $s_K(t|\omega)$ are functions of the default indicator $\mathbf{1}_{\{\tau_K(\omega)\leq t\}}$. Consequently, independence of the default indicators implies Eq.(5.52). The idea of Eq.(5.52) is that n defaults out of K names can be attained either by n defaults out of the first $K-1$ names and the survival of the K-th name, or by $n-1$ defaults out of the first $K-1$ and the default of the K-th name. For $n = 0$, it is clear that

$$\mathrm{P}(D_K(t) = 0|\omega) = \prod_{k=1}^{K} s_k(t|\omega), \tag{5.58}$$

for all K. The recursion starts from $K = 1$ with $\mathrm{P}(D_1(t) = 0) = s_1(t|\omega)$ and $\mathrm{P}(D_1(t) = 1) = d_1(t|\omega)$, and runs for $K = 2, ..., N$ with $\mathrm{P}(D(t) = n|\omega) = \mathrm{P}(D_N(t) = n|\omega)$ which is the loss distribution of the whole portfolio. Finally, the unconditional default distribution is

$$\mathrm{P}(D(t) = n) = \int_\Omega \mathrm{P}(D(t) = n|\omega)\mathrm{P}(d\omega), \tag{5.59}$$

an integral over all $\omega \in \Omega$. It is worth noting that this algorithm does not require homogeneity among the conditional survival probabilities $s_i(t|\omega)$, which means that it is applicable to both homogeneous and inhomogeneous portfolios. In additions, the number of terms need to be calculated in this algorithm is approximately $N^2/2$ which is much more efficient than the expression given by Eq.(5.49) when N is large.

5.3.2 Conditions of Independence

For the correlated intensity model proposed by Duffie and Gârleanu [26], a realization of the common part of the firms' default intensities, is employed as a condition of independence for default times. We could adopt the same method to find conditions of independence for the crisis model, as the crisis model can be identified as a special case of a more general class of correlated intensity models. However, this approach is not very efficient. Instead, we follow a totally different approach to choosing the conditions of independence for the crisis model. In fact, these conditions of independence arise quite naturally from the definition of the default time of the crisis model. We restate the definition of the default time of a firm which depends on two common factors,

$$\tau_i = \inf\left\{ s \geq 0 : \bar{N}_i(s) + \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t_j^{S(i)}\}} X_i^{S(i)}(s - t_j^{S(i)}) + \sum_{k=1}^{\infty} \mathbf{1}_{\{s > t_k^G\}} X_i^G(s - t_k^G) > 0 \right\}. \tag{5.60}$$

This is the particular form that we will use for calibration to market data. The correlations among default times come from the second and third sum of the definition. Specifically, the random jump times t_j^l of the Poisson processes N^l for $l = 0, ..., L$ are the sources of the dependence. Although each firm is affected by two common factors, the total number of sectors in a portfolio is $L+1$ (L industrial groups plus one global sector). The conditions of independence are basically the occurrences of the crises. Consider a scenario

$$\omega^u = \omega(u(L, \vec{m}_L, t)) = \left\{ \omega : t_j^l(\omega) = u_j^l \in (0, t],\ j = 1, ..., m_l,\ l = 0, ..., L \right\}, \tag{5.61}$$

where

$$u(L, \vec{m}_L, t) = \begin{pmatrix} u_1^0 & u_2^0 & \dots & u_{m_0}^0 & & \\ u_1^1 & u_2^1 & \dots & \dots & \dots & u_{m_1}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^L & u_2^L & \dots & \dots & u_{m_L}^L & \end{pmatrix} \quad \text{and} \quad \vec{m}_L = (m_0, m_1, ..., m_L). \tag{5.62}$$

$u(L, \vec{m}_L, t)$ is an array of $L+1$ rows and each row has m_l entries which specify jump times of N^l by time t. This is the scenario for which there are m_l crises occurring at $u_1^l, u_2^l, ...$ and $u_{m_l}^l$ all before time t in the l-th sector for $l = 0, 1, ..., L$. For a given ω^u, Eq.(5.60) becomes

$$\tau_i(\omega^u) = \inf_{s \geq 0} \Bigg\{ \bar{N}_i(s) + \sum_{j=1}^{m_{S(i)}} \mathbf{1}_{\{s > u_j^{S(i)}\}} X_i^{S(i)}(s - u_j^{S(i)}) + \sum_{k=1}^{m_G} \mathbf{1}_{\{s > u_k^G\}} X_i^G(s - u_k^G) \\ + \sum_{j=m_{S(i)+1}}^{\infty} \mathbf{1}_{\{s > t_j^{S(i)}\}} X_i^{S(i)}(s - t_j^{S(i)}) + \sum_{k=m_G+1}^{\infty} \mathbf{1}_{\{s > t_k^G\}} X_i^G(s - t_k^G) > 0 \Bigg\}, \tag{5.63}$$

where $t_j^{S(i)}$ and t_k^G are the random jump times of $X_i^{S(i)}$ and X_i^G after t respectively. Define

$$\tilde{\tau}_i(\omega^u) = \inf_{s \geq 0} \Bigg\{ \bar{N}_i(s) + \sum_{j=1}^{m_{S(i)}} \mathbf{1}_{\{s > u_j^{S(i)}\}} X_i^{S(i)}(s - u_j^{S(i)}) + \sum_{k=1}^{m_G} \mathbf{1}_{\{s > u_k^G\}} X_i^G(s - u_k^G) > 0 \Bigg\}, \tag{5.64}$$

which is almost identical to Eq.(5.63) except that the last two sums inside the brackets are deleted. Note that if $\tau_i(\omega^u) \leq t$, then

$$\tau_i(\omega^u) = \tilde{\tau}_i(\omega^u), \tag{5.65}$$

since a default must be triggered by a jump of $\bar{N}_i$, $X_i^{S(i)}$ or X_i^G before t and does not contribute to anything that happens after t. The default indicators under ω^u are

$$\mathbf{1}_{\{\tau_i(\omega^u) \leq t\}} = \mathbf{1}_{\{\tilde{\tau}_i(\omega^u) \leq t\}}, \tag{5.66}$$

for $i = 1, ..., N$, which are independent since all $\tilde{\tau}_i$ are defined by independent Poisson processes as indicated by Eq.(5.64). Consequently, with the identification of the independent conditions for the default indicators, we can apply the recursive algorithm outlined in the previous subsection to compute the conditional loss distribution of a portfolio.

5.3.3 Conditional Individual Survival Probability

This subsection is devoted to the derivation of the conditional survival probabilities, which are the building blocks of the conditional loss distribution, for every firm in a portfolio. The conditional survival probability of firm i for a given ω^u as specified by Eq.(5.61) is

$$\mathrm{P}\left(\tau_i(\omega^u) > t\right) = \mathrm{P}\left(\bar{\tau}_i(\omega^u) > t\right) \mathrm{P}\left(\bar{\tau}_i^{S(i)}(\omega^u) > t\right) \mathrm{P}\left(\bar{\tau}_i^{G}(\omega^u) > t\right). \tag{5.67}$$

Since the idiosyncratic default intensity $\bar{\tau}_i$ does not depend on the occurrences of the crises in the sectors,

$$\mathrm{P}\left(\bar{\tau}_i(\omega^u) > t\right) = \mathrm{P}\left(\bar{\tau}_i > t\right) \tag{5.68}$$

$$= \mathrm{E}\left(e^{-\int_0^t \bar{\lambda}_i(s)ds} | \bar{\lambda}_i(0)\right). \tag{5.69}$$

On the other hand,

$$\mathrm{P}\left(\bar{\tau}_i^l(\omega^u) > t\right) = \prod_{j_{m_l}=1}^{m_l} \left(1 - Z^l(t - u^l_{j_{m_l}})\right) \tag{5.70}$$

is the conditional survival probability that firm i is not killed by the m_l crises in the l-th sector before t. As a result,

$$\mathrm{P}\left(\tau_i(\omega^u) > t\right) = \mathrm{E}\left(e^{-\int_0^t \bar{\lambda}_i(s)ds} | \bar{\lambda}_i(0)\right) \prod_{j_{m_{S(i)}}=1}^{m_{S(i)}} \left(1 - Z^{S(i)}(t - u^{S(i)}_{j_{m_{S(i)}}})\right) \prod_{j_{m_G}=1}^{m_G} \left(1 - Z^{G}(t - u^{G}_{j_{m_G}})\right). \tag{5.71}$$

In particular, suppose the crises have immediate effect only, i.e. $Z^{S(i)}(\cdot) = p^{S(i)}$ and $Z^G(\cdot) = p^G$ which is just the stress event model. Then

$$\mathrm{P}(\tau_i(\omega^u) > t) = \mathrm{E}\left(e^{-\int_0^t \bar{\lambda}_i(s)ds} | \bar{\lambda}_i(0)\right) \left(1 - p^{S(i)}\right)^{m_{S(i)}} \left(1 - p^G\right)^{m_G}, \tag{5.72}$$

where the numbers of crises $m_{S(i)}$ and m_G depend on ω^u. It is worth noting that this conditional survival probability as well as the corresponding conditional loss distribution depend only upon the idiosyncratic intensities and the number of crises in each sector by time t, but not the occurrence times of the crises.

5.3.4 Unconditional Loss Distribution

In this subsection, we first present the closed form expression of the probability of the condition given by Eq.(5.61) and show explicitly that the sum of the conditional probabilities equals one. Then, we aggregate the conditional loss distributions and the density $\mathrm{P}(d\omega^u)$ to form the unconditional loss distribution of a portfolio. Finally, we provide a series expression for the unconditional loss distribution such that the terms of the series are enumerated in descending order of their 'sizes'. We present our calculations explicitly for the case where the crisis intensities λ^l are deterministic, and the corresponding stochastic versions could be easily extended by taking expectations.

We partition the probability space Ω as follows:

$$\mathrm{P}(\Omega) = \prod_{l=0}^{L} 1 \tag{5.73}$$

$$= \prod_{l=0}^{L} \left(\sum_{m_l=0}^{\infty} q^l_{m_l} \right), \tag{5.74}$$

where

$$q^l_{m_l} = e^{-\Lambda^l(t)} \frac{(\Lambda^l(t))^{m_l}}{m_l!}, \tag{5.75}$$

is the probability that there are m_l crises in the l-th sector by t. Furthermore, the probability that the $m_l > 0$ crises occur at u^l_1,...,$u^l_{m_l-1}$ and $u^l_{m_l}$ is

$$e^{-\Lambda^l(t)} \frac{(\Lambda^l(t))^{m_l}}{m_l!} \prod_{j_{m_l}=1}^{m_l} \left(\frac{\lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}}}{\Lambda^l(t)} \right) = \frac{e^{-\Lambda^l(t)}}{m_l!} \prod_{j_{m_l}=1}^{m_l} \lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}}. \tag{5.76}$$

It is possible that $m_l = 0$ which is the scenario that there is no crisis in the l-th sector. The probability of this occurrence is simply

$$q^l_0 = e^{-\Lambda^l(t)}. \tag{5.77}$$

For the sake of notational brevity, when $m_l = 0$, define

$$\prod_{j_0=1}^{0} \frac{\lambda^l(u^l_{j_0}) du^l_{j_0}}{0!} = 1, \tag{5.78}$$

and

$$\int_0^t \prod_{j_0=1}^{0} \frac{\lambda^l(u^l_{j_0})du^l_{j_0}}{0!} = 1. \tag{5.79}$$

Hence the probability that there are m_l crises occurring in the l-th sector for $l = 0, ..., L$ at times given by an array $u(L, \vec{m}_L, t)$ is

$$\mathrm{P}(d\omega^u) = \prod_{l=0}^{L} \prod_{j_{m_l}=1}^{m_l} \frac{e^{-\Lambda^l(t)}}{m_l!} \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}, \tag{5.80}$$

where ω^u is defined by Eq.(5.61). We can rearrange Eq.(5.74) as a sum of products

$$\begin{aligned}
\prod_{l=0}^{L}\left(\sum_{m_l=0}^{\infty} q^l_{m_l}\right) &= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(q^0_{m_0}\cdots q^L_{m_L}\right) && (5.81)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=0}^{L} q^l_{m_l}\right) && (5.82)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=0}^{L} e^{-\Lambda^l(t)}\frac{(\Lambda^l(t))^{m_l}}{m_l!}\right) && (5.83)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=0}^{L} e^{-\Lambda^l(t)}\frac{(\Lambda^l(t))^{m_l}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\left(\frac{\int_0^t \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}}{\Lambda^l(t)}\right)\right) && (5.84)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=0}^{L}\frac{e^{-\Lambda^l(t)}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\left(\int_0^t \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right)\right) && (5.85)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=0}^{L}\frac{e^{-\Lambda^l(t)}}{m_l!}\int_{(0,t]^{m_l}}\prod_{j_{m_l}=1}^{m_l}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) && (5.86)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_0\times\cdots\times m_L}}\prod_{l=0}^{L}\frac{e^{-\Lambda^l(t)}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) && (5.87)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_0\times\cdots\times m_L}}\prod_{l=0}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) && (5.88)\\
&= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_0\times\cdots\times m_L}}\mathrm{P}(d\omega^u)\right). && (5.89)
\end{aligned}$$

This derivation explicitly shows how the probabilities $\mathrm{P}(d\omega^u)$ aggregate to one. Since under each ω^u, default indicators $\mathbf{1}_{\{\tau_i(\omega^u)\leq t\}}$ are independent as shown in the previous subsection, the unconditional loss distribution is then

$$\mathrm{P}(D(t)=n) = \int_\Omega \mathrm{P}(D(t)=n|\omega^u)\mathrm{P}(d\omega^u) \tag{5.90}$$

$$= \sum_{m_0=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_0\times\cdots\times m_L}} \mathrm{P}(D(t)=n|\omega^u)\prod_{l=0}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right). \tag{5.91}$$

The big summation above is not a very useful expression for computing the unconditional loss distribution. We can rearrange the summation by ascending order in terms of the total number of crises occurring by t in all sectors, thus

$$\mathrm{P}(D(t)=n) = \sum_{k=0}^{\infty}\left(\sum_{\sum m_l=k}\left(\int_{(0,t]^{m_0\times\cdots\times m_L}} \mathrm{P}(D(t)=n|\omega^u)\prod_{l=0}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right)\right). \tag{5.92}$$

Define

$$\phi_k(t;n) = \sum_{\sum m_l=k}\left(\int_{(0,t]^{m_0\times\cdots\times m_L}} \mathrm{P}(D(t)=n|\omega^u)\prod_{l=0}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right), \tag{5.93}$$

which is the loss distribution generated by exactly k crises in all sectors and write

$$\phi(t;n) = \mathrm{P}(D(t)=n) \tag{5.94}$$

$$= \sum_{k=0}^{\infty}\phi_k(t;n). \tag{5.95}$$

We call $\phi_k(t;n)$ the k-th order term of the unconditional loss distribution. Furthermore, define

$$|\phi_k(t)| := \mathrm{P}(\text{total number of crises occurs by } t=k) \tag{5.96}$$

$$= \sum_{n=0}^{N}\phi_k(t;n) \tag{5.97}$$

$$= \sum_{\sum m_l=k}\left(\prod_{l=0}^{L}\frac{e^{-\Lambda^l(t)}(\Lambda^l(t))^{m_l}}{m_l!}\right), \tag{5.98}$$

which measures the 'size' of the k-th order term of the loss distribution $\phi_k(t;n)$. Since the crisis intensity $\lambda^l(s)$ of each sector is generally quite small, $|\phi_k(t)|$ decreases in k and is negligible for large k. Consequently, only the first few terms of the loss distribution $\phi_k(t;n)$ are needed to construct a good approximation to the full loss distribution. In addition, define

$$\epsilon_K(t) = \mathrm{P}(\text{total number of crises by } t > K) \tag{5.99}$$

$$= 1 - \sum_{k=0}^{K} |\phi_k(t)|, \tag{5.100}$$

which is a measure of the error of the K-th order approximation for the loss distribution. $\epsilon_K(t)$ is the probability of scenarios that are not considered in the K-th order approximation. The closer the value ϵ_K is to zero, the more accurate the approximation. Finally, it is easy to verify that

$$\sum_{k=0}^{\infty} |\phi_k(t)| = 1. \tag{5.101}$$

5.3.5 Zeroth Order and First Order Approximations

According to Eq.(5.93), the zeroth order term of the loss distribution is

$$\phi_0(t;n) = \prod_{l=0}^{L} e^{-\Lambda^l(t)} \mathrm{P}(D(t) = n | \omega^u = \text{no crisis by } t). \tag{5.102}$$

This is simply the product of the probability that there is no crisis in any sector with the corresponding conditional loss distribution. The conditional loss distribution can be computed by employing the conditional survival probabilities

$$s_i(t|\omega^u = \text{no crisis by } t) = \mathrm{E}(e^{-\int_0^t \bar{\lambda}_i(s)ds} | \bar{\lambda}_i(0)). \tag{5.103}$$

Thus, the conditional loss distribution $\phi_0(t;n)$ is generated solely by the independent idiosyncratic Poisson processes $\bar{N}_i$ and corresponds to the situation that there is no default correlation among firms. This zeroth order term filters out the loss distribution contributed

by firm-specific independent defaults and leaves the default correlation to higher order terms. Define

$$\Lambda^{\Sigma}(t) = \sum_{l=0}^{L} \Lambda^{l}(t) \tag{5.104}$$

as the sum of the cumulative crisis intensities of all sectors. It is easy to see that

$$|\phi_0(t)| = \prod_{l=0}^{L} e^{-\Lambda^{l}(t)} \tag{5.105}$$

$$= e^{-\Lambda^{\Sigma}(t)}, \tag{5.106}$$

is a non-increasing function of time and $|\phi_0(0)| = 1$. These two features imply that initially $\phi_0(t;n)$ is a good approximation for the loss distribution since it includes almost all probability, but the contribution from $\phi_0(t;n)$ begins to fade as t increases and the probability propagates to higher order terms. On the other hand, the zeroth order term is the main component of the unconditional loss distribution even if t is not small. For example, consider a CDO with 5-year maturity which is composed of six industrial sectors. Suppose the average crisis frequency in each industrial sector is 249 years and the average global crisis frequency is 763 years (these figures are estimated from an extensive data set of tranche prices of the CDX North American Investment Grade Index by Longstaff and Rajan [3]), then

$$|\phi_0(5)| = e^{-\frac{5}{763} - \frac{5}{249} \times 6} \approx 0.8807, \tag{5.107}$$

which accounts for more than 88% of the total probability. It is worth mentioning that the computational cost of $\phi_0(t;n)$ is very low. It only needs to call the recursive algorithm once for each t even when the intensities are stochastic.

The conditional loss distribution $\phi_1(t;n)$ is the first term that brings in default correlation. Furthermore, it is the most significant component for the unconditional loss distribution

among the correlated terms. It can be easily deduced from Eq.(5.94) that the first order term is

$$\phi_1(t;n) = \sum_{\sum m_l=1} \left(\int_{(0,t]} \mathrm{P}(D(t)=n|\omega^u) \prod_{l=0}^{L} e^{-\Lambda^l(t)} \prod_{j_{m_l}=1}^{m_l} \frac{1}{m_l!} \lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}} \right) \tag{5.108}$$

$$= \sum_{l=0}^{L} \left(\int_{(0,t]} \mathrm{P}(D(t)=n|\omega^u) e^{-\sum_{k=0}^{L}\Lambda^k(t)} \lambda^l(u^l_1) du^l_1 \right) \tag{5.109}$$

$$= e^{-\sum_{k=0}^{L}\Lambda^k(t)} \sum_{l=0}^{L} \left(\int_{(0,t]} \mathrm{P}(D(t)=n|\omega^u) \lambda^l(u^l_1) du^l_1 \right). \tag{5.110}$$

This consists of $L+1$ terms of one-dimensional integrals and each of them can be computed by numerical integration. In order to compute the first order term, we first have to compute the conditional loss distribution for each u^l_1 which can be accomplished by employing the the conditional survival probabilities

$$s_i(t|\text{one crisis in the } l\text{-th sector at } u^l_1) = \begin{cases} \mathrm{E}\left(e^{-\int_0^t \bar{\lambda}_i(s)ds}|\bar{\lambda}_i(0)\right)\left(1 - Z^l(t-u^l_1)\right), & \text{if } S(i)=l; \\ \mathrm{E}\left(e^{-\int_0^t \bar{\lambda}_i(s)ds}|\bar{\lambda}_i(0)\right), & \text{otherwise,} \end{cases} \tag{5.111}$$

for all $i = 1, ..., N$ and the recursive algorithm. Recall that $S(i) = l$ means that firm i is in the l-th sector. According to Eq.(5.98), the 'size' of the first order term is

$$|\phi_1(t)| = \sum_{\sum m_l=1} \left(\prod_{l=0}^{L} \frac{e^{-\Lambda^l(t)}(\Lambda^l(t))^{m_l}}{m_l!} \right) \tag{5.112}$$

$$= \prod_{l=0}^{L} e^{-\Lambda^l(t)} \sum_{k=0}^{L} \Lambda^k(t) \tag{5.113}$$

$$= e^{-\Lambda^\Sigma(t)} \Lambda^\Sigma(t). \tag{5.114}$$

It is easy to see that $|\phi_1(t)|$ starts from zero, keeps increasing initially, then attains its maximum and decreases to zero over time. Let's calculate the typical size of $|\phi_1(t)|$ by using the estimates as in the zeroth order case:

$$|\phi_1(5)| = e^{-\frac{5}{763}-\frac{5}{249}\times 6}\left(\frac{5}{763} + \frac{5}{249}\times 6\right) \approx 0.1119. \tag{5.115}$$

Thus the loss distribution contributed by the first two terms of the series expansion in Eq.(5.94) accounts for more than 99% of the total probability. Consequently, the first order approximation for the unconditional loss distribution covers almost all the possible scenarios for $t \leq 5$ and is a fairly accurate approximation for the full loss distribution. If the maturity of the CDO is 10 years instead, the probability covered by the first order approximation drops to 97%. This is still fairly close to unity. It is worth noting that the higher order terms we are dropping correspond to conditional loss distributions that are centered at higher losses which only have significant impact on the prices of senior tranches that are only affected when a significant number of firms default. This observation gives us an insight for how to put the missing probability back into the loss distribution. For example, we can add the unaccounted probability to the first order term such that the updated unconditional loss distribution is

$$\tilde{\phi}_1(t;n) = \left(\frac{1 - |\phi_0(t)|}{|\phi_1(t)|}\right) \phi_1(t;n), \tag{5.116}$$

and approximate the full loss distribution as

$$\mathrm{P}(D(t) = n) \approx \phi_0(t;n) + \tilde{\phi}_1(t;n). \tag{5.117}$$

Finally, if we want to maintain the accuracy of the loss distribution as well as the coverage of scenarios, say over 99%, then evaluations of higher order terms are needed. This is the subject of next subsection.

5.3.6 Higher Order Approximations

The loss distribution of the higher order terms are given by Eq.(5.93) and can be computed directly. However, there are two difficulties in evaluating higher order terms of the loss distribution.

Firstly, suppose we want to cover more than $Q\%$ of the scenarios, how many terms do we need? The answer to this question is indeed quite simple, we just need to find a smallest integer K such that

$$\sum_{k=0}^{K} |\phi_k(t)| \geq Q\%. \tag{5.118}$$

Note that each $|\phi_k(t)|$ depends on the time t as well as the crisis intensities $\Lambda^l(t)$. The real difficulty in determining the necessary number of terms comes from the evaluations of $|\phi_k(t)|$ for $k > 1$ since $|\phi_k(t)|$ is given by a complicated Eq.(5.98). However, according to the simple expressions of $|\phi_0(t)|$ and $|\phi_1(t)|$ presented in the previous subsection, one may guess that

$$|\phi_k(t)| = e^{-\Lambda^\Sigma(t)} \frac{\left(\Lambda^\Sigma(t)\right)^k}{k!}. \tag{5.119}$$

It turns out that this conjecture is correct. We first prove it by the multinomial theorem and then supply an intuitive argument which provides a better insight into the problem. According to Eq.(5.98),

$$|\phi_k(t)| = \sum_{\sum_{m_l=k}} \left(\prod_{l=0}^{L} \frac{e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l}}{m_l!} \right) \tag{5.120}$$

$$= e^{-\Lambda^\Sigma(t)} \sum_{\sum_{m_l=k}} \left(\prod_{l=0}^{L} \frac{\left(\Lambda^l(t)\right)^{m_l}}{m_l!} \right) \tag{5.121}$$

$$= \frac{e^{-\Lambda^\Sigma(t)}}{k!} \sum_{\sum_{m_l=k}} \left(k! \prod_{l=0}^{L} \frac{\left(\Lambda^l(t)\right)^{m_l}}{m_l!} \right) \tag{5.122}$$

$$= \frac{e^{-\Lambda^\Sigma(t)}}{k!} \left(\Lambda^0(t) + \Lambda^1(t) + ... + \Lambda^L(t)\right)^k \tag{5.123}$$

$$= \frac{e^{-\Lambda^\Sigma(t)}}{k!} \left(\Lambda^\Sigma(t)\right)^k. \tag{5.124}$$

The second argument is based on a property of Poisson processes. Since each crisis process $N^l(s)$ with intensity $\lambda^l(s)$ is a Poisson process, we can aggregate all the crisis processes to form a single process which is also a Poisson process with intensity

$$\lambda^\Sigma(s) = \sum_{l=0}^{L} \lambda^l(s). \tag{5.125}$$

Consequently, the probability that there are k jumps of the aggregated Poisson process by t is simply

$$|\phi_k(t)| = \mathrm{P}(\text{total number of crises occurs by } t = k) \tag{5.126}$$

$$= e^{-\Lambda^\Sigma(t)} \frac{\left(\Lambda^\Sigma(t)\right)^k}{k!}. \tag{5.127}$$

Having this simple expression for each $|\phi_k(t)|$ enables us to compute the errors $\epsilon_k(t)$ for different orders and maturities easily. In Table 1, we enumerate a list of errors $\epsilon_k(t)$ as a guideline in determining the necessary number of terms to compute the loss distribution of the CDX.NA.IG index. Although we compute the errors by assuming six industrial sectors in the portfolio, the figures stay the same even if the number of sectors changes. According to the empirical estimates from Longstaff and Rajan [3], the mean intensity of a global crisis is

$$\lambda^G = \lambda^0 = \frac{1}{763}, \tag{5.128}$$

and the mean intensity of a sectoral crisis is

$$\sum_{l=1}^{L} \lambda^l = \frac{1}{41.5}. \tag{5.129}$$

The mean intensity of a sectoral crisis is independent of the number of industrial sectors L. As a result, the overall mean crisis intensity is always

$$\lambda^\Sigma = \frac{1}{763} + \frac{1}{41.5}. \tag{5.130}$$

However, it is important to notice that the loss distribution of the portfolio depends both on the number of sectors and the number of firms in each sector.

Table 1: Errors of different order approximations to the loss distribution of the CDX.NA.IG index, where the crises intensities are estimated from the empirical study of Longstaff and Rajan [3].

t(year)	1	3	5	7	10
ϵ_0	0.0251	0.0734	0.1193	0.1629	0.2243
ϵ_1	0.0003	0.0028	0.0074	0.0141	0.0273
ϵ_2	3×10^{-6}	7×10^{-5}	0.0003	0.0008	0.0023
ϵ_3	2×10^{-8}	1×10^{-6}	1×10^{-5}	4×10^{-5}	0.0001

The second difficulty comes from the enumeration of the scenarios for each order of approximation. For the zeroth and the first order terms, it is easy to enumerate the scenarios since they have either zero or one crisis in the sectors. However, it is not straightforward to allocate k crises in $L+1$ sectors systematically. We present a simple algorithm which can be easily implemented in any programming language in Appendix C. Note also that for a given portfolio, we can generate all the possible scenarios for each order and then save them for future calculations. Thus the efficiency of the algorithm is not very important as we only need to generate the scenarios once.

In order to prevent the leak of probability over time due to the finite order approximation to the loss distribution, we can also include the uncounted probability to the highest order term in the calculation such that the updated unconditional loss distribution of the k-th order term is

$$\tilde{\phi}_k(t;n) = \left(\frac{1 - \sum_{j=0}^{k-1} |\phi_j(t)|}{|\phi_k(t)|} \right) \phi_k(t;n), \tag{5.131}$$

and approximate the full loss distribution as

$$\mathrm{P}(D(t) = n) \approx \sum_{j=0}^{k-1} \phi_j(t;n) + \tilde{\phi}_k(t;n). \tag{5.132}$$

Hence, the total probability of the loss distribution is unity for all t. Finally, the k-dimensional integrals of $\phi_k(t;n)$ are computationally expensive if k is not small. Note that all the integrals involve an integration of the unconditional loss distribution $\mathrm{P}(D(t) = n|\omega^u)$, which does not have a closed form expression. As as result, the crisis model is efficient only when the maturity of the contract is not too long or the crisis intensities are small so the first few order terms are enough to give an accurate loss distribution of the portfolio. For longer maturity or larger crisis intensities, we can use the stress event model as an approximation for the the crisis model and get rid of the computational expensive integrals appearing in the calculations of the loss distribution.

5.3.7 Loss Distribution of the Stress Event Model

We investigate the particular case that $Z^l(\cdot) = p^l$ for all l, which is just the stress event model. In addition, since there are two common factors in the default intensity for each firm, this particular model is a bottom-up approach version of the three factor model of Longstaff and Rajan [3]. According to Eq.(5.72) and the discussion thereafter, the conditional loss distribution $\mathrm{P}(D(t) = n|\omega^u)$ is independent of the occurring times of the crises. Consequently, the loss distribution for each order is

$$\phi_k(t;n) = \sum_{\sum m_l = k} \left(\int_{(0,t]^{m_0 \times \cdots \times m_L}} \mathrm{P}(D(t) = n|\omega^u) \prod_{l=0}^{L} \prod_{j_{m_l}=1}^{m_l} \frac{e^{-\Lambda^l(t)}}{m_l!} \lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}} \right) \tag{5.133}$$

$$= \sum_{\sum m_l = k} \left(\mathrm{P}(D(t) = n|\omega^u) \int_{(0,t]^{m_0 \times \cdots \times m_L}} \prod_{l=0}^{L} \prod_{j_{m_l}=1}^{m_l} \frac{e^{-\Lambda^l(t)}}{m_l!} \lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}} \right) \tag{5.134}$$

$$= \sum_{\sum m_l = k} \left(\mathrm{P}(D(t) = n|\omega^u) \prod_{l=0}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \left(\Lambda^l(t)\right)^{m_l} \right). \tag{5.135}$$

This is a tractable expression which can be computed efficiently since no numerical integration is needed. In fact, we earn much more than saving the computational time of the numerical integration. This tractable expression for the loss distribution retains its simplicity even when stochastic crisis intensities are considered. By taking the expectation over all λ^l, it is easy to see that

$$\phi_k(t;n) = \sum_{\sum m_l = k} \left(\mathrm{E}\left[\mathrm{P}(D(t) = n|\omega^u) \prod_{l=0}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^0(0), ..., \lambda^L(0) \right] \right) \tag{5.136}$$

$$= \sum_{\sum m_l = k} \left(\mathrm{P}(D(t) = n|\omega^u) \prod_{l=0}^{L} \frac{1}{m!} \mathrm{E}\left[e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \right). \tag{5.137}$$

The conditional loss distribution $\mathrm{P}(D(t) = n|\omega^u)$ is independent of the crisis intensities λ^l and can be computed by the conditional survival probabilities Eq.(5.72). The expectation

$$\mathrm{E}\left[e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \tag{5.138}$$

admits a closed form expression for a wide class of stochastic processes. We provide in Appendix D an explicit expression of Eq.(5.138) when λ^l is an affine-jump diffusion process.

Finally, let's look at the number of conditional loss distributions in each order of the approximation. The computational time for the full loss distribution $\phi(t;n)$ for each time t is directly proportional to the total number of independent conditions. Recall that an independent condition is a possible distribution of crises in the sectors. In the zeroth order term $\phi_0(t;n)$, there is only one independent condition which is the scenario that no crisis happens at all. In the first order term $\phi_1(t;n)$, there are $L+1$ independent conditions. These correspond to one crisis happening in the l-th sector and none in the other for $l = 0, ..., L$. In general, there are ${}_{L+k}C_k$ independent conditions in the k-th order term $\phi_k(t;n)$. The derivation of the number of independent conditions in each order is given in Appendix C. As an illustrative example, consider a portfolio composed of six sectors. The total number of independent conditions is

$$
\begin{aligned}
{}_{6+0}C_0 + {}_{6+1}C_1 + {}_{6+2}C_2 + {}_{6+3}C_3 &= 1 + 7 + 28 + 84 && (5.139)\\
&= 120 && (5.140)
\end{aligned}
$$

in a third order approximation. This number of conditions are quite small for a seven-sector (or equivalently seven-factor) model. The typical number of independence conditions needed in a one-factor Gaussian copula is about 50 and the number increases exponentially with the increase in the number of factors. Thus, the speed of new our model is very competitive.

5.3.8 Structure of the Loss Distribution for the Stress Event Model

Under each scenario, ω^u, the conditional loss distribution is a multi-nomial distribution with individual default probability $q_i(\omega^u, t) := (1 - \mathrm{P}(\tau_i(\omega^u) > t)$, where the conditional survival probability is given by Eq.(5.72). It is important to notice that $\sum_{i=1}^{N} q_i(\omega^u, t)/N$ is usually not close to 0 or 1. Hence the conditional loss distributions can be approximated by a normal distribution as suggested by Shelton [58] in which the first two moments of the conditional loss distribution are fitted. To fit the exact distribution, choose the mean and variance of

the normal distribution as follows:

$$\mu(\omega^u, t) = \sum_{i=1}^{N} q_i(\omega^u, t), \tag{5.141}$$

$$\sigma^2(\omega^u, t) = \sum_{i=1}^{N} q_i(\omega^u, t)\left(1 - q_i(\omega^u, t)\right). \tag{5.142}$$

In fact, except for the scenario that there is no crisis at all (since $q_i(\omega^u, 0) = 0$), each conditional loss distribution can be well approximated by a normal distribution. The Gaussian approximation for the conditional loss distribution not only provides an efficient scheme to compute the loss distribution, but also delineates the evolution of the loss distribution in terms of Gaussian packets. Basically, the loss distribution is a weighted sum of Gaussian packets. Since each $q_i(\omega^u, t)$ increases with t, the Gaussian packets move to the tail of the loss distribution. Furthermore, the probability for each scenario

$$\mathrm{P}\left(\bigcap_{l=0}^{L}\{m_l \text{ crises in the } l\text{-th sector}\}\right) = \mathrm{E}\left[\prod_{l=0}^{L}\frac{e^{-\Lambda^l(t)}(\Lambda^l(t))^{m_l}}{m_l!}\Big|\lambda^0(0), ..., \lambda^L(0)\right] \tag{5.143}$$

$$= \prod_{l=0}^{L}\frac{1}{m_l!}\mathrm{E}\left[e^{-\Lambda^l(t)}(\Lambda^l(t))^{m_l}|\lambda^l(0)\right] \tag{5.144}$$

quantifies the size of each Gaussian packet.

5.4 CALIBRATION TO SINGLE MATURITY

We perform calibrations to market data by assuming constant intensities for the idiosyncratic processes and crisis processes. We will relax these assumptions and carry out the calibrations for stochastic intensities in the next subsection. To put the crisis model in practice, we need to identify the sectors in a portfolio as well as the number of firms in each sector. According to the documentation from Markit, a global financial information services company, the distribution of firms among different industrial groups for CDX.NA.IG index and iTraxx.EUR index are illustrated in Table 2. In order to reduce the number of free parameters in the crisis model, we impose the following conditions on the parameters:

Table 2: Compositions of CDX.NA.IG index and iTraxx.EUR index.

Industrial Groups	CDX.NA.IG	iTraxx.EUR
Autos	0	10
Consumers	37	30
Energy	14	20
Industrials	28	20
Technology, Media and Telecommunications	22	20
Financials	24	25

- $\bar{\lambda} = \bar{\lambda}_i$ for all i
- $\lambda^S = \lambda^l$ for $l = 1, .., L$
- $p^S = p_i^l$ for all i, where p_i^l is the probability of default triggered by a crisis in the l-th sector
- $p^G = p_i^G$ for all i, where p_i^G is the probability of default triggered by a global crisis with intensity λ^G

In addition, a constant recovery rate $R = 35\%$ is used to be consistent with Mortensen [4]. We also assume a constant risk-free interest rate, which is taken as the 12-month LIBOR rate, for each calibration. Furthermore, we employ a special case of the crisis model, the stress event model, in the calibrations. Consequently, the reduced set of five parameters is

$$\Theta = \{\bar{\lambda}, \lambda^S, \lambda^G, p^S, p^G\}. \tag{5.145}$$

Define

$$\mathrm{RMSE}_{\mathrm{tr}} = \sqrt{\frac{1}{5}\sum_{j=1}^{5}\left(\frac{\tilde{S}_{\mathrm{tr,j}} - S_{\mathrm{tr,j}}}{S_{\mathrm{tr,j}}}\right)^2} \tag{5.146}$$

as a relative root mean square error, where $S_{\mathrm{tr,j}}$ and $\tilde{S}_{\mathrm{tr,j}}$ are the market mid price and model implied price of a tranche. We apply the first order approximation as described in

Table 3: Market prices of 5-year CDS index tranches and 5-year CDS spreads, extracted from Mortensen [4].

	iTraxx.EUR			CDX.NA.IG		
Tranches	$\underline{K} - \overline{K}$	Aug 23, 04	Dec 5, 05	$\underline{K} - \overline{K}$	Aug 23, 04	Dec 5, 05
Equity	0-3%	25.5%	26.3%	0-3%	40.0%	41.1%
Junior Mezzanine	3-6%	146.0	80.6	3-7%	312.5	117.5
Senior Mezzanine	6-9%	60.3	23.1	7-10%	122.5	32.9
Senior	9-12%	36.3	10.3	10-15%	42.5	15.8
Super Senior	12-22%	19.3	5.8	15-30%	12.5	7.9
Mean CDS spread		39	37		67	51
Median CDS spread		36	29		48	35

subsection 5.3.5 to calibrate the parameter set Θ by minimizing the RMSE_{tr}. Four different sets of market tranche prices are used for calibrations. The tranche prices as well as the mean and median CDS spreads are illustrated in Table 3. It turns out that the model implied tranche prices are virtually exact for all four sets of data as illustrated in Table 4. Furthermore, the model implied median and mean CDS spread S_{implied} (shown in Table 4) is between the market median and market mean for each of the four data set. What is striking is that the implied CDS spread S_{implied} in the first data sets, iTraxx.EUR on August 23 2004, is only lower than the market mean by 0.5 bps and is higher than the market median by 2.5 bps. It is worth noting that the current simplified version of the stress event model which implicitly assumes uniform CDS spreads for all credits in a portfolio has no way to match both the median and mean of the asymmetric CDS spreads exactly. In fact, the deviation from the average CDS spread can be removed if an extra global intensity is added in the model.

In addition, we also perform calibrations to the same set of market data with the crisis model where the delayed time distributions are the same for all firms and sector such that

$$Z_i^l(s) = Z(s) = p\left(1 - e^{-\xi s}\right), \tag{5.147}$$

where p is the probability of default triggered by a crisis and $1/\xi$ is the mean delayed time. We set $\xi = 2$, which is equivalent to a mean delayed time of half a year, in the calibration. It turns out that the implied tranche prices also match the market data perfectly, the calibrated parameters being slightly different from the stress event model.

Table 4: Calibrated parameters using first order approximation and the implied CDS spreads.

	iTraxx.EUR		CDX.NA.IG	
	Aug 23, 04	Dec 5, 05	Aug 23, 04	Dec 5, 05
$\bar{\lambda}$	0.0038554	0.0043466	0.0049829	0.0059891
λ^S	0.0026856	0.0015065	0.0074953	0.0017373
λ^G	0.0038409	0.0010142	0.0041731	0.0015166
p^S	0.40329	0.38300	0.29776	0.33736
p^G	0.25574	0.26200	0.43690	0.38518
$\epsilon_1(5)$	0.0047	0.0012	0.0189	0.0013
$\mathrm{RMSE}_{\mathrm{tr}}$	6.19×10^{-5}	8.73×10^{-5}	7.64×10^{-5}	6.37×10^{-5}
S_{implied}	38.49	33.73	58.75	46.54

5.5 CALIBRATION TO THE TERM STRUCTURE OF ITRAXX.EUR TRANCHES ON MULTIPLE DAYS

The data set that we are using for calibration in this section is obtained from the monthly Markit iTraxx Tranche Fixings (see www.creditfixings.com). They consists of four days of market data observed on March 30, April 30, May 31 and June 29 in 2007. On each day,

there are five standard tranches with maturities 5, 7 and 10 years. There are altogether 60 data point and they are shown in Table 5. The model parameters are then calibrated to this set by minimizing the root mean square of the relative errors, i.e.

$$\text{RMSE} = \sqrt{\frac{1}{60}\sum_{l=1}^{4}\sum_{k=1}^{3}\sum_{j=1}^{5}\left(\frac{\tilde{S}_{tr,j}^{T_k,t_l} - S_{tr,j}^{T_k,t_l}}{S_{tr,j}^{T_k,t_l}}\right)^2}, \tag{5.148}$$

where $T_1 = 5$, $T_2 = 7$ and $T_3 = 10$ are the maturities, t_l is the index for the observing date and j is the index for the tranche. This larger data set enables us to investigate the finer structure of the model by considering more dynamic intensity processes. A special case of the stochastic affine jump-diffusion process for each intensity with the mean reverting level $\theta = 0$, i.e.

$$d\lambda_t = -\kappa\lambda_t dt + \sigma\sqrt{\lambda_t}dB_t + dJ_t \quad \lambda_t = \lambda_0, \tag{5.149}$$

is used (See Appendix D for details about affine jump-diffusion process). Recall that $\Lambda(t) = \int_0^t \lambda_s ds$ and the probability that a scenario that has k crises in a sector is

$$\mathrm{E}\left[e^{-\Lambda(t)}\frac{(\Lambda^k(t))^k}{k!}\right], \tag{5.150}$$

which admits a closed form expression if λ_t is an affine jump-diffusion process. We report an closed form expression of Eq.(5.150) for $k = 0$ and derive an expression for any positive integer k in Appendix D. We then present techniques for computing Eq.(5.150) for general k in Appendix E.

Similar to the previous calibration example, we assume that every firm follows the same idiosyncratic intensity process and global intensity process. Besides, all sectoral intensity processes are also assumed to have the same parameters. Including the constant impact factors p^S and p^G, there are altogether 14 fixed parameters:

$$\Theta_{\text{fix}} = \{\bar{\kappa}, \bar{\sigma}, \bar{l}, \bar{\mu}, \kappa^S, \sigma^S, l^S, \mu^S, p^S, \kappa^G, \sigma^G, l^G, \mu^G, p^G\} \tag{5.151}$$

and 12 initial intensities corresponding to the three intensity processes on four different dates. Besides, a constant recovery rate 35% and a constant interest rate 5.35% are used in the calibration.

We perform a fifth order calculation for the loss distribution and the calibrated parameters and model implied tranche prices are presented in Table 6 and Table 7 respectively. The model implied tranche prices match fairly well with the market mid prices in general with the root mean square of relative errors RMSE = 5.76%. A closer look at the prices reveals that the model implied prices perform very well for the 5-year maturity tranches with the maximum relative error of 6.12%. For other tranches, the relative errors are less than 10% except for the first three dates of the 7-year senior tranches and the 10-year junior mezzanine tranches. In those exceptions, the performance is still relatively good, with the relative errors less than 16%.

For the calibrated parameters, we first compare the impact factors computed in the previous calibrations with those computed here as shown in Table 8. We find that the impact factors p^S and p^G appear to be quite stable over time especially the global impact factor p^G. On the other hand, we see that all the default intensities are explosive, i.e. the risk-neutral mean reverting rates $\bar{\kappa}$, κ^S and κ^G are negative. It appears that the negative mean reverting rates are necessary to give enough upward sloping of the default intensities when we are trying to match the term structure of tranche spreads. In the calibrations to the correlated intensity model, Eckner [27] also finds negative mean reverting rates of the default intensities. Besides, the calibrations to the Generalized-Poisson loss model performed by Brigo et al. [29] also indicate upward sloping of the default intensities. The upward sloping of default intensities may suggest that investors take a more pessimistic view about the future default intensities and expect an increase of default intensities over time. The volatility of the idiosyncratic intensity $\bar{\sigma}$ is significantly greater than those of sectoral and global crisis intensities. Jump rates of the intensities are quite similar, with a few per hundred years for each of them. Jump sizes of the intensities are moderate, ranging from 11 bps to 175 bps. These are significantly lower than the jump size found by Eckner [27] which is around 3000 bps. The jump size in the correlated intensity model needs to be high in order to give enough default correlation among firms, while jumps in our model have only minor effect on the correlation among firms.

5.6 CALIBRATION TO CDS INDEX TRANCHE AND THE UNDERLYING CDS SPREADS

We calibrate a three-factor stress event model to standard CDS index tranches and the underlying CDS spreads simultaneously in this section. We perform the calibrations to two different 5-year maturity investment grade CDS indexes, namely CDX.NA.IG series 13 and iTraxx.EUR series 13. Each data set contains the first five index tranche prices and the 125 underlying 5-year maturity CDS spreads on April 15 2010 which are obtained from Bloomberg terminal. The quotes of the index tranches are given in Table 9. We assume a constant recovery rate $R = 35\%$ which is consistent with empirical evidence for senior unsecured bonds reported by [59]. Furthermore, we assume a constant interest rate $r = 0.94\%$, which is the 12-month Libor rate, for cash flow discounting. For the model parameters, we assume the simplest time-independent intensities for all the Poisson processes. Thus, the default intensity for each firm i can be computed by the so-called credit triangle, i.e.

$$\lambda_i = \frac{S_i}{(1-R)}, \quad i = 1, ..., N, \tag{5.152}$$

where S_i is the 5-year CDS spread of firm i. Hence, $S_i/(1-R)$ imposes a constraint on other parameters of firm i as follows:

$$\frac{S_i}{(1-R)} = \lambda_i = \bar{\lambda}_i + p_i^1\lambda^1 + p_i^2\lambda^2 + p_i^3\lambda^3. \tag{5.153}$$

This model specification has $4N+3$ parameters and N constraints. We favor a parsimonious model which is flexible to match tranche spreads. Therefore, we choose a parameter set of six members

$$\Theta = \{\lambda^1, p^1, \lambda^2, p^2, \lambda^3, p^3\}, \tag{5.154}$$

for the calibration, where λ^l are the sectoral stress event intensities and p^l are representative impact factors. The detailed specifications of p_i^l and $\bar{\lambda}_i$ in terms of the parameters in Θ are provided in Appendix F. Table 9 reports the market tranche quotes on April 15 2010 for the two indexes, and Table 10 and Table 11 show the model implied tranche price and

the corresponding parameters respectively. The model implied CDX tranches are matched very well with all tranche prices within the bid-ask spreads. The iTraxx tranches are also matched quite well with all tranche prices within the bid-ask spread except the fifth tranche. It is interesting to notice that a further lowering of the recovery rate in these two data sets can improve the calibration results such that all data can be matched almost perfectly.

The results of our calibrations show that the multi-factor stress event model is flexible enough to match market tranche prices and the underlying CDS spreads really well.

5.7 CONCLUSION

Stress event model has been proposed for a long time. However, there was no efficient method to compute the loss distribution for the stress event model and the default events were usually generated by Monte Carlo simulation. We develop a new model, the crisis model, which is an extension of the stress event model. We then provide a methodology to compute the loss distribution of a portfolio for the crisis model. It turns out that our methodology, when applied to the stress event model, is remarkably efficient. We perform calibrations to market data and the results are very promising. In addition, the computational cost for additional common factors, unlike other bottom-up approaches, is mild. This allows an extra flexibility in our approach to match market data. As a result, our approach has provided a practical framework to study the complicated default dependence of a portfolio.

Table 5: Market Mid prices of Markit iTraxx.EUR Series 7 Version 1. 0-3% tranche quoted in percentage as an upfront with a fixed 500bps and all the other tranches are spreads in bps without upfront.

Maturity	$\underline{K} - \overline{K}$	Mar 30, 07	Apr 30, 07	May 31, 07	Jun 29, 07
5-year	0-3%	11.23%	9.94%	6.33%	11.75%
	3-6%	57.75	49.82	39.90	62.05
	6-9%	14.28	12.45	10.33	16.29
	9-12%	6.24	5.53	4.39	7.48
	12-22%	2.58	2.54	1.93	3.10
7-year	0-3%	25.77%	24.84%	20.61%	26.38%
	3-6%	133.79	121.2	105.08	137.13
	6-9%	37.25	31.99	27.04	37.39
	9-12%	17.33	15.75	13.05	17.00
	12-22%	5.85	5.67	5.20	7.50
10-year	0-3%	40.51%	38.95%	35.00%	40.53%
	3-6%	338.96	322.20	294.21	368.60
	6-9%	98.59	93.48	85.17	108.55
	9-12%	46.91	43.59	38.98	50.33
	12-22%	14.38	14.50	12.20	15.95

Table 6: Calibrated parameters using fifth order approximation.

$\overline{\kappa}$	$\overline{\sigma}$	$\overline{l}$	$\overline{\mu}$
-0.73336	0.29682	0.0595	0.00663

κ^S	σ^S	l^S	μ^S	p^S
-0.25776	0.06831	0.02110	0.01750	0.31663

κ^G	σ^G	l^G	μ^G	p^G
-0.55558	0.11698	0.05129	0.00113	0.23659

	Mar 30, 07	Apr 30, 07	May 31, 07	Jun 29, 07
$\overline{\lambda}_0$	0.00018115	0.00017337	0.00007624	0.00024907
λ_0^S	0.00018371	0.00007061	0.00000000	0.00028316
λ_0^G	0.00001711	0.00001783	0.00000000	0.00005790

Table 7: Model implied tranche prices of Markit iTraxx.EUR Series 7 Version 1.

Maturity	$\underline{K} - \overline{K}$	Mar 30, 07	Apr 30, 07	May 31, 07	Jun 29, 07
5-year	0-3%	11.28%	10.03%	6.33%	11.72%
	3-6%	58.31	49.86	41.51	61.27
	6-9%	14.35	12.68	10.32	15.43
	9-12%	6.07	5.74	4.66	7.22
	12-22%	2.56	2.46	1.92	3.19
7-year	0-3%	25.70%	24.45%	20.60%	26.52%
	3-6%	128.88	114.38	96.49	134.35
	6-9%	38.28	34.00	28.83	40.27
	9-12%	14.80	13.74	11.63	16.43
	12-22%	6.18	5.96	4.92	7.44
10-year	0-3%	39.38%	38.54%	35.70%	40.59%
	3-6%	299.18	277.98	247.3	310.57
	6-9%	106.62	95.98	85.04	111.54
	9-12%	46.51	42.24	37.27	48.76
	12-22%	14.68	14.01	12.51	16.33
$\epsilon_5(10)$		0.0080	0.0073	0.0060	0.0081

Table 8: Impact parameters of Markit iTraxx.EUR on different dates.

	Aug 23, 2004	Dec 5, 05	Mar-Jun 2007
p^S	0.40329	0.38300	0.31663
p^G	0.25574	0.26200	0.23659

Table 9: Tranche spreads of CDX.NA.IG series 13 and iTraxx.EUR series 13 on April 15 2010. The quoting conventions for CDX and iTraxx are different. For CDX.NA.IG series 13, all the quotes are upfronts in percentage with fixed 100bps running spread. For iTraxx.EUR series 13, tranche 0-3%, 3-6% and 6-9% are upfronts with fixed running spreads 500bps, 300bps and 100bps respectively, and tranche 9-12% and 12-22% are spreads in bps.

CDX	0-3%	$3-7\%$	$7-10\%$	$10-15\%$	$15-30\%$
Bid	51.530	16.000	4.888	-1.210	-3.100
Mid	52.185	16.605	5.345	-0.855	-2.880
Ask	52.840	17.210	5.810	-0.500	-2.660
iTraxx	0-3%	$3-6\%$	$6-9\%$	$9-12\%$	$12-22\%$
Bid	28.880	-3.710	-3.880	115.330	53.500
Mid	29.130	-3.460	-3.630	118.500	55.665
Ask	29.380	-3.210	-3.380	121.670	57.830

Table 10: Model implied tranche spreads CDX.NA.IG series 13 and iTraxx.EUR series 13 on April 15 2010 using fourth order calculations.

	0-3%	$3-7\%$	$7-10\%$	$10-15\%$	$15-30\%$
CDX	52.274	16.703	5.386	-0.836	-2.669
iTraxx	29.193	-3.416	-3.562	117.707	60.051

Table 11: Model parameters calibrated from tranche spreads of CDX.NA.IG series 13 and iTraxx.EUR series 13 on April 15 2010 using fourth order calculations.

	λ^1	p^1	λ^2	p^2	λ^3	p^3
CDX	0.0427828	0.0883129	0.0122258	0.122791	0.00391431	1.000000
iTraxx	0.0346416	0.0639359	0.0162614	0.129091	0.00523434	1.000000

APPENDIX A

CREDIT DEFAULT SWAPS PRICING

A credit default swap (CDS) is an insurance contract on the event of default of a reference entity between two counterparties. In the event of default before maturity of the contract, the protection seller pays the loss given default to the protection buyer. There are two possible settlements:

- The protection buyer delivers a defaulted asset to the protection seller for a payment of the face value. This is known as physical settlement.
- The protection seller pays the protection buyer the difference between the face value and the market price of underlying defaulted asset. This is known as cash settlement.

In exchange for the protection, the protection buyer makes a series of premium payments until default or maturity of the contract is reached.

For simplicity, assume the notional of the CDS is one. The market value of the payments in a T-year CDS made by the protection seller (commonly called the protection leg) is

$$\mathrm{Prot}(0,T) = \mathrm{E}\left(e^{-\int_0^\tau r_s ds}\mathbf{1}_{\{\tau\leq T\}}(1-R)\right), \tag{A.1}$$

where τ is the default time, r_s is the risk-free interest rate and R is the recovery rate. Under the independence assumptions between the recovery rate, the interest rates and the default indicator, the value of the protection leg is approximated by the following discretization,

$$\mathrm{Prot}(0,T) \approx (1-R)\sum_{j=1}^{M} B\left(0, \frac{t_j+t_{j-1}}{2}\right)\left(\mathbb{Q}(\tau > t_{j-1}) - \mathbb{Q}(\tau > t_j)\right), \tag{A.2}$$

where $B(0,t)$ is the current price of a riskless zero-coupon bound maturing at time t with payoff one. Premium payments are made conditional on survival of the reference entity, and in the event of default, an accrual premium payment is made for the period since the previous payment date. Suppose the CDS contract specifies that the annual premium S is paid in arrears at $t_1, t_2, ..., t_M = T$. The market value of the payments made by the protection buyer (commonly called the premium leg or fixed leg) is

$$\text{Prem}(0,T;S) = \text{E}\left(\sum_{j=1}^{M} e^{-\int_0^{t_j} r_s ds}\mathbf{1}_{\{\tau>t_j\}}(t_j - t_{j-1})S + e^{-\int_0^{\tau} r_s ds}\mathbf{1}_{\{t_{j-1}<\tau\leq t_j\}}(\tau - t_{j-1})S\right). \tag{A.3}$$

Using the approximation that defaults occur half-way between premium payment dates, the premium leg can be approximated as

$$\begin{aligned}\text{Prem}(0,T;S) \approx & S\sum_{j=1}^{M}(t_j - t_{j-t})B(0,t_j)\mathbb{Q}(\tau > t_j) \\ & + \frac{t_j - t_{j-1}}{2}B\left(0, \frac{t_j + t_{j-1}}{2}\right)\left(\mathbb{Q}(\tau > t_{j-1}) - \mathbb{Q}(\tau > t_j)\right).\end{aligned} \tag{A.4}$$

The fair CDS premium S is obtained by setting the protection leg equal to the premium leg, hence

$$S = \frac{(1-R)\sum_{j=1}^{M} B\left(0, \frac{t_j + t_{j-1}}{2}\right)\left(\mathbb{Q}(\tau > t_{j-1}) - \mathbb{Q}(\tau > t_j)\right)}{\sum_{j=1}^{M}(t_j - t_{j-t})B(0,t_j)\mathbb{Q}(\tau > t_j) + \frac{t_j - t_{j-1}}{2}B\left(0, \frac{t_j + t_{j-1}}{2}\right)\left(\mathbb{Q}(\tau > t_{j-1}) - \mathbb{Q}(\tau > t_j)\right)}. \tag{A.5}$$

APPENDIX B

CDS INDEX TRANCHE PRICING

CDS index tranches are synthetic collateralised debt obligations (CDOs) based on a CDS index, where each tranche references a different segment of the loss distribution of the underlying CDS index. The main advantage of index tranche relative to other CDOs is that they are standardized. Standardization applies to both the composition of the reference portfolio and the structure of the the tranches. Standardized tranches have been issued on several indices. The most trading to date have been concentrated in the CDX.NA.IG index and iTraxx.EUR. Each tranche is characterized by the following two quantities:

$\underline{K}_i$: This is the attachment point, also known as the lower strike of the tranche. The is the percentage loss on the reference portfolio below which the tranche loss is zero. Once the percentage portfolio loss L_t is over $\underline{K}_i$, the tranche experiences loss.

$\overline{K}_i$: This is the detachment point, also know as the upper strike. If $L_t \geq \overline{K}_i$, the tranche loss is 100%. The quantity $\overline{K}_i - \underline{K}_i$ is the tranche width.

The characteristics of the most liquid standard tranches are shown in Table 12. Between these two limits, the tranche loss is a linear function of L_t. If we define the fractional loss of the tranche indexed by i at time t as L_t^i, then

$$L_t^i = \frac{(L_t - \underline{K}_i)^+ - (L_t - \overline{K}_i)^+}{\overline{K}_i - \underline{K}_i}. \tag{B.1}$$

It is worth noting that the fractional tranche loss L_t^i is a deterministic function of L_t only.

Table 12: The attachment and detachment points of the CDX.NA.IG and iTraxx.EUR traches

		CDX.IG.NA		iTraxx.EUR	
i	Tranches	$\underline{K}_i(\%)$	$\overline{K}_i(\%)$	$\underline{K}_i(\%)$	$\overline{K}_i(\%)$
1	Equity	0	3	0	3
2	Junior mezzanine	3	7	3	6
3	Senior mezzanine	7	10	6	9
4	Senior	10	15	9	12
5	Super senior	15	30	12	22

The cash flows of an index tranche is similar to that of CDS. The market value of the protection leg is

$$\mathrm{Prot}^i(t,T) = \mathrm{E}\left(\int_t^T e^{-\int_t^s r_u du} dL_s^i | \mathcal{F}_t\right). \tag{B.2}$$

Under the independence assumptions between the interest rates and the loss process L_s, the protection leg can be approximated by the following discretization

$$\mathrm{Prot}^i(t,T) \approx \sum_{\{j:t_j>t\}}^{M} B\left(t, \frac{t_j + \max(t_{j-1},t)}{2}\right)\left(\mathrm{E}(L_{t_j}^i|\mathcal{F}_t) - \mathrm{E}(L_{\max(t_{j-1},t)}^i|\mathcal{F}_t)\right), \tag{B.3}$$

where the set $\{t_j\}$ is usually the premium payment dates with $t_0 = 0$ and $t_M = T$. On the other hand, the market value of the premium leg is

$$\mathrm{Prem}^i(t,T;S^i,U) = U^i + \mathrm{E}\left(\sum_{\{j:t_j>t\}}^{M} e^{-\int_t^{t_j}}(t_j - \max(t_{j-1},t))S^i \int_{\max(t_{j-1},t)}^{t_j} \frac{1-L_s^i}{t_j - \max(t_{j-1},t)} ds \Big| \mathcal{F}_t\right), \tag{B.4}$$

where U^i is an up-front payment and S^i is an annual premium. The integral represents the average remaining principal over the interval $\max(t_{j-1},t)$ to t_j. If we assume that the

payments are made on the notional remaining at each payment date t_j, in addition to the assumptions and discretization as above, the premium leg can be approximated by

$$\mathrm{Prem}^i(t,T;S^i,U) \approx U^i + S^i \mathrm{E}\left(\sum_{\{j:t_j>t\}}^{M} B(t,t_j)(t_j - \max(t_{j-1},t))\left(1 - \mathrm{E}(L^i_{t_j}|\mathcal{F}_t)\right)\right). \quad \text{(B.5)}$$

The fair tranche price can be computed by setting the protection leg equal to the premium leg. It is worth noting that there are two different kinds of payments in the premium leg, viz. U^i and S^i. Before the global financial crisis, the equity tranche price was quoted as the upfront payment U^1 plus a fixed running premium of 500 basis points (bps) and other tranche prices were quoted as the premium S^i with zero up-front payment. Recently, the quoting convention has changed and an upfront payment is also required even for more senior tranches.

It is worth noting that in the evaluation of the fair price of a tranche the key quantity that needs to be computed is $\mathrm{E}(L^i_t)$, where L^i_t is given by Eq.(B.1). Consider

$$\mathrm{E}((L_t - K)^+) = \int_0^1 (l-K)^+ f_t(l)dl, \quad \text{(B.6)}$$

where $f_t(l)$ is the density distribution of the percentage portfolio loss at time t. If we define

$$F_t(l) = \mathbb{Q}(L_t > l), \quad \text{(B.7)}$$

which is the probability the percentage loss exceeds l, then

$$\mathrm{E}((L_t - K)^+) = -\int_K^1 (l-K)dF_t(l) \quad \text{(B.8)}$$

$$= \int_K^1 F_t(l)dl. \quad \text{(B.9)}$$

Consequently, the expectation of the tranche loss is

$$\mathrm{E}(L^i_t) = \frac{1}{\overline{K}_i - \underline{K}_i}\int_{\underline{K}_i}^{\overline{K}_i} F_t(l)dl. \quad \text{(B.10)}$$

This explicitly indicates that the price of the i-th tranche does not depend on the distribution of the percentage loss for $L_t > \overline{K}_i$. In other words, once $F_t(l)$ is fixed in $[0, \overline{K}_i]$ the distribution of $F_t(l)$ beyond $\overline{K}_i$ has no effect on the tranche price.

APPENDIX C

AN ALGORITHM TO ALLOCATE k CRISES IN $L+1$ SECTORS

C.1 THE NUMBER OF WAYS OF ALLOCATION

Before presenting the algorithm, we first prove that the number of different ways of allocating k crises in $L+1$ sectors is ${}_{L+k}C_k$. Denote $\circ$ as a crisis and $|$ as a wall dividing two sectors. Therefore, there are L walls and k crises for each scenario. For example, naming the sectors from right to left starting from the zeroth sector.

$$\circ\,|\,\circ\,\circ\,||\,\circ\,\circ\,\circ\,|\,\circ\,|| \tag{C.1}$$

represents a scenario for which there are no crises in the zeroth, first and fourth sectors, one crisis in both the second and sixth sectors, two crises in the fifth sector and three crises in the third sector. Note that there are altogether $L+k$ objects in each representation. With these notations, the number of ways of having k crises in $L+1$ sectors is equivalent to the number of ways of choosing k objects (the crises) from $L+k$ objects which is ${}_{L+k}C_k$. This number can be used to verify the number of ways generated by any algorithm which allocates k crises in $L+1$ sectors.

C.2 THE ALGORITHM

We present a simple algorithm which is very easily implemented to allocate k crises in $L+1$ sectors. For each fixed k, we want to enumerate all vectors $\vec{m}_L = (m_0, m_1, ..., m_L)$ such that $\sum_{l=0}^{L} m_l = k$. First, we encapsulate $\vec{m}_L$ as a $(k+1)$-based number of $L+1$ digits, i.e.

$$m_L m_{L-1} ... m_1 m_0 = \sum_{l=0}^{L} m_l (k+1)^l. \tag{C.2}$$

Then we start $m_L m_{L-1} ... m_1 m_0$ at $00...0k$ with a step size k until $k0...00$. This enumeration includes all scenarios that have k crises in the $L+1$ sectors. For each $m_L m_{L-1} ... m_1 m_0$ in the enumeration, we check if the sum of the $L+1$ digits equals k. We illustrate this procedure in the following generic codes:

```
i=1
for j = k to k*(k+1)^L with increment k
    if (sum_of_all_digits(j)=k) then
        s[i]=j and i=i+1
    endif
end
```

After running these codes, the array `s` contains ${}_{L+k}C_k$ numbers and each of them represents a scenario that has k crises in the $L+1$ sectors. We can decode each `s[i]` by looking at its $(k+1)$-based representation. The number of crises in each sector can then be obtained directly from the corresponding digit in the $(k+1)$-based representation. Note that this algorithm is not efficient, especially for large k and L. If k and L are both large, we could find a more efficient method by following the line of thought in the derivation of the total number of combination ${}_{L+k}C_k$ from the previous section. However, for k and L that are not too large, this algorithm takes negligible time compared to other computations in the calculation of the loss distribution.

APPENDIX D

BASIC AFFINE JUMP DIFFUSIONS

A stochastic process λ_t on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathbb{Q})$ is called a basic Affine Jump Diffusions (AJD) if it satisfies the following SDE:

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t + dJ_t, \tag{D.1}$$

where B is a standard Brownian motion, and J is an independent compound Poisson process with jump intensity l and exponentially distributed jumps with mean μ. Duffie and Singleton [50] show that the moment generating function of the cumulative intensity $\Lambda(t) = \int_0^t \lambda_s ds$ admits a closed form solution

$$\mathrm{E}(e^{q\Lambda(t)}|\lambda_0) = e^{\alpha(t)+\beta(t)\lambda_0}, \tag{D.2}$$

where

$$\alpha(t) = -\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{c_1 + d_1 e^{-\gamma t}}{c_1 + d_1}\right) + \frac{\kappa\theta t}{c_1} \tag{D.3}$$

$$+ l\left(\frac{d_1/c_1 - d_2/c_2}{-\gamma d_2}\right)\log\left(\frac{c_2 + d_2 e^{-\gamma t}}{c_2 + d_2}\right) + \frac{l(1-c_2)t}{c_2} \tag{D.4}$$

$$\beta(t) = \frac{1 - e^{-\gamma t}}{c_1 + d_1 e^{-\gamma t}}, \tag{D.5}$$

and

$$\gamma = \sqrt{\kappa^2 - 2\sigma^2 q} \tag{D.6}$$

$$c_1 = (\kappa + \gamma)/(2q) \tag{D.7}$$

$$c_2 = 1 - \mu/c_1 \tag{D.8}$$

$$d_1 = (-\kappa + \gamma)/(2q) \tag{D.9}$$

$$d_2 = (d_1 + \mu)/c_1. \tag{D.10}$$

With the help of the closed form expression of the moment generating function, we can compute the expectation

$$\mathrm{E}\left(e^{-\Lambda(t)}|\lambda_0\right), \tag{D.11}$$

which is the form of the probability for which there is no crisis by time t, by plugging $q = -1$ in Eq.(D.2)-Eq.(D.10). Longstaff and Rajan [3] derive a recursive system of ordinary differential equation to compute

$$\mathrm{E}\left(e^{-\Lambda(t)}(\Lambda(t))^k|\lambda_0\right). \tag{D.12}$$

Their approach, although works, is not very appealing since it is quite time-consuming to solve the system of ODEs numerically. Besides, it is hard to control the error propagation in the recursive ODE.

In fact, Eq.(D.12) can be computed easily by differentiating Eq.(D.2) with respect to q k times, then

$$\frac{d^k}{dq^k}\left(e^{\alpha(t)+\beta(t)\lambda_0}\right) = \mathrm{E}\left(e^{q\Lambda(t)}(\Lambda(t))^k|\lambda_0\right). \tag{D.13}$$

Plugging $q = -1$ and dividing by $k!$ yields the probability that there are k crises in the sector, i.e.

$$\mathbb{Q}(\text{k crises by time } t) = \frac{1}{k!}\frac{d^k}{dq^k}\left(e^{\alpha(t)+\beta(t)\lambda_0}\right)\Big|_{q=-1}. \tag{D.14}$$

The validity of exchanging the order of differentiation and expectation in Eq.(D) can be verified if $\Lambda(t) \geq 0$ for all t, which is true in our consideration here since we are viewing $\Lambda(t)$

as the cumulative intensity which is always non-negative. As a result, in order to compute the scenario probability, $\mathbb{Q}$(k crises by time t), we just need to find the k-th derivative of the moment generating function Eq.(D.2) at $q = -1$.

There are a few available ways in computing the derivatives. Firstly, although the closed form expression of the moment generating function Eq.(D.2) is quit complicated, we can find the derivatives by using some symbolic programs like Mathematica or Maple to compute the derivatives explicitly. This method of course works, but the code size could be very large even for low order derivatives. The large size of the code undermines the computational efficiency and also means that it is difficult to transport the code to other programming language such as C++ or MATLAB which is usually the programming language of the main program for computing the loss distribution. Secondly, we can apply numerical differentiation, e.g. five-point stencil, to find the derivatives. However, numerical differentiation usually can give accurate results for the first few derivatives only. In some cases, we need the fifth or higher order derivative in the calculation, but results from numerical differentiations are not very reliable for these high order derivatives. The third way is to compute the derivatives by using Automatic Differentiation (AD), which is sometimes alternatively called algorithmic differentiation. However, AD is developed with the ability to compute mixed derivatives of a multi-variable function and is not very efficient in computing high order derivatives of a single-variable function. Besides, AD can only be implemented in Object-Oriented language like C++, while our main program is written in MATLAB which is a procedural programming language.

Since the available methods for computing the derivatives are either not efficient or accurate, we develop a simple and efficient numerical algorithm which can be implemented easily in any procedural language. Furthermore, derivatives computed by this algorithm are exact (up to the rounding error). The algorithm of this exact numerical differentiation is presented in Appendix E.

APPENDIX E

EXACT NUMERICAL ALGORITHM FOR N-TH ORDER DERIVATIVE OF A SINGLE VARIABLE FUNCTION

For any real valued function $f(q) \in \mathbb{C}^N(q)$, define

$$\mathbf{f}(q) = (f^{(0)}(q), f^{(1)}(q), ..., f^{(N)}(q)) \tag{E.1}$$

as the $(N+1)$ dimensional vector where $f^{(k)}(q)$ is the k-th derivative of f at q. We call $\mathfrak{D}^{N+1}(q)$ the set containing all $\mathbf{f}(q)$. For the sake of notational simplicity, we will drop the dependence of q for the sequel of the discussion. Note that $\mathbf{f}$ stores the values of f and its derivatives at q. We would like to have elementary operations of elements in $\mathfrak{D}^{N+1}$ to give not only the right functional values but also the correct values of derivatives. For any $f, g \in \mathfrak{D}^{N+1}$, the elementary operations on them are defined as follows:

- Multiplication

$$\mathbf{h} = \mathbf{f}\,\mathbf{g} \tag{E.2}$$

The n-th derivative of h is simply given by the Leibniz rule

$$h^{(n)} = (fg)^{(n)} = \sum_{k=0}^{n} {}_nC_k f^{(k)} g^{(n-k)}. \tag{E.3}$$

- Division

$$\mathbf{h} = \mathbf{f}/\mathbf{g} \tag{E.4}$$

Firstly, define $h^{(0)} = f^{(0)}/g^{(0)}$. Applying Leibniz rule to $hg = f$ yields

$$\sum_{k=0}^{n} {}_nC_k h^{(k)} g^{(n-k)} = f^{(n)} \tag{E.5}$$

$$h^{(n)} g^{(0)} + \sum_{k=0}^{n-1} {}_nC_k h^{(k)} g^{(n-k)} = f^{(n)} \tag{E.6}$$

$$h^{(n)} = \frac{f^{(n)} - \sum_{k=0}^{n-1} {}_nC_k h^{(k)} g^{(n-k)}}{g^{(0)}}. \tag{E.7}$$

Thus, defining $h^{(n)}$ by Eq. (E.7) for $n = 1, ..N$, give the right derivatives for the quotient. This recursive definition of $h^{(n)}$ requires not only all the derivatives of f and g, but also the derivatives of h up to the $(n-1)$-th order.

- Power

$$\mathbf{h} = \mathbf{f}^a \tag{E.8}$$

Clearly, define $h^{(0)} = (f^{(0)})^a$. Note that

$$h^{(1)} = a(f^{(0)})^{a-1} f^{(1)} \tag{E.9}$$

$$= a\frac{h^{(0)}}{f^{(0)}} f^{(1)} \tag{E.10}$$

$$h^{(1)} f^{(0)} = a h^{(0)} f^{(1)} \tag{E.11}$$

$$\sum_{k=0}^{n-1} {}_{n-1}C_k h^{(k+1)} f^{(n-1-k)} = a \sum_{k=0}^{n-1} {}_{n-1}C_k h^{(k)} f^{(n-k)} \tag{E.12}$$

$$h^{(n)} = \frac{a \sum_{k=0}^{n-1} {}_{n-1}C_k h^{(k)} f^{(n-k)} - \sum_{k=0}^{n-2} {}_{n-1}C_k h^{(k+1)} f^{(n-1-k)}}{f^{(0)}}. \tag{E.13}$$

It is worth noting that we assume that $f^{(0)} \neq 0$. If not, either f is not in $\mathbb{C}^{N+1}$ or all the derivatives of f equal zero at q. Therefore, the recursive definition of $h^{(n)}$ in Eq.(E.13) gives the correct values of the higher order derivatives for $n = 1, ..., N$.

- Logarithm

$$\mathbf{h} = \mathbf{log}(\mathbf{f}) \tag{E.14}$$

Define $h^{(0)} = \log f^{(0)}$. Taking the derivative and multiply both sides by $f^{(0)}$ yields

$$h^{(1)} f^{(0)} = f^{(1)}. \tag{E.15}$$

Similarly, Leibniz rule implies

$$h^{(n)} = \frac{f^{(n)} - \sum_{k=0}^{n-2} {}_{n-1}C_k h^{(k+1)} f^{(n-1-k)}}{f^{(0)}}. \tag{E.16}$$

- Exponential

$$\mathbf{h} = \mathbf{exp}(\mathbf{f}) \tag{E.17}$$

Define $h^{(0)} = e^{f^{(0)}}$. It is clear that $h^{(1)} = h^{(0)} f^{(1)}$, then Leibniz rule implies

$$h^{(n)} = \sum_{k=0}^{n-1} {}_{n-1}C_k h^{(k)} f^{(n-k)}. \tag{E.18}$$

Besides, addition and subtraction of $\mathbf{f}$ and $\mathbf{g}$ as well as scalar multiplication can be done in the exactly same way as vector algebra. Finally, we also need to provide two special elements in $\mathfrak{D}^{N+1}$. For the independent variable q, the corresponding representation is

$$\mathbf{q} = (q, 1, 0, ..., 0). \tag{E.19}$$

We also define an element

$$\mathbf{1} = (1, 0, ..., 0), \tag{E.20}$$

so that for any scalar b,

$$\mathbf{b} = b\mathbf{1} = (b, 0, ..., 0). \tag{E.21}$$

For completeness, we also include the trigonometric functions sine and cosine

- Sin and Cos

$$\mathbf{f} = \mathbf{sin}(\mathbf{h}) \quad \text{and} \quad \mathbf{g} = \mathbf{cos}(\mathbf{h}) \tag{E.22}$$

As usual, define

$$f^{(0)} = \sin(h^{(0)}) \quad \text{and} \quad g^{(0)} = \cos(h^{(0)}). \tag{E.23}$$

It is also clear that first order derivatives are

$$f^{(1)} = \cos(h^{(0)})h^{(1)} \tag{E.24}$$

$$= g^{(0)}h^{(1)} \tag{E.25}$$

and

$$g^{(1)} = -\sin(h^{(0)})h^{(1)} \tag{E.26}$$

$$= -f^{(0)}h^{(1)}. \tag{E.27}$$

Thus, applying Leibniz rule for both equations yield

$$f^{(n)} = \sum_{k=0}^{n-1} {}_{n-1}C_k g^{(k)} h^{(n-k)} \quad \text{and} \quad g^{(n)} = -\sum_{k=0}^{n-1} {}_{n-1}C_k f^{(k)} h^{(n-k)}, \tag{E.28}$$

for $n = 1, ..., N$. It is worthy mentioning that both $f^{(n)}$ and $g^{(n)}$ are computed by the lower order derivatives of g and f respectively up to order $n-1$.

In addition to the above elementary operations, addition, subtraction and scalar multiplication can be defined in natural extensions. With the elementary operations defined for elements in $\mathfrak{D}^{N+1}$, derivatives of compositions of elementary functions can be computed easily.

APPENDIX F

DETERMINATION OF $\bar{\lambda}_i$ AND p_i^l FROM CDS SPREADS

We will fix $\bar{\lambda}_i$ and p_i^l for each name of the portfolio by using the 5-year CDS spreads with the constraints

$$\bar{\lambda}_i \geq 0 \quad i = 1, ..., N, \tag{F.1}$$

$$0 \leq p_i^l \leq, 1 \quad l = 1, 2, 3, \quad i = 1, ..., N. \tag{F.2}$$

We start by defining a relative credit quality in terms of the 5-year CDS spreads as follows:

$$c_i = \frac{S_i}{\frac{1}{N}\sum_{j=1}^{N} S_j}, \tag{F.3}$$

Then, for $l = 1, 2$, define an auxiliary impact parameter

$$\tilde{p}_i^l = \min\{c_i p^l, 1\}, \quad i = 1, ..., N \tag{F.4}$$

where p^l is a representative impact parameter of the l-sector which is to be calibrated to the tranche quotes. For $l = 3$, choose

$$0 \leq \tilde{p}_i^3 = p^3 \leq 1 \tag{F.5}$$

for all i. For most of the situations, we can choose $\tilde{p}_i^l = p_i^l$. Recall that $\lambda_i = S_i/(1-R)$ and $\lambda^l \geq 0$ are parameters to be calibrated from the tranches, so the idiosyncratic default intensity is then

$$\bar{\lambda}_i = \lambda_i - \tilde{p}_i^1\lambda^1 - \tilde{p}_i^2\lambda^2 - \tilde{p}_i^3\lambda^3. \tag{F.6}$$

However, $\bar{\lambda}_i$ computed as above could be negative for some cases. For those cases, we lower the values of p_i^l proportionally, so

$$p_i^l = \begin{cases} \tilde{p}_i^l, & \text{if } \lambda_i - \tilde{p}_i^1\lambda^1 - \tilde{p}_i^2\lambda^2 - \tilde{p}_i^3\lambda^3 \geq 0; \\ \dfrac{\lambda_i \tilde{p}_i^l}{\tilde{p}_i^1\lambda^1 + \tilde{p}_i^2\lambda^2 + \tilde{p}_i^3\lambda^3}, & \text{otherwise,} \end{cases} \tag{F.7}$$

for all l and i, and

$$\bar{\lambda}_i = \lambda_i - p_i^1\lambda^1 - p_i^2\lambda^2 - p_i^3\lambda^3. \tag{F.8}$$

With a fixed set of parameters

$$\Theta = \{\lambda^1, p^1, \lambda^2, p^2, \lambda^3, p^3\}, \tag{F.9}$$

the CDS spreads S_i can be matched exactly by choosing p_i^l and $\bar{\lambda}_i$ by Eq.(F.7) and Eq.(F.8) respectively. For $l = 1, 2$, the specification of p_i^l basically follows the idea of [27] where the dependence on a factor is proportional to the relative credit quality c_i. For $l = 3$, we choose p_i^3 to be the same if possible to include the possibility of some catastrophic events that has a high probability to kill many firms.

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