FIXED POINT PROPERTIES FOR c_0-LIKE SPACES

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In 1981, Maurey proved that every weakly compact, convex subset C of c_0 is such that every nonexpansive (n.e.) mapping $T : C \longrightarrow C$ has a fixed point; i.e., C has the fixed point property (FPP). Dowling, Lennard, and Turett proved the converse of Maurey's result by showing each closed bounded convex non-weakly compact subset C of c_0 fails FPP for n.e. mappings. However, in general the mapping failing to have a fixed point is not affine.

In Chapter 2 and Chapter 3, we prove that for certain classes of closed bounded convex non-weakly compact subsets C of c_0, there exists an affine nonexpansive mapping $T\colon C \to C$ that fails to have a fixed point. Our result depends on our main theorem: if a Banach space contains an asymptotically isometric (a.i.) c_0-summing basic sequence $(x_i)_{i\in\mathbb{N}}$, then the closed convex hull of the sequence fails the FPP for affine nonexpansive mappings. In fact, in Chapter 3, we show that very large classes of c_0-summing basic sequences turn out to be L-scaled a.i. c_0-summing basic sequences.

In Chapter 4, we work on Lorentz-Marcinkiewicz spaces and explore the FPP for $l^0_{w,\infty}$ spaces. Using Borwein and Sims' technique we prove for certain classes of weight sequence w that $X := l^0_{w,\infty}$ has the weak fixed point property (w-FPP) by using the Riesz angle concept. Furthermore, we find a formula for the Riesz angle of X for any weight sequence. Next, we show that X has the w-FPP for any w, but fails the FPP for n.e. mappings.

In Chapter 5, we show that any closed non-reflexive vector subspace Y of $l^0_{w,\infty}$ contains an isomorphic copy of c_0 and so Y fails the FPP for strongly asymptotically nonexpansive maps. Also, we show that l^1 cannot be renormed to have the FPP for semi-strongly asymptotically nonexpansive maps, and that c_0 cannot be renormed to have the FPP for strongly asymptotically nonexpansive maps. Finally, we show that reflexivity for Banach lattices is equivalent to the FPP for affine semi-strongly asymptotically nonexpansive mappings.

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PREFACE

First of all, I would like to thank my wife, Esra who was the only person coming with me from my country to US. I was fortunate to marry her who had so much patience and love while I was adapting to a new life, a new culture, a new language, and much more powerful educational system which requires full concentration by avoiding from stress.

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1.0 INTRODUCTION

We begin with a brief history of metric fixed point theory. In 1912, Brouwer [7] proved that for every (non-empty) convex, norm compact subset C of $X = \mathbb{R}^n$, every norm-to-norm continuous map $f\colon C \to C$ has a fixed point. Schauder [36] generalized this result to arbitrary Banach spaces $(X, \|\cdot\|)$. The closed, bounded, convex sets involved here are quite "small" (norm compact), while the class of continuous maps is "large" (all of them).

On the other hand, Banach's Contraction Mapping Theorem [3] tells us that for a complete metric space (Z, d), every strict contraction $f\colon Z \to Z$ [i.e., there exists $k \in [0, 1)$ such that $d(f(x), f(y)) \leq kd(x, y)$, for all $x, y \in Z$] has a (unique) fixed point in Z.

In the setting of Banach spaces, it follows that for all closed, bounded, convex subsets C of a Banach space $(X, \|\cdot\|)$, every map $f\colon C \to C$ that is a strict contraction with respect to the metric d generated by the norm, has a fixed point. The class of closed, bounded, convex sets involved here is "large" (all of them), while the class of continuous maps is quite "small" (strict contractions).

In 1965, Browder [8] proved an interesting "intermediate" theorem analogous to both Schauder's theorem and Banach's theorem for Hilbert spaces: [♠] [For every closed, bounded, convex (non-empty) subset C of a Hilbert space $(X, \|\cdot\|)$, for all nonexpansive mappings $T\colon C \to C$ [i.e., $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$], T has a fixed fixed point in C.] Soon after, also in 1965, Browder [9] and Göhde [24] (independently) generalized the result [♠] to uniformly convex Banach spaces $(X, \|\cdot\|)$; e.g., $X = L^p,\ 1 < p < \infty$, with its usual norm $\|\cdot\|_p$.

Later in 1965, Kirk [26] further generalized [♠] to all reflexive Banach spaces X with so-called "normal structure": those spaces such that all non-trivial closed, bounded, convex sets C have a smaller radius than diameter. This is a very large class of spaces. Spaces

$(X, \|\cdot\|)$ with the property of Browder [♠] became known as spaces with the "fixed point property for nonexpansive mappings" (FPP (n.e.)).

Note that we can do better than property [♠] in uniformly convex spaces $(X, \|\cdot\|)$. Indeed Goebel and Kirk [22] showed that there exists a constant $K \in (1, \infty)$ such that for all closed, bounded, convex sets $C \subseteq X$, for all uniformly Lipschitzian maps $T\colon C \to C$ [i.e., there exists $\lambda \in (0, \infty)$ such that for all $n \in \mathbb{N}$, for all $x, y \in C$, $\|T^n x - T^n y\| \leq \lambda \|x - y\|$] with Lipschitz constant $\lambda < K$, T has a fixed point in C. E.g., for Hilbert space, $K \in [\sqrt{2}, \frac{\pi}{2}]$ (see [23], Ch. 16). Also for L^p, $2 < p < \infty$, the $K \geq (1 + \frac{1}{2^p})^{\frac{1}{p}}$ (see [22], [28]). A simple example of a fixed point free Lipschitz map T (with $\lambda = \sqrt{2}$) on a closed, bounded, convex set C contained in the Hilbert space $X = L^2[0,1]$ follows. Let $C := \{f \in L^2[0,1]\colon 0 \leq f \leq 1$ and $\int_0^1 f dm = 1\}$ (Here, m is Lebesgue measure). Fix an arbitrary $f \in C$. For all $t \in [0, \frac{1}{2})$, $(Tf)(t) := \min\{2f(2t), 1\}$. Also, for all $t \in [\frac{1}{2}, 1]$, $(Tf)(t) := \max\{2f(2t-1), 1\} - 1$. T is fixed point free. T is called Alspach's mapping [2]. It is straightforward to check that [$\|Tf - Tg\|_2 \leq \sqrt{2}\|f - g\|_2$, for all $f, g \in C$], and that $\sqrt{2}$ is the smallest possible constant.

Returning to Kirk's theorem, we may ask if further generalizations are possible. Even after 47 years, it remains an open question as to whether or not every reflexive Banach space $(X, \|\cdot\|)$ has the fixed point property for nonexpansive maps. This and related questions have been and still are central themes in metric fixed point theory.

Recently, Domínguez Benavides [12] proved that the following intriguing partial result: [Given a reflexive Banach space $(X, \|\cdot\|)$, there exists an equivalent norm $\|\cdot\|^\sim$ on X such that $(X, \|\cdot\|^\sim)$ has the fixed point property for nonexpansive mappings]. This improves a theorem of van Dulst [20] for separable reflexive Banach spaces.

In contrast to this result, the non-reflexive Banach space $(l^1, \|\cdot\|_1)$, the space of all absolutely summable sequences, with the absolute sum norm $\|\cdot\|_1$, fails the fixed point property for nonexpansive mappings. E.g., let $C :=$ {sequences $(t_n)_{n\in\mathbb{N}}$: each $t_n \geq 0$ and $\sum_{n=1}^\infty t_n = 1$}. C is a closed, bounded, convex subset of l^1. Let $T\colon C \to C$ be the right shift map on C; i.e., $T(t_1, t_2, t_3, \dots) := (0, t_1, t_2, t_3, \dots)$. T is clearly $\|\cdot\|_1$-nonexpansive (being an isometry) and fixed point free on C.

Recently, in a significant development, P. K. Lin [30] provided the first example of a non-reflexive Banach space $(X, \|\cdot\|)$ with the fixed point property for nonexpansive mappings.

Professor Lin verified this fact for $X = l^1$ with the equivalent norm $\|\cdot\|$ given by

$$\|x\| = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|, \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in l^1 .$$

What about $(c_0, \|\cdot\|_\infty)$, the Banach space of real-valued sequences that converge to zero, with the absolute supremum norm $\|\cdot\|_\infty$? This is another non-reflexive Banach space of great importance in Banach space theory. It also fails the fixed point property for nonexpansive mappings. E.g., let $C := \{$sequences $(t_n)_{n\in\mathbb{N}}$: each $t_n \geq 0,\ 1 = t_1 \geq t_2 \geq \cdots \geq t_n \geq t_{n+1} \to 0$, as $n \to \infty\}$. Let $U\colon C \to C$ be the natural right shift map. $U(t_1, t_2, t_3, \dots) := (1, t_1, t_2, t_3, \dots)$. Then U is a $\|\cdot\|_\infty$-nonexpansive (isometric, actually) map with no fixed points in C.

Let $e_n \in c_0$ be the sequence with 1 in the n^{th} position and 0 everywhere else. Let $\sigma_1 := e_1,\ \sigma_2 := e_1 + e_2\ , \dots,\ \sigma_n := e_1 + \cdots + e_n$, for all $n \in \mathbb{N}$. $(\sigma_n)_{n\in\mathbb{N}}$ is the so-called "summing basis" of c_0. It is easy to check that the subset C of c_0 above is the closed convex hull of the summing basis $(\sigma_n)_{n\in\mathbb{N}}$. It is also easy to check that for all finitely non-zero sequences of real numbers $t = (t_n)_{n\in\mathbb{N}}$ (i.e., for all $t \in c_{00}$),

$$\left\| \sum_{n=1}^{\infty} t_n \sigma_n \right\|_\infty = \sup_{n\geq 1} \left| \sum_{k=n}^{\infty} t_k \right| .$$

We will return to these ideas later.

It is natural to ask whether there is a c_0-analogue of P. K. Lin's theorem about l^1. It remains an open question as to whether or not there exists an equivalent norm $\|\cdot\|^\sim$ on $(c_0, \|\cdot\|_\infty)$ such that $(c_0, \|\cdot\|^\sim)$ has the fixed point property for nonexpansive mappings. However, if we weaken the nonexpansive condition to "asymptotically nonexpansive", then the answer is "no". In 2000, Dowling, Lennard and Turett [15] showed that for every equivalent renorming $\|\cdot\|^\sim$ of $(c_0, \|\cdot\|_\infty)$, there exists a closed, bounded, convex set C and an asymptotically nonexpansive mapping $T\colon C \to C$ [i.e., there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $[1, \infty)$ such that $k_n \underset{n}{\longrightarrow} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C$, $\|T^n x - T^n y\| \leq k_n \|x - y\|$] such that T has no fixed point.

In contrast to this, note that in 1972, Goebel and Kirk [21] proved that for all uniformly convex spaces $(X, \|\cdot\|)$ (e.g., a Hilbert space), for every closed, bounded, convex set $C \subseteq X$, for all asymptotically nonexpansive maps $T\colon C \to C$, T has a fixed point in C.

Due to the above example in $(c_0, \|\cdot\|_\infty)$ and theorem about c_0, we are interested to understand more about Banach spaces $(X, \|\cdot\|)$ that contain subspaces isomorphic to c_0. Equivalently, we are interested in Banach spaces that contain "c_0-summing basic sequences". A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space $(X, \|\cdot\|)$ is a c_0-summing basic sequence if there exist constants $0 < A \le B < \infty$ such that for all $t \in c_{00}$,

$$A \sup_{n\in\mathbb{N}} \left| \sum_{k=n}^{\infty} t_k \right| \le \left\| \sum_{n=1}^{\infty} t_n \, x_n \right\| \le B \sup_{n\in\mathbb{N}} \left| \sum_{k=n}^{\infty} t_k \right| .$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is a more general analogue of the summing basis $(\sigma_n)_{n\in\mathbb{N}}$ in c_0.

Reflexive Banach spaces $(X, \|\cdot\|)$ (e.g., L^p, $1 < p < \infty$, and Hilbert spaces) do not contain c_0-summing basic sequences. On the other hand, many non-reflexive Banach spaces do. E.g., $C(K)$, the space of continuous real-valued functions on an infinite compact Hausdorff space K, with the supremum norm. Another example is $K(H)$, the space of compact operators on an infinite-dimensional Hilbert space H, with the operator norm. Also, the Lorentz-Marcinkiewicz spaces $l^0_{w,\infty}$ discussed in Chapter 4 are of this type.

All spaces that contain an isomorphic copy of c_0 fail the fixed point property for asymptotically nonexpansive maps. In 2003, Dowling, Lennard and Turett [16] showed that when a space contains a "nicer" c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, its closed convex hull $C = \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$ is such that there exists a nonexpansive (affine) map $U\colon C \to C$ without a fixed point. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space $(X, \|\cdot\|)$ is an asymptotically isometric c_0-summing basic sequence if there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ such that for all sequences $(t_n)_{n\in\mathbb{N}} \in c_{00}$,

$$\sup_{n\ge1} \left(\frac{1}{1+\varepsilon_n} \right) \left| \sum_{j=n}^{\infty} t_j \right| \le \left\| \sum_{j=1}^{\infty} t_j \, x_j \right\| \le \sup_{n\ge1} \, (1+\varepsilon_n) \left| \sum_{j=n}^{\infty} t_j \right| .$$

In the Dowling, Lennard and Turett theorem mentioned above, the "nicer" sequence $(x_n)_{n\in\mathbb{N}}$ mentioned above is an asymptotically isometric c_0-summing basic sequence with $\varepsilon_n < 2^{-1}4^{-n}$ for all $n \ge 2$. Dowling, Lennard and Turett [16] also showed that when the Banach space $X = (c_0, \|\cdot\|_\infty)$, then every non-weakly compact closed, bounded, convex subsets C of c_0 contains an asymptotically isometric c_0-summing basic sequence of the above type.

Later, in Dowling, Lennard and Turett [17] constructed a nonexpansive (non-affine) mapping Ψ from C into $K = \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$ that enabled them to prove: every non-weakly compact closed, bounded, convex subset C of $(c_0, \|\cdot\|_\infty)$ is such that $T = U \circ \Psi \colon C \to K \subseteq C$ is nonexpansive and fixed point free, where U is the affine map mentioned above.

This provided a converse to the important theorem of B. Maurey [34] (1981) that for all weakly compact convex subsets C of $(c_0, \|\cdot\|_\infty)$, every nonexpansive map $T \colon C \to C$ has a fixed point. (Note that in general Banach spaces the analogue of Maurey's result may fail. E.g., $X = (L^1[0,1], \|\cdot\|_1)$, $C := \{f \in L^1[0,1] : 0 \leq f \leq 1\}$ = the same set we discussed previously for $X = L^2[0,1]$, and $T \colon C \to C$ is Alspach's mapping. We remark that with respect to the norm $\|\cdot\|_1$, T is nonexpansive.)

This leads us to the main motivating question for this thesis: given a non-weakly compact closed, bounded, convex subset C of $(c_0, \|\cdot\|_\infty)$, does there exist a fixed point free nonexpansive map $T \colon C \to C$ that is also affine [i.e., $T\big((1-\lambda)\,x + \lambda\,y\big) = (1-\lambda)\,T(x) + \lambda\,T(y)$, for all $x, y \in C$ and for all $\lambda \in [0,1]$]? While this question remains open, considering it has led us to prove some other interesting theorems.

We considered sets C in $(c_0, \|\cdot\|_\infty)$ that are the closed convex hull of an arbitrary asymptotically isometric c_0-summing basic sequence $(\eta_n)_{n\in\mathbb{N}}$. In Chapter 2 of this thesis, extending Dowling, Lennard and Turett [17], we prove that [♦] [for all such sets $C \subseteq (c_0, \|\cdot\|_\infty)$, there exists an affine nonexpansive map $U \colon C \to C$ that is fixed point free. Also, the map U is slightly more than nonexpansive: it is contractive: [i.e., for all $x, y \in C$ with $x \neq y$, $\|Ux - Uy\|_\infty < \|x - y\|_\infty$]].

In Chapter 3, we apply Theorem [♦] to c_0-summing basic sequences in $(c_0, \|\cdot\|_\infty)$ of the following general form: $\eta_n := \gamma_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n)$, for all $n \in \mathbb{N}$. We find that whenever $0 < b_n$ converges to 1 and $0 < \gamma_n$ converges to 1 and $(\gamma_n)_{n\in\mathbb{N}}$ does not "oscillate too wildly", then $E = \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$ is an asymptotically isometric c_0-summing basic sequence; and so Theorem [♦] tells us there exists an affine contractive map $U \colon E \to E$ without a fixed point.

These are the main results of our thesis. The results of Chapter 2 have appeared in [27].

1.1 PRELIMINARIES AND OVERVIEW

We now describe the results in our thesis in more detail. The symbols $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of positive integers, the set of rational numbers and the set of real numbers, respectively. Throughout this thesis our scalar field is $\mathbb{R}$.

Definition 1.1.1. Let C be a non-empty closed, bounded, convex (c.b.c.) subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \longrightarrow C$ is called *nonexpansive* if $\|T(x) - T(y)\| \le \|x - y\|$, for all $x, y \in C$. Further, we call a mapping $T : C \longrightarrow C$ *contractive* if $\|T(x) - T(y)\| < \|x - y\|$, for all $x, y \in C$ with $x \neq y$.

We say that C has the *fixed point property for nonexpansive mappings* [FPP(n.e.)] if for all nonexpansive mappings $T : C \longrightarrow C$, there exists $z \in C$ with $T(z) = z$.

Definition 1.1.2. Let C be a non-empty closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$. A mapping $U : C \longrightarrow C$ is said to be *affine* if for all $\lambda \in [0, 1]$, for all $x, y \in C$,

$$U\big((1 - \lambda)\, x + \lambda\, y\big) = (1 - \lambda)\, U(x) + \lambda\, U(y) \ .$$

We say that C has the *fixed point property for affine nonexpansive mappings* [FPP(affine, n.e.)] if for all affine nonexpansive mappings $U : C \longrightarrow C$, there exists $z \in C$ with $U(z) = z$.

Let $(X, \|\cdot\|)$ be a Banach space and $E \subseteq X$. We will denote the closed, convex hull of E by $\overline{\text{co}}(E)$. As usual, $(c_0, \|\cdot\|_\infty)$ is given by

$$c_0 := \Big\{x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0\Big\} \ .$$

Further, $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$; and $(\ell^1, \|\cdot\|_1)$ is defined by

$$\ell^1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\} \ .$$

Let $n \in \mathbb{N}$. The scalar sequence e_n, with domain $\mathbb{N}$, is defined to be 1 in its nth coordinate, and 0 in all other coordinates. Recall that the sequence $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$. Moreover, we denote the vector space of all scalar sequences that have only finitely many non-zero terms by c_{00}. In other words, c_{00} is the linear span of $\{e_n : n \in \mathbb{N}\}$ inside c_0 (and ℓ^1).

We recall now the definition of an *asymptotically isometric* c_0*-summing basic sequence* in a Banach space $(X, \|\cdot\|)$, from Definition 1 of Dowling, Lennard and Turett [16].

Definition 1.1.3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space $(X, \|\cdot\|)$. We define $(x_n)_{n\in\mathbb{N}}$ to be an *asymptotically isometric (ai)* c_0*-summing basic sequence* in $(X, \|\cdot\|)$ if there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ such that for all sequences $(t_n)_{n\in\mathbb{N}} \in c_{00}$,

$$(\dagger) \qquad \sup_{n\geq 1} \left(\frac{1}{1+\varepsilon_n}\right) \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{j=1}^{\infty} t_j\, x_j\right\| \leq \sup_{n\geq 1}\, (1+\varepsilon_n) \left|\sum_{j=n}^{\infty} t_j\right| \;.$$

Note that we have slightly modified the statement of this definition, to an equivalent one, that allows for some or all ε_n's to be 0. Note also that we may replace c_{00} by ℓ^1 in the above definition. Further, if $L > 0$, we will call a sequence $(z_n)_{n\in\mathbb{N}}$ an *L-scaled asymptotically isometric* c_0*-summing basic sequence* in $(X, \|\cdot\|)$ if the sequence $(z_n/L)_{n\in\mathbb{N}}$ is an *asymptotically isometric* c_0*-summing basic sequence.*

Now, let's see the other definitions that construct the results of Chapters of this thesis.

Definition 1.1.4. Lower c_0-summing estimate

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space $(X, \|\cdot\|)$. Assume $\exists\, K \in (0,\infty)$ s.t. $\forall \alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$

$$K \sup_{n\geq 1} \left|\sum_{j=n}^{\infty} \alpha_j\right| \leq \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\| \;.$$

Then, we will say $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate.

Definition 1.1.5. Asymptotically Nonexpansive Mapping

Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. A mapping $T : C \longrightarrow C$ is said to be asymptotically nonexpansive if $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$, where $(k_n)_{n\in\mathbb{N}}$ is a sequence in $[1,\infty)$ converging to 1.

Definition 1.1.6. Strongly Asymptotically Nonexpansive Mapping

Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. We will say a mapping $T\colon C \to C$ is strongly asymptotically nonexpansive if $\exists\, \{\beta_{n,m} :\; n, m \in \mathbb{N}, n \geq m \geq 0\} \subseteq [1,\infty)$ such that $\forall x, y \in C$ and $\forall\, n > m$, $\|T^n x - T^n y\| \leq \beta_{n,m} \|T^m x - T^m y\|$ where $[\beta_{n,m} \to 1$ as $n \geq m \to \infty]$ and $[\beta_{n,m} \to 1$ as $n \to \infty\;, \forall m]$.

Definition 1.1.7. Semi-strongly Asymptotically Nonexpansive Mapping

Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. We will say a mapping $T\colon C \to C$ is semi-strongly asymptotically nonexpansive if $\exists\{\lambda_{n,m} \,:\, n, m \in \mathbb{N}, n \geq m \geq 0\} \subseteq [1,\infty)$ such that $\forall x, y \in C$ and $\forall\, n > m$, $\|T^n x - T^n y\| \leq \lambda_{n,m}\|T^m x - T^m y\|$ where $[\lambda_{n,m} \to 1$ as $n \geq m \to \infty]$.

Definition 1.1.8. $l_{w,\infty}$ space

$$l_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \begin{array}{c} \|x\|_{w,\infty} := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} x_j{}^{\star}}{\sum_{j=1}^{n} w_j} < \infty \,, \text{where}\, x^{\star} := (x_n{}^{\star})_{n\in\mathbb{N}} \\ \text{is the decreasing rearrangement of } x \end{array} \right\} .$$

This is an analogue of l^∞ space. Indeed $(l_{w,\infty}, \|\cdot\|_{w,\infty})$ is a non-separable Banach space. Note that $x^\star :=$ the sequence whose terms contain all non-zero terms of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in non-increasing order (repeated according to multiplicity), followed by infinitely many zeros when $|x|$ has only finitely many non-zero terms.

Definition 1.1.9. $l^0_{w,\infty}$ space

$$l^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \begin{array}{c} \limsup_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} x_j{}^{\star}}{\sum_{j=1}^{n} w_j} = 0 \,, \;\text{where}\, x^{\star} := (x_n{}^{\star})_{n\in\mathbb{N}} \\ \text{is the decreasing rearrangement of } x \end{array} \right\} .$$

This is an analogue of c_0 space. It is a fact that $(l^0_{w,\infty}, \|\cdot\|_{w,\infty})$ is a separable subspace of $l_{w,\infty}$.

Definition 1.1.10. $l_{w,1}$ space

$$l_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \|x\|_{w,1} := \sum_{j=1}^{\infty} w_j\, x_j{}^{\star} < \infty \right\} .$$

This is an analogue of l^1 space. $(l_{w,1}, \|\cdot\|_{w,1})$ is a separable Banach space.

Note that $(l^0_{w,\infty})^* \cong l_{w,1}$ and $(l_{w,1})^* \cong l_{w,\infty}$ where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic. A standard reference for Lorentz spaces is Lindenstrauss and Tzafriri [31].

Definition 1.1.11. Banach lattice

A partially ordered Banach space $(X, \leq)$ over the reals is called a Banach lattice provided

(i) $x \leq y$ implies $x + z \leq y + z$, for every $x, y, z \in X$.

(ii) $ax \geq 0$ for every $x \geq 0$ in X and every $a \geq 0$.

(iii) for all $x, y \in X$ there exists a least upper bound (l.u.b) $x \vee y$ and a greatest lower bound (g.l.b.) $x \wedge y$.

(iv) $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

Definition 1.1.12. Riesz angle

The Riesz angle α of a Banach lattice X is defined by $\alpha(X) = \sup\{\||x| \vee |y|\| : \|x\| \leq 1, \|y\| \leq 1\}$. Note that for L^p space $(1 \leq p \leq \infty)$, then $\alpha(L^p) = 2^{\frac{1}{p}}$ and for also c_0 space, $\alpha(c_0) = 1$.

Definition 1.1.13. w-FPP

A Banach space is said to have the weak fixed point property (w-FPP) if every nonexpansive mapping on every nonempty weak compact convex set has a fixed point.

Now, consider the Banach space c_0, consisting of all scalar sequences that converge to zero. In 1981 Maurey [34] proved that every weakly compact, convex subset C of c_0 is such that every nonexpansive mapping $T : C \longrightarrow C$ has a fixed point; i.e., C has the fixed point property (FPP). In 1998 Llorens-Fuster and Sims [33] proved the following theorem (Proposition 4.6).

Theorem 1.1.14. *Let* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *be any decreasing sequence in* $(0, \infty)$ *(i.e.,* $b_n \geq b_{n+1}$*, for all* $n \in \mathbb{N}$*), such that* $b_n \downarrow_n \kappa > 0$*. We define the sequence* $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by setting* $f_n := b_n\, e_n$*, for all* $n \in \mathbb{N}$*. Next, define the closed, bounded, convex subset* $E = E_{\overrightarrow{b}}$ *of* c_0 *by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} .$$

Then, there exists an affine $\|\cdot\|_\infty$*-nonexpansive mapping* $U : E \longrightarrow E$ *that is fixed point free. Moreover, if* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *is strictly decreasing, then* U *is contractive.*

The proof of Llorens-Fuster and Sims shows that the usual right shift mapping U works. Here, U is defined by

$$U\left(\sum_{n=1}^{\infty} t_n\, f_n\right) := f_1 + \sum_{n=1}^{\infty} t_n\, f_{n+1} \ .$$

Llorens-Fuster and Sims [33] also conjectured that in c_0 the only closed, bounded, convex subsets with the FPP are those that are weakly compact. In 2004 Dowling, Lennard and Turett [17] verified this conjecture. Indeed, they showed that every non-weakly compact, closed, bounded, convex (c.b.c.) subset K of $(c_0, \|\cdot\|_\infty)$ is such that there exists a $\|\cdot\|_\infty$-nonexpansive mapping T on K that is fixed point free.

This mapping T is generally not affine. It is an open question as to whether or not on every non-weakly compact, c.b.c. subset K of $(c_0, \|\cdot\|_\infty)$ there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping S that is fixed point free.

In this thesis we begin to study this question. We prove that if a Banach space contains an asymptotically isometric (ai) c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, then the closed convex hull of $(x_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$, fails the fixed point property for affine nonexpansive mappings. Moreover, we can show that there exists an *affine contractive* mapping $U : E \longrightarrow E$ that is fixed point free.

In particular, an analogue of Proposition 4.6 of Llorens-Fuster and Sims (Theorem 1.1.14 above) is true for arbitrary sequences $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ that converge to some $\kappa > 0$. (See Theorem 2.3.2 below.) The general affine mapping U is not the right shift map when $\overrightarrow{b}$ is not decreasing. Instead U is a generalization of the map used in the proof of Theorem 2 of [16].

Furthermore, in Section 2.4 we prove that for all sequences $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $0 < m := \inf_{n\in\mathbb{N}} b_n$ and $M := \sup_{n\in\mathbb{N}} b_n < \infty$, the closed, bounded, convex subset $E = E_{\overrightarrow{b}}$ of c_0 defined by

$$E := \left\{\sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0\right\} \ ,$$

where each $f_n := b_n\, e_n$, is such that there exists an affine contractive mapping $U : E \longrightarrow E$ that is fixed point free. The results of Chapter 2 have appeared in [27].

Another paper closely related to the above results is Domínguez Benavides, Japón Pineda and Prus [13]. In [13] it is proven that a non-empty closed, bounded, convex subset C of c_0 is weakly compact if and only if there exists a constant $M > 1$ such that all of C's non-empty closed, convex subsets have the fixed point property for affine mappings that are uniformly Lipschitzian with constant M. Also, in [16] the analogous result with $M = 1$ is proved.

Then, in Chapter 3, we investigate the fixed point property for the closed convex hull of certain c_0-summing basic sequences in $(c_0, \|\cdot\|_\infty)$. Then, we find out the following application of our previous work. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that for some $\Gamma > 0$ such that $\Gamma \leq \gamma_N, \forall N \in \mathbb{N}$ with

$$\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$$

and let $(b_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(0, \infty)$. Define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\begin{aligned}
\eta_1 &:= \gamma_1 b_1 e_1 \\
\eta_2 &:= \gamma_2(b_1 e_1 + b_2 e_2) \\
\eta_3 &:= \gamma_3(b_1 e_1 + b_2 e_2 + b_3 e_3) \\
\eta_4 &:= \gamma_4(b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4) \\
&\vdots \\
\eta_n &:= \gamma_n(b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + ... + b_n e_n) \\
&\vdots
\end{aligned}$$

Also assume that $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate;
i.e., $\exists\, K \in (0, \infty)$ s.t. $\forall \alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$

$$K \sup_{n\geq 1} \left| \sum_{j=n}^{\infty} \alpha_j \right| \leq \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\| .$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an L-scaled asymptotically isometric c_0-summing basic sequence. Furthermore, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, then, there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

Note that before this result we prove in our set-up with converging γ_n's and b_n's that the usual right shift $T : E \longrightarrow E$ is semi-strongly asymptotically nonexpansive and fixed point free, where E is again the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$. Later, in Chapter 4, we work in Lorentz-Marcinkiewicz spaces $l^0_{w,\infty}$ and explore the fixed point property for these spaces. Using Borwein and Sims' technique we prove that for $X := l^0_{w,\infty}$ with $w = (\frac{1}{n^p})_{n\in\mathbb{N}}$ where $0 < p < 1$, Riesz angle of X, $\alpha(X) < 2$; and so X has the weak fixed point property (w-FPP). However, the Riesz angle method does not apply to prove w-FPP for $p = 1$ since in that case $\alpha(X) = 2$. Also, we find out $\forall w \in c_0 \setminus l^1$,

$$\alpha(l^0_{w,\infty}) = 2 \lim_{\nu\longrightarrow\infty} \frac{\sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j}$$

and we show $l^0_{w,\infty}$ has the w-FPP by a theorem of P.K. Lin. Later, we prove $l^0_{w,\infty}$ has a c.b.c subset E such that the right shift map $T : E \longrightarrow E$ is affine, $\|\cdot\|_{w,\infty}$-nonexpansive and fixed point free; and in fact, next we prove $l^0_{w,\infty}$ contains an a.i c_0 copy and so fails the FPP.

Furthermore, in Chapter 5, we show that l^1 cannot be renormed to have the FPP for semi-strongly asymptotically nonexpansive maps. Indeed, if X is a Banach space containing an isomorphic copy of l^1, then by Strong James Distortion Theorem, for any null sequence (ε_n) in $(0, 1)$ there exists a sequence (x_n) in X such that

$$(1 - \varepsilon_k) \sum_{n=k}^{\infty} |t_n| \leq \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \leq \sum_{n=k}^{\infty} |t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in l^1$ and for all $k \in \mathbb{N}$.

Then, we consider the closed convex hull of x_n, i.e. $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. This set fails the fixed point property for $\|\cdot\|$-semi-strongly asymptotically nonexpansive mappings. In fact, we show that there exists an *affine semi-strongly asymptotically nonexpansive* mapping $T : E \longrightarrow E$ that is fixed point free. Furthermore, T is the usual right shift mapping.

Moreover, by a similar proof of Theorem 10 of Dowling, Lennard and Turett [15], we show that c_0 cannot be renormed to have the FPP for strongly asymptotically nonexpansive maps. Indeed, if X is a Banach space containing an isomorphic copy of c_0, then by the Theorem 8 in [15], there exist a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and a sequence (x_n) in X so that

$$\sup_{n\geq k} |t_n| \leq \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \leq (1 + \varepsilon_k) \sup_{n\geq k} |t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in c_0$ and for all $k \in \mathbb{N}$.

Then, we consider the closed convex hull of x_n, i.e. $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. This set fails the fixed point property for $\|\cdot\|$-strongly asymptotically nonexpansive mappings. In fact, we show that there exists an *affine strongly asymptotically nonexpansive* mapping $T : E \longrightarrow E$ that is fixed point free. Furthermore, T is the usual right shift mapping.

We conclude that if $(X, \|\cdot\|)$ is a non-reflexive Banach lattice, then $(X, \|\cdot\|)$ fails the fixed point property for $\|\cdot\|$-semi-strongly asymptotically nonexpansive mappings.

At the end of that Chapter, Chapter 5, we also show that any closed non-reflexive vector subspace Y of $l^0_{w,\infty}$ contains an isomorphic copy of c_0 and so Y fails the fixed point property for strongly asymptotically nonexpansive maps. Finally, we prove that a Banach lattice X is reflexive if and only if for every closed bounded convex set C contained in X, for every affine semi-strongly asymptotically nonexpansive mapping $U\colon C \to C$, U has a fixed point in C.

In the final chapter, we explain our future projects.

2.0 THE CLOSED, CONVEX HULL OF AN AI c_0-SUMMING BASIC SEQUENCE FAILS THE FIXED POINT PROPERTY

2.1 AN EXAMPLE OF A C.B.C. SUBSET OF c_0 THAT FAILS THE FPP(AFFINE, N.E.)

Fix $b \in (0,1)$. We define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := b\,e_1$, $f_2 := b\,e_2$, and $f_n := e_n$, for all integers $n \geq 3$.

Next, define the closed, bounded, convex subset $E = E_b$ of c_0 by

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} .$$

Question (★). Is it true that for all affine, $\|\cdot\|_\infty$-nonexpansive mappings $U : E \longrightarrow E$, there exists $z \in E$ such that $U(z) = z$? Put differently, does E have the FPP (affine, n.e.)?

Let us define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way. Let $\eta_1 := f_1$ and $\eta_n := f_1 + \cdots + f_n$, for all integers $n \geq 2$.

It is straightforward to check that

$$E := \left\{ \sum_{n=1}^{\infty} \alpha_n\, \eta_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

The next result shows that the answer to Question (★) is "no" if b is large enough.

Theorem 2.1.1. *Let $b > 32/33$. Then $E = E_b$ is such that there exists an affine, $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free. Moreover, U is contractive; i.e.,*

$$\|U(x) - U(y)\|_\infty < \|x - y\|_\infty \ , \text{ for all } x, y \in E \text{ with } x \neq y \ .$$

Proof. Firstly, we will verify that $(\eta_n)_{n\in\mathbb{N}}$ is an *asymptotically isometric c_0-summing basic sequence* in $(c_0, \|\cdot\|_\infty)$. Fix an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \in c_{00}$. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} t_j\,\eta_j &= t_1\,f_1 + t_2\,(f_1 + f_2) + t_3\,(f_1 + f_2 + f_3) + \dots \\
&= (t_1 + t_2 + t_3 + \dots)\,f_1 + (t_2 + t_3 + t_4 + \dots)\,f_2 + (t_3 + t_4 + t_5 + \dots)\,f_3 + \dots \\
&= \left(\sum_{j=1}^{\infty} t_j\right) b\,e_1 + \left(\sum_{j=2}^{\infty} t_j\right) b\,e_2 + \left(\sum_{j=3}^{\infty} t_j\right) e_3 + \left(\sum_{j=4}^{\infty} t_j\right) e_4 + \dots \ .
\end{aligned}$$

Therefore,

$$\left\|\sum_{j=1}^{\infty} t_j\,\eta_j\right\|_\infty = \left|\sum_{j=1}^{\infty} t_j\right| b \ \vee \ \left|\sum_{j=2}^{\infty} t_j\right| b \ \vee \ \left|\sum_{j=3}^{\infty} t_j\right| \ \vee \ \left|\sum_{j=4}^{\infty} t_j\right| \ \vee \ \dots \ .$$

Define $\varepsilon_2 := \varepsilon_1 := 1/b - 1$ and $\varepsilon_n := 0$, for all $n \geq 3$. Clearly, $(\eta_n)_{n\in\mathbb{N}}$ is an *asymptotically isometric c_0-summing basic sequence.*

By the statement and proof of Theorem 2 of [16], it follows that whenever $b > (1 + 2^{-1}\,4^{-2})^{-1} = 32/33$; i.e., $1/b - 1 = \varepsilon_2 < 2^{-1}\,4^{-2}$, it follows that $E = E_b$ is such that there exists an affine, $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free. □

Question (★★). Can the proof of Theorem 2 of [16] be modified to show that the answer to Question (★) above is "no", for all $b \in (0, 1)$?

The answer to Question (★★) is "yes"...

Theorem 2.1.2. *Let $b \in (0, 1)$. Then $E = E_b$ is such that there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.*

Proof. Fix $b \in (0,1)$. Define the sequence $(\theta_n)_{n\in\mathbb{N}}$ in $(0,b]$ by

$$\theta_n := b\,(1-b)^{n-1}\ ,\ \text{for all } n \in \mathbb{N}\ .$$

It is easy to see that each $\theta_n \in (0,1)$ and $\sum_{n=1}^{\infty} \theta_n = 1$. Recall from Section 1 the usual right shift mapping $T : E \longrightarrow E$ given by $T\left(\sum_{n=1}^{\infty} t_n\, f_n\right) := f_1 + \sum_{n=1}^{\infty} t_n\, f_{n+1}$. Note that T is *not* nonexpansive. We define the affine mapping $U : E \longrightarrow E$ by firstly setting

$$U(\eta_n) := \sum_{j=1}^{\infty} \theta_j\, T^j(\eta_n) = \sum_{j=1}^{\infty} \theta_j\, \eta_{j+n}\ ,\ \text{for all } n \in \mathbb{N}\ .$$

Next, for all $x = \sum_{n=1}^{\infty} \alpha_n\, \eta_n \in E$, define

$$U(x) := \sum_{n=1}^{\infty} \alpha_n\, U(\eta_n) = \sum_{n=1}^{\infty} \alpha_n \left(\sum_{j=1}^{\infty} \theta_j\, T^j(\eta_n)\right) = \sum_{j=1}^{\infty} \theta_j\, T^j(x)\ .$$

In summary, $U := \sum_{j=1}^{\infty} \theta_j\, T^j$.

By a similar argument to that in the proof of Theorem 2 of [16], it follows that U is fixed point free on E. It remains to show that U is nonexpansive. Let $x = \sum_{n=1}^{\infty} t_n\, \eta_n$ and $y = \sum_{n=1}^{\infty} s_n\, \eta_n \in E$; so that $t_n, s_n \geq 0$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1$. Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$. Also, define $\varepsilon_2 := \varepsilon_1 := 1/b - 1$ and $\varepsilon_n := 0$, for all $n \geq 3$. Then,

$$\|U(x) - U(y)\|_\infty \leq \max_{m\geq 1} \frac{1}{1+\varepsilon_m} \left|\sum_{j=m}^{\infty} \alpha_j\right| \cdot Q = \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_\infty \cdot Q\ ;$$

where

$$\begin{aligned} Q &:= \sup_{n\geq 3} \big((1+\varepsilon_2)\,\theta_{n-2} + (1+\varepsilon_3)\,\theta_{n-3} + \cdots + (1+\varepsilon_{n-1})\,\theta_1\big) \\ &= (1+\varepsilon_2)\,\theta_1\ \vee\ \sup_{n\geq 4} \big((1+\varepsilon_2)\,\theta_{n-2} + \theta_{n-3} + \cdots + \theta_2 + \theta_1\big) \\ &= \frac{1}{b}\,b\ \vee\ \sup_{n\geq 4} \left(\frac{1}{b}\,\theta_{n-2} + \theta_{n-3} + \cdots + \theta_2 + \theta_1\right) \\ &= 1\ \vee\ \sup_{n\geq 4} \left(\frac{1}{b}\,b\,(1-b)^{n-3} + \sum_{k=1}^{n-3} b\,(1-b)^{k-1}\right) \\ &= 1\ \vee\ \sup_{n\geq 4} \big((1-b)^{n-3} + 1 - (1-b)^{n-3}\big) = 1. \end{aligned}$$

Thus,

$$\|U(x) - U(y)\|_\infty \leq \max_{m\geq 1} \frac{1}{1+\varepsilon_m} \left|\sum_{j=m}^{\infty} \alpha_j\right| = \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_\infty = \|x-y\|_\infty\ .$$

□

Question ($\bigstar_3$). Can the proof of Theorem 2.1.2 be modified to show that for all $b \in (0,1)$, $E = E_b$ fails the FPP(affine, n.e.) via a *contractive* mapping U?

The answer to Question ($\bigstar_3$) is "yes".

Proof. Fix $b \in (0,1)$. Let $c \in (0,b)$. Define the sequence $(\theta_n)_{n\in\mathbb{N}}$ in $(0,c]$ by

$$\theta_n := c\,(1-c)^{n-1} \ , \ \text{for all } n \in \mathbb{N} \ .$$

It is easy to see that

$$\text{each } \theta_n \in (0,b) \text{ and } \sum_{n=1}^{\infty} \theta_n = 1.$$

Then, similarly to the proof of Theorem 2.1.2, $E = E_b$ is such that there exists an affine, $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free. Indeed, we similarly define the affine mapping $U : E \longrightarrow E$ by firstly setting

$$U(\eta_n) := \sum_{j=1}^{\infty} \theta_j\, \eta_{j+n} \ , \ \text{for all } n \in \mathbb{N} \ .$$

Next, for all $x = \sum_{n=1}^{\infty} \alpha_n\, \eta_n \in E$, define

$$U(x) := \sum_{n=1}^{\infty} \alpha_n\, U(\eta_n) \ .$$

By the proof of Theorem 2.1.2 above, and a similar argument to that in the proof of Theorem 2 of [16], we can show that U is fixed point free on E. It remains to show that U is *contractive*. Let $x = \sum_{n=1}^{\infty} t_n\, \eta_n$ and $y = \sum_{n=1}^{\infty} s_n\, \eta_n \in E$; so that $t_n, s_n \geq 0$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1$. Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$. Also, define $\varepsilon_2 := \varepsilon_1 := 1/b - 1$ and $\varepsilon_n := 0$, for all $n \geq 3$. We note that $\frac{b}{c} > 1$. Then,

$$\|U(x) - U(y)\|_\infty \leq \max_{m\geq 1} \frac{1}{1 + \frac{b}{c}\varepsilon_m} \left| \sum_{j=m}^{\infty} \alpha_j \right| \cdot Q = \left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_\infty \cdot Q \ ;$$

where

$$\begin{aligned}
Q &:= \sup_{n\geq 3}(1+\varepsilon_n)\left((1+\frac{b}{c}\varepsilon_2)\,\theta_{n-2}+(1+\frac{b}{c}\varepsilon_3)\,\theta_{n-3}+\cdots+(1+\frac{b}{c}\varepsilon_{n-1})\,\theta_1\right)\\
&= \sup_{n\geq 3}(1+\varepsilon_n)\left((1+\frac{b}{c}\varepsilon_2)\,\theta_{n-2}+\theta_{n-3}+\cdots+\theta_2+\theta_1\right)\\
&= (1+\frac{b}{c}(\frac{1}{b}-1))\,\theta_1 \ \vee\ \sup_{n\geq 4}\left((1+\frac{b}{c}(\frac{1}{b}-1))\,\theta_{n-2}+\theta_{n-3}+\cdots+\theta_2+\theta_1\right)\\
&\leq (1+\frac{1}{c}-\frac{b}{c})\,c \ \vee\ \sup_{n\geq 4}\left(\frac{1}{c}\,\theta_{n-2}+\theta_{n-3}+\cdots+\theta_2+\theta_1\right)\\
&\leq 1 \ \vee\ \sup_{n\geq 4}\left(\frac{1}{c}\,\theta_{n-2}+\theta_{n-3}+\cdots+\theta_2+\theta_1\right)=1.
\end{aligned}$$

$$\begin{aligned}
\text{Hence, for all } x,y\in E \text{ with } x\neq y \quad &\Rightarrow\quad \alpha_n=t_n-s_n \text{ for some } n\in\mathbb{N}\\
&\Rightarrow\quad \sum_{j=m_0}^{\infty}\alpha_j\neq 0 \text{ for some } m_0\in\mathbb{N}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\max_{m\geq 1}\frac{1}{1+\frac{b}{c}\,\varepsilon_m}\left|\sum_{j=m}^{\infty}\alpha_j\right| &= \frac{1}{1+\frac{b}{c}\,\varepsilon_{m_1}}\left|\sum_{j=m_1}^{\infty}\alpha_j\right| \text{ for some } m_1\in\mathbb{N}\\
&< \frac{1}{1+\varepsilon_{m_1}}\left|\sum_{j=m_1}^{\infty}\alpha_j\right|\\
&\leq \max_{m\geq 1}\frac{1}{1+\varepsilon_m}\left|\sum_{j=m}^{\infty}\alpha_j\right|
\end{aligned}$$

Thus,

$$\|U(x)-U(y)\|_\infty<\max_{m\geq 1}\frac{1}{1+\varepsilon_m}\left|\sum_{j=m}^{\infty}\alpha_j\right|=\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty=\|x-y\|_\infty\,.$$

□

2.2 A MORE GENERAL RESULT

We can generalize the previous theorem in the following way.

Theorem 2.2.1. *Let* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *be any increasing sequence (i.e.,* $b_n \le b_{n+1}$*, for all* $n \in \mathbb{N}$*) in* $(0,1]$ *with* $b_n \uparrow_n 1$*. We define the sequence* $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by setting* $f_n := b_n\, e_n$*, for all* $n \in \mathbb{N}$*. Next, define the closed, bounded, convex subset* $E = E_{\overrightarrow{b}}$ *of* c_0 *by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \ge t_2 \ge \cdots \ge t_n \downarrow_n 0 \right\} .$$

Then, there exists an affine $\|\cdot\|_\infty$*-nonexpansive mapping* $U : E \longrightarrow E$ *that is fixed point free.*

Proof. Let $\eta_n := f_1 + \cdots + f_n$, for all $n \in \mathbb{N}$. As before, we have that

$$E := \left\{ \sum_{n=1}^{\infty} \alpha_n\, \eta_n : \text{ each } \alpha_n \ge 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

Analogously to the proof of Theorem 2.1.2 above, we can find a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $[0,1]$ and a fixed point free affine mapping $U : E \longrightarrow E$ that satisfy the following conditions.

$$\sum_{n=1}^{\infty} \theta_n = 1\,; \tag{2.1}$$

$$U(\eta_n) = \sum_{j=1}^{\infty} \theta_j\, \eta_{j+n} \text{ , for all } n \in \mathbb{N}\,; \tag{2.2}$$

$$U(x) = \sum_{n=1}^{\infty} \alpha_n\, U(\eta_n) \text{ for all } x = \sum_{n=1}^{\infty} \alpha_n\, \eta_n \in E\,; \tag{2.3}$$

$$\forall x, y \in E\ ,\ \|U(x) - U(y)\|_\infty \le \|x - y\|_\infty \cdot Q\,; \tag{2.4}$$

$$\text{where } Q := \sup_{n\ge 3} b_n \gamma_n \text{ and } \gamma_n := \frac{\theta_{n-2}}{b_2} + \frac{\theta_{n-3}}{b_3} + \cdots + \frac{\theta_1}{b_{n-1}}\ ,\ \forall\, n \ge 3\,; \tag{2.5}$$

$$\text{and } Q = 1\,. \tag{2.6}$$

To find a sequence $(\theta_n)_{n\in\mathbb{N}}$ as above, given $(b_n)_{n\in\mathbb{N}}$, we let $\gamma_n = 1$ for all $n \ge 3$.

Consider three special cases.

(1) $b_1 = b_2 = b \in (0,1)$, and $b_j = 1$, for all $j \geq 3$.

$$[\,\gamma_n = 1, \forall\, n \geq 3\,] \Longleftrightarrow [\,\theta_n = b(1-b)^{n-1}, \forall\, n \in \mathbb{N}\,]\ .$$

(2) $b_1 = b_2 = b_3 = b \in (0,1)$, and $b_j = 1$, for all $j \geq 4$.

$$[\,\gamma_n = 1, \forall\, n \geq 3\,] \Longleftrightarrow [\,\theta_{2n-1} = b(1-b)^{n-1} \text{ and } \theta_{2n} = 0, \forall\, n \in \mathbb{N}\,]\ .$$

(3) $b_1 = b_2 = b_3 = b_4 = b \in (0,1)$, and $b_j = 1$, for all $j \geq 5$.

$$[\,\gamma_n = 1, \forall\, n \geq 3\,] \Longleftrightarrow [\,\theta_{3j+1} = b(1-b)^j, \forall\, j \geq 0 \text{ and } \theta_n = 0, \forall\, n \in \mathbb{N}\backslash\{3j+1 : j \geq 0\}\,]\ .$$

For the general case, we now confirm that by setting $\gamma_j = 1$, for all $j \geq 3$, we obtain a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $[0,1]$ with $\sum_{n=1}^{\infty} \theta_n = 1$. We will use our hypothesis that $(b_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(0,1]$.

$$\begin{aligned}
\gamma_3 = 1 \quad &\Longleftrightarrow \quad \theta_1 = b_2 \in (0,1] \text{ by hypothesis.} \\
\gamma_4 = 1 \quad &\Longleftrightarrow \quad \frac{\theta_2}{b_2} + \frac{\theta_1}{b_3} = 1 \\
&\Longleftrightarrow \quad \frac{\theta_2}{b_2} + \frac{b_2}{b_3} = 1 \text{ and } \theta_1 = b_2 \\
&\Longleftrightarrow \quad \theta_2 = b_2\left(1 - \frac{b_2}{b_3}\right) \in [0,1) \text{ and } \theta_1 = b_2 \in (0,1]\ . \\
\gamma_5 = 1 \quad &\Longleftrightarrow \quad \frac{\theta_3}{b_2} + \frac{\theta_2}{b_3} + \frac{\theta_1}{b_4} = 1 \\
&\Longleftrightarrow \quad \frac{\theta_3}{b_2} + X_5 = 1\ ,\ \text{ where } X_5 := \frac{\theta_2}{b_3} + \frac{\theta_1}{b_4}\ .
\end{aligned}$$

Note that $X_5 \leq \frac{\theta_2}{b_2} + \frac{\theta_1}{b_3} = 1$. Thus, $\gamma_5 = 1 \Longleftrightarrow \theta_3 = b_2(1 - X_5) \in [0,1)$.

$$\begin{aligned}
\gamma_6 = 1 \quad &\Longleftrightarrow \quad \frac{\theta_4}{b_2} + \frac{\theta_3}{b_3} + \frac{\theta_2}{b_4} + \frac{\theta_1}{b_5} = 1 \\
&\Longleftrightarrow \quad \frac{\theta_4}{b_2} + X_6 = 1\ ,\ \text{ where } X_6 := \frac{\theta_3}{b_3} + \frac{\theta_2}{b_4} + \frac{\theta_1}{b_5}\ .
\end{aligned}$$

From above, $X_6 \leq \frac{\theta_3}{b_2} + \frac{\theta_2}{b_3} + \frac{\theta_1}{b_4} = 1$. Therefore, $\gamma_6 = 1 \Longleftrightarrow \theta_4 = b_2(1 - X_6) \in [0,1)$. Continuing inductively, we construct a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $[0,1]$ such that $\gamma_n = 1$ for each $n \geq 3$; i.e.,

$$1 = \frac{\theta_{n-2}}{b_2} + \frac{\theta_{n-3}}{b_3} + \cdots + \frac{\theta_1}{b_{n-1}}\ ,\ \text{ for all } n \geq 3\ .$$

Let's prove that this sequence satisfies condition (2.1) above. Fix $n \geq 3$. Since each $b_n \leq 1$,

$$1 \geq \frac{\theta_{n-2}}{1} + \frac{\theta_{n-3}}{1} + \cdots + \frac{\theta_1}{1} = \sum_{j=1}^{n-2} \theta_j \ .$$

Therefore, $\sum_{j=1}^{\infty} \theta_j \leq 1$. Moreover, let $n = 2m+1$, where $m \in \mathbb{N}$, $m \geq 2$. Then,

$$\begin{aligned} 1 &= \frac{\theta_{2m-1}}{b_2} + \frac{\theta_{2m-2}}{b_3} + \cdots + \frac{\theta_2}{b_{2m-1}} + \frac{\theta_1}{b_{2m}} \\ &= \frac{\theta_{2m-1}}{b_2} + \frac{\theta_{2m-2}}{b_3} + \cdots + \frac{\theta_{m+1}}{b_m} + \frac{\theta_m}{b_{m+1}} + \frac{\theta_{m-1}}{b_{m+2}} + \cdots + \frac{\theta_1}{b_{2m}} \ . \end{aligned}$$

Thus, since $(b_n)_{n\in\mathbb{N}}$ is increasing,

$$\begin{aligned} 1 &\leq \frac{\theta_{2m-1}}{b_2} + \frac{\theta_{2m-2}}{b_2} + \cdots + \frac{\theta_{m+1}}{b_2} + \frac{\theta_m}{b_{m+1}} + \frac{\theta_{m-1}}{b_{m+1}} + \cdots + \frac{\theta_1}{b_{m+1}} \\ &= \frac{1}{b_2} \sum_{j=m+1}^{2m-1} \theta_j + \frac{1}{b_{m+1}} \sum_{j=1}^{m} \theta_j \ . \end{aligned}$$

Letting $m \to \infty$, it follows that $1 \leq \sum_{j=1}^{\infty} \theta_j$.

Next, using the sequence $(\theta_n)_{n\in\mathbb{N}}$, we define a mapping $U : E \longrightarrow E$ via the conditions (2.2) and (2.3) above. We prove condition (2.4), given the definitions in (2.5), similarly to the proof of Theorem 2 of [16]. Indeed, let $x = \sum_{n=1}^{\infty} t_n \, \eta_n$ and $y = \sum_{n=1}^{\infty} s_n \, \eta_n \in E$; so that $t_n, s_n \geq 0$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1$. Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$. Note that for any $(\beta_n)_{n\in\mathbb{N}} \in \ell^1$,

$$\sum_{j=1}^{\infty} \beta_j \, \eta_j = \left(\sum_{j=1}^{\infty} \beta_j \right) b_1 \, e_1 + \left(\sum_{j=2}^{\infty} \beta_j \right) b_2 \, e_2 + \left(\sum_{j=3}^{\infty} \beta_j \right) b_3 \, e_3 + \ldots$$

Therefore,

$$\left\| \sum_{j=1}^{\infty} \beta_j \, \eta_j \right\|_{\infty} = \left| \sum_{j=1}^{\infty} \beta_j \right| b_1 \vee \left| \sum_{j=2}^{\infty} \beta_j \right| b_2 \vee \left| \sum_{j=3}^{\infty} \beta_j \right| b_3 \vee \ldots$$

Furthermore, $(\alpha_j)_{j\in\mathbb{N}} \in \ell^1$ and $\sum_{j=1}^{\infty} \alpha_j = 0$; and so

$$\|x - y\|_{\infty} = \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} = \sup_{k \geq 2} \left| \sum_{j=k}^{\infty} \alpha_j \right| b_k \ .$$

Also,

$$\begin{aligned} U(x) - U(y) &= \sum_{k=1}^{\infty} \alpha_k \, U(\eta_k) = \sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^{\infty} \theta_j \, \eta_{j+k} \\ &= \sum_{m=2}^{\infty} \sum_{\substack{j,k \in \mathbb{N} \\ j+k=m}} \theta_j \, \alpha_k \, \eta_m = \sum_{n=1}^{\infty} \sum_{\substack{j,k \in \mathbb{N} \\ j+k=n+1}} \theta_j \, \alpha_k \, \eta_{n+1} \\ &= \sum_{n=1}^{\infty} (\theta_n \, \alpha_1 \, + \theta_{n-1} \, \alpha_2 \, + \cdots + \theta_1 \, \alpha_n) \; \eta_{n+1} \end{aligned}$$

$$= \left(\theta_1 \sum_{j=2}^{\infty} \alpha_j\right) f_3 + \left(\theta_2 \sum_{j=2}^{\infty} \alpha_j + \theta_1 \sum_{j=3}^{\infty} \alpha_j\right) f_4 + \left(\theta_3 \sum_{j=2}^{\infty} \alpha_j + \theta_2 \sum_{j=3}^{\infty} \alpha_j + \theta_1 \sum_{j=4}^{\infty} \alpha_j\right) f_5 + \ldots,$$

because $\sum_{j=1}^{\infty} \alpha_j = 0$. Thus,

$$\begin{aligned} \|U(x) - U(y)\|_\infty &= \sup_{n \geq 3} b_n \left| \theta_{n-2} \sum_{j=2}^{\infty} \alpha_j + \theta_{n-3} \sum_{j=3}^{\infty} \alpha_j + \cdots + \theta_1 \sum_{j=n-1}^{\infty} \alpha_j \right| \\ &= \sup_{n \geq 3} b_n \left| \frac{\theta_{n-2} b_2 \sum_{j=2}^{\infty} \alpha_j}{b_2} + \frac{\theta_{n-3} b_3 \sum_{j=3}^{\infty} \alpha_j}{b_3} + \cdots + \frac{\theta_1 b_{n-1} \sum_{j=n-1}^{\infty} \alpha_j}{b_{n-1}} \right| \end{aligned}$$

But, $b_k \left| \sum_{j=k}^{\infty} \alpha_j \right| \leq \|x - y\|_\infty$, for each $k \geq 2$. Hence,

$$\begin{aligned} \|U(x) - U(y)\|_\infty &\leq \sup_{n \geq 3} b_n \left(\frac{\theta_{n-2}}{b_2} + \frac{\theta_{n-3}}{b_3} + \cdots + \frac{\theta_1}{b_{n-1}} \right) \cdot \|x - y\|_\infty \\ &= Q \cdot \|x - y\|_\infty \; ; \end{aligned}$$

where

$$Q := \sup_{n \geq 3} b_n \gamma_n \; , \text{with } \gamma_n := \frac{\theta_{n-2}}{b_2} + \frac{\theta_{n-3}}{b_3} + \cdots + \frac{\theta_1}{b_{n-1}} \; , \text{ for all } n \geq 3 \; .$$

From above, each $\gamma_n = 1$, and so $Q = 1$. Thus, $\|U(x) - U(y)\|_\infty \leq \|x - y\|_\infty$.

Finally, it is straightforward to check that U is fixed point free on E. $\square$

2.3 BANACH SPACES CONTAINING ASYMPTOTICALLY ISOMETRIC c_0-SUMMING BASIC SEQUENCES

Theorem 2.3.1. *Let $L \in (0, \infty)$. If a Banach space contains an L-scaled asymptotically isometric c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, then $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$ fails the fixed point property for affine nonexpansive mappings. Indeed, more is true. There exists an affine contractive mapping $U : E \longrightarrow E$ that is fixed point free.*

Proof. We may assume that $L = 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space $(X, \|\cdot\|)$ that is an asymptotically isometric c_0-summing basic sequence. Then, by Definition 1.1.3, there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ such that for all sequences $(t_n)_{n\in\mathbb{N}} \in c_{00}$,

$$(\dagger\dagger) \qquad \sup_{n\geq 1} \left(\frac{1}{1+\varepsilon_n}\right) \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{j=1}^{\infty} t_j\, x_j\right\| \leq \sup_{n\geq 1}\,(1+\varepsilon_n) \left|\sum_{j=n}^{\infty} t_j\right| .$$

Clearly, we may assume that every $\varepsilon_n > 0$. Next, we may replace $(\varepsilon_n)_{n\in\mathbb{N}}$ with a decreasing sequence in the following way. Define

$$\zeta_n := \max_{j\geq n} \varepsilon_j \ , \ \text{for all } n \in \mathbb{N} .$$

Thus,

$$(\dagger\dagger\dagger) \qquad \sup_{n\geq 1} \left(\frac{1}{1+\zeta_n}\right) \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{j=1}^{\infty} t_j\, x_j\right\| \leq \sup_{n\geq 1}\,(1+\zeta_n) \left|\sum_{j=n}^{\infty} t_j\right|$$

and $(\zeta_n)_{n\in\mathbb{N}}$ is a null sequence in $(0, \infty)$. Hence, without loss of generality, we may assume that there exists a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ such that $(\dagger\dagger)$ is satisfied. As above, we define the closed, bounded, convex subset E of X by $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. It is straightforward to check, using condition $(\dagger\dagger)$, that

$$E := \left\{\sum_{j=1}^{\infty} t_j\, x_j \ : \ \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1\right\} .$$

We proceed similarly to the proof of the previous theorem. Once we find an appropriate sequence $(\theta_n)_{n\in\mathbb{N}}$ in $[0, 1]$ such that $\sum_{n=1}^{\infty} \theta_n = 1$, we will define an affine mapping $U : E \longrightarrow E$ by setting

$$U(x_n) := \sum_{j=1}^{\infty} \theta_j\, x_{j+n} \ , \ \text{for all } n \in \mathbb{N} ;$$

and then, for all $x = \sum_{n=1}^{\infty} \alpha_n x_n \in E$,

$$\begin{aligned} U(x) &:= \sum_{n=1}^{\infty} \alpha_n U(x_n) = \sum_{n=1}^{\infty} \alpha_n \sum_{j=1}^{\infty} \theta_j x_{j+n} = \sum_{n,j=1}^{\infty} \alpha_n \theta_j x_{j+n} \\ &= \alpha_1 \theta_1 x_2 + (\alpha_1 \theta_2 + \alpha_2 \theta_1) x_3 + (\alpha_1 \theta_3 + \alpha_2 \theta_2 + \alpha_3 \theta_1) x_4 + \dots . \end{aligned}$$

Let's investigate how we can ensure that such a mapping U is nonexpansive. We define $\sigma_n := 1 + \varepsilon_n$, for each $n \in \mathbb{N}$. Fix $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and $y = \sum_{n=1}^{\infty} \beta_n x_n \in E$ with $x \neq y$. We have that each $\alpha_j, \beta_j \geq 0$, $\sum_{j=1}^{\infty} \alpha_j = 1$ and $\sum_{j=1}^{\infty} \beta_j = 1$. We set $\gamma_n := \alpha_n - \beta_n$, for each $n \in \mathbb{N}$. Note that $\sum_{n=1}^{\infty} \gamma_n = 0$. Then, by the second inequality of (††) and the fact that $\sum_{n=1}^{\infty} \gamma_n = 0$,

$$\begin{aligned} &\|U(x) - U(y)\| = \|\gamma_1 \theta_1 x_2 + (\gamma_1 \theta_2 + \gamma_2 \theta_1) x_3 + (\gamma_1 \theta_3 + \gamma_2 \theta_2 + \gamma_3 \theta_1) x_4 + \dots \| \\ \leq\ & \sigma_3 \left| \theta_1 \sum_{j=2}^{\infty} \gamma_j \right| \vee \sigma_4 \left| \theta_2 \sum_{j=2}^{\infty} \gamma_j + \theta_1 \sum_{j=3}^{\infty} \gamma_j \right| \vee \sigma_5 \left| \theta_3 \sum_{j=2}^{\infty} \gamma_j + \theta_2 \sum_{j=3}^{\infty} \gamma_j + \theta_1 \sum_{j=4}^{\infty} \gamma_j \right| + \dots \\ \leq\ & \sigma_3 \theta_1 \left| \sum_{j=2}^{\infty} \gamma_j \right| \vee \sigma_4 \left(\theta_2 \left| \sum_{j=2}^{\infty} \gamma_j \right| + \theta_1 \left| \sum_{j=3}^{\infty} \gamma_j \right| \right) \\ & \vee \sigma_5 \left(\theta_3 \left| \sum_{j=2}^{\infty} \gamma_j \right| + \theta_2 \left| \sum_{j=3}^{\infty} \gamma_j \right| + \theta_1 \left| \sum_{j=4}^{\infty} \gamma_j \right| \right) \vee \dots \end{aligned}$$

$$\begin{aligned} &= \sigma_3 \left(\frac{\sigma_2 \theta_1 \left| \sum_{j=2}^{\infty} \gamma_j \right|}{\sigma_2} \right) \vee \sigma_4 \left(\frac{\sigma_2 \theta_2 \left| \sum_{j=2}^{\infty} \gamma_j \right|}{\sigma_2} + \frac{\sigma_3 \theta_1 \left| \sum_{j=3}^{\infty} \gamma_j \right|}{\sigma_3} \right) \\ &\quad \vee \sigma_5 \left(\frac{\sigma_2 \theta_3 \left| \sum_{j=2}^{\infty} \gamma_j \right|}{\sigma_2} + \frac{\sigma_3 \theta_2 \left| \sum_{j=3}^{\infty} \gamma_j \right|}{\sigma_3} + \frac{\sigma_4 \theta_1 \left| \sum_{j=4}^{\infty} \gamma_j \right|}{\sigma_4} \right) \vee \dots \\ &\leq \big(\sigma_3 (\sigma_2 \theta_1) \vee \sigma_4 (\sigma_2 \theta_2 + \sigma_3 \theta_1) \vee \sigma_5 (\sigma_2 \theta_3 + \sigma_3 \theta_2 + \sigma_4 \theta_1) \vee \dots \big) \|x - y\| . \end{aligned}$$

The last inequality above follows from the first inequality in (††).

Let $Q := \sigma_3 (\sigma_2 \theta_1) \vee \sigma_4 (\sigma_2 \theta_2 + \sigma_3 \theta_1) \vee \sigma_5 (\sigma_2 \theta_3 + \sigma_3 \theta_2 + \sigma_4 \theta_1) \vee \dots$. To build a map U that is nonexpansive, it is enough to find θ_n's as above such that $Q = 1$. Note that $Q = \sup_{n \geq 3} \sigma_n \Gamma_n$, where each $\Gamma_n := \theta_{n-2}\sigma_2 + \theta_{n-3}\sigma_3 + \dots + \theta_1 \sigma_{n-1}$. We define $b_n := \frac{1}{\sigma_n}, \forall n \in \mathbb{N}$

and note that $(b_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ with $b_n \uparrow_n 1$ (since $\sigma_n \downarrow_n 1$). Let's try setting $\frac{\Gamma_n}{b_n} = 1$, for all $n \geq 3$.

$$\begin{aligned}
\Gamma_3 = b_3 \quad &\Longleftrightarrow \quad \theta_1 = b_2 \cdot b_3 \in (0,1)\,. \\
\Gamma_4 = b_4 \quad &\Longleftrightarrow \quad \frac{\theta_2}{b_2} + \frac{\theta_1}{b_3} = b_4 \\
&\Longleftrightarrow \quad \frac{\theta_2}{b_2} + b_2 = b_4 \\
&\Longleftrightarrow \quad \theta_2 = b_2(b_4 - b_2) \in [0,1)\ ,\ \text{since } (b_n)_{n\in\mathbb{N}} \text{ is increasing.} \\
\Gamma_5 = b_5 \quad &\Longleftrightarrow \quad \frac{\theta_3}{b_2} + \frac{\theta_2}{b_3} + \frac{\theta_1}{b_4} = b_5 \\
&\Longleftrightarrow \quad \frac{\theta_3}{b_2} + X_5 = b_5\ ,\ \text{where } X_5 := \frac{\theta_2}{b_3} + \frac{\theta_1}{b_4}\,.
\end{aligned}$$

Note that $X_5 \leq \frac{\theta_2}{b_2} + \frac{\theta_1}{b_3} = b_4 \leq b_5$. Thus, $\Gamma_5 = b_5 \Longleftrightarrow \theta_3 = b_2(b_5 - X_5) \in [0,1)$.

$$\begin{aligned}
\Gamma_6 = b_6 \quad &\Longleftrightarrow \quad \frac{\theta_4}{b_2} + \frac{\theta_3}{b_3} + \frac{\theta_2}{b_4} + \frac{\theta_1}{b_5} = b_6 \\
&\Longleftrightarrow \quad \frac{\theta_4}{b_2} + X_6 = b_6\ ,\ \text{where } X_6 := \frac{\theta_3}{b_3} + \frac{\theta_2}{b_4} + \frac{\theta_1}{b_5}\,.
\end{aligned}$$

From above, $X_6 \leq \frac{\theta_3}{b_2} + \frac{\theta_2}{b_3} + \frac{\theta_1}{b_4} = b_5 \leq b_6$. Therefore, $\Gamma_6 = b_6 \Longleftrightarrow \theta_4 = b_2(b_6 - X_6) \in [0,1)$. Hence, inductively, we can build a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $[0,1)$ such that $\Gamma_n = b_n$ for each $n \geq 3$; i.e., $b_n = \frac{\theta_{n-2}}{b_2} + \frac{\theta_{n-3}}{b_3} + \cdots + \frac{\theta_1}{b_{n-1}}$, for all $n \geq 3$.

Now, we just need to show that $\sum_{n=1}^{\infty} \theta_n = 1$. But this follows by a similar argument to that given in the proof of Theorem 2.2.1 above. Hence, we have constructed an affine nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

By using the same idea as in the last part of the proof of Theorem 2 of [16], if we replace the above decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ by $(2\,\varepsilon_n)_{n\in\mathbb{N}}$, the above construction yields an affine *contractive* mapping $U : E \longrightarrow E$ that is fixed point free. □

By using Theorem 2.3.1, we can prove a strengthening of Theorem 2.2.1, that also includes Proposition 4.6 of Llorens-Fuster and Sims [33] (recalled in Theorem 1.1.14 above).

Theorem 2.3.2. *Let* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *be any sequence in* $(0,\infty)$ *that converges to some* $\kappa > 0$. *We define the sequence* $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by setting* $f_n := b_n\, e_n$, *for all* $n \in \mathbb{N}$. *Next, define the closed, bounded, convex subset* $E = E_{\overrightarrow{b}}$ *of* c_0 *by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} .$$

Then, there exists an affine $\|\cdot\|_\infty$*-contractive mapping* $U : E \longrightarrow E$ *that is fixed point free.*

Proof. We may assume that $\kappa = 1$. Let, $X := c_0$, and $\|\cdot\| := \|\cdot\|_\infty$. Define $x_n := f_1 + f_2 + \cdots + f_n$, for all $n \in \mathbb{N}$; where $f_n := b_n\, e_n$, for all $n \in \mathbb{N}$. We will show that the sequence $(x_n)_{n\in\mathbb{N}}$ is an asymptotically isometric c_0-summing basic sequence in X. Fix an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \in c_{00}$. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} t_j\, x_j &= t_1\, f_1 + t_2\,(f_1 + f_2) + t_3\,(f_1 + f_2 + f_3) + \ldots \\
&= (t_1 + t_2 + t_3 + \ldots)\, f_1 + (t_2 + t_3 + t_4 + \ldots)\, f_2 + \ldots \\
&= \left(\sum_{j=1}^{\infty} t_j \right) b_1\, e_1 + \left(\sum_{j=2}^{\infty} t_j \right) b_2\, e_2 + \left(\sum_{j=3}^{\infty} t_j \right) b_3\, e_3 + \ldots .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} t_j\, x_j \right\|_\infty &= \left| \sum_{j=1}^{\infty} t_j \right| b_1 \;\vee\; \left| \sum_{j=2}^{\infty} t_j \right| b_2 \;\vee\; \left| \sum_{j=3}^{\infty} t_j \right| b_3 \;\vee\; \left| \sum_{j=4}^{\infty} t_j \right| b_4 \;\vee\; \ldots \\
&= \sup_{n\in\mathbb{N}} b_n \left| \sum_{j=n}^{\infty} t_j \right| .
\end{aligned}$$

Choose a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $\frac{1}{1+\varepsilon_n} < b_n < 1 + \varepsilon_n$, for all $n \in \mathbb{N}$. Then $(x_j)_{j\in\mathbb{N}}$ satisfies condition (†), and we are done by Theorem 2.3.1. $\square$

Theorem 2.4.1.

$$(\triangle)\ \textit{Fix}\ \overrightarrow{b} = (b_n)_{n\in\mathbb{N}}\ \textit{in}\ \mathbb{R}\ \textit{with}\ 0 < m := \inf_{n\in\mathbb{N}} b_n\ \textit{and}\ M := \sup_{n\in\mathbb{N}} b_n\ < \infty\ .$$

We define the sequence $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by setting* $f_n := b_n\, e_n$*, for all* $n \in \mathbb{N}$*. Next, define the closed, bounded, convex subset* $E = E_{\overrightarrow{b}}$ *of* c_0 *by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\}\ .$$

Then, there exists an affine $\|\cdot\|_\infty$*-nonexpansive mapping* $U : E \longrightarrow E$ *that is fixed point free. Moreover, we may arrange for* U *to be* $\|\cdot\|_\infty$*-contractive.*

Proof. Let $L := \limsup_{n\to\infty} b_n$ and note that $m \leq L \leq M$. By Theorem 2.3.2, if $b_n \underset{n}{\longrightarrow} L$ then there exists an affine, $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free. We will first extend this result to the general situation $(\triangle)$, by examining some cases.

Case 1: $J := \{n \in \mathbb{N} : b_n \geq L\}$ is infinite.

We can write $J = \{n_k : k \in \mathbb{N}\}$, where $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$. Note that $\lim_{k\to\infty} b_{n_k} = L$.

Case 1.a: $(b_{n_k})_{k\in\mathbb{N}}$ is decreasing.

Case 1.a.1: $J = \{2k : k \in \mathbb{N}\}$, $L = M$, $m < M$, and $[\, b_{2k} := M,\ b_{2k-1} := m$, for all $k \in \mathbb{N}\,]$.

The vector $x = \sum_{n=1}^{\infty} t_n\, f_n \in E_{\overrightarrow{b}} \iff x = (m\, t_1, M\, t_2, m\, t_3, M\, t_4, \dots)$. We define $T : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ by $T(x) := (m\cdot 1, M\cdot 1, m\, t_1, M\, t_2, m\, t_3, M\, t_4, \dots)$, which means exactly applying the right shift twice. Hence, T is affine and fixed point-free. Furthermore, T is non-expansive. Indeed, let $y = \sum_{n=1}^{\infty} s_n\, f_n \in E_{\overrightarrow{b}}$. Then,

$$\|x - y\|_\infty = m \sup_{k\geq 2} |t_{2k-1} - s_{2k-1}|\ \vee\ M \sup_{k\geq 1} |t_{2k} - s_{2k}|\ .$$

Clearly, $\|T(x) - T(y)\|_\infty = \|x - y\|_\infty$.

Now, just consider a little bit different mapping, a variation on which will allow us to handle the general Case 1.a below.

Consider $U : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ defined by

$U(x) := (m \cdot 1, M \cdot 1, m\, t_2, M\, t_2, m\, t_4, M\, t_4, m\, t_6, M\, t_6, \dots)$. Clearly, U is fixed point-free. Indeed, if there exists an $x \in c_0$ such that $U(x) = x$, then $t_2 = 1, t_4 = t_2, t_6 = t_4$, $t_8 = t_6$, $\dots \Longrightarrow t_{2k} = 1, \forall k \in \mathbb{N} \Longrightarrow x \notin c_0$; contradiction. Also, U is non-expansive. For arbitrary $x, y \in E_{\overrightarrow{b}}$ as above,

$$\begin{aligned} \|U(x) - U(y)\|_\infty &= m \sup_{k \geq 1} |t_{2k} - s_{2k}| \ \vee \ M \sup_{k \geq 1} |t_{2k} - s_{2k}| \\ &= M \sup_{k \geq 1} |t_{2k} - s_{2k}| \leq \|x - y\|_\infty \ . \end{aligned}$$

The general Case 1.a:

Let $x \in E_{\overrightarrow{b}}$ and note that for all $j \notin J$, $b_j < L$. Note also that

$x = (b_1 t_1, b_2 t_2, \dots\dots, b_{n_1 - 1} t_{n_1 - 1}, b_{n_1} t_{n_1}, b_{n_1+1} t_{n_1+1}, \dots\dots, b_{n_2-1} t_{n_2-1}, b_{n_2} t_{n_2},$

$b_{n_2+1} t_{n_2+1}, \dots\dots, b_{n_k-1} t_{n_k-1}, b_{n_k} t_{n_k}, b_{n_k+1} t_{n_k+1}, \dots\dots, b_{n_{k+1}-1} t_{n_{k+1}-1},$

$b_{n_{k+1}} t_{n_{k+1}}, b_{n_{k+1}+1} t_{n_{k+1}+1}, \dots\dots)$.

Then, for arbitrary $x, y \in E_{\overrightarrow{b}}$ as above,

$$\|x - y\|_\infty = \sup_{k \in \mathbb{N}} b_{n_k} |t_{n_k} - s_{n_k}| \ \vee \ \sup_{j \notin J} b_j |t_j - s_j| \ .$$

Define $U : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ by

($\Box\Box$) $U(x) := (b_1(1), b_2(1), \dots\dots, b_{n_1-1}(1), b_{n_1}(1), b_{n_1+1}(1), \dots\dots, b_{n_2-1}(1), b_{n_2} t_{n_1},$

$b_{n_2+1} t_{n_1}, \dots\dots, b_{n_k-1} t_{n_{(k-2)}}, b_{n_k} t_{n_{(k-1)}}, b_{n_k+1} t_{n_{(k-1)}}, \dots\dots, b_{n_{(k+1)}-1} t_{n_{(k-1)}},$

$b_{n_{(k+1)}} t_{n_k}, b_{n_{(k+1)}+1} t_{n_k}, \dots\dots)$.

It is clear that U is affine. We see that U is fixed point-free. Indeed, $x \in E_{\overrightarrow{b}}$ and $x = U(x)$ $\Longrightarrow t_{n_1} = 1, t_{n_2} = t_{n_1}, \dots, t_{n_k} = t_{n_{k-1}}, \dots \Longrightarrow t_{n_k} = 1, \forall k \in \mathbb{N} \Longrightarrow x \notin c_0$; contradiction. Also, similarly to previous ideas, for every $x, y \in E_{\overrightarrow{b}}$ as above,

$$\|U(x) - U(y)\|_\infty = \sup_{k \in \mathbb{N}} b_{n_{k+1}} |t_{n_k} - s_{n_k}| \ \vee \ \sup_{k \in \mathbb{N}} b_{q_k} |t_{n_k} - s_{n_k}| \ ,$$

where each $q_k \in \{n_{k+1} + 1, \dots, n_{k+2} - 1\}$ is defined to be the smallest integer for which

$$b_{q_k} = \max_{n_{k+1}+1 \leq j \leq n_{k+2}-1} b_j \ .$$

Note that $q_k \notin J$, and so $b_{q_k} < L \leq b_{n_{k+1}}$. Furthermore, $b_{n_{k+1}} \leq b_{n_k}$. Hence,

$$\begin{aligned} \|U(x) - U(y)\|_\infty &\leq \sup_{k \in \mathbb{N}} b_{n_{k+1}} |t_{n_k} - s_{n_k}| \\ &\leq \sup_{k \in \mathbb{N}} b_{n_k} |t_{n_k} - s_{n_k}| \leq \|x - y\|_\infty \ . \end{aligned}$$

Thus, there exists a mapping $U : E_b \longrightarrow E_b$ that is fixed point-free, affine, and $\|\cdot\|_\infty$-nonexpansive.

The general Case 1:

Let $M_1 := M$. There exists $k' \in \mathbb{N}$ such that $b_{n_{k'}} = M_1 = M$. Let $k_1 := \min\{k' \in \mathbb{N} : b_{n_{k'}} = M\}$. Then $b_{n_{k_1}} = M = M_1$ and $[\, b_j \leq b_{n_{k_1}}, \forall j \geq n_{k_1} + 1\,]$. Let $M_2 := \max\{b_{n_k} : k \geq k_1 + 1\}$ and $k_2 := \min\{k' \geq k_1 + 1 : b_{n_{k'}} = M_2\}$. Note that $b_{n_{k_2}} = M_2$, $[\, b_j \leq b_{n_{k_2}}, \forall j \geq n_{k_2} + 1\,]$ and $b_{n_{k_2}} \leq b_{n_{k_1}}$.

Similarly, set $M_3 := \max\{b_{n_k} : k \geq k_2 + 1\}$ and let $k_3 := \min\{k' \geq k_2 + 1 : b_{n_{k'}} = M_3\}$. Then $b_{n_{k_3}} = M_3$, $[\, b_j \leq b_{n_{k_3}}, \forall j \geq n_{k_3} + 1\,]$ and $b_{n_{k_3}} \leq b_{n_{k_2}}$. We continue this way, and construct a new subsequence. We may relabel the sequence $(b_{n_{k_\nu}})_{\nu \in \mathbb{N}}$ as $(b_{n_\nu})_{\nu \in \mathbb{N}}$. This sequence is decreasing and $(\star_\nu)$ $[\, b_j \leq b_{n_\nu}, \forall j \geq n_\nu + 1, \forall \nu \geq 1\,]$. Redefine J to be $\{n_\nu : \nu \geq 1\}$. Then we define $U : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ exactly as in the case just before this one. Hence, a very similar argument using just that

$$b_{q_k} = \max_{n_{k+1}+1 \leq j \leq n_{k+2}-1} b_j \leq b_{n_{k+1}} \ ,$$

via $(\star_{k+1})$, implies that U is fixed point-free, affine, and $\|\cdot\|_\infty$-nonexpansive.

Case 2: $J := \{n \in \mathbb{N} : b_n \geq L\}$ is finite.

There exists $j_0 \in \mathbb{N}$ such that $b_j < L$, for all $j \geq j_0$. We may assume j_0 is smallest possible. Let $n_1 := j_0$. Note that $b_{n_1} < L := \limsup_{n \to \infty} b_n$. There exists $j' \in \mathbb{N}$ with $j' \geq n_1 + 1$ such that $b_{n_1} < b_{j'} (< L)$. Let $n_2 := \min\{j' \geq n_1 + 1 : b_{n_1} < b_{j'}\}$. So, $b_{n_1} < b_{n_2}$. Also, for all $j \in \{n_1 + 1, \ldots, n_2 - 1\}$, $b_j \leq b_{n_1}$. Similarly, there exists $j' \geq n_2 + 1$ such that $b_{n_2} < b_{j'} (< L)$. Let $n_3 := \min\{j' \geq n_2 + 1 : b_{n_2} < b_{j'}\}$. So, $b_{n_2} < b_{n_3}$. Further, for all $j \in \{n_2 + 1, \ldots, n_3 - 1\}$, $b_j \leq b_{n_2}$. Continuing inductively, we construct a subsequence $(b_{n_k})_{k \in \mathbb{N}}$ of $(b_n)_{n \in \mathbb{N}}$. Note that $(b_{n_k})_{k \in \mathbb{N}}$ is strictly increasing and

$$(\triangle\triangle) \quad \forall k \in \mathbb{N}, \forall j \in \{n_k + 1, \ldots, n_{k+1} - 1\}, \, b_j \leq b_{n_k} \ .$$

An interesting Case 2 example is the following. For all $\nu \geq 0, \forall j \in \mathbb{N}$,

$$b_{2j-1} := \frac{1}{2}\ ,\ b_{2(2j-1)} := \frac{2}{3}\ ,\ b_{2^2(2j-1)} := \frac{3}{4}\ ,\ \ldots\ ,\ b_{2^\nu(2j-1)} := \frac{\nu+1}{\nu+2}\ .$$

Then, $b_1 = \frac{1}{2}$, $b_2 = \frac{2}{3}$, $b_3 = \frac{1}{2}$, $b_4 = \frac{3}{4}$, $b_5 = \frac{1}{2}$, $b_6 = \frac{2}{3}$, $b_7 = \frac{1}{2}$, $b_8 = \frac{4}{5}$,$b_9 = \frac{1}{2}$, $b_{10} = \frac{2}{3}$, $b_{11} = \frac{1}{2}$, $b_{12} = \frac{3}{4}$, $b_{13} = \frac{1}{2}$, $b_{14} = \frac{2}{3}$, $b_{15} = \frac{1}{2}$, $b_{16} = \frac{5}{6}$, ... Hence, for this example, we see that $L = 1$ and $n_k = 2^{k-1}$, for all $k \in \mathbb{N}$.

We return to our proof of Case 2. Let $J := \{n_k : k \in \mathbb{N}\}$. Fix $x, y \in E_{\overrightarrow{b}}$. As before, we may write $x = \sum_{j=1}^{\infty} t_j f_j$ and $y = \sum_{j=1}^{\infty} s_j f_j$. Then

$$\|x - y\|_\infty = \sup_{k\in\mathbb{N}} b_{n_k}|t_{n_k} - s_{n_k}|\ \vee\ \sup_{j\notin J} b_j|t_j - s_j|\ .$$

Let $\overrightarrow{c} := (b_{n_k})_{k\in\mathbb{N}}$ and define $c_k := b_{n_k}$, for all $k \in \mathbb{N}$. We will apply Theorem 2.2.1 to the sequence $\overrightarrow{c}$, and the corresponding set $\widetilde{E_{\overrightarrow{c}}}$ given by

$$\widetilde{E_{\overrightarrow{c}}} := \left\{ \widetilde{x} = \sum_{k=1}^{\infty} t_{n_k} b_{n_k} e_{n_k} : x = \sum_{j=1}^{\infty} t_j b_j e_j \in E_{\overrightarrow{b}} \right\}\ .$$

By Theorem 2.2.1 and its proof, there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $W : \widetilde{E_{\overrightarrow{c}}} \longrightarrow \widetilde{E_{\overrightarrow{c}}}$ that is fixed point free. We will write W as

$$\widetilde{x} = \sum_{k=1}^{\infty} t_{n_k} b_{n_k} e_{n_k} \mapsto W(\widetilde{x}) = \sum_{k=1}^{\infty} w_k(\widetilde{x}) b_{n_k} e_{n_k} = \sum_{k=1}^{\infty} w_k b_{n_k} e_{n_k}\ ,$$

where for each $\widetilde{x} \in \widetilde{E_{\overrightarrow{c}}}$, the coefficients $w_k = w_k(\widetilde{x})$ are such that $(w_k)_{k\in\mathbb{N}} \in c_0$. Analogously to (□□) above, we define $U : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ by

$U(x) := (b_1(1), b_2(1), \ldots\ldots, b_{n_1-1}(1), b_{n_1}w_1, b_{n_1+1}w_1, \ldots\ldots, b_{n_2-1}w_1, b_{n_2}w_2,$
$b_{n_2+1}w_2, \ldots\ldots, b_{n_k-1}w_{(k-1)}, b_{n_k}w_k, b_{n_k+1}w_k, \ldots\ldots, b_{n_{(k+1)}-1}w_k,$
$b_{n_{(k+1)}}w_{(k+1)}, b_{n_{(k+1)}+1}w_{(k+1)}, \ldots\ldots)\ .$

Then U is affine and fixed point free. Indeed, $x = U(x) \implies \widetilde{x} = W(\widetilde{x})$, which yields contradiction. Next, we show U is nonexpansive.

$$\|U(x) - U(y)\|_\infty = \sup_{k\in\mathbb{N}} b_{n_k}|w_k(\widetilde{x}) - w_k(\widetilde{y})|\ \vee\ \sup_{k\in\mathbb{N}} b_{q_k}|w_k(\widetilde{x}) - w_k(\widetilde{y})|\ ,$$

where each $q_k \in \{n_k + 1, \ldots, n_{k+1} - 1\}$ is defined to be the smallest integer for which

$$b_{q_k} = \max_{n_k+1\leq j\leq n_{(k+1)}-1} b_j\ .$$

Note that each $b_{q_k} \leq b_{n_k}$, by fact $(\triangle\triangle)$ above. So,

$$\begin{aligned} \|U(x) - U(y)\|_\infty &\leq \|W(\widetilde{x}) - W(\widetilde{y})\|_\infty \ \vee \ \|W(\widetilde{x}) - W(\widetilde{y})\|_\infty \\ &\leq \|\widetilde{x} - \widetilde{y}\|_\infty = \sup_{k \in \mathbb{N}} b_{n_k} |t_{n_k} - s_{n_k}| \\ &\leq \|x - y\|_\infty \ . \end{aligned}$$

In summary, in all possible cases, we have constructed an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ that is fixed point free.

By using Theorem 2.3.2 instead of Theorem 2.2.1 in Case 2 above, and analogously using Theorem 2.3.2 in Case 1 above, we can construct an affine $\|\cdot\|_\infty$-contractive mapping $U : E_{\overrightarrow{b}} \longrightarrow E_{\overrightarrow{b}}$ that is fixed point free. (When $x \neq y$, there are two possibilities: $\widetilde{x} \neq \widetilde{y}$ and $\widetilde{x} = \widetilde{y}$. In the second situation, the last inequality immediately above is strict.)

The proof is complete. □

Fix $0 < m < M < \infty$. Note that Theorem 2.4.1 applies to the example: $[\, b_n := r_n$, for all $n \in \mathbb{N}\,]$, where $(r_n)_{n\in\mathbb{N}}$ is an enumeration of $\mathbb{Q} \cap [m, M)$.

Open Question (1) Let $(X, \|\cdot\|)$ be a Banach space that contains a c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, and define the closed convex hull of $(x_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. Then can we find an affine $\|\cdot\|$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free?

Open Question (2) In 2004 Dowling, Lennard and Turett showed that every non-weakly compact, closed, bounded, convex (c.b.c.) subset K of $(c_0, \|\cdot\|_\infty)$ is such that there exists a $\|\cdot\|_\infty$-nonexpansive mapping T on K that is fixed point free. This mapping T is generally not affine. It is an open question as to whether or not on every non-weakly compact, c.b.c. subset K of $(c_0, \|\cdot\|_\infty)$ there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping S that is fixed point free.

3.0 EXPLORING FIXED POINT PROPERTIES FOR CERTAIN c_0-SUMMING BASIC SEQUENCES IN c_0.

In the paper of Llorens-Fuster and Sims, it has been shown that when $(b_n)_{n\in N}$ is a sequence in $[1,\infty)$ that is decreasing sequence and converges to 1, then for the closed, bounded, convex subset $E = E_{\overrightarrow{b}}$ of c_0 defined by $E := \{\sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0\}$, the right shift mapping $S : E \longrightarrow E$ given by $S(\sum_{j=1}^{\infty} t_j\, f_j) = f_1 + \sum_{j=1}^{\infty} t_j\, f_{j+1}$, is a fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping. To ponder different cases, we can consider different patterns. For example, as the simplest case, consider the oscillating sequence $(b_n)_{n\in N}$ defined by $(b_n)_{n\in N} := (1, b, 1, b, 1, b, ...)$. Then, the composite mapping $SoS : E \longrightarrow E$ is a fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping.

We want to ponder different types, and in future work aim to investigate what happens if a general Banach space contains a c_0-summing basic sequence. To gain some insight we will work on the following problems. From very simple cases to more general case as in the following, we will investigate c_0-summing basic sequences.

Now, let

$$\begin{aligned}
\eta_1 &:= \gamma_1 b_1 e_1 \\
\eta_2 &:= \gamma_2 (b_1 e_1 + b_2 e_2) \\
\eta_3 &:= \gamma_3 (b_1 e_1 + b_2 e_2 + b_3 e_3) \\
&\vdots \\
\eta_n &:= \gamma_n (b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + ... + b_n e_n) \\
&\vdots
\end{aligned}$$

where for some $\Gamma > 0$, $\Gamma \leq \gamma_N, \forall N \in \mathbb{N}$, and $\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$; while

$$\overrightarrow{b} = (b_n)_{n\in\mathbb{N}} \subset \mathbb{R} \text{ with } 0 < m := \inf_{n\in\mathbb{N}} b_n \text{ and } M := \sup_{n\in\mathbb{N}} b_n < \infty \ .$$

Fix $N \in \mathbb{N}$. Then,

$$|\gamma_1 - \gamma_N| = \left|\sum_{n=2}^{N} (\gamma_n - \gamma_{n-1})\right| \leq \sum_{n=2}^{N} |\gamma_n - \gamma_{n-1}| \leq \sigma < \infty$$

Hence, $\forall N \in \mathbb{N}$, $\gamma_1 - \sigma \leq \gamma_N \leq \gamma_1 + \sigma$.

Indeed, $(\gamma_n)_{n\in\mathbb{N}}$ *converges to some* $L \in [\gamma_1 - \sigma, \gamma_1 + \sigma]$, since $\forall m > n$ *in* $\mathbb{N}$,

$$\begin{aligned}
|\gamma_m - \gamma_n| &= |(\gamma_m - \gamma_{m-1}) + (\gamma_{m-1} - \gamma_{m-2}) + .. + (\gamma_{n+2} - \gamma_{n+1}) + (\gamma_{n+1} - \gamma_n)| \\
&\leq \sum_{j=n+1}^{m} |\, \gamma_j - \gamma_{j-1} \,| \\
&\leq \sum_{j=n+1}^{\infty} |\, \gamma_j - \gamma_{j-1} \,| \to 0 \ as\ n \to \infty \ .
\end{aligned}$$

Note that $L \geq \Gamma > 0$. We see that if the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is strictly decreasing, then $\sigma = \gamma_1 - L$ and $\sigma < \gamma_1$. If $(\gamma_n)_{n\in\mathbb{N}}$ is increasing, then $\sigma = L - \gamma_1$ similarly, and in that case $\sigma < \gamma_1 \Leftrightarrow L < 2\gamma_1$. We note that $\sigma < \gamma_1$ generally fails. Indeed, consider $\gamma_1 = 1$, $\gamma_2 = 1 + \frac{1}{2}$, $\gamma_3 = 1$, $\gamma_4 = 1 - \frac{1}{4}$, $\gamma_5 = 1$, $\gamma_6 = 1 + \frac{1}{8}$, $\gamma_7 = 1$, $\gamma_8 = 1 + \frac{1}{16}$, ... Then, $\sigma = 2 > \gamma_1 > \Gamma := \frac{3}{4}$.

Furthermore, we note that it is not true that every convergent $(\gamma_n)_{n\in\mathbb{N}}$ sequence in $[\Gamma, \infty)$ $(\Gamma > 0)$ is of the above form. Indeed, consider $\gamma_1 = 1$, $\gamma_2 = 1 + \frac{1}{2}$, $\gamma_3 = 1$, $\gamma_4 = 1 - \frac{1}{3}$, $\gamma_5 = 1$, $\gamma_6 = 1 + \frac{1}{4}$, $\gamma_7 = 1$, $\gamma_8 = 1 - \frac{1}{5}$, $\gamma_9 = 1, ...$ Then,

$$\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| = \sum_{n=2}^{\infty} \frac{2}{n} = \infty \, .$$

Now, under the influence of these observations, let $t = (t_j)_{j\in\mathbb{N}} \in c_{00}$. Then,

$$
\begin{aligned}
\Lambda \;&:=\; \left\|\sum_{j=1}^{\infty} t_j\,\eta_j\right\|_{\infty} = \|t_1\,\eta_1 + t_2\,\eta_2 + t_3\,\eta_3 + \ldots\|_{\infty} \\
&= \left\| \begin{array}{c} e_1\,(\gamma_1\,b_1\,t_1 + \gamma_2\,b_1\,t_2 + \gamma_3\,b_1\,t_3 + \gamma_4\,b_1\,t_4 + \ldots\ldots\ldots) \\ +e_2\,(\gamma_2\,b_2\,t_2 + \gamma_3\,b_2\,t_3 + \gamma_4\,b_2\,t_4 + \gamma_5\,b_2\,t_5 + \ldots\ldots\ldots) \\ +e_3\,(\gamma_3\,b_3\,t_3 + \gamma_4\,b_3\,t_4 + \gamma_5\,b_3\,t_5 + \gamma_6\,b_3\,t_6 + \ldots) + \ldots \end{array} \right\|_{\infty} \\
&= b_1\,|\gamma_1\,t_1 + \gamma_2\,t_2 + \gamma_3\,t_3 + \gamma_4\,t_4 + \ldots| \\
&\vee\; b_2\,|\gamma_2\,t_2 + \gamma_3\,t_3 + \gamma_4\,t_4 + \gamma_5\,t_5 + \ldots| \\
&\vee\; b_3\,|\gamma_3\,t_3 + \gamma_4\,t_4 + \gamma_5\,t_5 + \gamma_6\,t_6 + \ldots| \;\vee \ldots\;.
\end{aligned}
$$

But,

$$
\begin{aligned}
\sum_{j=1}^{\infty} \gamma_j\,t_j \;&=\; \gamma_1\,t_1 + \gamma_2\,t_2 + \gamma_3\,t_3 + \gamma_4\,t_4 + \ldots \\
&= \gamma_1(t_1 + t_2 + t_3 + t_4 + \ldots) \\
&\quad -\gamma_1(t_2 + t_3 + t_4 + \ldots) \\
&\quad +\gamma_2(t_2 + t_3 + t_4 + \ldots) \\
&\quad -\gamma_2(t_3 + t_4 + \ldots) \\
&\quad +\gamma_3(t_3 + t_4 + \ldots) \\
&\quad -\gamma_3(t_4 + t_5 + \ldots) \\
&\quad +\gamma_4(t_4 + t_5 + \ldots) \\
&\quad - \ldots \\
&= \gamma_1 \sum_{j=1}^{\infty} t_j \;+ (\gamma_2 - \gamma_1) \sum_{j=2}^{\infty} t_j \;+ (\gamma_3 - \gamma_2) \sum_{j=3}^{\infty} t_j \;+ (\gamma_4 - \gamma_3) \sum_{j=4}^{\infty} t_j \;+ \ldots
\end{aligned}
$$

Hence,

$$
\sum_{j=1}^{\infty} \gamma_j\,t_j \;=\; \gamma_1 \sum_{k=1}^{\infty} t_k + \sum_{n=2}^{\infty}(\gamma_n - \gamma_{n-1}) \sum_{k=n}^{\infty} t_k
$$

Similarly,

$$
\begin{aligned}
\sum_{j=2}^{\infty} \gamma_j\,t_j \;&=\; \gamma_2 \sum_{k=2}^{\infty} t_k + \sum_{n=3}^{\infty}(\gamma_n - \gamma_{n-1}) \sum_{k=n}^{\infty} t_k \\
&\;\;\vdots
\end{aligned}
$$

$\forall \nu \in \mathbb{N}$,

$$\sum_{j=\nu}^{\infty} \gamma_j \, t_j \;\; = \;\; \gamma_\nu \sum_{k=\nu}^{\infty} t_k \, + \sum_{n=\nu+1}^{\infty} (\gamma_n - \gamma_{n-1}) \sum_{k=n}^{\infty} t_k$$

Thus,

$$\begin{aligned} \Lambda \;\; &:= \;\; \left\| \sum_{j=1}^{\infty} t_j \, \eta_j \right\|_\infty \\ &= \;\; \sup_{\nu \in \mathbb{N}} b_\nu \left| \gamma_\nu \left(\sum_{k=\nu}^{\infty} t_k \right) + \sum_{n=\nu+1}^{\infty} (\gamma_n - \gamma_{n-1}) \sum_{k=n}^{\infty} t_k \right| \end{aligned}$$

Hence,

$$\begin{aligned} \Lambda \;\; &\leq \;\; \sup_{\nu \in \mathbb{N}} b_\nu \left[\gamma_\nu \left| \sum_{k=\nu}^{\infty} t_k \right| + \sum_{n=\nu+1}^{\infty} |\gamma_n - \gamma_{n-1}| \left| \sum_{k=n}^{\infty} t_k \right| \right] \\ &\leq \;\; \sup_{\nu \in \mathbb{N}} b_\nu \left[\gamma_\nu + \sum_{n=\nu+1}^{\infty} |\gamma_n - \gamma_{n-1}| \right] \sup_{\mu \geq \nu} \left| \sum_{k=\mu}^{\infty} t_k \right| \\ &\leq \;\; M \, (T + \sigma) \sup_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} t_k \right| \end{aligned}$$

where $\gamma_\nu \leq T < \infty \, , \forall \nu \in \mathbb{N}$, since the sequence $(\gamma_n)_{n \in \mathbb{N}}$ is convergent by our hypotheses. Let's define

$$\tau := \sup_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} t_k \right| \quad \text{and } \tau_\nu := \sup_{\mu \geq \nu} \left| \sum_{k=\mu}^{\infty} t_k \right| , \forall \, \nu \in \mathbb{N} \, .$$

On the other hand, for all $\nu \in \mathbb{N}$,

$$\Lambda \geq b_\nu \, \gamma_\nu \left| \sum_{k=\nu}^{\infty} t_k \right| - b_\nu \sum_{n=\nu+1}^{\infty} |\gamma_n - \gamma_{n-1}| \sup_{\mu \geq \nu+1} \left| \sum_{k=\mu}^{\infty} t_k \right| .$$

Now, note that for some $\nu_0 \in \mathbb{N}$

$$\tau = \sup_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} t_k \right| = \max_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} t_k \right| = \left| \sum_{k=\nu_0}^{\infty} t_k \right| \;\; .$$

Then,

$$\begin{aligned}
\Lambda &\geq b_{\nu_0}\,\gamma_{\nu_0}\left|\sum_{k=\nu_0}^{\infty} t_k\right| - b_{\nu_0}\sum_{n=\nu_0+1}^{\infty}|\gamma_n-\gamma_{n-1}|\sup_{\mu\geq\nu_0+1}\left|\sum_{k=\mu}^{\infty} t_k\right| \\
&\geq b_{\nu_0}\,\gamma_{\nu_0}\tau - b_{\nu_0}\sum_{n=\nu_0+1}^{\infty}|\gamma_n-\gamma_{n-1}|\,\tau \\
&= b_{\nu_0}\left[\gamma_{\nu_0} - \sum_{n=\nu_0+1}^{\infty}|\gamma_n-\gamma_{n-1}|\right]\tau \\
&\geq m\left[\gamma_{\nu_0} - \sum_{n=\nu_0+1}^{\infty}|\gamma_n-\gamma_{n-1}|\right]\tau\ .
\end{aligned}$$

Hence, if $\sigma<\Gamma$, then

$$\Lambda = \left\|\sum_{j=1}^{\infty} t_j\,\eta_j\right\|_{\infty} \geq m(\Gamma-\sigma)\sup_{s\in\mathbb{N}}\left|\sum_{k=s}^{\infty} t_k\right|\ .$$

In summary, suppose that

$$(\gamma_n)_{n\in\mathbb{N}}\,\text{is a sequence in}\,[\Gamma,\infty)\,\text{with}\,\Gamma>0\,\text{where}\,\Gamma\leq\gamma_n,\forall n\in\mathbb{N}\text{ such that}$$

$$\sigma := \sum_{n=2}^{\infty}|\gamma_n-\gamma_{n-1}| < \infty\,\text{with}\,\Gamma>\sigma\ ;$$

and also, suppose that

$$\overrightarrow{b} = (b_n)_{n\in\mathbb{N}} \subset \mathbb{R}\text{ with } 0<m:=\inf_{n\in\mathbb{N}} b_n\text{ and } M:=\sup_{n\in\mathbb{N}} b_n < \infty\ .$$

Then, $(\gamma_n)_{n\in\mathbb{N}}$ is convergent and there exists $T>0$ with $T\geq\gamma_n$, $\forall n\in\mathbb{N}$. Moreover,

$$m(\Gamma-\sigma)\sup_{s\in\mathbb{N}}\left|\sum_{k=s}^{\infty} t_k\right| \leq \left\|\sum_{j=1}^{\infty} t_j\,\eta_j\right\|_{\infty} \leq M\,(T+\sigma)\sup_{s\in\mathbb{N}}\left|\sum_{k=s}^{\infty} t_k\right|,$$

for all $t=(t_j)_{j\in\mathbb{N}}\in c_{00}$.

Hence, $(\eta_n)_{n\in\mathbb{N}}$ is a c_0-summing basic sequence.

3.1 OUR CONSTRUCTION WITH DECREASING γ_n'S AND b_n'S , AND ASYMPTOTICALLY NONEXPANSIVE MAPPINGS.

Definition 3.1.1. Asymptotically Nonexpansive Mapping

Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. A mapping $T : C \longrightarrow C$ is said to be asymptotically nonexpansive if $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$, where $(k_n)_{n\in\mathbb{N}}$ is a sequence in $[1, \infty)$ converging to 1.

Theorem 3.1.2. *Let* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *be any decreasing sequence (i.e.,* $b_n \geq b_{n+1}$*, for all* $n \in \mathbb{N}$*) in* $(1, \infty)$ *with* $b_n \downarrow_n 1$*.Let* $(\gamma_n)_{n\in\mathbb{N}}$ *be another sequence such that*

$$\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$$

with $\vec{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ *a decreasing sequence converging to 1 (i.e.,* $\gamma_n \geq \gamma_{n+1}$*, for all* $n \in \mathbb{N}$ *in* $(1, \infty)$ *with* $\gamma_n \downarrow_n 1$*). We define the sequence* $(\eta_n)_{n\in\mathbb{N}}$ *by setting*

$$\eta_n := \gamma_n(b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + ... + b_n e_n), \text{ for all } n \in \mathbb{N} .$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ *is a* c_0*-summing basic sequence. Furthermore, define the closed convex hull of* $(\eta_n)_{n\in\mathbb{N}}$*,* $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$*. Then, there exists an affine* $\|\cdot\|_\infty$*-asymptotically nonexpansive mapping* $T : E \longrightarrow E$ *that is fixed point free. Moreover,* T *is the usual right shift map.*

Proof. Similarly to our previous work, we can show that $\forall Q \in \mathbb{N}$, and $\forall (t_n)_{n\in\mathbb{N}} \in \ell^1$,

$$\Lambda_Q := \left\| \sum_{j=Q}^{\infty} t_j \, \eta_j \right\|_\infty \leq b_Q(\gamma_Q + \sigma_{Q+1}) \sup_{s\geq Q} \left| \sum_{k=s}^{\infty} t_k \right| ,$$

where

$$\sigma_{Q+1} = \sum_{n=Q+1}^{\infty} |\gamma_n - \gamma_{n-1}| .$$

Now, let

$$x = \sum_{n=1}^{\infty} t_n \, \eta_n \in E$$

and

$$y = \sum_{n=1}^{\infty} s_n \, \eta_n \in E \ ;$$

so that $t_n, s_n \geq 0$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1 \ .$$

Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$. Then, consider the right shift $T : E \longrightarrow E$ is given by

$$T\left(\sum_{j=1}^{\infty} t_j \, \eta_j\right) = \sum_{j=1}^{\infty} t_j \, \eta_{j+1} \ .$$

Then, for any $Q \in \mathbb{N}$, $U := T^Q : E \longrightarrow E$ is given by

$$U\left(\sum_{j=1}^{\infty} t_j \, \eta_j\right) = \sum_{j=1}^{\infty} t_j \, \eta_{j+Q} \ .$$

Since the sequences $(b_j)_{j\in\mathbb{N}}$ and $(\gamma_j)_{j\in\mathbb{N}}$ are decreasing sequences,

$$\begin{aligned}
\|U(x) - U(y)\|_\infty &= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_{j+Q} \right\|_\infty \\
&= \left\| \sum_{k=Q+1}^{\infty} \alpha_{k-Q} \, \eta_k \right\|_\infty \\
&\leq b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2}) \sup_{s \geq Q+1} \left| \sum_{k=s}^{\infty} \alpha_{k-Q} \right| \\
&= b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2}) \sup_{s \geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| \ .
\end{aligned}$$

Also, for some $\nu_0 \in \mathbb{N}$

$$\tau := \sup_{s\in\mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| = \left| \sum_{k=\nu_0}^{\infty} \alpha_k \right|$$

and so

$$
\begin{aligned}
\|x-y\|_\infty &= \left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_\infty \\
&\geq b_{\nu_0}\, \gamma_{\nu_0} \left|\sum_{k=\nu_0}^{\infty} t_k\right| - b_{\nu_0} \sum_{n=\nu_0+1}^{\infty} |\gamma_n - \gamma_{n-1}| \sup_{\mu \geq \nu_0+1} \left|\sum_{k=\mu}^{\infty} t_k\right| \\
&\geq b_{\nu_0}\, \gamma_{\nu_0} \tau \, - b_{\nu_0} \sum_{n=\nu_0+1}^{\infty} |\gamma_n - \gamma_{n-1}| \, \tau \\
&= b_{\nu_0} \left[\gamma_{\nu_0} \, - \sum_{n=\nu_0+1}^{\infty} |\gamma_n - \gamma_{n-1}|\right] \tau \\
&= b_{\nu_0} \left[\gamma_{\nu_0} \, - \sum_{n=\nu_0+1}^{\infty} (\gamma_{n-1} - \gamma_n)\right] \tau \text{ since } \gamma_n \downarrow_n 1\, ; \\
&= b_{\nu_0}\, \tau \geq \tau\, , \text{ since } b_n \downarrow_n \, 1\, .
\end{aligned}
$$

Then, since the sequences $b_n \downarrow_n \, 1$, $\gamma_n \downarrow_n \, 1$ and $\sigma_n \underset{n}{\longrightarrow} 0$, there exists a sequence $(\zeta_Q)_{Q\in\mathbb{N}}$ in $(0,\infty)$ such that $\zeta_Q := b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2}) \to 1$ as $Q \to \infty$, and

$$
\|T^Q(x) - T^Q(y)\|_\infty \leq \zeta_Q \|x-y\|_\infty \, \forall \, Q \in \mathbb{N}.
$$

Hence, T is fixed point free, affine, $\|\cdot\|_\infty$-asymptotically nonexpansive mapping. $\square$

3.2 OUR CONSTRUCTION WITH CONVERGING γ_n'S AND b_n'S , AND SEMI-STRONGLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS.

Definition 3.2.1. Strongly Asymptotically Nonexpansive Mapping

Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. We will say a mapping $T: C \to C$ is strongly asymptotically nonexpansive if $\exists \{\beta_{n,m} : n, m \in \mathbb{N}, n \geq m \geq 0\} \subseteq [1, \infty)$ such that $\forall x, y \in C$ and $\forall\, n > m$,

$\|T^n x - T^n y\| \leq \beta_{n,m} \|T^m x - T^m y\|$

where $[\beta_{n,m} \to 1$ as $n \geq m \to \infty]$ and $[\beta_{n,m} \to 1$ as $n \to \infty$, $\forall m]$.

Note that the examples $[\beta_{n,m} := \frac{(1+\frac{1}{2^n})}{(1-\frac{1}{2^{m+1}})}]$ and $[\beta_{m+1,m} := 2$, $\beta_{n,m} := 1, \forall\, n \neq m+1]$ show that neither of the above convergence conditions implies the other.

Definition 3.2.2. Semi-strongly Asymptotically Nonexpansive Mapping

Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. We will say a mapping $T: C \to C$ is semi-strongly asymptotically nonexpansive if $\exists \{\lambda_{n,m} : n, m \in \mathbb{N}, n \geq m \geq 0\} \subseteq [1, \infty)$ such that $\forall x, y \in C$ and $\forall\, n > m$, $\|T^n x - T^n y\| \leq \lambda_{n,m} \|T^m x - T^m y\|$ where $[\lambda_{n,m} \to 1$ as $n \geq m \to \infty]$.

Lemma 3.2.3. *Let $(X, \|\cdot\|)$ be a general Banach space, and suppose that $C \subseteq X$ is a closed bounded convex subset. Assume the mapping $T : C \longrightarrow C$ is strongly asymptotically nonexpansive.Then, T is asymptotically nonexpansive and also semi-strongly asymptotically nonexpansive.*

Proof. Indeed, it is clear that if T is strongly asymptotically nonexpansive, then it is semi-strongly asymptotically nonexpansive, and also by choosing $m = 0$, it is asymptotically nonexpansive. □

Theorem 3.2.4. *Consider our expanded "set-up", with η_n's defined in terms of sequences of constants: γ_n's and b_n's as in previous investigations and so $(\eta_n)_{n\in\mathbb{N}}$ turns out to be a c_0-summing basic sequence, and we define the c.b.c. subset E of c_0 by $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$. Suppose we are in the case where $b_n \underset{n}{\longrightarrow} 1$ and $\gamma_n \underset{n}{\longrightarrow} 1$ Then, the right shift map*

$T : E \longrightarrow E$ is semi-strongly asymptotically nonexpansive. Here, note that T is fixed-point free and affine. (If $b_n \downarrow_n 1$ and $\gamma_n \downarrow_n 1$, then the right shift map $T : E \longrightarrow E$ is strongly asymptotically nonexpansive and fixed-point free.)

Hence, we have the following. Let $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ be two sequences with $b_n \underset{n}{\longrightarrow} 1$ and $\gamma_n \underset{n}{\longrightarrow} 1$ such that

$$\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty \ .$$

We define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\eta_n := \gamma_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n), \ \textit{for all } n \in \mathbb{N} \ .$$

Also suppose $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate. Furthermore, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, Then, there exists an affine $\|\cdot\|_\infty$-semi-strongly asymptotically nonexpansive mapping $T : E \longrightarrow E$ that is fixed point free. Moreover, T is the usual right shift map.

Proof. Note that

$$\overrightarrow{b} = (b_n)_{n\in\mathbb{N}} \subset \mathbb{R} \text{ with } 0 < m := \inf_{n\in\mathbb{N}} b_n \text{ and } M := \sup_{n\in\mathbb{N}} b_n < \infty \ .$$

Case 1: Let $(b_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ be decreasing to 1. I.e. $b_n \downarrow_n 1$ and $\gamma_n \downarrow_n 1$ Then, similarly to the proof of Theorem 3.1.2, consider the right shift mapping $T : E \longrightarrow E$ and let

$$x = \sum_{n=1}^{\infty} t_n \, \eta_n \in E$$

and

$$y = \sum_{n=1}^{\infty} s_n \, \eta_n \in E \ ;$$

so that $t_n, s_n \geq 0$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1 \ .$$

Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$. Then, for any $Q \in \mathbb{N}$, since the sequences $(b_j)_{j\in\mathbb{N}}$ and $(\gamma_j)_{j\in\mathbb{N}}$ are decreasing sequences,

$$\begin{aligned}
\|T^Q(x) - T^Q(y)\|_\infty &= \left\| \sum_{j=1}^{\infty} \alpha_j \eta_{j+Q} \right\| \\
&= \left\| \sum_{k=Q}^{\infty} \alpha_{k-Q} \eta_k \right\| \\
&\leq b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2}) \sup_{s \geq Q+1} \left| \sum_{k=s}^{\infty} \alpha_{k-Q} \right| \\
&= b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2}) \sup_{s \geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| .
\end{aligned}$$

Thus, for any $Q \in \mathbb{N}$,

$$\|T^Q(x) - T^Q(y)\|_\infty \leq b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2}) \sup_{s \geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| . \tag{3.1}$$

Also, note that, for some $\nu_0 \in \mathbb{N}$

$$\tau := \sup_{s\in\mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| = \left| \sum_{k=\nu_0}^{\infty} \alpha_k \right| \quad \text{and } \tau_\nu := \sup_{\mu \geq \nu} \left| \sum_{k=\mu}^{\infty} \alpha_k \right| , \forall\, \nu \in \mathbb{N} .$$

Now, fix $m \in \mathbb{N}$.

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &:= \left\| \sum_{j=m+1}^{\infty} \alpha_{j-m}\, \eta_j \right\|_\infty \\
&= \sup_{\nu \geq m+1} b_\nu \left| \gamma_\nu \left(\sum_{k=\nu}^{\infty} \alpha_{k-m} \right) + \sum_{r=\nu+1}^{\infty} (\gamma_r - \gamma_{r-1}) \sum_{k=r}^{\infty} \alpha_{k-m} \right|
\end{aligned}$$

Then, for all $n > m$

$$\|T^m(x) - T^m(y)\|_\infty \geq b_n\, \gamma_n \left| \sum_{k=n}^{\infty} \alpha_{k-m} \right| - b_n \sum_{r=n+1}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu \geq n+1} \left| \sum_{k=\mu}^{\infty} \alpha_{k-m} \right|$$

Then; for example, if $n = m + 1$,

$$\begin{aligned}\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+1}\,\gamma_{m+1}\left|\sum_{k=m+1}^{\infty} \alpha_{k-m}\right| - b_{m+1}\sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu\geq m+2}\left|\sum_{k=\mu}^{\infty} \alpha_{k-m}\right| \\ &= b_{m+1}\,\gamma_{m+1}\left|\sum_{k=1}^{\infty} \alpha_k\right| - b_{m+1}\sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu\geq 2}\left|\sum_{k=\mu}^{\infty} \alpha_k\right| \\ &\geq b_{m+1}\,\gamma_{m+1}\left|\sum_{k=1}^{\infty} \alpha_k\right| - b_{m+1}\sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}|\,\tau\end{aligned}$$

Now, define for $s \in \mathbb{N}$

$$\rho_s := \left|\sum_{k=s}^{\infty} \alpha_k\right| \ .$$

Hence,

$$\|T^m(x) - T^m(y)\|_\infty \geq b_{m+1}\,\gamma_{m+1}\rho_1 \ - b_{m+1}\sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}|\,\tau.$$

if $n = m + 2$,

$$\begin{aligned}\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+2}\,\gamma_{m+2}\left|\sum_{k=m+2}^{\infty} \alpha_{k-m}\right| - b_{m+2}\sum_{r=m+3}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu\geq m+3}\left|\sum_{k=\mu}^{\infty} \alpha_{k-m}\right| \\ &= b_{m+2}\,\gamma_{m+2}\left|\sum_{k=2}^{\infty} \alpha_k\right| - b_{m+2}\sum_{r=m+3}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu\geq 3}\left|\sum_{k=\mu}^{\infty} \alpha_k\right| \\ &\geq b_{m+2}\,\gamma_{m+2}\left|\sum_{k=2}^{\infty} \alpha_k\right| - b_{m+2}\sum_{r=m+3}^{\infty} |\gamma_r - \gamma_{r-1}|\,\tau \\ &= b_{m+2}\,\gamma_{m+2}\rho_2 \ - b_{m+2}\sum_{r=m+3}^{\infty} |\gamma_r - \gamma_{r-1}|\,\tau \ .\end{aligned}$$

Hence, inductively we get for all $j \in \mathbb{N}$,

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+j}\,\gamma_{m+j}\left|\sum_{k=m+j}^{\infty}\alpha_{k-m}\right| - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\sup_{\mu\geq m+j+1}\left|\sum_{k=\mu}^{\infty}\alpha_{k-m}\right| \\
&= b_{m+j}\,\gamma_{m+j}\left|\sum_{k=j}^{\infty}\alpha_k\right| - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\sup_{\mu\geq j+1}\left|\sum_{k=\mu}^{\infty}\alpha_k\right| \\
&\geq b_{m+j}\,\gamma_{m+j}\left|\sum_{k=j}^{\infty}\alpha_k\right| - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+j}\,\gamma_{m+j}\rho_j - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+\nu_0}\,\gamma_{m+\nu_0}\,\rho_{\nu_0} - b_{m+\nu_0}\sum_{r=m+\nu_0+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+\nu_0}\,\gamma_{m+\nu_0}\,\tau - b_{m+\nu_0}\sum_{r=m+\nu_0+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+\nu_0}\,\tau,\ \text{since}\ \gamma_n \downarrow_n 1\,; \\
&\geq \tau,\ \text{since } b_n \text{ is decreasing to 1 };
\end{aligned}$$

Then, combining this result with (3.1), we have that for all $Q > m$,

$$\|T^Q x - T^Q y\|_\infty \leq b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2})\|T^m x - T^m y\|_\infty\ .$$

Thus, let $\beta_{Q,m} := b_{Q+1}(\gamma_{Q+1} + \sigma_{Q+2})$.

Then Definition 3.2.1 is satisfied. So, T is strongly asymptotically nonexpansive (and so T is semi-strongly asymptotically nonexpansive).

Case 2: Let $(b_n)_{n\in\mathbb{N}}$ be increasing to 1 and $(\gamma_n)_{n\in\mathbb{N}}$ be decreasing to 1. I.e. $b_n \uparrow_n 1$ and $\gamma_n \downarrow_n 1$. Then, for any $x, y \in E$, write

$$x = \sum_{n=1}^{\infty} t_n\,\eta_n \in E$$

and

$$y = \sum_{n=1}^{\infty} s_n \, \eta_n \in E \ ;$$

so that $t_n, s_n \geq 0$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1 \ .$$

Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$ Then, consider the right shift $T : E \longrightarrow E$ given by

$$T\left(\sum_{j=1}^{\infty} t_j \, \eta_j\right) = \sum_{j=1}^{\infty} t_j \, \eta_{j+1} \ .$$

Then, for any $n \in \mathbb{N}$, $T^n : E \longrightarrow E$ is given by

$$T^n\left(\sum_{j=1}^{\infty} t_j \, \eta_j\right) = \sum_{j=1}^{\infty} t_j \, \eta_{j+n} \ .$$

Since $(b_j)_{j \in \mathbb{N}}$ is increasing to 1 , $b_n \leq 1$, $\forall n \in \mathbb{N}$ and since $(\gamma_j)_{j \in \mathbb{N}}$ is decreasing,

$$\begin{aligned}
\|T^n(x) - T^n(y)\|_\infty &= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_{j+n} \right\|_\infty && (3.2)\\
&= \left\| \sum_{k=n+1}^{\infty} \alpha_{k-n} \, \eta_k \right\|_\infty && (3.3)\\
&\leq (\gamma_{n+1} + \sigma_{n+2}) \sup_{s \geq n+1} \left| \sum_{k=s}^{\infty} \alpha_{k-n} \right| && (3.4)\\
&= (\gamma_{n+1} + \sigma_{n+2}) \sup_{s \geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| ; && (3.5)
\end{aligned}$$

Furthermore, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\Lambda_n &:= \left\| \sum_{j=n}^{\infty} \alpha_j \, \eta_j \right\|_\infty \\
&= \sup_{\nu \geq n} b_\nu \left| \gamma_\nu \left(\sum_{k=\nu}^{\infty} \alpha_k \right) + \sum_{r=\nu+1}^{\infty} (\gamma_r - \gamma_{r-1}) \sum_{k=r}^{\infty} \alpha_k \right| \ .
\end{aligned}$$

Note that, for some $\nu_0 \in \mathbb{N}$,

$$\tau := \sup_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| = \left| \sum_{k=\nu_0}^{\infty} \alpha_k \right| \quad \text{and } \tau_\nu := \sup_{\mu \geq \nu} \left| \sum_{k=\mu}^{\infty} \alpha_k \right| , \forall \, \nu \in \mathbb{N} \, .$$

Now, fix $m \in \mathbb{N}$.

$$\begin{aligned}\|T^m(x) - T^m(y)\|_\infty &:= \left\| \sum_{j=m+1}^{\infty} \alpha_{j-m}\, \eta_j \right\|_\infty \\ &= \sup_{\nu \geq m+1} b_\nu \left| \gamma_\nu \left(\sum_{k=\nu}^{\infty} \alpha_{k-m} \right) + \sum_{r=\nu+1}^{\infty} (\gamma_r - \gamma_{r-1}) \sum_{k=r}^{\infty} \alpha_{k-m} \right|\end{aligned}$$

Then, for all $n > m$,

$$\|T^m(x) - T^m(y)\|_\infty \geq b_n\, \gamma_n \left| \sum_{k=n}^{\infty} \alpha_{k-m} \right| - b_n \sum_{r=n+1}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu \geq n+1} \left| \sum_{k=\mu}^{\infty} \alpha_{k-m} \right|$$

Then; for example, if $n = m+1$,

$$\begin{aligned}\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+1}\, \gamma_{m+1} \left| \sum_{k=m+1}^{\infty} \alpha_{k-m} \right| - b_{m+1} \sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu \geq m+2} \left| \sum_{k=\mu}^{\infty} \alpha_{k-m} \right| \\ &= b_{m+1}\, \gamma_{m+1} \left| \sum_{k=1}^{\infty} \alpha_k \right| - b_{m+1} \sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}| \sup_{\mu \geq 2} \left| \sum_{k=\mu}^{\infty} \alpha_k \right| \\ &\geq b_{m+1}\, \gamma_{m+1} \left| \sum_{k=1}^{\infty} \alpha_k \right| - b_{m+1} \sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}|\, \tau \ .\end{aligned}$$

Now, define for $s \in \mathbb{N}$,

$$\rho_s := \left| \sum_{k=s}^{\infty} \alpha_k \right| \ .$$

Hence,

$$\|T^m(x) - T^m(y)\|_\infty \geq b_{m+1}\, \gamma_{m+1} \rho_1 \ - b_{m+1} \sum_{r=m+2}^{\infty} |\gamma_r - \gamma_{r-1}|\, \tau \ .$$

if $n = m + 2$,

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+2}\,\gamma_{m+2}\left|\sum_{k=m+2}^{\infty} \alpha_{k-m}\right| - b_{m+2}\sum_{r=m+3}^{\infty}|\gamma_r - \gamma_{r-1}|\sup_{\mu\geq m+3}\left|\sum_{k=\mu}^{\infty}\alpha_{k-m}\right| \\
&= b_{m+2}\,\gamma_{m+2}\left|\sum_{k=2}^{\infty}\alpha_k\right| - b_{m+2}\sum_{r=m+3}^{\infty}|\gamma_r - \gamma_{r-1}|\sup_{\mu\geq 3}\left|\sum_{k=\mu}^{\infty}\alpha_k\right| \\
&\geq b_{m+2}\,\gamma_{m+2}\left|\sum_{k=2}^{\infty}\alpha_k\right| - b_{m+2}\sum_{r=m+3}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+2}\,\gamma_{m+2}\rho_2 - b_{m+2}\sum_{r=m+3}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau\ .
\end{aligned}$$

Hence, inductively we get for all $j \in \mathbb{N}$,

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+j}\,\gamma_{m+j}\left|\sum_{k=m+j}^{\infty}\alpha_{k-m}\right| - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\sup_{\mu\geq m+j+1}\left|\sum_{k=\mu}^{\infty}\alpha_{k-m}\right| \\
&= b_{m+j}\,\gamma_{m+j}\left|\sum_{k=j}^{\infty}\alpha_k\right| - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\sup_{\mu\geq j+1}\left|\sum_{k=\mu}^{\infty}\alpha_k\right| \\
&\geq b_{m+j}\,\gamma_{m+j}\left|\sum_{k=j}^{\infty}\alpha_k\right| - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+j}\,\gamma_{m+j}\rho_j - b_{m+j}\sum_{r=m+j+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+\nu_0}\,\gamma_{m+\nu_0}\,\rho_{\nu_0} - b_{m+\nu_0}\sum_{r=m+\nu_0+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+\nu_0}\,\gamma_{m+\nu_0}\,\tau - b_{m+\nu_0}\sum_{r=m+\nu_0+1}^{\infty}|\gamma_r - \gamma_{r-1}|\,\tau \\
&= b_{m+\nu_0}\,\tau,\ \text{since}\ \gamma_n \downarrow_n 1\,; \\
&\geq b_m\,\tau,\ \text{since } b_n \text{ is increasing}\ ;
\end{aligned}$$

Then, for all $m \in \mathbb{N}$,

$$\|T^m(x) - T^m(y)\|_\infty \geq b_m \, \tau \ .$$

Hence, we have

$$(\frac{1}{b_m}) \, \|T^m(x) - T^m(y)\|_\infty \geq \tau \ .$$

Hence, combining this result with (3.5), we have that for all $n \geq m$ and for all $x, y \in E$,

$$\|T^n(x) - T^n(y)\|_\infty \leq \frac{(\gamma_{n+1} + \sigma_{n+2})}{b_m} \, \|T^m(x) - T^m(y)\|_\infty \ .$$

Note that $\lambda_{n,m} := \frac{\gamma_{n+1}+\sigma_{n+2}}{b_m} \longrightarrow \frac{1+0}{1} = 1$, as $n \geq m \longrightarrow \infty$.

Case 3: Let $(b_n)_{n\in\mathbb{N}}$ be decreasing to 1 and $(\gamma_n)_{n\in\mathbb{N}}$ be increasing to 1. I.e. $b_n \downarrow_n 1$ and $\gamma_n \uparrow_n 1$. Then, we apply the similar argument as in previous case, and find, for all $n \in \mathbb{N}$, that

$$\begin{aligned}
\|T^n(x) - T^n(y)\|_\infty &= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_{j+n} \right\|_\infty && (3.6)\\
&= \left\| \sum_{k=n+1}^{\infty} \alpha_{k-n} \, \eta_k \right\|_\infty && (3.7)\\
&\leq b_{n+1} \, (\gamma_{n+1} + \sigma_{n+2}) \sup_{s \geq n+1} \left| \sum_{k=s}^{\infty} \alpha_{k-n} \right| && (3.8)\\
&= b_{n+1} \sup_{s \geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| && (3.9)\\
&= b_{n+1} \tau && (3.10)
\end{aligned}$$

Let $m \in \mathbb{N}$. Similarly to Case 2,

$$\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &= \left\| \sum_{j=m+1}^{\infty} \alpha_{j-m}\, \eta_j \right\|_\infty \\
&\geq b_{m+\nu_0}\gamma_{m+\nu_0}\tau - b_{m+\nu_0} \sum_{r=m+\nu_0+1}^{\infty} |\gamma_r - \gamma_{r-1}|\tau \\
&= b_{m+\nu_0}(2\gamma_{m+\nu_0} - 1)\tau\,,\ \text{since } \gamma_n \uparrow_n 1\ ; \\
&\geq (2\gamma_m - 1)\tau\ ,\ \text{since } b_n \downarrow_n 1\ ,\text{and } \gamma_n \uparrow_n 1\ .
\end{aligned}$$

Then, since $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate and the sequences $b_n \underset{n}{\longrightarrow} 1$ and $\gamma_m \underset{m}{\longrightarrow} 1$, there exists a sequence $(\lambda_{n,m})_{n\geq m\geq 1}$ in $[1,\infty)$ such that $\lambda_{n,m} := \frac{b_{n+1}}{2\gamma_m - 1}$ for all m large enough, and such that

$$\|T^n(x) - T^n(y)\|_\infty \leq \lambda_{n,m}\|T^m x - T^m y\|_\infty\ ,$$

for all $x, y \in E$. Note that $\lambda_{n,m} \longrightarrow 1$ as $n \geq m \longrightarrow \infty$. Hence, T is fixed point free, affine, semi-strongly asymptotically nonexpansive mapping.

Case 4: Let $(b_n)_{n\in\mathbb{N}}$ be increasing to 1 and $(\gamma_n)_{n\in\mathbb{N}}$ be increasing to 1. I.e. $b_n \uparrow_n 1$ and $\gamma_n \uparrow_n 1$. Then, we apply the similar argument as in Case 2, and we find for $n \in \mathbb{N}$,

$$\begin{aligned}
\|T^n(x) - T^n(y)\|_\infty &:= \left\| \sum_{j=n+1}^{\infty} \alpha_{j-n}\, \eta_j \right\|_\infty \\
&= \sup_{\nu\geq n+1} b_\nu \left| \gamma_\nu \left(\sum_{k=\nu}^{\infty} \alpha_{k-n} \right) + \sum_{r=\nu+1}^{\infty} (\gamma_r - \gamma_{r-1}) \sum_{k=r}^{\infty} \alpha_{k-n} \right| \\
&\leq (\gamma_{n+1} + \sigma_{n+2}) \sup_{s\geq n+1} \left| \sum_{k=s}^{\infty} \alpha_{k-n} \right| \\
&= (\gamma_{n+1} + \sigma_{n+2}) \sup_{s\geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| \\
&= \sup_{s\geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| \ .
\end{aligned}$$

Note that, for some $\nu_0 \in \mathbb{N}$

$$\tau := \sup_{s\in\mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| = \left| \sum_{k=\nu_0}^{\infty} \alpha_k \right| .$$

Now, fix $m \in \mathbb{N}$.

$$\begin{aligned} \|T^m(x) - T^m(y)\|_\infty &\geq b_{m+\nu_0}\, \gamma_{m+\nu_0}\, \rho_{\nu_0} - b_{m+\nu_0} \sum_{r=m+\nu_0+1}^{\infty} |\gamma_r - \gamma_{r-1}|\, \tau \\ &= b_{m+\nu_0}\, \gamma_{m+\nu_0}\, \tau - b_{m+\nu_0} \sum_{r=m+\nu_0+1}^{\infty} |\gamma_r - \gamma_{r-1}|\, \tau \\ &= b_{m+\nu_0}(2\gamma_{m+\nu_0} - 1)\tau \\ &\geq b_m\, (2\gamma_m - 1)\tau . \end{aligned}$$

Then,

$$\|T^m(x) - T^m(y)\|_\infty \geq b_m\, (2\gamma_m - 1)\tau .$$

Hence, combining the results, we have for all $n \geq m$ large enough,

$$\|T^n(x) - T^n(y)\|_\infty \leq \frac{1}{b_m\,(2\gamma_m - 1)} \|T^m(x) - T^m(y)\|_\infty \;, \forall\, x, y \in E .$$

Similarly to Case 3, we are done.

Case 5 (General Case): Let $(b_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ be two sequences convergent to 1. I.e. $b_n \longrightarrow 1$ and $\gamma_n \longrightarrow 1$. Choose a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ such that $\frac{1}{1+\varepsilon_n} < b_n < 1+\varepsilon_n$, for all $n \in \mathbb{N}$. Clearly, we may replace $(\varepsilon_n)_{n\in\mathbb{N}}$ with a decreasing sequence in the following way. Define

$$\mu_n := \max_{j\geq n} \varepsilon_j \;, \text{ for all } n \in \mathbb{N} .$$

Thus, there exist decreasing null sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $\frac{1}{1+\varepsilon_n} < b_n < 1+\varepsilon_n$ and $\frac{1}{1+\zeta_n} < \gamma_n < 1+\zeta_n$ for all $n \in \mathbb{N}$. Now, for any $n \in \mathbb{N}$,

$$
\begin{aligned}
\|T^n(x) - T^n(y)\|_\infty &:= \left\| \sum_{j=n+1}^{\infty} \alpha_{j-n}\, \eta_j \right\|_\infty \\
&= \sup_{\nu \geq n+1} b_\nu \left| \gamma_\nu \left(\sum_{k=\nu}^{\infty} \alpha_{k-n} \right) + \sum_{r=\nu+1}^{\infty} (\gamma_r - \gamma_{r-1}) \sum_{k=r}^{\infty} \alpha_{k-n} \right| \\
&\leq \sup_{\nu \geq n+1} (1+\varepsilon_\nu) \left(\gamma_\nu \left| \sum_{k=\nu}^{\infty} \alpha_{k-n} \right| + \sum_{r=\nu+1}^{\infty} |\gamma_r - \gamma_{r-1}| \left| \sum_{k=r}^{\infty} \alpha_{k-n} \right| \right) \\
&\leq \sup_{\nu \geq n+1} (1+\varepsilon_\nu) \left((1+\zeta_\nu) \left| \sum_{k=\nu}^{\infty} \alpha_{k-n} \right| + \sum_{r=\nu+1}^{\infty} |\gamma_r - \gamma_{r-1}| \left| \sum_{k=r}^{\infty} \alpha_{k-n} \right| \right) \\
&\leq (1+\varepsilon_{n+1})(1+\zeta_{n+1}+\sigma_{n+2}) \sup_{s \geq n+1} \left| \sum_{k=s}^{\infty} \alpha_{k-n} \right| \\
&= (1+\varepsilon_{n+1})(1+\zeta_{n+1}+\sigma_{n+2}) \sup_{s \geq 1} \left| \sum_{k=s}^{\infty} \alpha_k \right| .
\end{aligned}
$$

Furthermore, for some $\nu_0 \in \mathbb{N}$,

$$
\tau := \sup_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| = \left| \sum_{k=\nu_0}^{\infty} \alpha_k \right| .
$$

Now, fix $m \in \mathbb{N}$.

$$
\begin{aligned}
\|T^m(x) - T^m(y)\|_\infty &\geq b_{m+\nu_0}\, \gamma_{m+\nu_0}\, \rho_{\nu_0} - b_{m+\nu_0} \sum_{r=m+\nu_0+1}^{\infty} |\gamma_r - \gamma_{r-1}|\, \tau \\
&= b_{m+\nu_0}\, \gamma_{m+\nu_0}\, \tau - b_{m+\nu_0} \sum_{r=m+\nu_0+1}^{\infty} |\gamma_r - \gamma_{r-1}|\, \tau \\
&\geq \frac{1}{1+\varepsilon_{m+\nu_0}} \left[\frac{1}{1+\zeta_{m+\nu_0}} - \sigma_{m+\nu_0+1} \right] \tau \\
&\geq \frac{1}{1+\varepsilon_m} \left[\frac{1}{1+\zeta_m} - \sigma_m \right] \tau .
\end{aligned}
$$

Then,

$$\|T^m(x) - T^m(y)\|_\infty \geq \frac{1}{1+\varepsilon_m}\left[\frac{1}{1+\zeta_m} - \sigma_m\right] \tau \ .$$

Hence, combining the results, we have, for all $n \geq m$ large enough,

$$\|T^n(x) - T^n(y)\|_\infty \leq \frac{(1+\varepsilon_m)(1+\varepsilon_{n+1})(1+\zeta_{n+1}+\sigma_{n+2})}{\left(\frac{1}{1+\zeta_m} - \sigma_m\right)} \|T^m(x) - T^m(y)\|_\infty \ ,$$

for all $x, y \in E$. Hence, T is semi-strongly asymptotically nonexpansive map that is affine and fixed point free.

□

3.3 STRONGER RESULTS FOR OUR CLASS OF c_0-SUMMING BASIC SEQUENCES, WHICH TURN OUT TO BE ASYMPTOTICALLY ISOMETRIC c_0-SUMMING BASIC SEQUENCES

As previously, we will investigate c_0-summing basic sequences. But this time, we will see that there is a large class of these sequences which are actually asymptotically isometric c_0-summing basic sequences. Let's investigate these sequences by some specific examples, and later we will work on general cases. We will first investigate these sequences when the b_n's are 1 for each n.

Let

$$\eta_n := \gamma_n(e_1 + e_2 + e_3 + e_4 + ... + e_n)\,,\ \text{for all } n \in \mathbb{N}\ ;$$

$$\text{where } (\gamma_n)_{n\in\mathbb{N}}\ \in\ (0,\infty) \text{ with } \sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}|\ < \infty\ ,$$

and there exists $\Gamma \in (0,\infty)$ such that $\gamma_n \geq \Gamma$, for all $n \in \mathbb{N}$.

3.3.1 Our construction with converging γ_n's such that first finitely many terms are constant, and all b_n's are 1. We then confirm that η_n's are an asymptotically isometric c_0-summing basic sequence

Example 3.3.1. Let $\gamma_1 = \gamma \in (0,1)$ and $\gamma_{n+1} = 1, \forall n \in \mathbb{N}$. Then, define

$$\begin{aligned}
\eta_1 &:= \gamma e_1\\
\eta_2 &:= e_1 + e_2\\
\eta_3 &:= e_1 + e_2 + e_3\\
\eta_4 &:= e_1 + e_2 + e_3 + e_4\\
&\ \vdots\\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n\\
&\ \vdots
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda \;&:=\; \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_\infty = \|\alpha_1\,\eta_1 + \alpha_2\,\eta_2 + \alpha_3\,\eta_3 + \ldots\|_\infty \\
&= \left\| \begin{array}{l} e_1\,(\gamma\,\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots\ldots\ldots) \\ +e_2\,(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots\ldots\ldots) \\ +e_3\,(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots) + \ldots \end{array} \right\|_\infty
\end{aligned}$$

$$\begin{aligned}
\Lambda \;&=\; |\gamma\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \\
&\vee\; |\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots| \\
&\vee\; |\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots| \;\vee \ldots \\
&=\; |\gamma\,\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \vee \left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \ldots \\
&=\; |\gamma\,(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots) + (1-\gamma)\,(\alpha_2 + \alpha_3 + \alpha_4 + \ldots)| \vee \left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \ldots \\
&=\; \left|\gamma \sum_{j=1}^{\infty} \alpha_j + (1-\gamma) \sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \ldots
\end{aligned}$$

Hence,

$$\Lambda \;\leq\; \sup_{s\in\mathbb{N}} \left|\sum_{k=s}^{\infty} \alpha_k\right| \;.$$

Also,

$$\Lambda \geq \left(\gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| - (1-\gamma) \left|\sum_{k=2}^{\infty} \alpha_k\right|\right) \vee \sup_{k\geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right|$$

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \gamma \left| \sum_{k=1}^{\infty} \alpha_k \right| - (1-\gamma) \left| \sum_{k=2}^{\infty} \alpha_k \right| \tag{3.11}$$

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \left| \sum_{j=k}^{\infty} \alpha_j \right|, \forall k \geq 2 \tag{3.12}$$

Hence, using (3.12) for $k = 2$ in (3.11), we get

$$(2-\gamma) \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \gamma \left| \sum_{k=1}^{\infty} \alpha_k \right|$$

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \frac{\gamma}{2-\gamma} \left| \sum_{k=1}^{\infty} \alpha_k \right|$$

Hence,

$$\Lambda \geq \frac{\gamma}{2-\gamma} \left| \sum_{k=1}^{\infty} \alpha_k \right| \vee \sup_{k \geq 2} \left| \sum_{j=k}^{\infty} \alpha_j \right|$$

Thus, we can find a decreasing null sequence $(\varepsilon_k)_{k \in \mathbb{N}}$

where $\varepsilon_1 = 1 - \frac{\gamma}{2-\gamma} = \frac{2-2\gamma}{2-\gamma} = \frac{1-\gamma}{1-\frac{\gamma}{2}} \in (0,1)$ and $\varepsilon_k = 0, \forall k \geq 2$ such that

$$\sup_{k \in \mathbb{N}} (1-\varepsilon_k) \left| \sum_{j=k}^{\infty} \alpha_j \right| \leq \left\| \sum_{k=1}^{\infty} \alpha_k \, \eta_k \right\|_{\infty} \leq \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \alpha_j \right|$$

$\forall (\alpha_j)_{j \in \mathbb{N}} \in c_{00}$

Example 3.3.2. Case for a fixed position $\nu > 2$ Let $\gamma_1 = \gamma_2 = \cdots = \gamma_\nu = \gamma \in (0,1)$ and $\gamma_{n+\nu} = 1, \forall n \in \mathbb{N}$. Then, define

$$\begin{aligned}
\eta_1 &:= \gamma e_1 \\
\eta_2 &:= \gamma(e_1 + e_2) \\
\eta_3 &:= \gamma(e_1 + e_2 + e_3) \\
\eta_4 &:= \gamma(e_1 + e_2 + e_3 + e_4) \\
&\vdots \\
\eta_{\nu-1} &:= \gamma(e_1 + e_2 + e_3 + e_4 + \cdots + e_{\nu-1}) \\
\eta_\nu &:= \gamma(e_1 + e_2 + e_3 + e_4 + \cdots + e_{\nu-1} + e_\nu) \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda &:= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty = \|\alpha_1 \, \eta_1 + \alpha_2 \, \eta_2 + \alpha_3 \, \eta_3 + \cdots + \alpha_\nu \, \eta_\nu + \alpha_\nu \, \eta_\nu + \ldots\|_\infty \\
&= \left\| \begin{array}{c} e_1 \, (\gamma \, \alpha_1 + \gamma \, \alpha_2 + \gamma \, \alpha_3 + \cdots + \gamma \, \alpha_{\nu-1} + \gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ldots) \\ +e_2 \, (\gamma \, \alpha_2 + \gamma \, \alpha_3 + \cdots + \gamma \, \alpha_{\nu-1} + \gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ldots) \\ +e_3 \, (\gamma \, \alpha_3 + \cdots + \gamma \, \alpha_{\nu-1} + \gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ldots) + \ldots \\ +e_\nu \, (\gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ \alpha_{\nu+2} + \ \alpha_{\nu+3} + \ldots) \\ +e_{\nu+1} \, (\alpha_{\nu+1} + \ \alpha_{\nu+2} + \ \alpha_{\nu+3} + \ldots) + \ldots \end{array} \right\|_\infty \\
&= |\gamma \, \alpha_1 + \gamma \, \alpha_2 + \gamma \, \alpha_3 + \cdots + \gamma \, \alpha_{\nu-1} + \gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ldots| \\
&\vee \ |\gamma \, \alpha_2 + \gamma \, \alpha_3 + \cdots + \gamma \, \alpha_{\nu-1} + \gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ldots| \\
&\vee \ |\gamma \, \alpha_3 + \cdots + \gamma \, \alpha_{\nu-1} + \gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ldots) + \ldots| \\
&\vee \ \ldots \vee |\gamma \, \alpha_\nu + \ \alpha_{\nu+1} + \ \alpha_{\nu+2} + \ \alpha_{\nu+3} + \ldots| \vee |\alpha_{\nu+1} + \ \alpha_{\nu+2} + \ \alpha_{\nu+3} + \ldots| \\
&\vee \ \left| \sum_{j=\nu+1}^{\infty} \alpha_j \right| \vee \left| \sum_{j=\nu+2}^{\infty} \alpha_j \right| \vee \ldots
\end{aligned}$$

$$
\begin{aligned}
\Lambda &= |\gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\dots)+(1-\gamma)(\alpha_{\nu+1}+\alpha_{\nu+2}+\alpha_{\nu+3}+\dots)| \\
&\quad \vee |\gamma(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\dots)+(1-\gamma)(\alpha_{\nu+1}+\alpha_{\nu+2}+\alpha_{\nu+3}+\dots)| \vee \dots\dots \\
&\quad \vee |\gamma(\alpha_\nu+\alpha_{\nu+1}+\alpha_{\nu+2}+\alpha_{\nu+3}+\dots)+(1-\gamma)(\alpha_{\nu+1}+\alpha_{\nu+2}+\alpha_{\nu+3}+\dots)| \\
&\quad \left|\sum_{j=\nu+1}^{\infty}\alpha_j\right| \vee \left|\sum_{j=\nu+2}^{\infty}\alpha_j\right| \vee \dots \\
&= \left|\gamma\sum_{j=1}^{\infty}\alpha_j+(1-\gamma)\sum_{j=\nu+1}^{\infty}\alpha_j\right| \vee \left|\gamma\sum_{j=2}^{\infty}\alpha_j+(1-\gamma)\sum_{j=\nu+1}^{\infty}\alpha_j\right| \vee \dots \\
\vee & \left|\gamma\sum_{j=\nu}^{\infty}\alpha_j+(1-\gamma)\sum_{j=\nu+1}^{\infty}\alpha_j\right| \vee \sup_{k\geq\nu+1}\left|\sum_{j=k}^{\infty}\alpha_j\right| .
\end{aligned}
$$

Hence,

$$
\Lambda \;\leq\; \sup_{s\in\mathbb{N}}\left|\sum_{k=s}^{\infty}\alpha_k\right| .
$$

Also,

$$
\begin{aligned}
\Lambda \geq \left(\gamma\left|\sum_{j=1}^{\infty}\alpha_j\right|-(1-\gamma)\left|\sum_{j=\nu+1}^{\infty}\alpha_j\right|\right) \vee \left(\gamma\left|\sum_{j=2}^{\infty}\alpha_j\right|-(1-\gamma)\left|\sum_{j=\nu+1}^{\infty}\alpha_j\right|\right) \vee \dots \\
\vee \left(\gamma\left|\sum_{j=\nu}^{\infty}\alpha_j\right|-(1-\gamma)\left|\sum_{j=\nu+1}^{\infty}\alpha_j\right|\right) \vee \sup_{k\geq\nu+1}\left|\sum_{j=k}^{\infty}\alpha_j\right|
\end{aligned}
$$

$$
\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \;\geq\; \gamma\left|\sum_{j=1}^{\infty}\alpha_j\right|-(1-\gamma)\left|\sum_{j=\nu+1}^{\infty}\alpha_j\right| \tag{3.13}
$$

$$
\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \;\geq\; \gamma\left|\sum_{j=2}^{\infty}\alpha_j\right|-(1-\gamma)\left|\sum_{j=\nu+1}^{\infty}\alpha_j\right| \tag{3.14}
$$

$$
\vdots \tag{3.15}
$$

$$
\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \;\geq\; \gamma\left|\sum_{j=\nu}^{\infty}\alpha_j\right|-(1-\gamma)\left|\sum_{j=\nu+1}^{\infty}\alpha_j\right| \tag{3.16}
$$

$$
\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \;\geq\; \left|\sum_{j=k}^{\infty}\alpha_j\right|, \forall k\geq\nu+1 \tag{3.17}
$$

Hence, using (3.17) for $k = \nu + 1$ in (3.16), we get

$$\begin{aligned}(2-\gamma)\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty &\geq \gamma\left|\sum_{k=\nu}^{\infty}\alpha_k\right| \\ \left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty &\geq \frac{\gamma}{2-\gamma}\left|\sum_{k=\nu}^{\infty}\alpha_k\right|\end{aligned}$$

Next, using (3.17) and the $(\nu - 1)$ case in (3.15)

$$\begin{aligned}(2-\gamma)\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty &\geq \gamma\left|\sum_{k=\nu-1}^{\infty}\alpha_k\right| \\ \left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty &\geq \frac{\gamma}{2-\gamma}\left|\sum_{k=\nu-1}^{\infty}\alpha_k\right|\end{aligned}$$

Continuing in this manner, we finally get that

$$\Lambda \geq \frac{\gamma}{2-\gamma}\left|\sum_{k=1}^{\infty}\alpha_k\right| \vee \frac{\gamma}{2-\gamma}\left|\sum_{k=2}^{\infty}\alpha_k\right| \vee \cdots \vee \frac{\gamma}{2-\gamma}\left|\sum_{k=\nu}^{\infty}\alpha_k\right| \vee \sup_{k\geq\nu+1}\left|\sum_{j=k}^{\infty}\alpha_j\right|$$

Thus, we can find a decreasing null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$

where $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_\nu = 1 - \frac{\gamma}{2-\gamma} = \frac{2-2\gamma}{2-\gamma} = \frac{1-\gamma}{1-\frac{\gamma}{2}} \in (0,1)$ and $\varepsilon_k = 0$,

$\forall k \geq \nu + 1$ such that

$$\sup_{k\in\mathbb{N}}(1-\varepsilon_k)\left|\sum_{j=k}^{\infty}\alpha_j\right| \leq \left\|\sum_{k=1}^{\infty}\alpha_k\,\eta_k\right\|_\infty \leq \sup_{k\in\mathbb{N}}\left|\sum_{j=k}^{\infty}\alpha_j\right|$$

,$\forall(\alpha_j)_{j\in\mathbb{N}} \in c_{00}$.

3.4 OUR CONSTRUCTION WITH γ_n'S INCREASING TO 1 AND ALL b_n'S EQUAL TO 1. WE AGAIN CONFIRM THAT THE η_n'S ARE AN ASYMPTOTICALLY ISOMETRIC c_0-SUMMING BASIC SEQUENCE

3.4.1 The case where all but finitely many γ_n's equal 1.

Example 3.4.1. Let $0 < \gamma_1 \leq \gamma_2 < 1$ Then, define

$$\begin{aligned}
\eta_1 &:= \gamma_1 e_1 \\
\eta_2 &:= \gamma_2(e_1 + e_2) \\
\eta_3 &:= e_1 + e_2 + e_3 \\
\eta_4 &:= e_1 + e_2 + e_3 + e_4 \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n \\
&\vdots
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda &:= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty = \|\alpha_1 \, \eta_1 + \alpha_2 \, \eta_2 + \alpha_3 \, \eta_3 + \ldots\|_\infty \\
&= \left\| \begin{array}{c} e_1 \, (\gamma_1 \, \alpha_1 + \gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \ldots\ldots\ldots) \\ +e_2 \, (\gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots\ldots\ldots) \\ +e_3 \, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots) + \ldots \end{array} \right\|_\infty \\
&= |\gamma_1 \alpha_1 + \gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \\
&\vee \ |\gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots| \\
&\vee \ |\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots| \ \vee \ldots
\end{aligned}$$

$$
\begin{aligned}
\Lambda &= |\gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 + \alpha_3 + \alpha_4 + \dots| \vee |\gamma_2\,\alpha_2 + \alpha_3 + \alpha_4 + \dots| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots \\
&= |\gamma_1\,(\alpha_1 + \alpha_2 + \dots) + (\gamma_2 - \gamma_1)\,(\alpha_2 + \alpha_3 + \dots) + (1-\gamma_2)\,(\alpha_3 + \alpha_4 + \dots)| \\
&\quad \vee |\gamma_2\,(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \dots) + (1-\gamma_2)\,(\alpha_3 + \alpha_4 + \alpha_5 + \dots)| \\
&\quad \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots
\end{aligned}
$$

$$
\begin{aligned}
\Lambda &= \left|\gamma_1 \sum_{j=1}^{\infty} \alpha_j + (\gamma_2 - \gamma_1) \sum_{j=2}^{\infty} \alpha_j + (1-\gamma_2) \sum_{j=3}^{\infty} \alpha_j\right| \\
&\vee \left|\gamma_2 \sum_{j=2}^{\infty} \alpha_j + (1-\gamma_2) \sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots
\end{aligned}
$$

Hence,

$$
\Lambda \leq \sup_{s\in\mathbb{N}} \left|\sum_{k=s}^{\infty} \alpha_k\right| .
$$

Also,

$$
\begin{aligned}
\Lambda \geq & \left(\gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[(\gamma_2 - \gamma_1)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (1-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right]\right) \\
\vee & \left(\gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - (1-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right) \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right|
\end{aligned}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[(\gamma_2 - \gamma_1)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (1-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right] \tag{3.18}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - (1-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right| \tag{3.19}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \left|\sum_{j=k}^{\infty} \alpha_j\right|, \forall k \geq 3 \tag{3.20}
$$

Hence, using (3.20) for $k = 3$ in (3.19), we get

$$(2-\gamma_2)\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| \tag{3.21}$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \frac{\gamma_2}{2-\gamma_2} \left|\sum_{k=2}^{\infty} \alpha_k\right| \tag{3.22}$$

Next, using this result (3.22) above with (3.18)

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[(\gamma_2-\gamma_1)\,\frac{2-\gamma_2}{\gamma_2}\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} + (1-\gamma_2)\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty}\right] .$$

Hence,

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \frac{\gamma_1 \, \gamma_2}{(2-\gamma_2)(2\gamma_2-\gamma_1)} \left|\sum_{k=1}^{\infty} \alpha_k\right| .$$

Thus,

$$\Lambda \geq \frac{\gamma_1 \, \gamma_2}{(2-\gamma_2)(2\gamma_2-\gamma_1)} \left|\sum_{k=1}^{\infty} \alpha_k\right| \vee \frac{\gamma_2}{2-\gamma_2}\left|\sum_{k=2}^{\infty} \alpha_k\right| \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| .$$

Thus, we can find a decreasing null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$

where $\varepsilon_1 = 1 - \frac{\gamma_1 \, \gamma_2}{(2-\gamma_2)(2\gamma_2-\gamma_1)} = \frac{2[(1-\gamma_1) - (1-\gamma_2)^2]}{(2-\gamma_2)(2\gamma_2-\gamma_1)} \in (0,1)$,
$\varepsilon_2 = 1 - \frac{\gamma_2}{2-\gamma_2} = \frac{2-2\gamma_2}{2-\gamma_2} = \frac{1-\gamma_2}{1-\frac{\gamma_2}{2}} \in (0,1)$ and $\varepsilon_k = 0, \forall k \geq 3$ such that

$$\sup_{k\in\mathbb{N}} (1-\varepsilon_k)\left|\sum_{j=k}^{\infty} \alpha_j\right| \leq \left\|\sum_{k=1}^{\infty} \alpha_k \, \eta_k\right\|_{\infty} \leq \sup_{k\in\mathbb{N}}\left|\sum_{j=k}^{\infty} \alpha_j\right|$$

$\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$.

3.4.2 General case for γ_n increasing , with infinitely many γ_n 's equal to 1.

Example 3.4.2. Let $k \in \mathbb{N}$ and $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4 \leq \cdots \leq \gamma_k < 1$ and $\gamma_n = 1$ for all $n \geq k+1$.Then, define

$$\begin{aligned}
\eta_1 &:= \gamma_1 e_1 \\
\eta_2 &:= \gamma_2(e_1 + e_2) \\
\eta_3 &:= \gamma_3(e_1 + e_2 + e_3) \\
\eta_4 &:= \gamma_4(e_1 + e_2 + e_3 + e_4) \\
&\vdots \\
\eta_k &:= \gamma_k(e_1 + e_2 + e_3 + e_4 + \cdots + e_k) \\
\eta_{k+1} &:= e_1 + e_2 + e_3 + e_4 + \cdots + e_k + e_{k+1} \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n \\
&\vdots
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda &:= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty = \|\alpha_1 \, \eta_1 + \alpha_2 \, \eta_2 + \alpha_3 \, \eta_3 + \alpha_4 \, \eta_4 + \cdots + \alpha_k \, \eta_k + \ldots\|_\infty \\
&= \left\| \begin{array}{c} e_1 \left(\gamma_1 \, \alpha_1 + \gamma_2 \, \alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \alpha_{k+1} + \ldots\ldots\ldots \right) \\ +e_2 \left(\gamma_2 \, \alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_{k-1}\alpha_{k-1} + \gamma_k\alpha_k + \alpha_{k+1} + \ldots\ldots\ldots \right) \\ + \cdots + e_k \left(\gamma_k\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \ldots \right) \\ +e_{k+1} \left(\alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \ldots \right) + \ldots \end{array} \right\|_\infty
\end{aligned}$$

$$
\begin{aligned}
\Lambda \;&=\; |\gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\; |\gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_{k-1}\alpha_{k-1} + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\; \cdots \vee |\gamma_k\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots| \\
&\vee\; |\alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots| \;\vee \dots \\
&=\; |\gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\; |\gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_{k-1}\alpha_{k-1} + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\; \cdots \vee |\gamma_k\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots| \vee \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \vee \left|\sum_{j=k+2}^{\infty} \alpha_j\right| \vee \dots \\
&=\; \left| \begin{array}{c} \gamma_1\,\sum_{j=1}^{\infty} \alpha_j + (\gamma_2 - \gamma_1)\,\sum_{j=2}^{\infty} \alpha_j\, + (\gamma_3 - \gamma_2)\,\sum_{j=3}^{\infty} \alpha_j \\ + \cdots + (\gamma_k - \gamma_{k-1})\,\sum_{j=k}^{\infty} \alpha_j\, + (1 - \gamma_k)\,\sum_{j=k+1}^{\infty} \alpha_j \end{array} \right| \\
&\vee\; \left|\gamma_2 \sum_{j=2}^{\infty} \alpha_j + (\gamma_3 - \gamma_2) \sum_{j=3}^{\infty} \alpha_j\, + \cdots + (\gamma_k - \gamma_{k-1}) \sum_{j=k}^{\infty} \alpha_j\, + (1 - \gamma_k) \sum_{j=k+1}^{\infty} \alpha_j\right| \\
&\vee\; \cdots \vee \left|\gamma_k \sum_{j=k}^{\infty} \alpha_j + (1 - \gamma_k) \sum_{j=k+1}^{\infty} \alpha_j\right| \vee \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \vee \left|\sum_{j=k+2}^{\infty} \alpha_j\right| \vee \dots
\end{aligned}
$$

Hence,

$$
\Lambda \;\leq\; \sup_{s \in \mathbb{N}} \left|\sum_{k=s}^{\infty} \alpha_k\right| .
$$

Also,

$$
\begin{aligned}
\Lambda \;\geq\; &\left(\gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_2 - \gamma_1) \left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_3 - \gamma_2) \left|\sum_{k=3}^{\infty} \alpha_k\right| \\ + \cdots + (\gamma_k - \gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + (1 - \gamma_k) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array} \right] \right) \\
\vee\; &\left(\gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_3 - \gamma_2) \left|\sum_{k=3}^{\infty} \alpha_k\right| + (\gamma_4 - \gamma_3) \left|\sum_{k=4}^{\infty} \alpha_k\right| \\ + \cdots + (\gamma_k - \gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + (1 - \gamma_k) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array} \right] \right) \\
\vee\; &\cdots \vee \left(\gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| - (1 - \gamma_k) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \right) \vee \sup_{s \geq k+1} \left|\sum_{j=s}^{\infty} \alpha_j\right| .
\end{aligned}
$$

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty &\geq \gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_2-\gamma_1)\,|\sum_{k=2}^{\infty} \alpha_k\,| + (\gamma_3-\gamma_2)\,|\sum_{k=3}^{\infty} \alpha_k\,| \\ + \cdots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + (1-\gamma_k) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array}\right] \\
\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty &\geq \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_3-\gamma_2)\,|\sum_{k=3}^{\infty} \alpha_k\,| + (\gamma_4-\gamma_3)\,|\sum_{k=4}^{\infty} \alpha_k\,| \\ + \cdots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + (1-\gamma_k) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array}\right] \\
&\vdots \\
\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty &\geq \gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| - (1-\gamma_k) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \\
\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty &\geq \left|\sum_{j=s}^{\infty} \alpha_j\right|, \forall s \geq k+1
\end{aligned}$$

Hence, using the second-to-last inequality above with the last one for $j = k+1$, we get

$$(2-\gamma_k)\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \geq \gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| \tag{3.23}$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \geq \frac{\gamma_k}{2-\gamma_k} \left|\sum_{j=k}^{\infty} \alpha_j\right| \tag{3.24}$$

Next, using this result (3.24) with the block of inequalities above, we see that

$$\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \geq \gamma_{k-1} \left|\sum_{j=k-1}^{\infty} \alpha_j\right| - \left[(\gamma_k-\gamma_{k-1})\,\frac{2-\gamma_k}{\gamma_k} \left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty + (1-\gamma_k) \left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty\right]$$

Hence,

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty &\geq \frac{\gamma_{k-1}\,\gamma_k}{(2-\gamma_k)(2\gamma_k-\gamma_{k-1})} \left|\sum_{j=k-1}^{\infty} \alpha_j\right| \\
&= \frac{\gamma_{k-1}\,\gamma_k}{(2-\gamma_k)\lambda_2} \left|\sum_{j=k-1}^{\infty} \alpha_j\right|
\end{aligned}$$

where $\lambda_2 = 2\gamma_k - \gamma_{k-1}$. Also, define $\lambda_1 = 1$. We use the immediately preceding with the block of inequalities above, to get that

$$\left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \geq \gamma_{k-2} \left|\sum_{j=k-2}^{\infty} \alpha_j\right| - \left[\begin{array}{c} (\gamma_{k-1}-\gamma_{k-2})\frac{(2-\gamma_k)(2\gamma_k-\gamma_{k-1})}{\gamma_{k-1}\,\gamma_k} \left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \\ +(\gamma_k-\gamma_{k-1})\frac{2-\gamma_k}{\gamma_k} \left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \\ +(1-\gamma_k) \left\|\sum_{j=1}^{\infty} \alpha_j \eta_j\right\|_\infty \end{array}\right]$$

Thus,

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} &\geq \frac{\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)[\gamma_{k-1}\gamma_k + (\gamma_k - \gamma_{k-1})\gamma_{k-1} + (\gamma_{k-1} - \gamma_{k-2})(2\gamma_k - \gamma_{k-1})]} \left| \sum_{j=k-2}^{\infty} \alpha_j \right| \\
&= \frac{\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_3} \left| \sum_{j=k-2}^{\infty} \alpha_j \right|
\end{aligned}$$

where $\lambda_3 = [\gamma_{k-1}\gamma_k + (\gamma_k - \gamma_{k-1})\gamma_{k-1} + (\gamma_{k-1} - \gamma_{k-2})(2\gamma_k - \gamma_{k-1})]$.

The next step similarly yields

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \gamma_{k-3} \left| \sum_{j=k-3}^{\infty} \alpha_j \right| - \left[\begin{array}{c} \frac{(\gamma_{k-2}-\gamma_{k-3})(2-\gamma_k)\lambda_3}{\gamma_{k-2}\gamma_{k-1}\gamma_k} \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \\ + \frac{(\gamma_{k-1}-\gamma_{k-2})(2-\gamma_k)\lambda_2}{\gamma_{k-1}\gamma_k} \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \\ + \frac{(\gamma_k-\gamma_{k-1})(2-\gamma_k)\lambda_1}{\gamma_k} \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \\ +(1-\gamma_k) \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \end{array} \right].$$

Then,

$$\gamma_{k-3} \left| \sum_{j=k-3}^{\infty} \alpha_j \right| \leq (2-\gamma_k) \left[\frac{(\gamma_{k-2}-\gamma_{k-3})\lambda_3}{\gamma_{k-2}\gamma_{k-1}\gamma_k} + \frac{(\gamma_{k-1}-\gamma_{k-2})\lambda_2}{\gamma_{k-1}\gamma_k} + \frac{(\gamma_k-\gamma_{k-1})\lambda_1}{\gamma_k} + 1 \right] \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty}.$$

In other words

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} &\geq \frac{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k) \left[\frac{(\gamma_{k-2}-\gamma_{k-3})\lambda_3}{\gamma_{k-2}\gamma_{k-1}\gamma_k} + \frac{(\gamma_{k-1}-\gamma_{k-2})\lambda_2}{\gamma_{k-1}\gamma_k} + \frac{(\gamma_k-\gamma_{k-1})\lambda_1}{\gamma_k} + 1 \right]} \left| \sum_{j=k-3}^{\infty} \alpha_j \right| \\
&= \frac{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k) \left[\begin{array}{c} (\gamma_{k-2} - \gamma_{k-3})\lambda_3 + (\gamma_{k-1} - \gamma_{k-2})\lambda_2\gamma_{k-2} \\ +(\gamma_k - \gamma_{k-1})\lambda_1\gamma_{k-2}\gamma_{k-1} + \gamma_{k-2}\gamma_{k-1}\gamma_k \end{array} \right]} \left| \sum_{j=k-3}^{\infty} \alpha_j \right| \\
&= \frac{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_4} \left| \sum_{j=k-3}^{\infty} \alpha_j \right|
\end{aligned}$$

where

$$\begin{aligned}
\lambda_4 &= (\gamma_{k-2} - \gamma_{k-3})\lambda_3 + (\gamma_{k-1} - \gamma_{k-2})\lambda_2\gamma_{k-2} + (\gamma_k - \gamma_{k-1})\lambda_1\gamma_{k-2}\gamma_{k-1} \\
&\quad + \gamma_{k-2}\gamma_{k-1}\gamma_k
\end{aligned}$$

Continuing to the next step,

$$\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \;\geq\; \gamma_{k-4}\left|\sum_{j=k-4}^{\infty}\alpha_j\right| - \left[\begin{array}{c} \frac{(\gamma_{k-3}-\gamma_{k-2})(2-\gamma_k)\lambda_4}{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \\ +\frac{(\gamma_{k-2}-\gamma_{k-3})(2-\gamma_k)\lambda_3}{\gamma_{k-2}\gamma_{k-1}\gamma_k}\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \\ +\frac{(\gamma_{k-1}-\gamma_{k-2})(2-\gamma_k)\lambda_2}{\gamma_{k-1}\gamma_k}\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \\ +\frac{(\gamma_k-\gamma_{k-1})(2-\gamma_k)\lambda_1}{\gamma_k}\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \\ +(1-\gamma_k)\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \end{array}\right].$$

Then,

$$\gamma_{k-4}\left|\sum_{j=k-4}^{\infty}\alpha_j\right| \leq (2-\gamma_k)\left[\begin{array}{c} \frac{(\gamma_{k-3}-\gamma_{k-4})\lambda_4}{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}+\frac{(\gamma_{k-2}-\gamma_{k-3})\lambda_3}{\gamma_{k-2}\gamma_{k-1}\gamma_k} \\ +\frac{(\gamma_{k-1}-\gamma_{k-2})\lambda_2}{\gamma_{k-1}\gamma_k}+\frac{(\gamma_k-\gamma_{k-1})\lambda_1}{\gamma_k}+1 \end{array}\right]\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty$$

In other words

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty &\geq \frac{\gamma_{k-4}\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\left[\begin{array}{c} \frac{(\gamma_{k-3}-\gamma_{k-4})\lambda_4}{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}+\frac{(\gamma_{k-2}-\gamma_{k-3})\lambda_3}{\gamma_{k-2}\gamma_{k-1}\gamma_k} \\ +\frac{(\gamma_{k-1}-\gamma_{k-2})\lambda_2}{\gamma_{k-1}\gamma_k}+\frac{(\gamma_k-\gamma_{k-1})\lambda_1}{\gamma_k}+1 \end{array}\right]}\left|\sum_{j=k-3}^{\infty}\alpha_j\right| \\
&= \frac{\gamma_{k-4}\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\left[\begin{array}{c} (\gamma_{k-3}-\gamma_{k-4})\lambda_4+(\gamma_{k-2}-\gamma_{k-3})\lambda_3\gamma_{k-3} \\ +(\gamma_{k-1}-\gamma_{k-2})\lambda_2\gamma_{k-3}\gamma_{k-2} \\ +(\gamma_k-\gamma_{k-1})\lambda_1\gamma_{k-3}\gamma_{k-2}\gamma_{k-1} \\ +\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k \end{array}\right]}\left|\sum_{j=k-4}^{\infty}\alpha_j\right| \\
&= \frac{\gamma_{k-4}\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_5}\left|\sum_{j=k-4}^{\infty}\alpha_j\right|
\end{aligned}$$

where

$$\begin{aligned}
\lambda_5 &= (\gamma_{k-3}-\gamma_{k-4})\lambda_4+(\gamma_{k-2}-\gamma_{k-3})\lambda_3\gamma_{k-3}+(\gamma_{k-1}-\gamma_{k-2})\lambda_2\gamma_{k-3}\gamma_{k-2} \\
&\quad +(\gamma_k-\gamma_{k-1})\lambda_1\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}+\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k
\end{aligned}$$

Continuing in this way, inductively we get a finite sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{k-1}, \lambda_k)$ such that $\forall p \in \{6, 7, \ldots, k\}$,

$$\begin{aligned}
\lambda_p &= (\gamma_{k-\{p-2\}} - \gamma_{k-\{p-1\}})\lambda_{(p-1)} + (\gamma_{k-\{p-3\}} - \gamma_{k-\{p-2\}})\lambda_{(p-2)}\gamma_{k-\{p-2\}} \\
&+(\gamma_{k-\{p-4\}} - \gamma_{k-\{p-3\}})\lambda_{(p-3)}\gamma_{k-\{p-2\}}\gamma_{k-\{p-3\}} \\
&+(\gamma_{k-\{p-5\}} - \gamma_{k-\{p-4\}})\lambda_{(p-4)}\gamma_{k-\{p-2\}}\gamma_{k-\{p-3\}}\gamma_{k-\{p-4\}} + \ldots \\
&+(\gamma_k - \gamma_{k-1})\lambda_1\gamma_{k-\{p-2\}}\gamma_{k-\{p-3\}}\gamma_{k-\{p-4\}} \cdots \gamma_{k-3}\gamma_{k-2}\gamma_{k-1} \\
&+\gamma_{k-\{p-2\}}\gamma_{k-\{p-3\}}\gamma_{k-\{p-4\}} \cdots \gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k
\end{aligned}$$

and

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \frac{\gamma_{k-(p-1)}\gamma_{k-(p-2)} \cdots \gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_p} \left| \sum_{j=k-(p-1)}^{\infty} \alpha_j \right| .$$

Hence,

$$\Lambda \geq \left\{ \begin{array}{c}
\frac{\gamma_1\gamma_2\gamma_3\cdots\gamma_{(k-1)}\gamma_k}{(2-\gamma_k)\lambda_k} \left|\sum_{k=1}^{\infty} \alpha_k\right| \vee \frac{\gamma_2\,\gamma_3\cdots\gamma_{(k-1)}\gamma_k}{(2-\gamma_k)\lambda_{(k-1)}} \left|\sum_{k=2}^{\infty} \alpha_k\right| \vee \ldots \\
\vee \frac{\gamma_{k-4}\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_5} \left|\sum_{j=k-4}^{\infty} \alpha_j\right| \vee \frac{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_4} \left|\sum_{j=k-3}^{\infty} \alpha_j\right| \\
\vee \frac{\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_3} \left|\sum_{j=k-2}^{\infty} \alpha_j\right| \vee \frac{\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_2} \left|\sum_{j=k-1}^{\infty} \alpha_j\right| \\
\vee \frac{\gamma_k}{(2-\gamma_k)\lambda_1} \left|\sum_{j=k}^{\infty} \alpha_j\right| \vee \sup_{s \geq k+1} \left|\sum_{k=s}^{\infty} \alpha_k\right|
\end{array} \right\} .$$

Thus, we can find a decreasing null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$

where $\varepsilon_1 = 1 - \frac{\gamma_1\gamma_2\gamma_3\cdots\gamma_{(k-1)}\gamma_k}{(2-\gamma_k)\lambda_k} \in (0,1)$, $\varepsilon_2 = 1 - \frac{\gamma_2\,\gamma_3\cdots\gamma_{(k-1)}\gamma_k}{(2-\gamma_k)\lambda_{(k-1)}} \in (0,1)$,
$\varepsilon_3 = 1 - \frac{\gamma_{k-4}\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_5} \in (0,1)\ldots\ldots\varepsilon_{k-4} = 1 - \frac{\gamma_{k-4}\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_5} \in (0,1)$
$\varepsilon_{k-3} = 1 - \frac{\gamma_{k-3}\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_4} \in (0,1)$, $\varepsilon_{k-2} = 1 - \frac{\gamma_{k-2}\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_3} \in (0,1)$
$\varepsilon_{k-1} = 1 - \frac{\gamma_{k-1}\gamma_k}{(2-\gamma_k)\lambda_2} \in (0,1)$, $\varepsilon_k = 1 - \frac{\gamma_k}{(2-\gamma_k)\lambda_1} \in (0,1)$ and $\varepsilon_s = 0, \forall s \geq k+1$ such that

$$\sup_{k\in\mathbb{N}} (1-\varepsilon_k) \left| \sum_{j=k}^{\infty} \alpha_j \right| \leq \left\| \sum_{k=1}^{\infty} \alpha_k \, \eta_k \right\|_{\infty} \leq \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \alpha_j \right|$$

$\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$.

Now, we see more the general result.

3.4.3 General case for increasing convergent γ_n's when all b_n's are 1

Theorem 3.4.3. *Let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence such that*

$$\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$$

with $\vec{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ *an increasing sequence converging to 1 such that* $\gamma_1 > \frac{1}{2}$ *(i.e.,* $\gamma_n \leq \gamma_{n+1}$*, for all* $n \in \mathbb{N}$*) in* $(0,1)$ *with* $\gamma_n \uparrow_n 1$*. We define the sequence* $(\eta_n)_{n\in\mathbb{N}}$ *by setting*

$$\eta_n := \gamma_n(e_1 + e_2 + e_3 + e_4 + ... + e_n) \ , \ \textit{for all } n \in \mathbb{N} \ .$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ *is an asymptotically isometric* c_0*-summing basic sequence. Furthermore, define the closed convex hull of* $(\eta_n)_{n\in\mathbb{N}}$*,* $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$*, Then, there exists an affine* $\|\cdot\|_\infty$*-nonexpansive mapping* $U : E \longrightarrow E$ *that is fixed point free.*

Proof.

$$\begin{aligned}
\Lambda \ &:= \ \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty = \|\alpha_1 \, \eta_1 + \alpha_2 \, \eta_2 + \alpha_3 \, \eta_3 + \alpha_4 \, \eta_4 + \cdots + \alpha_k \, \eta_k + \ldots\|_\infty \\
&= \left\| \begin{array}{c} e_1 \left(\gamma_1 \, \alpha_1 + \gamma_2 \, \alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \ldots\ldots\ldots\right) \\ +e_2 \left(\gamma_2 \, \alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_k\alpha_k + \ldots\ldots\ldots\right) \\ + \cdots + e_k \left(\gamma_k\alpha_k + \gamma_{k+1}\alpha_{k+1} + \gamma_{k+2}\alpha_{k+2} + \gamma_{k+3}\alpha_{k+3} + \ldots\right) + \ldots \end{array} \right\|_\infty \\
&= |\gamma_1 \, \alpha_1 + \gamma_2 \, \alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \ldots| \\
&\vee \ |\gamma_2 \, \alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_k\alpha_k + \ldots| \\
&\vee \ \cdots \vee |\gamma_k\alpha_k + \gamma_{k+1}\alpha_{k+1} + \gamma_{k+2}\alpha_{k+2} + \gamma_{k+3}\alpha_{k+3} + \ldots)| \vee \ldots
\end{aligned}$$

$$\begin{aligned}
\Lambda \ = \ &\left| \gamma_1 \sum_{j=1}^{\infty} \alpha_j + (\gamma_2 - \gamma_1) \sum_{j=2}^{\infty} \alpha_j \ + (\gamma_3 - \gamma_2) \sum_{j=3}^{\infty} \alpha_j \ + \cdots + (\gamma_k - \gamma_{k-1}) \sum_{j=k}^{\infty} \alpha_j \ + \ldots \right| \\
\vee \ &\left| \gamma_2 \sum_{j=2}^{\infty} \alpha_j + (\gamma_3 - \gamma_2) \sum_{j=3}^{\infty} \alpha_j \ + \cdots + (\gamma_k - \gamma_{k-1}) \sum_{j=k}^{\infty} \alpha_j \ + \ldots \right| \\
\vee \ &\cdots \vee \left| \gamma_k \sum_{j=k}^{\infty} \alpha_j + (\gamma_{k+1} - \gamma_k) \sum_{j=k+1}^{\infty} \alpha_j \ + \ldots \right| \vee \ldots
\end{aligned}$$

Hence,

$$\Lambda \;\leq\; \sup_{s\in\mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| \qquad (\star^{\star}_{\star})$$

Also,

$$\begin{aligned}
\Lambda \;\geq\; & \left(\gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_2-\gamma_1)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_3-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right| \\ + \dots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + \dots \end{array} \right] \right) \\
\vee\; & \left(\gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_3-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right| + (\gamma_4-\gamma_3)\left|\sum_{k=4}^{\infty} \alpha_k\right| \\ + \dots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + \dots \end{array} \right] \right) \dots \vee \\
\vee\; & \left(\gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| - \left[(\gamma_{k+1}-\gamma_k)\left|\sum_{j=k+1}^{\infty} \alpha_j\right| + (\gamma_{k+2}-\gamma_{k+1})\left|\sum_{j=k+2}^{\infty} \alpha_j\right| + \dots \right] \right) \\
\vee\; & \dots
\end{aligned}$$

Hence,

$$\left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_{\infty} \;\geq\; \gamma_1 \left| \sum_{k=1}^{\infty} \alpha_k \right| - \left[\begin{array}{c} (\gamma_2-\gamma_1)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_3-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right| \\ + \dots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + \dots \end{array} \right] \tag{3.25}$$

$$\left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_{\infty} \;\geq\; \gamma_2 \left| \sum_{k=2}^{\infty} \alpha_k \right| - \left[\begin{array}{c} (\gamma_3-\gamma_2)\left|\sum_{k=3}^{\infty} \alpha_k\right| + (\gamma_4-\gamma_3)\left|\sum_{k=4}^{\infty} \alpha_k\right| \\ + \dots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + \dots \end{array} \right] \tag{3.26}$$

$$\left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_{\infty} \;\geq\; \gamma_3 \left| \sum_{k=3}^{\infty} \alpha_k \right| - \left[\begin{array}{c} (\gamma_4-\gamma_3)\left|\sum_{k=4}^{\infty} \alpha_k\right| + (\gamma_5-\gamma_4)\left|\sum_{k=5}^{\infty} \alpha_k\right| \\ + \dots + (\gamma_k-\gamma_{k-1}) \left|\sum_{j=k}^{\infty} \alpha_j\right| + \dots \end{array} \right] \tag{3.27}$$

$$\vdots$$

$$\left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_{\infty} \;\geq\; \gamma_k \left| \sum_{j=k}^{\infty} \alpha_j \right| - \left[\begin{array}{c} (\gamma_{k+1}-\gamma_k)\left|\sum_{j=k+1}^{\infty} \alpha_j\right| \\ +(\gamma_{k+2}-\gamma_{k+1}) \left|\sum_{j=k+2}^{\infty} \alpha_j\right| + \dots \end{array} \right]$$

$$\vdots$$

Now, for every $n \in \mathbb{N}$ define

$$\tau_n := \left| \sum_{j=n}^{\infty} \alpha_j \right| \;,\; \Psi_n := 1 - \frac{\gamma_n}{\gamma_{n+1}} \text{ and } \Lambda := \left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_{\infty} .$$

Hence, using the inequalities (3.25) and (3.26)

$$\begin{aligned}
\tau_1 &\le \frac{1}{\gamma_1}\Lambda + \frac{\gamma_2-\gamma_1}{\gamma_1}\left[\frac{1}{\gamma_2}\Lambda + \frac{\gamma_3-\gamma_2}{\gamma_2}\tau_3 + \frac{\gamma_4-\gamma_3}{\gamma_2}\tau_4 + \frac{\gamma_5-\gamma_4}{\gamma_2}\tau_5 + \dots\right] \\
&+ \frac{\gamma_3-\gamma_2}{\gamma_1}\tau_3 + \frac{\gamma_4-\gamma_3}{\gamma_1}\tau_4 + \frac{\gamma_5-\gamma_4}{\gamma_1}\tau_5 + \dots \\
&= \left[\frac{1}{\gamma_1} + \frac{1}{\gamma_1}(1-\frac{\gamma_1}{\gamma_2})\right]\Lambda + \left[(\frac{\gamma_2-\gamma_1}{\gamma_1})(\frac{\gamma_3-\gamma_2}{\gamma_2}) + \frac{\gamma_3-\gamma_2}{\gamma_1}\right]\tau_3 \\
&+ \left[\frac{\gamma_4-\gamma_3}{\gamma_1} + (\frac{\gamma_2-\gamma_1}{\gamma_1})(\frac{\gamma_4-\gamma_3}{\gamma_2})\right]\tau_4 + \left[\frac{\gamma_5-\gamma_4}{\gamma_1} + (\frac{\gamma_2-\gamma_1}{\gamma_1})(\frac{\gamma_5-\gamma_4}{\gamma_2})\right]\tau_5 + \dots \\
&= \frac{1}{\gamma_1}\left[1 + (1-\frac{\gamma_1}{\gamma_2})\right]\Lambda + \frac{\gamma_3-\gamma_2}{\gamma_1}\left[1 + (1-\frac{\gamma_1}{\gamma_2})\right]\tau_3 + \frac{\gamma_4-\gamma_3}{\gamma_1}\left[1 + (1-\frac{\gamma_1}{\gamma_2})\right]\tau_4 \\
&+ \frac{\gamma_5-\gamma_4}{\gamma_1}\left[1 + (1-\frac{\gamma_1}{\gamma_2})\right]\tau_5 + \frac{\gamma_6-\gamma_5}{\gamma_1}\left[1 + (1-\frac{\gamma_1}{\gamma_2})\right]\tau_6 + \dots \\
&= \frac{1}{\gamma_1}\left[1+\Psi_1\right]\Lambda + \frac{\gamma_3-\gamma_2}{\gamma_1}\left[1+\Psi_1\right]\tau_3 + \frac{\gamma_4-\gamma_3}{\gamma_1}\left[1+\Psi_1\right]\tau_4 \\
&+ \frac{\gamma_5-\gamma_4}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{\gamma_6-\gamma_5}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \dots
\end{aligned}$$

Furthermore, by (3.27)

$$\left|\sum_{k=3}^{\infty}\alpha_k\right| \le \frac{1}{\gamma_3}\left(\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_{\infty} + \left[\begin{array}{c}(\gamma_4-\gamma_3)\left|\sum_{k=4}^{\infty}\alpha_k\right| + (\gamma_5-\gamma_4)\left|\sum_{k=5}^{\infty}\alpha_k\right| \\ + \dots + (\gamma_k-\gamma_{k-1})\left|\sum_{j=k}^{\infty}\alpha_j\right| + \dots\end{array}\right]\right)$$

Thus,

$$\begin{aligned}
\tau_1 &\le \frac{1}{\gamma_1}\left[1+\Psi_1\right]\Lambda + \frac{\gamma_3-\gamma_2}{\gamma_1}\left[1+\Psi_1\right]\tau_3 + \frac{\gamma_4-\gamma_3}{\gamma_1}\left[1+\Psi_1\right]\tau_4 \\
&+ \frac{\gamma_5-\gamma_4}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{\gamma_6-\gamma_5}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \dots \\
&\le \frac{1}{\gamma_1}\left[1+\Psi_1\right]\Lambda + \frac{\gamma_3-\gamma_2}{\gamma_1}\left[1+\Psi_1\right]\left[\begin{array}{c}\frac{1}{\gamma_3}\Lambda + \left(\frac{\gamma_4-\gamma_3}{\gamma_3}\right)\tau_4 \\ + \left(\frac{\gamma_5-\gamma_4}{\gamma_3}\right)\tau_5 + \left(\frac{\gamma_6-\gamma_5}{\gamma_3}\right)\tau_6 + \left(\frac{\gamma_7-\gamma_6}{\gamma_3}\right)\tau_7 + \dots\end{array}\right] \\
&+ \frac{\gamma_4-\gamma_3}{\gamma_1}\left[1+\Psi_1\right]\tau_4 + \frac{\gamma_5-\gamma_4}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{\gamma_6-\gamma_5}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \frac{\gamma_7-\gamma_6}{\gamma_1}\left[1+\Psi_1\right]\tau_7 + \dots
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\tau_1 &\leq \left[\frac{1}{\gamma_1}[1+\Psi_1] + \left(\frac{\gamma_3-\gamma_2}{\gamma_1}\right)[1+\Psi_1]\,\frac{1}{\gamma_3}\right]\Lambda \\
&+ \left[\left(\frac{\gamma_3-\gamma_2}{\gamma_1}\right)[1+\Psi_1]\left(\frac{\gamma_4-\gamma_3}{\gamma_3}\right) + \left(\frac{\gamma_4-\gamma_3}{\gamma_1}\right)[1+\Psi_1]\right]\tau_4 \\
&+ \left[\left(\frac{\gamma_5-\gamma_4}{\gamma_1}\right)[1+\Psi_1] + \left(\frac{\gamma_3-\gamma_2}{\gamma_1}\right)[1+\Psi_1]\left(\frac{\gamma_5-\gamma_4}{\gamma_3}\right)\right]\tau_5 \\
&+ \left[\left(\frac{\gamma_6-\gamma_5}{\gamma_1}\right)[1+\Psi_1] + \left(\frac{\gamma_3-\gamma_2}{\gamma_1}\right)[1+\Psi_1]\left(\frac{\gamma_6-\gamma_5}{\gamma_3}\right)\right]\tau_6 \\
&+ \left[\left(\frac{\gamma_7-\gamma_6}{\gamma_1}\right)[1+\Psi_1] + \left(\frac{\gamma_3-\gamma_2}{\gamma_1}\right)[1+\Psi_1]\left(\frac{\gamma_7-\gamma_6}{\gamma_3}\right)\right]\tau_7 + \dots \\
&= \frac{1}{\gamma_1}[1+\Psi_1]\left[1+\left(\frac{\gamma_3-\gamma_2}{\gamma_3}\right)\right]\Lambda + \frac{1}{\gamma_1}[1+\Psi_1]\left[1+\left(\frac{\gamma_3-\gamma_2}{\gamma_3}\right)\right](\gamma_4-\gamma_3)\tau_4 \\
&+ \frac{1}{\gamma_1}[1+\Psi_1]\left[1+\left(\frac{\gamma_3-\gamma_2}{\gamma_3}\right)\right](\gamma_5-\gamma_4)\tau_5 + \frac{1}{\gamma_1}[1+\Psi_1]\left[1+\left(\frac{\gamma_3-\gamma_2}{\gamma_3}\right)\right](\gamma_6-\gamma_5)\tau_6 \\
&+ \frac{1}{\gamma_1}[1+\Psi_1]\left[1+\left(\frac{\gamma_3-\gamma_2}{\gamma_3}\right)\right](\gamma_7-\gamma_6)\tau_7 + \dots \\
&= \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{\gamma_4-\gamma_3}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_4 \\
&+ \frac{\gamma_5-\gamma_4}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{\gamma_6-\gamma_5}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
&+ \frac{\gamma_7-\gamma_6}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \dots
\end{aligned}
$$

But since

$$
\left|\sum_{k=4}^{\infty}\alpha_k\right| \leq \frac{1}{\gamma_4}\left(\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_{\infty} + \left[\begin{array}{c}(\gamma_5-\gamma_4)\left|\sum_{k=5}^{\infty}\alpha_k\right| + (\gamma_6-\gamma_5)\left|\sum_{k=6}^{\infty}\alpha_k\right| \\ +\dots+(\gamma_k-\gamma_{k-1})\left|\sum_{j=k}^{\infty}\alpha_j\right| + \dots\end{array}\right]\right)
$$

Using the above inequality,

$$
\begin{aligned}
\tau_1 &\leq \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{\gamma_4-\gamma_3}{\gamma_1}[1+\Psi_1][1+\Psi_2]\left[\begin{array}{c}\frac{1}{\gamma_4}\Lambda + \left(\frac{\gamma_5-\gamma_4}{\gamma_4}\right)\tau_5 + \left(\frac{\gamma_6-\gamma_5}{\gamma_4}\right)\tau_6 \\ +\left(\frac{\gamma_7-\gamma_6}{\gamma_4}\right)\tau_7 + \left(\frac{\gamma_8-\gamma_7}{\gamma_4}\right)\tau_8 + \dots\end{array}\right] \\
&+ \frac{\gamma_5-\gamma_4}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{\gamma_6-\gamma_5}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
&+ \frac{\gamma_7-\gamma_6}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \dots
\end{aligned}
$$

$$
\begin{aligned}
\tau_1 &\leq \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2][1+\Psi_3]\Lambda \\
&+ \left[\left(\frac{\gamma_5-\gamma_4}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{\gamma_4-\gamma_3}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{\gamma_5-\gamma_4}{\gamma_4}\right)\right]\tau_5 \\
&+ \left[\left(\frac{\gamma_6-\gamma_5}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{\gamma_4-\gamma_3}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{\gamma_6-\gamma_5}{\gamma_4}\right)\right]\tau_6 \\
&+ \left[\left(\frac{\gamma_7-\gamma_6}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{\gamma_4-\gamma_3}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{\gamma_7-\gamma_6}{\gamma_4}\right)\right]\tau_7 + \dots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\tau_1 &\leq \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2][1+\Psi_3]\Lambda \\
&+ \left(\frac{\gamma_5-\gamma_4}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_5 \\
&+ \left(\frac{\gamma_6-\gamma_5}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_6 \\
&+ \left(\frac{\gamma_7-\gamma_6}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_7 + \dots
\end{aligned}
$$

Note

$$
(1+\Psi_1)(1+\Psi_2)(1+\Psi_3)\cdots = P_1 := \prod_{j=1}^{\infty}(1+\Psi_j) < \infty \text{ because } \sum_{j=1}^{\infty}\Psi_j < \infty .
$$

So,

$$
\begin{aligned}
\tau_1 &\leq \frac{1}{\gamma_1}P_1\Lambda + P_1\left(\sup_{k\geq 5}\tau_k\right)\left[\frac{\gamma_5-\gamma_4}{\gamma_1}+\frac{\gamma_6-\gamma_5}{\gamma_1}+\frac{\gamma_7-\gamma_6}{\gamma_1}+\dots\right] \\
&\leq \frac{1}{\gamma_1}P_1\Lambda + P_1\left(\sup_{k\geq 5}\tau_k\right)\left[\frac{1-\gamma_4}{\gamma_1}\right]
\end{aligned}
$$

Note also that

$$
\sup_{k\geq 5}\tau_k \leq K\Lambda
$$

where $K = \frac{1}{2\gamma_1 - 1} > 0$ Similarly, we can show inductively that $\forall \nu \in \mathbb{N}$,

$$
\begin{aligned}
\tau_1 &\leq \frac{1}{\gamma_1}P_1\Lambda + P_1\left(\sup_{k\geq \nu}\tau_k\right)\left[\frac{1-\gamma_\nu}{\gamma_1}\right] \\
&\leq \frac{1}{\gamma_1}P_1\Lambda + P_1\left(K\Lambda\right)\left[\frac{1-\gamma_\nu}{\gamma_1}\right] .
\end{aligned}
$$

Then, let $\nu \longrightarrow \infty$, and get $\tau_1 \leq \frac{P_1 \Lambda}{\gamma_1}$.

Similarly, $\forall m \in \mathbb{N}$,

$$\tau_m \leq \frac{P_m \Lambda}{\gamma_m} \ ; \ P_m := \prod_{j=m}^{\infty} (1 + \Psi_j) \ .$$

Note: $P_m \longrightarrow 1$ and $\gamma_m \longrightarrow 1$ as $m \longrightarrow \infty$,

$$\Lambda \geq \frac{\gamma_m}{P_m} \tau_m = \frac{\gamma_m}{P_m} \left| \sum_{j=m}^{\infty} \alpha_j \right| \ , \forall m \in \mathbb{N} \ .$$

Hence,

$$\left\| \sum_{j=1}^{\infty} \alpha_j \eta_j \right\|_{\infty} \geq \sup_{m \in \mathbb{N}} \frac{\gamma_m}{P_m} \left| \sum_{j=m}^{\infty} \alpha_j \right| \ , \ \forall \alpha = (\alpha_j)_{j \in \mathbb{N}} \in c_{00}$$

such that $0 < \frac{\gamma_m}{P_m} \longrightarrow 1$ as $m \longrightarrow \infty$. In fact, $\frac{\gamma_m}{P_m} \uparrow_m 1$. Thus, using the fact $(\star^{\star}_{\star})$ and the previous result, $(\eta_n)_{n \in \mathbb{N}}$ is an a.i. c_0-summing basic sequence. Hence, if we define the closed convex hull of $(\eta_n)_{n \in \mathbb{N}}$, $C := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, there exists an affine $\| \cdot \|_{\infty}$-nonexpansive mapping (or even contractive) mapping $U : C \longrightarrow C$ that is fixed point free, by Theorem 2.3.1.

□

3.5 OUR CONSTRUCTION WITH γ_n'S DECREASING TO 1 AND ALL b_n'S EQUAL TO 1. WE AGAIN CONFIRM THAT THE η_n'S ARE AN ASYMPTOTICALLY ISOMETRIC c_0-SUMMING BASIC SEQUENCE.

Here, we will investigate what happens when we have the set-up with η_n's defined in terms of sequences of constants: b_n's that are all 1 and γ_n's that are decreasing to 1 ; i.e. $\gamma_n \downarrow_n 1$. Hence, we aim to investigate the set-up

$$\eta_n := \gamma_n (e_1 + e_2 + e_3 + e_4 + ... + e_n) \ , \text{ for all } n \in \mathbb{N} \ ;$$

$$\text{where } (\gamma_n)_{n \in \mathbb{N}} \subset [1, \infty) \text{ with } \sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| = \gamma_1 - 1 < \infty \text{ such that } \gamma_n \downarrow_n 1 \ .$$

Hence, step by step as in previous work, we will study this case.

3.5.1 The decreasing case where all but finitely many γ_n's equal 1.

Example 3.5.1. Let $\gamma_1 = \gamma \in [1, \infty)$ and $\gamma_{n+1} = 1, \forall n \in \mathbb{N}$. Then, define

$$\begin{aligned}
\eta_1 &:= \gamma e_1 \\
\eta_2 &:= e_1 + e_2 \\
\eta_3 &:= e_1 + e_2 + e_3 \\
\eta_4 &:= e_1 + e_2 + e_3 + e_4 \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n \\
&\vdots
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda &:= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} = \|\alpha_1 \, \eta_1 + \alpha_2 \, \eta_2 + \alpha_3 \, \eta_3 + \ldots\|_{\infty} \\
&= \left\| \begin{array}{c} e_1 \, (\gamma \, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots\ldots\ldots) \\ +e_2 \, (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots\ldots\ldots) \\ +e_3 \, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots) + \ldots \end{array} \right\|_{\infty} \\
&= |\gamma\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \\
&\vee \; |\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots| \\
&\vee \; |\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots| \; \vee \ldots
\end{aligned}$$

$$\begin{aligned}
\Lambda &= |\gamma\,\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \vee \left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \ldots \\
&= |\gamma\,(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \ldots) + (1-\gamma)\,(\alpha_2 + \alpha_3 + \alpha_4 + \ldots)| \vee \left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \ldots \\
&= \left|\gamma \sum_{j=1}^{\infty} \alpha_j + (1-\gamma) \sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \ldots
\end{aligned}$$

Hence,

$$\Lambda \geq \left(\gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| - (\gamma - 1) \left|\sum_{k=2}^{\infty} \alpha_k\right|\right) \vee \sup_{k \geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right|$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| - (\gamma - 1) \left|\sum_{k=2}^{\infty} \alpha_k\right| \tag{3.28}$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \left|\sum_{j=k}^{\infty} \alpha_j\right|, \forall k \geq 2 \tag{3.29}$$

Hence, using (3.29) for $k = 2$ in (3.28), we get

$$\begin{aligned}
\gamma \left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} &\geq \gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| \text{ and so} \\
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} &\geq \left|\sum_{k=1}^{\infty} \alpha_k\right|
\end{aligned}$$

Hence,

$$\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \sup_{s \in \mathbb{N}} \left|\sum_{k=s}^{\infty} \alpha_k\right|$$

On the other hand, as we had previously defined

$$\tau_s = \left|\sum_{k=s}^{\infty} \alpha_k\right|, \forall s \in \mathbb{N} .$$

Thus,

$$\begin{aligned}
\Lambda &= \left|\gamma \sum_{j=1}^{\infty} \alpha_j + (1-\gamma) \sum_{j=2}^{\infty} \alpha_j \right| \vee \sup_{k\geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&\leq \left(\gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| + (\gamma-1) \left|\sum_{k=2}^{\infty} \alpha_k\right|\right) \vee \sup_{k\geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&\leq (\gamma \max\{\tau_1, \tau_2\} + (\gamma-1)\max\{\tau_1,\tau_2\}) \vee \sup_{k\geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma-1)\max\{\tau_1,\tau_2\} \vee \sup_{k\geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma-1)\,\tau_1 \vee (2\gamma-1)\,\tau_2 \vee \sup_{k\geq 2} \left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma-1)\left|\sum_{j=1}^{\infty} \alpha_j\right| \vee (2\gamma-1)\left|\sum_{j=2}^{\infty} \alpha_j\right| \vee \sup_{k\geq 3} \left|\sum_{j=k}^{\infty} \alpha_j\right| .
\end{aligned}$$

Hence, define $1+\varepsilon_1 = 1+\varepsilon_2 := 2\gamma-1$ and $1+\varepsilon_n := 1$ for all $n \in \mathbb{N}$ with $n \geq 3$ and so there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_1 = \varepsilon_2 = 2(\gamma-1) \geq 0$ and $\varepsilon_n = 0,\ \forall n \geq 3$ such that

$$\sup_{k\in\mathbb{N}} \frac{1}{(1+\varepsilon_k)} \left|\sum_{j=k}^{\infty} \alpha_j\right| \leq \left\|\sum_{k=1}^{\infty} \alpha_k\,\eta_k\right\|_{\infty} \leq \sup_{k\in\mathbb{N}} (1+\varepsilon_k) \left|\sum_{j=k}^{\infty} \alpha_j\right|$$

$\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$. Hence, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Example 3.5.2. Let $\gamma_1 = \gamma_2 = \gamma \in [1,\infty)$ and $\gamma_{n+2} = 1, \forall n \in \mathbb{N}$. Then, define

$$\begin{aligned}
\eta_1 &:= \gamma e_1 \\
\eta_2 &:= \gamma(e_1 + e_2) \\
\eta_3 &:= e_1 + e_2 + e_3 \\
\eta_4 &:= e_1 + e_2 + e_3 + e_4 \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n \\
&\vdots
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda &:= \left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} = \|\alpha_1\,\eta_1 + \alpha_2\,\eta_2 + \alpha_3\,\eta_3 + \dots\|_{\infty} \\
&= \left\| \begin{matrix} e_1\,(\gamma\,\alpha_1 + \gamma\,\alpha_2 + \alpha_3 + \alpha_4 + \dots\dots\dots) \\ +e_2\,(\gamma\,\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \dots\dots\dots) \\ +e_3\,(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \dots) + \dots \end{matrix} \right\|_{\infty} \\
&= |\gamma\alpha_1 + \gamma\,\alpha_2 + \alpha_3 + \alpha_4 + \dots| \\
&\vee\ |\gamma\,\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \dots| \\
&\vee\ |\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \dots|\ \vee \dots \\
&= |\gamma\,\alpha_1 + \gamma\,\alpha_2 + \alpha_3 + \alpha_4 + \dots| \vee |\gamma\,\alpha_2 + \alpha_3 + \alpha_4 + \dots|\ \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots \\
&= |\gamma\,(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \dots) + (1-\gamma)\,(\alpha_3 + \alpha_4 + \alpha_5 + \dots)| \\
&\quad \vee |\gamma\,(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \dots) + (1-\gamma)\,(\alpha_3 + \alpha_4 + \alpha_5 + \dots)| \\
&\quad \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots
\end{aligned}$$

$$\begin{aligned}
\Lambda &= \left|\gamma\,\sum_{j=1}^{\infty} \alpha_j + (1-\gamma)\,\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\gamma\,\sum_{j=2}^{\infty} \alpha_j + (1-\gamma)\,\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots \\
&= \left|\gamma\,\sum_{j=1}^{\infty} \alpha_j + (1-\gamma)\,\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\gamma\,\sum_{j=2}^{\infty} \alpha_j + (1-\gamma)\,\sum_{j=3}^{\infty} \alpha_j\right| \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right|
\end{aligned}$$

Hence,

$$\Lambda \geq \left(\gamma\left|\sum_{k=1}^{\infty} \alpha_k\right| - (\gamma-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right) \vee \left(\gamma\left|\sum_{k=2}^{\infty} \alpha_k\right| - (\gamma-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right) \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right|$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| - (\gamma - 1)\left|\sum_{k=3}^{\infty} \alpha_k\right| \tag{3.30}$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \gamma \left|\sum_{k=2}^{\infty} \alpha_k\right| - (\gamma - 1)\left|\sum_{k=3}^{\infty} \alpha_k\right| \tag{3.31}$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \left|\sum_{j=k}^{\infty} \alpha_j\right|, \forall k \geq 3 \tag{3.32}$$

Hence, using (3.32) for $k = 3$ in (3.31), we get

$$\begin{aligned} \gamma \left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} &\geq \gamma \left|\sum_{k=2}^{\infty} \alpha_k\right| \\ \left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} &\geq \left|\sum_{k=2}^{\infty} \alpha_k\right| \end{aligned}$$

Next, using this result above with (3.30)

$$\begin{aligned} \gamma \left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} &\geq \gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| \\ \left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} &\geq \left|\sum_{k=1}^{\infty} \alpha_k\right| \end{aligned}$$

Hence,

$$\left\|\sum_{j=1}^{\infty} \alpha_j \, \eta_j\right\|_{\infty} \geq \sup_{s \in \mathbb{N}} \left|\sum_{k=s}^{\infty} \alpha_k\right|$$

On the other hand,

$$
\begin{aligned}
\Lambda &= \left|\gamma \sum_{j=1}^{\infty} \alpha_j + (1-\gamma) \sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\gamma \sum_{j=2}^{\infty} \alpha_j + (1-\gamma) \sum_{j=3}^{\infty} \alpha_j\right| \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&\leq \left(\gamma \left|\sum_{k=1}^{\infty} \alpha_k\right| + (\gamma-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right) \vee \left(\gamma \left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right) \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&\leq (\gamma \max\{\tau_1,\tau_2,\tau_3\} + (\gamma-1)\max\{\tau_1,\tau_2,\tau_3\}) \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma-1)\max\{\tau_1,\tau_2,\tau_3\} \vee \sup_{k\geq 4}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma-1)\,\tau_1 \vee (2\gamma-1)\,\tau_2 \vee (2\gamma-1)\,\tau_3 \vee \sup_{k\geq 4}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma-1)\left|\sum_{j=1}^{\infty} \alpha_j\right| \vee (2\gamma-1)\left|\sum_{j=2}^{\infty} \alpha_j\right| \vee (2\gamma-1)\left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \sup_{k\geq 4}\left|\sum_{j=k}^{\infty} \alpha_j\right| .
\end{aligned}
$$

Hence, define $1+\varepsilon_1 = 1+\varepsilon_2 = 1+\varepsilon_3 := 2\gamma-1$ and $1+\varepsilon_n := 1$ for all $n \in \mathbb{N}$ with $n \geq 4$ and so there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2(\gamma-1) \geq 0$ and $\varepsilon_n = 0,\ \forall n \geq 4$ such that

$$
\sup_{k\in\mathbb{N}} \frac{1}{(1+\varepsilon_k)}\left|\sum_{j=k}^{\infty} \alpha_j\right| \leq \left\|\sum_{k=1}^{\infty} \alpha_k\,\eta_k\right\|_{\infty} \leq \sup_{k\in\mathbb{N}}(1+\varepsilon_k)\left|\sum_{j=k}^{\infty} \alpha_j\right| ,
$$

$\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$. Hence, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Example 3.5.3. Case for a fixed position $\nu > 2$ Let $\gamma_1 = \gamma_2 = \cdots = \gamma_\nu = \gamma \in [1, \infty)$ and $\gamma_{n+\nu} = 1, \forall n \in \mathbb{N}$. Then, define

$$\begin{aligned}
\eta_1 &:= \gamma e_1 \\
\eta_2 &:= \gamma(e_1 + e_2) \\
\eta_3 &:= \gamma(e_1 + e_2 + e_3) \\
\eta_4 &:= \gamma(e_1 + e_2 + e_3 + e_4) \\
&\vdots \\
\eta_{\nu-1} &:= \gamma(e_1 + e_2 + e_3 + e_4 + \cdots + e_{\nu-1}) \\
\eta_\nu &:= \gamma(e_1 + e_2 + e_3 + e_4 + \cdots + e_{\nu-1} + e_\nu) \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then, by the similar way to above, we can show,

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty \geq \sup_{s\in\mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right|$$

On the other hand,

$$\Lambda \leq (2\gamma - 1) \max_{1\leq k\leq \nu} \tau_k \vee \sup_{k\geq \nu+1} \left| \sum_{j=k}^{\infty} \alpha_j \right| .$$

Hence, define $1 + \varepsilon_1 = 1 + \varepsilon_2 = \cdots = 1 + \varepsilon_\nu := 2\gamma - 1$ and $1 + \varepsilon_n := 1$ for all $n \in \mathbb{N}$ with $n \geq \nu + 1$ and so there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_\nu = 2(\gamma - 1) \geq 0$ and $\varepsilon_n = 0, \ \forall n \geq \nu + 1$ such that

$$\sup_{k\in\mathbb{N}} \frac{1}{(1+\varepsilon_k)} \left| \sum_{j=k}^{\infty} \alpha_j \right| \leq \left\| \sum_{k=1}^{\infty} \alpha_k \, \eta_k \right\|_\infty \leq \sup_{k\in\mathbb{N}} (1+\varepsilon_k) \left| \sum_{j=k}^{\infty} \alpha_j \right| ,$$

$\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$. Hence, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Example 3.5.4. Let $\gamma_1 \geq \gamma_2 \geq 1$. Define

$$\begin{aligned}
\eta_1 &:= \gamma_1 e_1 \\
\eta_2 &:= \gamma_2 (e_1 + e_2) \\
\eta_3 &:= e_1 + e_2 + e_3 \\
\eta_4 &:= e_1 + e_2 + e_3 + e_4 \\
&\vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n \\
&\vdots
\end{aligned}$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary.

Then,

$$\begin{aligned}
\Lambda &:= \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty = \|\alpha_1 \, \eta_1 + \alpha_2 \, \eta_2 + \alpha_3 \, \eta_3 + \ldots\|_\infty \\
&= \left\| \begin{array}{c} e_1 \, (\gamma_1 \, \alpha_1 + \gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \ldots \ldots \ldots) \\ +e_2 \, (\gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots \ldots \ldots) \\ +e_3 \, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots) + \ldots \end{array} \right\|_\infty \\
&= |\gamma_1 \alpha_1 + \gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \\
&\vee \; |\gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots| \\
&\vee \; |\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \ldots| \; \vee \ldots \\
&= |\gamma_1 \, \alpha_1 + \gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \vee |\gamma_2 \, \alpha_2 + \alpha_3 + \alpha_4 + \ldots| \; \vee \left| \sum_{j=3}^{\infty} \alpha_j \right| \vee \left| \sum_{j=4}^{\infty} \alpha_j \right| \vee \ldots \\
&= |\gamma_1 \, (\alpha_1 + \alpha_2 + \ldots) + (\gamma_2 - \gamma_1) \, (\alpha_2 + \alpha_3 + \ldots) + (1 - \gamma_2) \, (\alpha_3 + \alpha_4 + \ldots)| \\
&\quad \vee |\gamma_2 \, (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots) + (1 - \gamma_2) \, (\alpha_3 + \alpha_4 + \alpha_5 + \ldots)| \\
&\quad \vee \left| \sum_{j=3}^{\infty} \alpha_j \right| \vee \left| \sum_{j=4}^{\infty} \alpha_j \right| \vee \ldots
\end{aligned}$$

$$
\begin{aligned}
\Lambda &= \left|\gamma_1 \sum_{j=1}^{\infty} \alpha_j + (\gamma_2-\gamma_1) \sum_{j=2}^{\infty} \alpha_j + (1-\gamma_2) \sum_{j=3}^{\infty} \alpha_j\right| \\
&\vee \left|\gamma_2 \sum_{j=2}^{\infty} \alpha_j + (1-\gamma_2) \sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=3}^{\infty} \alpha_j\right| \vee \left|\sum_{j=4}^{\infty} \alpha_j\right| \vee \dots \\
&= \left|\gamma_1 \sum_{j=1}^{\infty} \alpha_j + (\gamma_2-\gamma_1) \sum_{j=2}^{\infty} \alpha_j + (1-\gamma_2) \sum_{j=3}^{\infty} \alpha_j\right| \\
&\vee \left|\gamma_2 \sum_{j=2}^{\infty} \alpha_j + (1-\gamma_2) \sum_{j=3}^{\infty} \alpha_j\right| \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Lambda &\geq \left(\gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[(\gamma_1-\gamma_2)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_2-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right]\right) \\
&\vee \left(\gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - (\gamma_2-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right) \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right|
\end{aligned}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[(\gamma_1-\gamma_2)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_2-1)\left|\sum_{k=3}^{\infty} \alpha_k\right|\right] \tag{3.33}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - (\gamma_2-1)\left|\sum_{k=3}^{\infty} \alpha_k\right| \tag{3.34}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \left|\sum_{j=k}^{\infty} \alpha_j\right|, \forall k \geq 3 \tag{3.35}
$$

Hence, using (3.35) for $k = 3$ in (3.34), we get

$$
\gamma_2 \left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| \tag{3.36}
$$

$$
\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \geq \left|\sum_{k=2}^{\infty} \alpha_k\right| \tag{3.37}
$$

Next, using this result (3.37) above with (3.37)

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} &\geq \gamma_1 \left| \sum_{k=1}^{\infty} \alpha_k \right| - \left[(\gamma_1 - \gamma_2) \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} + (\gamma_2 - 1) \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \right] \\
&= \gamma_1 \left| \sum_{k=1}^{\infty} \alpha_k \right| - (\gamma_1 - 1) \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} .
\end{aligned}$$

Hence,

$$\gamma_1 \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \left| \sum_{k=1}^{\infty} \alpha_k \right|$$

$$\text{i.e.} \quad \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \left| \sum_{k=1}^{\infty} \alpha_k \right| .$$

Thus,

$$\left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_{\infty} \geq \sup_{s \in \mathbb{N}} \left| \sum_{k=s}^{\infty} \alpha_k \right| .$$

On the other hand,

$$\begin{aligned}
\Lambda &= \left| \gamma_1 \sum_{j=1}^{\infty} \alpha_j + (\gamma_2 - \gamma_1) \sum_{j=2}^{\infty} \alpha_j + (1 - \gamma_2) \sum_{j=3}^{\infty} \alpha_j \right| \\
&\vee \left| \gamma_2 \sum_{j=2}^{\infty} \alpha_j + (\gamma_2 - 1) \sum_{j=3}^{\infty} \alpha_j \right| \vee \sup_{k \geq 3} \left| \sum_{j=k}^{\infty} \alpha_j \right| \\
&\leq \left(\gamma_1 \left| \sum_{j=1}^{\infty} \alpha_j \right| + (\gamma_1 - \gamma_2) \left| \sum_{j=2}^{\infty} \alpha_j \right| + (\gamma_2 - 1) \left| \sum_{j=3}^{\infty} \alpha_j \right| \right) \\
&\vee \left(\gamma_2 \left| \sum_{j=2}^{\infty} \alpha_j \right| + (\gamma_2 - 1) \left| \sum_{j=3}^{\infty} \alpha_j \right| \right) \vee \sup_{k \geq 3} \left| \sum_{j=k}^{\infty} \alpha_j \right| \\
&= (\gamma_1 \, \tau_1 + (\gamma_1 - \gamma_2) \, \tau_2 + (\gamma_2 - 1) \, \tau_3) \\
&\vee (\gamma_2 \, \tau_2 + (\gamma_2 - 1) \, \tau_3) \vee \sup_{k \geq 3} \left| \sum_{j=k}^{\infty} \alpha_j \right| \\
&\leq (\gamma_1 \max \{\tau_1, \tau_2, \tau_3\} + (\gamma_1 - \gamma_2) \max \{\tau_1, \tau_2, \tau_3\} + (\gamma_2 - 1) \max \{\tau_1, \tau_2, \tau_3\}) \\
&\vee (\gamma_2 \max \{\tau_2, \tau_3\} + (\gamma_2 - 1) \max \{\tau_2, \tau_3\}) \vee \sup_{k \geq 3} \left| \sum_{j=k}^{\infty} \alpha_j \right| \\
&\leq (\gamma_1 \max \{\tau_1, \tau_2, \tau_3\} + (\gamma_1 - \gamma_2) \max \{\tau_1, \tau_2, \tau_3\} + (\gamma_2 - 1) \max \{\tau_1, \tau_2, \tau_3\}) \\
&\vee (\gamma_2 \max \{\tau_1, \tau_2, \tau_3\} + (\gamma_2 - 1) \max \{\tau_1, \tau_2, \tau_3\}) \vee \sup_{k \geq 3} \left| \sum_{j=k}^{\infty} \alpha_j \right|
\end{aligned}$$

$$
\begin{aligned}
&= (2\gamma_1 - 1)\ \max\{\tau_1, \tau_2, \tau_3\} \ \vee (2\gamma_2 - 1)\ \max\{\tau_1, \tau_2, \tau_3\} \ \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&\leq (2\gamma_1 - 1)\ \max\{\tau_1, \tau_2, \tau_3\} \ \vee (2\gamma_1 - 1)\ \max\{\tau_1, \tau_2, \tau_3\} \ \vee \sup_{k\geq 3}\left|\sum_{j=k}^{\infty} \alpha_j\right| \\
&= (2\gamma_1 - 1)\ \tau_1 \ \vee (2\gamma_1 - 1)\ \tau_2 \ \vee (2\gamma_1 - 1)\ \tau_3 \ \vee \sup_{k\geq 4}\left|\sum_{j=k}^{\infty} \alpha_j\right| .
\end{aligned}
$$

Hence, define $1+\varepsilon_1 = 1+\varepsilon_2 = 1+\varepsilon_3 := 2\gamma_1 - 1$ and $1+\varepsilon_n := 1$ for all $n \in \mathbb{N}$ with $n \geq 4$ and so there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2(\gamma_1 - 1) \geq 0$ and $\varepsilon_n = 0, \ \forall n \geq 4$ such that

$$
\sup_{k\in\mathbb{N}} \frac{1}{(1+\varepsilon_k)}\left|\sum_{j=k}^{\infty} \alpha_j\right| \leq \left\|\sum_{k=1}^{\infty} \alpha_k\, \eta_k\right\|_{\infty} \leq \sup_{k\in\mathbb{N}} (1+\varepsilon_k)\left|\sum_{j=k}^{\infty} \alpha_j\right|
$$

$\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$. Hence, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

3.5.2 General case for decreasing $(\gamma_n)_{n\in\mathbb{N}}$ with infinitely many γ_n's equal to 1.

Example 3.5.5. Let $3 \leq k \in \mathbb{N}$ and $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \cdots \geq \gamma_k \geq \gamma_n = 1$, $\forall n > k$. Define

$$
\begin{aligned}
\eta_1 &:= \gamma_1 e_1 \\
\eta_2 &:= \gamma_2(e_1 + e_2) \\
\eta_3 &:= \gamma_3(e_1 + e_2 + e_3) \\
\eta_4 &:= \gamma_4(e_1 + e_2 + e_3 + e_4) \\
&\ \vdots \\
\eta_k &:= \gamma_k(e_1 + e_2 + e_3 + e_4 + \cdots + e_k) \\
\eta_{k+1} &:= e_1 + e_2 + e_3 + e_4 + \cdots + e_k + e_{k+1} \\
&\ \vdots \\
\eta_n &:= e_1 + e_2 + e_3 + e_4 + ... + e_n \\
&\ \vdots
\end{aligned}
$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Let $(\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ be arbitrary. Then,

$$
\begin{aligned}
\Lambda &:= \left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_\infty = \|\alpha_1\,\eta_1 + \alpha_2\,\eta_2 + \alpha_3\,\eta_3 + \alpha_4\,\eta_4 + \cdots + \alpha_k\,\eta_k + \dots\|_\infty \\
&= \left\| \begin{array}{c} e_1\left(\gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \alpha_{k+1} + \dots\dots\dots\right) \\ +e_2\left(\gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_{k-1}\alpha_{k-1} + \gamma_k\alpha_k + \alpha_{k+1} + \dots\dots\dots\right) \\ + \cdots + e_k\left(\gamma_k\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots\right) \\ +e_{k+1}\left(\alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots\right) + \dots \end{array} \right\|_\infty \\
&= |\gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\ |\gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_{k-1}\alpha_{k-1} + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\ \cdots \vee |\gamma_k\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots| \\
&\vee\ |\alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots|\ \vee \dots \\
&= |\gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\ |\gamma_2\,\alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_{k-1}\alpha_{k-1} + \gamma_k\alpha_k + \alpha_{k+1} + \dots| \\
&\vee\ \cdots \vee |\gamma_k\alpha_k + \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} + \dots| \vee \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \vee \left|\sum_{j=k+2}^{\infty} \alpha_j\right| \vee \dots \\
&= \left| \begin{array}{c} \gamma_1\,\sum_{j=1}^{\infty}\alpha_j + (\gamma_2-\gamma_1)\,\sum_{j=2}^{\infty}\alpha_j\ + (\gamma_3-\gamma_2)\,\sum_{j=3}^{\infty}\alpha_j \\ + \cdots + (\gamma_k-\gamma_{k-1})\,\sum_{j=k}^{\infty}\alpha_j\ + (1-\gamma_k)\,\sum_{j=k+1}^{\infty}\alpha_j \end{array} \right| \\
&\vee\ \left|\gamma_2\sum_{j=2}^{\infty}\alpha_j + (\gamma_3-\gamma_2)\sum_{j=3}^{\infty}\alpha_j\ + \cdots + (\gamma_k-\gamma_{k-1})\sum_{j=k}^{\infty}\alpha_j\ + (1-\gamma_k)\sum_{j=k+1}^{\infty}\alpha_j\right| \\
&\vee\ \cdots \vee \left|\gamma_k\sum_{j=k}^{\infty}\alpha_j + (1-\gamma_k)\ \sum_{j=k+1}^{\infty}\alpha_j\right| \vee \left|\sum_{j=k+1}^{\infty}\alpha_j\right| \vee \left|\sum_{j=k+2}^{\infty}\alpha_j\right| \vee \dots
\end{aligned}
$$

Hence, as in previous examples,

$$
\begin{aligned}
\Lambda\ &\geq\ \left(\gamma_1\left|\sum_{k=1}^{\infty}\alpha_k\right| - \left[\begin{array}{c} (\gamma_1-\gamma_2)\left|\sum_{k=2}^{\infty}\alpha_k\right| + (\gamma_2-\gamma_3)\left|\sum_{k=3}^{\infty}\alpha_k\right| \\ + \cdots + (\gamma_{k-1}-\gamma_k)\left|\sum_{j=k}^{\infty}\alpha_j\right| + (\gamma_k-1)\left|\sum_{j=k+1}^{\infty}\alpha_j\right| \end{array}\right]\right) \\
&\vee\ \left(\gamma_2\left|\sum_{k=2}^{\infty}\alpha_k\right| - \left[\begin{array}{c} (\gamma_2-\gamma_3)\left|\sum_{k=3}^{\infty}\alpha_k\right| + (\gamma_3-\gamma_4)\left|\sum_{k=4}^{\infty}\alpha_k\right| \\ + \cdots + (\gamma_{k-1}-\gamma_k)\left|\sum_{j=k}^{\infty}\alpha_j\right| + (\gamma_k-1)\left|\sum_{j=k+1}^{\infty}\alpha_j\right| \end{array}\right]\right) \\
&\vee\ \cdots \vee \left(\gamma_k\left|\sum_{j=k}^{\infty}\alpha_j\right| - (\gamma_k-1)\left|\sum_{j=k+1}^{\infty}\alpha_j\right|\right) \vee \sup_{s\geq k+1}\left|\sum_{j=s}^{\infty}\alpha_j\right|
\end{aligned}
$$

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_1-\gamma_2)\,|\sum_{k=2}^{\infty} \alpha_k\,| + (\gamma_2-\gamma_3)\,|\sum_{k=3}^{\infty} \alpha_k\,| \\ + \cdots + (\gamma_{k-1}-\gamma_k)\, \left|\sum_{j=k}^{\infty} \alpha_j\right| + (\gamma_k - 1)\, \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array} \right] \\
\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| - \left[\begin{array}{c} (\gamma_2-\gamma_3)\,|\sum_{k=3}^{\infty} \alpha_k\,| + (\gamma_3-\gamma_4)\,|\sum_{k=4}^{\infty} \alpha_k\,| \\ + \cdots + (\gamma_{k-1}-\gamma_k)\, \left|\sum_{j=k}^{\infty} \alpha_j\right| + (\gamma_k - 1)\, \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array} \right] \\
&\vdots \\
\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| - (\gamma_k - 1) \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \\
\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \left|\sum_{j=s}^{\infty} \alpha_j\right|, \forall s \geq k+1
\end{aligned}$$

Hence, using the steps above with the last one for $j = k+1$, we get

$$\gamma_k \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} \geq \gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| \tag{3.38}$$

$$\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} \geq \left|\sum_{j=k}^{\infty} \alpha_j\right| \tag{3.39}$$

Next, using this result (3.39) above with the block of inequalities above, we see that

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \gamma_{k-1} \left|\sum_{j=k-1}^{\infty} \alpha_j\right| - \left[(\gamma_{k-1}-\gamma_k) \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} + (\gamma_k - 1) \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty}\right] \\
&= \gamma_{k-1} \left|\sum_{j=k-1}^{\infty} \alpha_j\right| - (\gamma_{k-1} - 1) \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} .
\end{aligned}$$

Hence,

$$\begin{aligned}
\gamma_{k-1} \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \gamma_{k-1} \left|\sum_{j=k}^{\infty} \alpha_j\right| \\
\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} &\geq \left|\sum_{j=k-1}^{\infty} \alpha_j\right| .
\end{aligned}$$

Thus, we inductively get

$$\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} \geq \sup_{n \in \mathbb{N}} \left|\sum_{j=n}^{\infty} \alpha_j\right| .$$

On the other hand,

$$\begin{aligned}
\Lambda \;\le\; & \left(\begin{array}{c} \gamma_1 \left|\sum_{k=1}^{\infty} \alpha_k\right| + (\gamma_1-\gamma_2)\left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_2-\gamma_3)\left|\sum_{k=3}^{\infty} \alpha_k\right| \\ + \cdots + (\gamma_{k-1}-\gamma_k)\left|\sum_{j=k}^{\infty} \alpha_j\right| + (\gamma_k - 1)\left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array} \right) \\
\vee \; & \left(\begin{array}{c} \gamma_2 \left|\sum_{k=2}^{\infty} \alpha_k\right| + (\gamma_2-\gamma_3)\left|\sum_{k=3}^{\infty} \alpha_k\right| + (\gamma_3-\gamma_4)\left|\sum_{k=4}^{\infty} \alpha_k\right| \\ + \cdots + (\gamma_{k-1}-\gamma_k)\left|\sum_{j=k}^{\infty} \alpha_j\right| + (\gamma_k - 1)\left|\sum_{j=k+1}^{\infty} \alpha_j\right| \end{array} \right) \\
\vee \; & \cdots \vee \left(\gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| + (\gamma_k-1)\left|\sum_{j=k+1}^{\infty} \alpha_j\right| \right) \vee \sup_{s\ge k+1} \left|\sum_{j=s}^{\infty} \alpha_j\right| \\
= \; & \left(\gamma_1\tau_1 + (\gamma_1-\gamma_2)\tau_2 + (\gamma_2-\gamma_3)\tau_3 + \cdots + (\gamma_{k-1}-\gamma_k)\,\tau_k + (\gamma_k-1)\,\tau_{k+1} \right) \\
\vee \; & \left(\gamma_2\tau_2 + (\gamma_2-\gamma_3)\tau_3 + (\gamma_3-\gamma_4)\tau_4 + \cdots + (\gamma_{k-1}-\gamma_k)\,\tau_k + (\gamma_k-1)\,\tau_{k+1} \right) \\
\vee \; & \cdots \vee (\gamma_k\tau_k + (\gamma_k-1)\tau_{k+1}) \vee \sup_{s\ge k+1} \left|\sum_{j=s}^{\infty} \alpha_j\right|
\end{aligned}$$

$$\begin{aligned}
\Lambda \;\le\; & \gamma_1 \max_{1\le j\le k+1} \tau_j + (\gamma_1-\gamma_2) \max_{1\le j\le k+1} \tau_j + (\gamma_2-\gamma_3) \max_{1\le j\le k+1} \tau_j \\
& \qquad + \cdots + (\gamma_{k-1}-\gamma_k) \max_{1\le j\le k+1} \tau_j + (\gamma_k-1) \max_{1\le j\le k+1} \tau_j \\
\vee \; & \gamma_2 \max_{1\le j\le k+1} \tau_j + (\gamma_2-\gamma_3) \max_{1\le j\le k+1} \tau_j + (\gamma_3-\gamma_4) \max_{1\le j\le k+1} \tau_j \\
& \qquad + \cdots + (\gamma_{k-1}-\gamma_k) \max_{1\le j\le k+1} \tau_j + (\gamma_k-1) \max_{1\le j\le k+1} \tau_j \\
\vee \; & \cdots \vee \gamma_k \max_{1\le j\le k+1} \tau_j + (\gamma_k-1) \max_{1\le j\le k+1} \tau_j \vee \sup_{s\ge k+1} \left|\sum_{j=s}^{\infty} \alpha_j\right| \\
= \; & (2\gamma_1-1) \max_{1\le j\le k+1} \tau_j \vee (2\gamma_2-1) \max_{1\le j\le k+1} \tau_j \vee \cdots \vee (2\gamma_k-1) \max_{1\le j\le k+1} \tau_j \\
\vee \; & \sup_{s\ge k+1} \left|\sum_{j=s}^{\infty} \alpha_j\right| \\
\le \; & (2\gamma_1-1) \max_{1\le j\le k+1} \tau_j \vee \sup_{s\ge k+2} \left|\sum_{j=s}^{\infty} \alpha_j\right| .
\end{aligned}$$

Hence, define $1+\varepsilon_1 = 1+\varepsilon_2 = \cdots = 1+\varepsilon_k = 1+\varepsilon_{k+1} := 2\gamma_1 - 1$ and $1+\varepsilon_n := 1$ for all $n \in \mathbb{N}$ with $n \ge k+2$ and so there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = \varepsilon_{k+1} = 2(\gamma_1 - 1) \ge 0$ and $\varepsilon_n = 0,\ \forall n \ge k+2$ such that

$$\sup_{s\in\mathbb{N}} \frac{1}{(1+\varepsilon_s)} \left|\sum_{j=s}^{\infty} \alpha_j\right| \le \left\|\sum_{s=1}^{\infty} \alpha_j\,\eta_j\right\|_{\infty} \le \sup_{s\in\mathbb{N}} (1+\varepsilon_s) \left|\sum_{j=s}^{\infty} \alpha_j\right| ,$$

$\forall(\alpha_j)_{j\in\mathbb{N}} \in c_{00}$. Hence, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence.

Next, we will see a more general result.

3.5.3 General case for decreasing convergent γ_n's when all b_n's are 1

Theorem 3.5.6. *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$\sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$$

with $\vec{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ a decreasing sequence converging to 1 (i.e., $\gamma_n \geq \gamma_{n+1}$, for all $n \in \mathbb{N}$) in $(1, \infty)$ with $\gamma_n \downarrow_n 1$. We define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\eta_n \;:=\; \gamma_n(e_1 + e_2 + e_3 + e_4 + ... + e_n)\ ,\ \textit{for all } n \in \mathbb{N}\ .$$

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an asymptotically isometric c_0-summing basic sequence. Furthermore, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, Then, there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

Proof.

$$\begin{aligned}
\Lambda \;&:=\; \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_\infty = \|\alpha_1\, \eta_1 + \alpha_2\, \eta_2 + \alpha_3\, \eta_3 + \alpha_4\, \eta_4 + \cdots + \alpha_k\, \eta_k + \ldots\|_\infty \\
&=\; \left\| \begin{array}{c} e_1\,(\gamma_1\, \alpha_1 + \gamma_2\, \alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \ldots\ldots\ldots) \\ +e_2\,(\gamma_2\, \alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_k\alpha_k + \ldots\ldots\ldots) \\ + \cdots + e_\nu\,(\gamma_\nu\alpha_\nu + \gamma_{\nu+1}\alpha_{\nu+1} + \gamma_{\nu+2}\alpha_{\nu+2} + \gamma_{\nu+3}\alpha_{\nu+3} + \ldots) + \ldots \end{array} \right\|_\infty \\
&=\; |\gamma_1\, \alpha_1 + \gamma_2\, \alpha_2 + \gamma_3\alpha_3 + \gamma_4\alpha_4 + \cdots + \gamma_k\alpha_k + \ldots| \\
&\vee\;\; |\gamma_2\, \alpha_2 + \gamma_3\alpha_3 + \cdots + \gamma_k\alpha_k + \ldots| \\
&\vee\;\; \cdots \vee |\gamma_\nu\alpha_\nu + \gamma_{\nu+1}\alpha_{\nu+1} + \gamma_{\nu+2}\alpha_{\nu+2} + \gamma_{\nu+3}\alpha_{\nu+3} + \ldots)| \vee \ldots \\
&=\; \left|\gamma_1 \sum_{j=1}^{\infty} \alpha_j + (\gamma_2 - \gamma_1) \sum_{j=2}^{\infty} \alpha_j\, + (\gamma_3 - \gamma_2) \sum_{j=3}^{\infty} \alpha_j\, + \cdots + (\gamma_k - \gamma_{k-1}) \sum_{j=k}^{\infty} \alpha_j\, + \ldots\right| \\
&\vee\;\; \left|\gamma_2 \sum_{j=2}^{\infty} \alpha_j + (\gamma_3 - \gamma_2) \sum_{j=3}^{\infty} \alpha_j\, + \cdots + (\gamma_k - \gamma_{k-1}) \sum_{j=k}^{\infty} \alpha_j\, + \ldots\right| \\
&\vee\;\; \cdots \vee \left|\gamma_\nu \sum_{j=\nu}^{\infty} \alpha_j + (\gamma_{\nu+1} - \gamma_\nu) \sum_{j=\nu+1}^{\infty} \alpha_j\, + \ldots\right| \vee \ldots
\end{aligned}$$

Recall from our first investigations considering b_n's that are 1 for each $n \in \mathbb{N}$, we get that $\forall (\alpha_j)_{j\in\mathbb{N}} \in c_{00}$ and $\forall \nu \in \mathbb{N}$,

$$(2\gamma_\nu - 1) \sup_{s\geq\nu} \left| \sum_{k=s}^{\infty} \alpha_k \right| \geq \left\| \sum_{j=1}^{\infty} \alpha_j \, \eta_j \right\|_\infty \geq \sup_{s\geq\nu} \left| \sum_{k=s}^{\infty} \alpha_k \right| . \qquad (\star^{\star}_{\star}\star^{\star}_{\star})$$

Hence, here we can easily see that usual right shift mapping is asymptotically nonexpansive, affine and fixed point free on $C := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$. However, we aim to get stronger result as our theorem states. Now, we prove this. Fix $(\alpha_j)_{j\in\mathbb{N}} \in c_{00}$.

$$\begin{aligned}
\Lambda \leq & \left[\gamma_1 \left| \sum_{j=1}^{\infty} \alpha_j \right| + (\gamma_1 - \gamma_2) \left| \sum_{j=2}^{\infty} \alpha_j \right| + (\gamma_2 - \gamma_3) \left| \sum_{j=3}^{\infty} \alpha_j \right| \right. \\
& \qquad \left. + \dots + (\gamma_{k-1} - \gamma_k) \left| \sum_{j=k}^{\infty} \alpha_j \right| + \dots \right] \\
\vee & \left[\gamma_2 \left| \sum_{j=2}^{\infty} \alpha_j \right| + (\gamma_2 - \gamma_3) \left| \sum_{j=3}^{\infty} \alpha_j \right| + (\gamma_3 - \gamma_4) \left| \sum_{j=4}^{\infty} \alpha_j \right| \right. \\
& \qquad \left. + \dots + (\gamma_{k-1} - \gamma_k) \left| \sum_{j=k}^{\infty} \alpha_j \right| + \dots \right] \vee \dots \\
\vee & \left[\gamma_\nu \left| \sum_{j=\nu}^{\infty} \alpha_j \right| + (\gamma_\nu - \gamma_{\nu+1}) \left| \sum_{j=\nu+1}^{\infty} \alpha_j \right| + (\gamma_{\nu+1} - \gamma_{\nu+2}) \left| \sum_{j=\nu+2}^{\infty} \alpha_j \right| + \dots \right] \\
\vee & \dots \\
= & \left[\gamma_1 \tau_1 + (\gamma_1 - \gamma_2)\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + \dots + (\gamma_{k-1} - \gamma_k)\, \tau_k + \dots \right] \\
\vee & \left[\gamma_2 \tau_2 + (\gamma_2 - \gamma_3)\tau_3 + (\gamma_3 - \gamma_4)\tau_4 + \dots + (\gamma_{k-1} - \gamma_k)\, \tau_k + \dots \right] \vee \dots \\
\vee & \left[\gamma_\nu \tau_\nu + (\gamma_\nu - \gamma_{\nu+1})\tau_{\nu+1} + (\gamma_{\nu+1} - \gamma_{\nu+2})\tau_{\nu+2} + \dots \right] \\
\vee & \dots
\end{aligned}$$

Case 1: Suppose $\gamma_1 - 1 < 1 \Leftrightarrow \gamma_1 < 2$.

Consider the sub-case where

$$\Lambda \leq \gamma_1 \tau_1 + (\gamma_1 - \gamma_2)\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + \dots$$

But recall

$$\Lambda \geq \sup_{k\in\mathbb{N}} \tau_k \ .$$

Then,

$$\begin{aligned}\Lambda &\leq \gamma_1\tau_1 + (\gamma_1 - \gamma_2)\Lambda + (\gamma_2 - \gamma_3)\Lambda + \dots \\ &= \gamma_1\tau_1 + (\gamma_1 - 1)\Lambda \ .\end{aligned}$$

Hence,

$$\Lambda \leq \frac{\gamma_1}{2-\gamma_1}\tau_1 \ .$$

Next, consider the sub-case where

$$\Lambda \leq \gamma_2\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + (\gamma_3 - \gamma_4)\tau_4 + \dots$$

Then,

$$\begin{aligned}\Lambda &\leq \gamma_2\tau_2 + (\gamma_2 - \gamma_3)\Lambda + (\gamma_3 - \gamma_4)\Lambda + \dots \\ &= \gamma_2\tau_2 + (\gamma_2 - 1)\Lambda \ .\end{aligned}$$

Hence,

$$\Lambda \leq \frac{\gamma_2}{2-\gamma_2}\tau_2 \ .$$

We see that for some $k \in \mathbb{N}$ (depending on $(\alpha_j)_j$)

$$\Lambda \leq \frac{\gamma_k}{2-\gamma_k}\tau_k \ .$$

Hence,

$$\Lambda \leq \sup_{\mu \geq 1} \frac{\gamma_\mu}{2-\gamma_\mu} \left| \sum_{k=\mu}^{\infty} \alpha_k \right| \ .$$

Case 2: Suppose $\gamma_1 \geq 2 > \gamma_2$.

Consider the sub-case where

$$\Lambda \leq \gamma_1\tau_1 + (\gamma_1 - \gamma_2)\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + \dots$$

Then,

$$\begin{aligned}
\Lambda &\leq \gamma_1\tau_1 + (\gamma_1 - \gamma_2)\tau_2 + (\gamma_2 - \gamma_3)\Lambda + (\gamma_3 - \gamma_4)\Lambda + \ldots \\
&= \gamma_1\tau_1 + (\gamma_1 - \gamma_2)\tau_2 + (\gamma_2 - 1)\Lambda \\
&\leq \gamma_1 \max\{\tau_1, \tau_2\} + (\gamma_1 - \gamma_2)\max\{\tau_1, \tau_2\} + (\gamma_2 - 1)\Lambda \\
&\leq (2\gamma_1 - \gamma_2)\tau_1 \vee \tau_2 + (\gamma_2 - 1)\Lambda \ .
\end{aligned}$$

Hence,

$$\begin{gathered}
(2 - \gamma_2)\Lambda \leq (2\gamma_1 - \gamma_2)\tau_1 \vee (2\gamma_1 - \gamma_2)\tau_2 \\
\Rightarrow \ \Lambda \leq \left(\frac{2\gamma_1 - \gamma_2}{2 - \gamma_2}\right)\tau_1 \vee \left(\frac{2\gamma_1 - \gamma_2}{2 - \gamma_2}\right)\tau_2 \ .
\end{gathered}$$

Next, consider the sub-case where

$$\Lambda \leq \gamma_2\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + (\gamma_3 - \gamma_4)\tau_4 + \ldots$$

Then,

$$\begin{aligned}
\Lambda &\leq \gamma_2\tau_2 + (\gamma_2 - \gamma_3)\Lambda + (\gamma_3 - \gamma_4)\Lambda + \ldots \\
&= \gamma_2\tau_2 + (\gamma_2 - 1)\Lambda \ .
\end{aligned}$$

Hence,

$$\Lambda \leq \frac{\gamma_2}{2 - \gamma_2}\tau_2 \ .$$

For all the other sub-cases, i.e. if $3 \leq k \in \mathbb{N}$,

$$\Lambda \leq \frac{\gamma_k}{2 - \gamma_k}\tau_k \ .$$

Hence,

$$\begin{aligned}
\Lambda &\leq \left(\frac{2\gamma_1 - \gamma_2}{2 - \gamma_2}\right)\tau_1 \vee \left(\frac{2\gamma_1 - \gamma_2}{2 - \gamma_2}\right)\tau_2 \vee \sup_{\mu \geq 2} \frac{\gamma_\mu}{2 - \gamma_\mu}\left|\sum_{k=\mu}^{\infty} \alpha_k\right| \\
&= \left(\frac{2\gamma_1 - \gamma_2}{2 - \gamma_2}\right)\tau_1 \vee \left(\frac{2\gamma_1 - \gamma_2}{2 - \gamma_2}\right)\tau_2 \vee \sup_{\mu \geq 3} \frac{\gamma_\mu}{2 - \gamma_\mu}\left|\sum_{k=\mu}^{\infty} \alpha_k\right|
\end{aligned}$$

Case 3: Now, fix $\nu \in \mathbb{N}, \nu > 1$. Suppose $\gamma_\nu \geq 2 > \gamma_{\nu+1}$.

Consider the sub-case where

$$\Lambda \leq \gamma_1\tau_1 + (\gamma_1 - \gamma_2)\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + \dots$$

Then,

$$\begin{aligned}
\Lambda &\leq \gamma_1\tau_1 + (\gamma_1 - \gamma_2)\tau_2 + \cdots + (\gamma_\nu - \gamma_{\nu+1})\tau_{\nu+1} + (\gamma_{\nu+1} - 1)\Lambda \\
&\leq \gamma_1 \max\{\tau_1, \tau_2, \dots, \tau_{\nu+1}\} + (\gamma_1 - \gamma_2)\max\{\tau_1, \tau_2, \dots, \tau_{\nu+1}\} + \dots \\
&+ (\gamma_\nu - \gamma_{\nu+1})\max\{\tau_1, \tau_2, \dots, \tau_{\nu+1}\} + (\gamma_{\nu+1} - 1)\Lambda \\
&\leq (2\gamma_1 - \gamma_{\nu+1})\{\tau_1 \vee \tau_2 \vee \cdots \vee \tau_{\nu+1}\} + (\gamma_{\nu+1} - 1)\Lambda\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
(2 - \gamma_{\nu+1})\Lambda \leq (2\gamma_1 - \gamma_{\nu+1})\tau_1 \vee (2\gamma_1 - \gamma_{\nu+1})\tau_2 \vee \cdots \vee (2\gamma_1 - \gamma_{\nu+1})\tau_{\nu+1} \\
\Rightarrow\ \Lambda \leq \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_1 \vee \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_2 \vee \cdots \vee \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_{\nu+1} \\
\Leftrightarrow\ \Lambda \leq \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\max_{1\leq \mathrm{J}\leq \nu+1}\tau_j\ .
\end{aligned}$$

Next, consider the sub-case where

$$\Lambda \leq \gamma_2\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + (\gamma_3 - \gamma_4)\tau_4 + \dots$$

Then,

$$\begin{aligned}
\Lambda &\leq \gamma_2\tau_2 + (\gamma_2 - \gamma_3)\tau_3 + \cdots + (\gamma_\nu - \gamma_{\nu+1})\tau_{\nu+1} + (\gamma_{\nu+1} - 1)\Lambda \\
&\leq \gamma_2 \max\{\tau_2, \tau_3, \dots, \tau_{\nu+1}\} + (\gamma_2 - \gamma_3)\max\{\tau_2, \tau_3, \dots, \tau_{\nu+1}\} + \dots \\
&+ (\gamma_\nu - \gamma_{\nu+1})\max\{\tau_2, \tau_3, \dots, \tau_{\nu+1}\} + (\gamma_{\nu+1} - 1)\Lambda \\
&\leq (2\gamma_2 - \gamma_{\nu+1})\{\tau_2 \vee \tau_3 \vee \cdots \vee \tau_{\nu+1}\} + (\gamma_{\nu+1} - 1)\Lambda\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
(2 - \gamma_{\nu+1})\Lambda &\leq (2\gamma_2 - \gamma_{\nu+1})\tau_2 \vee (2\gamma_2 - \gamma_{\nu+1})\tau_3 \vee \cdots \vee (2\gamma_2 - \gamma_{\nu+1})\tau_{\nu+1} \\
\Rightarrow\ \Lambda &\leq \left(\frac{2\gamma_2 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_2 \vee \left(\frac{2\gamma_2 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_3 \vee \cdots \vee \left(\frac{2\gamma_2 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_{\nu+1} \\
\Leftrightarrow\ \Lambda &\leq \left(\frac{2\gamma_2 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\max_{2\leq \mathrm{J}\leq \nu+1}\tau_j \\
&\leq \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\max_{1\leq \mathrm{J}\leq \nu+1}\tau_j
\end{aligned}$$

Inductively, we arrive at the sub-case

$$\Lambda \leq \gamma_\nu \tau_\nu + (\gamma_\nu - \gamma_{\nu+1})\tau_{\nu+1} + (\gamma_{\nu+1} - \gamma_{\nu+2})\tau_{\nu+2} + \dots$$

Then,

$$\begin{aligned} \Lambda &\leq \gamma_\nu \tau_\nu + (\gamma_\nu - \gamma_{\nu+1})\tau_{\nu+1} + (\gamma_{\nu+1} - 1)\Lambda \\ &\leq \gamma_\nu \max\{\tau_\nu, \tau_{\nu+1}\} + (\gamma_\nu - \gamma_{\nu+1}) \max\{\tau_\nu, \tau_{\nu+1}\} + (\gamma_{\nu+1} - 1)\Lambda \\ &\leq (2\gamma_\nu - \gamma_{\nu+1})\{\tau_\nu \vee \tau_{\nu+1}\} + (\gamma_{\nu+1} - 1)\Lambda \ . \end{aligned}$$

Hence,

$$\begin{aligned} (2 - \gamma_{\nu+1})\Lambda &\leq (2\gamma_\nu - \gamma_{\nu+1})\tau_\nu \vee (2\gamma_\nu - \gamma_{\nu+1})\tau_{\nu+1} \\ \Rightarrow \Lambda &\leq \left(\frac{2\gamma_\nu - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_\nu \vee \left(\frac{2\gamma_\nu - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\tau_{\nu+1} \\ \Leftrightarrow \Lambda &\leq \left(\frac{2\gamma_\nu - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\max_{\nu \leq \mathrm{J} \leq \nu+1} \tau_j \\ &\leq \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\right)\max_{1 \leq \mathrm{J} \leq \nu+1} \tau_j \ . \end{aligned}$$

The next sub-case is

$$\Lambda \leq \gamma_{\nu+1}\tau_{\nu+1} + (\gamma_{\nu+1} - \gamma_{\nu+2})\tau_{\nu+2} + (\gamma_{\nu+2} - \gamma_{\nu+3})\tau_{\nu+3} + \dots$$

Then,

$$\Lambda \leq \gamma_{\nu+1}\tau_{\nu+1} + (\gamma_{\nu+1} - 1)\Lambda \ .$$

Hence,

$$\Lambda \leq \frac{\gamma_{\nu+1}}{2 - \gamma_{\nu+1}}\tau_{\nu+1} \ .$$

For all the other sub-cases, i.e. if $\nu + 2 \leq k \in \mathbb{N}$,

$$\Lambda \leq \frac{\gamma_k}{2 - \gamma_k}\tau_k \ .$$

Hence,

$$\begin{aligned}\Lambda &\leq \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2-\gamma_{\nu+1}}\right) \max_{1\leq \text{J} \leq \nu+1} \tau_j \vee \sup_{\mu \geq \nu+1} \frac{\gamma_\mu}{2-\gamma_\mu} \left|\sum_{k=\mu}^{\infty} \alpha_k\right| \\ &= \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2-\gamma_{\nu+1}}\right) \max_{1\leq \text{J} \leq \nu+1} \tau_j \vee \sup_{\mu \geq \nu+2} \frac{\gamma_\mu}{2-\gamma_\mu} \left|\sum_{k=\mu}^{\infty} \alpha_k\right| .\end{aligned}$$

Hence, we conclude that if $\gamma_n \downarrow_n 1$ and $\gamma_\nu \geq 2 > \gamma_{\nu+1}$ for some $\nu \in \mathbb{N}$,

$$\sup_{s\geq 1} \tau_s \leq \Lambda \leq \left(\frac{2\gamma_1 - \gamma_{\nu+1}}{2-\gamma_{\nu+1}}\right) \max_{1\leq \text{J} \leq \nu+1} \tau_j \vee \sup_{\mu \geq \nu+2} \frac{\gamma_\mu}{2-\gamma_\mu} \left|\sum_{k=\mu}^{\infty} \alpha_k\right| .$$

We have covered all possibilities for $\gamma_n \downarrow_n 1$. Therefore, $(\eta_n)_{n\in\mathbb{N}}$ is an a.i. c_0-summing basic sequence. □

Note that we can compare our result with the Example 3.5.5, and in that case we would have $\gamma_1 \geq \ldots \gamma_k \geq 1 = \gamma_{k+1} = \gamma_{k+2} = \ldots$ for some $k \in \mathbb{N}$, and find the same conditions.

3.6 GENERAL CASE FOR CONVERGENT γ_n'S WHEN ALL b_n'S ARE 1

Recall the following definition.

Definition 3.6.1. Lower c_0-summing estimate

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence. Assume $\exists\, K \in (0,\infty)$ s.t. $\forall \alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$

$$K \sup_{n\geq 1} \left|\sum_{j=n}^{\infty} \alpha_j\right| \leq \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\| .$$

Then, we will say $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate.

Note that for our set-up, this is true generally when $\Gamma - \sigma > 0$; when $(\gamma_j)_{j\in\mathbb{N}}$ is decreasing; and when γ_j is increasing to L with $\gamma_1 > \frac{L}{2}$.

Theorem 3.6.2. *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$\text{for some } \Gamma > 0\,,\ \Gamma \leq \gamma_N, \forall N \in \mathbb{N}\,,\ \text{and } \sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty\,.$$

Define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\eta_n := \gamma_n(e_1 + e_2 + e_3 + e_4 + ... + e_n)\ ,\ \text{for all } n \in \mathbb{N}\ .$$

Also assume that $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate.

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an L-scaled asymptotically isometric c_0-summing basic sequence. Furthermore, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\mathrm{co}}(\{\eta_n : n \in \mathbb{N}\})$, Then, there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

Proof. As we have seen earlier, note that $(\gamma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to some $L \in [\gamma_1 - \sigma, \gamma_1 + \sigma]$. Without loss of generality we may assume $L = 1$. Fix $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ and for any $k \in \mathbb{N}$ define

$$\beta_k := \sum_{j=k}^{\infty} \alpha_j\ .$$

We obtain that

$$\Lambda := \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_\infty = \sup_{k\geq 1} |\gamma_k\, \beta_k + (\gamma_{k+1} - \gamma_k)\, \beta_{k+1} + (\gamma_{k+2} - \gamma_{k+1})\, \beta_{k+2} + \ldots|\ .$$

Now, we recall that by our earlier investigation, and since $(\eta_k)_{k\in\mathbb{N}}$ satisfies a lower c_0-summing estimate, there exist constants $u \in [0, 1)$ and $u_2 \in [0, \infty)$, independent of α,

$$(1 - u) \sup_{k\in\mathbb{N}} \tau_k \leq \Lambda \leq (1 + u_2) \sup_{k\in\mathbb{N}} \tau_k$$

Furthermore, $\exists\, l \in \mathbb{N}$ such that

$$\begin{aligned} \Lambda &\leq |\gamma_l\, \beta_l + (\gamma_{l+1} - \gamma_l)\, \beta_{l+1} + (\gamma_{l+2} - \gamma_{l+1})\, \beta_{l+2} + \ldots| \\ &\leq \gamma_l \tau_l + |\gamma_{l+1} - \gamma_l|\tau_{l+1} + |\gamma_{l+2} - \gamma_{l+1}|\tau_{l+2} + \ldots \qquad (\clubsuit) \end{aligned}$$

Also, by hypothesis, since

$$\sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$$

it follows that

$$\delta_k := \frac{1}{1-u}\sum_{j=k}^{\infty} |\gamma_{j+1} - \gamma_j| \longrightarrow 0 \text{ as } k \longrightarrow \infty \ .$$

So, $\exists\, l_0 \in \mathbb{N}$ such that $0 \leq \delta_k < 1$, $\forall k \geq l_0$.

Hence, if $l \geq l_0$ then by (♣),

$$\begin{aligned} \Lambda &\leq \gamma_l \tau_l + \frac{|\gamma_{l+1} - \gamma_l|}{1-u}\Lambda + \frac{|\gamma_{l+2} - \gamma_{l+1}|}{1-u}\Lambda + \dots \\ &= \gamma_l \tau_l + \delta_l \Lambda \\ \Rightarrow \qquad \Lambda &\leq \frac{\gamma_l}{1-\delta_l}\tau_l \leq \max_{k \geq l}\left(\frac{\gamma_k}{1-\delta_k}\right)\tau_k \ . \end{aligned}$$

But if $l < l_0$,

$$\begin{aligned} \Lambda \leq\ & \gamma_l \tau_l + |\gamma_{l+1} - \gamma_l|\tau_{l+1} + |\gamma_{l+2} - \gamma_{l+1}|\tau_{l+2} + \dots + |\gamma_{l_0} - \gamma_{l_0-1}|\tau_{l_0} \\ +\ & \frac{|\gamma_{l_0+1} - \gamma_{l_0}|}{1-u}\Lambda + \frac{|\gamma_{l_0+2} - \gamma_{l_0+1}|}{1-u}\Lambda + \dots \end{aligned}$$

Then,

$$\begin{aligned} \Lambda(1-\delta_{l_0}) &\leq \left(\sup_{1 \leq j \leq l_0} \tau_j\right) \sup_{1 \leq s < l_0} [\gamma_s + |\gamma_{s+1} - \gamma_s| + \dots + |\gamma_{l_0} - \gamma_{l_0-1}|] \\ &= \sup_{1 \leq j \leq l_0} \tau_j \, Z_{l_0} \end{aligned}$$

where

$$Z_{l_0} := \sup_{1 \leq s < l_0} [\gamma_s + |\gamma_{s+1} - \gamma_s| + \dots + |\gamma_{l_0} - \gamma_{l_0-1}|] \in (0, \infty)$$

Hence,

$$\Lambda \leq \left(\sup_{1 \leq j \leq l_0 - 1} \tau_j\right) \frac{Z_{l_0}}{1-\delta_{l_0}} \vee \sup_{j \geq l_0}\left(\frac{\gamma_j}{1-\delta_j} \vee 1\right)\tau_j \ .$$

Now, on the other hand, analogously to previous work, for every $n \in \mathbb{N}$ define

$$\Psi_n := \frac{|\gamma_{n+1} - \gamma_n|}{\gamma_{n+1}} \ .$$

Then

$$\begin{aligned}\Lambda &= \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} \geq \sup_{k\geq 1}\left[\gamma_k \left|\sum_{j=k}^{\infty} \alpha_j\right| - \left[\begin{array}{c} |\gamma_{k+1}-\gamma_k| \left|\sum_{j=k+1}^{\infty} \alpha_j\right| \\ +|\gamma_{k+2}-\gamma_{k+1}| \left|\sum_{j=k+2}^{\infty} \alpha_j\right| + \dots \end{array}\right]\right] \\ &= \sup_{k\geq 1}\Big[\gamma_k\tau_k - \big[\ |\gamma_{k+1}-\gamma_k|\tau_{k+1} + |\gamma_{k+2}-\gamma_{k+1}|\tau_{k+2} + \dots\ \big]\Big]\end{aligned}$$

Hence,

$$\begin{aligned}\tau_1 &\leq \frac{1}{\gamma_1}\Lambda + \frac{|\gamma_2-\gamma_1|}{\gamma_1}\left[\frac{1}{\gamma_2}\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_2}\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_2}\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_2}\tau_5 + \dots\right] \\ &+ \frac{|\gamma_3-\gamma_2|}{\gamma_1}\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_1}\tau_5 + \dots \\ &= \left[\frac{1}{\gamma_1} + \frac{1}{\gamma_1}\frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\Lambda + \left[\frac{|\gamma_2-\gamma_1|}{\gamma_1}\frac{|\gamma_3-\gamma_2|}{\gamma_2} + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\right]\tau_3 \\ &+ \left[\frac{|\gamma_4-\gamma_3|}{\gamma_1} + \frac{|\gamma_2-\gamma_1|}{\gamma_1}\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right]\tau_4 + \left[\frac{|\gamma_5-\gamma_4|}{\gamma_1} + \frac{|\gamma_2-\gamma_1|}{\gamma_1}\frac{|\gamma_5-\gamma_4|}{\gamma_2}\right]\tau_5 + \dots \\ &= \frac{1}{\gamma_1}\left[1 + \frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1 + \frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1 + \frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_4 \\ &+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1 + \frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1 + \frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_6 \\ &= \frac{1}{\gamma_1}\left[1+\Psi_1\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\Psi_1\right]\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\Psi_1\right]\tau_4 \\ &+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \dots\end{aligned}$$

Furthermore,

$$\left|\sum_{k=3}^{\infty} \alpha_k\right| \leq \frac{1}{\gamma_3}\left(\left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_{\infty} + \left[\begin{array}{c} |\gamma_4-\gamma_3| \left|\sum_{k=4}^{\infty} \alpha_k\right| + |\gamma_5-\gamma_4| \left|\sum_{k=5}^{\infty} \alpha_k\right| \\ + \dots + |\gamma_k-\gamma_{k-1}| \left|\sum_{j=k}^{\infty} \alpha_j\right| + \dots \end{array}\right]\right)$$

Thus,

$$
\begin{aligned}
\tau_1 &\le \frac{1}{\gamma_1}\left[1+\Psi_1\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\Psi_1\right]\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\Psi_1\right]\tau_4 \\
&+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \dots \\
&\le \frac{1}{\gamma_1}\left[1+\Psi_1\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\Psi_1\right]\left[\begin{array}{c} \frac{1}{\gamma_3}\Lambda + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_3}\right)\tau_4 \\ + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_3}\right)\tau_5 + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_3}\right)\tau_6 + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_3}\right)\tau_7 + \dots \end{array}\right] \\
&+ \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\Psi_1\right]\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_1}\left[1+\Psi_1\right]\tau_7 + \dots \\
&= \left[\frac{1}{\gamma_1}[1+\Psi_1] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\frac{1}{\gamma_3}\right]\Lambda \\
&+ \left[\left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_4-\gamma_3|}{\gamma_3}\right) + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)\left[1+\Psi_1\right]\right]\tau_4 \\
&+ \left[\left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_3}\right)\right]\tau_5 \\
&+ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_3}\right)\right]\tau_6 \\
&+ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_3}\right)\right]\tau_7 + \dots \\
&= \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]\Lambda + \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_4-\gamma_3|\tau_4 \\
&+ \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_5-\gamma_4|\tau_5 + \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_6-\gamma_5|\tau_6 \\
&+ \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_7-\gamma_6|\tau_7 + \dots \\
&= \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{\gamma_4-\gamma_3}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_4 \\
&+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
&+ \frac{|\gamma_7-\gamma_6|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \dots
\end{aligned}
$$

But since

$$
\left|\sum_{k=4}^{\infty}\alpha_k\right| \le \frac{1}{\gamma_4}\left(\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_{\infty} + \left[\begin{array}{c} (|\gamma_5-\gamma_4|)\left|\sum_{k=5}^{\infty}\alpha_k\right| + |\gamma_6-\gamma_5|\left|\sum_{k=6}^{\infty}\alpha_k\right| \\ + \dots + |\gamma_k-\gamma_{k-1}|\left|\sum_{j=k}^{\infty}\alpha_j\right| + \dots \end{array}\right]\right)
$$

Using the above inequality,

$$\begin{aligned}
\tau_1 &\le \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{|\gamma_4-\gamma_3|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\left[\begin{array}{c} \frac{1}{\gamma_4}\Lambda + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right)\tau_5 + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\tau_6 \\ + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\tau_7 + \left(\frac{|\gamma_8-\gamma_7|}{\gamma_4}\right)\tau_8 + \dots \end{array}\right] \\
&+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
&+ \frac{|\gamma_7-\gamma_6|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \dots \\
&= \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2][1+\Psi_3]\Lambda \\
&+ \left[\left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right)\right]\tau_5 \\
&+ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\right]\tau_6 \\
&+ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\right]\tau_7 + \dots
\end{aligned}$$

Hence,

$$\begin{aligned}
\tau_1 &\le \frac{1}{\gamma_1}[1+\Psi_1][1+\Psi_2][1+\Psi_3]\Lambda \\
&+ \left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_5 \\
&+ \left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_6 \\
&+ \left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_7 + \dots
\end{aligned}$$

Note that

$$(1+\Psi_1)(1+\Psi_2)(1+\Psi_3)\cdots = P_1 := \prod_{j=1}^{\infty}(1+\Psi_j) \ < \infty \text{ because } \sum_{j=1}^{\infty} \Psi_j \ < \infty \ .$$

So,

$$\begin{aligned}
\tau_1 &\le \frac{1}{\gamma_1}P_1\Lambda + P_1\left(\sup_{k\ge 5}\tau_k\right)\left[\frac{|\gamma_5-\gamma_4|}{\gamma_1} + \frac{|\gamma_6-\gamma_5|}{\gamma_1} + \frac{|\gamma_7-\gamma_6|}{\gamma_1} + \dots\right] \\
&\le \frac{1}{\gamma_1}P_1\Lambda + P_1\left(\sup_{k\ge 5}\tau_k\right)\frac{\sigma_4}{\gamma_1} \ ;
\end{aligned}$$

where recall for $n \in \mathbb{N}$,

$$\sigma_n = \sum_{j=n}^{\infty} |\gamma_{j+1} - \gamma_j| \ .$$

Also note that

$$\sup_{k \geq 5} \tau_k \leq K\Lambda \ ,$$

for some $K > 0$ independent of α. Similarly, we can show inductively that $\forall \nu \in \mathbb{N}$,

$$\begin{aligned} \tau_1 &\leq \frac{1}{\gamma_1} P_1 \Lambda + P_1 \left(\sup_{k \geq \nu+1} \tau_k \right) \frac{\sigma_\nu}{\gamma_1} \\ &\leq \frac{1}{\gamma_1} P_1 \Lambda + P_1 (K\Lambda) \frac{\sigma_\nu}{\gamma_1} \ . \end{aligned}$$

Letting $\nu \longrightarrow \infty$, we get $\tau_1 \leq \frac{P_1 \Lambda}{\gamma_1}$.

Similarly, $\forall m \in \mathbb{N}$,

$$\tau_m \leq \frac{P_m \Lambda}{\gamma_m} \ ; \ P_m := \prod_{j=m}^{\infty} (1 + \Psi_j) \ .$$

Note: $P_m \longrightarrow 1$ and $\gamma_m \longrightarrow 1$ as $m \longrightarrow \infty$. So,

$$\Lambda \geq \frac{\gamma_m}{P_m} \tau_m = \frac{\gamma_m}{P_m} \left| \sum_{j=m}^{\infty} \alpha_j \right| \ , \forall m \in \mathbb{N}$$

Hence,

$$\left\| \sum_{j=1}^{\infty} \alpha_j \eta_j \right\|_\infty \geq \sup_{m \in \mathbb{N}} \frac{\gamma_m}{P_m} \left| \sum_{j=m}^{\infty} \alpha_j \right| \ , \ \forall \alpha = (\alpha_j)_{j \in \mathbb{N}} \in c_{00}$$

where $0 < \frac{\gamma_m}{P_m} \longrightarrow 1$ as $m \longrightarrow \infty$. In fact, $\frac{\gamma_m}{P_m} \uparrow_m 1$. Thus, using this fact and the previous result, $(\eta_n)_{n \in \mathbb{N}}$ is an a.i. c_0-summing basic sequence. Hence, if we define the closed convex hull of $(\eta_n)_{n \in \mathbb{N}}$, $C := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, there exists an affine $\| \cdot \|_\infty$-nonexpansive mapping (or even contractive) mapping $U : C \longrightarrow C$ that is fixed point free, by Theorem 2.3.1.

□

3.7 CONVERGENT γ_n'S AND b_n'S

Theorem 3.7.1. *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$\text{for some } \Gamma > 0\,,\ \Gamma \le \gamma_N, \forall N \in \mathbb{N};\ \text{and } \sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty\ .$$

Also, let $(b_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(0,\infty)$, with limit $M \in (0,\infty)$. Define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\eta_n := \gamma_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n)\ ,\ \text{for all } n \in \mathbb{N}\ .$$

Also assume that $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate.

Then, $(\eta_n)_{n\in\mathbb{N}}$ is an L-scaled asymptotically isometric c_0-summing basic sequence. Furthermore, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, Then, there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free. (Note that this result strengthen our results given in Theorem 3.1.2 and Theorem 3.2.4.)

Proof. Now, as in previous proofs, $(\gamma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to some $L \in [\gamma_1 - \sigma, \gamma_1 + \sigma]$. Without loss of generality we may assume $L = 1$. Also, similarly assume $b_n \underset{n}{\longrightarrow} 1$. Fix $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ and for any $k \in \mathbb{N}$ define

$$\beta_k := \sum_{j=k}^{\infty} \alpha_j$$

and

$$\tau_k := \left|\sum_{j=k}^{\infty} \alpha_j\right|\ .$$

We obtain

$$\begin{aligned}
\Lambda &:= \left\|\sum_{j=1}^{\infty} \alpha_j\, \eta_j\right\|_\infty \\
&= \sup_{\nu\ge1} b_\nu \left|\gamma_\nu \sum_{k=\nu}^{\infty} \alpha_k + \sum_{r=\nu}(\gamma_{r+1} - \gamma_r)\beta_{r+1}\right| \\
&\le \sup_{\nu\ge1} b_\nu \left(\gamma_\nu\tau_\nu + \sum_{r=\nu}^{\infty} |\gamma_{r+1} - \gamma_r|\tau_{r+1}\right)\ .
\end{aligned}$$

Let

$$\kappa := \min_{1\le l<\infty} \frac{1}{b_l}\ .$$

Then, $\exists\, l \in \mathbb{N}$ s.t.

$$\Lambda \le b_l \left(\gamma_l \tau_l + \sum_{r=l}^{\infty} |\gamma_{r+1} - \gamma_r| \tau_{r+1} \right)\ .$$

Now, we recall that by our earlier investigation and since $(\eta_k)_{k\in\mathbb{N}}$ satisfies a lower c_0-summing estimate, there exist constants $u \in [0,1)$ and $u_2 \in [0,\infty)$, independent of α, such that

$$(1-u) \sup_{k\in\mathbb{N}} \tau_k \le \Lambda \le (1+u_2) \sup_{k\in\mathbb{N}} \tau_k\ .$$

Then,

$$\delta_k := \frac{1}{1-u} \sum_{j=k}^{\infty} |\gamma_{j+1} - \gamma_j| \underset{k}{\longrightarrow} 0\ ;$$

and so $\exists\, l_0$ s.t. $\delta_k < \kappa, \forall k \ge l_0$.

Case 1: $l \ge l_0$.

Recall that

$$\begin{aligned}
\Lambda &\le b_l \left(\gamma_l \tau_l + \sum_{r=l}^{\infty} |\gamma_{r+1} - \gamma_r| \tau_{r+1} \right) \\
\frac{\Lambda}{b_l} - \delta_l \Lambda &\le \gamma_l \tau_l \\
\Lambda(\kappa - \delta_l) &\le \frac{\Lambda}{b_l} - \delta_l \Lambda \le \gamma_l \tau_l\ .
\end{aligned}$$

Hence,

$$\Lambda \le \frac{\gamma_l \tau_l}{(\kappa - \delta_l)} \quad \le \quad \max_{k\ge l} \left(\frac{\gamma_k}{\kappa - \delta_k} \right) \tau_k\ .$$

Case 2: $l < l_0$.

$$\Lambda \quad \le \quad b_l \left(\gamma_l \tau_l + \sum_{r=l}^{\infty} |\gamma_{r+1} - \gamma_r| \tau_{r+1} \right)\ ;$$

and so

$$\begin{aligned}\Lambda &\leq b_l\gamma_l\tau_l + b_l|\gamma_{l+1}-\gamma_l|\tau_{l+1} + b_l|\gamma_{l+2}-\gamma_{l+1}|\tau_{l+2} + \cdots + b_l|\gamma_{l_0}-\gamma_{l_0-1}|\tau_{l_0} \\ &+ b_l\frac{|\gamma_{l_0+1}-\gamma_{l_0}|}{1-u}\Lambda + b_l\frac{|\gamma_{l_0+2}-\gamma_{l_0+1}|}{1-u}\Lambda + \dots \\ &= b_l\left(\gamma_l\tau_l + \sum_{r=l}^{l_0-1}|\gamma_{r+1}-\gamma_r|\tau_{r+1} + \sum_{r=l_0}^{\infty}|\gamma_{r+1}-\gamma_r|\frac{\Lambda}{1-u}\right) \\ &= b_l\left(\gamma_l\tau_l + \sum_{r=l}^{l_0-1}|\gamma_{r+1}-\gamma_r|\tau_{r+1} + \delta_{l_0}\Lambda\right) \ .\end{aligned}$$

Then,

$$\begin{aligned}(\kappa-\delta_{l_0})\Lambda \leq \Lambda(\frac{1}{b_l}-\delta_{l_0}) &\leq \left(\sup_{1\leq j\leq l_0}\tau_j\right)\sup_{1\leq s<l_0}\left[\gamma_s + |\gamma_{s+1}-\gamma_s| + \cdots + |\gamma_{l_0}-\gamma_{l_0-1}|\right] \\ &= \left(\sup_{1\leq j\leq l_0}\tau_j\right)\sup_{1\leq s<l_0}\left(\gamma_s + \sum_{r=s}^{l_0-1}|\gamma_{r+1}-\gamma_r|\right) \\ &= \sup_{1\leq j\leq l_0}\tau_j\, Z_{l_0}\end{aligned}$$

where

$$Z_{l_0} := \sup_{1\leq s<l_0}\left[\gamma_s + |\gamma_{s+1}-\gamma_s| + \cdots + |\gamma_{l_0}-\gamma_{l_0-1}|\right] = \sup_{1\leq s<l_0}\left[\gamma_s + \sum_{r=s}^{l_0-1}|\gamma_{r+1}-\gamma_r|\right] \in (0,\infty) \ .$$

Hence,

$$\Lambda \leq \left(\sup_{1\leq j\leq l_0-1}\tau_j\right)\frac{Z_{l_0}}{\kappa-\delta_{l_0}} \vee \sup_{j\geq l_0}\left(\frac{\gamma_j}{\kappa-\delta_j}\vee 1\right)\tau_j \ .$$

Now, on the other hand, as in previous work, for every $n\in\mathbb{N}$ define

$$\Psi_n := \frac{|\gamma_{n+1}-\gamma_n|}{\gamma_{n+1}} \ .$$

Then

$$\begin{aligned}\Lambda &= \left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \geq \sup_{k\geq 1} b_k\left[\gamma_k\left|\sum_{j=k}^{\infty}\alpha_j\right| - \left[\begin{array}{c}|\gamma_{k+1}-\gamma_k|\left|\sum_{j=k+1}^{\infty}\alpha_j\right| \\ +|\gamma_{k+2}-\gamma_{k+1}|\left|\sum_{j=k+2}^{\infty}\alpha_j\right| + \dots\end{array}\right]\right] \\ &= \sup_{k\geq 1} b_k\left[\gamma_k\tau_k - \left[\ |\gamma_{k+1}-\gamma_k|\tau_{k+1} + |\gamma_{k+2}-\gamma_{k+1}|\tau_{k+2} + \dots\ \right]\right] \ .\end{aligned}$$

Thus,

$$\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \geq b_1\gamma_1\left|\sum_{k=1}^{\infty}\alpha_k\right| - b_1\left[\begin{array}{c}|\gamma_2-\gamma_1|\,|\sum_{k=2}^{\infty}\alpha_k\,| + |\gamma_3-\gamma_2|\,|\sum_{k=3}^{\infty}\alpha_k\,| \\ +\cdots+|\gamma_k-\gamma_{k-1}|\,\left|\sum_{j=k}^{\infty}\alpha_j\right|+\dots\end{array}\right] \quad (3.40)$$

$$\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \geq b_2\gamma_2\left|\sum_{k=2}^{\infty}\alpha_k\right| - b_2\left[\begin{array}{c}|\gamma_3-\gamma_2|\,|\sum_{k=3}^{\infty}\alpha_k\,| + |\gamma_4-\gamma_3|\,|\sum_{k=4}^{\infty}\alpha_k\,| \\ +\cdots+|\gamma_k-\gamma_{k-1}|\,\left|\sum_{j=k}^{\infty}\alpha_j\right|+\dots\end{array}\right] \quad (3.41)$$

$$\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \geq b_3\gamma_3\left|\sum_{k=3}^{\infty}\alpha_k\right| - b_3\left[\begin{array}{c}|\gamma_4-\gamma_3|\,|\sum_{k=4}^{\infty}\alpha_k\,| + |\gamma_5-\gamma_4|\,|\sum_{k=5}^{\infty}\alpha_k\,| \\ +\cdots+|\gamma_k-\gamma_{k-1}|\,\left|\sum_{j=k}^{\infty}\alpha_j\right|+\dots\end{array}\right] \quad (3.42)$$

$$\vdots \quad (3.43)$$

$$\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_\infty \geq b_k\gamma_k\left|\sum_{j=k}^{\infty}\alpha_j\right| - b_k\left[\begin{array}{c}|\gamma_{k+1}-\gamma_k|\,\left|\sum_{j=k+1}^{\infty}\alpha_j\right| \\ +|\gamma_{k+2}-\gamma_{k+1}|\,\left|\sum_{j=k+2}^{\infty}\alpha_j\right|+\dots\end{array}\right] \quad (3.44)$$

$$\vdots \quad (3.45)$$

Hence,

$$\begin{aligned}
\tau_1 &\leq \frac{1}{b_1\gamma_1}\Lambda + \frac{|\gamma_2-\gamma_1|}{\gamma_1}\left[\frac{1}{b_2\gamma_2}\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_2}\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_2}\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_2}\tau_5 + \dots\right] \\
&+ \frac{|\gamma_3-\gamma_2|}{\gamma_1}\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_1}\tau_5 + \dots \\
&= \left[\frac{1}{b_1\gamma_1} + \frac{1}{\gamma_1}\frac{|\gamma_2-\gamma_1|}{b_2\gamma_2}\right]\Lambda + \left[\frac{|\gamma_2-\gamma_1|}{\gamma_1}\frac{|\gamma_3-\gamma_2|}{\gamma_2} + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\right]\tau_3 \\
&+ \left[\frac{|\gamma_4-\gamma_3|}{\gamma_1} + \frac{|\gamma_2-\gamma_1|}{\gamma_1}\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right]\tau_4 + \left[\frac{|\gamma_5-\gamma_4|}{\gamma_1} + \frac{|\gamma_2-\gamma_1|}{\gamma_1}\frac{|\gamma_5-\gamma_4|}{\gamma_2}\right]\tau_5 + \dots
\end{aligned}$$

$$\begin{aligned}
\tau_1 &\leq \frac{1}{\gamma_1}\left[\frac{1}{b_1} + \frac{|\gamma_2-\gamma_1|}{b_2\gamma_2}\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_4 \\
&+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\frac{|\gamma_2-\gamma_1|}{\gamma_2}\right]\tau_6 + \dots \\
&= \frac{1}{\gamma_1}\left[\frac{1}{b_1} + \frac{\Psi_1}{b_2}\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\Psi_1\right]\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\Psi_1\right]\tau_4 \\
&+ \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \dots
\end{aligned}$$

Furthermore,

$$\left|\sum_{k=3}^{\infty} \alpha_k\right| \le \frac{1}{b_3\gamma_3}\left\|\sum_{j=1}^{\infty} \alpha_j\,\eta_j\right\|_\infty + \frac{1}{\gamma_3}\left[\begin{array}{c} |\gamma_4-\gamma_3|\,|\sum_{k=4}^{\infty} \alpha_k| + |\gamma_5-\gamma_4|\,|\sum_{k=5}^{\infty} \alpha_k| \\ +\cdots+|\gamma_k-\gamma_{k-1}|\left|\sum_{j=k}^{\infty} \alpha_j\right| + \ldots \end{array}\right] .$$

Let

$$m_1 := \min_{j\ge 1} b_j \ .$$

We get that

$$\begin{aligned}
\tau_1 \ &\le\ \frac{1}{\gamma_1}\left[\frac{1}{b_1}+\frac{\Psi_1}{b_2}\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\Psi_1\right]\tau_3 + \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\Psi_1\right]\tau_4 \\
&+\ \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \ldots \\
&\le\ \frac{1}{\gamma_1}\left[\frac{1}{b_1}+\frac{\Psi_1}{b_2}\right]\Lambda + \frac{|\gamma_3-\gamma_2|}{\gamma_1}\left[1+\Psi_1\right]\left[\begin{array}{c} \frac{1}{b_3\gamma_3}\Lambda + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_3}\right)\tau_4 \\ +\left(\frac{|\gamma_5-\gamma_4|}{\gamma_3}\right)\tau_5 + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_3}\right)\tau_6 + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_3}\right)\tau_7 + \ldots \end{array}\right] \\
&+\ \frac{|\gamma_4-\gamma_3|}{\gamma_1}\left[1+\Psi_1\right]\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_1}\left[1+\Psi_1\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}\left[1+\Psi_1\right]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_1}\left[1+\Psi_1\right]\tau_7 + \ldots
\end{aligned}$$

$$\begin{aligned}
\tau_1 \ &\le\ \left[\frac{1}{\gamma_1}\left[\frac{1}{b_1}+\frac{\Psi_1}{b_2}\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\frac{1}{b_3\gamma_3}\right]\Lambda \\
&+\ \left[\left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_4-\gamma_3|}{\gamma_3}\right) + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)\left[1+\Psi_1\right]\right]\tau_4 \\
&+\ \left[\left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_3}\right)\right]\tau_5 \\
&+\ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_3}\right)\right]\tau_6 \\
&+\ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_3}\right)\right]\tau_7 + \ldots \\
&\le\ \left[\frac{1}{\gamma_1}\left[\frac{1}{m_1}+\frac{\Psi_1}{m_1}\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\frac{1}{m_1\gamma_3}\right]\Lambda \\
&+\ \left[\left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_4-\gamma_3|}{\gamma_3}\right) + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)\left[1+\Psi_1\right]\right]\tau_4 \\
&+\ \left[\left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_3}\right)\right]\tau_5 \\
&+\ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_3}\right)\right]\tau_6 \\
&+\ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)\left[1+\Psi_1\right] + \left(\frac{|\gamma_3-\gamma_2|}{\gamma_1}\right)\left[1+\Psi_1\right]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_3}\right)\right]\tau_7 + \ldots
\end{aligned}$$

Hence,

$$
\begin{aligned}
\tau_1 \quad \leq \quad & \frac{1}{m_1\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]\Lambda + \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_4-\gamma_3|\tau_4 \\
+ \quad & \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_5-\gamma_4|\tau_5 + \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_6-\gamma_5|\tau_6 \\
+ \quad & \frac{1}{\gamma_1}\left[1+\Psi_1\right]\left[1+\left(\frac{|\gamma_3-\gamma_2|}{\gamma_3}\right)\right]|\gamma_7-\gamma_6|\tau_7 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{m_1\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{|\gamma_4-\gamma_3|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_4 \\
+ \quad & \frac{|\gamma_5-\gamma_4|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
+ \quad & \frac{|\gamma_7-\gamma_6|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \ldots
\end{aligned}
$$

But since

$$
\left|\sum_{k=4}^{\infty}\alpha_k\right| \leq \frac{1}{b_4\gamma_4}\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_{\infty} + \frac{1}{\gamma_4}\left[\begin{array}{c} |\gamma_5-\gamma_4|\,|\sum_{k=5}^{\infty}\alpha_k\,| + |\gamma_6-\gamma_5|\,|\sum_{k=6}^{\infty}\alpha_k\,| \\ + \cdots + |\gamma_k-\gamma_{k-1}|\,\left|\sum_{j=k}^{\infty}\alpha_j\right| + \ldots \end{array}\right]
$$

Using the above inequality,

$$
\begin{aligned}
\tau_1 \quad \leq \quad & \frac{1}{m_1\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{|\gamma_4-\gamma_3|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\left[\begin{array}{c} \frac{1}{b_4\gamma_4}\Lambda + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right)\tau_5 + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\tau_6 \\ + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\tau_7 + \left(\frac{|\gamma_8-\gamma_7|}{\gamma_4}\right)\tau_8 + \ldots \end{array}\right] \\
+ \quad & \frac{|\gamma_5-\gamma_4|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
+ \quad & \frac{|\gamma_7-\gamma_6|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \ldots \\
\leq \quad & \frac{1}{m_1\gamma_1}[1+\Psi_1][1+\Psi_2]\Lambda + \frac{|\gamma_4-\gamma_3|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\left[\begin{array}{c} \frac{1}{m_1\gamma_4}\Lambda + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right)\tau_5 + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\tau_6 \\ + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\tau_7 + \left(\frac{|\gamma_8-\gamma_7|}{\gamma_4}\right)\tau_8 + \ldots \end{array}\right] \\
+ \quad & \frac{|\gamma_5-\gamma_4|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_6 \\
+ \quad & \frac{|\gamma_7-\gamma_6|}{\gamma_1}[1+\Psi_1][1+\Psi_2]\tau_7 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\tau_1 &\le \frac{1}{m_1\gamma_1}[1+\Psi_1][1+\Psi_2][1+\Psi_3]\Lambda \\
&+ \left[\left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right)\right]\tau_5 \\
&+ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\right]\tau_6 \\
&+ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\right]\tau_7 + \dots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\tau_1 &\le \frac{1}{m_1\gamma_1}[1+\Psi_1][1+\Psi_2][1+\Psi_3]\Lambda \\
&+ \left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_5 \\
&+ \left(\frac{|\gamma_6-\gamma_5|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_6 \\
&+ \left(\frac{|\gamma_7-\gamma_6|}{\gamma_1}\right)[1+\Psi_1][1+\Psi_2][1+\Psi_3]\tau_7 + \dots
\end{aligned}
$$

Note that

$$
(1+\Psi_1)(1+\Psi_2)(1+\Psi_3)\cdots = P_1 := \prod_{j=1}^{\infty}(1+\Psi_j) < \infty \text{ because } \sum_{j=1}^{\infty}\Psi_j < \infty .
$$

So,

$$
\begin{aligned}
\tau_1 &\le \frac{1}{m_1\gamma_1}P_1\Lambda + P_1\left(\sup_{k\ge 5}\tau_k\right)\left[\frac{|\gamma_5-\gamma_4|}{\gamma_1} + \frac{|\gamma_6-\gamma_5|}{\gamma_1} + \frac{|\gamma_7-\gamma_6|}{\gamma_1} + \dots\right] \\
&\le \frac{1}{m_1\gamma_1}P_1\Lambda + P_1\left(\sup_{k\ge 5}\tau_k\right)\frac{\sigma_4}{\gamma_1} ;
\end{aligned}
$$

where recall for $n \in \mathbb{N}$,

$$
\sigma_n = \sum_{j=n}^{\infty}|\gamma_{j+1}-\gamma_j| .
$$

Also note that

$$
\sup_{k\ge 5}\tau_k \le K\Lambda ,
$$

for some $K > 0$ independent of α. Similarly, we can show inductively that $\forall \nu \in \mathbb{N}$, $\nu \geq 4$,

$$\begin{aligned} \tau_1 &\leq \frac{1}{m_1\gamma_1} P_1 \Lambda + P_1 \left(\sup_{k \geq \nu+1} \tau_k \right) \frac{\sigma_\nu}{\gamma_1} \\ &\leq \frac{1}{m_1\gamma_1} P_1 \Lambda + P_1 \left(K\Lambda \right) \frac{\sigma_\nu}{\gamma_1} \end{aligned}$$

Letting $\nu \longrightarrow \infty$, we get $\tau_1 \leq \frac{P_1\Lambda}{m_1\gamma_1}$.

Let

$$m_2 := \min_{j \geq 2} b_j \ .$$

Then

$$\begin{aligned} \tau_2 &\leq \frac{1}{\gamma_2}\left[\frac{1}{b_2} + \frac{\Psi_2}{b_3}\right]\Lambda + \frac{|\gamma_4-\gamma_3|}{\gamma_2}\left[1+\Psi_2\right]\tau_4 + \frac{|\gamma_5-\gamma_4|}{\gamma_2}\left[1+\Psi_2\right]\tau_5 \\ &+ \frac{|\gamma_6-\gamma_5|}{\gamma_2}\left[1+\Psi_2\right]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_2}\left[1+\Psi_2\right]\tau_7 + \ldots \\ &\leq \frac{1}{\gamma_2}\left[\frac{1}{b_2} + \frac{\Psi_2}{b_3}\right]\Lambda + \frac{|\gamma_4-\gamma_3|}{\gamma_2}\left[1+\Psi_2\right]\left[\begin{array}{c} \frac{1}{b_4\gamma_4}\Lambda + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right)\tau_5 \\ + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\tau_6 + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\tau_7 + \left(\frac{|\gamma_8-\gamma_7|}{\gamma_4}\right)\tau_8 + \ldots \end{array}\right] \\ &+ \frac{|\gamma_5-\gamma_4|}{\gamma_2}\left[1+\Psi_2\right]\tau_5 + \frac{|\gamma_6-\gamma_5|}{\gamma_2}\left[1+\Psi_2\right]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_2}\left[1+\Psi_2\right]\tau_7 + \frac{|\gamma_8-\gamma_7|}{\gamma_2}\left[1+\Psi_2\right]\tau_8 + \ldots \\ &= \left[\frac{1}{\gamma_2}\left[\frac{1}{b_2} + \frac{\Psi_2}{b_3}\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_1}\right)\left[1+\Psi_2\right]\frac{1}{b_4\gamma_4}\right]\Lambda \\ &+ \left[\left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right) + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_2}\right)\left[1+\Psi_2\right]\right]\tau_5 \\ &+ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_2}\right)\left[1+\Psi_2\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\right]\tau_6 \\ &+ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_2}\right)\left[1+\Psi_2\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\right]\tau_7 \\ &+ \left[\left(\frac{|\gamma_8-\gamma_7|}{\gamma_2}\right)\left[1+\Psi_2\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_8-\gamma_7|}{\gamma_4}\right)\right]\tau_8 + \ldots \\ &\leq \left[\frac{1}{\gamma_2}\left[\frac{1}{m_2} + \frac{\Psi_1}{m_2}\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\frac{1}{m_2\gamma_4}\right]\Lambda \\ &+ \left[\left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_5-\gamma_4|}{\gamma_4}\right) + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_2}\right)\left[1+\Psi_2\right]\right]\tau_5 \\ &+ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_2}\right)\left[1+\Psi_2\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_4}\right)\right]\tau_6 \\ &+ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_2}\right)\left[1+\Psi_2\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_4}\right)\right]\tau_7 \\ &+ \left[\left(\frac{|\gamma_8-\gamma_7|}{\gamma_2}\right)\left[1+\Psi_2\right] + \left(\frac{|\gamma_4-\gamma_3|}{\gamma_2}\right)\left[1+\Psi_2\right]\left(\frac{|\gamma_8-\gamma_7|}{\gamma_4}\right)\right]\tau_8 + \ldots \end{aligned}$$

Hence,

$$\begin{aligned}
\tau_2 &\leq \frac{1}{m_2\gamma_2}\left[1+\Psi_2\right]\left[1+\left(\frac{|\gamma_4-\gamma_3|}{\gamma_4}\right)\right]\Lambda + \frac{1}{\gamma_2}\left[1+\Psi_2\right]\left[1+\left(\frac{|\gamma_4-\gamma_3|}{\gamma_4}\right)\right]|\gamma_5-\gamma_4|\tau_5 \\
&+ \frac{1}{\gamma_2}\left[1+\Psi_2\right]\left[1+\left(\frac{|\gamma_4-\gamma_3|}{\gamma_4}\right)\right]|\gamma_6-\gamma_5|\tau_6 + \frac{1}{\gamma_2}\left[1+\Psi_2\right]\left[1+\left(\frac{|\gamma_4-\gamma_3|}{\gamma_4}\right)\right]|\gamma_7-\gamma_6|\tau_7 \\
&+ \frac{1}{\gamma_2}\left[1+\Psi_2\right]\left[1+\left(\frac{|\gamma_4-\gamma_3|}{\gamma_4}\right)\right]|\gamma_8-\gamma_7|\tau_8 + \dots
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m_2\gamma_2}[1+\Psi_2][1+\Psi_3]\Lambda + \frac{|\gamma_5-\gamma_4|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_5 \\
&+ \frac{|\gamma_6-\gamma_5|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_7 \\
&+ \frac{|\gamma_8-\gamma_7|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_8 + \dots
\end{aligned}$$

But since

$$\left|\sum_{k=5}^{\infty}\alpha_k\right| \leq \frac{1}{b_5\gamma_5}\left\|\sum_{j=1}^{\infty}\alpha_j\,\eta_j\right\|_{\infty} + \frac{1}{\gamma_5}\left[\begin{array}{c} |\gamma_6-\gamma_5|\,|\sum_{k=6}^{\infty}\alpha_k\,| + |\gamma_7-\gamma_6|\,|\sum_{k=7}^{\infty}\alpha_k\,| \\ + \dots + |\gamma_k-\gamma_{k-1}|\left|\sum_{j=k}^{\infty}\alpha_j\right| + \dots \end{array}\right]$$

Using the above inequality,

$$\begin{aligned}
\tau_2 &\leq \frac{1}{m_2\gamma_2}[1+\Psi_2][1+\Psi_3]\Lambda + \frac{|\gamma_5-\gamma_4|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\left[\begin{array}{c} \frac{1}{b_5\gamma_5}\Lambda + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_5}\right)\tau_6 + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_5}\right)\tau_7 \\ + \left(\frac{|\gamma_8-\gamma_7|}{\gamma_5}\right)\tau_8 + \left(\frac{|\gamma_9-\gamma_8|}{\gamma_5}\right)\tau_9 + \dots \end{array}\right] \\
&+ \frac{|\gamma_6-\gamma_5|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_7 \\
&+ \frac{|\gamma_8-\gamma_7|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_8 + \dots \\
&\leq \frac{1}{m_2\gamma_2}[1+\Psi_2][1+\Psi_3]\Lambda + \frac{|\gamma_5-\gamma_4|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\left[\begin{array}{c} \frac{1}{m_2\gamma_5}\Lambda + \left(\frac{|\gamma_6-\gamma_5|}{\gamma_5}\right)\tau_6 + \left(\frac{|\gamma_7-\gamma_6|}{\gamma_5}\right)\tau_7 \\ + \left(\frac{|\gamma_8-\gamma_7|}{\gamma_5}\right)\tau_8 + \left(\frac{|\gamma_9-\gamma_8|}{\gamma_5}\right)\tau_9 + \dots \end{array}\right] \\
&+ \frac{|\gamma_6-\gamma_5|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_6 + \frac{|\gamma_7-\gamma_6|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_7 \\
&+ \frac{|\gamma_8-\gamma_7|}{\gamma_2}[1+\Psi_2][1+\Psi_3]\tau_8 + \dots
\end{aligned}$$

$$
\begin{aligned}
\tau_2 &\leq \frac{1}{m_2\gamma_2}[1+\Psi_2][1+\Psi_3][1+\Psi_4]\Lambda \\
&+ \left[\left(\frac{|\gamma_6-\gamma_5|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3] + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_1}\right)[1+\Psi_2][1+\Psi_3]\left(\frac{|\gamma_6-\gamma_5|}{\gamma_5}\right)\right]\tau_6 \\
&+ \left[\left(\frac{|\gamma_7-\gamma_6|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3] + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3]\left(\frac{|\gamma_7-\gamma_6|}{\gamma_5}\right)\right]\tau_7 \\
&+ \left[\left(\frac{|\gamma_8-\gamma_7|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3] + \left(\frac{|\gamma_5-\gamma_4|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3]\left(\frac{|\gamma_8-\gamma_7|}{\gamma_5}\right)\right]\tau_8 + \ldots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\tau_2 &\leq \frac{1}{m_2\gamma_2}[1+\Psi_2][1+\Psi_3][1+\Psi_4]\Lambda \\
&+ \left(\frac{|\gamma_6-\gamma_5|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3][1+\Psi_4]\tau_6 \\
&+ \left(\frac{|\gamma_7-\gamma_6|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3][1+\Psi_4]\tau_7 \\
&+ \left(\frac{|\gamma_8-\gamma_7|}{\gamma_2}\right)[1+\Psi_2][1+\Psi_3][1+\Psi_4]\tau_8 + \ldots
\end{aligned}
$$

Note that

$$
(1+\Psi_2)(1+\Psi_3)(1+\Psi_4)\cdots = P_2 := \prod_{j=2}^{\infty}(1+\Psi_j) < \infty \text{ because } \sum_{j=2}^{\infty} \Psi_j < \infty .
$$

So,

$$
\begin{aligned}
\tau_2 &\leq \frac{1}{m_2\gamma_2}P_2\Lambda + P_2\left(\sup_{k\geq 6}\tau_k\right)\left[\frac{|\gamma_6-\gamma_5|}{\gamma_2} + \frac{|\gamma_7-\gamma_6|}{\gamma_2} + \frac{|\gamma_8-\gamma_7|}{\gamma_2} + \ldots\right] \\
&\leq \frac{1}{m_2\gamma_2}P_2\Lambda + P_2\left(\sup_{k\geq 6}\tau_k\right)\frac{\sigma_5}{\gamma_2} \quad ;
\end{aligned}
$$

where recall for $n \in \mathbb{N}$,

$$
\sigma_n = \sum_{j=n}^{\infty} |\gamma_{j+1}-\gamma_j| .
$$

Also note that

$$
\sup_{k\geq 6}\tau_k \leq K\Lambda ,
$$

for the same $K > 0$ independent of α as in the previous step. Similarly, we can show inductively that $\forall \nu \in \mathbb{N}$, $\nu \geq 5$,

$$\begin{aligned} \tau_2 &\leq \frac{1}{m_2\gamma_2} P_2\Lambda + P_2 \left(\sup_{k \geq \nu+1} \tau_k \right) \frac{\sigma_\nu}{\gamma_2} \\ &\leq \frac{1}{m_2\gamma_2} P_2\Lambda + P_2 \left(K\Lambda \right) \frac{\sigma_\nu}{\gamma_2} \ . \end{aligned}$$

Letting $\nu \longrightarrow \infty$, we get $\tau_2 \leq \frac{P_2\Lambda}{m_2\gamma_2}$.

Similarly, $\forall s \in \mathbb{N}$,

$$m_s := \min_{j \geq s} b_j \longrightarrow 1$$

as $s \longrightarrow \infty$, and

$$\tau_s \leq \frac{P_s\Lambda}{m_s\gamma_s} \ ; \ P_s := \prod_{j=s}^{\infty} (1 + \Psi_j) \ .$$

Note: $P_s \longrightarrow 1$, $m_s \longrightarrow 1$ and $\gamma_s \longrightarrow 1$ as $s \longrightarrow \infty$. So,

$$\Lambda \geq \frac{m_s\gamma_s}{P_s} \tau_s = \frac{m_s\gamma_s}{P_s} \left| \sum_{j=s}^{\infty} \alpha_j \right| \ , \forall s \in \mathbb{N}$$

Hence,

$$\left\| \sum_{j=1}^{\infty} \alpha_j \eta_j \right\|_\infty \geq \sup_{s \in \mathbb{N}} \frac{m_s\gamma_s}{P_s} \left| \sum_{j=s}^{\infty} \alpha_j \right| \ , \ \forall \alpha = \ (\alpha_j)_{j \in \mathbb{N}} \in c_{00} \ ;$$

where $0 < \frac{m_s\gamma_s}{P_s} \longrightarrow 1$ as $s \longrightarrow \infty$. In fact, $\frac{m_s\gamma_s}{P_s} \uparrow_s 1$. Thus, using this fact and the previous result, $(\eta_n)_{n \in \mathbb{N}}$ is an a.i. c_0-summing basic sequence. Hence, if we define the closed convex hull of $(\eta_n)_{n \in \mathbb{N}}$, $C := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, there exists an affine $\| \cdot \|_\infty$-nonexpansive mapping (or even contractive) mapping $U : C \longrightarrow C$ that is fixed point free.

$\square$

Open Question (3) Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that

$$\text{for some } \Gamma > 0\,,\ \Gamma \leq \gamma_N, \forall N \in \mathbb{N}\,;\ \text{and}\ \ \sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}|\ < \infty\ .$$

Also, let $(b_n)_{n\in\mathbb{N}}$ be a bounded sequence in $[m, M]$, where $0 < m \leq M < \infty$ are constants. Define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\eta_n := \gamma_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n)\ ,\ \text{for all } n \in \mathbb{N}\ .$$

Also, assume that $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate. Then, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$. It is an open question whether or not there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

Notice that this open question includes section 2.4 (all $\gamma_n = 1$), where we know this question has a positive answer.

4.0 LORENTZ-MARCINKIEWICZ SPACES AND EXPLORING THE FIXED POINT PROPERTY FOR $l^0_{w,\infty}$ SPACES

In this chapter, we will investigate the fixed point property for $l^0_{w,\infty}$ spaces.

First of all, we will give some preliminaries.

4.1 PRELIMINARIES

Fix the so-called weight sequence $w \in (c_0 \setminus l^1)^+$, $w_1 = 1$, such that $(w_n)_{n\in\mathbb{N}}$ is decreasing; i.e., $w = (w_n)_{n\in\mathbb{N}}$, $w_n > 0, \forall n \in \mathbb{N}$ such that $1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_n \geq w_{n+1} \geq \ldots, \forall n \in \mathbb{N}$ with $w_n \underset{n}{\longrightarrow} 0$ and

$$\sum_{n=1}^{\infty} w_n = \infty \ .$$

E.g., $w_n = \frac{1}{n}, \forall n \in \mathbb{N}$.

Definition 4.1.1. $l_{w,\infty}$ space

$$l_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \begin{array}{c} \|x\|_{w,\infty} := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} {x_j}^{\star}}{\sum_{j=1}^{n} w_j} < \infty\,, \quad \text{where}\, x^{\star} := ({x_n}^{\star})_{n\in\mathbb{N}} \\ \text{is the decreasing rearrangement of } x \end{array} \right\} .$$

This is an analogue of l^{∞} space. Indeed $(l_{w,\infty}, \|\cdot\|_{w,\infty})$ is a non-separable Banach space. Note that $x^{\star} :=$ the sequence whose terms contain all non-zero terms of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in non-increasing order repeated according to multiplicity, followed by infinitely many zeros when $|x|$ has only finitely many non-zero terms.

Definition 4.1.2. $l^0_{w,\infty}$ space

$$l^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \begin{array}{c} \limsup_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} {x_j}^\star}{\sum_{j=1}^{n} w_j} = 0 \text{, where } x^\star := ({x_n}^\star)_{n\in\mathbb{N}} \\ \text{is the decreasing rearrangement of } x \end{array} \right\}.$$

This is an analogue of c_0 space. It is a fact that $(l^0_{w,\infty}, \|\cdot\|_{w,\infty})$ is a separable subspace of $l_{w,\infty}$.

Definition 4.1.3. $l_{w,1}$ space

$$l_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \;\middle|\; \|x\|_{w,1} := \sum_{j=1}^{n} w_j \, {x_j}^\star \; < \infty \right\} .$$

This is an analogue of l^1 space. $(l_{w,1}, \|\cdot\|_{w,1})$ is a separable Banach space.

Note that $(l^0_{w,\infty})^* \cong l_{w,1}$ and $(l_{w,1})^* \cong l_{w,\infty}$ where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic. A standard reference for Lorentz spaces is Lindenstrauss and Tzafriri [31].

In this chapter, we will investigate the weak fixed point property and fixed point property for this space. We note that Maurey proved c_0 has the w-FPP for nonexpansive mappings in [34], and later Borwein and Sims extended this result in [6]. Also, Aksoy and Khamsi describe Borwein and Sims' results in [1]. First of all we will give some preliminaries.

Definition 4.1.4. weakly orthogonal

Let X be a Banach lattice. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence weakly convergent to some $x_0 \in X$. Then, $(x_n)_{n\in\mathbb{N}}$ is weakly orthogonal if

$$\liminf_{n\longrightarrow\infty} \liminf_{m\longrightarrow\infty} \| |x_n - x_0| \wedge |x_m - x_0| \| = 0 \, .$$

A subset C of X is a weakly orthogonal set if every weakly convergent sequence of points of C is weakly orthogonal. Then, we say X is weakly orthogonal if every weakly compact convex subset of X is weakly orthogonal.

See the proof of Theorem 4.3.1 for the definition of an unconditional basis. Also, see Definition 5.0.2 for the definition of a Banach lattice.

Definition 4.1.5. R.A.P

A Banach lattice X has the Riesz approximation property (R.A.P) if there exists a family $\mathcal{P}$ of linear projections such that $P|x| = |P(x)|$, $\forall P \in \mathcal{P}$, $P(X)$ is a finite dimensional ideal, $\forall P \in \mathcal{P}$, and $\forall x \in X$,

$$\inf_{P\in\mathcal{P}} \|x - P(x)\| = 0$$

Proposition 4.1.6. *R.A.P implies Weakly Orthogonality*

If X is a Banach lattice with R.A.P, then X is weakly orthogonal.

Proof. A proof can be found in [6]. □

Definition 4.1.7. Riesz angle

The Riesz angle α of a Banach lattice X is defined by $\alpha(X) = \sup\{\||x| \vee |y|\| : \|x\| \leq 1, \|y\| \leq 1\}$. Note that for L^p space $(1 \leq p \leq \infty)$, then $\alpha(L^p) = 2^{\frac{1}{p}}$ and for also c_0 space, $\alpha(c_0) = 1$.

Definition 4.1.8. w-FPP

A Banach space is said to have the weak fixed point property (w-FPP) if every nonexpansive mapping on every nonempty weakly compact convex set has a fixed point.

Theorem 4.1.9. *R.A.WFPP*

Let X be a Banach lattice with Riesz angle $\alpha(X) < 2$ and let C be a weakly compact convex subset of X which is weakly orthogonal. Then, C has the weak fixed point property.

Proof. A proof can be found in [6]. □

Theorem 4.1.10. *P.K. Lin unconditionality*

Let X be a Banach space. If X has an unconditional basis (e_n) with unconditional constant $\lambda < \frac{\sqrt{33}-3}{2}$, then X has the wFPP.

Proof. A proof can be found in [1]. □

4.2 RIESZ ANGLE FOR $l^0_{w,\infty}$

Using Borwein and Sims' techniques, we want to explore the w-FPP and stronger properties for $l^0_{w,\infty}$ space.

Proposition 4.2.1. *Consider $X := l^0_{w,\infty}$. Then, X has R.A.P and so by Proposition 4.1.6, $l^0_{w,\infty}$ space is weakly orthogonal.*

Proof. First of all, $(X, \|\cdot\|) \subseteq c_0$ is a Banach lattice under the usual pointwise ordering; i.e. $x \leq u \Leftrightarrow [x_j \leq u_j, \forall j \in \mathbb{N}]$ Now, $\forall E \subseteq \mathbb{N}$, E finite, $\forall x \in c_0$,

$$(P_E(x))_n := \begin{cases} x_n & , \text{if}\, n \in E \\ 0 & , \text{if}\, n \notin E \end{cases}$$

Then, $\|x - P_{[1,n]}(x)\| \underset{n}{\longrightarrow} 0$.

The proof details are well-known. □

Theorem 4.2.2. *Consider $X := l^0_{w,\infty}$ with $w = (\frac{1}{n^p})_{n\in\mathbb{N}}$ where $0 < p < 1$. Then, $\alpha(X) < 2$ and so X has the w-FPP.*

Proof.

Case 1: $p = \frac{1}{2}$ Consider $X := l^0_{w,\infty}$ with $w = (\frac{1}{n^{\frac{1}{2}}})_{n\in\mathbb{N}}$. Then, $\alpha(X) < 2$ and so X has the w-FPP.

Let's denote the closed ball for any normed space Y by

$\mathcal{B}_Y := \{x \in Y | \|x\|_{w,\infty} \leq 1\}$. Now, let $x := (x_n)_{n\in\mathbb{N}}, y := (y_n)_{n\in\mathbb{N}} \in \mathcal{B}_X$ be given, then we will consider

$\||x| \vee |y|\|_{w,\infty} = \|(|x_1| \vee |y_1|, |x_2| \vee |y_2|, |x_3| \vee |y_3|, \ldots, |x_n| \vee |y_n|, \ldots)\|_{w,\infty}$. Now, define for each $n \in \mathbb{N}$, $z_n = |x_n| \vee |y_n|$ and consider the decreasing rearrangement for the sequence $z = (z_n)_{n\in\mathbb{N}}$. From the definition of decreasing rearrangement, $\exists$ a 1-1 mapping $\pi : \mathbb{N} \longrightarrow \mathbb{N}$

and $\exists(\varepsilon_j)_{j\in\mathbb{N}}$ s.t. each $\varepsilon_{\pi(j)} \in \{-1,1\}$ and then $(z^\star)_k = |z_{\pi(k)}| = \varepsilon_{\pi(k)}\, z_{\pi(k)}, \forall k \in \mathbb{N}$. Let $\Pi_{\mathbb{N}}$ be the set of all 1-1 functions $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ Then, fix $n \in \mathbb{N}$ and define $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

$$\begin{aligned} \sum_{j=1}^{n} z_j^\star &= \sum_{j=1}^{n} |z_\pi(j)|, \text{ for some } \pi \in \Pi_{\mathbb{N}} \\ &= \sum_{\substack{j=1 \\ j \in A_n}}^{n} |x_\pi(j)| + \sum_{\substack{j=1 \\ j \in B_n}}^{n} |y_\pi(j)| \end{aligned}$$

where $A_n := \{k \in \mathbb{N}_n \,:\, |x_\pi(k)| \geq |y_\pi(k)|\}$ and $B_n := \mathbb{N}_n - A_n$

Now, define $M := \#(A_n) \in \{0, 1, 2, \ldots, n\}$ and $N := \#(A_n) \in \{0, 1, 2, \ldots, n\}$. Clearly, $M + N = n$.

Then,

$$\sum_{j=1}^{n} z_j^\star \leq \sum_{j=1}^{M} x_j^\star + \sum_{j=1}^{N} y_j^\star \tag{4.1}$$

$$\leq \|x\|_{w,\infty} \sum_{j=1}^{M} w_j + \|y\|_{w,\infty} \sum_{j=1}^{N} w_j \tag{4.2}$$

$$\leq \sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j \tag{4.3}$$

Hence,

$$\frac{\sum_{j=1}^{n} z_j^\star}{\sum_{j=1}^{n} w_j} \leq \frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j}$$

$$\sum_{j=1}^{M} w_j \leq 1 + \int_1^M \frac{1}{x^{\frac{1}{2}}}\, dx = 2M^{\frac{1}{2}} - 1$$

and

$$\sum_{j=1}^{n} w_j \geq \int_1^{n+1} \frac{1}{x^{\frac{1}{2}}}\, dx = 2((n+1)^{\frac{1}{2}} - 1)\ .$$

Then,

$$\frac{\sum_{j=1}^{n} z_j^\star}{\sum_{j=1}^{n} w_j} \leq \frac{2M^{\frac{1}{2}} - 1 + 2N^{\frac{1}{2}} - 1}{2((n+1)^{\frac{1}{2}} - 1)} = \frac{M^{\frac{1}{2}} + N^{\frac{1}{2}} - 1}{(n+1)^{\frac{1}{2}} - 1}\ .$$

Note that for the function f defined by

$$f(x) = x^{\frac{1}{2}} + (n-x)^{\frac{1}{2}}, \forall x \in [0,n] \ ,$$

f is maximized at $\frac{n}{2}$ and so

$$f(x) \le f\left(\frac{n}{2}\right) = \sqrt{2}\sqrt{n}, \forall x \in [0,n] \ .$$

Hence,

$$\frac{\sum_{j=1}^{n} z_j^\star}{\sum_{j=1}^{n} w_j} \le \frac{\sqrt{2}\sqrt{n}-1}{(n+1)^{\frac{1}{2}}-1} \ < 2, \forall n \in \mathbb{N} \ .$$

Also,

$$\lim_{n \longrightarrow \infty} \frac{\sqrt{2}\sqrt{n}-1}{(n+1)^{\frac{1}{2}}-1} = \sqrt{2} < 2 \ .$$

Hence,

$$\||x| \vee |y|\|_{w,\infty} = \sup_{n \in \mathbb{N}} \frac{\sum_{j=1}^{n} z_j^\star}{\sum_{j=1}^{n} w_j} < 2 \ .$$

General Case: $p \in (0,1)$ As in the previous specific case, let $x := (x_n)_{n\in\mathbb{N}}$, $y := (y_n)_{n\in\mathbb{N}} \in \mathcal{B}_X$ be given, then we will define for each $n \in \mathbb{N}$, $z_n = |x_n| \vee |y_n|$ and fix $n \in \mathbb{N}$ then notice for some $M, N \in \mathbb{N}_n$ with $M + N = n$

$$\frac{\sum_{j=1}^{n} z_j^\star}{\sum_{j=1}^{n} w_j} \le \frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j} \qquad [\spadesuit\spadesuit]$$

Now, note that for any $k \in \mathbb{N}$,

$$\sum_{j=1}^{k} w_j \le 1 + \int_1^k \frac{1}{x^p}\,dx \ \text{ and}$$

$$\int_1^{k+1} \frac{1}{x^p}\,dx \le \sum_{j=1}^{k} w_j \ .$$

Hence,

$$\frac{\sum_{j=1}^{k} w_j}{1 + \int_1^k \frac{1}{x^p}\,dx} \le 1 \le \frac{\sum_{j=1}^{k} w_j}{\int_1^{k+1} \frac{1}{x^p}\,dx} \ .$$

In fact, it can be also seen that

$$\frac{\sum_{j=1}^{k} w_j}{\int_0^k \frac{1}{x^p}\,dx} \le 1 \le \frac{\sum_{j=1}^{k} w_j}{\int_1^k \frac{1}{x^p}\,dx}$$

$$\Leftrightarrow \frac{\sum_{j=1}^{k} w_j}{\frac{1}{1-p} + \int_1^k \frac{1}{x^p}\,dx} \le 1 \le \frac{\sum_{j=1}^{k} w_j}{\int_1^k \frac{1}{x^p}\,dx}\ .$$

Hence, for k large enough,

$$\frac{\sum_{j=1}^{k} w_j}{\int_1^k \frac{1}{x^p}\,dx}$$

is asymptotically equivalent to 1; i.e.,

$$\frac{\sum_{j=1}^{k} w_j}{\int_1^k \frac{1}{x^p}\,dx} \sim 1\ .$$

In other words,

$$\sum_{j=1}^{k} w_j \ \text{ and } \ \int_1^k \frac{1}{x^p}\,dx$$

are asymptotically equivalent. Now, let's go back to the inequality [♠♠]. If $\exists\,\nu \in \mathbb{N}$ s.t. $n = 2\nu$, then

$$\begin{aligned}
\frac{\sum_{j=1}^{\nu-1} w_j + \sum_{j=1}^{\nu+1} w_j}{\sum_{j=1}^{2\nu} w_j} &= \frac{\sum_{j=1}^{\nu-1} w_j + \sum_{j=1}^{\nu} w_j + w_{\nu+1}}{\sum_{j=1}^{2\nu} w_j} \\
&\le \frac{\sum_{j=1}^{\nu-1} w_j + \sum_{j=1}^{\nu} w_j + w_{\nu}}{\sum_{j=1}^{2\nu} w_j} \text{ since } w_{\nu+1} \le w_\nu \\
&= \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j}\ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\sum_{j=1}^{\nu-2} w_j + \sum_{j=1}^{\nu+2} w_j}{\sum_{j=1}^{2\nu} w_j} &= \frac{\sum_{j=1}^{\nu-2} w_j + \sum_{j=1}^{\nu} w_j + w_{\nu+1} + w_{\nu+2}}{\sum_{j=1}^{2\nu} w_j} \\
&\le \frac{\sum_{j=1}^{\nu-2} w_j + \sum_{j=1}^{\nu} w_j + w_{\nu-1} + w_{\nu}}{\sum_{j=1}^{2\nu} w_j} \\
&= \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j}\ .
\end{aligned}$$

Inductively, if $\nu > 2$, for $0 \leq s < \nu$, we obtain

$$\begin{aligned}
\frac{\sum_{j=1}^{\nu-s} w_j + \sum_{j=1}^{\nu+s} w_j}{\sum_{j=1}^{2\nu} w_j} &= \frac{\sum_{j=1}^{\nu-s} w_j + \sum_{j=1}^{\nu} w_j + w_{\nu+1} + w_{\nu+2} + \cdots + +w_{\nu+s}}{\sum_{j=1}^{2\nu} w_j} \\
&\leq \frac{\sum_{j=1}^{\nu-s} w_j + \sum_{j=1}^{\nu} w_j + w_{\nu-s+1} + w_{\nu-s+2} + \cdots + w_{\nu}}{\sum_{j=1}^{2\nu} w_j} \\
&= \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j} .
\end{aligned}$$

Hence, we obtain that

$$\frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j} \leq \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j} = \frac{2\sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j} .$$

On the other hand, if $n = 2\nu + 1$, similarly, we can show

$$\frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j} \leq \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu+1} w_j}{\sum_{j=1}^{2\nu+1} w_j} .$$

In fact, using the facts above, we will be able to give a general formula for Riesz angle. However, for our particular case, for $\nu \in \mathbb{N}$, define

$$Q_p(2\nu) := \frac{2\sum_{j=1}^{\nu} \frac{1}{j^p}}{\sum_{j=1}^{2\nu} \frac{1}{j^p}} \sim \frac{2\int_0^{\nu} x^{-p}\,dx}{\int_0^{2\nu} x^{-p}\,dx} = 2^p .$$

Also,

$$\begin{aligned}
Q_p(2\nu+1) := \frac{\sum_{j=1}^{\nu} \frac{1}{j^p} + \sum_{j=1}^{\nu+1} \frac{1}{j^p}}{\sum_{j=1}^{2\nu+1} \frac{1}{j^p}} \sim \frac{\int_0^{\nu} x^{-p}\,d + \int_0^{\nu+1} x^{-p}\,dx}{\int_0^{2\nu+1} x^{-p}\,dx} &= \frac{(\nu+1)^{1-p} + \nu^{1-p}}{(2\nu+1)^{1-p}} \\
&= \frac{(1+\frac{1}{\nu})^{1-p} + 1}{(2+\frac{1}{\nu})^{1-p}} .
\end{aligned}$$

Hence, for both cases

$$\frac{\sum_{j=1}^{n} z_j^{\star}}{\sum_{j=1}^{n} w_j} \leq Q_p(n) < 2, \forall n \in \mathbb{N}$$

and

$$\lim_{n \longrightarrow \infty} Q_p(n) = 2^p < 2 .$$

Thus,

$$\alpha(l^0_{w,\infty}) = \sup_{x,y \in \mathcal{B}_{l^0_{w,\infty}}} \| |x| \vee |y| \|_{w,\infty} < 2 .$$

$\square$

Case 3: p=1

Let $w_j := \frac{1}{j}, \forall j \in \mathbb{N}$, then $\alpha(X) = 2$ and so the Riesz angle proposition does not apply to show the w-FPP in this case. Indeed, consider

$x := (w_1, 0, w_2, 0, w_3, 0, w_4, 0, \dots), y := (0, w_1, 0, w_2, 0, w_3, 0, w_4, 0, \dots) \in l_{w,\infty}$ Then, $z := x \vee y = (w_1, w_1, w_2, w_2, w_3, w_3, w_4, w_4, \dots)$ with $|x| = x; |y| = y$,

$x^\star = w; y^\star = w$ and so

$$\|x\|_{w,\infty} = \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^n x_j^\star}{\sum_{j=1}^n w_j} = \|w\|_{w,\infty} = 1$$

and similarly, $\|y\|_{w,\infty} = 1$, $z^\star = z$

$$\|z\|_{w,\infty} = \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^n z_j^\star}{\sum_{j=1}^n w_j} \le 2 \ .$$

Then, our calculations with the function Q_p for p=1 would work here again, and

$$Q_1(2n) = \frac{2\sum_{j=1}^n w_j}{\sum_{j=1}^{2n} w_j} = \frac{2\sum_{j=1}^n \frac{1}{j}}{\sum_{j=1}^{2n} \frac{1}{j}} \underset{\text{large n}}{\sim} 2\frac{\ln n}{\ln 2n} \longrightarrow 2, \text{ as } n \longrightarrow \infty \ .$$

So, $\|z\|_{w,\infty} = 2$. We can also consider for any $\nu \in \mathbb{N}$,

$x^{(\nu)} := (w_1, 0, w_2, 0, w_3, 0, w_4, 0, \dots, 0, w_\nu, 0, 0, 0, \dots)$,

$y^{(\nu)} := (0, w_1, 0, w_2, 0, w_3, 0, w_4, 0, \dots, 0, w_{\nu-1}, 0, w_\nu, 0, 0, 0, \dots) \in l^0_{w,\infty}$

Then, $z^{(\nu)} := |x^{(\nu)}| \vee |y^{(\nu)}| = (w_1, w_1, w_2, w_2, w_3, w_3, w_4, w_4, \dots, w_\nu, w_\nu, 0, 0, 0, \dots)$ such that

$$1 \ge \|x^{(\nu)}\|_{w,\infty} \ge \frac{w_1}{w_1} = 1 \ .$$

Similarly $\|y^{(\nu)}\|_{w,\infty} = 1$. Furthermore,

$$2 > \|z^{(\nu)}\|_{w,\infty} \ge \frac{2\sum_{j=1}^\nu w_j}{\sum_{j=1}^{2\nu} w_j} \longrightarrow 2 \text{ as } \nu \longrightarrow \infty \ .$$

Hence,

$$\alpha(l^0_{w,\infty}) := \sup_{x,y\in\mathcal{B}_{l^0_{w,\infty}}} \||x| \vee |y|\|_{w,\infty} = 2 \ .$$

Theorem 4.2.3. $\forall w \in c_0 \setminus l^1$,

$$\alpha(l^0_{w,\infty}) = 2 \lim_{\nu\longrightarrow\infty} \frac{\sum_{j=1}^\nu w_j}{\sum_{j=1}^{2\nu} w_j}$$

Proof. let $x := (x_n)_{n\in\mathbb{N}}, y := (y_n)_{n\in\mathbb{N}} \in \mathcal{B}_X$ be given, then we will consider

$\||x| \vee |y|\|_{w,\infty} = \|(|x_1| \vee |y_1|, |x_2| \vee |y_2|, |x_3| \vee |y_3|, \dots, |x_n| \vee |y_n|, \dots)\|_{w,\infty}$ Now, define for each $n \in \mathbb{N}$, $z_n = |x_n| \vee |y_n|$ Then, fix $n \in \mathbb{N}$ and define $\mathbb{N}_n = \{1, 2, \dots, n\}$ Then notice for some $M, N \in \mathbb{N}_n$ with $M + N = n$

$$\frac{\sum_{j=1}^{n} z_j^\star}{\sum_{j=1}^{n} w_j} \leq \frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j} .$$

As in previous investigation, if $n = 2\nu$ then

$$\frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j} \leq \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j} = \frac{2\sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j}$$

and if $n = 2\nu$ then

$$\frac{\sum_{j=1}^{M} w_j + \sum_{j=1}^{N} w_j}{\sum_{j=1}^{n} w_j} \leq \frac{\sum_{j=1}^{\nu} w_j + \sum_{j=1}^{\nu+1} w_j}{\sum_{j=1}^{2\nu+1} w_j} \underset{\text{large } \nu}{\sim} \frac{2\sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j} := Q(2\nu) .$$

Furthermore, as in previous case, for any $\nu \in \mathbb{N}$, define

$x^{(\nu)} := (w_1, 0, w_2, 0, w_3, 0, w_4, 0, \dots, 0, w_\nu, 0, 0, 0, \dots)$,

$y^{(\nu)} := (0, w_1, 0, w_2, 0, w_3, 0, w_4, 0, \dots, 0, w_{\nu-1}, 0, w_\nu, 0, 0, 0, \dots) \in l^0_{w,\infty}$

Then, $z^{(\nu)} := |x^{(\nu)}| \vee |y^{(\nu)}| = (w_1, w_1, w_2, w_2, w_3, w_3, w_4, w_4, \dots, w_\nu, w_\nu, 0, 0, 0, \dots)$ such that

$$1 \geq \|x^{(\nu)}\|_{w,\infty} \geq \frac{w_1}{w_1} = 1 .$$

Similarly $\|y^{(\nu)}\|_{w,\infty} = 1$. Moreover,

$$\lim_{\nu\longrightarrow\infty} Q(2\nu) \geq \alpha(X) \geq \|z^{(\mu)}\|_{w,\infty} \geq \frac{2\sum_{j=1}^{\mu} w_j}{\sum_{j=1}^{2\mu} w_j}, \forall \mu \in \mathbb{N} .$$

□

Note that formula holds indeed for $1 \leq p < \infty$

if $p = 1$, then

$$\alpha(X) = 2 \lim_{\nu\longrightarrow\infty} \frac{\sum_{j=1}^{\nu} w_j}{\sum_{j=1}^{2\nu} w_j} = 2 \lim_{\nu\longrightarrow\infty} \frac{\int_1^{\nu} \frac{1}{j}\, dj}{\sum_{j=1}^{2\nu} \frac{1}{j}} = \lim_{\nu\longrightarrow\infty} \frac{2\ln(\nu)}{\ln(2\nu)} = \lim_{\nu\longrightarrow\infty} \frac{2\ln(\nu)}{\ln(2) + \ln(\nu)} = 2 .$$

4.3 $l^0_{w,\infty}$ HAS THE W-FPP

Theorem 4.3.1. *$l^0_{w,\infty}$ has the w-FPP by P.K. Lin [29] Define $X := l^0_{w,\infty}$ and let $\|\cdot\| = \|\cdot\|_{w,\infty}$. Then, $(X, \|\cdot\|)$ has the weak fixed point property.*

Proof. We will show that X satisfies the hypothesis of the Theorem 4.1.10. Let's recall the definition of the unconditional constant of an unconditional basis $(x_n)_{n\in\mathbb{N}}$. We shall often assume the sequence $(x_n)_{n\in\mathbb{N}}$ normalized, that is $\|x_n\| = 1, \forall n \in \mathbb{N}$. We say $(x_n)_{n\in\mathbb{N}}$ is an unconditional basis if there exists a constant $\lambda \geq 1$ such that for every sequence of signs $(\varepsilon_n)_{n\in\mathbb{N}} (\varepsilon_n = \pm 1)$ and for every convergent series

$$\sum_{j=1}^{\infty} t_j \, x_j$$

(where $(t_j)_{j\in\mathbb{N}}$ is a sequence of scalars) such that

$$\left\| \sum_{j=1}^{\infty} t_j \, \varepsilon_j \, x_j \right\| \leq \lambda \left\| \sum_{j=1}^{\infty} t_j \, x_j \right\|.$$

Then, the smallest λ satisfying the above condition, i.e.

$$\lambda := \sup_{\substack{\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}} \\ \varepsilon_n = \pm 1}} \frac{\| \sum_{j=1}^{\infty} t_j \, \varepsilon_j \, x_j \|}{\| \sum_{j=1}^{\infty} t_j \, x_j \|}$$

is called the unconditionality constant of the unconditional basis. Now, for our space, the usual basis $(x_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$ is a 1-symmetric basis. Define $S_{\mathbb{N}} := \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi \text{ 1-1 and onto map}\}$ set of all permutations. Then, $\forall \varepsilon_j = \pm 1$, $\forall \pi \in S_{\mathbb{N}}$, and for any sequence of scalars $(t_j)_{j\in\mathbb{N}}$,

$$\left\| \sum_{n=1}^{\infty} \varepsilon_n \, t_{\pi(n)} \, e_n \right\|_{w,\infty} = \left\| \sum_{n=1}^{\infty} t_n \, e_n \right\|_{w,\infty} = \left\| \sum_{n=1}^{\infty} t_n^{\star} \, e_n \right\|_{w,\infty}.$$

Hence, $\lambda = 1 < \frac{\sqrt{33}-3}{2}$. □

4.4 $l^0_{w,\infty}$ FAILS THE FPP FOR AFFINE, $\|\cdot\|_{w,\infty}$-NONEXPANSIVE MAPPINGS

Theorem 4.4.1. *$l^0_{w,\infty}$ fails the FPP*

Define $C := \{x = (t_1\, w_1, t_2\, w_2, t_3\, w_3, \dots) \mid t \in c_0 \ , 1 = t_1 \geq t_2 \geq \dots \geq 0\}$.

Then, $C \subseteq l^0_{w,\infty}$ is a convex, closed, bounded set, and $\exists\, T : C \to C$ s.t. T is fixed point free, $\|\cdot\|_{w,\infty}$-nonexpansive, affine mapping.

Proof. It is clear that C is convex. We need to show $C \subseteq l^0_{w,\infty}$. Let $x \in C$ be given. Then, $x^\star = x$ and

$$\|x\|_{w,\infty} = \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^n x_j^\star}{\sum_{j=1}^n w_j} \ .$$

$$\Psi_n(x) = \frac{\sum_{j=1}^n t_j\, w_j}{\sum_{j=1}^n w_j} \leq \frac{\sum_{j=1}^n 1\, w_j}{\sum_{j=1}^n w_j} = 1 \ .$$

Hence, $\|x\|_{w,\infty} \leq 1, \forall x \in C$. Thus, $C \subseteq l_{w,\infty}$ and in fact, $C \subseteq \mathcal{B}_{l_{w,\infty}}$. Now, to show $C \subseteq l^0_{w,\infty}$, we need to show for $x \in C$, $\Psi_n(x) \underset{n}{\longrightarrow} 0$ and then we would prove $x \in l^0_{w,\infty}$.

Now, first of all,

$$0 \leq \Psi_n(x) = \sum_{j=1}^n t_j\, \gamma_j^{(n)}$$

where

$$\gamma_j^{(n)} = \frac{w_j}{\sum_{k=1}^n w_k} > 0 \text{ and } \sum_{j=1}^n \gamma_j^{(n)} = 1 \ .$$

Then, fix $\varepsilon > 0$, and choose $N = N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon$, $t_j < \frac{\varepsilon}{2}$.

Let $n > N_\varepsilon$, n arbitrary.

$$\begin{aligned}
\Psi_n(x) &= \sum_{j=1}^{N} t_j\gamma_j^{(n)} + \sum_{j=N+1}^{n} t_j\gamma_j^{(n)} \\
&\leq (1)\sum_{j=1}^{N} \gamma_j^{(n)} + \sum_{j=N+1}^{n} t_j\gamma_j^{(n)} \\
&< \sum_{j=1}^{N} \gamma_j^{(n)} + \frac{\varepsilon}{2}\sum_{j=N+1}^{n} \gamma_j^{(n)} \text{ but since } \sum_{j=N+1}^{n} \gamma_j^{(n)} < 1 \\
&< \sum_{j=1}^{N} \gamma_j^{(n)} + \frac{\varepsilon}{2} \\
&= \sum_{j=1}^{N} \left(\frac{w_j}{\sum_{k=1}^{n} w_k}\right) + \frac{\varepsilon}{2} \\
&= \frac{\sum_{j=1}^{N_\varepsilon} w_j}{\sum_{k=1}^{n} w_k} + \frac{\varepsilon}{2} =: \frac{K(\varepsilon)}{\sum_{k=1}^{n} w_k} + \frac{\varepsilon}{2} \ .
\end{aligned}$$

Choose $M_\varepsilon > N_\varepsilon$ s.t. $\forall n \geq M_\varepsilon$,

$$\frac{K(\varepsilon)}{\sum_{k=1}^{n} w_k} < \frac{\varepsilon}{2} \ .$$

Then, $\forall n \geq M_\varepsilon \, (> N_\varepsilon)$,

$0 \leq \Psi_n(x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and so

$$\lim_{n \longrightarrow \infty} \Psi_n(x) = 0 \ .$$

Hence, $C \subseteq l^0_{w,\infty}$.

Claim 4.4.2. *C is closed.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in C norm convergent to an $x_0 \in l_{w,\infty}$ i.e. $x_n \underset{n}{\longrightarrow} x_0$ in norm then $x_n \underset{n}{\longrightarrow} x_0$ coordinatewise.

Say $x_0 := (x_{0,1}, x_{0,2}, x_{0,3}, \ldots, x_{0,k}, \ldots)$ and $x_n := (t_1^n w_1, t_2^n w_2, t_3^n w_3, \ldots, t_k^n w_k, \ldots)$. Then, $x_{0,1} = w_1$ and $t_k^n w_k \underset{n}{\longrightarrow} x_{0,k}$ pointwise. Then, $t_k^n \underset{n}{\longrightarrow} \frac{x_{0,k}}{w_k} =: s_k$ $s_1 = \frac{x_{0,1}}{w_1} = 1$; and each $t_k^n \geq 0 \Rightarrow s_k \geq 0$, by pointwise convergence. Now, fix $k \in \mathbb{N}$, and we can show $s_k \geq s_{k+1}$ by taking limit as n goes to ∞ using the inequality $t_k^n \geq t_{k+1}^n$, since $t_k^n \underset{n}{\longrightarrow} s_k$. Hence, $s_k \geq s_{k+1}, \forall k \in \mathbb{N}$. Now, we need to show $s_k \underset{k}{\longrightarrow} 0$. Since $x_0 = (x_{0,k})_{k\in\mathbb{N}} \in l^0_{w,\infty} \subseteq c_0$, $x_{0,k} \underset{k}{\longrightarrow} 0$.

Now, we know that $x_0 = (s_1 w_1, s_2 w_2, \ldots, s_k w_k, \ldots)$ $1 = s_1 \geq s_2 \geq \ldots s_k \geq \cdots \geq 0$ and so $x_0^{\star} = x_0$. Recall that $x_0 \in l^0_{w,\infty}$ by hypothesis. Equivalently,

$$\lim_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n} (x_0^{\star})_k}{\sum_{k=1}^{n} w_k} = 0 \ .$$

$$\text{Set for each n} \in \mathbb{N}, \ R_n := \frac{\sum_{k=1}^{n} (x_0^{\star})_k}{\sum_{k=1}^{n} w_k} \ .$$

$$\text{Then}, \ R_n = \frac{\sum_{k=1}^{n} s_k w_k}{\sum_{k=1}^{n} w_k} \underset{n}{\longrightarrow} 0.$$

Suppose to get a contradiction that $s_k \underset{k}{\not\longrightarrow} 0$.

But we know $s_k \downarrow_k$ and $s_k \geq 0, \forall k \in \mathbb{N}$. So, $\exists\, L > 0$ s.t. $s_k \underset{k}{\longrightarrow} L$. Then,

$$R_n = \frac{\sum_{k=1}^{n} s_k w_k}{\sum_{k=1}^{n} w_k} \geq R_n = \frac{\sum_{k=1}^{n} L\, w_k}{\sum_{k=1}^{n} w_k} = L > 0, \forall n \in \mathbb{N} \ .$$

But, we know that $R_n \underset{n}{\longrightarrow} 0$, and we are getting $\forall n \in \mathbb{N}, R_n \geq L > 0$, which is a contradiction. In conclusion, $C \subseteq l^0_{w,\infty}$ is a closed bounded convex subset. □

Now, we will prove the following claim.

Claim 4.4.3. $\exists\, T : C \longrightarrow C$ $\ \|\cdot\|_{w,\infty}$-*nonexpansive and fixed point free.*

Proof. Indeed, consider T as the right shift.

I.e. $T : x = (t_j w_j)_{j \in \mathbb{N}} \longrightarrow Tx = (1w_1, t_1 w_2, t_2 w_3, t_3 w_4, \ldots) = (1w_1, 1w_2, t_2 w_3, t_3 w_4, \ldots)$

T is clearly fixed point free. Indeed, assume $\exists\, x \in C$ s.t. $Tx = x$ but then that means $1 = t_1 = t_2 = t_3 = \cdots = t_k \Leftrightarrow x = (w_1, w_2, w_3, \ldots) \in l_{w,\infty} - l^0_{w,\infty} \Rightarrow x \notin C$ which would be a contradiction. Now, let's see T is nonexpansive. Hence, let $x, y \in C$ and say $x = (t_1 w_1, t_2 w_2, t_3 w_3, \ldots)$, $y = (s_1 w_1, s_2 w_2, s_3 w_3, \ldots)$ for some scalars t_n and s_n.

Then, $Tx = (1w_1, t_1 w_2, t_2 w_3, t_3 w_4, \ldots) = (1w_1, 1w_2, t_2 w_3, t_3 w_4, \ldots)$ and

$Ty = (1w_1, s_1 w_2, s_2 w_3, s_3 w_4, \ldots) = (1w_1, 1w_2, s_2 w_3, s_3 w_4, \ldots)$.

Note that if $\beta = (\beta_j)_{j \in \mathbb{N}} \in l_{w,\infty}$ is s.t. $0 \leq \alpha_j \leq \beta_j, \forall j \in \mathbb{N}$,

then $\alpha = (\alpha_j)_{j \in \mathbb{N}} \in l_{w,\infty}$ and $[\|\alpha\|_{w,\infty} \leq \|\beta\|_{w,\infty}]$ [♣♣]

Also,

$$\forall u \in c_0, \forall n \in \mathbb{N}, \sum_{j=1}^{n} u_j^{\star} = \max_{\substack{F \subseteq \mathbb{N} \\ \#(F) = n}} \sum_{k \in F} |u_k| \qquad [\clubsuit\clubsuit\clubsuit]$$

and so It follows that $[\clubsuit\clubsuit\clubsuit] \Rightarrow [\clubsuit\clubsuit]$. Hence,

$$\begin{aligned}
\|x-y\|_{w,\infty} &= \|(0,(t_2-s_2)w_2,(t_3-s_3)w_3,(t_4-s_4)w_4,\dots)\|_{w,\infty} \\
&= \|(|t_2-s_2|w_2,|t_3-s_3|w_3,|t_4-s_4|w_4,\dots)\|_{w,\infty} \\
&= \sup_{n\in\mathbb{N}}\left[\frac{\sum_{k=1}^{n}|t_{k+1}-s_{k+1}|w_{k+1}}{\sum_{k=1}^{n} w_j}\right] .
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|Tx-Ty\|_{w,\infty} &= \|(0,0,(t_2-s_2)w_3,(t_3-s_3)w_4,(t_4-s_4)w_5,\dots)\|_{w,\infty} \\
&= \|(0,0,|t_2-s_2|w_3,|t_3-s_3|w_4,|t_4-s_4|w_5,\dots)\|_{w,\infty} \\
&= \|(|t_2-s_2|w_3,|t_3-s_3|w_4,|t_4-s_4|w_5,\dots)\|_{w,\infty} \\
&= \sup_{n\in\mathbb{N}}\left[\frac{\sum_{k=1}^{n}|t_{k+1}-s_{k+1}|w_{k+2}}{\sum_{k=1}^{n} w_k}\right]
\end{aligned}$$

Thus, $\|Tx-Ty\|_{w,\infty} \le \|x-y\|_{w,\infty}$. $\square$

The proof of Theorem 4.4.1 is complete. $\square$

4.5 $l^0_{w,\infty}$ CONTAINS AN ASYMPTOTICALLY ISOMETRIC COPY OF c_0.

Theorem 4.5.1. *$l^0_{w,\infty}$ contains an a.i c_0 copy and so fails the FPP.*

$\forall w \in c_0 \setminus l^1$, there exists a subset $Y \subseteq l^0_{w,\infty}$ such that Y is an asymptotically isometric copy of c_0 and so $l^0_{w,\infty}$ fails the fixed point property for affine, $\|\cdot\|_{w,\infty}$-nonexpansive mappings.

Proof. Let $w \in c_0 \setminus l^1$ be given. Fix $\varepsilon_j \downarrow_j 0, \varepsilon_j \in (0,1)$. Then, choose a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ so that $1 \leq k_1 < k_2 < k_3 < \dots$ and each term is large enough such that

$$\frac{\sum_{j=1}^{k_n} w_j}{\sum_{j=1}^{k_{n+1}} w_j} \leq \varepsilon_{n+1}, \forall n \in \mathbb{N} \ \text{(since } \sum_{j=1}^{\infty} w_j = \infty) \ ;$$

i.e., increasing to ∞ fast enough so that

$$\frac{\sum_{j=1}^{k_n} w_j}{\sum_{j=1}^{k_{n+1}} w_j} \leq \varepsilon_{n+1}, \forall n \in \mathbb{N} \ .$$

Define $\eta_1 = f_1 = (w_1, w_2, w_3, \dots, w_{k_1}, 0, 0, 0, \dots)$.

Then notice $\|\eta_1\|_{w,\infty} = \|f_1\|_{w,\infty} = 1$,

next define

$$f_2 = (0, 0, 0, \dots, \underset{\substack{\uparrow \\ k_1 \ ^{st} \text{ term}}}{0}, w_{k_1+1}, w_{k_1+2}, \dots, w_{k_2-1}, w_{k_2}, 0, 0, 0, \dots)$$

and then define $\eta_2 := f_1 + f_2$ and notice $\|\eta_2\|_{w,\infty} = \|f_1 + f_2\|_{w,\infty} = 1$, next define

$$f_3 = (0, 0, 0, \dots, \underset{\substack{\uparrow \\ k_2 \ ^{nd} \text{ term}}}{0}, w_{k_2+1}, w_{k_2+2}, \dots, w_{k_3-1}, w_{k_3}, 0, 0, 0, \dots)$$

and then define $\eta_3 := f_1 + f_2 + f_3$ and notice $\|\eta_3\|_{w,\infty} = \|f_1 + f_2 + f_3\|_{w,\infty} = 1$. Continuing in this way, we obtain a sequence $(f_n)_{n\in\mathbb{N}}$ and so $(\eta_n)_{n\in\mathbb{N}}$ such that $\|\eta_n\|_{w,\infty} = \|f_1 + f_2 + f_3 + \dots + f_n\|_{w,\infty} = 1, \forall n \in \mathbb{N}$. Then, define the set

$$Y := \{x = t_1 f_1 + t_2 f_2 + t_3 f_3 + \dots = \sum_{n=1}^{\infty} t_n f_n \ \mid t = (t_n)_{n\in\mathbb{N}} \in c_0\} \ .$$

Then, let $t \in c_0$ and $x \in Y$ be arbitrarily given. Then,

$$\begin{aligned}
\|x\|_{w,\infty} &= \|t_1 f_1 + t_2 f_2 + t_3 f_3 + \dots\|_{w,\infty} = \left\| \sum_{n=1}^{\infty} t_n f_n \right\|_{w,\infty} \\
&= \||t_1| f_1 + |t_2| f_2 + |t_3| f_3 + \dots\|_{w,\infty} = \left\| \sum_{n=1}^{\infty} |t_n| f_n \right\|_{w,\infty} \\
&\leq \|\|t\|_\infty f_1 + \|t\|_\infty f_2 + \|t\|_\infty f_3 + \dots\|_{w,\infty} = \left\| \sum_{n=1}^{\infty} \|t\|_\infty f_n \right\|_{w,\infty} \\
&= \|t\|_\infty \|f_1 + f_2 + f_3 + \dots\|_{w,\infty} = \|t\|_\infty \left\| \sum_{n=1}^{\infty} f_n \right\|_{w,\infty} = \|t\|_\infty \|w\|_{w,\infty} \\
&= \|t\|_\infty \ .
\end{aligned}$$

Also,

$$\|x\|_{w,\infty} \geq \frac{|t_1| \sum_{j=1}^{k_1} w_j}{\sum_{j=1}^{k_1} w_j} = |t_1| \geq |t_1|(1-\varepsilon_1) \ .$$

Next,

$$\begin{aligned}
\|x\|_{w,\infty} &\geq \|t_2 f_2\|_{w,\infty} \\
&\geq \frac{|t_2| \sum_{j=k_1+1}^{k_2} w_j}{\sum_{j=1}^{k_2-k_1} w_j} = |t_2| \frac{\sum_{j=1}^{k_2} w_j - \sum_{j=1}^{k_1} w_j}{\sum_{j=1}^{k_2-k_1} w_j} \geq |t_2| \frac{\sum_{j=1}^{k_2} w_j - \sum_{j=1}^{k_1} w_j}{\sum_{j=1}^{k_2} w_j} \\
&\geq |t_2|(1-\varepsilon_2) \ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|x\|_{w,\infty} &\geq \|t_3 f_3\|_{w,\infty} \\
&\geq \frac{|t_3| \sum_{j=k_2+1}^{k_3} w_j}{\sum_{j=1}^{k_3-k_2} w_j} = |t_3| \frac{\sum_{j=1}^{k_3} w_j - \sum_{j=1}^{k_2} w_j}{\sum_{j=1}^{k_3-k_2} w_j} \geq |t_3| \frac{\sum_{j=1}^{k_3} w_j - \sum_{j=1}^{k_2} w_j}{\sum_{j=1}^{k_3} w_j} \\
&\geq |t_3|(1-\varepsilon_3) \ .
\end{aligned}$$

Then, inductively, we obtain $\|x\|_{w,\infty} \geq |t_n|(1-\varepsilon_n), \forall n \in \mathbb{N}$. Hence,

$$\sup_{\nu \in \mathbb{N}} (1-\varepsilon_\nu)|t_\nu| \leq \|x\|_{w,\infty} \leq \sup_{\nu \in \mathbb{N}} |t_\nu| \leq \sup_{\nu \in \mathbb{N}} (1+\varepsilon_\nu)|t_\nu|$$

$$\sup_{\nu \in \mathbb{N}} (1-\varepsilon_\nu)|t_\nu| \leq \left\| \sum_{\nu=1}^{\infty} t_\nu w_\nu \right\|_{w,\infty} \leq \sup_{\nu \in \mathbb{N}} (1+\varepsilon_\nu)|t_\nu| \ .$$

□

Note that at the end of next chapter, Chapter 5, we show that any closed non-reflexive vector subspace Y of $l^0_{w,\infty}$ contains an isomorphic copy of c_0 and so Y fails the fixed point property for asymptotically nonexpansive maps.

More Open Questions In 1998, Dowling, Lennard and Turett show in [14] that every infinite-dimensional subspace of c_0 fails to have the fixed point property for nonexpansive mappings on closed, bounded, convex subsets. Later, In 2010 they define an equivalent norm $\|\cdot\|$ to c_0's canonical norm and prove in [18] that every infinite-dimensional subspace of $(c_0, \|\cdot\|)$ fails to have the fixed point property. Hence, for our space $l^0_{w,\infty}$ that we work on now, we are wondering the analogue of these questions. Hence, the following is an open question.

Open question (4) Does every infinite-dimensional (or non-reflexive) subspace of $l^0_{w,\infty}$ fail the FPP for nonexpansive mappings (or even for asymptotically nonexpansive mappings) on closed, bounded, convex subsets?

Furthermore, in 1999, Dowling and Randrianantoanina [19] investigated the fixed point property for spaces of compact operators $K(H)$ on a Hilbert space. They found out that every nonreflexive subspace Y of $K(H)$ fails the FPP. M. Besbes [5] showed every weakly compact subset of $K(H)$ has the FPP.

Open question (5) It is another open question whether or not every closed bounded convex subset C of $K(H)$ has the FPP for nonexpansive mappings if and only if C is weakly compact.

5.0 SOME RESULTS FOR ANY EQUIVALENT RENORMING OF l^1, RENORMING OF c_0, AND REFLEXIVITY.

It is well-known (Theorems 1.c.12 in [31] and 1.c.5 in [32]) that a Banach lattice or a Banach space with an unconditional basis is reflexive if and only if it contains no isomorphic copies of c_0 or l^1. Hence, if it can be shown that neither c_0 nor l^1 can be renormed to have the fixed point property, it would follow that the fixed point property in either a Banach lattice or in a Banach space with an unconditional basis would imply reflexivity. In 2005, P.K. Lin in [30] proved that $\exists$ an equivalent norm $\|\cdot\|$ on l^1 s.t. $(l^1, \|\cdot\|)$ has the FPP for nonexpansive mappings. However, it is an open question whether or not l^1 can be renormed to have the fixed point property for asymptotically nonexpansive mappings. In 2000, Dowling, Lennard and Turett in [15] showed that l^1 cannot be renormed to have the fixed point property for uniformly lipschitzian mappings using Strong James Distortion Theorem. Furthermore, they prove c_0 cannot be renormed to have the fixed point property for asymptotically nonexpansive mappings. First of all, we will give some preliminaries.

Definition 5.0.2. Banach lattice

A partially ordered Banach space $(X, \leq)$ over the reals is called a Banach lattice provided

(i) $x \leq y$ implies $x + z \leq y + z$, for every $x, y, z \in X$.

(ii) $ax \geq 0$ for every $x \geq 0$ in X and every $a \geq 0$.

(iii) for all $x, y \in X$ there exists a least upper bound (l.u.b) $x \vee y$ and a greatest lower bound (g.l.b) $x \wedge y$.

(iv) $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

James [25] proved the following theorem which is called James Distortion Theorem.

Theorem 5.0.3. *James Distortion Theorem*

A Banach space X contains an isomorphic copy of l^1 if and only if, for every $0 < \varepsilon < 1$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$(1-\varepsilon)\sum_{n=1}^{\infty}|t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sum_{n=1}^{\infty}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in l^1$.

A Banach space X contains an isomorphic copy of c_0 if and only if, for every $0 < \varepsilon < 1$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$(1-\varepsilon)\sup_{n\in\mathbb{N}}|t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq (1+\varepsilon)\sup_{n\in\mathbb{N}}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in c_0$.

Dowling, Lennard and Turett realized this theorem can be strengthened to

Theorem 5.0.4. *Strong James Distortion Theorem*

A Banach space X contains an isomorphic copy of l^1 if and only if, for every null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$(1-\varepsilon_k)\sum_{n=k}^{\infty}|t_n| \leq \left\|\sum_{n=k}^{\infty} t_n x_n\right\| \leq \sum_{n=k}^{\infty}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in l^1$ and for all $k \in \mathbb{N}$.

A Banach space X contains an isomorphic copy of c_0 if and only if, for every null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$(1-\varepsilon_k)\sup_{n\geq k}|t_n| \leq \left\|\sum_{n=k}^{\infty} t_n x_n\right\| \leq (1+\varepsilon_k)\sup_{n\geq k}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in c_0$ and for all $k \in \mathbb{N}$.

Theorem 5.0.5. *Lindenstrauss , Tzafriri 1.c.5*

The following properties are equivalent for every Banach lattice X.

(i) *X is reflexive.*

(ii) *No subspace of X is isomorphic to l^1 or to c_0.*

Then, we can prove the following fact.

Theorem 5.0.6. *Renorming of l^1 and Semi-strongly Asymptotically Nonexpansive Maps.*

If X is a Banach space containing an isomorphic copy of l^1 , then there exists a closed, bounded, convex subset E of X and an affine, semi-strongly asymptotically nonexpansive $T : E \to E$ such that T has no fixed point. Consequently, l^1 cannot be renormed to have the fixed point property for semi-strongly asymptotically nonexpansive mappings.

Proof. Let X be a Banach space containing an isomorphic copy of l^1 and consider a null sequence (ε_n) in $(0, 1)$ and then by the Strong James Distortion Theorem there exists a sequence (x_n) in X such that

$$(1-\varepsilon_k)\sum_{n=k}^{\infty}|t_n| \le \left\|\sum_{n=k}^{\infty} t_n x_n\right\| \le \sum_{n=k}^{\infty}|t_n| \ ,$$

for all $(t_n)_{n\in\mathbb{N}} \in l^1$ and for all $k \in \mathbb{N}$.

Then the closed convex hull of x_n, i.e. $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$, fails the fixed point property for $\|\cdot\|$-semi-strongly asymptotically nonexpansive mappings. In fact, we show that there exists an *affine semi-strongly asymptotically nonexpansive* mapping $T : E \longrightarrow E$ that is fixed point free. Furthermore, T is the usual right shift mapping. Note that

$$E = \left\{\sum_{n=1}^{\infty} t_n\, x_n : 0 \le t_n, \forall n \in \mathbb{N}, \sum_{n=1}^{\infty} t_n = 1\right\} \ .$$

For any $x, y \in E$, write

$$x = \sum_{n=1}^{\infty} t_n\, x_n \in E$$

and

$$y = \sum_{n=1}^{\infty} s_n\, x_n \in E \ ;$$

so that $t_n, s_n \ge 0$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1 \ .$$

Let $\alpha_n := t_n - s_n$, for all $n \in \mathbb{N}$. Then, consider the right shift $T : E \longrightarrow E$ given by

$$T\left(\sum_{j=1}^{\infty} t_j\, x_j\right) = \sum_{j=1}^{\infty} t_j\, x_{j+1} \ .$$

Then, for any $n \in \mathbb{N}$, $T^n : E \longrightarrow E$ given by

$$T^n\left(\sum_{j=1}^{\infty} t_j\, x_j\right) = \sum_{j=1}^{\infty} t_j\, x_{j+n}\ .$$

$$\begin{aligned} \|T^n(x) - T^n(y)\| &= \left\|\sum_{j=1}^{\infty} \alpha_j\, x_{j+n}\right\| \\ &= \left\|\sum_{k=n+1}^{\infty} \alpha_{k-n}\, x_k\right\| \\ &\leq \sum_{k=n+1}^{\infty} |\alpha_{k-n}|\ . \end{aligned}$$

Now, fix $m \in \mathbb{N}$

$$\begin{aligned} \|T^m(x) - T^m(y)\| &= \left\|\sum_{j=m+1}^{\infty} \alpha_{j-m}\, x_j\right\| \\ &\geq (1 - \varepsilon_{m+1}) \sum_{k=m+1}^{\infty} |\alpha_{k-m}|\ . \end{aligned}$$

Then, for all $n > m$

$$\begin{aligned} \|T^n(x) - T^n(y)\| &\leq \sum_{k=n+1}^{\infty} |\alpha_{k-n}| \\ &= \sum_{k=m+1}^{\infty} |\alpha_{k-m}| \\ &\leq \frac{1}{(1 - \varepsilon_{m+1})} \|T^m(x) - T^m(y)\| \end{aligned}$$

Define $\lambda_{n,m} = \frac{1}{1-\varepsilon_{m+1}}, \forall\, n \geq m \geq 1$. Then $\lambda_{n,m} \longrightarrow 1$ as $n \geq m \longrightarrow \infty$, and $\|T^n(x) - T^n(y)\| \leq \lambda_{n,m} \|T^m(x) - T^m(y)\|$ for all $x, y \in E$. Thus, T is $\|\cdot\|$-semi-strongly asymptotically nonexpansive, and it is easily checked that T has no fixed point. $\square$

Theorem 5.0.7. *Renorming of c_0 and Strongly Asymptotically Nonexpansive Maps.*

If X is a Banach space containing an isomorphic copy of c_0 , then there exists a closed, bounded, convex subset E of X and an affine, strongly asymptotically nonexpansive $T : E \to E$ such that T has no fixed point. Consequently, c_0 cannot be renormed to have the fixed point property for strongly asymptotically nonexpansive mappings.

Proof. Let X be a Banach space containing an isomorphic copy of c_0 and consider a null sequence (ε_n) in $(0,1)$. Dowling, Lennard and Turett proved in [15] that the usual right shift mapping is asymptotically nonexpansive and fixed point free. We will use similar method and show that right shift mapping is strongly asymptotically nonexpansive. Now, by Theorem 8 in [15], there exist a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and a sequence (x_n) in X so that

$$\sup_{n\geq k}|t_n| \leq \left\|\sum_{n=k}^{\infty} t_n x_n\right\| \leq (1+\varepsilon_k)\sup_{n\geq k}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in c_0$ and for all $k\in\mathbb{N}$. Let $f_n = x_1+\cdots+x_n$, for all $n\in\mathbb{N}$.

Now, the closed convex hull of f_n, i.e. $E := \overline{\text{co}}(\{f_n : n\in\mathbb{N}\})$, fails the fixed point property for $\|\cdot\|$-strongly asymptotically nonexpansive mappings. In fact, we show that there exists an *affine strongly asymptotically nonexpansive* mapping $T : E \longrightarrow E$ that is fixed point free. Furthermore, T is the usual right shift mapping. Note that

$$\begin{aligned} E &= \left\{\sum_{n=1}^{\infty}\alpha_n\, f_n : 0\leq\alpha_n, \forall n\in\mathbb{N}, \sum_{n=1}^{\infty}\alpha_n = 1\right\} \\ &= \left\{\sum_{n=1}^{\infty} t_n\, x_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0\right\} . \end{aligned}$$

For any $x, y\in E$, write

$$x = \sum_{n=1}^{\infty} t_n\, x_n \in E$$

and

$$y = \sum_{n=1}^{\infty} s_n\, x_n \in E\ ;$$

so that $t_n, s_n \geq 0$ for all $n\in\mathbb{N}$, and $1 = t_1\geq t_2\geq\cdots\geq t_n\downarrow_n 0$ and $1 = s_1\geq s_2\geq\cdots\geq s_n\downarrow_n 0$. Let $\alpha_n := t_n - s_n$, for all $n\in\mathbb{N}$. Then, consider the right shift $T : E\longrightarrow E$ given by

$$T\left(\sum_{j=1}^{\infty} t_j\, x_j\right) = x_1 + \sum_{j=1}^{\infty} t_j\, x_{j+1}\ .$$

Then, for any $n\in\mathbb{N}$, $T^n : E\longrightarrow E$ given by

$$T^n\left(\sum_{j=1}^{\infty} t_j\, x_j\right) = x_1+\cdots+x_n + \sum_{j=1}^{\infty} t_j\, x_{j+n}\ .$$

$$\begin{aligned} \|T^n(x) - T^n(y)\| &= \left\| \sum_{j=1}^{\infty} \alpha_j \, x_{j+n} \right\| \\ &= \left\| \sum_{k=n+1}^{\infty} \alpha_{k-n} \, x_k \right\| \\ &\leq (1+\varepsilon_{n+1}) \sup_{k \geq n+1} |\alpha_{k-n}| \\ &= (1+\varepsilon_{n+1}) \sup_{k \geq 1} |\alpha_k| \ . \end{aligned}$$

Now, fix $m \in \mathbb{N}$.

$$\begin{aligned} \|T^m(x) - T^m(y)\| &= \left\| \sum_{j=m+1}^{\infty} \alpha_{j-m} \, x_j \right\| \\ &\geq \sup_{k \geq m+1} |\alpha_{k-m}| \\ &= \sup_{k \geq 1} |\alpha_k| \ . \end{aligned}$$

Then, for all $n > m$

$$\begin{aligned} \|T^n(x) - T^n(y)\| &\leq (1+\varepsilon_{n+1}) \sup_{k \geq 1} |\alpha_k| \\ &\leq (1+\varepsilon_{n+1}) \|T^m(x) - T^m(y)\| \ . \end{aligned}$$

Let $\beta_{n,m} = (1+\varepsilon_{n+1}), \forall\, n \geq m \geq 0$. Then $\beta_{n,m} \longrightarrow 1$ as $n \longrightarrow \infty\,, \forall\, m \in \mathbb{N}$, $\beta_{n,m} \longrightarrow 1$, as $n \geq m \longrightarrow \infty$, and $\|T^n(x) - T^n(y)\| \leq \beta_{n,m} \|T^m(x) - T^m(y)\|$ for all $x, y \in E$. Thus, T is $\|\cdot\|$-strongly asymptotically nonexpansive and it is easily checked that T has no fixed point. □

Remark 5.0.8. We can conclude the following.

Let $(X, \|\cdot\|)$ be a nonreflexive Banach lattice. Then, $(X, \|\cdot\|)$ fails the fixed point property for $\|\cdot\|$-semi-strongly asymptotically nonexpansive mappings.

Note that it is a well known fact if $(X, \|\cdot\|)$ is a nonreflexive Banach lattice, then X contains an isomorphic copy of l^1 or c_0 (see Theorem 5.0.5). Hence, $(X, \|\cdot\|)$ fails the fixed point property for $\|\cdot\|$-semi-strongly asymptotically nonexpansive mappings.

Now, using the above facts, we can give some results for our previous Chapter 4 about Lorentz-Marcinkiewicz spaces $l^0_{w,\infty}$.

5.1 ANY CLOSED NON-REFLEXIVE VECTOR SUBSPACE Y OF $l^0_{w,\infty}$ CONTAINS AN ISOMORPHIC COPY OF c_0

For this section, first we will give some preliminaries about duality for Banach spaces and separable Banach spaces. Here, we will give some propositions proved, for example, in Beauzamy [4].

Proposition 5.1.1. *the dual of a subspace and quotient space*

Let E be a normed space,and F a subspace of E, endowed with the induced norm. Then:

(a) *The dual $F^\star$ of F can be isometrically identified with the quotient $E^\star/F^\perp$; the duality mapping is defined by:*

(1) $\dot{\xi}(x) = \xi(x)$, if $x \in F$, $\dot{\xi} \in E^\star/F^\perp$, and $\xi \in E^\star$ is any element in the class $\dot{\xi}$.

(b) *The weak topology $\sigma(F, E^\star/F^\perp)$ is the topology induced on F by $\sigma(E, E^\star)$.*

Proposition 5.1.2. *the dual of a quotient space and annihilator of subspace*

Let E be a normed space,and F a subspace of E, endowed with the induced norm. Then:

(a) *The dual of E/F can be isometrically identified with $F^\perp$; the duality mapping is defined by:*

(2) $< \dot{x}, \xi >= \xi(x)$, if $\dot{x} \in E/F$, x being any element in the class $\dot{x}$, $x \in E$, and $\xi \in F^\perp$.

(b) *The topology $\sigma(E/F, F^\perp)$ is identical with the quotient topology of $\sigma(E, E^\star)$ by F.*

Proposition 5.1.3. *E is reflexive if and only if its unit ball $\mathcal{B}_E$ is $\sigma(E, E^\star)$-compact.*

Corollary 5.1.4. *subspace of reflexive space*

If E is reflexive, then all closed subspaces of E are reflexive.

Theorem 5.1.5. *Let $(X, \|\cdot\|)$ be a separable Banach space. Let Z be a closed vector subspace of X. Then $(X/Z, \|\cdot\|_{X/Z})$ is a separable Banach space.*

Proof. By hypothesis, there exists a dense sequence $(u_n)_{n\in\mathbb{N}}$ in $(X, \|\cdot\|)$.

Let $[x] = x + Z \in X/Z$. Fix $\varepsilon > 0$. There exists $k \in \mathbb{N}$ such that

$$\|x - u_k\| < \varepsilon \ .$$

Let θ be the zero element in X. Note that $\theta \in Z$. Thus,

$$\begin{aligned} \|[x] - [u_k]\|_{X/Z} &= \|[x - u_k]\|_{X/Z} := \inf\{\|x - u_k - z\| \mid z \in Z\} \\ &\leq \|x - u_k - \theta\| = \|x - u_k\| < \varepsilon \ . \end{aligned}$$

Therefore, $([u_n])_{n\in\mathbb{N}}$ is a dense sequence in $(X/Z, \|\cdot\|_{X/Z})$. □

Theorem 5.1.6. *Lindenstrauss , Tzafriri 1.c.12*

(a) *A Banach space X with an unconditional basis which does not have subspaces isomorphic to c_0 or l^1 must be reflexive. In particular, if X has an unconditional basis and $X^{\star\star}$ is separable then X is reflexive.*

(b) *A weakly sequentially complete Banach space with an unconditional basis is isomorphic to a conjugate space.*

(c) *If X has an unconditional basis and $X^\star$ is separable then $X^\star$ has an unconditional basis.*

Open question We plan to investigate if $l^0_{w,\infty}$ does not have any infinite-dimensional reflexive subspace.

However, using the facts above and the Theorem 1.c.12 in [31], we can give the following nice result.

Theorem 5.1.7. *Let Y be any closed, non-reflexive vector subspace of $l^0_{w,\infty}$. Then, Y contains an isomorphic copy of c_0 and so $(Y, \|\cdot\|_{w,\infty})$ fails the fixed point property for strongly asymptotically nonexpansive maps.*

Proof. We know that if Y is non-reflexive then by Theorem 5.1.6, ∃ an isomorphic copy of l^1 or c_0 inside Y.

Claim 5.1.8. *Y does not contain an isomorphic copy of l^1 and so it contains an isomorphic copy of c_0.*

Proof. By contradiction, assume not. I.e. $l^1 \lesssim l^0_{w,\infty}$. Then, by the previous propositions, there exists a subspace Z of the dual space of $(l^0_{w,\infty})^\star$; i.e., $Z \leq (l^0_{w,\infty})^\star$ such that $(l^1)^\star$ is isometrically identical to $(l^0_{w,\infty})^\star/Z$ i.e., $(l^1)^\star \cong (l^0_{w,\infty})^\star/Z$. But also $(l^1)^\star \cong l^\infty$ and $(l^0_{w,\infty})^\star/Z \cong (l_{w,1}/Z)$. However, we know that $l_{w,1}$ is separable and so is the quotient space

$l_{w,1}/Z$. However, l^∞ is non-separable. But, this tells us $(l^1)^\star \cong (l^0_{w,\infty})^\star/Z$ is a contradiction. Hence, Y cannot contain an isomorphic copy of l^1 and so it has to contain an isomorphic copy of c_0. □

The proof of Theorem 5.1.7 is complete. □

5.2 REFLEXIVITY AND SEMI-STRONGLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS.

Theorem 5.2.1. *[Mil'man and Mil'man [35], Section 4]*

Let $(X, \|\cdot\|)$ be a Banach space and C be a weakly compact convex set in X. Let $U: C \to C$ be an affine norm-to-norm continuous map. Then U has a fixed point in C.

Theorem 5.2.2. *Reflexivity and Semi-strongly Asymptotically Nonexpansive Maps.*

Let $(X, \|\cdot\|)$ be a Banach lattice or a Banach space with an unconditional basis. Then the following are equivalent:

(1) *X is reflexive*

(2) *For every closed bounded convex set C contained in X, for every affine semi-strongly asymptotically nonexpansive mapping $U: C \to C$, U has a fixed point in C.*

Proof. To prove (2) implies (1) , we prove the equivalent statement below:

If X is a nonreflexive Banach lattice or a Banach space with an unconditional basis, then there exists a closed bounded convex subset K in X and there exists an affine semi-strongly asymptotically nonexpansive mapping $T: K \to K$ such that T is fixed point free.

Indeed, this result can be easily concluded from our Remark 5.0.8 since the right shift mapping is affine. We can give proof steps as in the following though.

Now, if $(X, \|\cdot\|)$ is nonreflexive Banach lattice or a Banach space with an unconditional basis, it contains an isomorphic copy of l^1 or c_0 by the Theorem 5.0.5 and Theorem 5.1.6 (a).

By our Theorem 5.0.6, if it contains an isomorphic copy of l^1 , then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $K := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$, fails the fixed point property for $\|\cdot\|$-

semi-strongly asymptotically nonexpansive mappings. In fact, we show that there exists an *affine semi-strongly asymptotically nonexpansive* mapping $T\colon K \longrightarrow K$ that is fixed point free. Moreover, T is the usual right shift mapping.

Furthermore, by our other Theorem 5.0.7, if it contains an isomorphic copy of c_0 , then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $K := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$, fails the fixed point property for $\|\cdot\|$-strongly asymptotically nonexpansive mappings. In fact, we show that there exists an *affine strongly asymptotically nonexpansive* mapping $T\colon K \longrightarrow K$ that is fixed point free. Also, T is the usual right shift mapping and semi-strongly asymptotically nonexpansive.

Now we prove (1) implies (2).

If X is reflexive, then B_X is weakly compact by the proposition 5.1.3. So, every closed bounded convex set C contained in X must also be weakly compact. Hence, Theorem 5.2.1 implies the result. □

Open question (6) Whether or not if Y is a non-reflexive subspace of $l^0_{w,\infty}$, then Y contains an asymptotically isometric copy of c_0.

Open question (7) Whether or not every infinite-dimensional subspace Y of $l^0_{w,\infty}$ is non-reflexive.

Open question (8) (which is equivalent to (6) and (7) together) Whether or not every infinite-dimensional subspace Y of $l^0_{w,\infty}$ contains an asymptotically isometric copy of c_0.

6.0 FUTURE PROJECTS

I see my research taking two different directions in the future. First and mainly, I would like to continue working on fixed point theory. Hence, I would like to use our tools in this thesis and work on the following open questions. However, I would also be interested in frame theory as my second possible direction. Although in this thesis we have not worked on frame theory, we did investigate perturbation of frames. Now, in conclusion, I would like to list the open questions that I plan to focus on mainly on fixed point theory, and one question that could get me into frame theory when I get a chance in the future.

Open Question (1) Let $(X, \|\cdot\|)$ be a Banach space that contains a c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, then define the closed convex hull of $(x_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. Can we find an affine $\|\cdot\|$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free?

Open Question (2) In 2004 Dowling, Lennard and Turett showed that every non-weakly compact, closed, bounded, convex (c.b.c.) subset K of $(c_0, \|\cdot\|_\infty)$ is such that there exists a $\|\cdot\|_\infty$-nonexpansive mapping T on K that is fixed point free. This mapping T is generally not affine. It is an open question as to whether or not on every non-weakly compact, c.b.c. subset K of $(c_0, \|\cdot\|_\infty)$ there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping S that is fixed point free.

Open Question (3) Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence such that

$$\text{for some } \Gamma > 0\,,\ \Gamma \leq \gamma_N, \forall N \in \mathbb{N}\,;\ \text{and}\ \sigma := \sum_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty\ .$$

Also, let $(b_n)_{n\in\mathbb{N}}$ be a bounded sequence in $[m, M]$, where $0 < m \leq M < \infty$ are constants. Define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting

$$\eta_n \;:=\; \gamma_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n) \;, \text{ for all } n \in \mathbb{N} \;.$$

Also, assume that $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate. Then, define the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$. It is an open question whether or not there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

Notice that this open question includes section 2.4 (all $\gamma_n = 1$), where we know this question has a positive answer.

More Open Questions In 1998, Dowling, Lennard and Turett show in [14] that every infinite-dimensional subspace of c_0 fails to have the fixed point property for nonexpansive mappings on closed, bounded, convex subsets. Later, In 2010 they define an equivalent norm $\|\cdot\|$ to c_0's canonical norm and prove in [18] that every infinite-dimensional subspace of $(c_0, \|\cdot\|)$ fail to have the fixed point property. Hence, for our space $l^0_{w,\infty}$ that we work on now, we are wondering the analogue of these questions. Hence, the following is an open question.

Open question (4) Does every infinite-dimensional (or non-reflexive) subspace of $l^0_{w,\infty}$ fail the FPP for nonexpansive mappings (or even for asymptotically nonexpansive mappings) on closed, bounded, convex subsets?

Furthermore, in 1999, Dowling and Randrianantoanina [19] investigated the fixed point property for spaces of compact operators $K(H)$ on a Hilbert space. They found out that every nonreflexive subspace Y of $K(H)$ fails the FPP. M. Besbes [5] showed every weakly compact subset of $K(H)$ has the FPP.

Open question (5) It is another open question whether or not every closed bounded convex subset C of $K(H)$ has the FPP for nonexpansive mappings if and only if C is weakly compact.

Open question (6) Whether or not if Y is a non-reflexive subspace of $l^0_{w,\infty}$, then Y contains an asymptotically isometric copy of c_0.

Open question (7) Whether or not every infinite-dimensional subspace Y of $l^0_{w,\infty}$ is non-reflexive.

Open question (8) (which is equivalent to (6) and (7) together) Whether or not every infinite-dimensional subspace Y of $l^0_{w,\infty}$ contains an asymptotically isometric copy of c_0.

Open Question (9) Borwein and Sims, in 1984, showed that if K is a weak compact and convex subset of $(c, \|\cdot\|_\infty)$, then K has the FPP for nonexpansive mappings. (Note that weak compactness and convexity implies c.b.c., but c.b.c. doesn't imply weak compactness.)

Let K be a subset of $(c, \|\cdot\|_\infty)$. Assume K is closed, bounded and convex. Moreover, assume K has the FPP for nonexpansive mappings. Is it true that K is weakly compact?

Open Question (10) Let H be a Hilbert space with inner product $< \cdot, \cdot >$ and derived norm $\|\cdot\|$. Assume I is a countable index set. A family of elements $(f_i)_{i\in I}$ in H is called a frame (with bounds A and B) if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |< f, f_i >|^2 \leq B\|f\|^2, \forall f \in H.$$

We call $(f_i)_{i\in I}$ a frame sequence in H if it is a frame for its closed linear span $[f_i]$. Let K and L be two subspaces of H. When $K \neq \{0\}$, the gap from K to L is given by

$$\delta(K, L) := \sup_{x\in K\ \|x\|=1} \inf_{y\in L} \|x - y\|$$

and we define $\delta(K, L) = 0$ when $K = \{0\}$. It can be easily seen that $\delta(K, L) = \|P_K\, P_L^\perp\|$ where P_K denotes orthogonal projection onto K.

In 1999, Ole Christensen perturbs frame sequences with the following theorem in [10].

Theorem 6.0.3. *Let $(f_i)_{i\in I}$ be a frame sequence with bounds A, B in Hilbert space H, and let $(f_i)_{i\in I}$ be a family in H. Let $(g_i)_{i\in I}$ be a family in H. Assume that there exist constants $\lambda_2 \in [0, 1)$ and $\lambda_1, \mu \geq 0$ such that*

$$\left\|\sum_{i\in F} c_i(f_i - g_i)\right\| \leq \lambda_1 \left\|\sum_{i\in F} c_i\, f_i\right\| + \lambda_2 \left\|\sum_{i\in F} c_i\, g_i\right\| + \mu \left(\sum_{i\in F} |c_i|^2\right)^{\frac{1}{2}}$$

for all finite scalar sequences $(c_i)_{i\in F}, F \subseteq I$. Then, $(g_i)_{i\in I}$ is a Bessel sequence with upper bound $B\big(1 + \frac{\lambda_1+\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\big)^2$. Also, define $\delta_N = \delta(N_T, N_U)$ where T and U are preframe operators for Bessel sequences $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ respectively. Note that here N is the kernel *for the operators. Then, if* $\delta_N < 1$ *and* $\lambda_1 + \frac{\mu}{\sqrt{A}(1-\delta_N{}^2)^{\frac{1}{2}}} < 1$, *then,* $(g_i)_{i\in I}$ *is a frame sequence with lower bound*

$$A(1-\delta_N{}^2)\left(1 - \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}(1-\delta_N{}^2)^{\frac{1}{2}}}}{1+\lambda_2}\right)^2 .$$

In 2000, Ole Christensen, Chris Lennard and Christine Lewis developed new frame perturbations in [11], and showed the following theorem.

Theorem 6.0.4. *Let* $(f_i)_{i\in I}$ *be a frame sequence with bounds* A, B *in Hilbert space* H, *and let* $(g_i)_{i\in I}$ *be a family in* H. *Let* $K := [g_i]$ *and* $L := [f_i]$. *Assume that there exist constants* $\lambda_2 \in [0,1)$ *and* $\lambda_1, \mu \geq 0$ *such that*

$$\lambda_1 + \frac{\mu}{\sqrt{A}} < \sqrt{1-\delta(K,L)^2}$$

and

$$\left\|\sum_{i\in F} c_i(f_i-g_i)\right\| \leq \lambda_1 \left\|\sum_{i\in F} c_i\, f_i\right\| + \lambda_2 \left\|\sum_{i\in F} c_i\, g_i\right\| + \mu\left(\sum_{i\in F}|c_i|^2\right)^{\frac{1}{2}}$$

for all finite scalar sequences $(c_i)_{i\in F}, F\subseteq I$. *Then,* $(g_i)_{i\in I}$ *is a frame sequence with bounds*

$$A\left(1-\frac{\lambda_1+\lambda_2+\mu/\sqrt{A}}{1+\lambda_2}\right)^2 \text{ and } B\left(1+\frac{\lambda_1+\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\right)^2$$

Moreover, $[g_i]$ *is isomorphic to* $[f_i]$ *and* $[g_i]^\perp$ *is isomorphic to* $[f_i]^\perp$.

As we see, both papers have similar conclusion with different hypothesis.

Our aim, or idea would be to find (i.e., invent) a unifying hypothesis [call it (**)], weaker than the hypotheses in both papers, such that we still have that: (**) implies the frame sequence perturbation conclusion of both papers. Also, we are planning to investigate what we would have if we work in Banach spaces. That means, can we perturb frames in Banach spaces?

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