CONFORMAL MAPPINGS AND ISOMETRIC IMMERSIONS UNDER SECOND ORDER SOBOLEV REGULARITY

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ABSTRACT

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We consider two classes of vector valued functions with conformal constraint- conformal mappings from an n-dimensional domain into $\mathbb{R}^n$ and isometric immersions of an n-dimensional domain into $\mathbb{R}^{n+1}$ (co-dimension one) for $n \geq 3$.

Iwaniec and Martin proved that in even dimensions $n \geq 3$, $W^{1,n/2}_{\text{loc}}$ conformal mappings are Möbius transformations and they conjectured that it should also be true in odd dimensions. In the first part of this manuscript, we prove this theorem for a conformal map $f \in W^{1,1}_{\text{loc}}$ in dimension $n \geq 3$ under one additional assumption that the norm of the first order derivative $|Df|$ satisfies $|Df|^p \in W^{1,2}_{\text{loc}}$ for $p \geq (n-2)/4$. This is optimal in the sense that if $|Df|^p \in W^{1,2}_{\text{loc}}$ for $p < (n-2)/4$, it may not be a Möbius transform. This result shows the necessity of the Sobolev exponent in the Iwaniec-Martin conjecture.

In the second part, we prove the developability and $C^{1,1/2}_{\text{loc}}$ regularity of $W^{2,2}$ isometric immersions of n-dimensional domains into $\mathbb{R}^{n+1}$ for $n \geq 3$. The result is sharp in the sense that $W^{1,p}, 1 \leq p \leq \infty$ and $W^{2,p}, 1 \leq p < 2$ isometric immersions may not be developable. Based on this result, we also prove that if the domain is C^1 and convex, smooth isometric immersions are strongly dense in this space.

Keywords: Conformal mappings, isometric immersions, Sobolev Spaces.

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PREFACE

I would like to express my greatest thanks to my advisor- Professor Piotr Hajłasz, who brought me into the world of mathematics and taught me step by step how to start research. He provided me tremendous amount of help and support during my PhD studies and thesis writing process, and gave me a great many stimulating discussions on my research topics and methodology with lots and lots of extremely useful comments and suggestions. I cannot successfully finish my PhD without Professor Hajłasz's guidance.

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1.0 INTRODUCTION

Function spaces with constraints are important tools in the study of qualitative properties of solutions to various nonlinear and geometric partial differential equations such as regularity, rigidity, compactness and convergence. An important feature in the study of the mappings in these spaces is the interaction between their analytical, geometrical, or topological properties. In this manuscript we consider two classes of Sobolev mappings with conformal constraints and investigate their analytical and geometrical properties.

In classical geometry, a C^1 diffeomorphism $f : \Omega \to \mathbb{R}^N$ for $\Omega \subset \mathbb{R}^n, 2 \leq n \leq N$ is conformal if it preserves the angle of any two curves at each point of its domain. An equivalent analytical definition is a C^1 vectored valued function $f : \Omega \to \mathbb{R}^N$ for $\Omega \subset \mathbb{R}^n, 2 \leq n \leq N$ is *conformal* if its derivatives satisfies the relations

$$Df^T(x)Df(x) = |J(x, f)|^{2/n} \cdot \mathrm{I} \quad \text{for all } x \in \Omega \text{ and}$$

$$J(x, f) \neq 0 \quad \text{for all } x \in \Omega.$$

Here $Df(x)$ denotes the $N \times n$ matrix of all partial derivatives of f at x and $J(x, f)$ is the general Jacobian (for definition see, eg, [16], Chapter 3). As a special case, we call f an *isometry* if $J(x, f) \equiv 1$, meaning that f not only preserve the angles of any two curves at each point, but also preserve the length of each curve.

The case most widely investigated is when $n = N$, in which case we call this vector valued function a *conformal mapping*, while conventionally for the case $n + 1 \leq N$, we call it a *conformal immersion*.

Conformal mappings were first introduced in complex analysis, where every holomorphic function with non-vanishing derivative is conformal, hence the plane is rich in conformal maps. However, in dimension $n \geq 3$, the only conformal maps of class C^1 are Möbius transformations, that is, mappings generated by translations, rotations, dilatations, reflections, and inversions in spheres. In particular, Möbius transformations are C^∞ smooth. For C^3 conformal diffeomorphisms in $\mathbb{R}^3$, Liouville [34] established this result in 1850. This is a strong rigidity theorem. In particular, it is a strong contrast to the situation in the plane. From then on, there have been extensive studies of conformal mappings and a lot of deep and interesting results have been developed. Capelli [7] in 1886 extended the Liouville theorem on conformal mappings to all higher dimensions for maps also of class C^3. Another well-known proof is the one given by Nevanlinna [44], [15] in 1960 under a C^4 smoothness assumption using elementary tools of analysis.

Since to define conformal maps we only need C^1 regularity, a natural question to ask is if the Liouville theorem holds under C^1 regularity assumption. This, however, turns out to be difficult- it was until almost a hundred year later in 1947, Hartman [22] proved the theorem for C^2 conformal diffeomorphisms and later in 1958 [23] for the C^1 case. Another proof of the Liouville theorem for C^2 conformal diffeomorphisms was given by Sarvas [53] in 1978.

The Liouville theorem, first introduced as a theorem in geometry, turns out to have profound applications in the theory of quasiconformal mappings and in the non-convex calculus of variations. Therefore, there is a need for proving the theorem under still weaker assumptions. The right setting turns out to be the setting of Sobolev spaces $W^{1,p}_{\text{loc}}$, which we will introduce in Chapter 2. In Sobolev spaces we can define conformal mappings in an analogous way, that is, $f \in W^{1,p}(\Omega, \mathbb{R}^n), 1 \leq p \leq \infty$ where $\Omega \subset \mathbb{R}^n$ is conformal if

$$Df^T(x)Df(x) = |J(x,f)|^{\frac{2}{n}} \cdot \mathrm{I} \quad \text{for a.e. } x \in \Omega \text{ and}$$

$$J(x,f) \geq 0 \text{ a.e. in } \Omega \text{ or } J(x,f) \leq 0 \text{ a.e. in } \Omega.$$

The case of 1-quasiconformal mappings, that is, conformal homeomorphisms in the Sobolev space $W^{1,n}_{\text{loc}}$, was treated by Gehring [17] in 1962 and later by Reshetnyak [51] without the homeomorphism assumption. Both authors refer to difficult results in nonlinear PDEs, geometry, and the theory of quasiconformal mappings. An elementary, but rather involved proof of Reshetnyak's result was given by Bojarski and Iwaniec [4] in 1982, also see [27] and [29].

More recent developments on the Liouville Theorem arise from the work of Iwaniec [26], which proved that in all dimensions we can relax the assumption to $W^{1,n-\epsilon}$ for some $\epsilon > 0$. This gives rise to the following interesting question: what is the optimal Sobolev exponent for the Liouville theorem to hold? In Euclidean spaces of even dimensions $n \geq 3$, Iwaniec and Martin [28], proved the Liouville theorem for $W^{1,n/2}_{\mathrm{loc}}$. Meanwhile, they gave a counterexample showing that in all dimensions $n \geq 3$, conformal maps in the space $W^{1,p}_{\mathrm{loc}}$ for $p < n/2$ may not be Möbius. Hence they gave an answer to the above question in even dimensions and they conjectured that the same Sobolev exponent $n/2$ still holds in odd dimensions. This conjecture, which is known as the Iwaniec-Martin conjecture remains one of the most challenging open problems in this area until today. Besides theory of conformal mappings, plenty of deep results have also been developed for some wider classes, for example, quasiconformal and quasiregular mappings. In Chapter 3 we discuss some fundamental theorems in the areas of conformal mappings, quasiconformal mappings and quasiregular mappings. In particular, we present one short proof of the classical result of the Liouville Theorem for $W^{1,n}$ conformal mappings due to Gehring [17]. For details and further results we refer the reader to manuscripts and books by Bojarski and Iwaniec [3], Iwaniec and Martin [27] and Reshetnyak [52].

In Chapter 4 of this manuscript we prove the Liouville theorem for a conformal map $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, $n \geq 3$ under one additional assumption that the norm of the first order derivative $|Df|$ satisfies $|Df|^p \in W^{1,2}_{\mathrm{loc}}$ for $p \geq (n-2)/4$. This is optimal in the sense that if $|Df|^p \in W^{1,2}_{\mathrm{loc}}$ for $p < (n-2)/4$, the map f may not be a Möbius transform. Actually, this result shows the necessity of the Sobolev exponent in the Iwaniec-Martin conjecture. Meanwhile, we show that the Iwaniec-Martin conjecture can be reduced to a conjecture (conjecture 4.1.1) about a Caccioppoli type estimate. In particular, this Caccioppoli estimate suggests why the exponent $n/2$ is critical.

In the special case when $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ is an isometry, (i.e. $J(x, f) = 1$ a.e.), f is Lipschitz ($f \in W^{1,\infty}$). The Liouville Theorem for $W^{1,n}_{\text{loc}}$ conformal mappings automatically imply that f is Möbius. In fact, elementary computations show that f is just an affine transformation, i.e. mappings generated by translations and rotations only. Hence for any isometry from a domain of $\mathbb{R}^n$ to $\mathbb{R}^n$, everything is known even under the weakest regularity assumption. It is then natural to ask if the situation is as simple for isometrc immersions from a subset of $\mathbb{R}^n$ to $\mathbb{R}^N$ when $N \geq n + 1$. It turns out that the situation is extremely complicated even for the case $N = n + 1$.

Even in classical geometry, the rigidity results for isometric immersions depends heavily on the regularity of the given mapping. For example, Kuiper show that that are C^1 smooth isometric embeddings (i.e. immersions without self-intersection) of the unit sphere $\mathbb{S}^2$ into arbitrarily small balls in $\mathbb{R}^3$ [32], while Hilbert has already shown that C^2 isometric immersions from $\mathbb{S}^2$ into $\mathbb{R}^3$ is a rigid motion. Since C^1 isometric immersions from a 2-dimensional bounded region $\mathbb{R}^3$ are not rigid motion, of course the cases of all $W^{1,p}, 1 \leq p \leq \infty$ isometric immersions cannot be rigid motion, either. Therefore, a natural setting for studying the rigid properties of isometric immersions is the Sobolev space $W^{2,p}$, which has an intermediate regularity between the C^1 and C^2 class.

Regarding $W^{2,p}$ isometric immersions from a subset of $\mathbb{R}^2$ into $\mathbb{R}^3$, Pakzad [46] proved that if $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain, any isometric immersions $u \in W^{2,2}(\Omega, \mathbb{R}^3)$ is indeed of class C^1 and is developable, i.e., for any $x \in \Omega$, either u is affine in a neighborhood of x, or it is affine on a segment passing through x and joining the boundary of Ω at both ends. In particular, it is a rigid motion. The Sobolev exponent $p = 2$ is the borderline regularity for which the result holds. For example, the mapping $u(r\cos\theta, r\sin\theta) = r(1/2\cos 2\theta, 1/2\sin 2\theta, \sqrt{3}/2)$ is an isometric immersion of the unit disk $B^2(0, 1) \subset \mathbb{R}^2$ with a conic singularity at the origin and satisfies $u \in W^{2,p}(B^2(0, 1), \mathbb{R}^3)$ regularity for all $p < 2$, but it is not developable, nor can it be approximated by smooth isometries.

Based on the regularity-developability result, Pakzad also proved that smooth isometries are dense in the space of $W^{2,2}$ isometries if the domain is convex. We would like to point out here that density of good functions in a space is a valuable tool in the calculus of variations. For example, the density result can be used in proving regularity results for the critical points or in controlling the energy of the recovery sequences in the context of Γ-convergence such as when convergence is studied in the reduction from thin three-dimensional nonlinear elasticity to two-dimensional plate or shell theories. In several instances, this question is naturally connected to the topological and geometric rigidity properties of the smoother functions. A major indication of a positive answer to the density question is when the classical rigidity results are true for mappings of Sobolev type.

Now we come to the question whether we can extend Pakzad's result to higher dimensions and to what degree. A natural step is the extension to $W^{2,n}(\Omega, \mathbb{R}^{n+1})$ isometric immersions where $\Omega \subset \mathbb{R}^n$. In fact, from Morrey's inequality, which we shall see in Chapter 2, $W^{2,p}$ mappings are C^1 and has Hölder continuous derivatives if $p > n$. The case $p = n$ is at the borderline and we are still able to obtain a lot of topological properties as for the case $p > n$, if some additional geometric constraints are imposed on the mapping. For example, Reshetnyak [50] proved that $W^{1,n}$ homeomorphisms from a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ have the Lusin property, i.e., they map measure zero sets into measure zero sets. Another example is that $W^{1,n}$ mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$ with almost everywhere positive Jacobian are open (i.e., they map open sets to open sets) and continuous [58]. In fact, this argument was used to prove the C^1 regularity in Pakzad's result. Of course, we also have that the Liouville Theorem holds for $W^{1,n}$ conformal mappings from a subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. Actually, it was shown in [47] that $W^{2,n}$ isometric immersions are C^1 by applying an argument similar to the proof of weak monotonicity of deformations with positive Jacobian due to J. Manfredi [39]. Then similar developability and density results as to the $\mathbb{R}^2$ can be proved based on this C^1 regularity property. We refer the readers to Section 3.6 for a survey of main results in the area of isometric immersion of co-dimension one.

However, just like the Liouville Theorem case, $p = n$ may not be the optimal Sobolev exponent for the rigidity result to hold unless we can find a counter-example for the case $p < n$. On the other hand, is a well-established fact in differential geometry that higher dimensional manifolds are generally more rigid. Indeed, as was shown in [47], Remark 1, it is impossible to construct an isometric immersion in $W^{2,2}(B^3, \mathbb{R}^4)$ with a conic singularity at the origin as mentioned above because a $W^{2,2}$ regularity for an isometry of co-dimension 1 implies that all sectional curvatures of the image vanish as L^1 functions, and removes the possibility of conic singularities. These observations bring up the question if the regularity-developability and density results are true for $W^{2,2}(B^3, \mathbb{R}^4)$ isometric immersions. We not only give a positive answer to this question but also proves it holds in all dimensions: in Chapter 5 we prove that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, any isometric immersion $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ for $n \geq 3$ is of class C^1 and developable, i.e., for any $x \in \Omega$, either u is affine in a neighborhood of x, or there exists a unique $n-1$ dimensional hyperplane P of $\mathbb{R}^n$ passing thought x such that u is affine on the connected component of x in $P \cap \Omega$. We also prove the density result based on the developability result if the domain is C^1 and convex. We do not know if the density result is true for Lipschitz domains. The Sobolev exponent $p = 2$ is far away from the borderline case $p = n$ whenever $n \geq 3$, hence the argument to prove $W^{2,n}$ isometric immersions is C^1 does not apply in this situation. In Section 3.6 we will state the results that $W^{2,n}, n \geq 2$ isometric immersions is C^1 due to Pakzad [46] [47] to give the readers some sense that the major difficult of going from $W^{2,n}$ to $W^{2,2}$ is the lack of C^1 regularity on the first hand. To prove these results without the help of C^1 condition, we developed a new "slicing argument" which slices a n dimensional domain into $k = 2, ..., n$ dimensional slices and prove developability by induction on lower dimensional slices. Once we prove the developability result, we then prove the C^1 regularity- a reverse order of the argument for isometric immersions under $W^{2,n}$ assumption. In light of the counter example $u : B^n(0,1) \to \mathbb{R}^{n+1}$ defined by $u(r\cos\theta, r\sin\theta, x_3, ..., x_n) = (r/2\cos 2\theta, r/2\sin 2\theta, \sqrt{3}r/2, x_3, ..., x_n)$, our result for the Sobolev exponent $p = 2$ is is sharp.

Conformal mappings and isometric immersions in Sobolev space are just two special cases of the entire class of vector valued Sobolev functions with conformal constraints, but yet every results for these two classes are deep. Even for the class of conformal mappings, the Iwaniec and Martin conjecture still remains open, while for isometric immersions, the situation is not yet clear for immersions of higher co-dimension. Very little has been known for other class of vector valued Sobolev functions with conformal constraint and these areas remain wide open. Our research results, first, provide some developments in the areas of conformal mappings and isometric immersions under Sobolev setting; second, attempt to attract more attention to these larger unknown areas.

2.0 A BACKGROUND IN SOBOLEV SPACES

Sobolev spaces are a fundamental tool in the calculus of variations and partial differential equations. Actually, the scope of applications of Sobolev spaces is very wide and it goes far beyond the calculus of variations and partial differential equations. We present a brief review of some basic concepts in Sobolev spaces, in particular, those needed in later chapters. The major source of this brief survey is Evans and Gariepy [16], and Gilbarg and Trudinger [18]. All the proofs of the results in this section can be found in these two textbooks, or any other book on the calculus of variations and partial differential equations.

2.1 BASIC THEORY OF SOBOLEV SPACES

2.1.1 L^p spaces.

Let Ω be an open subset of $\mathbb{R}^n$. If there is a set $E \subset \Omega$ with its Lebesgue measure $|E| = 0$ such that a property $P(x)$ is satisfied for all $x \in \Omega \setminus E$, then we say that the property $P(x)$ is satisfied *almost everywhere* (a.e.).

For $1 \leq p < \infty$, Let $\tilde{L}^p(\Omega)$ denote the class of all measurable functions $f : \Omega \to \mathbb{R}$ such that,

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty$$

and we define $L^p(\Omega) = \tilde{L}^p(\Omega)/\sim$, where $f \sim g$ if $f = g$ a.e. in Ω. That is, we do not distinguish functions that are equal almost everywhere. For $p = \infty$ we define $\tilde{L}^\infty(\Omega)$ to be the class of all essentially bounded measurable functions, i.e., there is $M > 0$ with $f(x) \leq M$

for a.e. $x \in \Omega$. We denote the smallest value of such M by $\|f\|_{L^\infty(\Omega)}$. Finally we set $L^\infty(\Omega) = \tilde{L}^\infty(\Omega)/\sim$.

For $1 \le p \le \infty$, we define $L^p_{\text{loc}}(\Omega)$ as

$$L^p_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} : f \in L^p(\Omega') \text{ for each open subset } \Omega' \subset\subset \Omega\}.$$

Here $\Omega' \subset\subset \Omega$ means the closure $\overline{\Omega'} \subset \Omega$.

Two of the most important inequality about L^p spaces is the *Hölder inequality*:

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}, \quad 1 \le p, q \le \infty \text{ such that } \frac{1}{p} + \frac{1}{q} = 1$$

and the *Minkowski inequality*:

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}, \qquad 1 \le p \le \infty.$$

2.1.2 Definition of Sobolev spaces.

Denote $C^\infty(\Omega)$ the space of all smooth functions in Ω and $C^\infty_0(\Omega)$ the space of all smooth functions vanishing outside a compact subset of Ω.

Definition 2.1.1 *Assume $f \in L^1_{\text{loc}}(\Omega)$. We say $g_i \in L^1_{\text{loc}}(\Omega), 1 \le i \le n$ is the weak partial derivative of f with respect to x_i in Ω if*

$$\int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega g_i \phi\, dx$$

for all $\phi \in C^\infty_0(\Omega)$.

Weak partial derivative with respect to x_i, if it exists, is unique by the following result,

Lemma 2.1.1 *If $g \in L^1_{\text{loc}}(\Omega)$ and $\int_\Omega g\phi\, dx = 0$ for all $\phi \in C^\infty_0(\Omega)$, then $g = 0$ a.e.*

Therefore, we write,

$$\frac{\partial f}{\partial x_i} \equiv g_i, \quad i = (1, ..., n) \quad \text{and}$$

$$\nabla f \equiv \Big(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\Big).$$

Definition 2.1.2 *Let* $1 \leq p \leq \infty$.

1. *The function* f *belongs to the Sobolev space* $W^{1,p}(\Omega)$ *if* $f \in L^p(\Omega)$ *and all the weak partial derivative* $\partial f / \partial x_i, i = 1, ..., n$ *exist and belong to* $L^p(\Omega)$. *In particular, it satisfies the integration by part formula,*

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} \, dx = - \int_\Omega \frac{\partial f}{\partial x_i} \phi \, dx$$

for all $\phi \in C_0^\infty(\Omega)$.

2. *The function* f *belongs to* $W^{1,p}_{\text{loc}}(\Omega)$ *if* $f \in W^{1,p}(\Omega')$ *for each open set* $\Omega' \subset\subset \Omega$.
3. *We say* f *is a Sobolev function is* $f \in W^{1,p}(\Omega)$ *for some* $1 \leq p \leq \infty$.
4. *If* $f \in W^{1,p}(\Omega)$ *for some* $1 \leq p \leq \infty$, *we define the Sobolev norm*

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Remark 2.1.1 *It is also conventional to abbreviate the Sobolev norm as* $\|f\|_{1,p}$ *if there is no confusion about which domain the Sobolev space is on.*

Definition 2.1.3 *We say* $f_k \to f$ *in* $W^{1,p}(\Omega)$ *provided,*

$$\|f_k - f\|_{W^{1,p}(\Omega)} \to 0.$$

Similarly, $f_k \to f$ *in* $W^{1,p}_{\text{loc}}(\Omega)$ *provided*

$$\|f_k - f\|_{W^{1,p}(\Omega')} \to 0 \quad \textit{for each } \Omega' \subset\subset \Omega.$$

We can also extend the definitions to higher order Sobolev space,

Definition 2.1.4 *Assume* $f, g \in L^1_{\text{loc}}(\Omega)$. *Let* α *be a multi-index. We say* $D^\alpha f = g$ *in the weak sense if*

$$\int_\Omega f D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega g \phi \, dx$$

for all $\phi \in C_0^\infty(\Omega)$.

Definition 2.1.5 *The function f belongs to the Sobolev space $W^{m,p}(\Omega)$ for some positive integer m and $1 \leq p \leq \infty$ if $f \in L^p(\Omega)$ and all the weak partial derivative $D^\alpha f, |\alpha| \leq m$ exist and belong to $L^p(\Omega)$. The $W^{m,p}$ norm is defined by,*

$$\|f\|_{W^{m,p}(\Omega)} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}.$$

We then define $W^{m,p}_{\text{loc}}(\Omega)$, $W^{m,p}(\Omega, \mathbb{R}^N)$, $W^{m,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$ is an obvious analogous way.

From the definition of Sobolev spaces, we know it is a subspace of L^p. Moreover, we have,

Theorem 2.1.1 *$W^{m,p}(\Omega)$ is a Banach space, i.e., it is a complete space with respect to the $W^{m,p}$ norm.*

In particular, it is a closed subspace of $L^p(\Omega)$ and hence it inherits many important property of L^p. One important property is that $W^{m,p}$ space for $1 < p < \infty$ is reflexive. As a consequence,

Theorem 2.1.2 *Every bounded sequence in $W^{m,p}(\Omega), 1 < p < \infty$ has a weakly convergent subsequence.*

2.1.3 Basic properties of Sobolev spaces.

We define the C^∞ function $\varphi : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\varphi(x) =: \begin{cases} \exp\left(\frac{c}{|x|^2-1}\right) & \text{if} \quad |x| < 1 \\ 0 \quad \text{if} \quad |x| \geq 1 \end{cases}.$$

The constant c is chosen so that

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 1.$$

For $\epsilon > 0$, define $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(x/\epsilon)$. φ_ϵ is called the *standard mollifier* (with respect to parameter ϵ). Write $\Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$. If $f \in L^1_{\text{loc}}(\Omega)$, define the convolution of f with the standard mollifier as,

$$f^\epsilon(x) := \varphi_\epsilon * f(x) = \int_{B(x,\epsilon)} \varphi_\epsilon(x-y) f(y)\, dy, \quad x \in \Omega_\epsilon.$$

Theorem 2.1.3 *1. For each $\epsilon > 0$, $f^\epsilon \in C^\infty(\Omega_\epsilon)$.*

2. *If f is continuous, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.*
3. *If $f \in L^p_{\text{loc}}(\Omega)$ for some $1 \le p < \infty$, then $f^\epsilon \to f$ in $L^p_{\text{loc}}(\Omega)$.*
4. *$f^\epsilon \to f$ a.e. in Ω_ϵ.*
5. *If $f \in W^{m,p}_{\text{loc}}(\Omega)$ for some $1 \le p \le \infty$, then,*

$$D^\alpha f^\epsilon = \varphi_\epsilon * D^\alpha f, \quad |\alpha| \le m.$$

6. *In particular, if $f \in W^{m,p}_{\text{loc}}(\Omega)$ for some $1 \le p < \infty$, then $f^\epsilon \to f$ in $W^{m,p}_{\text{loc}}(\Omega)$.*

Actually, using partition of unity we can derive a global approximation result,

Theorem 2.1.4 *The subspace $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ for all $1 \le p < \infty$, i.e., for every functions $f \in W^{m,p}(\Omega)$, there is a sequence of smooth function $f^\epsilon \to f$ in $W^{m,p}(\Omega)$.*

In view of Theorem 2.1.3, we can approximation Sobolev functions by smooth function, and consequently we can verify that many of the usual calculus rules hold for weak derivatives.

Theorem 2.1.5 *1. If $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$, $1/p + 1/q = 1$, then $fg \in W^{1,1}(\Omega)$ and*

$$\nabla(fg) = (\nabla f)g + f(\nabla g) \quad a.e.$$

2. *If $f \in W^{1,p}(\Omega)$ and $F \in C^1(\mathbb{R})$, (extra assumption $F(0) = 0$ is required if Ω is unbounded), then $F(f) \in W^{1,p}(\Omega)$ and*

$$\nabla F(f) = F'(f)\nabla f \quad a.e.$$

3. *If $f \in W^{1,p}(\Omega)$, then $f^+ = \max\{f, 0\}, f^- = \max\{-f, 0\}, |f| \in W^{1,p}(\Omega)$ with*

$$\nabla f^+ = \begin{cases} \nabla f(x) & a.e. \text{ on } \{f > 0\} \\ 0 & a.e. \text{ on } \{f \le 0\} \end{cases}$$

$$\nabla f^- = \begin{cases} 0 & a.e. \text{ on } \{f \ge 0\} \\ -\nabla f & a.e. \text{ on } \{f < 0\} \end{cases}$$

$$\nabla|f| = \begin{cases} \nabla f & a.e. \ on \ \{f > 0\} \\ 0 & a.e. \ on \ \{f = 0\} \\ -\nabla f & a.e. \ on \ \{f < 0\} \end{cases} .$$

4. In particular, $\nabla f = 0$ a.e. on $\{f = 0\}$.

Corollary 2.1.1 *Suppose $f, g \in W^{m,p}(\Omega)$ agree on a set $E \subset \Omega$, then $D^\alpha f(x) = D^\alpha g(x)$ for a.e. $x \in E$ and all $|\alpha| \leq m$.*

2.2 THREE CHARACTERIZATIONS OF SOBOLEV SPACES

In this section we summarize three ways to characterize Sobolev spaces $W^{1,p}$.

2.2.1 Approximation.

In Theorem 2.1.3, we know that for every function $f \in W^{1,p}(\Omega)$, $1 \leq p < \infty$ there is a sequence of smooth functions $f^\epsilon \to f$ in $W^{1,p}(\Omega)$. In particular, this implies there is a sequence of C^1 Functions $f^\epsilon \to f$ in $W^{1,p}(\Omega)$. The reverse is also true,

Lemma 2.2.1 *$f \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$ if and only if there is a sequence of C^1 functions $f^\epsilon \to f$ in $W^{1,p}(\Omega)$.*

If $1 < p < \infty$, then by using Theorem 2.1.2, we have a slightly stronger result,

Lemma 2.2.2 *$f \in W^{1,p}(\Omega)$ for some $1 < p < \infty$ if and only if there is a sequence of C^1 functions $f^\epsilon \to f$ in $L^p(\Omega)$ and $\sup_\epsilon \|\nabla f^\epsilon\|_{L^p(\Omega)} < \infty$.*

One application of Lemma 2.2.1 is the Fubini Theorem for Sobolev functions,

Theorem 2.2.1 (Fubini for Sobolev Functions) *Let $f \in W^{1,p}((0,1)^n), 1 \leq p < \infty$ and a sequence of functions $f_k \to f$ in $W^{1,p}((0,1)^n)$. Denote points in the cube by $(t, x) \in (0,1) \times (0,1)^{n-1}$. Then for a.e. $t \in (0,1)$, $f(t, \cdot) \in W^{1,p}((0,1)^{n-1})$. Moreover, there is a subsequence, still denoted by f_k, such that $f_k(t, \cdot) \to f(t, \cdot)$ in $W^{1,p}((0,1)^{n-1})$.*

A version of this theorem is also true if $n-1$ dimensional slices are replaced by m dimensional slices for all $1 \leq m \leq n-1$.

2.2.2 Difference quotient.

For $f \in L^1(\Omega)$ and $0 < |h| < \mathrm{dist}(x, \partial\Omega)$ we define the difference quotient as,

$$\Delta^h f(x) := \frac{|f(x+h) - f(x)|}{|h|}.$$

Lemma 2.2.3 *If $f \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$, then $\Delta^h f \in L^p(\Omega')$ for any h and $\Omega' \subset\subset \Omega$ satisfying $|h| < \mathrm{dist}(\Omega', \partial\Omega)$, Moreover,*

$$\|\Delta^h f\|_{L^p(\Omega')} \leq \|\nabla f\|_{L^p(\Omega)}.$$

If $1 < p < \infty$, again using Theorem 2.1.2, we can prove that the converse is also true:

Lemma 2.2.4 *If $f \in L^p(\Omega), 1 < p < \infty$, and suppose there is a constant K such that $\Delta^h f \in L^p(\Omega')$ and $\|\Delta^h f\|_{L^p(\Omega')} \leq K$ for all h and $\Omega' \subset\subset \Omega$ satisfying $|h| < \mathrm{dist}(\Omega', \partial\Omega)$, then $f \in W^{1,p}(\Omega)$ and its weak derivative ∇f satisfies $\|\nabla f\|_{L^p(\Omega)} \leq K$.*

2.2.3 ACL characterization.

Definition 2.2.1 *We say that a continuous function f defined on an interval $[a, b]$ is absolutely continuous if for every $\epsilon > 0$, there is $\delta > 0$ such that if $I_1, ..., I_k$ are pairwise disjoint segments contained in $[a, b]$ with $\sum_{i=1}^k |I_i| < \delta$, then $\sum_{i=1}^k |f(I_i)| < \epsilon$.*

We denote the class of absolutely continuous functions on $[a, b]$ by $AC[a, b]$. It is easy to see that the function $f(x) = c + \int_a^x h(t)\, dt$, where $h \in L^1(a, b)$ and c is any constant, is absolutely continuous. The following lemma says these are the only absolutely continuous functions.

Lemma 2.2.5 *If $f \in AC[a, b]$, then f' exists a.e., $f' \in L^1(a, b)$ and $f(x) = f(a) + \int_a^x f'(t)\, dt$ for all $x \in [a, b]$.*

The above lemma implies integration by part formula holds for absolutely continuous functions.

Lemma 2.2.6 *If $f, g \in AC[a, b]$, then the formula for integration by parts holds*

$$\int_a^b f(x)g'(x)\, dx = fg\big|_a^b - \int_a^b f'(x)g(x)\, dx.$$

Definition 2.2.2 *Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a measurable function f is absolutely continuous on lines, denoted by $f \in ACL(\Omega)$ if f is absolutely continuous on almost every line parallel to coordinate axes.*

Since from Lemma 2.2.5 absolutely continuous are differentiable a.e., $f \in ACL(\Omega)$ has partial derivatives a.e. and hence the classical gradient ∇f is defined a.e.

Definition 2.2.3 *We say $f \in ACL^p(\Omega)$ if $f \in L^p(\Omega) \cap ACL(\Omega)$ and $\nabla f \in L^p(\Omega)$.*

It is easy to see $ACL^p(\Omega) \subset W^{1,p}(\Omega)$ from the fact that integration by parts holds for absolutely continuous functions, the Fubini Theorem (Theorem 2.2.1) and the definition of weak derivatives. It turns out that every $f \in W^{1,p}$ can be alternated on a set of measure zero in a way that the resulting function belongs to $ACL^p(\Omega)$. This characterization of Sobolev spaces goes back to Nikodym.

Theorem 2.2.2 $W^{1,p}(\Omega) = ACL^P(\Omega)$ *for all* $1 \leq p \leq \infty$.

The following results are direct consequences on ACL characterization.

Corollary 2.2.1 *Functions in the space $W^{1,\infty}$ are locally Lipschitz continuous. If in addition Ω is a bounded Lipschitz domain, then $W^{1,\infty}(\Omega) = \mathrm{Lip}(\Omega)$.*

Corollary 2.2.2 *If $f \in W^{1,p}(\Omega)$ where Ω is connected and $\nabla f = 0$ a.e. on Ω, then f is constant.*

Furthermore, Theorem 2.1.5 and Corollary 2.1.1 can also be obtained as a consequence of the ACL characterization.

2.3 EMBEDDING THEOREMS

The Sobolev embedding theorem is a fundamental tool in analysis of Sobolev spaces. For $1 \leq p < n$, define

$$p^* = \frac{np}{n-p}.$$

p^* is called the *Sobolev conjugate* of p.

Theorem 2.3.1 (Gagliardo-Nirenberg-Sobolev) *Assume $1 < p < n$, then $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$. Moreover, there exists a constant $C_1 = C_1(n,p)$ such that for all $f \in W^{1,p}(\mathbb{R}^n)$,*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_1 \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

The Sobolev-embedding theorem also holds for bounded Lipschitz domains,

Theorem 2.3.2 *If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $1 \leq p < n$, then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Moreover, there exists $C_2 = C_2(\Omega, p)$ such that for all $f \in W^{1,p}(\Omega)$,*

$$\|f\|_{L^{p^*}(\Omega)} \leq C_2(\Big(\|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}\Big).$$

One of the most result in the theory of Sobolev spaces is the following theorem,

Theorem 2.3.3 (Rellich-Kondrachov) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ is compact if $q < p^*$ and $1 < p < n$, or $q < \infty$ and $n \leq p < \infty$.*

As an application of the Sobolev embedding theorem and the Rellich Kondrachov Theorem we have,

Theorem 2.3.4 (Sobolev-Poincaré inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p < n$, then there exists $C_3 = C_3(\Omega, p)$ such that for all $f \in W^{1,p}(\Omega)$,*

$$\|f - f_\Omega\|_{L^{p^*}(\mathbb{R}^n)} \leq C_3 \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

where $f_\Omega = \fint_\Omega f(x)\, dx$.

We also have the following version of Poincaré inequality on balls,

Theorem 2.3.5 (Poincaré inequality) *Assume $1 < p < n$, then there exists a constant $C_4 = C_4(n,p)$ such that for all $f \in W^{1,p}(B(x,r))$,*

$$\left(\fint_{B(x,r)} \left| f(y) - \fint_{B(x,r)} f(z)\, dz \right|^{p^*} dy\right)^{\frac{1}{p^*}} \leq C_4 r \left(\fint_{B(x,r)} |\nabla f(y)|^p dy\right)^{\frac{1}{p}}.$$

For the case $n \geq p$ and the domain Ω is bounded Lipschitz, if $f \in W^{1,p}(\Omega)$, then $f \in W^{1,q}$ for all $q < n$. Therefore, the Sobolev embedding theorem asserts that f is integrable with any exponent less than ∞. However, we also have a much better result due to Morrey, which says $W^{1,p}$ functions for $p > n$ are Hölder continuous.

Definition 2.3.1 *A function $f : \Omega \to \mathbb{R}^n$ is Hölder continuous with exponent α (i.e. $f \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$) for $0 < \alpha < 1$ provided,*

$$\sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

Theorem 2.3.6 (Morrey's inequality) *If $f \in W^{1,p}(B), n < p < \infty$, then $f \in C^{0,1-n/p}(B)$. Moreover, there exists a constant $C_5 = C_5(n, p)$ such that*

$$|f(x) - f(y)| \leq C_5|x - y|^{1-n/p}\|\nabla f\|_{L^p(B)}.$$

Corollary 2.3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $n < p < \infty$, then $W^{1,p}(\Omega) \subset C^{0,1-n/p}(\Omega)$.*

Corollary 2.3.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p < \infty$, if $f \in W^{m,p}(\Omega)$ for all $m = 1, 2, ...,$ then $f \in C^\infty(\Omega)$.*

Lastly we mention the integrability of $W^{1,n}$ space due to Trudinger,

Theorem 2.3.7 (Trudinger) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, then there exists $C_6 = C_6(\Omega, n, p)$ and $C_7 = C_7(\Omega, n, p)$ such that*

$$⨍_\Omega \left(\frac{f - f_\Omega}{C_6\|\nabla f\|_{L^n(\Omega)}} \right) \leq C_7.$$

3.0 ON SOME VECTOR VALUED SOBOLEV FUNCTIONS WITH CONSTRAINT

Similar to Definition 2.1.2 and 2.1.3 for scalar valued Sobolev function, we can also defined the Sobolev space for vectored valued function.

If $f = (f^1, ..., f^N)$ is a vector valued function with $f(x) \in \mathbb{R}^N$ for a.e. $x \in \Omega$, we say $f \in W^{m,p}(\Omega, \mathbb{R}^N)$ (or $f \in W^{m,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$) where m is any positive integer, if each component $f^j \in W^{m,p}(\Omega)$ (or $f^j \in W^{m,p}_{\text{loc}}(\Omega)$). If $f \in W^{m,p}(\Omega, \mathbb{R}^N)$, the Sobolev norm is defined as $\sum_j \|f^j\|_{W^{m,p}(\Omega)}$. Similarly, we say $f_k \to f$ in $W^{m,p}(\Omega, \mathbb{R}^N)$ (or $W^{m,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$ if each component $f^j_k \to f^j$ in $W^{m,p}(\Omega)$ (or $W^{m,p}_{\text{loc}}(\Omega)$).

It is obvious from the definition that all the theory mentioned so far for scalar valued Sobolev functions apply to each component of vector valued Sobolev functions, so there is indeed no difference in basic Sobolev space theory. However, in areas such as calculus of variations, there are fundamental different results for scalar and vector valued Sobolev functions. For instance, one commonly used trick in obtaining higher regularity is to slice a scalar functions into different level sets, while we cannot apply the same trick to vector valued functions. On the other hand, a lot of interesting results arise from the interpolations between components of vector valued functions, for example, Alhfors' deformation theorem. In this chapter we will discuss some special cases of vector valued Sobolev functions. We present some results that are closely related to our problems and survey on other well-known results. The main reference of the survey is Bojarski and Iwaniec [3], Iwaniec and Martin [27] and Reshetnyak [52].

3.1 AHLFORS'S DEFORMATION THEOREM

For a mapping $f \in C^1(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, Ahlfors [1], introduced a linear *Cauchy-Riemann operator*

$$Sf = \frac{1}{2}(Df + D^T f) - \left(\frac{1}{n}\mathrm{div} f\right) \cdot \mathrm{I}.$$

The mapping f is called a *trivial deformation* if $Sf = 0$. Ahlfors proved that a trivial deformation is a polynomial of degree 2. The Ahlfors operator is the first degree (linear) approximation of the nonlinear system of equations near the identity map for conformal mappings. We present here this elegant Theorem. Note that the original version requires the mapping to be C^1, but if we approximate distributions by Schwatz function these two versions are equivalent.

Theorem 3.1.1 *If X is a distributional vector field in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ which satisfies*

$$DX + DX^T = \frac{2}{n}\mathrm{div} X \cdot \mathrm{I}, \tag{3.1.1}$$

then X is a polynomial of degree 2 and is of the form

$$X(x) = a + Bx + 2(c \cdot x)x - |x|^2 c$$

where $a, c \in \mathbb{R}^n$ and $B = [b_{ij}] : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping satisfying $b_{ij} = -b_{ji}$ for $i \neq j$ and $b_{ii} = b_{jj}$ for all i, j.

Proof. In order to prove that X is a polynomial of degree 2 it suffices to show that all distributional partial derivatives of order 3 are equal zero.

Let $X = (X_1, \ldots, X_n)$, $X_{i,j} = \frac{\partial}{\partial x_j} X_i$, $X_{i,jk} = \frac{\partial}{\partial x_k} X_{i,j}$ in the distributional sense, and so on. From (3.1.1) one immediately gets that $X_{i,j} = -X_{j,i}$ for $i \neq j$, and $X_{i,i} = X_{j,j}$ for all i, j.

Since $n \geq 3$ we take i, j, k distinct and then,

$$X_{i,jk} = X_{i,kj} = -X_{k,ij} = -X_{k,ji} = X_{j,ki} = X_{j,ik} = -X_{i,jk}$$

Hence $X_{i,jk} = 0$ for i, j, k distinct.

We will show that $X_{i,jk\ell} = 0$ for all i, j, k, ℓ. If we have at least 3 distinct indices among $\{i, j, k, \ell\}$, we can always permute them to have the first three indices distinct and $X_{i,jk\ell} = 0$ is obvious. If there are only two distinct indices, say, $\{i, j, k, \ell\} = \{i, j\}, i \neq j$ then we have two cases $X_{i,ijj}$ and $X_{i,jjj}$ (plus permutation of indices). We have

$$X_{i,ijj} = X_{i,jij} = -X_{j,iij} = -X_{j,jii}. \tag{3.1.2}$$

Since $n \geq 3$, there is k different from i, j and hence

$$X_{i,ijj} = -X_{j,jii} = -X_{k,kii} = X_{i,ikk} = X_{j,jkk} = -X_{k,kjj} = -X_{i,ijj} = 0,$$

where we repeatedly use (3.1.2). In the case $X_{i,jjj}$, we again find k different from i, j

$$X_{i,jjj} = -X_{j,ijj} = -X_{j,jij} = -X_{k,kij} = -X_{k,ijk} = 0.$$

The last case is when all indices are equal, but in that case

$$X_{i,iii} = X_{j,jii} = 0$$

by the case proved above.

Thus X is a polynomial of degree 2 and hence

$$X_i = a_i + \sum_j b_{ij} x_j + \sum_{j,k} c_{ijk} x_j x_k.$$

We may assume $c_{ijk} = c_{ikj}$. Thus

$$X_{i,j} = b_{ij} + 2 \sum_k c_{ijk} x_k, \quad X_{i,jk} = c_{ijk}.$$

Since $X_{i,j} = -X_{j,i}$ for $i \neq j$ and $X_{i,i} = X_{j,j}$ for all i, j, $b_{ij} = -b_{ji}$ for $i \neq j$ and $b_{ii} = b_{jj}$ for all i, j. If i, j, k are distinct, then $c_{ijk} = X_{i,jk} = 0$, so

$$\begin{aligned} X_i &= a_i + \sum_j b_{ij} x_j + \sum_k c_{iik} x_i x_k + \sum_{k \neq i} c_{iki} x_k x_i + \sum_{k \neq i} c_{ikk} x_k^2 \\ &= a_i + \sum_j b_{ij} x_j + 2 \sum_k c_{iik} x_i x_k - c_{iii} x_i^2 + \sum_{k \neq i} c_{ikk} x_k^2. \end{aligned}$$

Since $X_{i,i} = X_{j,j}$ for all i, j,

$$c_{iik} = c_{jjk} := c_k \quad \text{for all } i, j, k$$

and since $X_{i,k} = -X_{k,i}$ for $i \neq k$,

$$c_{ikk} = -c_{kik} = -c_{kki} = -c_i \quad \text{for } i \neq k.$$

Thus

$$\begin{aligned} X_i &= a_i + \sum_j b_{ij} x_j + 2\left(\sum_k c_k x_k\right) x_i - c_i \sum_k x_k^2 \\ &= a_i + \sum_j b_{ij} x_j + 2(c \cdot x) x_i - |x|^2 c_i. \end{aligned}$$

The proof is complete. □

3.2 p-HARMONIC MAPPINGS

For a vector valued mappings $f \in W^{1,p}(\Omega, \mathbb{R}^N)$ where $\Omega \subset \mathbb{R}^n$, define the weak differential $Df : \Omega \to \mathbb{R}^{N \times n}$ as

$$Df(x) = \Big(\frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_n}\Big).$$

We say that $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$ is a weak solution to the p-harmonic system, i.e. f is p-harmonic, if

$$\int_\Omega |Df|^{p-2} \langle Df, D\varphi \rangle = 0 \text{ for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^N) \tag{3.2.1}$$

where $\langle \cdot \rangle$ denotes the pointwise Hilbert Smith matrix product,

$$\langle M, N \rangle = \text{trace}(M^T N).$$

We abbreviate (3.2.1) as,

$$\text{div}(|Df|^{p-2} Df) = 0. \tag{3.2.2}$$

Here $|Df| = (\sum_i |\partial f / \partial x_i|^2)^{1/2}$ is the Hilbert-Smith norm for matrices.

Equation (3.2.2) is the Euler-Lagrange equation of the p-Dirichlet integral

$$I(u) = \frac{1}{p} \int_\Omega |Du|^p.$$

There has been plenty of deep results concerning the regularity of p-harmonic mappings. For the case $N = 1$, Ural'tseva [56] proved that for $p > 2$, p-harmonic functions have Hölder continuous derivatives (i.e., $C^{1,\alpha}$ functions for some $\alpha \in (0,1)$). This result was later extended to cover the case $1 < p < 2$ by Lewis [33] and DiBenedetto [12]. For the case $N \geq 1$, $p > 2$, Uhlenbeck [55] proved the $C^{1,\alpha}_{\text{loc}}$ regularity for p-harmonic mappings, and Dibenedetto and Friedman [14] generalized Uhlenbeck's result to all $1 < p < \infty$. As a summary of all above results, we have,

Theorem 3.2.1 *If $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ is p-harmonic, then $f \in C^{1,\alpha}_{\text{loc}}(\Omega, \mathbb{R}^N)$ for some $\alpha \in (0,1)$.*

Note that in general, p-harmonic mappings do not have any better regularity than $C^{1,\alpha}_{\text{loc}}$, hence the above regularity result is optimal.

If we assume $p \geq 2$, then $W^{1,p}$ p-harmonic mappings actually enjoy some second order differentiability regularity gain. This result was proved by Bojarski and Iwaniec [4] following the Nireberg method of difference quotients,

Theorem 3.2.2 *If $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$ is p-harmonic for $p \geq 2$, then*

$$|Df|^{\frac{p-2}{2}} Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{N \times n}).$$

This result plays a fundamental role in the development of conformal mappings. For its importance and elegance we present the proof here.

Proof. Let $F(x) = |Du(x)|^{(p-2)/2} Du(x)$. Clearly $F \in L^2_{\text{loc}}(\Omega, \mathbb{R}^{m \times n})$. According to a difference quotient characterization of $W^{1,2}_{\text{loc}}$ it suffices to prove that for any $\varphi \in C^\infty_0(\Omega)$

$$\left(\int_\Omega \varphi^2(x) \left| F(x+h) - F(x) \right|^2 \, dx \right)^{1/2} \leq C|h| \quad \text{for small } h \in \mathbb{R}^n.$$

Let $G(x) = |Du(x)|^{p-2}Du(x)$. Taking

$$\psi(x) = \varphi^2(x)\,(u(x+h) - u(x))$$

as a test function we have

$$\int_\Omega \langle G(x+h) - G(x), D\psi(x)\rangle \, dx = 0$$

and hence

$$\begin{aligned}&\int_\Omega \varphi^2(x)\,\langle G(x+h) - G(x), Du(x+h) - Du(x)\rangle \, dx \\ &\quad = \; -2\int_\Omega \varphi(x)(u(x+h) - u(x))\,\langle G(x+h) - G(x), D\varphi(x)\rangle \, dx.\end{aligned}$$

The elementary inequalities for vectors $\xi, \zeta \in \mathbb{R}^k$ (valid for $p \geq 2$)

$$\left\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\right\rangle \geq C_1(p)\left||\xi|^{(p-2)/2}\xi - |\zeta|^{(p-2)/2}\zeta\right|^2,$$

$$\left||\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right| \leq C_2(p)\,(|\xi|^p + |\zeta|^p)^{(p-2)/(2p)}\left||\xi|^{(p-2)/2}\xi - |\zeta|^{(p-2)/2}\zeta\right|$$

applied to matrices regarded as vectors give

$$\begin{aligned}&\int_\Omega \varphi^2(x)|F(x+h) - F(x)|^2\,dx \\ &\leq \; C\int_\Omega |\varphi(x)||u(x+h) - u(x)|\,|D\varphi(x)| \times \\ &\qquad (|Du(x+h)|^p + |Du(x)|^p)^{(p-2)/(2p)}\,|F(x+h) - F(x)|\,dx. \\ &\leq \; C\left(\int_\Omega |\varphi(x)|^2|F(x+h) - F(x)|^2\,dx\right)^{1/2} \times \\ &\quad \left(\int_\Omega |u(x+h) - u(x)|^2|D\varphi(x)|^2\,(|Du(x+h)|^p + |Du(x)|^p)^{(p-2)/p}\,dx\right)^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_\Omega \varphi^2(x)|F(x+h) - F(x)|^2\,dx \\ &\leq \; C\int_\Omega |u(x+h) - u(x)|^2|D\varphi(x)|^2\,(|Du(x+h)|^p + |Du(x)|^p)^{(p-2)/p}\,dx \\ &\leq \; \left(\int_\Omega |u(x+h) - u(x)|^p|D\varphi(x)|^p\,dx\right)^{2/p} \times \\ &\quad \left(\int_{\operatorname{supp}\varphi} |Du(x+h)|^p + |Du(x)|^p\,dx\right)^{(p-2)/p}\end{aligned}$$

and it suffices to observe that the first integral on the right hand side is bounded by $C|h|^2$, while the second integral is bounded by a constant independent of (small) h. $\square$

Corollary 3.2.1 *If $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^m)$ is p-harmonic, $p \geq 2$, then for any $p/2 \leq s \leq p$*

$$|Du|^{s-1}Du \in W^{1,p/s}_{\text{loc}}.$$

Proof. For $s = p/2$ this is the previous result, so we can assume that $p/2 < s \leq p$. The matrix function

$$\Phi_\alpha(A) = |A|^\alpha A, \quad \alpha > 0$$

is of class C^1 and

$$|Du|^{s-1}Du = \Phi_{\frac{2s-p}{p}}\left(|Du|^{(p-2)/2}Du\right).$$

Since $|Du|^{(p-2)/2}Du \in W^{1,2}_{\text{loc}}$, the result follows from the chain rule. $\square$

Finally, it is worthwhile to remark here that to define p-harmonic mappings in the distribution sense as in (3.2.1), we only need the mapping to be of class $W^{1,p-1}_{\text{loc}}$. However, very little has been known about p-harmonic mappings in this class.

3.3 ADJOINT DIFFERENTIAL

The determinant $\det : \mathbb{R}^{n\times n}$ is a polynomial of degree n with its variables being entries of a matrix $A = [a_{ij}]$. Its gradient,

$$A_{ij} = \frac{\partial \det A}{\partial a_{ij}}$$

are in fact co-factors of A,

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the submatrix of A obtained by deleting the ith row and jth column. Denote $A^\sharp = [A_{ij}]$ and A^T as the transpose of A, from the familiar Crammer's rule we have the following relation,

$$A^T A^\sharp = A^\sharp A^T = \det A \cdot \mathrm{I}.$$

For the same weak differential defined in Section 3.2, define the Jacobian as,

$$J(x, f) = \det Df(x).$$

Now the adjoint differential of f is the matrix valued function defined by $D^\sharp f(x) = Df(x)^\sharp$. It is a polynomial of degree $n-1$ with respect to the entries $\partial f^i / \partial x_j$. Therefore,

$$D^\sharp : W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n) \to L^{p/(n-1)}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n}).$$

For a vector valued function $f^\epsilon \in C^\infty(\Omega, \mathbb{R}^n)$, $D^\sharp f^\epsilon$ is also smooth, hence we can differentiate to obtain for every $x \in \Omega$,

$$\text{div} D^\sharp f^\epsilon(x) := \sum_{i=1}^{n} \frac{\partial \left[\left(D^\sharp f^\epsilon(x) \right)_i \right]}{\partial x_i} = 0$$

where $\left(D^\sharp f^\epsilon(x)\right)_i$ denote the ith column of $D^\sharp f^\epsilon(x)$ due to all the mixed partials cancel each other. Let $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$, integration by part gives,

$$0 = \int_\Omega \langle \text{div} D^\sharp f^\epsilon, \phi \rangle = -\int_\Omega \langle D^\sharp f^\epsilon, D\phi \rangle.$$

where in the left hand side, $\langle \cdot \rangle$ denotes the pointwise vector product, while in the right hand side $\langle \cdot \rangle$ denotes the pointwise Hilbert Smith matrix product as defined in Section 3.2. We then approximate $f \in W^{1,p}(\Omega, \mathbb{R}^n), p \geq n-1$ by a sequence of smooth function and pass to the limit to obtain,

Lemma 3.3.1 *For $p \geq n-1$, the adjoint differential operator $D^\sharp : W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n) \to L^{p/(n-1)}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is divergence free, i.e.,*

$$\int_\Omega \langle D^\sharp f, D\phi \rangle = 0.$$

Remark 3.3.1 *We abbreviate the divergence free condition in Lemma 3.3.1 as*

$$\text{div} D^\sharp f = 0.$$

From the relation

$$D^T f(x) D^\sharp f(x) = D^\sharp f(x) D^T f(x) = J(x, f) \cdot \text{I}$$

we also obtain the following identity for the Jacobian which plays an important role in the analytic degree theory of Sobolev mappings,

Lemma 3.3.2 *If $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ vanishes on $\partial\Omega$, then*

$$\int_\Omega J(x, f)\, dx = 0.$$

3.4 QUASICONFORMAL, QUASIREGULAR AND WEAKLY-QUASIREGULAR MAPPINGS

We first consider a wider class of conformal mappings-namely, quasiconformal mappings, quasiregular mappings and weakly quasiregular mappings. These mappings also play a crucial role in the geometric function theory.

We start the introduction of quasiconformal mappings through its geometric definition. A homeomorphism $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is *-quasiconformal* if there is a constant $H \geq 1$ such that,

$$H(x, f) = \limsup_{r \to 0} \frac{\sup_{|y-x|=r} |f(y) - f(x)|}{\inf_{|y-x|=r} |f(y) - f(x)|} \leq H \quad \text{for every } x \in \Omega.$$

If in addition for a constant K, we have $H(x, f) \leq K$ a.e., we say that f is K-*quasiconformal.*

The geometric meaning of quasiconformal mappings is that there exists a constant H, $1 \leq H < \infty$ such that an infinitesimally small sphere is transformed by the mapping into either a point or an infinitesimally small ellipsoid for which the ratio of the largest semiaxis to the smallest does not exceed the constant H. The above definition is quit geometrical, however, using tools in Sobolev spaces we can show a lot of analytical properties of quasiconformal mappings. A first deep result is,

Theorem 3.4.1 *If $f : \Omega \to \mathbb{R}^n$ is K-quasiconformal, then*

1. *f is differentiable a.e.,*
2. *$f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$, and*
3. $\max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi|$ *for a.e. $x \in \Omega$.*

The second fundamental result, shows that we can find an equivalent analytical definition for quasiconformal mappings,

Theorem 3.4.2 *Let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism. Then the following conditions are equivalent,*

1. *f is quasiconformal,*
2. *$f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, its Jacobian does not change sign, i.e., $J(x, f) \leq 0$ a.e. or $J(x, f) \geq 0$ a.e. and there is a constant $K \geq 1$ such that,*

   $$|Df(x)|^n \leq K|J(x, f)| \text{ for a.e. } x \in \Omega$$

   where $|Df(x)|$ is the operator norm of the matrix $Df(x)$.

We refer the readers to [52], Chapter 8 for the proofs of the above theorems.

The first equivalent statement is known as the geometric definition, while the second statement is known as the analytical definition of quasiconformal mappings. The geometric definition is useful in proving geometrical properties of quasiconformal mapping. For example, from the geometric definition, it is easy to see that the inverse of a quasiconformal mapping is again quasiconformal, as a consequence, the Jacobian is nonzero almost everywhere. This cannot be easily seen from the analytical definition. The analytical definition, on the other hand, is extremely useful in proving analytical and topological properties such as regularity and analytical degree theory for Sobolev mappings.

Quasiregular mappings are defined as mappings satisfying the analytical definition stated in Theorem 3.4.2, but without the homeomorphism assumption. In particular, we call this mapping K-*quasiregular*. In this definition, the Sobolev space $W^{1,n}$ with the exponent $p = n$ is the natural assumption since it allows the Jacobian to be integrable. Regarding our definition of conformal mappings, it is easy to see that $W^{1,n}_{\mathrm{loc}}$ conformal mappings is 1-quasiregular. Conversely Reshetnyak's result on the Liouville theorem proves 1-quasiregular mappings are indeed conformal mappings.

The class of quasiregular mappings, although wider than the class of quasiconformal mappings, still enjoys the same properties. In fact, quasiregular mappings are indeed local homeomorphisms outside a closed branch set of measure zero [3]. In the sequel we briefly state some of the important properties of quasiregular mappings.

Theorem 3.4.3 *Quasiregular mappings are in the space* $W^{1,p}_{\text{loc}}$ *for some* $p > n$.

As a consequence, we obtain that,

Corollary 3.4.1 *Quasiregular mappings are differentiable almost everywhere.*

Corollary 3.4.2 *Quasiregular mappings have the Lusin property, i.e., they map measure zero sets to measure zero sets. In particular, change of variable formula applies for quasiregular mappings.*

Theorem 3.4.4 *Nonconstant Quasiregular mappings are open and discrete.*

The above theorem is crucial in the degree theory of quasiregular mappings. However, it is beyond the scope of this manuscript and we refer the readers to [3] for detailed discussions.

A $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n), p < n, \Omega \subset \mathbb{R}^n$ mapping is called *weakly-K-quasiregular* if $J(x, f) \leq 0$ a.e. or $J(x, f) \geq 0$ a.e. and there is a constant $K \geq 1$ such that,

$$|Df(x)|^n \leq K|J(x, f)| \text{ for a.e. } x \in \Omega.$$

It was an remarkable finding of Iwaniec [26] that weakly quasiconformal mappings have self-improving integrability, i.e.,

Theorem 3.4.5 *There exists an* $\epsilon > 0$ *depending on* n *such that every weakly quasiregular mappings* $f \in W^{1,n-\epsilon}_{\text{loc}}(\Omega, \mathbb{R}^n)$ *is quasiregular.*

This theorem, combined with Theorem 3.4.3 implies that every weakly quasiregular mappings $f \in W^{1,n-\epsilon}_{\text{loc}}$ is in the space $W^{1,p}_{\text{loc}}$ for some $p > n$.

Moreover, Iwanice and Martin [28] proved a sharp result for weakly-1-quasiregular mappings in even dimensions. We will state this result in the next section in the context of conformal mappings.

3.5 THE LIOUVILLE THEOREM FOR CONFORMAL MAPPINGS

Let Ω be a domain in $\mathbb{R}^n$. A function $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is *conformal* or a *weak solution* to the *Cauchy-Riemann System* if,

$$\big(Df(x)\big)^T Df(x) = |J(x,f)|^{2/n} \cdot \mathrm{I} \quad \text{and,} \tag{3.5.1}$$

$$J(x,f) \geq 0 \quad \text{a.e.} \quad \text{or} \quad J(x,f) \leq 0 \quad \text{a.e.,} \tag{3.5.2}$$

where Df is the weak differential of f, i.e., the matrix of weak partial derivatives of f, and $J(x,f) = \det Df(x)$. We call f with $J(x,f) \geq 0$ a.e. *sense preserving* and $J(x,f) \leq 0$ a.e. *sense reversing*. Note that in dimensions $n \geq 3$, this is an over-determined system. That is why the situation is more rigid than in the plane.

A natural setting for the study of Sobolev conformal mappings is the Sobolev space $W^{1,n}$ since in this space, the Jacobian is integrable. For conformal mappings in this space, A first observation is that $W^{1,n}$ conformal mappings are n-harmonic. Indeed, (3.5.1), (3.5.2) and our discussion about the adjoint differential easily give,

$$Df^\sharp = n^{\frac{2-n}{2}} |Df|^{n-2} Df \quad \text{a.e., or } Df^\sharp = -n^{\frac{2-n}{2}} |Df|^{n-2} Df \quad \text{a.e.,}$$

and we know from Lemma 3.3.1 that $\mathrm{div} D^\sharp f = 0$. As a corollary to Theorem 3.2.2 and Corollary 3.2.1 we have,

Lemma 3.5.1 *$W^{1,n}_{loc}$ conformal map enjoys the following partial second order differentiability,*

$$|Df|^{\frac{n-2}{2}} Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n}) \quad \textit{and}$$

$$|Df|^{s-1} Df \in W^{1,p/s}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n}) \quad \textit{for any} \quad p/2 \leq s \leq p.$$

This result plays a fundamental role in the advance of the Liouville Theorem.

A *Möbius transform* is a composition of translations, dilations, rotations, reflections, and inversions with respect to spheres. More precisely, it has the following form,

$$f(x) = b + \frac{r^2 A(x-a)}{|x-a|^\alpha} \tag{3.5.3}$$

where $b \in \mathbb{R}^n$, $r \in \mathbb{R} \setminus \{0\}$, $a \in \mathbb{R}^n \setminus \Omega$, A an orthogonal matrix, and α is either 0 or 2.

For $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$, Reshetnyak [51] (See also [4], [27], and [29]) proved the following result,

Theorem 3.5.1 *Every function $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 3$ satisfying (4.1.1) and (4.1.2) is either constant or a Möbius transform restricted to Ω.*

Moreover, Iwaniec [26] proved the Liouville theorem for functions of class $f \in W^{1,n-\epsilon}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some small ϵ depending on n. It is natural to inquire what the optimal Sobolev exponent is for the Liouville theorem to hold. Regarding this question, Iwaniec and Martin [28] gave an answer in even dimensions, that is,

Theorem 3.5.2 *Every function $f \in W^{1,n/2}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for even dimensions $n \geq 4$ satisfying (4.1.1) and (4.1.2) is either constant or a Möbius transform restricted to Ω.*

Meanwhile, they gave a counter-example (Example 4.5.2) showing that in all dimensions $n \geq 3$, conformal maps in the space $W^{1,2p}$ for $p < n/4$ are not necessarily Möbius. Therefore, Iwaniec and Martin made the following conjecture:,

Conjecture 3.5.1 *[Iwaniec-Martin] Every function $f \in W^{1,n/2}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 3$ satisfying (4.1.1) and (4.1.2) is either constant or a Möbius transformation restricted to Ω.*

In addition, Malý [37] constructed another example of a conformal map in $W^{1,2p}$ for $p < n/4$ that is not a Möbius transform. The remarkable fact is that this map is Hölder continuous. This result shows that continuity of the mapping is not enough to replace the crucial condition $p \geq n/2$.

3.5.1 The Liouville Theorem for 1-quasiconformal mappings.

In this section we present a short proof of Gerhing's result [17]. This proof is due to Liu [35].

Theorem 3.5.3 *Let $f : \Omega \to \mathbb{R}^n$ be a 1-quasiconformal mapping in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Then f is a Möbius transformation in $\mathbb{R}^n$ restricted to Ω.*

Proof. Let $V = f(\Omega)$ and let

$$g := f^{-1} : V \to \Omega,$$

be the inverse mapping. From the properties of quasiconformal mappings in Section 3.4, g is 1-quasiconformal and differentiable a.e. Hence $\mathrm{I} = D(g(f(x))) = (Dg)(f(x))Df(x)$ a.e. Thus

$$(Dg)(f(x)) = [Df(x)]^{-1} \quad \text{a.e.}$$

Note that here we use the fact that both f and g have the Lusin property and $Jf \neq 0$ a.e.

Fix $e_i = (0, \ldots, 1, \ldots, 0)$ and for a compactly contained domain $A \Subset \Omega$ define

$$f_t(x) := g(f(x) + te_i),$$

for $x \in A$ and $|t| < \mathrm{dist}(f(A), \partial V)$. It is again a well defined 1-quasiconformal mapping.

Note that for a.e. $x \in \Omega$ we have

$$\lim_{t\to 0} \frac{f_t(x) - f_0(x)}{t} = \lim_{t\to 0} \frac{g(f(x) + te_i) - x}{t} = Dg(f(x))e_i = [Df(x)]^{-1}e_i.$$

The proof of Theorem 3.5.3 is based on the following result which is of independent interest.

Theorem 3.5.4 *Let $f : \Omega \to \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$, be 1-quasiconformal and $g = f^{-1}$. For a compact domain $A \Subset \Omega$ define $f_t(x) := g(f(x) + te_i)$ for $x \in A$ and $|t| < \mathrm{dist}(f(A), \partial V)$. Let*

$$X(x) := \lim_{t\to 0} \frac{f_t(x) - f_0(x)}{t} = [Df(x)]^{-1}e_i$$

Then

$$X \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$$

and

$$DX + DX^T = \left(\frac{2}{n}\text{div}X\right) \cdot \text{I}. \tag{3.5.4}$$

Note that in dimension two, these are exactly the Cauchy-Riemann equations. According to Ahlfors' deformation theorem every distributional vector field in dimension at least three that satisfies (3.5.4) is a polynomial of degree 2. This will allow us to complete the proof of Theorem 3.5.3 by adapting the argument of Sarvas [53] that he originally used in the C^2 case.

3.5.1.1 Proof of Theorem 3.5.4.

Recall that $f : \Omega \to \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$, is 1-quasiconformal. Let $V = f(\Omega)$ and $g := f^{-1} : V \to \Omega$. For a compact domain $A \Subset \Omega$ define $f_t(x) := g(f(x) + te_i)$ for $x \in A$ and $|t| < \text{dist}(f(A), \partial V)$. Let

$$X_t(x) := \frac{f_t(x) - x}{t} \in W^{1,n}_{\text{loc}}(\Omega)\,.$$

We claim that

Lemma 3.5.2 *For every compact set $A \subset \Omega$*

$$X_t(x) = \frac{f_t(x) - x}{t} \to X(x) \quad \textit{in } L^1(A) \textit{ as } t \to 0.$$

Proof. Since we have a.e. convergence, by a generalized version of Dominated Convergence theorem ([11], Theorem 21, page 23), the above lemma follows easily from the following lemma.

Lemma 3.5.3 *The family of functions $X_t(x)$ is equi-integrable in any compact subset of Ω.*

Proof. Note that $g = f^{-1}$ is 1-quasiconfomal and hence n-harmonic. Thus Lemma 3.5.1 implies that

$$Jg = \pm|Jg| = \pm n^{-\frac{n}{2}}|Dg|^n \in W^{1,1}_{\text{loc}}. \tag{3.5.5}$$

We first note that by (3.5.5) and the Sobolev embedding theorem $Jg \in L^{\frac{n}{n-1}}_{\text{loc}}(V)$. Let A be a compact set of Ω and E be any measurable subset of A. Since g is 1-quasiconformal and thus has Lusin property, we can apply change of variable formula [21] to obtain

$$\int_E \left|\frac{f_t(x)-x}{t}\right| dx = \int_{f(E)} \left|\frac{g(y+te_i)-g(y)}{t}\right| |Jg(y)|\, dy$$
$$\leq \left\|\frac{g(y+te_i)-g(y)}{t}\right\|_{L^n(f(E))} \|Jg\|_{L^{\frac{n}{n-1}}(f(E))} \leq M\|Jg\|_{L^{\frac{n}{n-1}}(f(E))}, \quad (3.5.6)$$

because $g \in W^{1,n}_{\text{loc}}$ and hence the difference quotients of g are bounded in L^n on compact subsets of V. Let $\epsilon > 0$ be given, since $Jg \in L^{\frac{n}{n-1}}(f(A))$, by absolute continuity of the integral, there is $c > 0$ such that $\|Jg\|_{L^{n/(n-1)}(f(E))} < \epsilon M^{-1}$ whenever $|f(E)| < c$. Since $|f(E)| = \int_E |Jf|\, dx$, there is $\delta > 0$ such that $|f(E)| < c$ whenever $|E| < \delta$. Thus, for $|E| < \delta$, the left hand side of (3.5.6) is less than ϵ. The proof is complete. □

Now we will prove that the derivatives of X_t,

$$DX_t = \frac{Df_t - \mathrm{I}}{t} \in L^n_{\text{loc}}(\Omega)$$

converge in the distributional sense to a function in L^1_{loc}

Lemma 3.5.4 *There exists $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ such that*

$$\int_\Omega DX_t(x)\varphi(x)\, dx \to \int_\Omega u(x)\varphi(x)\, dx$$

as $t \to 0$ for all $\varphi \in C_0^\infty(\Omega)$.

Proof. Without loss of generality we may assume that $Jg \geq 0$ a.e. By the change of variables,

$$\int_\Omega \frac{Df_t(x)-\mathrm{I}}{t}\varphi(x)\, dx = \int_\Omega \frac{Dg(f(x)+te_i)Df(x)-\mathrm{I}}{t}\varphi(x)\, dx$$
$$= \int_V \frac{Dg(y+te_i)Df(g(y))-\mathrm{I}}{t} Jg(y)\varphi(g(y))\, dy$$
$$= \int_V \frac{Dg(y+te_i)[Dg(y)]^{-1}-\mathrm{I}}{t} Jg(y)\varphi(g(y))\, dy \quad (3.5.7)$$

for $Df(g(y)) = [Dg(y)]^{-1}$ a.e. From the formula for the inverse matrix we have that $[Dg(y)]^{-1}Jg(y) = (Dg^{\#}(y))^T$ if $Jg(y) \neq 0$. Hence (3.5.7) is equal to,

$$\begin{aligned}
\int_V & \frac{Dg(y+te_i)(Dg^{\#}(y))^T - Jg(y)\cdot \mathrm{I}}{t}\,\varphi(g(y))\,dy \\
&= \int_V Dg(y+te_i)\frac{[(Dg^{\#}(y)) - Dg^{\#}(y+te_i)]^T}{t}\,\varphi(g(y))\,dy \\
&\qquad + \int_V \frac{Jg(y+te_i) - Jg(y)}{t}\cdot \mathrm{I}\,\varphi(g(y))\,dy \qquad (3.5.8)
\end{aligned}$$

The last equality follows from $Dg(y)(Dg^{\#})^T(y) = Jg(y)\cdot \mathrm{I}$.

Since by Lemma 3.5.1,

$$\pm n^{\frac{n-2}{2}}(Dg)^{\#} = |Dg|^{n-2}Dg \in W^{1,\frac{n}{n-1}}_{\mathrm{loc}} \tag{3.5.9}$$

it is an elementary fact, [20, page 265], that

$$\frac{Dg^{\#}(y) - Dg^{\#}(y+te_i)}{t} \to -\frac{\partial}{\partial y_i}Dg^{\#}(y) \quad \text{in } L^{\frac{n}{n-1}}_{\mathrm{loc}}(V).$$

On the other hand, $Dg \in L^n_{\mathrm{loc}}(V)$, $Dg(y+te_i)$ is a translation of $Dg(y)$ and $\varphi(g(y))$ is bounded with compact support, so $Dg(y+te_i)\varphi(g(y)) \to Dg(y)\varphi(g(y))$ in $L^n(V)$ as $t \to 0$. We thus obtain convergence for the first integral on the right hand side of (3.5.8)

$$\begin{aligned}
\int_V & Dg(y+te_i)\frac{[(Dg^{\#}(y)) - Dg^{\#}(y+te_i)]^T}{t}\,\varphi(g(y))\,dy \to \\
&\qquad - \int_V Dg(y)\left[\frac{\partial}{\partial y_i}Dg^{\#}(y)\right]^T \varphi(g(y))\,dy. \qquad (3.5.10)
\end{aligned}$$

Since by (3.5.5), $Jg(y) \in W^{1,1}_{\mathrm{loc}}(V)$,

$$\frac{Jg(y+te_i) - Jg(y)}{t} \to \frac{\partial}{\partial y_i}Jg(y) \quad \text{in } L^1_{\mathrm{loc}}(V).$$

Hence we obtain convergence for the second integral on the right hand side of (3.5.8)

$$\int_V \frac{Jg(y+te_i) - Jg(y)}{t}\cdot \mathrm{I}\,\varphi(g(y))\,dy \to \int_V \frac{\partial}{\partial y_i}Jg(y)\cdot \mathrm{I}\,\varphi(g(y))\,dy \tag{3.5.11}$$

Thus

$$\int_\Omega \frac{Df_t(x)-\mathrm{I}}{t}\varphi(x)\,dx \to -\int_V Dg(y)\left[\frac{\partial}{\partial y_i}Dg^{\#}(y)\right]^T \varphi(g(y))\,dy \\ + \int_V \frac{\partial}{\partial y_i}Jg(y)\cdot \mathrm{I}\,\varphi(g(y))\,dy = \int_\Omega u(x)\varphi(x)\,dx \quad (3.5.12)$$

where

$$u(x) = \left[-Dg(f(x)\left[(\frac{\partial}{\partial y_i}Dg^{\#})(f(x))\right]^T + \left(\frac{\partial}{\partial y_i}Jg\right)(f(x))\cdot \mathrm{I}\right] Jf(x) \in L^1_{\mathrm{loc}}(\Omega),$$

since

$$-Dg(y)\left[\frac{\partial}{\partial y_i}Dg^{\#}(y)\right]^T + \frac{\partial}{\partial y_i}Jg(y)\cdot \mathrm{I} \in L^1_{\mathrm{loc}}(V).$$

The proof is complete. □

Corollary 3.5.1 *$DX = u \in L^1_{\mathrm{loc}}$ and hence $X \in W^{1,1}_{\mathrm{loc}}(\Omega)$.*

Proof. By Lemma 3.5.2 and 3.5.4,

$$\begin{aligned}\int_\Omega X(x)\frac{\partial\varphi}{\partial x_j}(x)\,dx &= \lim_{t\to 0}\int_\Omega X_t(x)\frac{\partial\varphi}{\partial x_j}(x)\,dx = -\lim_{t\to 0}\int_\Omega \frac{\partial X_t}{\partial x_j}(x)\varphi(x)\,dx \\ &= -\lim_{t\to 0}\int_\Omega DX_t(x)e_j\varphi(x)\,dx = -\int_\Omega u(x)e_j\varphi(x)\,dx\,.\end{aligned}$$

Thus $DX = u \in L^1_{\mathrm{loc}}$. The proof is complete. □

Since

$$Df_t^T Df_t = Jf_t^{2/n}\cdot \mathrm{I} \text{ a.e.}, \quad \text{and} \quad Jf_t > 0 \text{ a.e.}$$

we have

$$\frac{Jf_t^{1/n}-1}{t}\cdot \mathrm{I} = \frac{\frac{Df_t^T Df_t}{Jf_t^{1/n}} - \mathrm{I}}{t} = \frac{(Df_t - Jf_t^{1/n}\cdot \mathrm{I})^T Df_t}{tJf_t^{1/n}} + \frac{Df_t - \mathrm{I}}{t}.$$

Observe that

$$\frac{(Df_t - Jf_t^{1/n}\cdot \mathrm{I})^T Df_t}{tJf_t^{1/n}} = \frac{Jf_t^{1/n}-1}{t}\cdot \mathrm{I} - \frac{Df_t - \mathrm{I}}{t} \in L^n_{\mathrm{loc}}. \quad (3.5.13)$$

Lemma 3.5.5 *There exists $v(x) \in L^1_{\text{loc}}(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$*

$$\int_\Omega \frac{(Df_t - Jf_t^{1/n} \cdot \mathrm{I})^T Df_t}{tJf_t^{1/n}}\,\varphi(x)\,dx \to \int_\Omega u^T(x)\varphi(x)\,dx - \int_\Omega v(x)\cdot \mathrm{I}\,\varphi(x)\,dx$$

as $t \to 0$, where u is the same as in Lemma 3.5.4.

Proof. Recall that

$$Df_t(x) = Dg(f(x)+te_i)Df(x), \quad Jf_t(x) = Jg(f(x)+te_i)Jf(x),$$

and $Df(g(y)) = [Dg(y)]^{-1}$. Hence the change of variables formula yields

$$\int_\Omega \frac{(Df_t - Jf_t^{1/n} \cdot \mathrm{I})^T Df_t}{tJf_t^{1/n}}\,\varphi(x)\,dx = \int_V \varphi(g(y))Jg(y)\times$$
$$\times\,\frac{(Dg(y+te_i)[Dg(y)]^{-1}Jg(y)^{\frac{1}{n}} - Jg(y+te_i)^{\frac{1}{n}}\cdot \mathrm{I})^T Dg(y+te_i)[Dg(y)]^{-1}}{tJg(y+te_i)^{\frac{1}{n}}}\,dy\,.$$

Since

$$[Dg]^{-1}Jg = (Dg^{\#})^T, \quad Dg^TDg = Jg^{\frac{2}{n}}\cdot \mathrm{I}, \quad [Dg]^{-1} = Dg^T/Jg^{\frac{2}{n}},$$

one easily checks the above is equal to,

$$\int_V \varphi(g(y))\left[\frac{[(Dg^{\#}(y) - Dg^{\#}(y+te_i)]Jg(y+te_i)^{\frac{1}{n}}Dg^T(y)}{tJg(y)^{\frac{1}{n}}}\right.$$
$$\left.+\frac{[Jg(y+te_i)^{1-\frac{1}{n}} - Jg(y)^{1-\frac{1}{n}}]Dg(y+te_i)Dg^T(y)}{tJg(y)^{\frac{1}{n}}}\right]dy \quad (3.5.14)$$

We know from the proof of Lemma 3.5.4 that

$$\frac{Dg^{\#}(y) - Dg^{\#}(y+te_i)}{t} \to -\frac{\partial}{\partial y_i}Dg^{\#}(y) \quad \text{in } L^{\frac{n}{n-1}}_{\text{loc}}.$$

We will show now that

$$\frac{Jg(y+te_i)^{\frac{1}{n}}Dg^T(y)}{Jg(y)^{\frac{1}{n}}} \to Dg^T(y) \quad \text{in } L^n_{\text{loc}}(V).$$

Indeed, $n^{\frac{n}{2}} Jg(y) = |Dg|^n$, so $|Dg(y)Jg(y)^{-\frac{1}{n}}| = n^{\frac{1}{2}}$. Hence for any compact set $K \subset V$,

$$\int_K \left| \frac{Jg(y+te_i)^{\frac{1}{n}} Dg^T(y)}{Jg(y)^{\frac{1}{n}}} - Dg^T(y) \right|^n dy$$
$$\leq \int_K |Jg(y+te_i)^{\frac{1}{n}} - Jg(y)^{\frac{1}{n}}|^n |Dg^T(y)Jg(y)^{-\frac{1}{n}}|^n \, dy$$
$$= n^{\frac{n}{2}} \int_K |Jg(y+te_i)^{\frac{1}{n}} - Jg(y)^{\frac{1}{n}}|^n \, dy \to 0. \quad (3.5.15)$$

This implies convergence of the first half of (3.5.14)

$$\int_V \varphi(g(y)) \frac{[(Dg^\#(y) - Dg^\#(y+te_i)]Jg(y+te_i)^{\frac{1}{n}} Dg^T(y)}{tJg(y)^{\frac{1}{n}}} \, dy$$
$$\to - \int_V \varphi(g(y)) \left[\frac{\partial}{\partial y_i} Dg^\#(y) \right] Dg^T(y) \, dy. \quad (3.5.16)$$

By (3.5.9) and (3.5.5) $Jg(y)^{1-\frac{1}{n}} = c|Dg^\#(y)| \in W^{1,\frac{n}{n-1}}_{\text{loc}}(V)$, thus,

$$\frac{Jg(y+te_i)^{1-\frac{1}{n}} - Jg(y)^{1-\frac{1}{n}}}{t} \to \frac{\partial}{\partial y_i}[Jg(y)^{1-\frac{1}{n}}] \quad \text{in } L^{\frac{n}{n-1}}_{\text{loc}}(V).$$

By the same argument as in (3.5.15),

$$Dg(y+te_i)Dg^T(y)Jg(y)^{-\frac{1}{n}} \to Jg(y)^{\frac{1}{n}} \cdot \mathrm{I} \quad \text{in } L^n_{\text{loc}}(V).$$

Hence we have convergence for the second half of (3.5.14),

$$\int_V \varphi(g(y)) \frac{Jg(y+te_i)^{1-\frac{1}{n}} - Jg(y)^{1-\frac{1}{n}}}{t} Dg(y+te_i)Dg^T(y)Jg(y)^{-\frac{1}{n}} \, dy \to$$
$$\int_V \varphi(g(y)) \frac{\partial}{\partial y_i}[Jg(y)^{1-\frac{1}{n}}]Jg(y)^{\frac{1}{n}} \cdot \mathrm{I} \, dy = \frac{n-1}{n} \int_V \varphi(g(y)) \frac{\partial}{\partial y_i} Jg(y) \cdot \mathrm{I} \, dy. \quad (3.5.17)$$

The last equality follows from $Jg(y) = [Jg(y)^{1-\frac{1}{n}}]^{\frac{n}{n-1}}$ and the chain rule for Sobolev functions. Now (3.5.14), (3.5.16) and (3.5.17) yield

$$\int_\Omega \frac{(Df_t - Jf_t(x)^{1/n} \cdot \mathrm{I})^T Df_t}{tJf_t(x)^{1/n}} \varphi(x) \, dx$$
$$\to - \int_V \varphi(g(y)) \left[\frac{\partial}{\partial y_i} Dg^\#(y) \right] Dg^T(y) \, dy + \frac{n-1}{n} \int_V \varphi(g(y)) \frac{\partial}{\partial y_i} Jg(y) \cdot \mathrm{I} \, dy$$
$$= \int_\Omega u^T(x)\varphi(x) \, dx - \int_\Omega v(x) \cdot \mathrm{I} \, \varphi(x) \, dx, \quad (3.5.18)$$

where $u(x) \in L^1_{\text{loc}}(\Omega)$ is the same matrix valued function as in Lemma 3.5.4 and $v(x) = \frac{1}{n}(\frac{\partial}{\partial y_i} Jg)(f(x))Jf(x) \in L^1_{\text{loc}}(\Omega)$ is a scalar function. The proof is complete. □

Lemma 3.5.6

$$\int_\Omega \frac{Jf_t(x)^{1/n} - 1}{t} \cdot \mathrm{I}\,\varphi(x)\,dx \to n\int_\Omega v(x)\cdot \mathrm{I}\,\varphi(x)\,dx \quad \textit{as } t\to 0,$$

where $v \in L^1_{\mathrm{loc}}(\Omega)$ *is the same as in Lemma 3.5.5*

Proof. Recall again that $Jf_t(x) = Jg(f(x)+te_i)Jf(x)$. Hence the change of variables formula yields,

$$\begin{aligned}\int_\Omega &\frac{Jg(f(x)+te_i)Jf(x)-1}{t}\cdot \mathrm{I}\,\varphi(x)\,dx\\ &= \int_V \frac{Jg(y+te_i)Jf(g(y))-1}{t} Jg(y)\cdot \mathrm{I}\,\varphi(g(y))\,dy\\ &= \int_V \frac{Jg(y+te_i)-Jg(y)}{t}\cdot \mathrm{I}\,\varphi(g(y))\,dy\\ &\to \int_V \frac{\partial}{\partial y_i} Jg(y)\cdot \mathrm{I}\,\varphi(g(y))\,dy = n\int_\Omega v(x)\cdot \mathrm{I}\,\varphi(x)\,dx.\end{aligned}$$

The proof is complete. □

Applying Lemma 3.5.4, 3.5.5 and 3.5.6 to (3.5.13) we obtain,

$$\int_\Omega u^T(x)\varphi(x)\,dx - \int_\Omega v(x)\cdot \mathrm{I}\,\varphi(x)\,dx = n\int_\Omega v(x)\cdot \mathrm{I}\,\varphi(x)\,dx - \int_\Omega u(x)\varphi(x)\,dx\,.$$

Since $DX = u$ we get

$$\int_\Omega (DX + DX^T)\varphi(x)\,dx = (n+1)\int_\Omega v(x)\cdot \mathrm{I}\,\varphi(x)\,dx.$$

Hence $X_{i,i} = (n+1)/2\,v(x)$, $i = 1,2,\dots,n$ so $(n/2)(n+1)v(x) = \operatorname{div} X$. Since the equality is true for any $\varphi \in C_0^\infty(\Omega)$ we conclude

$$DX + DX^T = \left(\frac{2}{n}\operatorname{div} X\right)\cdot \mathrm{I}.$$

The proof of Theorem 3.5.4 is complete. □

3.5.1.2 Proof of Theorem 3.5.3. Once we obtain Theorem 3.5.4, the proof of the Liouville Theorem follows from any well-known proofs under C^3 or C^4 assumption. Indeed, Theorem 3.1.1 tells us that $[Df]^{-1}(x)e_i$ is C^∞ smooth for every $i = 1, ..., n$, hence by conformality,

$$\frac{1}{Jf^{2/n}} = [Df]^{-1}e_i \cdot [Df]^{-1}e_i$$

is also smooth and is a polynomial of degree 4. Let $\tilde{\Omega}$ be the open subset of Ω with the roots of $1/(Jf^{2/n})$ removed. It then follows Jf is smooth in the open set $\tilde{\Omega}$. Now by conformality again, $Df^T = [Df]^{-1}Jf^{2/n}$ is C^∞ smooth in $\tilde{\Omega}$. We can then apply, say Nevanlinna's argument [44] to obtain that f is a Möbius transformation in $\tilde{\Omega}$. By the fact that f is a homeomprphism in Ω we can actually conclude that f is a Möbius transformation in Ω.

However, here we also provide another interesting proof due to Sarvas [53]: Given f 1-quasiconformal, we can assume $0 \in \Omega$, $f(0) = 0$, and $Df(0) = \mathrm{I}$. Indeed, we can compose f with translations and dilation and note that the composition is again a 1-quasiconformal mapping. Therefore, $f(x) = x + |x|\epsilon(x)$ with $\epsilon(x) \to 0$ as $x \to 0$. Let $h(x) = \frac{x}{|x|^2}$ be an inversion with respect to the unit sphere. Let $\tilde{\Omega} = \Omega \setminus \{0\}$. $\tilde{\Omega}$ is open and $f(x) \neq 0$ on $\tilde{\Omega}$. Then $F : \tilde{\Omega} \to \mathbb{R}^n$, $F = h \circ f$ is also 1-quasiconformal. Now $DF(x) = |f(x)|^{-2}(I - 2Q_{f(x)})Df(x), x \neq 0$, where $Q_x : \mathbb{R}^n \to \mathbb{R}^n$ is given by $Q_x y = |x|^{-2}(y \cdot x)x$ and $(I - 2Q_x)^{-1} = (I - 2Q_x)$. Hence $[DF(x)]^{-1}e_i = |f(x)|^2[Df(x)]^{-1}(I - 2Q_{f(x)})e_i = |f(x)|^2[Df(x)]^{-1}(e_i - 2(f(x) \cdot e_i)f(x))$. Note that $[Df(x)]^{-1}e_i$ is a polynomial of degree 2. In particular, it is defined for $x = 0$ since $Df(0) = I$ and f a homeomorphism with $f(0) = 0$, hence $[DF(0)]^{-1}e_i = 0$. Now since by Theorem 3.1.1, $[DF(x)]^{-1}e_i = a + Bx + 2(c \cdot x)x - |x|^2 c$. The condition $[DF(0)]^{-1}e_i = 0$ gives $a = 0$. Let $e \in \mathbb{R}^n$, $|e| = 1$. Inserting $x = se$ for small $s > 0$, we get

$$\begin{aligned} sBe = B(se) &= -s^2[2(c \cdot e)e - c] + [DF(se)]^{-1}e_i \\ &= -s^2[2(c \cdot e)e - c] + |f(se)|^2[Df(se)]^{-1}(I - 2Q_{f(se)})e_i \end{aligned}$$

Substituting $f(x) = x + |x|\epsilon(x)$ we get

$$sBe = -s^2[2(c \cdot e)e - c] + s^2|e + \epsilon(se)|^2([Df(se)]^{-1}(I - 2Q_{f(se)})e_i$$

Dividing by s and let $s \to 0$ gives $Be = 0$. Since e is an arbitrary unit vector, this yields $B = 0$. Therefore $[DF(x)]^{-1}e_i = 2(c \cdot x)x - |x|^2 c$. This implies c cannot be zero. Otherwise, $DF^{-1} = 0$ everywhere, violating $J(x, F) \neq 0$ a.e. for quasiconformal mappings. Putting $x = sc$ for small $s > 0$ we obtain,

$$s^2|c|^2 c = DF^{-1}(sc)e_i = s^2|c + |c|\epsilon(sc)|^2[Df(sc)]^{-1}(I - 2Q_{f(sc)})e_i$$

Divide by s^2 and then let $s \to 0$, then $Q_{f(sc)}e_i \to Q_c e_i$ since $f(x) = x + |x|\epsilon(x)$, and so the above implies $c = (I - 2Q_c)e_i$, and this implies $e_i = (I - 2Q_c)c = -c$. Finally,

$$[DF(x)]^{-1}e_i = -(2(e_i \cdot x)x - |x|^2 e_i) = |x|^2(I - 2Q_x)e_i$$

Since i as for e_i is arbitrary, we conclude that $DF^{-1}(x) = |x|^2(I - 2Q_x) = Dh^{-1}(x)$ for $x \in \tilde{\Omega}$, or $DF = Dh$ for all $x \in \tilde{\Omega}$. Thus $F = h + d$ for some constant vector d and for all $x \in \tilde{\Omega}$. Note that $F = h \circ f$, thus $f(x) = h^{-1}(h(x) + d) = h(h(x) + d)$. In the above argument we do not distinguish a.e. equivalent functions, but this is not a problem since f is a homeomorphism so they must equal everywhere. The proof is complete. $\square$

3.6 ISOMETRIC IMMERSIONS

It has been known since at least the 19th century that any smooth surface with zero Gaussian curvature is locally ruled, i.e. passing through any point of the surface is a straight segment lying on the surface. Such surfaces were called developable surfaces. This terminology was used as an indication that any such surface is in isometric equivalence with the plane, i.e. any piece of it can be *developed* on the flat plane without any stretching or compressing. Meanwhile, it was already suspected that there exist somewhat regular surfaces applicable to the plane, but yet not developable. However, it was not until the work of John Nash at the zenith of the last century that the existence of such unintuitive phenomena was rigorously established.

In his pioneering work, Nash settled several questions. He established that any Riemannian manifold can be isometrically embedded in an Euclidean space [43]. Moreover, if the dimension of the space is large enough, this embedding can be done in a manner so that the diameter of the image is as small as one wishes. As for the lower dimensional embeddings, Nash [42] and Kuiper [32], established the existence of a C^1 isometric embedding of any Riemannian manifold into another manifold of one higher dimension. Their method, which is now famously re-cast in the framework of convex integration [19], involved iterated perturbations of a given short mapping of the manifold towards realizing an isometry.

A surprising corollary of these results is the existence of a C^1 flat torus in $\mathbb{R}^3$. Another one is that there are C^1 isometric embeddings of the two dimensional unit sphere into three dimensional space with arbitrarily small diameter. By contrast, it was established by Hartman and Nirenberg any flat C^2 surface in $\mathbb{R}^3$ must be developable [24], while Hilbert had already shown that any C^2 isometric immersion of the sphere must be a rigid motion. On the other hand, the former result was generalized by Pogorelov's for C^1 isometries with total zero curvature in [48, Chapter II] and [49, Chapter IX].

A natural question arises in this context for the analyst: What about isometric immersions of intermediate regularity, say of Hölder or Sobolev type? Regarding Hölder regularity, rigidity of $C^{1,\alpha}$ isometries of 2 dimensional flat domains has been established for $\alpha \geq 2/3$ [5, 6], while their flexibility in the sense of Nash and Kuiper is known for $\alpha < 1/7$ [6, 9]. The critical value for α is conjectured to be $1/2$ in this case. As for the regularity of Sobolev isometries, following the results of Kirchheim in [30] on $W^{2,\infty}$ solutions to degenerate Monge-Ampère equations, the rigidity of $W^{2,2}$ isometries of a flat domain was established in [46]. More precisely, it was established that such mappings are developable in the classical sense, i.e.

Theorem 3.6.1 *Let $u \in I^{2,2}(\Omega, \mathbb{R}^3)$, where Ω is a Lipschitz domain in $\mathbb{R}^2$. Then for every point $x \in \Omega$, there exists either a neighborhood U of x, or a segment passing through x and joining $\partial\Omega$ at both end, on which u is affine.*

Based on this developability is the density result,

Theorem 3.6.2 *If in addition Ω is convex, then $I^{2,2}(\Omega, \mathbb{R}^3) \cap C^\infty(\Omega, \mathbb{R}^3)$ is strongly dense in $I^{2,2}(\Omega, \mathbb{R}^3)$.*

It can be shown that each component of $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$ satisfies $\operatorname{rank} D^2 u^\ell \leq 1$, $1 \leq \ell \leq n+1$ (eg. Lemma 5.2.1 in Section 5.2). Here we would like to present two results due to Pakzad [46], Lemma 2.1 and [47], Theorem 1 to give the readers some sense of the difference spirit between $W^{2,n}$ and $W^{2,2}$ isometric immersions.

Lemma 3.6.1 *Let $f \in W^{1,2}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, such that Df is symmetric and singular (i.e., of zero determinant), then f is continuous in Ω.*

Proof. For $\delta > 0$ define $f_\delta(x, y) := f(x, y) + \delta(-y, x)$. Then $f_\delta \to f$ uniformly as $\delta \to 0$ and $\det(Df_\delta) = \delta^2$. So f_δ is a map in $W^{1,2}(\Omega, \mathbb{R}^2)$ with positive Jacobian, hence open and continuous [58]. Passing to the limit we obtain continuity of f. □

Lemma 3.6.2 *Let $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, such that $\operatorname{rank} Df \leq 1$ and Df is symmetric a.e. in Ω , then f is continuous in Ω.*

The proof used an argument similar to proof of weak monotonicity of deformations with positive Jacobian due to Manfredi [39].

Pakzad's result proves the interior C^1 regularity of isometric immersions, a regularity result up to boundary is given by Müller and Pakzad [40],

Theorem 3.6.3 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with $C^{1,\alpha}$ boundary for some $\alpha > 0$. Let $u \in I^{2,2}(\Omega, \mathbb{R}^3)$, then u is C^1 up to the boundary, with a logarithmic modulus of continuity. More precisely, there exists a constant $C(\Omega)$ and $R_0(\Omega)$ depending on Ω such that for every $r < R/4 < R_0(\Omega)$ and for every $x \in \Omega$,*

$$\operatorname{osc}_{B(x,r)\cap\Omega} \nabla u \leq C(\Omega) \ln^{-1/2}(R/r) \|\nabla^2 u\|_{B(x,r)\cap\Omega}.$$

They also proved that if Ω is just a Lipschitz domain, each component of u, u^i can be approximated in $W^{2,2}$ norm by functions $u^i_k \in W^{1,\infty} \cap W^{2,2}$ with $\det \nabla^2 u^i_k = 0$.

On the other hand, the surface of L^2 integrable second fundamental form have been as the curvature functionals in elasticity. Toro [54] proved the existence of bilipschitz parametrizations of graph of $W^{2,2}$ functions on $\mathbb{R}^2$. Müller and Šverák [41] improved Taro's result by showing the existence of conformal parametrizations with continuous metric for these graphs. Sobolev isometries also arise in the study of nonlinear elastic thin films. Kirchhoff's plate model put forward in the 19th century [31] consists in minimizing the L^2 norm of the second fundamental form of isometric immersions of a 2d domain into $\mathbb{R}^3$ under suitable forces or boundary conditions. In other words, using the modern terminology, the space of admissible maps for this model is that of $W^{2,2}$ isometric immersions. Using the developability results mentioned above for this class of mappings, Hornung has studied the regularity of the critical points of the Kirchhoff's functional in [25]. For other applications in nonlinear elasticity of both the developability and density results for Sobolev isometric immersions of flat domains see [8, 36].

4.0 THE LIOUVILLE THEOREM UNDER SECOND ORDER DIFFERENTIABILITY ASSUMPTION

4.1 INTRODUCTION

The history and development of the Liouville Theorem has been discussed in Chapter 1 and Section 3.5. In this chapter, we will present our results of the Liouville theorem and its relation with the I-M conjecture. Let us recall the definition of conformal mappings in Sobolev space one more time.

Let Ω be a domain in $\mathbb{R}^n$. A function $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is *conformal* or a *weak solution* to the *Cauchy-Riemann System* if,

$$\big(Df(x)\big)^T Df(x) = |J(x, f)|^{2/n} \cdot \mathrm{I} \quad \text{and,} \tag{4.1.1}$$

$$J(x, f) \geq 0 \quad \text{a.e.} \quad \text{or} \quad J(x, f) \leq 0 \quad \text{a.e.,} \tag{4.1.2}$$

where Df is the weak differential of f, i.e., the matrix of weak partial derivatives of f, and $J(x, f) = \det Df(x)$. We call f with $J(x, f) \geq 0$ a.e. *sense preserving* and $J(x, f) \leq 0$ a.e. *sense reversing.*

As our main finding, we prove the Liouville theorem for a conformal map $f \in W^{1,1}_{\text{loc}}$ in dimension $n \geq 3$ under one additional assumption that the norm of the first order derivative $|Df|$ satisfies $|Df|^p \in W^{1,2}_{\text{loc}}$ for $p \geq (n-2)/4$.

Theorem 4.1.1 (Liouville) *Let Ω be a domain in $\mathbb{R}^n$ for $n \geq 3$. Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies (4.1.1) and (4.1.2). Assume further that $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$. Then f is either constant or a Möbius transform restricted to Ω for $p \geq (n-2)/4$.*

This result, gives a necessary condition for the Sobolev exponent in the Iwaniec-Martin conjecture (Conjecture 3.5.1) because Sobolev inequality exactly gives,

$$f \in W^{1,1}_{\text{loc}}, |Df|^p \in W^{1,2}_{\text{loc}} \Rightarrow |Df|^p \in L^{\frac{2n}{n-2}}_{\text{loc}} \Rightarrow f \in W^{1,\frac{2pn}{n-2}}_{\text{loc}}.$$

Thus $f \in W^{1,n/2}_{\text{loc}}$ if $|Df|^{(n-2)/4} \in W^{1,2}_{\text{loc}}$.

Meanwhile, we also discover a Caccioppoli estimate that suggests that the condition $p \geq n/4$ is sufficient for the Liouville Theorem to hold. However, this setting still requires an additional assumption of second order differentiability $|Df|^p \in W^{1,2}_{\text{loc}}$. The main result is as follows:

Theorem 4.1.2 *Let Ω be a domain in $\mathbb{R}^n$. Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies (4.1.1) and (4.1.2). Assume further that $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ for either $p \geq 1/2$ or $p \geq (n-2)/4$ and $n \geq 3$. Then $|Df|^{p-1}Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. Furthermore, if $p > n/4$, then for any $\phi \in C^\infty_0(\Omega)$, we have the Caccioppoli type estimate,*

$$\sum_{i,j=1}^{n} \int_\Omega \left|(|Df|^{p-1} f_i)_j(x)\right|^2 \phi^2(x)\, dx \leq C(n,p) \int_\Omega |Df(x)|^{2p} |\nabla\phi(x)|^2\, dx. \tag{4.1.3}$$

If $p \geq n/4$, $|Df|^{2p}$ is a subharmonic distribution,

$$\int_\Omega |Df(x)|^{2p} \Delta\phi(x)\, dx \geq 0 \quad \textit{whenever} \quad \phi \geq 0. \tag{4.1.4}$$

Observe that Theorem 4.1.2 implies the Liouville theorem for $p \geq n/4$. Indeed, $|Df|^{2p}$ is subharmonic and hence locally bounded, according to the following well-known result [27], Lemma 4.9.1 (A detailed proof can be found in [2], Theorem 2.6.4.2.),

Lemma 4.1.1 *Any subharmonic distribution $h \in \mathcal{D}'(\Omega)$ can be represented by a locally integrable function and such function satisfies the mean value property,*

$$h(x) \leq \fint_{B(x,r)} h(y) dy$$

for a.e. $x \in \Omega$. Consequently, h is bounded from above locally.

Thus Theorem 4.1.2 and the Lemma imply $f \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^n)$. The Liouville theorem follows from Theorem 3.5.1.

Although Theorem 4.1.2 holds under the assumption $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$, this assumption does not appear in (4.1.3) and (4.1.4). Therefore, we conjecture that $f \in W^{1,2p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for $p \geq n/4$ should be enough to conclude subharmonicity. Therefore, the following conjecture, if shown to be true, would give a positive answer to the Iwaniec-Martin conjecture.

Conjecture 4.1.1 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. Let $f \in W^{1,2p}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $p \geq 1/2$, satisfies (4.1.1) and (4.1.2). Then if $p > n/4$, for any $\phi \in C_0^\infty(\Omega)$, we have the Caccioppoli type estimate,*

$$\sum_{i,j=1}^{n} \int_\Omega \left|(|Df|^{p-1} f_i)_j(x)\right|^2 \phi^2(x)\, dx \leq C(n,p) \int_\Omega |Df(x)|^{2p} |\nabla \phi(x)|^2\, dx.$$

If $p \geq n/4$, $|Df|^{2p}$ is a subharmonic distribution,

$$\int_\Omega |Df(x)|^{2p} \Delta\phi(x)\, dx \geq 0 \quad \textit{whenever} \quad \phi \geq 0.$$

Theorem 4.1.2 gives us second order differentiability $|Df|^{p-1}Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ for appropriate p. The fundamental difference after this point is that now we can eventually differentiate and permute the Cauchy Riemann system, while we were not allowed to do that before we prove Theorem 4.1.2. This leads us to the Liouville theorem for even smaller p beyond Theorem 4.1.2, i.e. for $(n-2)/4 \leq p < n/4$. This result does not contradict Example 4.5.2 which shows that conformal maps in the space $W^{1,2p}$ for $p < n/4$ may not be Möbius, because as argument before, Sobolev inequality exactly gives $f \in W^{1,n/2}_{\text{loc}}$ if $|Df|^{(n-2)/4} \in W^{1,2}_{\text{loc}}$. At the same time, we also give another example (Example 4.5.1) to show that under such additional assumption, $p = (n-2)/4$ is the optimal exponent for the Liouville theorem to hold.

This chapter is organized as follows. In section 4.2 we discuss the Liouville theorem for conformal diffeomorphisms of class C^2. This provides some motivations for us to generalize our result to the Sobolev space. In section 4.3 we discuss some preliminary results that would be used in the proof of the main theorems. In particular, we show how to approximate fractions of integrable functions by fractions of smooth functions. In section 4.4 we prove the main results, and finally in section 4.5, we give some examples to discuss the cruciality of the Sobolev exponents.

4.2 THE C^2 CASE

In this section we present two simple proofs of the Liouville theorem for C^2 conformal diffeomorphisms. Differently from the proofs by Hartman [22] and by Sarvas [53] under the same regularity assumption, our proofs are related to the Nevanlina argument, but the argument is somewhat new since it allows for lower regularity than the Nevanlina's proof. We present the proofs in the C^2 case, because it will clarify the proof under Sobolev regularity. In the Sobolev case we will follow the same direction and the proofs in the C^2 case will show what technical difficulties are directly associated with the Sobolev regularity.

Theorem 4.2.1 (Liouville) *Let Ω be an open domain in $\mathbb{R}^n$, $n \geq 3$. Any conformal diffeomorphism $f \in C^2(\Omega, \mathbb{R}^n)$ is a restriction of a Möbius transform to Ω .*

The most important step is to prove that f is C^∞ smooth. We will state this as a separate lemma,

Lemma 4.2.1 *Any conformal diffeomorphism $f \in C^2(\Omega, \mathbb{R}^n)$, $n \geq 3$ is $C^\infty(\Omega)$ smooth.*

We will discuss two proofs. The first proof is shorter. Moreover, it connects the nonlinear Cauchy-Riemann system and the Ahlfors' deformation theorem. However, due to the integrability issue, it cannot be extended to functions in the Sobolev space. Hence we provide the second proof which will allow us to extend this result to functions in the Sobolev space. We will need the well-known Weyl lemma,

Lemma 4.2.2 (Weyl) *Every locally integrable function u defined on a domain Ω satisfying*

$$\int_\Omega u \Delta \varphi = 0$$

for every $\varphi \in C_0^\infty(\Omega)$ is a C^∞ functions.

Note: Throughout the rest of this section, $\cdot_j$ will denote the partial derivative with respect x_j for $1 \leq j \leq n$ for a single function. And $(\cdot)_j$ will denote the partial derivative with respect x_j for $1 \leq j \leq n$ for the expressions inside the parentheses. Also since all the identities will be true for every $x \in \Omega$, in most of the cases, we will *omit* x for simplicity of notation.

First proof. Without loss of generality, we can assume $J(x, f) > 0$ for every $x \in \Omega$. The negative case can be treated by composing f with a reflection and the composition is still a conformal diffeomorphism. Let

$$\lambda(x) := J(x, f)^{\frac{1}{n}}.$$

λ is called the coefficient of conformality. Denote $f_i = \partial f / \partial x_i$. The Cauchy-Riemann system is then equivalent to

$$\langle f_i, f_j \rangle = \lambda^2 \delta_{ij} \quad 1 \leq i, j \leq n \tag{4.2.1}$$

where we use $\langle \cdot, \cdot \rangle$ to denote the dot product for vector valued functions at each point. Let p be any real number, then the Cauchy-Riemann system also gives

$$\langle \lambda^{p-1} f_i, \lambda^{p-1} f_j \rangle = \lambda^{2p} \delta_{ij}, \quad 1 \leq i, j \leq n. \tag{4.2.2}$$

Note that $\lambda^{p-1} f_i$ is C^1 since f is C^2 and $\lambda \in C^1$ is positive.

We now suppose $p \neq 0$. The following identity easily follows from the product rule,

$$(\lambda^{p-1} f_i)_j - (\lambda^{p-1} f_j)_i = \frac{p-1}{p} \Big((\lambda^p)_j \frac{f_i}{\lambda} - (\lambda^p)_i \frac{f_j}{\lambda} \Big), \quad 1 \leq i, j \leq n. \tag{4.2.3}$$

We then differentiate (4.2.2) to obtain,

$$\langle (\lambda^{p-1} f_i)_k, \lambda^{p-1} f_j \rangle + \langle \lambda^{p-1} f_i, (\lambda^{p-1} f_j)_k \rangle = (\lambda^{2p})_k \delta_{ij}. \tag{4.2.4}$$

Permutation of indices i, j, k yields,

$$\langle (\lambda^{p-1} f_i)_j, \lambda^{p-1} f_k \rangle + \langle \lambda^{p-1} f_i, (\lambda^{p-1} f_k)_j \rangle = (\lambda^{2p})_j \delta_{ik}, \tag{4.2.5}$$

$$\langle (\lambda^{p-1} f_k)_i, \lambda^{p-1} f_j \rangle + \langle \lambda^{p-1} f_k, (\lambda^{p-1} f_j)_i \rangle = (\lambda^{2p})_i \delta_{jk}. \tag{4.2.6}$$

If we add (4.2.4) and (4.2.5), then subtract (4.2.6), combining with the identity (4.2.3), we obtain the following,

$$\begin{aligned} 2\left\langle\lambda^{p-1}f_i,(\lambda^{p-1}f_j)_k\right\rangle+\frac{p-1}{p}\left\langle\lambda^{p-1}f_i,(\lambda^p)_j\frac{f_k}{\lambda}-(\lambda^p)_k\frac{f_j}{\lambda}\right\rangle \\ +\frac{p-1}{p}\left\langle\lambda^{p-1}f_j,(\lambda^p)_k\frac{f_i}{\lambda}-(\lambda^p)_i\frac{f_k}{\lambda}\right\rangle+\frac{p-1}{p}\left\langle\lambda^{p-1}f_k,(\lambda^p)_j\frac{f_i}{\lambda}-(\lambda^p)_i\frac{f_j}{\lambda}\right\rangle \\ =2\lambda^p(\lambda^p)_k\delta_{ij}+2\lambda^p(\lambda^p)_j\delta_{ik}-2\lambda^p(\lambda^p)_i\delta_{jk}. \end{aligned} \tag{4.2.7}$$

Now the fact that λ as well as its partial derivatives are scalar functions thus can be taken out of the dot products, together with the following identity easily adapted from (4.2.1),

$$\left\langle\frac{f_i}{\lambda},\frac{f_j}{\lambda}\right\rangle=\delta_{ij},1\leq i,j\leq n \tag{4.2.8}$$

give,

$$\left\langle\lambda^{p-1}f_i,(\lambda^{p-1}f_j)_k\right\rangle=\lambda^p(\lambda^p)_k\delta_{ij}+\frac{1}{p}\lambda^p(\lambda^p)_j\delta_{ik}-\frac{1}{p}\lambda^p(\lambda^p)_i\delta_{jk}. \tag{4.2.9}$$

By (4.2.8), the value of $\{f_i/\lambda\}$, $1\leq i\leq n$ at each point in its domain is an orthonormal basis of $\mathbb{R}^n$. Thus we can write

$$(\lambda^{p-1}f_j)_k=\sum_i\left\langle(\lambda^{p-1}f_j)_k,\frac{f_i}{\lambda}\right\rangle\frac{f_i}{\lambda}=\sum_i\frac{1}{\lambda^p}\left\langle(\lambda^{p-1}f_j)_k,\lambda^{p-1}f_i\right\rangle\frac{f_i}{\lambda} \tag{4.2.10}$$

This and (4.2.9) yields the following,

$$(\lambda^{p-1}f_j)_k=(\lambda^p)_k\frac{f_j}{\lambda}+\frac{1}{p}(\lambda^p)_j\frac{f_k}{\lambda}-\frac{1}{p}\sum_i(\lambda^p)_i\frac{f_i}{\lambda}\delta_{jk}. \tag{4.2.11}$$

If we take $j=k$ and sum up, we also get,

$$\sum_i(\lambda^{p-1}f_i)_i=\frac{p+1-n}{p}\sum_i(\lambda^p)_i\frac{f_i}{\lambda}. \tag{4.2.12}$$

In particular, if we take $p=-1$, then we obtain,

$$(\lambda^{-2}f_j)_k=(\lambda^{-1})_k\frac{f_j}{\lambda}-(\lambda^{-1})_j\frac{f_k}{\lambda}+\sum_i(\lambda^{-1})_i\frac{f_i}{\lambda}\delta_{jk}, \tag{4.2.13}$$

and

$$\sum_i(\lambda^{-2}f_i)_i=n\sum_i(\lambda^{-1})_i\frac{f_i}{\lambda}. \tag{4.2.14}$$

Thus, we have the following identity,

$$(\lambda^{-2} f_j)_k + (\lambda^{-2} f_k)_j = \frac{2}{n} \sum_i (\lambda^{-2} f_i)_i \delta_{jk} \,. \tag{4.2.15}$$

Let f^ν to be the νth component of f, then (4.2.15) says,

$$(\lambda^{-2} f_j^\nu)_k + (\lambda^{-2} f_k^\nu)_j = \frac{2}{n} \sum_i (\lambda^{-2} f_i^\nu)_i \delta_{jk} \,. \tag{4.2.16}$$

Hence $u = (\lambda^{-2} f_1^\nu, ..., \lambda^{-2} f_n^\nu)$ satisfies,

$$Du + Du^T = \frac{2}{n} \mathrm{div}(u) \cdot \mathrm{I} \,.$$

By Ahlfors' deformation theorem discussed in Section 3.1, u is a polynomial of degree two. Now since

$$\langle \lambda^{-2} f_j, \lambda^{-2} f_j \rangle = \lambda^{-2}$$

is a polynomial of degree 4, which means λ^2 is smooth. Thus $f_i^\nu = \lambda^2(\lambda^{-2} f_i^\nu)$ is smooth. The first proof is complete. □

Second proof. We will start from (4.2.9) in the first proof.

For any $\phi \in C_0^\infty(\Omega)$, (4.2.9) gives

$$\sum_{i,j} \int_\Omega \langle (\lambda^{p-1} f_i)_j, \lambda^{p-1} f_j \rangle \phi_i = \frac{p-1+n}{2p} \sum_i \int_\Omega (\lambda^{2p})_i \phi_i, \tag{4.2.17}$$

and,

$$\sum_{i,j} \int_\Omega \langle (\lambda^{p-1} f_i)_i, \lambda^{p-1} f_j \rangle \phi_j = \frac{p+1-n}{2p} \sum_i \int_\Omega (\lambda^{2p})_i \phi_i. \tag{4.2.18}$$

We cannot integrate by parts on the left hand sides of (4.2.17) and (4.2.18) directly because the functions $(\lambda^{p-1} f_i)_j, 1 \le i, j \le n$ are continuous only. However, if we approximate $\lambda^{p-1} f_i$ smoothly by convolution, then integration by parts and passing to the limit give,

$$\sum_{i,j} \int_\Omega \langle (\lambda^{p-1} f_i)_j, \lambda^{p-1} f_j \rangle \phi_i - \langle (\lambda^{p-1} f_i)_i, \lambda^{p-1} f_j \rangle \phi_j$$
$$= \sum_{i,j} \int_\Omega \langle (\lambda^{p-1} f_i)_i, (\lambda^{p-1} f_j)_j \rangle \phi - \langle (\lambda^{p-1} f_i)_j, (\lambda^{p-1} f_j)_i \rangle \phi, \tag{4.2.19}$$

because $(\cdot)_{ij}$ and $(\cdot)_{ji}$ will cancel each for smooth function before passing to the limits. By (4.2.12) and (4.2.8),

$$\sum_{i,j}\big\langle(\lambda^{p-1}f_i)_i,(\lambda^{p-1}f_j)_j\big\rangle=\Big\langle\sum_i(\lambda^{p-1}f_i)_i,\sum_j(\lambda^{p-1}f_j)_j\Big\rangle$$
$$=\frac{(p+1-n)^2}{p^2}\Big\langle\sum_i(\lambda^p)_i\frac{f_i}{\lambda},\sum_j(\lambda^p)_j\frac{f_j}{\lambda}\Big\rangle=\frac{(p+1-n)^2}{p^2}\sum_k(\lambda^p)_k(\lambda^p)_k,$$

and by (4.2.11) and (4.2.8)

$$\sum_{i,j}\big\langle(\lambda^{p-1}f_i)_j,(\lambda^{p-1}f_j)_i\big\rangle=\sum_{i,j}\Big\langle(\lambda^p)_j\frac{f_i}{\lambda}+\frac{1}{p}(\lambda^p)_i\frac{f_j}{\lambda}-\frac{1}{p}\sum_k(\lambda^p)_k\frac{f_k}{\lambda}\delta_{ij},$$
$$(\lambda^p)_i\frac{f_j}{\lambda}+\frac{1}{p}(\lambda^p)_j\frac{f_i}{\lambda}-\frac{1}{p}\sum_\ell(\lambda^p)_\ell\frac{f_\ell}{\lambda}\delta_{ij}\Big\rangle=\frac{p^2+2np-2p+n-1}{p^2}\sum_k(\lambda^p)_k(\lambda^p)_k.$$

Substituting these two identities into (4.2.19) yields,

$$\sum_{i,j}\int_\Omega\big\langle(\lambda^{p-1}f_i)_j,\lambda^{p-1}f_j\big\rangle\phi_i-\big\langle(\lambda^{p-1}f_i)_i,\lambda^{p-1}f_j\big\rangle\phi_j$$
$$=\Big(\frac{(p+1-n)^2}{p^2}-\frac{p^2+2np-2p+n-1}{p^2}\Big)\int_\Omega\sum_k(\lambda^p)_k(\lambda^p)_k\phi$$
$$=\frac{(1-n)(4p+2-n)}{p^2}\int_\Omega|\nabla\lambda^p|^2\phi. \quad (4.2.20)$$

Combine (4.2.20), (4.2.17) and (4.2.18) we obtain that,

$$\frac{(1-n)(4p+2-n)}{p^2}\int_\Omega|\nabla\lambda^p|^2\phi=\frac{n-1}{p}\int_\Omega\sum_i(\lambda^{2p})_i\phi_i. \quad (4.2.21)$$

In particular, if we pick $p=(n-2)/4$, note that since $n\geq 3$, p will not be zero, we obtain the harmonicity of λ^{2p}, i.e.,

$$\int_\Omega\lambda^{\frac{n-2}{2}}\Delta\phi=0. \quad (4.2.22)$$

The Weyl lemma then implies λ is a smooth function. Now it follows from (4.2.11) and a bootstrap argument that f is smooth. The second proof is complete. □

Proof of Theorem 4.2.1. Let us consider (4.2.9) again. We only consider the case $p=-1$. If i,j,k are distinct,

$$\big\langle\lambda^{-2}f_i,(\lambda^{-2}f_j)_k\big\rangle=0. \quad (4.2.23)$$

Differentiating with respect to j gives,

$$\left\langle(\lambda^{-2}f_i)_j, (\lambda^{-2}f_j)_k\right\rangle + \left\langle\lambda^{-2}f_i, (\lambda^{-2}f_j)_{kj}\right\rangle = 0. \tag{4.2.24}$$

Also by (4.2.9),

$$\left\langle\lambda^{-2}f_i, (\lambda^{-2}f_j)_j\right\rangle = \lambda^{-1}(\lambda^{-1})_i. \tag{4.2.25}$$

Differentiating with respect to k gives,

$$\begin{aligned}&\left\langle(\lambda^{-2}f_i)_k, (\lambda^{-2}f_j)_j\right\rangle + \left\langle\lambda^{-2}f_i, (\lambda^{-2}f_j)_{jk}\right\rangle \\ &= (\lambda^{-1})_i(\lambda^{-1})_k + \lambda^{-1}(\lambda^{-1})_{ik}\end{aligned} \tag{4.2.26}$$

Since we are in the smooth setting, $(\lambda^{-2}f_j)_{kj} = (\lambda^{-2}f_j)_{jk}$, taking the difference of (4.2.26) and (4.2.24) gives,

$$\left\langle(\lambda^{-2}f_i)_k, (\lambda^{-2}f_j)_j\right\rangle - \left\langle(\lambda^{-2}f_i)_j, (\lambda^{-2}f_j)_k\right\rangle = (\lambda^{-1})_i(\lambda^{-1})_k + \lambda^{-1}(\lambda^{-1})_{ik} \tag{4.2.27}$$

Now from (4.2.13), we know

$$(\lambda^{-2}f_i)_k = (\lambda^{-1})_k\frac{f_i}{\lambda} - (\lambda^{-1})_i\frac{f_k}{\lambda}, \quad (\lambda^{-2}f_j)_j = \sum_\ell(\lambda^{-1})_\ell\frac{f_\ell}{\lambda},$$

$$(\lambda^{-2}f_i)_j = (\lambda^{-1})_j\frac{f_i}{\lambda} - (\lambda^{-1})_i\frac{f_j}{\lambda}, \quad (\lambda^{-2}f_j)_k = (\lambda^{-1})_k\frac{f_j}{\lambda} - (\lambda^{-1})_j\frac{f_k}{\lambda}.$$

Substituting them into (4.2.27), and using the fact that i, j, k are distinct, thus $\left\langle f_i, f_j\right\rangle = 0$, $\left\langle f_i, f_k\right\rangle = 0$ and $\left\langle f_j, f_k\right\rangle = 0$, we obtain that,

$$(\lambda^{-1})_i(\lambda^{-1})_k = (\lambda^{-1})_i(\lambda^{-1})_k + \lambda^{-1}(\lambda^{-1})_{ik}.$$

Hence,

$$(\lambda^{-1})_{ik} = 0 \quad \text{for} \quad i \neq k. \tag{4.2.28}$$

This is true in any Euclidean coordinate system. That means the quadratic form

$$d^2(\lambda^{-1})(v_1, v_2) = 0$$

whenever we evaluate it on two orthonormal vectors v_1, v_2. Taking

$$v_1 = \frac{e_i + e_k}{\sqrt{2}}, \quad v_2 = \frac{e_i - e_k}{\sqrt{2}},$$

we also obtain

$$0 = d^2(\lambda^{-1})\Big(\frac{e_i + e_k}{\sqrt{2}}, \frac{e_i - e_k}{\sqrt{2}}\Big) = \frac{1}{2}\big((\lambda^{-1})_{ii} - (\lambda^{-1})_{kk}\big). \tag{4.2.29}$$

Thus, (4.2.28) and (4.2.29) imply that

$$(\lambda^{-1})_{ij} = a(x)\delta_{ij}$$

for i, j not necessarily distinct. We claim that $a(x)$ is constant. Indeed, for $i \neq j$,

$$a_j = (\lambda^{-1})_{iij} = (\lambda^{-1})_{iji} = 0.$$

Hence $(\lambda^{-1})_{ij} = a\delta_{ij}$, with a a constant and thus,

$$\lambda^{-1}(x) = \frac{a}{2}\sum_i x_i^2 + \sum_i b_i x_i + c, \quad a, b_i, c \text{ constants.} \tag{4.2.30}$$

We want to prove f is Möbius based on (4.2.30). Pick $i \neq j$ and differentiate (4.2.25) with respect to i, we obtain,

$$\big\langle(\lambda^{-2}f_i)_i, (\lambda^{-2}f_j)_j\big\rangle + \big\langle\lambda^{-2}f_i, (\lambda^{-2}f_j)_{ji}\big\rangle = (\lambda^{-1})_i(\lambda^{-1})_i + \lambda^{-1}(\lambda^{-1})_{ii}. \tag{4.2.31}$$

(4.2.9) gives,

$$\big\langle\lambda^{-2}f_i, (\lambda^{-2}f_j)_i\big\rangle = -(\lambda^{-1})(\lambda^{-1})_j.$$

Differentiating with respect to j gives,

$$\big\langle(\lambda^{-2}f_j)_i, (\lambda^{-2}f_i)_j\big\rangle + \big\langle\lambda^{-2}f_i, (\lambda^{-2}f_j)_{ij}\big\rangle = -(\lambda^{-1})_j(\lambda^{-1})_j - \lambda^{-1}(\lambda^{-1})_{jj}. \tag{4.2.32}$$

Subtracting (4.2.32) from (4.2.31), together with the fact that $(\lambda^{-1})_{ii} = (\lambda^{-1})_{jj}$ for all i, j yields,

$$\begin{aligned}
&\big\langle(\lambda^{-2}f_i)_i, (\lambda^{-2}f_j)_j\big\rangle - \big\langle(\lambda^{-2}f_j)_i, (\lambda^{-2}f_i)_j\big\rangle \\
&\qquad = (\lambda^{-1})_i(\lambda^{-1})_i + \lambda^{-1}(\lambda^{-1})_{ii} + (\lambda^{-1})_j(\lambda^{-1})_j + \lambda^{-1}(\lambda^{-1})_{jj} \\
&\qquad\qquad = (\lambda^{-1})_i(\lambda^{-1})_i + (\lambda^{-1})_j(\lambda^{-1})_j + 2\lambda^{-1}(\lambda^{-1})_{ii}.
\end{aligned} \tag{4.2.33}$$

We know from (4.2.13) that

$$(\lambda^{-2}f_i)_i = (\lambda^{-2}f_j)_j = \sum_\ell (\lambda^{-1})_\ell \frac{f_\ell}{\lambda},$$

$$(\lambda^{-2}f_i)_j = (\lambda^{-1})_j \frac{f_i}{\lambda} - (\lambda^{-1})_i \frac{f_j}{\lambda}, \quad (\lambda^{-2}f_j)_i = (\lambda^{-1})_i \frac{f_j}{\lambda} - (\lambda^{-1})_j \frac{f_i}{\lambda}.$$

Put them back into (4.2.33) and use the fact that $\langle f_i, f_j \rangle = 0$ for $i \neq j$, we obtain,

$$\sum_\ell (\lambda^{-1})_\ell (\lambda^{-1})_\ell = 2\lambda^{-1}(\lambda^{-1})_{ii}. \tag{4.2.34}$$

Coming back to (4.2.30), we first consider the case $a \neq 0$. In this case rewrite λ^{-1} as

$$\lambda^{-1}(x) = \frac{a}{2}|x - x_0|^2 + k, \quad k \text{ constant}, \quad x_0 \in \mathbb{R}^n. \tag{4.2.35}$$

We want to show $k = 0$. Substitute this expression for λ^{-1} into (4.2.34),

$$a^2 \sum_\ell (x_\ell - x_{0_\ell})^2 = 2(\frac{a}{2}|x - x_0|^2 + k) \cdot a. \tag{4.2.36}$$

Hence $a \neq 0$ implies $k = 0$ which is what we wanted to show.

Now we consider inversion:

$$g(x) = \frac{x - x_0}{|x - x_0|^2} + x_0.$$

It is easy to check that its coefficient of conformality is $1/|x - x_0|^2$. Clearly the coefficient of conformality of f^{-1} at $f(x)$ is

$$\lambda^{-1}(x) = \frac{a}{2}|x - x_0|^2.$$

Hence $h = g \circ f^{-1}$ is a conformal transformation whose coefficient of comformality at $f(x)$ is

$$\frac{a}{2}|x - x_0|^2 \frac{1}{|x - x_0|^2} = \frac{a}{2}$$

which is constant. Therefore, h is an isometry followed by a dilation. Hence $f = h^{-1} \circ g$ is an inversion followed by a dilatation, followed by an isometry.

For the case $a = 0$, from (4.2.30), $\lambda^{-1}(x) = \sum_i b_i x_i + c$. Substituting this into (4.2.34) yields,

$$\sum_\ell b_\ell^2 = 2(\sum_i b_i x_i + c) \cdot 0 = 0 \tag{4.2.37}$$

Thus λ is just a constant, so f is an isometry, followed by dilatation. The proof is complete.

□

4.3 PRELIMINARY LEMMAS

4.3.1 Notations.

Let $f \in W^{1,\max\{1,2p\}}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a weak solution to the Cauchy-Riemann System, i.e., f which satisfies (4.1.1) and (4.1.2).

Without loss of generality, we always assume f is sense preserving, i.e.,

$$J(x,f) \geq 0 \quad \text{a.e.}$$

For a matrix A, we define its norm as the *Hilbert Schmidt norm*:

$$|A| = \sqrt{\mathrm{trA^TA}}\,.$$

It is then easy to see from the conformality of f that

$$\frac{1}{\sqrt{n}}|Df(x)| = J(x,f)^{\frac{1}{n}} \text{ a.e.}$$

We define

$$\lambda(x) := \frac{1}{\sqrt{n}}|Df(x)| = J(x,f)^{\frac{1}{n}}$$

and

$$\rho(x) := \lambda^p(x)$$

Let $f_i = \frac{\partial f}{\partial x_i}$, f_i is also the ith column of Df. Relation (4.1.1) implies

$$|f_i(x)| = \lambda(x) = \rho^{\frac{1}{p}}(x) \quad \text{for } i = 1, ..., n. \tag{4.3.1}$$

where $|\cdot|$ is the usual vector norm for vector functions. For the case $0 < p < 1$, we define

$$\rho^{\frac{p-1}{p}} f_i(x) = 0 \quad \text{whenever} \quad \rho(x) = 0.$$

Then for any $p > 0$, we have,

$$\left|\rho^{\frac{p-1}{p}} f_i(x)\right| = \rho(x),$$

and from the assumption on f,

$$\rho^{\frac{p-1}{p}} f_i \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \quad \text{for } i = 1, ..., n.$$

For each $x \in \Omega$, let

$$R(x) := \mathrm{Cof}\big(Df(x)\big)$$

be the cofactor matrix. That is, R satisfies the pointwise relation,

$$(Df(x))^T R(x) = J(x, f) \cdot \mathrm{I}\,.$$

The assumption that f is conformal and $J(x, f) \geq 0$ implies

$$R(x) = \lambda^{n-2}(x) Df(x) = \rho^{\frac{n-2}{p}}(x) Df(x) \quad \text{a.e.}$$

For $p < n - 1$, we define

$$\frac{R(x)}{\rho^{(n-p-1)/p}(x)} = 0 \quad \text{whenever} \quad \rho(x) = 0.$$

It is easy to see that

$$\rho^{\frac{p-1}{p}}(x) Df(x) = \frac{R(x)}{\rho^{(n-p-1)/p}(x)} \quad \text{a.e.} \tag{4.3.2}$$

Write λ^ϵ, ρ^ϵ and f^ϵ as the convolution of λ, ρ and f with the standard mollifier with parameter ϵ. Put

$$R^\epsilon(x) = \mathrm{Cof}\big(Df^\epsilon(x)\big).$$

Note that R^ϵ is *not* the convolution of R. Actually, R may not even be integrable.

4.3.2 Approximating fractional functions by smooth mappings.

Let f, Df, R, λ, ρ; Df^ϵ, R^ϵ, λ^ϵ, and ρ^ϵ be as defined in the previous section. Since $\lambda \geq 0$, $\rho \geq 0$, λ^ϵ and ρ^ϵ are nonnegative as well. We want to approximate $\rho^{\frac{p-1}{p}} Df$ by $(\rho^\epsilon)^{\frac{p-1}{p}} Df^\epsilon$. This is not obvious since $p-1$ may be negative so we have fractional functions. We have to prove it under more careful estimates.

Lemma 4.3.1 *Let Df, R, λ, ρ; Df^ϵ, R^ϵ, λ^ϵ, and ρ^ϵ be as above. We have the following,*

1. *For $p \geq 1$, $(\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} Df^\epsilon \to \rho^{\frac{p-1}{p}} Df$ in $L^2_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ as $\epsilon \to 0$.*
2. *For $p \geq 1$, $\frac{R^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}} \to \rho^{\frac{p-1}{p}} Df$ in $L^2_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ as $\epsilon \to 0$.*
3. *For $0 < p < 1$, $(\lambda^\epsilon + \epsilon)^{p-1} Df^\epsilon \to \rho^{\frac{p-1}{p}} Df$ in $L^2_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ as $\epsilon \to 0$.*
4. *For $0 < p < 1$, $\frac{R^\epsilon}{(\lambda^\epsilon+\epsilon)^{n-p-1}} \to \rho^{\frac{p-1}{p}} Df$ in $L^2_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$ as $\epsilon \to 0$.*

Remark 4.3.1 *Case (1) and (2) are still true if we replace $(\rho^\epsilon)^{1/p}$ by λ^ϵ. However, they are not needed in later proofs. On the other hand, case (4) and (3) may not be true if we replace λ^ϵ by $(\rho^\epsilon)^{1/p}$. The reason is stated in the proof.*

Proof. Case (1) is easy. In the proof based on the Hölder's inequality one needs to use the estimate

$$\left|x^{\frac{p-1}{p}} - y^{\frac{p-1}{p}}\right| \leq |x-y|^{\frac{p-1}{p}} \quad \text{for} \quad x, y \geq 0.$$

We start by considering case (2). It is enough to consider the case $p < n-1$. For the case $p \geq n-1$ we can prove directly using Hölder's inequality because we do not have to deal with fractions.

Since R^ϵ is homogeneous polynomial of degree $n-1$ of the partials $\partial (f^\epsilon)^i / \partial x_j$, we have, for each $x \in \Omega$,

$$\frac{|R^\epsilon|}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}(x) \leq C(n) \frac{|Df^\epsilon|^{n-1}}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}(x) = C|Df^\epsilon|^p \frac{|Df^\epsilon|^{n-1-p}}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}(x),$$

where C is some constant which may vary even in the same line. By the definition of λ, ρ, (4.3.1) and Hölder's inequality (because $p \geq 1$), we have the following pointwise inequality,

$$
\begin{aligned}
|Df^\epsilon(x)| = \left|\int_{B(x,\epsilon)} \varphi_\epsilon(x-y) Df(y) dy\right| &\leq \int_{B(x,\epsilon)} \varphi_\epsilon(x-y)|Df(y)| dy \\
= \sqrt{n}\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\lambda(y) dy &\leq \sqrt{n}\Big(\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\rho(y) dy\Big)^{\frac{1}{p}} \\
&= \sqrt{n}\big(\rho^\epsilon(x)\big)^{\frac{1}{p}} \leq \sqrt{n}\big(\rho^\epsilon(x)+\epsilon\big)^{\frac{1}{p}}. \quad (4.3.3)
\end{aligned}
$$

Hence, for each $x \in \Omega$,

$$
|Df^\epsilon|^p \frac{|Df^\epsilon|^{n-1-p}}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}(x) \leq C(n,p)|Df^\epsilon(x)|^p.
$$

Together we obtain the pointwise inequality,

$$
\left|\frac{R^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}\right|^2 \leq C|Df^\epsilon|^{2p} \tag{4.3.4}
$$

Note that for any compact subset K of Ω, $|Df^\epsilon|^{2p} \to |Df|^{2p}$ in $L^1(K)$. On the other hand,

$$
\frac{R^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}} \to \frac{R}{\rho^{(n-p-1)/p}} \text{ a.e.}
$$

Indeed, for a.e. x, we have $R^\epsilon(x) \to R(x)$, $(\rho^\epsilon+\epsilon)^{(n-p-1)/p}(x) \to \rho^{(n-p-1)/p}(x)$, $|Df^\epsilon(x)|^p \to |Df(x)|^p$. We pick one such x. If $\rho(x) > 0$, then the pointwise convergence is obvious. When $\rho(x) = 0$, recall that in the previous section we defined $\rho^{(p+1-n)/p}(x)R(x)$ to be zero. Hence by (4.3.4),

$$
\begin{aligned}
\left|\frac{R^\epsilon(x)}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}(x)} - \frac{R(x)}{\rho^{(n-p-1)/p}(x)}\right| &= \left|\frac{R^\epsilon(x)}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}(x)}\right| \\
&\leq C|Df^\epsilon(x)|^p \to C|Df(x)|^p = C\rho(x) = 0.
\end{aligned}
$$

We have the following version of the Dominated Convergence Theorem ([13], Proposition 10.1c),

Lemma 4.3.2 *Let $|h_k| \leq g_k$, $g_k \to g$ in L^1 and $h_k \to h$ a.e. Then $\int h_k \to \int h$.*

Lemma 4.3.2 then gives

$$\left\| \frac{R^\epsilon}{(\rho^\epsilon + \epsilon)^{(n-p-1)/p}} \right\|_{L^2(K)} \to \left\| \frac{R}{\rho^{(n-p-1)/p}} \right\|_{L^2(K)}.$$

It follows from [13], Proposition 7.1c that norm convergence and a.e. convergence imply convergence in L^2, that is,

$$\frac{R^\epsilon}{(\rho^\epsilon + \epsilon)^{(n-p-1)/p}} \to \frac{R}{\rho^{(n-p-1)/p}} \quad \text{in} \quad L^2_{\text{loc}}(\Omega).$$

The proves case (2).

In the cases (4) and (3), we basically follow the same argument as in the case (2). The only difference in (4.3.3) is that for $p < 1$,

$$\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\lambda(y)dy \leq \left(\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\rho(y)dy \right)^{\frac{1}{p}}$$

is no longer true. However, the argument in (4.3.3) still gives us for $1 \leq i \leq n$,

$$|Df^\epsilon(x)| \leq C(n)\lambda^\epsilon(x) \text{ for each } x. \tag{4.3.5}$$

Therefore, for each $x \in \Omega$, in case (4),

$$|(\lambda^\epsilon + \epsilon)^{p-1} Df^\epsilon(x)|^2 = \left| (\lambda^\epsilon + \epsilon)^p \frac{Df^\epsilon}{\lambda^\epsilon + \epsilon}(x) \right|^2 \leq C(n)(\lambda^\epsilon + \epsilon)^{2p}(x),$$

and in case (3)

$$\left| \frac{R^\epsilon}{(\lambda^\epsilon + \epsilon)^{n-p-1}}(x) \right|^2 \leq C(n) \Big(|Df^\epsilon|^p \frac{|Df^\epsilon|^{n-1-p}}{(\lambda^\epsilon + \epsilon)^{n-p-1}}(x) \Big)^2 \leq C(n,p)|Df^\epsilon(x)|^{2p}.$$

The rest follows from the same Dominated Convergence argument. The proof is complete. □

By a similar argument, we prove in advance here a lemma that will be used in the proof of Theorem 4.1.1 later,

Lemma 4.3.3 *Let f and ρ be as above. Suppose further $\rho = |Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ for either $p \geq 1/2$ or $p \geq (n-2)/4$ and $n \geq 3$. Then for any $\phi \in C_0^\infty(\Omega)$,*

$$\int_\Omega \rho^{\frac{p-1}{p}} f_i \phi_j - \rho^{\frac{p-1}{p}} f_j \phi_i = \frac{p-1}{p} \int_\Omega \Big(\rho_i \frac{f_j}{\rho^{1/p}} - \rho_j \frac{f_i}{\rho^{1/p}} \Big) \phi. \tag{4.3.6}$$

where $\cdot_j$ denotes the partial derivative in the weak sense with respect to x_j.

Remark 4.3.2 *The right hand side of (4.3.6) is well defined since $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ and $|f_i| = \rho^{1/p}$ for all i. Further, the weak partial derivative $\rho_j = 0$ a.e. in the set where $\rho = 0$, thus the value of $f_i/\rho^{\frac{1}{p}}(x)$ when $\rho(x) = 0$ does not matter and we simply define this to be e_i.*

Remark 4.3.3 *The above lemma holds for any general mapping satisfying the required regularity conditions. No conformal structure is needed.*

Proof. We again first consider the case in all dimensions and $p \geq 1$. $(\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon$ for $i = 1, ..., n$ is smooth. Identity (4.3.6) is obviously true for smooth functions using integration by parts, i.e.

$$\int_\Omega (\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon \phi_j - (\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_j^\epsilon \phi_i = \frac{p-1}{p} \int_\Omega \Big(\rho_i^\epsilon \frac{f_j^\epsilon}{(\rho^\epsilon + \epsilon)^{1/p}} - \rho_j^\epsilon \frac{f_i^\epsilon}{(\rho^\epsilon + \epsilon)^{1/p}} \Big) \phi. \tag{4.3.7}$$

The left hand side converges to the left hand side of (4.3.6) by case (1) of Lemma 4.3.1. Since by (4.3.3), $|f_i^\epsilon| \leq C(n)(\rho^\epsilon)^{\frac{1}{p}}$ and $\rho \in W^{1,2}_{\text{loc}}(\Omega)$, The right hand side also converges to the right hand side of (4.3.6) by the Dominated Convergence Theorem (Lemma 4.3.2).

We now consider the case $1/2 \leq p < 1$ and in all dimensions. $(\lambda^\epsilon + \epsilon)^{p-1} f_i^\epsilon$ is smooth and we can apply integration by parts to get the following identity:

$$\begin{aligned} &\int_\Omega (\lambda^\epsilon + \epsilon)^{p-1} f_i^\epsilon \phi_j - (\lambda^\epsilon + \epsilon)^{p-1} f_j^\epsilon \phi_i \\ &\qquad = \frac{p-1}{p} \int_\Omega \Big(\big((\lambda^\epsilon + \epsilon)^p \big)_i \frac{f_j^\epsilon}{\lambda^\epsilon + \epsilon} - \big((\lambda^\epsilon + \epsilon)^p \big)_j \frac{f_i^\epsilon}{\lambda^\epsilon + \epsilon} \Big) \phi. \end{aligned} \tag{4.3.8}$$

The left hand side converges to the left hand side of (4.3.6) by case (4) of Lemma 4.3.1. For the right hand side, we claim that for each $x \in \Omega$,

$$\big| \big((\lambda^\epsilon + \epsilon)^p \big)_j (x) \big|^2 \leq \big(|\rho_j|^2 \big)^\epsilon (x), \tag{4.3.9}$$

where $\left(|\rho_j|^2\right)^\epsilon$ is the convolution of $|\rho_j|^2 \in L^1_{\text{loc}}$ with the standard mollifier.

Indeed, since $1/2 \le p < 1$, $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ implies $\lambda \in W^{1,1}_{\text{loc}}(\Omega)$ and its weak derivative equals,

$$\lambda_j = \frac{1}{p}\rho^{\frac{1-p}{p}}\rho_j.$$

Hence, by Hölder's inequality with the assumption that $1 < 1/p \le 2$,

$$\begin{aligned}
\left|\lambda_j^\epsilon(x)\right| = \left|\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\lambda_j(y)dy\right| &= \left|\frac{1}{p}\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\rho^{\frac{1-p}{p}}(y)\rho_j(y)dy\right| \\
\le \frac{1}{p}\left(\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)\rho^{\frac{1}{p}}(y)dy\right)^{1-p} &\left(\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)|\rho_j(y)|^{\frac{1}{p}}dy\right)^p \\
&= \frac{1}{p}(\lambda^\epsilon)^{1-p}(x)\left(\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)|\rho_j(y)|^{\frac{1}{p}}dy\right)^p \\
&\le \frac{1}{p}(\lambda^\epsilon)^{1-p}(x)\left(\int_{B(x,\epsilon)} \varphi_\epsilon(x-y)|\rho_j(y)|^2dy\right)^{\frac{1}{2}} \\
&= \frac{1}{p}(\lambda^\epsilon)^{1-p}(x)\left(\left(|\rho_j|^2\right)^\epsilon(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, for each point in Ω,

$$\left|\left((\lambda^\epsilon+\epsilon)^p\right)_j\right| = p\left|(\lambda^\epsilon+\epsilon)^{p-1}\lambda_j^\epsilon\right| \le \frac{(\lambda^\epsilon)^{1-p}}{(\lambda^\epsilon+\epsilon)^{1-p}}\left(\left(|\rho_j|^2\right)^\epsilon\right)^{\frac{1}{2}} \le \left(\left(|\rho_j|^2\right)^\epsilon\right)^{\frac{1}{2}}.$$

which proves our claim.

Now (4.3.5) and (4.3.9) together give, pointwisely,

$$\left|\left((\lambda^\epsilon+\epsilon)^p\right)_j\frac{f_i^\epsilon}{\lambda^\epsilon+\epsilon}\right|^2 \le C\left(|\rho_j|^2\right)^\epsilon.$$

Observe that $\left(|\rho_j|^2\right)^\epsilon$ converges to $|\rho_j|^2$ in L^1_{loc}. We also have

$$\left((\lambda^\epsilon+\epsilon)^p\right)_j\frac{f_i^\epsilon}{\lambda^\epsilon+\epsilon} \to \left(\lambda^p\right)_j\frac{f_i}{\lambda} = \rho_j\frac{f_i}{\rho^{1/p}} \text{ a.e.}$$

Indeed, for $\lambda(x) > 0$, this is obvious. We know that $\left(\lambda^p\right)_j = 0$ and $\rho_j = 0$ a.e. in the set where $\lambda^p = \rho = 0$. Hence in such a set,

$$\begin{aligned}
\left|\left((\lambda^\epsilon+\epsilon)^p\right)_j\frac{f_i^\epsilon}{\lambda^\epsilon+\epsilon} - \left(\lambda^p\right)_j\frac{f_i}{\lambda}\right| &= \left|\left((\lambda^\epsilon+\epsilon)^p\right)_j\frac{f_i^\epsilon}{\lambda^\epsilon+\epsilon}\right| \\
&\le \left(|\rho_j|^2\right)^\epsilon \to |\rho_j|^2 = 0 \quad \text{a.e.}
\end{aligned}$$

The Dominated Convergence Theorem thus gives us norm convergence, which, together with a.e. convergence yields,

$$\big((\lambda^\epsilon+\epsilon)^p\big)_j \frac{f_i^\epsilon}{\lambda^\epsilon+\epsilon} \to \rho_j \frac{f_i}{\rho^{1/p}} \text{ in } L^2_{\text{loc}}(\Omega) \quad 1 \le i, j \le n. \tag{4.3.10}$$

Thus the right hand side of (4.3.8) converges to the right hand side of (4.3.6).

Finally, we deal with the case $(n-2)/4 \le p < 1$ and $n \ge 3$. Fix $c > 0$, $(\lambda^\epsilon + c)^{p-1} f_i^\epsilon$ is smooth and we can apply integration by parts to get the following identity:

$$\begin{aligned}\int_\Omega (\lambda^\epsilon+c)^{p-1} f_i^\epsilon \phi_j - (\lambda^\epsilon+c)^{p-1} f_j^\epsilon \phi_i \\ = \frac{p-1}{p}\int_\Omega \Big(\big((\lambda^\epsilon+c)^p\big)_i \frac{f_j^\epsilon}{\lambda^\epsilon+c} - \big((\lambda^\epsilon+c)^p\big)_j \frac{f_i^\epsilon}{\lambda^\epsilon+c}\Big)\phi.\end{aligned} \tag{4.3.11}$$

The left hand side converges to

$$\int_\Omega (\rho^{1/p}+c)^{p-1} f_i \phi_j - (\rho^{1/p}+c)^{p-1} f_j \phi_i$$

by Dominated Convergence Theorem and case (4) of Lemma 4.3.1 because for ϵ sufficiently large,

$$(\lambda^\epsilon+c)^{p-1} f_i^\epsilon \le (\lambda^\epsilon+\epsilon)^{p-1} f_i^\epsilon \to \lambda^{p-1} f_i.$$

For the right hand side, note that $\lambda = \rho^{\frac{1}{p}} \in W^{1,1}_{\text{loc}}(\Omega)$ and its weak derivative equals,

$$\lambda_j = \frac{1}{p}\rho^{\frac{1-p}{p}}\rho_j.$$

Indeed, $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ implies $\rho \in L^{2n/(n-2)}_{\text{loc}}(\Omega)$ by Sobolev inequality, so Hölder's inequality and the assumption $3 \le n$ and $(n-2)/4 \le p < 1$ imply $\lambda_j \in L^1_{\text{loc}}$ for all $1 \le j \le n$. Therefore,

$$\Big|\big((\lambda^\epsilon+c)^p\big)_j \frac{f_i^\epsilon}{\lambda^\epsilon+c}\Big| = p\Big|\frac{\lambda_j^\epsilon(x)}{(\lambda^\epsilon+c)^{1-p}}\frac{f_i^\epsilon}{\lambda^\epsilon+c}\Big| \le \frac{p}{c^{1-p}}\lambda_j^\epsilon(x) \to \frac{p}{c^{1-p}}\lambda_j(x) \quad \text{in } L^1_{\text{loc}}.$$

Moreover, since $\lambda^\epsilon + c \ge c > 0$ everywhere and $\lambda + c \ge c > 0$ a.e., easy computation gives,

$$p\frac{\lambda_j^\epsilon(x)}{(\lambda^\epsilon+c)^{1-p}}\frac{f_i^\epsilon}{\lambda^\epsilon+c} \to p\frac{\lambda_j(x)}{(\lambda+c)^{1-p}}\frac{f_i}{\lambda+c} = \frac{\rho^{(1-p)/p}\rho_j}{(\rho^{1/p}+c)^{1-p}}\frac{f_i}{\rho^{1/p}+c} \text{ a.e.}$$

Dominated Convergence Theorem then gives us norm convergence, which, together with a.e. convergence yields,

$$\big((\lambda^\epsilon + c)^p\big)_j \frac{f_i^\epsilon}{\lambda^\epsilon + c} \to \frac{\rho^{(1-p)/p}\rho_j}{(\rho^{1/p}+c)^{1-p}} \frac{f_i}{\rho^{1/p}+c} \text{ in } L^1_{\text{loc}}(\Omega). \tag{4.3.12}$$

Letting $\epsilon \to 0$ in (4.3.11) then gives,

$$\int_\Omega (\rho^{1/p}+c)^{p-1} f_i \phi_j - (\rho^{1/p}+c)^{p-1} f_j \phi_i$$
$$= \frac{p-1}{p} \int_\Omega \Big(\frac{\rho^{(1-p)/p}\rho_i}{(\rho^{1/p}+c)^{1-p}} \frac{f_j}{\rho^{1/p}+c} - \frac{\rho^{(1-p)/p}\rho_j}{(\rho^{1/p}+c)^{1-p}} \frac{f_i}{\rho^{1/p}+c} \Big) \phi.$$

Finally since $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ and $\rho_j = 0$ whenever $\rho = 0$, Dominated Convergence Theorem allows us to let $c \to 0$ to obtain (4.3.6). The proof is complete. □

4.4 PROOF OF THE MAIN THEOREMS

4.4.1 Motivation.

We would like to mimic the second proof in the C^2 case for the Liouville Theorem. Under the assumption $f \in W^{1,\max\{1,2p\}}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $|Df|^{2p}$ is integrable and we want to show it is harmonic or subharmonic. In the C^2 case, (4.2.21) exactly proves that $|Df|^{2p}$ is harmonic or subharmonic for p sufficiently large. However, the key identities to obtain (4.2.21) are (4.2.11) and (4.2.12). Actually these are the key identities for any proof of the Liouville Theorem. These identities are obtained by differentiating the Cauchy Riemann system and permuting indices. Hence to mimic the proof, we at least need f to be, loosely speaking, under some second order differentiability condition. Therefore, the first step toward proving the Liouville theorem would be to prove the second order differentiability of f.

When we investigate (4.2.21), it is not difficult to observe that this is a Caccioppoli type estimate. This suggests that the conformal system itself, is related to some Caccioppoli type estimate that will give us second order differentiability. However, (4.2.21) seems to indicate that such a Caccioppoli estimate holds for any p, which is apparently not true since we have counter examples (Example 4.5.2) for $p < n/4$. The reason is that in order to get this estimate, we already assume second order differentiability of f, which has ruled out some p like those in the counter example. But still, it gives us strong hints that Caccioppoli estimates would be the approach based on such conformal structure, even though the role of $p \geq n/4$ is unclear in (4.2.21).

Let us investigate the conformal system again. As mentioned in Section 3.1, the Cauchy Riemann system and Jacobian is nonnegative imply that the cofactor matrix $R = \mathrm{Cof}(Df)$ satisfies,

$$R = \lambda^{n-2} Df \quad \text{a.e.},$$

where $\lambda = n^{-1/2}|Df|$ is as before. If f is C^2, direct computation shows it satisfies the divergence free condition:

$$\sum_i (R_i)_i = \sum_i \frac{\partial R_i}{\partial x_i} = 0,$$

where R_i is the ith column of R. If we differentiate $\lambda^{(p+1-n)} R = \lambda^{(p-1)} Df$ using the product rule, we have,

$$\sum_i (\lambda^{p-1} f_i)_i = \sum_i \left(\frac{R_i}{\lambda^{n-1-p}} \right)_i = \frac{p+1-n}{p} \sum_i (\lambda^p)_i \frac{R_i}{\lambda^{n-1}} = \frac{p+1-n}{p} \sum_i (\lambda^p)_i \frac{f_i}{\lambda},$$

which is exactly (4.2.12). The difference is that to obtain (4.2.12), we needed to assume second differentiability of f so that we could differentiate and permute the Cauchy Riemann system, while in the above approach, we can approximate R by smooth cofactor matrix and obtain the above equality under only the assumption that λ^p, which is the norm of Df to the power of p, is weakly differentiable. This builds a bridge for us to go beyond the assumption that f has to be second differentiable. We can proceed by only assuming $|Df|^p$ is weakly differentiable. More importantly, it justifies why Jacobian does not change sign is essential in the Liouville theorem. Note that the Liouville theorem is not true if we allow Jacobian to change sign. For example, $f(x) = (x_1, ..., x_n)$ if $x_n \geq 0$ and $f(x) = (x_1, ..., -x_n)$ if $x_n < 0$ satisfies (4.1.1), but it is not Möbius.

Indeed, we prove in Theorem 4.4.1 that $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ implies $|Df|^{p-1}Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. Furthermore, the estimate has a critical coefficient (see (4.4.1)) that suggests the important role of $p = n/4$ (Theorem 4.1.2): we obtain a Caccioppoli estimate for $|Df|^{p-1}Df$ for $p > n/4$ and the subharmonicity of $|Df|^{2p}$ for $p \geq n/4$. As mentioned in Section 1, the Liouville will follow from the subharmonicity of $|Df|^{2p}$.

It is also mentioned in Section 1 that although the role of $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ does not appear in (4.1.3) and (4.1.4), it is needed in obtaining the critical coefficient. The reason is that $\langle f_i(x), f_j(x)\rangle = 0$ for $i \neq j$ so we have a lot of cancellation to keep the coefficient small. Indeed, it cannot be any smaller since we have equality in (4.4.1). However, it only vanishes when f_i and f_j are at the same point. If we estimate by any kind of difference quotient or approximation, before passing to the limit, we are unable to keep f_i and f_j at the same point so their dot product does not vanish–that is why we need to assume $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ so that we can pass to the limit. We hope that it is just a technical assumption and can be removed under better approximation.

4.4.2 Proof of the main theorems.

We need the following theorem to prove Theorem 4.1.2.

Theorem 4.4.1 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies (4.1.1) and (4.1.2). Assume further that $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ for either $p \geq 1/2$ or $p \geq (n-2)/4$ and $n \geq 3$. Then $|Df|^{p-1}Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. Moreover, for any $\phi \in C_0^\infty(\Omega)$, we have the equality,*

$$\begin{aligned}&\sum_{i,j=1}^n \int_\Omega \Big|(|Df|^{p-1}f_i)_j(x)\Big|^2 \phi(x)\,dx \qquad (4.4.1)\\ &\quad= \Big(1 - \frac{(n-1)(4p-n)}{np^2}\Big)\int_\Omega |\nabla |Df|^p(x)|^2\phi(x)\,dx + \frac{n-1}{pn}\int_\Omega |Df(x)|^{2p}\Delta\phi(x)\,dx.\end{aligned}$$

Let us first show how to prove Theorem 4.1.2 from the above theorem. Let $\rho = \big(n^{-1/2}|Df|\big)^p$. Note that $\rho(x) = \big|\rho^{\frac{p-1}{p}} f_i(x)\big|$ for a.e. x and for each $i = 1, ..., n$ since f is conformal. Thus,

$$|\rho_j(x)| \leq \big|(\rho^{\frac{p-1}{p}} f_i)_j(x)\big| \quad \text{for all } 1 \leq i,j \leq n \qquad (4.4.2)$$

Actually, (4.4.2) is true for scalar function if we replace inequality by equality. However, in our case when $\rho^{\frac{p-1}{p}} f_i$ is a vector valued function, we only have inequality. (4.4.2) implies,

$$n|\nabla\rho|^2 = n\sum_j |\rho_j(x)|^2 \leq \sum_{i,j} \big|(\rho^{\frac{p-1}{p}} f_i)_j(x)\big|^2.$$

Substituting $\rho = \big(n^{-1/2}|Df|\big)^p$ we have,

$$\big|\nabla|Df|^p\big|^2 \leq \sum_{i,j} \big|(|Df|^{p-1} f_i)_j(x)\big|^2.$$

Observe that

$$1 - \frac{(n-1)(4p-n)}{np^2} = \frac{(p+1-n)^2 + (n-1)(p-1)^2}{np^2} \geq 0.$$

Thus, for $\phi \geq 0$, (4.4.1) gives,

$$\sum_{i,j=1}^n \int_\Omega \big|(|Df|^{p-1} f_i)_j(x)\big|^2 \phi(x)\,dx \tag{4.4.3}$$

$$\leq \Big(1 - \frac{(n-1)(4p-n)}{np^2}\Big) \sum_{i,j=1}^n \int_\Omega \big|(|Df|^{p-1} f_i)_j(x)\big|^2 \phi(x)\,dx \tag{4.4.4}$$

$$+\frac{n-1}{pn} \int_\Omega |Df(x)|^{2p} \Delta\phi(x)\,dx$$

Hence, for $p \geq n/4$, the first term on the right hand side of (4.4.3) is less than or equal to the term on the left hand side, which yields,

$$\int_\Omega |Df(x)|^{2p} \Delta\phi(x)\,dx \geq 0.$$

Furthermore, for $p > n/4$, the first term on the right hand side of (4.4.3) is strictly less than the term on the left hand side. Hence,

$$\sum_{i,j=1}^n \int_\Omega \big|(|Df|^{p-1} f_i)_j(x)\big|^2 \phi(x)\,dx \leq C(n,p) \int_\Omega |Df(x)|^{2p} \Delta\phi(x)\,dx \tag{4.4.5}$$

Also note that (4.4.5) is true if we replace the test function ϕ by ϕ^2, and

$$\Big|\int_\Omega |Df(x)|^{2p} \Delta\phi^2(x)\,dx\Big| = \Big|\sum_i 4 \int_\Omega |Df(x)|^p \big(|Df|^p\big)_i(x)\phi(x)\phi_i(x)\,dx\Big|$$

$$\leq 4\Big(\int_\Omega \sum_{i,j} \big|(|Df|^{p-1} f_i)_j(x)\big|^2 \phi^2(x)\,dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |Df(x)|^{2p} \big|\nabla\phi(x)\big|^2\,dx\Big)^{\frac{1}{2}}$$

Thus, (4.1.3) follows for $p > n/4$ from dividing both sides of (4.4.5) the term

$$\left(\int_\Omega \sum_{i,j} \left|(|Df|^{p-1} f_i)_j(x)\right|^2 \phi^2(x)\,dx\right)^{\frac{1}{2}}$$

The proof is complete. $\square$

Proof of Theorem 4.4.1. Throughout the entire proof, unless specified (indeed there will be one exceptional case which occurs to cofactor matrix), $(\cdot)_j$ will always denote the partial derivative with respect to x_j of the expressions inside the parenthesis (we use $\cdot_j$ for a single term). Also, unless specified, $(\cdot)^\epsilon$ will always denote the convolution of the terms inside the parenthesis (we use $\cdot^\epsilon$ for a single term) with the standard mollifier with parameter ϵ.

We first consider the case $p \geq 1$. Again let $\rho = \left(n^{-1/2}|Df|\right)^p$. Let $\left(\rho^{\frac{p-1}{p}} f_i\right)^h$ be the convolution of $\rho^{\frac{p-1}{p}} f_i$ with the standard mollifier with parameter h, i.e.

$$\left(\rho^{\frac{p-1}{p}} f_i\right)^h(x) = \int_{B(x,h)} \varphi_h(x-y)\rho^{\frac{p-1}{p}} f_i(y)dy,$$

and let $\left(\rho^{\frac{p-1}{p}} f_i\right)^h_j$ be its partial derivative with respect to x_j. We want to show all such terms for indices $1 \leq i, j \leq n$ are bounded sequences (with respect to h) in L^2_{loc}. This will imply $\rho^{\frac{p-1}{p}} Df \in W^{1,2}_{\text{loc}}$.

To begin, for any $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, we switch indices i and j to obtain,

$$\begin{aligned}
&\sum_{i,j} \int_\Omega \left|\left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x)\right|^2 \phi(x)\,dx \qquad (4.4.6)\\
&= \sum_{i,j} \int_\Omega \left\langle \left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x), \left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x)\right\rangle \phi(x)\,dx\\
&= \sum_{i,j} \int_\Omega \left\langle \left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x), \left(\rho^{\frac{p-1}{p}} f_j\right)^h_i(x)\right\rangle \phi(x)\,dx\\
&+ \sum_{i,j} \int_\Omega \left\langle \left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x), \left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x) - \left(\rho^{\frac{p-1}{p}} f_j\right)^h_i(x)\right\rangle \phi(x)\,dx\\
&= \sum_{i,j} \int_\Omega \left\langle \left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x), \left(\rho^{\frac{p-1}{p}} f_j\right)^h_i(x)\right\rangle \phi(x)\,dx\\
&+ \frac{1}{2} \sum_{i,j} \int_\Omega \left|\left(\rho^{\frac{p-1}{p}} f_i\right)^h_j(x) - \left(\rho^{\frac{p-1}{p}} f_j\right)^h_i(x)\right|^2 \phi(x)\,dx,
\end{aligned}$$

Let

$$I_1 = \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)_i^h(x) \Big\rangle \phi(x)\, dx, \tag{4.4.7}$$

and

$$I_2 = \frac{1}{2} \sum_{i,j} \int_\Omega \Big| \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x) - \big(\rho^{\frac{p-1}{p}} f_j\big)_i^h(x) \Big|^2 \phi(x)\, dx. \tag{4.4.8}$$

We estimate I_2 first. By Lemma 4.3.1,

$$(\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon \to \rho^{\frac{p-1}{p}} f_i \quad \text{in} \quad L^2_{\text{loc}}(\Omega) \quad \text{for} \quad 1 \le i \le n.$$

Thus for each fixed sufficiently small h, the *double* convolution,

$$\begin{aligned} &\int_\Omega \Big| \big((\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon\big)_j^h(x) - \big((\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_j^\epsilon\big)_i^h(x) \Big|^2 \phi(x)\, dx \\ &\qquad \to \int_\Omega \Big| \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x) - \big(\rho^{\frac{p-1}{p}} f_j\big)_i^h(x) \Big|^2 \phi(x)\, dx \quad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{4.4.9}$$

Indeed, if we write out the expression for convolution, we have,

$$\begin{aligned} &\int_\Omega \Big| \big((\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon\big)_j^h(x) - \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x) \Big|^2 \phi(x)\, dx \\ &= \int_\Omega \Big| \int_{B(0,h)} (\varphi_h)_j(y) \Big((\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon - \rho^{\frac{p-1}{p}} f_i \Big)(x-y)\, dy \Big|^2 \phi(x)\, dx \\ &\le \big\| (\varphi_h)_j \big\|_\infty^2 \int_\Omega \Big| \int_{B(x,h)} \big| (\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon(y) - \rho^{\frac{p-1}{p}} f_i(y) \big|\, dy \Big|^2 |\phi(x)|\, dx \\ &\le \big\| (\varphi_h)_j \big\|_\infty^2 \|\phi\|_1 \Big(\int_{(\operatorname{supp} \phi)_h} \big| (\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon(x) - \rho^{\frac{p-1}{p}} f_i(x) \big|\, dx \Big)^2 \\ &\to 0, \end{aligned}$$

where $(\operatorname{supp} \phi)_h$ is the h neighborhood of $\operatorname{supp} \phi$. We need h to be small enough so that $(\operatorname{supp} \phi)_h$ is contained in a compact subset of Ω. Then (4.4.9) follows from the triangle inequality.

Note that since $(\rho^\epsilon+\epsilon)^{\frac{p-1}{p}} f_i^\epsilon$ is smooth, we can differentiate $(\rho^\epsilon+\epsilon)^{\frac{p-1}{p}} f_i^\epsilon$ inside the convolution to obtain,

$$\begin{aligned}\int_\Omega \Big| \big((\rho^\epsilon+\epsilon)^{\frac{p-1}{p}} f_i^\epsilon\big)_j^h(x) - \big((\rho^\epsilon+\epsilon)^{\frac{p-1}{p}} f_j^\epsilon\big)_i^h(x)\Big|^2 \phi(x)\,dx \\ = \int_\Omega \Big| \Big(\big((\rho^\epsilon+\epsilon)^{\frac{p-1}{p}} f_i^\epsilon\big)_j - \big((\rho^\epsilon+\epsilon)^{\frac{p-1}{p}} f_j^\epsilon\big)_i \Big)^h(x)\Big|^2 \phi(x)\,dx \\ = \frac{(p-1)^2}{p^2} \int_\Omega \Big| \Big(\rho_j^\epsilon \frac{f_i^\epsilon}{(\rho^\epsilon+\epsilon)^{1/p}} - \rho_i^\epsilon \frac{f_j^\epsilon}{(\rho^\epsilon+\epsilon)^{1/p}} \Big)^h(x)\Big|^2 \phi(x)\,dx. \quad (4.4.10)\end{aligned}$$

For each x, by (4.3.3) in the proof of Lemma 4.3.1,

$$\Big| \rho_j^\epsilon \frac{f_i^\epsilon}{(\rho^\epsilon+\epsilon)^{1/p}}(x) \Big| \le C|\rho_j^\epsilon(x)|.$$

Since $\rho \in W^{1,2}_{\text{loc}}(\Omega)$, its convolution $\rho_j^\epsilon \to \rho_j$ in L^2_{loc} for all $1 \le j \le n$. We can conclude from the Dominated Convergence Theorem that

$$\rho_j^\epsilon \frac{f_i^\epsilon}{(\rho^\epsilon+\epsilon)^{1/p}} \to \rho_j \frac{f_i}{\rho^{1/p}} \quad \text{in} \quad L^2_{\text{loc}}(\Omega) \quad \text{for all} \quad i,j. \quad (4.4.11)$$

Note that the same argument has been used in the proof of Lemma 4.3.3.

Combining (4.4.9), (4.4.10) and (4.4.11) we can pass $\epsilon \to 0$ to obtain,

$$I_2 = \frac{1}{2}\frac{(p-1)^2}{p^2} \sum_{i,j} \int_\Omega \Big| \Big(\rho_j \frac{f_i}{\rho^{1/p}} - \rho_i \frac{f_j}{\rho^{1/p}} \Big)^h(x) \Big|^2 \phi(x)\,dx. \quad (4.4.12)$$

We now consider I_1. Using the product rule and integration by part twice,

$$
\begin{aligned}
I_1 &= \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)_i^h(x) \Big\rangle \phi(x)\, dx \\
&= \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \Big(\big(\rho^{\frac{p-1}{p}} f_j\big)^h(x)\phi(x)\Big)_i \Big\rangle\, dx \\
&- \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_i(x)\, dx \\
&= \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \Big(\big(\rho^{\frac{p-1}{p}} f_j\big)^h(x)\phi(x)\Big)_j \Big\rangle\, dx \\
&- \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_i(x)\, dx \\
&= \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)_j^h(x) \Big\rangle \phi(x)\, dx \\
&+ \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_j(x)\, dx \\
&- \sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_i(x)\, dx.
\end{aligned}
\tag{4.4.13}
$$

We consider each term separately. We first consider

$$
\sum_{i,j} \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)_j^h(x) \Big\rangle \phi(x)\, dx = \int_\Omega \Big| \sum_i \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x) \Big|^2 \phi(x)\, dx. \tag{4.4.14}
$$

Again, denote $R^\epsilon(x) = \mathrm{Cof}(Df^\epsilon(x))$ and R_i^ϵ is its ith column. Note that this is the *only* exception to our notations for derivatives and convolution with the standard mollifier. By Lemma 4.3.1,

$$
\frac{R_i^\epsilon}{(\rho^\epsilon + \epsilon)^{(n-p-1)/p}} \to \rho^{\frac{p-1}{p}} f_i \quad \text{in} \quad L^2_{\mathrm{loc}}(\Omega) \quad \text{for} \quad 1 \le i \le n.
$$

By the same argument as in (4.4.9),

$$
\begin{aligned}
&\int_\Omega \Big| \sum_i \Big(\frac{R_i^\epsilon}{(\rho^\epsilon + \epsilon)^{(n-p-1)/p}} \Big)_i^h(x) \Big|^2 \phi(x)\, dx \\
&\qquad \to \int_\Omega \Big| \sum_i \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x) \Big|^2 \phi(x)\, dx \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
\tag{4.4.15}
$$

Recall that smooth cofactor matrix R^ϵ satisfies the divergence free condition, that is

$$\sum_i (R_i^\epsilon)_i(x) = 0 \quad \text{for all } x.$$

Hence, for all x,

$$\begin{aligned}\sum_i \Big(\frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}\Big)_i^h(x) &= \Big(\sum_i \Big(\frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}\Big)_i\Big)^h(x) \\ &= \frac{(p+1-n)}{p}\Big(\sum_i \rho_i^\epsilon \frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-1)/p}}\Big)^h(x). \quad (4.4.16)\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega \Big|\sum_i \Big(\frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}}\Big)_i^h(x)\Big|^2 \phi(x)\,dx \\ &\qquad = \frac{(p+1-n)^2}{p^2}\int_\Omega \Big|\Big(\sum_i \rho_i^\epsilon \frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-1)/p}}\Big)^h(x)\Big|^2 \phi(x)\,dx. \quad (4.4.17)\end{aligned}$$

For each x, we have

$$\Big|\rho_i^\epsilon \frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-1)/p}}(x)\Big| \le C|\rho_i^\epsilon(x)|.$$

Since $\rho \in W^{1,2}_{\text{loc}}(\Omega)$, its convolution $\rho_i^\epsilon \to \rho_i$ in L^2_{loc} for all $1 \le i \le n$. We can conclude from the Dominated Convergence Theorem that

$$\rho_i^\epsilon \frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-1)/p}} \to \rho_i \frac{R_i}{\rho^{(n-1)/p}} = \rho_i \frac{f_i}{\rho^{1/p}} \quad \text{in} \quad L^2_{\text{loc}}(\Omega) \quad \text{for all} \quad i. \quad (4.4.18)$$

Combining (4.4.14), (4.4.15), (4.4.17), and (4.4.18), after passing to the limit as $\epsilon \to 0$ we obtain,

$$\begin{aligned}&\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)_j^h(x)\Big\rangle \phi(x)\,dx \\ &\qquad = \frac{(p+1-n)^2}{p^2}\int_\Omega \Big|\Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)^h(x)\Big|^2 \phi(x)\,dx. \quad (4.4.19)\end{aligned}$$

We are left with the estimate for the remaining two terms in I_1. A similar argument as in (4.4.15) shows that

$$\int_\Omega \Big\langle \Big(\frac{R_i^\epsilon}{(\rho^\epsilon+\epsilon)^{(n-p-1)/p}} \Big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_j(x)\,dx \to \int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_j(x)\,dx \quad 1\le i,j\le n \quad \text{as} \quad \epsilon\to 0. \tag{4.4.20}$$

Switching indices i and j in the first step and applying the above convergence along with (4.4.16) and (4.4.18) in the last step gives,

$$\begin{aligned}
&\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_j(x)\,dx \\
&-\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_i(x)\,dx \\
&=\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_j(x)\,dx \\
&-\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_j\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_i\big)^h(x) \Big\rangle \phi_j(x)\,dx \\
&=2\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_i\big)_i^h(x), \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x) \Big\rangle \phi_j(x)\,dx \\
&-\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x), \big(\rho^{\frac{p-1}{p}} f_i\big)^h(x) \Big\rangle_i \phi_j(x)\,dx \\
&=\frac{2(p+1-n)}{p}\int_\Omega \Big\langle \Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)^h(x), \sum_j \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x)\phi_j(x) \Big\rangle\,dx \\
&+\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x), \big(\rho^{\frac{p-1}{p}} f_i\big)^h(x) \Big\rangle \phi_{ij}(x)\,dx.
\end{aligned} \tag{4.4.21}$$

Together we get,

$$\begin{aligned}
I_1 &= \frac{(p+1-n)^2}{p^2}\int_\Omega \Big| \Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)^h(x) \Big|^2 \phi(x)\,dx \\
&+\frac{2(p+1-n)}{p}\int_\Omega \Big\langle \Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)^h(x), \sum_j \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x)\phi_j(x) \Big\rangle\,dx \\
&+\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x), \big(\rho^{\frac{p-1}{p}} f_i\big)^h(x) \Big\rangle \phi_{ij}(x)\,dx.
\end{aligned} \tag{4.4.22}$$

Finally we combine the expression of (4.4.22) and (4.4.12) for I_1 and I_2 and obtain,

$$\begin{aligned}
&\sum_{i,j}\int_\Omega \Big|\big(\rho^{\frac{p-1}{p}} f_i\big)_j^h(x)\Big|^2 \phi(x)\,dx \qquad (4.4.23)\\
&= \frac{(p+1-n)^2}{p^2}\int_\Omega \Big|\Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)^h(x)\Big|^2 \phi(x)\,dx\\
&+\frac{2(p+1-n)}{p}\int_\Omega \Big\langle \Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)^h(x), \sum_j \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x)\phi_j(x)\Big\rangle\,dx\\
&+\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_j\big)^h(x), \big(\rho^{\frac{p-1}{p}} f_i\big)^h(x)\Big\rangle \phi_{ij}(x)\\
&+\frac{1}{2}\frac{(p-1)^2}{p^2}\sum_{i,j}\int_\Omega \Big|\Big(\rho_j\frac{f_i}{\rho^{1/p}} - \rho_i\frac{f_j}{\rho^{1/p}}\Big)^h(x)\Big|^2 \phi(x)\,dx.
\end{aligned}$$

Since $\rho \in W^{1,2}_{\text{loc}}(\Omega)$, the right hand side integrals of (4.4.23) converge as $h \to 0$, so the expression on the left hand side is bounded for all h as well. This allows us to conclude that $\rho^{\frac{p-1}{p}} Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. Letting $h \to 0$, we have,

$$\begin{aligned}
&\sum_{i,j}\int_\Omega \Big|\big(\rho^{\frac{p-1}{p}} f_i\big)_j(x)\Big|^2 \phi(x)\,dx \qquad (4.4.24)\\
&= \frac{(p+1-n)^2}{p^2}\int_\Omega \Big|\Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)(x)\Big|^2 \phi(x)\,dx\\
&+\frac{2(p+1-n)}{p}\int_\Omega \Big\langle \Big(\sum_i \rho_i \frac{f_i}{\rho^{1/p}}\Big)(x), \sum_j \big(\rho^{\frac{p-1}{p}} f_j\big)(x)\phi_j(x)\Big\rangle\,dx\\
&+\sum_{i,j}\int_\Omega \Big\langle \big(\rho^{\frac{p-1}{p}} f_j\big)(x), \big(\rho^{\frac{p-1}{p}} f_i\big)(x)\Big\rangle \phi_{ij}(x)\\
&+\frac{1}{2}\frac{(p-1)^2}{p^2}\sum_{i,j}\int_\Omega \Big|\Big(\rho_j\frac{f_i}{\rho^{1/p}} - \rho_i\frac{f_j}{\rho^{1/p}}\Big)(x)\Big|^2 \phi(x)\,dx.
\end{aligned}$$

Now we use the fact that

$$\Big\langle \frac{f_i}{\rho^{1/p}}(x), \frac{f_j}{\rho^{1/p}}(x)\Big\rangle = \delta_{ij}.$$

Then (4.4.24) becomes

$$\begin{aligned}
&\sum_{i,j}\int_\Omega \Big|\big(\rho^{\frac{p-1}{p}} f_i\big)_j(x)\Big|^2 \phi(x)\,dx \qquad (4.4.25)\\
&= \frac{(p+1-n)^2}{p^2}\int_\Omega \sum_i |\rho_i(x)|^2\phi(x)\,dx\\
&+\frac{2(p+1-n)}{p}\int_\Omega \sum_i \rho(x)\rho_i(x)\phi_i(x)\,dx + \int_\Omega \rho^2(x)\sum_i \phi_{ii}(x)\\
&+\frac{(n-1)(p-1)^2}{p^2}\int_\Omega \sum_i |\rho_i(x)|^2\phi(x)\,dx\\
&= \Big(n - \frac{(n-1)(4p-n)}{p^2}\Big)\int_\Omega |\nabla\rho(x)|^2\phi(x)\,dx\\
&+\frac{n-1}{p}\int_\Omega \rho^2(x)\Delta\phi(x)\,dx.
\end{aligned}$$

Substituting $\rho = \big(n^{-1/2}|Df|\big)^p$ gives (4.4.1) for the case $p \geq 1$.

As we see in the proof for the first case, the reason we need to distinguish the case $p \geq 1$, $1/2 \leq p < 1$ and $(n-2)/4 \leq p < 1$ and $n \geq 3$ is the different approximation in (4.4.9), (4.4.11), (4.4.15) and (4.4.18) using Lemma 4.3.1 and Lemma 4.3.3. All the rest are exactly the same. Therefore, all we need to to justify the convergence of each approximation in (4.4.9), (4.4.11), (4.4.15) and (4.4.18) for the rest two cases.

Now for the case $1/2 \leq p < 1$, we follow exactly the same method, but instead of $\big((\rho^\epsilon + \epsilon)^{\frac{p-1}{p}} f_i^\epsilon\big)_j^h$ and $\big(R_i^\epsilon/(\rho^\epsilon+\epsilon)^{(n-p-1)/p}\big)_j^h$, we approximate $\big(\rho^{\frac{p-1}{p}} f_i\big)_j^h$ by $\big((\lambda^\epsilon+\epsilon)^{p-1} f_i^\epsilon\big)_j^h$ and $\big(R_i^\epsilon/(\lambda^\epsilon+\epsilon)^{n-p-1}\big)_j^h$, where λ^ϵ is the convolution of $\lambda = \rho^{1/p}$ which is the same as the one defined in Lemma 4.3.1. The reason is the same as why we distinguish these two cases in Lemma 4.3.1 (see the paragraph above (4.3.5) for detail). Note that in this case, we can again pass to the limit because of Lemma 4.3.1, (4.3.10) in the proof of Lemma 4.3.3 and the same argument as (4.4.9).

Finally, for the case $(n-2)/4 \leq p < 1$ and $n \geq 3$, we also follow a similar method as the case $1/2 \leq p < 1$, but instead of $\big((\lambda^\epsilon+\epsilon)^{p-1} f_i^\epsilon\big)_j^h$ and $\big(R_i^\epsilon/(\lambda^\epsilon+\epsilon)^{n-p-1}\big)_j^h$, we first consider approximation by $\big((\lambda^\epsilon+c)^{p-1} f_i^\epsilon\big)_j^h$ in (4.4.9) and (4.4.11), and $\big(R_i^\epsilon/(\lambda^\epsilon+c)^{n-p-1}\big)_j^h$ in (4.4.15) and (4.4.18) for some fixed $c > 0$. In (4.4.9) when approximated by $\left((\lambda^\epsilon + c)^{p-1} f_i^\epsilon\right)_j^h$, we can pass $\epsilon \to 0$ because of Lemma 4.3.1 and the same argument as (4.4.9). In (4.4.11), we can again pass $\epsilon \to 0$ by (4.3.12) in the proof of Lemma 4.3.3 and by the same argument as proving (4.4.9). Note that the proof of (4.4.9) only requires L^1_{loc} convergence of the approximation sequence, which is exactly what (4.3.12) gives. Since $\rho \in W^{1,2}_{\text{loc}}(\Omega)$, and $\rho_j = 0$ whenever $\rho = 0$, Dominated Convergence Theorem allows us to pass $c \to 0$ to achieve the same result as (4.4.12). The convergence of other case as in (4.4.15) and (4.4.18) follows exactly the same argument. The proof of the theorem for all cases is complete. □

Proof of Theorem 4.1.1. It is enough to prove Lemma 4.4.1 below. Indeed, if we have proved Lemma 4.4.1, then by Lemma 4.1.1, $|Df|$ is locally bounded, so $f \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^n)$, then the Liouville Theorem follows from Theorem 3.5.1.

In addition, for the case $p = (n-2)/4$, $|Df|^{2p}$ is harmonic, hence smooth by the Weyl Lemma. There is a simple argument for f to be Möbius: If f is not constant, we consider the open domain $\Omega^+ := \{x \in \Omega : |Df(x)| > 0\}$. Since $|Df|^{2p}$ is smooth, $\lambda = |Df|$ is also smooth in Ω^+. Now it follows from (4.2.11) and a bootstrap argument that f is smooth in Ω^+. We then deduced from Theorem 4.2.1 that f is Möbius on Ω^+. However, the formula (3.5.3) for Möbius transforms gives $|Df| = r^2|x-a|^{-2}$ for some $r \neq 0$ and $a \in \mathbb{R}^n$, which does not vanish on the closure of Ω^+, so Ω^+ is open and closed with respect to Ω. Thus $\Omega^+ = \Omega$, which proves f is Möbius on Ω. □

We are left to prove the following lemma,

Lemma 4.4.1 *Under the assumptions of Theorem 4.1.1, $|Df|^{2p}$ is harmonic for $p = (n-2)/4$ and subharmonic for $p > (n-2)/4$.*

Proof. Again let $\rho = \left(n^{-1/2}|Df|\right)^p$. From Theorem 4.4.1 we know that $\rho^{\frac{p-1}{p}} Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. We can now apply an argument similar to the C^2 case. We differentiate

$$\left\langle \rho^{\frac{p-1}{p}} f_i, \rho^{\frac{p-1}{p}} f_j \right\rangle = \rho^2 \delta_{ij}$$

using the product rule and obtain,

$$\left\langle (\rho^{\frac{p-1}{p}} f_i)_k, \rho^{\frac{p-1}{p}} f_j \right\rangle + \left\langle \rho^{\frac{p-1}{p}} f_i, (\rho^{\frac{p-1}{p}} f_j)_k \right\rangle = (\rho^2)_k \delta_{ij}. \tag{4.4.26}$$

Permute the indices i, j, k we also obtain,

$$\left\langle (\rho^{\frac{p-1}{p}} f_i)_j, \rho^{\frac{p-1}{p}} f_k \right\rangle + \left\langle \rho^{\frac{p-1}{p}} f_i, (\rho^{\frac{p-1}{p}} f_k)_j \right\rangle = (\rho^2)_j \delta_{ik}. \tag{4.4.27}$$

$$\left\langle (\rho^{\frac{p-1}{p}} f_k)_i, \rho^{\frac{p-1}{p}} f_j \right\rangle + \left\langle \rho^{\frac{p-1}{p}} f_k, (\rho^{\frac{p-1}{p}} f_j)_i \right\rangle = (\rho^2)_i \delta_{jk}. \tag{4.4.28}$$

Since $\rho^{\frac{p-1}{p}} Df \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$, Lemma 4.3.3 now becomes the pointwise identity,

$$(\rho^{\frac{p-1}{p}} f_i)_j - (\rho^{\frac{p-1}{p}} f_j)_i = \frac{p-1}{p}\Big(\rho_j \frac{f_i}{\rho^{1/p}} - \rho_i \frac{f_j}{\rho^{1/p}}\Big) \quad 1 \le i, j \le n. \tag{4.4.29}$$

Adding (4.4.26) and (4.4.27), then subtracting (4.4.28), together with (4.4.29) gives,

$$\begin{aligned} 2\big\langle \rho^{\frac{p-1}{p}} f_i, (\rho^{\frac{p-1}{p}} f_j)_k \big\rangle &+ \frac{p-1}{p}\big\langle \rho^{\frac{p-1}{p}} f_i, \rho_j \frac{f_k}{\rho^{1/p}} - \rho_k \frac{f_j}{\rho^{1/p}} \big\rangle \\ &+ \frac{p-1}{p}\big\langle \rho^{\frac{p-1}{p}} f_j, \rho_k \frac{f_i}{\rho^{1/p}} - \rho_i \frac{f_k}{\rho^{1/p}} \big\rangle + \frac{p-1}{p}\big\langle \rho^{\frac{p-1}{p}} f_k, \rho_j \frac{f_i}{\rho^{1/p}} - \rho_i \frac{f_j}{\rho^{1/p}} \big\rangle \\ &= 2\rho\rho_k \delta_{ij} + 2\rho\rho_j \delta_{ik} - 2\rho\rho_i \delta_{jk}. \end{aligned} \tag{4.4.30}$$

We use the fact that ρ as well as its partial derivatives are scalar functions thus can be taken out of the dot products and also the identity,

$$\big\langle \frac{f_i}{\rho^{1/p}}, \frac{f_j}{\rho^{1/p}} \big\rangle = \delta_{ij}, 1 \le i, j \le n \tag{4.4.31}$$

to obtain

$$\big\langle \rho^{\frac{p-1}{p}} f_i, (\rho^{\frac{p-1}{p}} f_j)_k \big\rangle = \rho\rho_k \delta_{ij} + \frac{1}{p}\rho\rho_j \delta_{ik} - \frac{1}{p}\rho\rho_i \delta_{jk}. \tag{4.4.32}$$

If $\rho(x) > 0$,

$$\Big\{ \frac{f_i}{\rho^{1/p}}(x) \Big\}_{i=1,\dots,n}$$

form an orthonormal basis for $\mathbb{R}^n$. Hence, if $\rho(x) > 0$, by (4.4.32),

$$
\begin{aligned}
(\rho^{\frac{p-1}{p}} f_j)_k(x) &= \sum_i \big\langle (\rho^{\frac{p-1}{p}} f_j)_k(x), \frac{f_i}{\rho^{1/p}}(x) \big\rangle \frac{f_i}{\rho^{1/p}}(x) \\
&= \sum_i \frac{1}{\rho(x)} \big\langle (\rho^{\frac{p-1}{p}} f_j)_k(x), \rho^{\frac{p-1}{p}} f_i(x) \big\rangle \frac{f_i}{\rho^{1/p}}(x) \\
&= \rho_k \frac{f_j}{\rho^{1/p}}(x) + \frac{1}{p} \rho_j \frac{f_k}{\rho^{1/p}}(x) - \frac{1}{p} \sum_i \rho_i \frac{f_i}{\rho^{1/p}}(x) \delta_{jk}. \quad (4.4.33)
\end{aligned}
$$

On the other hand, $\rho(x) = 0$ implies $\rho^{\frac{p-1}{p}} f_j(x) = 0$. Hence $\rho_k = 0$ a.e. and $(\rho^{\frac{p-1}{p}} f_j)_k = 0$ a.e. for all k on the set where $\rho = 0$. Therefore, on the set where $\rho = 0$, identity (4.4.33) still holds. Thus (4.4.33) is true for a.e. $x \in \Omega$. That is,

$$
(\rho^{\frac{p-1}{p}} f_j)_k = \rho_k \frac{f_j}{\rho^{1/p}} + \frac{1}{p} \rho_j \frac{f_k}{\rho^{1/p}} - \frac{1}{p} \sum_i \rho_i \frac{f_i}{\rho^{1/p}} \delta_{jk}. \quad (4.4.34)
$$

Therefore,

$$
\sum_{i,j} \big|(\rho^{\frac{p-1}{p}} f_i)_j\big|^2 = \Big(n + \frac{2n-2}{p^2}\Big) |\nabla \rho|^2. \quad (4.4.35)
$$

Now we can substitute the expression (4.4.35) into (4.4.1) in Theorem 4.4.1 to obtain,

$$
\frac{(n-1)(4p+2-n)}{p^2} \int_\Omega |\nabla \rho(x)|^2 \phi(x)\, dx = \frac{n-1}{p} \int_\Omega \rho^2(x) \Delta \phi(x)\, dx. \quad (4.4.36)
$$

which is exactly (4.2.21) in the C^2 case.

Therefore, for $p = (n-2)/4$, ρ^2 is harmonic, hence smooth by the Weyl Lemma. For $p > (n-2)/4$, ρ^2 is subharmonic. The proof of the Lemma as well as that of Theorem 4.1.1 is complete. □

4.5 EXAMPLES

In this section we provide two examples, the first one is to show that under the assumption of Theorem 4.1.1, f may not be a Möbius transform if $p < (n-2)/4$, which implies that under second differentiability assumption $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$, the Liouville theorem cannot be improved.

The second example due to Iwaniec and Martin [28] shows conformal maps in the space $W^{1,2p}$ for $p < n/4$ in all dimensions may not be Möbius. This example does not satisfy the second differentiability assumption $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$. It implies, perhaps, the assumption $f \in W^{1,n/2}_{\text{loc}}$ is enough to rule out the situation like those in Example 4.5.2. Thus, from this perspective, the second differentiability assumption $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$ is not the weakest.

Also note that both examples are counter-examples to Conjecture 4.1.1 for the case $p < n/4$.

Example 4.5.1 *Let Ω be the unit ball $B^n(0,1)$. We define $f : B^n(0,1) \to \mathbb{R}^n$ as follows,*

$$f(x) = \begin{cases} \frac{x}{|x|^2} & \text{if} \quad x \neq 0 \\ 0 \quad \text{if} & x = 0 \end{cases}.$$

Then f satisfies (4.1.1) and (4.1.2), $f \in W^{1,2p}_{\text{loc}}\big(B^n(0,1), \mathbb{R}^n\big)$ and $|Df|^p \in W^{1,2}_{\text{loc}}(B^n(0,1))$ for $p < (n-2)/4$, but f is not Möbius.

Proof. The mapping f is an inversion and hence a Möbius transformation in $B^n(0,1) \setminus \{0\}$. However 0 is included in our domain and therefore f is *not* Möbius in $B^n(0,1)$. The mapping f is discontinuous at 0. Actually it diverges to infinity as x approaches 0. On the other hand since f is a Möbius transform in $B^n(0,1) \setminus \{0\}$, the condition (4.1.1) and (4.1.2) are satisfied in $B^n(0,1) \setminus \{0\}$ and hence a.e. in $B^n(0,1)$. It easily follows from the integration in spherical coordinates that $f \in L^{2p}\big(B^n(0,1), \mathbb{R}^n\big)$ for $p < n/2$ and in particular for for $p < (n-2)/4$. It is obvious that f is absolutely continuous on almost all lines. Therefore to prove that $f \in W^{1,2p}_{\text{loc}}\big(B^n(0,1), \mathbb{R}^n\big)$, it suffices to show that the pointwise derivative of f which is defined everywhere but at the origin satisfies $Df \in L^{2p}$. Since f is the inversion in

$B^n(0,1)\setminus\{0\}$, its conformality coefficient is $1/|x|^2$ and hence $|Df(x)| = \sqrt{n}/|x|^2$ for $x \neq 0$. Again, integration in spherical coordinates shows that $Df \in L^{2p}\big(B^n(0,1), \mathbb{R}^{n\times n}\big)$ for $p < n/4$ and hence for $p < (n-2)/4$. Now $|Df(x)|^p = n^{p/2}/|x|^{2p}$, so

$$\big|\nabla |Df|^p\big| = \frac{C}{|x|^{2p+1}} \in L^2_{\mathrm{loc}}(\Omega)$$

provided $2(2p+1) < n$, i.e. $p < (n-2)/4$. The proof is complete. □

Example 4.5.2 *Let us recall the original example of Iwaniec and Martin. Then we will discuss how it is related to our results. Let Ω be a domain in $\mathbb{R}^n$. For any $1/2 \leq p < n/4$, Iwaniec and Martin [28] constructed a mapping $f \in W^{1,2p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for $p < n/4$ that satisfies (4.1.1) and (4.1.2), yet not a Möbius transform. The construction goes as follows: Let $\Phi : B^n(a,r) \to \mathbb{R}^n$ be defined by*

$$\Phi(x) = \begin{cases} a + (x-a)\frac{r^2}{|x-a|^2} & \text{if} \quad x \neq a, \\ a \quad \text{if} \quad x = a. \end{cases}$$

We know from previous example that $\Phi \in W^{1,2p}(B^n(a,r), \mathbb{R}^n)$ for $p < n/4$ and that Φ satisfies (4.1.1) and (4.1.2). Moreover the integration in spherical coordinates gives,

$$\int_{B^n(a,r)} |\Phi(x) - x|^{2p}\, dx = \frac{C(n,p) r^{2p}}{n-2p} |B^n(a,r)|, \tag{4.5.1}$$

and

$$\int_{B^n(a,r)} |D\Phi(x)|^{2p}\, dx = \frac{C(n,p)}{n-4p} |B^n(a,r)|. \tag{4.5.2}$$

By Vitali's covering theorem, we can find a family of disjoint open balls, each of radius less than 1, $\mathcal{F} = \{B_j\}_{j=1}^\infty$, $B_j \subset\subset \Omega$ such that $|\Omega \setminus \bigcup_{j=1}^\infty B_j| = 0$.

In each ball $B_j = B^n(a_j, r_j)$ we define,

$$\Phi_j(x) = \begin{cases} a_j + (x-a_j)\frac{r_j^2}{|x-a_j|^2} & \text{if} \quad x \neq a_j \\ a_j \quad \text{if} \quad x = a_j \end{cases}.$$

Finally we define $f : \Omega \to \mathbb{R}^n$ *as,*

$$f(x) = \begin{cases} \Phi_j(x) & \text{if} \quad x \in B_j \\ x & \text{otherwise} \end{cases}.$$

From (4.5.1) we know

$$\begin{aligned}\int_\Omega |f(x) - x|^{2p}\, dx &= \sum_{j=1}^{\infty} \int_{B_j} |\Phi_j(x) - x|^{2p}\, dx \\ &= \sum_{j=1}^{\infty} \frac{C(n,p) r_j^{2p}}{n - 2p} |B_j| \le C(n,p) \sum_{j=1}^{\infty} |B_j| = C(n,p)|\Omega|. \quad (4.5.3)\end{aligned}$$

Since $f(x) - x$ *vanishes on the boundary of each* B_j *and* $\Phi_j \in W^{1,2p}(B_j, \mathbb{R}^n)$*, we have, for* $\phi \in C_0^\infty(\Omega)$*, with the notation* $D_i = \partial/\partial x_i$*,*

$$\begin{aligned}\int_\Omega f(x) D_i\phi(x)\, dx &= \int_\Omega (f(x) - x) D_i \phi(x)\, dx + \int_\Omega D_i\phi(x) x\, dx \\ &= \sum_{j=1}^{\infty} \int_{B_j} (\Phi_j(x) - x) D_i\phi(x)\, dx - \int_\Omega \phi(x) e_i\, dx \\ &= -\sum_{j=1}^{\infty} \int_{B_j} (D_i\Phi_j(x) - e_i)\phi(x)\, dx - \int_\Omega \phi(x) e_i\, dx \\ &= -\int_\Omega (g(x) - I) e_i \phi(x)\, dx - \int_\Omega \phi(x) e_i\, dx \\ &= -\int_\Omega g(x) e_i \phi(x)\, dx.\end{aligned}$$

This implies $\partial f/\partial x_i = g(x) e_i$*. Thus* $Df = g$*. Now from (4.5.2),*

$$\int_\Omega |Df(x)|^{2p}\, dx = \sum_{j=1}^{\infty} \int_{B_j} |D\Phi_j(x)|^{2p}\, dx = \sum_{j=1}^{\infty} \frac{C(n,p)}{n - 4p} |B_j| = \frac{C(n,p)}{n - 4p} |\Omega|, \quad (4.5.4)$$

(4.5.3) and (4.5.4) thus implies $f \in W^{1,2p}_{\text{loc}}(\Omega, \mathbb{R}^n)$*, for* $p < n/4$*. Since each* Φ_j *satisfies the Cauchy-Riemann system (4.1.1) and all their Jacobian* $J(x, \Phi_j) \le -1$ *a.e., the map* f *satisfies (4.1.1) and (4.1.2) as well.*

We will show now that $|Df|^p \notin W^{1,2}_{\text{loc}}(\Omega)$ *for any* p*, in particular for* $1/2 \le p < n/4$*.*

$$\big|\nabla |D\Phi|^p\big| = C(n,p) \Big| \nabla \frac{r^{2p}}{|x - a|^{2p}} \Big| = C(n,p) \frac{r^{2p}}{|x|^{2p+1}},$$

and

$$\int_{B^n(0,1)} |\nabla |D\Phi|^p|^2 dx = C(n,p) \int_{B^n(0,1)} \frac{r^{4p}}{|x|^{2(2p+1)}} dx$$
$$= C(n,p) r^{4p} \int_0^r t^{n-1-4p-2} dt = \frac{C(n,p)}{n-4p-2} r^{n-2}.$$

Hence,

$$\int_\Omega |\nabla |Df|^p|^2 \, dx = \sum_{j=1}^{\infty} \int_{B_j} |\nabla |D\Phi_j|^p|^2 \, dx = \sum_{j=1}^{\infty} \frac{C(n,p)}{n-4p-2} r_j^{n-2}. \tag{4.5.5}$$

We want to show $\sum_{j=1}^{\infty} r_j^{n-2} = \infty$. *Fix* $\epsilon > 0$, *consider the family* $\mathcal{F}' = \{B_j = B^n(a_j, r_j), r_j \leq \epsilon\}$. *Let* P_{n-1} *be the projection onto the* $n-1$ *dimensional subspace* $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$.

We claim that

$$P_{n-1}(\Omega) \subset \bigcup_{B_j \in \mathcal{F}'} P_{n-1}(B_j).$$

Let $\Omega' = \bigcup\{B_j : B_j \in \mathcal{F}'\}$ and $\Omega'' = \bigcup\{B : B \in \mathcal{F} \setminus \mathcal{F}'\}$. Since $B_j \neq \Omega$ for all $B_j \in \mathcal{F}$ and the balls are pairwise disjoint, every line orthogonal to this $n-1$ dimensional subspace that intersects Ω must also intersect $\Omega \setminus \Omega''$. Since Ω' has full measure in $\Omega \setminus \Omega''$, from the Fubini theorem, almost every such a line must intersect Ω'. Thus the claim follows. The claim then implies the $n-1$-dimensional Lebesgue measure of $P_{n-1}(\Omega)$ satisfies,

$$0 < |P_{n-1}(\Omega)| \leq C \sum_{\{j : B_j \in \mathcal{F}'\}} r_j^{n-1}.$$

Then,

$$|P_{n-1}(\Omega)| \leq C \sum_{\{j : B_j \in \mathcal{F}'\}} r_j^{n-1} \leq C\epsilon \sum_{\{j : B_j \in \mathcal{F}'\}} r_j^{n-2} \leq C\epsilon \sum_{j=1}^{\infty} r_j^{n-2}.$$

Hence,

$$C \sum_{j=1}^{\infty} r_j^{n-2} \geq \frac{1}{\epsilon} |P_{n-1}(\Omega)|$$

Since ϵ can be arbitrarily small, $\sum_{j=1}^{\infty} r_j^{n-2} = \infty$. This together with (4.5.5) show that

$$\int_\Omega |\nabla |Df|^p|^2 \, dx = \infty.$$

Hence $|Df|^p \notin W^{1,2}_{\text{loc}}(\Omega)$. Of course Theorem 4.1.1 implies this result. However, here we find a concrete example.

Finally, it is worth mentioning that the Hölder continuous example constructed by Malý [37] is based on the Iwaniec and Martin type construction, thus it also fails to satisfies $|Df|^p \in W^{1,2}_{\text{loc}}(\Omega)$.

5.0 RIGIDITY, REGULARITY AND DENSITY OF CO-DIMENSION ONE $W^{2,2}$ ISOMETRIC IMMERSIONS

5.1 INTRODUCTION

The history and development of isometric immersions of co-dimension one has been discussed in Chapter 1 and Section 3.6. In this chapter, we present our results on regularity, developability and density of $W^{2,2}$ isometric immersions from a subset of $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ for $n \geq 3$, where the case $n = 2$ has been established by Pakzad [46].

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n, n \geq 2$. We define the class of Sobolev isometric immersions from Ω to $\mathbb{R}^{n+1}$ as,

$$I^{2,2}(\Omega, \mathbb{R}^{n+1}) := \{u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) : (Du)^T Du = \mathrm{I} \text{ a.e.}\} \tag{5.1.1}$$

As our main finding, we have,

Theorem 5.1.1 *Let $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Then $u \in C^{1,1/2}_{\text{loc}}(\Omega, \mathbb{R}^{n+1})$. Moreover, for every $x \in \Omega$, either u is affine in a neighborhood of x, or there exists a unique $(n-1)$-dimensional hyperplane $P \ni x$ of $\mathbb{R}^n$ such that u is affine on the connected component of x in $P \cap \Omega$.*

The argument, differently from the two dimensional case, where we could prove C^1 regularity on the first hand, use a "slicing argument" which slices a n dimensional domain into $k = 2, ..., n$ dimensional slices and prove developability by induction on lower dimensional slices. C^1 regularity is obtained at the very end of this proof.

Based on this result, we also prove that smooth isometric immersions are dense in $I^{2,2}$ if the domain is regular enough.

Theorem 5.1.2 *If Ω is a C^1 convex domain, then for every $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, there is a sequence of mappings $u_m \in I^{2,2} \cap C^\infty(\Omega, \mathbb{R}^{n+1}) \to u$ in $W^{2,2}$ norm.*

This chapter is organized as follows. In section 5.2 we prove that $W^{2,2}$ isometric immersions enjoy the property that the Hessian of each component has rank less than or equal to one almost everywhere. This argument is similar to the Liouville Theorem argument that we differentiate the constraint and permute indices. In section 5.3 we present the proof of Theorem 5.1.1. In section 5.4 we present the proof of Theorem 5.1.2.

5.2 PRELIMINARY

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n, n \geq 2$. Recall the class of Sobolev isometric immersions from Ω to $\mathbb{R}^{n+1}$ is,

$$I^{2,2}(\Omega, \mathbb{R}^{n+1}) := \{u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) : (Du)^T Du = \mathrm{I} \text{ a.e.}\} \tag{5.2.1}$$

Note that the condition $(Du)^T Du = \mathrm{I}$ implies that u is Lipschitz continuous, thus,

$$u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1}). \tag{5.2.2}$$

Given $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, let $u^j, 1 \leq j \leq n+1$, be the j-th component of u and let $u_{,i} = \partial u / \partial x_i, 1 \leq i \leq n$, be the partial derivative of u in the $\mathbf{e}_i$ direction. We will also use such notations for any functions afterwards.

For a.e. $x \in \Omega$, consider the cross product

$$\mathbf{n}(x) = u_{,1}(x) \times \cdots \times u_{,n}(x).$$

That is, $\mathbf{n}(x)$ is the unique unit vector orthogonal to $u_{,i}(x)$ for all $1 \leq i \leq n$ such that

$$\mathbf{n}(x), u_{,1}(x), ..., u_{,n}(x)$$

form a positive basis of $\mathbb{R}^{n+1}$.

Note that $\mathbf{n}$ can also be identified as differential forms: consider the 1-form,

$$\omega_i = \sum_{j=1}^{n+1} u^j_{,i} dx_j.$$

Then

$$\mathbf{n} = *(\omega_1 \wedge \cdots \wedge \omega_n). \tag{5.2.3}$$

because for any $\xi \in \bigwedge^1(\mathbb{R}^{n+1})$,

$$\langle \xi, \mathbf{n} \rangle = \langle \xi, *(\omega_1 \wedge \cdots \wedge \omega_n) \rangle = \xi \wedge \omega_1 \wedge \cdots \wedge \omega_n = \det[\xi, u_{,1}, ..., u_{,n}].$$

Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1}) \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1})$, it follows from (5.2.3) that $\mathbf{n} \in W^{1,2}(\Omega, \mathbb{R}^{n+1})$.

As u is isometric immersion, $\langle u_{,i}, u_{,j} \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, we can differentiate using the product rule to obtain,

$$\langle u_{,ik}, u_{,j} \rangle + \langle u_{,i}, u_{,jk} \rangle = 0 \quad \text{a.e.} \tag{5.2.4}$$

Permutation of indices i, j, k yields,

$$\langle u_{,ij}, u_{,k} \rangle + \langle u_{,i}, u_{,kj} \rangle = 0 \quad \text{a.e.} \tag{5.2.5}$$

$$\langle u_{,ki}, u_{,j} \rangle + \langle u_{,k}, u_{,ji} \rangle = 0 \quad \text{a.e.} \tag{5.2.6}$$

Using the fact that $u_{,ij} = u_{,ji}$ for all i, j, we add (5.2.4) and (5.2.5), then subtract (5.2.6) to obtain,

$$\langle u_{,i}, u_{,jk} \rangle = 0 \quad \text{a.e.} \quad \text{for all} \quad 1 \leq i, j, k \leq n. \tag{5.2.7}$$

Since for a.e. points in the domain, $\mathbf{n}, u_{,1}, , , u_{,j}$ form a basis for $\mathbb{R}^{n+1}$, we can write,

$$u_{,jk} = \sum_{i=1}^{n} \langle u_{,jk}, u_{,i} \rangle u_{,i} + \langle u_{,jk}, \mathbf{n} \rangle \mathbf{n}.$$

(5.2.7) then gives,

$$u_{,jk} = \langle u_{,jk}, \mathbf{n} \rangle \mathbf{n} \quad \text{a.e.} \quad \text{for all} \quad 1 \leq j, k \leq n. \tag{5.2.8}$$

Note that $A_{jk} := \langle u_{,jk}, \mathbf{n} \rangle$ is the element in row j and column k of the second fundamental form A, which is an $n \times n$ matrix. In particular, (5.2.8) holds for each component of $u_{,jk}$ and $\mathbf{n}$, i.e.,

$$u^{\ell}_{,jk} = A_{jk} \mathbf{n}^{\ell} \quad \text{for all} \quad 1 \leq \ell \leq n+1, \quad 1 \leq j, k \leq n.$$

Thus, the Hessian of u^{ℓ} satisfies,

$$D^2 u^{\ell} = \mathbf{n}^{\ell} A, \quad 1 \leq \ell \leq n+1. \tag{5.2.9}$$

Lemma 5.2.1 *The second fundamental form $A \in M^{n \times n}$ has the following properties,*

$$\frac{\partial A_{ij}}{\partial x_k} = \frac{\partial A_{ik}}{\partial x_j} \quad \textit{in distributional sense for all} \quad 1 \leq i, j, k \leq n, \tag{5.2.10}$$

and

$$A_{ij} A_{kl} - A_{il} A_{kj} = 0 \quad \textit{for all} \quad 1 \leq i, j, k, l \leq n. \tag{5.2.11}$$

Proof. For a smooth immersion $v : \Omega \to \mathbb{R}^{n+1}$, not necessarily isometric, let $g_{ij} = \langle v_{,i}, v_{,j} \rangle$ be the first fundamental forms, then by differentiating g_{ij} twice,

$$g_{ij,kl} = \langle v_{,ikl}, v_{,j} \rangle + \langle v_{,ik}, v_{,jl} \rangle + \langle v_{,il}, v_{,jk} \rangle + \langle v_{,i}, v_{,jkl} \rangle.$$

Making the summation over the proper permutations of i, j, k, l gives,

$$g_{ij,kl} + g_{kl,ij} - g_{il,kj} - g_{kj,il} = -2 \langle v_{,ij}, v_{,kl} \rangle + 2 \langle v_{,il}, v_{,kj} \rangle. \tag{5.2.12}$$

Given any other smooth immersion $w : \Omega \to \mathbb{R}^{n+1}$, the following identity is also obvious,

$$\langle v_{,ij}, w \rangle_{,k} - \langle v_{,ik}, w \rangle_{,j} = \langle v_{,ij}, w_{,k} \rangle - \langle v_{,ik}, w_{,j} \rangle. \tag{5.2.13}$$

Now we let a sequence of smooth immersions $u_m \to u$ in $W^{2,2}(\Omega, \mathbb{R}^{n+1})$ and $\mathbf{n}_m \to \mathbf{n}$ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$. Writing the left hand sides of (5.2.12) and (5.2.13) as distributional derivatives and passing to the limit we get,

$$0 = -2 \langle u_{,ij}, u_{,kl} \rangle + 2 \langle u_{,il}, u_{,kj} \rangle. \tag{5.2.14}$$

because $\langle u_{,i}, u_{,j}\rangle = \delta_{ij}$ for all i, j. In addition, since $\mathbf{n}$ is a unit vector, $\langle \mathbf{n}_{,k}, \mathbf{n}\rangle = 0$. Then by (5.2.8), $\langle u_{,ij}, \mathbf{n}_{,k}\rangle = 0$ for all i, j, k, thus,

$$\langle u_{,ij}, \mathbf{n}\rangle_{,k} - \langle u_{,ik}, \mathbf{n}\rangle_{,j} = 0 \tag{5.2.15}$$

The two identities in the lemma follow easily from $A_{ij} = \langle u_{,ij}, \mathbf{n}\rangle$, (5.2.14), and (5.2.15). The proof is complete. □

Corollary 5.2.1 *The second fundamental form A satisfies* $\operatorname{rank} \mathrm{A} \leq 1$ *and A is symmetric a.e. in Ω. Moreover, the Hessian of each component of u satifies* $\operatorname{rank} D^2u^\ell \leq 1$ *for all $1 \leq \ell \leq n+1$ a.e. on Ω.*

Proof. By identity (5.2.11), all 2×2 minors of A vanish, hence the rank of A is less than or equal to 1. By (5.2.9), $\operatorname{rank} D^2u^\ell = n^\ell \operatorname{rank} A \leq 1$ and A is symmetric a.e since D^2u^ℓ is symmetric a.e. The proof is complete. □

5.3 DEVELOPABILITY AND REGULARITY

Our main result follows from the following proposition,

Proposition 5.3.1 *Let $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let A be the second fundamental form of u. Let P_k be a k-dimensional plane of $\mathbb{R}^n$, $k \leq n$. Suppose on $P_k \cap \Omega$ we have the following properties,*

1. *There exists a sequence of smooth functions u^ϵ defined in the domain Ω such that*

$$\int_{P_k \cap \Omega} |u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^k \to 0.$$

   Here Du^ϵ, Du, D^2u^ϵ and D^2u denote the full gradient with respect to the domain Ω.
2. *The full gradient Du satisfies $Du^T Du = \mathrm{I}$ a.e. on $P_k \cap \Omega$.*
3. *$D^2u^\ell = \mathbf{n}^\ell A, 1 \leq \ell \leq n+1$ a.e. on $P_k \cap \Omega$.*
4. $\operatorname{rank} A \leq 1$ *and A is symmetric a.e. on $P_k \cap \Omega$.*

Then $u \in C^{1,1/2}_{\text{loc}}(P_k, \mathbb{R}^{n+1})$*. Moreover, for every* $x \in P_k \cap \Omega$*, either* Du *is constant in a neighborhood in* $P_k \cap \Omega$ *of* x*, or there exists a unique* $(k-1)$*-dimensional hyperplane* $P^x_{k-1} \ni x$ *of* P_k *such that* Du *is constant on the connected component of* x *in* $P^x_{k-1} \cap \Omega$*.*

The proof is based on induction on lower dimensional slices. Before we prove the theorem, we will show that it implies our main result (Theorem 5.1.1),

Corollary 5.3.1 *Let* $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$*, where* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^n$*. Then* $u \in C^{1,1/2}_{\text{loc}}(\Omega, \mathbb{R}^{n+1})$*. Moreover, for every* $x \in \Omega$*, either* Du *is constant in a neighborhood of* x*, or there exists a unique* $(n-1)$*-dimensional hyperplane* $P \ni x$ *of* $\mathbb{R}^n$ *such that* Du *is constant on the connected component of* x *in* $P \cap \Omega$*.*

Proof. We simply take $k = n$ in Proposition 5.3.1, in which case $P_n \cap \Omega = \Omega$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, the convolution of u with the standard mollifier u^ϵ apparently satisfies assumption (1). By the fact that $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, $Du^T Du = \mathrm{I}$ a.e. in Ω, which is property (2). Property (3) follows from equation (5.2.9) and property (4) follows from Corollary 5.2.1. Therefore, all the assumptions of Proposition 5.3.1 are satisfied, and hence the conclusion of Theorem 5.3.1 follows from the conclusion of Proposition 5.3.1. The proof is complete. □

Corollary 5.3.2 *Let* $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$*, where* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^n$*. Then for every* k *dimensional slices* $P_k \cap \Omega$*,* Du *is constant either on* k *dimensional neighborhoods of* $P_k \cap \Omega$*, or constant on* $k-1$*-dimensional slices of* $P_k \cap \Omega$*.*

Proof. Since assumptions (1)-(4) of Proposition 5.3.1 are satisfied a.e. in Ω. By Fubini Theorem, assumptions (1)-(4) also holds in $\mathcal{H}^{n-k}$ a.e. k-dimensional slices. Thus the conclusion of Proposition 5.3.1 holds for a.e. k-dimensional slices. Since ∇u is continuous, by a simply approximation argument, it holds on *every* k-dimensional slices. The proof is complete. □

Assumptions (2), (3) and (4) are properties of isometric immersions, while (1) is for any general Sobolev functions. One may wonder the use of assumption (1). Actually, it is for allowing the use of the chain rule which involves the full gradient even in lower dimensional slices. To be precise, we prove the following lemma which will play an important role everywhere in the proof of Proposition 5.3.1.

Lemma 5.3.1 (Chain Rule) *Let $\Psi \in W^{1,2}(\Omega, \mathbb{R}^N)$, $N \geq 1$. Suppose for any k-dimensional domain $\Sigma \subset \Omega$ there exist a sequence of smooth functions $\Psi^\epsilon \in C^\infty(\Omega, \mathbb{R}^N)$ such that*

$$\int_\Sigma |\Psi^\epsilon - \Psi|^2 + |D\Psi^\epsilon - D\Psi|^2 \, d\mathcal{H}^k \to 0, \tag{5.3.1}$$

where $D\Psi$ denotes the full gradient with respect to the domain Ω. Let $\mathbf{v}$ be any directional vector of Σ, then the chain rule,

$$\frac{d}{dt}\Big|_{t=0} \Psi(\cdot + t\mathbf{v}) = D\Psi\mathbf{v}$$

holds in the weak sense over the domain Σ. In particular,

$$\Psi \in W^{1,2}(\Sigma, \mathbb{R}^N).$$

Proof. Let $\phi \in C_0^\infty(\Sigma)$, then,

$$\int_\Sigma \frac{d}{dt}\Big|_{t=0} \Psi^\epsilon(x + t\mathbf{v})\phi(x) d\mathcal{H}^k = -\int_\Sigma \Psi^\epsilon(x) \frac{d}{dt}\Big|_{t=0} \phi(x + t\mathbf{v}) d\mathcal{H}^k.$$

Since Ψ^ϵ is smooth in Ω, we have,

$$\int_\Sigma \frac{d}{dt}\Big|_{t=0} \Psi^\epsilon(x + t\mathbf{v})\phi(x) d\mathcal{H}^k = \int_\Sigma D\Psi^\epsilon(x)\mathbf{v}\phi(x) d\mathcal{H}^k.$$

By (5.3.1) we pass to the limit to conclude that,

$$\int_\Sigma D\Psi(x)\mathbf{v}\phi(x) d\mathcal{H}^k = -\int_\Sigma \Psi(x) \frac{d}{dt}\Big|_{t=0} \phi(x + t\mathbf{v}) d\mathcal{H}^k.$$

Thus the chain rule as stated in the Lemma hold in the weak sense over the domain Σ. The proof is complete. □

Remark 5.3.1 *Note that the above lemma involve the full gradient of Ψ. The assumption $\Psi \in W^{1,2}(C, \mathbb{R}^k)$ by itself is not enough to conclude the chain rule.*

5.3.1 Base case-regularity on 2-dimensional slices.

Suppose for a 2-dimensional plane P_2 all the assumptions (1)-(4) in Proposition 5.3.1 are satisfied. Without loss of generality, we can assume P_2 is parallel to the space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Indeed, it is easy to see that assumption (1)-(4) in Proposition 5.3.1 are invariant under change of coordinate system. We denote P_2 by $P_{\mathbf{e}_1\mathbf{e}_2}$ to remind ourselves such a fact.

Let $f = \nabla u^\ell \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for some arbitrary $1 \leq \ell \leq n+1$. Define,

$$g := (f^1, f^2)|_{P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega} \in W^{1,2}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^2).$$

Lemma 5.3.2 *Let C be a line segment in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ such that*

$$\int_C |f^\epsilon - f|^2 + |Df^\epsilon - Df|^2 d\mathcal{H}^1 \to 0. \tag{5.3.2}$$

Moreover $\operatorname{rank} Df \leq 1$ *and Df is symmetric for $\mathcal{H}^1$ a.e. points on C. Then if g is constant on C, so is f.*

Proof. Let $\mathbf{v}$ be the unit directional vector of C. Since $\mathbf{v}$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, 0, ..., 0).$$

Let $\tilde{\mathbf{v}} = (\mathbf{v}_1, \mathbf{v}_2)$, then the first two component of f satisfies $\nabla f^1 \cdot \mathbf{v} = \nabla g^1 \cdot \tilde{\mathbf{v}}$ a.e. on C and $\nabla f^2 \cdot \mathbf{v} = \nabla g^2 \cdot \tilde{\mathbf{v}}$ a.e. on C.

Since f satisfy the assumption of Lemma 5.3.1, the chain rule,

$$\frac{d}{dt}\Big|_{t=0} f(\cdot + t\mathbf{v}) = Df\mathbf{v}$$

holds in the weak sense on C. In particular, it holds for it first two component f^1 and f^2 and, of course, g.

As g is constant on C,

$$0 = \frac{d}{dt}\Big|_{t=0} g(\cdot + t\tilde{\mathbf{v}}) \tag{5.3.3}$$

in the weak sense. Hence,

$$Dg\tilde{\mathbf{v}} = 0 \quad \text{a.e. on } C.$$

This implies,

$$\nabla f^1 \cdot \mathbf{v} = 0 \quad \text{and} \quad \nabla f^2 \cdot \mathbf{v} = 0 \quad \text{a.e. on } C.$$

For $z \in C$ such that $\nabla f^1(z) \cdot \mathbf{v} = 0$ and $\nabla f^2(z) \cdot \mathbf{v} = 0$, $\operatorname{rank} Df(z) \leq 1$ and $Df(z)$ is symmetric, we have two cases: 1)$\nabla f^1(z) \neq 0$ or $\nabla f^2(z) \neq 0$, 2)$\nabla f^1(z) = \nabla f^2(z) = 0$ In the first case, we can assume with loss of generality that $\nabla f^1(z) \neq 0$. Therefore, $\operatorname{rank} Df(z) = 1$ and

$$\nabla f^i(z) = a_z^i \nabla f^1(z) \quad \text{for all } i > 1$$

It then follows that

$$\nabla f^i(z) \cdot \mathbf{v} = a_z^i \big(\nabla f^1(z) \cdot \mathbf{v}\big) = 0 \quad \text{for all } i > 1.$$

In the second case, by symmetry,

$$f^i_{,j}(z) = f^j_{,i}(z) = 0, \quad \text{for } j = 1, 2, \text{ and } i = 1, ..., n.$$

As $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, 0, ..., 0)$,

$$\nabla f^i(z) \cdot \mathbf{v} = 0 \quad \text{for all } i = 1, ..., n$$

Therefore, in either cases, we have proved

$$\nabla f^i \cdot \mathbf{v} = 0 \quad \text{a.e. on } C \quad \text{for all } i = 1, ..., n.$$

Therefore, f is constant on C by the chain rule in (5.3.3). The proof is complete. □

Corollary 5.3.3 *If g is constant on a 2-dimensional region U in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, f is constant on U as well.*

Proof. Observe that if U is a 2-dimensional region of $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, which has strictly positive 2 Hausdorff measure, then the assumptions (1) and (4) of Proposition 5.3.1 imply,

$$\int_U |f^\epsilon - f|^2 + |Df^\epsilon - Df|^2 d\mathcal{H}^2 \to 0.$$

$\operatorname{rank} Df \le 1$ and Df is symmetric for $\mathcal{H}^2$ a.e. points on U. Thus the same argument for line segments in Lemma 5.3.2 gives for any directional vector $\mathbf{v}$ of U, $\nabla f^i \cdot \mathbf{v} = 0$ a.e. on U for all $i = 1, ..., n$, hence the chain rule implies f is constant on U. The proof is complete. □

Lemma 5.3.3 *Let $f = \nabla u^\ell \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some arbitrary $1 \le \ell \le n+1$. Then $f \in C^{0,1/2}_{\text{loc}}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^n)$. Moreover, for every point of x, either there exists a neighborhood in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ of x, or a unique line segment in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through x and joining $\partial\Omega$ at both ends, on which f is constant.*

Proof. By assumption (4) of Proposition 5.3.1, Df satisfies $\operatorname{rank} Df \le 1$ and $Df = D^2u^\ell$ is symmetric a.e. on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Therefore, $g := (f^1, f^2)|_{P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega} \in W^{1,2}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^2)$ also satisfies $\operatorname{rank} Dg \le 1$ and Dg is symmetric a.e. on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. We employ [46], Proposition 1, which we state for the readers' convenience,

Theorem 5.3.1 (Pakzad) *Let $g \in W^{1,2}(\Sigma, \mathbb{R}^2)$, where Σ is a Lipschitz domain in $\mathbb{R}^2$, be a map with almost everywhere symmetric singular (i.e., of zero determinant), then g is continuous on Σ. Furthermore, for every point of x, either there exists a neighborhood of x, or a unique segment passing through x and joining $\partial\Sigma$ at both ends, on which g is constant.*

Apparently our g satisfies the assumption of Theorem 5.3.1 on the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. We take a closer look of g on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Suppose g is constant on some maximal neighborhood $U \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, by continuity of g, it is also constant on its closure $\overline{U} \cap \Omega$. Now if $x \in \partial U \cap \Omega$, then x is not contained in a constant neighborhood of g, therefore by Theorem 5.3.1, there exists a unique line segment $C^U_x \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through x and joining $\partial\Omega$ at both ends on which g is constant, which implies $\partial U \cap \Omega \subset \bigcup_{x\in\partial U\cap\Omega} C^U_x$. We emphasize here $\partial U \cap \Omega$ because the entire ∂U may contain some part belonging to $\partial\Omega$, but ∂U inside Ω will not. On the other hand, suppose g is constant on some line segment C^U_x passing through $x \in \partial U \cap \Omega$ and joining $\partial\Omega$ at both end, since g is constant on U and C^U_x, which intersect at x, it must be constant on the convex hull of U and C_x^U inside Ω by continuity of g. But U is maximal, hence $\bigcup_{x\in\partial U\cap\Omega} C_x^U \subset \partial U\cap\Omega$. Therefore,

$$\partial U\cap\Omega = \bigcup_{x\in\partial U\cap\Omega} C_x^U.$$

Moreover, continuity of g ensures for $x, z \in \partial U\cap\Omega$, $C_x^U = P_z^U$ if $z \in C_x^U$ and $C_x^U \cap C_z^U \cap \Omega = \emptyset$ if $z \notin C_x^U$ (Figure 1).

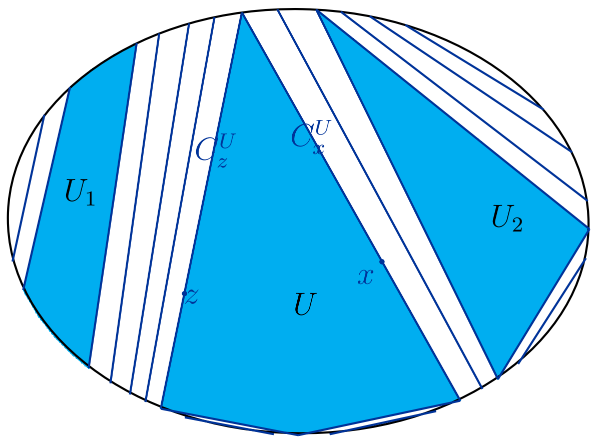

Figure 1:

Let $x_0 \in P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega$. We can choose small enough $\delta > 0$ so that for any region U on which g is constant, the 2-dimensional ball $B^2(x_0,\delta) \subset P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega$ intersects ∂U at no more than *two* line segments belonging to ∂U, Indeed, since for any maximal constant region U, all line segments in ∂U do not intersect inside Ω, suppose for some maximal constant region U containing or near x_0, the angles between two line segments $C_{x_1}^U$ and $C_{x_2}^U$ (if they are nonparallel) in ∂U is large, or the distance between them (if they are parallel) is large, we can choose δ small enough so $B^2(x_0,\delta)$ intersects at most one of them (Figure 2). Suppose there is a sequence of maximal constant regions U_m converging to x_0 in distance, in which case there are two line segments $C_{x_1}^{U_m}$ and $C_{x_2}^{U_m}$ in ∂U_m whose angle (if they are nonparallel) or distance (if they are parallel) goes to zero. Then all the other line segments in ∂U_m must be arbitrarily close to $\partial\Omega$, we can again choose δ small enough so that $B^2(x_0,\delta)$ is away from

$\partial\Omega$ and hence it does not intersect a third line segment in ∂U_m (Figure 3).

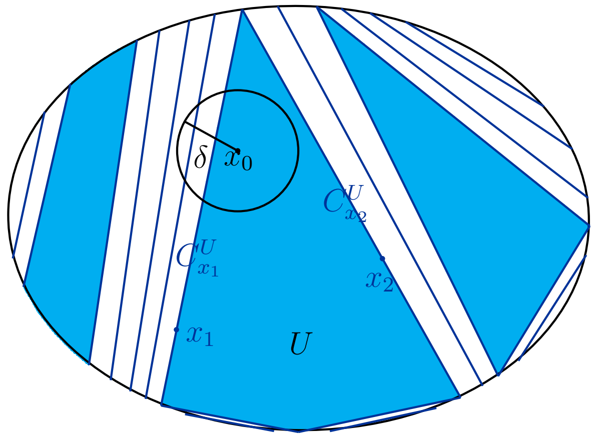

Figure 2:

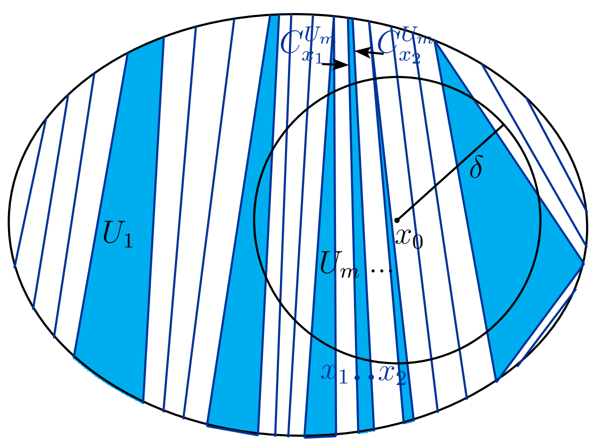

Figure 3:

We now focus on $B^2(x_0, \delta) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. For any $x \in B^2(x_0, \delta)$, we want to construct a line segment C_x in $B^2(x_0, \delta)$ passing through x and joining $\partial\Omega$ at both ends on which g is constant and $C_x \cap C_z \cap B^2(x_0, \delta) = \emptyset$ if $z \notin C_x$. For those x not contained in a constant region of g, this line segment is given automatically by Theorem 5.3.1. If x is contained in a constant maximal region U of g, then it is constant on every line segment in U that passes through it so we have to choose the appropriate one: 1) If $B^2(x_0, \delta)$ intersect only one line segment C^U in Ω that belongs to ∂U, then we define C_x to be the line segments inside $B^2(x_0, \delta)$ passing through x and parallel to C^U; 2) If $B^2(x_0, \delta)$ intersects two line segments C_1^U, C_2^U in Ω that belongs to ∂U, let L_1 and L_2 be the two lines that contain C_1^U and C_2^U. If L_1 and L_2 are not parallel, let $O := L_1 \cap L_2$ and let C_x be the segment of the line passing through O and x inside $B^2(x_0, \delta)$. If L_1 and L_2 are parallel, then we let C_x be the line segment inside $B^2(x_0, \delta)$ passing through x and parallel to L_1. (Figure 4).

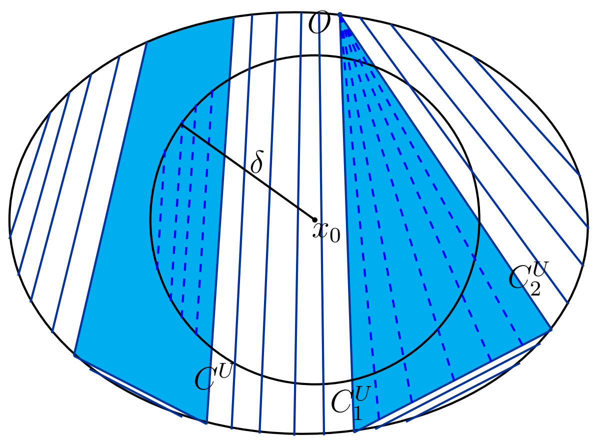

Figure 4:

In this way, we construct a family of line segments $\{C_x\}_{x \in B^2(x_0,\delta)}$ in $B^2(x_0, \delta)$ on which g is constant and $C_x \cap C_z \cap B^2(x_0, \delta) = \emptyset$ if $z \notin C_x$. For every $x \in B^2(x_0, \delta)$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector in $B^2(x_0, \delta)$ orthogonal to C_x. Since none of the C_xs intersect inside $B^2(x, \delta)$, they approach each other in an Lipschitz angle. Therefore, we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector fields. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

then has a unique solution $\gamma : (a, b) \to B^2(x_0, \delta)$ for some interval $(a, b) \subset \mathbb{R}$ containing 0. Moreover, $\cup\{C_{\gamma(t)}\}_{t \in (a,b)} = B^2(x_0, \delta)$. Therefore, $\{C_{\gamma(t)}\}_{t \in (a,b)}$ is a foliation of $B^2(x_0, \delta)$

(Figure 5).

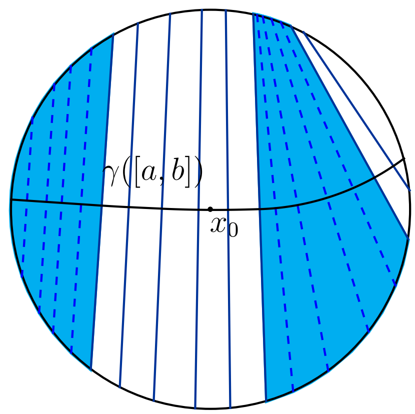

Figure 5:

We define the function $h : B^2(x_0, \delta) \to B^2(x_0, \delta)$ as

$$h(x) = \gamma(t) \quad \text{if } x \in C_{\gamma(t)}.$$

Since none of the $C_{\gamma(t)}$ intersect inside $B^2(x_0, \delta)$, h is well defined and h is constant along each $C_{\gamma(t)}$, i.e. $h^{-1}(\gamma(t)) = C_{\gamma(t)}$. Since γ is Lipschitz, h is Lipschitz as well.

We now want to show the assumptions of Lemma 5.3.2 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a, b)$. Let E_0 be the set of all $x \in B^2(x_0, \delta)$ such that rank $Df(x) > 1$ or $Df(x)$ is not symmetric. By assumption (4) of Proposition 5.3.1 on f, $|E_0| = 0$. As h is Lipschitz, we can apply the coarea formula to h to obtain,

$$\begin{aligned} 0 = \int_{E_0} |J_h(x)| dx &= \int_\gamma \mathcal{H}^1\big(E_0 \cap h^{-1}(w)\big) d\mathcal{H}^1(w) \\ &= \int_a^b \mathcal{H}^1\big(E_0 \cap h^{-1}(\gamma(t))\big) |\gamma'(t)|\, dt = \int_a^b \mathcal{H}^1\big(E_0 \cap C_{\gamma(t)}\big) |\gamma'(t)|\, dt. \end{aligned}$$

Therefore, for a.e. $t \in (a,b)$, $\mathcal{H}^1\big(E_0 \cap C_{\gamma(t)}\big) = 0$ by the fact $|\gamma'| = 1$. Moreover, by change of variable formula,

$$\begin{aligned}
&\int_{B^2(x_0,\delta)} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) J_h \\
&\qquad = \int_\gamma \int_{h^{-1}(w)} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 d\mathcal{H}^1(w) \\
&\qquad = \int_a^b \int_{h^{-1}(\gamma(t))} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 |\gamma'(t)|\, dt \\
&\qquad = \int_a^b \int_{C_{\gamma(t)}} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 |\gamma'(t)|\, dt.
\end{aligned}$$

Since $|J_h| = |\gamma'| = 1$, together with assumption (1) in Proposition 5.3.1, we then have for a.e. $t \in (a,b)$,

$$\int_{C_{\gamma(t)}} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 \to 0.$$

Therefore, the assumptions of Lemma 5.3.2 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a,b)$. It follows that f is constant on $C_{\gamma(t)}$ for a.e. $t \in (a,b)$. Then, if necessary, we choose an initial value for γ arbitrary close to x_0 such that f is absolutely continuous on γ. Hence we conlcude f is $C^{0,1/2}$ on γ by the Sobolev embedding theorem.

Let F be the set of $t \in (a,b)$ such that f is not constant along $C_{\gamma(t)}$, then $\mathcal{H}^1(F) = 0$. We modify f to be constant along $C_{\gamma(t)}$ for each $t \in F$. Note that,

$$\mathcal{H}^1(\{C_{\gamma(t)} : t \in F\}) \le 2\delta \mathcal{H}^1(\{\gamma(t) : t \in F\}) = 2\delta \mathcal{H}^1(F) = 0.$$

Hence f is $C^{0,1/2}$ up to modification of a set of measure zero in $B^2(x_0,\delta)$. Moreover, f is constant on $C_{\gamma(t)}$ for all t, which foliates $B^2(x_0,\delta)$. In addition, by Corollary 5.3.3, f is constant on every 2-dimensional region in $B^2(x_0,\delta)$ on which g is constant. Therefore, f is either constant on a line segment joining $\partial B^2(x_0,\delta)$ at both ends, or constant on a 2-dimensional region in $B^2(x_0,\delta)$. This proves Lemma 5.3.3 for the ball $B^2(x_0,\delta)$.

Now we prove the lemma for the entire domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Indeed, suppose for some $x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, x is not contained in a constant region of f. Then by what we have proved, there is a line segment passing through x and joins the boundary of $B^2(x,\delta_x)P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ for some $\delta_x > 0$, on which f is constant. Let $\overline{y_1y_2}$ be the largest line segment containing this segment on which f is constant. Suppose $y_1 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, then from what we have proved, f is either constant on 2-dimensional regions or line segments passing through y_1 joining the boundary or $B^2(y_1, \delta_{y_1}) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ for some $\delta_{y_1} > 0$. First y_1 cannot be contained in a constant region of f, otherwise we can prolong the segment $[y_1, y_2]$. Thus, there must be a line segment $\overline{z_1z_2}$ passing through y_1 and joining the boundary of $B^2(y_1, \delta_{y_1})$ at both end on which f is constant. Second, $\overline{z_1z_2}$ cannot have the same direction as $\overline{y_1y_2}$, otherwise, we can again prolong the segment $\overline{y_1y_2}$. Then we consider the region Δ bounded by $\overline{y_2z_1}$, $\overline{z_1z_2}$ and $\overline{z_2y_2}$. Since g is constant on $\overline{y_1y_2}$ and $\overline{z_1z_2}$, by Theorem 5.3.1, g must be constant on Δ because no line segment can join the boundary of $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through a point inside Δ without intersecting either $\overline{y_1y_2}$ or $\overline{z_1z_2}$ (Figure 6). Hence by Corollary 5.3.7, f is constant on Δ as well, contradiction to our assumption x is not contained in a constant region of f. The proof is complete. □

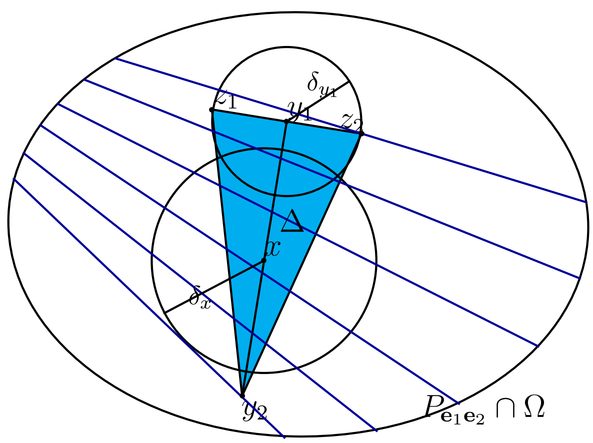

Figure 6:

Now we are ready to prove Proposition 5.3.1 for the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Since we take $f = \nabla u^\ell$ for arbitrary $1 \le \ell \le n+1$, Lemma 5.3.3 gives all ∇u^ℓ are continuous on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ and constant either on 2-dimensional neighborhoods or line segments in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ joining $\partial\Omega$ at both ends. Therefore, what is left is to prove that they are constant on the *same* neighborhoods or line segments in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$.

Recall equation (5.2.3) that $\mathbf{n}$ is the wedge product of entries of Du, hence is continuous. Let

$$\Delta_\ell = \{x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega : \mathbf{n}^\ell(x) \neq 0\}.$$

Apparently each Δ_ℓ is open by continuity. Moreover, since $|\mathbf{n}| = 1$ everywhere,

$$\bigcup_{1 \leq \ell \leq n+1} \Delta_\ell = P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega.$$

Let $x_0 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, then $x_0 \in \Delta_\ell$ for some ℓ, without loss of generality, Δ_1. Then as the same argument as in the proof of Lemma 5.3.3, there exist $B^2(x_0, \delta) \subset \Delta_1$ for some $\delta > 0$, on which we can construct a foliation $\{C_{\gamma(t)}\}_{t \in (a,b)}$, i.e. $\cup\{C_{\gamma(t)}\}_{t \in (a,b)} = B^2(x_0, \delta)$ and $C_\gamma(t) \cap C_\gamma(t') \cap B^2(x_0, \delta) = \emptyset$ for $t' \neq t$. Moreover, D^2u^1 is constant on $C_\gamma(t)$ for every $t \in (a, b)$. Assumption (1) and (3) in Proposition 5.3.1, together with the same argument using coarea formula and change of variable formula as in the proof of Lemma 5.3.3 yield for a.e. $t \in (a, b)$

$$\int_{C_{\gamma(t)}} |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^1 \to 0$$

and $D^2u^\ell = (\mathbf{n}^\ell/\mathbf{n}^1)D^2u^1, 2 \leq \ell \leq n+1$ $\mathcal{H}^1$ a.e on $C_{\gamma(t)}$.

On one such $C_{\gamma(t)}$, let $\mathbf{v}$ be its directional vector, then the chain rule in Lemma 5.3.1 and ∇u^1 is constant on $C_{\gamma(t)}$ imply

$$0 = \frac{d}{dt}\Big|_{t=0} \nabla u^1(\cdot + t\mathbf{v}) = (D^2u^1)\mathbf{v}$$

in the weak sense in $C_{\gamma(t)}$. Therefore,

$$\big(D^2u^\ell\big)\mathbf{v} = \frac{\mathbf{n}^\ell}{\mathbf{n}^1}(D^2u^1)\mathbf{v} = 0, \quad 2 \leq \ell \leq n+1 \text{ a.e. on } C_{\gamma(t)}.$$

Hence again by the chain rule in Lemma 5.3.1, $\nabla u^\ell, 2 \leq \ell \leq n+1$ is constant on $C_{\gamma(t)}$. Therefore, each ∇u^ℓ are constant on $C_{\gamma(t)}$ for a.e. $t \in (a, b)$. Furthermore, since for each $1 \leq \ell \leq n+1$, ∇u^ℓ is continuous, we conclude that ∇u^ℓ for all $1 \leq \ell \leq n+1$ are constant on all $C_{\gamma(t)}$ that foliates $B^2(x_0, \delta)$. On the other hand, each 2-dimensional region U of $B^2(x_0, \delta)$

automatically satisfies all the assumptions (1) and (3) in Proposition 5.3.1, hence the same argument for each $C_{\gamma(t)}$ gives ∇u^ℓ for all $2 \leq \ell \leq n+1$ are constant on the same region on which ∇u^1 is constant. This proves Du is either constant on 2-dimensional regions or constant on line segments in $B^2(x_0, \delta)$ joining the boundary. The proof of Proposition 5.3.1 for the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ follows from exactly the same argument as the last paragraph of the proof of Lemma 5.3.3.

Since assumptions (1)-(4) in Proposition 5.3.1 are invariant under change of coordinate system, the proof of Proposition 5.3.1 for any 2-dimensional plane P_2 follows from a change of coordinates. The proof for the base case is complete. □

5.3.2 Inductive step-developability and regularity on k-dimensional slices.

5.3.2.1 Developability. Based on our inductive assumption, we first prove a weaker result in k-dimensional slices of Ω than Proposition 5.3.1. That is, we prove that u is developable on any k-dimensional slices satisfying assumptions (1)-(4) of Proposition 5.3.1.

Proposition 5.3.2 *Suppose Proposition 5.3.1 is true for any $(k-1)$-dimensional slices of Ω on which assumptions (1)-(4) are satisfied. Let P_k be any k-dimensional plane such that assumptions (1)-(4) for u holds on $P_k \cap \Omega$, Then for every $x \in \Omega$, either u is affine in a neighborhood in $P_k \cap \Omega$ of x, or there exists a unique $(k-1)$-dimensional hyperplane $P^x_{k-1} \ni x$ of P_k such that u is affine on the connected component of x in $P^x_{k-1} \cap \Omega$.*

Before we prove the proposition, we need to define a terminology that is higher dimensional analogy of "line segments joining the boundary of some domain at both ends".

Definition 5.3.1 *By a k-plane P in Σ, we mean a connected component of a k-dimensional plane $P \cap \Sigma$, where Σ is any N-dimensional region with $N \geq k \geq 1$.*

Remark 5.3.2 *We emphasize here that such k-plane P in Σ refers to not the entire plane, but just the part inside a region. On the other hand, it refers to the entire connected part inside this region.*

Proof. Let $\mathbf{v}$ be any unit directional vector of P_k, let $\mathbf{v}_1, ..., \mathbf{v}_{k-1}$ be a set of *linearly independent unit* vectors of P_k perpendicular to $\mathbf{v}$, we parametrize the family of $(k-1)$-dimensional planes parallel to the space spanned by these vectors as follows:

$$P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} = \{z : z = y + s_1\mathbf{v}_1 + \cdots s_{k-1}\mathbf{v}_{k-1}, s_1, ..., s_{k-1} \in \mathbb{R}, y \in \text{span}\langle\mathbf{v}\rangle\}.$$

Lemma 5.3.4 *Given direction* $\mathbf{v}$*, for a.e.* $y \in \text{span}\langle\mathbf{v}\rangle$*,* u *is* $C^{1,1/2}_{\text{loc}}$ *and is an isometry on* $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$*. Moreover for every* $x \in P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$*,* u *is either affine on a* $(k-1)$*-dimensional region in* $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ *containing* x*, or affine on an* $(k-2)$*-plane in* $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ *passing through* x*.*

Proof. Since u satisfy assumptions (1)-(4) on $P_k \cap \Omega$, by Fubini Theorem, for a.e. $y \in \text{span}\langle\mathbf{v}\rangle$, assumptions (1)-(4) are also satisfied on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. Hence by our inductive assumption on $(k-1)$ slices of Ω, ∇u is $C^{1,1/2}_{\text{loc}}$ on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. Since by assumption (2) $\nabla u^T \nabla u = \mathrm{I}$ a.e., and hence everywhere in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ by continuity. Therefore, by assumption (1) and the chain rule in Lemma 5.3.1, u is an isometry on $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$.

Moreover by our inductive assumption, for every $x \in P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$, Du is either constant on a $(k-1)$-dimensional region in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ containing x, or constant on an $(k-2)$-plane in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ passing through x. Hence by the the chain rule in Lemma 5.3.1, u is either affine on $(k-1)$ dimensional regions in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$, or affine on $(k-2)$-planes in $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$. The proof is complete. □

Now we want to show that a substantial part of Lemma 5.3.4 is true for *every* rather than a.e. $(k-1)$-dimensional planes in Ω.

Lemma 5.3.5 *Given direction* $\mathbf{v}$*, for every* $y \in \text{span}\langle\mathbf{v}\rangle$ *and for every* $x \in P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$*,* u *is either an affine isometry on a* $(k-1)$*-dimensional region in* $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ *containing* x*, or an affine isometry on an* $(k-2)$*-plane in* $P^y_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}} \cap \Omega$ *passing through* x*.*

Remark 5.3.3 *We obtain from the proof of Lemma 5.3.4 that* u *is* C^1 *on a.e. planes. However, Lemma 5.3.4 does not imply* u *is* C^1 *on every plane because even though* Du *is* *continuous on a.e. planes, we cannot conclude from here that Du is continuous in Ω, so we cannot pass to the limit.*

Proof. Given $y \in \text{span}\langle \mathbf{v} \rangle$, Lemma 5.3.4 guarantees a sequence $y_m \in \text{span}\langle \mathbf{v} \rangle$, $y_m \to y$ such that Lemma 5.3.4 is true on $P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ for every m.

Let $x \in P^{y}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$, we divide the proof into the following two cases:

1. There is a sequence of $(k-2)$-planes P_m in $P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ on which u is an affine isometry and P_m converges to x in distance.
2. There does not exist such a sequence of $(k-2)$-planes.

Suppose we are in case 1, then the limit of P_m must also be a $(k-2)$-plane P in $P^{y}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ passing through x. Also since u is Lipschitz continuous, u must also be an affine isometry on P, which proves the Lemma in this case (Figure 7).

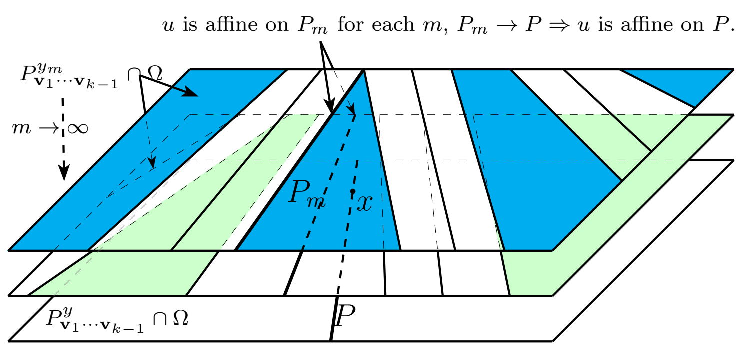

Figure 7:

Suppose now we are in case 2. If we cannot find such a sequence of $(k-2)$-planes, then we must find $x_m \in P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$, $x_m \to x$ with the property that there is $\epsilon > 0$ such that u is an affine isometry on $B^{k-1}(x_m, \epsilon) \subset P^{y_m}_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$. Otherwise, there will again be a sequence of $(k-2)$-planes (i.e. the boundaries of the maximal affine regions containing x_m) converging to x in distance, contradiction to the fact that we are in case 2. Continuity of u then must force u to be an affine isometry on $B^{k-1}(x, \epsilon) \subset P^{y}_{\mathbf{v}_1 \cdots \mathbf{v}_{n-1}} \cap \Omega$, which again proves the lemma in this case (Figure 8). The proof is complete. ☐

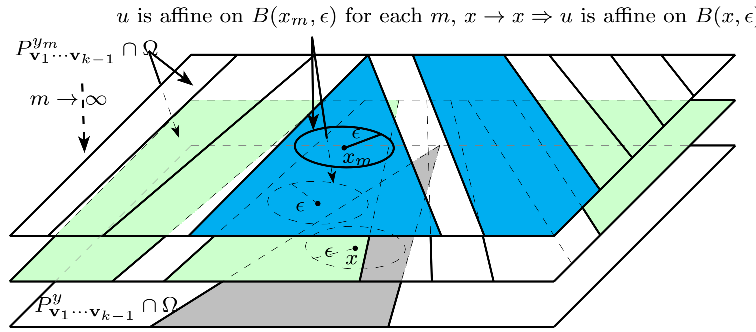

Figure 8:

Lemma 5.3.6 *Suppose u is an affine isometry on two line segments C_1 and C_2 in $P_k \cap \Omega$ intersecting at x. Moreover, x is in the interior of both C_1 and C_2. Let H be the convex hull of the line segments C_1 and C_2, then u is an affine isometry on $H \cap \Omega$.*

Remark 5.3.4 *The assumption x is in the interior is essential. This is why this proof fails for the counterexample of a conic with a singularity at zero.*

Proof. We parameterize C_1 and C_2 by $\{x + t\mathbf{v}_1, t \in [-a, b]\}$ and $\{x + s\mathbf{v}_2, s \in [-c, d]\}$, respectively, with both $\mathbf{v}_1$ and $\mathbf{v}_2$ unit vectors. We can assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, otherwise, the conclusion of the lemma is obvious. Since u is affine on both C_1 and C_2, $u(C_1)$ and $u(C_2)$ are both line segments in $\mathbb{R}^{n+1}$. We can again parameterize the lines that contains the line segments $u(C_1)$ and $u(C_2)$ by $u(x) + t\tilde{\mathbf{v}}_1$ and $u(x) + s\tilde{\mathbf{v}}_2$, both $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ are unit vectors due to the isometry assumption.

Let $y \in H \cap \Omega$, we can of course assume that y is neither in C_1 nor C_2, otherwise, there is nothing to prove. In this way, we can find a line L_3 passing through y and intersect C_1 at only one point, denoted x_{13}; and C_2 at only one point, denoted x_{23} and the segment $\overline{x_{13}x_{23}}$ lies inside Ω. Since $x_{13} \in C_1$, $x_{13} = x + t_0\mathbf{v}_1$ for some $t_0 \in [-a, b]$. Similarly $x_{23} = x + s_0\mathbf{v}_2$

for some $s_0 \in [-c, d]$. Then since

$$y = wx_{13} + (1-w)x_{23} \quad \text{for some } w \in [0,1], \tag{5.3.4}$$

It follows

$$y = x + wt_0\mathbf{v}_1 + (1-w)s_0\mathbf{v}_2.$$

To prove u is affine isometry on H, we need to prove

$$u(y) = u(x) + wt_0\tilde{\mathbf{v}}_1 + (1-w)s_0\tilde{\mathbf{v}}_2. \tag{5.3.5}$$

We first claim that the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between C_1 and C_2. Since x is in the interior of C_1 and C_2, we can construct a parallelogram $ABCD$ centered at x, with $A, C \in C_1$ and $B, D \in C_2$. Since u is an affine isometry on C_1 and C_2, $|u(A) - u(x)| = |A - x|, |u(B) - u(x)| = |B - x|, |u(C) - u(x)| = |C - x|$ and $|u(D) - u(x)| = |D - x|$. On the other hand, $|u(A) - u(B)| \le |A - B|$ and $|u(B) - u(C)| \le |B - C|$ since u is 1-Lipschitz (Figure 9).

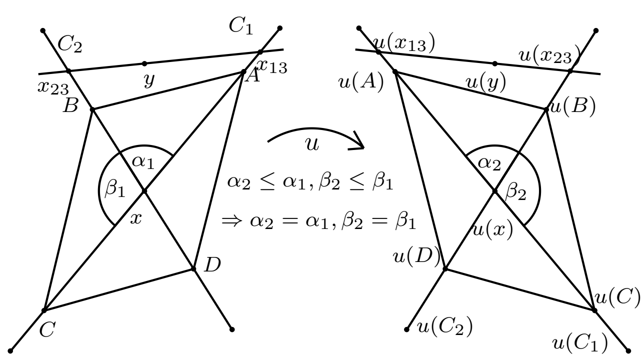

Figure 9:

This implies the angle α_2 between the line segments $\overline{u(x)u(A)}$ and $\overline{u(x)u(B)}$ must be smaller than or equal to the angle α_1 between $\overline{xA}$ and $\overline{xB}$, and the angle β_2 between the line segments $\overline{u(x)u(B)}$ and $\overline{u(x)u(C)}$ must be smaller than or equal to the angle β_1 between $\overline{xB}$ and $\overline{xC}$. Hence $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. This proves our claim.

Since by assumption, u is an affine isometry on $\overline{x_{13}x}$ and $\overline{x_{23}x}$, we have

$$u(x_{13}) - u(x) = t_0\tilde{\mathbf{v}}_1 \quad \text{and} \quad u(x_{23}) - u(x) = s_0\tilde{\mathbf{v}}_2.$$

for the same t_0, s_0 and unit vector $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$ as defined before. In particular, $|u(x_{13}) - u(x)| = |x_{13} - x|$ and $|u(x_{23}) - u(x)| = |x_{23} - x|$. Moreover, since the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between C_1 and C_2, we have $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$.

On the other hand, $u(\overline{x_{13}x_{23}})$ is a 1-Lipschitz curve, hence the length the the curve $u(\overline{x_{13}x_{23}})$, denoted by $|u(\overline{x_{13}x_{23}})|$, satisfies $|u(\overline{x_{13}x_{23}})| \leq |x_{13} - x_{23}|$. Altogether we have

$$|u(x_{13}) - u(x_{23})| \leq |u(\overline{x_{13}x_{23}})| \leq |x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|.$$

This implies

$$|u(\overline{x_{13}x_{23}})| = |u(x_{13}) - u(x_{23})|.$$

Hence the curve $u(\overline{x_{13}x_{23}})$ must coincide with line segment $\overline{u(x_{13})u(x_{23})}$. Therefore, u also maps the line segment $\overline{x_{13}x_{23}}$ onto a line segment $\overline{u(x_{13})u(x_{23})}$, which means u is affine on $\overline{x_{13}x_{23}}$.

Finally, since u is 1-Lipschitz, $|u(x_{13}) - u(y)| \leq |x_{13} - y|$ and $|u(x_{23}) - u(y)| \leq |x_{23} - y|$. However, since u is affine on $\overline{x_{13}x_{23}}$,

$$|u(x_{13}) - u(x_{23})| = |u((x_{13}) - u(y)| + |u(y) - u((x_{23})| \leq |x_{13} - y| + |y - x_{23}| = |x_{13} - x_{23}|.$$

But we have just proved that $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$. Hence $|u((x_{13}) - u(y)| = |x_{13} - y|$ and $|u((x_{23}) - u(y)| = |x_{23} - y|$. Therefore,

$$u(y) = wu(x_{13}) + (1 - w)u(x_{23})$$

for the same w as (5.3.4), which yields (5.3.5). The proof is complete. $\square$

Corollary 5.3.4 *Given a ℓ-dimensional ($\ell \le k$) neighborhood U in $P_k \cap \Omega$, and a line segment C in $P_k \cap \Omega$ such that $x = C \cap U$ is in the interior of both U and C, if u is an affine isometry on both U and C, then u is an affine isometry on the convex hull H of U and C inside Ω*

Proof. Let $y \in H \cap \Omega$. We need to show that $u(y) = u(x) + t\tilde{\mathbf{v}}$ for some $\tilde{\mathbf{v}}$ as a linear combination of directional vectors in $u(U)$ and $u(C)$ and $|t\tilde{\mathbf{v}}| = |y - x|$. Let P_y be a 2-dimensional plane that contains y and C. Then P_y intersects U at some line segment C_y. Since u is an affine isometry on both C and C_y, by Lemma 5.3.6, u is an affine isometry on the convex hull of C and C_y (Figure 10).

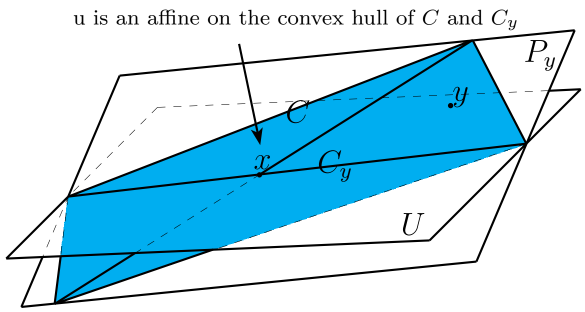

Figure 10:

Since this convex hull contains both y and x, this implies $u(y) = u(x) + t\tilde{\mathbf{v}}$ for some vector $\tilde{\mathbf{v}}$, $|t\tilde{\mathbf{v}}| = |y - x|$ and $\tilde{\mathbf{v}}$ is a linear combination of directional vectors of $u(C)$ and $u(C_y)$. Our claim then follows because $C_y \subset U$ and u is an affine isometry on U, so any vectors of $u(C_y)$ is a linear combination of vectors in $u(U)$. The proof is complete. □

By obvious induction we then have,

Corollary 5.3.5 *Suppose U_1 and U_2 are k_1 and k_2-dimensional neighborhoods ($k_1, k_2 \le k$) in $P_k \cap \Omega$ with nonempty intersections. Moreover, there is a point $x \in U_1 \cap U_2$ belonging to the interior of both U_1 and U_2. If u is an affine isometry on both U_1 and U_2, then u is an affine isometry on the convex hull of U_1 and U_2 inside Ω.*

Now we are ready to prove Proposition 5.3.2. Given $x \in P_k \cap \Omega$, we first claim that there exists a $(k-1)$-dimensional hyperplane P_0^x in P_k and a $(k-1)$-dimensional isometric affine neighborhood $U_0^x \subset P_0^x \cap \Omega$ such that $x \in U_0^x$. Suppose not, for all $(k-1)$-dimensional hyperplanes in $P_k \cap \Omega$ that pass through x, x is not contained in any $(k-1)$-dimensional affine neighborhood. In particular, let $\mathbf{v}_1, ..., \mathbf{v}_k$ be linearly independent vectors of P_k and Let $P^x_{\mathbf{v}_1 \cdots \hat{\mathbf{v}}_i \cdots \mathbf{v}_k}$, $i = 1, ..., n$ be the $(k-1)$-dimensional hyperplanes in Ω passing through x and parallel to the space spanned by $\mathbf{v}_1, ..., \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, ..., \mathbf{v}_k$. Then x is not contained in any $(k-1)$-dimensional affine neighborhood in $P^x_{\mathbf{v}_1 \cdots \hat{\mathbf{v}}_i \cdots \mathbf{v}_k} \cap \Omega$. Thus, by Lemma 5.3.5 there exists $(k-2)$-planes $P_{\hat{i}} \ni x$ in $P^x_{\mathbf{v}_1 \cdots \hat{\mathbf{v}}_i \cdots \mathbf{v}_k} \cap \Omega$ and u is an affine isometry on $P_{\hat{i}}$ for each i. By Corollary 5.3.5, u is an affine isometry on the convex hull of $P_{\hat{i}}$ for all $1 \leq i \leq k$ (Figure 11 Case 1). Let $\mathbf{v}_{\hat{i}}$ be a directional vector of $P_{\hat{i}}$. Since $P_{\hat{i}} \subset P^x_{\mathbf{v}_1 \cdots \hat{\mathbf{v}}_i \cdots \mathbf{e}_k}$, which is orthogonal to $\mathbf{v}_i$ Therefore, at least $k-1$ out of these k vectors are linearly independent. This convex hull has $k-1$ linearly independent directional vectors, hence it must be a $(k-1)$-dimensional neighborhood, contradiction to our assumption, which proves our claim.

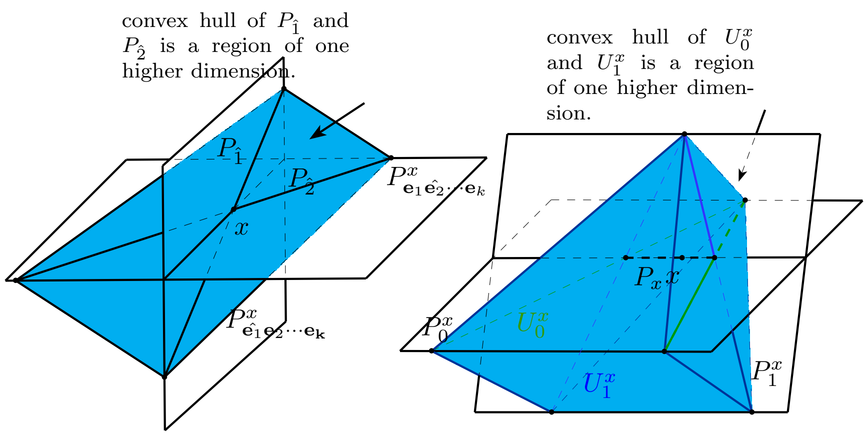

Figure 11:

Therefore, x must be contained in a $(k-1)$-dimensional isometric affine neighborhood $U_0^x \subset P_0^x \cap \Omega$ for some $(k-1)$-dimensional hyperplane P_0^x. If U_0^x is the entire connected component containing x in $P_0^x \cap \Omega$, then the conclusion of the proposition is achieved. Suppose not, then we can find a $(k-2)$-plane P_x in U_0^x, but *not* a $(k-2)$-plane in $P_0^x \cap \Omega$, i.e., it is away from $\partial\Omega$, on which u is an affine isometry. Let P_1^x be a different $(k-1)$-dimensional hyperplane such that $P_x = U_0^x \cap P_1^x$. Since $P_x \subset P_1^x \cap \Omega$ is not a $(n-2)$-plane in $P_1^x \cap \Omega$, by Lemma 5.3.5, x must be contained in a $(k-1)$-dimensional neighborhood $U_1^x \subset P_1^x \cap \Omega$ on which u is an affine isometry (Figure 11 Case 2). By Corollary 5.3.5, u is affine on the convex hull of U_0^x and U_1^x, whose interior is a k-dimensional neighborhood, which also achieve the conclusion of Proposition 5.3.2. The proof is complete. □

5.3.2.2 Regularity. We will prove the following lemma, which is the higher dimensional version of Lemma 5.3.2.

Lemma 5.3.7 *Suppose on a k-plane P $(1 \le k \le n)$ in Ω we have the following:*

1. *There is a sequence of smooth functions $u^\epsilon \in C^\infty(\Omega, \mathbb{R}^{n+1})$ such that*

$$\int_P |u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^k \to 0.$$

2. rank $D^2u^\ell \le 1$ *and D^2u^ℓ is symmetric a.e. on P for all $1 \le \ell \le n+1$.*

Then if u is affine on P, Du is constant on P.

Proof. Let $\mathbf{v}$ be any unit directional vector in P. By assumption 1 and the chain rule in Lemma 5.3.1, u is affine on P implies

$$Du(x)\mathbf{v} = \text{constant} \quad \text{for a.e. } x \in P.$$

Take the direction derivative one more time, together with assumption 1 we obtain,

$$(\mathbf{v})^T D^2u^\ell \mathbf{v} = 0 \quad \text{for a.e. } x \in P \tag{5.3.6}$$

for all $1 \le \ell \le n+1$. However, to show Du is constant on P, we need a conclusion stronger than (5.3.6), i.e.,

$$D^2u^\ell \mathbf{v} = 0 \quad \text{for a.e. } x \in P \tag{5.3.7}$$

for all $1 \le \ell \le n+1$. We will show under our assumptions, (5.3.6) implies (5.3.7). Indeed, By assumption 2, we can write D^2u^ℓ as

$$D^2u^\ell(x) = \lambda(x)\mathbf{b}(x) \otimes \mathbf{b}(x) \quad \text{a.e.}$$

for some scalar function λ and $\mathbf{b} \in \mathbb{S}^{n-1}$. Then (5.3.6) implies,

$$(\mathbf{v})^T \lambda(x)\mathbf{b}(x) \otimes \mathbf{b}(x)\mathbf{v} = \lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle^2 = 0 \quad \text{a.e.}$$

This then implies

$$\lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle = 0 \quad \text{a.e.}$$

Therefore,

$$D^2 u^\ell \mathbf{v} = \lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle \mathbf{b}(x) = 0 \quad \text{a.e.}$$

which is exactly (5.3.7). The proof of the lemma is complete. □

Proof of Theorem 5.3.1. The proof is exactly the same as the proof of Lemma 5.3.3. For the sake of completeness we repeat the proof here. Let P_k be any k-dimensional plane such that assumptions (1)-(4) in Proposition 5.3.1 for u holds on $P_k \cap \Omega$.

Suppose u is affine on some maximal neighborhood $U \subset P_k \cap \Omega$, by continuity of u, it is also affine on its closure $\overline{U} \cap \Omega$. Now if $x \in \partial U \cap \Omega$, then x is not contained in an affine neighborhood of u, therefore by Theorem 5.3.2, it is affine on a unique $(k-1)$-plane P_x^U in $P_k \cap \Omega$ passing through x, which implies $\partial U \cap \Omega \subset \bigcup_{x \in \partial U \cap \Omega} P_x^U$. We emphasize here $\partial U \cap \Omega$ because the entire ∂U may contain some part belonging to $\partial \Omega$, but ∂U inside Ω will not. On the other hand, suppose u is affine on some $(k-1)$-plane P_x^U in $P_k \cap \Omega$ passing through $x \in \partial U \cap \Omega$. Since u is affine on U and P_x^U, which intersect at x, it must be affine on the convex hull of U and P_x^U inside Ω by Corollary 5.3.5. But U is maximal, hence $\bigcup_{x \in \partial U \cap \Omega} P_x^U \subset \partial U \cap \Omega$. Therefore,

$$\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P_x^U.$$

Moreover, Corollary 5.3.5 ensures for $x, z \in \partial U \cap \Omega$, $P_x^U = P_z^U$ if $z \in P_x^U$ and $P_x^U \cap P_z^U \cap \Omega = \emptyset$ if $z \notin P_x^U$.

As was argued in the proof of Lemma 5.3.3 (Figure 6), it suffices to prove Proposition 5.3.1 locally. Let $x_0 \in P_k \cap \Omega$. We can choose small enough $\delta > 0$ so that for any region U on which u is affine, the k-dimensional ball $B^k(x_0, \delta) \subset P_k \cap \Omega$ intersects ∂U at no more than *two* $(k-1)$-planes belonging to ∂U. Indeed, since for any maximal constant region U, all $(k-1)$-planes in ∂U do not intersect inside Ω. Suppose for some maximal constant region U containing or near x_0, the angles between two $(k-1)$-planes $P^U_{x_1}$ and $P^U_{x_2}$ (if they are nonparallel) on ∂U is large, or the distance between them (if they are parallel) is large, we can choose δ small enough so $B^k(x_0, \delta)$ intersects at most one of them (Figure 2). Suppose there is a sequence of maximal affine regions U_m converging to x_0 in distance, in which case there are two $(k-1)$-planes $P^{U_m}_{x_1}$ and $P^{U_m}_{x_2}$ on ∂U_m whose angle (if they are nonparallel) or distance (if they are parallel) goes to zero. Then all the other $(k-1)$-planes on ∂U_m must be arbitrarily close to $\partial\Omega$, we can again choose δ small enough so that $B^k(x_0, \delta)$ is away from $\partial\Omega$ and hence it does not intersect a third $(k-1)$-planes on ∂U_m (Figure 3).

We now focus on $B^k(x_0, \delta) \subset P_k \cap \Omega$. For any $x \in B^k(x_0, \delta)$, we want to construct a $(k-1)$-plane P_x in $B^k(x_0, \delta)$ passing through x on which u is affine and $P_x \cap P_z \cap B^K(x_0, \delta) = \emptyset$ if $z \notin P_x$. For those x not contained in an affine region of u, this $(k-1)$-plane is given automatically by Theorem 5.3.2 and Corollary 5.3.5. If x is contained in an affine maximal region U of u, then it is affine on every $(k-1)$-planes in U that passes through it so we have to choose the appropriate one: 1) If $B^k(x_0, \delta)$ intersect only one $(k-1)$-plane P^U in $P_k \cap \Omega$ that belongs to ∂U, then we define P_x to be the $(k-1)$-plane in $B^k(x_0, \delta)$ passing through x and parallel to P^U; 2) If $B^k(x_0, \delta)$ intersects two $(k-1)$-planes P^U_1, P^U_2 in $P_k \cap \Omega$ that belongs to ∂U, let P_1 and P_2 be the two $(k-1)$-dimensional hyperplane that contain P^U_1 and P^U_2. If P_1 and P_2 are not parallel, let $O := P_1 \cap P_2$ and let P_x be the $(k-1)$-plane in $B^k(x_0, \delta)$ passing through x whose extension goes through O. If P_1 and P_2 are parallel, then we let P_x be the $(k-1$-plane $B^k(x_0, \delta)$ inside $B^k(x_0, \delta)$ passing through x and parallel to P_1. (Figure 4).

In this way, we construct a family of $(k-1)$-planes $\{P_x\}_{x \in B^k(x_0,\delta)}$ in $B^k(x_0, \delta)$ on which u is affine and $P_x \cap P_z \cap B^k(x_0, \delta) = \emptyset$ if $z \notin P_x$. For every $x \in B^k(x_0, \delta)$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector in $B^k(x_0, \delta)$ orthogonal to P_x. Since none of the P_xs intersect inside $B^k(x, \delta)$, they approach each other in an Lipschitz angle. Therefore, we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector fields. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

then has a unique solution $\gamma : (a,b) \to B^k(x_0,\delta)$ for some interval $(a,b) \in \mathbb{R}$ containing 0. Moreover, $\cup\{P_{\gamma(t)}\}_{t\in(a,b)} = B^k(x_0,\delta)$. Therefore, $\{P_{\gamma(t)}\}_{t\in(a,b)}$ is a foliation of $B^k(x_0,\delta)$ (Figure 5).

We define the function $h : B^k(x_0,\delta) \to B^k(x_0,\delta)$ as

$$h(x) = \gamma(t) \quad \text{if } x \in P_{\gamma(t)}.$$

Since none of the $P_{\gamma(t)}$ intersect inside $B^k(x_0,\delta)$, h is well defined and h is constant along each $P_{\gamma(t)}$, i.e., $h^{-1}(\gamma(t)) = P_{\gamma(t)}$. Since γ is Lipschitz, h is Lipschitz as well.

We now want to show the assumptions of Lemma 5.3.7 are satisfied along $P_{\gamma(t)}$ for a.e. $t \in (a,b)$. Let E_0 be the set of all $x \in B^k(x_0,\delta)$ such that $\operatorname{rank} D^2u^\ell(x) > 1$ or $D^2u^\ell(x)$ is not symmetric for any $1 \le \ell \le n+1$. By our assumptions in Proposition 5.3.1 on u, $|E_0| = 0$. As h is Lipschitz, we can apply the coarea formula for h to obtain,

$$\begin{aligned} 0 = \int_{E_0} |J_h(x)|dx &= \int_\gamma \mathcal{H}^{k-1}\big(E_0 \cap h^{-1}(w)\big)d\mathcal{H}^1(w) \\ &= \int_a^b \mathcal{H}^{k-1}\big(E_0 \cap h^{-1}(\gamma(t))\big)|\gamma'(t)|\,dt = \int_a^b \mathcal{H}^{k-1}\big(E_0 \cap P_{\gamma(t)}\big)|\gamma'(t)|\,dt. \end{aligned}$$

Therefore, for a.e. $t \in (a,b)$, $\mathcal{H}^{k-1}\big(E_0 \cap P_{\gamma(t)}\big) = 0$ by the fact $|\gamma'| = 1$. Moreover, by change of variable formula,

$$\begin{aligned} &\int_{B^k(x_0,\delta)} \big(|u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2\big) J_h \\ &\qquad = \int_\gamma \int_{h^{-1}(w)} |u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^{k-1} d\mathcal{H}^1(w) \\ &\qquad = \int_a^b \int_{h^{-1}(\gamma(t))} |u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^{k-1}|\gamma'(t)|\,dt \\ &\qquad = \int_a^b \int_{P_{\gamma(t)}} |u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^{k-1}|\gamma'(t)|\,dt. \end{aligned}$$

Since $|J_h| = |\gamma'| = 1$, together with assumption (1) in Proposition 5.3.1, we then have for a.e. $t \in (a,b)$,

$$\int_{P_{\gamma(t)}} |u^\epsilon - u|^2 + |Du^\epsilon - Du|^2 + |D^2u^\epsilon - D^2u|^2 d\mathcal{H}^{k-1} \to 0.$$

Therefore, the assumptions of Lemma 5.3.7 are satisfied along $P_{\gamma(t)}$ for a.e. $t \in (a,b)$. It follows that Du is constant on $P_{\gamma(t)}$ for a.e. $t \in (a,b)$. Then, if necessary, we choose an initial value for γ arbitrary close to x_0 such that Du is absoluately continuous on γ. Hence we conclude Du is $C^{0,1/2}$ on γ by the Sobolev embedding theorem.

Let F be the set of $t \in (a,b)$ such that Du is not constant along $P_{\gamma(t)}$, then $\mathcal{H}^1(F) = 0$. We modify Du to be constant along $P_{\gamma(t)}$ for each $t \in F$. Note that,

$$\mathcal{H}^k(\{P_{\gamma(t)} : t \in F\}) \leq (2\delta)^{k-1}\mathcal{H}^1(\{\gamma(t) : t \in F\}) = (2\delta)^{k-1}\mathcal{H}^1(F) = 0.$$

Hence Du is $C^{0,1/2}$ up to modification of a set of measure zero in $B^k(x_0,\delta)$. Moreover, Du is constant on $P_{\gamma(t)}$ for all t, which foliates $B^k(x_0,\delta)$. In addition, each k dimensional affine region of u automatically satisfies all the assumptions in Lemma 5.3.7. Thus Du is constant on any k dimensional region on which u is affine. Therefore, Du is either constant on a $(k-1)$-plane or k-dimensional region in $B^k(x_0,\delta)$. This proves Proposition 5.3.1. The proof is complete. □

5.4 DENSITY

In this section we show smooth isometric immersions are strongly dense in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ if Ω is a convex C^1 domain.

Theorem 5.4.1 *If Ω is a C^1 convex domain, then for every $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, there is a sequence of mappings $u_m \in I^{2,2} \cap C^\infty(\Omega, \mathbb{R}^{n+1}) \to u$ in $W^{2,2}$ norm.*

5.4.1 Foliations of the domain.

We have argued in the proof of Theorem 5.3.1 in section 5.3.2.2 that for every maximal region $U \subset \Omega$ on which u is affine, $\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P_x^U$, where P_x^U is some $(n-1)$-plane in Ω containing x with the property that for $x_1, x_2 \in \partial U \cap \Omega$, $P_{x_1}^U = P_{x_2}^U$ if $x_2 \in P_{x_1}^U$ and $P_{x_1}^U \cap P_{x_2}^U \cap \Omega = \emptyset$ if $x_2 \notin P_{x_1}^U$.

We say a maximal region on which u is affine is a *body* if its boundary contains more than *two* different $(n-1)$-planes in Ω.

Lemma 5.4.1 *It is sufficient to prove Theorem 5.4.1 for a function in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ with finite number of bodies.*

Proof. We will show that we can approximate a function $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$ by maps in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ with finite number of bodies. First we can assume that all bodies are pairwise disjoint, otherwise, they must be contained in one body since Du is continuous and constant on all bodies. Suppose there are infinitely many such disjoint bodies. Since they must be countable, we label them B_i. As Ω is bounded, $\sum |B_i| < \infty$. Given B_i for i large enough, there must be two $(n-1)$-planes $P_1^{B_i}$ and $P_2^{B_i}$ in ∂B_i whose angle between them is arbitrarily small if they are nonparallel, or whose distance is arbitrarily small if they are parallel. There all other $(n-1)$ planes in ∂B_i must be arbitrarily close to $\partial\Omega$. We call the regions bounded by $P_1^{B_i}$ and $P_2^{B_i}$ and $\partial\Omega$ U_i. From what we have argued, $R_i := U_i \setminus B_i$ (Figure 12 left) is arbitrarily small.

Moreover, since Ω is convex, $|R_i| \leq |B_i|$ for i sufficiently large. Hence for every $\epsilon > 0$, there is M such that $\sum_{i \geq M} |R_i| < \epsilon$. Then for every $i \geq M$, since u is affine on B_i, we can modify u to u^ϵ by affine extension to R_i (Figure 12 right). Obviously the new map u^ϵ satisfies $u^\epsilon \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ with $\|u^\epsilon\|_{W^{2,2}} \leq \|u\|_{W^{2,2}}$ and $Du^\epsilon \in O(n, n+1)$ a.e., which means $u^\epsilon \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$. Since $\sum_{i \geq M} |R_i| < \epsilon$ which is arbitrarily small, $\|u^\epsilon - u\|_{W^{2,2}}$ is arbitrarily small by absolute continuity of integrable functions. The proof is complete. □

Now we can just assume $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$ has finite number of bodies. Since each body as

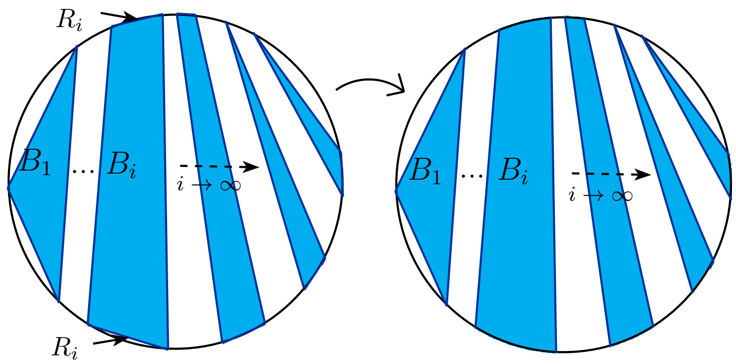

Figure 12:

the inverse image of a point, must be closed. Thus the finite union of them is closed. We now consider its complement, an open set $\widetilde{\Omega}$. Note that now for every n-dimensional region $U \subset \widetilde{\Omega}$, $\partial U \cap \widetilde{\Omega}$ consists of at most two $(n-1)$-planes.

We construct exactly the same foliation as in the proof of Lemma 5.3.3. First, for every $x \in \widetilde{\Omega}$, we will construct a $(n-1)$-plane P_x in $\widetilde{\Omega}$ passing through it on which Du is constant and $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$. For those x not contained in a constant region of Du, this $(n-1)$-plane in $\widetilde{\Omega}$ is given automatically by Theorem 5.3.1. If x is contained in a constant maximal region U of Du, then it is constant on every $(n-1)$-plane in U that passes through it so we have to choose the appropriate one: 1) If $\partial U \cap \widetilde{\Omega}$ consists of only one $(n-1)$-plane P^U in $\widetilde{\Omega}$, we define P_x to be the $(n-1)$-plane in $\widetilde{\Omega}$ passing through x and parallel to P^U; 2) If $\partial U \cap \widetilde{\Omega}$ consists of two $(n-1)$-planes P_1^U, P_2^U in $\widetilde{\Omega}$, let P_1 and P_2 be the two $(n-1)$ dimensional hyperplanes that contain P_1^U and P_2^U. If P_1 and P_2 are not parallel, let $A := P_1 \cap P_2$ and let P_x be the $(n-2)$-plane in $\widetilde{\Omega}$ passing through A and x. If P_1 and P_2 are parallel, then we let P_x be the $(n-2)$-plane passing through x and parallel to P_1. The component $P_x \cap \Omega$ is then our desired $(n-1)$-plane in Ω (and we still denote it P_x). In this way, we construct a family of $(n-1)$-plane $\{P_x\}_{x \in \widetilde{\Omega}}$ in $\widetilde{\Omega}$ on which Du is constant and $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$ (Figure 13).

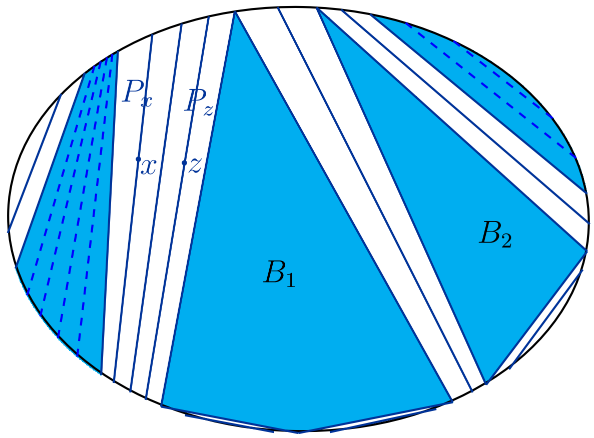

Figure 13:

For every $x \in \widetilde{\Omega}$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector orthogonal to the family P_x constructed above. Since none of the P_xs intersect inside $\widetilde{\Omega}$, which has a Lipschitz boundary, they approach each other in an Lipschitz angle. Therefore, we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector fields. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0$$

has a unique solution $\gamma : (a, b) \to \widetilde{\Omega}$ for some interval $(a, b) \subset \mathbb{R}$ containing 0. Note that $P_x = P_{\gamma(t)}$ if $x \in P_{\gamma(t)}$, therefore, $\{P_{\gamma(t)}\}_{t \in (a,b)}$ is a local foliation of $\widetilde{\Omega}$ such that Du is constant on $P_{\gamma(t)}$ for all $t \in (a, b)$ (Figure 14).

5.4.2 Leading curve, leading fronts, leading $(n-1)$-planes, covered domain, and moving frame.

We define some terminologies in this section and prove some of their properties.

Definition 5.4.1 *Let $\{P_x\}_{x \in \widetilde{\Omega}}$ be a family of $(n-1)$-planes in $\widetilde{\Omega}$ passing through x on which Du is constant and $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$ We say that a twice differentiable curve $\gamma : [0, \ell] \to \widetilde{\Omega}$ parametrized by arclength and has bounded second derivative is a leading curve*

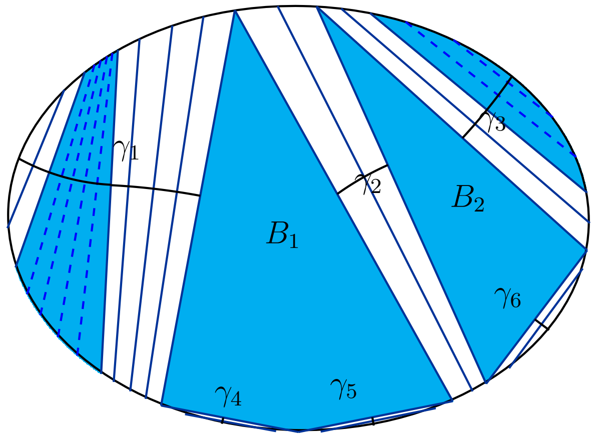

Figure 14:

if it is orthogonal at the point of intersection to all $P_x, x \in \widetilde{\Omega}$ such that $\gamma([0,\ell]) \cap P_x \neq \emptyset$ (Figure 15).

It is easy to see that γ constructed in Subsection 5.4.1 when restricted to the interval $[0,\ell]$ is a Leading curve, since by the ODE

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

$|\gamma'| = 1$ and $|\gamma''|$ is bounded as $\mathbf{N}$ is Lipschitz.

Definition 5.4.2 *The $(n-1)$-dimensional hyperplane $F_\gamma(t)$ orthogonal to $\gamma(t)$ at $t \in [0,\ell]$ is called the Leading front of γ at $t \in [0,\ell]$ (Figure 15).*

Remark 5.4.1 *It then follows from the definition of the Leading curve that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \widetilde{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0,\ell]$ such that $t \neq \tilde{t}$. Moreover, $F_\gamma(t) \cap \widetilde{\Omega} = F_\gamma(t) \cap \Omega$, otherwise, $F_\gamma(t) \cap B \neq \emptyset$ where B is one of the bodies in $\Omega \setminus \widetilde{\Omega}$. Since Du, being continuous, is constant on $F_\gamma(t) \cap \widetilde{\Omega}$ and B, it must be constant on their convex hull, which is again a body, contradiction to that a body is a maximal region. Therefore, $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \Omega = \emptyset$ for all $t, \tilde{t} \in [0,\ell], t \neq \tilde{t}$.* □

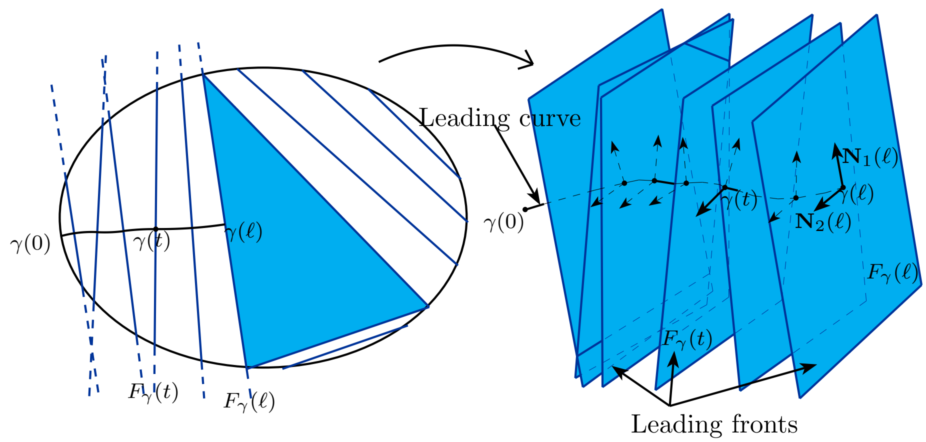

Figure 15:

We say that a curve γ covers the domain $A \subset \Omega$ if

$$A \subset \bigcup \{F_\gamma(t) : t \in [0, \ell]\}.$$

By $\Omega(\gamma)$ we refer to the biggest set covered by γ in Ω. We now restrict our attention to the covered domain $\Omega(\gamma)$. It is obvious that $\Omega(\gamma)$ is convex since it is bounded by $F_\gamma(0)$, $F_\gamma(\ell)$ and $\partial\Omega$.

From the construction in Subsection 5.4.1, the $(n-1)$-planes $P_{\gamma(t)}$ in $\widetilde{\Omega}$, $t \in [0, \ell]$ which constitute a local foliation of $\widetilde{\Omega}$ are global foliations of $\Omega(\gamma)$. Moreover, $P_{\gamma(t)} = F_\gamma(t) \cap \Omega(\gamma) = F_\gamma(t) \cap \Omega$ for all $t \in [0, \ell]$. We relabel them $P_\gamma(t)$ to be in consistence of notation and we name them:

Definition 5.4.3 *The component $P_\gamma(t) := F_\gamma(t) \cap \Omega$ is called the Leading $(n-1)$-planes in Ω of γ at $t \in [0, \ell]$.*

Let $\mathbf{N}_1(t), \mathbf{N}_2(t), ..., \mathbf{N}_{n-1}(t)$ be an orthonormal basis for the Leading front $F_\gamma(t)$ (Figure 15) such that $\mathbf{N}_i$ is Lipschitz for all $1 \le i \le n-1$ and $\det[\mathbf{N}_1(\tilde{t}), ..., \mathbf{N}_{n-1}(\tilde{t}), \gamma'(t)] = 1$. It is obvious such orthonormal basis exists because we can pick $\mathbf{N}_1(0), \mathbf{N}_2(0), ..., \mathbf{N}_{n-1}(0)$ as an othonormal basis for $F_\gamma(0)$ that form a positive orientation with $\gamma'(0)$ and then move this frame along γ in an orientation preserving way (note that γ is not a closed curve so this is possible). Let $\Phi : [0, \ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as,

$$\Phi(t, s) := \gamma(t) + s_1 \mathbf{N}_1(t) + \cdots + s_{n-1} \mathbf{N}_{n-1}(t), \tag{5.4.1}$$

where $s = (s_1, ..., s_{n-1})$. Then we can represent the Leading front at $t \in [0, \ell]$ as,

$$F_\gamma(t) = \{\Phi(t, s), s = (s_1, ..., s_{n-1}) \in \mathbb{R}^{n-1}\}. \tag{5.4.2}$$

For each $t \in [0, \ell]$, define the open set,

$$\Sigma^\gamma(t) = \{s = (s_1, ..., s_{n-1}) \in \mathbb{R}^{n-1} : \Phi(t, s) \in \Omega\}. \tag{5.4.3}$$

It is obvious that $0 \in \Sigma^\gamma(t)$, hence it is non-empty open subset of $\mathbb{R}^{n-1}$. Then we can also parametrize the Leading planes as

$$P_\gamma(t) = \{\Phi(t, s), s = (s_1, ..., s_{n-1}) \in \Sigma^\gamma(t)\}. \tag{5.4.4}$$

Now define,

$$\Sigma^\gamma := \{(t, s), \Phi(t, s) \in \Omega\}. \tag{5.4.5}$$

Of course we can also write,

$$\Sigma^\gamma = \{(t, s), t \in [0, \ell], s = (s_1, ..., s_{n-1}) \in \Sigma^\gamma(t)\}.$$

We will focus on the restriction of Φ in Σ^γ. However, if no confusion is caused, we still denote such restriction Φ. It is easy to see Φ maps Σ^γ into $\Omega(\gamma)$. Indeed, if $x = \Phi(t, s)$ for some $(t, s) \in \Sigma^\gamma$, by definition of Σ^γ, $\Phi(t, s) \in \Omega$. On the other hand, $\Phi(t, s) \in F_\gamma(t)$, thus, $x = \Phi(t, s) \in F_\gamma(t) \cap \Omega \subset \Omega(\gamma)$.

Lemma 5.4.2 $\Phi : \Sigma^\gamma \to \Omega(\gamma)$ *is one-to-one and onto. In particular,*

$$\Omega(\gamma) = \{\Phi(t, s), (t, s) \in \Sigma^\gamma\} = \bigcup \{P_\gamma(t) : t \in [0, \ell]\}.$$

Proof. We first show one-to-one. Suppose $\Phi(t_1, s_1) = \Phi(t_2, s_2)$ while $(t_1, s_1) \neq (t_2, s_2)$. Since $s \to \Phi(t, s)$ is obviously one-to-one by the definition of Φ, it must be $t_1 \neq t_2$. We have argued in Remark 5.4.1 that $F_\gamma(t_1) \cap F_\gamma(t_2) \cap \Omega = \emptyset$. Therefore, $F_\gamma(t_1) \cap F_\gamma(t_2) \cap \Omega(\gamma) = \emptyset$ since $\Omega(\gamma) \subset \Omega$. However, $\Phi(t_1, s_1) \in F_\gamma(t_1)$ and $\Phi(t_2, s_2) \in F_\gamma(t_2)$, contradiction to $\Phi(t_1, s_1) = \Phi(t_2, s_2)$.

We will now show onto. Let $x \in \Omega(\gamma)$, then $x = \Phi(t, s)$ for some $t \in [0, \ell]$ and $s \in \mathbb{R}^{n-1}$. Since $x \in \Omega(\gamma)$, $\Phi(t, s) \in \Omega(\gamma) \subset \Omega$, hence $(t, s) \in \Sigma^\gamma$. The proof is complete. □

Apparently we can rewrite $\Phi(t, s) := \gamma(t) + s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t), t \in [0, \ell], s \in \mathbb{R}^{n-1}$ as

$$\Phi(t, S, s) = \gamma(t) + S\big(s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t)\big), t \in [0, \ell], s \in \mathbb{S}^{n-2}, S \geq 0.$$

We then rewrite the representation of Leading front in (5.4.2) in an equivalent way:

$$F_\gamma(t) = \{\Phi(t, S, s), S \geq 0, s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}\}. \tag{5.4.6}$$

For each $t \in [0, \ell]$ and $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$, define the scalar function,

$$S_s^\gamma(t) = \sup\{S \geq 0 : \Phi(t, S, s) \in \Omega\}. \tag{5.4.7}$$

That is, $S_s^\gamma(t)$ is the distance from $\gamma(t)$ to $\partial\Omega$ in the direction $s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t)$. From the definition of $\Sigma^\gamma(t)$ and Σ^γ,

$$\Sigma^\gamma(t) = \{(S, s) : s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\} \quad \text{and,} \tag{5.4.8}$$

$$\Sigma^\gamma = \{(t, S, s), t \in [0, \ell], s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\}. \tag{5.4.9}$$

Let us compute the moving frame for γ: Since $|\gamma'(t)| = 1$, $\gamma''(t) \cdot \gamma'(t) = 0$, we can then write

$$\gamma''(t) = \kappa_1(t)\mathbf{N}_1(t) + \cdots + \kappa_{n-1}(t)\mathbf{N}_{n-1}(t)$$

Similarly we can also write

$$\mathbf{N}_i' = \kappa_{i_0}\gamma' + \kappa_{i_1}\mathbf{N}_1 + \cdots + \kappa_{i_{n-1}}\mathbf{N}_{n-1}$$

It is easy to see that $\kappa_{i_0} = -\kappa_i$, $\kappa_{i_i} = 0$ and $\kappa_{i_j} = -\kappa_{j_i}$. The matrix would be

$$\begin{pmatrix} \gamma' \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & \ldots & \kappa_{n-1} \\ -\kappa_1 & 0 & \kappa_{1_2} & \ldots & \kappa_{1_{n-1}} \\ -\kappa_2 & -\kappa_{1_2} & 0 & \ldots & \kappa_{2_{n-1}} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -\kappa_{n-1} & -\kappa_{1_{n-1}} & -\kappa_{2_{n-1}} & \ldots & 0 \end{pmatrix} \begin{pmatrix} \gamma' \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \end{pmatrix}$$

Given two *non-parallel* leading front $F_\gamma(t)$ and $F_\gamma(\tilde{t})$, denote their intersection-a $(n-2)$ dimensional plane $F(t, \tilde{t})$. Given $s = (s_1, ...s_{n-1}) \in \mathbb{S}^{n-2}$, define $L_s(t, \tilde{t})$ as the distance from $\gamma(t)$ to $F(t, \tilde{t})$ along the direction $s_1\mathbf{N}_1(t) + ... + s_{n-1}\mathbf{N}_{n-1}(t)$ (we set $L_s(t, \tilde{t}) = +\infty$ if it does not hit $F(t, \tilde{t})$ along a certain direction $s_1\mathbf{N}_1(t) + ... + s_{n-1}\mathbf{N}_{n-1}(t)$) (Figure 16). We then define,

$$L^\gamma_s(t) := \inf\{L_s(t, \tilde{t}) : \tilde{t} \neq t\}. \tag{5.4.10}$$

Since all $F(t, \tilde{t})$ are outside Ω, $L^\gamma_s(t) \geq S^\gamma_s(t)$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$ (Figure 16).

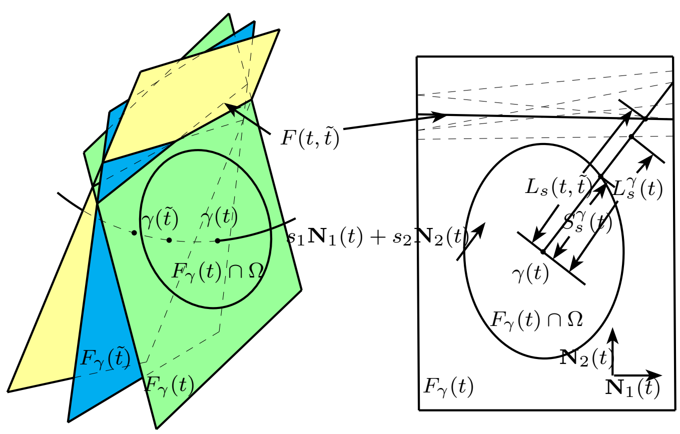

Figure 16:

Lemma 5.4.3 $L^\gamma_s(t)\big(s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)\big) \leq 1$ *for all* $t \in [0, \ell]$ *and* $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$.

Proof. Suppose $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel, we equate the representation of these two Leading fronts,

$$\gamma(t) + s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t) = \gamma(\tilde{t}) + r_1\mathbf{N}_1(\tilde{t}) + \cdots + r_{n-1}\mathbf{N}_{n-1}(\tilde{t}).$$

This is a linear system of n equations and $2n-2$ unknowns $(s_1, ..., s_{n-1}, r_1, ..., r_{n-1})$. Solution for this system of equations exists because the two Leading front are not parallel. Then direct computation using Crammer's rule gives the formula for $F(t,\tilde{t})$ explicitly,

$$F(t,\tilde{t}) = \{x \in F_\gamma(t) : (x - \gamma(t)) \cdot \Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}\mathbf{N}_1(t) - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}\mathbf{N}_{n-1}(t)\Big) = 1\},$$

where

$$h_i(t,\tilde{t}) := \det[\mathbf{N}_1(\tilde{t}), ..., \mathbf{N}_{n-1}(\tilde{t}), \mathbf{N}_i(t)]$$

for $1 \le i \le n-1$. and

$$H(t,\tilde{t}) = \det[\mathbf{N}_1(\tilde{t}), ..., \mathbf{N}_{n-1}(\tilde{t}), \gamma(t) - \gamma(\tilde{t})]$$

Note that $H(t,\tilde{t}) \neq 0$ since $\gamma(t) - \gamma(\tilde{t})$ is not parallel to $F_\gamma(\tilde{t})$.

We first claim that

$$L_s(t,\tilde{t})\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) \le 1. \tag{5.4.11}$$

Indeed, we divide the situation into two cases. In the first case, suppose we travel from $\gamma(t)$ along a given direction $s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t)$ and hit $F(t,\tilde{t})$, then for $x \in F(t,\tilde{t})$,

$$x - \gamma(t) = L_s(t,\tilde{t})\big(s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t)\big).$$

Therefore,

$$\begin{aligned} L_s(t,\tilde{t})\big(s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t)\big) \cdot \Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}\mathbf{N}_1(t) - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}\mathbf{N}_{n-1}(t)\Big) \\ = L_s(t,\tilde{t})\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) = 1. \end{aligned} \tag{5.4.12}$$

Suppose for a certain direction $s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t)$ we do not hit $F(t,\tilde{t})$, in which case we set $L_s(t,\tilde{t}) = +\infty$, then we must hit $F(t,\tilde{t})$ through the direction $-s_1\mathbf{N}_1(t) - \cdots - s_{n-1}\mathbf{N}_{n-1}(t)$, therefore, by (5.4.12),

$$L_{-s}(t,\tilde{t})\Big(\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 + \cdots + \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) = 1.$$

In particular, since $L_{-s}(t,\tilde{t}) > 0$,

$$\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 + \cdots + \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1} > 0.$$

We then must have,

$$L_s(t,\tilde{t})\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) < 0. \tag{5.4.13}$$

(5.4.12) and 5.4.13) together gives in either cases, (5.4.11) holds, which proves our claim.

We secondly claim,

$$L_s^\gamma(t)\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) \leq 1 \tag{5.4.14}$$

for all $t, \tilde{t} \in [0, \ell]$ and $s \in \mathbb{S}^{n-2}$. Indeed, if for a given $t, \tilde{t}$ and $s \in \mathbb{S}^{n-2}$, $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel, and

$$\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) \geq 0,$$

then,

$$\begin{aligned} L_s^\gamma(t)\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) \\ \leq L_s(t,\tilde{t})\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) = 1 \end{aligned}$$

which gives (5.4.14) for this case. If for a certain $t, \tilde{t}$ and $s \in \mathbb{S}^{n-2}$, $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are still not parallel, but

$$-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1} < 0,$$

then

$$L_s^\gamma(t)\Big(-\frac{h_1(t,\tilde{t})}{H(t,\tilde{t})}s_1 - \cdots - \frac{h_{n-1}(t,\tilde{t})}{H(t,\tilde{t})}s_{n-1}\Big) < 0,$$

hence (5.4.14) is obviously satisfied. Finally, if $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are parallel, then $h_i(t,\tilde{t}) = 0$ for all $1 \le i \le n-1$, hence the (5.4.14) is again satisfied. Therefore (5.4.14) is true for all for all $t, \tilde{t} \in [0, \ell]$ and $s \in \mathbb{S}^{n-2}$.

We thirdly claim that

$$-\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \to \kappa_i(t), 1 \le i \le n-1. \tag{5.4.15}$$

as $\tilde{t} \to t$. Indeed, $H(t,\tilde{t}) \approx t - \tilde{t}$ as $\tilde{t} \to t$. Moreover,

$$h_i(t,t) = \det[\mathbf{N}_1(t), ..., \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] = 0.$$

Then,

$$-\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \approx -\frac{h_i(t,\tilde{t}) - h_i(t,t)}{t-\tilde{t}} \to \det[\mathbf{N}_1'(t), ..., \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] + \cdots + \det[\mathbf{N}_1(t), ..., \mathbf{N}_{n-1}'(t), \mathbf{N}_i(t)]. \tag{5.4.16}$$

Recall that

$$\mathbf{N}_i' = \kappa_{i_0}\gamma' + \kappa_{i_1}\mathbf{N}_1 + \cdots + \kappa_{i_{n-1}}\mathbf{N}_{n-1}$$

with $\kappa_{i_0} = -\kappa_i$, $\kappa_{i_i} = 0$ and $\kappa_{i_j} = -\kappa_{j_i}$. Plug this expression into (5.4.16) and it is easy to see that all other terms vanish except

$$\det[\mathbf{N}_1(t), ..., \mathbf{N}_i'(t), ..., \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] = -\kappa_i \det[\mathbf{N}_1(t), ..., \gamma'(t), ..., \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)] = \kappa_i$$

because $\det[\mathbf{N}_1(t), ..., \mathbf{N}_{n-1}(t), \gamma'(t)] = 1$. This is proves (5.4.15).

Passing (5.4.14) to the limit we obtain the lemma. The proof is complete. □

Recall $S_s^\gamma(t)$ as defined in (5.4.7) satisfies $S_s^\gamma(t) \le L_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ due to the fact that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \Omega = \emptyset$ for all $t, \tilde{t} \in [0,\ell], \tilde{t} \ne t$, we then have,

Corollary 5.4.1 *$S_s^\gamma(t)\big(s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)\big) \le 1$ for all $t \in [0,\ell]$ and $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$.*

Proof. If $s_1\kappa_1(t) + ... + s_{n-1}\kappa_{n-1}(t) \geq 0$, then

$$S_s^\gamma(t)\big(s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)\big) \leq L_s^\gamma(t)\big(s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)\big) \leq 1. \quad (5.4.17)$$

If $s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t) < 0$, then (5.4.17) is obviously true. Thus we have (5.4.17) for all $t \in [0, \ell]$ and $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$. The proof is complete. □

From the definition of Φ in (5.4.1), Φ is Lipschtiz, hence its Jacobian $J_\Phi = \det D\Phi$ exists a.e. on Σ^γ, where Σ^γ has two equivalent representation (5.4.5) and (5.4.9). We will show the Corollary 5.4.1 implies $J_\Phi > 0$ a.e. on Σ^γ where

Lemma 5.4.4 $J_\Phi(t, s) = 1 - s_1\kappa_1(t) - \cdots - s_{n-1}\kappa_{n-1}(t) > 0$ *for all* $(t, s) \in \Sigma^\gamma$.

Proof. Differentiating $\Phi(t, s)$ with respect to $(t, s_1, ..., s_{n-1})$ gives,

$$J_\Phi(t, s) = \det[\gamma'(t) + s_1\mathbf{N}_1'(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}'(t), \mathbf{N}_1(t), ..., \mathbf{N}_{n-1}(t)]. \quad (5.4.18)$$

Substitute the expression of $\gamma'(t), \mathbf{N}_1'(t), ..., \mathbf{N}_{n-1}'(t)$ as linear combinations of $\gamma(t), \mathbf{N}_1'(t), ..., \mathbf{N}_{n-1}(t)$ into (5.4.18), we obtain, after Gaussian elimination, that,

$$J_\Phi(t, s) = 1 - s_1\kappa_1(t) - \cdots - s_{n-1}\kappa_{n-1}(t). \quad (5.4.19)$$

If $s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t) \leq 0$, then obviously $J_\Phi(t, s) > 0$. Suppose now $s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t) > 0$. By (5.4.9), $\Sigma^\gamma = \{(t, S, s), t \in [0, \ell], s = (s_1, ...s_{n-1}) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\}$, thus,

$$\begin{aligned} s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t) &= |s|\Big(\frac{s_1}{|s|}\kappa_{1,m}(t) + \cdots + \frac{s_{n-1}}{|s|}\kappa_{n-1,m}(t)\Big) \\ &< S_s^\gamma(t)\Big(\frac{s_1}{|s|}\kappa_{1,m}(t) + \cdots + \frac{s_{n-1}}{|s|}\kappa_{n-1,m}(t)\Big) \leq 1 \end{aligned}$$

by Corollary 5.4.1. Therefore, $J_\Phi(t, s) > 0$ for all $(t, s) \in \Sigma^\gamma$. The proof is complete. □

5.4.3 Moving frames in the target space.

We are now in a position to define the moving frame in the target space $\mathbb{R}^{n+1}$. Let $\mathbf{N}_i(t), 1 \leq i \leq n-1$ be as in Subsection 5.4.2. Define the leading curve corresponding to γ in $u(\Omega(\gamma))$ to be

$$\tilde{\gamma} := u \circ \gamma.$$

Recall we defined in (5.4.1) that

$$\Phi(t,s) = \gamma(t) + s_1\mathbf{N}_1(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}(t),$$

and in (5.4.4) that

$$P_\gamma(t) = \{\Phi(t,s), s = (s_1, ..., s_{n-1}) \in \Sigma^\gamma(t)\}.$$

We also recall from Subsection 5.4.1 that Du is constant on $P_\gamma(t)$ for each $t \in [0,\ell]$. Hence for each $t \in [0,\ell]$, $Du \circ \Phi$ is constant on $\Sigma^\gamma(t)$.

Consider the moving frame $(\tilde{\gamma}', \mathbf{v}_1, ..., \mathbf{v}_{n-1}, \mathbf{n})$,

$$\begin{aligned}
\tilde{\gamma}'(t) &= \tilde{\gamma}'(t) \\
\mathbf{v}_1(t) &= Du(\gamma(t))\mathbf{N}_1(t) \\
&\cdots \\
\mathbf{v}_{n-1}(t) &= Du(\gamma(t))\mathbf{N}_{n-1}(t) \\
\mathbf{n}(t) &= \tilde{\gamma}'(t) \times \mathbf{v}_1(t) \times ... \times \mathbf{v}_{n-1}(t)
\end{aligned}$$

Since u is an isometric affine map along $P_\gamma(t)$ for each $t \in [0,\ell]$ we obtain

$$u(\Phi(t,s)) = \tilde{\gamma}(t) + s_1\mathbf{v}_1(t) + \cdots + s_{n-1}\mathbf{v}_{n-1}(t). \tag{5.4.20}$$

for all $t \in [0,\ell]$ and $s \in \Sigma^\gamma(t)$. Differentiating with respect to t to get

$$\begin{aligned}
&Du(\Phi(t,s))(\gamma'(t) + s_1\mathbf{N}_1'(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}'(t)) \\
&\qquad\qquad = \tilde{\gamma}'(t) + s_1\mathbf{v}_1'(t) + \cdots + s_{n-1}\mathbf{v}_{n-1}'(t) \qquad (5.4.21)
\end{aligned}$$

and differentiating with respect to $(s_1, ..., s_{n-1})$ to get,

$$Du(\Phi(t,s))\mathbf{N}_i(t) = \mathbf{v}_i(t) \tag{5.4.22}$$

By the matrix of moving frame in Subsection 5.4.2

$$\mathbf{N}_i' = -\kappa_i\gamma' - \kappa_{1_i}\mathbf{N}_1 - \cdots + 0\cdot\mathbf{N}_i + \kappa_{i_{i+1}}\mathbf{N}_{i+1} + \cdots + \kappa_{i_{n-1}}\mathbf{N}_{n-1},$$

together with (5.4.21) and (5.4.22) we have

$$\begin{aligned} Du(\Phi(t,s))(1 - s_1\kappa_1(t) - \cdots - s_{n-1}\kappa_{n-1}(t))\gamma'(t) \\ + s_1(\kappa_{1_2}(t)\mathbf{v}_1(t) + \cdots + \kappa_{1_{n-1}}(t)\mathbf{v}_{n-1}(t)) \\ + \cdots + s_{n-1}(-\kappa_{1_{n-1}}(t)\mathbf{v}_1(t) + \cdots + 0\cdot\mathbf{v}_{n-1}(t)) \\ = \tilde{\gamma}'(t) + s_1\mathbf{v}_1'(t) + \cdots + s_{n-1}\mathbf{v}_{n-1}'(t) \end{aligned} \tag{5.4.23}$$

(5.4.21) with $s = 0$ gives,

$$Du(\Phi(t,0))\gamma'(t) = \tilde{\gamma}'(t).$$

Since $Du \circ \Phi$ is constant on $\Sigma^\gamma(t)$ for each $t \in [0,\ell]$,

$$Du(\Phi(t,s))\gamma'(t) = Du(\Phi(t,0))\gamma'(t) = \tilde{\gamma}'(t) \tag{5.4.24}$$

Matching coefficients in (5.4.23) gives,

$$\mathbf{v}_i' = -\kappa_i\tilde{\gamma}' - \kappa_{1_i}\mathbf{v}_1 - \cdots + 0\cdot\mathbf{v}_i + \kappa_{i_{i+1}}\mathbf{v}_{i+1} + \cdots + \kappa_{i_{n-1}}\mathbf{v}_{n-1}.$$

The Darboux frame of $\tilde{\gamma}$ would be

$$\begin{pmatrix} \tilde{\gamma}' \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & ... & \kappa_{n-1} & \kappa_{\mathbf{n}} \\ -\kappa_1 & 0 & \kappa_{1_2} & ... & \kappa_{1_{n-1}} & 0 \\ -\kappa_2 & -\kappa_{1_2} & 0 & ... & \kappa_{2_{n-1}} & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ -\kappa_{n-1} & -\kappa_{1_{n-1}} & -\kappa_{2_{n-1}} & ... & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\gamma}' \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{n} \end{pmatrix}$$

(5.4.22) and (5.4.24) shows at each point in $\Omega(\gamma)$, Du maps an orthonormal frame to another orthonormal frame and this orthonormal frame only depends on t.

5.4.4 Change of variable formula.

Recall that $\Phi : \Sigma^\gamma \to \Omega(\gamma)$ is one-to-one and onto, where Σ^γ was defined in (5.4.5), For $(t, s) \in \Sigma^\gamma$, let $u_i(t, s) := (\frac{\partial}{\partial x_i} u) \circ \Phi(t, s)$, note that u_i is the ith column of $Du \circ \Phi$. The following holds for all $(t, s) \in \Sigma^\gamma$ and for simplicity we omit it. Since $Du^T \mathbf{n} \cdot \gamma' = \mathbf{n} \cdot Du\gamma' = \mathbf{n} \cdot \tilde{\gamma}' = 0$ and $Du^T \mathbf{n} \cdot \mathbf{N}_j = \mathbf{n} \cdot Du\mathbf{N}_j = \mathbf{n} \cdot \mathbf{v}_j = 0$ for all $1 \le j \le n-1$, we have $Du^T \mathbf{n} = 0$, i.e. $u_i \cdot \mathbf{n} = 0$ for all $1 \le i \le n$. Thus,

$$\begin{aligned} u_i &= \big(u_i \cdot \tilde{\gamma}'\big)\tilde{\gamma}' + \sum_{j=1}^{n-1}\big(u_i \cdot \mathbf{v}_j\big)\mathbf{v}_j + \big(u_i \cdot \mathbf{n}\big)\mathbf{n} \\ &= (u_i \cdot \tilde{\gamma}')\tilde{\gamma}' + \sum_{j=1}^{n-1}(u_i \cdot \mathbf{v}_j)\mathbf{v}_j = (u_i \cdot Du\gamma')\tilde{\gamma}' + \sum_{j=1}^{n-1}(u_i \cdot Du\mathbf{N}_j)\mathbf{v}_j \\ &= (Du^T u_i \cdot \gamma')\tilde{\gamma}' + \sum_{j=1}^{n-1}(Du^T u_i \cdot \mathbf{N}_j)\mathbf{v}_j = (\mathbf{e}_i \cdot \gamma')\tilde{\gamma}' + \sum_{j=1}^{n-1}(\mathbf{e}_i \cdot \mathbf{N}_j)\mathbf{v}_j \end{aligned} \tag{5.4.25}$$

where $\mathbf{e}_i = (0, .., 1, ..., 0)$. Note that the right hand side of (5.4.25) is independent of s, by differentiating with respect to $s = (s_1, ..., s_{n-1})$ we have,

$$(D\frac{\partial}{\partial x_i}u)(\Phi(t, s))\mathbf{N}_i(t) = 0 \text{ for all } i \tag{5.4.26}$$

Differentiating u_i with respect to t we obtain,

$$\begin{aligned} &(D\frac{\partial}{\partial x_i}u)(\Phi(t, s))(\gamma'(t) + s_1\mathbf{N}_1'(t) + \cdots + s_{n-1}\mathbf{N}_{n-1}'(t)) \\ &= (\mathbf{e}_i \cdot \gamma''(t))\tilde{\gamma}'(t) + (\mathbf{e}_i \cdot \gamma'(t))\tilde{\gamma}''(t) + \sum_{j=1}^{n-1}(\mathbf{e}_i \cdot \mathbf{N}_j'(t))\mathbf{v}_j(t) + \sum_{j=1}^{n-1}(\mathbf{e}_i \cdot \mathbf{N}_j(t))\mathbf{v}_j'(t). \end{aligned} \tag{5.4.27}$$

If we write out $\mathbf{N}_i'$ as a linear combination of γ' and $\mathbf{N}_i, i = 1, ...n-1$, the left hand side of (5.4.27) becomes

$$(1 - s_1\kappa_1(t) - \cdots - s_{n-1}\kappa_{n-1}(t))(D\frac{\partial}{\partial x_i}u)(\Phi(t, s))\gamma'(t).$$

If we write out $\tilde{\gamma}''$ and $\mathbf{v}_j', j = 1, ..., n-1$ as a linear combination of $\tilde{\gamma}'$ and $\mathbf{v}_j, j = 1, ..., n-1$ and $\mathbf{n}$ by the matrix of the moving frame in target space in Subsection 5.4.3, it is easy to see that all terms on the right hand side of (5.4.27) cancel each other, only the term

$(\mathbf{e}_i\cdot\gamma'(t))\kappa_{\mathbf{n}}(t)\mathbf{n}(t)$ left. By Lemma 5.4.4, $1-s_1\kappa_1(t)-\cdots-s_{n-1}\kappa_{n-1}(t)>0$ for all $(t,s)\in\Sigma^\gamma$. Therefore,

$$(D\frac{\partial}{\partial x_i}u)(\Phi(t,s))\gamma'(t)=\frac{(\mathbf{e}_i\cdot\gamma'(t))\kappa_{\mathbf{n}}(t)\mathbf{n}(t)}{1-s_1\kappa_1(t)-\cdots-s_{n-1}\kappa_{n-1}(t)} \tag{5.4.28}$$

Since Φ is Lipschitz with $J_\Phi(t,s)=1-s_1\kappa_1(t)-\cdots-s_{n-1}\kappa_{n-1}(t)>0$, Change of variable $x=\Phi(t,s)$ with (5.4.20) and (5.4.28) give,

$$\begin{aligned}\int_{\Omega(\gamma)}|u(x)|^2dx=\int_0^\ell\int_{\Sigma^\gamma(t)}&|\tilde{\gamma}(t)+s_1\mathbf{v}_1(t)+\cdots+s_{n-1}\mathbf{v}_{n-1}(t)|^2\\&\cdot\big(1-s_1\kappa_1(t)-\cdots-s_{n-1}\kappa_{n-1}(t)\big)d\mathcal{H}^{n-1}(s)dt.\end{aligned} \tag{5.4.29}$$

$$\int_{\Omega(\gamma)}|Du(x)|^2dx=n|\Omega(\gamma)|. \tag{5.4.30}$$

$$\begin{aligned}\int_{\Omega(\gamma)}|D^2u(x)|^2dx&=\int_0^\ell\int_{\Sigma^\gamma(t)}\frac{\sum_i(\mathbf{e}_i\cdot\gamma'(t))^2\kappa_{\mathbf{n}}^2(t)}{\big(1-s_1\kappa_1(t)+\cdots+s_{n-1}\kappa_{n-1}(t)\big)}d\mathcal{H}^{n-1}(s)dt\\&=\int_0^\ell\int_{\Sigma^\gamma(t)}\frac{\kappa_{\mathbf{n}}^2(t)}{\big(1-s_1\kappa_1(t)-\cdots-s_{n-1}\kappa_{n-1}(t)\big)}d\mathcal{H}^{n-1}(s)dt.\end{aligned} \tag{5.4.31}$$

5.4.5 Approximation process for one covered domain.

Recall from (5.4.10) that for a given $t\in[0,\ell]$ and $s\in\mathbb{S}^{n-2}$,

$$L_s^\gamma(t):=\inf\{L_s(t,\tilde{t}):\tilde{t}\neq t\}. \tag{5.4.32}$$

where $L_s(t,\tilde{t})$ is the distance from $\gamma(t)$ to the intersection of two leading fronts $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ along direction $s_1\mathbf{N}_1(t)+\cdots+s_{n-1}\mathbf{N}_{n-1}(t)$. Also recall $S_s^\gamma(t)$ defined in (5.4.7) is the distance from $\gamma(t)$ to $\partial\Omega$ in the direction $s_1\mathbf{N}_1+\cdots+s_{n-1}\mathbf{N}_{n-1}$. Since all Leading fronts meet outside Ω, $L_s^\gamma(t)\geq S_s^\gamma(t)$ for all $s\in\mathbb{S}^{n-2}$ and $t\in[0,\ell]$.

Lemma 5.4.5 *There exists a sequence of isometries $u_m\in W^{2,2}(\Omega(\gamma),\mathbb{R}^{n+1})$ converging strongly to u with the property that each u_m has a suitable leading curve $\gamma_m:[0,\ell_m]\to\mathbb{R}^n$ for which $L_s^{\gamma_m}(t)-S_s^{\gamma_m}(t)>\rho_m>0$ for all $s\in\mathbb{S}^{n-2}$ and $t\in[0,\ell_m]$.*

Proof. The proof is exactly the same as the 2-dimensional case, [46], proposition 3.2, because its proof is independent of dimensions. For the sake of completeness we include it here.

Consider $D_m : \mathbb{R}^n \to \mathbb{R}^n$ as the dilation centered at $x_0 = \gamma(0)$ by

$$D_m(x) := \frac{m}{m-1}(x - x_0) + x_0.$$

and as a correspondence, $\tilde{D}_m : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as the dilation centered at $y_0 = u(x_0)$ by

$$\tilde{D}_m(y) := \frac{m}{m-1}(y - y_0) + y_0.$$

Let $\Omega_m(\gamma) = D_m(\Omega(\gamma))$ and $\tilde{u}_m : \Omega_m(\gamma) \to \mathbb{R}^{n+1}$ as

$$\tilde{u}_m := \tilde{D}_m \circ u \circ D_m^{-1}$$

Notice that $\Omega(\gamma) \subset \Omega_m(\gamma)$ (Figure 17), so $\tilde{u}_m$ is well defined over $\Omega(\gamma)$, the sequence $\tilde{u}_m \to u$ in $W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ and it is an isometric immersion. However, we still need some further construction to have a suitable leading curve γ_m that satisfies $L_s^{\gamma_m}(t) - S_s^{\gamma_m}(t) > \rho_m > 0$. The curve

$$\gamma_m(t) := D_m \circ \gamma(\frac{m-1}{m}t).$$

defined on $[0, \frac{m}{m-1}\ell]$ is a leading curve for $\tilde{u}_m$, put,

$$\ell_m^* := \sup\{t : \gamma_m(t) \in \Omega(\gamma) \text{ and } F_{\gamma_m}(t) \cap F_\gamma(\ell) \cap \overline{\Omega} = \emptyset\}.$$

Finally we define our desired sequence of isometric immersion u_m as $\tilde{u}_m$ for the region of $\Omega(\gamma)$ covered by $F_{\gamma_m}(t), 0 \le t \le \ell_m^* - 1/m$ and extend by affine extension to the entire $\Omega(\gamma)$ (Figure 17), i.e.,

$$u_m(x) = \begin{cases} \tilde{u}_m(x) \quad \text{if } x \in F_{\gamma_m}(t) \text{ for } 0 \le t \le \ell_m^* - \frac{1}{m} \\ D\tilde{u}_m\Big(\gamma_m\big(\ell_m^* - \frac{1}{m}\big)\Big)\Big(x - \gamma_m\big(\ell_m^* - \frac{1}{m}\big)\Big) \\ +\tilde{u}_m\Big(\gamma_m\big(\ell_m^* - \frac{1}{m}\big)\Big) \quad \text{otherwise.} \end{cases} \tag{5.4.33}$$

It is obvious each u_m admits a leading curve (still denoted by γ_m) satisfying $L_s^{\gamma_m}(t) - S_s^{\gamma_m}(t) > \rho_m > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell_m]$. The proof is complete. □

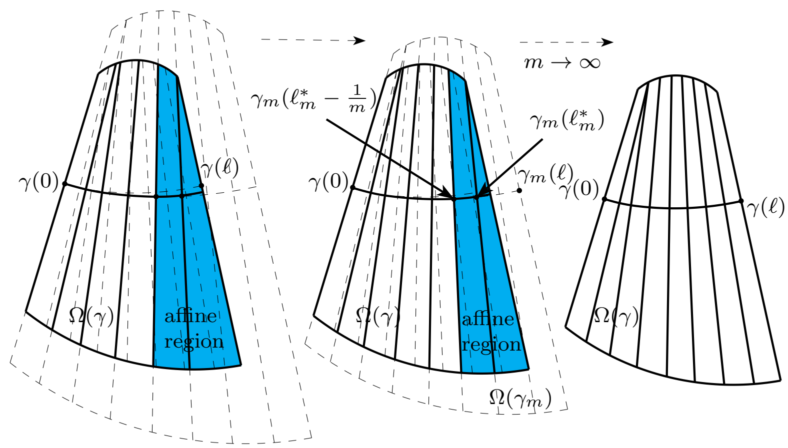

Figure 17:

Remark 5.4.2 *The construction in (5.4.33) is called affine extension to some region. From now on, we will simply use the term "affine extension" without giving the explicit formula.* □

Remark 5.4.3 *By the above Lemma, we can just assume u has a suitable Leading curve γ that satisfies $L_s^\gamma(t) - S_s^\gamma(t) > \rho > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$.* □

Lemma 5.4.6 *Suppose $L_s^\gamma(t) - S_s^\gamma(t) > \rho > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$. Then we can construct a sequence of smooth maps in $I^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ converging strong to u.*

Proof. The idea of construction is to construct a smooth curve γ_m Approximating γ. We do not know yet this curve is a Leading curve of u_m or not, so we cannot call the $(n-2)$-dimensional hyperplane orthogonal to γ_m at t Leading fronts. Instead we call them *orthogonal fronts* and denote them $F_{\gamma_m}(t)$. If we manage to show all such orthogonal fronts meet outside $\Omega(\gamma_m)$, γ_m becomes a Leading curve for u_m and $F_{\gamma_m}(t)$ are actually the Leading fronts. We then define u_m to be isometric affine mapping along each Leading front $F_{\gamma_m}(t)$. Since all the Leading fronts intersect outside Ω, u_m is well-defined.

We first need the following lemma,

Lemma 5.4.7 *There exists smooth curve γ_m such that $\gamma_m(t) \to \gamma(t)$ in $W^{2,p}([0,\ell],\mathbb{R}^n)$ for all $1 \le p < \infty$ and satisfies $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0,\ell]$.*

Proof. The construction needs six steps:

Step 1. Recall from the matrix of moving frame defined in Subsection 5.4.2 that $\gamma''(t) = \kappa_1(t)\mathbf{N}_1(t) + \cdots + \kappa_{n-1}(t)\mathbf{N}_{n-1}(t)$, with κ_i bounded. We can choose uniformly bounded smooth functions $\tilde{\kappa}_{i,m} \to \kappa_i$ a.e. on $[0,\ell]$, and hence in measure due to $[0,\ell]$ is bounded. Since the sequence $\tilde{\kappa}_{i,m}$ are uniformly bounded, it follows $\tilde{\kappa}_{i,m} \to \kappa_i$ in L^p for all $1 \le p < \infty$. Similarly we can find uniformly bounded smooth functions $\kappa_{i_j,m} \to \kappa_{i_j}$ a.e. on $[0,\ell]$ (hence in L^p for all $1 \le p < \infty$) for $\kappa_{i_j}, 1 \le i,j \le n-1$. By solving ordinary differential equations with respect to the moving frame

$$\begin{pmatrix} \Gamma'_m \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}' = \begin{pmatrix} 0 & \tilde{\kappa}_{1,m} & \tilde{\kappa}_{2,m} & \dots & \tilde{\kappa}_{n-1,m} \\ -\tilde{\kappa}_{1,m} & 0 & \kappa_{1_2,m} & \dots & \kappa_{1_{n-1},m} \\ -\tilde{\kappa}_{2,m} & -\kappa_{1_2,m} & 0 & \dots & \kappa_{2_{n-1},m} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -\tilde{\kappa}_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \dots & 0 \end{pmatrix} \begin{pmatrix} \Gamma'_m \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}$$

We obtain a unique orthogonal frame $(\Gamma'_m(t), \mathbf{N}_{1,m}(t), ..., \mathbf{N}_{n-1,m}(t))$ with initial condition $\Gamma'_m(0) = \gamma'(0)$, and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. We can then define

$$\Gamma_m(t) = \Gamma(0) + \int_0^t \Gamma'_m(\tau)d\tau.$$

We want to show $(\Gamma'_m, \mathbf{N}_{1,m}, ..., \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, ..., \mathbf{N}_{n-1})$ uniformly. This result is given by the following theorem due to Opial, [45], Theorem 1.

Lemma 5.4.8 (Opial) *Suppose the linear systems of differential equations,*

$$x'(t) = A_k(t)x(t), \quad x(0) = a_k, \quad k = 0, 1, 2, ... \tag{5.4.34}$$

admit a solution $x_k(t)$ in $[0,\ell]$ for all k. Suppose $a_k \to a_0$,

$$\int_0^t A_k(s)\,ds \to \int_0^t A_0(s)\,ds$$

uniformly for all $t \in [0,\ell]$ and A_k is a bounded sequence in L^1, i.e. $\sup_k \|A_k\|_{L^1([0,\ell])} < \infty$, *then the solutions*

$$x_k(t) \to x_0(t) \quad \textit{uniformly.}$$

Since $\tilde{\kappa}_{i,m} \to \kappa_i$ and $\kappa_{i_j,m} \to \kappa_{i_j}$ in L^p for all $1 \le p < \infty$, in particular for $p = 1$, the conditions in Lemma 5.4.8 are satisfied, hence $(\Gamma'_m, \mathbf{N}_{1,m}, ..., \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, ..., \mathbf{N}_{n-1})$ uniformly. Since $\Gamma''_m = \tilde{\kappa}_{1,m}\mathbf{N}_{1,m} + \cdots + \tilde{\kappa}_{n-1,m}\mathbf{N}_{n-1,m}$, Γ''_m are uniformly bounded, and $\Gamma''_m \to \gamma''$ a.e. (and hence in L^p for all $1 \le p < \infty$), Poincaré inequality for intervals show that $\Gamma_m \to \gamma$ in $W^{2,p}[0,\ell], \mathbb{R}^n)$ for all $1 \le p < \infty$.

However Γ_m is not our desired curve since we cannot guarantee all its leading fronts intersect outside $\overline{\Omega}$. This happens if Γ_m is too "curvy". We need to "flatten" its curvature continuously. This needs to be done in several steps:

Step 2. We construct $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, ..., \tilde{\tilde{\kappa}}_{n-1,m})$ continuous on $t \in [0,\ell]$ and for each $t \in [0,\ell]$ and $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{2})(s_1\tilde{\tilde{\kappa}}_{1,m}(t) + \cdots + s_{n-1}\tilde{\tilde{\kappa}}_{n-1,m}(t)) \le 1 \tag{5.4.35}$$

where

$$S_s^{\Gamma_m}(t) = \sup\{S \ge 0 : \Gamma_m(t) + S\big(s_1\mathbf{N}_{1,m}(t) + \cdots + s_{n-1}\mathbf{N}_{n-1,m}(t)\big) \in \Omega\}.$$

We first need the following lemma using implicit function theorem for C^1 functions.

Lemma 5.4.9 *$S_s^{\Gamma_m}(t)$ is uniformly continuous on $(s,t) \in \mathbb{S}^{n-2} \times [0,\ell]$ and $S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t)$ uniformly on $(s,t) \in \mathbb{S}^{n-2} \times [0,\ell]$.*

Proof. Let $t_0 \in [0,\ell]$ and $s^0 = (s_1^0, ..., s_{n-1}^0) \in \mathbb{S}^{n-1}$ be arbitrary. We parameterize locally $\mathbb{S}^{n-2}$ by the polar coordinates: $s_i = s_i(\theta)$ where $\theta = (\theta_1, ..., \theta_{n-2}) \in U_1 \subset [0,\pi)^{n-3} \times [0, 2\pi)$. Let $\theta^0 \in U_1$ be such that $s_i^0 = s_i(\theta^0)$.

Let $\gamma^0 = \gamma(t_0)$ and $\mathbf{N}_i^0 = \mathbf{N}_i(t_0)$. Let x_0 be the intersection of the line segment $L = \{\gamma^0 + S\big(s_1^0\mathbf{N}_1^0 + ... + s_{n-1}^0\mathbf{N}_{n-1}^0\big), 0 \le S\}$ and $\partial\Omega$. Then $x_0 = \gamma^0 + S_0\big(s_1^0\mathbf{N}_1^0 + ... + s_{n-1}^0\mathbf{N}_{n-1}^0\big)$

for some $S_0 > 0$.

Since Ω is a C^1 domain, there exits an open subset of $U_2 \subset \mathbb{R}^{n-1}$ and a C^1 function $\alpha : U_2 \to \partial\Omega$ and $\alpha(\eta_1^0, ..., \eta_{n-1}^0) = x_0$ for some $(\eta_1^0, ..., \eta_{n-1}^0) \in U_2$.

Consider $F : \mathbb{R}^n \times \mathbb{R}^{n \times n-1} \times U_1 \times \mathbb{R} \times U_2 \to \mathbb{R}^n$

$$F(\gamma, \mathbf{N}_1, ...\mathbf{N}_{n-1}, \theta, S, \eta_1, ..., \eta_{n-1}) = \gamma + S\big(s_1(\theta)\mathbf{N}_1 + \cdots + s_{n-1}(\theta)\mathbf{N}_{n-1}\big) - \alpha(\eta_1, ..., \eta_{n-1}).$$

Since $x_0 \in \partial\Omega \cap L$,

$$F(\gamma^0, \mathbf{N}_1^0, ..., \mathbf{N}_{n-1}^0, \theta^0, S^0, \eta_1^0, ..., \eta_{n-1}^0) = 0.$$

Let

$$\mathbf{x} = (\gamma, \mathbf{N}_1, ..., \mathbf{N}_{n-1}, \theta)$$

$$\mathbf{y} = (S, \eta_1, ..., \eta_{n-1})$$

Denote

$$\alpha_k := \frac{\partial \alpha}{\partial \eta_k}, 1 \le k \le n-1.$$

Then

$$\det\big[(\frac{\partial F}{\partial \mathbf{y}}(\gamma^0, \mathbf{N}_1^0, ..., \mathbf{N}_{n-1}^0, \theta^0, S^0, \eta_1^0, ..., \eta_{n-1}^0)\big] = \det\big[s_1(\theta^0)\mathbf{N}_1^0 + \cdots + s_{n-1}(\theta^0)\mathbf{N}_{n-1}^0, \alpha_1(\eta_1^0, ..., \eta_{n-1}^0), ..., \alpha_{n-1}(\eta_1^0, ..., \eta_{n-1}^0)\big] \neq 0.$$

Otherwise, the line segment L would be parallel to the tangent plane of $\partial\Omega$ at x_0, which is not possible since Ω is convex.

By implicit function theorem, there is an open neighborhood $V_1 \subset \mathbb{R}^n \times \mathbb{R}^{n \times n-1} \times U_1$ of $\mathbf{x}_0 = (\gamma^0, \mathbf{N}_1^0, ..., \mathbf{N}_{n-1}^0, \theta^0)$, $V_2 \subset \mathbb{R} \times U_2$ of $\mathbf{y}_0 = (S^0, \eta_1^0, ..., \eta_{n-1}^0)$, and a C^1 diffeomorphism $\mathbf{y} : V_1 \to V_2$ such that

$$F(\mathbf{x}, \mathbf{y}(\mathbf{x})) = F(\mathbf{x}, S(\mathbf{x}), \eta_1(\mathbf{x}), ..., \eta_{n-1}(\mathbf{x})) = 0.$$

for all $\mathbf{x} \in V_1$.

Since γ, $\mathbf{N}_i, 1 \le i \le n-1$ are Lipschitz on $[0,\ell]$ and $\Gamma_m \to \gamma$ uniformly and $\mathbf{N}_{i,m} \to \mathbf{N}_i$ uniformly on $[0,\ell]$ for all $1 \le i \le n-1$, there exists an open interval $O \subset \mathbb{R}$ containing t_0, an open subset $\Delta \subset U_1$ containing θ_0 and an integer M such that for all $t \in [0,\ell] \cap O$, $\theta \in \Delta$ and $m \ge M$,

$$\mathbf{x}(t,\theta) = (\gamma(t), \mathbf{N}_1(t), ..., \mathbf{N}_{n-1}(t), \theta) \in V_1 \quad \text{and,}$$

$$\mathbf{x}_m(t,\theta) = (\Gamma_m(t), \mathbf{N}_{1,m}(t), ..., \mathbf{N}_{n-1,m}(t), \theta) \in V_1.$$

Apparently $\mathbf{x}_m(t,\theta) \to \mathbf{x}(t,\theta)$ uniformly for all $t \in [0,\ell] \cap O$ and $\theta \in \Delta$. Since S is C^1 on $\mathbf{x} \in V_1$,

$$S(\mathbf{x}_m(t,\theta)) \to S(\mathbf{x}(t,\theta)) \text{ uniformly on } t \in [0,\ell] \cap O \text{ and } \theta \in \Delta. \tag{5.4.36}$$

Moreover, since S is C^1 on $\mathbf{x} \in V_1$ and $\mathbf{x}$ is uniformly continuous on $t \in [0,\ell] \cap O$ and $s \in s(\Delta)$, S is uniformly continuous on $t \in [0,\ell] \cap O$ and $s \in s(\Delta)$.

Now note that since $F\big(\mathbf{x}(t,\theta), \mathbf{y}(\mathbf{x}(t,\theta))\big) = 0$ and $F\big(\mathbf{x}_m(t,\theta), \mathbf{y}(\mathbf{x}_m(t,\theta))\big) = 0$, for each $s = s(\theta) \in s(\Delta) \subset \mathbb{S}^{n-2}$, we have $S_s^{\gamma}(t) = S(\mathbf{x}(t,\theta))$ and $S_s^{\Gamma_m}(t) = S(\mathbf{x}_m(t,\theta))$. Thus by (5.4.36),

$$S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t) \text{ uniformly on } t \in [0,\ell] \cap O \text{ and } s \in s(\Delta),$$

and $S_s^{\Gamma_m}(t)$ is uniformly continuous on $t \in [0,\ell] \cap O$ and $s \in s(\Delta)$.

It remains to observe that both $[0,\ell]$ and $\mathbb{S}^{n-2}$ are both compact can be covered by a finite union of neighborhoods on which (5.4.36) holds. The proof is complete. □

Define

$$\lambda_m(t) := \min\Big\{1, \frac{1}{\sup_{|s|=1}\{(S_s^{\Gamma_m}(t) + \frac{\varrho}{2})(s_1\tilde{\kappa}_{1,m}(t) + \cdots + s_{n-1}\tilde{\kappa}_{n-1,m}(t))\}}\Big\}$$

where $\tilde{\kappa}_{i,m}, 1 \le i \le n-1$ are those found in Step 1. A first observation is $0 < \lambda_m \le 1$. Indeed, there must exist $s \in \mathbb{S}^{n-2}$ such that $s_1\tilde{\kappa}_{1,m}(t) + ...s_{n-1}\tilde{\kappa}_{n-1,m}(t) \ge 0$ so the supreme over all $s \in \mathbb{S}^{n-2}$ must be nonnegative. On the other hand, $S_s^{\Gamma_m}$ as well as all $\tilde{\kappa}_{i,m}$ are bounded so λ_m is bounded below by a positive number.

A second observation is that λ_m is continuous. Indeed, by Lemma 5.4.9, $(S_s^{\Gamma_m}(t)+\frac{\rho}{2})(s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t))$ is uniformly continuous on $(s,t) \in \mathbb{S}^{n-2} \times [0,\ell]$. Hence the supreme over $\mathbb{S}^{n-2}$ is attained and a simple argument gives $h(t) := \sup_{|s|=1}\{(S_s^{\Gamma_m}(t)+\frac{\rho}{2})(s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t))\}$ is continuous.

We then define a vector valued function $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, ..., \tilde{\tilde{\kappa}}_{n-1,m})$ as

$$(\tilde{\tilde{\kappa}}_{1,m}(t), ..., \tilde{\tilde{\kappa}}_{n-1,m}(t)) := \lambda_m(t)(\tilde{\kappa}_{1,m}(t), ..., \tilde{\kappa}_{n-1,m}(t))$$

$\tilde{\tilde{\kappa}}_m$ is obviously continuous. It remains to show $\tilde{\tilde{\kappa}}_m$ satisfies (5.4.35). Indeed, for any $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$,

$$\begin{aligned}(S_s^{\Gamma_m}(t)+\frac{\rho}{2})(s_1\tilde{\tilde{\kappa}}_{1,m}(t)+\cdots+s_{n-1}\tilde{\tilde{\kappa}}_{n-1,m}(t))\\ = \lambda_m(S_s^{\Gamma_m}(t)+\frac{\rho}{2})(s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t)).\end{aligned}$$

If $s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t) \geq 0$, then by the definition of λ_m,

$$\begin{aligned}\lambda_m(t)(S_s^{\Gamma_m}(t)+\frac{\rho}{2})\big(s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t)\big)\\ \leq \min\big\{(S_s^{\Gamma_m}(t)+\frac{\rho}{2})\big(s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t)\big),\, 1\big\} \leq 1.\end{aligned}$$

If $s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t) < 0$, then

$$\lambda_m(t)(S_s^{\Gamma_m}(t)+\frac{\rho}{2})(s_1\tilde{\kappa}_{1,m}(t)+\cdots+s_{n-1}\tilde{\kappa}_{n-1,m}(t)) < 0 \leq 1.$$

Thus (5.4.35) is satisfied.

Step 3. We want to show that $(\tilde{\tilde{\kappa}}_{1,m}, ..., \tilde{\tilde{\kappa}}_{n-1,m}) \to (\kappa_1, ..., \kappa_{n-1})$ a.e. Indeed, we know that $\tilde{\kappa}_m = (\tilde{\kappa}_{1,m}, ..., \tilde{\kappa}_{n-1,m}) \to (\kappa_1, ..., \kappa_{n-1})$ a.e.. Therefore, all we need to show is $\lambda_m \to 1 \quad$ a.e..

By possibly replacing λ_m by its subsequence, it suffices to prove $\lambda_m \to 1$ in measure. From the definition of λ_m, it is enough to show the Lebesgue measure of the set:

$$E_m = \{t \in [0, l], \exists s \in \mathbb{S}^{n-2}, (S_s^{\Gamma_m}(t) + \frac{\rho}{2})(s_1\tilde{\kappa}_{1,m}(t) + \cdots + s_{n-1}\tilde{\kappa}_{n-1,m}(t)) > 1\}$$

goes to zero.

First by assumption, $L_s^{\gamma}(t) - S_s^{\gamma}(t) > \rho > 0$ and by Lemma 5.4.3, $L_s^{\gamma}(t)(\kappa_1(t)s_1 + \cdots + \kappa_{n-1}(t)s_{n-1}) \leq 1$, thus,

$$(S_s^{\gamma}(t) + \rho)(s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)) \leq 1. \tag{5.4.37}$$

for all $t \in [0, \ell]$ and $s \in \mathbb{S}^{n-2}$. Indeed, if $s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)) < 0$, (5.4.37) is obvious. If $s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)) \geq 0$,

$$(S_s^{\gamma}(t) + \rho)(s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)) \leq L_s^{\gamma}(t)(\kappa_1(t)s_1 + \cdots + \kappa_{n-1}(t)s_{n-1}) \leq 1$$

which again gives (5.4.37).

If $t \in E_m$, there is $s \in \mathbb{S}^{n-2}$ such that

$$s_1\tilde{\kappa}_{1,m}(t) + \cdots + s_{n-1}\tilde{\kappa}_{n-1,m}(t) > \frac{1}{S_s^{\Gamma_m}(t) + \rho/2}.$$

Therefore all $t \in E_m$ and our choice of $s = s(t)$ as above, we have,

$$\begin{aligned} |\tilde{\kappa}_m(t) - \kappa(t)| &\geq s_1\tilde{\kappa}_{1,m}(t) + \cdots + s_{n-1}\tilde{\kappa}_{n-1,m}(t) - (s_1\kappa_1(t) + \cdots + s_{n-1}\kappa_{n-1}(t)) \\ &> \frac{\rho/2 + S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)}{(S_s^{\Gamma_m}(t) + \rho/2)(S_s^{\gamma}(t) + \rho)} \geq \frac{\rho/2 - |S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)|}{\rho^2/2}. \end{aligned}$$

By Lemma 5.4.9,

$$S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t) \text{ uniformly on } s \in \mathbb{S}^{n-2} \text{ and } t \in [0, \ell],$$

then we can find m sufficiently large so that $|S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)| < \rho/4$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$. Since $\tilde{\kappa}_m \to \kappa$ a.e.,

$$\lim_{m\to\infty} |E_m| \leq \lim_{m\to\infty} |\{t : |\tilde{\kappa}_m(t) - \kappa(t)| \geq \frac{1}{2\rho}\}| = 0$$

which is what we wanted to show.

Step 4. Since $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, ..., \tilde{\tilde{\kappa}}_{n-1,m})$ are continuous, for each m we can find κ_m smooth and $|\tilde{\tilde{\kappa}}_m - \kappa_m| \to 0$ uniformly on $t \in [0, \ell]$. Hence for m sufficiently large,

$$(S_s^{\Gamma_m}(t) + \frac{\rho}{4})(s_1\kappa_{1,m}(t) + \cdots + s_{n-1}\kappa_{n-1,m}(t)) \leq 1 \tag{5.4.38}$$

Step 5. We now define our desired curve γ_m. Given $\kappa_m = (\kappa_{1,m}, ...\kappa_{n-1,m})$ smooth as found in Step 4, and $\kappa_{i_j,m} \to \kappa_{i_j}$ found in step 1, we again solve the ODE with respect to the moving frame,

$$\begin{pmatrix} \gamma_m' \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_{1,m} & \kappa_{2,m} & \dots & \kappa_{n-1,m} \\ -\kappa_{1,m} & 0 & \kappa_{1_2,m} & \dots & \kappa_{1_{n-1},m} \\ -\kappa_{2,m} & -\kappa_{1_2,m} & 0 & \dots & \kappa_{2_{n-1},m} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -\kappa_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \dots & 0 \end{pmatrix} \begin{pmatrix} \gamma_m' \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}$$

We can fine a unique orthogonal frame $(\gamma_m'(t), \mathbf{N}_{1,m}(t), ...\mathbf{N}_{n-1,m}(t))$ with initial condition $\gamma_m'(0) = \gamma'(0)$ and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. Moreover, by Lemma 5.4.8, $(\gamma_m'(t), \mathbf{N}_{1,m}(t), ...\mathbf{N}_{n-1,m}(t)) \to (\gamma'(t), \mathbf{N}_1(t), ...\mathbf{N}_{n-1}(t))$ uniformly. Let

$$\gamma_m(t) = \gamma(0) + \int_0^t \gamma_m'(\tau)d\tau.$$

We claim γ_m satisfies for m sufficiently large,

$$(S_s^{\gamma_m}(t) + \frac{\rho}{8})(s_1\kappa_{1,m}(t) + \cdots + s_{n-1}\kappa_{n-1,m}(t)) \leq 1 \tag{5.4.39}$$

Indeed, by the same argument of Lemma 5.4.9 using implicit function theorem, $S_s^{\gamma_m}$ also converges to S_s^{γ} uniformly. Together with Lemma 5.4.9 we obtain $|S_s^{\gamma_m} - S_s^{\Gamma_m}| \to 0$ uniformly. Thus the claim follows from (5.4.38).

Step 6. Finally, we claim that orthogonal fronts satisfy $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0, \ell]$.

For γ_m and its moving frame $(\gamma_m', \mathbf{N}_{1,m}, ..., \mathbf{N}_{n-1,m})$ found in Step 5, let $\Phi_m : [0,\ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as

$$\Phi_m(t,s) = \gamma_m(t) + s_1\mathbf{N}_{1,m}(t) + \cdots + s_{n-1}\mathbf{N}_{n-1,m}(t),$$

Hence the orthogonal front

$$F_{\gamma_m}(t) = \{\Phi_m(t,s), t \in [0,\ell], s \in \mathbb{R}^{n-1}\}.$$

Let $\Sigma^{\gamma_m} = \{(t,s) : \Phi_m(t,s) \in \overline{\Omega}\}$. By the same argument as Lemma 5.4.2, Φ_m maps Σ^{γ_m} onto $\overline{\Omega(\gamma_m)}$ where $\Omega(\gamma_m)$ is the subset of Ω covered by all orthogonal front $F_{\gamma_m}(t), t \in [0,\ell]$.

By the same computation as (5.4.19),

$$J_{\Phi_m}(t,s) = 1 - s_1\kappa_{1,m}(t) - \cdots - s_{n-1}\kappa_{n-1,m}(t).$$

Let $d := \mathrm{diam}(\Omega)$, we claim that

$$1 - s_1\kappa_{1,m}(t) - \cdots - s_{n-1}\kappa_{n-1,m}(t) \geq \min\{\rho/16d, 1/2\}$$

for all $(t,s) \in \Sigma^{\gamma_m}$. Indeed, If $(s_1/|s|)\kappa_{1,m}(t) + \cdots + (s_{n-1}/|s|)\kappa_{n-1,m}(t) \geq 1/2d$, then by (5.4.39),

$$\begin{aligned} 1 - |s|\Big(\frac{s_1}{|s|}\kappa_{1,m}(t) + \cdots + \frac{s_{n-1}}{|s|}\kappa_{n-1,m}(t)\Big) & \\ \geq 1 - S_s^{\gamma_m}(t)\Big(\frac{s_1}{|s|}\kappa_{1,m}(t) + \cdots + \frac{s_{n-1}}{|s|}\kappa_{n-1,m}(t)\Big) & \\ \geq \frac{\rho}{8}\Big(\frac{s_1}{|s|}\kappa_{1,m}(t) + \cdots + \frac{s_{n-1}}{|s|}\kappa_{n-1,m}(t)\Big) \geq \frac{\rho}{8}\cdot\frac{1}{2d}. & \end{aligned}$$

If $(s_1/|s|)\kappa_{1,m}(t) + \cdots + (s_{n-1}/|s|)\kappa_{n-1,m}(t) < 1/2d$, then

$$1 - |s|\Big(\frac{s_1}{|s|}\kappa_{1,m}(t) + \cdots + \frac{s_{n-1}}{|s|}\kappa_{n-1,m}(t)\Big) > 1 - \frac{|s|}{2d} \geq \frac{1}{2}.$$

Hence, the claim follows. By Inverse function theorem due to Clarke [10], Φ admits a local Lipschitz inverse, actually a *global* Lipschitz inverse $\Phi_m^{-1} : \overline{\Omega(\gamma_m)} \to \Sigma^{\gamma_m}$ since the Jacobian is everywhere bounded below by a positive constant in Σ^{γ_m}. In particular, Φ_m is one-to-one on Σ^{γ_m}. This implies all orthogonal front $F_{\gamma_m}(t), t \in [0,\ell]$ meets outside $\overline{\Omega}$. The proof of

Lemma 5.4.7 is complete. □

We also need to define the curves $\tilde{\gamma}_m$ in the target space $u(\Omega(\gamma))$ corresponding to γ_m. Recall $\kappa_{\mathbf{n}}$ defined in the moving frame in the target space is bounded, we choose a sequence of uniformly bounded smooth function $\tilde{\kappa}_{\mathbf{n},m}$ such that $\tilde{\kappa}_{\mathbf{n},m} \to \kappa_{\mathbf{n}}$ a.e. in $[0,\ell]$, (and hence in L^p for all $1 \leq p < \infty$).

We need to flatten $\tilde{\kappa}_{\mathbf{n},m}$ around the end points 0 and ℓ $\tilde{\kappa}_{\mathbf{n},m}$ for two reasons: first, it might happen that $\Omega(\gamma) \nsubseteq \Omega(\gamma_m)$ so we need to extend the isometric immersion defined on $\Omega(\gamma_m)$ smoothly to the region of $\Omega(\gamma)$ outside $\Omega(\gamma_m)$. Second, so far all the construction is on one covered domain $\Omega(\gamma)$ and our final goal is to glue all the different covered domains together smoothly. By flattening $\tilde{\kappa}_{\mathbf{n},m}$ around the end point 0 and ℓ, u_m constructed later is affine near the Leading planes $P_\gamma(0)$ and $P_\gamma(\ell)$ (for definition of leading planes see Definition 5.4.3) so that we can join all the piece smoothly. The modification goes as follows: by (5.4.28), the second derivative of u vanishes whenever $\kappa_{\mathbf{n}} = 0$. Put

$$\ell_m^* = \begin{cases} \ell \quad \text{if} \quad \Omega(\gamma) \subset \Omega(\gamma_m) \text{ and,} \\ \sup\{t \in [0,\ell], F_{\gamma_m}(t) \cap F_\gamma(\ell) \cap \overline{\Omega}(\gamma) = \emptyset\} \text{ otherwise} \end{cases}$$

By step 1 of Lemma 5.4.7, $F_{\gamma_m}(t) \to F_\gamma(t)$ uniformly, hence $\ell_m^* \to \ell$ as $m \to \infty$.

Let ψ_1 be any smooth positive function which is 0 on $[-1,\infty)$ and 1 on $(-\infty,-2)$. Let ψ_2 be any smooth positive function which is 0 on $(-\infty,1]$ and 1 on $(2,\infty)$. We put,

$$\kappa_{\mathbf{n},m}(t) := \psi_1(m(t-\ell_m^*))\psi_2(mt)\tilde{\kappa}_{\mathbf{n},m}(t), t \in [0,\ell]$$

and we solve the following linear system for initial values $\tilde{\gamma}_m'(0) = \tilde{\gamma}'(0)$, $\mathbf{v}_{i,m}(0) = \mathbf{v}_i(0)$, and $\mathbf{n}_m(0) = \mathbf{n}(0)$:

$$\begin{pmatrix} \tilde{\gamma}_m' \\ \mathbf{v}_{1,m} \\ \mathbf{v}_{2,m} \\ \vdots \\ \mathbf{v}_{n-1,m} \\ \mathbf{n}_m \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_{1,m} & \kappa_{2,m} & \dots & \kappa_{n-1,m} & \kappa_{\mathbf{n},m} \\ -\kappa_{1,m} & 0 & \kappa_{1_2,m} & \dots & \kappa_{1_{n-1},m} & 0 \\ -\kappa_{2,m} & -\kappa_{1_2,m} & 0 & \dots & \kappa_{2_{n-1},m} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -\kappa_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \dots & 0 & 0 \\ -\kappa_{\mathbf{n},m} & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\gamma}_m' \\ \mathbf{v}_{1,m} \\ \mathbf{v}_{2,m} \\ \vdots \\ \mathbf{v}_{n-1,m} \\ \mathbf{n}_m \end{pmatrix}$$

We define

$$\tilde{\gamma}_m(t) = \tilde{\gamma}(0) + \int_0^t \tilde{\gamma}_m'(\tau)d\tau.$$

The same argument as in step 1 in the proof of Lemma 5.4.7, $\tilde{\gamma}_m \to \tilde{\gamma}$ in $W^{2,p}([0,\ell],\mathbb{R}^{n+1})$ and the moving frame $(\tilde{\gamma}_m', \mathbf{v}_{1,m}, ..., \mathbf{v}_{n-1,m}, \mathbf{n}_m) \to (\tilde{\gamma}', \mathbf{v}_1, ..., \mathbf{v}_{n-1}, \mathbf{n})$ uniformly.

Eventually, we define our approximating sequence u_m on $\Omega(\gamma_m)$:

$$\begin{aligned} u_m(\gamma_m(t) + s_1\mathbf{N}_{1,m}(t) + \dots + s_{n-1}\mathbf{N}_{n-1,m}(t)) \\ = \tilde{\gamma}_m(t) + s_1\mathbf{v}_{1,m}(t) + \dots + s_{n-1}\mathbf{v}_{n-1,m}(t) \end{aligned} \quad (5.4.40)$$

where γ_m is defined in Lemma 5.4.7. Such γ_m assures that all its leading fronts intersect outside $\overline{\Omega}$, hence u_m is well-defined and smooth over $\Omega(\gamma) \cap \Omega(\gamma_m)$.

As before, let $\Phi_m : [0,\ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as

$$\Phi_m(t,s) = \gamma_m(t) + s_1\mathbf{N}_{1,m}(t) + \dots + s_{n-1}\mathbf{N}_{n-1,m}(t),$$

and let $\Delta^{\gamma_m} = \{(t,s) : \Phi_m(t,s) \in \Omega(\gamma)\}$. Same argument as Step 6 in Lemma 5.4.7 gives $\Phi_m(t,s)$ is a bi-Lipschitz mapping of Δ^{γ_m} onto $\Omega(\gamma) \cap \Omega(\gamma_m)$. By differentiating with respect to $t, s_1, ...s_{n-1}$, same as (5.4.24) and (5.4.22), we see that at each point of x, $Du_m(x)$ maps an othonormal frame to an othonormal frame. Hence $Du_m(x) \in O(\mathbb{R}^n, \mathbb{R}^{n+1})$. Moreover, u_m is affine near $P_{\gamma_m}(\ell)$ and can be extended by an affine isometry over $\Omega(\gamma)$. Therefore,

$u_m \in I^{2,2}(\Omega(\gamma), \mathbb{R}^n)$. Everything we have proved for isometric immersions of course applies, in particular, by (5.4.25), (5.4.28) (5.4.26) we have,

$$\frac{\partial}{\partial x_i} u_m \circ \Phi_m(t,s) = (\mathbf{e}_i \cdot \gamma_m'(t))\tilde{\gamma}_m'(t) + \sum_{j=1}^{n-1} (\mathbf{e}_i \cdot \mathbf{N}_{j,m}(t))\mathbf{v}_{j,m}(t), \tag{5.4.41}$$

$$(D\frac{\partial}{\partial x_i} u_m)(\Phi_m(t,s))\gamma_m'(t) = \frac{(\mathbf{e}_i \cdot \gamma_m'(t))\kappa_{\mathbf{n},m}(t)\mathbf{n}(t)}{1 - s_1\kappa_{1,m}(t) - \cdots - s_{n-1}\kappa_{n-1,m}(t)}, \text{ and} \tag{5.4.42}$$

$$(D\frac{\partial}{\partial x_i} u_m)(\Phi_m(t,s))\mathbf{N}_{i,m}(t) = 0, \quad 1 \le i \le n-1. \tag{5.4.43}$$

for all $t \in [0, \ell]$ and $s = (s_1, ..., s_{n-1}) \in \Delta^{\gamma_m}(t)$.

Moreover, by (5.4.29),

$$\begin{aligned}
\int_{\Omega(\gamma)} |u_m(x)|^2 dx &= \int_{\Omega(\gamma)\cap\Omega(\gamma_m)} |u_m(x)|^2 dx + \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} |u_m(x)|^2 dx \\
&= \int_0^\ell \int_{\Delta^{\gamma_m}(t)} |\tilde{\gamma}_m(t) + s_1\mathbf{v}_{1,m}(t) + \cdots + s_{n-1}\mathbf{v}_{n-1,m}(t)|^2 \\
&\quad \cdot \big(1 - s_1\kappa_{1,m}(t) - \cdots - s_{n-1}\kappa_{n-1,m}(t)\big) d\mathcal{H}^{n-1}(s)dt \\
&\quad + \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} |u_m(\ell) + Du_m(\ell)(x - \gamma_m(\ell)|^2 dx.
\end{aligned} \tag{5.4.44}$$

$$\int_{\Omega(\gamma)} |Du_m(x)|^2 dx = n|\Omega(\gamma)|. \tag{5.4.45}$$

$$\begin{aligned}
\int_{\Omega(\gamma)} |D^2u_m(x)|^2 dx &= \int_{\Omega(\gamma)\cap\Omega(\gamma_m)} |D^2u_m(x)|^2 dx + \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} |D^2u_m(x)|^2 dx \\
&= \int_0^\ell \int_{\Delta^{\gamma_m}(t)} \frac{\kappa_{\mathbf{n},m}^2(t)}{\big(1 - s_1\kappa_{1,m}(t) - \cdots - s_{n-1}\kappa_{n-1,m}(t)\big)} d\mathcal{H}^{n-1}(s)dt + 0.
\end{aligned} \tag{5.4.46}$$

It is easy to see $u_m \to u$ in $W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ because $(\gamma_m', \mathbf{N}_{1,m}, ...\mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, ...\mathbf{N}_{n-1})$ uniformly, $(\tilde{\gamma}_m', \mathbf{v}_{1,m}, ..., \mathbf{v}_{n-1,m}, \mathbf{n}_m) \to (\tilde{\gamma}', \mathbf{v}_1, ..., \mathbf{v}_{n-1}, \mathbf{n})$ uniformly, $\kappa_{\mathbf{n},m} \to \kappa_{\mathbf{n}}$, $\kappa_{i,m} \to \kappa_i, 1 \le i \le n-1$ in $L^p([0,\ell])$ for all $1 \le p < \infty$, $1 - s_1\kappa_{1,m}(t) - \cdots - s_{n-1}\kappa_{n-1,m}(t) \ge \min\{\rho/16d, 1/2\}$, $\Delta^{\gamma_m}(t) \to \Sigma^\gamma(t)$ for all $t \in [0,\ell]$ and $|\Omega(\gamma) \setminus \Omega(\gamma_m)| \to 0$. The proof is complete. □

Combining proposition 5.4.5 and 5.4.6 we get a smooth approximation sequence for any isometry u in $\Omega(\gamma)$.

5.4.6 Approximation for the entire domain.

The proof is exactly the same as the proof in section 3.3 in [46]. For the sake of completeness we include the proof here.

Recall that we defined a maximal region on which u is affine a *body* if its boundary contains more than *two* different $(n-1)$-planes in Ω (recall Definition 5.3.1 for the definition of $(n-1)$-planes in Ω) and we have shown that we can assume Ω has only a finite number of bodies and is partitioned into bodies and covered domains. We call the maximal subdomain covered by some Leading curve γ an *arm*. Similar to Lemma 5.4.1 we also have,

Lemma 5.4.10 *It is sufficient to prove Theorem 5.4.1 for a function in* $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ *with finite number of arms*

Proof. Since we have a finite number of bodies, the complement of bodies in $\tilde{\Omega}$ is a finite union of connected components $\cup_{j=1}^{N}\Delta_j$. Suppose one such region Δ is between two bodies B_1 and B_2, we want to show $\overline{\Delta}$ can be covered by a finite number of Leading curves.

Let us recall the definition of Leading planes (Definition 5.4.3). From our definition, each Leading plane is an open set with respect to the Leading front it belongs. Here we slight change the definition and still denote a Leading plane as its closure with respect to the Leading front it belongs. Since each $x \in \overline{\Delta}$ is covered by some leading curve, with our new definition of Leading planes, $\overline{\Delta}$ is a union of Leading planes by obvious modification of Lemma 5.4.2. For each Leading plane P, let $B^{n-1}(x^P, r^P)$ be the largest $n-1$ dimensional ball contain in P and we denote x^P the *center* of P. Since Δ is between two bodies and Δ is a convex domain,

$$r := \inf_P r^P > 0.$$

Let $\mathbf{N}$ be the normal vector field orthogonal to these Leading planes everywhere. Since none of the Leading planes intersect inside Δ by the definition of Leading curve, which has a Lipschitz boundary, the normal vector field approach each other in an Lipschitz angle. Therefore, we can choose an orientation of such that $\mathbf{N}$ is a Lipschitz vector fields.

Note that $\overline{\Delta} \cap B_1$ is a Leading plane and we denote it P_0. Let x_0 be the center of P_0 and let $\gamma_1 : [0, \ell_1] \to \overline{\Delta}$ be the unique maximal solution to the ODE,

$$\gamma_1'(t) = \mathbf{N}(\gamma_1(t)) \quad \gamma_1(0) = x_0. \tag{5.4.47}$$

If $\gamma_1(\ell_1) \in \Delta$, we can always find a unique leading curve $\gamma : [-\delta, \delta] \to \Delta$ with $\gamma(0) = \gamma_1(\ell_1)$ for some $\delta > 0$. Therefore, we can always prolong γ_1 inside Δ as long as it does not touch $\partial\Delta$, contraction to γ_1 being the maximal solution. Therefore, $\gamma_1(\ell_1) \in \partial\Delta$. Note that $\partial\Delta$ consists of components of ∂B_1, ∂B_2, and $\partial\Omega$. If $\gamma_1(\ell_1) \in \partial B_2$, then the entire Δ is covered by γ_1 and we are done. The only situation a different Leading curve is needed is when $\gamma_1(\ell_1) \in \partial\Omega \setminus \partial B_2$ (Figure 18).

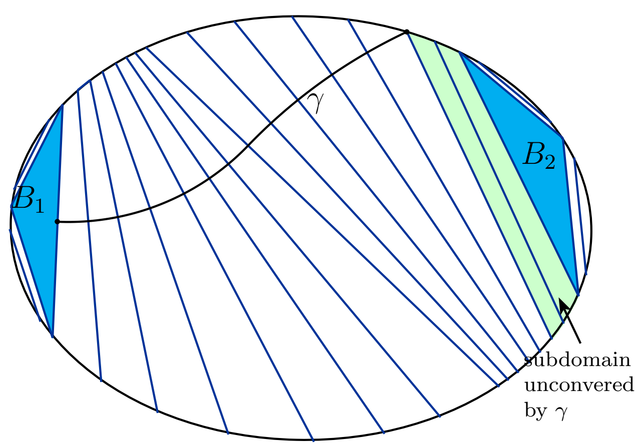

Figure 18:

In the latter case we consider the Leading plane P_1 in Ω passing through the point $\gamma_1(\ell_1)$. Such Leading plane P_1 is uniquely defined by Lemma 5.4.5. Let x_1 be the center of P_1 and let $\gamma_2 : [0, \ell_2]$ be the unique maximal solution to the ODE in (5.4.47) with initial condition $\gamma_2(0) = x_1$ and then we repeat the same argument as above.

We claim that a finite number of such $\gamma_i : [0, \ell_i]$ cover Δ. Indeed, denote the length of the curve $\gamma_i([0, \ell_i])$ by $|\gamma_i([0, \ell_i])|$, then as γ_i is parametrized by arc-length, $|\gamma_i([0, \ell_i])| = \ell_i$. Since $\gamma_i(0) = x_{i-1}$, which is the center of P_{i-1}, and $\gamma_i(\ell_i) \in \partial\Omega$, the distance $|\gamma(\ell_i) - \gamma(0)| \geq \min\{r, \mathrm{dist}(x_0, \partial\Omega)\} =: c > 0$. Altogether we have,

$$\ell_i = |\gamma_i([0, \ell_i])| \geq |\gamma(\ell_i) - \gamma(0)| \geq c.$$

Now by the same change of variable formula (5.4.29) and Remark 5.4.3 that $1 - s_1\kappa_1(t) - \cdots - s_{n-1}\kappa_{n-1}(t) > \rho > 0$,

$$\Omega(\gamma_i) \quad = \quad \int_0^{\ell_i} \int_{\Sigma^{\gamma_i}(t)} 1 \; - \; s_1\kappa_1(t) \; - \; \cdots \; - \; s_{n-1}\kappa_{n-1}(t)\big) d\mathcal{H}^{n-1}(s)dt \quad \geq \quad C(n)cr^{n-1}\rho.$$

Hence i must be a finite number since Δ is bounded.

If Δ has common boundary with one body, then there exist a sequence of isometric immersions $u_m \to u$ in $W^{2,2}(\Omega, \mathbb{R}^{n+1})$ and each u_m admits a finite number of arms that covers Δ. To do this we simply cut off the Leading planes corresponding to $\gamma_0(t), 0 \leq t \leq 1/m$ and apply affine extension to such region. The idea is clear from Figure 19.

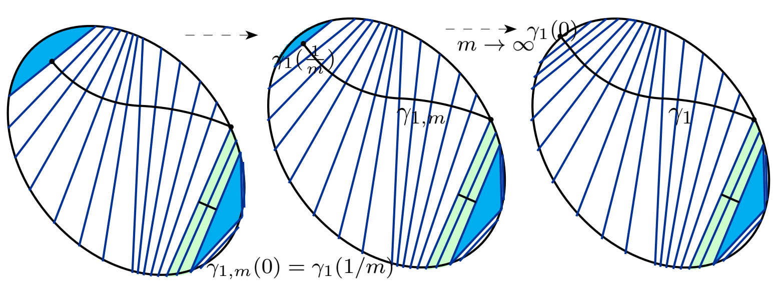

Figure 19:

If Δ does not have common boundary with any bodies, then we divide Δ into two regions Δ_1 and Δ_2 by one Leading plane and apply the cut off argument to Δ_1 and Δ_2 individually. The proof is complete. □

Now since Ω is convex and simply-connected, we claim that two bodies are connected through one chain of bodies and arms: It suffices to consider the graph obtained by retracting bodies to vertexes and arms to edges. This graph is simply connected because it is a deformation retract of Ω. Therefore every two vertexes are connected through only one chain of edges, which proves the claim (Figure 20).

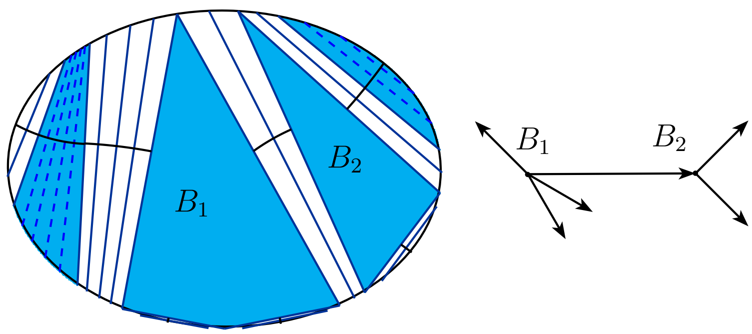

Figure 20:

We begin by a central body B_1 and define our approximation sequence on each arm as in Subsection 5.4.5. Note that for this final purpose, we have constructed our approximation smooth isometric immersion to be affine near both ends, this allows us to apply affine transformation to the target space of each arm so that the affine regions near its ends join together smoothly all the way till we reach B_2. Meanwhile, we also apply affine transformation to $u(B_2)$ so that it joins the last arm smoothly. It is easy to see from the uniform convergence of each term in representation (5.4.41) such affine transformation goes to identity as $m \to 0$. Now we continue our construction using B_2 as a new starting point. Note that we will never come back to B_1 because they are connected through only one chain of arms. The construction of the entire domain Ω is complete. □

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