TIME-PERIODIC SOLUTIONS OF MAGNETOELASTIC SYSTEMS AND EMBEDDING OF THE ATTRACTOR OF 2-DIMENSIONAL NAVIER-STOKES EQUATIONS INTO EUCLIDEAN SPACES

by

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Mahdi Mohebbi, PhD

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In this work we address two rather independent problems. The first part is dedicated to the existence of periodic solutions for a magnetoelastic system modeling the interaction of a linear elastic body with a nonlinear dissipation and a magnetic field. The resulting system is a coupled Hyperbolic-Parabolic system of PDE's.

In the second part, for 2-D Navier-Stokes equations on a C^2 bounded domain Ω and a time independent force $\boldsymbol{f}$, a class of nonlinear homeomorphisms is constructed from the attractor of Navier-Stokes to curves in $\mathbb{R}^N$, for sufficiently large N. The construction uses an ε-net on Ω (so does not use the information "near" the boundary) and is more physically perceivable compared to abstract common embeddings.

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PREFACE

... how He extends the shadow, and if
He willed, He could have made it
stationary ...
— He, 25:45

I was extremely fortunate for the opportunity to have Dr. Galdi as my Ph.D. advisor. I knew, when I started working with him, that learning rigorous mathematical thinking is a difficult task for someone like me (with a purely engineering background). At this point, I have realized that teaching these skills to someone like me, was even extremely more difficult. I am infinitely grateful to him for his patience, which made his vast knowledge available to me (and others). On the way to professional training, I have also significantly benefited from his mannerism, which makes me indebted to him even more. Yet, I understand what I have witnessed relative to my capacity, is certainly just a portion of the whole.

The main strategies and proofs in the first part, are the result of a joint work with Dr. J. C. Oliveira during his visit from Pittsburgh, which was the best period of my graduate studies. I am beholden to him as I have enormously profited from his remarkably precise and persistent approach. He also assisted me in understanding many ideas, not originally clear or known to me.

I have to also thank my colleagues who contributed to our group with their great ideas, in particular, Dr. G. Mazzone, for many useful discussions on various subjects and also my committee members Dr. A. M. Robertson, Dr. W. S. Slaughter, Dr. P. Zunino and Dr. D. Wang for their support and their cooperation with my bad habit of doing things in the last minute.

1.0 TIME-PERIODIC SOLUTIONS TO A MAGNETOELASTIC SYSTEM

In this chapter we investigate the existence of periodic solutions for a magnetoelastic system in bounded, simply connected, 3-D domains with boundaries of class C^2. The mathematical model includes a nonlinear mechanical dissipation and a periodic forcing function of period T. We prove the existence of T-periodic weak solutions under some assumptions for the dissipation term.

The system of governing equations consist of a hyperbolic equation (the elastic part) and a parabolic equation (the magnetic part), coupled through a nonlinear coupling. This different nature of equations along the nonlinear behavior, causes interconnected difficulties.

In section 1.1 we will give a formulation of the problem obtained from Maxwell's equations coupled with the equations of linearized elasticity. Issues regarding frame invariance and magnetic force inside the elastic material are addressed in this section. In formulating the electromagnetic equations, we closely follow Penfield and Haus (1967), while the final coupled form of the equations is in the line of Eringen and Maugin (1990).

Section 1.2.2 deals with some difficulties regarding the existence of periodic solutions to hyperbolic equations. We also, briefly, discuss the more popular existing methods of finding periodic solutions of PDE's and examine their applicability to our problem. Finally, in section 1.2.5, we explain a technique that enables us to find time periodic solutions in such mixed hyperbolic-parabolic systems.

The mathematical analysis is a joint work with Dr. J. C. Oliveira (Mohebbi and Oliveira, 2012).

1.1 FORMULATION

In free space, the Maxwell's equations are governing the relation among electromagnetic fields, charges and currents as follows (in SI units) (Penfield and Haus, 1967),

$$\begin{aligned} &\textit{Gauss' law:} && \epsilon_0 \operatorname{div} \boldsymbol{E} = q_f, \\ &\textit{Faraday's law:} && \operatorname{curl} \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \\ &\textit{Conservation of magnetic flux:} && \mu_0 \operatorname{div} \boldsymbol{H} = 0, \\ &\textit{Ampère's law:} && \operatorname{curl} \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{J}_f, \end{aligned}$$

where $\boldsymbol{E}$ is the electric field intensity and $\boldsymbol{H}$ is the magnetic field intensity. $\boldsymbol{J}_f$ is the free current and q_f is the free charge density. The universal constants $\epsilon_0 = 8.85 \times 10^{-12}$ (farad/meter) and $\mu_0 = 1.26 \times 10^{-6}$ (henry/meter), are the permittivity and permeability of free space, respectively. They are related to the speed of light, c, by

$$c = (\epsilon_0 \mu_0)^{-2} = 2.99 \times 10^8 \text{ meter/second.}$$

In the presence of matter the above equations should be modified. Since there is no practical method of measurement of field quantities inside matter, several modifications exist (of course, all are consistent and meaningful) and none can be designated as the "correct" one. Here, we adopt the Minkowski formulation which starts by introducing two auxiliary fields into the original Maxwell's equations: the electric displacement field, $\boldsymbol{D}$, and the magnetic flux, $\boldsymbol{B}$,

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \qquad \boldsymbol{B} = \mu_0 \boldsymbol{H}. \tag{1.1}$$

With the help of the above definitions, one can rewrite the original Maxwell's equations in the form

$$\operatorname{div} \boldsymbol{D} = q_f, \tag{1.2a}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.2b}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{1.2c}$$

$$\operatorname{curl} \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_f. \tag{1.2d}$$

We remark that other popular formulations also involve introducing these auxiliary fields, the difference is the physical interpretation of the field variables in each formulation. For example, see Penfield and Haus (1967), Section 3.4, for a comparison of Minkowski and Chu formulations.

Consider a body, $\mathcal{B}$, of matter (which can be polarizable and magnetizable). Minkowski suggested that, when $\mathcal{B}$ is at rest, the electromagnetic field equations in $\mathcal{B}$ are still governed by (1.2) with (1.1) replaced with new constitutive equations, say

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}), \qquad \boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}). \tag{1.3}$$

In general, $\boldsymbol{D}$ and $\boldsymbol{B}$ can be nonlinear functions of $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively. In the case of a (electromagnetically) linear material, we have

$$\boldsymbol{D} = \epsilon_{\mathcal{B}}\boldsymbol{E}, \qquad \boldsymbol{B} = \mu_{\mathcal{B}}\boldsymbol{H}, \tag{1.4}$$

where now $\epsilon_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are the permittivity and permeability of the medium $\mathcal{B}$. Both $\epsilon_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are positive and if $\mathcal{B}$ is (electromagnetically) isotropic and homogeneous will also be constant.

When the body, $\mathcal{B}$ is moving and deforming, the Minkowski formulation is generalized by assuming that (1.3) (or (1.4)) holds in a frame which is stationary with respect to $\mathcal{B}$. This frame is called the “rest frame”. This is, of course, understood that this frame is local to the points of $\mathcal{B}$, in the sense that if $\mathcal{B}$ is not moving rigidly then we have a rest frame associated with each point* of $\mathcal{B}$ and at each time t. Indicating the field quantities in this frame by superscript 0, the above assumption gives that the following equations govern the electromagnetic fields in $\mathcal{B}$,

$$\operatorname{div}^0 \boldsymbol{D}^0 = q_f^0, \tag{1.5a}$$

$$\operatorname{curl}^0 \boldsymbol{E}^0 = -\frac{\partial \boldsymbol{B}^0}{\partial t^0}, \tag{1.5b}$$

$$\operatorname{div}^0 \boldsymbol{B}^0 = 0, \tag{1.5c}$$

$$\operatorname{curl}^0 \boldsymbol{H}^0 = \frac{\partial \boldsymbol{D}^0}{\partial t^0} + \boldsymbol{J}_f^0, \tag{1.5d}$$

*Indeed this point is the representative of the material element in its neighbourhood.

$$\boldsymbol{D}^0 = \boldsymbol{D}^0(\boldsymbol{E}^0), \qquad \boldsymbol{B}^0 = \boldsymbol{B}^0(\boldsymbol{H}^0), \tag{1.5e}$$

where, we have indicated the space and time coordinates in the rest frame by $(\boldsymbol{x}^0, t^0)$ and the differentiation with respect to these coordinates is also labeled with a superscript to avoid confusion. To complete the formulation we need transformation laws that give us the field quantities in the rest frame in terms of our "observation frame". Denoting by $\boldsymbol{v}(\boldsymbol{x}, t)$, $(\boldsymbol{x}, t) \in \mathcal{B} \times [0, \infty)$, the velocity of points of $\mathcal{B}$, these transformation laws in their non-relativistic form are given by

$$\begin{aligned}
\boldsymbol{E}^0 &= \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \\
\boldsymbol{D}^0 &= \boldsymbol{D} + \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{H}, \\
\boldsymbol{H}^0 &= \boldsymbol{H} - \boldsymbol{v} \times \boldsymbol{D}, \\
\boldsymbol{B}^0 &= \boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}. \\
\boldsymbol{J}_f^0 &= \boldsymbol{J}_f - q_f \boldsymbol{v}, \\
\boldsymbol{q}_f^0 &= q_f.
\end{aligned} \tag{1.6}$$

Given the above transformation laws, we should elaborate more on what we mean by the "rest frame:" In a non-relativistic setting, the rest frame at the point $(\boldsymbol{x}, t)$ is a frame moving with a constant velocity equal to $\boldsymbol{v}(\boldsymbol{x}, t)$ with respect to the observation frame. The transformation between these rest frames and the observation frame is a Galilean transformation, that is

$$t^0 = t, \qquad \boldsymbol{x}^0 = \boldsymbol{x} - \boldsymbol{v}t. \tag{1.7}$$

Where we have abused the notation by using again $\boldsymbol{v}$ for the constant relative velocity of a particular rest frame and the observation frame.

1.1.1 Quasi-static magnetic field system and Galilean invariance

When the dimensions of $\mathcal{B}$ is small compared to the electromagnetic wavelength corresponding to the time scale of changes in $\mathcal{B}$, (1.5) can be more simplified to a quasi-static formulation. Assume $\mathcal{B}$ is conducting (so that the effects of charges, q_f, can be ignored) and that the magnetic field is dominating in the system, that is the electric field induced according to (1.5b) does not itself contribute to changing the magnetic field (and so we can ignore the electric displacement $\boldsymbol{D}$), then (1.5) reduces to

$$\operatorname{curl} \boldsymbol{E}^0 = -\frac{\partial \boldsymbol{B}^0}{\partial t}, \tag{1.8a}$$

$$\operatorname{div} \boldsymbol{B}^0 = 0, \tag{1.8b}$$

$$\operatorname{curl} \boldsymbol{H}^0 = \boldsymbol{J}_f^0, \tag{1.8c}$$

$$\boldsymbol{B}^0 = \boldsymbol{B}^0(\boldsymbol{H}^0). \tag{1.8d}$$

With the following modifications to the transformation laws (1.6):

$$\boldsymbol{E}^0 = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \tag{1.9a}$$

$$\boldsymbol{H}^0 = \boldsymbol{H}, \tag{1.9b}$$

$$\boldsymbol{B}^0 = \boldsymbol{B}, \tag{1.9c}$$

$$\boldsymbol{J}_f^0 = \boldsymbol{J}_f. \tag{1.9d}$$

Note that since we ignored the effects of electric displacement no transformation law is given for $\boldsymbol{D}$ in the above. Also, notice that (1.9c) should be given as above for consistency with (1.8d) and (1.9b).

It is easy to see that the system (1.8) together with (1.9) is Galilean invariant and particularly that (1.8) holds also in the observation frame with the same form. Indeed, fix a particular rest frame and let $Q^0(\boldsymbol{x}^0, t^0)$ and $Q(\boldsymbol{x}, t)$ be representations of the same quantity in this rest frame and the observation frame, respectively. From (1.7) we have

$$\begin{aligned}
\frac{\partial Q^0}{\partial t^0} &= \frac{\partial Q}{\partial t}\frac{\partial t}{\partial t^0} + \frac{\partial Q}{\partial x_i}\frac{\partial x_i}{\partial t}\frac{\partial t}{\partial t^0} = \frac{\partial Q}{\partial t} + \boldsymbol{v} \cdot \nabla Q, \\
\frac{\partial Q^0}{\partial x_i^0} &= \frac{\partial Q}{\partial t}\frac{\partial t}{\partial t^0}\frac{\partial t^0}{\partial x_i^0} + \frac{\partial Q}{\partial x_k}\frac{\partial x_k}{\partial x_k^0} = \frac{\partial Q}{\partial x_i}.
\end{aligned}$$

Using the above relations and (1.9), we can write (1.8) in the observation frame as follows

$$\operatorname{curl} \boldsymbol{E} + \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}) = -\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{v} \cdot \nabla \boldsymbol{B},$$

$$\operatorname{div} \boldsymbol{B} = 0,$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}_f,$$

$$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}).$$

Reminding our abuse of notation that $\boldsymbol{v}$, when used in transformations between rest frames and the observation frame, is assumed to be constant (for a fixed rest frame), we have that $\operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}) = -\boldsymbol{v} \cdot \nabla \boldsymbol{B}$. Since the above holds for every rest frame we conclude that the governing equations of a quasi-magnetic system in the observation frame (and any other frame obtained by a Galilean transformation of it) are

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.10a}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{1.10b}$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}_f, \tag{1.10c}$$

$$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}). \tag{1.10d}$$

1.1.2 Electromagnetic constitutive equations and the equation of magnetic field

To close the system (1.10) we need constitutive equations for $\boldsymbol{B}$ and $\boldsymbol{J}_f$. Assuming that the body $\mathcal{B}$ is electromagnetically linear from (1.4) we have

$$\boldsymbol{B} = \mu_{\mathcal{B}} \boldsymbol{H}.$$

For the current density we use the generalized Ohm's law which assumes a linear relationship between the current density, $\boldsymbol{J}_f$, and the electric field intensity, $\boldsymbol{E}$, in the rest frame (Eringen and Maugin, 1990, p. 115)

$$\boldsymbol{J}_f^0 = \sigma \boldsymbol{E}^0,$$

where σ is the conductivity of $\mathcal{B}$ and is assumed to be a positive constant.† With the help of (1.9a) and (1.9d) we get

$$\boldsymbol{J}_f = \sigma \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right). \tag{1.11}$$

†Invoking the assumption that $\mathcal{B}$ is electromagnetically isotropic and homogeneous.

Using these equations in (1.10) we have

$$\begin{aligned}
\operatorname{curl} \boldsymbol{E} &= -\mu_{\mathcal{B}} \frac{\partial \boldsymbol{H}}{\partial t}, \\
\operatorname{div} \boldsymbol{H} &= 0, \\
\operatorname{curl} \boldsymbol{H} &= \sigma \left(\boldsymbol{E} + \mu_{\mathcal{B}} \boldsymbol{v} \times \boldsymbol{H} \right).
\end{aligned}$$

Taking the curl of the last equation in the above and using the first one we get

$$\begin{aligned}
\operatorname{curl} \boldsymbol{E} &= -\mu_{\mathcal{B}} \frac{\partial \boldsymbol{H}}{\partial t}, \\
\operatorname{div} \boldsymbol{H} &= 0, \\
\operatorname{curl} \operatorname{curl} \boldsymbol{H} &= -\sigma \mu_{\mathcal{B}} \frac{\partial \boldsymbol{H}}{\partial t} + \sigma \mu_{\mathcal{B}} \operatorname{curl} \left(\boldsymbol{v} \times \boldsymbol{H} \right),
\end{aligned}$$

and so for the case of quasi-static magnetic systems, governing equations can be written as two decoupled systems for $\boldsymbol{H}$ and $\boldsymbol{E}$:

$$\begin{cases} \mu_{\mathcal{B}} \dfrac{\partial \boldsymbol{H}}{\partial t} + \varrho \operatorname{curl} \operatorname{curl} \boldsymbol{H} = \mu_{\mathcal{B}} \operatorname{curl} \left(\boldsymbol{v} \times \boldsymbol{H} \right), \\ \operatorname{div} \boldsymbol{H} = 0, \end{cases} \tag{1.12}$$

$$\operatorname{curl} \boldsymbol{E} = -\mu_{\mathcal{B}} \frac{\partial \boldsymbol{H}}{\partial t}, \tag{1.13}$$

Where $\varrho = 1/\sigma$ is the electrical resistivity of $\mathcal{B}$. With the above result it is clear that when dealing with a quasi-static magnetic system the focus should be in (1.12), since once the magnetic field intensity, $\boldsymbol{H}$, is known one easily obtains the electric field intensity, $\boldsymbol{E}$, from (1.13), if interested.

1.1.3 The fundamental system of continuum mechanics with electromagnetic forces

Transformations of $\mathcal{B}$, from a mechanical perspective, conform with the fundamental system of continuum mechanics, i.e.,

$$\begin{aligned} \textit{Conservation of mass:} \qquad & \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \textit{Balance of linear momentum:} \qquad & \rho \boldsymbol{a} = \operatorname{div} \boldsymbol{T} + \rho \boldsymbol{f}_{\mathcal{B}}, \end{aligned}$$

where $\rho(\boldsymbol{x}, t)$ is the mass density and $\boldsymbol{v}(\boldsymbol{x}, t)$ and $\boldsymbol{a}(\boldsymbol{x}, t)$ are velocity and acceleration of the material elements of $\mathcal{B}$, respectively. $\boldsymbol{T}(\boldsymbol{x}, t)$ is the Cauchy stress tensor and $\boldsymbol{f}_{\mathcal{B}}(\boldsymbol{x}, t)$ is the external body force per unit volume acting on $\mathcal{B}$. The reader immediately notices that the balance of angular momentum has been left out of the above system. The reason is that, in general, when $\mathcal{B}$ is polarizable and magnetizable the balance of angular momentum takes a complicated form which includes these effects (Eringen and Maugin, 1990, p. 79). Since we already assumed that $\mathcal{B}$ is conducting we can ignore the effects of polarization and assuming that $\mathcal{B}$ is non-ferromagnetic we can also disregard magnetization effects. Then, balance of angular momentum reduces to the well known classical result that

$$\textit{Balance of angular momentum:} \qquad \boldsymbol{T} = \boldsymbol{T}^T,$$

that is, the Cauchy stress tensor is symmetric.

Regarding the body force, $\boldsymbol{f}_{\mathcal{B}}$, it is more convenient to distinguish body forces of an electromagnetic origin among others. Let us write $\boldsymbol{f}_{\mathcal{B}} = \boldsymbol{f} + \boldsymbol{f}_{em}$, where $\boldsymbol{f}_{em}$ is the body force due to electromagnetic effects. In general, calculating $\boldsymbol{f}_{em}$, is an elaborate task and depending on various phenomena, may be very involved. For a very general treatment we refer the reader to the monograph of Penfield and Haus (1967). However, when polarization and magnetization are negligible, $\boldsymbol{f}_{em}$ takes rather a simple form which is given by the Lorenz force law

$$\boldsymbol{f}_{em} = q_f \boldsymbol{E} + \boldsymbol{J}_f \times \boldsymbol{B}.$$

Noting that in our quasi-static magnetic approximation we ignored the effects of charges, using (1.4) and (1.10c), we get

$$\boldsymbol{f}_{em} = \mu_{\mathcal{B}} \operatorname{curl} \boldsymbol{H} \times \boldsymbol{H},$$

and so the fundamental system of continuum mechanics in this setting reads

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0, \tag{1.14a}$$

$$\rho \boldsymbol{a} = \operatorname{div} \boldsymbol{T} + \rho \mu_{\mathcal{B}} \operatorname{curl} \boldsymbol{H} \times \boldsymbol{H} + \rho \boldsymbol{f}, \tag{1.14b}$$

$$\boldsymbol{T} = \boldsymbol{T}^T. \tag{1.14c}$$

1.1.4 The linearized equations of elasticity and the final magnetoelastic system

To close (1.14) a constitutive equation is needed for $\boldsymbol{T}$. Here, we adopt the well known linearized elasticity model[‡] which yields the following system

$$\rho = \rho_0, \tag{1.15}$$

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla \operatorname{div} \boldsymbol{u} + \rho \mu_{\mathcal{B}} \operatorname{curl} \boldsymbol{H} \times \boldsymbol{H} + \rho \boldsymbol{f}, \tag{1.16}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the displacement of the points of $\mathcal{B}$ with respect to its natural reference configuration, while μ and λ are Lamé constants with $\mu > 0$ and $2\mu + 3\lambda > 0$.

Note that in the above the density, ρ, is a known quantity approximately equal to the initial density distribution. Therefore, using (1.12) and noting that $\boldsymbol{v} = \dfrac{\partial \boldsymbol{u}}{\partial t}$, the system of equations whose solution completely determines the state of a *conducting*, *non-ferromagnetic*, *homogeneous*[§] and *isotropic*[§] medium is

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \frac{\mu}{\rho} \Delta \boldsymbol{u} - \frac{\lambda + \mu}{\rho} \nabla \operatorname{div} \boldsymbol{u} = \mu_{\mathcal{B}} \operatorname{curl} \boldsymbol{H} \times \boldsymbol{H} + \boldsymbol{f}, \tag{1.17a}$$

$$\mu_{\mathcal{B}} \frac{\partial \boldsymbol{H}}{\partial t} + \varrho \operatorname{curl} \operatorname{curl} \boldsymbol{H} = \mu_{\mathcal{B}} \operatorname{curl}(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}), \tag{1.17b}$$

$$\operatorname{div} \boldsymbol{H} = 0. \tag{1.17c}$$

[‡]This requires further, that we assume $\mathcal{B}$ is also mechanically isotropic, homogeneous and linear (in the sense of the relation between stress and strain).

[§]Both form electromagnetic and mechanical perspectives.

1.1.5 Boundary conditions

To provide proper boundary conditions for (1.17) we need to consider a more specific physical situation. Assume that $\mathcal{B}$ occupies a simply connected bounded domain $\Omega \subset \mathbb{R}^3$ of class C^2. Let us fix the particles of $\mathcal{B}$ on the boundary of Ω. This immediately translates to the following boundary condition for $\boldsymbol{u}$:

$$\boldsymbol{u}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{1.18}$$

Now, assume that we put this body in an external "constant" magnetic field, $\boldsymbol{H}_0$ and that the body comes into equilibrium with $\boldsymbol{H}_0$, i.e., the charges are redistributed in such a way to balance $\boldsymbol{H}_0$. If to reach such an equilibrium there are deformations with respect to the stress free configuration of the body, we assume they are negligible. An external body force, $\boldsymbol{f}$, on the body causes additional deformation of $\mathcal{B}$. Consequently, the magnetic field inside the body will change to some field $\boldsymbol{H}$ which, in turn, deforms $\mathcal{B}$ in Ω. The deformation represented by displacement $\boldsymbol{u}$, and the magnetic field $\boldsymbol{H}$, has to satisfy (1.17).

As long as the boundary conditions for magnetic field are concerned, note that the conservation of magnetic flux (Eq. 1.17c) holds universally in the space. Applying this to a proper control volume at the boundary of Ω, using the divergence theorem, the mean value theorem (for integrals) and letting the control volume to shrink to a point $\boldsymbol{x} \in \partial\Omega$, we find the following jump condition for the normal component of $\boldsymbol{H}$ (see e.g., Senior and Volakis, 1995, Eq. 2.4),

$$\mu_{\mathcal{B}} \boldsymbol{n} \cdot \boldsymbol{H} = \mu_{S} \boldsymbol{n} \cdot \boldsymbol{H}_0, \quad \boldsymbol{x} \in \partial\Omega. \tag{1.19}$$

where μ_S is the permittivity of the space outside of $\mathcal{B}$, which we assume is homogeneous and isotropic too. $\boldsymbol{n}$ is the outward normal to $\partial\Omega$. Similarly (see the above reference), the following jump condition should hold for the tangential component of the electric field (using an analogous argument as above with Faraday's law (1.13) and Stokes theorem instead),

$$\boldsymbol{n} \times (\boldsymbol{E} - \boldsymbol{E}_S) = \boldsymbol{J}_M, \quad \boldsymbol{x} \in \partial\Omega,$$

where $\boldsymbol{J}_M$ is the magnetization current and $\boldsymbol{E}_{\mathcal{S}}$ is the electric field outside $\mathcal{B}$. Since we ignored magnetization effects and since the electric field outside $\mathcal{B}$ is zero, we get

$$\boldsymbol{n} \times \boldsymbol{E} = 0, \quad \boldsymbol{x} \in \partial\Omega.$$

As our system of equations does not explicitly involves $\boldsymbol{E}$, we should translate the above boundary condition to an equivalent condition on $\boldsymbol{H}$. To achieve this, we use (1.11) at the boundary along with (1.18) and the above equation to obtain

$$\boldsymbol{n} \times \operatorname{curl} \boldsymbol{H} = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{1.20}$$

Since it is more favorable to work with homogeneous boundary conditions, let us set $\boldsymbol{H} = \boldsymbol{h}(\boldsymbol{x}, t) + \frac{\mu_{\mathcal{S}}}{\mu_{\mathcal{B}}} \boldsymbol{H}_0$. Then, from (1.19) and (1.20) we have

$$\boldsymbol{n} \cdot \boldsymbol{h} = 0, \quad \boldsymbol{n} \times \operatorname{curl} \boldsymbol{h} = 0, \qquad \boldsymbol{x} \in \partial\Omega.$$

Introducing $\boldsymbol{h}$ in (1.17), we get

$$\begin{aligned}
&\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \frac{\mu}{\rho} \Delta \boldsymbol{u} - \frac{\lambda + \mu}{\rho} \nabla \operatorname{div} \boldsymbol{u} = \mu_{\mathcal{B}} \operatorname{curl} \boldsymbol{h} \times (\boldsymbol{h} + \frac{\mu_{\mathcal{S}}}{\mu_{\mathcal{B}}} \boldsymbol{H}_0) + \boldsymbol{f}, \\
&\mu_{\mathcal{B}} \frac{\partial \boldsymbol{h}}{\partial t} + \varrho \operatorname{curl} \operatorname{curl} \boldsymbol{h} = \mu_{\mathcal{B}} \operatorname{curl} \left(\frac{\partial \boldsymbol{u}}{\partial t} \times [\boldsymbol{h} + \frac{\mu_{\mathcal{S}}}{\mu_{\mathcal{B}}} \boldsymbol{H}_0] \right), \\
&\operatorname{div} \boldsymbol{h} = 0,
\end{aligned}$$

and so the governing equations of our magneto-elastic system with proper boundary conditions will be as follows,

$$\left.\begin{aligned}
&\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \frac{\mu}{\rho} \Delta \boldsymbol{u} - \frac{\lambda + \mu}{\rho} \nabla \operatorname{div} \boldsymbol{u} = \mu_{\mathcal{B}} \operatorname{curl} \boldsymbol{h} \times (\boldsymbol{h} + \frac{\mu_{\mathcal{S}}}{\mu_{\mathcal{B}}} \boldsymbol{H}_0) + \boldsymbol{f}, \\
&\mu_{\mathcal{B}} \frac{\partial \boldsymbol{h}}{\partial t} + \varrho \operatorname{curl} \operatorname{curl} \boldsymbol{h} = \mu_{\mathcal{B}} \operatorname{curl} \left(\frac{\partial \boldsymbol{u}}{\partial t} \times [\boldsymbol{h} + \frac{\mu_{\mathcal{S}}}{\mu_{\mathcal{B}}} \boldsymbol{H}_0] \right), \\
&\operatorname{div} \boldsymbol{h} = 0,
\end{aligned}\right\} \text{in } \Omega \times [t_1, t_2], \tag{1.21a}$$

$$\boldsymbol{u} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{h} = 0, \quad \boldsymbol{n} \times \operatorname{curl} \boldsymbol{h} = 0, \qquad \text{on } \partial\Omega \times [t_1, t_2]. \tag{1.21b}$$

1.2 MATHEMATICAL ANALYSIS

Equations (1.21) have independently received little attention in the mathematical literature. This is partly due to the nature of the coupling between the first two equations of (1.21a). For certain questions of interest, this coupling makes the level of difficulty of this system, the same as that of Navier-Stokes equations. In fact, for bounded domains, questions like the existence of global weak solutions (Botsenyuk, 1992) and the existence of regular solutions for small data (Botsenyuk, 1996) are answered with techniques which are very similar to those used for Navier-Stokes. In other cases where the nature of the equations also plays an important role, one will, generally, face more difficulties due to mixed hyperbolic-parabolic nature of the equations.

Besides the above mentioned results on existence and regularity, some variation of these equations (majorly in a direction to simplify the mathematical analysis) has been also studied. Menzala and Zuazua (1998) proved the asymptotic stability using LaSalle's invariance principle for the system with linearized coupling. That is, when $\boldsymbol{h} + \frac{\mu_{\mathcal{S}}}{\mu_{\mathcal{B}}}\boldsymbol{H}_0$, in (1.21a), is replaced with some known vector function $\boldsymbol{b}$. The proof used LaSalle's invariance principle and did not provide a decay rate for the system. Charão, Oliveira, and Menzala (2009) proved that the total energy of this system tends to zero as $t \to \infty$ when a nonlinear dissipation $\boldsymbol{\nu}(\boldsymbol{x}, \boldsymbol{u}')$ is effective on a small subregion of the domain. The rate of decay is given and depends on the behavior of $\boldsymbol{\nu}$ with respect to the second variable: algebraic decay for strong solutions and exponential decay if the behavior is close to linear.

1.2.1 Function spaces and notation

We use the following notation for basic function spaces on $\Omega \subset \mathbb{R}^N$ with $1 \leq p \leq \infty$ and $0 \leq m$. Below, X is a Banach space with norm $\|\cdot\|_X$. The Euclidean norm is simply denoted by $|\cdot|$. Without confusion $|\Omega|$, is also used for the N-dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$. When we talk about derivatives of order m, we have (mixed) partial derivatives in mind.

$L^p(\Omega)$: The usual Lebesgue spaces.

$W^{m,p}(\Omega)$: The usual Sobolev spaces.

$C^m(\Omega)$: The space of functions on Ω continuously differentiable up to (and including) order m. ($C^\infty(\Omega) = \bigcap_{m\geq 0} C^m(\Omega)$.)

$C^m(\overline{\Omega})$: The space of functions on Ω with bounded and uniformly continuous derivatives up to (and including) order m. ($C^\infty(\overline{\Omega}) = \bigcap_{m\geq 0} C^m(\overline{\Omega})$.)

$C_0^\infty(\Omega)$: The space of all functions in $C^\infty(\Omega)$ with compact support in Ω.

$W_0^{m,p}(\Omega)$: The closure of $C_0^\infty(\Omega)$ in $W^{m,p}$-norm.

$W^{-m,p}(\Omega)$: The dual space of $W_0^{m,p}(\Omega)$.

$C_T^m(X)$ The space of all functions $\eta\colon \mathbb{R} \longrightarrow X$, m-times continuously differentiable such that $\eta(t) = \eta(t+T)$, $\forall t \in \mathbb{R}$.

$L^p(a,b;X)$: The space of all functions $f\colon (a,b) \longrightarrow X$ such that $\|f\|_{L^p(a,b;X)} < \infty$:

$$\|f\|_{L^p(a,b;X)} := \begin{cases} \left(\displaystyle\int_a^b \|f\|_X^p \, dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{a<t<b} \|f\|_X, & p = \infty. \end{cases}$$

For a vector function $\boldsymbol{u}$, $\boldsymbol{u} \in X$ means that each component of $\boldsymbol{u}$ belongs to X. For simplicity, we drop the subscript from the L^2-norm, so $\|\boldsymbol{u}\|$ is the norm of $\boldsymbol{u}$ in L^2. Also, the inner product in $L^2(\Omega)$ is denoted by

$$(\boldsymbol{u}, \boldsymbol{v})_2 = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}, \qquad \boldsymbol{u}, \boldsymbol{v} \in L^2(\Omega).$$

Occasionally, when X is a Hilbert space, we use $(\boldsymbol{u}, \boldsymbol{v})_X$, for the inner product in X. For $\boldsymbol{u}$ in a real Banach space, X, and $\boldsymbol{L}$ in the its dual, X', the value of $\boldsymbol{L}$ at $\boldsymbol{u}$ is indicated by $\langle \boldsymbol{L}, \boldsymbol{u} \rangle$.

Let

$$\mathcal{D} = \{\boldsymbol{v} \in C_0^\infty(\Omega)\colon \operatorname{div} \boldsymbol{v} = 0, \, \boldsymbol{n} \cdot \boldsymbol{v}|_{\partial\Omega} = 0\},$$
$$\tilde{\mathcal{D}} = \{\boldsymbol{\psi} \in C^\infty(\overline{\Omega})\colon \operatorname{div} \boldsymbol{\psi} = 0, \, \boldsymbol{n} \cdot \boldsymbol{\psi}|_{\partial\Omega} = 0\}.$$

We denote by H^m and $\tilde{H}^m$ the closure of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ in $W^{m,2}$, respectively. For simplicity, we set $H := H^0$ and $\tilde{H} := \tilde{H}^0$. In agreement with the above, H^{-m} and $\tilde{H}^{-m}$ show the dual spaces of H^m and $\tilde{H}^m$, respectively.

To avoid cumbersome notation in what follows, let us introduce some conventions that simplify the presentation of (1.21a). We use $'$ to denote differentiation with respect to time, e.g., $\boldsymbol{u}'' = \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2}$. Also, we set

$$\begin{aligned}
\mathcal{L}\,\boldsymbol{u} &:= -\frac{\mu}{\rho}\,\Delta\,\boldsymbol{u} - \frac{\lambda+\mu}{\rho}\,\nabla\,\mathrm{div}\,\boldsymbol{u},\\
\tilde{\mathcal{L}}\,\boldsymbol{h} &:= \varrho\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{h}.
\end{aligned}\tag{1.22}$$

Without loss of generality we assume $\mu_{\mathcal{B}} = 1$ and use $\boldsymbol{B}_0 = \mu_{\mathcal{S}}\boldsymbol{H}_0$ to rewrite (1.21) as

$$\left.\begin{aligned}
&\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} = \mathrm{curl}\,\boldsymbol{h} \times (\boldsymbol{h} + \boldsymbol{B}_0) + \boldsymbol{f},\\
&\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} = \mathrm{curl}\,(\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0])\,,\\
&\mathrm{div}\,\boldsymbol{h} = 0,
\end{aligned}\right\}\ \text{in } \Omega \times [t_1, t_2],\tag{1.23a}$$

$$\boldsymbol{u} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{h} = 0, \quad \boldsymbol{n}\times\mathrm{curl}\,\boldsymbol{h} = 0, \qquad \text{on } \partial\Omega \times [t_1, t_2].\tag{1.23b}$$

1.2.2 Some notes on periodic solutions

When it comes to periodic solutions of (1.23) one must be careful about the hyperbolic equation, i.e., the first equation of (1.23a). To see some difficulties related to this equation, let us forget the coupling term for the moment and consider the equation

$$\begin{aligned}
&\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} = \boldsymbol{f}, \qquad \text{in } \Omega,\\
&\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega,\\
&\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t+T), \qquad \boldsymbol{u}'(\boldsymbol{x}, t) = \boldsymbol{u}'(\boldsymbol{x}, t+T), \qquad \text{in } \Omega,
\end{aligned}\tag{1.24}$$

where $\boldsymbol{f}$ is periodic with period T. Regarding the strong solutions of the above equation we have

Theorem 1.1. (Haraux, 1981, p. 163) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with regular boundary. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of eigenvalues of $\mathcal{L}$ in $W^{2,2}(\Omega) \bigcap W_0^{1,2}(\Omega)$ and $\boldsymbol{\vartheta}_n(\boldsymbol{x})$ be the corresponding eigenvectors. Let $\omega = \dfrac{2\pi}{T}$, then* (1.24) *has a strong solution if and only if*

1. *For all* $(m,n) \in \mathbb{Z} \times \mathbb{N}$ *such that* $m^2\omega^2 = \lambda_n$, $\boldsymbol{f}$ *satisfies*

$$f_{mn} := \int_0^T \int_\Omega \boldsymbol{f}(\boldsymbol{x},t) \cdot \boldsymbol{\vartheta}_n \, e^{\imath m \omega t} = 0,$$

2. $\displaystyle \sum_{\{(m,n) | m^2\omega^2 \neq \lambda_n\}} m^2 \lambda_n \left(\frac{|f_{mn}|}{m^2\omega^2 - \lambda_n} \right)^2 < \infty.$

Here, $\imath$ is the imaginary unit. These conditions essentially avoid the phenomena of resonance. The first one says that the force $\boldsymbol{f}$ should not contain an excitation frequency that (itself or any integer multiple of it) matches an eigenvalue of $\boldsymbol{\mathcal{L}}$. The second condition says that even if $\boldsymbol{f}$ satisfies the first condition, its excitation modes (and their integer multiples) should be far enough from any eigenvalue of $\boldsymbol{\mathcal{L}}$ such that the sum is finite. The finiteness of the sum guaranties the convergence of the Fourier series of the solution.

These conditions as one may see, are too restrictive. In general, if we consider the coupling term as well, asserting such conditions for $\operatorname{curl} \boldsymbol{h} \times \boldsymbol{h}$ which, as an unknown, itself is obtained as part of the solution, would be very difficult if not impossible. To avoid the complications regarding resonance, authors have often considered a dissipative term in the parabolic equation. From physical point of view, this amounts to assuming the body is not perfectly elastic and has some viscoelastic characteristics. Following Prodi (1966), we consider a nonlinear dissipative term, $\boldsymbol{\nu}(\boldsymbol{u}')$, in the equation of elastic body, and so our system becomes

$$\left.\begin{aligned} &\boldsymbol{u}'' + \boldsymbol{\mathcal{L}}\,\boldsymbol{u} + \boldsymbol{\nu}(\boldsymbol{u}') = \operatorname{curl} \boldsymbol{h} \times (\boldsymbol{h} + \boldsymbol{B}_0) + \boldsymbol{f}, \\ &\boldsymbol{h}' + \tilde{\boldsymbol{\mathcal{L}}}\,\boldsymbol{h} = \operatorname{curl}\left(\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0]\right), \\ &\operatorname{div} \boldsymbol{h} = 0, \end{aligned}\right\} \text{ in } \Omega \times [0,T], \tag{1.25a}$$

$$\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega \times [0,T], \tag{1.25b}$$

$$\boldsymbol{n} \cdot \boldsymbol{h} = 0, \qquad \text{on } \partial\Omega \times [0,T], \tag{1.25c}$$

$$\boldsymbol{n} \times \operatorname{curl} \boldsymbol{h} = 0, \qquad \text{on } \partial\Omega \times [0,T], \tag{1.25d}$$

where $\boldsymbol{\nu}$ satisfies (H3) in Section 1.2.4.

There are several techniques used to prove the existence of periodic solutions to hyperbolic and parabolic equations. For hyperbolic equations, for example, see the work of Prouse

(1964) who considered periodic solutions to a nonlinear abstract second order equation and Prodi (1966) who generalizes the dissipative term to what we have adopted here. Lions (1969) also, used the ideas of Prodi and an elliptic regularity argument for finding periodic solutions in the same context. Below we will outline this idea so that the reader sees why the same approach is not as promising for the system (1.25). Prodi considered the following system, for some T-periodic forcing $\boldsymbol{f}$:

$$\boldsymbol{u}'' - \Delta\,\boldsymbol{u} + \boldsymbol{\nu}(\boldsymbol{u}') = \boldsymbol{f},$$

and decomposed $\boldsymbol{u}$ into a time dependent component and a steady average part

$$\boldsymbol{u} = \boldsymbol{v} + \overline{\boldsymbol{u}}, \qquad \overline{\boldsymbol{u}} := \frac{1}{T}\int_0^T \boldsymbol{u}.$$

Taking the average of the equation and subtracting from itself, he obtained the following equations for $\boldsymbol{v}$ and $\overline{\boldsymbol{u}}$

$$\begin{aligned} -\Delta\,\overline{\boldsymbol{u}} + \overline{\boldsymbol{\nu}(\boldsymbol{v}')} &= \overline{\boldsymbol{f}}, \\ \boldsymbol{v}'' - \Delta\,\boldsymbol{v} + \boldsymbol{\nu}(\boldsymbol{v}') - \overline{\boldsymbol{\nu}(\boldsymbol{v}')} &= \boldsymbol{f} - \overline{\boldsymbol{f}}. \end{aligned}$$

To obtain the above, he has used the property $\boldsymbol{u}' = \boldsymbol{v}'$ to make the equation for $\boldsymbol{v}$ (the second equation above), independent of the equation for $\overline{\boldsymbol{u}}$. The equation he obtained for $\boldsymbol{v}$ in this fashion, is very similar to the original equation but he now can enjoy the fact that $\overline{\boldsymbol{v}} = 0$, which provides him with a Poincaré inequality in time that was not available for $\boldsymbol{u}$. Once he obtains $\boldsymbol{v}$, finding $\overline{\boldsymbol{u}}$ form the first equation is trivial. For the case of the system (1.25), one can easily see that such a decomposition to averages and time dependent parts will not yield to independent equations. Indeed, such a decomposition in (1.25) will lead to a system of four coupled equations which, in principle, is not easier to solve than the original system.

For periodic solutions to parabolic equation (the second equation of (1.25a) without the coupling term, of course) almost all methods used to find periodic solutions to Navier-Stokes equations might be applicable. One might see Serrin (1959); Yudovič (1960); Prodi (1960); Lions (1960); Prouse (1963); Kozono and Nakao (1996); Kato (1997) and Galdi and Silvestre (2006) for details and various applications of such methods. Unfortunately, these techniques are not successful when applied to (1.25), either. Apart from the method of Kato (1997), they are based on the Poincaré map and some fixed point theorem which is inappropriate for (1.25), as this system lacks a "suitable" energy inequality that either would produce a contraction (for Banach fixed point theorem), or would guarantee the mapping of a proper bounded set into itself (for Brouwer fixed-point theorem). The method of Kato (1997) also requires regularity results which are not available for our system.

1.2.3 Preliminaries

We begin by the following Lemma which provides us with an equivalent norm for $\tilde{H}^1$ and a Poincaré type inequality

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and simply connected domain of class C^2. Then there exists a real positive constant $\tilde{c}_p$ such that for every $\boldsymbol{h} \in \tilde{H}^1$,*

$$\|\boldsymbol{h}\|_{\tilde{H}^1} \leq \tilde{c}_p \|\operatorname{curl} \boldsymbol{h}\| , \tag{1.26}$$

$$\|\boldsymbol{h}\| \leq \tilde{c}_p \|\operatorname{curl} \boldsymbol{h}\| . \tag{1.27}$$

Proof. This is indeed Lemma 1.6, page 465 of Temam (1979), as our $\tilde{H}^1$ is a subspace of his $H^1 \bigcap H_0$. □

Next we undertake a more precise definition of some useful operators. For any fixed $\boldsymbol{u} \in W_0^{1,2}$ we define the linear bounded operator $\mathcal{L}\colon W_0^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega)$ and its associated bilinear form by

$$\langle \mathcal{L}\, \boldsymbol{u}, \boldsymbol{v} \rangle = a_I(\boldsymbol{u}, \boldsymbol{v}) := \frac{\mu}{\rho}(\nabla\, \boldsymbol{u}, \nabla\, \boldsymbol{v})_2 + \frac{\lambda + \mu}{\rho}(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v})_2, \qquad \text{for all } \boldsymbol{v} \in W_0^{1,2}.$$

Similarly, for any $\boldsymbol{h} \in \tilde{H}^1$ the linear bounded operator $\tilde{\mathcal{L}}\colon \tilde{H}^1 \longrightarrow \tilde{H}^{-1}$ and its associated bilinear form is defined by

$$\langle \tilde{\mathcal{L}}\, \boldsymbol{h}, \boldsymbol{b} \rangle = a_{II}(\boldsymbol{h}, \boldsymbol{b}) := \varrho(\operatorname{curl} \boldsymbol{h}, \operatorname{curl} \boldsymbol{b})_2. \qquad \text{for all } \boldsymbol{b} \in \tilde{H}^1. \tag{1.28}$$

Note that $a_I(\boldsymbol{u}, \boldsymbol{v})$ and $a_{II}(\boldsymbol{h}, \boldsymbol{b})$ define inner products on $W_0^{1,2}(\Omega)$ and $\tilde{H}^1$, respectively, with associated norms equivalent to those of $W_0^{1,2}(\Omega)$ and $\tilde{H}^1$.

For $\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{b} \in C^\infty(\overline{\Omega})$, let

$$b(\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{b}) = \int_\Omega \boldsymbol{u} \times \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{b},$$

Note that by properties of the triple product

$$b(\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{b}) = -b(\boldsymbol{h}, \boldsymbol{u}, \boldsymbol{b}). \tag{1.29}$$

We also have the following standard result (see e.g., Constantin and Foiaş, 1988, p. 49, Proposition 6.1)

Lemma 1.3. *Let Ω be an open bounded domain in $\mathbb{R}^3$ of class C^2. Let $0 \leq s_1 \leq 2$, $0 \leq s_2 \leq 2$ and $0 \leq s_3 \leq 1$, such that*

$$\text{a) } s_1 + s_2 + s_3 \geq \frac{3}{2}, \qquad\qquad \text{if } s_i \neq \frac{3}{2}, \quad i = 1, 2, 3,$$

or

$$\text{b) } s_1 + s_2 + s_3 > \frac{3}{2}, \qquad\qquad \text{if } s_i = \frac{3}{2}, \text{ for some } i = 1, 2, 3.$$

Then there is a constant $c_1 = c_1(s_1, s_2, s_3, \Omega)$ such that

$$|b(\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{b})| \leq c_1 \|\boldsymbol{u}\|_{W^{s_1,2}} \|\boldsymbol{h}\|_{W^{s_2,2}} \|\operatorname{curl} \boldsymbol{b}\|_{W^{s_3,2}}. \tag{1.30}$$

With the help of the above lemma, we define the following linear bounded operators: For given $(\boldsymbol{h}, \boldsymbol{b}) \in W^{1,2}(\Omega) \times W^{\frac{1}{2},2}(\Omega)$ we define $\mathcal{B}_I(\boldsymbol{h}, \boldsymbol{b})\colon W^{1,2}(\Omega) \times W^{\frac{1}{2},2}(\Omega) \longrightarrow W^{-1,2}(\Omega)$ by

$$\langle \mathcal{B}_I(\boldsymbol{h}, \boldsymbol{b}), \boldsymbol{v} \rangle = b(\boldsymbol{b}, \boldsymbol{v}, \boldsymbol{h}), \qquad \forall \boldsymbol{v} \in W_0^{1,2}(\Omega),$$

and for given $(\boldsymbol{u}, \boldsymbol{h}) \in L^2(\Omega) \times W^{1,2}(\Omega)$ we define $\mathcal{B}_{II}(\boldsymbol{u}, \boldsymbol{h})\colon L^2(\Omega) \times W^{1,2}(\Omega) \longrightarrow \tilde{H}^{-2}$ by,

$$\langle \mathcal{B}_{II}(\boldsymbol{u}, \boldsymbol{h}), \boldsymbol{b} \rangle = b(\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{b}), \qquad \forall \boldsymbol{b} \in \tilde{H}^2(\Omega).$$

Let $S_1 = \{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots\}$ be an orthogonal basis for $W_0^{1,2}(\Omega)$, orthonormal in $L^2(\Omega)$. Also, let $S_2 = \{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots\}$ be an orthonormal basis for $\tilde{H}$. Indeed, since C_0^∞ is dense in $W_0^{1,2}$

and $W_0^{1,2}$ is a separable Hilbert space we can choose S_1 such that $S_1 \subset C_0^\infty$. Similarly we choose S_2 such that $S_2 \subset \tilde{\mathcal{D}}$. Let

$$S_1^n = \mathrm{span}\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}, \qquad S_2^n = \mathrm{span}\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n\}.$$

Before giving the weak formulation and the main theorem, we prove the following Gronwall type lemma for periodic functions. The essential idea goes back at least to Prouse (1964).

Lemma 1.4. *Let $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ be T-periodic continuous functions and $\mathcal{G} \colon \mathbb{R} \to \mathbb{R}$ be a T-periodic function such that $\mathcal{G} \in C^1(\mathbb{R})$ and the following inequalities hold:*

$$\int_0^T \mathcal{G}(\tau) \leq c_2,$$

and

$$\frac{d\mathcal{G}}{dt} \leq g_1(t) + g_2(t)\, \mathcal{G}(t), \qquad \forall t \in \mathbb{R},$$

where c_2 is a positive constant. Then,

$$\sup_{t\in[0,T]} \mathcal{G}(t) \leq \frac{c_2}{T} + 2T \sup_{t\in[0,T]} g_1(t) + 2c_2 \sup_{t\in[0,T]} g_2(t).$$

Proof. We know that there exists $z \in (0, T)$ such that $\int_0^T \mathcal{G}(\tau)d\tau = T\mathcal{G}(z)$. Thus, using the hypothesis we infer that

$$\mathcal{G}(z) \leq \frac{C_1}{T}.$$

Now, we integrate the differential inequality from z to $t \in [z, T]$ and obtain

$$\begin{aligned}
\mathcal{G}(t) &\leq \mathcal{G}(z) + \int_z^t g_1(\tau)\, d\tau + \int_z^t g_2(\tau)\mathcal{G}(\tau)\, d\tau \\
&\leq \frac{c_2}{T} + T \sup_{t\in[0,T]} g_1(t) + \sup_{t\in[0,T]} g_2(t) \int_0^T \mathcal{G}(\tau)\, d\tau \\
&\leq \frac{c_2}{T} + T \sup_{t\in[0,T]} g_1(t) + c_2 \sup_{t\in[0,T]} g_2(t), \qquad (1.31)
\end{aligned}$$

which holds for every $t \in [z, T]$. In order to get an estimate for $\mathcal{G}(t)$ on the interval $[0, z]$, we integrate the differential inequality over the interval $[0, t]$ for $t \in [0, z]$, use the previous estimate and obtain:

$$\begin{aligned}
\mathcal{G}(t) &\le \mathcal{G}(0) + \int_0^t g_1(\tau)\, d\tau + \int_0^t g_2(\tau)\mathcal{G}(\tau)\, d\tau \\
&\le \mathcal{G}(T) + T \sup_{t\in[0,T]} g_1(t) + \sup_{t\in[0,T]} g_2(t) \int_0^T \mathcal{G}(\tau)\, d\tau \\
&\le \frac{c_2}{T} + 2T \sup_{t\in[0,T]} g_1(t) + 2c_2 \sup_{t\in[0,T]} g_2(t).
\end{aligned}$$

This inequality holds for any $t \in [0, z]$, which in combination with (1.31) completes the proof. □

1.2.4 Weak formulation

Before we proceed with the definition of weak solutions let us gather our assumptions in one place:

(H0) $\Omega \subset \mathbb{R}^3$ is a bounded simply connected domain of class C^2.

(H1) $p \in [3, 4]$ and $q = \dfrac{p+2}{p+1}$.

(H2) $\boldsymbol{f} \in C([0, T]; L^2(\Omega))$ [¶] with $\boldsymbol{f}(0) = \boldsymbol{f}(T)$. $\boldsymbol{B}_0$ is a constant vector.

(H3) $\boldsymbol{\nu} \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is continuous, monotone and satisfies the following properties

(a) There exists a positive constant K_0, such that

$$\boldsymbol{x} \cdot \boldsymbol{\nu}(\boldsymbol{x}) \ge K_0 \left|\boldsymbol{x}\right|^{p+2}, \quad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

(b) There exist positive constants r_ν and K_1, such that

$$|\boldsymbol{\nu}(\boldsymbol{x})| \le K_1 \left|\boldsymbol{x}\right|^{p+1} \text{ if } |\boldsymbol{x}| > r_\nu.$$

Note that as an example of a function which satisfies (H3), we have $\boldsymbol{\nu}(\boldsymbol{x}) = |\boldsymbol{x}|^p\, \boldsymbol{x}$.

Definition 1.5. We say that $(\boldsymbol{u}, \boldsymbol{h})$ is a *weak T-periodic solution* of (1.25) if

1. $\boldsymbol{u} \in L^\infty(0, T; W_0^{1,2}(\Omega))$ with $\boldsymbol{u}' \in L^\infty(0, T; L^2(\Omega)) \cap L^{p+2}(0, T; L^{p+2}(\Omega))$,

[¶] From (H1) it follows that $\boldsymbol{f}$ belongs also to $C([0, T]; L^q(\Omega))$.

2. $\boldsymbol{h} \in L^\infty(0,T;\tilde{H}(\Omega)) \cap L^2(0,T;\tilde{H}^1(\Omega))$,

3. $\boldsymbol{u}$ and $\boldsymbol{h}$ satisfy

$$\int_0^T (\boldsymbol{u},\boldsymbol{\varphi})_2\,\eta'' + \int_0^T a_I(\boldsymbol{u},\boldsymbol{\varphi})\,\eta + \int_0^T (\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{\varphi})_2\,\eta = \int_0^T (\operatorname{curl}\boldsymbol{h}\times[\boldsymbol{h}+\boldsymbol{B}_0],\boldsymbol{\varphi})_2\,\eta + \int_0^T (\boldsymbol{f},\boldsymbol{\varphi})_2\,\eta, \tag{1.32}$$

$$-\int_0^T (\boldsymbol{h},\boldsymbol{\psi})_2\,\eta' + \int_0^T a_{II}(\boldsymbol{h},\boldsymbol{\psi})\,\eta = \int_0^T (\boldsymbol{u}'\times[\boldsymbol{h}+\boldsymbol{B}_0],\operatorname{curl}\boldsymbol{\psi})_2\,\eta \tag{1.33}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$, $\boldsymbol{\psi} \in \tilde{\mathcal{D}}$ and $\eta \in C_T^2(\mathbb{R})$.

It is easy to verify that if $\boldsymbol{u}$ and $\boldsymbol{h}$ are smooth enough T-periodic functions satisfying (1.25) then they also satisfy (1.32) and (1.33), respectively, by taking proper inner products and integrating by parts in time and space variables. They also satisfy 1 and 2 above.

Conversely, assume that $\boldsymbol{u}$ and $\boldsymbol{h}$ are smooth enough weak solutions. We wish to show that they satisfy (1.25). First, let us consider $\boldsymbol{u}$. Since $\boldsymbol{u}$ is smooth enough in $L^\infty(0,T;W_0^{1,2})$ it follows that $\boldsymbol{u}(t) \in W_0^{1,2}$ for all $t \in [0,T]$ and hence the boundary condition (1.25b) is satisfied. Next, integrating (1.32) by parts twice in time in the first term, and once in space variables in the second term, we get

$$(\boldsymbol{u}(T),\boldsymbol{\varphi})_2\,\eta'(T) - (\boldsymbol{u}(0),\boldsymbol{\varphi})_2\,\eta'(0) - (\boldsymbol{u}'(T),\boldsymbol{\varphi})_2\,\eta(T) + (\boldsymbol{u}'(0),\boldsymbol{\varphi})_2\,\eta(0) + \int_0^T \left(\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \boldsymbol{\nu}(\boldsymbol{u}') - \operatorname{curl}\boldsymbol{h}\times(\boldsymbol{h}+\boldsymbol{B}_0) - \boldsymbol{f},\boldsymbol{\varphi}\right)_2 \eta = 0. \tag{1.34}$$

For all $\eta \in C_T^2(\mathbb{R})$ and $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$. Since the above holds for all $\eta \in C_T^2(\mathbb{R})$, in particular it holds for all $\tilde{\eta} \in C_0^2(0,T)$. Consequently,

$$\int_0^T \left(\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \boldsymbol{\nu}(\boldsymbol{u}') - \operatorname{curl}\boldsymbol{h}\times(\boldsymbol{h}+\boldsymbol{B}_0) - \boldsymbol{f},\boldsymbol{\varphi}\right)_2 \tilde{\eta} = \int_0^T \mathcal{F}_1(t)\,\tilde{\eta} = 0, \qquad \forall \tilde{\eta} \in C_0^2(0,T),$$

so $\mathcal{F}_1(t)$ should belong to the orthogonal complement of $C_0^2(0,T)$ in $L^2(0,T)$, which is the set $\{0\}$, [‖] and hence we get

$$\left(\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \boldsymbol{\nu}(\boldsymbol{u}') - \operatorname{curl}\boldsymbol{h}\times(\boldsymbol{h}+\boldsymbol{B}_0) - \boldsymbol{f},\boldsymbol{\varphi}\right)_2 = (\boldsymbol{\mathcal{F}}_2,\boldsymbol{\varphi})_2 = 0, \qquad \boldsymbol{\varphi} \in C_0^\infty(\Omega).$$

‖This follows by the density of $C_0^2(0,T)$ in $L^2(0,T)$.

By a similar reasoning we use the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ to conclude that $\boldsymbol{\mathcal{F}}_2 = 0$, and so the first equation of (1.25a) also holds.

As far as $\boldsymbol{u}$ is concerned, it remains to show that $\boldsymbol{u}(0) = \boldsymbol{u}(T)$ and $\boldsymbol{u}'(0) = \boldsymbol{u}'(T)$. For this, we use (1.34) with the knowledge that $\mathcal{F}_1 = 0$ to get

$$(\boldsymbol{u}(T), \boldsymbol{\varphi})_2\, \eta'(T) - (\boldsymbol{u}(0), \boldsymbol{\varphi})_2\, \eta'(0) - (\boldsymbol{u}'(T), \boldsymbol{\varphi})_2\, \eta(T) + (\boldsymbol{u}'(0), \boldsymbol{\varphi})_2\, \eta(0) = 0, \qquad \forall \eta \in C_T^2(\mathbb{R}).$$

Now choosing $\eta \in C^2(\mathbb{R})$ such that $\eta(0) = \eta(T) = 0$ and $\eta'(0) = \eta'(T) = a$, for some arbitrary $a \in \mathbb{R}$, we get

$$(\boldsymbol{u}(T) - \boldsymbol{u}(0), \boldsymbol{\varphi})_2\, a = 0, \qquad \forall a \in \mathbb{R} \text{ and } \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega).$$

which implies that $(\boldsymbol{u}(T) - \boldsymbol{u}(0), \boldsymbol{\varphi})_2 = 0$ for all $C_0^\infty(\Omega)$, so $\boldsymbol{u}(T) - \boldsymbol{u}(0) = 0$ (which is the orthogonal complement of $C_0^\infty(\Omega)$ in $L^2(\Omega)$). To conclude $\boldsymbol{u}'(T) = \boldsymbol{u}'(0)$ we argue analogously.

Turning our attention to $\boldsymbol{h}$, we see that the $\boldsymbol{h}$ component of the tuple $(\boldsymbol{u}, \boldsymbol{h})$ of a weak solution, belongs to $\tilde{H}^1$ for almost all $t \in [0, T]$, and hence satisfies $\operatorname{div} \boldsymbol{h} = 0$ in the weak sense, ** and $\boldsymbol{n} \cdot \boldsymbol{h} = 0$ in the trace sense. If the weak solution is smooth enough, these will be satisfied in the usual sense and so (1.25a) and (1.25c) are recovered.

It is important to observe that the boundary condition (1.25d) is not satisfied in any sense by weak solutions. Indeed, in formulating the weak solutions, we ignored such a boundary condition in our function spaces and just implicitly used it in the definition of $\tilde{\boldsymbol{\mathcal{L}}}$ in (1.28). We now show that if $\boldsymbol{h}$ is smooth enough, this boundary condition, as well as the second equation of (1.25a) are satisfied. Integrating the first term of (1.33) by parts in time, we get

$$\begin{aligned}(\boldsymbol{h}(0), \boldsymbol{\psi})_2 \eta(0) - (\boldsymbol{h}(T), \boldsymbol{\psi})_2 \eta(T) \\ + \int_0^T \left[(\boldsymbol{h}', \boldsymbol{\psi})_2 + a_{II}(\boldsymbol{h}, \boldsymbol{\psi}) - (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{\psi})_2 \right] \eta = 0,\end{aligned}$$

$\forall \eta \in C_T^2(\mathbb{R})$ and $\forall \boldsymbol{\psi} \in \tilde{\boldsymbol{\mathcal{D}}}$. By arguments similar to preceding paragraphs we conclude that

$$(\boldsymbol{h}(0) - \boldsymbol{h}(T), \boldsymbol{\psi})_2 = 0, \qquad \forall \boldsymbol{\psi} \in \tilde{\boldsymbol{\mathcal{D}}}, \tag{1.35}$$

**When weak derivatives are used in computing $\operatorname{div} \boldsymbol{h}$.

and that

$$(\boldsymbol{h}', \boldsymbol{\psi})_2 + \varrho(\operatorname{curl}\boldsymbol{h}, \operatorname{curl}\boldsymbol{\psi})_2 - (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl}\boldsymbol{\psi})_2 = 0, \qquad \forall \boldsymbol{\psi} \in \tilde{\mathcal{D}}.$$

We proceed by integrating by parts in the second and the third term above, to obtain

$$\Big(\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} - \operatorname{curl}(\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0]), \boldsymbol{\psi}\Big)_2 + \int_{\partial\Omega} \operatorname{curl}\boldsymbol{h} \times \boldsymbol{n} \cdot \boldsymbol{\psi} = 0, \qquad \forall \boldsymbol{\psi} \in \tilde{\mathcal{D}}. \tag{1.36}$$

Note that since $\boldsymbol{u}(t) = 0$ on $\partial\Omega$ (for all t) we have that $\boldsymbol{u}'(t) = 0$ on $\partial\Omega$ and so the term involving $\boldsymbol{u}'$ does not contribute to the boundary integral. In particular, the above relation holds for all $\boldsymbol{\psi} \in \mathcal{D} \subset \tilde{\mathcal{D}}$ and so

$$\Big(\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} - \operatorname{curl}(\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0]), \boldsymbol{\psi}\Big)_2 = (\boldsymbol{\mathcal{F}}_3, \boldsymbol{\psi})_2 = 0, \qquad \forall \boldsymbol{\psi} \in \mathcal{D}.$$

Let us remind that by Helmholtz-Weyl decomposition $L^2 = H \oplus G$, where H is as defined before and $G = \{\boldsymbol{w} \in L^2(\Omega) \colon \boldsymbol{w} = \nabla\pi \text{ for some } \pi \in W^{1,2}_{loc}(\Omega)\}$ (Galdi, 2011). Since $\mathcal{D}$ is dense in H it follows from the above relation that $\boldsymbol{\mathcal{F}}_3 = \nabla\pi_1 \in G$. Using this information in (1.36) we get

$$\int_{\partial\Omega} \operatorname{curl}\boldsymbol{h} \times \boldsymbol{n} \cdot \boldsymbol{\psi} = 0, \qquad \forall \boldsymbol{\psi} \in \tilde{\mathcal{D}},$$

which by noticing that $\operatorname{curl}\boldsymbol{h} \times \boldsymbol{n}$ and $\boldsymbol{\psi}$ both have zero normal component on $\partial\Omega$, we deduce that $\operatorname{curl}\boldsymbol{h} \times \boldsymbol{n} = 0$ on $\partial\Omega$ and so (1.25d) is recovered.

It remains to show that $\boldsymbol{\mathcal{F}}_3 = 0$, to conclude that the second equation of (1.25a) holds. Some lines before, we showed that $\boldsymbol{\mathcal{F}}_3 \in G$. We now show that $\boldsymbol{\mathcal{F}}_3$ belongs to H as well, and hence it should be zero. For any $\nabla\pi_2 \in G$, notice that

$$(\boldsymbol{h}', \nabla\pi_2)_2 + \varrho(\operatorname{curl}\boldsymbol{h}, \operatorname{curl}\nabla\pi_2)_2 - (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl}\nabla\pi_2)_2 = 0,$$

since $\boldsymbol{h}' \in \tilde{H}$ and the curl of a gradient is always zero. Integrating the above by parts in Ω and using the fact that $\boldsymbol{u}' = \operatorname{curl}\boldsymbol{h} \times \boldsymbol{n} = 0$ on $\partial\Omega$, we obtain

$$(\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} - \operatorname{curl}(\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0]), \nabla\pi_2)_2 = 0.$$

The fact that $\boldsymbol{h}(0) = \boldsymbol{h}(T)$, follows from (1.35) and a similar argument as above. Once we established our weak formulation in the above, we have

Theorem 1.6. *Under the assumptions* (H0)–(H3), (1.25) *admits at least one weak T-periodic solution.*

1.2.5 Proof of Theorem 1.6: Existence of solutions to Faedo-Galerkin approximations

For any fixed $n > 0$, we are seeking approximate solutions of the form

$$\boldsymbol{u}_n = \sum_{i=1}^{n} c_{ni}(t)\boldsymbol{\varphi}_i, \qquad \boldsymbol{h}_n = \sum_{i=1}^{n} d_{ni}(t)\boldsymbol{\psi}_i,$$

where c_{ni} and d_{ni} satisfy the following system of ordinary differential equations, $1 \leq i \leq n$:

$$\begin{aligned} &c''_{ni} + c_{nj}a_I(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) + (\boldsymbol{\nu}(c_{nj}\boldsymbol{\varphi}_j), \boldsymbol{\varphi}_i)_2 = d_{nj}(\operatorname{curl}\boldsymbol{\psi}_j \times [d_{nk}\operatorname{curl}\boldsymbol{\psi}_k + \boldsymbol{B}_0], \boldsymbol{\varphi}_i)_2 + (\boldsymbol{f}, \boldsymbol{\varphi}_i)_2, \\ &d'_{ni} + d_{nj}a_{II}(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) = c'_{nj}(\boldsymbol{\varphi}_j \times [d_{nk}\boldsymbol{\psi}_k + \boldsymbol{B}_0], \operatorname{curl}\boldsymbol{\psi}_i)_2, \\ &\qquad c_{ni}(0) = c_{ni}(T), \qquad c'_{ni}(0) = c'_{ni}(T), \qquad d_{ni}(0) = d_{ni}(T), \end{aligned} \tag{1.37}$$

with summation on repeated indices j and k. In operator form the above is written as

$$\begin{aligned} &\boldsymbol{u}''_n + \mathcal{L}\,\boldsymbol{u}_n + \boldsymbol{\nu}(\boldsymbol{u}'_n) = \mathcal{B}_I(\boldsymbol{h}_n, \boldsymbol{h}_n + \boldsymbol{B}_0) + \boldsymbol{f}, \\ &\boldsymbol{h}'_n + \tilde{\mathcal{L}}\,\boldsymbol{h}_n = \mathcal{B}_{II}(\boldsymbol{u}'_n, \boldsymbol{h}_n + \boldsymbol{B}_0), \\ &\boldsymbol{u}_n(0) = \boldsymbol{u}_n(T), \qquad \boldsymbol{u}'_n(0) = \boldsymbol{u}'_n(T), \qquad \boldsymbol{h}_n(0) = \boldsymbol{h}_n(T), \end{aligned} \tag{1.38}$$

and we are looking for solutions $\boldsymbol{u}_n \in C^2_T(S^n_1)$ and $\boldsymbol{h}_n \in C^1_T(S^n_2)$. Let us fix n and in what follows drop the index n from $\boldsymbol{u}_n$ and $\boldsymbol{h}_n$ for a lighter notation. To show the existence of a solution to the nonlinear system (1.37), or equivalently (1.38), we first consider the following linear system

$$\begin{aligned} &\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \boldsymbol{\nu}(\boldsymbol{v}') + \alpha\boldsymbol{u}' = \alpha\boldsymbol{v}' + \mathcal{B}_I(\boldsymbol{h}, \boldsymbol{b} + \boldsymbol{B}_0) + \boldsymbol{f}, \\ &\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} = \mathcal{B}_{II}(\boldsymbol{u}', \boldsymbol{b} + \boldsymbol{B}_0), \\ &\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \boldsymbol{u}'(0) = \boldsymbol{u}'(T), \qquad \boldsymbol{h}(0) = \boldsymbol{h}(T). \end{aligned} \tag{1.39}$$

where $\alpha > 0$ and $\boldsymbol{v} \in C^1_T(S^n_1)$ and $\boldsymbol{b} \in C_T(S^n_2)$ are given T-periodic functions. The existence of T-periodic solutions to the above system follows from the following theorem (see e.g., Burton, 1985, Theorem 1.2.1):

Theorem 1.7. *Consider the linear system of equations*

$$\boldsymbol{x}' = \boldsymbol{A}(t) \cdot \boldsymbol{x} + \boldsymbol{B}(t), \tag{1.40}$$

where $\boldsymbol{A}(t)$ is an $N \times N$ matrix of continuous functions from $\mathbb{R} \longrightarrow \mathbb{R}$, and $\boldsymbol{B}(t)\colon \mathbb{R} \longrightarrow \mathbb{R}^N$ is continuous. Let both $\boldsymbol{A}(t)$ and $\boldsymbol{B}(t)$ be periodic with period T. Then (1.40) has a unique periodic solution of period T, if and only if the only T-periodic solution to the corresponding homogeneous system is the trivial solution.

Let us show that (1.39) satisfies the conditions of the above theorem. The corresponding homogeneous equation is

$$\begin{aligned} &\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \alpha \boldsymbol{u}' = \mathcal{B}_I(\boldsymbol{h}, \boldsymbol{b} + \boldsymbol{B}_0), \\ &\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} = \mathcal{B}_{II}(\boldsymbol{u}', \boldsymbol{b} + \boldsymbol{B}_0). \end{aligned} \tag{1.41}$$

Assume that it has a T-periodic solution $(\boldsymbol{u}, \boldsymbol{h})$. Taking the L^2-inner product of the first equation with $\boldsymbol{u}'$ and the second equation with $\boldsymbol{h}$ and adding the two equations (using (1.29)) we get

$$\frac{d}{dt}\left\{\|\boldsymbol{u}\|^2 + a_I(\boldsymbol{u}, \boldsymbol{u}) + \|\boldsymbol{h}\|^2\right\} + 2\varrho\,\|\operatorname{curl} \boldsymbol{h}\|^2 + 2\alpha\,\|\boldsymbol{u}'\|^2 = 0.$$

Integrating the above in $[0, T]$ and using the assumption that $\boldsymbol{u}$ and $\boldsymbol{h}$ are periodic we obtain

$$\int_0^T \|\operatorname{curl} \boldsymbol{h}\|^2 = 0, \qquad \int_0^T \|\boldsymbol{u}'\|^2 = 0,$$

which immediately gives that $\boldsymbol{u}' = 0$ and using (1.27) also gives $\boldsymbol{h} = 0$. To see that we also have $\boldsymbol{u} = 0$, set $\boldsymbol{h} = 0$ and $\boldsymbol{u}' = 0$ in the first equation of (1.41) to get that $\boldsymbol{u}$ must satisfy $\mathcal{L}\,\boldsymbol{u} = 0$, which has the unique solution $\boldsymbol{u} = 0$.

Let us define the mapping

$$\Phi\colon C^1_T(S^n_1) \times C_T(S^n_2) \longrightarrow C^2_T(S^n_1) \times C^1_T(S^n_2), \tag{1.42}$$

which maps any $(\boldsymbol{v}, \boldsymbol{b}) \in C^1_T(S^n_1) \times C_T(S^n_2)$ to the unique solution of (1.39), $(\boldsymbol{u}, \boldsymbol{h}) \in C^2_T(S^n_1) \times C^1_T(S^n_2)$. It is clear that if Φ has a fixed point, $(\boldsymbol{u}, \boldsymbol{h})$, then $(\boldsymbol{u}, \boldsymbol{h})$ solves (1.38). So our goal is to show the existence of a fixed point for Φ. For this, we use the Leray-Schauder principle formulated as follows (see e.g., Zeidler, 1986, Theorem 6.A):

Theorem 1.8. (Leray-Schauder Principle) *Let X be a Banach space and $\Phi\colon X \longrightarrow X$ a compact operator. Assume that there exists $r > 0$ such that if $x = \beta\,\Phi(x)$, with $0 < \beta < 1$, then $\|x\|_X \le r$. Then Φ has a fixed point.*

Note that since $C^2_T(S^n_1) \times C^1_T(S^n_2) \subset C^1_T(S^n_1) \times C_T(S^n_2)$, we can view Φ as a mapping from $C^1_T(S^n_1) \times C_T(S^n_2)$ into itself, so we may take X to be the domain of Φ. To show the compactness of Φ, note that $C^{m+1}_T(S^n_i), i = 1, 2$ is compactly embedded in $C^m_T(S^n_i), i = 1, 2$, respectively. To see this latter point, restrict the domain of $C^{m+1}_T(S^n_i)$ from $\mathbb{R}$ to $[0, T]$. Now $[0, T]$ is convex and bounded so by Adams and Fournier (2003, Theorem 1.34) this restriction of $C^{m+1}_T(S^n_i)$ is compactly embedded in $C^m_T(S^n_i)$ when its domain is also restricted to $[0, T]$. Once the proper subsequence is extracted, we may extend the functions periodically back to $\mathbb{R}$.

Its range being compactly embedded in its domain, for Φ to be compact, it just has to be continuous. The continuity of Φ is shown in the Appendix. We now turn our attention to the a priori bound needed in Theorem 1.8.

One easily sees that if there is $(\boldsymbol{u}, \boldsymbol{h})$ such that $(\boldsymbol{u}, \boldsymbol{h}) = \beta\Phi(\boldsymbol{u}, \boldsymbol{h})$, then $(\boldsymbol{u}, \boldsymbol{h}) \in C^2_T(S^n_1) \times C^1_T(S^n_2)$, i.e., the range of Φ. (In particular, all fixed points of Φ should belong to its range.) Also, $(\boldsymbol{u}, \boldsymbol{h})$ satisfies

$$\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \beta\boldsymbol{\nu}(\boldsymbol{v}') + \alpha(1-\beta)\boldsymbol{u}' = \mathcal{B}_I(\boldsymbol{h}, \boldsymbol{b} + \boldsymbol{B}_0) + \beta\boldsymbol{f}, \tag{1.43}$$

$$\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} = \mathcal{B}_{II}(\boldsymbol{u}', \boldsymbol{b} + \boldsymbol{B}_0), \tag{1.44}$$

We proceed to show the existence of $r > 0$ such that any $(\boldsymbol{u}, \boldsymbol{h})$ satisfying the above equations also satisfy

$$\|(\boldsymbol{u}, \boldsymbol{h})\| := \sup_{t\in[0,T]} \left\{ \|\boldsymbol{u}'\| + a_I(\boldsymbol{u}, \boldsymbol{u})^{\frac{1}{2}} + \|\boldsymbol{h}\| \right\} \le r, \tag{1.45}$$

where we have used a natural product norm for the *domain* of Φ. Taking the L^2-inner product of (1.43) by $\boldsymbol{u}'$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{ \|\boldsymbol{u}'\|^2 + a_I(\boldsymbol{u}, \boldsymbol{u}) \right\} + \alpha(1-\beta)\,\|\boldsymbol{u}'\|^2 + \beta\,(\boldsymbol{\nu}(\boldsymbol{u}'), \boldsymbol{u}')_2 = b(\boldsymbol{h} + \boldsymbol{B}_0, \boldsymbol{u}', \boldsymbol{h}) + \beta\,(\boldsymbol{f}, \boldsymbol{u}')_2. \tag{1.46}$$

Taking the L^2-inner product of (1.44) by $\boldsymbol{h}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}\|^2 + \varrho\,\|\mathrm{curl}\,\boldsymbol{h}\|^2 = b(\boldsymbol{u}', \boldsymbol{h} + \boldsymbol{B}_0, \boldsymbol{h}). \tag{1.47}$$

We also take the L^2-inner product of (1.43) by $\frac{\epsilon}{2}\boldsymbol{u}$, for some $\epsilon > 0$, which we will specify later, to get

$$\frac{d}{dt}\left\{\frac{\epsilon}{2}(\boldsymbol{u}', \boldsymbol{u})_2\right\} + \frac{\epsilon}{2}a_I(\boldsymbol{u}, \boldsymbol{u}) + \frac{\epsilon\,\beta}{2}(\boldsymbol{\nu}(\boldsymbol{u}'), \boldsymbol{u})_2 + \frac{\epsilon}{2}\alpha(1-\beta)(\boldsymbol{u}', \boldsymbol{u})_2 \\ = \frac{\epsilon}{2}\|\boldsymbol{u}'\|^2 + \frac{\epsilon}{2}\,b(\boldsymbol{h} + \boldsymbol{B}_0, \boldsymbol{u}, \boldsymbol{h}) + \frac{\epsilon\,\beta}{2}(\boldsymbol{f}, \boldsymbol{u})_2. \tag{1.48}$$

Adding (1.46) and (1.47), with the help of (1.29) we obtain

$$\frac{d\mathcal{E}}{dt} + \varrho\,\|\mathrm{curl}\,\boldsymbol{h}\|^2 + \beta\,(\boldsymbol{\nu}(\boldsymbol{u}'), \boldsymbol{u}')_2 \le \beta\,\|\boldsymbol{f}\|_{L^q}\,\|\boldsymbol{u}'\|_{L^{p+2}}\,,$$

where $q = (p+2)/(p+1)$ and we have ignored the non-negative term $\alpha(1-\beta)\,\|\boldsymbol{u}\|^2$ on the left hand side. $\mathcal{E}$ is the natural energy of the system and is given by

$$\mathcal{E} := \frac{1}{2}\left(\|\boldsymbol{u}'\|^2 + a_I(\boldsymbol{u}, \boldsymbol{u}) + \|\boldsymbol{h}\|^2\right).$$

Using the assumption $(\mathrm{H2})_{\mathrm{a}}$ on $\boldsymbol{\nu}$ and Young's inequality we get

$$\frac{d\mathcal{E}}{dt} + \varrho\,\|\mathrm{curl}\,\boldsymbol{h}\|^2 + \beta K_0\int_\Omega |\boldsymbol{u}'|^{p+2} \le c_3\beta\,\|\boldsymbol{f}\|_{L^q}^q + \frac{\beta K_0}{2}\,\|\boldsymbol{u}'\|_{L^{p+2}}^{p+2}\,,$$

or

$$\frac{d\mathcal{E}}{dt} + \varrho\,\|\mathrm{curl}\,\boldsymbol{h}\|^2 + \frac{\beta K_0}{2}\,\|\boldsymbol{u}'\|_{L^{p+2}}^{p+2} \le c_3\beta\,\|\boldsymbol{f}\|_{L^q}^q\,, \tag{1.49}$$

which upon integration in $[0, T]$ implies that (note that $\boldsymbol{u}$ and $\boldsymbol{h}$ are periodic)

$$\int_0^T \|\mathrm{curl}\,\boldsymbol{h}\|^2 \le c_3\,\|\boldsymbol{f}\|_{L^q(0,T;L^q)}^q\,, \tag{1.50}$$

and

$$\int_0^T \|\boldsymbol{u}'\|_{L^{p+2}}^{p+2} \le c_4\,\|\boldsymbol{f}\|_{L^q(0,T;L^q)}^q\,. \tag{1.51}$$

Remark 1.9. Equation (1.49) is the natural energy estimate of the system. Although, we are working on the a priori bound required by Leray-Schauder principle (Theorem 1.8), through (1.43) and (1.44), this can be obtained for a general Galerkin approximation of (1.37). Note that this energy inequality cannot be cast in the form

$$\frac{d\mathcal{E}}{dt} + c_5\mathcal{E} \le c_6,$$

which is encountered in dissipative parabolic equations. Also note that we do not have an estimate for $\boldsymbol{u}$ in which case the proof would go quite simpler (for example, might have enabled us to apply Lemma 1.4 immediately).

Let us proceed to obtain more estimates by adding (1.46), (1.47) and (1.48), to obtain

$$\begin{aligned}\frac{d\mathcal{G}^2}{dt} + \frac{\epsilon}{2}a_I(\boldsymbol{u},\boldsymbol{u}) + \varrho\,\|\mathrm{curl}\,\boldsymbol{h}\|^2 + \beta\,(\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{u}')_2 &\le \frac{\epsilon\,\beta}{2}\,|(\boldsymbol{f},\boldsymbol{u})_2| + \beta\,\,|(\boldsymbol{f},\boldsymbol{u}')_2| + \frac{\epsilon}{2}\,\|\boldsymbol{u}'\|^2 \\ &+ \frac{\epsilon}{2}\,|b(\boldsymbol{h}+\boldsymbol{B}_0,\boldsymbol{u},\boldsymbol{h})| + \frac{\epsilon\,\beta}{2}\,|(\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{u})_2| + \frac{\epsilon}{2}\alpha(1-\beta)\,|(\boldsymbol{u}',\boldsymbol{u})_2|\,, \quad (1.52)\end{aligned}$$

where $\mathcal{G}$ is defined by

$$\mathcal{G}^2 := \frac{1}{2}\left(\|\boldsymbol{u}'\|^2 + a_I(\boldsymbol{u},\boldsymbol{u}) + \|\boldsymbol{h}\|^2 + \epsilon(\boldsymbol{u}',\boldsymbol{u})_2\right).$$

This is the time to specify our ϵ such that $\mathcal{G}$ is well defined, more specifically, such that the right hand side of the above is non-negative. This is undertaken in the following lemma

Lemma 1.10. *Let c_p be the Poincaré constant for Ω and $0 < \epsilon < \frac{4}{c_p}$. Then there are $c_7, c_8 > 0$ such that*

$$c_7\,\mathcal{E} \le \mathcal{G}^2 \le c_8\,\mathcal{E}. \quad (1.53)$$

Proof. Choose $c_7 = 1 - \frac{c_p\epsilon}{4}$ and $c_8 = 1 + \frac{c_p\epsilon}{4}$, since

$$\frac{\epsilon}{2}\,|(\boldsymbol{u}',\boldsymbol{u})| \le \frac{\epsilon}{2}\,\|\boldsymbol{u}'\|\,\|\boldsymbol{u}\| \le \frac{c_p\epsilon}{4}\,\|\boldsymbol{u}'\|^2 + \frac{c_p\epsilon}{4}a_I(\boldsymbol{u},\boldsymbol{u}) \le \frac{c_p\epsilon}{4}\mathcal{E}.$$

□

We proceed to estimate the terms on the right hand side of (1.52). From (1.30) and (1.26) we have

$$
\begin{aligned}
|b(\boldsymbol{h}+\boldsymbol{B}_0,\boldsymbol{u},\boldsymbol{h})| &\le \|\boldsymbol{h}+\boldsymbol{B}_0\|_{W^{1,2}}\|\boldsymbol{u}\|_{W_0^{1,2}}\|\operatorname{curl}\boldsymbol{h}\| \\
&\le \|\boldsymbol{h}\|_{W^{1,2}}\|\operatorname{curl}\boldsymbol{h}\|\|\boldsymbol{u}\|_{W_0^{1,2}}+\|\boldsymbol{B}_0\|_{W^{1,2}}\|\operatorname{curl}\boldsymbol{h}\|\|\boldsymbol{u}\|_{W_0^{1,2}} \\
&\le \|\operatorname{curl}\boldsymbol{h}\|^2\|\boldsymbol{u}\|_{W_0^{1,2}}+\frac{1}{2}\left(\|\boldsymbol{B}_0\|^2_{W^{1,2}}+\|\operatorname{curl}\boldsymbol{h}\|^2\right)\|\boldsymbol{u}\|_{W_0^{1,2}} \\
&\le c_9\|\operatorname{curl}\boldsymbol{h}\|^2 a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+c_{10}a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}.
\end{aligned}
$$

Using $(\mathrm{H3})_{\mathrm{b}}$ and continuous embedding of $W_0^{1,2}(\Omega)$ into $L^{p+2}(\Omega)$, for $p+2\in[5,6]$ (from (H1)), we get

$$
\begin{aligned}
|(\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{u})_2| &\le K_1\int_{\Omega_1}|\boldsymbol{u}'|^{p+1}|\boldsymbol{u}|+\left(\sup_{|\boldsymbol{x}|\le r_\nu}|\boldsymbol{\nu}(\boldsymbol{x})|\right)\int_{\Omega_2}|\boldsymbol{u}| \\
&\le K_1\left(\int_\Omega|\boldsymbol{u}'|^{p+2}\right)^{\frac{p+1}{p+2}}\left(\int_\Omega|\boldsymbol{u}|^{p+2}\right)^{\frac{1}{p+2}}+c_{11}\|\boldsymbol{u}\| \\
&\le c_{12}\|\boldsymbol{u}'\|^{p+1}_{L^{p+2}}a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+c_{14}\,a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}},
\end{aligned}
$$

where $\Omega_1:=\{\boldsymbol{x}\in\Omega\colon|\boldsymbol{u}'(\boldsymbol{x},t)|>r_\nu\}$ and $\Omega_2=\Omega\backslash\Omega_1$. Note that we used one of our restrictive assumptions on p, in this estimate.

We are now ready to complete the work on (1.52). We also remark that as the final estimates should not depend on β we disregard the non-negative term $\beta(\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{u}')$ on the left hand side and instead add $\frac{\epsilon}{2}\|\boldsymbol{u}'\|^2$ to both sides of the inequality. We also set the β's in the right hand side equal to 0 or 1 in favor of increasing the term on this side. With these efforts (1.52) becomes

$$
\begin{aligned}
\frac{d\mathcal{G}^2}{dt}+\frac{\epsilon}{2}a_I(\boldsymbol{u},\boldsymbol{u})+\varrho\|\operatorname{curl}\boldsymbol{h}\|^2+\frac{\epsilon}{2}\|\boldsymbol{u}'\|^2 &\le \frac{c_p\epsilon}{2}\|\boldsymbol{f}\|a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+\|\boldsymbol{f}\|\|\boldsymbol{u}'\|+\frac{\epsilon}{2}\|\boldsymbol{u}'\|^2 \\
&\quad+\frac{c_9\epsilon}{2}\|\operatorname{curl}\boldsymbol{h}\|^2a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+\frac{\epsilon}{2}(c_{10}+c_{14})\,a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}} \\
&\quad+\frac{c_{12}\epsilon}{2}\|\boldsymbol{u}'\|^{p+1}_{L^{p+2}}a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+\frac{c_p\epsilon\alpha}{2}\|\boldsymbol{u}'\|a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}.
\end{aligned}
$$

We observe that we can decrease the left hand side by disregarding a non-negative term $(1-\frac{\epsilon}{2})\left\|\boldsymbol{h}\right\|^2$, and clean up the right hand side with proper positive constants, to obtain

$$2\mathcal{G}\frac{d\mathcal{G}}{dt}+\frac{\epsilon}{2}\mathcal{E}\leq c_{15}\left\|\boldsymbol{f}\right\|\left(a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+\left\|\boldsymbol{u}'\right\|\right)+c_{16}\left\|\boldsymbol{u}'\right\|\left(\left\|\boldsymbol{u}'\right\|+a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}\right)$$
$$+c_{17}\left(\left\|\operatorname{curl}\boldsymbol{h}\right\|^2+\left\|\boldsymbol{u}'\right\|^{p+1}_{L^{p+2}}\right)a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+c_{18}\,a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}.$$

Assume that $\mathcal{G}(t)\neq 0,\ \forall t\in\mathbb{R}$, since if there is a t_0 such that $\mathcal{G}(t_0)=\mathcal{E}(t_0)=0$ then we would go back to (1.49) and integrate from t_0 to $t<t_0+T$ to conclude the existence of a constant r such that $\mathcal{E}(t)<r,\ t\in\mathbb{R}$. Dividing both sides of the above inequality by $\mathcal{G}$ and using (1.53), we get

$$2\frac{d\mathcal{G}}{dt}+\frac{\epsilon}{2c_8}\mathcal{G}\leq c_{15}\left\|\boldsymbol{f}\right\|\frac{a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}+\left\|\boldsymbol{u}'\right\|}{\mathcal{G}}+c_{16}\left\|\boldsymbol{u}'\right\|\frac{\left\|\boldsymbol{u}'\right\|+a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}}{\mathcal{G}}$$
$$+c_{17}\left(\left\|\operatorname{curl}\boldsymbol{h}\right\|^2+\left\|\boldsymbol{u}'\right\|^{p+1}_{L^{p+2}}\right)\frac{a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}}{\mathcal{G}}+c_{18}\frac{a_I(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}}{\mathcal{G}}.$$

Note that by (1.53) all the terms with denominator $\mathcal{G}$ are bounded by $\sqrt{\frac{6}{c_7}}$ and so with other suitable constants we have

$$\frac{d\mathcal{G}}{dt}+c_{19}\mathcal{G}\leq c_{20}\left\|\boldsymbol{f}\right\|+c_{21}\left(\left\|\operatorname{curl}\boldsymbol{h}\right\|^2+\left\|\boldsymbol{u}'\right\|^{p+1}_{L^{p+2}}+\left\|\boldsymbol{u}'\right\|\right)+c_{22}.$$

Integrating the above in $[0,T]$ and using the fact that $\mathcal{G}$ is T-periodic we obtain

$$\int_0^T\mathcal{G}\leq c_{23}\int_0^T\left\|\boldsymbol{f}\right\|+c_{24}\int_0^T\left(\left\|\operatorname{curl}\boldsymbol{h}\right\|^2+\left\|\boldsymbol{u}'\right\|^{p+1}_{L^{p+2}}+\left\|\boldsymbol{u}'\right\|\right)+c_{25}T,$$

which, in view of (1.50) and (1.51) and Hölder inequality, yields

$$\int_0^T\mathcal{G}\leq c_{26},$$

and in particular by (1.53)

$$\int_0^T\mathcal{E}^{\frac{1}{2}}\leq c_{27}. \tag{1.54}$$

We now go back to the natural energy estimate of the system, but this time with a different treatment. The energy equation was obtained by adding (1.46) and (1.47):

$$\frac{d\mathcal{E}}{dt}+\varrho\left\|\operatorname{curl}\boldsymbol{h}\right\|^2+\beta\,(\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{u}')_2=\beta\,(\boldsymbol{f},\boldsymbol{u}')_2.$$

Let us disregard the positive terms in the left hand side and increase β to 1 on the right hand side with the following estimate

$$\frac{1}{2}\mathcal{E}^{\frac{1}{2}}\frac{d\mathcal{E}^{\frac{1}{2}}}{dt} \leq \|\boldsymbol{f}\| \, \|\boldsymbol{u}'\|\,,$$

noting that $\|\boldsymbol{u}'\| + \|\boldsymbol{h}\| \leq \sqrt{6}\,\mathcal{E}^{\frac{1}{2}}$ we obtain after dividing by $\mathcal{E}^{\frac{1}{2}}$,

$$\frac{d\mathcal{E}^{\frac{1}{2}}}{dt} \leq c_{28} \sup_{0\leq t\leq T} \|\boldsymbol{f}\| \leq c_{29}.$$

The above estimate together with (1.54) and Lemma 1.4, yield to

$$\sup_{0\leq t\leq T} \mathcal{E}^{\frac{1}{2}} \leq c_{30}, \tag{1.55}$$

which is equivalent to (1.45) and completes all the requirements of the Leray-Schauder principle.

To summarize, we showed that for each $n > 0$, there exists a unique T-periodic solution $(\boldsymbol{u}_n, \boldsymbol{h}_n) \in C^2_T(S^n_1) \times C^1_T(S^n_1)$ to the approximate equation (1.38). Although all the estimates we found were obtained in the process of using Leray-Schauder principle, one may check that most of them are also true for the solution of (1.38). In particular,

(i) $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;W^{1,2}_0)$. (1.55)

(ii) $\{\boldsymbol{u}'_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2)$. (1.55)

(iii) $\{\boldsymbol{u}'_n\}_{n\in\mathbb{N}}$ is bounded in $L^{p+2}(0,T;L^{p+2})$. (1.51)

(iv) $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;\tilde{H})$. (1.55)

(v) $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;\tilde{H}^1)$. (1.50)

1.2.6 Proof of Theorem 1.6: Passing to the limit

From (i) and the Banach-Alaoglu theorem, it follows that there is a $\boldsymbol{u} \in L^\infty(0,T;W_0^{1,2})$ and a subsequence of $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$, which we denote again by $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$, such that $\boldsymbol{u}_n \xrightarrow{\text{w}^*} \boldsymbol{u}$. Since $L^\infty(0,T;W_0^{1,2})$ is continuously embedded in $L^2(0,T;W_0^{1,2})$, from (i) it also follows that $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;W_0^{1,2})$. So there is $\tilde{\boldsymbol{u}} \in L^2(0,T;W_0^{1,2})$ such that along a (further) subsequence $\boldsymbol{u}_n \xrightarrow{\text{w}} \tilde{\boldsymbol{u}}$. By the previously mentioned embedding of L^∞ in L^2 it is easy to see that $\tilde{\boldsymbol{u}} = \boldsymbol{u}$. So we infer the existence of $\boldsymbol{u} \in L^\infty(0,T;W_0^{1,2})$ such that

(I) $\boldsymbol{u}_n \xrightarrow{\text{w}} \boldsymbol{u}$ in $L^2(0,T;W_0^{1,2})$.

With similar arguments to the above, we can also show the existence of $\boldsymbol{u}' \in L^\infty(0,T;L^2) \cap L^{p+2}(0,T;L^{p+2})$ and $\boldsymbol{h} \in L^\infty(0,T;\tilde{H}) \cap L^2(0,T;\tilde{H}^1)$ such that

(II) $\boldsymbol{u}_n' \xrightarrow{\text{w}} \boldsymbol{u}'$ in $L^2(0,T;L^2)$,

(III) $\boldsymbol{u}_n' \xrightarrow{\text{w}} \boldsymbol{u}'$ in $L^{p+2}(0,T;L^{p+2})$,

(IV) $\boldsymbol{h}_n \xrightarrow{\text{w}} \boldsymbol{h}$ in $L^2(0,T;\tilde{H}^1)$.

Note that $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{h}$ satisfy 1 and 2 of Definition 1.5.

We next show that in view of (ii) and (v) we also have the following strong convergence

(V) $\boldsymbol{h}_n \longrightarrow \boldsymbol{h}$ in $L^2(0,T;\tilde{H}(\Omega))$.

To see this, we explicitly write the second equation of (1.38) as a functional equation in $\tilde{H}^{-2}$, that is

$$\langle \boldsymbol{h}_n', \boldsymbol{\psi}\rangle + \langle \tilde{\mathcal{L}}\,\boldsymbol{h}_n, \boldsymbol{\psi}\rangle = \langle \mathcal{B}_{II}(\boldsymbol{u}', \boldsymbol{h}_n + \boldsymbol{B}_0), \boldsymbol{\psi}\rangle, \qquad \forall \boldsymbol{\psi} \in \tilde{H}^2.$$

Using (1.30) we have the following estimate

$$\langle \boldsymbol{h}_n', \boldsymbol{\psi}\rangle = -\varrho(\operatorname{curl}\boldsymbol{h}_n, \operatorname{curl}\boldsymbol{\psi})_2 + b(\boldsymbol{u}_n', \boldsymbol{h}_n + \boldsymbol{B}_0, \boldsymbol{\psi}) \le \varrho\,\|\boldsymbol{h}_n\|_{\tilde{H}^1}\,\|\boldsymbol{\psi}\|_{\tilde{H}^2} + \|\boldsymbol{u}_n'\|\;\|\boldsymbol{h}_n + \boldsymbol{B}_0\|_{W^{1,2}}\,\|\boldsymbol{\psi}\|_{\tilde{H}^2}\,.$$

Dividing both sides by $\|\boldsymbol{\psi}\|_{\tilde{H}^2}$ and taking the supremum over $\boldsymbol{\psi} \in \tilde{H}^2$, we obtain

$$\|\boldsymbol{h}_n'\|_{\tilde{H}^{-2}} \le \varrho\,\|\boldsymbol{h}_n\|_{\tilde{H}^1} + \|\boldsymbol{u}_n'\|\;\|\boldsymbol{h}_n\|_{\tilde{H}^1} + \|\boldsymbol{u}_n'\|\;\|\boldsymbol{B}_0\|_{W^{1,2}}\,,$$

which after squaring both sides and integrating in $[0,T]$, yields

$$\int_0^T \|\boldsymbol{h}_n'\|_{\tilde{H}^{-2}}^2 \leq 2\varrho \int_0^T \|\boldsymbol{h}_n\|_{\tilde{H}^1}^2 + 2\,\|\boldsymbol{u}_n'\|_{L^\infty(0,T;L^2)}^2 \int_0^T \|\boldsymbol{h}_n\|_{\tilde{H}^1}^2 + c_{31} \int_0^T \|\boldsymbol{u}_n'\|^2 .$$

From the above relation and (ii) and (v), it follows that $\{\boldsymbol{h}_n'\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;\tilde{H}^{-2})$, which together with (v) and Aubin-Lions compactness lemma gives (V).

Multiplying (1.37) or (1.38) by $\eta \in C_T^2(\mathbb{R})$ and integrating by parts in time we have that for every $n > 0$ and for every $i > 0$, $(\boldsymbol{u}_n, \boldsymbol{h}_n)$ satisfies

$$\begin{aligned}\int_0^T (\boldsymbol{u}_n, \boldsymbol{\varphi}_i)_2\, \eta'' + \int_0^T a_I(\boldsymbol{u}_n, \boldsymbol{\varphi}_i)\, \eta + \int_0^T (\boldsymbol{\nu}(\boldsymbol{u}_n'), \boldsymbol{\varphi}_i)_2\, \eta \\ = \int_0^T (\operatorname{curl} \boldsymbol{h}_n \times [\boldsymbol{h}_n + \boldsymbol{B}_0], \boldsymbol{\varphi}_i)_2\, \eta + \int_0^T (\boldsymbol{f}, \boldsymbol{\varphi}_i)_2\, \eta,\end{aligned} \tag{1.56}$$

$$-\int_0^T (\boldsymbol{h}_n, \boldsymbol{\psi}_i)_2\, \eta' + \int_0^T a_{II}(\boldsymbol{h}_n, \boldsymbol{\psi}_i)\, \eta = \int_0^T (\boldsymbol{u}_n' \times [\boldsymbol{h}_n + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{\psi}_i)_2\, \eta. \tag{1.57}$$

We next take the limit of the above equations as $n \longrightarrow \infty$. From (I) and (IV) we have that

$$\begin{aligned}\lim_{n\to\infty} \int_0^T (\boldsymbol{u}_n, \boldsymbol{\varphi}_i)_2\, \eta'' = \int_0^T (\boldsymbol{u}, \boldsymbol{\varphi}_i)_2\, \eta'', \quad \lim_{n\to\infty} \int_0^T a_I(\boldsymbol{u}_n, \boldsymbol{\varphi}_i)\, \eta = \int_0^T a_I(\boldsymbol{u}, \boldsymbol{\varphi}_i)\, \eta, \\ \lim_{n\to\infty} \int_0^T (\boldsymbol{h}_n, \boldsymbol{\psi}_i)_2\, \eta' = \int_0^T (\boldsymbol{h}, \boldsymbol{\psi}_i)_2\, \eta', \quad \lim_{n\to\infty} \int_0^T a_{II}(\boldsymbol{h}_n, \boldsymbol{\psi}_i)\, \eta = \int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{\psi}_i)\, \eta.\end{aligned} \tag{1.58}$$

For the first coupling term we have

$$\begin{aligned}\lim_{n\to\infty} \left| \int_0^T (\operatorname{curl} \boldsymbol{h}_n \times [\boldsymbol{h}_n + \boldsymbol{B}_0] - \operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{\varphi}_i)_2\, \eta \right| \\ \leq \lim_{n\to\infty} \left| \int_0^T (\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h}) \times [\boldsymbol{h}_n + \boldsymbol{B}_0], \boldsymbol{\varphi}_i)_2\ \eta \right| \\ + \lim_{n\to\infty} \left| \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h}_n - \boldsymbol{h}], \boldsymbol{\varphi}_i)_2\ \eta \right| .\end{aligned} \tag{1.59}$$

For the first term on the right hand side we have

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\int_0^T (\operatorname{curl}(\boldsymbol{h}_n-\boldsymbol{h})\times[\boldsymbol{h}_n+\boldsymbol{B}_0],\boldsymbol{\varphi}_i)_2\ \eta\right| \\
&\qquad\qquad \le \lim_{n\to\infty}\left|\int_0^T (\operatorname{curl}(\boldsymbol{h}_n-\boldsymbol{h})\times[\boldsymbol{h}_n-\boldsymbol{h}],\boldsymbol{\varphi}_i)_2\ \eta\right| \\
&\qquad\qquad\qquad + \lim_{n\to\infty}\left|\int_0^T (\operatorname{curl}(\boldsymbol{h}_n-\boldsymbol{h})\times[\boldsymbol{h}+\boldsymbol{B}_0],\boldsymbol{\varphi}_i)_2\ \eta\right| \\
&\qquad\qquad \le \lim_{n\to\infty}\Big(\|\boldsymbol{h}_n-\boldsymbol{h}\|_{L^2(0,T;\tilde{H}^1)}\,\|\boldsymbol{h}_n-\boldsymbol{h}\|_{L^2(0,T;\tilde{H})}\,\|\boldsymbol{\varphi}_i\|_{W^{2,2}}\,\|\eta\|_{L^\infty(0,T)}\Big) \\
&\qquad\qquad\qquad + \lim_{n\to\infty}\left|\int_0^T (\operatorname{curl}(\boldsymbol{h}_n-\boldsymbol{h}),[\boldsymbol{h}+\boldsymbol{B}_0]\times\boldsymbol{\varphi}_i\,\eta)_2\right| \\
&[\text{by (v)}] \qquad\quad \le c_{32}\lim_{n\to\infty}\|\boldsymbol{h}_n-\boldsymbol{h}\|_{L^2(0,T;\tilde{H})} \\
&[\boldsymbol{\varphi}_i|_{\partial\Omega}=0] \qquad\quad + \left|\lim_{n\to\infty}\int_0^T (\boldsymbol{h}_n-\boldsymbol{h},\operatorname{curl}([\boldsymbol{h}+\boldsymbol{B}_0]\times\boldsymbol{\varphi}_i\,\eta))_2\right| \\
&\qquad\qquad = 0,
\end{aligned}
$$

where the last step follows from (V) and the fact that

$$\operatorname{curl}([\boldsymbol{h}+\boldsymbol{B}_0]\times\boldsymbol{\varphi}_i\eta)\in L^2(0,T;L^2).$$

Handling the second term on the right hand side of (1.59) is more straightforward:

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\int_0^T (\operatorname{curl}\boldsymbol{h}\times[\boldsymbol{h}_n-\boldsymbol{h}],\boldsymbol{\varphi}_i)_2\ \eta\right| \\
&\qquad \le \lim_{n\to\infty}\Big(\|\boldsymbol{h}\|_{L^2(0,T;\tilde{H}^1)}\,\|\boldsymbol{h}_n-\boldsymbol{h}\|_{L^2(0,T;\tilde{H})}\,\|\boldsymbol{\varphi}_i\|_{W^{2,2}}\,\|\eta\|_{L^\infty(0,T)}\Big) \\
&\qquad\qquad\qquad\qquad \le c_{33}\lim_{n\to\infty}\|\boldsymbol{h}_n-\boldsymbol{h}\|_{L^2(0,T;\tilde{H})}=0.
\end{aligned}
$$

So from (1.59) we have

$$\lim_{n\to\infty}\int_0^T (\operatorname{curl}\boldsymbol{h}_n\times[\boldsymbol{h}_n+\boldsymbol{B}_0],\boldsymbol{\varphi}_i)_2\,\eta=\int_0^T (\operatorname{curl}\boldsymbol{h}\times[\boldsymbol{h}+\boldsymbol{B}_0],\boldsymbol{\varphi}_i)_2\,\eta. \tag{1.60}$$

A similar argument can be done for the other coupling term to yield

$$\lim_{n\to\infty}\int_0^T (\boldsymbol{u}_n'\times[\boldsymbol{h}_n+\boldsymbol{B}_0],\operatorname{curl}\boldsymbol{\psi}_i)_2\,\eta=\int_0^T (\boldsymbol{u}'\times[\boldsymbol{h}+\boldsymbol{B}_0],\operatorname{curl}\boldsymbol{\psi}_i)_2\,\eta. \tag{1.61}$$

We now investigate the convergence properties of the nonlinear dissipative term. Let q be as defined in (H1) and note that by (H3) we have

$$\|\boldsymbol{\nu}(\boldsymbol{u}_n')\|_{L^q(0,T;L^q)}^q = \int_0^T \int_\Omega |\boldsymbol{\nu}(\boldsymbol{u}_n')|^q = \int_0^T \int_{\Omega_1} |\boldsymbol{\nu}(\boldsymbol{u}_n')|^q + \int_0^T |\Omega_2| \left(\sup_{|x|\le r_\nu} |\boldsymbol{\nu}(\boldsymbol{x})| \right)^q$$
$$\le K_1 \int_0^T \int_\Omega |\boldsymbol{u}'|^{p+2} + c_{34} = K_1 \, \|\boldsymbol{u}'\|_{L^{p+2}(0,T;L^{p+2})}^{p+2} + c_{34},$$

so by (iii), $\{\boldsymbol{\nu}(\boldsymbol{u}_n')\}_{n\in\mathbb{N}}$ is bounded in $L^q(0,T;L^q)$, and there is $\boldsymbol{\xi} \in L^q(0,T;L^q)$ and a (further) subsequence along which

(VI) $\boldsymbol{\nu}(\boldsymbol{u}_n') \xrightarrow{\text{w}} \boldsymbol{\xi}$ in $L^q(0,T;L^q)$.

Using (1.58), (1.60), (1.61) and the above convergence, after letting $n \longrightarrow \infty$, (1.56) and (1.57) read

$$\int_0^T (\boldsymbol{u}, \boldsymbol{\varphi}_i)_2 \, \eta'' + \int_0^T a_I(\boldsymbol{u}, \boldsymbol{\varphi}_i) \, \eta + \int_0^T (\boldsymbol{\xi}, \boldsymbol{\varphi}_i)_2 \, \eta$$
$$= \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{\varphi}_i)_2 \, \eta + \int_0^T (\boldsymbol{f}, \boldsymbol{\varphi}_i)_2 \, \eta,$$
$$-\int_0^T (\boldsymbol{h}, \boldsymbol{\psi}_i)_2 \, \eta' + \int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{\psi}_i) \, \eta = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{\psi}_i)_2 \, \eta.$$

Note that the above also holds for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$ and $\boldsymbol{\psi} \in \tilde{\mathcal{D}}$ in place of $\boldsymbol{\varphi}_i$ and $\boldsymbol{\psi}_i$. The reason is that every such $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ can be approximated with $\boldsymbol{\varphi}_i$ and $\boldsymbol{\psi}_i$ in C^m-norm for arbitrary $m > 0$. For general ideas on the proof see Galdi (2000, Lemma 2.3). In a similar manner as above, passing to the limit in this new approximation we will obtain

$$\int_0^T (\boldsymbol{u}, \boldsymbol{\varphi})_2 \, \eta'' + \int_0^T a_I(\boldsymbol{u}, \boldsymbol{\varphi}) \, \eta + \int_0^T (\boldsymbol{\xi}, \boldsymbol{\varphi})_2 \, \eta$$
$$= \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{\varphi})_2 \, \eta + \int_0^T (\boldsymbol{f}, \boldsymbol{\varphi})_2 \, \eta, \tag{1.62}$$
$$-\int_0^T (\boldsymbol{h}, \boldsymbol{\psi})_2 \, \eta' + \int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{\psi}) \, \eta = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{\psi})_2 \, \eta, \tag{1.63}$$

which will be the item 3 in Definition 1.5 once we show that $\boldsymbol{\xi} = \boldsymbol{\nu}(\boldsymbol{u}')$ and the proof of Theorem 1.6 will be completed. Note that in the case that the dissipation is linear, i.e., $\boldsymbol{\nu}(\boldsymbol{u}') = c\boldsymbol{u}'$, we would not have (III) and (VI), but (II) will be enough to conclude the proof at this point without further work. The rest of this section will be spent in showing that

$\boldsymbol{\xi} = \boldsymbol{\nu}(\boldsymbol{u}')$. The procedure is classical, by now, and is known as "Minty's trick" (Minty, 1963). For a clear demonstration, see also Galdi (2008, p. 202).

Step 1. We wish to obtain an energy equality directly from (1.62) and (1.63). Formally, it is equivalent to replacing $\boldsymbol{u}'$ for $\boldsymbol{\varphi}\eta$ in the first equation above, and $\boldsymbol{h}$ for $\boldsymbol{\psi}\eta$ in the second. Rigorously, it is done as follows:

Definition 1.11. Let $\boldsymbol{w} \in L^s(0,T;X)$, $1 \leq s < \infty$. For any $0 < \ell < T$, we define the (time-) mollification of $\boldsymbol{w}$ by

$$\boldsymbol{w}_\ell(t) = \int_0^T \zeta_\ell(t-\tau)\boldsymbol{w}(\tau)\, d\tau,$$

for some even and positive $\zeta_\ell \in C_0^\infty(-\ell,\ell)$, such that $\displaystyle\int_{-\infty}^{\infty} \zeta_\ell = 1$.

We remark that by properties of convolutions we have $(\boldsymbol{u}')_\ell = (\boldsymbol{u}_\ell)'$, so the notation $\boldsymbol{u}'_\ell$ used in the sequel is not ambiguous. We have the following lemma directly from the definition of mollifications (see also part 1 of Lemma 1.13):

Lemma 1.12. *Let $\boldsymbol{w} \in L^s(0,T;X)$, $1 \leq s < \infty$ and $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}} \subset L^s(0,T;X)$ such that $\lim_{n\to\infty} \|\boldsymbol{w}_n - \boldsymbol{w}\|_{L^s(0,T;X)} = 0$. Then*

$$\lim_{n\to\infty} \left\| \frac{\partial^m (\boldsymbol{w}_n)_\ell}{\partial t^m} - \frac{\partial^m \boldsymbol{w}_\ell}{\partial t^m} \right\|_{L^s(0,T;X)} = 0, \qquad m \geq 0.$$

Proof. Let $c_{35} = \sup_{-\ell<t<\ell} \left| \dfrac{d^m \zeta_\ell}{dt^m} \right|$. The proof follows by taking the limit as $n \longrightarrow \infty$ of the inequality below:

$$\begin{aligned}\left\| \frac{\partial^m (\boldsymbol{w}_n)_\ell}{\partial t^m} - \frac{\partial^m \boldsymbol{w}_\ell}{\partial t^m} \right\|^s_{L^s(0,T;X)} &= \int_0^T \left\| \int_0^T \frac{d^m \zeta_\ell(t-\tau)}{dt^m} (\boldsymbol{w}_n(\tau) - \boldsymbol{w}(\tau))\, d\tau \right\|^s_X dt \\ &\leq \int_0^T c_{35}^s T^{1-s} \int_0^T \|\boldsymbol{w}_n(\tau) - \boldsymbol{w}(\tau)\|^s_X \, d\tau\, dt \leq (c_{35}T)^s \|\boldsymbol{w}_n - \boldsymbol{w}\|^s_{L^s(0,T;X)} .\end{aligned}$$

□

Now consider our weak solution $(\boldsymbol{u}, \boldsymbol{h})$. We wish to construct sequences $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}}$ with proper convergence properties and such that we can appropriately use them as test functions. Let

$$\boldsymbol{u}_k = \sum_{i=1}^{k} (\boldsymbol{u}, \boldsymbol{\varphi}_i)_{W^{1,2}} \boldsymbol{\varphi}_i,$$

It is clear that $\lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{W^{1,2}} = 0$ for almost all t. Also, since $\boldsymbol{u} \in L^\infty(0,T; W_0^{1,2}) \subset L^2(0,T; W_0^{1,2})$ it follows from the dominated convergence theorem that

(a) $\lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{L^2(0,T;W^{1,2})} = 0.$

With a similar argument and since $L^\infty(0,T; W_0^{1,2}) \subset L^{p+2}(0,T; L^{p+2})$ we have

(b) $\lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{L^{p+2}(0,T;L^{p+2})} = 0.$

For $\boldsymbol{h}$, let $\{\boldsymbol{\Psi}_i\}_{i\in\mathbb{N}} \subset \tilde{\mathcal{D}}$ be an orthonormal basis of $\tilde{H}^1$ and let

$$\boldsymbol{h}_k = \sum_{i=1}^{k} (\boldsymbol{u}, \boldsymbol{\Psi}_i)_{\tilde{H}^1} \boldsymbol{\Psi}_i.$$

This time since $\boldsymbol{h} \in L^2(0,T; \tilde{H}^1)$, from the Lebesgue monotone convergence theorem, we have

(c) $\lim_{k\to\infty} \|\boldsymbol{h}_k - \boldsymbol{h}\|_{L^2(0,T;\tilde{H}^1)} = 0.$

Let us take $\boldsymbol{\varphi}\,\eta = \boldsymbol{u}_{\ell,k} := (\boldsymbol{u}_k)_\ell$ in (1.62) and $\boldsymbol{\psi}\,\eta = \boldsymbol{h}_{\ell,k} := (\boldsymbol{h}_n)_\ell$ in (1.63), to obtain

$$\begin{aligned}\int_0^T (\boldsymbol{u}, \boldsymbol{u}'''_{\ell,k})_2 + \int_0^T a_I(\boldsymbol{u}, \boldsymbol{u}'_{\ell,k}) + \int_0^T (\boldsymbol{\xi}, \boldsymbol{u}'_{\ell,k})_2 \\ = \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{u}'_{\ell,k})_2 + \int_0^T (\boldsymbol{f}, \boldsymbol{u}'_{\ell,k})_2,\end{aligned}$$

$$-\int_0^T (\boldsymbol{h}, \boldsymbol{h}'_{\ell,k})_2 + \int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{h}_{\ell,k}) = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h}_{\ell,k})_2.$$

We wish to take the limit as $k \longrightarrow \infty$ in the above equations and obtain

$$\begin{aligned}\int_0^T (\boldsymbol{u}, \boldsymbol{u}'''_{\ell})_2 + \int_0^T a_I(\boldsymbol{u}, \boldsymbol{u}'_{\ell}) + \int_0^T (\boldsymbol{\xi}, \boldsymbol{u}'_{\ell})_2 \\ = \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{u}'_{\ell})_2 + \int_0^T (\boldsymbol{f}, \boldsymbol{u}'_{\ell})_2,\end{aligned} \tag{1.64}$$

$$-\int_0^T (\boldsymbol{h}, \boldsymbol{h}'_{\ell})_2 + \int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{h}_{\ell}) = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h}_{\ell})_2. \tag{1.65}$$

The details are as follows: From (a), (c) and Lemma 1.12 we have

$$\lim_{k\to\infty}\int_0^T (\boldsymbol{u},\boldsymbol{u}'''_{\ell,k})_2=\int_0^T (\boldsymbol{u},\boldsymbol{u}'''_{\ell})_2, \qquad \lim_{k\to\infty}\int_0^T a_I(\boldsymbol{u},\boldsymbol{u}'_{\ell,k})=\int_0^T a_I(\boldsymbol{u},\boldsymbol{u}'_{\ell}),$$
$$\lim_{k\to\infty}\int_0^T (\boldsymbol{\xi},\boldsymbol{u}'_{\ell,k})_2=\int_0^T (\boldsymbol{\xi},\boldsymbol{u}'_{\ell})_2, \qquad \lim_{k\to\infty}\int_0^T (\boldsymbol{f},\boldsymbol{u}'_{\ell,k})_2=\int_0^T (\boldsymbol{f},\boldsymbol{u}'_{\ell})_2,$$
$$\lim_{k\to\infty}\int_0^T (\boldsymbol{h},\boldsymbol{h}'_{\ell,k})_2=\int_0^T (\boldsymbol{h},\boldsymbol{h}'_{\ell})_2, \qquad \lim_{k\to\infty}\int_0^T a_{II}(\boldsymbol{h},\boldsymbol{h}_{\ell,k})=\int_0^T a_{II}(\boldsymbol{h},\boldsymbol{h}_{\ell,k}).$$

For coupling terms we are going to use a little finer estimates with respect to those we were using until now. At this level they might not be necessary, but since we also wish to take the limit as $\ell \longrightarrow 0$ later, this will save us from redundancy:

$$\begin{aligned}\left|\int_0^T (\operatorname{curl}\boldsymbol{h}\times[\boldsymbol{h}+\boldsymbol{B}_0],\boldsymbol{u}'_{\ell,k}-\boldsymbol{u}'_\ell)_2\right| &\le \int_0^T \|\operatorname{curl}\boldsymbol{h}\|_{L^2}\,\|\boldsymbol{h}+\boldsymbol{B}_0\|_{L^{\frac{2(p+2)}{p}}}\,\left\|\boldsymbol{u}'_{\ell,k}-\boldsymbol{u}'_\ell\right\|_{L^{p+2}}\\ &\le c_{36}\int_0^T \|\operatorname{curl}\boldsymbol{h}\|_{L^2}\,\|\boldsymbol{h}\|_{\tilde{H}^1}^{\frac{3}{p+2}}\,\|\boldsymbol{h}\|_{L^2}^{\frac{p-1}{p+2}}\,\left\|\boldsymbol{u}'_{\ell,k}-\boldsymbol{u}'_\ell\right\|_{L^{p+2}}\\ &\le c_{37}\left(\int_0^T \|\operatorname{curl}\boldsymbol{h}\|^{\frac{p+5}{p+1}}\right)^{\frac{p+1}{p+2}}\left\|\boldsymbol{u}'_{\ell,k}-\boldsymbol{u}'_\ell\right\|_{L^{p+2}(0,T;L^{p+2})},\end{aligned} \tag{1.66}$$

where in the first step we used extended Hölder inequality and in the second the interpolation inequality

$$\|\boldsymbol{w}\|_{L^s}\le c\,\|\boldsymbol{w}\|_{W^{1,2}}^{\frac{3s-6}{2s}}\,\|\boldsymbol{w}\|^{\frac{6-s}{2s}}, \qquad 2\le s\le 6.$$

In the last step we used the fact that $\boldsymbol{h}\in L^\infty(0,T;\tilde{H})$ and another Hölder inequality. Since by (H1), $p\in[3,4]$ we have

$$\int_0^T \|\operatorname{curl}\boldsymbol{h}\|^{\frac{p+5}{p+1}}\le c_{38}\int_0^T \|\operatorname{curl}\boldsymbol{h}\|^2.$$

Note that we used our second restrictive assumption (that $p>3$), at this point. In view of (1.50), (b) and Lemma 1.12, by taking the limit of both sides of (1.66) as $k\longrightarrow\infty$, we obtain

$$\lim_{k\to\infty}\int_0^T (\operatorname{curl}\boldsymbol{h}\times[\boldsymbol{h}+\boldsymbol{B}_0],\boldsymbol{u}'_{\ell,k})_2=\int_0^T (\operatorname{curl}\boldsymbol{h}\times[\boldsymbol{h}+\boldsymbol{B}_0],\boldsymbol{u}'_\ell)_2.$$

For the second coupling we can follow the same steps until we get

$$\left|\int_0^T (\boldsymbol{u}'\times[\boldsymbol{h}+\boldsymbol{B}_0]\,,\operatorname{curl}[\boldsymbol{h}_{\ell,k}-\boldsymbol{h}_\ell])_2\right|\le c_{39}\int_0^T \|\boldsymbol{u}'\|_{L^{p+2}}\,\|\boldsymbol{h}\|_{\tilde{H}^1}^{\frac{3}{p+2}}\,\|\operatorname{curl}[\boldsymbol{h}_{\ell,k}-\boldsymbol{h}_\ell]\|_{L^2},$$

and continue the estimate differently, with an extended Hölder inequality:

$$\leq c_{40} \left\|\boldsymbol{u}'\right\|_{L^{p+2}(0,T;L^{p+2})} \left(\int_0^T \left\|\operatorname{curl} \boldsymbol{h}\right\|^{\frac{6}{p}}\right)^{\frac{p}{2(p+2)}} \left\|\boldsymbol{h}_{\ell,k} - \boldsymbol{h}_\ell\right\|_{L^2(0,T;\tilde{H}^1)}.$$

Then, by (1.50) and (c), taking the limit of both sides of the above as $k \longrightarrow \infty$, we obtain

$$\lim_{k\to\infty} \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h}_{\ell,k})_2 = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h}_\ell)_2.$$

Once we have (1.64) and (1.65), one may notice that since ζ_ℓ is an even function, then ζ_ℓ''' is an odd function, so by the symmetry of the inner product we obtain

$$\begin{aligned}
\int_0^T (\boldsymbol{u}, \boldsymbol{u}_\ell''')_2 &= \int_0^T \int_0^T \zeta_\ell'''(t-\tau)\,(\boldsymbol{u}(t), \boldsymbol{u}(\tau))_2\, d\tau\, dt \\
&= -\int_0^T \int_0^T \zeta_\ell'''(\tau-t)\,(\boldsymbol{u}(t), \boldsymbol{u}(\tau))_2\, d\tau\, dt \\
&= -\int_0^T \int_0^T \zeta_\ell'''(t-\tau)\,(\boldsymbol{u}(\tau), \boldsymbol{u}(t))_2\, d\tau\, dt \qquad [\text{renaming } \tau \Rightarrow t,\ t \Rightarrow \tau] \\
&= 0.
\end{aligned}$$

Similarly,

$$\int_0^T a_I(\boldsymbol{u}, \boldsymbol{u}_\ell') = 0, \qquad \text{and} \qquad \int_0^T (\boldsymbol{h}, \boldsymbol{h}_\ell')_2 = 0,$$

therefore, (1.64) and (1.65) read

$$\int_0^T (\boldsymbol{\xi}, \boldsymbol{u}_\ell')_2 = \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{u}_\ell')_2 + \int_0^T (\boldsymbol{f}, \boldsymbol{u}_\ell')_2, \tag{1.67}$$

$$\int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{h}_\ell) = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h}_\ell)_2. \tag{1.68}$$

We now let $\ell \longrightarrow 0$ in the above equations. For this, we need convergence properties equivalent to those we obtained from (a)–(c) and Lemma 1.12, which are, altogether, provided by the following lemma (Hille and Phillips, 1957):

Lemma 1.13. *For $\boldsymbol{w} \in L^s(0,T;X)$, $1 \leq q < \infty$, we have*

1. $\boldsymbol{w}_\ell \in C^\infty([0,T];X)$,
2. $\lim_{\ell\to 0} \left\|\boldsymbol{w}_\ell - \boldsymbol{w}\right\|_{L^s(0,T;X)} = 0.$

Using estimates similar to the ones we used when taking the limit as $k \longrightarrow \infty$, and in view of the above lemma, (1.67) and (1.68) in the limit $\ell \longrightarrow 0$ will be

$$\int_0^T (\boldsymbol{\xi}, \boldsymbol{u}')_2 = \int_0^T (\operatorname{curl} \boldsymbol{h} \times [\boldsymbol{h} + \boldsymbol{B}_0], \boldsymbol{u}')_2 + \int_0^T (\boldsymbol{f}, \boldsymbol{u}')_2,$$
$$\int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{h}) = \int_0^T (\boldsymbol{u}' \times [\boldsymbol{h} + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h})_2.$$

Adding the above equations together and using (1.29), we get the energy equality

$$\int_0^T (\boldsymbol{\xi}, \boldsymbol{u}')_2 + \int_0^T a_{II}(\boldsymbol{h}, \boldsymbol{h}) = \int_0^T (\boldsymbol{f}, \boldsymbol{u}')_2. \tag{1.69}$$

Step 2. In this step we obtain an energy inequality from the Galerkin approximations. Replacing η in (1.56) by c'_{ni} and in (1.57) by d_{ni} and summing over $1 \leq i \leq n$, we get

$$-\int_0^T (\boldsymbol{u}'_n, \boldsymbol{u}''_n)_2 + \int_0^T a_I(\boldsymbol{u}_n, \boldsymbol{u}'_n) + \int_0^T (\boldsymbol{\nu}(\boldsymbol{u}'_n), \boldsymbol{u}'_n)_2$$
$$= \int_0^T (\operatorname{curl} \boldsymbol{h}_n \times [\boldsymbol{h}_n + \boldsymbol{B}_0], \boldsymbol{u}'_n)_2 + \int_0^T (\boldsymbol{f}, \boldsymbol{u}'_n)_2,$$
$$-\int_0^T (\boldsymbol{h}_n, \boldsymbol{h}'_n)_2 + \int_0^T a_{II}(\boldsymbol{h}_n, \boldsymbol{h}_n) = \int_0^T (\boldsymbol{u}'_n \times [\boldsymbol{h}_n + \boldsymbol{B}_0], \operatorname{curl} \boldsymbol{h}_n)_2.$$

Note that we have integrated the first term of (1.56) by parts and used the periodicity of η and $\boldsymbol{u}_n$ before replacing η by c'_{ni}. This is because c_{ni} is only twice differentiable. Also note that by periodicity of $\boldsymbol{u}_n$ and $\boldsymbol{h}_n$, we have

$$\int_0^T (\boldsymbol{u}'_n, \boldsymbol{u}''_n)_2 = 0, \qquad \int_0^T a_I(\boldsymbol{u}_n, \boldsymbol{u}'_n) = 0, \quad \text{and} \quad \int_0^T (\boldsymbol{h}_n, \boldsymbol{h}'_n)_2 = 0,$$

so, adding the above equations and using (1.29) we get

$$\int_0^T (\boldsymbol{\nu}(\boldsymbol{u}'_n), \boldsymbol{u}'_n)_2 + \int_0^T a_{II}(\boldsymbol{h}_n, \boldsymbol{h}_n) = \int_0^T (\boldsymbol{f}, \boldsymbol{u}'_n)_2.$$

Taking the $\liminf$ of both sides as $n \longrightarrow \infty$ and using the properties of inferior limits, we obtain

$$\liminf_{n\to\infty} \int_0^T (\boldsymbol{\nu}(\boldsymbol{u}'_n), \boldsymbol{u}'_n)_2 + \liminf_{n\to\infty} \int_0^T a_{II}(\boldsymbol{h}_n, \boldsymbol{h}_n) \leq \int_0^T (\boldsymbol{f}, \boldsymbol{u}')_2. \tag{1.70}$$

The last step. From (1.69) and (1.70) one findes

$$\liminf_{n\to\infty}\int_0^T(\boldsymbol{\nu}(\boldsymbol{u}_n'),\boldsymbol{u}_n')_2+\liminf_{n\to\infty}\int_0^T a_{II}(\boldsymbol{h}_n,\boldsymbol{h}_n)\le\int_0^T(\boldsymbol{\xi},\boldsymbol{u}')_2+\int_0^T a_{II}(\boldsymbol{h},\boldsymbol{h}).$$

Since $\boldsymbol{h}_n \xrightarrow{\text{w}} \boldsymbol{h}$ in $L^2(0,T;\tilde{H}^1)$, by properties of weak limits we have

$$\int_0^T\|\operatorname{curl}\boldsymbol{h}\|^2\le\liminf_{n\to\infty}\int_0^T\|\operatorname{curl}\boldsymbol{h}_n\|^2,$$

and so from the above two relations we conclude

$$\liminf_{n\to\infty}\int_0^T(\boldsymbol{\nu}(\boldsymbol{u}_n'),\boldsymbol{u}_n')_2\le\int_0^T(\boldsymbol{\xi},\boldsymbol{u}')_2.$$

Also, for any $\boldsymbol{\phi}\in L^{p+2}(0,T;L^{p+2})$, from (III) and (VI) we have

$$\lim_{n\to\infty}\int_0^T(\boldsymbol{\nu}(\boldsymbol{\phi}),\boldsymbol{u}_n')_2=\int_0^T(\boldsymbol{\nu}(\boldsymbol{\phi}),\boldsymbol{u}')_2,$$

$$\lim_{n\to\infty}\int_0^T(\boldsymbol{\nu}(\boldsymbol{u}_n'),\boldsymbol{\phi})_2=\int_0^T(\boldsymbol{\xi},\boldsymbol{\phi})_2,$$

$$\lim_{n\to\infty}\int_0^T(\boldsymbol{\nu}(\boldsymbol{\phi}),\boldsymbol{\phi})_2=\int_0^T(\boldsymbol{\nu}(\boldsymbol{\phi}),\boldsymbol{\phi})_2.$$

Adding the last four equations and using the fact that $\boldsymbol{\nu}$ is monotone, we obtain

$$0\le\liminf_{n\to\infty}\int_0^T(\boldsymbol{\nu}(\boldsymbol{u}_n')-\boldsymbol{\nu}(\boldsymbol{\phi}),\boldsymbol{u}_n'-\boldsymbol{\phi})_2\le\int_0^T(\boldsymbol{\xi}-\boldsymbol{\nu}(\boldsymbol{\phi}),\boldsymbol{u}'-\boldsymbol{\phi})_2,\qquad\forall\boldsymbol{\phi}\in L^{p+2}(0,T;L^{p+2}).$$

For some $\delta>0$ and $\boldsymbol{w}\in L^{p+2}(0,T;L^{p+2})$, choose $\boldsymbol{\phi}=\boldsymbol{u}'-\delta\,\boldsymbol{w}$ in the above, to get

$$0\le\delta\int_0^T(\boldsymbol{\xi}-\boldsymbol{\nu}(\boldsymbol{u}'-\delta\,\boldsymbol{w}),\boldsymbol{w})_2.$$

Dividing by δ and taking the limit as $\delta\longrightarrow 0$ (using the dominated convergence theorem and continuity of $\boldsymbol{\nu}$) we get

$$0\le\int_0^T(\boldsymbol{\xi}-\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{w})_2. \tag{1.71}$$

Now take $\boldsymbol{\phi}=\boldsymbol{u}'+\delta\,\boldsymbol{w}$, and we similarly find

$$0\le-\delta\int_0^T(\boldsymbol{\xi}-\boldsymbol{\nu}(\boldsymbol{u}'-\delta\,\boldsymbol{w}),\boldsymbol{w})_2.$$

Dividing by $-\delta$ and taking the limit as $\delta\longrightarrow 0$, we have

$$0\ge\int_0^T(\boldsymbol{\xi}-\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{w})_2,$$

which, together with (1.71), yields

$$\int_0^T(\boldsymbol{\xi}-\boldsymbol{\nu}(\boldsymbol{u}'),\boldsymbol{w})_2=0,\qquad\forall\boldsymbol{w}\in L^{p+2}(0,T;L^{p+2}).$$

That is, $\boldsymbol{\xi}=\boldsymbol{\nu}(\boldsymbol{u}')$.

2.0 ON THE EMBEDDING OF THE ATTRACTOR OF 2-D NAVIER-STOKES EQUATIONS INTO $\mathbb{R}^N$

The problem of existence of an attractor for 2-D Navier-Stokes (N-S) equations, initiated by Hopf (1948), was first addressed by the works of Foiaş and Prodi (1967) and Ladyzhenskaya (1975). In its most general case this attractor exists in H and has finite box-counting dimensions. As the attractor hosts the ultimate dynamics of a N-S system, there is a natural interest in mappings from the attractor into $\mathbb{R}^N$ since such an embedding might yield to a dynamically equivalent finite dimensional system of ODE's.

When a dynamical system possesses an inertial manifold (so that the mapping of the attractor into $\mathbb{R}^N$ is through a Lipschitz manifold on which the attractor lies), constructing such an equivalent finite dimensional system of ODE's is relatively easy (Eden, Foiaş, Nicolaenko, and Temam, 1994). To obtain finite dimensional dynamics in systems like 2-D Navier-Stokes, for which existence of an inertial manifold is not known, an alternate method should be devised. One such method with partial success is that of Mañé's projections. This method is a result of studying the more general problem of embedding a compact finite-dimensional subset, K, of a Banach space, X, into $\mathbb{R}^N$.

First results in this direction, when X is finite dimensional are due to Mañé (1981), Ben-Artzi, Eden, Foiaş, and Nicolaenko (1993) and Eden et al. (1994). They considered projections into finite dimensional subspaces of X. Foiaş and Olson (1996) extend this result to the case that X is not necessarily finite dimensional. Projecting the original PDE using these maps, a finite system of ordinary differential equations is obtained. This system of ODE's, however, lacks uniqueness of solutions, in general, and hence does not yield to a dynamical system (Eden et al., 1994). This suggests a "relaxed" definition of dynamical systems as "generalized" dynamical system with equivalent "generalized" dynamics compared to the underlying PDE. To obtain the original "generalized" dynamics from the projected ODE's, the projection mappings should have Hölder inverses. This fact is shown to be true in a very general setting by Hunt and Kaloshin (1999), although most of the above cited references also obtain such results for their mappings.

One problem with Mañé's projections is that the mappings and the corresponding system of ODE's do not have a physically meaningful interpretation (Robinson, 2001, Section 16.1.1). In this paper we construct nonlinear homeomorphisms between the attractor generated by N-S equations, $\mathcal{A}$, and curves in $\mathbb{R}^N$. Being nonlinear and having the range as curves in $\mathbb{R}^n$, these homeomorphisms do not fall in the category of the mappings considered in the works above. This class of mappings posses a physically tangible explanation and give a different characterization of the attractor. Finding an equivalent finite dimensional dynamics through such new descriptions might be easier, as they are occurring in $\mathbb{R}^N$ rather than Hilbert spaces that $\mathcal{A}$ lies in. The class considered here has two important properties: First, the construction does not use the fact that $\mathcal{A}$ has finite dimensions and depends mainly on the invariance of $\mathcal{A}$ with respect to the solution operator; Second, the mapping is obtained without using the information of the flow "near" boundaries, suggesting that the ultimate dynamics occur inside the domain of the flow. This result is in itself appealing as it continues to hold in the case of non-homogeneous (time-independent) boundary conditions (see Temam (1997), and Section 2.4 below).

The mapping is constructed using the idea of "determining nodes" introduced by Foiaş and Temam (1984). They showed if two solutions of 2-D Navier-Stokes equations converge to each other on a (suitable) set of finite points in the domain, then the two solutions will converge; and conjectured that these nodal values might uniquely determine the elements of $\mathcal{A}$. For the case of a periodic domain with analytic force Friz and Robinson (2001) showed that each point on the attractor can be identified by its values at N distinct points (N sufficiently large) in the domain, from which Foiaş and Temam's conjecture follows (see also Friz, Kukavica, and Robinson, 2001; Robinson, 2005, for related results). Our main theorem, Theorem 2.4, shows that under mild assumptions on the forcing term, a similar result to this conjecture holds with the condition that the "trajectories" passing through two elements of $\mathcal{A}$ coincide on (one of) these sets of finite points. As the values of a given trajectory at these finite number of points and at different times can be viewed as a curve in $\mathbb{R}^N$ (with N determined by the number of points), we obtain a mapping between a point of $\mathcal{A}$ and a curve in $\mathbb{R}^N$.

The results of this chapter has been reported in Mohebbi.

2.1 PRELIMINARIES

We keep the notation and function spaces consistent with the previous chapter so one may consult Section 1.2.1 for reference.

Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary. For homogeneous boundary conditions the N-S equations can be written in functional form as,

$$\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + P(\boldsymbol{u}.\nabla\boldsymbol{u}) = \boldsymbol{f}. \tag{2.1}$$

Here, $A = -P\Delta$ is the Stokes operator and P is the orthogonal projection operator from $\left(L^2(\Omega)\right)^2$ into H. The above, of course, needs to be augmented with an initial condition, $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0 \ \in H$. It is well known (see e.g., Temam, 1979; Galdi, 2000) that for any $\boldsymbol{u}_0 \in H$, (2.1) has a unique weak solution $\boldsymbol{u} \in L^2(0, T; H^1) \bigcap L^\infty(0, T; H)$ for any $T > 0$ and $\boldsymbol{f} \in L^\infty(0, \infty; H)$. These solutions can be redefined on a set of times of measure zero such that $\boldsymbol{u}(t) \in H$ for all $t \in [0, T)$. We define the solution operator $S(t) : H \to H$, for all $t \geq 0$, as $S(t)\boldsymbol{v} = \boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is the solution to (2.1) with $\boldsymbol{u}_0 = \boldsymbol{v}$. For $X \subset H$, $S(t)X := \bigcup_{\boldsymbol{v}\in X} S(t)\boldsymbol{v}$. To make the solutions operator, S, form a continuous semigroup, we shall require that the system is autonomous, i.e., $\boldsymbol{f} \in H$ and is independent of time. For such autonomous system with unique solutions, we have $S(t)S(s) = S(s)S(t) = S(s + t)$.

2.1.1 The global attractor of the Navier-Stokes equations

Definition 2.1. A set $A \subset H$ is called *absorbing* for the N-S equations if for any bounded set $X \in H$, there is $\tau = \tau(A, X)$ such that for any $t > \tau$, $S(t)X \subset A$.

Definition 2.2. The *global attractor*, $\mathcal{A}$, (of the N-S equations) is "the" set $\mathcal{A} \in H$ with the following properties:

1. $\mathcal{A}$ is compact.
2. $\mathcal{A}$ is *invariant*; That is $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$.
3. For all bounded $X \subset H$,

$$\mathrm{dist}(S(t)X, \mathcal{A}) := \sup_{x \in S(t)X} \inf_{a \in \mathcal{A}} \|x - a\|_H \to 0 \qquad \text{as } t \to \infty.$$

4. $\mathcal{A}$ is *maximal*, in the sense that there is no proper subset of $\mathcal{A}$ with the above properties.

For a bounded Ω of class C^2 as considered above, it has been shown by Foiaş and Prodi Foiaş and Prodi (1967) (see also Ladyzhenskaya Ladyzhenskaya (1975)) that

$$\|S(t)\boldsymbol{u}_0\|_H \leq c_0(\|\boldsymbol{f}\|_H, \nu, \Omega), \tag{2.2}$$

$$\|S(t)\boldsymbol{u}_0\|_{H^1} \leq c_1(\varepsilon, \|\boldsymbol{f}\|_H, \nu, \Omega), \tag{2.3}$$

for all $t > \epsilon$ and $\boldsymbol{u}_0 \in H$. So $K = \{\boldsymbol{u} \in H : \|\boldsymbol{u}\|_H \leq c_0 \text{ and } \|\boldsymbol{u}\|_{H^1} \leq c_1\}$ is a compact absorbing set in H (considering a fixed ε), and hence the global attractor for the Navier-Stokes equations exits (see e.g., Ladyzhenskaya, 1991). The attractor generated by Navier-Stokes equations for a given ν and $\boldsymbol{f}$ on a domain Ω is referred to as $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$.

2.1.2 Properties of the global attractor

As will be demonstrated, if a property holds for the solutions of the Navier-Stokes equations independent of the initial condition, it also holds for all points of the attractor, $\mathcal{A}$. We now use this fact to arrive at an attractor smooth enough to prove our following theorems. In the case that $\boldsymbol{f}$ is independent of t, by Galdi (2000, Theorem 5.6) and the Sobolev embedding theorem (see also Ladyzhenskaya (1975))

$$\sup_{t \geq \varepsilon} \|S(t)\boldsymbol{u}\|_{H^2} \leq c_2' \left(\|\boldsymbol{f}\|_H, \nu, \Omega, \varepsilon\right),$$

$$\sup_{\boldsymbol{x} \in \Omega,\, t \geq \varepsilon} \left| \frac{\partial S(t)\boldsymbol{u}}{\partial t} \right| \leq c_3' \left(\|\boldsymbol{f}\|_H, \nu, \Omega, \varepsilon\right),$$

for all $\varepsilon > 0$ and all $\boldsymbol{u} \in H$. Since the attractor is invariant, for any $t > 0$ and any point $\boldsymbol{u} \in \mathcal{A}$, we may find at least one $\boldsymbol{u}_{-t} \in \mathcal{A}$ (as will be clear in the next paragraph $\boldsymbol{u}_{-t}$ is indeed unique) such that $\boldsymbol{u} = S(t)\boldsymbol{u}_{-t}$. So if we fix ε in the above inequalities and choose a proper t, above inequalities hold for any $\boldsymbol{u} \in \mathcal{A}$ for that fixed ε. This removes the dependence of c_2' and c_3' on ε and we get,

$$\|S(t)\boldsymbol{u}\|_{H^2} \leq c_2\left(\|\boldsymbol{f}\|_H, \nu, \Omega\right), \tag{2.4}$$

$$\sup_{\boldsymbol{x}\in\Omega,\, t\in\mathbb{R}} \left|\frac{\partial S(t)\boldsymbol{u}}{\partial t}\right| \leq c_3\left(\|\boldsymbol{f}\|_H, \nu, \Omega\right), \tag{2.5}$$

for all $\boldsymbol{u} \in \mathcal{A}$. Similarly, (2.2) and (2.3) hold for all $\boldsymbol{u}_0 \in \mathcal{A}$.

One of the central ideas in what follows is the notion of time on the attractor. In fact, for initial conditions $\boldsymbol{u}_0 \in \mathcal{A}$, it is possible to extend the definition of the solution operator, $S(t)$, to include negative values of time. One consequence of continuous dependence result of Knops and Payne (1968) (see Lemma 2.8) is backward uniqueness for solutions of 2-D Navier-Stokes equations (at least for the ones that are as smooth as the points of the attractor constructed above) which concludes that $S(t)$ is injective: $\boldsymbol{u}_0 = \boldsymbol{v}_0$ if $S(t)\boldsymbol{u}_0 = S(t)\boldsymbol{v}_0$, for some $t > 0$. Since the attractor is invariant in time, for any $t > 0$ and $\boldsymbol{u} \in \mathcal{A}$, we set $S(-t)\boldsymbol{u} = \boldsymbol{u}_{-t}$ where $\boldsymbol{u}_{-t} \in \mathcal{A}$ satisfies $S(t)\boldsymbol{u}_{-t} = \boldsymbol{u}$. Note that this does not imply that the Navier-Stokes equations are solvable backwards in time.

2.2 CONSTRUCTION OF THE MAPPINGS

In this section we construct a mapping from the attractor into curves in $\mathbb{R}^N$ for N sufficiently large. For $\boldsymbol{u} \in \mathcal{A}$, the following Lemma due to Foiaş and Temam, provide us with an estimate for $\|\boldsymbol{u}\|_H$ in terms of values of $\boldsymbol{u}$ on a set of discrete points in Ω.

Lemma 2.3. Foiaş and Temam (1984) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, with $\mathcal{E}_\Omega$ an ε-net over Ω, then for any $\boldsymbol{u} \in W^{m,2}$, $m > 1$,*

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq c_4\, \eta^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) + c_5\, \varepsilon^\alpha\ \|\boldsymbol{u}\|_{W^{m,2}(\Omega)}\,, \tag{2.6}$$

where c_i's are positive constants, $0 < \alpha \leq m-1$ and

$$\eta^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) = \max_{\boldsymbol{x}_n \in \mathcal{E}_\Omega} |\boldsymbol{u}(\boldsymbol{x}_n)|\,. \tag{2.7}$$

In view of the above, we show in the next theorem that we can find a finite number of points in Ω, away from the boundary, such that if the trajectories (for $t < 0$) passing through two points, $\boldsymbol{u}$ and $\boldsymbol{v}$, coincide on these points, then $\boldsymbol{u} = \boldsymbol{v}$.

Theorem 2.4. *Let $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$ be the attractor generated by the Navier-Stokes equations under the assumptions of Section 2.1.2. Then there is an ε-net, $\mathcal{E}_\Omega$, over Ω such that if for $\boldsymbol{u}^*$, $\boldsymbol{v}^* \in \mathcal{A}$,*

$$\eta^{(\mathcal{E}_\Omega)}\left(S(-t)\boldsymbol{u}^* - S(-t)\boldsymbol{v}^*\right) = 0, \qquad \forall\, t > t_0 \geq 0,$$

then $\boldsymbol{u}^* = \boldsymbol{v}^*$.

Proof. The proof is in line with the proof of Theorem 3.1 of Foiaş and Temam (1984). For $\tau > t_0$, consider the following Navier-Stokes equations on the attractor for $0 \leq t \leq \tau - t_0$:

$$\begin{aligned} &\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + P(\boldsymbol{u}.\nabla\boldsymbol{u}) = \boldsymbol{f}, &\qquad &\frac{d\boldsymbol{v}}{dt} + \nu A\boldsymbol{v} + P(\boldsymbol{v}.\nabla\boldsymbol{v}) = \boldsymbol{f},\\ &\boldsymbol{u}(0) = S(-\tau)\boldsymbol{u}^* = \boldsymbol{u}_0, & &\boldsymbol{v}(0) = S(-\tau)\boldsymbol{v}^* = \boldsymbol{v}_0. \end{aligned}$$

Subtracting the two equations, with $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{w}_0 = \boldsymbol{u}_0 - \boldsymbol{v}_0$, we get

$$\begin{aligned} &\frac{d\boldsymbol{w}}{dt} + \nu A\boldsymbol{w} + P(\boldsymbol{u}.\nabla\boldsymbol{w}) + P(\boldsymbol{w}.\nabla\boldsymbol{v}) = 0,\\ &\boldsymbol{w}(0) = \boldsymbol{w}_0. \end{aligned} \tag{2.8}$$

Taking the L^2-inner product of the above equation with $A\boldsymbol{w}$ over Ω, we find

$$(\frac{d\boldsymbol{w}}{dt}, A\boldsymbol{w})_2 + \nu(A\boldsymbol{w}, A\boldsymbol{w})_2 + \left(P(\boldsymbol{u}.\nabla\boldsymbol{w}), A\boldsymbol{w}\right)_2 + \left(P(\boldsymbol{w}.\nabla\boldsymbol{v}), A\boldsymbol{w}\right)_2 = 0.$$

Since for smooth enough solutions under consideration

$$(\frac{d\boldsymbol{w}}{dt}, A\boldsymbol{w})_2 = \frac{1}{2}\frac{d}{dt}\left\|\nabla \boldsymbol{w}\right\|^2,$$

for all t, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\nabla \boldsymbol{w}\right\|^2 + \nu\left\|A\boldsymbol{w}\right\|^2 \leq & \left|(\boldsymbol{u}.\nabla\boldsymbol{w}, A\boldsymbol{w})_2\right| + \left|(\boldsymbol{w}.\nabla\boldsymbol{v}, A\boldsymbol{w})_2\right| \\ & \leq c_1'\left\|\boldsymbol{u}\right\|_{W^{2,2}(\Omega)}\left\|\boldsymbol{w}\right\|_{W^{1,2}(\Omega)}\left\|A\boldsymbol{w}\right\| \\ & + c_1'\left\|\boldsymbol{w}\right\|_{W^{1,2}(\Omega)}\left\|\boldsymbol{v}\right\|_{W^{2,2}(\Omega)}\left\|A\boldsymbol{w}\right\|\end{aligned}$$

where we have used the following variation of the inequality (1.30):

$$\left|(\boldsymbol{u}_1.\nabla\boldsymbol{u}_2, \boldsymbol{u}_3)_2\right| \leq c'\left\|\boldsymbol{u}_1\right\|_{W^{m_1,2}}\left\|\nabla\,\boldsymbol{u}_2\right\|_{W^{m_2,2}}\left\|\boldsymbol{u}_3\right\|_{W^{m_3,2}}, \tag{2.9}$$

for $m_1 + m_2 + m_3 > 1$. Note that $\left\|A\boldsymbol{w}\right\|$ is a norm equivalent to $\left\|\boldsymbol{w}\right\|_{W^{2,2}(\Omega)}$ (Temam, 1979) and by (2.4) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\nabla \boldsymbol{w}\right\|^2 + \nu\left\|A\boldsymbol{w}\right\|^2 &\leq c_2'\left\|\boldsymbol{w}\right\|_{W^{1,2}(\Omega)}\left\|A\boldsymbol{w}\right\| \\ &\leq \frac{(c_2')^2}{2\varepsilon_1}\left\|\boldsymbol{w}\right\|^2_{W^{1,2}(\Omega)} + \frac{\varepsilon_1}{2}\left\|A\boldsymbol{w}\right\|^2,\end{aligned}$$

where we have used Young's inequality in the last step. Now choosing ε_1 such that $c_4' = \nu - \varepsilon_1/2 > 0$ and using the interpolation inequality

$$\left\|\boldsymbol{w}\right\|^2_{W^{1,2}(\Omega)} \leq c_5'\left\|\boldsymbol{w}\right\|\left\|\boldsymbol{w}\right\|_{W^{2,2}(\Omega)},$$

and Lemma 2.3, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla \boldsymbol{w}\right\|^2 + c_4'\left\|A\boldsymbol{w}\right\|^2 \\ &\qquad\leq \frac{(c_2')^2 c_5'}{2\varepsilon_1}\left\|\boldsymbol{w}\right\|\left\|\boldsymbol{w}\right\|_{W^{2,2}(\Omega)} \\ &\qquad\leq \frac{(c_2')^2 c_5'}{2\varepsilon_1}\left(c_4\eta^{(\varepsilon_\Omega)}(\boldsymbol{w}) + c_5\varepsilon\left\|\boldsymbol{w}\right\|_{W^{2,2}(\Omega)}\right)\left\|\boldsymbol{w}\right\|_{W^{2,2}(\Omega)} \\ &\qquad\leq \frac{(c_2')^2 c_5' c_4}{2\varepsilon_1}\eta^{(\varepsilon_\Omega)}(\boldsymbol{w})\left\|\boldsymbol{w}\right\|_{W^{2,2}(\Omega)} + c_6'\varepsilon\left\|A\boldsymbol{w}\right\|^2.\end{aligned} \tag{2.10}$$

If we pick ε such that $c_7' = c_4' - c_6'\varepsilon > 0$ and if on such ε-net the assumption of the theorem is satisfied (that is, (2.4) holds), then it follows that $\eta^{(\mathcal{E}_\Omega)}(\boldsymbol{w}) = 0$ for all $0 \le t < \tau - t_0$. So the above inequality reads

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2 + c_7'\|A\boldsymbol{w}\|^2 \le 0, \qquad 0 \le t < \tau - t_0,$$

or

$$\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2 + c_8'\|\nabla \boldsymbol{w}\|^2 \le 0, \qquad 0 \le t < \tau - t_0,$$

which upon integration over $[0, \tau - t_0)$ gives

$$e^{c_8'(\tau - t_0)}\|\nabla \boldsymbol{w}(\tau - t_0)\|^2 - \|\nabla \boldsymbol{w}(0)\|^2 \le 0.$$

Since $\boldsymbol{w}(\tau - t_0) = S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*$ and $\boldsymbol{w}(0) = S(-\tau)\boldsymbol{u}^* - S(-\tau)\boldsymbol{v}^*$ we get

$$e^{c_8'(\tau - t_0)}\|\nabla\left(S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*\right)\|^2 \le \|\nabla\left(S(-\tau)\boldsymbol{u}^* - S(-\tau)\boldsymbol{v}^*\right)\|^2,$$

and by (2.3) it follows that

$$\|\nabla\left(S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*\right)\|^2 \le c_2\, e^{-c_8'(\tau - t_0)},$$

which in the limit $\tau \to \infty$, gives $\|\nabla\left(S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*\right)\| = 0$ and by Poincaré inequality yields $S(-t_0)\boldsymbol{u}^* = S(-t_0)\boldsymbol{v}^*$; but then

$$\boldsymbol{u}^* = S(t_0)S(-t_0)\boldsymbol{u}^* = S(t_0)S(-t_0)\boldsymbol{v}^* = \boldsymbol{v}^*.$$

□

Proof of Theorem 2.4 gives conditions for ε such that if (2.4) holds (on $\mathcal{E}_\Omega$), then the statement of the theorem follows and hence leads to the following definition:

Definition 2.5. An ε-net on Ω is called a *qualified net* if $\varepsilon < (\nu/c_6')$, where c_6' is given in the proof of Theorem 2.4.

Consider the attractor $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$ generated by the Navier-Stokes equations and a qualified net $\mathcal{E}_\Omega = \{\boldsymbol{x}_n\}_{1\le n\le N}$. In the following, we construct a mapping $\gamma : \boldsymbol{u} \longrightarrow \gamma(\boldsymbol{u})$, where $\gamma(\boldsymbol{u})$ is a curve in $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$, parametrized naturally by time t. At each fixed time, the point of $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ that is on the curve, has coordinates that specify the values of the solution $\boldsymbol{u}$, on the points of $\mathcal{E}_\Omega$. Also, note that it follows immediately from the continuity of the solution of 2-D N-S equations that $\gamma(\boldsymbol{u})$ is continuous with respect to its parameter t.

For $\boldsymbol{u} \in \mathcal{A}$ and $t \in \mathbb{R}$, let $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t) \in (\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$, be the point of $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ whose $(2i-1)$-component is given by the first component of the vector $(S(t)\boldsymbol{u})(\boldsymbol{x}_n)$, and whose $(2i)$-component is the second component of $(S(t)\boldsymbol{u})(\boldsymbol{x}_n)$, $1 \le i \le N(\mathcal{E}_\Omega)$. This gives a map from $(\boldsymbol{u}, t) \in \mathcal{A} \times \mathbb{R}$ into $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t) \in (\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$. Since

$$\begin{aligned}\eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u}) = \max_{\boldsymbol{x}_n\in\mathcal{E}_\Omega} |(S(t)\boldsymbol{u})(\boldsymbol{x}_n)| &\le \sqrt{\sum_{\boldsymbol{x}_n\in\mathcal{E}_\Omega} |(S(t)\boldsymbol{u})(\boldsymbol{x}_n)|^2} = \left|\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t)\right| \\ &\le \sqrt{N(\mathcal{E}_\Omega)}\ \max_{\boldsymbol{x}_n\in\mathcal{E}_\Omega} |(S(t)\boldsymbol{u})(\boldsymbol{x}_n)| = \sqrt{N(\mathcal{E}_\Omega)}\ \eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u}), \quad (2.11)\end{aligned}$$

it follows that on the set of points $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t)$, for any $\boldsymbol{u}$ and t, $\eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u})$ is a norm equivalent to the Euclidean norm.

Let $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) = \bigcup_{t<0} \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t)$, so $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})$ is a curve in $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$, which is obtained by projection on $\mathcal{E}_\Omega$ of the trajectory passing through $\boldsymbol{u}$ at $t = 0$. We write $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) = \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{v})$ if $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t) = \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{v})(t)$ for all $t < t_0 \le 0$. Let Γ be the set of all curves $\lambda : (-\infty, 0) \to (\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ of class C^1, then by (2.11) and Theorem 2.4, we observe that the mapping

$$\gamma^{(\mathcal{E}_\Omega)} : \boldsymbol{u} \in \mathcal{A} \longrightarrow \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) \in \Gamma$$

is injective.

Since it is possible to have many qualified nets, $\mathcal{E}_\Omega$, and a corresponding $\gamma^{(\mathcal{E}_\Omega)}$, we obtain a class of mappings which then by the results of next section are unique up to a homeomorphism. When there is no confusion about the underlying qualified net for a mapping we use a simpler notation $\gamma(\boldsymbol{u})$ to refer to the image of $\boldsymbol{u}$.

Remark 2.6. It is always possible to choose a *qualified net*, $\mathcal{E}_\Omega$, such that

$$\operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon, \qquad \text{for all } \boldsymbol{x} \in \mathcal{E}_\Omega.$$

So to construct the mapping, no information is needed "near" the boundary.

2.3 PROPERTIES OF THE MAPPINGS

Here we show continuity of $\gamma^{(\mathcal{E}_\Omega)}$ and continuity of its inverse. Continuity has the major consequence that the range of the mappings is a compact subset of $\mathbb{R}^N$, and together with the continuity of the inverse they show that the range of $\gamma^{(\mathcal{E}_\Omega)}$ and $\gamma^{(\mathcal{E}'_\Omega)}$ corresponding to two qualified nets $\mathcal{E}_\Omega$ and $\mathcal{E}'_\Omega$ are homeomorphic. To proceed with continuity results, we first introduce a topology on Γ. The following construction is along the ideas of Galdi and Rionero (1979).

For any $-\tau \in (-\infty, 0)$,

$$d_{-\tau}(\gamma_1, \gamma_2) = \int_{-\tau}^{0} |\gamma_1(t) - \gamma_2(t)|\, dt, \qquad \gamma_1, \gamma_2 \in \Gamma,$$

(as can be easily verified) defines a pseudo-metric in Γ (the set of all C^1 curves in $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ defined on $(-\infty, 0)$).

Let $h : (0,1) \to (-\infty, 0)$ be defined by $h(s) = 1 - 1/s^\alpha$ for some $0 < \alpha$.* For any $\gamma \in \Gamma$ and $0 < \varepsilon < 1$ let $B_\varepsilon^{(\gamma)} = \{\gamma' \in \Gamma : d_{h(\varepsilon)}(\gamma, \gamma') < \varepsilon\}$, and consider the following family of sets:

$$\mathcal{B} = \{B_\varepsilon^{(\gamma)} : \gamma \in \Gamma,\ 0 < \varepsilon < 1\}.$$

Next, observe that $\mathcal{B}$ can serve as a base for a topology on Γ Galdi and Rionero (1979). Designating the empty set and all sets representable as union of sets of $\mathcal{B}$ as "open" we arrive at a topology $\mathcal{T}$ on Γ.

The family of pseudo-metrics $d_{h(s)}$, $0 < s < 1$, can be used to define the following metric on Γ

$$d(\gamma_1, \gamma_2) = \int_0^1 \frac{d_{h(s)}(\gamma_1, \gamma_2)}{1 + d_{h(s)}(\gamma_1, \gamma_2)} ds.$$

Let us denote by $\mathcal{T}_w$ the topology induced by the metric d on Γ, then we have Galdi and Rionero (1979),

Lemma 2.7. *$\mathcal{T}_w$ is weaker than $\mathcal{T}$.*

The following logarithmic convexity argument of Knops and Payne Knops and Payne (1968) plays a key role in proving continuity:

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*h can be any non-decreasing function as long as the the construction of a topology is concerned. The specific form assumed here is merely for ease of algebraic operations in the theorems that follow.

Lemma 2.8. *Let $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$ be the attractor generated by Navier-Stokes equations (under the assumptions of Section 2.1.2). For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, $\boldsymbol{w}(t) = S(t)\boldsymbol{u} - S(t)\boldsymbol{v}$ satisfies*

$$\|\boldsymbol{w}(t)\|_{L^2(\Omega)} \leq \exp\left(\frac{c_7}{2c_6}\left(t - \lambda t_1 - (1-\lambda)t_2\right)\right) \|\boldsymbol{w}(t_1)\|_{L^2(\Omega)}^{\lambda} \|\boldsymbol{w}(t_2)\|_{L^2(\Omega)}^{(1-\lambda)},$$

for any $t_1 < t < t_2$ with $\lambda = \dfrac{e^{c_6 t} - e^{c_6 t_2}}{e^{c_6 t_1} - e^{c_6 t_2}}$.

In view of the above Lemma we immediately have the following continuity result:

Theorem 2.9. *The mapping $\gamma^{(\mathcal{E}_\Omega)} : \mathcal{A} \to \Gamma$ is continuous if $\mathcal{A}$ is endowed with the topology of $L^2(\Omega)$ and Γ with the topology, $\mathcal{T}$.*

Proof. Let $B \in \mathcal{T}$ be an open neighborhood of $\gamma(\boldsymbol{u})$, then since $\mathcal{B}$ is a base for $\mathcal{T}$, it follows that there is $0 < \varepsilon$ (< 1) such that $B_\varepsilon^{(\gamma(\boldsymbol{u}))} \subset B$. So we need to show that there is $\delta = \delta(\varepsilon)$ such that

$$\boldsymbol{v} : \ \|\boldsymbol{u} - \boldsymbol{v}\| < \delta \Rightarrow d_{h(\varepsilon)}(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v})) < \varepsilon,$$

but by (2.11) and Sobolev embedding theorem, we have

$$\begin{aligned} d_{h(\varepsilon)}(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v})) &= \int_{h(\varepsilon)}^{0} |\gamma(\boldsymbol{u})(t) - \gamma(\boldsymbol{v})(t)|\, dt \\ &\leq \sqrt{N(\mathcal{E}_\Omega)} \int_{h(\varepsilon)}^{0} \eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u} - S(t)\boldsymbol{v}) dt \leq \sqrt{N(\mathcal{E}_\Omega)} \int_{h(\varepsilon)}^{0} c_9' \|S(t)\boldsymbol{u} - S(t)\boldsymbol{v}\|_{W^{m,2}(\Omega)}\, dt. \end{aligned}$$

for some $1 < m < 2$. Using the more compact notation $\boldsymbol{w}(t) = S(t)\boldsymbol{u} - S(t)\boldsymbol{v}$ and the interpolation inequality

$$\|\boldsymbol{w}\|_{W^{m,2}(\Omega)} \leq c_{10}' \|\boldsymbol{w}\|_{W^{2,2}(\Omega)}^{\frac{m}{2}} \|\boldsymbol{w}\|^{1-\frac{m}{2}},$$

we obtain

$$\begin{aligned} d_{h(\varepsilon)}(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v})) &\leq \sqrt{N(\mathcal{E}_\Omega)} \int_{h(\varepsilon)}^{0} c_9' c_{10}' \|\boldsymbol{w}(t)\|_{W^{2,2}(\Omega)}^{\frac{m}{2}} \|\boldsymbol{w}(t)\|^{1-\frac{m}{2}}\, dt \\ &\leq c_{11}' \int_{h(\varepsilon)}^{0} \|\boldsymbol{w}(t)\|^{1-\frac{m}{2}}\, dt, \end{aligned}$$

where we have used (2.4) in the last step.

For the given ε, let us use lemma 2.8 with $t_1 = h(\varepsilon)$, $t_2 = 0$ and $\lambda = \dfrac{e^{c_6 t} - 1}{e^{c_6 t_1} - 1}$ to get

$$\begin{aligned} d_{h(\varepsilon)}(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v})) &\leq c'_{11} \int_{h(\varepsilon)}^{0} \left[\exp\left(\frac{c_7}{2c_6}(t - \lambda t) \right) \|\boldsymbol{w}(t_1)\|^{\lambda} \|\boldsymbol{w}(0)\|^{1-\lambda} \right]^{1-\frac{m}{2}} dt \\ &\leq c'_{12}(t_1)\ \|\boldsymbol{w}(0)\| \int_{h(\varepsilon)}^{0} e^{c_6 t} \|\boldsymbol{w}(0)\|^{(1-\frac{m}{2})\dfrac{1-e^{c_6 t}}{e^{c_6 t_1}-1}} dt \\ &\leq c'_{13}(t_1) \frac{1}{|\ln \|\boldsymbol{w}(0)\||}, \end{aligned}$$

again, with the help of (2.2) and bounds for various powers of e in the interval $(t_1, 0)$. Noting that $\|\boldsymbol{w}(0)\| = \|\boldsymbol{u} - \boldsymbol{v}\|$, choosing

$$\delta < \exp\left(\min\{-1, -\frac{c'_{13}(t_1)}{\varepsilon}\} \right),$$

completes the proof. □

Remark 2.10. By lemma 2.7, the mapping $\gamma^{(\mathcal{E}_\Omega)}$ is also continuous when Γ is furnished with the topology $\mathcal{T}_w$. Also, the topology on $\mathcal{A}$ can be replaced by a stronger topology, most interesting of them is that of H^1.

Theorem 2.11. *The inverse of mapping $\gamma^{(\mathcal{E}_\Omega)}$ is continuous when Γ is furnished with the topology $\mathcal{T}_w$ and $\mathcal{A}$ with the topology of H^1.*

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$ Then for $\boldsymbol{w}(t) = S(t)\boldsymbol{u} - S(t)\boldsymbol{v}$ by (2.10) and (2.4) we have

$$\frac{1}{2}\frac{d}{dt} \|\nabla \boldsymbol{w}(t)\|^2 + c'_7 \|A\boldsymbol{w}(t)\|^2 \leq c'_{14} \eta^{(\mathcal{E}_\Omega)}(\boldsymbol{w}(t)),$$

or †

$$\frac{1}{2}\frac{d}{dt} \|\nabla \boldsymbol{w}(t)\|^2 + c'_8 \|\nabla \boldsymbol{w}(t)\|^2 \leq c'_{14} \eta(\boldsymbol{w}(t)).$$

Integrating the above for $h(s) \leq t < 0$ with the initial condition $\boldsymbol{w}(h(s)) = S(h(s))\boldsymbol{u} - S(h(s))\boldsymbol{v}$ we obtain

$$\|\nabla \boldsymbol{w}(0)\|^2 - e^{c'_8 h(s)} \|\nabla \boldsymbol{w}(h(s))\|^2 \leq c'_{14} \int_{h(s)}^{0} e^{c'_8 t} \eta(\boldsymbol{w}(t)) dt \leq c'_{14} \int_{h(s)}^{0} \eta(\boldsymbol{w}(t)) dt.$$

† We will use the simpler notation $\eta(\boldsymbol{w}(t))$ when the underlying ε-net on Ω is fixed, without any confusions.

Noting that $\boldsymbol{w}(0) = \boldsymbol{u} - \boldsymbol{v}$ and using (2.3), the above inequality yields

$$\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2 \le c'_{15} e^{c'_8 h(s)} + c'_{14} \int_{h(s)}^{0} \eta(\boldsymbol{w}(t))dt.$$

For any $0 < s' < s$, since $h(s') < h(s)$ we can increase the last term to obtain

$$\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2 \le c'_{15} e^{c'_8 h(s)} + c'_{14} \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt,$$

and hence dividing by $1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt$ yields

$$\frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} \le \frac{c'_{15} e^{c'_8 h(s)}}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} + c'_{14} \frac{\int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}. \tag{2.12}$$

Using (2.4), (2.2) and the Sobolev embedding theorem, it follows that $\eta(\boldsymbol{w}(t)) \le c'_{16}$, so for the term on the left hand side of the above we have

$$\frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} \ge \frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 - c'_{16} h(s')} \ge \frac{1}{c'_{17}} \frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 - h(s')} = \frac{(s')^{\alpha}}{c'_{17}} \|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2,$$

hence, (2.12) implies,

$$\frac{(s')^{\alpha}}{c'_{17}} \|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2 \le c'_{15} e^{c'_8 h(s)} + c'_{14} \frac{\int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}.$$

Taking the square root of both sides and integrating for $0 < s' < s$ we obtain

$$\begin{aligned}(s')^{1+\alpha/2} \|\nabla(\boldsymbol{u}-\boldsymbol{v})\| &\le c'_{18}\, s'\, e^{c'_8 h(s)/2} + c'_{19} \int_0^s \left[\frac{\int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} \right]^{\frac{1}{2}} ds' \\ &\le c'_{18}\, s'\, e^{c'_8 h(s)/2} + c'_{20} \left(\int_0^s ds' \right)^{\frac{1}{2}} [d(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v}))]^{\frac{1}{2}},\end{aligned}$$

where we have used Hölder inequality and (2.11) in the last step along with the fact that since the integrand in the last integral is positive the limit of the integral can be increased from s to 1. So,

$$\|\nabla(\boldsymbol{u}-\boldsymbol{v})\| \leq c'_{18}\,\frac{e^{c'_8 h(s)/2}}{s^{(\alpha/2)}} + \frac{c'_{20}}{s^{(\alpha+1)/2}}\,[d(\gamma(\boldsymbol{u}),\gamma(\boldsymbol{v}))]^{\frac{1}{2}}\,,$$

and continuity follows once for a given ε we choose s such that

$$c'_{18}\,\frac{e^{c'_8 h(s)/2}}{s^{(\alpha/2)}} < \frac{\varepsilon}{2},$$

and δ such that,

$$\delta < \frac{\varepsilon^2\, s^{\alpha+1}}{(2c'_{20})^2}.$$

□

2.4 NON-HOMOGENEOUS BOUNDARY CONDITIONS

For a Lipschitz domain $\Omega \subset \mathbb{R}^2$, let $B^{m-\frac{1}{2},2}(\partial\Omega)$, be the space of traces of functions in $W^{m,2}(\Omega)$, $1 \leq m < \infty$. Consider the non-homogeneous Navier-Stokes equations (as in Section 2.1.2 on a bounded, C^2 domain):

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla\boldsymbol{u} &= \nabla p + \nu\Delta\boldsymbol{u} + \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned} \tag{2.13}$$

with initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in H$ and boundary condition

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^*(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega,\ t > 0.$$

For compatibility, we require

$$\int_{\partial\Omega} \boldsymbol{u}^*.\boldsymbol{n} = 0,$$

and to avoid unnecessary complications, let us assume $\partial\Omega$ is connected (Galdi, 2011, Section IX.4). To show the existence of weak solutions to (2.13), let us assume $\boldsymbol{f} \in L^2(\Omega)$

and following Temam (1997), a time independent boundary condition $\boldsymbol{u}^* \in B^{\frac{1}{2},2}(\partial\Omega)$. This ensures that for any $\alpha > 0$, $\boldsymbol{u}^*$ has an extension, $\boldsymbol{V} \in W^{1,2}(\Omega)$, such that (Galdi, 2011)

$$-(\boldsymbol{w}.\nabla\boldsymbol{V}, \boldsymbol{w}) \leq \alpha \|\boldsymbol{w}\|_{H^1}, \qquad \text{for all } \boldsymbol{w} \in H^1(\Omega).$$

If we write $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}$, then (2.13) has a weak solution if there is a weak solution, $\boldsymbol{v}$, to

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} - \nu\Delta\boldsymbol{v} + \boldsymbol{v}.\nabla\boldsymbol{v} &= \nabla p + \nu\Delta\boldsymbol{V} - \boldsymbol{v}.\nabla\boldsymbol{V} - \boldsymbol{V}.\nabla\boldsymbol{v} - \boldsymbol{V}.\nabla\boldsymbol{V} + \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{v} &= 0, \end{aligned} \tag{2.14}$$

with initial condition $\boldsymbol{v}(0) = \boldsymbol{u}_0 - \boldsymbol{V}$ and homogeneous boundary conditions. A weak solution to (2.14) is defined (similar to Navier-Stokes equations) as a function $\boldsymbol{v} \in L^2(0,T;H^1) \cap L^\infty(0,T;H)$ such that it satisfies the above after taking its inner product with a test function and integrating by parts. The existence of such a solution, satisfying (2.2) and (2.3), and hence the existence of the attractor is shown by Temam (1997).

To obtain the regularity results of (2.4) and (2.5), assume $\boldsymbol{u}^* \in B^{\frac{3}{2},2}(\partial\Omega)$ which guarantees that $\boldsymbol{V}$ can be chosen such that $\boldsymbol{V} \in W^{2,2}(\Omega)$. Then (2.4) and (2.5) follow by the same argument as the homogeneous case, once we note that Lemmas 5.4 and 5.5 of Galdi (2000) continue to hold when Navier-Stokes equations is replaced by (2.14), after obvious modifications of the proofs. Then, all other theorems and lemmas in previous sections will be valid without any change, as they are either based on the equation for evolution of the difference between two solutions of N-S equations (and the boundary condition for such an equation is always homogeneous) and/or they use estimates (2.2), (2.3), (2.4) and (2.5).

APPENDIX

CONTINUITY OF THE MAP Φ

We will prove that Φ defined in (1.42), is continuous at $(\boldsymbol{v}_1, \boldsymbol{b}_1) \in C_T^1(S_1^n) \times C_T(S_2^n)$ with respect to the norm introduced in (1.45). The proof is standard, by an ε-δ argument. Let

$$(\boldsymbol{u}_1, \boldsymbol{h}_1) = \Phi(\boldsymbol{v}_1, \boldsymbol{b}_1), \qquad (\boldsymbol{v}_2, \boldsymbol{h}_2) = \Phi(\boldsymbol{v}_2, \boldsymbol{b}_2), \tag{A.1}$$

for some $(\boldsymbol{v}_2, \boldsymbol{b}_2)$ in δ-neighborhood of $(\boldsymbol{v}_1, \boldsymbol{b}_1)$. To relax the notation we set

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}_2 - \boldsymbol{v}_1, & \boldsymbol{b} &= \boldsymbol{b}_2 - \boldsymbol{b}_1, \\ \boldsymbol{u} &= \boldsymbol{u}_2 - \boldsymbol{u}_1, & \boldsymbol{h} &= \boldsymbol{h}_2 - \boldsymbol{h}_1. \end{aligned}$$

The goal is to show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|(\boldsymbol{v}, \boldsymbol{b})\| < \delta$, then

$$\|(\boldsymbol{u}, \boldsymbol{h})\| < \varepsilon. \tag{A.2}$$

Subtracting the first equation in (A.1) from the second, with (1.39) in mind, we observe that $(\boldsymbol{u}, \boldsymbol{h})$ satisfies

$$\begin{aligned} &\boldsymbol{u}'' + \mathcal{L}\,\boldsymbol{u} + \alpha\boldsymbol{u}' + \boldsymbol{\nu}(\boldsymbol{v}_2) - \boldsymbol{\nu}(\boldsymbol{v}_1) = \alpha\boldsymbol{v}' + \operatorname{curl}\boldsymbol{h} \times (\boldsymbol{b}_2 + \boldsymbol{B}_0) + \operatorname{curl}\boldsymbol{h}_1 \times \boldsymbol{b}, \\ &\boldsymbol{h}' + \tilde{\mathcal{L}}\,\boldsymbol{h} = \operatorname{curl}(\boldsymbol{u}' \times [\boldsymbol{b}_2 + \boldsymbol{B}_0]) + \operatorname{curl}(\boldsymbol{u}_1' \times \boldsymbol{b}). \end{aligned} \tag{A.3}$$

We then take the L^2-inner product of the first equation with $\boldsymbol{u}'$ and the second equation with $\boldsymbol{h}$ and adding the resulting equations together. Using (1.29) we obtain

$$\frac{d}{dt}\left\{\|\boldsymbol{u}'\|^2 + a_I(\boldsymbol{u},\boldsymbol{u}) + \|\boldsymbol{h}\|^2\right\} + \varrho\,\|\operatorname{curl}\boldsymbol{h}\|^2 + \alpha\,\|\boldsymbol{u}'\|^2 + (\boldsymbol{\nu}(\boldsymbol{v}_2') - \boldsymbol{\nu}(\boldsymbol{v}_1'), \boldsymbol{u}')_2$$
$$= \alpha(\boldsymbol{v}', \boldsymbol{u}')_2 + (\operatorname{curl}\boldsymbol{h}_1 \times \boldsymbol{b}, \boldsymbol{u}')_2 + (\boldsymbol{u}_1' \times \boldsymbol{b}, \operatorname{curl}\boldsymbol{h})_2.$$

By (1.30) and Hölder inequality we have

$$\frac{d\mathcal{E}}{dt} + \varrho\,\|\operatorname{curl}\boldsymbol{h}\|^2 + \alpha\,\|\boldsymbol{u}'\|^2 \le \|\boldsymbol{\nu}(\boldsymbol{v}_2') - \boldsymbol{\nu}(\boldsymbol{v}_1')\|_{L^q}\,\|\boldsymbol{u}'\|_{L^{p+2}} + \alpha\,\|\boldsymbol{v}'\|\,\|\boldsymbol{u}'\|$$
$$+ \|\operatorname{curl}\boldsymbol{h}_1\|\,\|\boldsymbol{b}\|_{W^{2,2}}\,\|\boldsymbol{u}'\| + \|\boldsymbol{u}_1'\|\,\|\boldsymbol{b}\|_{W^{2,2}}\,\|\operatorname{curl}\boldsymbol{h}\|. \quad \text{(A.4)}$$

Note that since all the functions above are finite dimensional in space, all the spatial norms are equivalent. For example, consider $\boldsymbol{b}$ which at any time t has the following representation

$$\boldsymbol{b}(t) = \sum_{i=1}^{n} b_{ni}(t)\boldsymbol{\psi}_i.$$

Then, for instance,

$$\|\boldsymbol{b}\|_{W^{2,2}} = \sum_{i=1}^{n} |b_{ni}|\,\|\boldsymbol{\psi}_i\|_{W^{2,2}},$$

hence, letting

$$c_1 = \max_{1\le i\le n} \frac{\|\boldsymbol{\psi}_i\|_{W^{2,2}}}{\|\boldsymbol{\psi}_i\|},$$

it follows that $\|\boldsymbol{b}\|_{W^{2,2}} \le c_1\,\|\boldsymbol{b}\|$. Note that as $\{\boldsymbol{\psi}_i\}_{i\in\mathbb{N}}$ is a basis of $\tilde{H}$, without loss of generality we have assumed $\boldsymbol{\psi}_i \ne 0$, $\forall i$. Similarly, we can find other proper constants such that

$$\|\operatorname{curl}\boldsymbol{h}_1\| \le c_2\,\|\boldsymbol{h}_1\|, \qquad \|\boldsymbol{u}'\|_{L^{p+2}} \le c_3\,\|\boldsymbol{u}'\|, \quad \text{and} \quad \|\boldsymbol{v}'\|_{C(\Omega)} \le c_4\,\|\boldsymbol{v}'\|, \quad \text{(A.5)}$$

so from (A.4), with suitable positive constants depending only on n and Ω, we deduce

$$\frac{d\mathcal{E}}{dt} + \varrho\,\|\operatorname{curl}\boldsymbol{h}\|^2 + \alpha\,\|\boldsymbol{u}'\|^2 \le c_3\,\|\boldsymbol{\nu}(\boldsymbol{v}_2') - \boldsymbol{\nu}(\boldsymbol{v}_1')\|_{L^q}\,\|\boldsymbol{u}'\| + \alpha\,\|\boldsymbol{v}'\|\,\|\boldsymbol{u}'\|$$
$$+ c_5\,\|\boldsymbol{h}_1\|\,\|\boldsymbol{b}\|\,\|\boldsymbol{u}'\| + c_1\,\|\boldsymbol{u}_1'\|\,\|\boldsymbol{b}\|\,\|\operatorname{curl}\boldsymbol{h}\|.$$

Since $(\boldsymbol{v}_1, \boldsymbol{b}_1)$ is the point at which we want to show the continuity, its image under Φ, $(\boldsymbol{u}_1, \boldsymbol{h}_1)$, is fixed. Setting

$$c_6 = \sup_{t\in[0,T]} \|\boldsymbol{u}_1\|, \qquad \textit{and} \qquad c_7 = \sup_{t\in[0,T]} \|\boldsymbol{h}_1\|,$$

we obtain (for other suitable positive constants),

$$\frac{d\mathcal{E}}{dt} + \varrho \|\mathrm{curl}\, \boldsymbol{h}\|^2 + \alpha \|\boldsymbol{u}'\|^2 \le c_3 \|\boldsymbol{\nu}(\boldsymbol{v}_2') - \boldsymbol{\nu}(\boldsymbol{v}_1')\|_{L^q} \|\boldsymbol{u}'\| + \alpha \|\boldsymbol{v}'\| \, \|\boldsymbol{u}'\| \\ + c_8 \|\boldsymbol{b}\| \, \|\boldsymbol{u}'\| + c_9 \|\boldsymbol{b}\| \, \|\mathrm{curl}\, \boldsymbol{h}\|.$$

Using Young's inequality in the above, we get

$$\frac{d\mathcal{E}}{dt} + c_{10} \|\mathrm{curl}\, \boldsymbol{h}\|^2 + c_{11} \|\boldsymbol{u}'\|^2 \le c_4 \|\boldsymbol{\nu}(\boldsymbol{v}_2') - \boldsymbol{\nu}(\boldsymbol{v}_1')\|_{L^q}^2 + c_{12} \|\boldsymbol{v}'\|^2 + c_{14} \|\boldsymbol{b}\|^2.$$

For every $\varepsilon_1 > 0$, by continuity of $\boldsymbol{\nu}$, there is a $\delta_1 > 0$ such that if $\sup_{t\in[0,T]} \|\boldsymbol{v}'\|_{C(\Omega)} < \delta_1$, then

$$|\boldsymbol{\nu}(\boldsymbol{v}_2'(\boldsymbol{x}, t)) - \boldsymbol{\nu}(\boldsymbol{v}_1'(\boldsymbol{x}, t))| < \varepsilon_1. \qquad \forall \boldsymbol{x} \in \Omega \text{ and } \forall t \in [0, T].$$

By (A.5), it follows that the above holds, in particular, when $\|(\boldsymbol{v}, \boldsymbol{b})\| < \delta_1$. Let us restrict ourselves to this δ_1 neighborhood of $(\boldsymbol{v}_1, \boldsymbol{b}_1)$, then

$$\|\boldsymbol{\nu}(\boldsymbol{v}_2') - \boldsymbol{\nu}(\boldsymbol{v}_1')\|_{L^q} < |\Omega|^{\frac{1}{q}} \varepsilon_1, \qquad \forall t \in [0, T],$$

and so

$$\frac{d\mathcal{E}}{dt} + c_{10} \|\mathrm{curl}\, \boldsymbol{h}\|^2 + c_{11} \|\boldsymbol{u}'\|^2 < c_{15}\, \varepsilon_1^2 + c_{16}\, \delta_1.$$

From the above we get that in particular,

$$\frac{d\mathcal{E}}{dt} < c_{15}\, \varepsilon_1^2 + c_{16}\, \delta_1, \tag{A.6}$$

$$\int_0^T \|\mathrm{curl}\, \boldsymbol{h}\|^2 + \int_0^T \|\boldsymbol{u}'\|^2 < c_{17}\, \varepsilon_1^2 + c_{18}\, \delta_1. \tag{A.7}$$

Next we take the L^2-inner product of (A.3) by $\boldsymbol{u}$ and integrate over the interval $[0,T]$ to obtain

$$\int_0^T a_I(\boldsymbol{u},\boldsymbol{u}) + \int_0^T (\boldsymbol{\nu}(\boldsymbol{v}_2) - \boldsymbol{\nu}(\boldsymbol{v}_1), \boldsymbol{u})_2 = \int_0^T \|\boldsymbol{u}'\|^2 + \alpha \int_0^T (\boldsymbol{v}', \boldsymbol{u})_2 \\ + \int_0^T (\operatorname{curl} \boldsymbol{h} \times (\boldsymbol{b}_2 + \boldsymbol{B}_0), \boldsymbol{u})_2 + \int_0^T (\operatorname{curl} \boldsymbol{h}_1 \times \boldsymbol{b}, \boldsymbol{u})_2.$$

Using a similar approach as above we get

$$c_{22} \int_0^T a_I(\boldsymbol{u},\boldsymbol{u}) \leq \int_0^T \|\boldsymbol{\nu}(\boldsymbol{v}_2) - \boldsymbol{\nu}(\boldsymbol{v}_1)\|^2_{L^q} + \int_0^T \|\boldsymbol{u}'\|^2 + c_{19} \int_0^T \|\boldsymbol{v}'\|^2 \\ + c_{20} \int_0^T \|\operatorname{curl} \boldsymbol{h}\|^2 \, \|\boldsymbol{b}_2 + \boldsymbol{B}_0\|^2 + c_{21} \int_0^T \|\boldsymbol{b}\|^2.$$

Since we have already restricted $(\boldsymbol{v}_2, \boldsymbol{b}_2)$ in a δ_1-neighborhood of $(\boldsymbol{v}_1, \boldsymbol{b}_1)$, it follows that there exists a constant c_{23} depending on $\boldsymbol{b}_1$ and $\boldsymbol{B}_0$ such that $\|\boldsymbol{b}_2 + \boldsymbol{B}_0\| \leq c_{23}$. Using this information along with (A.7), we have

$$\int_0^T a_I(\boldsymbol{u},\boldsymbol{u}) \leq c_{24}\, \varepsilon_1^2 + c_{25}\, \delta_1,$$

which, again in view of (A.7), yields

$$\mathcal{E} \leq c_{26}\, \varepsilon_1^2 + c_{27}\, \delta_1.$$

From the above inequality and (A.6), by Lemma 1.4, we conclude

$$\sup_{t\in[0,T]} \mathcal{E} \leq c_{28}\, \varepsilon_1^2 + c_{29}\, \delta_1.$$

For any $\varepsilon > 0$, pick $\varepsilon_1 \leq \sqrt{\dfrac{\varepsilon}{2c_{28}}}$ and fix the corresponding δ_1. Choosing $\delta < \min\{\delta_1, \dfrac{\varepsilon}{2c_{29}}\}$, completes the proof.

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