STRUCTURAL MODEL FOR CREDIT DEFAULT IN ONE AND HIGHER DIMENSIONS

by

BO SHI

B.S., Northern Jiaotong University, 2002

M.A., University of Hawaii, 2006

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This dissertation was presented

by

BO SHI

It was defended on

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and approved by

John Chadam, Mathematics Department

Juan Manfredi, Mathematics Department

David Swigon, Mathematics Department

Satish Iyengar, Statistics Department

Dissertation Director: John Chadam, Mathematics Department

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BO SHI, PhD

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In this thesis, we provide a new structural model for default of a single name which is an extension in several directions of Merton's seminal work [41] and also propose a new hierarchical model in higher dimensions in a heterogeneous setting.

Our new model takes advantage of the fact that currently much more data is readily available about the equity (stock) markets, and through our analysis, can be translated to the much less transparent credit markets. We show how this can be used to provide volatilities for the default indices in structural models for these same stocks. More importantly, we use the equity data to obtain an implied probability distribution for the firms' liabilities, a quantity that is only reported quarterly, and often with questionable reliability. This completes the structural model for a single firm by specifying (probabilistically) the absorbing default barrier. In particular, we can then obtain the default probability of this firm and capture its Credit Default Swap(CDS) spreads. For several companies selected from different industry sectors, the values that our model obtain are in good agreement with the credit market data. Furthermore, we are able to extend this approach to higher dimensional models (e.g., with 125 firms) where the correlations among the firms are essential. Specifically, we use hierarchical models for which each firm's default boundary a linear combination of a systematic factor (e.g, the Dow Jones Industrial Average) and an idiosyncratic factor, with firm-to-firm correlations obtained through their correlations with the systemic factor. Once again the parameters for these high dimensional structural models are obtained from equity data and the resulting values for the tranche spreads for the CDX: NAIG Series 17

Collateralized Debt Obligations (CDO) compare favorably with actual market data.

In the course of this work we also provide results for the probabilistic inverse first passage problem for a Brownian motion default index: given a default probability, find the probability distribution for linear default barriers (equivalently initial distributions) that reproduce the given default probability.

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PREFACE

I am indebted to many people for the successful completion of this thesis. First and foremost I wish to express my sincere gratitude from the bottom of my heart to my advisor Professor John Chadam for inspiring my research interest in financial mathematics, sharing his incomparably profound knowledge with me, and patiently guiding me throughout my research. As a respectful mentor and great friend, Professor Chadam not only teaches me the way to solve the mathematical problems, but also the way to think, learn and understand them. Without his enormous contributions of time and suggestions, this thesis would not have been possible.

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1.0 INTRODUCTION

Since 2008, the global economy has been hit hard due to the sub-prime mortgage crisis. While the housing and credit bubbles were growing, a series of factors caused the financial system to become increasingly fragile. From then on, the government and all financial institutions have paid more and more attention to credit risk and risk management. In particular, the Office of the Comptroller of the Currency(OCC) issued several important regulative policies including the "SUPERVISORY GUIDANCE ON MODEL RISK MANAGEMENT" which is the guideline for model development and validation. One of the most important reasons to issue such policy is that many financial institutions use very sophisticated models to estimate the price but many of them underestimate the risk associated with some extreme market event. Therefore, it is crucial to understand the infrastructure of the single company as well as the entire market. Credit default models have been a hot topic especially during the recent global financial crisis. Some previously widely used models such as the Gaussian Copula model, which is used to measure the risk and evaluate some credit derivatives in complicated credit portfolios, now has been criticized because of its tendacy to underestimate the risk in some extreme scenarios. Therefore, there are many extensions and improvements to the original work of Li [39], who first applied the copula approach in finance.

There are two major types of models to capture the dynamics of the default process in the credit risk world for a single name: structural and intensity models (reduced form models). Structural models are more intuitive from the point of finance and they usually use the asset and liability to determine the time of default. Merton [41] was credited to be the first to consider this approach and others such as Black and Cox [5] and Duffie and Lando [18] have extended his idea. On the other hand, intensity models do not consider endogenous definition of default via asset and liability values as in structural models. In fact, they model the default time directly as an event of an exogenously given jump process.

Modeling the default time and the correlations among the firms in a large portfolio, usually containing more than 100 firms, is the most challenging problem for academic researchers and market practioners. Firstly, modeling and calibrating the correlation structure is difficult and requires careful specification. Secondly, in order to price the most liquid credit derivatives like CDOs, we must evaluate the loss distributions for the entire portfolio consisting of more than 100 obligors. This poses significant computational problems unless realistic simplifications are introduced.

1.1 CREDIT DEFAULT MODELS

In this section, we will briefly describe the three major types of models in the introduction. There is a huge literature on the modeling of default times, portfolio losses, valuation of credit derivatives and measurement of the risk. References can be found in Schonbucher [52] and Bielecki & Rutkouski [8].

1.1.1 Intensity Models

In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. Basically, they directly model defaults through a jump process with an intensity which is sometimes referred to as the hazard rate. An advantage of the intensity-based framework is that it is computationally more efficient than structural models and easier to include different dependence structures.

Duffie and Singleton [20] consider default intensities with an idiosyncratic component as well as a common intensity process which could be interpreted as the global economic factor or sector factor. Research papers in this vein include Mortensen [45].

1.1.2 Structural Models

Structural default models provide a link between the credit quality of a firm and the firmś economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the intensity (reduced) approach.

Merton [41] applied this idea in bond pricing that includes default risk. He considers that a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. In particular, he considers that the firm's value, V_t, follows a geometric Brownian motion and the firm's debt is a zero-coupon bond with maturity T. If a firm's value at maturity, V_T, is below the face value of the bond, this firm is defined as in default at maturity, T. A major problem with this definition of default is that the default only can occur at the maturity T. This is not realistic and has been extended by other researchers. One of the contributions of Black-Cox [5] is to introduce a new definition of the default, the so called first passage time, which specifies the default as the first time the firmś value hits a lower barrier, allowing default to take place at any time before maturity, T. Other extensions include models with stochastic interest rates as well as models which determine the lower threshold endogenously, e.g., as an optimal level from the shareholders' perspective; see Duffie and Lando [18] and Leland and Toft [40].

One of the criticisms of structural models is that they cannot capture credit spreads over risk-free Treasury bonds in a short maturity. The main reason for this discrepancy is due to the predictability of the default time in the short term. Several researchers have improved structural models in various ways to overcome this deficiency. Zhou [60] includes Poisson jumps in the Black-Cox model and finds that one can have significantly non-zero spreads in a short maturity. Another type of modification is to introduce the notion of incomplete or imperfect information and then the default time becomes unpredictable. References in this direction include Duffie and Lando [18], Schmidt [53] and Giesecke [22]. Recently Giesecke and Goldberg [24], Yi, Tchernitser and Hurd [58] introduce the randomness in structural models, either in the initial status of a firm's value or in the location of its default barrier.

As for higher dimensional extensions, Iyengar [29] and Zhou [60], provide the default probability for two firms which follow correlated Brownian motion. Hull and White [27] and

Overbeck and Schmidt [48] extend the Black-Cox model to n firms. Both papers use Monte Carlo schemes for valuation and observe that model prices are quite similar to those obtained from a Gaussian copula model. Instead of considering the dependence through the Brownian motions, McLeish and Metzler [44] develop a model which introduces a "systematic risk" factor that controls the dynamics of credit qualities.

1.1.3 Copula and Factor Models

When the marginal distributions are given, it is natural to use a copula approach to construct its multivariate model with the correlation embedded. The most often seen copulas are the Gaussian Copula and class of Archimedean Copulas such as the Clayton Copula, Frank Copula and Gumbel Copula [55]. The Gaussian Copula was introduced by Li [39] as the market standard, and then became a popular tool to evaluate the tranches prices for CDO and for other complicated credit derivatives. However, Brigo, Pallavicini, and Torresetti [7] pointed out that there are big differences between market index tranche quotes and equally correlated Gaussian Copula.

Due to its easy implementation and computational efficiency, it has, nevertheless, become a major tool to evaluate CDO tranche prices among academic researchers as well as market practioners. Research papers in this vein include Burtschell, Gregory and Laurent [6]. This computational efficiency is primarily due to the conditional independence of the factors which provides several well-established techniques, to obtain accurate approximations for the portfolio loss distribution and tranche losses. References can be found in Andersen and Sidenius [2], Hull and White [27] and O'Kane [32].

1.2 OUTLINE OF THESIS

We introduce the concept of intensity models, define the survival probability and provide its connections with structural models in Chapter 2. For constant intensity, we provide the closed-form distribution for the default barrier to reproduce the given default probability.

In Chapter 3, we define default in the Vasicek's model as well as the Black-Cox first-passage time structural model and provide the extension of Merton's ideas connecting equity data and credit markets to this first passage time setting. Most importantly, we use our new model to derive the closed form solution to the value of European Call/Put option and use that to estimate the distribution of liabilities. We conclude this chapter by estimating CDS spreads using our model for several selected firms and compare our results with credit market data. In chapter 4, we extend the idea in our single name model to our hierarchical model, where we made the default barrier as a linear combination of a systemic factor and an idiosyncratic factor with firm-to-firm correlations obtained through their correlations with systemic factor. In particular, we consider both homogeneous and inhomogeneous models. For the homogeneous model, we apply the same technique as others to calibrate the parameters in our model to minimize the Root Mean Square Error(RMSE) of tranche prices in CDX NA IG Series 17. On the other hand, we directly estimated the parameters from equity data in our inhomogeneous model without calibration and then calculated the tranche prices to compare the results in market data. To better improve our estimates, we use the idea of VIX, which uses the option data to estimate the volatility of global index, to estimate the volatility in the asset process. In the end of each section in this chapter, we compare our model results with market results.

2.0 INTENSITY MODELS

Intensity models consider the default time as the first jump in a point process; e.g., a Poisson process, N_t. A homogeneous Poisson process with constant intensity $\lambda > 0$ satisfies

$$\mathbb{P}[N_t - N_s = k] = \frac{1}{k!}(t-s)^k \lambda^k e^{-\lambda(t-s)} \tag{2.1}$$

In a more general setting, we might consider the intensity, λ_t, to follow a stochastic process. If we define the default time as

$$\tau = \inf\{t > 0 : N_t > 0\}, \tag{2.2}$$

then the survival probability is defined as

$$\mathbb{P}[\tau > t] = \mathbb{P}[N_t = 0] = \mathbb{E}[e^{-\int_0^t \lambda_s ds}]. \tag{2.3}$$

Conversely, the associated time-dependent default intensity is the conditional default arrival rate

$$\lambda_t = \lim_{h \to 0} \frac{\mathbb{P}[\tau \le t + h | \tau > t]}{h}. \tag{2.4}$$

In other words, λ_t represents the instantaneous default probability, the very short-term default risk.

2.1 INTENSITY BASED IMPLIED DEFAULT BARRIER

In recent years, some researchers have tried to connect the two major credit default models: structural and intensity models. Giesecke [22] have shown the connection for structural models with incomplete information. Dionne and Laajimi [19] and Chadam, Cheng, Chen and Saunders consider the related inverse problem which is to determine the deterministic barrier in a structural model if any default probability is given. However, the boundary obtained typically has a complicated dependence on time requiring numerical simulation if it is to be used. Here we consider the problem of determining the random behavior of a time-independent boundary (for which an analytical expression for the survival probability is known; see below) associated with a given intensity model.

With constant intensity, the survival probability is $\mathbb{P}[\tau > t] = e^{-\lambda t}$. We assume the firm's asset, V_t, follows Brownian motion with drift starting at v_0,

$$dV_t = \mu dt + \sigma dW_t, \tag{2.5}$$

where W_t is the standard Brownian motion. Suppose the default barrier is a constant $b = x$. It has been showed by Black-Cox [5] that the first passage default probability is

$$\mathbb{P}[\tau < t | b = x] = \Phi(\frac{x - v_0 - \mu t}{\sigma\sqrt{t}}) + \exp(\frac{2\mu(x - v_0)}{\sigma^2})\Phi(\frac{x - v_0 + \mu t}{\sigma\sqrt{t}}). \tag{2.6}$$

where $\Phi(.)$ is the CDF for a standard normal random variable. Therefore the first passage default probability for a random boundary with distribution $f_b(x)$ is

$$\begin{aligned}\mathbb{P}[\tau < t] &= \int_{-\infty}^{v_0} \mathbb{P}[\tau_b < t | b = x] f_b(x) dx & (2.7)\\ &= \int_{-\infty}^{v_0} [\Phi(\frac{x - v_0 - \mu t}{\sigma\sqrt{t}}) + \exp(\frac{2\mu(x - v_0)}{\sigma^2})\Phi(\frac{x - v_0 + \mu t}{\sigma\sqrt{t}})] f_b(x) dx. & (2.8)\end{aligned}$$

Thus for the given survival probability, one has the following relationship

$$\begin{aligned}\mathbb{P}[\tau > t] &= e^{-\lambda t} & (2.9)\\ &= \int_{-\infty}^{v_0} [\Phi(\frac{-x + v_0 + \mu t}{\sigma\sqrt{t}}) - \exp(\frac{2\mu(x - v_0)}{\sigma^2})\Phi(\frac{x - v_0 + \mu t}{\sigma\sqrt{t}})] f_b(x) dx. & (2.10)\end{aligned}$$

To solve this integral equation, we use Laplace transforms as well as an identity in Abramowitz and Stegun [3], page 1026.

$$\mathcal{L}\{e^{-\lambda t}\}(u) = \mathcal{L}\{\int_{-\infty}^{v_0} [\Phi(\frac{-x+v_0+\mu t}{\sigma\sqrt{t}}) - \exp(\frac{2\mu(x-v_0)}{\sigma^2})\Phi(\frac{x-v_0+\mu t}{\sigma\sqrt{t}})] f_b(x)dx\}(u) \tag{2.11}$$

$$\frac{1}{u+\lambda} = \mathcal{L}\{\int_{-\infty}^{0} [\Phi(\frac{-y+\mu t}{\sigma\sqrt{t}}) - \exp(\frac{2\mu y}{\sigma^2})\Phi(\frac{y+\mu t}{\sigma\sqrt{t}})] f_b(v_0+y)dy\}(u) \tag{2.12}$$

$$\frac{1}{u+\lambda} = \frac{1}{u} - \frac{1}{u}\int_{-\infty}^{0} e^{(\mu/\sigma+\sqrt{\mu^2/\sigma^2+2u})y/\sigma} f_b(v_0+y)dy \tag{2.13}$$

$$\frac{\lambda}{u+\lambda} = \int_{-\infty}^{0} e^{(\mu/\sigma+\sqrt{\mu^2/\sigma^2+2u})y/\sigma} f_b(v_0+y)dy \tag{2.14}$$

$$\frac{\lambda}{u+\lambda} = \mathcal{L}\{f_b(v_0-y)\}((\mu/\sigma+\sqrt{\mu^2/\sigma^2+2u})/\sigma). \tag{2.15}$$

To simplify the calculation, we take $\sigma = 1$.

$$\frac{1}{u/\lambda+1} = \mathcal{L}\{f_b(-x)\}(\mu+\sqrt{\mu^2+2u}) \tag{2.16}$$

$$\mathcal{L}\{f_b(v_0-y)\}(s) = \frac{1}{(((s-\mu)^2-\mu^2)/2\lambda)+1} \tag{2.17}$$

$$\mathcal{L}\{f_b(v_0-y)\}(s) = \frac{2\lambda}{(s-\mu)^2-(\mu^2-2\lambda)} \tag{2.18}$$

$$\mathcal{L}\{f_b(v_0-y)\}(s) = \frac{\sqrt{2\lambda}}{(s-\mu)-\sqrt{\mu^2-2\lambda}}\frac{\sqrt{2\lambda}}{(s-\mu)+\sqrt{\mu^2-2\lambda}}, \tag{2.19}$$

where the last identity holds if $\mu^2 - 2\lambda > 0$. Now using the property of the convolution $(f * g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$ and the inverse Laplace transform

$$\mathcal{L}^{-1}\{\frac{\sqrt{2\lambda}}{(s-\mu)-\sqrt{\mu^2-2\lambda}}\}(z) = \sqrt{2\lambda}e^{(\mu+\sqrt{\mu^2-2\lambda})z}, \tag{2.20}$$

$$\mathcal{L}^{-1}\{\frac{\sqrt{2\lambda}}{(s-\mu)+\sqrt{\mu^2-2\lambda}}\}(z) = \sqrt{2\lambda}e^{(\mu-\sqrt{\mu^2-2\lambda})z}, \tag{2.21}$$

$$f_b(v_0-y) = \int_0^y 2\lambda e^{(\mu+\sqrt{\mu^2-2\lambda})(y-z)}e^{(\mu-\sqrt{\mu^2-2\lambda})z}dz \tag{2.22}$$

$$= 2\lambda e^{(\mu+\sqrt{\mu^2-2\lambda})y}\int_0^y e^{-2\sqrt{\mu^2-2\lambda}z}dz \tag{2.23}$$

$$= 2\lambda e^{(\mu+\sqrt{\mu^2-2\lambda})y}\frac{1-e^{2\sqrt{\mu^2-2\lambda}y}}{2\sqrt{\mu^2-2\lambda}} \tag{2.24}$$

$$= \lambda\frac{e^{(\mu+\sqrt{\mu^2-2\lambda})y}-e^{(\mu-\sqrt{\mu^2-2\lambda})y}}{\sqrt{\mu^2-2\lambda}}. \tag{2.25}$$

Hence the probability density distribution is explicitly obtained:

$$f_b(x) = \begin{cases} 0, & x \geq v_0; \\ \lambda \dfrac{e^{-(\mu+\sqrt{\mu^2-2\lambda})(x-v_0)} - e^{-(\mu-\sqrt{\mu^2-2\lambda})(x-v_0)}}{\sqrt{\mu^2-2\lambda}}, & x < v_0. \end{cases} \tag{2.26}$$

Please see the verification in Appendix A that (2.26) is a real probability density function with integral equal to 1 provided that $\mu < -\sqrt{2\lambda}$. Moreover

Proposition 1. *If the Laplace transform of the survival probability of the first passage time* τ *has the forma* $\mathcal{L}\{S_\tau(t)\}(u) = \sum_{i=1}^n \dfrac{c_i}{\lambda_i(u+\lambda_i)}$*, for some positive* λ_i, k_i, c_i *and* $\sum_{i=1}^n \dfrac{c_i}{\lambda_i} = 1$*, then if the drift in the Brownian motion* $\mu < \min\limits_{1\leq i\leq n} -\sqrt{2\lambda_i}$*, there is a probability density* $f_b(x)$ *associated with the given survival probability, i.,e* $f_b(x) \geq 0$ *and* $\int_{-\infty}^0 f_b(x)dx = 1$*.*

Proof: This follows immediately from the linearity of the Laplace transform and the above calculation.

This suggests that the level of randomness coherent in intensity models for default can be captured in structural models with simple default boundaries provided that one allows the boundaries to be random.

3.0 STRUCTURAL MODELS

The major advantage of structural models is their conceptual setting based on a firm's asset/liability dynamics. However, many researchers have noticed the deficiencies in the standard structural model with respect to the specification of parameters including the volatility, initial asset value and liability default barrier. For example, Albanese and Chen [1], Sato [50] and Schmidt [53] study modifications to the default barrier while Hurd and Kuznetsov [28] consider more general processes for the firms' assets. Furthermore, there is convincing evidence that many investors who have no insider information or are not closely connected to the market have incomplete information or delayed information about parameter values. Duffie and Lando [18], Giesecke [22] and Jarrow and Protter [30] address issues of this kind.

3.1 EQUIVALENCE OF RANDOM DEFAULT BARRIER AND RANDOM INITIAL STATE

Suppose the firm's asset, V_t, follows a diffusion process with initial state v_0:

$$dV_t = \mu dt + \sigma dW_t, \tag{3.1}$$

$$V_0 = v_0 \tag{3.2}$$

where W_t stands for standard Brownian Motion. Let $b < 0$ denote the constant default boundary. Then using the Itô's Lemma, the transition density $u(x, t)$ satisfies the following initial-boundary value problem for the forward Kolmogorov equation:

$$\frac{\partial u}{\partial t} = -\mu\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \tag{3.3}$$

$$u(x,0) = \delta(x - v_0) \tag{3.4}$$

$$u(b,t) = 0 \tag{3.5}$$

$$u(\infty,t) = 0 \tag{3.6}$$

The fundamental solution of this problem is $\frac{1}{\sigma\sqrt{2\pi t}}e^{-\frac{(x-v_0-\mu t)^2}{2\sigma^2 t}}$. Then using Green's function, we find the explicit solution to our problem,

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}}e^{-\frac{(x-v_0-\mu t)^2}{2\sigma^2 t}} - e^{\frac{2\mu(b-v_0)}{\sigma^2}}\frac{1}{\sigma\sqrt{2\pi t}}e^{-\frac{(x-v_0-\mu t-2b)^2}{2\sigma^2 t}} \tag{3.7}$$

and the survival probability is:

$$P[\tau > t] = S(t; b, v_0) = \int_b^\infty u(x,t)dx \tag{3.8}$$

$$= \Phi(\frac{v_0 - b + \mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu(b-v_0)}{\sigma^2}}\Phi(\frac{b - v_0 + \mu t}{\sigma\sqrt{t}}), \tag{3.9}$$

with the usual definition of default time τ

$$\tau = \inf\{t \geq 0; V_t \leq b\}. \tag{3.10}$$

Now consider the following two cases:

1. For a random boundary $b < v_0$ with probability density distribution $f_b(y)$ and a constant initial state v_0, the survival probability is

$$P[\tau > t] = S_b(t) = \int_{-\infty}^{v_0} f_b(y)S(t; y, v_0)dy \tag{3.11}$$

$$= \int_{-\infty}^{v_0} f_b(y)[\Phi(\frac{v_0 - y + \mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu(y-v_0)}{\sigma^2}}\Phi(\frac{y - v_0 + \mu t}{\sigma\sqrt{t}})]dy \tag{3.12}$$

$$= \int_{-\infty}^{0} f_b(z + v_0)[\Phi(\frac{-z + \mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu z}{\sigma^2}}\Phi(\frac{z + \mu t}{\sigma\sqrt{t}})]dz \tag{3.13}$$

2. For a random initial state $v_0 > b$ with distribution $g_{v_0}(y)$ and a constant default boundary $b < 0$, the survival probability is

$$P[\tau > t] = S_{v_0}(t) = \int_b^\infty g_{v_0}(y)S(t;b,y)dy \tag{3.14}$$

$$= \int_b^\infty g_{v_0}(y)[\Phi(\frac{y-b+\mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu(b-y)}{\sigma^2}}\Phi(\frac{b-y+\mu t}{\sigma\sqrt{t}})]dy \tag{3.15}$$

$$= \int_{-\infty}^0 g_{v_0}(b-z)[\Phi(\frac{-z+\mu t}{\sigma\sqrt{t}}) - e^{\frac{2\mu z}{\sigma^2}}\Phi(\frac{z+\mu t}{\sigma\sqrt{t}})]dz \tag{3.16}$$

Thus we have,

Proposition 2. *The random default boundary(RDB) with constant initial state v_0 is equivalent to the random initial state(RIS) with constant default boundary b if $f_b(y+v_0) = g_{v_0}(b-y)$, where f and g are the pdf for RDB and RIS with $y < 0$, respectively*

3.2 MODIFIED BLACK-SCHOLES MODEL

Motivated by Finger [21], which summarizes the approach used by J.P, MORGAN, we will present a new model with a random default barrier. To model the firm's asset value, V_t, it is customary to use geometric Brownian motion to avoid negative values of assets. Furthermore, we will assume the liability L_t grows exponentially as, for example, a simple zero coupon bond with constant interest rate. In a risk neutral world, both the drift term in the firm's asset and the growth rate in the liability are the same and equal to the risk free interest rate r. More precisely with V_t denoting the asset per share and L_t the Liability per share,

$$\frac{dV_t}{V_t} = rdt + \sigma_v dW_t \tag{3.17}$$

$$L_t = L_0 e^{rt}, \tag{3.18}$$

where σ_v stands for the asset volatility and L_0 is the current liability. One of the important contributions of Black and Scholes [9] is that they derived a partial differential equation, now called the Black-Scholes equation, which governs the price of options over time. Merton [41] expanded the mathematical understanding of the option pricing model.

We now incorporate their ideas in our proposed model. First we determine the process for the equity, $S_t = V_t - L_t$:

$$dS_t = dV_t - dL_t \tag{3.19}$$

$$= rV_t dt + \sigma_v V_t dW_t - rL_t dt \tag{3.20}$$

$$= rS_t + \sigma_v(L_t + S_t)dW_t. \tag{3.21}$$

Then the corresponding Black-Scholes equation for a European call option can be derived using Ito's calculus and the concept of a replicating portfolio

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma_v^2(S_t + L_t)^2\frac{\partial^2 C(S,t)}{\partial S^2} + rS_t\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0, \tag{3.22}$$

subject to

$$C(S,T) = \max(S - E, 0) \tag{3.23}$$

$$C(0,t) = 0, \tag{3.24}$$

where E is the call option's strike price and T is the option's expiry. An explicit formula is available for the solution of problem (3.22)–(3.24) (see Appendix B for details).

$$C(S,t;E,T,r,\sigma_v,L_0) = L_0 e^{rt}[e^{\xi}\Phi(d_1) - \Phi(d_2) - \Phi(d_3) + e^{\xi}\Phi(d_4)] - Ee^{-r(T-t)}[\Phi(d_2) - e^{\xi}\Phi(d_4)] \tag{3.25}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-\frac{y^2}{2}} dy$, the standard normal CDF and,

$$\xi = \ln(\frac{S}{L_0 e^{rt}} + 1) \tag{3.26}$$

$$\hat{\xi} = \ln(\frac{E}{L_0 e^{rT}} + 1) \tag{3.27}$$

$$\tau = \frac{1}{2}\sigma_v^2(T - t) \tag{3.28}$$

$$d_1 = \frac{\tau + \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{3.29}$$

$$d_2 = \frac{-\tau + \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{3.30}$$

$$d_3 = \frac{\tau - \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{3.31}$$

$$d_4 = \frac{-\tau - \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{3.32}$$

When $L_0 \to 0$, one retrieves the original version outlined by Merton [41].

$$\xi - \hat{\xi} = \ln(\frac{S}{L_0 e^{rt}} + 1) - \ln(\frac{E}{L_0 e^{rT}} + 1) \tag{3.33}$$

$$\to \ln\frac{S}{E} + r(T-t) \tag{3.34}$$

$$d_3 \to -\infty \tag{3.35}$$

$$d_4 \to -\infty \tag{3.36}$$

Therefore $C(S,t;E,T,r,\sigma_v,L_0) \to L_0 e^{rt}(\frac{S}{L_0 e^{rt}} + 1)\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2)$

$$\to S\Phi(\frac{\tau + \ln\frac{S}{E} + r(T-t)}{\sqrt{2\tau}}) - Ee^{-r(T-t)}\Phi(\frac{-\tau + \ln\frac{S}{E} + r(T-t)}{\sqrt{2\tau}}) \tag{3.37}$$

3.3 APPLICATION TO CREDIT DEFAULT SWAPS

While a firm's asset and liability data are only available from quarterly reports (and often purposely misleading), its equity data is publicly available. Indeed, we have direct access to a wealth of equity-related data including option values and implied volatilities. In the following two subsections, we will use our proposed model to study credit default swaps(CDS) on several companies and compare our analytical results with credit data.

To calculate the CDS spreads, we begin by deriving the survival probability using our model. Here the default time is defined as first-passage time with random barrier and its associated survival probability is

$$\mathrm{P}[\tau > t] = \int_{-\infty}^{\infty} [\Phi(\frac{\ln(V_0) - \ln(x) - \sigma_v^2/2t}{\sigma_v\sqrt{t}}) - \frac{V_0}{L_0}\Phi(\frac{-\ln(V_0) + \ln(x) - \sigma_v^2/2t}{\sigma_v\sqrt{t}})] f_l(x)dx \tag{3.38}$$

$$= \int_{-\infty}^{\infty} [\Phi(\frac{\ln(1+\frac{S_0}{x}) - \sigma_v^2/2t}{\sigma_v\sqrt{t}}) - \frac{S_0 + x}{x}\Phi(\frac{-\ln(1+\frac{S_0}{x}) - \sigma_v^2/2t}{\sigma_v\sqrt{t}})] f_l(x)dx \tag{3.39}$$

which follows from the previous results (3.11) with S_0, the current equity price. We then calculate the survival probability using the formula for CDS spreads $s(t)$ with accrual premium payment under the assumption that default occurs half-way between premium payment dates (see J. Hull, M. Predescu, and A. White [26], page 11),

$$s(t) = \frac{(1-R)\sum_{i=1}^{N} B(0, \frac{t_i+t_{i+1}}{2})[P(\tau > t_{i-1}) - P(\tau > t_i)]}{\sum_{i=1}^{N}(t_i - t_{i-1})B(0,t_i)P(\tau > t_i) + \frac{t_i - t_{i-1}}{2}B(0, \frac{t_i+t_{i+1}}{2})[P(\tau > t_{i-1}) - P(\tau > t_i)]}, \tag{3.40}$$

where t_i is the date of the quarterly payment, R is the recovery rate and $B(0,t)$ is the current price of a riskless zero-coupon bond maturing at time t with payoff \$1.

3.3.1 Data Source and Implementation

From (3.38) and (3.40), the parameters required in order to calculate the survival probability, and therefore $s(t)$, are $\{\sigma_v, r, R, L_0, V_0\}$. First of all, some related financial data can be readily obtained:

- Historical stock prices S_0
- 3-month At-The-Money(ATM) Option Values including European Call and Put and values associated ATM Implied Volatility σ_{imp} on each trading day;
- Market Credit Spreads on each trading day with tenor T : 1-,2-,3-,4-,5-,7-,10- year;
- US Treasury Bills Interest Rates on each trading day with 1-,3-,6- month and 1-,2-,3-,4-,5-,7-,10- year maturity, as risk free interest rate r.

Then, the only parameters remaining are σ_v and L_0 since r and R can be observed directly and $V_0 = S_0 + L_0$. For our model, the most important issue is how to correctly model the liability. In practice, we use the following steps to estimate L_0, or more precisely its distribution $f_l(x)$. In fact there is one other important relation between the volatility of the asset, σ_v, and volatility of the equity, σ_s, that we can use (see page 17 of Giesecke and Goldberg [24])

$$S\sigma_s = V\sigma_v\Delta \tag{3.41}$$

$$\sigma_v = \frac{S\sigma_s}{V\Delta} \tag{3.42}$$

where Δ is the equity's hedging Delta (see Shreve [54], page 159) defined by

$$\Delta = \Phi(\frac{\ln(S/E) + (r - \sigma_s^2/2)(T-t)}{\sigma_s(T-t)}). \tag{3.43}$$

Therefore the option value only depends on L_0 if we use σ_{imp} to replace σ_s in (3.41) and find σ_v in terms from $\frac{S\sigma_s}{V\Delta}$ in (3.25). Hence the only remaining problem is to estimate the distribution $f_l(x)$ of L_0.

The following are the steps to estimate this distribution, $f_l(x)$, of L_0 using option data, and then CDS 5-year spreads with market data.

1. The solution to the PDE of European Call Option has been derived. From Bloomberg, the 3-month(corporate reporting period) ATM call option values and its associated implied volatility are given, so we can use those to calculate the implied value of L_0 on each trading day by (3.25).
2. Repeat the calculation in the first step to find the latest 3-month L_0 from latest 3-month option values from Bloomberg.
3. Once the values of L_0 have been calculated by the previous two steps, we use the software EASYFIT to fit the data for L_0 from among several distributions and choose the best as the distribution of $f_l(x)$.
4. Then using (3.38) we obtain the survival probability from our first passage time model
5. Apply (3.40) to calculate the current 5-year CDS spread from the model
6. Apply steps 1-5 to calculate the historical CDS spreads during the time horizon in our study

Of course, there are several possible choices for the distribution of L_0. Using EASYFIT we determine that the Gamma distribution is our best choice based on three Goodness of Fit tests: Kolmogorov-Smirnov, Anderson-Darling and Chi-Square. Thus, in the following section of the case study, we use the Gamma distribution for the purposes of comparison.

3.3.2 Case Studies

The previous 6-step procedure can be used to estimate the historical CDS spreads for any company whose equity data is publicly available. In particular, we will conduct our empirical studies for the following four companies: HEINZ, KODAK, J.P. MORGAN and BANK OF AMERICA. We choose those four companies for the following reasons:

- To compare firms with different credit ratings: HEINZ and J.P.MORGAN is are quite healthy companies while KODAK and BANK OF AMERICA are in financial distress
- To choose companies from different industrial sectors, HEINZ AND KODAK are from industrials while J.P. MORGAN and BANK OF AMERICA are financial institutions.

From Figure 1 for HEINZ, we can see that the spreads obtained from our model fit the overall pattern of the market observations well, especially capturing some major spikes and the time when the credit crisis started (around the middle of time horizon: late 2008). Although HEINZ is a relatively healthy company, the credit crisis also hits the company's sales and marketing after late 2008 and the credit spreads from the graph also reflect this phenomena. KODAK was a company that was quite stable before the credit crisis and currently is experiencing financial distress due to the rapid development of digital products (intrinsic risk factor) and the credit crisis starting in 2008 (global risk factor). Our model does capture the huge spike in the middle and the end of the time horizon, and also reflects the relative stable period before 2008. In addition, we conducted the same calculations for two major financial institutions, J.P. MORGAN and BANK OF AMERICA from Oct 2007 to Oct 2011. From Figure 3 and 4, it is worth noting that the major spikes are captured by the model especially during the credit crisis of 2008 and the most recent financial trouble in Europe. Moreover, Due to the BANK OF AMERICA's mortgage problem (intrinsic risk factor) also triggered the recent CDS spread jump, which is also captured by our model. In conclusion, our model matches the credit spreads for the four diverse companies in the study. Most importantly, the model captures the timing of the huge spikes in the spreads. As such, it can be considered as an important new credit risk indicator for predicting future spikes.

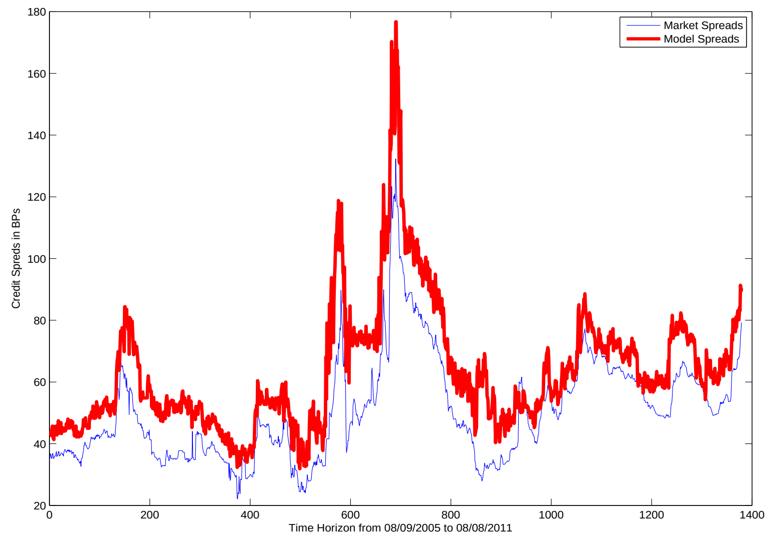

Figure 1: HEINZ's 5-year Credit Spreads Comparison with L_0 following a Gamma Distribution; DATA SOURCE: From Bloomberg 08/09/2005-08/08/2011

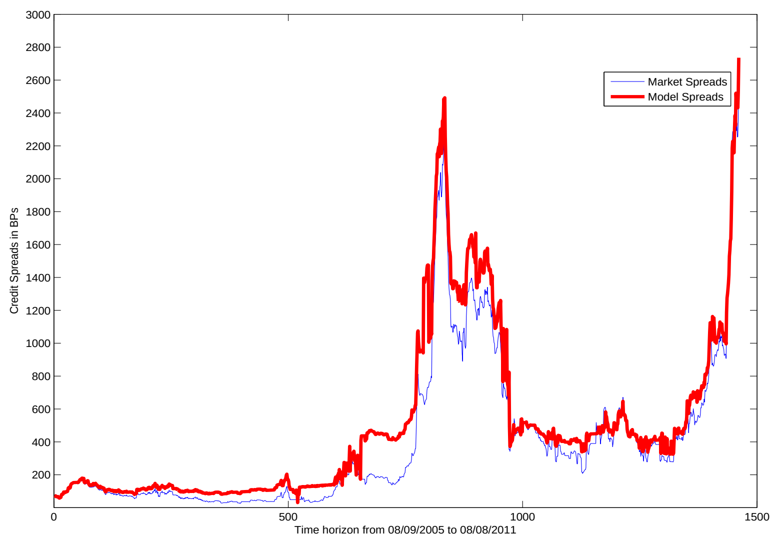

Figure 2: KODAK's 5-year Credit Spreads Comparison with L_0 following a Gamma Distribution; DATA SOURCE: From Bloomberg 08/09/2005-08/08/2011

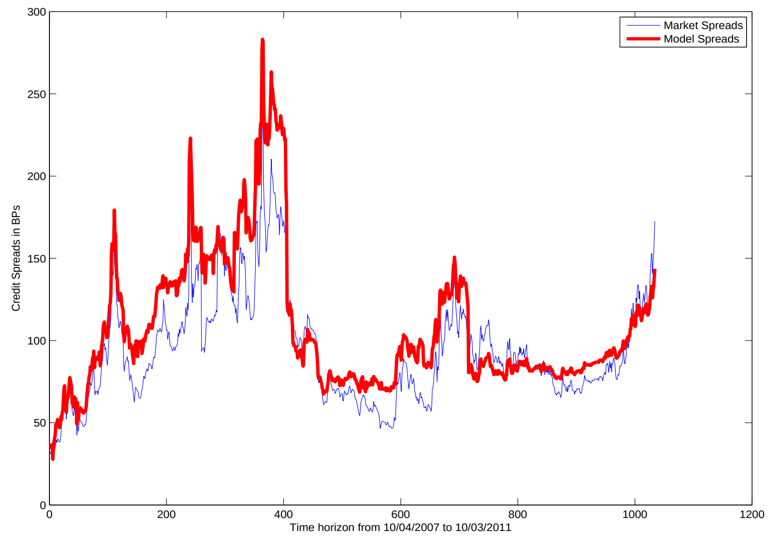

Figure 3: J.P. MORGAN's 5-year Credit Spreads Comparison with L_0 following a Gamma Distribution; DATA SOURCE: From Bloomberg 10/04/2007-10/02/2011

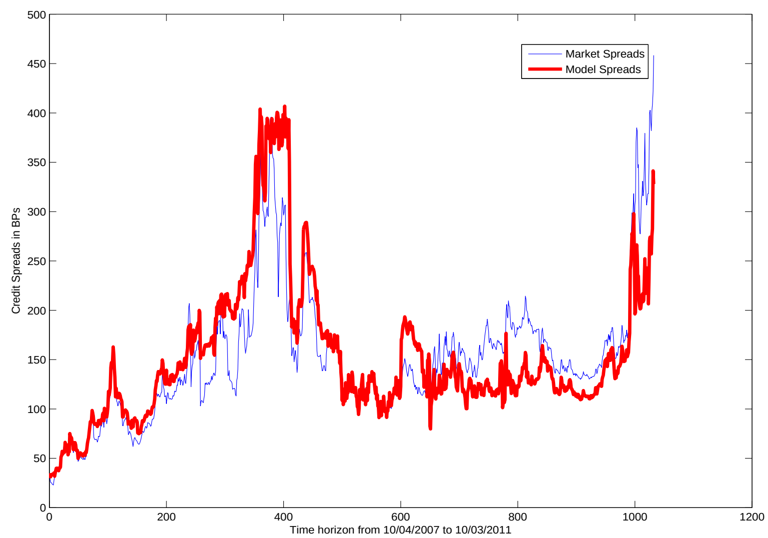

Figure 4: BANK OF AMERICA's 5-year Credit Spreads Comparison with L_0 following a Gamma Distribution; DATA SOURCE: From Bloomberg 10/04/2007-10/02/2011

4.0 CORRELATION MODELS

The only successful way to manage credit risk is by diversifying the exposures and avoiding concentration. For example, Philip J. Schonbucher [53] remarks "In a large portfolio of credits, idiosyncratic risk is fully diversified, and the only source of portfolio loss uncertainty is the uncertainty in the portfolio default rate driven by common factor". Diversification of credit risk can only be successful if an adequate portfolio credit risk model is in place that can quantify the risks in the portfolio.

To capture the correlation-dependent effects, we will use a hierarchical model that extends the single firm models discussed in the previous chapter. This approach is quite standard in credit risk modeling and has been studied and extended in many different directions. In the case of homogeneous portfolios, it is often coupled with large portfolio approximation techniques. Here we will propose a new heterogenous model with different correlations among the individuals and the market index.

4.1 HIERARCHICAL MODELS

A two factor latent variable Gaussian model for an individual firm's assets, is modeled by Li [39],

$$V_i = \rho_i M + \sqrt{1 - \rho_i^2} Z_i, \tag{4.1}$$

with both the latent factor M and the idiosyncratic random variable Z_i taken as independent standard normal processes. The latent factor M will be interpreted as the macro economy status and Z_i is the firm-specific factor. The correlation between the firm i and the latent factor is ρ_i and the correlation between V_i and V_j is thus $\rho_i \rho_j$ if $i \neq j$.

To begin, we quickly recall the existing results using this approach. In the Vasicek setting where default occurs if the firm's value falls below the barrier $B_i(T)$ at maturity, T, the conditional default probability of firm i under the Vasicek's framework (default occurring if a firm's value hits the barrier $B_i(t)$ at the maturity, T) is

$$P[V_i(T) < B_i(T)|M(T) = m] = P[Z_i(T) < \frac{B_i(T) - \rho_i m}{\sqrt{1-\rho_i^2}}] \tag{4.2}$$

$$= \Phi(\frac{B_i(T) - \rho_i m}{\sqrt{1-\rho_i^2}}). \tag{4.3}$$

In a homogeneous portfolio, one takes $\rho_i = \rho_j$ if $i \neq j$. Under these assumptions, given the market situation $M = m$, all the firms have the same conditional risk-neutral default probability. Moreover, for a given value of the market component M, the defaults are mutually independent for all the underlying companies. Letting $D^N(t)$ be the total defaults in the portfolio with $N = 125$ names that have occurred by time t conditional on the market condition $M = m$, then $D^N(t)$ follows a binomial distribution

$$P[D^N(T) = k|M(T) = m] = \binom{N}{k} P[V_i(T) < B_i(T)|M(T) = m]^k (1 - P[V_i(T) < B_i(T)|M = m])^{125-k} \tag{4.4}$$

Hence, the unconditional probability of exactly k defaults by time t

$$P[D^N(T) = k] = \int_{-\infty}^{\infty} P[D^N(T) = k|M = m]\phi(m)dm. \tag{4.5}$$

Therefore we can estimate the expected loss in a given large portfolio such CDS NA IG series and ITraxx Euro series with the approximation algorithm provided by Vasicek [56].

Due to its easy implementation and approximation, the homogeneous large portfolio model has been widely used. However, many market practioners and academic researchers noticed there are three major drawbacks of the factor models:

1. not all correlations are the same
2. the factors M and Z_i are not normal
3. default can occur anytime up to maturity

A. Mortensen [45] removes the first and second assumptions while J. Hull, M. Predescu, and A. White [26] use the first-passage time as the default time. From their numerical results in the estimation of CDX prices, one observes that the modified models improve the results when compared with actual market quotations.

4.2 HOMOGENEOUS CORRELATION MODEL

First we will propose a homogeneous correlation model which extends our single firm model in the preceding chapter. From now on, we take the default time to be the first-passage time as in our single firm model. Thus the assets and liabilities follow the process for each company i under risk-neutral probability

$$\frac{dV_t^i}{V_t^i} = rdt + \sigma_i dW_i \tag{4.6}$$

$$L^i = L_0^i e^{rt}, \tag{4.7}$$

equivalently, after taking the difference of logarithm of asset and liability one has

$$\ln \frac{V_t^i}{L_t^i} = \ln \frac{V_0^i}{L_0^i} - \frac{\sigma_i^2}{2} t + \sigma_i dW_i. \tag{4.8}$$

Note that, the default barrier for this new process is 0. We now assume that $\ln \dfrac{V_0^i}{L_0^i} := LV_0^i$ follows the model

$$LV_0^i = \rho_i M + \sqrt{1-\rho_i^2} Z_i, \tag{4.9}$$

where M and Z_i are time-independent random variables and

- M and Z_i are independent random variables with pdf $f(x)$ and $g_i(x)$
- Z_i and Z_j are independent random variables if $i \neq j$
- ρ_i is the correlation of LV_0^i and M

Therefore the survival probability conditioned on both M and Z_i is

$$P[\tau_i > t|M = m, Z_i = z_i] = \Phi(\frac{\rho_i m + \sqrt{1-\rho_i^2}z_i - \frac{\sigma_i^2}{2}t}{\sigma_i\sqrt{t}}) - e^{\rho_i m+\sqrt{1-\rho_i^2}z_i}\Phi(\frac{-\rho_i m - \sqrt{1-\rho_i^2}z_i - \frac{\sigma_i^2}{2}t}{\sigma_i\sqrt{t}}), \tag{4.10}$$

In the homogeneous model, we will assume

- $\sigma_i = \sigma$ and $Z_i = Z$. In other words, each firm has the same default probability;
- correlation between individual firm and the market is same. In other words, $\rho_i = \rho$;
- each firm has the same weight $\dfrac{1}{N}$ in the portfolio with N names.

Hence the probability of k defaults in N-names basket given M and Z is

$$P[D^N(t) = k|M = m, Z = z_i] = \binom{N}{k} P[\tau_i > t|M = m, Z_i = z]^k(1 - P[\tau_i > t|M = m, Z_i = z])^{N-k} \tag{4.11}$$

$$:= PD(m, z) \tag{4.12}$$

and its unconditional default probability is

$$P[D^N(t) = n] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\binom{N}{k} PD(m, z)^n(1 - PD(m, z)])^{N-k}dF(m)dG(z) \tag{4.13}$$

$$P[D^N(t) = n] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\binom{N}{k} s^n(1 - s)^{N-k}dF(PD^{-1}(p, z))dG(z). \tag{4.14}$$

As mentioned earlier, it is essential to estimate the loss distribution in order to calculate the tranche losses of a CDO (see Appendix C). Since it is common that the number of firms in the portfolio is very large, usually more than 100, therefore the estimation to calculate the loss distribution is very crucial. As suggested by Vasicek [56],

$$\lim_{N\to\infty}\sum_{k=0}^{[N\gamma]}\binom{N}{n} s^n(1 - s)^{N-n} = \begin{cases} 0, & \gamma < s; \\ 1, & \gamma > s. \end{cases} \tag{4.15}$$

Thus

$$P[Loss \leq \gamma] = \sum_{k=0}^{[N\gamma]} P[D^N(t) = k] \tag{4.16}$$

$$= \sum_{k=0}^{[N\gamma]} -\int_{-\infty}^{\infty} \int_{0}^{1} \binom{N}{k} s^k (1-s)^{N-k} dF(PD^{-1}(p,z)) dG(z) \tag{4.17}$$

$$\approx -\int_{-\infty}^{\infty} \int_{0}^{\gamma} dF(PD^{-1}(p,z)) dG(z) \tag{4.18}$$

$$= \int_{-\infty}^{\infty} 1 - F(PD^{-1}(\gamma,z)) dG(z) \tag{4.19}$$

$$= 1 - \int_{-\infty}^{\infty} F(PD^{-1}(\gamma,z)) dG(z). \tag{4.20}$$

As showed in the Appendix C, we are able to estimate the expected losses for each tranche in the given portfolio based on the above estimation (4.16).

4.2.1 Calibration Results

To compare the results of market quotes and our model, we used standard calibration methods to estimate the parameters in the model. The relative root mean square error is given by

$$\text{RRMSE} = \sqrt{\frac{1}{N_{tr}} \sum_{i=1}^{N_{tr}} (\frac{\bar{s_i} - s_i}{s_i})^2}, \tag{4.21}$$

where N_{tr} stands for the number of tranches in the CDS index tranches, and $\bar{s_i}$ and s_i are the model tranche prices and market quotes of tranche prices. In the calibrations, we use three different distributions for the factors M and Z_i: Gamma, Log-Normal and t_4 with two unknown parameters r_1, r_2 in each distribution. For instance, r_1 and r_2 are the mean and standard deviation for the t_4-distribution. The basic idea of calibration is to minimize the RRMSE based on the parameters in the model, and in our model the parameters involved are contained in the set

$$\Theta = \{\sigma, \rho, r_1, r_2\}. \tag{4.22}$$

The following Table 1 shows the comparison between market quotes and model results and Table 2 compares the results of error. Although the model calibration results do not have very good estimation based on relative error, they reflect the spreads pattern reasonably well, especially when compared to previous work from Mortensen [45].

Table 1: iTraxx 5-year index tranches comparison on 23-August-2004, data source: from Mortensen [45]

	Market	Gamma	t_4	Log-Normal
0-3%	**25.5%**	28.2%	27.7%	26.8%
3-6%	**146bp**	162.72bp	166.31bp	178.48bp
6-9%	**60.3bp**	52.37bp	67.83bp	71.94bp
9-12%	**36.3bp**	30.83bp	40.25bp	22.51bp
12-22%	**19.3bp**	14.13bp	17.32bp	10.13bp
ρ		0.183	0.231	0.256

Table 2: iTraxx 5-year index tranches relative error on 23-August-2004

	Gamma	t_4	Log-Normal
0-3%	10.59%	8.63%	5.10%
3-6%	11.45%	13.91 %	22.25%
6-9%	13.15%	12.49%	19.30%
9-12%	15.07%	10.88%	37.99%
12-22%	26.79%	10.26%	47.51%

4.3 INHOMOGENEOUS CORRELATION MODEL

In this section we will present the heterogenous extension of the model.

Let us restate our model of (4.9)

$$\ln \frac{V_t^i}{L_t^i} = \ln \frac{V_0^i}{L_0^i} - \frac{\sigma_i^2}{2} t + \sigma_i dW_i \tag{4.23}$$

$$\ln \frac{V_0^i}{L_0^i} = \rho_i M + \sqrt{1-\rho_i^2} Z_i, \tag{4.24}$$

with heterogenous assumptions

- $\rho_i \neq \rho_j$ if $i \neq j$
- Z_i and Z_j are independent and follow similar kind of distributions but with different parameters
- $\sigma_i \neq \sigma_j$ if $i \neq j$

This model is the higher dimensional extension of our one-dimension model where we calculate $\ln \frac{V_0^i}{L_0^i}$ and σ_i from the one-dimensional problem using equity option values. To determine the factor M and the correlation ρ_i, we use a scaled market index as M and the calculated correlation coefficient with $\ln \frac{V_0^i}{L_0^i}$ as our ρ_i. For example, to estimate the tranche price in CDX.NA.IG, we use a scaled Dow Jones Industrial Average as M. Also we fit the distributions of M and Z_i to the realized values of these random variables. In other words, we use the daily scaled market index as the realized values of the factor M. Once we know M and ρ_i (using the realized values of $\ln \frac{V_0^i}{L_0^i} := LV_0^i$ for each firm), we take the realizations of Z_i as

$$z_i = \frac{LV_0^i - \rho_i m}{\sqrt{1-\rho_i^2}} \tag{4.25}$$

In practice, we perform the calculations based on the following procedures

1. Apply the method in one dimension to calculate L_0^i and σ_i which were backed out from option values for each i and take $V_0^i = S_0^i + L_0^i$.
2. Use the scaled Dow Jones Index Average(DJX) as M, in the real application using DJX/50000 as the realization of M
3. Calculate the correlation coefficient ρ_i for $\ln \frac{V_0^i}{L_0^i}$ and DJX/50000
4. Solve z_i from (4.25)

Thus, we use historical equity information to get the realizations of M and Z_i. By results from EASYFIT, we choose to use Log-Normal distributions for M and Gamma distributions for Z_i.

4.3.1 Loss Distribution

Based on the heterogenous model we proposed, the survival probability conditioned on M is $P[\tau_i > t|M = m]$

$$= \int_{-\infty}^{\infty} [\Phi(\frac{\rho_i m + \sqrt{1-\rho_i^2} z_i - \frac{\sigma^2}{2} t}{\sigma\sqrt{t}}) - e^{\rho_i m + \sqrt{1-\rho_i^2} z_i} \Phi(\frac{-\rho_i m - \sqrt{1-\rho_i^2} z_i - \frac{\sigma^2}{2} t}{\sigma\sqrt{t}})] g_i(z_i) dz_i \tag{4.26}$$

Since defaults are conditionally independent, the conditional probability of observing k defaults in a N-basket can be obtained in Andersen, Sidenius, and Basu [4] through the following recursive algorithm for $k = 1...N$,

$$P[D^N(t) = k|M = m] = P[D^{N-1}(t) = k|M = m] P[\tau_N > t|M = m]$$

$$+ P[D^{N-1}(t) = k - 1|M = m] P[\tau_N < t|M = m] \tag{4.27}$$

For $k = 0$ the last term obviously disappears. This recursion algorithm starts from $P[D^0(t) = k|M = m) = 1_{\{k=0\}}$. In order to calculate the unconditional loss distribution,

$$P[D^N(t) = k] = \int_{\Omega} P[D^N(t) = k|M = m] dF(m). \tag{4.28}$$

According to the document stated by MARKIT, each company in the CDX NA IG series (125 companies) has the same weight $w_i = w = \frac{1}{125}$, so the loss at time t

$$L(t) = \sum_{i=1}^{N} w_i (1 - R_i) 1_{\{\tau_i \leq t\}} = w(1 - R) \sum_{i=1}^{N} 1_{\{\tau_i \leq t\}} = w(1 - R) D^N(t), \tag{4.29}$$

where $R_i = R$ is the individual recovery rate. Specifically, the ith tranche loss with attachment point $\underline{K}_i$ and detachment point $\overline{K}_i$ is

$$L^i(t) = \frac{\max\{L(t) - \underline{K}_i, 0\} - \max\{L(t) - \overline{K}_i, 0\}}{\overline{K}_i - \underline{K}_i} \tag{4.30}$$

Table 3: Compositions of CDX.NA.IG index and iTraxx.EUR index.

Industrial Groups	CDX.NA.IG	iTraxx.EUR
Autos	0	10
Consumers	37	30
Energy	14	20
Industrials	28	20
Technology, Media and Telecommunications	22	20
Financial institutions	24	25

4.3.2 Application to CDX Index Tranche Pricing

There are two standard CDX index tranches in the market and the above table shows the distributions of industries in the portfolios. To calculate the index tranche prices, we need equity data for all 125 firms in the portfolio. In the real application, there are two companies whose financial data are not publicly available in CDX.NA.IG Series 17:**National Rural Utilities Cooperative Finance Corporation** and **Cox Communications, Inc**. So we use the data from the remaining 123 companies in conducting the numerical calculation.

Under the independence assumptions between the interest rates and the loss process $L(t)$, the protection leg can be approximated by the following discretization

$$\text{Prot}^i(t,T) \approx \sum_{\{j:t_j>t\}}^{M} B\left(t, \frac{t_j + \max(t_{j-1},t)}{2}\right)\left(\text{E}(L^i_{t_j}|\mathcal{F}_t) - \text{E}(L^i_{\max(t_{j-1},t)}|\mathcal{F}_t)\right), \qquad (4.31)$$

where the set $\{t_j\}$ is usually the premium payment dates with $t_0 = 0$ and $t_M = T$. On the other hand, the market value of the premium leg is

$$\text{Prem}^i(t,T;S^i,U) = U^i + \text{E}\left(\sum_{\{j:t_j>t\}}^{M} e^{-\int_t^{t_j}}(t_j - \max(t_{j-1},t))S^i \int_{\max(t_{j-1},t)}^{t_j} \frac{1-L_s^i}{t_j - \max(t_{j-1},t)} ds \Big| \mathcal{F}_t\right) \tag{4.32}$$

$$\approx U^i + S^i\text{E}\left(\sum_{\{j:t_j>t\}}^{M} B(t,t_j)(t_j - \max(t_{j-1},t))\left(1 - \text{E}(L_{t_j}^i|\mathcal{F}_t)\right)\right). \tag{4.33}$$

where U^i is an up-front payment and S^i is an annual premium. Therefore, we can apply (4.31) and (4.32) to conduct numerically calculate the tranche prices and compare the results with market observations. Table 4 shows the comparison of the Up-Front Payment if the running spread is fixed and Table 5 shows the comparison of the running spread if the Up-Front payment is fixed. Figure 5 reflects the histogram of ρ_i for 123 companies in CDX.NA.IG Series 17 and the average ρ is calculated in the following two tables. As we can see from those two tables, the estimation always overshoot the the market quotations both for up-front payment and running spreads. One of the important reasons lies in the fact that is our computed correlation ρ_i is much bigger than the implied correlation suggested by Bloomberg, which is around 0.23. As we can see from the histogram of ρ_i, most companies are strongly positive correlated with the market index and it makes perfect sense since MARKIT will always choose companies which are healthy and have major impacts on the market in its index portfolio. We compare our results to those of J. Hull, M. Predescu, and A. White [26]. Theirs is a first passage time model similar to ours with a constant barrier but with constant or stochastic correlation. Comparing their results in Table 6 with ours we find that their error is slightly better than our model in some tranches but has major drawback in the most senior tranche since it has low calibrated correlation. In our model, we have relatively good estimates for the senior tranche due to strong positive correlation.

Table 4: Comparison of our model prices with the market quotes for the 5-year CDX NA IG Series 17 index tranches. The market quotes are on Oct 19, 2011. Bloomberg has changed the format of quotes. The quotes in the following table are the up-front percentage with fixed basis points in each tranche. $0-3\% : 500bps, 3-7\% : 100bps, 7-15\% : 100bps, and 15-100\% : 25bps$. In the calculation we are given the spreads and try to calculate the up-front.

	Market Upper Front	Model Upper Front	Abs Error(Rel)
0-3%	**40.285%**	56.34%	16.055%(39.85%)
3-7%	**28.035%**	33.50%	5.465%(19.49%)
7-15%	**9.530%**	12.81%	3.28(34.42%)
15-100%	**1.5%**	1.89%	0.39(26.00%)
Average ρ		0.6453	

4.4 VIX-LIKE VOLATILITY

Since there is a slightly higher relative error in the tranche quotes, we are trying to improve our estimations of the parameters in our model. One of the parameters that could be readily improved is the volatility σ_i. In the previous discussion, we are using the current volatility as σ_i for all firms. For example, to estimate the tranche price on Oct 19th, 2011, we used the volatility calculated from option values on Oct 19th, 2011 as σ_i. One of the disadvantage is that there might be huge difference if we choose options with different strikes, so we apply the same idea used in VIX [11] from the work by Kresimir Demeterfi, Emanuel Derman, Michael Kamal and Joseph Zou [14]. The generalized formula and one hypothetical example of VIX, σ, can be found from [11] (see its derivation [14])

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta E_i}{E_i^2} e^{rT} Q(E_i) - \frac{1}{T}[\frac{F}{E_0} - 1]^2, \tag{4.34}$$

where T is the time to expiry, F is the forward index level derived from option indices, E_0 is the first strike below the forward index level, E_i is the strike price of ith out-of-the money option (a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$), $\Delta E_i = \dfrac{E_{i+1} - E_i}{2}$, r is the risk free interest rate and $Q(E_i)$ is the midpoint of the bid-ask spread for each option with strike E_i.

Table 5: Comparison of the model prices with the market quotes for the 5-year CDX NA IG Series 17 index tranches. The market quotes are on Oct 19, 2011. Bloomberg has changed the format of quotes. The quotes in the following table are the running spread in each tranche. $0-3\% : 500bps, 3-7\% : 100bps, 7-15\% : 100bps, and 15-100\% : 25bps$ with up-front $40.285\%, 28.035\%, 9.530\%$ and 0.015%, respectively. In other words, we are given the up-front and calculate the running spreads.

	Market Running Spread	Model Running Spread	Abs Error(Rel)
0-3%	**500bps**	607bps	107bps%(21.4%)
3-7%	**100bps**	128bps	28bps(28%)
7-15%	**100bps**	124bps	24bps(24%)
15-100%	**25bps**	30bps	5bps(20%)
Average ρ		0.6453	

Although the VIX is often called the "fear index", a high VIX is not necessarily bearish for stocks. Instead, the VIX is a measure of market perceived volatility in either direction, including to the upside. In practical terms, when investors anticipate large upside volatility, they are unwilling to sell upside call stock options unless they receive a large premium. Option buyers will be willing to pay such high premiums only if similarly anticipating a large upside move. The resulting aggregate of increases in upside stock option call prices raises the VIX just as does the aggregate growth in downside stock put option premiums that occurs when option buyers and sellers anticipate a likely sharp move to the downside. When the market is believed as likely to soar as to plummet, writing any option that will cost the writer in the event of a sudden large move in either direction may look equally risky. Hence high VIX readings mean investors see significant risk that the market will move sharply, whether downward or upward. The highest VIX readings occur when investors anticipate that huge moves in either direction are likely. Only when investors perceive neither significant downside risk nor significant upside potential will the VIX be low.

VIX can be regarded as the optimal estimate of future volatility. In our model, we are using this idea to calculate σ_i for each firm in our portfolio where F is the forward price. As we can see from the Table 7, the results are slightly better than the results from Table 4.

Table 6: J. Hull, M. Predescu, and A. White [26] compared the model prices with the market quotes for the 5-year CDX NA IG index tranches. The market quotes are the average quotes for the 60 days in 2004 for which a complete set of data is available. This table shows the results from the two models in their paper [26]: the base case model (Base Case) and the structural model with stochastic correlation (Stochastic Corr.)

	Market	Base	Abs Error(Rel)	Stochastic Correlation	Abs Error(Rel)
0-3%	**38.56%**	38.56%	0	38.56%	0
3-7%	**283.54bps**	371.72bps	88.18(31.1%)	251.23bps	32.31(11.40%)
7-10%	**110.75bps**	98.53bps	12.22(11.03%)	89.93bps	20.82(18.80%)
10-15%	**39.39bps**	26.38bps	13.02(33.05%)	49.55bps	10.16(25.8%)
15-30%	**11.95bps**	1.89bps	10.06(84.18%)	19.1bps	7.15(59.83%)
ρ *calibrated*		0.156		0.083	

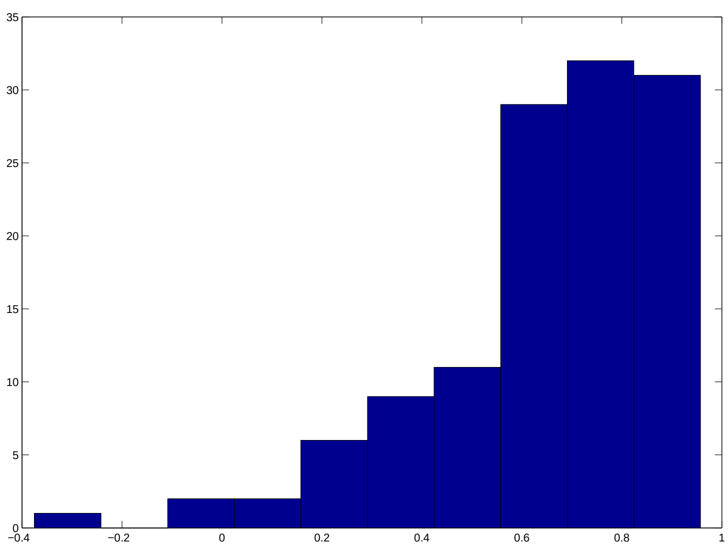

Figure 5: Histogram of ρ for 123 companies

Table 7: Comparison of our model prices with the market quotes for the 5-year CDX NA IG Series 17 index tranches using VIX-like σ_i. The market quotes are on Oct 19, 2011. Bloomberg has changed the format of quotes. The quotes in the following table are the up-front percentage with fixed basis points in each tranche. $0-3\% : 500bps, 3-7\% : 100bps, 7-15\% : 100bps, and 15-100\% : 25bps$. In the calculation we are given the spreads and try to calculate the up-front.

	Market Up-Front	Model Up-Front	Abs Error(Rel)
0-3%	**40.285%**	54.28%	13.995%(34.74%)
3-7%	**28.035%**	33.05%	5.015%(17.89%)
7-15%	**9.530%**	12.47%	2.94(30.85%)
15-100%	**1.5%**	1.88%	0.38(25.33%)
Average ρ		0.6453	

5.0 CONCLUSIONS

This thesis mainly addressed two aspects of credit default models. First we propose a new model to derive a close form solution of European Option values and then use this formula to estimate CDS credit spreads with the equity data which is public available. As we have shown in this thesis, the CDS spreads from four selected firms match the overall paten of the market data. More importantly, those four plots also capture the major spikes from the market data. Thus, our model can be regarded as an indicator of credit risk. Secondly we extend this idea and propose a heterogeneous hierarchical model in higher dimension to estimate the CDX index tranche prices and compare the market quotations. The major advantage of our higher dimensional heterogeneous model is that we have derived a formula which relates credit parameters to transparent equity data thus greatly reducing the need for extensive numerical simulations. Compared with the other hierarchical model which is also using the first-passage time as the definition of default, proposed by J. Hull, M. Predescu, and A. White [26], our results have similar deviations in most tranches from the market data and have much better estimation in the most senior tranche. The reason that our estimates always overshoot the market quotes is due to the existence of strong positive correlation (see Figure 5). To better improve our estimates, we consider an alternative way to estimate the volatility for each firm in our portfolio with the similar idea used to calculate the VIX since we notice that our results always overestimate the market quotes. It turns out that the tranche prices are slightly better than our original proposed model.

APPENDIX A

VERIFICATION OF THAT $f_b(x)$ IS A PDF

For $\mu < -\sqrt{2\lambda}$, the implied default barrier is

$$f_b(x) = \begin{cases} 0, & x \geq v_0; \\ \lambda \dfrac{e^{-(\mu+\sqrt{\mu^2-2\lambda})(x-v_0)} - e^{-(\mu-\sqrt{\mu^2-2\lambda})(x-v_0)}}{\sqrt{\mu^2-2\lambda}}, & x < v_0. \end{cases} \tag{A.1}$$

First of all, $f_b(x)$ is positive since $\mu < 0$. Furthermore

$$\int_{-\infty}^{\infty} f_b(x)dx = \int_{-\infty}^{v_0} \lambda \frac{e^{-(\mu+\sqrt{\mu^2-2\lambda})(x-v_0)} - e^{-(\mu-\sqrt{\mu^2-2\lambda})(x-v_0)}}{\sqrt{\mu^2-2\lambda}} dx \tag{A.2}$$

$$= \int_{-\infty}^{0} \lambda \frac{e^{-(\mu+\sqrt{\mu^2-2\lambda})y} - e^{-(\mu-\sqrt{\mu^2-2\lambda})y}}{\sqrt{\mu^2-2\lambda}} dx \tag{A.3}$$

$$= \frac{\lambda}{\sqrt{\mu^2-2\lambda}} \left(\frac{-1}{\mu+\sqrt{\mu^2-2\lambda}} - \frac{-1}{\mu-\sqrt{\mu^2-2\lambda}}\right) \tag{A.4}$$

$$= 1. \tag{A.5}$$

APPENDIX B

SOLUTION TO THE MODIFIED BLACK-SCHOLES EQUATION

The PDE for the Call Option is

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma_v^2(S_t + L_t)^2\frac{\partial^2 C(S,t)}{\partial S^2} + rS_t\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0 \tag{B.1}$$

subject to

$$C(S,T) = \max(S - E, 0) \tag{B.2}$$

$$C(0,t) = 0, \tag{B.3}$$

where E is the strike. Compared with the standard Black-Scholes equation, the only difference is the coefficient of the second partial derivative. To solve the above equation, apply the transformations below

$$C(S,t) = e^{-\frac{2r\tau}{\sigma_v^2} - \frac{\tau}{4} + \frac{\xi}{2}}u(\xi,\tau) \tag{B.4}$$

$$\xi = \ln(\frac{S}{L_0e^{rt}} + 1) \tag{B.5}$$

$$\tau = \frac{1}{2}\sigma_v^2(T - t) \tag{B.6}$$

Computing all the derivatives in the new coordinates

$$\frac{\partial \xi}{\partial S} = \frac{1}{S + L_0 e^{rt}} = \frac{1}{S+L} \tag{B.7}$$

$$\frac{\partial \xi}{\partial t} = -r\frac{S}{S + L_0 e^{rt}} = -r\frac{S}{S+L} \tag{B.8}$$

$$\frac{\partial \tau}{\partial t} = -\frac{\sigma_v^2}{2} \tag{B.9}$$

$$\frac{\partial C}{\partial t} = e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(-\frac{2r}{\sigma_v^2}\frac{\partial \tau}{\partial t} - \frac{1}{4}\frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial \xi}{\partial t})u(\xi,\tau) + e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t}) \tag{B.10}$$

$$= e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(r - \frac{1}{2}\frac{rS}{S+L} + \frac{1}{8}\sigma_v^2)u(\xi,\tau) + e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(-\frac{rS}{S+L}\frac{\partial u}{\partial \xi} - \frac{\sigma_v^2}{2}\frac{\partial u}{\partial \tau}) \tag{B.11}$$

$$\frac{\partial C}{\partial S} = e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}\frac{1}{2}\frac{\partial \xi}{\partial S}u(\xi,\tau) + e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial S} \tag{B.12}$$

$$= \frac{1}{S+L}e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(\frac{u(\xi,\tau)}{2} + \frac{\partial u}{\partial \xi}) \tag{B.13}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{-1}{(S+L)^2}e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(\frac{u(\xi,\tau)}{2} + \frac{\partial u}{\partial \xi}$$

$$+\frac{1}{S+L}e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}\frac{1}{2}\frac{\partial \xi}{\partial S}(\frac{u(\xi,\tau)}{2} + \frac{\partial u}{\partial \xi}) + \frac{1}{S+L}e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(\frac{1}{2}\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial S} + \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial S}) \tag{B.14}$$

$$= -\frac{1}{2(S+L)^2}e^{-\frac{2r\tau}{\sigma_v^2}-\frac{\tau}{4}+\frac{\xi}{2}}(-\frac{1}{4}u(\xi,\tau) + \frac{\partial^2 u}{\partial \xi^2}) \tag{B.15}$$

Therefore the original PDE (3.22) is equivalent to the following heat equation,

$$\frac{\partial u(\xi,\tau)}{\partial \tau} = \frac{\partial^2 u(\xi,\tau)}{\partial \xi^2} \tag{B.16}$$

subject to

$$u(\xi,0) = e^{-\frac{\xi}{2}}\max((e^{\xi}-1)L - E, 0) = e^{-\frac{\xi}{2}}\max((e^{\xi}-1)L_0 e^{rT} - E, 0) \tag{B.17}$$

$$= e^{-\frac{\xi}{2}}\max((e^{\xi}-1)L_0 e^{rT} - E, 0) \tag{B.18}$$

$$u(0,\tau) = 0 \tag{B.19}$$

Now using the Green's function

$$G(\xi,\tau;x,y) = g(\xi,\tau;x,y) - g(\xi,\tau;-x,y) \tag{B.20}$$

$$g(\xi,\tau;x,y) = \frac{1}{\sqrt{4\pi(\tau-y)}}e^{-\frac{(\xi-x)^2}{4(\tau-y)}}, \tag{B.21}$$

the solution to (B.16) is

$$u(\xi,\tau) = \int_0^\infty G(\xi,\tau;x,0)u(x,0)dx \tag{B.22}$$

$$\begin{aligned}
u(\xi,\tau) &= \int_0^\infty G(\xi,\tau;x,0)u(x,0)dx && \text{(B.23)}\\
&= \int_0^\infty \frac{1}{4\pi\tau}[e^{-\frac{(\xi-x)^2}{4\tau}} - e^{-\frac{(\xi+x)^2}{4\tau}}]e^{-\frac{x}{2}}\max((e^x-1)L_0e^{rT}-E,0)dx && \text{(B.24)}\\
&= \int_{\ln(1+\frac{E}{L_0e^{rT}})}^\infty \frac{1}{\sqrt{4\pi\tau}}[e^{-\frac{(\xi-x)^2}{4\tau}} - e^{-\frac{(\xi+x)^2}{4\tau}}]e^{-\frac{x}{2}}((e^x-1)L_0e^{rT}-E)dx && \text{(B.25)}\\
&= \int_{\ln(1+\frac{E}{L_0e^{rT}})}^\infty \frac{1}{\sqrt{4\pi\tau}}[e^{-\frac{(\xi-x)^2}{4\tau}} - e^{-\frac{(\xi+x)^2}{4\tau}}](L_0e^{rT}e^{\frac{x}{2}} - (E+L_0e^{rT})e^{-\frac{x}{2}})dx && \text{(B.26)}\\
&= I_1 - I_2 - I_3 + I_4, && \text{(B.27)}
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{\ln(1+\frac{E}{L_0e^{rT}})}^\infty \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(\xi-x)^2}{4\tau}}L_0e^{rT}e^{\frac{x}{2}}dx && \text{(B.28)}\\
&= L_0e^{rT}\int_{\ln(1+\frac{E}{L_0e^{rT}})}^\infty \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(\xi-x)^2-2\tau x}{4\tau}}dx && \text{(B.29)}\\
&= L_0e^{rT}\int_{\ln(1+\frac{E}{L_0e^{rT}})}^\infty \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-\xi-\tau)^2-\tau(\tau+2\xi)}{4\tau}}dx && \text{(B.30)}\\
&= L_0e^{rT+\frac{\tau+2\xi}{4}}\int_{\ln(1+\frac{E}{L_0e^{rT}})}^\infty \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-\xi-\tau)^2}{4\tau}}dx && \text{(B.31)}\\
&= L_0e^{rT+\frac{\tau+2\xi}{4}}\int_{-d_1}^\infty \frac{1}{\sqrt{2\tau}}e^{-\frac{z^2}{2}}dz && \text{(B.32)}\\
&= L_0e^{rT+\frac{\tau+2\xi}{4}}\Phi(d_1) && \text{(B.33)}
\end{aligned}$$

$$I_2 = \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(\xi+x)^2}{4\tau}} L_0 e^{rT} e^{\frac{x}{2}} dx \tag{B.34}$$

$$= L_0 e^{rT} \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(\xi+x)^2-2\tau x}{4\tau}} dx \tag{B.35}$$

$$= L_0 e^{rT} \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x+\xi-\tau)^2+\tau(-\tau+2\xi)}{4\tau}} dx \tag{B.36}$$

$$= L_0 e^{rT+\frac{\tau-2\xi}{4}} \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x+\xi-\tau)^2}{4\tau}} dx \tag{B.37}$$

$$= L_0 e^{rT+\frac{\tau-2\xi}{4}} \int_{-d_3}^{\infty} \frac{1}{\sqrt{2\tau}} e^{-\frac{z^2}{2}} dz \tag{B.38}$$

$$= L_0 e^{rT+\frac{\tau-2\xi}{4}} \Phi(d_3) \tag{B.39}$$

$$I_3 = \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(\xi-x)^2}{4\tau}} (E + L_0 e^{rT}) e^{-\frac{x}{2}} dx \tag{B.40}$$

$$= (E + L_0 e^{rT}) \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(\xi-x)^2+2\tau x}{4\tau}} dx \tag{B.41}$$

$$= (E + L_0 e^{rT}) \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x-\xi+\tau)^2+\tau(-\tau+2\xi)}{4\tau}} dx \tag{B.42}$$

$$= (E + L_0 e^{rT}) e^{\frac{\tau-2\xi}{4}} \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x-\xi+\tau)^2}{4\tau}} dx \tag{B.43}$$

$$= (E + L_0 e^{rT}) e^{\frac{\tau-2\xi}{4}} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\tau}} e^{-\frac{z^2}{2}} dz \tag{B.44}$$

$$= (E + L_0 e^{rT}) e^{\frac{\tau-2\xi}{4}} \Phi(d_2) \tag{B.45}$$

$$I_4 = \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(\xi+x)^2}{4\tau}} (E + L_0 e^{rT}) e^{-\frac{x}{2}} dx \tag{B.46}$$

$$= (E + L_0 e^{rT}) \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(\xi+x)^2+2\tau x}{4\tau}} dx \tag{B.47}$$

$$= (E + L_0 e^{rT}) \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x+\xi+\tau)^2-\tau(\tau+2\xi)}{4\tau}} dx \tag{B.48}$$

$$= (E + L_0 e^{rT}) e^{\frac{\tau+2\xi}{4}} \int_{\ln(1+\frac{E}{L_0 e^{rT}})}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x+\xi+\tau)^2}{4\tau}} dx \tag{B.49}$$

$$= (E + L_0 e^{rT}) e^{\frac{\tau+2\xi}{4}} \int_{-d_4}^{\infty} \frac{1}{\sqrt{2\tau}} e^{-\frac{z^2}{2}} dz \tag{B.50}$$

$$= (E + L_0 e^{rT}) e^{\frac{\tau+2\xi}{4}} \Phi(d_4) \tag{B.51}$$

Therefore

$$C(S,t;E,T,r,\sigma_v,L_0) = L_0e^{rt}[e^{\xi}\Phi(d_1) - \Phi(d_2) - \Phi(d_3) + e^{\xi}\Phi(d_4)] - Ee^{-r(T-t)}[\Phi(d_2) - e^{\xi}\Phi(d_4)] \tag{B.52}$$

where

$$\xi = \ln(\frac{S}{L_0e^{rt}} + 1) \tag{B.53}$$

$$\hat{\xi} = \ln(\frac{E}{L_0e^{rT}} + 1) \tag{B.54}$$

$$\tau = \frac{1}{2}\sigma_v^2(T-t) \tag{B.55}$$

$$d_1 = \frac{\tau + \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{B.56}$$

$$d_2 = \frac{-\tau + \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{B.57}$$

$$d_3 = \frac{\tau - \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{B.58}$$

$$d_4 = \frac{-\tau - \xi - \hat{\xi}}{\sqrt{2\tau}} \tag{B.59}$$

Similarly for the European put option $P(S,t)$,

$$\frac{\partial P(S,t)}{\partial t} + \frac{1}{2}\sigma_v^2(S_t + L_t)^2\frac{\partial^2 P(S,t)}{\partial S^2} + rS_t\frac{\partial P(S,t)}{\partial S} - rP(S,t) = 0 \tag{B.60}$$

subject to

$$P(S,T) = \max(E - S, 0) \tag{B.61}$$

$$P(0,t) = Ee^{-r(T-t)}. \tag{B.62}$$

With the same transformation as for the call option and the same Green's function,

$$u(\xi,\tau) = \int_0^{\infty} G(\xi,\tau;x,0)u(x,0)dx + \int_0^{\tau} G_x(\xi,\tau;0,s)u(0,s)ds. \tag{B.63}$$

The only difference in the calculation is the second integral involving G_x, and it is not very hard to find the anti-derivative. Therefore the European Put option also has the analytical formula below

$$P(S,t;E,T,r,\sigma,L_0) = L_0e^{rt}[\Phi(-d_2) - \Phi(-d_4) - \Phi(-d_1) + \Phi(-d_3)]$$

$$- Ee^{-r(T-t)}[\Phi(-d_2) - \Phi(d_4)] + S[\Phi(d_1) - \Phi(-d_4) + \frac{E}{L_0 e^{rT}}\Phi(d_4)]. \tag{B.64}$$

In fact, it is easy to check the answer using put-call parity

$$C(S, t; E, T, r, \sigma, L_0) - P(S, t; E, T, r, \sigma, L_0) = S - Ee^{-r(T-t)} \tag{B.65}$$

APPENDIX C

CDX INDEX TRANCHE LOSS ESTIMATION

CDS index tranches are synthetic CDOs based on a CDS index, where each tranche references a different segment of the loss distribution of the underlying CDS index.
Each tranche is characterized by the following two quantities:

- $\underline{K_i}$: This is the attachment point, also known as the lower strike of the tranche, which is the percentage loss on the reference portfolio below which the tranche loss is zero. Once the percentage portfolio loss is over $\underline{K_i}$, the tranche experiences loss.

- $\overline{K_i}$: This is the detachment point, also know as the upper strike. If Loss $\geq \overline{K_i}$, the tranche loss is 100%. The quantity $\overline{K_i} - \underline{K_i}$ is the tranche width.

The expected loss of the tranche i with attachment point $\underline{K_i}$ and detachment point $\overline{K_i}$ can be formulated

$$EL_i = \frac{1}{\overline{K_i} - \underline{K_i}}[\int_{\underline{K_i}}^{1}(x - \underline{K_i})dL(x) - \int_{\overline{K_i}}^{1}(x - \overline{K_i})dL(x)], \tag{C.1}$$

where $L(x)$ is the cumulative loss distribution. This can be rewritten as follows.

$$EL_i = \frac{1}{\overline{K_i} - \underline{K_i}}[\int_{\underline{K_i}}^{\overline{K_i}} xdL(x) - \underline{K_i}(L(1) - L(\underline{K_i})) + \overline{K_i}(L(1) - L(\overline{K_i}))] \tag{C.2}$$

$$= \frac{1}{\overline{K_i} - \underline{K_i}}[xL(x)|_{(\underline{K_i},\overline{K_i})} - \int_{\underline{K_i}}^{\overline{K_i}} L(x)dx - \underline{K_i}(L(1) - L(\underline{K_i})) + \overline{K_i}(L(1) - F(\overline{K_i}))] \tag{C.3}$$

$$= \frac{1}{\overline{K_i} - \underline{K_i}}[\overline{K_i}L(\overline{K_i}) - \underline{K_i}L(\underline{K_i}) - \int_{\underline{K_i}}^{\overline{K_i}} L(x)dx - \underline{K_i}(L(1) - L(\underline{K_i})) + \overline{K_i}(L(1) - L(\overline{K_i}))] \tag{C.4}$$

$$= \frac{1}{\overline{K_i} - \underline{K_i}}[-\int_{\underline{K_i}}^{\overline{K_i}} L(x)dx + L(1)(\overline{K_i} - \underline{K_i})] \tag{C.5}$$

For the homogeneous model proposed in chapter 4, this can be reduced to the following double integral (4.16)

$$El_i = \frac{1}{\overline{K_i} - \underline{K_i}} \int_{\underline{K_i}}^{\overline{K_i}} \int_{-\infty}^{\infty} F(PD^{-1}(x,z))dG(z)dx \tag{C.6}$$

To determine the value of tranche prices, we need to calculate the protection leg and premium leg. Under the independence assumptions between interest rates and the loss process $L(x)$, the protection leg and premium leg can be approximated by the following discretization formula [23]

$$\text{Prot}_i = \sum_{j=0}^{N} e^{-\frac{t_j+t_{j-1}}{2}}(EL_i(t_j) - EL_i(t_j - 1)) \tag{C.7}$$

$$\text{Prem}_i = u_i + s_i \sum_{j=0}^{N} e^{-t_j}(1 - EL_i(t_j))\Delta t, \tag{C.8}$$

The fair tranche price can be computed by setting the protection leg equal to the premium leg. It is worth noting that there are two different kinds of payments in the premium leg, viz. u_i and s_i. Before the global financial crisis, the equity tranche price was quoted as the up-front payment u_i plus a fixed running premium of 500 basis points (bps) and other tranche prices were quoted as the premium s_i with zero up-front payment. Recently, the quoting convention has changed and an upfront payment is also required even for more senior tranches. According to the data source from Bloomberg, it has changed the format of quotations, the market quotes for CDX NA IG Series being fixed running spreads for the following tranches $0-3\%:500bps, 3-7\%:100bps, 7-15\%:100bps, and 15-100\%:25bps$ with up-front payment varied due to market conditions.

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