CONTAGION IN FINANCIAL BANKING SYSTEMS

by

Cristina M. Megas

B. S., Mathematics, University Of Toronto, 2006

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This thesis was presented

by

Cristina M. Megas

It was defended on

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and approved by

Bogdan Ion, PhD, Professor

Thesis Advisors: John M. Chadam, PhD, Professor and

Gregory M. Constantine, PhD, Professor

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This paper takes a financial network, applies a shock to the system and looks at the resulting institutions that fail. It considers the propagation of contagion through the financial network by employing various techniques. The first method calculates unique clearing payments for all the banks in the system. It also defines fundamental default and contagious failure of any financial institution and differentiates between these two important concepts.

The second method uses mean-field approximations to make all the banks in the system identical. It reduces an institution's external assets so that it defaults and looks at the subsequent failures that spread through the financial network. This technique provides criteria for shocks and for initial and successive defaults. It considers cases with and without liquidity shocks.

This paper presents a connectivity measure using Kirchhoff's theorem. It computes the Kirchhoff number of all the banks in a financial network and finds the most and least vulnerable institutions. Finally, this paper tests the described connectivity measure by performing simulations on financial systems with a varying number of large banks and analyzing the results.

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1.0 INTRODUCTION

Within the last few years, bank crashes have led to a significant disturbance of the financial system in the United States and many other countries around the world. Financial failures have spread out to negatively affect many aspecst of the larger economy ([2], p. 3). The crisis that started in 2008 has made it clear that we need a better understanding of financial networks and systemic risk.

Regulators have found it challenging to foresee the effect of defaults as a result of a lack of transparency on the structure of the financial system and adequate methods to screen systemic risk ([2], p. 3). The complexity of current financial systems around the globe makes it difficult to create indicators that accurately assess the systemic risk of any institution([2],p. 3).

One of the most significant aspects that has been emphasized in the past several years has been the interconnectedness of banks in the financial network ([2], p. 3). This has lead to an increase in the probability of **contagion**, a scenario in which small shocks, that initially only impact a fraction of the institutions in the system, spread to the entire network ([2], p. 3).

2.0 NETWORKS

In order to understand the structure of the financial system, we first consider various general aspects of networks. Networks that have been studied so far include the Internet, the World Wide Web, social networks of connections between people, networks of business relations between different companies, food webs, distribution networks (blood vessels, postal delivery routes), networks of citations ([8], p. 168-169). A **network** is characterized by a number of **nodes** or **vertices** and the **edges** or **connections** between them ([8], p. 168). While in the past analysis was focused on small graphs and the characteristics of its vertices, today scientists have begun to investigate large scale properties of networks ([8], p. 169). This new tactic has been the result of computers' ability to analyze large sets of data and by the fact that many real-world networks contain millions of vertices and edges ([8], p. 169, 171).

The simplest type of network is a collection of vertices linked by edges ([8], p. 171). However, many networks are more complex than this ([8], p. 171). They may have different types of vertices and edges, each with various properties ([8], p. 171). As an example, consider a social network of individuals (Facebook), where the nodes could correspond to men or women, people of distinct nationalities, locations, ages, incomes or jobs ([8], p. 171). The edges might represent friendship, work contacts or geographical closeness ([8], p. 171). The edges might also be weighted to show how well any two people in the network are acquainted with each other ([8], p. 171-172). Networks can be **directed** or **undirected** ([8], p. 172). In a directed graph, every edge has a direction, with some edges that run both ways ([8], p. 172). A graph of phone calls between people is directed ([8], p. 172). In an undirected network, all edges run in both directions ([8], p. 172). Graphs may transform over time, with some nodes or edges appearing or vanishing; some values on the edges and vertices might also change ([8], p. 172).

The **degree** of a vertex is the number of edges connected to it ([8], p. 173). Every vertex in a directed graph has a degree, but also an in-degree and an out-degree ([8], p. 173). The in-degree represents the number of incoming edges, while the out-degree is the number of outgoing connections ([8], p. 173). In a directed graph, the degree of a vertex is the sum of its in-degree and out-degree ([8], p. 173). Vertices in undirected graphs are only described by their degrees ([8], p. 173).

In any network of n nodes, p_k represents the fraction of vertices that have degree k ([8], p. 185). It is the probability that any randomly chosen node in the network has degree k ([8], p. 185). We plot p_k by creating a histogram of the degrees of vertices ([8], p. 185). This histogram shows the degree distribution of the network ([8], p. 185). However, real-world networks do not exhibit a binomial distribution, but are, instead, right-skewed, showing a long right tail in their distribution ([8], p. 185). Measuring the right tail of real-world graphs is challenging since histograms are noisy ([8], p. 185). As an alternative, the (complementary) cumulative distribution function is plotted to find the type of degree distribution of the network ([8], p. 185). It has the advantage of diminishing the noise in the tail of the distribution and is defined by

$$P_k = P(d \geq k) = \sum_{d=k}^{n} p_d$$

([8], p. 185-186)

When plotting the cumulative distributions, power-law and exponential degree distributions are observed ([8], p. 186). The former exhibit power laws in their tails, such that

$$p_k \sim k^{-\alpha}$$

and

$$P_k \sim \sum_{d=k}^{n} d^{-\alpha} \sim k^{-(\alpha-1)}$$

for some constant, α ([8], p. 186). The latter distributions show exponential tails, such that

$$p_k \sim e^{-\frac{k}{\beta}}$$

and

$$P_k \sim \sum_{d=k}^{n} p_d \sim \sum_{d=k}^{n} e^{-\frac{d}{\beta}} \sim e^{-\frac{k}{\beta}}$$

for some constant, β ([8], p. 186). These two degree distributions are found by plotting the corresponding (complementary) cumulative distributions on logarithmic scales (for power laws) or semi-logarithmic scales (for exponentials) ([8], p. 186).

When it comes to directed networks, degree distributions become more intricate ([8], p. 186). These types of networks have both an **in-degree** and an **out-degree**, so the degree distribution is a function p_{jk} of two variables, taking into account the fraction of nodes that concurrently have in-degree j and out-degree k ([8], p. 186). The in-degree, out-degree and degree distributions can all be derived for directed graphs ([8], p. 186).

Networks with power-law distributions are called **scale-free** ([8], p. 186). Real-world scale-free networks include the World Wide Web, the Internet and metabolic networks ([8], p. 186, 188). Scale-free graphs show that few nodes have many connections, while most have a low number of edges directly attached to them. Networks with exponential distributions are **random** networks ([8], p. 188). The power-grid and railway networks are examples of graphs with exponential distributions ([8], p. 188).

Various models have been created to accurately reproduce power-law and exponential degree distributions of real-world networks. Scale-free network models include **Price's model** and the **Barabási and Albert model** ([8], p. 213-221), while random networks were represented by the **Erdös-Rényi** or **Poisson random graphs** ([8], p. 197-199).

Price studied the graph of citations between scientific papers in 1965 and discovered that the network's in-degrees and out-degrees have power-law distributions ([8], p. 213). **Price's model** consists of a directed network of n nodes where p_k is the fraction of vertices with in-degree k such that

$$\sum_{k=1}^{n} p_k = 1$$

([8], p. 213). New nodes are being appended to the graph, at a non-constant rate, such that each added vertex has an out-degree which is permanently fixed at the beginning ([8], p. 213). The out-degree of vertices vary, but the mean out-degree, m, is constant ([8], p. 213). m is also the mean in-degree because $\sum_{k=1}^{n} kp_k = m$ ([8], p. 214). It is possible that $m < 1$ ([8], p. 214). Price defined the probability of attachment of one of the new edges to an old vertex to be proportional to $k + k_0$, where k is the old node's in-degree and k_0 is a constant ([8], p. 214). Following various calculations, Price found that

$$p_k \sim k^{-(2+\frac{1}{m})}$$

which means that the in-degree distribution exhibits a power tail with exponent $2 + \frac{1}{m}$ ([8], p. 214). Price's model is also called cumulative advantage or preferential attachment ([8], p. 213).

The **model of Barab*á*si and Albert** is similar to the preferential attachment method ([8], p. 215). Vertices are added to a network of degree m and every edge is appended to another node with probability proportional to the degree of that vertex ([8], p. 215). However, the graph of the Barab*á*si and Albert model is undirected, so that in-degrees and out-degrees coincide ([8], p. 215). They found that

$$p_k \sim k^{-3}$$

which means that the degree distribution shows a power tail with a single fixed exponent, 3 ([8], p. 216-217).

Solomonoff and Rapoport and, independently, Erdös and Rényi constructed a simple model of a random graph ([8], p. 197). **Erdös and Rényi** called their models $G_{n,p}$ and $G_{n,m}$ ([8], p. 197). They considered n vertices and connected each pair in the network with probability p ([8], p. 197). $G_{n,p}$ is a collection of all such networks in which a graph having m edges shows up with probability $p^m(1-p)^{M-m}$, where $M = \frac{n(n-1)}{2}$ represents the maximum number of possible edges ([8], p. 197). $G_{n,m}$ denotes all graphs of n nodes and exactly m edges, such that each graph has equal probability of materializing ([8], p. 197). In the $G_{n,p}$ model, the mean degree of the network, $z = p(n-1)$, is constant, so that the degree distribution of the graph is Poisson ([8], p. 197). This means that the probability of vertex having degree k is

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{z^k e^{-z}}{k!}$$

([8], p. 198).

3.0 THE FINANCIAL SYSTEM

The financial system can be described by a weighted, directed network, identified by a triplet (V, E, c) that consists of:

- a set, V, of financial institutions, whose number is n.
- an nxn matrix E of bilateral exposures
- a vector, c, of capital amounts

([2], p. 7).

The matrix of bilateral exposures

$$E = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}$$

has entries e_{ij} which represent the exposure of bank i to bank j; each e_{ij} shows the liabilities of institution i to institution j, $\forall\, i, j \in V$ ([4], p. 5-6, [10], p. 834). It is also the greatest short-term loss of bank j in case bank i defaults ([2], p. 7). Since no bank can borrow or lend money to itself, the bilateral exposures matrix becomes

$$E = \begin{pmatrix} 0 & e_{12} & \cdots & e_{1n} \\ e_{21} & 0 & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & 0 \end{pmatrix}$$

where $e_{ij} = 0$ if $i = j$, $\forall\ i,\ j \in V$ ([10], p. 835).

The vector of capital is

$$c = (c_1, c_2, \ldots, c_n)$$

where each c_i is the capital of institution i, $\forall\ i \in V$ ([2], p. 7). It is sometimes called the shareholders' equity, the equity capital or net worth of bank i and symbolizes a financial institution's ability to absorb losses ([2], p. 7, [4], p. 5).

3.1 THE FINANCIAL NETWORK

A financial network is heterogeneous and consists of n banks of different sizes whose edges are represented by e_{ij} and the matrix of bilateral exposures, E ([2], p. 7-8). It is directed and some of the connections between financial institutions are pointed in both directions ([2], p. 7-8). Some banks are highly connected, acting as 'hubs' of the networks, while most institutions only have a few links ([2], p. 8). The **in-degree**, $k_{in}(i)$, of a vertex $i \in V$ is defined to be the number of its debtors such that

$$k_{in}(i) = \sum_{j=1}^{n} \mathbb{1}_{\{e_{ji}>0\}}$$

([2], p. 8). The **out-degree**, $k_{out}(i)$, of a node $i \in V$ is the number of its creditors

$$k_{out}(i) = \sum_{j=1}^{n} \mathbb{I}_{\{e_{ij}>0\}}$$

([2], p. 8). The **degree**, $k(i)$ of a vertex $i \in V$ denotes its connectivity and is

$$k(i) = k_{in}(i) + k_{out}(i)$$

([2], p. 8).

The total interbank assets of institution i are

$$a_i = \sum_{j=1}^{n} e_{ji}$$

which corresponds to the sum of column i in the matrix E ([10], p. 834). Its total interbank liabilities are

$$l_i = \sum_{j=1}^{n} e_{ij}$$

which is the sum of row i in E, $\forall\, i \in V$ ([10], p. 834).

Any bank i in the network has the balance sheet shown in table 3.1 ([4], p. 13, [6], p. 824, [7], p.29, [9], p. 2038-2040). The equity capital, c_i, is the difference between bank i's assets and liabilities, such that

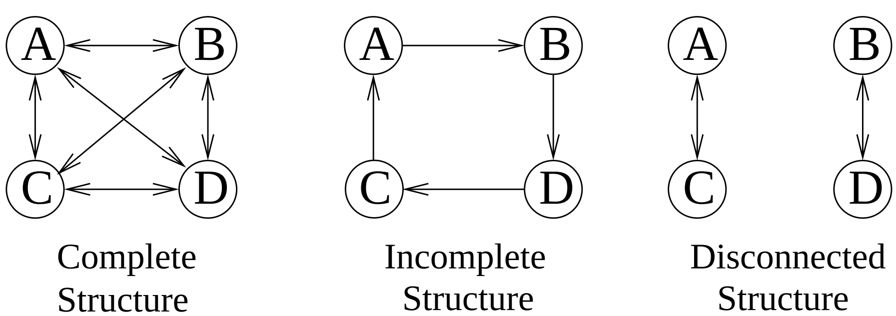
$$c_i = (a_i + e_i) - (l_i + d_i) \tag{3.1}$$

([2], p. 7, [9], p. 2039). A bank i in the network is **solvent** if its assets exceed its liabilities, so that $c_i > 0$ ([6], p. 824).

Table 3.1: The general structure of a bank's balance sheet

Assets	Liabilities
Interbank Assets, a_i	Interbank Liabilities, l_i
Other Assets (External Assets, e_i)	Other Liabilities (Customer Deposits, d_i)

Various graph structures were used to model interbank networks. In one of the first papers published on this topic, Allen and Gale studied a simple system of four banks which were connected in different ways ([1], [7], p. 27). They analyzed three types of financial networks: complete, incomplete and disconnected structures ([1], [7], p. 27). Freixas et al. studied the money centre structure, while Nier et al. focused their attention on the Erdös-Rényi structure ([7], p. 27). Moussa and Cont et al. examined scale-free structures ([2], p.3, [7]).



Figures 3.1, 3.2 and 3.3. Complete, incomplete and disconnected structures.

Real-world banking systems have also been studied in the last decade ([7], p. 26). Upper and Worms have have analyzed the German banking system in 2004 and concluded that it is two-tiered: the larger banks are on the upper tier, while the smaller institutions lie in the lower tier ([7], p. 26, [10], p. 837). The low-level banks have few connections with institutions in their locations; most of the links they established are with banks in the upper tier ([10], p. 837). Toivanen considered the financial system in Finland and deduced that it was a three-tier structure ([7], p. 26). Boss et al., Moussa and Cont explored the Austrian and the Brazilian systems, respectively, and discovered that they are scale free ([7], p. 26).

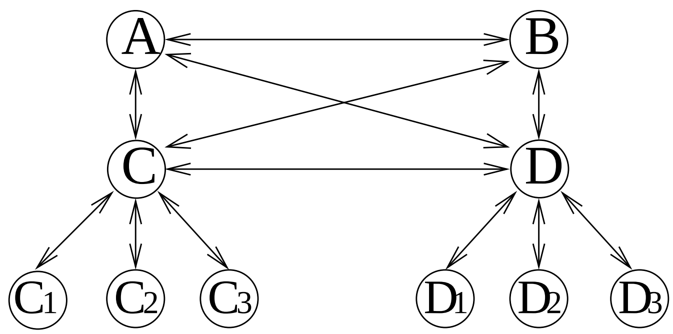


Figure 3.4. German network: two-tier structure.

4.0 THE PROBLEM

This paper analyzes the following issue: we have a financial network depicted by the triplet (V, E, c) and we want to jolt the system and find the number of banks that default as a result of the shock. We need to define the shock in this financial network, examine how it propagates and discover its effects in terms of the bank failures it produces.

Two types of defaults are considered:

- **Fundamental default** is the default that results when a bank is no longer able to honor its promises, given that all other institutions fulfill their obligations ([4], p. 6-7).
- **Contagious default** is the default that occurs only when other financial institutions are unable to keep their promises ([4], p. 7).

Fundamental default of a bank, as a result of a shock, may lead to contagious defaults of other institutions; however, contagious default cannot occur if there is no fundamental failure.

4.1 THE METHOD OF CLEARING PAYMENTS

The method of clearing payments was first introduced by Eisenberg and Noe ([4], p. 7). It finds out how much a financial institution is able to pay at a particular point in time. This procedure was further explained and used by Elsinger et al. ([4], p. 6-7, 31-32).

To employ the clearing payment technique, Elsinger et al. defined:

- a vector, $y \in \Re^n_+$ of the total liabilities of banks towards the rest of the system

  $$y = (y_1, y_2, \ldots, y_n)$$

  such that

  $$y_i = \sum_{j=1}^{n} e_{ij}$$

  for all financial institutions i in the system ([4], p. 6)

- a matrix $\Pi \in [0,1]^{nxn}$ which is obtained from the matrix E of bilateral exposures by normalizing the entries by total liabilities; Π shows how much debt any bank can hold towards all the other institutions in the network; the entries of Π are

  $$\Pi_{ij} = \begin{cases} \frac{e_{ij}}{y_i} & \text{if } y_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

  ([4], p. 6).

- **a clearing payment vector,** $p^* \in \Re^n_+$ that shows the total payments made by the institutions in the system under the clearing algorithm

  $$p^* = (p_1, p_2, \ldots, p_n)$$

  such that

  $$p^*_i = min\{y_i, max[\sum_{j=1}^{n} \Pi_{ji} p^*_j + c_i, 0]\}, \quad \forall i \in V \tag{4.1}$$

  ([4], p. 6).

The clearing payment vector provides us with two significant insights ([4], p.6). First of all, it points out the insolvent banks in the system ([4], p.6). This occurs if $p_i^* < y_i$ for any financial institution $i \in V$ ([4], p.6). Second, the recovery rate for any defaulting bank $i \in V$ can be calculated by the formula $\frac{p_i^*}{y_i}$ ([4], p.6). Eisenberg and Noe proved that there exists a unique clearing payment vector, p^* for each c ([4], p. 7); see theorem 2 ([3], p. 242, 248-249). Notice, that in the definition of p_i^*, the quantity $\sum_{j=1}^{n} \Pi_{ji} p_j^* + c_i$ can sometimes be negative. This is due to the fact that $c_i \in \Re$ can be negative for some institutions in the system; some banks may have a negative balance sheet, whereby their liabilities exceed their assets ([4], p.6).

Going back to the two concepts of default, Elsinger et al. state that **fundamental default** occurs if

$$\sum_{j=1}^{n} \Pi_{ji} y_j + c_i - y_i < 0, \quad \forall i \in V \tag{4.2}$$

([4], p. 6-7). The first two terms in (4.2) above represent the assets of bank i. The first term shows the sum of money that institution i should get from other banks, while c_i and y_i represent its capital and total debt. Bank i experiences **contagious default** if

$$\sum_{j=1}^{n} \Pi_{ji} y_j + c_i - y_i \geq 0 \tag{4.3}$$

but

$$\sum_{j=1}^{n} \Pi_{ji} p_j^* + c_i - y_i < 0 \tag{4.4}$$

([4], p. 7). (4.3) above shows that bank i would be solvent if all the other banks were able to make their payments. However, in (4.4), $\sum_{j=1}^{n} \Pi_{ji} p_j^*$ is the actual amount that bank i receives from the other institutions.; this means that bank i is effectively insolvent due to the fact that the other banks make payments that are smaller than the amounts they owe.

4.2 A TOY EXAMPLE

To exemplify the method of clearing payments discussed above, consider a system with three banks, $n = 3$ ([4], p. 31). Suppose that the matrix of bilateral exposures is

$$E = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

and the vector of capital is $c = (1, 1, 1)$ ([4], p. 31). We can depict the banking network with the following diagram.

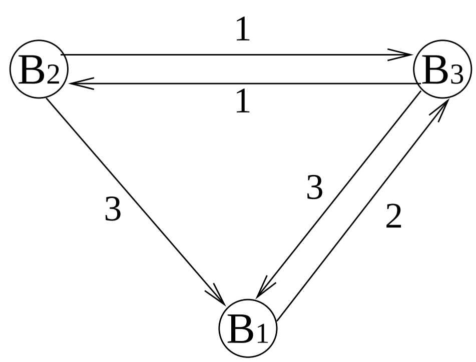


Figure 4.1. Graphical representation of the toy-model network of interbank exposures.

B_1, B_2, B_3 represent the three institutions in our system and the number on the arrows are their exposures. For example, B_3 owes an amount of 3 to B_1 and a sum of 1 to B_2 ([4], p. 31).

We can calculate $y = (2, 4, 4)$ by summing up each row of E ([4], p. 31). We can also find

$$\Pi = \begin{pmatrix} 0 & 0 & 1 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

This can be described by a new diagram of the system.

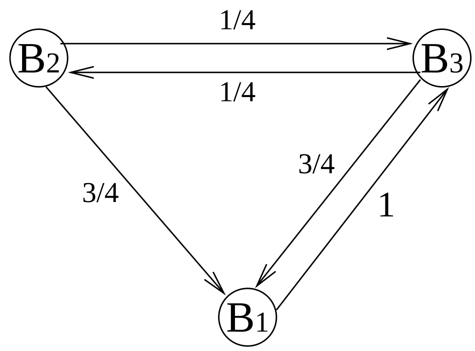


Figure 4.2. Graphical representation of the toy-model network of normalized interbank exposures.

The clearing payments can now be computed using (4.1). We now obtain the system

$$\begin{cases} p_1^* = min\{2, max[\frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1, 0]\} \\ p_2^* = min\{4, max[\frac{1}{4}p_3^* + 1, 0]\} \\ p_3^* = min\{4, max[p_1^* + \frac{1}{4}p_2^* + 1, 0]\} \end{cases}$$

Since p_1^*, p_2^*, $p_3^* \geq 0$, the system becomes

$$\begin{cases} p_1^* = min\{2, \frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1\} \\ p_2^* = min\{4, \frac{1}{4}p_3^* + 1\} \\ p_3^* = min\{4, p_1^* + \frac{1}{4}p_2^* + 1\} \end{cases}$$

p_1^* can be written as

$$p_1^* = \begin{cases} 2 & \text{if } \frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1 \geq 2 \\ \frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1 & \text{if } \frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1 < 2 \end{cases}$$

If $p_1^* = 2$, then

$$\begin{cases} p_2^* = min\{4, \frac{1}{4}p_3^* + 1\} \\ p_3^* = min\{4, \frac{1}{4}p_2^* + 3\} \end{cases}$$

p_2^* can be written as

$$p_2^* = \begin{cases} 4 & \text{if } \frac{1}{4}p_3^* + 1 \geq 4 \\ \frac{1}{4}p_3^* + 1 & \text{if } \frac{1}{4}p_3^* + 1 < 4 \end{cases}$$

If $p_2^* = 4$, then $p_3^* = 4$, but $\frac{1}{4}p_3^* + 1 = 2 \geq 4$ (contradiction). If $p_2^* = \frac{1}{4}p_3^* + 1$, then $p_3^* = min\{4, \frac{1}{16}p_3^* + \frac{13}{4}\}$, or we can write

$$p_3^* = \begin{cases} 4 & \text{if } \frac{1}{16}p_3^* + \frac{13}{4} \geq 4 \\ \frac{1}{16}p_3^* + \frac{13}{4} & \text{if } \frac{1}{16}p_3^* + \frac{13}{4} < 4 \end{cases}$$

If $p_3^* = 4$, then $p_2^* = 2$, but $\frac{1}{4}p_2^* + 3 = \frac{7}{2} \geq 4$ (contradiction). If $p_3^* = \frac{1}{16}p_3^* + \frac{13}{4}$, then $p_3^* = \frac{52}{15}$ and $p_2^* = \frac{28}{15}$. We check the conditions:

$$\frac{1}{4}p_3^* + 1 = \frac{28}{15} < 4$$

$$\frac{1}{16}p_3^* + \frac{13}{4} = \frac{52}{15} < 4$$

$$\frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1 = 5 > 2$$

If $p_1^* = \frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1$, then

$$\begin{cases} p_2^* = min\{4, \frac{1}{4}p_3^* + 1\} \\ p_3^* = min\{4, p_2^* + \frac{3}{4}p_3^* + 2\} \end{cases}$$

p_2^* can be written as

$$p_2^* = \begin{cases} 4 & \text{if } \frac{1}{4}p_3^* + 1 \geq 4 \\ \frac{1}{4}p_3^* + 1 & \text{if } \frac{1}{4}p_3^* + 1 < 4 \end{cases}$$

If $p_2^* = 4$, then $p_3^* = min\{4, 6 + \frac{3}{4}p_3^*\}$, or we can write

$$p_3^* = \begin{cases} 4 & \text{if } 6 + \frac{3}{4}p_3^* \geq 4 \\ 6 + \frac{3}{4}p_3^* & \text{if } 6 + \frac{3}{4}p_3^* < 4 \end{cases}$$

If $p_3^* = 4$, then $\frac{1}{4}p_3^* + 1 = 2 \geq 4$ (contradiction). If $p_3^* = 6 + \frac{3}{4}p_3^*$, then $p_3^* = 24$, but $6 + \frac{3}{4}p_3^* = 24 < 4$ (contradiction). If $p_2^* = \frac{1}{4}p_3^* + 1$, then $p_3^* = min\{4, p_3^* + 3\}$, or we can write

$$p_3^* = \begin{cases} 4 & \text{if } p_3^* + 3 \geq 4 \\ p_3^* + 3 & \text{if } p_3^* + 3 < 4 \end{cases}$$

If $p_3^* = 4$, then $p_2^* = 2$, but $p_2^* + p_3^* = 6 < \frac{4}{3}$ (contradiction). If $p_3^* = p_3^* + 3$, a solution cannot be found for p_3^*, p_2^* or p_1^*. Therefore, the unique solution of the system is

$$\begin{cases} p_1^* = 2 \\ p_2^* = \frac{28}{15} \\ p_3^* = \frac{52}{15} \end{cases}$$

We can calculate and graphically represent the clearing payments that each bank is able to make to every other institution in the system. For example, B_2 owed $\frac{3}{4}$ of its debt to B_1, so that B_2 makes a payment of $\frac{3}{4}p_2^* = \frac{7}{5}$ to B_1.

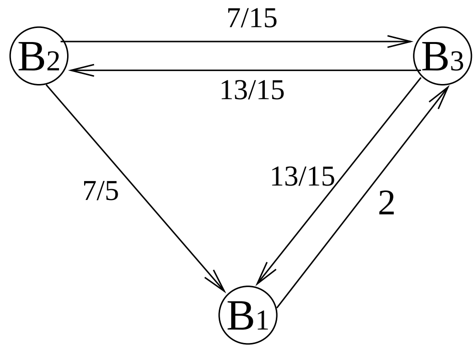


Figure 4.3 Graphical representation of the toy-model network showing clearing payments

We need to check equations (4.2), (4.3) and (4.4) to see if any of the banks in our system are defaulting. For the first bank, B_1, we have

$$\frac{3}{4}y_2 + \frac{3}{4}y_3 + c_1 - y_1 = 5 > 0$$

$$\frac{3}{4}p_2^* + \frac{3}{4}p_3^* + c_1 - y_1 = 3 > 0$$

For the second bank, B_2, we obtain

$$\frac{1}{4}y_3 + c_2 - y_2 = -2 < 0$$

$$\frac{1}{4}p_3^* + c_2 - y_2 = \frac{-32}{15} < 0$$

For the third bank, B_3, we get

$$y_1 + \frac{1}{4}y_2 + c_3 - y_3 = 0 \geq 0$$

$$p_1^* + \frac{1}{4}p_2^* + c_3 - y_3 = \frac{-8}{15} < 0$$

This leads us to conclude that the first bank is solvent ([4], p.31). B_2 experiences **fundamental default**, while B_3 finds itself in **contagious default** ([4], p.31). The fundamental default of the second bank leads the third institution into insolvency ([4], p.31). We can also check this by looking at the assets and liabilities of each bank. For the first bank,

$$1 + \frac{7}{5} + \frac{13}{5} - 2 = 3 > 0$$

which means that the institution can meet its obligations without difficulties. However, B_2 and B_3 are insolvent since they have no capital to dispose of in their transactions:

$$1 + \frac{13}{15} - \frac{7}{15} - \frac{7}{5} = 0$$

$$1 + 2 + \frac{7}{15} - \frac{13}{15} - \frac{13}{5} = 0$$

Consider the same scenario, with the same n, E, y, Π, but with a different vector, $c' = (1, 3, 2)$ ([4], p.31). We use the same algorithm described by equation (4.1) and detailed in the previous pages to solve the system

$$\begin{cases} p_1^* = min\{2, \frac{3}{4}p_2^* + \frac{3}{4}p_3^* + 1\} \\ p_2^* = min\{4, \frac{1}{4}p_3^* + 3\} \\ p_3^* = min\{4, p_1^* + \frac{1}{4}p_2^* + 2\} \end{cases}$$

The unique solution is

$$\begin{cases} p_1^* = 2 \\ p_2^* = 4 \\ p_3^* = 4 \end{cases}$$

([4], p.32).

Since $p_1^* = y_1$, $p_2^* = y_2$, $p_3^* = y_3$, every bank is able to meet its obligations, so that no institution in the system defaults ([4], p.31). We can formally check equations (4.2), (4.3) and (4.4). For B_1, we have

$$\frac{3}{4}y_2 + \frac{3}{4}y_3 + c_1' - y_1 = 5 > 0$$

$$\frac{3}{4}p_2^* + \frac{3}{4}p_3^* + c_1' - y_1 = 5 > 0$$

For B_2, we obtain

$$\frac{1}{4}y_3 + c_2' - y_2 = 0 \geq 0$$

$$\frac{1}{4}p_3^* + c_2' - y_2 = 0 \geq 0$$

For the third bank, B_3, we get

$$y_1 + \frac{1}{4}y_2 + c_3' - y_3 = 1 \geq 0$$

$$p_1^* + \frac{1}{4}p_2^* + c_3' - y_3 = 1 \geq 0$$

5.0 SCENARIO

We consider the following scenario. Suppose we have a random Erdös-Rényi network of n banks which are connected with probability p ([6], p. 824). Every institution in this financial system has a balance sheet described by table 3.1 ([6], p. 824). A bank's average number of incoming or outgoing connections is

$$z = p(n-1) \tag{5.0}$$

([6], p. 824). An institution $i \in V$ has a number of incoming links, $z_i^{(in)}$, which represent the number of interbank assets; it also has a number of outgoing connections, $z_i^{(out)}$, which is the number of interbank liabilities ([6], p. 824). $z_i^{(in)}$ and $z_i^{(out)}$ vary for the institutions in the financial system, but z is constant ([6], p. 824).

Suppose that θ_i is the ratio of bank i's interbank assets to total assets,

$$\theta_i = \frac{a_i}{a_i + e_i} \tag{5.1}$$

([6], p. 824-825). We let θ represent the ratio of the financial system's interbank assets to all assets, such that

$$\theta = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i + e_i} \tag{5.2}$$

and $0 < \theta < 1$ ([6], p. 824, [9], p. 2039). The interbank assets of the system and θ are fixed ([6], p. 824, [9], p. 2039). We also define w to be the average value of each individual interbank asset or loan given out by an institution

$$w = \frac{\frac{\theta \sum_{i=1}^{n} a_i + e_i}{n}}{z} \tag{5.3}$$

such that, for any bank $i \in V$, $a_i + e_i = e_i + z_i^{(out)} w \iff a_i = z_i^{(out)} w$ ([6], p. 824). We obtain $\theta_i = \frac{z_i^{(out)} w}{a_i + e_i}$ ([6], p. 825). According to this algorithm, θ_i, $a_i + e_i$ (the total assets of a bank) and the net worth, c_i, will vary from institution to institution, depending on $z_i^{(out)}$ ([6], p. 825).

To better analyze the situation, we use **mean-field approximations** in our financial network ([6], p. 825). We let

$$a_i + e_i = 1, \quad e_i = e, \quad a_i = a, \quad l_i = l, \quad d_i = d$$

$$c_i = c = 1 - (l + d)$$

for all banks $i \in V$ ([6], p. 825). This means that all the institutions in the system have total assets normalized to unity; they have the same external and interbank assets and the same levels of capital ([6], p. 825). From equations (5.1), (5.2) and (5.3), we can conclude that

$$\theta_i = \theta = a$$

$$z_i^{(out)} = z$$

Every bank has the same value of c, θ, w and z ([6], p. 825). Furthermore, since $a + e = 1$, $\theta = a$, we find that $e = 1 - \theta$ and $l = \theta$ ([6], p. 825). We now have a homogeneous financial network in which all banks are the same.

5.1 SHOCKS AND CONTAGION

Various authors define shocks in different ways. The shock to the system might shave off a percentage of a bank's external assets ([6], [9]), or it may take off a percentage of assets that cannot be recovered in the case of bankruptcy ([10]). The shock might reduce a bank's capital through various uncertainty scenarios like changes in foreign exchange or interest rates ([2], [4], [7]).

We can shock the entire financial network by applying the shock to:

- a single bank and counting the number of defaults that occur; this is repeated for all the institutions in the system ([6], [9], [10]).
- all the banks at the same time and tallying the number of institutions that fail ([2], [4], [7]).

5.1.1 SHOCKS AND CONTAGION SPREAD BY INTERBANK LOANS

Following the explained scenario, we consider the impact of a shock hitting any single institution in our network by wiping out a fraction, f, of its external assets; f is a number between 0 and 1 ([6], p. 825). We define the **phase I shock** as

$$s(I) = f(e) = f(1 - \theta) \tag{5.4}$$

([6], p. 825). The financial institution will fail if

$$s(I) > c \tag{5.5}$$

([6], p. 825). We assume that no liquidity shock factor affects this defaulting institution, nor the value of the assets of the banks in the financial system ([6], p. 825). The loss $s(I) - c$ is shared equally among the defaulting bank's creditors (since all institutions in the network are the same and have the same value of w) ([6], p. 825). This happens as long as the loss, $s(I) - c$, is less than the defaulting institution's liabilities, θ ([6], p. 825). If not, each creditor losses its loan and the failing bank's customers also lose some of their deposits ([6], p. 825).

The **phase I default** of a single bank leads each of its z creditors to experience a **phase II shock** of strength

$$s(II) = \frac{min\{\theta, s(I) - c\}}{z} = \frac{min\{\theta, f(1-\theta) - c\}}{z} \tag{5.6}$$

([6], p. 825-826). The z creditor institutions will **default in phase II** if

$$s(II) > c \tag{5.7}$$

([6], p. 826).

In turn, this may lead to a **phase III shock** of magnitude

$$s(III) = \frac{min\{\theta, s(II) - c\}}{z} = \frac{min\{\theta, \frac{min\{\theta, f(1-\theta)-c\}}{z} - c\}}{z} \tag{5.8}$$

([6], p. 826). **Phase III defaults** might occur if

$$k_c s(III) > c \tag{5.9}$$

for a constant, k_c ([6], p. 826). This may lead to phase IV, phase V, and so on, as the shocks and subsequent defaults cascade through the financial system ([6], p. 826).

Equations (5.4) and (5.5) give us the condition for phase I failure

$$f(1-\theta) - c > 0 \tag{5.10}$$

([6], p. 826). We know that $c > 0$ and, since $0 < \theta < 1$, then $c < f$.

From equation (5.6) and (5.7), we obtain the criterion for phase II failure of each of the z creditor banks of the initial defaulting institution

$$min\{\theta, f(1-\theta) - c\} > zc \tag{5.11}$$

([6], p. 826). We find θ_c, the critical value of θ, by equating the quantities inside the minimum function of (5.11).

$$\theta = f(1-\theta) - c \iff \theta = f - f\theta - c$$

$$\theta + f\theta = f - c$$

$$\theta_c = \frac{f-c}{1+f} \tag{5.11a}$$

([6], p. 826). If $f \to 0$, $\theta_c \to 0$ and if $f \to 1$, $\theta_c \to \frac{1}{2}$. We reconsider equation (5.11). If $\theta < \theta_c$, then

$$min\{\theta, f(1-\theta) - c\} > zc \iff \theta > zc \tag{5.12}$$

([6], p. 826). If $\theta > \theta_c$, equation (5.11) becomes

$$f(1-\theta) - c > zc \iff f - f\theta - c > zc$$

$$f\theta < f - c - zc$$

$$\theta_c < 1 - \frac{c(1+z)}{f} \tag{5.13}$$

([6], p. 826).

We can now graph the lines $\theta_c = \frac{f-c}{1+f}$, $\theta = zc$, and $\theta_c = 1 - \frac{c(1+z)}{f}$ ([6], p. 826). We obtain the $(1, 0, f)$ triangle with c and θ as the x- and y-axes ([6], p. 826-827). When $\theta < \theta_c$ and $\theta > zc$, the phase II failure of the z creditors is described by the $(\frac{f}{1+f}, 0, A)$ triangle. When $\theta > \theta_c$ and $\theta < 1 - \frac{c(1+z)}{f}$, phase II default occurs in the $(1, \frac{f}{1+f}, A)$ triangle. In conclusion, phase II failures can only occur to the left of point A, in the $(1, 0, A)$ triangle ([6], p. 827). No phase II default can occur to the right of point A and outside the $(1, 0, A)$ triangle ([6], p. 827). A is the intersection of three lines, whose x-coordinate is

$$\frac{f-c}{1+f} = zc \iff f - c = zc + zcf$$

$$c(1 + z + zf) = f$$

$$c = \frac{f}{1 + z(1 + f)}$$

([6], p. 827). Phase II failure does not occur if

$$c > \frac{f}{1 + z(1 + f)} \tag{5.14}$$

([6], p. 827).

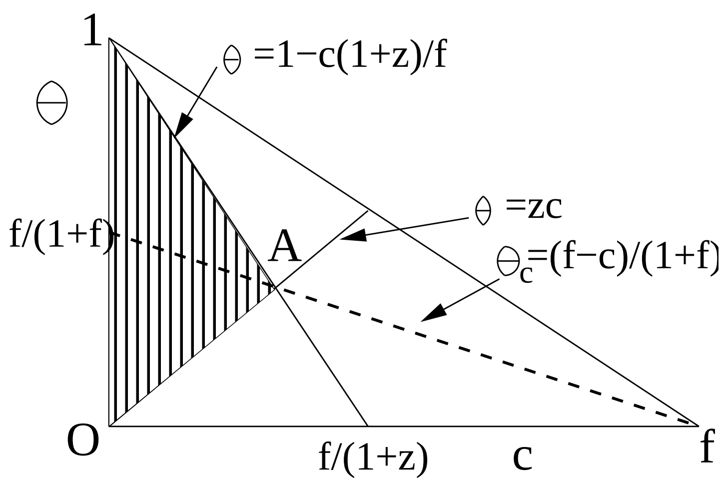


Figure 5.1. The shaded area represents the region where creditors of the initially defaulting bank will also fail in phase II.

After phase II has ended, the z creditor banks of the initial failing institution have defaulted ([6], p. 827). We separate the banks that have failed (z banks from phase II and 1 bank from phase I) from the institutions that have not yet defaulted ([6], p. 827). We calculate the number of unaffected banks

$$
\begin{aligned}
n - (z+1) &= n - [p(n-1)+1] \\
&= (n-1)(1-p)
\end{aligned}
$$

([6], p. 827). Each of the unaffected institutions will face $k = 0, 1, 2, \ldots, z$ hits from the z phase II failing banks with probability

$$
P(k) = \frac{z!}{(z-k)!k!} p^k (1-p)^{z-k}
$$

([6], p. 827). z may not necessarily be an integer; in practice, z is approximated by the nearest integer ([6], p. 827). For each unhit institution, we calculate $P(k)$, $\forall k$, and we find the largest such probability ($P(k)$ closest to 1) and the corresponding k ([6], p. 827). We let k_c be the maximum of all values k for each of the institutions in our financial network ([6], p. 827). k_c is the largest value of k such that at least one of the previously unhit banks will probably be impacted k times ([6], p. 827). $k_c = 1, 2, \ldots, z$ ([6], p. 828).

Since it is likely that one or more banks will experience k_c of these shocks, the condition for phase III default is

$$
\begin{aligned}
k_c s(III) > c \iff s(III) &> \frac{c}{k_c} \\
min\{\theta, \frac{min\{\theta, f(1-\theta) - c\}}{z} - c\} &> \frac{zc}{k_c} \qquad (5.15)
\end{aligned}
$$

([6], p. 827). If $\theta < \theta_c$, equation (5.15) becomes

$$
\begin{aligned}
min\{\theta, \frac{\theta}{z} - c\} > \frac{zc}{k_c} \iff \frac{\theta}{z} - c &> \frac{zc}{k_c} \\
\theta &> zc + \frac{z^2 c}{k_c} \\
\theta &> zc(1 + z^*) \qquad (5.16)
\end{aligned}
$$

where $z^* = \frac{z}{k_c}$ ([6], p. 828). If $\theta > \theta_c$, then

$$
\begin{aligned}
min\{\theta, \frac{f(1-\theta)-c}{z} - c\} > \frac{zc}{k_c} \iff & \frac{f(1-\theta)-c}{z} - c > \frac{zc}{k_c} \\
& \theta > zc + \frac{z^2c}{k_c} \\
& f(1-\theta) > c + zc + \frac{z^2c}{k_c} \\
& \theta < 1 - \frac{c + zc + zz^*c}{f} \\
& \theta < 1 - \frac{c(1+z+zz^*)}{f} \qquad (5.17)
\end{aligned}
$$

where $z^* = \frac{z}{k_c}$ ([6], p. 828).

We can now graph the lines $\theta_c = \frac{f-c}{1+f}$, $\theta = zc(1+z^*)$, and $\theta = 1 - \frac{c(1+z+zz^*)}{f}$. When $\theta < \theta_c$ and $\theta > zc(1+z^*)$, phase III failure is observed in the $(\frac{f}{1+f}, 0, B)$. When $\theta > \theta_c$ and $\theta < 1 - \frac{c(1+z+zz^*)}{f}$, phase III default occurs in the $(1, \frac{f}{1+f}, B)$ triangle. As a result, phase III failures can only occur to the left of point B, in the $(1, 0, B)$ triangle ([6], p. 827-828). No phase III failures take place to the right of point B and outside the $(1, 0, B)$ triangle ([6], p. 828). We notice that B is the intersection of three lines and the x-coordinate of B is

$$
\begin{aligned}
\frac{f-c}{1+f} = zc(1+z^*) \iff & f - c = zc + zz^*c + zcf + zz^*cf \\
& c(1 + z + zz^* + zf + zz^*f) = f \\
& c = \frac{f}{1 + z(1+z^*)(1+f)}
\end{aligned}
$$

We conclude that phase III failure does not happen if

$$
c > \frac{f}{1 + z(1+z^*)(1+f)} \qquad (5.18)
$$

([6], p. 828).

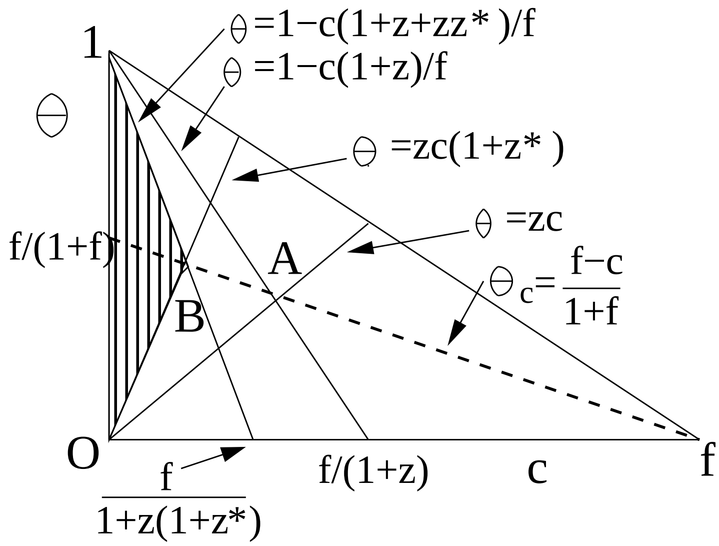


Figure 5.2. The shaded area is the region where phase III shocks produce failure.

From figure 5.2, we observe that the region where phase III shocks cause default is smaller than the area where phase II shocks bring about default ([6], p. 827). We conclude that subsequent shocks have lower and lower impact as they diffuse throughout the system. Phase III failure might lead to phase IV, phase V, and so on until the process ends; the obtained results can be extended for subsequent defaults with resulting equations that are more complex than the ones already derived ([6], p. 826, 830).

5.1.2 LIQUIDITY SHOCKS

Up until now, we have supposed that banks' defaults had no impact on the assets of other banks, with the exception of the losses suffered by the creditor institutions ([6], p. 829). However, this is not the case in reality; in the event of a crisis, many banks resort to selling some of their assets which leads to a depreciation in the price of the assets of the entire financial network ([6], p. 829).

We assume that the value of any institution's external assets is further decreased by a factor

$$q = e^{-\alpha x} \tag{5.19}$$

where x is the fraction of defaulting banks of the entire system ([6], p. 829). α is a constant that corresponds to a decrease in the asset price of the financial network ([6], p. 829). In their paper, Nier et al. state that α represents "the speed at which the price for banking assets declines with the amount of assets sold" ([9], p. 2047). When $\alpha = 0$, the financial system is not affected by liquidity problems ([9], p. 2048). However, when $\alpha > 0$, liquidity issues impact the network ([9], p. 2049). Nier et al. consider several value of α: $\alpha = 0, 1.5, 3$ ([9], p. 2049-2050).

We now reconsider our scenario, but also incorporate liquidity problems ([6], p. 829). The initial bank in our network still fails according to equation (5.11); this causes the external assets of the remaining $n-1$ solvent institutions to depreciate by the factor given in equation (5.19) ([6], p. 829). The other $n-1$ banks experience a **phase II liquidity shock** defined by

$$s^*(II) = \beta_1(1-\theta) \tag{5.20}$$

([6], p. 829). Here, $\beta_1 = 1 - e^{-\alpha x}$ represents the value of the assets of the remaining institutions after depreciation ([6], p. 829). In this case, $x = \frac{1}{n}$ ([6], p. 829). Since $\alpha > 0$, $\beta_1 = 1 - e^{\frac{-\alpha}{n}}$ and $0 < \beta_1 < 1$. It is important to note that $s^*(II)$ (which affects $n-1$ banks) is distinct from the phase II shock, *s(II)* of equation (5.6) (which only impacts the defaulting institution's z creditors) ([6], p. 829).

We consider two cases. In the first case, the remaining $n-1$ banks **default in phase II** if the liquidity shock is large

$$s^*(II) > c \iff \beta_1(1-\theta) > c \tag{5.21}$$

([6], p. 829). We can graph the line $\beta_1(1-\theta) = c$ in the $(1, 0, f)$ triangle, illustrated below in figure 5.3. Equation (5.21) corresponds to the failure of the entire financial network if $\beta_1 > c$, provided that θ is small ([6], p. 829). This can be seen in the $(1, 0, \beta_1)$ triangle([6], p. 829-830).

In the second case, the phase II liquidity shock may not be too large; that is, $\beta_1(1-\theta) < c$ which means that $\beta_1 < c$, given that θ is small ([6], p. 829). The criteria for **phase II failure** of z creditor banks is

$$s^*(II) + s(II) > c \iff \beta_1(1-\theta) + \frac{min\{\theta, f(1-\theta) - c\}}{z} > c \tag{5.22}$$

([6], p. 829). If $\theta < \theta_c$, then equation (5.22) becomes

$$\begin{aligned} \beta_1(1-\theta) + \frac{\theta}{z} > c \iff & \beta_1 z - \beta_1 z\theta + \theta > zc \\ & \theta(1 - \beta_1 z) > zc - \beta_1 z \\ & \theta > \frac{z(c - \beta_1)}{1 - \beta_1} \end{aligned} \tag{5.23}$$

([6], p. 830). If $\theta > \theta_c$, then

$$\begin{aligned} \beta_1(1-\theta) + \frac{f(1-\theta) - c}{z} > c \iff & \beta_1 z - \beta_1 z\theta + f - f\theta - c > zc \\ & \theta(f + \beta_1 z) < f + \beta_1 z - c - zc \\ & \theta < 1 - \frac{c(1+z)}{f + \beta_1 z} \end{aligned} \tag{5.24}$$

([6], p. 830).

We can graph the lines $\theta = \frac{z(c-\beta_1)}{1-\beta_1}$ and $\theta = 1 - \frac{c(1+z)}{f+\beta_1 z}$. When $\theta < \theta_c$ and $\theta > \frac{z(c-\beta_1)}{1-\beta_1 z}$, phase II failure (from a combination of liquidity and interbank loan shocks) is observed in the (A_0, β_1, D) triangle. When $\theta > \theta_c$ and $\theta < 1 - \frac{c(1+z)}{f+\beta_1 z}$, phase II default is illustrated by the $(1, A_0, D)$ triangle. Phase II failure occurs in the $(1, \beta_1, D)$ triangle and only to the left of point D ([6], p. 830). Defaults resulting from phase II shocks do not take place to the right of point D and outside the $(1, \beta_1, D)$ triangle ([6], p. 830). As previously discussed, the x-coordinate of D is

$$\frac{f-c}{1+f} = \frac{z(c-\beta_1)}{1-\beta_1 z} \iff f - f\beta_1 z - c + \beta_1 zc = zc + fzc - z\beta_1 - f\beta_1 z$$

$$c(z + fz + 1 + \beta_1 z) = f + \beta_1 z$$

$$c = \frac{f + \beta_1 z}{1 + z(1 + f - \beta_1)}$$

([6], p. 830). Phase II failure does not happen if

$$c > \frac{f + \beta_1 z}{1 + z(1 + f - \beta_1)} \tag{5.25}$$

([6], p. 830). Figure 5.3 also points out the fact that liquidity shocks make the situation worse; financial institutions fail more easily once liquidity effects are taken into account ([6], p. 830).

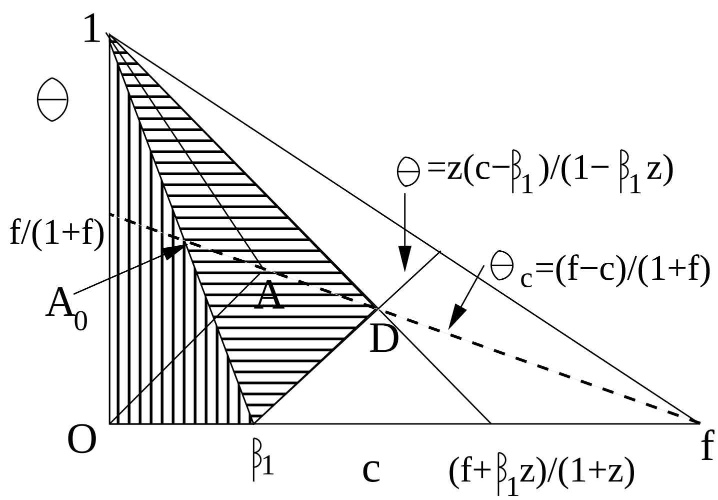


Figure 5.3. The first shaded area (on the left) shows phase II failure of all other banks from liquidity shocks, while the second region (on the right) represents the default of z creditors from liquidity and interbank loan shocks.

Supposing that the phase II liquidity shock does not bring down the entire financial system but only affects z creditors, a **phase III liquidity shock** is experienced by the remaining

$(n-1)(1-p)$ solvent banks

$$s^*(III) = \beta_2(1-\theta) \tag{5.26}$$

([6], p. 830). $\beta_2 = 1 - e^{-\alpha x_2}$ shows the value of the assets of the solvent financial institutions ([6], p. 830). In this case, $x_2 = \frac{1+z}{n}$ such that $0 < \beta_2 < 1$ ([6], p. 830). $\beta_2 > \beta_1$ because $x_2 > x$.

There are two possibilities. In the first instance, the $(n-1)(1-p)$ banks all **default in phase III** if the liquidity shock is too large

$$s^*(III) > c \iff \beta_2(1-\theta) > c \tag{5.27}$$

([6], p. 830). We graph the line $\beta_2(1-\theta) = c$ in the $(1, 0, f)$ triangle ([6], p. 830). (5.26) provides evidence for the failure of the entire financial network if $\beta_2 > c$, given that θ is negligible ([6], p. 830). The crash of the whole system is shown by the $(1, D, \beta_1, \beta_2)$ region in figure 5.4 where $\hat{\beta} = \frac{f+z\beta_1}{1+z}$ ([6], p. 830).

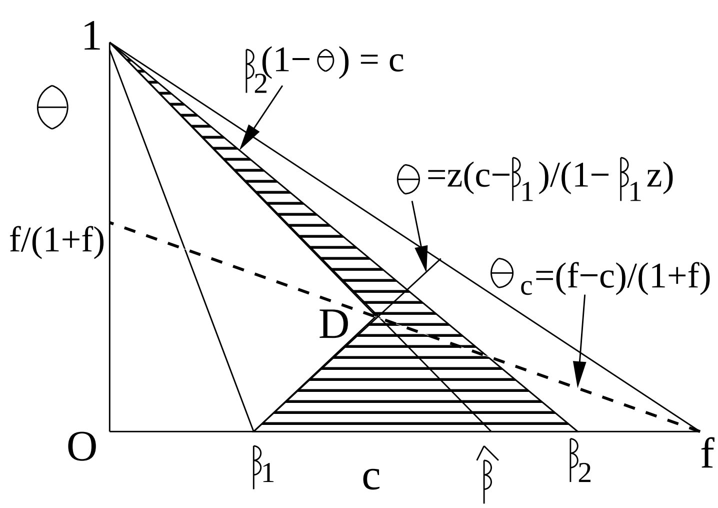


Figure 5.4. The shaded area represents phase III failure of all other banks from liquidity shocks.

The second possibility is to analyze what occurs if the phase III liquidity shock is not that forceful; in other words, $\beta_2(1-\theta) < c$, which means that $\beta_2 < c$ for small values of θ ([6], p. 830). The condition for **phase III failure** resulting from liquidity and interbank loan shocks is

$$\frac{k_c}{z}s^*(II) + s^*(III) + k_c s(III) > c \iff \frac{s^*(II)}{z} + \frac{s^*(III)}{k_c} + s(III) > \frac{c}{k_c}$$

$$\beta_1(1-\theta) + \frac{\beta_2(1-\theta)z}{k_c} +$$

$$+ min\{\theta, \frac{min\{\theta, f(1-\theta)-c\}}{z} - c\} > \frac{zc}{k_c} \tag{5.28}$$

([6], p. 830). If $\theta < \theta_c$, (5.28) becomes

$$\beta_1(1-\theta) + \frac{\beta_2(1-\theta)z}{k_c} + min\{\theta, \frac{\theta}{z} - c\} > \frac{zc}{k_c}$$

$$\iff \beta_1 - \beta_1\theta + \beta_2 z^* - \beta_2\theta z^* + \frac{\theta}{z} - c > z^*c$$

$$\theta > \frac{z[c(1+z^*) - \beta_1 - z^*\beta_2]}{1 - z\beta_1 - zz^*\beta_2} \tag{5.29}$$

([6], p. 830). If $\theta > \theta_c$, then

$$\beta_1(1-\theta) + \frac{\beta_2(1-\theta)z}{k_c} + min\{\theta, \frac{f(1-\theta)-c}{z} - c\} > \frac{zc}{k_c}$$

$$\iff \beta_1 z - \beta_1\theta z + \beta_2 zz^* - \beta_2\theta zz^* + f - f\theta - c - zc > zz^*c$$

$$\theta(z\beta_1 + zz^*\beta_2 + f) < z\beta_1 + zz^*\beta_2 + f - c - zc - zz^*c$$

$$\theta < 1 - \frac{c(1+z+zz^*)}{f + z\beta_1 + zz^*\beta_2} \tag{5.30}$$

([6], p. 830).

Graphing the lines $\theta = \frac{z[c(1+z^*)-\beta_1-z^*\beta_2]}{1-z\beta_1-zz^*\beta_2}$ and $\theta = 1 - \frac{c(1+z+zz^*)}{f+z\beta_1+zz^*\beta_2}$, we obtain figure 5.5. When $\theta < \theta_c$, and $\theta > \frac{z[c(1+z^*)-\beta_1-z^*\beta_2]}{1-z\beta_1-zz^*\beta_2}$, phase III failure (from a mix of liquidity and interbank loan shocks) occurs in the (D_0, E_0, E) triangle. We keep track of the conditions resulting from phase II failure from liquidity and interbank loan effects. When $\theta > \theta_c$ and $\theta < 1 - \frac{c(1+z+zz^*)}{f+z\beta_1+zz^*\beta_2}$, phase III default is shown in the $(1, D_0, E)$ triangle. We conclude that phase

III failure is illustrated in the $(1, E_0, E)$ triangle and only to the left of point E ([6], p. 830). Phase III failure cannot occur to the right of point E and outside the $(1, E_0, E)$ triangle ([6], p. 830). We find the x-coordinate of point E to be

$$\frac{f-c}{1+f} = \frac{z[c(1+z^*) - \beta_1 - z^*\beta_2]}{1 - z\beta_1 - zz^*\beta_2}$$

$$\iff f - fz\beta_1 - fzz^*\beta_2 = zc + zz^*c - z\beta_1 - zz^*\beta_2 + fzc + fzz^*c - fz\beta_1 - fzz^*\beta_2$$

$$c = \frac{f + z\beta_1 + zz^*\beta_2}{z(1+z^*)(1+f)}$$

Phase III failure cannot occur if $c > \frac{f+z\beta_1+zz^*\beta_2}{z(1+z^*)(1+f)}$.

Notice that, once liquidity shocks get started, the whole financial network can collapse, even for larger values of c that would guard the system against interbank loan shocks ([6], p. 830). This is in contrast to the propagation of regular shocks illustrated in section 5.1.1 and in figures 5.1 and 5.2 ([6], p. 830).

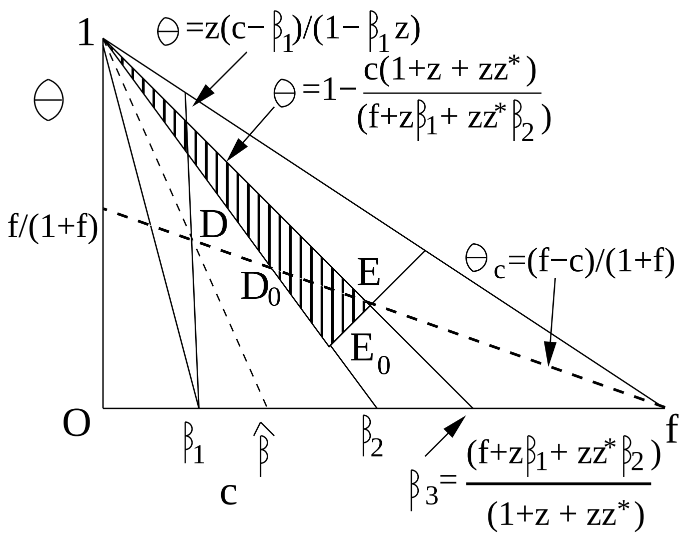


Figure 5.5. Phase III failures as a result of a combination of liquidity and interbank loan shocks are shown in the shaded area.

6.0 A CONNECTIVITY MEASURE USING KIRCHHOFF'S THEOREM

We consider an undirected graph G with n vertices v_1, v_2, ..., v_n. G is without loops and has at most one edge between any two vertices. A **tree** is a connected subgraph of G that contains no cycles ([5], p. 31). A **spanning tree** is a tree that includes every vertex of G ([5], p. 34).

Suppose that we want to find out how many trees exist in a graph. **Cayley's theorem** states that the number of spanning trees in a complete graph with n vertices is n^{n-2} ([11], p. 48). **Kirchhoff's theorem**, also known as the **Matrix Tree Theorem**, asserts that the number of spanning trees in G is equal to any cofactor of the nxn matrix $C(G)$, where the entries in $C(G)$ are

$$c_{ij} = \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } v_i \text{ and } v_j \text{ are connected to each other and } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

([11], p. 50). K is a submatrix of C which is obtained by deleting the last row and column of C. The absolute value of the determinant of K represents the number of spanning trees of the graph G. Note that we obtain the same number of spanning trees of G no matter what column i and row i we eliminate ($i \in \{1, 2, \dots, n\}$).

To every vertex v_k in G, we associate the number t_k in the following way: we eliminate v_k and all the edges connected to it and we obtain a subgraph G_k. t_k represents the number of spanning trees in G_k computed using Kirchhoff's theorem as described above. Let $t_i = min\{t_1, t_2, \dots, t_n\}$ and $t_j = max\{t_1, t_2, \dots, t_n\}$. We define the vertex v_i as the most vulnerable vertex of G and v_j as the least vulnerable node in the network. In most cases, v_i is the vertex with the largest degree (as shown in examples 6.1 and 6.2); however, there may be instances when this is not true (as in example 6.3).

To illustrate this connectivity measure in a financial setting, we consider a network V of ten nodes or banks labelled B_1 through B_{10} connected by 16 edges. We follow the procedure described above to find the most and least vulnerable financial institutions in each of the three networks.

Example 6.1

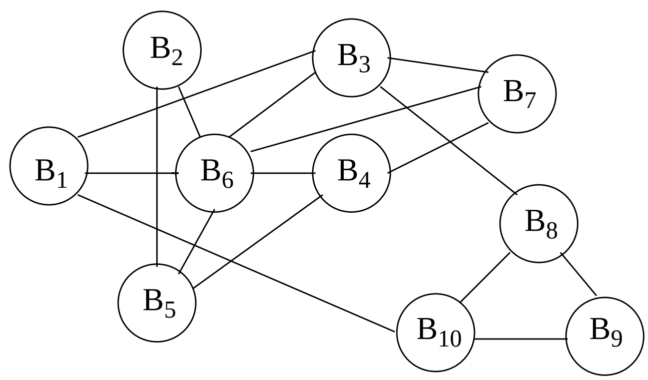


Figure 6.1. A banking network with a node of larger degree.

In this network, the financial institution B_6 has degree 6 and more connections than the rest of the banks. The other nodes in the system have degree 2 or 3. When we eliminate B_1, we obtain

$$K_1 = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 5 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{pmatrix}$$

We repeat the procedure for every financial institution in the system and find that B_6 is the most vulnerable bank while B_9 is the least susceptible institution.

Table 6.1: The Kirchhoff number for all banks in example 6.1

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
165	542	228	444	141	11	24	144	1084	144

Example 6.2

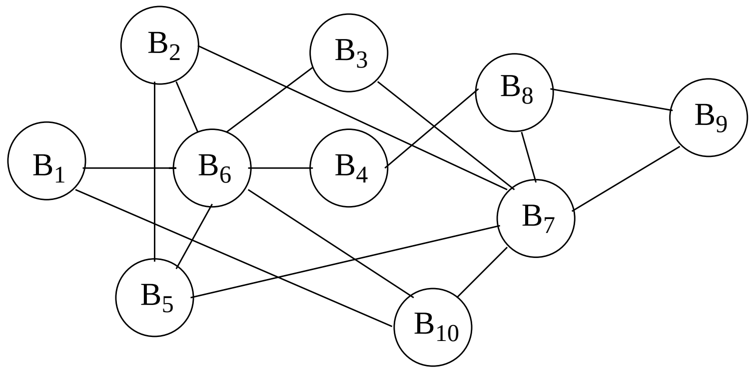


Figure 6.2. A banking network with two nodes of larger degree.

In this network, B_6 and B_7 each have degree 6 and the largest number of connections. The other financial institutions have degree 2 or 3. Using the described connectivity measure, we obtain

Table 6.2: The Kirchhoff number for all banks in example 6.2

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
624	316	411	504	28	9	9	168	584	488

This means that B_6 and B_7 are the most vulnerable banks in the system while B_1 is the least.

Example 6.3

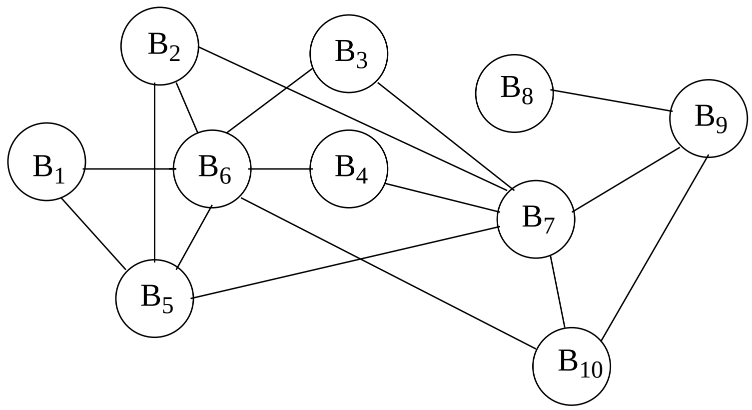


Figure 6.3. A banking network with two nodes of larger degree.

In this network, B_6 and B_7 each have degree 6, the largest number of links. The others have degree 2, 3 or 4. We calculate the connectivity measure.

Table 6.3: The Kirchhoff number for all banks in example 6.3

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
320	220	419	419	84	6	8	1028	0	0

Here, B_9 and B_{10} are the most vulnerable financial institutions in the network even though they have degree 3. The second and third most susceptible banks are B_6 and B_7, the most connected nodes in the network. The least vulnerable institution is B_8.

Note that this connectivity measure using Kirchhoff's theorem only considers how connected a node is in the network. It does not take into account how large a bank is, what its assets and liabilities are, or what its exposures in the interbank market are. Consider B_9 and B_{10} of example 6.3. According to this connectivity measure, they are the most vulnerable banks in the system; however, if all the other variables were taken into account, this would not be the case. This is also shown by the simulations in chapter 7.

7.0 SIMULATIONS

Following the ideas and analysis presented by Nier et al., the Flow Network simulator was used to model various scenarious ([9]). It was developed by Amadeo Alentorn at the Bank of England and downloaded from http://www.amadeo.name/flow.htm.

This simulator generates a random Erdös-Rényi network when the user specifies a number of nodes, n. It specifically creates a financial network of n banks which are labelled B_0 through B_{n-1}, each with their own balance sheet (as described by table 3.1). We have the option of choosing between homogeneous and heterogeneous networks. In a homogeneous system, the n banks in the system have approximately the same assets and liabilities and are connected with a probability p, $0 < p \leq 1$. A heterogeneous system consists of n banks such that m of these financial institutions ($m \leq n$) are larger than the rest ([9], p. 2051). We have to specify p, the probability that a small bank is connected to any other institution in the system; we also have to input q which is the probability that a large institution is linked to the system such that $q \geq p$ ([9], p. 2051). k represents the percentage of total assets that is assigned to the large banks ([9], p. 2052).

No matter the type of network that is created, the simulator needs to have the value of the external assets of the system ([9], p. 2041). Users also have to indicate the percentage of total external assets in the total assets of the network and the net worth, γ, as a percentage of the total assets of the entire system ([9], p. 2041).

Once these various parameters are specified, Flow Network creates balance sheets for all the financial institutions in our network using the following algorithm ([9], p. 2038-2040). The total assets of a bank $i \in V$ are $a_i + e_i$ and its total liabilities are $l_i + d_i$ ([9], p. 2039). The total assets of the financial network are $A = \sum_{i=1}^{n}(a_i + e_i)$ and its total external assets are $E = \sum_{i=1}^{n} e_i$ ([9], p. 2039).We let $\delta = \frac{E}{A}$ be the percentage of external assets in total assets ([9], p. 2039). We know that $A = E + I$, where I is the system's interbank assets such that $I = \sum_{i=1}^{n} a_i$ ([9], p. 2039). Then $\theta = 1 - \delta$ is equal to the percentage of interbank assets so that $\theta = \frac{I}{A}$ and equivalent to (5.2); $\theta A = I$ and $w = \frac{I}{Z}$ where Z is the total number of links in the network ([9], p. 2039); w represents how much one bank lends to another ([9], p. 2039).

Given E, δ, and c, we can calculate A, I, θ, w ([9], p. 2039). We find that $A = \frac{E}{\delta}$ and $I = A - E$ ([9], p. 2039). We compute θ, w, l_i and a_i ([9], p. 2039). Any bank in the network is able to function only if its external assets are no less than its net interbank borrowing, which means that $e_i \geq l_i - a_i$ ([9], p. 2039). We first compute $\tilde{e_i} = l_i - a_i$, then $\hat{e_i} = \frac{E - \sum_{i=1}^{n} \tilde{e_i}}{n}$ ([9], p. 2039-2040). We have $e_i = \tilde{e_i} + \hat{e_i}$ ([9], p. 2040). We know that $\gamma = \frac{c_i}{a_i + e_i}$ is specified for the network and for every bank in particular ([9], p. 2040). We can compute c_i for all the institutions $i \in V$ in the financial network ([9], p. 2040). This leads us to find $d_i = (a_i + e_i) - (l_i + c_i)$ by rearranging the terms of equation (3.1) ([9], p. 2040).

Nier et al. set $n = 25$ with probability $p = 0.2$, $E = 100000$, $\gamma = 5\%$ (0.05) and $\theta = 30\%$ (0.3) ([9], p. 2053). They performed various experiments on homogeneous networks and also briefly looked at heterogeneous systems with one larger bank ([9], p. 2041- 2053). I kept the same setup and parameters as Nier et al., but further analyzed heterogeneous networks. I looked at financial institutions with one, two or three larger banks. I kept $p = 0.2$ constant, but varied q from 0.2 to 1 and performed 20 trials for each (p, q) combination ([9], p. 2052). Each larger institution holds 12.5% or 0.125 of the total assets of the system. Unlike Nier et al. who kept the number of links in the heterogeneous system constant, I wanted to see what happens if the number of connections increases ([9], p. 2052). The banks can only be shocked one at a time, with the number of defaulting institutions being counted; this is done until all the banks in the network have been shocked; the shock wipes off 100% of the institution's external assets, such that $f = 1$. No liquidity effects are taken into consideration in the simulations.

In the case where one large bank dominates the system, we observe in figure 7.1 that the average number of defaults created by the greater institution increases with q. The mean number of failures generated by the small banks remains approximately constant.

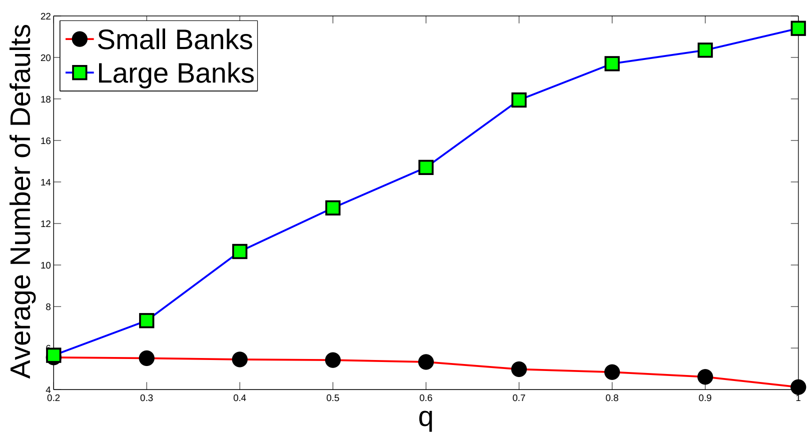


Figure 7.1. Average number of defaults as a function of the connection probability q of the large bank for small and large banks, for 20 experiments for each parameter. Parameter values are (γ, θ, n, E)= (0.05, 0.3, 25, 100000). $m = 1$, $p = 0.2$, q varies from 0.2 to 1, $k = 0.125$.

When two or three large institutions are part of the financial system, the situation is completely different. The average number of failures produced by the large banks increases up to $q = 0.7$, then decreases. Notice that the mean number of defaults spawned by the small institutions also slightly decreases. Figures 7.2 and 7.3 illustrate a more stable banking network than figure 7.1 in the sense that the average number of defaults cannot bring the whole system down.

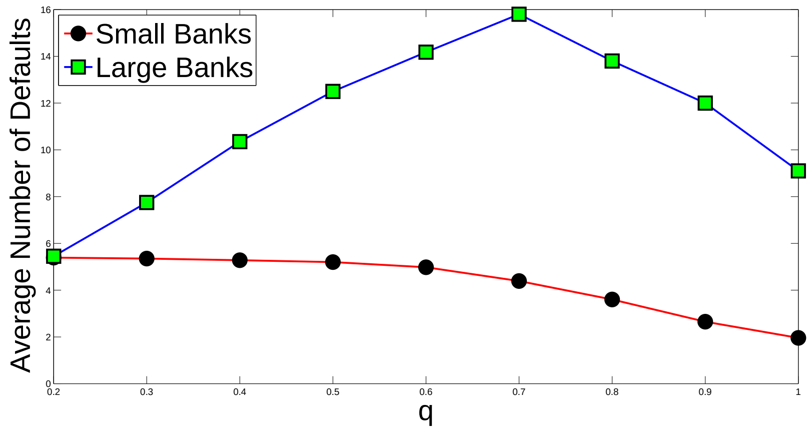


Figure 7.2. Average number of defaults as a function of the connection probability q of the two large banks for small and large institutions, for 20 experiments for each parameter. Parameter values are (γ, θ, n, E)= (0.05, 0.3, 25, 100000). $m = 2$, $p = 0.2$, q varies from 0.2 to 1, $k = 0.25$.

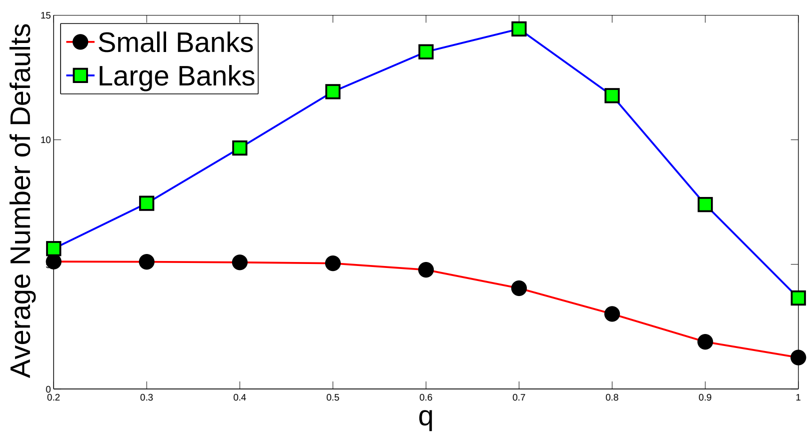


Figure 7.3. Average number of defaults as a function of the connection probability q of the three large banks for small and large institutions, for 20 experiments for each parameter. Parameter values are (γ, θ, n, E)= (0.05, 0.3, 25, 100000). $m = 3$, $p = 0.2$, q varies from 0.2 to 1, $k = 0.375$.

8.0 CONCLUSIONS

This paper analyzes how the structure of a financial network affects systemic risk by studying a scenario and performing simulations. However, there are some issues with the ideas presented here. First of all, the scenario and simulations use a random Erdös-Rényi network, while real-world financial networks are known to be scale-free ([2], p. 3-4, [7], p. 25-28).

Second, the given scenario and simulations use a simplistic balance sheet and only depreciate external assets. It would be more realistic if the assets and liabilities of financial institutions were fluctuated utilizing uncertainty models; these use historical simulations to vary the foreign exchange and interest rates, changes in equity prices and losses from loans to non-banks ([4], p. 11-15). Furthermore, this might also be achieved using a model of correlated market shocks proposed by Cont et al. ([2], p. 21-22).

The third issue is that we shock the system one bank at a time; this is not credible since financial shocks may affect more than one institution at a particular moment in time. This minimizes contagion ([2], p. 5).

In addition to this, our scenario and simulations make use of institutions whose net worth, γ, is a fixed proportion of the total assets for every bank in the network ([9], p. 2040). Nevertheless, in scale-free networks, the bigger institutions tend to be risk-takers; as a result, their capital buffers are at a lower level than those of smaller banks ([6], p. 825).

Real-world banking systems that have been analyzed so far suffer from the limitation that some of the entries of the matrix E of bilateral exposures are unknown ([4], p. 10, [10], p. 834-835). Financial institutions do not need to declare their exact exposure to every other bank in the system, only their total monthly exposure in the market to the central bank ([4], p. 8, [10], p. 832). Since the German and Austrian systems are tiered, banks also have to state their exposure with the head institution and with other banks in the system ([4], p. 8-9, [10], p. 832-833). The maximum entropy principle is used to determine the missing entries of E ([4], p. 10-11, [10], p. 835). However, this method spreads the institutions' interbank exposures as evenly as possible which leads to an underestimation of the resulting number of defaults and contagion ([10], p. 847).

An interesting extension of this paper would be to continue the simulations with Flow Network by looking at financial systems with an increasing number of large institutions $(4, 5, 6, \ldots)$. Will the network get to a critical stage beyond which the average number of defaults generated by the large banks will increase to bring the entire financial system down? Or will the network become increasingly stable with a rise in the number of large institutions?

This paper computes a measure to quantify the most vulnerable bank in the financial system. This could be improved by also considering banks' exposures in the interbank market and their balance sheets. It would be interesting to analyze contagion in a system where the most vulnerable institutions hold more capital than the rest of the banks. Cont et al. suggest that this would be a good strategy to make the financial system more resilient against systemic breakdown ([2], p. 38-39).

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