NUMERICAL STUDY OF THE CONVEXITY OF THE EXERCISE BOUNDARY OF THE AMERICAN PUT OPTION ON A DIVIDEND-PAYING ASSET

by

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ABSTRACT

Numerical evidence is provided to show that the optimal exercise boundary for American put options with continuous dividend rate d is convex for values $d \leq r$, where r is the risk-free rate. For $d > r$ the boundary is not convex. As d increases beyond r, the non-convex region moves away from expiry and increases in size. A front-fixing method has been used to transform the American put problem into a nonlinear parabolic differential equation posed on a fixed domain. Explicit and implicit finite-difference methods are used to simulate the problem numerically. As a test, both the explicit and implicit method has been compared and the finite-difference methods give stable results.

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PREFACE

I am grateful to my advisor, Dr. John Chadam, for his guidance, support and encouragement over the course of my work. His insightful comments and suggestions and his practical approach were extremely helpful in my research.

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1.0 INTRODUCTION

Since the papers of Chen and Chadam proving the convexity of the early exercise boundary when the dividend rate is zero have been published [1], there has been considerable interest in examining the situation for dividend-paying assets.

This thesis provides a numerical study of this problem. We find that the early exercise boundary is convex when the constant dividend rate, d, satisfies $0 \le d \le r$, where r is the risk-free interest rate and it loses convexity where d > r. For d slightly larger than r, the non-convex region is small and very close to expiry.

Knowing that the exercise boundary is convex has important consequences. For example, Chen and Chadam used convex interpolation together with the near expiry and far from expiry behavior for the early exercise boundary for the American put (on a zero–dividend–paying asset) to obtain accurate estimates for its location for all time [1].

In this work the problem is cast in the Partial Differential Equation setting with the early exercise boundary arising as a moving free boundary. We begin by transforming the free boundary problem into one with fixed boundaries at the expense of having the unknown boundary appear in the drift term. The problem in this form is studied numerically using finite-difference methods. In particular, both the explicit and implicit schemes are provided to separately confirm the results.

The remainder of the thesis is organized as follows:

The second chapter covers numerical models used in solving the nonlinear partial differential equation which has been transformed using the front–fixing method to one with a fixed domain. Numerical approximations have been done using implicit and explicit finite–difference method.

In the next chapter we present the results obtained from the numerical simulations in the form of graphs. The graphs confirm the fact that the results from the implicit and explicit methods match perfectly and give stable results.

Finally, we conclude on the note that the early exercise boundary for the American put option is convex for $d \leq r$ and fails convexity for $d > r$, where d is the dividend rate and r is the risk-free interest rate.

2.0 MODEL FOR AMERICAN PUT OPTION

2.1 MATHEMATICAL MODEL

An option is a specific financial derivative (contingent claim) that represents a contract sold by one party (option writer) to another party (option holder). The contract offers the buyer the right, but not the obligation, to buy (call) or sell (put) a security or other financial asset at an agreed-upon price - the strike price - during a certain period of time or on a specific date (exercise date). An option that can be exercised anytime during its life is called an American option.

Since investors have the freedom to exercise their American options at any point during the life of the contract, it is imperative to know at any given time, the asset price at which it is optimal to exercise the option. This relationship describes the early exercise boundary. In this paper, American options on assets paying a constant dividend rate have been modeled and we study when the exercise boundary is convex with the help of numerical simulations.

2.1.1 Standard Model

To begin, the standard Black-Scholes model is considered for determining the price of an American put option on an asset paying dividends at a constant rate.

If we assume the asset S follows a geometric Brownian motion [2], given by the equation,

$$\frac{dS}{S} = (\mu - d)\, S\, dt + \sigma S\, dW(t)$$

where, μ is the real-world growth rate, d is the dividend rate and σ is the volatility.

Then the price of the American put, P(S, t), at time t, for t in the range $(0 \le t \le T)$ and asset S satisfies the following parabolic free boundary problem [1]

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-d)S\frac{\partial P}{\partial S} - rP = 0 \text{ for } S > S_f(t) \text{ and } 0 < t < T \qquad (1)$$

$$P(S, T) = \max(E - S, 0) \text{ for } S \ge 0 \qquad (2)$$

$$P = E - S_f\,,\ \frac{\partial P}{\partial S}(S_f(t), t) = -1\,, \qquad at\ \ S = S_f(t) \qquad (3)$$

$$\lim_{S \to \infty} P(S, t) = 0 \qquad (4)$$

$$S_f(t) = \begin{cases} E & if \quad d \le r \\ \dfrac{rE}{d} & if \quad d > r \end{cases} \qquad (5)$$

Here P(S, T) = max (E – S, 0) is the price at expiry at T, S is the log-normal underlying asset price, E is the exercise (strike) price, σ is the volatility, d is the dividend rate and r is the risk-free interest rate. The early exercise boundary $S = S_f(t)$ is an unknown in the problem and is the focus of our attention here.

2.1.2 The Transformed Model

The basic idea of the front-fixing method is to make a change of variables, so that the moving free boundary of the American option is transformed to a fixed boundary.

The transformations of the variables made are as follows:

$$x = S / S_f \,, \qquad S = x S_f \tag{6}$$

$$\text{and, } p(x, t) = P(S, t) = P(x S_f(t), t) \text{ where } x \ \varepsilon\ [1, \infty) \text{ for } S \ \varepsilon [S_f(t), \infty) \tag{7}$$

This leads to a new set of equations for the unknown p(x, t) in $x \geq 1$ *and* $0 \leq t \leq T$.

Using equation (6), the final condition takes the form

$$p(x, T) = P(S, T) = P(x S_f(T), T) \tag{8}$$

$$p(x, T) = \max(E - x S_f(T), 0) = \max(E - xE, 0) = E \max(1-x, 0) = 0$$

$$\text{for } x \geq 1$$

Using equation (5) when $S_f(t) = E$ for $d \leq r$ in equation (8)

Now, we need to derive the boundary conditions. The corresponding calculations are as follows.

From equations (3), (4) and (5), we find that

$$p\,(1, t) = P\,(S_f(t),\, t) = E - S_f(t) \tag{9}$$

and,

$$\lim_{x \to \infty} p(x,\, t) = \lim_{x \to \infty} P(x S_f(t),\, t) = 0 \tag{10}$$

Then, differentiating equation (7) with respect to x gives,

$$\frac{\partial p}{\partial x} = \frac{\partial P}{\partial S}\frac{\partial S}{\partial x} = S_f(t)\frac{\partial P}{\partial S} \tag{11}$$

From equation (6), we know that,

$$S = x S_f$$

Thus equation (3) implies that

$$\frac{\partial p}{\partial x}(1,\, t) = S_f(t)\frac{\partial P}{\partial S}(S_f(t),\, t) = -\, S_f(t) \tag{12}$$

Finally we need to derive a partial differential equation for p(x, t) from equation (1) which governs P(S, t). In order to do this we need to express $\partial P / \partial t$, $\partial P / \partial S$, $\partial^2 P / \partial S^2$ in terms of p and its derivatives in x and t.

Using (11), we get

$$\frac{\partial P}{\partial S} = \frac{1}{S_f(t)} \frac{\partial p}{\partial x} \tag{13}$$

Differentiating equation (11) with respect to x, gives

$$\frac{\partial^2 p}{\partial x^2} = S_f(t) \frac{\partial^2 P}{\partial S^2} \frac{\partial S}{\partial x} = S_f^2(t) \frac{\partial^2 P}{\partial S^2}$$

or

$$\frac{\partial^2 P}{\partial S^2} = \frac{1}{S_f^{\ 2}(t)} \frac{\partial^2 p}{\partial x^2} \tag{14}$$

Now differentiating equation (7) with respect to t, we get

$$\frac{\partial p}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial S} \frac{\partial S}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial S} x S_f(t) \tag{15}$$

Thus, equation (13) yields,

$$\frac{\partial P}{\partial t} = \frac{\partial p}{\partial t} - x \frac{S_f(t)}{S(t)} \frac{\partial p}{\partial x} \tag{16}$$

Hence, it follows from equations (1), (6), (13), (14) and (16) that p(x, t) must satisfy

$$\frac{\partial p}{\partial t} - x \frac{S_f(t)}{S(t)} \frac{\partial p}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} + (r - d) \frac{\partial p}{\partial x} - rp = 0 \tag{17}$$

To summarize, it follows from equations (8) and (12) that the two unknown's, p and S_f, are governed by the following system.

$$\frac{\partial p}{\partial t} + x(r - d - \frac{S'_f(t)}{S(t)})\frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} - rp = 0 \tag{18}$$

$$p(x, T) = 0 \text{ for } x \geq 1 \tag{19}$$

$$p(1, t) = E - S_f(t) \tag{20}$$

$$\frac{\partial p}{\partial x}(1, t) = - S_f(t) \tag{21}$$

$$\lim_{x \to \infty} p(x, t) = 0 \tag{22}$$

$$S_f(T) = \begin{cases} E & \text{if } d \leq r \\ \frac{rE}{d} & \text{if } d > r \end{cases} \tag{23}$$

2.2 NUMERICAL METHODS

As mentioned earlier, we will study equations (18) through (23) numerically using both explicit and implicit finite–difference methods. We simulate both methods to estimate the location of the exercise boundary to corroborate our results in testing the convexity of the early exercise boundary.

The idea underlying finite-difference methods is to replace the partial derivatives occurring in partial differential equations by approximations based on Taylor series expansions of functions near the point or points of interest. For example, the partial derivative $\partial p / \partial \tau$ may be defined to be the limiting difference.

$$\frac{\partial p}{\partial \tau}(x,t) = \lim_{\partial t \to 0} \frac{p(x, t+\Delta t) - p(x, t)}{\Delta t}$$

Δt is regarded as non-zero but small, i.e.,

$$\frac{\partial p}{\partial t}(x, t) \approx \frac{p(x, t+\Delta t) - p(x, t)}{\Delta t} + O(\Delta t)$$

This is called the forward finite-difference scheme for $\partial p / \partial t$. As the $O(\Delta t)$ term suggests, the smaller Δt is, the more accurate the approximation. In the two methods that we have used, Δt is of the order $10^{(-6)}$ to make the approximation accurate and also to meet the stability criterion.

We will also use the central differences for the spatial derivatives of p, i.e.,

$$\frac{\partial p}{\partial x}(x,\ t) \ \approx \frac{p(x+\Delta x,\ t) - \ p\ (x-\Delta x,\ t)}{2\Delta x} + O((\Delta x)^2)$$

2.2.1 The Implicit Scheme

Implicit finite-difference methods allow us to use large spatial meshes without excessively small time steps. We will use the centered implicit scheme, which uses the backward-difference approximation for the $\partial p / \partial t$ term and the symmetric central difference approximation for the $\partial p / \partial x$, $\partial^2 p / \partial x^2$ terms.

In order to solve the system of equations from (18) through (23) numerically, the large value of x, is approximated by x_∞ where the boundary condition given by equation (4) is imposed, that is, we put,

$$p(x_\infty,\ t) = 0 \tag{24}$$

Now for a given set of positive integers M and N, we define Δx (change in x) and Δt (change in t) as follows:

$$\Delta x = \frac{x_\infty\ -\ 1}{M\ +\ 1}\ , \qquad \Delta t\ = \frac{T}{N+1}$$

$$x_m\ = 1 +\ m\Delta x \quad for\ \ m\ =\ 0 \ldots M+1 \tag{24'}$$

$$t_n\ =\ n\,\Delta t \qquad for\ \ n\ =\ 0 \ldots N+1$$

The main objective of the transformation in the above equation (24′) is to derive an approximation.

$$p_m^n \approx p(x_m, t_n) \ \ for\ m = 0, 1..., M+1 \ and\ n = N,\ N-1, ...0$$

In this scheme, p_n^{m+1}, p_{n-1}^{m+1} *and* p_{n+1}^{m+1} will all depend on p_n^m in an implicit manner. The final conditions equations (19) and (23) give,

$$p_m^{N+1} = 0, \qquad m = 0, 1, ..., M+1 \tag{25}$$

and

$$S_f^{N+1} = \begin{cases} E & for\ \ 0 \le d \le r \\ \dfrac{rE}{d} & for\ \ d > r \end{cases} \tag{26}$$

The boundary conditions in the equations (20) and (24) imply that:

$$p_0^n = E - S_f^n \qquad for\ \ n = N,\ N-1, ...0 \tag{27}$$

$$p_{M+1}^n = 0 \qquad for\ n = N,\ N-1, ..., 0 \tag{28}$$

A finite-difference approximation of equation (21) is given by,

$$\frac{p_1^n - p_0^n}{\Delta x} = -S_f^n \qquad for\ \ n = N,\ N-1, ..., 0$$

Using equation (27), we have

$$p_1^n = E - (1+\Delta x) S_f^n \qquad for\ n = N,\ N-1, ..., 0 \tag{29}$$

We use a centered implicit scheme to discretize equation (**18**)

$$\frac{p_m^{n+1} - p_m^n}{\Delta t} + \frac{1}{2}\sigma^2 x_m^2 \frac{p_{m-1}^n - 2p_m^n + p_{m+1}^n}{(\Delta x)^2} + x_m\left(r - d - \frac{S_f^{n+1} - S_f^n}{\Delta t\, S_f^{n+1}}\right)\frac{p_{m+1}^n - p_{m-1}^n}{2\Delta x} - rp_m^n = 0 \qquad (30)$$

for m = 1,2,…M and n = N, N-1,…, 0

Here, p^{n+1} and S_f^{n+1} are known and we want to calculate p^n and S_f^n.

From equation (**30**), it follows that:

$$\left[\frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 p_{m-1}^n - \frac{r\,x_m}{2\Delta x} p_{m-1}^n + \frac{d\,x_m}{2\,\Delta\,x} p_{m-1}^n + x_m\left(\frac{S_f^{n+1} - S_f^n}{\Delta t\, S_f^{n+1}}\right)\frac{p_{j-1}^n}{2\Delta x}\right] +$$

$$\left[\frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 p_{m+1}^n + \frac{r\,x_m}{2\Delta x} p_{m+1}^n - \frac{d\,x_m}{2\,\Delta\,x} p_{m+1}^n - x_m\left(\frac{S_f^{n+1} - S_f^n}{\Delta t\, S_f^{n+1}}\right)\frac{p_{m+1}^n}{2\Delta x}\right] +$$

$$\frac{p_m^{n+1}}{\Delta t} - \left[+\frac{p_m^n}{\Delta t} + \frac{\sigma^2 x_m^2}{(\Delta x)^2} p_m^n + rp_m^n\right] = 0$$

Grouping like terms gives the following equations.

$$\left[\frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 - \frac{r\,x_m}{2\Delta x} + \frac{d\,x_m}{2\,\Delta\,x} + d\left(\frac{S_f^{n+1} - S_f^{n}}{\Delta t\,S_f^{n+1}}\right)\frac{1}{2\Delta x}\right] p_{m-1}^{n} +$$

$$\left[\frac{1}{2(\Delta x)^2}\sigma^2 x_j^2 + \frac{r\,x_j}{2\Delta x} - \frac{d\,x_m}{2\,\Delta\,x} - x_m\left(\frac{S_f^{n+1} - S_f^{n}}{\Delta t\,S_f^{n+1}}\right)\frac{1}{2\Delta x}\right] p_{m+1}^{n} +$$

$$\left[-\frac{1}{\Delta t} - \frac{\sigma^2 x_m^2}{(\Delta x)^2} - r\right] p_m^n \quad + \quad \frac{p_m^{n+1}}{\Delta t} \quad = 0$$

So,

$$p_j^{n+1} = \left[\frac{-\Delta t}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{(d-r)\,x_m\Delta t}{2\,\Delta\,x} - x_m\left(\frac{S_f^{n+1} - S_f^{n}}{S_f^{n+1}}\right)\frac{1}{2\Delta x}\right] p_{m-1}^{n} +$$

$$\left[\frac{-\Delta t}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{(d-r)\,x_m\Delta t}{2\,\Delta\,x} + x_m\left(\frac{S_f^{n+1} - S_f^{n}}{S_f^{n+1}}\right)\frac{1}{2\Delta x}\right] p_{m+1}^{n} + \qquad (31)$$

$$\left[1 + \frac{\sigma^2 x_m^2\Delta t}{(\Delta x)^2} + r\Delta t\right] p_m^n$$

Thus equation (31) takes the form,

$$\beta_m^n \; p_{m-1}^n + \alpha_m^n \; p_m^n \; + \gamma_m^n \; p_{m+1}^n \; = \; b_m^n \tag{32}$$

where,

$$\beta_m^n \; = \; \frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 \; - \; \frac{r\, x_m}{2\Delta x} \; + \; \frac{d\, x_m}{2\,\Delta\, x} \; + \; x_m\left(\frac{S_f^{n+1} - S_f^n}{\Delta t\, S_f^{n+1}}\right)\frac{1}{2\Delta x}$$

$$\alpha_m^n \; = \; 1 + \frac{\sigma^2 x_m^2 \Delta t}{(\Delta x)^2} + r\Delta t$$

$$\gamma_m^n \; = \; \frac{-\Delta t}{2(\Delta x)^2}\sigma^2 x_m^2 \; + \; \frac{(d-r)\, x_m \Delta t}{2\,\Delta\, x} \; + \; x_m\left(\frac{S_f^{n+1} - S_f^n}{S_f^{n+1}}\right)\frac{1}{2\Delta x}$$

$$b_m^n \; = \; p_m^{n+1}$$

for m = 1, 2, ... M and n = N, N-1,…., 0

Now, evaluating equation (32) for different values of j, we get the following iterative scheme.

Putting, m = 1 in equation (32) we get,

$$\gamma_1^n \; p_2^n \quad = \; b_1^n - \beta_1^n \; p_0^n + \alpha_1^n \; p_1^n \tag{33}$$

And using equations (27) and (29), the equation further simplifies into the following.

$$\gamma_1^n \; p_2^n \quad = \; b_1^n - \beta_1^n \; (E - S_f^n) + \alpha_1^n \left[E - (1 + \Delta\, x) S_f^n\right] \tag{34}$$

Putting m = 2 in equation (32), we get

$$\gamma_2^n\ p_3^n\ + \alpha_2^n\ p_2^n\ =\ b_2^n - \beta_2^n\ p_1^n\ = b_2^n\ -\ \beta_2^n\left[E - (1 + \Delta x) S_f^n\right] \tag{35}$$

Putting m = M in equation (32), we get

$$\gamma_M^n\ p_{M+1}^n + \beta_M^n\ p_{M-1}^n + \alpha_M^n\ p_M^n\ \ =\ b_M^n \tag{36}$$

But, from equation (28) we know that,

$$p_{M+1}^n\ =\ 0$$

So equation (36) gets transformed into

$$\beta_M^n\ p_{M-1}^n + \alpha_M^n\ p_M^n\ \ =\ b_M^n$$

Finally, for m = 3, 4, .. M-1, we have the following equations:

$$\gamma_m^n\ p_{m+1}^n + \beta_m^n\ p_{m-1}^n + \alpha_m^n\ p_m^n\ \ =\ b_m^n \tag{37}$$

We now have a system of M unknowns' p_2^n , p_3^n ,....p_M^n and S_f^n and M equations - (34), (35), (36) and (37) - for each time step $t_n = n\Delta t$. We then apply the Newton-Raphson method to solve these M equations and M unknowns.

We define a matrix A = A (S_f^n) ε $R^{M,M-1}$ given by

$$A(S_f^n) = \begin{pmatrix} \gamma_1^n & & & & & \\ \alpha_2^n & \gamma_2^n & & & & \\ \beta_3^n & \alpha_3^n & \gamma_3^n & & & \\ & & & & & \\ & & & & & \\ & & & \beta_{M-1}^n & \alpha_{M-1}^n & \gamma_{M-1}^n \\ & & & & \beta_M^n & \alpha_M^n \end{pmatrix} \quad (38)$$

where,

$$\gamma_1^n = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_1^2 + \frac{(d-r)}{2\,\Delta x}(x_1 \Delta t) + \frac{x_1}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^n}{S_f^n}\right)$$

$$\alpha_2^n = 1 + \frac{\Delta t\ \sigma^2 x_2^2}{(\Delta x)^2} + r\,\Delta t$$

$$\gamma_2^n = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_2^2 + \frac{(d-r)}{2\,\Delta x}(x_2 \Delta t) + \frac{x_2}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^n}{S_f^n}\right)$$

$$\beta_3^n = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_3^2 + \frac{(r-d)}{2\,\Delta x}(x_3 \Delta t) - \frac{x_3}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^n}{S_f^n}\right)$$

$$\alpha_3^n = 1 + \frac{\Delta t\ \sigma^2 x_3^2}{(\Delta x)^2} + r\,\Delta t$$

$$\gamma_3^n = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_3^2 + \frac{(d-r)}{2\,\Delta x}(x_3 \Delta t) + \frac{x_3}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^n}{S_f^n}\right)$$

$$\beta_{M-1}^{n} = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_{M-1}^2 + \frac{(r-d)}{2\,\Delta x}\left(x_{M-1}\Delta t\right) - \frac{x_{M-1}}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^{n}}{S_f^{n}}\right)$$

$$\beta_{M}^{n} = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_{M}^2 + \frac{(r-d)}{2\,\Delta x}\left(x_{M}\Delta t\right) - \frac{x_{M}}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^{n}}{S_f^{n}}\right)$$

$$\alpha_{M-1}^{n} = 1 + \frac{\Delta t\ \sigma^2 x_{M-1}^2}{(\Delta x)^2} + r\,\Delta t$$

$$\alpha_{M}^{n} = 1 + \frac{\Delta t\ \sigma^2 x_{M}^2}{(\Delta x)^2} + r\,\Delta t$$

$$\gamma_{3}^{n} = -\frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\sigma^2 x_{M-1}^2 + \frac{(d-r)}{2\,\Delta x}\left(x_{M-1}\Delta t\right) + \frac{x_{M-1}}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^{n}}{S_f^{n}}\right)$$

And the mapping f = f(S_f^n) : $R \to R^M$ is given by,

$$\mathrm{f}\,(\mathrm{S}_f^n) = \begin{bmatrix} b_1^n - \beta_1^n\left(E - S_f^n\right) - \alpha_1^n\left[E - (1+\Delta x)S_f^n\right] \\ b_2^n - \beta_2^n\left[E - (1+\Delta x)S_f^n\right] \\ b_3^n \\ \\ \\ \\ b_M^n \end{bmatrix} \qquad (39)$$

So the system of equations (34) through (37) can be written in the form,

$$\mathrm{F}\,(\mathrm{p}^n, S_f^n) = \mathrm{A}(S_f^n)\,p^n - f(S_f^n) = 0,$$

$$\text{where}\quad \mathrm{p}^n = (\,p_2^n, p_3^n, \ldots\ldots\ldots, p_M^n\,) \qquad (40)$$

This is the non-linear problem we solve by using the Newton-Raphson method.

We define a new variable, z = $(p_2^n, p_3^n, ..., p_M^n, S_f^n)$ and define the iteration as

$$z_{k+1} = z_k + J^{-1}(z_k)F(z_k)$$

where, J is the Jacobian of F. (41)

We calculate the Jacobian using a Matlab code.

Having computed $p_2^n, p_3^n, ..., p_M^n$ and S_f^n by equation (41), we calculate p_0^n, p_1^n and p_{M+1}^n using the equations (27), (28) and (29).

In this paper we compute the value of the American put option for different dividends. The simulations were performed in Matlab initially for a zero dividend rate, in order to compare the results with known computations [3]. It will also be compared with our explicit scheme to follow in the subsequent section. After the testing, the code will be run for arbitrary dividend rates and the convexity of the early exercise boundary will be observed.

2.2.2 The Centered Explicit Scheme

Explicit methods are more efficient than implicit methods in terms of arithmetical operations per time-step. On the other hand, they recognize finer meshes and suffer from stability constraints. In the explicit method, if at time step m, we know p_m^n for all values of n we can explicitly calculate p_m^{n+1} i.e. p_m^{n+1} depends only on p_{m+1}^n, p_m^m and p_{m-1}^n

The equations from (18) through (23) are not approximated using a centered explicit scheme.

$$\frac{p_m^{n+1} - p_m^n}{\Delta t} + \frac{1}{2}\sigma^2 x_m^2 \frac{p_{m-1}^{n+1} - 2p_m^{n+1} + p_{m+1}^{n+1}}{(\Delta x)^2} + x_m\left(r - d - \frac{S_f^{n+1} - S_f^n}{\Delta t\, S_f^{n+1}}\right)\frac{p_{m+1}^{n+1} - p_{m-1}^{n+1}}{2\Delta x} - rp_m^{n+1} = 0$$

$$\text{for} \quad m = 1, 2, \ldots M \quad \text{and} \quad n = N, N\text{-}1, \ldots, 0 \tag{42}$$

Here p^{n+1} and S_f^{n+1} are known and we want to calculate p^n and S_f^n.

Rearranging the terms in equation (42) we have,

$$\frac{p_m^{n+1}}{\Delta t} - \frac{p_m^n}{\Delta t} + \frac{1}{2}\frac{\sigma^2 x_m^2}{(\Delta x)^2} p_m^{n+1} - \frac{\sigma^2 x_m^2}{(\Delta x)^2} p_m^{n+1} + \frac{1}{2}\frac{\sigma^2 x_m^2}{(\Delta x)^2} p_{m+1}^{n+1} + \frac{rx_m}{2\Delta x} p_{m+1}^{n+1} - \frac{dx_j}{2\Delta x} p_{m+1}^{n+1} - \frac{rx_m}{2\Delta x} p_{m-1}^{n+1} + \frac{dx_m}{2\Delta x} p_{m-1}^{n+1} -$$
$$\frac{x_m}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^n}{\Delta t S_f^n}\right) p_{m+1}^{n+1} + \frac{x_m}{2\Delta x}\left(\frac{S_f^{n+1} - S_f^n}{\Delta t S_f^n}\right) p_{m-1}^{n+1} - rp_m^{n+1} = 0$$

Grouping like terms together, we get:

$$\left[\frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{x_m}{2\Delta x}(d - r + \frac{1}{\Delta t})\right] p_{m-1}^{n+1} + \left[1 - \frac{\Delta t}{(\Delta x)^2}\sigma^2 x_m^2 - r\,\Delta t\right] p_m^{n+1} +$$

$$\left[\frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{x_m}{2\Delta x}(r - d + \frac{1}{\Delta t})\right] p_{m+1}^{n+1} + \left[\frac{x_m}{2\Delta x}\left(\frac{p_{m+1}^n - p_{m-1}^n}{S_f^n}\right)\right] +$$

$$p_m^n = 0$$

Thus, the above equations can be written in a more compact form.

$$- p_m^n \; + \; \mathrm{D}_m^{n+1} S_f^n \; + \; A_m p_{m-1}^{n+1} + B_m p_m^{n+1} + C_m p_{m+1}^{n+1} = 0$$

$$p_m^n \; - \; \mathrm{D}_m^{n+1} S_f^n \; = A_j p_{m-1}^{n+1} + B_m p_m^{n+1} + C_m p_{m+1}^{n+1} \tag{43}$$

for m = 1, 2,….,M and n = N, N-1,….,0

where,

$$A_m = \frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{x_m}{2\Delta x}(d - r + \frac{1}{\Delta t})$$

$$B_m = 1 - \frac{\Delta t}{(\Delta x)^2}\sigma^2 x_m^2 - r\,\Delta t$$

$$C_m = \frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{x_m}{2\Delta x}(r - d + \frac{1}{\Delta t})$$

$$D_m^n = \frac{x_m}{2\Delta x}\left(\frac{p_{m+1}^n - p_{m-1}^n}{S_f^n}\right)$$

In equation (**43**), putting m = 1, we get the following equation:

$$p_1^n \; - \; \mathrm{D}_1^{n+1} S_f^n \; = A_1 p_0^{n+1} + B_1 p_1^{n+1} + C_1 p_2^{n+1} \tag{44}$$

And applying the boundary conditions

$$p_m^{N+1} = 0, \qquad m = 0,1,........., M+1$$
$$S_f^{N+1} = E$$

$$p_0^n = E - S_f^n \qquad for \;\; n = N,\; N-1,........0$$

$$p_{M+1}^n = 0 \qquad for \;\; n = N,\; N-1,........0$$

$$\frac{p_1^n - p_0^n}{\Delta x} = -S_f^n \qquad for \;\; n = N,\; N-1,........0$$

$$p_1^n = E - (1+\Delta x)S_f^n \qquad for \;\; n = N,\; N-1,........0$$

we get a formula S_f^n

$$S_f^n = \frac{E - (A_1 p_0^{n+1} + B_1 p_1^{n+1} + C_1 p_2^{n+1})}{D_1^{n+1} + (1+\Delta x)} \qquad (45)$$

This formula, given by equation (**45**), holds for dividend d < r, where r is the risk-free interest rate. However, for the dividend yield rate d > r this formula does not work, as it makes the denominator $D_1^{n+1} + (1+\Delta x)$ equal to zero and hence makes the explicit code blow up.

To avoid this and to make the convergence possible we introduce a different formula for S_f^n for d > r,

$$S_f^n = \left(\frac{r}{r_f}\right) * E * \left(1 - \sigma * alpha * \sqrt{2*(T-t)}\right) \qquad (46)$$

where, alpha = 0.4517

This expression is the near–expiry behavior of the early exercise boundary derived by Keller [4]. By making necessary changes to the explicit method we carry out simulations and see that the results of the explicit and implicit method match perfectly.

3.0 NUMERICAL SIMULATIONS

3.1 ALGORITHMS

3.1.1 Explicit method algorithm

1. for m = 1, 2,…….M+1 do $p_j^{N+1} = 0$

2. $S_f^{N+1} = E$

3. for n = N+1, N,………0 do $p_{M+1}^{n} = 0$

4. for m = 1, 2,……..M do

$$A_m = \frac{1}{2(\Delta x)^2}\sigma^2 x_j^2 + \frac{x_j}{2\Delta x}(r_f - r + \frac{1}{\Delta t})$$

$$B_j = 1 - \frac{\Delta t}{(\Delta x)^2}\sigma^2 x_m^2 - r\,\Delta t$$

$$C_m = \frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{x_m}{2\Delta x}(r - d + \frac{1}{\Delta t})$$

5. for n = N, N-1,……………0

a) $D_m^n = \frac{x_m}{2\Delta x}\left(\frac{p_{m+1}^n - p_{m-1}^n}{S_f^n}\right)$

$$b) \quad S_f^n = \frac{E - (A_1 p_0^{n+1} + B_1 p_1^{n+1} + C_1 p_2^{n+1})}{D_1^{n+1} + (1+\Delta x)}$$

$$c) \quad p_0^n = E - S_f^n \quad and \quad p_1^n = E - (1+\Delta x) S_f^n$$

d) for m = 2, 3,………………..M do

$$p_m^n = A_m p_{m-1}^{n+1} + B_m p_m^{n+1} + C_m p_{m+1}^{n+1} + D_m^{n+1} S_f^n$$

The algorithm as stated does not work at the initial step. It possesses a discontinuity in the spatial derivative. We use a standard trick to remove this discontinuity by modifying the initial step at the expense of introducing an error of order O (Δx); i.e., we take

$$p_1^n = E - (1+\Delta x) S_f^n \ , \quad p_1^n = -(\Delta x) S_f^n \quad for \ S_f^n = E$$

The adjustment to the algorithm for all dividend rates can be achieved by replacing 2 with the following:

2. If $d > r$

$$S_f^{N+1} = \left(\frac{r}{d}\right) E$$

else

$$S_f^{N+1} = \begin{cases} E & for \ 0 \le d \le r \\ \dfrac{rE}{d} & for \ d > r \end{cases}$$

3.1.2 Implicit method algorithm

For the implicit method, the algorithm is slightly different but it does not require the adjustment to p_1^n outlined above (see **4** below). The algorithm used for the implicit scheme is as follows:

1. for m = 1,2,…….M+1 do $p_j^{N+1} = 0$
2. if d > r

   $$S_f^{N+1} = \left(\frac{r}{r_f}\right) E$$

   else

   $$S_f^{N+1} = E$$

3. for n = N+1, N,……0 do $p_{M+1}^n = 0$
4. $p_0^n = E - S_f^n \quad and \quad p_1^n = E - (1+\Delta x) S_f^n$
5. for m = 1, 2,…..M do

$$\beta_m^n = \frac{1}{2(\Delta x)^2}\sigma^2 x_m^2 - \frac{r\, x_m}{2\Delta x} + \frac{d\, x_m}{2\,\Delta\, x} + x_m\left(\frac{S_f^{n+1} - S_f^n}{\Delta t\, S_f^{n+1}}\right)\frac{1}{2\Delta x}$$

$$\alpha_m^n = 1 + \frac{\sigma^2 x_m^2 \Delta t}{(\Delta x)^2} + r\Delta t$$

$$\gamma_m^n = \frac{-\Delta t}{2(\Delta x)^2}\sigma^2 x_m^2 + \frac{(d-r)\,x_m \Delta t}{2\,\Delta\, x} + x_m\left(\frac{S_f^{n+1} - S_f^{n}}{S_f^{n+1}}\right)\frac{1}{2\Delta x}$$

6. Define a function $F(p^n, S_f^n) = A(S_f^n)v^n - f(S_f^n) = 0$

7. Find the Jacobian J

8. for m = 2, 3,……..M do

$$z_{k+1} = z_k + J^{-1}(z_k)F(z_k),$$

where, z = $(p_2^n, p_3^n, \ldots\ldots\ldots, p_M^n, S_f^n)$

The initial guess for the implicit method is

$$S_f^{N+1} = \left(\frac{r}{r_f}\right)E \quad for\ \ d > r$$

3.2 SIMULATION RESULTS

The graphs are plotted with the early exercise boundary as a function of time. At expiry, the graph approaches $S_f(T) = E = 1$ with an infinite slope as existing theoretical results show [3]. All three simulations - implicit, explicit and that of Nielson [3] - agree precisely in the case of d = 0. This provides experimental verification that the implicit and explicit codes outlined in the previous sections give correct results.

We summarize our simulations for the following parameters.

E = 1,

T = 1,

r = 0.1,

$\Delta t = 5*(10)^{(-6)}$,

$\Delta x = 0.001$,

$x_\infty = 2$,

$\sigma = 0.2$.

3.2.1 Graph of exercise boundary S_f versus time to expiry with the dividend rate d = 0

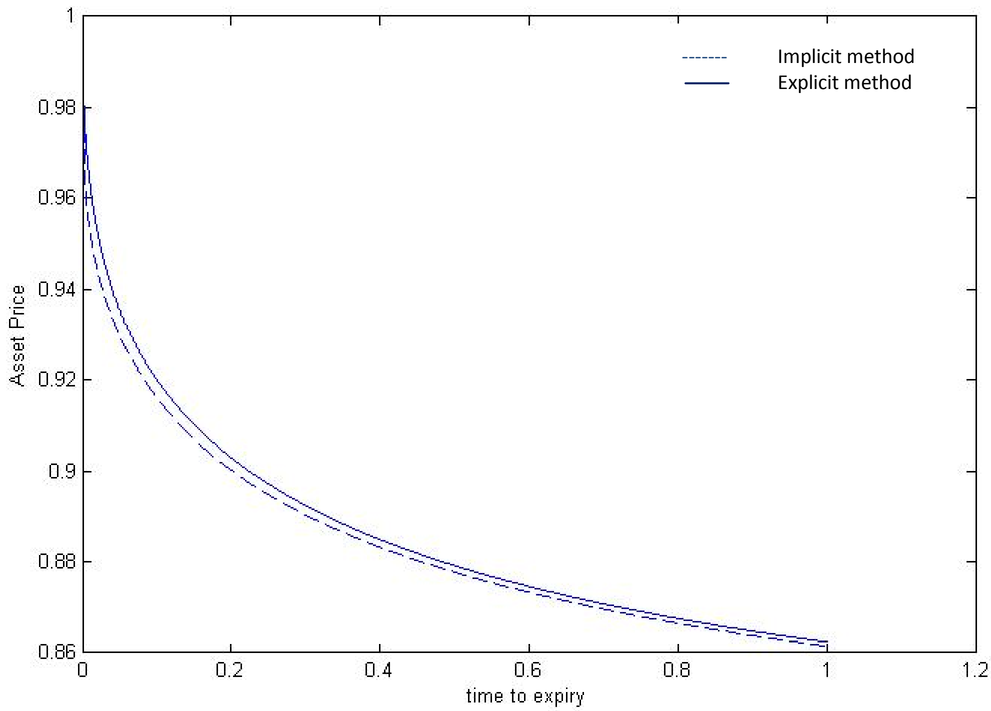

Figure 1

Explicit method with r = 0.1, $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001, x_∞ = 2, $\sigma = 0.2$.

Implicit method with r = 0.1, $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001, x_∞ = 3, $\sigma = 0.2$.

The location of the early exercise boundary at t=1, is S_f^n = 0.8623 for the explicit method and 0.8616 for the implicit method.

Remarks

We see that when the dividend rate is zero, the explicit and implicit method match perfectly. This result matches perfectly with the results of another simulation carried out in a paper written by Nielsen [3]. **Figure 2** in the next page displays the results of the paper and one can easily see that it matches with the results of the explicit and implicit code displayed in **Figure 1** above.

3.2.2 Graph of exercise boundary S_f as shown in the paper by Nielsen with dividend rate d = 0

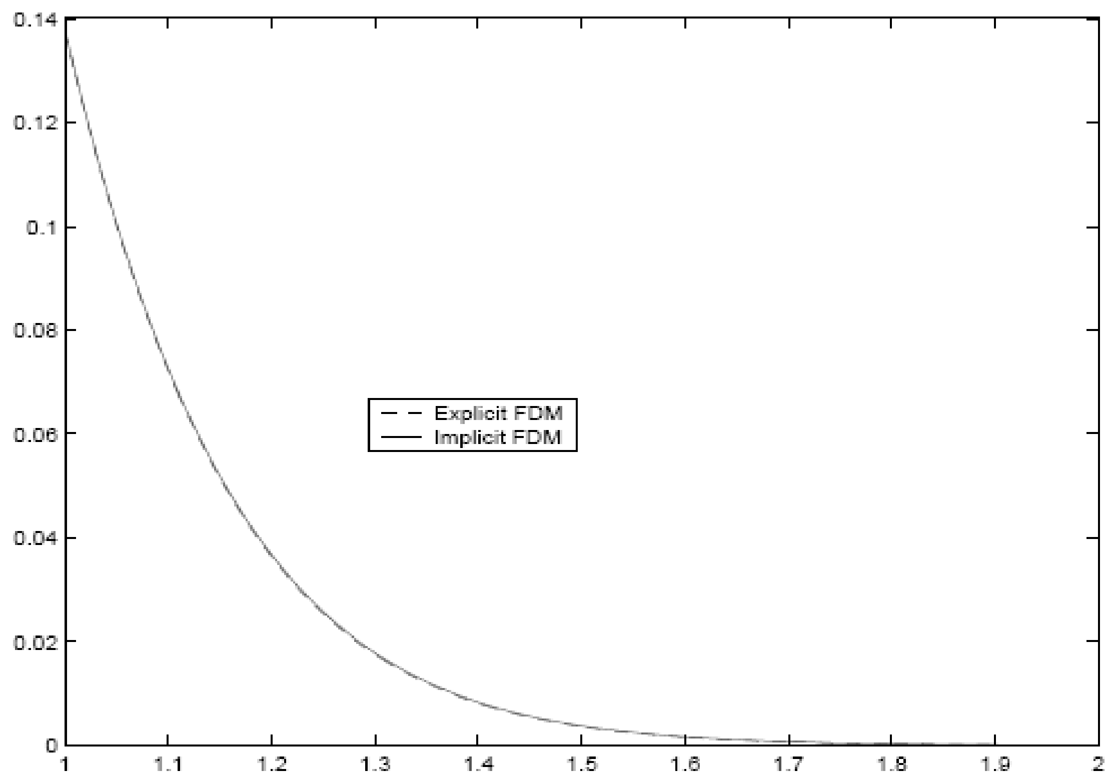

Figure 2

The location of the early exercise boundary is S_f^n = 0.8623 for the explicit method and 0.8619 for the implicit method.

3.2.3 Graph of exercise boundary S_f versus time to expiry with dividend rate d = 0.045 ≤ r = 0.05

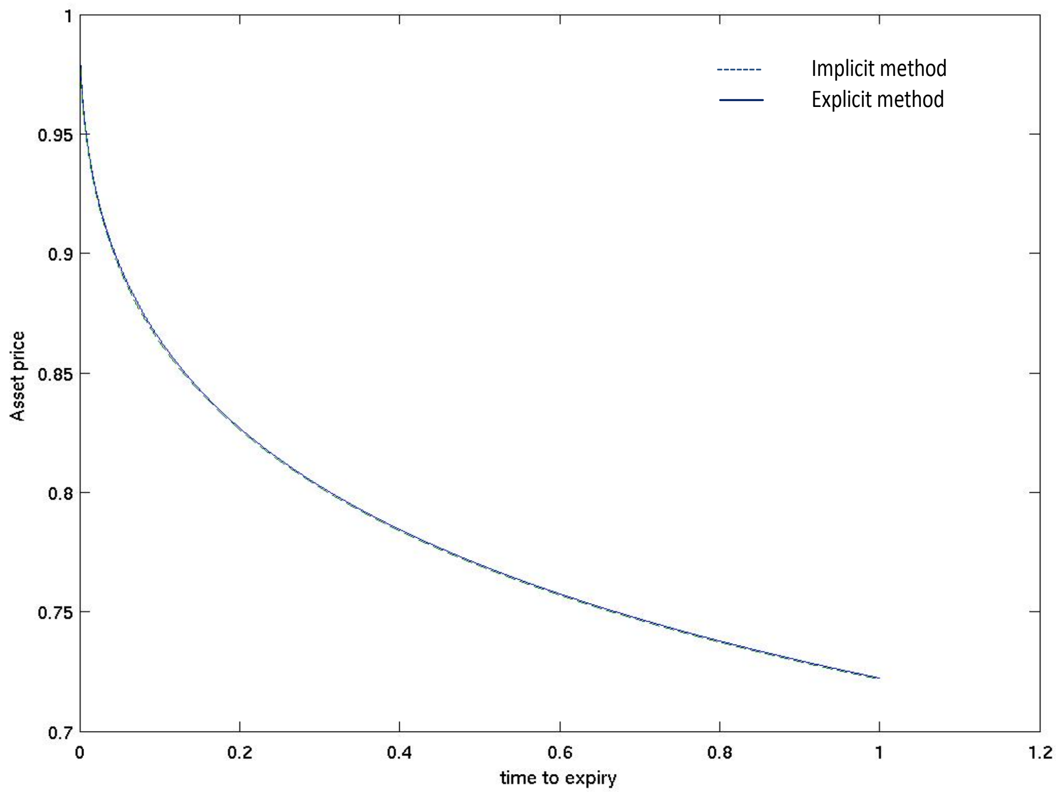

Figure 3

The parameters are $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001.

The location of the early exercise boundary at t =1 are S_f^n = 0.7217 for the explicit method and 0.7223 for the implicit method.

3.2.4 Graph of exercise boundary S_f versus time to expiry with dividend rate d = 0.05 = r

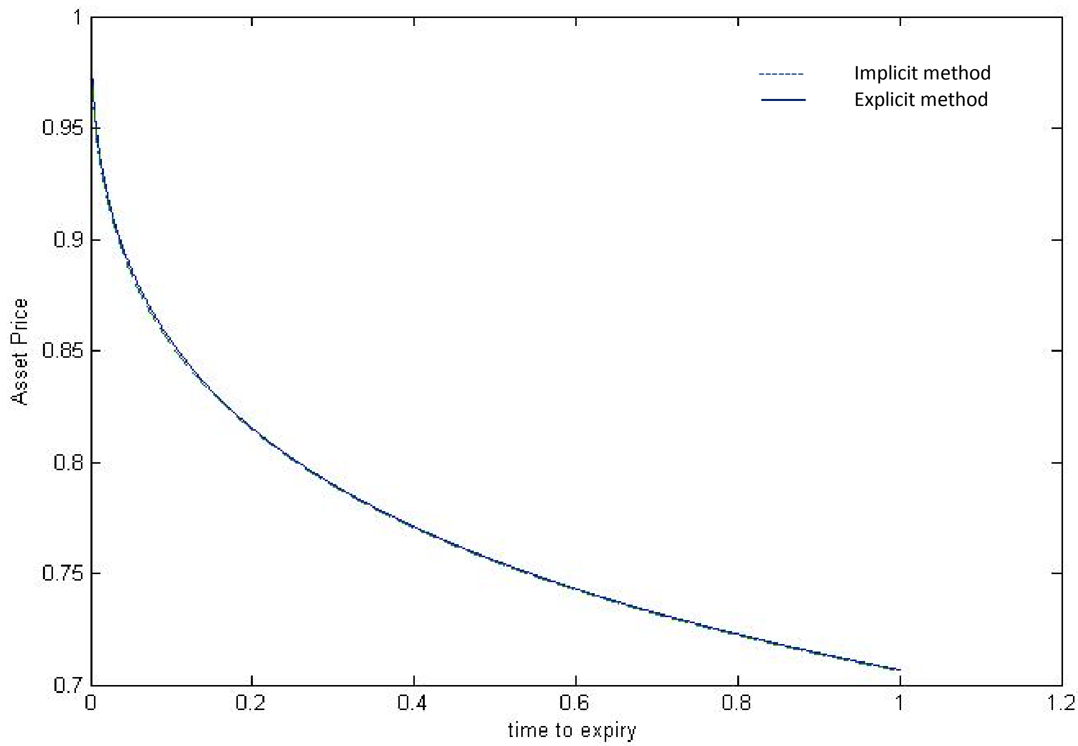

Figure 4

The parameters are $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001.

The location of the early exercise boundary at t = 1 are S_f^n = 0.7062 for the explicit method and 0.7068 for the implicit method.

3.2.5 Graph of exercise boundary S_f versus time to expiry with dividend rate d = 0.055 > r = 0.05

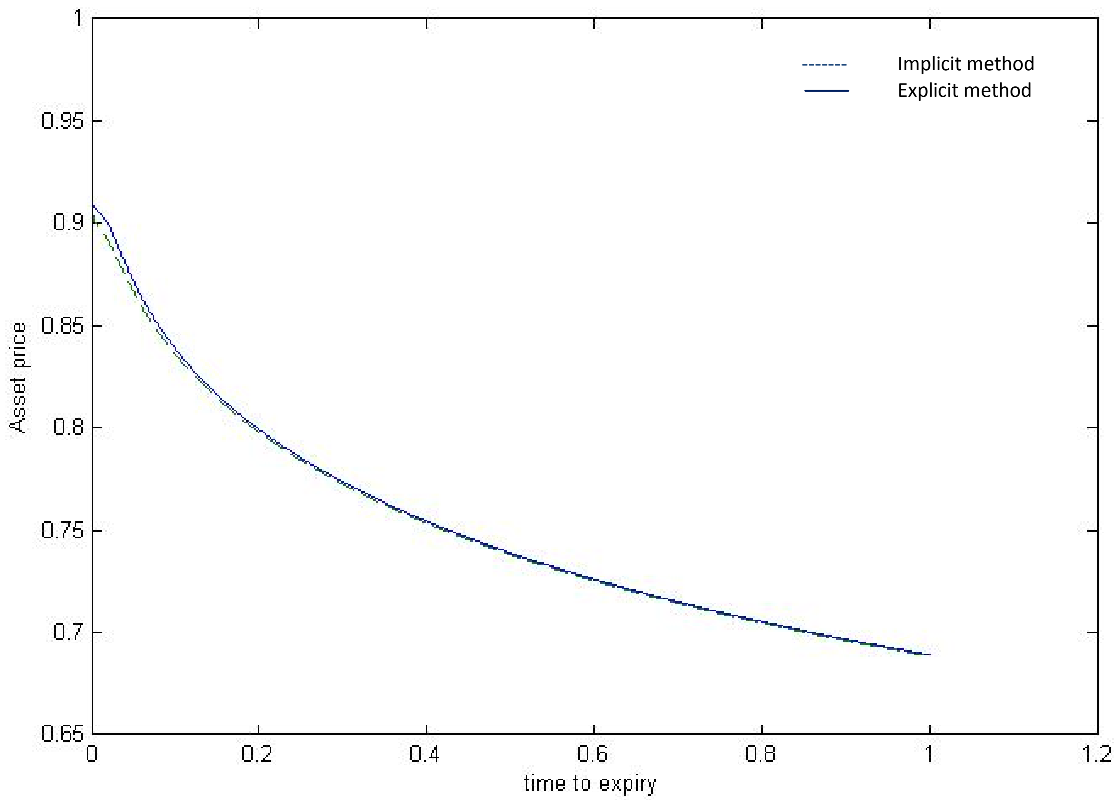

Figure 5

The parameters are $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001.

The location of the early exercise boundary at t = 1 is S_f^n = 0.6883 for the explicit method and 0.6891 for the implicit method.

3.2.6 Graph of exercise boundary S_f versus time to expiry with dividend rate d = 0.06 > r = 0.05

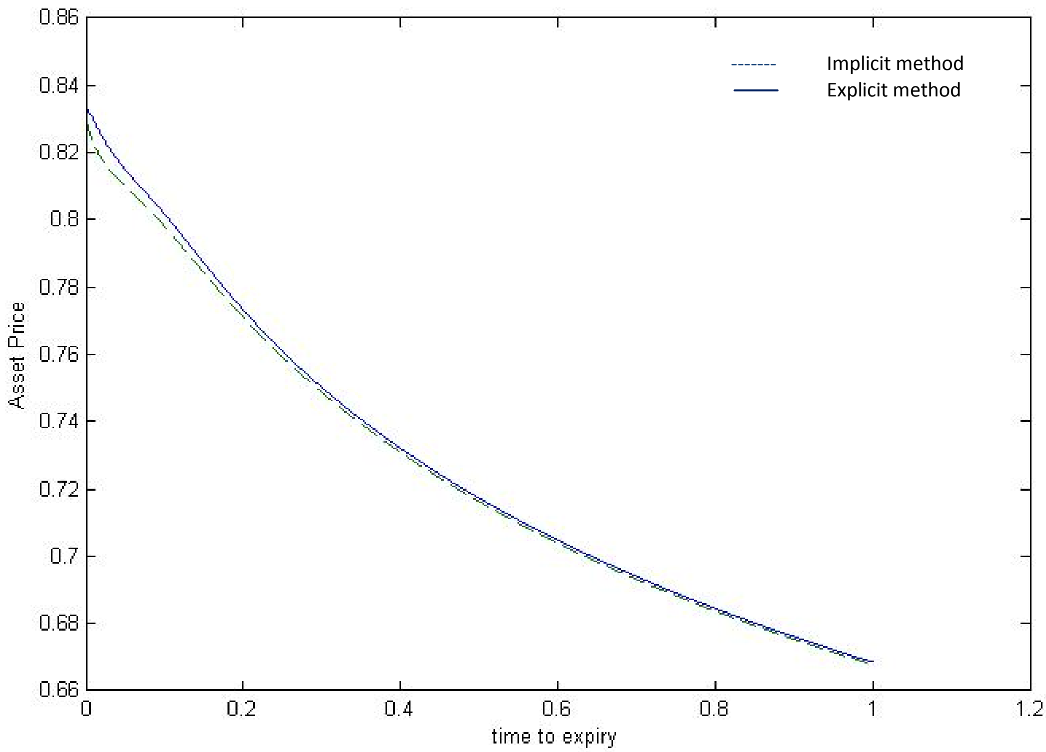

Figure 6

The parameters are $\Delta t = 5*(10)^{(-6)}$, $\Delta x = 0.001$.

The location of the early exercise boundary at t = 1 is $S_f^n = 0.6677$ for the explicit method and 0.6686 for the implicit method.

Remarks

From all the figures above, we see that the implicit and explicit scheme match up to the second decimal place.

We can also notice the fact that for d $\leq$ r the early exercise boundary is convex but for d $>$ r it is not convex. We see a bump in both the implicit and explicit schemes for d $>$ r and the bump can be seen clearly from the figure of the first derivative of the early exercise boundary S_f^n as shown in **Figure 7** and **Figure 8.**

Another fact that can be seen from **Figure 9** is that for d $>$ r, the bump spreads out as we increase d from 0.06 to 0.07, keeping r = 0.05.

3.2.7 Graph of the first derivative of the early exercise boundary S_f versus time with dividend rate d = 0.55 > r = 0.05

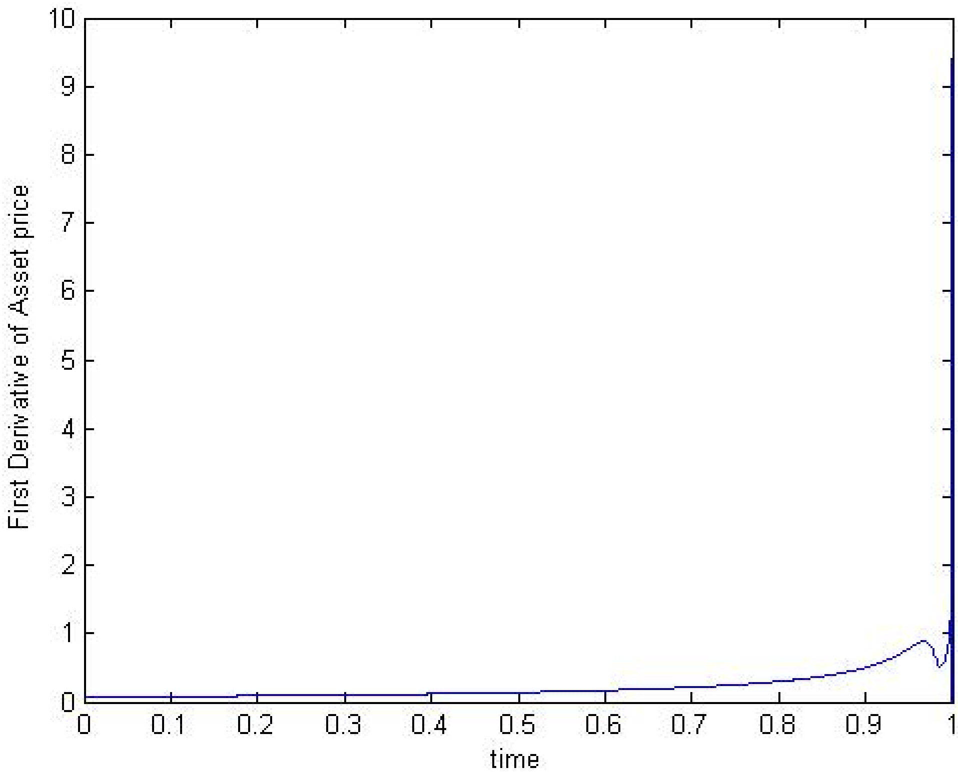

Figure 7

The parameters are given by: r = 0.05, d = 0.055, $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001.

3.2.8 Graph of the first derivative of the early exercise boundary S_f versus time with dividend rate d = 0.57 > r = 0.05

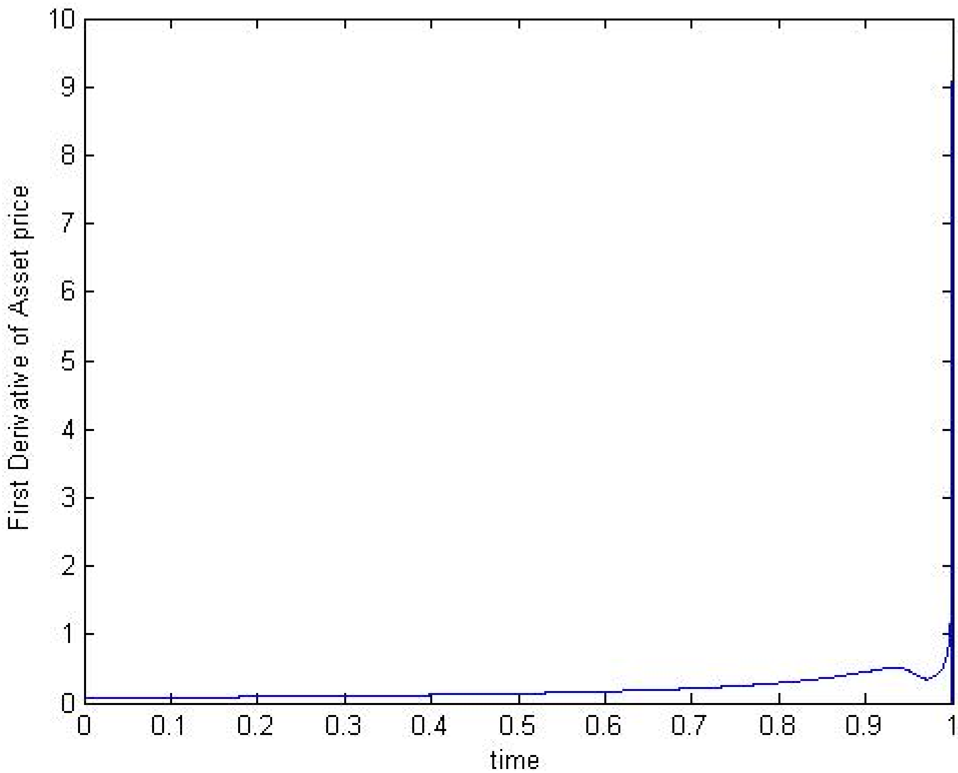

Figure 8

The parameters are given by: $\Delta t = 5*(10)^{(-6)}$, Δx = 0.001.

3.2.9 Graph of the first derivative of the early exercise boundary S_f versus time with dividend rate d = 0.06 > r = 0.05

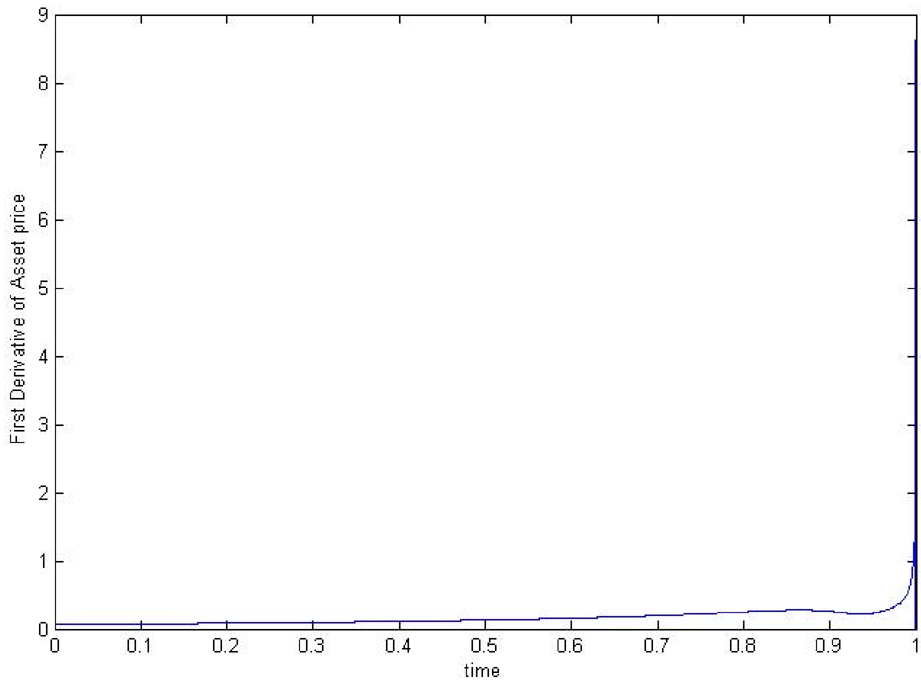

Figure 9

The parameters are given by: $\Delta t = 5*(10)^{(-6)}$, $\Delta x = 0.001$.

4.0 CONCLUSION

In this paper we show numerically that the behavior of the American put option is convex for all values of $d \leq r$ (where d is the dividend rate and r is the risk-free interest rate) and is not convex for d > r.

In modeling this problem, quite a number of obstacles were overcome. Firstly the explicit code would not run until we introduced the boundary condition given by the equation $p_1^n = -(\Delta x) S_f^n$ for S_f^n = E in the Matlab code. While trying to run the explicit code for values d > r, we also faced the problem of convergence, and every time the code was being executed it blew up. The reason for this was that the denominator $D_1^{n+1} + (1 + \Delta x)$ given in equation (45) would become zero for values d > r.

To solve this problem we introduced a different formula for S_f^n for values d > r obtained from the equation used below [4].

$$S_f^n = \left(\frac{r}{d}\right) * E * \left(1 - \sigma * alpha * \sqrt{2 * (T - t)}\right) \text{, where alpha } = 0.4517$$

In the implicit code we did not encounter this problem and started with the formula of S_f^n .We used the Newton Raphson's method in the iteration step in the implicit code.

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