# FIXED POINTS OF NONEXPANSIVE MAPS ON CLOSED, BOUNDED, CONVEX SETS IN $\ell^1$.

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In 1965, W.A. Kirk proved that all reflexive Banach spaces $(X, \|\cdot\|)$ with normal structure are such that for all nonempty, closed, bounded, and convex subsets $C \subseteq X$, every nonexpansive map $T : C \longrightarrow C$ has a fixed point, i.e. $(X, \|\cdot\|)$ has the *fixed point property for nonexpansive mappings* (FPP(n.e.)).

In 1979, K. Goebel and T. Kuczumow constructed "very irregular" closed, bounded, convex, non-weak$^*$-compact subsets $K$ of $\ell^1$, and showed that such $K$ have the FPP(n.e.). We show that we may perturb the sets of Goebel and Kuczumow to construct a new and larger class of sets that have the FPP(n.e.).

Ultimately, we would like to answer the following: which isomorphic $\ell^1$-basic sequences $(x_n)_{n\in\mathbb{N}}$ are such that their closed convex hulls have the FPP(n.e.)? Theorem 2.2.1, Theorem 2.3.15, and Theorem 2.3.18 give new and interesting isomorphic $\ell^1$-basic sequences in $(\ell^1, \|\cdot\|_1)$ whose closed convex hulls have the FPP(n.e.).

In 2003, W. Kaczor and S. Prus showed that under a certain assumption, the sets constructed by Goebel and Kuczumow have the fixed point property for asymptotically nonexpansive mappings and that this is equivalent to the sets having the fixed point property for mappings of asymptotically nonexpansive type.

In the second part of this thesis, we prove a theorem (Theorem 3.4.1) that provides an estimate for the $\ell^1$-distance of a point to a simplex. As a corollary, we prove an interesting special case of the theorem of Kaczor and Prus.

We further calculate the best uniform-Lipschitz constant of the right shift $R$ on one of

the sets $K$ of Goebel and Kuczumow.

We also consider another closed, bounded, convex, non-weak$^*$-compact subset $G$ of the positive face of the usual unit sphere $S$ in $\ell^1$. We show that, in contrast to the sets $K$ above, $G$ fails to have the fixed point property for asymptotically nonexpansive mappings.

## TABLE OF CONTENTS

## PREFACE

First and foremost, I would like to thank Dr. Chris Lennard. Chris has been my guide in mathematics for the last several years. He and his wife Cathy have made my time at the University of Pittsburgh feel like I've been a member of a family, which is far beyond what is expected from an academic advisor. I would like to thank Chris for teaching me about functional analysis, showing me how to be a good teacher, and being a good friend whenever I needed one.

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# 1.0 INTRODUCTION

We begin by giving a brief introduction to metric fixed point theory, and also to the work of Goebel and Kuczumow that inspired much of the work in this thesis.

In 1912, L.E.J. Brouwer [3] determined that for $n \in \mathbb{N}$, for $C$ equal to the closed unit ball of $\mathbb{R}^n$, every norm-to-norm continuous map $f : C \longrightarrow C$ has a fixed point. This result was later extended to every compact convex subset of $\mathbb{R}^n$. In this form, the theorem was extended by J. Schauder [14] in 1930 to every Banach space $(X, \|\cdot\|)$. The class of norm compact, convex sets is small. On the other hand, the class of continuous mappings involved is large.

In 1922, S. Banach [1] introduced the Banach Contraction Mapping Theorem. If $(X, d)$ is a complete metric space, and $f : X \longrightarrow X$ is a strict contraction, then $f$ has a unique fixed point in $X$. In terms of Banach spaces, it follows that for a nonempty, closed, bounded, and convex subset $C$ of a Banach space $(X, \|\cdot\|)$, if $T : C \longrightarrow C$ is a strict contraction for the metric $d = d_{\|\cdot\|}$ generated by the norm, then $T$ has a fixed point. Here the class of closed, bounded, and convex sets is large. On the other hand, the class of continuous maps only includes the strict contractions.

In terms of the sizes of the classes involved, F. Browder [4] provided a more balanced theorem in 1965. It stated that for all nonempty, closed, bounded, and convex subsets $C$ of a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with associated norm $\|\cdot\|$, every nonexpansive map $T : C \longrightarrow C$ has a fixed point in $C$. Here, $T$ is nonexpansive means that $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$.

Later in 1965, Browder [5] and D. Göhde [10] each generalized the previous theorem to all uniformly convex Banach spaces. A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if for each $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that for $x, y \in X$,

$$\left.\begin{array}{l}\|x\| \leq 1\\ \|y\| \leq 1\\ \|x-y\| > \varepsilon\end{array}\right\} \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

An example of a uniformly convex Banach space is $L^p$, $1 < p < \infty$ with the standard norm $\|\cdot\|_p$.

Even later still in 1965, W.A. Kirk [12] further generalized the theorem of Browder to all reflexive Banach spaces $(X, \|\cdot\|)$ with normal structure, i.e. those spaces such that all non-singleton closed, bounded, and convex sets have a greater diameter than radius. Banach spaces $(X, \|\cdot\|)$ such that for all nonempty, closed, bounded, and convex subsets $C \subseteq X$, every nonexpansive map $T : C \longrightarrow C$ has a fixed point, are called spaces with the *fixed point property for nonexpansive mappings*. We often abbreviate this and write FPP(n.e.). We also note that the sequence spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$ are both nonreflexive and do not have the FPP(n.e.).

For a long time, it was unclear whether or not all Banach spaces with the property FPP(n.e.) were reflexive. In 2008, P.K. Lin [13] provided the first example of a nonreflexive Banach space with the FPP(n.e.). Lin's example is the space $\ell^1$ endowed with a norm that is equivalent to the usual norm.

The primary motivation for our work occurred in 1979. K. Goebel and T. Kuczumow [9] constructed "very irregular" closed, bounded, convex, non-weak$^*$-compact subsets $K$ of $\ell^1$ (with its usual norm), and showed that such $K$ have the FPP(n.e.): i.e., every nonexpansive mapping $T : K \longrightarrow K$ has a fixed point. In this thesis we show that we may perturb the sets of Goebel and Kuczumow to construct a new and larger class of sets that have the FPP(n.e.). Note that in Theorem 2.0.4, Goebel and Kuczumow use $b_i = 1 + a_i > 0$. Our results in Theorem 2.3.1, Theorem 2.3.6, Theorem 2.3.11, Theorem 2.3.15, and Theorem 2.3.18 allow for each $b_i$ to be real-valued.

Ultimately, we would like to answer the following question. Precisely which isomorphic $\ell^1$-basic sequences $(x_n)_{n\in\mathbb{N}}$ (or asymptotically isometric $\ell^1$-basic sequences $(x_n)_{n\in\mathbb{N}}$) in $(\ell^1, \|\cdot\|_1)$ are such that their closed convex hulls have the FPP(n.e.)? Our theorems (Theorem 2.2.1, Theorem 2.3.15, and Theorem 2.3.18) give new and interesting isomorphic $\ell^1$-basic sequences in $(\ell^1, \|\cdot\|_1)$ whose closed convex hulls have the FPP(n.e.); which is a step towards a solution of this open problem.

In 2003, W. Kaczor and S. Prus [11] showed that under a certain assumption, the sets constructed by Goebel and Kuczumow have the fixed point property for asymptotically nonexpansive mappings and that this is equivalent to the sets having the fixed point property for mappings of asymptotically nonexpansive type.

In the second part of this thesis, we prove a theorem (Theorem 3.4.1) that provides an estimate for the $\ell^1$-distance of a point to a simplex. As a corollary to Theorem 3.4.1 we prove an interesting special case of the theorem of Kaczor and Prus. We remark that our proof technique seems to be rather different to the method used in their paper.

We further calculate the best uniform-Lipschitz constant of the right shift $R$ on one of the sets $K$ of Goebel and Kuczumow. Note that $R$ is fixed point free on $K$, and so must fail to be asymptotically nonexpansive.

We also consider another closed, bounded, convex, non-weak*-compact subset $G$ of the positive face of the usual unit sphere $S$ in $\ell^1$. Dowling, Lennard and Turett [7] recently showed that $G$ has the fixed point property for nonexpansive mappings. In this thesis we show that, in contrast to the sets $K$ above, $G$ fails to have the fixed point property for asymptotically nonexpansive mappings.

## 1.1 PRELIMINARIES AND OVERVIEW

We denote the set of all positive integers and the set of all real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. We define the Banach space $(\ell^1, \|\cdot\|_1)$ by

$$\ell^1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

For all $n \in \mathbb{N}$, let $e_n = (e_{n,k})_{k\in\mathbb{N}}$ be defined by setting $e_{n,n} := 1$ and $e_{n,k} := 0$, for all $k \in \mathbb{N}$ with $k \neq n$. Of course, each $e_n \in \ell^1$. We will often write $\|\cdot\|_1$ as $\|\cdot\|$.

Also, the Banach space $(c_0, \|\cdot\|_\infty)$ is given by

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0 \right\} ;$$

where $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Of course, $\ell^1$ is the dual of $c_0$.

The subspace $(c_{00}, \|\cdot\|_\infty)$ of $(c_0, \|\cdot\|_\infty)$ is defined as

$$c_{00} := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } x_n = 0 \text{ for all but finitely many } n \in \mathbb{N} \right\} .$$

**Definition 1.1.1.** Let $C$ be a nonempty closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : C \longrightarrow C$ be a mapping.

(1) We say that $T$ is *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\| \ , \text{ for all } x, y \in C \ .$$

(2) We say that $T$ is *asymptotically nonexpansive* if there exists a sequence $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{R}$ decreasing to 1, such that for all $m \in \mathbb{N}$,

$$\|T^m(x) - T^m(y)\| \leq \lambda_m \|x - y\| \ , \text{ for all } x, y \in C \ .$$

(3) We say that $T$ is *uniformly Lipschitzian* if there exists $M \in [1, \infty)$, such that for all $m \in \mathbb{N}$,

$$\|T^m(x) - T^m(y)\| \leq M \|x - y\| \ , \text{ for all } x, y \in C \ .$$

We call $M$ a *uniform Lipschitz constant* for $T$.

(4) We define $T$ to be *affine* if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \ ,$$

for all $x_1, x_2 \in C$ with $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

(5) We say that $T$ has an *approximate fixed point sequence* if there exists $(x_n)_{n\in\mathbb{N}}$ in $C$ such that $\|T(x_n) - x_n\| \underset{n}{\longrightarrow} 0$.

Clearly, $[(1) \implies (2) \implies (3)]$. Of course, the converses are not generally true. Some of our examples below illustrate this. It is also well known that $[(1) \implies (5)]$. (See, for example, [8].) We note that it is an open question as to whether $[(2) \implies (5)]$. It is also well known that $[(4) \implies (5)]$. We include a proof, for the sake of completeness.

**Lemma 1.1.2.** *Let $(X, \|\cdot\|)$ be a Banach space and $M \subseteq X$ be a nonempty, closed, bounded, and convex set. Let $T : M \longrightarrow M$ be an affine mapping. Then there exists an approximate fixed point sequence $(x_n)_{n\in\mathbb{N}}$ for $T$ in $M$.*

*Proof.* Fix $x_0 \in M$. Define

$$x_n := \left(\frac{I + T + T^2 + \cdots + T^n}{n+1}\right)(x_0) \quad , \text{ for all } n \in \mathbb{N} .$$

Each $x_n$ is in $M$, because $M$ is convex. Let

$$d := \operatorname{diam}(M) := \sup_{u,v\in M} \|u - v\| \in [0, \infty) .$$

Since $T$ is affine, we have that

$$\begin{aligned}
\|Tx_n - x_n\| &= \left\| T\left(\frac{I + T + T^2 + \cdots + T^n}{n+1}\right)(x_0) - \left(\frac{I + T + T^2 + \cdots + T^n}{n+1}\right)(x_0) \right\| \\
&= \frac{1}{n+1} \left\| (T + T^2 + T^3 + \cdots T^{n+1})(x_0) - (I + T + T^2 + \cdots + T^n)(x_0) \right\| \\
&= \frac{1}{n+1} \left\| T^{n+1}x_0 - x_0 \right\| \\
&\leq \frac{d}{n+1} \longrightarrow 0
\end{aligned}$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence for $T$. □

We will sometimes use "c.b.c." as an abbreviation for the phrase "closed, bounded and convex". Also for a given collection of mappings $\mathcal{E}$ on a c.b.c. set $C$ in a Banach space $(X, \|\cdot\|)$, we say that that $C$ has the fixed point property with respect to $\mathcal{E}$ (FPP($\mathcal{E}$)) if for all $T \in \mathcal{E}$, $T$ has a fixed point in $C$.

**Definition 1.1.3.** We say that a Banach space $(X, \|\cdot\|)$ is *asymptotically isometric* to $(\ell^1, \|\cdot\|_1)$ if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ such that
$X = \overline{\text{linear span}}\{x_n : n \in \mathbb{N}\}$ and there exists a sequence of scalars $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0, 1)$ such that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$ and $\forall \alpha \in c_{00}$,

$$\sum_{n=1}^{\infty}(1 - \varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| \leq \sum_{n=1}^{\infty}(1 + \varepsilon_n)|a_n|$$

Note that

(1) When each $\varepsilon_n = 0$, $(X, \|\cdot\|)$ is an isometric copy of $(\ell^1, \|\cdot\|_1)$.

(2) $1 - \varepsilon_n \le \|x_n\| \le 1 + \varepsilon_n, \forall n \in \mathbb{N}$.

(3) We may replace "$\forall \alpha \in c_{00}$" by "$\forall \alpha \in \ell^1$."

(4) Without loss of generality, each $\|x_n\| = 1$ and we replace the right hand inequality in the definition by

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| \le \sum_{n=1}^{\infty} |a_n|.$$

## 2.0 RESULTS FOR NONEXPANSIVE MAPPINGS

In 1979, Goebel and Kuczumow [9] proved the following theorem.

**Theorem 2.0.4.** *Let $(b_j)_{j\in\mathbb{N}}$ be a bounded sequence of positive real numbers with $\Gamma := \inf_{j\in\mathbb{N}} b_j > 0$ and put $f_j = b_j e_j$. Let $C$ be the non-weak-star compact, closed, bounded, and convex set defined by*

$$C = \left\{ x = \sum_{j=1}^{\infty} t_j f_j : \textit{each } t_j \geq 0 \textit{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}$$

*Let $N_0 := \{j \in \mathbb{N} : b_j = \Gamma\}$. Then $C$ has the FPP(n.e.) if and only if $N_0$ is nonempty and finite.*

In Chapters 3 and 4, we aim to prove similar theorems for more general sets that can be considered to be perturbed Goebel and Kuczumow sets. Two of our theorems, Theorem 2.3.15 and Theorem 2.3.18, will include Theorem 2.0.4 as a special case. We begin by exploring some basic examples (Examples 2.1.1, 2.1.2, 2.1.3, and 2.1.4). All examples except for Example 2.1.2 are new results. Example 2.1.2 is a special case of Theorem 2.0.4 (with $b_1 = b$, $b_2 = b$, and $b_n = 1, \forall n \geq 3$) that we include for the purpose of comparison with our three new examples. Example 2.1.4 includes Example 2.1.1, Example 2.1.2, and Example 2.1.3 as special cases.

## 2.1 EXAMPLES

**Example 2.1.1.** To begin, let $b$ and $c$ be real numbers such that $0 \leq c < b < 1$ and $b+c < 1$. Define $f_1 := be_1 + ce_2$ and $f_n := e_n$, $\forall n \geq 2$, where $\{e_j : j \in \mathbb{N}\}$ is the usual basis for $\ell^1$.

Next, we define the following closed, bounded, and convex subset of $\ell^1$.

$$K_{b,c} := \left\{ x = \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}$$

We will show that

**Theorem** The set $K_{b,c}$ has the FPP(n.e.)

*Proof.* Note that for $\alpha = (\alpha_j)_{j \in \mathbb{N}} \in \ell^1$ and $x = \sum_{j=1}^{\infty} \alpha_j f_j$,

$$\begin{aligned}
\|x\|_1 &= \|\alpha_1(be_1 + ce_2) + \alpha_2 e_2 + \alpha_3 e_3 + \cdots\|_1 \\
&= \|\alpha_1 b e_1 + (\alpha_1 c + \alpha_2) e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \cdots\|_1 \\
&= |\alpha_1| b + |\alpha_1 c + \alpha_2| + |\alpha_3| + |\alpha_4| + \cdots \\
&\leq |\alpha_1|(b + c) + \sum_{j=2}^{\infty} |\alpha_j| \\
&\leq (1 \vee (b + c)) \sum_{j=1}^{\infty} |\alpha_j|.
\end{aligned}$$

Also,

$$\begin{aligned}
\|x\|_1 &\geq |\alpha_1| b + (|\alpha_2| - |\alpha_1| c) + |\alpha_3| + |\alpha_4| + \cdots \\
&= |\alpha_1|(b - c) + |\alpha_2| + |\alpha_3| + |\alpha_4| + \cdots \\
&\geq (1 \wedge (b - c)) \sum_{j=1}^{\infty} |\alpha_j|.
\end{aligned}$$

Hence,

$$(b-c)\sum_{j=1}^{\infty}|\alpha_j| \le \|x\|_1 \le \sum_{j=1}^{\infty}|\alpha_j|$$

and so $(f_j)_{j\in\mathbb{N}}$ is an $\ell^1$-basic sequence for $\ell^1$. Note that, from above, $(f_j)_{j\in\mathbb{N}}$ is also an asymptotically isometric $\ell^1$-basic sequence in $\ell^1$.

Note that $K_{b,c} = \overline{co}\{f_j : j \in \mathbb{N}\}$.

Let $T : K_{b,c} \longrightarrow K_{b,c}$ be nonexpansive. Then there exists $(x^{(n)})_{n\in\mathbb{N}} \subseteq K_{b,c}$ such that

$$\|x^{(n)} - Tx^{(n)}\|_1 \underset{n}{\longrightarrow} 0.$$

Without loss of generality, passing to a subsequence if necessary, there exists $z \in \ell^1$ such that $x^{(n)} \underset{n}{\longrightarrow} z$ weak-star. Hence we have that $z \in W_{b,c}$, where

$$W_{b,c} := \overline{K_{b,c}}^{w^*} = \left\{ \sum_{j=1}^{\infty}\gamma_j f_j : \text{each } \gamma_j \ge 0 \text{ and } \sum_{j=1}^{\infty}\gamma_j \le 1 \right\}.$$

We now show that $T$ must have a fixed point in $K_{b,c}$.

<u>Case 1:</u> $z \in K_{b,c}$.

Define

$$r(y) := \limsup_{n\longrightarrow\infty} \|x^{(n)} - y\|_1, \forall y \in \ell^1.$$

In [9], Goebel and Kuczumow show that $\forall y \in \ell^1$,

$$r(y) = r(z) + \|z - y\|_1. \quad (\bigstar)$$

Then, by $(\bigstar)$ we have that $r(Tz) = r(z) + \|z - Tz\|_1$. However,

$$
\begin{aligned}
r(Tz) &= \limsup_{n \longrightarrow \infty} \|Tz - x^{(n)}\|_1 \\
&\leq \limsup_{n \longrightarrow \infty} \|Tz - Tx^{(n)}\|_1 + \limsup_{n \longrightarrow \infty} \|Tx^{(n)} - x^{(n)}\|_1 \\
&\leq r(z) + 0 \\
&= r(z).
\end{aligned}
$$

Therefore $\|z - Tz\| \leq 0$ and so $Tz = z$.

<u>Case 2:</u> $z \in W_{b,c} \backslash K_{b,c}$.

Then $z$ is of the form $z = \sum_{j=1}^{\infty} \gamma_j f_j$, such that $\sum_{j=1}^{\infty} \gamma_j < 1$. Define $\delta := 1 - \sum_{j=1}^{\infty} \gamma_j$.

Next we define $h_\lambda := (\gamma_1 + \lambda\delta) f_1 + (\gamma_2 + (1-\lambda)\delta) f_2 + \sum_{j=1}^{\infty} \gamma_j f_j$. We wish for $h_\lambda$ to be in $K_{b,c}$, so we restrict values of $\lambda$ to be in $[-\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta} + 1]$.

Note that, for $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
\|h_\lambda - z\|_1 &= \|\lambda b \delta e_1 + [\lambda\delta c + (1-\lambda)\delta] e_2\|_1 \\
&= |\lambda| b\delta + |\lambda\delta c + (1-\lambda)\delta| \\
&= |\lambda| b\delta + \delta|\lambda c + (1-\lambda)| \\
&= \begin{cases} \delta(-(b-c)-1)\lambda + \delta, & \text{if } \lambda < 0; \\ \delta((b+c)-1)\lambda + \delta & \text{if } 0 \leq \lambda < \frac{1}{1-c}; \\ \delta((b-c)+1)\lambda - \delta & \text{if } \frac{1}{1-c} \leq \lambda. \end{cases}
\end{aligned}
$$

Since $0 \leq c < b < 1$ and $b + c < 1$, $\|h_\lambda - z\|_1$ is minimized when $\lambda = \frac{1}{1-c}$. However, we must consider two cases; when $\frac{1}{1-c} < \frac{\gamma_2}{\delta} + 1$ and when $\frac{1}{1-c} \geq \frac{\gamma_2}{\delta} + 1$.

<u>Sub-Case $(a)$:</u> $\frac{1}{1-c} < \frac{\gamma_2}{\delta} + 1$.

$\|h_\lambda - z\|_1$ is minimized when $\lambda = \lambda_0 = \frac{1}{1-c}$. Note that

$$h_{\lambda_0} = \left(\gamma_1 + \frac{\delta}{1-c}\right) f_1 + \left(\gamma_2 - \frac{c\delta}{1-c}\right) f_2 + \sum_{j=3}^{\infty} \gamma_j f_j,$$

and $\|h_{\lambda_0} - z\|_1 = \delta \frac{b}{1-c}$.

Next, fix $y \in K_{b,c}$ of the form $y = \sum_{j=1}^{\infty} t_j f_j$. Observe that

$$\begin{aligned}
\|y - z\|_1 &= |t_1 - \gamma_1| b + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)| + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= |t_1 - \gamma_1| b + \frac{b}{1-c} |(t_1 - \gamma_1)c + (t_2 - \gamma_2)| + \frac{b}{1-c} \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&+ \left(1 - \frac{b}{1-c}\right) |(t_1 - \gamma_1)c + (t_2 - \gamma_2)| + \left(1 - \frac{b}{1-c}\right) \sum_{j=3}^{\infty} |t_j - \gamma_j|.
\end{aligned}$$

Let

$$Q := \left(1 - \frac{b}{1-c}\right) |(t_1 - \gamma_1)c + (t_2 - \gamma_2)| + \left(1 - \frac{b}{1-c}\right) \sum_{j=3}^{\infty} |t_j - \gamma_j|.$$

Then,

$$\begin{aligned}
\|y - z\|_1 &= |t_1 - \gamma_1| b + \frac{b}{1-c} |(t_1 - \gamma_1)c + (t_2 - \gamma_2)| + \frac{b}{1-c} \sum_{j=3}^{\infty} |t_j - \gamma_j| + Q \\
&\geq \left| (t_1 - \gamma_1) b + \frac{b}{1-c} \left[ (t_1 - \gamma_1)c + (t_2 - \gamma_2) \right] + \frac{b}{1-c} \sum_{j=3}^{\infty} (t_j - \gamma_j) \right| + Q \\
&= \left| \frac{b}{1-c}(t_1 - \gamma_1) + \frac{b}{1-c}(t_2 - \gamma_2) + \frac{b}{1-c} \sum_{j=3}^{\infty} (t_j - \gamma_j) \right| + Q \\
&= \frac{b}{1-c} \left| \sum_{j=1}^{\infty} t_j - \sum_{j=1}^{\infty} \gamma_j \right| + Q \\
&= \frac{b}{1-c} \left| 1 - (1 - \delta) \right| + Q \\
&= \delta \frac{b}{1-c} + Q \\
&\geq \delta \frac{b}{1-c}
\end{aligned}$$

with equality in the last inequality if and only if $Q = 0$.

Note that in the case of $Q = 0$ we must have that both $t_j = \gamma_j, \forall j \geq 3$ and $|(t_1 - \gamma_1)c + (t_2 - \gamma_2)| = 0$. However,

$$|(t_1 - \gamma_1)c + (t_2 - \gamma_2)| = 0 \Longrightarrow (t_1 - \gamma_1)c + (t_2 - \gamma_2) = 0.$$

Now, since $t_j = \gamma_j, \forall j \geq 3$, $\sum_{j=1}^{\infty} t_j = 1$, and $\sum_{j=1}^{\infty} \gamma_j = 1 - \delta$, some simple calculations show that

$$t_1 + t_2 = \gamma_1 + \gamma_2 + \delta \tag{2.1}$$

$$ct_1 + t_2 = c\gamma_1 + \gamma_2 \tag{2.2}$$

Solving these equations gives

$$t_1 = \gamma_1 + \frac{\delta}{1 - c} \tag{2.3}$$

$$t_2 = \gamma_2 - \frac{c\delta}{1 - c} \tag{2.4}$$

Hence $Q = 0$ if and only if $y = h_{\lambda_0}$.

Therefore,

$$\|h_{\lambda_0} - z\|_1 = \min_{y \in K_{b,c}} \|y - z\|_1$$

and this minimizer $h_{\lambda_0}$ is unique.

Let $h = h_{\lambda_0}$. From above, $r(Th) = r(z) + \|z - Th\|_1$. Also,

$$
\begin{aligned}
r(Th) &= \limsup_{n\longrightarrow\infty} \|x^{(n)} - Th\|_1 \\
&\leq \limsup_{n\longrightarrow\infty} \left[\|x^{(n)} - Tx^{(n)}\|_1 + \|Tx^{(n)} - Th\|_1\right] \\
&\leq 0 + \limsup_{n\longrightarrow\infty} \|Tx^{(n)} - Th\|_1 \\
&\leq \limsup_{n\longrightarrow\infty} \|x^{(n)} - h\|_1 \\
&= r(h) \\
&= r(z) + \|z - h\|_1.
\end{aligned}
$$

This implies that $\|z - Th\|_1 \leq \|z - h\|_1$.

Hence, since $Th \in K_{b,c}$,

$$
\frac{\delta b}{1-c} \leq \|z - Th\|_1 \leq \|z - h\|_1 = \frac{\delta b}{1-c}.
$$

Therefore, since the minimizer is unique, $Th = h$.

<u>Sub-Case $(b)$:</u> $\frac{1}{1-c} \geq \frac{\gamma_2}{\delta} + 1$.

Let $\lambda_0 := \frac{\gamma_2}{\delta} + 1$. Since $0 \leq c < b < 1$, $b + c < 1$, and

$$
\|h_\lambda - z\|_1 = \begin{cases} \delta(-(b-c)-1)\lambda + \delta, & \text{if } \lambda < 0; \\ \delta((b+c)-1)\lambda + \delta & \text{if } 0 \leq \lambda < \frac{1}{1-c}; \\ \delta((b-c)+1)\lambda - \delta & \text{if } \frac{1}{1-c} \leq \lambda, \end{cases}
$$

we have that

$$
\min_{\lambda \in \left[-\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta}+1\right]} \|h_\lambda - z\|_1 = \|h_{\lambda_0} - z\|_1 = \delta\left[(b+c) - (1-(b+c))\frac{\gamma_2}{\delta}\right]
$$

Define $\Gamma := \delta\left[(b+c) - (1-(b+c))\frac{\gamma_2}{\delta}\right]$. Note that

$$\frac{\delta b}{1-c} < \Gamma \leq \delta(b+c)$$

and since $\delta > 0$,

$$0 < \frac{b}{1-c} < \frac{\Gamma}{\delta} \leq b + c < 1.$$

Next, fix a general $y \in K_{b,c}$ of the form $y = \sum_{j=1}^{\infty} t_j f_j$, where each $t_j \geq 0$ and $\sum_{j=1}^{\infty} t_j = 1$. Then

$$\|y - z\|_1 = |t_1 - \gamma_1|b + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)| + \sum_{j=3}^{\infty} |t_j - \gamma_j|$$

<u>Sub-Case $(b)(i)$:</u> Both $t_1 \geq \gamma_1$ and $t_2 \geq \gamma_2$.

Then we have

$$\begin{aligned}
\|y - z\|_1 &= (t_1 - \gamma_1)b + (t_1 - \gamma_1)c + (t_2 - \gamma_2) + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= (t_1 - \gamma_1)(b + c) + (t_2 - \gamma_2) + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= (t_1 - \gamma_1)(b + c)\left[\frac{\Gamma}{\delta(b+c)}\right] + (t_1 - \gamma_1)(b + c)\left[1 - \frac{\Gamma}{\delta(b+c)}\right] \\
&+ (t_2 - \gamma_2)\left[\frac{\Gamma}{\delta}\right] + (t_2 - \gamma_2)\left[1 - \frac{\Gamma}{\delta}\right] \\
&+ \left[\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty} |t_j - \gamma_j| + \left[1 - \frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= \left|(t_1 - \gamma_1)(b + c)\left[\frac{\Gamma}{\delta(b+c)}\right]\right| + \left|(t_2 - \gamma_2)\left[\frac{\Gamma}{\delta}\right]\right| + \frac{\Gamma}{\delta}\sum_{j=3}^{\infty} |t_j - \gamma_j| + Q
\end{aligned}$$

where

$$Q = (t_1 - \gamma_1)(b + c)\left[1 - \frac{\Gamma}{\delta(b+c)}\right] + (t_2 - \gamma_2)\left[1 - \frac{\Gamma}{\delta}\right] + \left[1 - \frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty} |t_j - \gamma_j| \geq 0.$$

Then

$$\begin{aligned}\|y-z\|_1 &\geq \left|(t_1-\gamma_1)(b+c)\left[\frac{\Gamma}{\delta(b+c)}\right]+(t_2-\gamma_2)\frac{\Gamma}{\delta}+\sum_{j=3}^{\infty}\frac{\Gamma}{\delta}(t_j-\gamma_j)\right|+Q\\ &=\frac{\Gamma}{\delta}\left|\sum_{j=1}^{\infty}(t_j-\gamma_j)\right|+Q\\ &=\Gamma+Q\end{aligned}$$

Hence $\|y-z\|_1 \geq \Gamma$ with equality if and only if $Q=0$. However,

$$Q=0 \Longleftrightarrow t_j=\gamma_j, \forall j \in \mathbb{N}$$

and $\sum_{j=1}^{\infty} t_j = 1$, whereas $\sum_{j=1}^{\infty} \gamma_j < 1$. Therefore in the case that both $t_1 \geq \gamma_1$ and $t_2 \geq \gamma_2$, we have that $\|y-z\|_1 > \Gamma$.

Sub-Case $(b)(ii)$: $t_1 < \gamma_1$.

Then

$$\begin{aligned}\|y-z\|_1 &= (\gamma_1-t_1)b+|(t_1-\gamma_1)c+(t_2-\gamma_2)|+\sum_{j=3}^{\infty}|t_j-\gamma_j|\\ &\geq (\gamma_1-t_1)b+(t_1-\gamma_1)c+(t_2-\gamma_2)+\sum_{j=3}^{\infty}(t_j-\gamma_j)\\ &=(\gamma_1-t_1)(b-c)+\sum_{j=2}^{\infty}(t_j-\gamma_j)\\ &=(\gamma_1-t_1)(b-c)+(1-t_1)+(\delta-1+\gamma_1)\\ &=(\gamma_1-t_1)(1+(b-c))+\delta\\ &>\delta\\ &>\Gamma.\end{aligned}$$

Hence, in the case where $t_1 < \gamma_1$, $\|y-z\|_1 > \Gamma$.

Sub-Case $(b)(iii)$: $t_1 \geq \gamma_1$, $t_2 < \gamma_2$, and $t_1-\gamma_1 \leq \delta+\gamma_2$.

In this case,

$$
\begin{aligned}
\|y-z\|_1 &= (t_1-\gamma_1)b + |(t_1-\gamma_1)c + (t_2-\gamma_2)| + \sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&\geq (t_1-\gamma_1)(b+c) + (t_2-\gamma_2) + \sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= (t_1-\gamma_1)(b+c)\left[\frac{\Gamma}{\delta(b+c)}\right] + (t_1-\gamma_1)(b+c)\left[1-\frac{\Gamma}{\delta(b+c)}\right] \\
&+ (t_2-\gamma_2)\left[\frac{\Gamma}{\delta}\right] + (t_2-\gamma_2)\left[1-\frac{\Gamma}{\delta}\right] \\
&+ \left[\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| + \left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= (t_1-\gamma_1)\left[\frac{\Gamma}{\delta}\right] + (t_2-\gamma_2)\left[\frac{\Gamma}{\delta}\right] + \left[\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&+ (t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}\right] + (t_2-\gamma_2)\left[1-\frac{\Gamma}{\delta}\right] + \left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&\geq (t_1-\gamma_1)\left[\frac{\Gamma}{\delta}\right] + (t_2-\gamma_2)\left[\frac{\Gamma}{\delta}\right] + \left[\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}(t_j-\gamma_j) + Q
\end{aligned}
$$

where

$$
Q = (t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}\right] + (t_2-\gamma_2)\left[1-\frac{\Gamma}{\delta}\right] + \left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j|.
$$

Hence,

$$
\begin{aligned}
\|y-z\|_1 &\geq \left[\frac{\Gamma}{\delta}\right]\sum_{j=1}^{\infty}(t_j-\gamma_j) + Q \\
&= \Gamma + Q
\end{aligned}
$$

Hence $\|y-z\|_1 \geq \Gamma$ with equality if and only if $Q = 0$.

Note that

$$
\begin{aligned}
Q &\geq (t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}\right]+\left[1-\frac{\Gamma}{\delta}\right](t_2-\gamma_2)+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}(t_j-\gamma_j)\\
&=(t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}\right]+\left[1-\frac{\Gamma}{\delta}\right](t_2-\gamma_2)\\
&+\left[1-\frac{\Gamma}{\delta}\right][(1-t_1-t_2)-(1-\delta-\gamma_1-\gamma_2)]\\
&=(t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}-1+\frac{\Gamma}{\delta}\right]+(t_2-\gamma_2)\left[1-\frac{\Gamma}{\delta}-1+\frac{\Gamma}{\delta}\right]+\delta\left[1-\frac{\Gamma}{\delta}\right]\\
&=(t_1-\gamma_1)\left[(b+c)-1\right]+\delta-\Gamma.
\end{aligned}
$$

Let $Q' := (t_1-\gamma_1)\left[(b+c)-1\right]+\delta-\Gamma$. Then

$$
\begin{aligned}
Q'\geq 0 &\iff (t_1-\gamma_1)[(b+c)-1]\geq \Gamma-\delta\\
&\iff (t_1-\gamma_1)\leq \frac{\delta-\Gamma}{1-(b+c)}
\end{aligned}
$$

Also,

$$
\frac{\delta-\Gamma}{1-(b+c)}=\delta+\gamma_2,
$$

and so,

$$
Q'\geq 0 \iff t_1-\gamma_1\leq \delta+\gamma_2.
$$

Therefore $Q=0 \Longrightarrow Q'=0 \Longrightarrow t_1=\gamma_1+\gamma_2+\delta$. In this case,

$$
\begin{aligned}
Q &\geq (\gamma_2+\delta)\left[b+c-\frac{\Gamma}{\delta}\right]+\left[1-\frac{\Gamma}{\delta}\right](-\gamma_2)+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= \gamma_2[b+c-1]+\delta(b+c)-\Gamma+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= \Gamma-\Gamma+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= \left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&\geq 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q=0 &\Longrightarrow \left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j|=0 \\
&\Longrightarrow t_j=\gamma_j, \forall j\geq 3.
\end{aligned}
$$

Then,

$$
\begin{aligned}
t_2 &= 1-t_1-\sum_{j=3}^{\infty}t_j \\
&= 1-(\gamma_1+\gamma_2+\delta)-\sum_{j=3}^{\infty}\gamma_j \\
&= 1-\delta-\sum_{j=1}^{\infty}\gamma_j \\
&= 1-\delta-(1-\delta) \\
&= 0
\end{aligned}
$$

Therefore, in summary, we have that $\|y-z\|_1 \geq \Gamma$ with equality when $y=(\gamma_1+\gamma_2+\delta)f_1+0\cdot f_2+\sum_{j=3}^{\infty}\gamma_j f_j$, i.e. when $y=h_{\lambda_0}$.

Sub-Case $(b)(iv)$: $t_1\geq\gamma_1$, $t_2<\gamma_2$, and $t_1-\gamma_1>\delta+\gamma_2$.

Observe here that

$$\begin{aligned}\|y-z\|_1 &\geq (t_1-\gamma_1)(b+c)+t_2-\gamma_2+\sum_{j=3}^{\infty}|t_j-\gamma_j| \\ &> (\delta+\gamma_2)(b+c)-\gamma_2+\sum_{j=3}^{\infty}|t_j-\gamma_j| \\ &= \delta(b+c)-(1-(b+c))\gamma_2+\sum_{j=3}^{\infty}|t_j-\gamma_j| \\ &= \Gamma+\sum_{j=3}^{\infty}|t_j-\gamma_j| \\ &\geq \Gamma.\end{aligned}$$

Therefore, due to the strict inequality in this calculation, $\|y-z\|_1 > \Gamma$.

Now, having covered all of the cases in $2(b)$, we see that $h = h_{\lambda_0}$ is the unique minimizer of $\{\|y-z\|_1 : y \in K_{b,c}\}$. Therefore, just as in case $2(a)$, we see that $Th = h$.

Hence, $K_{b,c}$ has the Fixed Point Property for nonexpansive maps, as desired. □

**Example 2.1.2.** Let $0 < b < 1$ and define $f_1 := be_1$, $f_2 := be_2$, and $f_n := e_n$, $\forall n \geq 3$.

Next, define the closed, bounded, and convex subset of $\ell^1$,

$$K_b := \left\{x = \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1\right\}.$$

We will show that

**Theorem** [Goebel and Kuczumow] The set $K_b$ has the FPP(n.e.)

*Proof.* Note that for $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in \ell^1$ and $x = \sum_{j=1}^{\infty} \alpha_j f_j$,

$$\begin{aligned}\|x\|_1 &= \|\alpha_1 be_1 + \alpha_2 be_2 + \alpha_3 e_3 + \cdots\|_1 \\ &= |\alpha_1|b + |\alpha_2|b + |\alpha_3| + |\alpha_4| + \cdots \\ &\leq \sum_{j=1}^{\infty} |\alpha_j|.\end{aligned}$$

Also,

$$\begin{aligned}\|x\|_1 &\geq |\alpha_1|b + |\alpha_2|b + |\alpha_3|b + |\alpha_4|b + \cdots \\ &= b\sum_{j=1}^{\infty} |\alpha_j|.\end{aligned}$$

In summary,

$$b\sum_{j=1}^{\infty} |\alpha_j| \leq \|x\|_1 \leq \sum_{j=1}^{\infty} |\alpha_j|.$$

Note that $(f_j)_{j\in\mathbb{N}}$ is an isomorphic $\ell^1$-basic sequence for $\ell^1$, and $K_b = \overline{co}\{f_j : j \in \mathbb{N}\}$. Note that, from above, $(f_j)_{j\in\mathbb{N}}$ is also an asymptotically isomorphic $\ell^1$-basic sequence in $\ell^1$.

Let $T : K_b \longrightarrow K_b$ be nonexpansive. Then there exists $(x^{(n)})_{n\in\mathbb{N}} \subseteq K_b$ such that

$$\|x^{(n)} - Tx^{(n)}\|_1 \underset{n}{\longrightarrow} 0.$$

Without loss of generality, passing to a subsequence if necessary, there exists $z \in \ell^1$ such that $x^{(n)} \underset{n}{\longrightarrow} z$ weak-star. Hence we have that $z \in W_b$, where

$$W_b := \overline{K_b}^{w^*} = \left\{ \sum_{j=1}^{\infty} \gamma_j f_j : \text{each } \gamma_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \gamma_j \leq 1 \right\}.$$

We now show that $T$ must have a fixed point in $K_b$.

<u>Case 1:</u> $z \in K_b$.

Note that the proof of Case 1 in Example 2.1.1 also demonstrates that $T$ has a fixed point in this case as well.

Case 2: $z \in W_b \backslash K_b$.

In this case, $z = \sum_{j=1}^{\infty} \gamma_j f_j$ such that each $\gamma_j \geq 0$ and $\sum_{j=1}^{\infty} \gamma_j < 1$. Let $\delta := 1 - \sum_{j=1}^{\infty} \gamma_j \in (0,1]$.

Note that for an arbitrary $y \in K_b$ of the form $\sum_{j=1}^{\infty} t_j f_j$,

$$\begin{aligned}
\|y - z\|_1 &= \left\| (t_1 - \gamma_1) f_1 + (t_2 - \gamma_2) f_2 + \sum_{j=3}^{\infty} (t_j - \gamma_j) f_j \right\|_1 \\
&= \left\| (t_1 - \gamma_1) b e_1 + (t_2 - \gamma_2) b e_2 + \sum_{j=3}^{\infty} (t_j - \gamma_j) e_j \right\|_1 \\
&= b|t_1 - \gamma_1| + b|t_2 - \gamma_2| + \sum_{j=3}^{\infty} |t_j - \gamma_j|.
\end{aligned}$$

For $\lambda \in \left[ -\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta} + 1 \right]$, define

$$h_\lambda := (\gamma_1 + \lambda\delta) f_1 + (\gamma_2 + (1-\lambda)\delta) f_2 + \sum_{j=3}^{\infty} \gamma_j f_j.$$

Then,

$$\begin{aligned}
\|h_\lambda - z\|_1 &= \|\lambda \delta f_1 + (1-\lambda)\delta f_2\|_1 \\
&= |\lambda| b\delta + |1-\lambda| b\delta \\
&= \begin{cases} -2b\delta\lambda + b\delta & \text{if } \lambda \in \left[ -\frac{\gamma_1}{\delta}, 0 \right); \\ b\delta & \text{if } \lambda \in \left[ 0, 1 \right]; \\ 2b\delta\lambda - b\delta & \text{if } \lambda \in \left( 1, \frac{\gamma_2}{\delta} + 1 \right]. \end{cases}
\end{aligned}$$

Hence, $\|h_\lambda - z\|_1$ is minimized for $\lambda \in [0,1]$, in which case $\|h_\lambda - z\|_1 = b\delta$.

Note that

$$
\begin{aligned}
\|y-z\|_1 &= b|t_1-\gamma_1| + b|t_2-\gamma_2| + \sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= b|t_1-\gamma_1| + b|t_2-\gamma_2| + b\sum_{j=3}^{\infty}|t_j-\gamma_j| + (1-b)\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&\geq b\left|\sum_{j=1}^{\infty}(t_j-\gamma_j)\right| + (1-b)\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= b\delta + (1-b)\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&\geq b\delta
\end{aligned}
$$

with equality in the last inequality if and only if $(1-b)\sum_{j=3}^{\infty}|t_j-\gamma_j| = 0$. Since $b \in (0,1)$, this occurs if and only if $t_j = \gamma_j, \forall j \geq 3$. Therefore, $\|y-z\|_1 = b\delta$ if and only if $y = h_\lambda$ for some $\lambda \in [0,1]$.

Let $\Lambda := \{h_\lambda : \lambda \in [0,1]\} \subseteq K_b$. Note that $\Lambda$ is compact as it is the continuous image of a compact set. It is also easy to check that $\Lambda$ is convex.

Note that for $h \in \Lambda$,

$$
\begin{aligned}
r(Th) &= \limsup_{n\longrightarrow\infty} \|x^{(n)} - Th\|_1 \\
&= \limsup_{n\longrightarrow\infty} \|x^{(n)} - Tx^{(n)} + Tx^{(n)} - Th\|_1 \\
&\leq \limsup_{n\longrightarrow\infty} \|x^{(n)} - Tx^{(n)}\|_1 + \limsup_{n\longrightarrow\infty} \|Tx^{(n)} - Th\|_1 \\
&\leq \limsup_{n\longrightarrow\infty} \|x^{(n)} - Tx^{(n)}\|_1 + \limsup_{n\longrightarrow\infty} \|x^{(n)} - h\|_1 \\
&= 0 + \limsup_{n\longrightarrow\infty} \|x^{(n)} - h\|_1 \\
&= r(h).
\end{aligned}
$$

Also, $r(Th) = z + \|z - Th\|_1$ and $r(h) = z + \|z-h\|_1$. Hence,

$$\begin{aligned}\|z - Th\|_1 \le \|z-h\|_1 &\implies \|z - Th\|_1 = \|z-h\|_1\\ &\implies Th \in \Lambda.\end{aligned}$$

Therefore, $T(\Lambda) \subseteq \Lambda$, and since $T$ is continuous, Brouwer's Fixed Point Theorem tells us that $T$ has a fixed point.

Hence, $K_b$ has the Fixed Point Property for nonexpansive maps, as desired. □

**Example 2.1.3.** Let $0 \le c < b < 1$ with $b + c < 1$. Let $f_1 := be_1 + ce_2$, $f_2 := be_2$, and $f_n := e_n$, $\forall j \ge 3$.

Let

$$K_{b,c} := \left\{ \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \ge 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}.$$

and

$$W_{b,c} := \overline{K_{b,c}}^{w^*} = \left\{ \sum_{j=1}^{\infty} \gamma_j f_j : \text{each } \gamma_j \ge 0 \text{ and } \sum_{j=1}^{\infty} \gamma_j \le 1 \right\}.$$

Note that for $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in \ell^1$ and $x = \sum_{j=1}^{\infty} \alpha_j f_j$,

$$\begin{aligned}\|x\|_1 &= \|\alpha_1(be_1 + ce_2) + \alpha_2 be_2 + \alpha_3 e_3 + \cdots\|_1\\ &= |\alpha_1|b + |\alpha_2 b + \alpha_1 c| + |\alpha_3| + |\alpha_4| + \cdots\\ &\le |\alpha_1|(b+c) + |\alpha_2|b + |\alpha_3| + |\alpha_4| + \cdots\\ &\le \sum_{j=1}^{\infty} |\alpha_j|.\end{aligned}$$

Also,

$$\begin{aligned}\|x\|_1 &\ge |\alpha_1|(b-c) + |\alpha_2|b + |\alpha_3| + |\alpha_4| + \cdots\\ &\ge (b-c)\sum_{j=1}^{\infty} |\alpha_j|.\end{aligned}$$

In summary,

$$(b-c)\sum_{j=1}^{\infty}|\alpha_j| \leq \|x\|_1 \leq \sum_{j=1}^{\infty}|\alpha_j|.$$

Note that $(f_j)_{j\in\mathbb{N}}$ is an isomorphic $\ell^1$-basic sequence for $\ell^1$, and $K_b = \overline{co}\{f_j : j \in \mathbb{N}\}$. Note that, from above, $(f_j)_{j\in\mathbb{N}}$ is also an asymptotically isomorphic $\ell^1$-basic sequence in $\ell^1$.

We will show that

**Theorem** The set $K_{b,c}$ has the FPP(n.e.)

*Proof.* Let $T : K_{b,c} \longrightarrow K_{b,c}$ be nonexpansive. Then there exists $(x^{(n)})_{n\in\mathbb{N}} \subseteq K_{b,c}$ such that

$$\|x^{(n)} - Tx^{(n)}\|_1 \underset{n}{\longrightarrow} 0.$$

Without loss of generality, passing to a subsequence if necessary, there exists $z \in W_{b,c}$ such that $x^{(n)} \underset{n}{\longrightarrow} z$ weak-star. We will now show that $T$ has a fixed point in $K_{b,c}$.

<u>Case 1:</u> $z \in K_{b,c}$.

This case proceeds in exactly the same manner as in Examples 2.1.1 and 2.1.2.

<u>Case 2:</u> $z \in W_{b,c}\backslash K_{b,c}$.

Then $z$ has the form $z = \sum_{j=1}^{\infty}\gamma_j f_j$ such that each $\gamma_j \geq 0$ and $\sum_{j=1}^{\infty}\gamma_j < 1$. Let $\delta := 1 - \sum_{j=1}^{\infty}\gamma_j$.

For $\lambda \in \left[-\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta}+1\right]$, define

$$h_\lambda := (\gamma_1 + \lambda\delta)f_1 + (\gamma_2 + (1-\lambda)\delta)f_2 + \sum_{j=3}^{\infty}\gamma_j f_j.$$

Then,

$$
\begin{aligned}
\|h_\lambda - z\|_1 &= \|\lambda\delta f_1 + (1-\lambda)\delta f_2\|_1 \\
&= \|\lambda\delta(be_1 + ce_2) + (1-\lambda)b\delta e_2\|_1 \\
&= \|\lambda b\delta e_1 + (\lambda\delta c + (1-\lambda)b\delta))e_2\|_1 \\
&= \delta\Big[|\lambda|b + |\lambda c + (1-\lambda)b|\Big] \\
&= \begin{cases} -\delta(2b-c)\lambda + b\delta & \text{if } \lambda \in \Big[-\frac{\gamma_1}{\delta}, 0\Big); \\ \delta c\lambda + b\delta & \text{if } \lambda \in \Big[0, \frac{b}{b-c}\Big]; \\ \delta(2b-c)\lambda - b\delta & \text{if } \lambda \in \Big(\frac{b}{b-c}, \frac{\gamma_2}{\delta} + 1\Big]. \end{cases}
\end{aligned}
$$

Hence $\|h_\lambda - z\|_1$ is minimized when $\lambda = 0$, in which case $\|h_\lambda - z\|_1 = b\delta$.

Note that for an arbitrary $y \in K_b$ of the form $\sum_{j=1}^{\infty} t_j f_j$,

$$
\begin{aligned}
\|y - z\|_1 &= |t_1 - \gamma_1|b + |(t_2 - \gamma_2)b + (t_1 - \gamma_1)c| + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= \left(1 - \frac{c}{b}\right)|t_1 - \gamma_1|b + \left(\frac{c}{b}\right)|t_1 - \gamma_1|b + |(t_2 - \gamma_2)b + (t_1 - \gamma_1)c| \\
&+ b\sum_{j=3}^{\infty} |t_j - \gamma_j| + (1-b)\sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&\geq \left|(b-c)(t_1 - \gamma_1) + (t_2 - \gamma_2)b + (t_1 - \gamma_1)c + b\sum_{j=3}^{\infty} |t_j - \gamma_j|\right| \\
&+ \left(\frac{c}{b}\right)|t_1 - \gamma_1|b + (1-b)\sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= \left|b\sum_{j=1}^{\infty}(t_j - \gamma_j)\right| + \left(|t_1 - \gamma_1|c + (1-b)\sum_{j=3}^{\infty} |t_j - \gamma_j|\right)
\end{aligned}
$$

Let $Q := |t_1 - \gamma_1|c + (1-b)\sum_{j=3}^{\infty} |t_j - \gamma_j|$. Then

$$
\|y - z\|_1 \geq b\delta + Q \geq b\delta
$$

with equality in the last inequality if and only if $Q = 0$. Let us examine this case.

Note that if $c = 0$, then our current example is the same as Example 2.1.2, so let us assume that $c > 0$. Then

$$Q = 0 \Longrightarrow t_1 = \gamma_1 \quad \text{and} \quad t_j = \gamma_j, \forall j \geq 3.$$

Hence $\|y - z\|_1 \geq b\delta$ with equality if and only if $y = \gamma_1 f_1 + (\gamma_2 + \delta) f_2 + \sum_{j=3}^{\infty} \gamma_j f_j$, i.e. if and only if $y = h_0$.

Therefore, in this case there is a unique minimizer, and so just as in case 2 in Example 2.1.1, $T$ has a fixed point.

Hence, $K_{b,c}$ has the Fixed Point Property for nonexpansive maps, as desired. □

**Example 2.1.4.** To begin, let $b$, $b_2$, and $c$ be real numbers such that $0 \leq c < b \leq b_2 \leq 1$ and $b + c < 1$. Define $f_1 := be_1 + ce_2$, $f_2 := b_2 e_2$, and $f_n := e_n$, $\forall n \geq 3$, where $\{e_j : j \in \mathbb{N}\}$ is the usual basis for $\ell^1$.

Next, we define the following closed, bounded, and convex subset of $\ell^1$.

$$K_{b,c} := \left\{ x = \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}$$

We will show that

**Theorem** The set $K_{b,c}$ has the FPP(n.e.)

*Proof.* Note that for $\alpha = (\alpha_j)_{j \in \mathbb{N}} \in \ell^1$ and $x = \sum_{j=1}^{\infty} \alpha_j f_j$,

$$
\begin{aligned}
\|x\|_1 &= \|\alpha_1(be_1 + ce_2) + \alpha_2 b_2 e_2 + \alpha_3 e_3 + \cdots\|_1 \\
&= \|\alpha_1 b e_1 + (\alpha_1 c + \alpha_2 b_2) e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \cdots\|_1 \\
&= |\alpha_1| b + |\alpha_1 c + \alpha_2 b_2| + |\alpha_3| + |\alpha_4| + \cdots \\
&\le |\alpha_1|(b + c) + |\alpha_2| b_2 \sum_{j=3}^{\infty} |\alpha_j| \\
&\le \sum_{j=1}^{\infty} |\alpha_j|.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|x\|_1 &\ge |\alpha_1| b + (|\alpha_2| b_2 - |\alpha_1| c) + |\alpha_3| + |\alpha_4| + \cdots \\
&= |\alpha_1|(b - c) + |\alpha_2| b_2 + |\alpha_3| + |\alpha_4| + \cdots \\
&\ge (b - c) \sum_{j=1}^{\infty} |\alpha_j|.
\end{aligned}
$$

Hence,

$$
(b - c) \sum_{j=1}^{\infty} |\alpha_j| \le \|x\|_1 \le \sum_{j=1}^{\infty} |\alpha_j|
$$

and so $(f_j)_{j \in \mathbb{N}}$ is an $\ell^1$-basic sequence for $\ell^1$. Note that, from above, $(f_j)_{j \in \mathbb{N}}$ is also an asymptotically isometric $\ell^1$-basic sequence in $\ell^1$.

Note that $K_{b,c} = \overline{co}\{f_j : j \in \mathbb{N}\}$. We will show that $K_{b,c}$ has the fixed point property for nonexpansive maps.

Let $T : K_{b,c} \longrightarrow K_{b,c}$ be nonexpansive. Then there exists $(x^{(n)})_{n \in \mathbb{N}} \subseteq K_{b,c}$ such that

$$
\|x^{(n)} - Tx^{(n)}\|_1 \underset{n}{\longrightarrow} 0.
$$

Without loss of generality, passing to a subsequence if necessary, there exists $z \in \ell^1$ such that $x^{(n)} \underset{n}{\longrightarrow} z$ weak-star. Hence we have that $z \in W_{b,c}$, where

$$W_{b,c} := \overline{K_{b,c}}^{w^*} = \left\{ \sum_{j=1}^{\infty} \gamma_j f_j : \text{each } \gamma_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \gamma_j \leq 1 \right\}.$$

We now show that $T$ must have a fixed point in $K_{b,c}$.

Case 1: $z \in K_{b,c}$.

This case proceeds in exactly the same manner as in the previous examples.

Case 2: $z \in W_{b,c} \backslash K_{b,c}$.

Then $z$ is of the form $z = \sum_{j=1}^{\infty} \gamma_j f_j$, such that $\sum_{j=1}^{\infty} \gamma_j < 1$. Define $\delta := 1 - \sum_{j=1}^{\infty} \gamma_j$.

Next we define $h_\lambda := (\gamma_1 + \lambda\delta) f_1 + (\gamma_2 + (1-\lambda)\delta) f_2 + \sum_{j=1}^{\infty} \gamma_j f_j$. We want $h_\lambda$ to be in $K_{b,c}$, so we restrict values of $\lambda$ to be in $\left[ -\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta} + 1 \right]$.

Note that, for $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\|h_\lambda - z\|_1 &= \delta \|\lambda b e_1 + [\lambda c + (1-\lambda) b_2] e_2\|_1 \\
&= \delta \left[ |\lambda| b + |\lambda c + (1-\lambda) b_2| \right] \\
&= \begin{cases} \delta[-(b_2 + b - c)\lambda + b_2], & \text{if } \lambda < 0; \\ \delta[(b + c - b_2)\lambda + b_2], & \text{if } 0 \leq \lambda < \frac{b_2}{b_2 - c}; \\ \delta[(b_2 + b - c)\lambda - b_2], & \text{if } \frac{b_2}{b_2 - c} \leq \lambda. \end{cases}
\end{aligned}$$

Sub-Case $(a)$: $b_2 \leq b + c$.

Then,

$$\min_{\lambda \in \left[ -\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta} + 1 \right]} \|h_\lambda - z\|_1 = \delta b_2,$$

which occurs when $\lambda = 0$ if $b_2 < b + c$, and for all $\lambda \in \left[0, \min\left\{\frac{b_2}{b_2-c}, \frac{\gamma_2}{\delta} + 1\right\}\right]$ if $b_2 = b + c$.

Note that, for $y \in K_{b,c}$ of the form $y = \sum_{j=1}^{\infty} t_j f_j$,

$$\begin{aligned}
\|y - z\|_1 &= |t_1 - \gamma_1|b + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= \left(\frac{b_2 - c}{b}\right)|t_1 - \gamma_1|b + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + b_2 \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&+ \left(1 - \frac{b_2 - c}{b}\right)|t_1 - \gamma_1|b + (1 - b_2) \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= (b_2 - c)|t_1 - \gamma_1| + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + b_2 \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&+ (b + c - b_2)|t_1 - \gamma_1| + (1 - b_2) \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&\geq b_2 \left| \sum_{j=1}^{\infty} (t_j - \gamma_j) \right| + Q \\
&= \delta b_2 + Q \\
&\geq \delta b_2
\end{aligned}$$

where $Q := (b + c - b_2)|t_1 - \gamma_1| + (1 - b_2) \sum_{j=3}^{\infty} |t_j - \gamma_j| \geq 0$ since $0 \leq \frac{b_2 - c}{b} \leq 1$.

Note that we have equality in the last inequality if and only if $Q = 0$. In this case, we have to consider two possibilities.

Sub-Case $(a)(i)$: $b_2 < b + c$ and $b_2 < 1$.

Then $Q = 0$ implies that $t_1 = \gamma_1$ and $t_j = \gamma_j$ for all $j \geq 3$. Hence

$$\begin{aligned}
t_2 &= 1 - \sum_{j=3}^{\infty} t_j - t_1 \\
&= 1 - \sum_{j=3}^{\infty} \gamma_j - \gamma_1 \\
&= \left(1 - \sum_{j=1}^{\infty} \gamma_j\right) + \gamma_2 \\
&= \delta + \gamma_2
\end{aligned}$$

Hence $\|y - z\|_1 = \delta b_2$ if and only if $y = h_0 = \gamma_1 f_1 + (\gamma_2 + \delta) f_2 + \sum_{j=3}^{\infty} \gamma_j f_j$.

Let $h = h_0$. From above, $r(Th) = r(z) + \|z - Th\|_1$. Also,

$$\begin{aligned}
r(Th) &= \limsup_{n \longrightarrow \infty} \|x^{(n)} - Th\|_1 \\
&\leq \limsup_{n \longrightarrow \infty} \left[\|x^{(n)} - Tx^{(n)}\|_1 + \|Tx^{(n)} - Th\|_1\right] \\
&\leq 0 + \limsup_{n \longrightarrow \infty} \|Tx^{(n)} - Th\|_1 \\
&\leq \limsup_{n \longrightarrow \infty} \|x^{(n)} - h\|_1 \\
&= r(h) \\
&= r(z) + \|z - h\|_1.
\end{aligned}$$

This implies that $\|z - Th\|_1 \leq \|z - h\|_1$.

Hence, since $Th \in K_{b,c}$,

$$\delta b_2 \leq \|z - Th\|_1 \leq \|z - h\|_1 = \delta b_2,$$

and so

$$\|z - Th\|_1 = \|z - h\|_1 = \delta b_2.$$

Therefore, since the minimizer is unique, $Th = h$.

Sub-Case $(a)(ii)$: $b_2 = b + c$ and $b_2 < 1$.

In this case $Q = 0$ implies that $t_j = \gamma_j$ for all $j \geq 3$. Then

$$\begin{aligned} t_1 + t_2 &= 1 - \sum_{j=3}^{\infty} t_j \\ &= 1 - \sum_{j=3}^{\infty} \gamma_j \\ &= \delta + \gamma_1 + \gamma_2. \end{aligned}$$

Hence $\|y - z\|_1 = \delta b_2$ if and only if $y = h_\lambda$ for $\lambda \in \left[0, \min\left\{\frac{b_2}{b_2 - c}, \frac{\gamma_2}{\delta} + 1\right\}\right]$.

Let $\Lambda := \left\{h_\lambda : \lambda \in \left[0, \min\left\{\frac{b_2}{b_2 - c}, \frac{\gamma_2}{\delta} + 1\right\}\right]\right\} \subseteq K_b$. Note that $\Lambda$ is compact as it is the continuous image of a compact set. It is also easy to check that $\Lambda$ is convex.

Note that for $h \in \Lambda$,

$$\begin{aligned} r(Th) &= \limsup_{n \longrightarrow \infty} \|x^{(n)} - Th\|_1 \\ &= \limsup_{n \longrightarrow \infty} \|x^{(n)} - Tx^{(n)} + Tx^{(n)} - Th\|_1 \\ &\leq \limsup_{n \longrightarrow \infty} \|x^{(n)} - Tx^{(n)}\|_1 + \limsup_{n \longrightarrow \infty} \|Tx^{(n)} - Th\|_1 \\ &\leq \limsup_{n \longrightarrow \infty} \|x^{(n)} - Tx^{(n)}\|_1 + \limsup_{n \longrightarrow \infty} \|x^{(n)} - h\|_1 \\ &= 0 + \limsup_{n \longrightarrow \infty} \|x^{(n)} - h\|_1 \\ &= r(h). \end{aligned}$$

Also, $r(Th) = z + \|z - Th\|_1$ and $r(h) = z + \|z - h\|_1$. Hence,

$$\begin{aligned} \|z - Th\|_1 \leq \|z - h\|_1 &\Longrightarrow \|z - Th\|_1 = \|z - h\|_1 \\ &\Longrightarrow Th \in \Lambda. \end{aligned}$$

Therefore, $T(\Lambda) \subseteq \Lambda$, and since $T$ is continuous, Brouwer's Fixed Point Theorem tells us that $T$ has a fixed point.

Sub-Case $(b)$: $b + c < b_2$.

Then,

$$\min_{\lambda \in \left[-\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta}+1\right]} \|h_\lambda - z\|_1 = \begin{cases} \frac{b_2 b}{b_2 - c}\delta, & \text{if } \frac{b_2}{b_2-c} \leq \frac{\gamma_2}{\delta} + 1; \\ (b + c - b_2)(\gamma_2 + \delta) + b_2\delta, & \text{if } \frac{\gamma_2}{\delta} + 1 < \frac{b_2}{b_2-c}. \end{cases}$$

Sub-Case $(b)(i)$: $\frac{b_2}{b_2-c} \leq \frac{\gamma_2}{\delta} + 1$.

Note that, for $y \in K_{b,c}$ of the form $y = \sum_{j=1}^{\infty} t_j f_j$,

$$\begin{aligned}
\|y - z\|_1 &= |t_1 - \gamma_1|b + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= |t_1 - \gamma_1|b + \left(\frac{b}{b_2 - c}\right)|(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \left(\frac{b_2 b}{b_2 - c}\right)\sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&+ \left(1 - \frac{b}{b_2 - c}\right)|(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \left(1 - \frac{b_2 b}{b_2 - c}\right)\sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&\geq \frac{b_2 b}{b_2 - c}\left|\sum_{j=1}^{\infty}(t_j - \gamma_j)\right| \\
&+ \left(1 - \frac{b}{b_2 - c}\right)|(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \left(1 - \frac{b_2 b}{b_2 - c}\right)\sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&= \frac{b_2 b}{b_2 - c}\delta + Q \\
&\geq \frac{b_2 b}{b_2 - c}\delta
\end{aligned}$$

where $Q := \left(1 - \frac{b}{b_2-c}\right)|(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \left(1 - \frac{b_2 b}{b_2-c}\right)\sum_{j=3}^{\infty} |t_j - \gamma_j| \geq 0$.

Note that we have equality in the last inequality if and only if $Q = 0$. Since both $\frac{b}{b_2-c} \in (0,1)$ and $\frac{b_2 b}{b_2-c} \in (0,1)$, we must have

$$(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2 = 0 \tag{2.5}$$

$$t_j = \gamma_j \quad \text{for all } j \geq 3 \tag{2.6}$$

Since $t_j = \gamma_j$, $\forall j \geq 3$, $\sum_{j=1}^{\infty} t_j = 1$, and $\sum_{j=1}^{\infty} \gamma_j = 1 - \delta$ this last system is equivalent to

$$t_1 c + t_2 b_2 = \gamma_1 c + \gamma_2 b_2 \tag{2.7}$$

$$t_1 + t_2 = \gamma_1 + \gamma_2 + \delta \tag{2.8}$$

Solving these equations gives

$$t_1 = \gamma_1 + \frac{\delta b_2}{b_2 - c} \tag{2.9}$$

$$t_2 = \gamma_2 - \frac{\delta c}{b_2 - c} \tag{2.10}$$

Hence, $\|y - z\|_1 = \frac{b_2 b}{b_2 - c}\delta$ if and only if $y = h_\lambda$ for $\lambda = \frac{b_2}{b_2 - c}$.

Therefore, since the minimizer is unique, just as in Sub-Case $(A)(i)$, $Th_\lambda = h_\lambda$.

Sub-Case $(b)(ii)$: $\frac{\gamma_2}{\delta} + 1 < \frac{b_2}{b_2 - c}$.

For simplicity, let $\Gamma := (b + c - b_2)(\gamma_2 + \delta) + b_2\delta$, i.e.

$$\Gamma = \min_{\lambda \in \left[-\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta} + 1\right]} \|h_\lambda - z\|_1.$$

Also, define $\lambda_0 = \frac{\gamma_2}{\delta} + 1$.

Note that $0 < \frac{\Gamma}{\delta} < 1$, $0 < \frac{\Gamma}{\delta(b+c)} < 1$, and $0 < \frac{\Gamma}{\delta b_2} < 1$.

Fix a general $y \in K_{b,c}$ of the form $y = \sum_{j=1}^{\infty} t_j f_j$, where each $t_j \geq 0$ and $\sum_{j=1}^{\infty} t_j = 1$. Then

$$\|y - z\|_1 = |t_1 - \gamma_1| b + |(t_1 - \gamma_1)c + (t_2 - \gamma_2)b_2| + \sum_{j=3}^{\infty} |t_j - \gamma_j|$$

Sub-Case $(b)(ii)(\alpha)$: Both $t_1 \geq \gamma_1$ and $t_2 \geq \gamma_2$.

Then we have

$$\begin{aligned}
\|y-z\|_1 &\geq (b+c)(t_1-\gamma_1) + (t_2-\gamma_2)b_2 + \sum_{j=3}^{\infty} |t_j-\gamma_j| \\
&= \left(\frac{\Gamma}{\delta(b+c)}\right)(b+c)(t_1-\gamma_1) + \left(\frac{\Gamma}{\delta b_2}\right)(t_2-\gamma_2)b_2 + \left(\frac{\Gamma}{\delta}\right)\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&+ \left(1-\frac{\Gamma}{\delta(b+c)}\right)(b+c)(t_1-\gamma_1) + \left(1-\frac{\Gamma}{\delta b_2}\right)(t_2-\gamma_2)b_2 \\
&+ \left(1-\frac{\Gamma}{\delta}\right)\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&\geq \frac{\Gamma}{\delta}\sum_{j=1}^{\infty}(t_j-\gamma_j) + \left(1-\frac{\Gamma}{\delta(b+c)}\right)(b+c)(t_1-\gamma_1) \\
&+ \left(1-\frac{\Gamma}{\delta b_2}\right)(t_2-\gamma_2)b_2 + \left(1-\frac{\Gamma}{\delta}\right)\sum_{j=3}^{\infty}|t_j-\gamma_j| \\
&= \Gamma + Q \\
&\geq \Gamma
\end{aligned}$$

where

$$Q = \left(1-\frac{\Gamma}{\delta(b+c)}\right)(b+c)(t_1-\gamma_1) + \left(1-\frac{\Gamma}{\delta b_2}\right)(t_2-\gamma_2)b_2 + \left(1-\frac{\Gamma}{\delta}\right)\sum_{j=3}^{\infty}|t_j-\gamma_j|.$$

Hence $\|y-z\|_1 \geq \Gamma$ with equality if and only if $Q=0$. However,

$$Q = 0 \Longleftrightarrow t_j = \gamma_j, \forall j \in \mathbb{N}$$

and $\sum_{j=1}^{\infty} t_j = 1$, whereas $\sum_{j=1}^{\infty}\gamma_j < 1$. Therefore in the case that both $t_1 \geq \gamma_1$ and $t_2 \geq \gamma_2$, we have that $\|y-z\|_1 > \Gamma$.

Sub-Case $(b)(ii)(\beta)$: $t_1 < \gamma_1$.

In this case

$$
\begin{aligned}
\|y-z\|_1 &= (\gamma_1 - t_1)b + |(t_1-\gamma_1)c + (t_2-\gamma_2)b_2| + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\
&\geq (\gamma_1 - t_1)b + (t_1-\gamma_1)c + (t_2-\gamma_2)b_2 + b_2\sum_{j=3}^{\infty}(t_j-\gamma_j) \\
&= (\gamma_1 - t_1)(b-c) + b_2\sum_{j=2}^{\infty}(t_j-\gamma_j) \\
&= (\gamma_1 - t_1)(b-c) + b_2(1-t_1) + b_2(\delta - 1 + \gamma_1) \\
&= (\gamma_1 - t_1)(b-c+b_2) + b_2\delta \\
&> b_2\delta \\
&> \Gamma.
\end{aligned}
$$

Hence, in the case where $t_1 < \gamma_1$, $\|y-z\|_1 > \Gamma$.

Sub-Case $(b)(ii)(\gamma)$: $t_1 \geq \gamma_1$, $t_2 < \gamma_2$, and $t_1 - \gamma_1 \leq \delta + \gamma_2$.

Then as in Sub-Case $(b)(ii)(\alpha)$,

$$
\begin{aligned}
\|y-z\|_1 &\geq \Gamma + Q \\
&\geq \Gamma
\end{aligned}
$$

where

$$
Q = \left(1 - \frac{\Gamma}{\delta(b+c)}\right)(b+c)(t_1-\gamma_1) + \left(1 - \frac{\Gamma}{\delta b_2}\right)(t_2-\gamma_2)b_2 + \left(1 - \frac{\Gamma}{\delta}\right)\sum_{j=3}^{\infty}|t_j-\gamma_j|.
$$

Hence $\|y-z\|_1 \geq \Gamma$ with equality if and only if $Q = 0$.

Note that

$$
\begin{aligned}
Q &\geq (t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}\right]+(t_2-\gamma_2)\left[b_2-\frac{\Gamma}{\delta}\right]+\left[b_2-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}(t_j-\gamma_j)\\
&=(t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}\right]+\left[b_2-\frac{\Gamma}{\delta}\right](t_2-\gamma_2)\\
&+\left[b_2-\frac{\Gamma}{\delta}\right][(1-t_1-t_2)-(1-\delta-\gamma_1-\gamma_2)]\\
&=(t_1-\gamma_1)\left[(b+c)-\frac{\Gamma}{\delta}-b_2+\frac{\Gamma}{\delta}\right]+(t_2-\gamma_2)\left[b_2-\frac{\Gamma}{\delta}-b_2+\frac{\Gamma}{\delta}\right]+\delta\left[b_2-\frac{\Gamma}{\delta}\right]\\
&=(t_1-\gamma_1)\left[(b+c)-b_2\right]+\delta b_2-\Gamma.
\end{aligned}
$$

Let $Q' := (t_1-\gamma_1)\left[(b+c)-b_2\right]+\delta b_2-\Gamma$. Then

$$
\begin{aligned}
Q' \geq 0 &\iff (t_1-\gamma_1)[(b+c)-b_2] \geq \Gamma-\delta b_2\\
&\iff (t_1-\gamma_1) \leq \frac{\delta b_2-\Gamma}{b_2-(b+c)}=\gamma_2+\delta
\end{aligned}
$$

Therefore $Q=0 \implies Q'=0 \implies t_1=\gamma_1+\gamma_2+\delta$. In this case,

$$
\begin{aligned}
0=Q &\geq (\gamma_2+\delta)\left[b+c-\frac{\Gamma}{\delta}\right]+\left[b_2-\frac{\Gamma}{\delta}\right](-\gamma_2)+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j|\\
&=\gamma_2[b+c-b_2]+\delta(b+c)-\Gamma+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j|\\
&=\Gamma-\Gamma+\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j|\\
&=\left[1-\frac{\Gamma}{\delta}\right]\sum_{j=3}^{\infty}|t_j-\gamma_j|\\
&\geq 0.
\end{aligned}
$$

Hence,

$$Q = 0 \Longrightarrow \left[1 - \frac{\Gamma}{\delta}\right] \sum_{j=3}^{\infty} |t_j - \gamma_j| = 0$$

$$\Longrightarrow t_j = \gamma_j, \forall j \geq 3.$$

Then,

$$\begin{aligned} t_2 &= 1 - t_1 - \sum_{j=3}^{\infty} t_j \\ &= 1 - (\gamma_1 + \gamma_2 + \delta) - \sum_{j=3}^{\infty} \gamma_j \\ &= 1 - \delta - \sum_{j=1}^{\infty} \gamma_j \\ &= 1 - \delta - (1 - \delta) \\ &= 0 \end{aligned}$$

Therefore, in summary, we have that $\|y - z\|_1 \geq \Gamma$ with equality when $y = (\gamma_1 + \gamma_2 + \delta) f_1 + 0 \cdot f_2 + \sum_{j=3}^{\infty} \gamma_j f_j$, i.e. when $y = h_{\lambda_0}$.

Sub-Case $(b)(ii)(\delta)$: $t_1 \geq \gamma_1$, $t_2 < \gamma_2$, and $t_1 - \gamma_1 > \delta + \gamma_2$.

Observe here that

$$\begin{aligned} \|y - z\|_1 &\geq (t_1 - \gamma_1)(b + c) + (t_2 - \gamma_2) b_2 + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\ &> (\gamma_2 + \delta)(b + c) - \gamma_2 b_2 + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\ &= \Gamma + \sum_{j=3}^{\infty} |t_j - \gamma_j| \\ &\geq \Gamma. \end{aligned}$$

Therefore, due to the strict inequality in this calculation, in this case, $\|y - z\|_1 > \Gamma$.

Now, having covered all of the sub-cases, we see that $h = h_{\lambda_0}$ is the unique minimizer of $\{\|y - z\|_1 : y \in K_{b,c}\}$. Therefore, as in Sub-Case $(a)(i)$, we see that $Th = h$.

Hence, $K_{b,c}$ has the FPP(n.e.), as desired. $\square$

## 2.2 FIRST THEOREM

We now prove a theorem that generalizes the perturbation idea in Example 2.1.1 to an arbitrary finite number of coordinates. We begin with the necessary definitions.

Fix $n \in \mathbb{N}$ and let $b, c \in \mathbb{R}$ such that $0 \leq c < b < 1$ and $b + c < 1$. Define $f_k := be_k + ce_{k+1}, \forall k \in \{1, \ldots, n\}$ and $f_k := e_k, \forall k > n$, where (as usual) $\{e_j : j \in \mathbb{N}\}$ is the standard basis for $\ell^1$.

Next, we define the following closed, bounded, and convex subset of $\ell^1$.

$$K_{b,c} := \left\{ x = \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}$$

Note that for $\alpha = (\alpha_j)_{j \in \mathbb{N}} \in \ell^1$ and $x = \sum_{j=1}^{\infty} \alpha_j f_j$,

$$\begin{aligned}
\|x\|_1 &= \|\alpha_1(be_1 + ce_2) + \cdots + \alpha_{n-1}(be_{n-1} + ce_n) + \alpha_n(be_n + ce_{n+1}) \\
&+ \alpha_{n+1}e_{n+1} + a_{n+2}e_{n+2} + \cdots\|_1 \\
&= \|\alpha_1 be_1 + (\alpha_1 c + \alpha_2 b)e_2 + \cdots + (\alpha_{n-1}c + \alpha_n b)e_n \\
&+ (\alpha_n c + \alpha_{n+1})e_{n+1} + \alpha_{n+2}e_{n+2} + \cdots\|_1 \\
&= |\alpha_1|b + |\alpha_1 c + \alpha_2 b| + \cdots + |\alpha_{n-1}c + \alpha_n b| + |\alpha_n c + \alpha_{n+1}| + \sum_{j=n+2}^{\infty} |\alpha_j| \\
&\leq |\alpha_1|(b+c) + |\alpha_2|(b+c) + \cdots + |\alpha_{n-1}(b+c) + |\alpha_n|(b+c) + \sum_{j=n+1}^{\infty} |\alpha_j| \\
&\leq \sum_{j=1}^{\infty} |\alpha_j|
\end{aligned}$$

Also,

$$\begin{aligned}
\|x\|_1 &\geq |\alpha_1|(b-c) + |\alpha_2|(b-c) + \cdots |\alpha_n|(b-c) + \sum_{j=n+1}^{\infty} |\alpha_j| \\
&\geq (b-c) \sum_{j=1}^{\infty} |\alpha_j|.
\end{aligned}$$

Hence,

$$(b-c) \sum_{j=1}^{\infty} |\alpha_j| \leq \|x\|_1 \leq \sum_{j=1}^{\infty} |\alpha_j|$$

and so $(f_j)_{j\in\mathbb{N}}$ is an isomorphic $\ell^1$-basic sequence for $\ell^1$. Indeed, from above, $(f_j)_{j\in\mathbb{N}}$ is also an asymptotically isometric $\ell^1$-basic sequence in $\ell^1$.

Note that $K_{b,c} = \overline{co}\{f_j : j \in \mathbb{N}\}$.

**Theorem 2.2.1.** *The set $K_{b,c}$ has the FPP(n.e.).*

*Proof.* Let $T : K_{b,c} \longrightarrow K_{b,c}$ be nonexpansive. Then there exists $(x^{(n)})_{n\in\mathbb{N}} \subseteq K_{b,c}$ such that

$$\|x^{(n)} - Tx^{(n)}\|_1 \underset{n}{\longrightarrow} 0.$$

Without loss of generality, passing to a subsequence if necessary, there exists $z \in \ell^1$ such that $x^{(n)} \underset{n}{\longrightarrow} z$ weak-star. Hence we have that $z \in W_{b,c}$, where

$$W_{b,c} := \overline{K_{b,c}}^{w^*} = \left\{ \sum_{j=1}^{\infty} \gamma_j f_j : \text{each } \gamma_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \gamma_j \leq 1 \right\}.$$

<u>Case 1:</u> $z \in K_{b,c}$.

Note that the proof of Case 1 in Example 2.1.1 works here without any changes to show that $Tz = z$.

<u>Case 2:</u> $z \in W_{b,c} \backslash K_{b,c}$.

Then $z = \sum_{j=1}^{\infty} \gamma_j f_j$ such that each $\gamma_j \geq 0$ and $\sum_{j=1}^{\infty} \gamma_j < 1$. Let $\delta = 1 - \sum_{j=1}^{\infty} \gamma_j \in (0, 1]$.

<u>Sub-Case 2(a):</u> $\frac{1}{1-c} \leq \frac{\gamma_{n+1}}{\delta} + 1$.

**Lemma 2.2.2.** ($\heartsuit$) *Let* $\Gamma := \frac{\delta b}{1-c}$. *For all* $y = \sum_{j=1}^{\infty} t_j f_j \in K_{b,c}$,

$$\|y - z\|_1 \geq \Gamma + (1 - (b + c)) \sum_{j=n+2}^{\infty} |t_j - \gamma_j|.$$

*Proof.*

$$\|y - z\|_1 = U + (1 - (b + c)) \sum_{j=n+2}^{\infty} |t_j - \gamma_j|$$

where

$$U = |t_1 - \gamma_1|b + \sum_{j=2}^{n} |(t_{j-1} - \gamma_{j-1})c + (t_j - \gamma_j)b|$$
$$+ |(t_n - \gamma_n)c + (t_{n+1} - \gamma_{n+1})| + (b+c)\sum_{j=n+2}^{\infty} |t_j - \gamma_j|.$$

We must show that $U \geq \Gamma$.

Let $q_1, q_2, \ldots, q_n$ be defined as follows.

$$q_n = 1 \text{ and } q_{j-1} = \frac{b - q_j c + q_j c^2}{b(1-c)}, \forall j \in \{2, \ldots, n\}.$$

**Claim 2.2.3.** ($\bigstar_1$) *For* $j \in \{1, \ldots, n\}$, $q_j \in \left[\frac{b}{1-c}, 1\right]$.

*Proof of Claim* ($\bigstar_1$). By induction (counting down), when $j = n$,

$$q_n = 1 \in \left[\frac{b}{1-c}, 1\right].$$

When $j = n-1$,

$$q_{n-1} = \frac{b - c + c^2}{b(1-c)}$$

Hence,

$$\begin{aligned} q_{n-1} \leq 1 &\iff b - c + c^2 \leq b(1-c) \\ &\iff b \leq b(1-c) + c(1-c) = (b+c)(1-c) \\ &\iff \frac{b}{1-c} \leq b + c \end{aligned}$$

Note that this last inequality is indeed true. Also,

$$\begin{aligned} q_{n-1} \geq \frac{b}{1-c} &\iff b - c + c^2 \geq b^2 \\ &\iff b - c \geq b^2 - c^2 = (b+c)(b-c) \\ &\iff 1 \geq b + c \end{aligned}$$

This last inequality is also true. Hence, $q_{n-1} \in \left[\frac{b}{1-c}, 1\right]$.

Now assume that $q_j \in \left[\frac{b}{1-c}, 1\right]$ for $j \in \{\nu, \ldots, n\}$ ($\nu \geq 2$). To finish the proof of the claim, we show that this implies that $q_{\nu-1} \in \left[\frac{b}{1-c}, 1\right]$.

Note that

$$q_{\nu-1} = \frac{b - q_\nu c + q_\nu c^2}{b(1-c)}.$$

Then,

$$\begin{aligned} q_{\nu-1} \leq 1 &\iff b - q_\nu c + q_\nu c^2 \leq b - bc \\ &\iff bc \leq q_\nu c(1-c) \\ &\iff \frac{b}{1-c} \leq q_\nu \end{aligned}$$

and this last inequality is true by our inductive hypothesis.

Also,

$$\begin{aligned} q_{\nu-1} \geq \frac{b}{1-c} &\iff b - q_\nu c + q_\nu c^2 \geq b^2 \\ &\iff b - q_\nu c \geq b^2 - q_\nu c^2 \\ &\iff b - q_\nu c \geq (b - q_\nu c)c - bc + b^2 \\ &\iff (b - q_\nu c)(1-c) \geq b(b-c) \\ &\iff \frac{b - q_\nu c}{b-c} \geq \frac{b}{1-c} \end{aligned}$$

However, since $q_\nu \leq 1$ by assumption,

$$\frac{b}{1-c} \leq 1 \leq 1 + \frac{c(1-q_\nu)}{b-c} = \frac{b - q_\nu c}{b-c}.$$

Hence, $q_{\nu-1} \in \left[\frac{b}{1-c}, 1\right]$. □

**Claim 2.2.4.** ($\bigstar_2$) *For $j \in \{1, \ldots, n-1\}$, $bq_j + cq_{j+1} = \frac{b}{1-c}$.*

*Proof of Claim* ($\bigstar_2$). Fix $j \in \{1, \dots, n-1\}$.

$$\begin{aligned}
bq_j + cq_{j+1} &= b\frac{b - q_{j+1}c + q_{j+1}c^2}{b(1-c)} + cq_{j+1} \\
&= \frac{b - q_{j+1}c + q_{j+1}c^2}{1-c} + \frac{(1-c)cq_{j+1}}{1-c} \\
&= \frac{b - q_{j+1}c + q_{j+1}c^2 + q_{j+1}c - q_{j+1}c^2}{1-c} \\
&= \frac{b}{1-c}
\end{aligned}$$

□

Then, using these claims, we have that

$$\begin{aligned}
U &\geq |t_1 - \gamma_1| bq_1 + \sum_{j=2}^{n} |(t_{j-1} - \gamma_{j-1})c + (t_j - \gamma_j)b| q_j \\
&\quad + |(t_n - \gamma_n)c + (t_{n+1} - \gamma_{n+1})| \left(\frac{b}{1-c}\right) + \left(\frac{b}{1-c}\right) \sum_{j=n+2}^{\infty} |t_j - \gamma_j| \\
&\geq \sum_{j=1}^{n-1} (t_j - \gamma_j)(bq_j + cq_{j+1}) + (t_n - \gamma_n)\left(bq_n + c\frac{b}{1-c}\right) \\
&\quad + \left(\frac{b}{1-c}\right) \sum_{j=n+1}^{\infty} (t_j - \gamma_j) \\
&= \frac{b}{1-c} \sum_{j=1}^{\infty} (t_j - \gamma_j) \\
&= \frac{b\delta}{1-c} \\
&= \Gamma.
\end{aligned}$$

□

Note also that

$$
\begin{aligned}
h = \gamma_1 f_1 + \gamma_2 f_2 + \cdots + \gamma_{n-1} f_{n-1} \\
+ \left(\gamma_n + \frac{\delta}{1-c}\right) f_n + \left(\gamma_{n+1} - \frac{c\delta}{1-c}\right) f_{n+1} + \sum_{j=n+2}^{\infty} \gamma_j f_j \in K_{b,c}
\end{aligned}
$$

and

$$\|h - z\|_1 = \Gamma.$$

Let $Q := \left\{ y \in K_{b,c} : \|y - z\|_1 \leq \Gamma \right\}$. Note that $h \in Q$ and so $Q \neq \phi$. Note also that $Q$ is closed, bounded, and convex, as well as $T$-invariant. By Lemma (♡), for all $y \in Q$,

$$\Gamma + (1 - (b + c)) \sum_{j=n+2}^{\infty} |t_j - \gamma_j| \leq \|y - z\|_1 \leq \Gamma,$$

which implies that $t_j = \gamma_j$, $\forall j \geq n + 2$. Hence,

$$Q \subseteq \left\{ y = \sum_{j=1}^{\infty} t_j f_j \in K_{b,c} : t_j = \gamma_j, \forall j \geq n + 2 \right\}$$

which is a norm compact set. Therefore $Q$ is norm compact, and so by Brouwer's Fixed Point Theorem, $T$ has a fixed point in $Q \subseteq K_{b,c}$.

Sub-Case 2(b): $\frac{\gamma_{n+1}}{\delta} + 1 < \frac{1}{1-c}$.

**Lemma 2.2.5.** (◇) *Let* $\tilde{\Gamma} := (b + c)\delta - (1 - (b + c))\gamma_{n+1}$. *For all* $y = \sum_{j=1}^{\infty} t_j f_j \in K_{b,c}$,

$$\|y - z\|_1 \geq \tilde{\Gamma} + (1 - (b + c)) \sum_{j=n+2}^{\infty} |t_j - \gamma_j|.$$

*Proof.*

$$\begin{aligned}
\|y-z\|_1 &= |t_1-\gamma_1|b + \left(\sum_{j=2}^{n}\Big|(t_{j-1}-\gamma_{j-1})c + (t_j-\gamma_j)b\Big|\right) \\
&\quad + \Big|(t_n-\gamma_n)c + (t_{n+1}-\gamma_{n+1})\Big| + \left(\sum_{j=n+2}^{\infty}\Big|t_j-\gamma_j\Big|\right) \\
&= |t_1-\gamma_1|b + \left(\sum_{j=2}^{n}\Big|(t_{j-1}-\gamma_{j-1})c + (t_j-\gamma_j)b\Big|\right) \\
&\quad + \Big|(t_n-\gamma_n)c + (t_{n+1}-\gamma_{n+1})\Big| \\
&\quad + (b+c)\Bigg(\sum_{j=n+2}^{\infty}\Big|t_j-\gamma_j\Big|\Bigg) + (1-(b+c))\Bigg(\sum_{j=n+2}^{\infty}\Big|t_j-\gamma_j\Big|\Bigg)
\end{aligned}$$

Let

$$\begin{aligned}
U &:= |t_1-\gamma_1|b + \sum_{j=2}^{n}\Big|(t_{j-1}-\gamma_{j-1})c + (t_j-\gamma_j)b\Big| + \Big|(t_n-\gamma_n)c + (t_{n+1}-\gamma_{n+1})\Big| \\
&\quad + (b+c)\sum_{j=n+2}^{\infty}\Big|t_j-\gamma_j\Big|.
\end{aligned}$$

If we can show that $U \geq \tilde{\Gamma}$ then we are done.

$$
\begin{aligned}
U &= |t_1-\gamma_1|b + \sum_{j=2}^{n} \left|(t_{j-1}-\gamma_{j-1})c + (t_j-\gamma_j)b\right| \\
&+ \left|(t_n-\gamma_n)c + (t_{n+1}-\gamma_{n+1})\right| + (b+c)\sum_{j=n+2}^{\infty}\left|t_j-\gamma_j\right| \\
&\geq (t_1-\gamma_1)b + \sum_{j=2}^{n}\Big((t_{j-1}-\gamma_{j-1})c + (t_j-\gamma_j)b\Big) \\
&+ \Big((t_n-\gamma_n)c + (t_{n+1}-\gamma_{n+1})\Big) + (b+c)\sum_{j=n+2}^{\infty}\Big(t_j-\gamma_j\Big) \\
&= \sum_{j=1}^{n}(t_j-\gamma_j)(b+c) + (t_{n+1}-\gamma_{n+1}) + (b+c)\sum_{j=n+2}^{\infty}\Big(t_j-\gamma_j\Big) \\
&= (b+c)\Bigg[\sum_{j=1}^{\infty}(t_j-\gamma_j)\Bigg] + (1-(b+c))(t_{n+1}-\gamma_{n+1}) \\
&= (b+c)\delta + (1-(b+c))(t_{n+1}-\gamma_{n+1}) \\
&= \tilde{\Gamma} + (1-(b+c))\gamma_{n+1} + (1-(b+c))(t_{n+1}-\gamma_{n+1}) \\
&= \tilde{\Gamma} + (1-(b+c))t_{n+1} \\
&\geq \tilde{\Gamma}.
\end{aligned}
$$

□

Note here that

$$
\begin{aligned}
h &= \gamma_1 f_1 + \gamma_2 f_2 + \cdots + \gamma_{n-1} f_{n-1} \\
&\quad + (\gamma_n + \gamma_{n+1} + \delta) f_n + 0 f_{n+1} + \sum_{j=n+2}^{\infty} \gamma_j f_j \in K_{b,c}
\end{aligned}
$$

and

$$\|h - z\|_1 = \tilde{\Gamma}.$$

Let $Q := \left\{ y \in K_{b,c} : \|y - z\|_1 \leq \tilde{\Gamma} \right\}$. Note that $h \in Q$ and so $Q \neq \phi$. Note also that $Q$ is closed, bounded, and convex, as well as $T$-invariant. By Lemma ($\diamondsuit$), for all $y \in Q$,

$$\tilde{\Gamma} + (1-(b+c)) \sum_{j=n+2}^{\infty} |t_j - \gamma_j| \leq \|y - z\|_1 \leq \tilde{\Gamma},$$

which implies that $t_j = \gamma_j$, $\forall j \geq n+2$. Hence,

$$Q \subseteq \left\{ y = \sum_{j=1}^{\infty} t_j f_j \in K_{b,c} : t_j = \gamma_j, \forall j \geq n+2 \right\}$$

which is a norm compact set. Therefore $Q$ is norm compact, and so by Brouwer's Fixed Point Theorem, $T$ has a fixed point in $Q \subseteq K_{b,c}$. □

## 2.3 A NEW APPROACH TO EXTENDING THE THEOREMS OF GOEBEL AND KUCZUMOW TO A LARGER CLASS OF SUBSETS OF $\ell^1$.

In Theorem 2.0.4, Goebel and Kuczumow showed that $C$ has the FPP(n.e.) precisely when $N_0$ is finite and nonempty. In our examples and in Theorem 2.2.1 in Chapter 3, we assumed without loss of generality that $N_0 = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Note that these examples, and this theorem allowed for perturbations of one "$N_0$-coordinate" (namely, $n$) with an "$\mathbb{N} \backslash N_0$-coordinate" $(n+1)$.

In this chapter we will prove two theorems (Theorem 2.3.15 and Theorem 2.3.18) that generalize Theorem 2.0.4 but do not include the examples of Chapter 3, nor Theorem 2.2.1 as special cases. This is because, in contrast to the material in Chapter 3, "$N_0$-coordinates" are perturbed strictly amongst themselves. Also, in the following theorems, we are again assuming that $N_0$ is finite and nonempty, and we typically assume, without loss of generality, that $N_0 = \{1, 2, \ldots, \nu, \nu+1\}$ for some $\nu \in \mathbb{N}$.

To begin, fix $\nu \in \mathbb{N}$. Let $b_1, b_2, \ldots, b_{\nu+1} \in (-1, 1)$ and $c_1, c_2, \ldots, c_{\nu+1} \in (-1, 1)$. Also assume that $\forall j \in \{1, 2, \ldots, \nu+1\}$,

$$|b_j| + |c_j| \in (0, 1)$$

$$|b_j| - |c_j| > 0$$

Let

$$f_j = b_j e_j + c_j e_{j+1}, \forall j \in \{1, 2, \dots, \nu\}$$
$$f_{\nu+1} = b_{\nu+1} e_{\nu+1} + c_{\nu+1} e_1$$
$$f_j = e_j, \forall j \geq \nu + 2.$$

Note that for all $x = \sum_{j\in\mathbb{N}} \alpha_j f_j$,

$$\begin{aligned}
\|x\|_1 &= \|\alpha_1(b_1 e_1 + c_1 e_2) + \cdots + \alpha_\nu(b_\nu e_\nu + c_\nu e_{\nu+1}) + \alpha_{\nu+1}(b_{\nu+1}e_{\nu+1} + c_{\nu+1}e_1) \\
&\quad + \sum_{j=\nu+2}^{\infty} \alpha_j e_j\|_1 \\
&= \|(\alpha_1 b_1 + \alpha_{\nu+1}c_{\nu+1})e_1 + (\alpha_2 b_2 + \alpha_1 c_1)e_2 + (\alpha_3 b_3 + \alpha_2 c_2)e_3 + \cdots \\
&\quad + (\alpha_\nu b_\nu + \alpha_{\nu-1}c_{\nu-1})e_\nu + (\alpha_{\nu+1}b_{\nu+1} + \alpha_\nu c_\nu)e_{\nu+1} + \sum_{j=\nu+2}^{\infty} \alpha_j e_j\|_1 \\
&= |\alpha_1 b_1 + \alpha_{\nu+1}c_{\nu+1}| + |\alpha_2 b_2 + \alpha_1 c_1| + \cdots + |\alpha_{\nu+1}b_{\nu+1} + \alpha_\nu c_\nu| \\
&\quad + \sum_{j=\nu+2}^{\infty} |\alpha_j|
\end{aligned}$$

Then

$$\min_{k\in\{1,\dots,\nu+1\}} (|b_k| - |c_k|) \sum_{j\in\mathbb{N}} |\alpha_j| \leq \|x\|_1 \leq \sum_{j\in\mathbb{N}} |\alpha_j|$$

Note that $(f_j)_{j\in\mathbb{N}}$ is an $\ell^1$-basic sequence.

Let $C \subseteq \ell^1$ be defined by

$$C := \overline{co}\{f_j : j \in \mathbb{N}\} = \left\{ \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}$$

**Theorem 2.3.1.** *The set $C$ has the FPP(n.e.).*

*Proof.* Fix a nonexpansive map $T : C \longrightarrow C$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be an approximate fixed point sequence for $T$, i.e., $\|Tx^{(k)} - x^{(k)}\|_1 \underset{k}{\longrightarrow} 0$.

Let $W$ be the weak$^* = \sigma(\ell^1, c_0)$ - closure of $C$ in $\ell^1$. Then,

$$W = \left\{ \sum_{j=1}^{\infty} \beta_j f_j : \text{each } \beta_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \beta_j \leq 1 \right\}.$$

Note that $\mathcal{B}_{\ell^1}$ is weak$^*$-sequentially compact. Hence, there exists a subsequence $(x^{(k_j)})_{j\in\mathbb{N}}$ of $(x^{(k)})_{k\in\mathbb{N}}$ and $z \in W$ such that $x^{(k_j)} \underset{j}{\longrightarrow} z$ weak$^*$. Without loss of generality, $x^{(k)} \underset{k}{\longrightarrow} z$ weak$^*$.

<u>Case 1:</u> $z \in C$.

The same method used in Chapter 3 shows that $Tz = z$. (See Example 2.1.1).

<u>Case 2:</u> $z \in W \backslash C$.

In this case, $z = \sum_{j=1}^{\infty} \gamma_j f_j$, where each $\gamma_j \geq 0$ and $\sum_{j=1}^{\infty} \gamma_j < 1$. Let $\delta := 1 - \sum_{j=1}^{\infty} \gamma_j \in (0, 1]$.

For all $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu+1})$ with each $\lambda_j \in \mathbb{R}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{\nu+1} = 1$ (i.e. $\vec{\lambda} \in \Lambda$), define

$$h_{\vec{\lambda}} := (\gamma_1 + \lambda_1\delta)f_1 + (\gamma_2 + \lambda_2\delta)f_2 + \cdots + (\gamma_\nu + \lambda_\nu\delta)f_\nu + (\gamma_{\nu+1} + \lambda_{\nu+1}\delta)f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j.$$

Let $H := \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \right\}$.

Let

$$Q := H \cap C = \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and } \gamma_j + \lambda_j\delta \geq 0, \forall j \in \{1, 2, \ldots, \nu + 1\} \right\}.$$

Clearly, $Q \supseteq \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and each } \lambda_j \geq 0 \right\} \neq \phi$.

**Claim 2.3.2.**

$$Q = \left\{ p = s_1 f_1 + s_2 f_2 + \cdots + s_\nu f_\nu + s_{\nu+1} f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j : \textit{each } s_j \geq 0 \right.$$
$$\left. \textit{and } \sum_{j=1}^{\nu+1} s_j + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \right\} =: P$$

*Proof.* $Q \subseteq P$ is clear.

To show $P \subseteq Q$, let $p \in P$. Define $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu+1}) \in \mathbb{R}^{\nu+1}$ implicitly via:

$$\gamma_j + \lambda_j \delta = s_j, \forall j \in \{1, 2, \ldots, \nu+1\}.$$

So, each $\lambda_j := \frac{s_j - \gamma_j}{\delta} \in \mathbb{R}$.

Note that

$$\begin{aligned}
\sum_{j=1}^{\nu+1} \lambda_j &= \frac{1}{\delta} \left( \sum_{j=1}^{\nu+1} s_j - \sum_{j=1}^{\nu+1} \gamma_j \right) \\
&= \frac{1}{\delta} \left( 1 - \sum_{j=\nu+2}^{\infty} \gamma_j - \sum_{j=1}^{\nu+1} \gamma_j \right) \\
&= \frac{1}{\delta} \left( 1 - \sum_{j=1}^{\infty} \gamma_j \right) \\
&= \frac{1}{\delta} \cdot \delta \\
&= 1
\end{aligned}$$

Hence, $p \in Q$.

□

Let $y \in C$ be arbitrary, i.e., $y = \sum_{j=1}^{\infty} t_j f_j$, where each $t_j \geq 0$ and $\sum_{j=1}^{\infty} t_j = 1$. Then,

$$\begin{aligned}
\|y-z\|_1 &= \Big\| \sum_{j=1}^{\infty} t_j f_j - \sum_{j=1}^{\infty} \gamma_j f_j \Big\|_1 \\
&= \Big\| \sum_{j=1}^{\infty} (t_j - \gamma_j) f_j \Big\|_1 \\
&= \Big\| \sum_{j=1}^{\nu} (t_j - \gamma_j)(b_j e_j + c_j e_{j+1}) + (t_{\nu+1} - \gamma_{\nu+1})(b_{\nu+1} e_{\nu+1} + c_{\nu+1} e_1) \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) e_j \Big\|_1 \\
&= \Big\| \sum_{j=1}^{\nu+1} (t_j - \gamma_j) b_j e_j + \Big[ \sum_{j=1}^{\nu} (t_j - \gamma_j) c_j e_{j+1} + (t_{\nu+1} - \gamma_{\nu+1}) c_{\nu+1} e_1 \Big] \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) e_j \Big\|_1 \\
&= \Big\| \sum_{j=1}^{\nu+1} (t_j - \gamma_j) b_j e_j + \sum_{k=2}^{\nu+1} (t_{k-1} - \gamma_{k-1}) c_{k-1} e_k + c_{\nu+1}(t_{\nu+1} - \gamma_{\nu+1}) e_1 \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) e_j \Big\|_1 \\
&= \Big\| \sum_{j=2}^{\nu+1} \Big[ (t_j - \gamma_j) b_j + (t_{j-1} - \gamma_{j-1}) c_{j-1} \Big] e_j \\
&+ \Big[ (t_1 - \gamma_1) b_1 + (t_{\nu+1} - \gamma_{\nu+1}) c_{\nu+1} \Big] e_1 + \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) e_j \Big\|_1
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y-z\|_1 = \sum_{j=2}^{\nu+1} & \Big| (t_j - \gamma_j) b_j + (t_{j-1} - \gamma_{j-1}) c_{j-1} \Big| \\
& + \Big| (t_1 - \gamma_1) b_1 + (t_{\nu+1} - \gamma_{\nu+1}) c_{\nu+1} \Big| + \sum_{j=\nu+2}^{\infty} |t_j - \gamma_j|.
\end{aligned}$$

**Claim 2.3.3.** *Suppose that theres exists $\mu \geq \nu + 2$ such that $t_\mu > \gamma_\mu$. Then there exists $w \in C$, i.e., $w = \sum_{j=1}^{\infty} \sigma_j f_j$ such that each $\sigma_j \geq 0$ and $\sum_{j=1}^{\infty} \sigma_j = 1$, for which $\sigma_j \leq \gamma_j$, for all $j \geq \nu + 2$, and $\|w - z\|_1 < \|y - z\|_1$.*

*Proof.* Let

$$A := \left\{ k \geq \nu + 2 : t_k > \gamma_k \right\}.$$

Note that $A \neq \phi$ since $\mu \in A$.

Let $\tau := \sum_{k \in A}(t_k - \gamma_k) > 0$, and let

$$(\sigma_j)_{j \in \mathbb{N}} = \Big( t_1 + \tau, t_2, \ldots, t_\nu, t_{\nu+1}, \sigma_{\nu+2}, \sigma_{\nu+3}, \ldots \Big),$$

where for all $k \in A$, $\sigma_k := \gamma_k$, and for all $k \in B := F \backslash A$, $\sigma_k := t_k$, where $F := \{\nu + 2, \nu + 3, \ldots\}$.

Note that for all $k \in B$, $t_k \leq \gamma_k$. So, for all $k \geq \nu + 2$, $\sigma_k \leq \gamma_k$. Also, for all $j \in \mathbb{N}$, $\sigma_j \geq 0$. Further,

$$\begin{aligned}
\sum_{j=1}^{\infty} \sigma_j &= t_1 + \tau + \sum_{j=2}^{\nu+1} t_j + \sum_{k \geq \nu+2} \sigma_k \\
&= \tau + \sum_{j=1}^{\nu+1} t_j + \sum_{k \in A} \sigma_k + \sum_{k \in B} \sigma_k \\
&= \sum_{k \in A}(t_k - \gamma_k) + \sum_{j=1}^{\nu+1} t_j + \sum_{k \in A} \gamma_k + \sum_{k \in B} t_k \\
&= \sum_{k \geq \nu+2} t_k + \sum_{j=1}^{\nu+1} t_j \\
&= \sum_{j=1}^{\infty} t_j \\
&= 1.
\end{aligned}$$

$$\begin{aligned}\|w-z\|_1 = &|(t_1+\tau-\gamma_1)b_1+(t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| \\ &+|(t_2-\gamma_2)b_2+(t_1+\tau-\gamma_1)c_1| \\ &+\sum_{j=3}^{\nu+1}|(t_j-\gamma_j)b_j+(t_{j-1}-\gamma_{j-1})c_{j-1}| \\ &+\sum_{j=\nu+2}^{\infty}|\sigma_j-\gamma_j|.\end{aligned}$$

Note that $\sum_{j=\nu+2}^{\infty}|\sigma_j-\gamma_j| = \sum_{j\in B}|t_j-\gamma_j|$.

Then,

$$\begin{aligned}\|w-z\|_1 \le &|(t_1-\gamma_1)b_1+(t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}|+\tau|b_1|+|(t_2-\gamma_2)b_2+(t_1-\gamma_1)c_1|+\tau|c_1| \\ &+\sum_{j=3}^{\nu+1}|(t_j-\gamma_j)b_j+(t_{j-1}-\gamma_{j-1})c_{j-1}|+\sum_{j=\nu+2}^{\infty}|t_j-\gamma_j| \\ &-\sum_{j\in A}|t_j-\gamma_j|.\end{aligned}$$

So,

$$\begin{aligned}\|w-z\|_1 &\le \|y-z\|_1+\tau(|b_1|+|c_1|)-\sum_{j\in A}(t_j-\gamma_j) \\ &= \|y-z\|_1+(|b_1|+|c_1|-1)\tau \\ &< \|y-z\|_1.\end{aligned}$$

□

**Claim 2.3.4.** *Suppose that* $[t_j \le \gamma_j, \forall j \ge \nu+2](\star)$ *and that there exists* $\mu \ge \nu+2$ *such that* $t_\mu < \gamma_\mu$. *Then there exists* $p \in Q$ *such that*

$$\|p-z\|_1 < \|y-z\|_1.$$

*Proof.* Let $E := \{k \geq \nu+2 : t_k < \gamma_k\}$. Note $E \neq \phi$ since $\mu \in E$. Let $\tau := \sum_{k \in E}(\gamma_k - t_k) > 0$.

Note that an arbitrary $p \in Q$ has the form $p = \sum_{j=1}^{\infty} \eta_j f_j$ such that each $\eta_j \geq 0$, $\sum_{j=1}^{\infty} \eta_j = 1$, and $\eta_j = \gamma_j, \forall j \geq \nu + 2$.

Define

$$(\eta_j)_{j \in \mathbb{N}} := (t_1(1-\varepsilon), t_2(1-\varepsilon), \ldots, t_\nu(1-\varepsilon), t_{\nu+1}(1-\varepsilon), \gamma_{\nu+2}, \gamma_{\nu+3}, \ldots, \gamma_k, \ldots),$$

where $\varepsilon \in [0, 1)$ is fixed (to be chosen later).

Note that for all $j \in \mathbb{N}$, $\eta_j \geq 0$.

Further,

$$\begin{aligned}\sum_{j=1}^{\infty} \eta_j &= \sum_{j=1}^{\nu+1} t_j(1-\varepsilon) + \sum_{j=\nu+2}^{\infty} \gamma_j \\ &= (1-\varepsilon)\sum_{j=1}^{\nu+1} t_j + \sum_{j=\nu+2}^{\infty} \gamma_j\end{aligned}$$

We want $\sum_{j=1}^{\infty} \eta_j = 1$. This is true if and only if

$$\begin{aligned}(1-\varepsilon)\Bigg(1 - \sum_{j=\nu+2}^{\infty} t_j\Bigg) + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \\ \iff 1 - \varepsilon = \frac{1 - \sum_{j=\nu+2}^{\infty} \gamma_j}{1 - \sum_{j=\nu+2}^{\infty} t_j}\end{aligned}$$

Note that

$$1 - \sum_{j=\nu+2}^{\infty} \gamma_j \geq 1 - \sum_{j=1}^{\infty} \gamma_j = \delta > 0.$$

Also,

$$\begin{aligned}1 - \sum_{j=\nu+2}^{\infty} \gamma_j &\leq 1 - \sum_{j=\nu+2}^{\infty} t_j \\ &\iff \Bigg[\sum_{j=\nu+2}^{\infty} t_j \leq \sum_{j=\nu+2}^{\infty} \gamma_j\Bigg] (\dagger)\end{aligned}$$

By hypothesis ($\star$), statement ($\dagger$) is true. Therefore,

$$0 < \delta \leq 1 - \sum_{j=\nu+2}^{\infty} \gamma_j \leq 1 - \sum_{j=\nu+2}^{\infty} t_j$$

is a true statement.

Consequently, $1 - \varepsilon$ above is well-defined and $1 - \varepsilon \in (0, 1]$.

Hence $\varepsilon \in [0, 1)$ and

$$\begin{aligned}\varepsilon &= 1 - \left(\frac{1 - \sum_{j=\nu+2}^{\infty} \gamma_j}{1 - \sum_{j=\nu+2}^{\infty} t_j}\right) \\ &= \frac{\sum_{j=\nu+2}^{\infty}(\gamma_j - t_j)}{1 - \sum_{j=\nu+2}^{\infty} t_j}\end{aligned}$$

Hence,

$$\begin{aligned}\varepsilon &= \frac{\sum_{k\in E}(\gamma_k - t_k)}{\sum_{j=1}^{\nu+1} t_j} \\ &= \frac{\tau}{\sum_{j=1}^{\nu+1} t_j}\end{aligned}$$

and so $\varepsilon \in (0, 1)$.

Then,

$$
\begin{aligned}
\|p-z\|_1 &= |(t_1(1-\varepsilon)-\gamma_1)b_1 + (t_{\nu+1}(1-\varepsilon)-\gamma_{\nu+1})c_{\nu+1}| \\
&\quad + \sum_{j=2}^{\nu+1} |(t_j(1-\varepsilon)-\gamma_j)b_j + (t_{j-1}(1-\varepsilon)-\gamma_{j-1})c_{j-1}| \\
&\quad + \sum_{j=\nu+2}^{\infty} |\gamma_j - \gamma_j| \\
&\leq |(t_1-\gamma_1)b_1 + (t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| + t_1\varepsilon|b_1| + t_{\nu+1}\varepsilon|c_{\nu+1}| \\
&\quad + \sum_{j=2}^{\nu+1} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}| + \sum_{j=2}^{\nu+1} t_j\varepsilon|b_j| \\
&\quad + \sum_{j=2}^{\nu+1} t_{j-1}\varepsilon|c_{j-1}| \\
&= \|y-z\|_1 - \sum_{j=\nu+2}^{\infty} |t_j-\gamma_j| + \varepsilon\left[\sum_{j=1}^{\nu+1} t_j|b_j| + \sum_{k=1}^{\nu+1} t_k|c_k|\right] \\
&= \|y-z\|_1 - \tau + \varepsilon \sum_{j=1}^{\nu+1} t_j(|b_j|+|c_j|) \\
&\leq \|y-z\|_1 - \tau + \frac{\tau}{\sum_{j=1}^{\nu+1} t_j} \max_{1\leq j\leq \nu+1}(|b_j|+|c_j|) \sum_{j=1}^{\nu+1} t_j \\
&= \|y-z\|_1 + \tau(\max\{|b_1|+|c_1|,\ldots,|b_{\nu+1}|+|c_{\nu+1}|\}-1) \\
&< \|y-z\|_1
\end{aligned}
$$

□

**Claim 2.3.5.**

$$J_C := \inf_{y\in C}\|y-z\|_1 \geq \inf_{p\in Q}\|p-z\|_1 =: J_Q$$

*and*

$$\left[\|y_0-z\|_1 = \inf_{y\in C}\|y-c\|_1 \Longrightarrow y_0\in Q\right](\ddagger).$$

*Proof.* First note that by Claim 2.3.3 and Claim 2.3.4,

$$\|y_0 - z\|_1 = \inf_{y \in C} \|y - c\|_1 \implies y_0 \in Q.$$

To show that $J_C \geq J_Q$, fix $y \in C$. If $y \in Q$, then $\|y - z\|_1 \geq \|y - z\|_1$. So suppose that $y \notin Q$. Then there exists $\mu \geq \nu + 2$ such that $t_\mu \neq \gamma_\mu$.

<u>Case 1:</u> There exists $\mu \geq \nu + 2$ such that $t_\mu > \gamma_\mu$.

By Claim 2.3.3, there exists $w \in \sum_{j=1}^{\infty} \sigma_j f_j \in C$ such that

$$[\sigma_j \leq \gamma_j, \forall j \geq \nu + 2] \text{ and } \|y - z\|_1 > \|w - z\|_1.$$

If $\sigma_j = \gamma_j$ for all $j \geq \nu + 2$ then $w \in Q$. Otherwise, there exists $\psi \geq \nu + 2$ such that $t_\psi < \gamma_\psi$.

By Claim 2.3.4, there exists $p \in Q$ such that

$$\|w - z\|_1 > \|p - z\|_1$$
$$\implies \|y - z\|_1 > \|p - z\|_1.$$

<u>Case 2:</u> For all $\mu \geq \nu + 2$, $t_\mu \leq \gamma_\mu$.

By Case 1, there exists $q \in Q$ such that

$$\|y - z\|_1 \geq \|q - z\|_1.$$

In all cases, $\forall y \in C$, there exists $p \in Q$ such that

$$\|y - z\|_1 \geq \|p - z\|_1.$$

Hence, $J_C \geq J_Q$. □

However, $Q$ is a nonempty, norm compact (convex) subset of $\ell^1$. Thus, there exists $p_0 \in Q \subseteq C$ such that

$$\|y - z\|_1 \geq \|p_0 - z\|_1, \forall y \in C.$$

Let

$$\tilde{Q} := \{y \in C : \|y - z\|_1 \leq \|p_0 - z\|_1\}.$$

Then by (‡) in Claim 2.3.5, $\tilde{Q} \subseteq Q \subseteq C$.

Note that $\tilde{Q}$ is a closed, bounded, and convex set in $\ell^1$ that is nonempty and norm compact. However, since

$$r(y) = r(z) + \|y - z\|_1, \forall y \in \ell^1,$$

we have that $r(Ty) \leq r(y)$, $\forall y \in C$. Therefore, $y \in \tilde{Q} \Longrightarrow Ty \in \tilde{Q}$.

Thus $T$ maps $\tilde{Q}$ into $\tilde{Q}$ and so by Brouwer's (or Schauder's) Fixed Point Theorem, there exists $v \in \tilde{Q} \subseteq Q \subseteq C$ such that $Tv = v$. □

In our next theorem, we show that we can remove the restriction that $b_j = 1, \forall j \notin F$.

Let $(b_j)_{j\in\mathbb{N}}$ be a bounded sequence of real numbers. Assume that there exists $F \subseteq \mathbb{N}$ that is finite and nonempty such that

$$\hat{\Gamma} := \max_{j\in F}(|b_j| + |c_j|) < |b_k|, \forall k \in \mathbb{N}\backslash F$$

where $(c_j)_{j\in F}$ is a sequence of real numbers such that

$$|b_j| - |c_j| > 0, \forall j \in F.$$

For what follows, we may assume without loss of generality, that $F = \{1, 2, \ldots, \nu + 1\}$ for some $\nu \in \mathbb{N}$.

Let

$$f_j = b_j e_j + c_j e_{j+1}, \forall j \in \{1, 2, \ldots, \nu\}$$

$$f_{\nu+1} = b_{\nu+1} e_{\nu+1} + c_{\nu+1} e_1$$

$$f_j = b_j e_j, \forall j \geq \nu + 2.$$

Note that for all $x = \sum_{j\in\mathbb{N}} \alpha_j f_j$,

$$\begin{aligned}
\|x\|_1 &= \|\alpha_1(b_1e_1 + c_1e_2) + \cdots + \alpha_\nu(b_\nu e_\nu + c_\nu e_{\nu+1}) + \alpha_{\nu+1}(b_{\nu+1}e_{\nu+1} + c_{\nu+1}e_1) \\
&\quad + \sum_{j=\nu+2}^{\infty} \alpha_j b_j e_j\|_1 \\
&= \|(\alpha_1 b_1 + \alpha_{\nu+1}c_{\nu+1})e_1 + (\alpha_2 b_2 + \alpha_1 c_1)e_2 + (\alpha_3 b_3 + \alpha_2 c_2)e_3 + \cdots \\
&\quad + (\alpha_\nu b_\nu + \alpha_{\nu-1}c_{\nu-1})e_\nu + (\alpha_{\nu+1}b_{\nu+1} + \alpha_\nu c_\nu)e_{\nu+1} + \sum_{j=\nu+2}^{\infty} \alpha_j b_j e_j\|_1 \\
&= |\alpha_1 b_1 + \alpha_{\nu+1}c_{\nu+1}| + |\alpha_2 b_2 + \alpha_1 c_1| + \cdots + |\alpha_{\nu+1}b_{\nu+1} + \alpha_\nu c_\nu| \\
&\quad + \sum_{j=\nu+2}^{\infty} |\alpha_j||b_j|
\end{aligned}$$

Then

$$\min_{k\in\{1,\ldots,\nu+1\}}(|b_k| - |c_k|)\sum_{j\in\mathbb{N}}|\alpha_j| \leq \|x\|_1 \leq \sup_{k\in\mathbb{N}}|b_k|\sum_{j\in\mathbb{N}}|\alpha_j|$$

and so $(f_j)_{j\in\mathbb{N}}$ is an $\ell^1$-basic sequence.

Let $C \subseteq \ell^1$ be defined by

$$C := \overline{co}\{f_j : j \in \mathbb{N}\} = \left\{\sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1\right\}.$$

**Theorem 2.3.6.** *The set $C$ has the FPP(n.e.).*

*Proof.* Fix a nonexpansive map $T : C \longrightarrow C$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be an approximate fixed point sequence for $T$, i.e., $\|Tx^{(k)} - x^{(k)}\|_1 \underset{k}{\longrightarrow} 0$.

Let $W$ be the weak$^* = \sigma(\ell^1, c_0)$ - closure of $C$ in $\ell^1$. Then,

$$W = \left\{ \sum_{j=1}^{\infty} \beta_j f_j : \text{each } \beta_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \beta_j \leq 1 \right\}.$$

Note that $\mathcal{B}_{\ell^1}$ is weak$^*$-sequentially compact. Hence, there exists a subsequence $(x^{(k_j)})_{j\in\mathbb{N}}$ of $(x^{(k)})_{k\in\mathbb{N}}$ and $z \in W$ such that $x^{(k_j)} \underset{j}{\longrightarrow} z$ weak$^*$. Without loss of generality, $x^{(k)} \underset{k}{\longrightarrow} z$ weak$^*$.

Case 1: $z \in C$.

Then as before, $Tz = z$.

Case 2: $z \in W \backslash C$.

In this case, $z = \sum_{j=1}^{\infty} \gamma_j f_j$, where each $\gamma_j \geq 0$ and $\sum_{j=1}^{\infty} \gamma_j < 1$. Let
$\delta := 1 - \sum_{j=1}^{\infty} \gamma_j \in (0, 1]$.

For all $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{\nu+1})$ with each $\lambda_j \in \mathbb{R}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{\nu+1} = 1$ (i.e. $\vec{\lambda} \in \Lambda$), define

$$h_{\vec{\lambda}} := (\gamma_1 + \lambda_1\delta)f_1 + (\gamma_2 + \lambda_2\delta)f_2 + \cdots + (\gamma_\nu + \lambda_\nu\delta)f_\nu + (\gamma_{\nu+1} + \lambda_{\nu+1}\delta)f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j.$$

Let $H := \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \right\}$.

Let

$$Q := H \cap C = \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and } \gamma_j + \lambda_j\delta \geq 0, \forall j \in \{1, 2, \dots, \nu+1\} \right\}.$$

Clearly, $Q \supseteq \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and each } \lambda_j \geq 0 \right\} \neq \phi$.

As in the previous section,

$$Q = \Bigg\{ p = s_1 f_1 + s_2 f_2 + \cdots + s_\nu f_\nu + s_{\nu+1} f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j : \text{each } s_j \geq 0$$
$$\text{and } \sum_{j=1}^{\nu+1} s_j + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \Bigg\}$$

Let $y \in C$ be arbitrary, i.e., $y = \sum_{j=1}^{\infty} t_j f_j$, where each $t_j \geq 0$ and $\sum_{j=1}^{\infty} t_j = 1$. Then,

$$\begin{aligned}
\|y - z\|_1 &= \left\| \sum_{j=1}^{\infty} t_j f_j - \sum_{j=1}^{\infty} \gamma_j f_j \right\|_1 \\
&= \left\| \sum_{j=1}^{\infty} (t_j - \gamma_j) f_j \right\|_1 \\
&= \Bigg\| \sum_{j=1}^{\nu} (t_j - \gamma_j)(b_j e_j + c_j e_{j+1}) + (t_{\nu+1} - \gamma_{\nu+1})(b_{\nu+1} e_{\nu+1} + c_{\nu+1} e_1) \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) b_j e_j \Bigg\|_1 \\
&= \Bigg\| \sum_{j=1}^{\nu+1} (t_j - \gamma_j) b_j e_j + \left[ \sum_{j=1}^{\nu} (t_j - \gamma_j) c_j e_{j+1} + (t_{\nu+1} - \gamma_{\nu+1}) c_{\nu+1} e_1 \right] \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) b_j e_j \Bigg\|_1 \\
&= \Bigg\| \sum_{j=1}^{\nu+1} (t_j - \gamma_j) b_j e_j + \sum_{k=2}^{\nu+1} (t_{k-1} - \gamma_{k-1}) c_{k-1} e_k + c_{\nu+1} (t_{\nu+1} - \gamma_{\nu+1}) e_1 \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) b_j e_j \Bigg\|_1 \\
&= \Bigg\| \sum_{j=2}^{\nu+1} \Big[ (t_j - \gamma_j) b_j + (t_{j-1} - \gamma_{j-1}) c_{j-1} \Big] e_j + \Big[ (t_1 - \gamma_1) b_1 + (t_{\nu+1} - \gamma_{\nu+1}) c_{\nu+1} \Big] e_1 \\
&+ \sum_{j=\nu+2}^{\infty} (t_j - \gamma_j) b_j e_j \Bigg\|_1
\end{aligned}$$

Hence,

$$\|y - z\|_1 = \sum_{j=2}^{\nu+1} \left|(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}\right| + \left|(t_1 - \gamma_1)b_1 + (t_{\nu+1} - \gamma_{\nu+1})c_{\nu+1}\right|$$
$$+ \sum_{j=\nu+2}^{\infty} |(t_j - \gamma_j)b_j|.$$

**Claim 2.3.7.** *Suppose that theres exists $\mu \geq \nu + 2$ such that $t_\mu > \gamma_\mu$. Then there exists $w \in C$, i.e., $w = \sum_{j=1}^{\infty} \sigma_j f_j$ such that each $\sigma_j \geq 0$ and $\sum_{j=1}^{\infty} \sigma_j = 1$, for which $\sigma_j \leq \gamma_j$, for all $j \geq \nu + 2$, and $\|w - z\|_1 < \|y - z\|_1$.*

*Proof.* Let

$$A := \left\{k \geq \nu + 2 : t_k > \gamma_k\right\}.$$

Note that $A \neq \phi$ since $\mu \in A$.

Let $\tau := \sum_{k \in A}(t_k - \gamma_k) > 0$, and let

$$(\sigma_j)_{j\in\mathbb{N}} = \left(t_1 + \tau, t_2, \ldots, t_\nu, t_{\nu+1}, \sigma_{\nu+2}, \sigma_{\nu+3}, \ldots\right),$$

where for all $k \in A$, $\sigma_k := \gamma_k$, and for all $k \in B := F \backslash A$, $\sigma_k := t_k$, where $F := \{\nu + 2, \nu + 3, \ldots\}$.

Note that for all $k \in B$, $t_k \leq \gamma_k$. So, for all $k \geq \nu + 2$, $\sigma_k \leq \gamma_k$. Also, for all $j \in \mathbb{N}$, $\sigma_j \geq 0$. Further,

$$\begin{aligned}\sum_{j=1}^{\infty}\sigma_j &= t_1+\tau+\sum_{j=2}^{\nu+1}t_j+\sum_{k\geq\nu+2}\sigma_k\\ &= \tau+\sum_{j=1}^{\nu+1}t_j+\sum_{k\in A}\sigma_k+\sum_{k\in B}\sigma_k\\ &= \sum_{k\in A}(t_k-\gamma_k)+\sum_{j=1}^{\nu+1}t_j+\sum_{k\in A}\gamma_k+\sum_{k\in B}t_k\\ &= \sum_{k\geq\nu+2}t_k+\sum_{j=1}^{\nu+1}t_j\\ &= \sum_{j=1}^{\infty}t_j\\ &= 1.\end{aligned}$$

$$\begin{aligned}\|w-z\|_1 &= |(t_1+\tau-\gamma_1)b_1+(t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}|\\ &\quad+|(t_2-\gamma_2)b_2+(t_1+\tau-\gamma_1)c_1|\\ &\quad+\sum_{j=3}^{\nu+1}|(t_j-\gamma_j)b_j+(t_{j-1}-\gamma_{j-1})c_{j-1}|\\ &\quad+\sum_{j=\nu+2}^{\infty}|(\sigma_j-\gamma_j)b_j|.\end{aligned}$$

Note that $\sum_{j=\nu+2}^{\infty}|(\sigma_j-\gamma_j)b_j|=\sum_{j\in B}|(t_j-\gamma_j)b_j|$.

Then,

$$\begin{aligned}\|w-z\|_1 \leq\ & |(t_1-\gamma_1)b_1+(t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}|+\tau|b_1|+|(t_2-\gamma_2)b_2+(t_1-\gamma_1)c_1|\\ &+\tau|c_1|+\sum_{j=3}^{\nu+1}|(t_j-\gamma_j)b_j+(t_{j-1}-\gamma_{j-1})c_{j-1}|+\sum_{j=\nu+2}^{\infty}|t_j-\gamma_j||b_j|\\ &-\sum_{j\in A}|t_j-\gamma_j||b_j|.\end{aligned}$$

So,

$$
\begin{aligned}
\|w-z\|_1 &\leq \|y-z\|_1 + \tau(|b_1|+|c_1|) - \sum_{j\in A}(t_j-\gamma_j)|b_j| \\
&= \|y-z\|_1 + \sum_{j\in A}(t_j-\gamma_j)(|b_1|+|c_1|-|b_j|) \\
&< \|y-z\|_1.
\end{aligned}
$$

□

**Claim 2.3.8.** *Suppose that* $[t_j \leq \gamma_j, \forall j \geq \nu+2](\star)$ *and that there exists* $\mu \geq \nu+2$ *such that* $t_\mu < \gamma_\mu$. *Then there exists* $p \in Q$ *such that*

$$
\|p-z\|_1 < \|y-z\|_1.
$$

*Proof.* Let $E := \{k \geq \nu+2 : t_k < \gamma_k\}$. Note $E \neq \phi$ since $\mu \in E$. Let $\tau := \sum_{k\in E}(\gamma_k - t_k) > 0$.

Note that an arbitrary $p \in Q$ has the form $p = \sum_{j=1}^{\infty} \eta_j f_j$ such that each $\eta_j \geq 0$, $\sum_{j=1}^{\infty} \eta_j = 1$, and $\eta_j = \gamma_j, \forall j \geq \nu+2$.

Define

$$
(\eta_j)_{j\in\mathbb{N}} := (t_1(1-\varepsilon), t_2(1-\varepsilon), \ldots, t_\nu(1-\varepsilon), t_{\nu+1}(1-\varepsilon), \gamma_{\nu+2}, \gamma_{\nu+3}, \ldots, \gamma_k, \ldots),
$$

where $\varepsilon \in [0,1)$ is fixed (to be chosen later).

Note that for all $j \in \mathbb{N}$, $\eta_j \geq 0$.

Further,

$$
\begin{aligned}
\sum_{j=1}^{\infty} \eta_j &= \sum_{j=1}^{\nu+1} t_j(1-\varepsilon) + \sum_{j=\nu+2}^{\infty} \gamma_j \\
&= (1-\varepsilon)\sum_{j=1}^{\nu+1} t_j + \sum_{j=\nu+2}^{\infty} \gamma_j
\end{aligned}
$$

We want $\sum_{j=1}^{\infty} \eta_j = 1$. This is true if and only if

$$\begin{aligned}(1-\varepsilon)\left(1-\sum_{j=\nu+2}^{\infty} t_j\right)+\sum_{j=\nu+2}^{\infty}\gamma_j &= 1\\ \Longleftrightarrow 1-\varepsilon &= \frac{1-\sum_{j=\nu+2}^{\infty}\gamma_j}{1-\sum_{j=\nu+2}^{\infty}t_j}\end{aligned}$$

Note that

$$1-\sum_{j=\nu+2}^{\infty}\gamma_j \geq 1-\sum_{j=1}^{\infty}\gamma_j = \delta > 0.$$

Also,

$$\begin{aligned}1-\sum_{j=\nu+2}^{\infty}\gamma_j \leq 1-\sum_{j=\nu+2}^{\infty}t_j\\ \Longleftrightarrow\left[\sum_{j=\nu+2}^{\infty}t_j \leq \sum_{j=\nu+2}^{\infty}\gamma_j\right](\dagger)\end{aligned}$$

By hypothesis ($\star$), statement ($\dagger$) is true. Therefore,

$$0 < \delta \leq 1-\sum_{j=\nu+2}^{\infty}\gamma_j \leq 1-\sum_{j=\nu+2}^{\infty}t_j$$

is a true statement.

Consequently, $1-\varepsilon$ above is well-defined and $1-\varepsilon \in (0,1]$.

Hence $\varepsilon \in [0,1)$ and

$$\begin{aligned}\varepsilon &= 1-\left(\frac{1-\sum_{j=\nu+2}^{\infty}\gamma_j}{1-\sum_{j=\nu+2}^{\infty}t_j}\right)\\ &= \frac{\sum_{j=\nu+2}^{\infty}(\gamma_j-t_j)}{1-\sum_{j=\nu+2}^{\infty}t_j}\end{aligned}$$

Hence,

$$\varepsilon = \frac{\sum_{k\in E}(\gamma_k - t_k)}{\sum_{j=1}^{\nu+1} t_j} = \frac{\tau}{\sum_{j=1}^{\nu+1} t_j}$$

and so $\varepsilon \in (0,1)$.

Then,

$$\begin{aligned}
\|p-z\|_1 &= |(t_1(1-\varepsilon)-\gamma_1)b_1 + (t_{\nu+1}(1-\varepsilon)-\gamma_{\nu+1})c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(t_j(1-\varepsilon)-\gamma_j)b_j + (t_{j-1}(1-\varepsilon)-\gamma_{j-1})c_{j-1}| \\
&+ \sum_{j=\nu+2}^{\infty} |(\gamma_j-\gamma_j)b_j| \\
&\le |(t_1-\gamma_1)b_1 + (t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| + t_1\varepsilon|b_1| + t_{\nu+1}\varepsilon|c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}| + \sum_{j=2}^{\nu+1} t_j\varepsilon|b_j| \\
&+ \sum_{j=2}^{\nu+1} t_{j-1}\varepsilon|c_{j-1}| \\
&= \|y-z\|_1 - \sum_{j=\nu+2}^{\infty} |(t_j-\gamma_j)b_j| + \varepsilon\left[\sum_{j=1}^{\nu+1} t_j|b_j| + \sum_{k=1}^{\nu+1} t_k|c_k|\right] \\
&= \|y-z\|_1 - \sum_{k\in E}(\gamma_k-t_k)|b_k| + \varepsilon\sum_{j=1}^{\nu+1} t_j(|b_j|+|c_j|) \\
&\le \|y-z\|_1 - \sum_{k\in E}(\gamma_k-t_k)|b_k| + \frac{\tau}{\sum_{j=1}^{\nu+1} t_j}\max_{1\le j\le\nu+1}(|b_j|+|c_j|)\sum_{j=1}^{\nu+1} t_j \\
&= \|y-z\|_1 - \sum_{k\in E}(\gamma_k-t_k)|b_k| + \tau\left(\max_{1\le j\le\nu+1}(|b_j|+|c_j|)\right) \\
&= \|y-z\|_1 + \sum_{k\in E}(\gamma_k-t_k)\left(\max_{1\le j\le\nu+1}(|b_j|+|c_j|) - |b_k|\right) \\
&< \|y-z\|_1
\end{aligned}$$

□

**Claim 2.3.9.**

$$J_C := \inf_{y \in C} \|y - z\|_1 \geq \inf_{p \in Q} \|p - z\|_1 =: J_Q$$

*and*

$$\left[ \|y_0 - z\|_1 = \inf_{y \in C} \|y - c\|_1 \Longrightarrow y_0 \in Q \right] (\ddagger).$$

*Proof.* First note that by Claim 2.3.7 and Claim 2.3.8,

$$\|y_0 - z\|_1 = \inf_{y \in C} \|y - c\|_1 \Longrightarrow y_0 \in Q.$$

To show that $J_C \geq J_Q$, fix $y \in C$. If $y \in Q$, then $\|y - z\|_1 \geq \|y - z\|_1$. So suppose that $y \notin Q$. Then there exists $\mu \geq \nu + 2$ such that $t_\mu \neq \gamma_\mu$.

<u>Case 1:</u> There exists $\mu \geq \nu + 2$ such that $t_\mu > \gamma_\mu$.

By Claim 2.3.7, there exists $w \in \sum_{j=1}^{\infty} \sigma_j f_j \in C$ such that

$$[\sigma_j \leq \gamma_j, \forall j \geq \nu + 2] \text{ and } \|y - z\|_1 > \|w - z\|_1.$$

If $\sigma_j = \gamma_j$ for all $j \geq \nu + 2$ then $w \in Q$. Otherwise, there exists $\psi \geq \nu + 2$ such that $t_\psi < \gamma_\psi$.

By Claim 2.3.8, there exists $p \in Q$ such that

$$\begin{aligned} \|w - z\|_1 &> \|p - z\|_1 \\ \Longrightarrow \|y - z\|_1 &> \|p - z\|_1. \end{aligned}$$

<u>Case 2:</u> For all $\mu \geq \nu + 2$, $t_\mu \leq \gamma_\mu$.

By Case 1, there exists $q \in Q$ such that

$$\|y - z\|_1 \geq \|q - z\|_1.$$

In all cases, $\forall y \in C$, there exists $p \in Q$ such that

$$\|y - z\|_1 \geq \|p - z\|_1.$$

Hence, $J_C \geq J_Q$. □

However, $Q$ is a nonempty, norm compact (convex) subset of $\ell^1$. Thus, there exists $p_0 \in Q \subseteq C$ such that

$$\|y - z\|_1 \geq \|p_0 - z\|_1, \forall y \in C.$$

Let

$$\tilde{Q} := \{y \in C : \|y - z\|_1 \leq \|p_0 - z\|_1\}.$$

Then by (‡) in Claim 2.3.9, $\tilde{Q} \subseteq Q \subseteq C$.

Note that $\tilde{Q}$ is a closed, bounded, and convex set in $\ell^1$ that is nonempty and norm compact. However, since

$$r(y) = r(z) + \|y - z\|_1, \forall y \in \ell^1,$$

we have that $r(Ty) \leq r(y)$, $\forall y \in C$. Therefore, $y \in \tilde{Q} \implies Ty \in \tilde{Q}$.

Thus $T$ maps $\tilde{Q}$ into $\tilde{Q}$ and so by Brouwer's (or Schauder's) Fixed Point Theorem, there exists $v \in \tilde{Q} \subseteq Q \subseteq C$ such that $Tv = v$. □

Now we give an example to demonstrate that the class of sets we have considered in Theorem 2.3.1 and Theorem 2.3.6 is different from the class of sets in Theorem 2.0.4.

**Example 2.3.10.** Let $(b_j)_{j\in\mathbb{N}}$ be a bounded sequence of positive numbers and $(c_j)_{j\in\{1,\ldots,\nu+1\}}$ be such that for all $j \in \{1, \ldots, \nu + 1\}$,

$$b_j + |c_j| < 1$$
$$b_j - |c_j| > 0.$$

Define $f_j = b_j e_j + c_j e_{j+1}$ for $j \in \{1, \ldots, \nu\}$, $f_{\nu+1} = b_{\nu+1}e_{\nu+1} + c_{\nu+1}e_1$, and $f_j = e_j$, for all $j \geq \nu + 2$. We have seen previously that $(f_j)_{j\in\mathbb{N}}$ is an $\ell^1$-basic sequence. As before we define

$$C = \left\{ \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}.$$

Also, recall that Goebel and Kuczumow sets in Theorem 2.0.4 have the following form. For $(d_k)_{k\in\mathbb{N}}$ such that $0 < d_k < 1$ for all $k \in \{1, \ldots, \nu + 1\}$, define

$$\begin{aligned} g_k &= d_k e_k, \quad \forall k \in \{1, \ldots, \nu + 1\} \\ g_k &= e_k, \qquad \forall k \geq \nu + 2. \end{aligned}$$

Then

$$D = \left\{ \sum_{k=1}^{\infty} s_k g_k : \text{each } s_k \geq 0 \text{ and } \sum_{k=1}^{\infty} s_k = 1 \right\}.$$

We wish to show that there are examples of sets $C$ that are not equal to some Goebel and Kuczumow set $D$.

Assuming that $\nu = 1$, fix a set $C$ as above. If $C = D$ for some set $D$ as above, then $C \subseteq D$. Therefore we have

$$f_1 = \sum_{k=1}^{\infty} s_k g_k.$$

This implies that $b_1 = s_1 d_1$, $c_1 = s_2 d_2$, and $s_k = 0$ for all $k \geq 3$. Therefore,

$$\frac{b_1}{d_1} + \frac{c_1}{d_2} = 1.$$

Similarly,

$$f_2 = \sum_{k=1}^{\infty} u_k g_k.$$

which implies that $b_2 = u_2 d_2$, $c_2 = u_1 d_1$, and $u_k = 0$ for all $k \geq 3$. Hence

$$\frac{c_2}{d_1} + \frac{b_2}{d_2} = 1.$$

Let $q_1 = d_1^{-1}$ and $q_2 = d_2^{-1}$. Then we have the following system.

$$b_1 q_1 + c_1 q_2 = 1$$
$$c_2 q_1 + b_2 q_2 = 1.$$

The solution of this system should be such that $0 < d_1, d_2 < 1$, i.e. $q_1, q_2 > 1$. From linear algebra, the solution is

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{b_1 b_2 - c_1 c_2} \begin{pmatrix} b_2 - c_1 \\ b_1 - c_2 \end{pmatrix}.$$

(Note that $b_1 b_2 - c_1 c_2 > 0$).

However, by making a choice of $b_1, b_2, c_1, c_2$ such that $c_1 < b_1 < c_2 < b_2$ we see that this forces $q_1 > 0$ and $q_2 < 0$, a contradiction. Therefore such a set $C$ cannot be written as a Goebel and Kuczumow set $D$.

Next fix a Goebel and Kuczumow set $D$. Suppose that $D \subseteq C$ for some $C$ as above. Then

$$\begin{aligned} g_1 = d_1 e_1 &= t_1 f_1 + t_2 f_2 + t_3 f_3 + \cdots \\ &= t_1(b_1 e_1 + c_1 e_2) + t_2(b_2 e_2 + c_2 e_1) + t_3 e_3 + \cdots . \end{aligned}$$

Also,

$$\begin{aligned} g_2 = d_2 e_2 &= w_1 f_1 + w_2 f_2 + w_3 f_3 + \cdots \\ &= w_1(b_1 e_1 + c_1 e_2) + w_2(b_2 e_2 + c_2 e_1) + w_3 e_3 + \cdots . \end{aligned}$$

Hence $t_j = w_j = 0$ for all $j \geq 3$, and

$$
\begin{aligned}
d_1 &= t_1 b_1 + t_2 c_2 \\
0 &= t_1 c_1 + t_2 b_2 \\
d_2 &= w_1 c_1 + w_2 b_2 \\
0 &= w_1 b_1 + w_2 c_2.
\end{aligned}
$$

However, if for example $b_1, b_2, c_1, c_2 > 0$, this implies that $t_1 = t_2 = 0$ and $w_1 = w_2 = 0$, a contradiction.

Therefore, there are examples of sets $C$ that are not subsets of any Goebel and Kuczumow set $D$, and also examples of sets $D$ that cannot be subsets of a set that has the form $C$.

Next we further generalize Theorem 2.3.6.

Let $(b_j)_{j\in\mathbb{N}}$ and $(c_j)_{j\in\mathbb{N}}$ be bounded sequences of real numbers such that

$$|b_j| - |c_j| > 0, \forall j \in \mathbb{N}.$$

Assume that there exists $F \subseteq \mathbb{N}$ that is finite and nonempty such that

$$0 < \hat{\Gamma} := \max_{j\in F}(|b_j| + |c_j|) < |b_k| - |c_k|, \forall k \in \mathbb{N}\backslash F$$

$$|b_j| - |c_j| > 0, \forall j \in F.$$

For what follows, we may assume without loss of generality, that $F = \{1, 2, \ldots, \nu + 1\}$ for some $\nu \in \mathbb{N}$.

Let

$$
\begin{aligned}
f_j &= b_j e_j + c_j e_{j+1}, \forall j \in \{1, 2, \ldots, \nu\} \\
f_{\nu+1} &= b_{\nu+1} e_{\nu+1} + c_{\nu+1} e_1 \\
f_j &= b_j e_j + c_j e_{j+1}, \forall j \geq \nu + 2.
\end{aligned}
$$

Note that for all $x = \sum_{j\in\mathbb{N}} \alpha_j f_j$,

$$\begin{aligned}
\|x\|_1 &= \|\alpha_1(b_1e_1 + c_1e_2) + \cdots + \alpha_\nu(b_\nu e_\nu + c_\nu e_{\nu+1}) + \alpha_{\nu+1}(b_{\nu+1}e_{\nu+1} + c_{\nu+1}e_1) \\
&+ \sum_{j=\nu+2}^{\infty} \alpha_j(b_je_j + c_je_{j+1})\|_1 \\
&= \|(\alpha_1b_1 + \alpha_{\nu+1}c_{\nu+1})e_1 + (\alpha_2b_2 + \alpha_1c_1)e_2 + (\alpha_3b_3 + \alpha_2c_2)e_3 + \cdots \\
&+ (\alpha_\nu b_\nu + \alpha_{\nu-1}c_{\nu-1})e_\nu + (\alpha_{\nu+1}b_{\nu+1} + \alpha_\nu c_\nu)e_{\nu+1} + (\alpha_{\nu+2}b_{\nu+2})e_{\nu+2} \\
&+ \sum_{j=\nu+3}^{\infty} (\alpha_jb_j + \alpha_{j-1}c_{j-1})e_j\|_1 \\
&= |\alpha_1b_1 + \alpha_{\nu+1}c_{\nu+1}| + \sum_{j=2}^{\nu+1} |\alpha_jb_j + \alpha_{j-1}c_{j-1}| + |\alpha_{\nu+2}b_{\nu+2}| \\
&+ \sum_{j=\nu+3}^{\infty} |\alpha_jb_j + \alpha_{j-1}c_{j-1}|
\end{aligned}$$

Then

$$\min_{k\in\{1,\ldots,\nu+1\}}(|b_k| - |c_k|)\sum_{j\in\mathbb{N}}|\alpha_j| \le \|x\|_1 \le \sup_{k\in\mathbb{N}}(|b_k| + |c_k|)\sum_{j\in\mathbb{N}}|\alpha_j|$$

and so $(f_j)_{j\in\mathbb{N}}$ is an $\ell^1$-basic sequence.

Let $C \subseteq \ell^1$ be defined by

$$C := \overline{co}\{f_j : j \in \mathbb{N}\} = \left\{ \sum_{j=1}^{\infty} t_jf_j : \text{each } t_j \ge 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}.$$

**Theorem 2.3.11.** *Suppose that*

$$(\circledast)\, \forall k \ge \nu + 2, [b_{k+1} > 0 \textit{ and } c_k \le 0] \textit{ or } [b_{k+1} < 0 \textit{ and } c_k \ge 0]$$

*If $T : C \longrightarrow C$ is a nonexpansive map, then $T$ has a fixed point in $C$.*

*Proof.* Fix a nonexpansive map $T : C \longrightarrow C$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be an approximate fixed point sequence for $T$, i.e., $\|Tx^{(k)} - x^{(k)}\|_1 \underset{k}{\longrightarrow} 0$.

Let $W$ be the weak$^* = \sigma(\ell^1, c_0)$ - closure of $C$ in $\ell^1$. Then,

$$W = \left\{ \sum_{j=1}^{\infty} \beta_j f_j : \text{each } \beta_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \beta_j \leq 1 \right\}.$$

Note that $\mathcal{B}_{\ell^1}$ is weak$^*$-sequentially compact. Hence, there exists a subsequence $(x^{(k_j)})_{j\in\mathbb{N}}$ of $(x^{(k)})_{k\in\mathbb{N}}$ and $z \in W$ such that $x^{(k_j)} \underset{j}{\longrightarrow} z$ weak$^*$. Without loss of generality, $x^{(k)} \underset{k}{\longrightarrow} z$ weak$^*$.

Note that if $z \in C$, then as we have seen previously, $Tz = z$. Assume that $z = \sum_{j=1}^{\infty} \gamma_j f_j \in W \backslash C$.

For all $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu+1})$ with each $\lambda_j \in \mathbb{R}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{\nu+1} = 1$ (i.e. $\vec{\lambda} \in \Lambda$), define

$$h_{\vec{\lambda}} := (\gamma_1 + \lambda_1\delta)f_1 + (\gamma_2 + \lambda_2\delta)f_2 + \cdots + (\gamma_\nu + \lambda_\nu\delta)f_\nu + (\gamma_{\nu+1} + \lambda_{\nu+1}\delta)f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j.$$

Let $H := \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \right\}$.

Let

$$Q := H \cap C = \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and } \gamma_j + \lambda_j\delta \geq 0, \forall j \in \{1, 2, \ldots, \nu+1\} \right\}.$$

As we have shown previously,

$$Q = \left\{ p = s_1 f_1 + s_2 f_2 + \cdots + s_\nu f_\nu + s_{\nu+1} f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j : \text{each } s_j \geq 0 \right.$$
$$\left. \text{and } \sum_{j=1}^{\nu+1} s_j + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \right\}$$

Let $y = \sum_{j=1}^{\infty} t_j f_j \in C$ be arbitrary. Note that

$$\|y-z\|_1 = |(t_1-\gamma_1)b_1 + (t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| + \sum_{j=2}^{\nu+1} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}|$$
$$+ |(t_{\nu+2}-\gamma_{\nu+2})b_{\nu+2}| + \sum_{j=\nu+3}^{\infty} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}|$$

**Claim 2.3.12.** *Suppose that theres exists $\mu \geq \nu+2$ such that $t_\mu > \gamma_\mu$. Then there exists $w \in C$, i.e., $w = \sum_{j=1}^{\infty} \sigma_j f_j$ such that each $\sigma_j \geq 0$ and $\sum_{j=1}^{\infty} \sigma_j = 1$, for which $\sigma_j \leq \gamma_j$, for all $j \geq \nu+2$, and $\|w-z\|_1 < \|y-z\|_1$.*

*Proof.* Let

$$A := \Big\{ k \geq \nu+2 : t_k > \gamma_k \Big\}.$$

Note that $A \neq \phi$ since $\mu \in A$.

Let $\tau := \sum_{k\in A}(t_k - \gamma_k) > 0$, and let

$$(\sigma_j)_{j\in\mathbb{N}} = \Big( t_1+\tau, t_2, \ldots, t_\nu, t_{\nu+1}, \sigma_{\nu+2}, \sigma_{\nu+3}, \ldots \Big),$$

where for all $k \in A$, $\sigma_k := \gamma_k$, and for all $k \in B := F\backslash A$, $\sigma_k := t_k$, where $F := \{\nu+2, \nu+3, \ldots\}$.

Note that for all $k \in B$, $t_k \leq \gamma_k$. So, for all $k \geq \nu+2$, $\sigma_k \leq \gamma_k$. Also, for all $j \in \mathbb{N}$, $\sigma_j \geq 0$. Further,

$$\begin{aligned}
\sum_{j=1}^{\infty} \sigma_j &= t_1 + \tau + \sum_{j=2}^{\nu+1} t_j + \sum_{k\geq\nu+2} \sigma_k \\
&= \tau + \sum_{j=1}^{\nu+1} t_j + \sum_{k\in A} \sigma_k + \sum_{k\in B} \sigma_k \\
&= \sum_{k\in A}(t_k-\gamma_k) + \sum_{j=1}^{\nu+1} t_j + \sum_{k\in A}\gamma_k + \sum_{k\in B} t_k \\
&= \sum_{k\geq\nu+2} t_k + \sum_{j=1}^{\nu+1} t_j \\
&= \sum_{j=1}^{\infty} t_j \\
&= 1
\end{aligned}$$

Let $w = \sum_{j=1}^{\infty} \sigma_j f_j$. Then,

$$\|w - z\|_1 = |(t_1 + \tau - \gamma_1)b_1 + (t_{\nu+1} - \gamma_{\nu+1})c_{\nu+1}| + |(t_2 - \gamma_2)b_2 + (t_1 + \tau - \gamma_1)c_1| \\ + \sum_{j=3}^{\nu+1} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| + |(\sigma_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| \\ + \sum_{j=\nu+3}^{\infty} |(\sigma_j - \gamma_j)b_j + (\sigma_{j-1} - \gamma_{j-1})c_{j-1}|$$

Using the triangle inequality along with the definition of $\sigma_j$ for $j \in A$ versus $j \in B$,

$$\|w - z\|_1 \leq |(t_1 - \gamma_1)b_1 + (t_{\nu+1} - \gamma_{\nu+1})c_{\nu+1}| + |(t_2 - \gamma_2)b_2 + (t_1 - \gamma_1)c_1| \\ + \sum_{j=3}^{\nu+1} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| + |(\sigma_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| \\ + \sum_{\substack{j=\nu+3 \\ j,j-1 \in B}}^{\infty} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| \\ + \sum_{\substack{j=\nu+3 \\ j \in B, j-1 \in A}}^{\infty} |(t_j - \gamma_j)b_j| + \sum_{\substack{j=\nu+3 \\ j \in A, j-1 \in B}}^{\infty} |(t_{j-1} - \gamma_{j-1})c_{j-1}| \\ + \tau(|b_1| + |c_1|)$$

Hence, if $(\star) = \|y - z\|_1 - \|w - z\|_1$, then

$$
\begin{aligned}
(\star) &\geq -\tau(|b_1| + |c_1|) \\
&+ |(t_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| - |(\sigma_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| \\
&+ \sum_{j=\nu+3}^{\infty} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| \\
&- \sum_{\substack{j=\nu+3 \\ j,j-1\in B}}^{\infty} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| \\
&- \sum_{\substack{j=\nu+3 \\ j\in B, j-1\in A}}^{\infty} |(t_j - \gamma_j)b_j| \\
&- \sum_{\substack{j=\nu+3 \\ j\in A, j-1\in B}}^{\infty} |(t_{j-1} - \gamma_{j-1})c_{j-1}| \\
&= -\tau(|b_1| + |c_1|) + \sum_{\substack{j=\nu+2 \\ j\in A}}^{\nu+2} |(t_j - \gamma_j)b_j| \\
&+ \sum_{\substack{j=\nu+3 \\ j,j-1\in A}}^{\infty} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| \\
&+ \sum_{\substack{j=\nu+3 \\ j\in B, j-1\in A}}^{\infty} \Big[ |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| - |(t_j - \gamma_j)b_j| \Big] \\
&+ \sum_{\substack{j=\nu+3 \\ j\in A, j-1\in B}}^{\infty} \Big[ |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| - |(t_{j-1} - \gamma_{j-1})c_{j-1}| \Big]
\end{aligned}
$$

Now since $b_j$ and $c_{j-1}$ have opposite signs for each $j$, we can evaluate the absolute values in each of the last two sums. Doing so, we obtain the following.

$$\begin{aligned}\|y-z\|_1 - \|w-z\|_1 \geq -\tau(|b_1|+|c_1|) &+ \sum_{\substack{j=\nu+2 \\ j\in A}}^{\nu+2} |(t_j-\gamma_j)b_j| \\ &+ \sum_{\substack{j=\nu+3 \\ j,j-1\in A}}^{\infty} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}| \\ &+ \sum_{\substack{j=\nu+3 \\ j\in B, j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}| \\ &+ \sum_{\substack{j=\nu+3 \\ j\in A, j-1\in B}}^{\infty} |(t_j-\gamma_j)b_j|\end{aligned}$$

Using the reverse triangle inequality,

$$\begin{aligned}\|y-z\|_1 - \|w-z\|_1 \geq -\tau(|b_1|+|c_1|) &+ \sum_{\substack{j=\nu+2 \\ j\in A}}^{\nu+2} |(t_j-\gamma_j)b_j| \\ &+ \sum_{\substack{j=\nu+3 \\ j,j-1\in A}}^{\infty} |(t_j-\gamma_j)b_j| - \sum_{\substack{j=\nu+3 \\ j,j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}| \\ &+ \sum_{\substack{j=\nu+3 \\ j\in B, j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}| \\ &+ \sum_{\substack{j=\nu+3 \\ j\in A, j-1\in B}}^{\infty} |(t_j-\gamma_j)b_j|\end{aligned}$$

Noting that

$$\sum_{\substack{j=\nu+3 \\ j\in B, j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}| \geq 0$$

and

$$\sum_{\substack{j=\nu+3 \\ j,j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}| \leq \sum_{\substack{j=\nu+3 \\ j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}|$$

and combining

$$\sum_{\substack{j=\nu+3\\ j,j-1\in A}}^{\infty} |(t_j-\gamma_j)b_j| + \sum_{\substack{j=\nu+3\\ j\in A, j-1\in B}}^{\infty} |(t_j-\gamma_j)b_j|$$

we observe that

$$\begin{aligned}
\|y-z\|_1 - \|w-z\|_1 \geq& -\tau(|b_1|+|c_1|)\\
&+ \sum_{\substack{j=\nu+2\\ j\in A}}^{\nu+2} |(t_j-\gamma_j)b_j| + \sum_{\substack{j=\nu+3\\ j\in A}}^{\infty} |(t_j-\gamma_j)b_j|\\
&+ 0 - \sum_{\substack{j=\nu+3\\ j-1\in A}}^{\infty} |(t_{j-1}-\gamma_{j-1})c_{j-1}|
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y-z\|_1 - \|w-z\|_1 \geq& -\left(\sum_{k\in A}(t_k-\gamma_k)\right)(|b_1|+|c_1|)\\
&+ \sum_{\substack{j=\nu+2\\ j\in A}}^{\infty} |(t_j-\gamma_j)b_j| - \sum_{\substack{k=\nu+2\\ k\in A}}^{\infty} |(t_k-\gamma_k)c_k|\\
&\geq \sum_{k\in A}(t_k-\gamma_k)\left[|b_k|-|c_k|-\hat{\Gamma}\right]\\
&> 0
\end{aligned}$$

Therefore, $\|y-z\|_1 > \|w-z\|_1$, as desired. □

**Claim 2.3.13.** *Suppose that $[t_j \leq \gamma_j, \forall j \geq \nu+2](\star)$ and that there exists $\mu \geq \nu+2$ such that $t_\mu < \gamma_\mu$. Then there exists $p \in Q$ such that*

$$\|p-z\|_1 < \|y-z\|_1.$$

*Proof.* Let $E := \Big\{k \geq \nu + 2 : t_k < \gamma_k\Big\}$. Note that $E \neq \phi$ since $\mu \in E$. Let $\tau := \sum_{k\in E}(\gamma_k - t_k) > 0$. Note that an arbitrary $p \in Q$ has the form $p = \sum_{j=1}^{\infty} \eta_j f_j$ such that each $\eta_j \geq 0$ and $\sum_{j=1}^{\infty} \eta_j = 1$, and $\eta_j = \gamma_j$, $\forall j \geq \nu + 2$.

Define

$$(\eta_j)_{j\in\mathbb{N}} := \Big(t_1(1-\varepsilon), t_2(1-\varepsilon), \ldots, t_\nu(1-\varepsilon), t_{\nu+1}(1-\varepsilon), \gamma_{\nu+2}, \gamma_{\nu+3}, \ldots, \gamma_k, \ldots\Big)$$

where $\varepsilon \in (0, 1)$ is fixed, and to be chosen later.

Note that $\forall j \in \mathbb{N}, \eta_j \geq 0$. Also

$$\begin{aligned}\sum_{j=1}^{\infty} \eta_j &= \sum_{j=1}^{\nu+1} t_j(1-\varepsilon) + \sum_{j=\nu+2}^{\infty} \gamma_j \\ &= (1-\varepsilon)\sum_{j=1}^{\nu+1} t_j + \sum_{j=\nu+2}^{\infty} \gamma_j\end{aligned}$$

We want $\sum_{j=1}^{\infty} \eta_j = 1$. This is true if and only if

$$\begin{aligned}(1-\varepsilon)\Big(1 - \sum_{j=\nu+2}^{\infty} t_j\Big) + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \\ \iff 1 - \varepsilon = \frac{1 - \sum_{j=\nu+2}^{\infty} \gamma_j}{1 - \sum_{j=\nu+2}^{\infty} t_j}\end{aligned}$$

Note that

$$1 - \sum_{j=\nu+2}^{\infty} \gamma_j \geq 1 - \sum_{j=1}^{\infty} \gamma_j = \delta > 0.$$

Also,

$$\begin{aligned}1 - \sum_{j=\nu+2}^{\infty} \gamma_j \leq 1 - \sum_{j=\nu+2}^{\infty} t_j \\ \iff \Big[\sum_{j=\nu+2}^{\infty} t_j \leq \sum_{j=\nu+2}^{\infty} \gamma_j\Big](\dagger)\end{aligned}$$

By hypothesis $(\star)$, statement $(\dagger)$ is true. Further, $\exists\, \mu \geq \nu + 2$ such that $t_\mu < \gamma_\mu$. Therefore,

$$0 < \delta \leq 1 - \sum_{j=\nu+2}^{\infty} \gamma_j < 1 - \sum_{j=\nu+2}^{\infty} t_j$$

is a true statement.

Consequently, $1 - \varepsilon$ above is well-defined and $1 - \varepsilon \in (0, 1)$.

Hence $\varepsilon \in (0, 1)$ and

$$\begin{aligned}
\varepsilon &= 1 - \left( \frac{1 - \sum_{j=\nu+2}^{\infty} \gamma_j}{1 - \sum_{j=\nu+2}^{\infty} t_j} \right) \\
&= \frac{\sum_{j=\nu+2}^{\infty} (\gamma_j - t_j)}{1 - \sum_{j=\nu+2}^{\infty} t_j} \\
&= \frac{\tau}{1 - \sum_{j=\nu+2}^{\infty} t_j}
\end{aligned}$$

Hence, $p \in Q$. Also,

$$
\begin{aligned}
\|p - z\|_1 &= |(t_1(1-\varepsilon) - \gamma_1)b_1 + (t_{\nu+1}(1-\varepsilon) - \gamma_{\nu+1})c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(t_j(1-\varepsilon) - \gamma_j)b_j + (t_{j-1}(1-\varepsilon) - \gamma_{j-1})c_{j-1}| \\
&+ |(\gamma_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| + \sum_{j=\nu+3}^{\infty} |(\gamma_j - \gamma_j)b_j + (\gamma_{j-1} - \gamma_{j-1})c_{j-1}| \\
&= |(t_1(1-\varepsilon) - \gamma_1)b_1 + (t_{\nu+1}(1-\varepsilon) - \gamma_{\nu+1})c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(t_j(1-\varepsilon) - \gamma_j)b_j + (t_{j-1}(1-\varepsilon) - \gamma_{j-1})c_{j-1}| \\
&\leq |(t_1 - \gamma_1)b_1 + (t_{\nu+1} - \gamma_{\nu+1})c_{\nu+1}| + t_1\varepsilon|b_1| + t_{\nu+1}\varepsilon|c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| + \sum_{j=2}^{\nu+1} t_j\varepsilon|b_j| \\
&+ \sum_{j=2}^{\nu+1} t_{j-1}\varepsilon|c_{j-1}| \\
&= \|y - z\|_1 - |(t_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| \\
&- \sum_{j=\nu+3}^{\infty} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}| \\
&+ \varepsilon\left[\sum_{j=1}^{\nu+1} t_j|b_j| + \sum_{k=1}^{\nu+1} t_k|c_k|\right] \\
&\leq \|y - z\|_1 - |(t_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| \\
&- \left(\sum_{j=\nu+3}^{\infty}\Big[|(t_j - \gamma_j)b_j| - |(t_{j-1} - \gamma_{j-1})c_{j-1}|\Big]\right) \\
&+ \varepsilon\sum_{j=1}^{\nu+1}(|b_j| + |c_j|)t_j \\
&\leq \|y - z\|_1 - \sum_{j=\nu+2}^{\infty} |(t_j - \gamma_j)b_j| \\
&+ \sum_{k=\nu+2}^{\infty} |(t_k - \gamma_k)c_k| \\
&+ \frac{\tau}{\sum_{\alpha=1}^{\nu+1} t_\alpha}\left[\max_{l\in\{1,\ldots,\nu+1\}}(|b_l| + |c_l|)\right]\sum_{j=1}^{\nu+1} t_j
\end{aligned}
$$

Hence,

$$\begin{aligned}\|p-z\|_1 &\le \|y-z\|_1 - \sum_{k=\nu+2}^{\infty}(\gamma_k - t_k)(|b_k|-|c_k|)\\ &+ \sum_{k\in E}(\gamma_k - t_k)\hat{\Gamma}\\ &= \|y-z\|_1 - \sum_{k\in E}(\gamma_k - t_k)(|b_k|-|c_k|) + \sum_{k\in E}(\gamma_k - t_k)\hat{\Gamma}\\ &= \|y-z\|_1 - \sum_{k\in E}(\gamma_k - t_k)\left[|b_k|-|c_k|-\hat{\Gamma}\right]\\ &< \|y-z\|_1\end{aligned}$$

□

**Claim 2.3.14.**

$$J_C := \inf_{y\in C}\|y-z\|_1 \ge \inf_{p\in Q}\|p-z\|_1 =: J_Q$$

*and*

$$\left[\|y_0 - z\|_1 = \inf_{y\in C}\|y-c\|_1 \Longrightarrow y_0 \in Q\right](\ddagger).$$

*Proof.* First note that by Claim 2.3.12 and Claim 2.3.13,

$$\|y_0 - z\|_1 = \inf_{y\in C}\|y-c\|_1 \Longrightarrow y_0 \in Q.$$

To show that $J_C \ge J_Q$, fix $y \in C$. If $y \in Q$, then $\|y-z\|_1 \ge \|y-z\|_1$. So suppose that $y \notin Q$. Then there exists $\mu \ge \nu+2$ such that $t_\mu \ne \gamma_\mu$.

Case 1: There exists $\mu \ge \nu+2$ such that $t_\mu > \gamma_\mu$.

By Claim 2.3.12, there exists $w \in \sum_{j=1}^{\infty}\sigma_j f_j \in C$ such that

$$[\sigma_j \le \gamma_j, \forall j \ge \nu+2] \text{ and } \|y-z\|_1 > \|w-z\|_1.$$

If $\sigma_j = \gamma_j$ for all $j \ge \nu+2$ then $w \in Q$. Otherwise, there exists $\psi \ge \nu+2$ such that $t_\psi < \gamma_\psi$.

By Claim 2.3.13, there exists $p \in Q$ such that

$$\begin{aligned} &\|w - z\|_1 > \|p - z\|_1 \\ \implies &\|y - z\|_1 > \|p - z\|_1. \end{aligned}$$

Case 2: For all $\mu \geq \nu + 2$, $t_\mu \leq \gamma_\mu$.

By Case 1, there exists $q \in Q$ such that

$$\|y - z\|_1 \geq \|q - z\|_1.$$

In all cases, $\forall y \in C$, there exists $p \in Q$ such that

$$\|y - z\|_1 \geq \|p - z\|_1.$$

Hence, $J_C \geq J_Q$. $\square$

However, $Q$ is a nonempty, norm compact (convex) subset of $\ell^1$. Thus, there exists $p_0 \in Q \subseteq C$ such that

$$\|y - z\|_1 \geq \|p_0 - z\|_1, \forall y \in C.$$

Let

$$\tilde{Q} := \{y \in C : \|y - z\|_1 \leq \|p_0 - z\|_1\}.$$

Then by (‡) in Claim 2.3.14, $\tilde{Q} \subseteq Q \subseteq C$.

Note that $\tilde{Q}$ is a closed, bounded, and convex set in $\ell^1$ that is nonempty and norm compact. However, since

$$r(y) = r(z) + \|y - z\|_1, \forall y \in \ell^1,$$

we have that $r(Ty) \leq r(y)$, $\forall y \in C$. Therefore, $y \in \tilde{Q} \implies Ty \in \tilde{Q}$.

Thus $T$ maps $\tilde{Q}$ into $\tilde{Q}$ and so by Brouwer's (or Schauder's) Fixed Point Theorem, there exists $v \in \tilde{Q} \subseteq Q \subseteq C$ such that $Tv = v$. $\square$

Note that in Theorem 2.3.11 we need special assumptions about the signs of each $b_j$ and $c_j$. We now show that we may remove those restrictions.

As in Theorem 2.3.11, let $(b_j)_{j\in\mathbb{N}}$ and $(c_j)_{j\in\mathbb{N}}$ be bounded sequences of real numbers such that

$$|b_j| - |c_j| > 0, \forall j \in \mathbb{N}.$$

Assume that there exists $F \subseteq \mathbb{N}$ that is finite and nonempty such that

$$0 < \hat{\Gamma} := \max_{j\in F}(|b_j| + |c_j|) < |b_k| - |c_k|, \forall k \in \mathbb{N}\backslash F$$

$$|b_j| - |c_j| > 0, \forall j \in F.$$

For what follows, we may assume without loss of generality, that $F = \{1, 2, \ldots, \nu + 1\}$ for some $\nu \in \mathbb{N}$.

Let

$$\begin{aligned} f_j &= b_j e_j + c_j e_{j+1}, \forall j \in \{1, 2, \ldots, \nu\} \\ f_{\nu+1} &= b_{\nu+1} e_{\nu+1} + c_{\nu+1} e_1 \\ f_j &= b_j e_j + c_j e_{j+1}, \forall j \geq \nu + 2. \end{aligned}$$

Note that as before, $\forall x = \sum_{j\in\mathbb{N}} \alpha_j f_j$,

$$\min_{k\in\{1,\ldots,\nu+1\}}(|b_k| - |c_k|) \sum_{j\in\mathbb{N}} |\alpha_j| \leq \|x\|_1 \leq \sup_{k\in\mathbb{N}}(|b_k| + |c_k|) \sum_{j\in\mathbb{N}} |\alpha_j|$$

and so $(f_j)_{j\in\mathbb{N}}$ is an $\ell^1$-basic sequence.

Let $C \subseteq \ell^1$ be defined by

$$C := \overline{co}\{f_j : j \in \mathbb{N}\} = \left\{ \sum_{j=1}^{\infty} t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}.$$

**Theorem 2.3.15.** *The set $C$ has the FPP(n.e.).*

*Proof.* Fix a nonexpansive map $T : C \longrightarrow C$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be an approximate fixed point sequence for $T$, i.e., $\|Tx^{(k)} - x^{(k)}\|_1 \underset{k}{\longrightarrow} 0$.

Let $W$ be the weak$^*$ $= \sigma(\ell^1, c_0)$ - closure of $C$ in $\ell^1$. Then,

$$W = \left\{ \sum_{j=1}^{\infty} \beta_j f_j : \text{each } \beta_j \geq 0 \text{ and } \sum_{j=1}^{\infty} \beta_j \leq 1 \right\}.$$

Note that $\mathcal{B}_{\ell^1}$ is weak$^*$-sequentially compact. Hence, there exists a subsequence $(x^{(k_j)})_{j\in\mathbb{N}}$ of $(x^{(k)})_{k\in\mathbb{N}}$ and $z \in W$ such that $x^{(k_j)} \underset{j}{\longrightarrow} z$ weak$^*$. Without loss of generality, $x^{(k)} \underset{k}{\longrightarrow} z$ weak$^*$.

Note that if $z \in C$, then as we have seen previously, $Tz = z$. Assume that $z = \sum_{j=1}^{\infty} \gamma_j f_j \in W \backslash C$.

For all $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu+1})$ with each $\lambda_j \in \mathbb{R}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{\nu+1} = 1$ (i.e. $\vec{\lambda} \in \Lambda$), define

$$h_{\vec{\lambda}} := (\gamma_1 + \lambda_1\delta)f_1 + (\gamma_2 + \lambda_2\delta)f_2 + \cdots + (\gamma_\nu + \lambda_\nu\delta)f_\nu + (\gamma_{\nu+1} + \lambda_{\nu+1}\delta)f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j.$$

Let $H := \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \right\}$.

Let

$$Q := H \cap C = \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and } \gamma_j + \lambda_j\delta \geq 0, \forall j \in \{1, 2, \ldots, \nu+1\} \right\}.$$

As we have shown previously,

$$Q = \left\{ p = s_1 f_1 + s_2 f_2 + \cdots + s_\nu f_\nu + s_{\nu+1} f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j : \text{each } s_j \geq 0 \right.$$
$$\left. \text{and } \sum_{j=1}^{\nu+1} s_j + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \right\}$$

Fix $y \in C$ of the form $y = \sum_{k=1}^{\infty} t_k f_k$. Note that

$$\|y - z\|_1 = |(t_1 - \gamma_1)b_1 + (t_{\nu+1} - \gamma_{\nu+1})c_{\nu+1}| + \sum_{j=2}^{\nu+1} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}|$$
$$+ |(t_{\nu+2} - \gamma_{\nu+2})b_{\nu+2}| + \sum_{j=\nu+3}^{\infty} |(t_j - \gamma_j)b_j + (t_{j-1} - \gamma_{j-1})c_{j-1}|$$

**Claim 2.3.16.** *Suppose that $t_\mu \neq \gamma_\mu$ for at least one $\mu \geq \nu + 2$. Then $\exists p \in Q$ such that $\|p - z\|_1 < \|y - z\|_1$.*

*Proof.* Let

$$A := \{k \geq \nu + 2 : t_k > \gamma_k\}$$

and

$$B := \{\nu + 2, \nu + 3, \ldots\} \backslash A = \{k \geq \nu + 2 : t_k \leq \gamma_k\}.$$

Define $\tau := \sum_{k \in A}(t_k - \gamma_k)$. Note that $\tau \geq 0$ and $\tau = 0 \iff A = \phi$.

Let $w = \phi(y) := \sum_{j=1}^{\infty} \sigma_j f_j$, where $(\sigma_j)_{j \in \mathbb{N}}$ is defined as follows.

$$(\sigma_j)_{j \in \mathbb{N}} = (t_1 + \tau, t_2, \ldots, t_\nu, t_{\nu+1}, \sigma_{\nu+2}, \sigma_{\nu+3}, \ldots)$$

where $\forall k \in A, \sigma_k := \gamma_k$ and $\forall k \in B, \sigma_k := t_k$.

Note that $\sigma_j \leq \gamma_j, \forall j \geq \nu + 2$.

Define $E := \{k \geq \nu + 2 : \sigma_k < \gamma_k\}$ and $\xi := \sum_{k \in E}(\gamma_k - \sigma_k)$. Note that $\xi \geq 0$ and $\xi = 0 \iff E = \phi$. Also note that $E \subseteq B$ and so $\xi = \sum_{k \in B}(\gamma_k - \sigma_k) = \sum_{k \in B}(\gamma_k - t_k)$.

Note that, by hypothesis, either $\tau > 0$ or $\xi > 0$

Next, let $p = \psi(w) = (\psi \circ \phi)(y) := \sum_{j=1}^{\infty} \eta_j f_j$, where $(\eta_j)_{j \in \mathbb{N}}$ is defined as follows.

$$(\eta_j)_{j \in \mathbb{N}} := (\sigma_1(1 - \varepsilon), \sigma_2(1 - \varepsilon), \ldots, \sigma_\nu(1 - \varepsilon), \sigma_{\nu+1}(1 - \varepsilon), \gamma_{\nu+2}, \gamma_{\nu+3}, \ldots, \gamma_k, \ldots)$$

where

$$\varepsilon := \frac{\sum_{j=\nu+2}^{\infty}(\gamma_j - \sigma_j)}{\sum_{j=1}^{\nu+1} \sigma_j} = \frac{\xi}{\sum_{j=1}^{\nu+1} t_j + \tau}$$

It is easily checked that $\varepsilon \in [0, 1)$ and $\sum_{j=1}^{\infty} \eta_j = 1$, i.e. $p \in Q$.

Then, we have that

$$\begin{aligned}
\|p-z\|_1 &= |(\eta_1-\gamma_1)b_1 + (\eta_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(\eta_j-\gamma_j)b_j + (\eta_{j-1}-\gamma_{j-1})c_{j-1}| \\
&+ |(\eta_{\nu+2}-\gamma_{\nu+2})b_{\nu+2}| \\
&+ \sum_{j=\nu+3}^{\infty} |(\eta_j-\gamma_j)b_j + (\eta_{j-1}-\gamma_{j-1})c_{j-1}| \\
&= |(\eta_1-\gamma_1)b_1 + (\eta_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| \\
&+ \sum_{j=2}^{\nu+1} |(\eta_j-\gamma_j)b_j + (\eta_{j-1}-\gamma_{j-1})c_{j-1}| \\
&= |((t_1+\tau)(1-\varepsilon)-\gamma_1)b_1 + (t_{\nu+1}(1-\varepsilon)-\gamma_{\nu+1})c_{\nu+1}| \\
&+ |(t_2(1-\varepsilon)-\gamma_2)b_2 + ((t_1+\tau)(1-\varepsilon)-\gamma_1)c_1| \\
&+ \sum_{j=3}^{\infty} |(t_j(1-\varepsilon)-\gamma_j)b_j + (t_{j-1}(1-\varepsilon)-\gamma_{j-1})c_{j-1}|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|p-z\|_1 &\leq \tau(1-\varepsilon)|b_1| + t_1\varepsilon|b_1| + t_{\nu+1}\varepsilon|c_{\nu+1}| \\
&+ |(t_1-\gamma_1)b_1 + (t_{\nu+1}-\gamma_{\nu+1})c_{\nu+1}| \\
&+ t_2\varepsilon|b_2| + \tau(1-\varepsilon)|c_1| + t_1\varepsilon|c_1| + |(t_2-\gamma_2b_2 + (t_1-\gamma_1)c_1| \\
&+ \sum_{j=3}^{\nu+1} t_j\varepsilon|b_j| + \sum_{j=3}^{\infty} t_{j-1}\varepsilon|c_{j-1}| + \sum_{j=3}^{\nu+1} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}| \\
&= \|y-z\|_1 - |(t_{\nu+2}-\gamma_{\nu+2})b_{\nu+2}| \\
&- \left(\sum_{j=\nu+3}^{\infty} |(t_j-\gamma_j)b_j + (t_{j-1}-\gamma_{j-1})c_{j-1}|\right) \\
&+ \varepsilon\sum_{j=1}^{\nu+1} t_j(|b_j|+|c_j|) + \tau(1-\varepsilon)(|b_1|+|c_1|)
\end{aligned}$$

Hence,

$$
\begin{aligned}
\|p-z\|_1 &\leq \|y-z\|_1 - \sum_{j=\nu+2}^{\infty} |(t_j-\gamma_j)b_j| + \sum_{k=\nu+2}^{\infty} |(t_k-\gamma_k)c_k| \\
&+ \frac{\xi}{\sum_{j=1}^{\nu+1} t_j + \tau}\left[\max_{1\leq l\leq \nu+1}(|b_l|+|c_l|)\right]\sum_{j=1}^{\nu+1} t_j \\
&+ \tau(1-\varepsilon)\hat{\Gamma} \\
&= \|y-z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k-\gamma_k|(|b_k|-|c_k|) \\
&+ \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau}\left(\xi\sum_{j=1}^{\nu+1} t_j + (\tau-\tau\varepsilon)\left(\sum_{j=1}^{\nu+1} t_j + \tau\right)\right) \\
&= \|y-z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k-\gamma_k|(|b_k|-|c_k|) \\
&+ \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau}\left((\xi+\tau)\sum_{j=1}^{\nu+1} t_j - \tau\varepsilon\sum_{j=1}^{\nu+1} t_j + \tau^2 - \tau^2\varepsilon\right) \\
&= \|y-z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k-\gamma_k|(|b_k|-|c_k|) \\
&+ \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau}\left((\xi+\tau)\left[\sum_{j=1}^{\nu+1} t_j + \tau\right] - (\xi+\tau)\tau - \tau\varepsilon\sum_{j=1}^{\nu+1} t_j + \tau^2 - \tau^2\varepsilon\right) \\
&= \|y-z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k-\gamma_k|(|b_k|-|c_k|) \\
&+ \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau}\left((\xi+\tau)\left[\sum_{j=1}^{\nu+1} t_j + \tau\right] - \xi\tau - \tau\varepsilon\sum_{j=1}^{\nu+1} t_j - \tau^2\varepsilon\right) \\
&\leq \|y-z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k-\gamma_k|(|b_k|-|c_k|) \\
&+ \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau}\left((\xi+\tau)\left[\sum_{j=1}^{\nu+1} t_j + \tau\right]\right) \\
&= \|y-z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k-\gamma_k|(|b_k|-|c_k|) + \hat{\Gamma}(\xi+\tau)
\end{aligned}
$$

Note that

$$\begin{aligned}\xi + \tau &= \sum_{k\in B}(\gamma_k - t_k) + \sum_{k\in A}(t_k - \gamma_k) \\ &= \sum_{k=\nu+2}^{\infty} |t_k - \gamma_k|\end{aligned}$$

Hence,

$$\begin{aligned}\|p - z\|_1 &\leq \|y - z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k - \gamma_k|(|b_k| - |c_k|) + \hat{\Gamma} \sum_{k=\nu+2}^{\infty} |t_k - \gamma_k| \\ &= \|y - z\|_1 - \sum_{k=\nu+2}^{\infty} |t_k - \gamma_k| \left[(|b_k| - |c_k|) - \hat{\Gamma}\right]\end{aligned}$$

Note that since, by hypothesis, $\exists\, \mu \geq \nu + 2$ with $t_\mu \neq \gamma_\mu$,

$$\sum_{k=\nu+2}^{\infty} |t_k - \gamma_k| \left[(|b_k| - |c_k|) - \hat{\Gamma}\right] > 0$$

and hence $\|p - z\|_1 < \|y - z\|_1$, as desired. □

**Claim 2.3.17.**

$$J_C := \inf_{y\in C} \|y - z\|_1 \geq \inf_{p\in Q} \|p - z\|_1 =: J_Q$$

*and*

$$\left[\|y_0 - z\|_1 = \inf_{y\in C} \|y - c\|_1 \Longrightarrow y_0 \in Q\right](\ddagger).$$

*Proof.* First note that by Claim 2.3.16,

$$\|y_0 - z\|_1 = \inf_{y\in C} \|y - c\|_1 \Longrightarrow y_0 \in Q.$$

To show that $J_C \geq J_Q$, fix $y \in C$. If $y \in Q$, then $\|y - z\|_1 \geq \|y - z\|_1$. So suppose that $y \notin Q$. Then there exists $\mu \geq \nu + 2$ such that $t_\mu \neq \gamma_\mu$.

Again, by Claim 2.3.16, there exists $p \in Q$ such that

$$\|y - z\|_1 > \|p - z\|_1$$

Hence, $J_C \geq J_Q$. □

However, $Q$ is a nonempty, norm compact (convex) subset of $\ell^1$. Thus, there exists $p_0 \in Q \subseteq C$ such that

$$\|y - z\|_1 \geq \|p_0 - z\|_1, \forall y \in C.$$

Let

$$\tilde{Q} := \{y \in C : \|y - z\|_1 \leq \|p_0 - z\|_1\}.$$

Then by (‡) in Claim 2.3.17, $\tilde{Q} \subseteq Q \subseteq C$.

Note that $\tilde{Q}$ is a closed, bounded, and convex set in $\ell^1$ that is nonempty and norm compact. However, since

$$r(y) = r(z) + \|y - z\|_1, \forall y \in \ell^1,$$

we have that $r(Ty) \leq r(y)$, $\forall y \in C$. Therefore, $y \in \tilde{Q} \implies Ty \in \tilde{Q}$.

Thus $T$ maps $\tilde{Q}$ into $\tilde{Q}$ and so by Brouwer's (or Schauder's) Fixed Point Theorem, there exists $v \in \tilde{Q} \subseteq Q \subseteq C$ such that $Tv = v$. □

In Theorem 2.2.1, Theorem 2.3.1, Theorem 2.3.6, Theorem 2.3.11, and Theorem 2.3.15, the perturbations involved only two coordinates at a time. In Theorem 2.3.18, we construct a set $C$ in a similar way to Theorem 2.3.15, but we allow for "$N_0$-coordinates" to be arbitrarily perturbed amongst themselves.

Define for all $j \in \mathbb{N}$

$$f_j := \sum_{k=1}^{\infty} b_{j,k} e_k$$

Assume that there exists $F \subseteq \mathbb{N}$ that is finite and nonempty such that $\forall j \in F, [b_{j,k} = 0, \forall k \notin F]$. In other words, $\forall j \in F$,

$$f_j = \sum_{k\in F} b_{j,k} e_k$$

Without loss of generality, we may assume that $F = \{1, 2, \ldots, \nu, \nu+1\}$, for some $\nu \in \mathbb{N}$. I.e. $\forall j \in F$,

$$f_j = \sum_{k=1}^{\nu+1} b_{j,k} e_k$$

Assume that

$$0 < \hat{\Gamma} := \max_{1\le j\le \nu+1} \left( \sum_{k=1}^{\infty} |b_{j,k}| \right) < |b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}|, \forall m \in \mathbb{N}\setminus F$$

Note that

$$\hat{\Gamma} := \max_{1\le j\le \nu+1} \left( \sum_{k=1}^{\nu+1} |b_{j,k}| \right)$$

Also assume

$$|b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}| > 0,\ \forall m \in F$$

which implies that $\exists\, \tilde{\Gamma} > 0$ such that $|b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}| \ge \tilde{\Gamma}, \forall m \in \mathbb{N}$.

There exists constants $U$ and $V$ such that

$$0 < U \le |b_{m,m}| \le V < \infty, \forall m \in \mathbb{N}$$

Each $f_j \in \ell^1$. Let $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in \ell^1$.

$$\begin{aligned}
\|\sum_{j=1}^{\infty} \alpha_j f_j\|_1 &= \|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}\alpha_j e_k\|_1 \\
&= \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} b_{j,k}\alpha_j \right| \\
&\le 2V\|\alpha\|_1
\end{aligned}$$

Also,

$$
\begin{aligned}
\|\sum_{j=1}^{\infty}\alpha_j f_j\|_1 &= \sum_{k=1}^{\infty}\left|b_{k,k}\alpha_k + \sum_{j\in\mathbb{N}\setminus\{k\}} b_{j,k}\alpha_j\right| \\
&\geq \sum_{k=1}^{\infty}\left(|b_{k,k}||\alpha_k| - \sum_{\substack{j=1\\ j\neq k}}^{\infty}|b_{j,k}||\alpha_j|\right) \\
&= \sum_{k=1}^{\infty}|b_{k,k}||\alpha_k| - \sum_{k=1}^{\infty}\sum_{\substack{j=1\\ j\neq k}}^{\infty}|b_{j,k}||\alpha_j| \\
&= \sum_{k=1}^{\infty}|b_{k,k}||\alpha_k| - \sum_{j=1}^{\infty}\left(\sum_{\substack{k=1\\ k\neq j}}^{\infty}|b_{j,k}|\right)|\alpha_j| \\
&= \sum_{m=1}^{\infty}|b_{m,m}||\alpha_m| - \sum_{m=1}^{\infty}\left(\sum_{k\in\mathbb{N}\setminus\{m\}}|b_{m,k}|\right)|\alpha_m| \\
&= \sum_{m=1}^{\infty}\left(|b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}}|b_{m,k}|\right)|\alpha_m| \\
&\geq \sum_{m=1}^{\infty}\tilde{\Gamma}|\alpha_m| \\
&= \tilde{\Gamma}\|\alpha\|_1
\end{aligned}
$$

Hence, $(f_j)_{j\in\mathbb{N}}$ spans an isomorphic copy of $\ell^1$ inside of $(\ell^1, \|\cdot\|_1)$.

Let $C \subseteq \ell^1$ be defined by

$$C := \overline{co}\{f_j : j\in\mathbb{N}\} = \left\{\sum_{j=1}^{\infty}t_j f_j : \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty}t_j = 1\right\}.$$

Note that $C$ is a closed, bounded, and convex subset of $\ell^1$.

**Theorem 2.3.18.** *The set $C$ has the FPP(n.e.).*

*Proof.* Fix a nonexpansive map $T : C \longrightarrow C$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be an approximate fixed point sequence for $T$, i.e., $\|Tx^{(k)} - x^{(k)}\|_1 \underset{k}{\longrightarrow} 0$.

Let $W$ be the weak$^*$ $= \sigma(\ell^1, c_0)$ - closure of $C$ in $\ell^1$. Then,

$$W = \left\{\sum_{j=1}^{\infty}\beta_j f_j : \text{each } \beta_j \geq 0 \text{ and } \sum_{j=1}^{\infty}\beta_j \leq 1\right\}.$$

Note that $\mathcal{B}_{\ell^1}$ is weak*-sequentially compact. Hence, there exists a subsequence $(x^{(k_j)})_{j\in\mathbb{N}}$ of $(x^{(k)})_{k\in\mathbb{N}}$ and $z \in W$ such that $x^{(k_j)} \underset{j}{\longrightarrow} z$ weak*. Without loss of generality, $x^{(k)} \underset{k}{\longrightarrow} z$ weak*.

Note that if $z \in C$, then as we have seen previously, $Tz = z$. Assume that $z = \sum_{j=1}^{\infty} \gamma_j f_j \in W \backslash C$.

For all $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu+1})$ with each $\lambda_j \in \mathbb{R}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{\nu+1} = 1$ (i.e. $\vec{\lambda} \in \Lambda$), define

$$h_{\vec{\lambda}} := (\gamma_1 + \lambda_1\delta)f_1 + (\gamma_2 + \lambda_2\delta)f_2 + \cdots + (\gamma_\nu + \lambda_\nu\delta)f_\nu + (\gamma_{\nu+1} + \lambda_{\nu+1}\delta)f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j.$$

Let $H := \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \right\}$.

Let

$$Q := H \cap C = \left\{ h_{\vec{\lambda}} : \vec{\lambda} \in \Lambda \text{ and } \gamma_j + \lambda_j\delta \geq 0, \forall j \in \{1, 2, \ldots, \nu+1\} \right\}.$$

As we have shown previously,

$$Q = \left\{ p = s_1 f_1 + s_2 f_2 + \cdots + s_\nu f_\nu + s_{\nu+1} f_{\nu+1} + \sum_{j=\nu+2}^{\infty} \gamma_j f_j : \text{each } s_j \geq 0 \right.$$
$$\left. \text{and } \sum_{j=1}^{\nu+1} s_j + \sum_{j=\nu+2}^{\infty} \gamma_j = 1 \right\}$$

Fix $y \in C$ of the form $y = \sum_{k=1}^{\infty} t_k f_k$. Note that

$$\begin{aligned} \|y - z\|_1 &= \left\| \sum_{j=1}^{\infty} (t_j - \gamma_j) f_j \right\|_1 \\ &= \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} b_{j,k}(t_j - \gamma_j) \right| \end{aligned}$$

**Claim 2.3.19.** *Suppose that $t_\mu \neq \gamma_\mu$ for at least one $\mu \geq \nu + 2$. Then $\exists\, p \in Q$ such that $\|p - z\|_1 < \|y - z\|_1$.*

*Proof.* Let

$$A := \{k \geq \nu + 2 : t_k > \gamma_k\}$$

and

$$B := \{\nu + 2, \nu + 3, \ldots\} \backslash A = \{k \geq \nu + 2 : t_k \leq \gamma_k\}.$$

Define $\tau := \sum_{k \in A} (t_k - \gamma_k)$. Note that $\tau \geq 0$ and $\tau = 0 \iff A = \phi$.

Let $w = \phi(y) := \sum_{j=1}^{\infty} \sigma_j f_j$, where $(\sigma_j)_{j \in \mathbb{N}}$ is defined as follows.

$$(\sigma_j)_{j \in \mathbb{N}} = (t_1 + \tau, t_2, \ldots, t_\nu, t_{\nu+1}, \sigma_{\nu+2}, \sigma_{\nu+3}, \ldots)$$

where $\forall k \in A, \sigma_k := \gamma_k$ and $\forall k \in B, \sigma_k := t_k$.

Note that $\sigma_j \leq \gamma_j, \forall j \geq \nu + 2$.

Define $E := \{k \geq \nu + 2 : \sigma_k < \gamma_k\}$ and $\xi := \sum_{k \in E} (\gamma_k - \sigma_k)$. Note that $\xi \geq 0$ and $\xi = 0 \iff E = \phi$. Also note that $E \subseteq B$ and so $\xi = \sum_{k \in B} (\gamma_k - \sigma_k) = \sum_{k \in B} (\gamma_k - t_k)$.

Note that, by hypothesis, either $\tau > 0$ or $\xi > 0$

Next, let $p = \psi(w) = (\psi \circ \phi)(y) := \sum_{j=1}^{\infty} \eta_j f_j$, where $(\eta_j)_{j \in \mathbb{N}}$ is defined as follows.

$$(\eta_j)_{j \in \mathbb{N}} := (\sigma_1(1 - \varepsilon), \sigma_2(1 - \varepsilon), \ldots, \sigma_\nu(1 - \varepsilon), \sigma_{\nu+1}(1 - \varepsilon), \gamma_{\nu+2}, \gamma_{\nu+3}, \ldots, \gamma_k, \ldots)$$

where

$$\varepsilon := \frac{\sum_{j=\nu+2}^{\infty} (\gamma_j - \sigma_j)}{\sum_{j=1}^{\nu+1} \sigma_j} = \frac{\xi}{\sum_{j=1}^{\nu+1} t_j + \tau}$$

It is easily checked that $\varepsilon \in [0, 1)$ and $\sum_{j=1}^{\infty} \eta_j = 1$, i.e. $p \in Q$.

Then, we have that

$$
\begin{aligned}
\|p-z\|_1 &= \sum_{k=1}^{\infty}\left|\sum_{j=1}^{\infty} b_{j,k}(\eta_j-\gamma_j)\right| \\
&= \sum_{k=1}^{\infty}\left|\sum_{j=1}^{\nu+1} b_{j,k}(\eta_j-\gamma_j)\right| \\
&= \sum_{k=1}^{\infty}\left|b_{1,k}((t_1+\tau)(1-\varepsilon)-\gamma_1)+\sum_{j=2}^{\nu+1} b_{j,k}(t_j(1-\varepsilon)-\gamma_j)\right| \\
&\leq \sum_{k=1}^{\infty}\tau(1-\varepsilon)|b_{1,k}|+\sum_{k=1}^{\infty}|b_{1,k}|t_1|\varepsilon+\sum_{k=1}^{\infty}\sum_{j=2}^{\nu+1}|b_{j,k}|t_j\varepsilon \\
&+\sum_{k=1}^{\infty}\left|b_{1,k}(t_1-\gamma_1)+\sum_{j=2}^{\nu+1} b_{j,k}(t_j-\gamma_j)\right| \\
&= \tau(1-\varepsilon)\sum_{k=1}^{\infty}|b_{1,k}|+\varepsilon\sum_{k=1}^{\infty}\sum_{j=1}^{\nu+1}|b_{j,k}|t_j+\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\nu+1} b_{j,k}(t_j-\gamma_j)\right|
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|p-z\|_1 &\leq \|y-z\|_1-\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\infty} b_{j,k}(t_j-\gamma_j)\right| \\
&+\tau(1-\varepsilon)\hat{\Gamma}+\hat{\Gamma}\varepsilon\sum_{j=1}^{\nu+1}t_j+\sum_{k=1}^{\infty}\left|\sum_{j=1}^{\nu+1} b_{j,k}(t_j-\gamma_j)\right| \\
&= \|y-z\|_1-\sum_{k=1}^{\nu+1}\left|\sum_{j=1}^{\infty} b_{j,k}(t_j-\gamma_j)\right|-\sum_{k=\nu+2}^{\infty}\left|\sum_{j=1}^{\infty} b_{j,k}(t_j-\gamma_j\right| \\
&+\tau(1-\varepsilon)\hat{\Gamma}+\hat{\Gamma}\varepsilon\sum_{j=1}^{\nu+1}t_j+\sum_{k=1}^{\nu+1}\left|\sum_{j=1}^{\nu+1} b_{j,k}(t_j-\gamma_j)\right| \\
&+\sum_{k=\nu+2}^{\infty}\left|\sum_{j=1}^{\nu+1} b_{j,k}(t_j-\gamma_j)\right|
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|p-z\|_1 \le \|y-z\|_1 &- \sum_{k=\nu+2}^{\infty}\left|b_{k,k}(t_k-\gamma_k)+\sum_{j\in\mathbb{N}\setminus\{k\}} b_{j,k}(t_j-\gamma_j)\right| \\
&+ \sum_{k=1}^{\nu+1}\left|\sum_{j=\nu+2}^{\infty} b_{j,k}(t_j-\gamma_j)\right| + \tau(1-\varepsilon)\hat{\Gamma} \\
&+ \sum_{k=\nu+2}^{\infty}\left|\sum_{j=1}^{\nu+1} b_{j,k}(t_j-\gamma_j)\right| + \hat{\Gamma}\varepsilon\sum_{j=1}^{\nu+1} t_j \\
&\le \|y-z\|_1 - \left[\sum_{k=\nu+2}^{\infty}|b_{k,k}||t_k-\gamma_k| - \sum_{k=\nu+2}^{\infty}\sum_{j\in\mathbb{N}\setminus\{k\}}|b_{j,k}||t_j-\gamma_j|\right] \\
&+ \sum_{k=1}^{\nu+1}\sum_{j=\nu+2}^{\infty}|b_{j,k}||t_j-\gamma_j| + \sum_{k=\nu+2}^{\infty}\sum_{j=1}^{\nu+1}|b_{j,k}||t_j-\gamma_j| \\
&+ \tau(1-\varepsilon)\hat{\Gamma} + \hat{\Gamma}\varepsilon\sum_{j=1}^{\nu+1} t_j
\end{aligned}
$$

Let

$$
\begin{aligned}
\Theta &:= \sum_{k=\nu+2}^{\infty}\sum_{j\in\mathbb{N}\setminus\{k\}}|b_{j,k}||t_j-\gamma_j| \\
\Omega &:= \sum_{k=1}^{\nu+1}\sum_{j=\nu+2}^{\infty}|b_{j,k}||t_j-\gamma_j| \\
Z &:= \sum_{k=\nu+2}^{\infty}\sum_{j=1}^{\nu+1}|b_{j,k}||t_j-\gamma_j|
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|p-z\|_1 \le \|y-z\|_1 - \left[\sum_{k=\nu+2}^{\infty}|b_{k,k}||t_k-\gamma_k| - \Theta\right] \\
+ \Omega + Z + \tau(1-\varepsilon)\hat{\Gamma} + \hat{\Gamma}\varepsilon\sum_{j=1}^{\nu+1} t_j
\end{aligned}
$$

Note that

$$\Theta = \sum_{k=\nu+2}^{\infty} \sum_{j=1}^{\nu+1} |b_{j,k}||t_j - \gamma_j| + \sum_{k=\nu+2}^{\infty} \sum_{\substack{j=\nu+2 \\ j\neq k}}^{\infty} |b_{j,k}||t_j - \gamma_j|$$

$$= Z + \sum_{j=\nu+2}^{\infty} \sum_{\substack{k=\nu+2 \\ k\neq j}}^{\infty} |b_{j,k}||t_j - \gamma_j|$$

Recall that $\forall j \in \{1, 2, \ldots, \nu+1\}, [b_{j,k} = 0, \forall k \geq \nu+2]$. Hence $Z = 0$, and

$$\Theta = \sum_{j=\nu+2}^{\infty} \left( \sum_{\substack{k=\nu+2 \\ k\neq j}}^{\infty} |b_{j,k}| \right) |t_j - \gamma_j|$$

Also,

$$\Omega = \sum_{j=\nu+2}^{\infty} \left( \sum_{k=1}^{\nu+1} |b_{j,k}| \right) |t_j - \gamma_j|$$

and so

$$\|p - z\|_1 \leq \|y - z\|_1 - \left[ \sum_{m=\nu+2}^{\infty} |b_{m,m}||t_m - \gamma_m| - (\Omega + \Theta) \right]$$
$$+ \tau(1-\varepsilon)\hat{\Gamma} + \hat{\Gamma}\varepsilon \sum_{j=1}^{\nu+1} t_j$$

Note that

$$\Omega + \Theta = \sum_{m=\nu+2}^{\infty} \left( \sum_{k=1}^{\nu+1} |b_{m,k}| \right) |t_m - \gamma_m|$$
$$+ \sum_{m=\nu+2}^{\infty} \left( \sum_{\substack{k=\nu+2 \\ k\neq m}}^{\infty} |b_{m,k}| \right) |t_m - \gamma_m|$$
$$= \sum_{m=\nu+2}^{\infty} \left( \sum_{\substack{k=1 \\ k\neq m}}^{\infty} |b_{m,k}| \right) |t_m - \gamma_m|$$

Hence,

$$\|p - z\|_1 \leq \|y - z\|_1 - \sum_{m=\nu+2}^{\infty} \left[ |b_{m,m}| - \sum_{\substack{k=1 \\ k \neq m}}^{\infty} |b_{m,k}| \right] |t_m - \gamma_m| + \hat{\Gamma} \left[ \varepsilon \sum_{j=1}^{\nu+1} t_j + (\tau - \tau\varepsilon) \right]$$

Call

$$W := \hat{\Gamma} \left[ \varepsilon \sum_{j=1}^{\nu+1} t_j + (\tau - \tau\varepsilon) \right]$$

Note that

$$\begin{aligned}
W &= \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau} \left[ \xi \sum_{j=1}^{\nu+1} t_j + (\tau - \tau\varepsilon) \left( \sum_{j=1}^{\nu+1} t_j + \tau \right) \right] \\
&= \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau} \left[ (\xi + \tau) \sum_{j=1}^{\nu+1} t_j - \tau\varepsilon \sum_{j=1}^{\nu+1} t_j + \tau^2 - \tau^2 \varepsilon \right] \\
&= \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau} \left[ (\xi + \tau) \left( \sum_{j=1}^{\nu+1} t_j + \tau \right) - (\xi + \tau)\tau - \tau\varepsilon \sum_{j=1}^{\nu+1} t_j + \tau^2 - \tau^2 \varepsilon \right] \\
&= \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau} \left[ (\xi + \tau) \left[ \sum_{j=1}^{\nu+1} t_j + \tau \right] - \xi\tau - \tau\varepsilon \sum_{j=1}^{\nu+1} t_j - \tau^2 \varepsilon \right] \\
&\leq \frac{\hat{\Gamma}}{\sum_{j=1}^{\nu+1} t_j + \tau} (\xi + \tau) \left[ \sum_{j=1}^{\nu+1} t_j + \tau \right] \\
&= \hat{\Gamma} \left[ \sum_{k \in B} (\gamma_k - t_k) + \sum_{k \in A} (t_k - \gamma_k) \right] \\
&= \hat{\Gamma} \left[ \sum_{k \in B} |t_k - \gamma_k| + \sum_{k \in A} |t_k - \gamma_k| \right] \\
&= \hat{\Gamma} \sum_{k=\nu+2}^{\infty} |t_k - \gamma_k|
\end{aligned}$$

Thus,

$$
\begin{aligned}
\|p-z\|_1 &\leq \|y-z\|_1 - \sum_{m=\nu+2}^{\infty}\left[|b_{m,m}| - \sum_{\substack{k=1\\k\neq m}}^{\infty}|b_{m,k}|\right]|t_m-\gamma_m| \\
&+ \hat{\Gamma}\sum_{m=\nu+2}^{\infty}|t_m-\gamma_m| \\
&= \|y-z\|_1 - \sum_{m=\nu+2}^{\infty}\left(\left[|b_{m,m}| - \sum_{\substack{k=1\\k\neq m}}^{\infty}|b_{m,k}|\right] - \hat{\Gamma}\right)|t_m-\gamma_m|
\end{aligned}
$$

Note that since, by hypothesis, $\exists\, \mu \geq \nu+2$ with $t_\mu \neq \gamma_\mu$,

$$
\sum_{m=\nu+2}^{\infty}\left(\left[|b_{m,m}| - \sum_{\substack{k=1\\k\neq m}}^{\infty}|b_{m,k}|\right] - \hat{\Gamma}\right)|t_m-\gamma_m| > 0
$$

Hence, $\|p-z\|_1 < \|y-z\|_1$, as desired. □

**Claim 2.3.20.**

$$
J_C := \inf_{y\in C}\|y-z\|_1 \geq \inf_{p\in Q}\|p-z\|_1 =: J_Q
$$

*and*

$$
\left[\|y_0-z\|_1 = \inf_{y\in C}\|y-c\|_1 \implies y_0 \in Q\right](\ddagger).
$$

*Proof.* First note that by Claim 2.3.19,

$$
\|y_0-z\|_1 = \inf_{y\in C}\|y-c\|_1 \implies y_0 \in Q.
$$

To show that $J_C \geq J_Q$, fix $y \in C$. If $y \in Q$, then $\|y-z\|_1 \geq \|y-z\|_1$. So suppose that $y \notin Q$. Then there exists $\mu \geq \nu+2$ such that $t_\mu \neq \gamma_\mu$.

Again, by Claim 2.3.19, there exists $p \in Q$ such that

$$
\|y-z\|_1 > \|p-z\|_1
$$

Hence, $J_C \geq J_Q$. □

However, $Q$ is a nonempty, norm compact (convex) subset of $\ell^1$. Thus, there exists $p_0 \in Q \subseteq C$ such that

$$\|y - z\|_1 \geq \|p_0 - z\|_1, \forall y \in C.$$

Let

$$\tilde{Q} := \{y \in C : \|y - z\|_1 \leq \|p_0 - z\|_1\}.$$

Then by (‡) in Claim 2.3.20, $\tilde{Q} \subseteq Q \subseteq C$.

Note that $\tilde{Q}$ is a closed, bounded, and convex set in $\ell^1$ that is nonempty and norm compact. However, since

$$r(y) = r(z) + \|y - z\|_1, \forall y \in \ell^1,$$

we have that $r(Ty) \leq r(y)$, $\forall y \in C$. Therefore, $y \in \tilde{Q} \Longrightarrow Ty \in \tilde{Q}$.

Thus $T$ maps $\tilde{Q}$ into $\tilde{Q}$ and so by Brouwer's (or Schauder's) Fixed Point Theorem, there exists $v \in \tilde{Q} \subseteq Q \subseteq C$ such that $Tv = v$. □

# 3.0 RESULTS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

## 3.1 A C.B.C. NON-WEAK$^*$-COMPACT SUBSET OF $\ell^1$ THAT HAS THE FIXED POINT PROPERTY FOR CERTAIN ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Fix $b \in (0,1)$ and define $f_1 := b\,e_1$, and $f_n := e_n$ for all $n \geq 2$. We also, define the closed, bounded, convex subset $K$ of $\left(\ell^1, \|\cdot\|_1\right)$ by

$$K := \left\{ \sum_{j=1}^{\infty} t_j\, f_j : \sum_{j=1}^{\infty} t_j = 1 \text{ and each } \ t_j \geq 0 \right\} .$$

In 1979 Goebel and Kuczumow [9] proved that for all mappings $T : K \longrightarrow K$ that are $\|\cdot\|_1$-nonexpansive, $T$ has a fixed point. Our first result is an extension of this theorem.

**Theorem 3.1.1.** *Let $K$ be as defined as above. Let $T : K \longrightarrow K$ be a mapping that is asymptotically nonexpansive and has an approximate fixed point sequence. Then $T$ has a fixed point.*

*In particular, all asymptotically nonexpansive, affine mappings $T : K \longrightarrow K$ have a fixed point.*

*Proof.* Consider the weak$^* = \sigma(\ell^1, c_0)$ topology on $\ell^1$. Recall that on bounded subsets of $\ell^1$, such as $K$, this topology is equivalent to the topology of coordinate-wise convergence. Also, $c_0$ is separable. Consequently, applying the Banach-Alaoglu theorem, we see that the closed unit ball $B_{\ell^1}$ is weak$^*$-sequentially compact.

We define $W :=$ the weak$^*$-closure of $K$. It is straightforward to check that

$$W = \left\{ \sum_{j=1}^{\infty} \beta_j\, f_j \ : \ \sum_{j=1}^{\infty} \beta_j \leq 1 \text{ and each } \ \beta_j \geq 0 \right\} .$$

Let $(x_n)_{n\in\mathbb{N}}$ be an approximate fixed point sequence for $T$ in $K$. By passing to a subsequence if necessary, we may assume that there exists an $x_0 \in W$ such that $x_n \underset{n}{\longrightarrow} x_0$ weak*.

<u>Case 1: $x_0 \in K$.</u>

We define $r(z)$, the asymptotic radius about $z$ with respect to $(x_n)_{n\in\mathbb{N}}$, by

$$r(z) := \limsup_{n\in\mathbb{N}} \|x_n - z\| \ , \text{ for all } z \in \ell^1 \ .$$

Goebel and Kuczumow [9], Lemma 3, proved that

$$r(z) = r(x_0) + \|z - x_0\| \ , \text{ for all } z \in \ell^1 \ . \quad (\dagger)$$

Since $T$ is asymptotically nonexpansive, there exists a sequence $(\lambda_m)_{m\in\mathbb{N}}$ in $[1,\infty)$, decreasing to 1, such that

$$\|T^m u - T^m v\| \leq \lambda_m \|u - v\| \ , \text{ for all } u, v \in K \ .$$

Fix $m \in \mathbb{N}$. For all $n \in \mathbb{N}$ we have that

$$\|T^m x_n - x_n\| \leq \sum_{j=1}^{m} \|T^j x_n - T^{j-1} x_n\| \leq \|T x_n - x_n\| \left( \sum_{j=1}^{m} \lambda_{j-1} \right) \longrightarrow 0 \ ,$$

as $n \longrightarrow \infty$. Next, fix $z \in K$ and $m \in \mathbb{N}$. Then,

$$\begin{aligned} r(T^m z) &= \limsup_{n\in\mathbb{N}} \|T^m z - x_n\| \\ &\leq \limsup_{n\in\mathbb{N}} \Big( \|T^m z - T^m x_n\| + \|T^m x_n - x_n\| \Big) \\ &\leq \limsup_{n\in\mathbb{N}} \|T^m z - T^m x_n\| + \limsup_{n\in\mathbb{N}} \|T^m x_n - x_n\| \\ &\leq \lambda_m \limsup_{n\in\mathbb{N}} \|z - x_n\| \end{aligned}$$

Thus,

$$r(T^m z) \leq \lambda_m \, r(z) \ , \text{ for all } z \in K \text{ and for all } m \in \mathbb{N} \ . \quad (\ddagger)$$

From $(\dagger)$ and$(\ddagger)$ we have that

$$r(x_0) + \|T^m x_0 - x_0\| = r(T^m x_0) \leq \lambda_m \, r(x_0) \ ;$$

and hence,

$$\|T^m x_0 - x_0\| \leq (\lambda_m - 1) r(x_0) \ .$$

Since $\lambda_m \longrightarrow 1$ as $m \longrightarrow \infty$, we see that

$$\|T^{m+1} x_0 - x_0\| \underset{m}{\longrightarrow} 0$$

and

$$\|T^{m+1} x_0 - T x_0\| \leq \lambda_1 \|T^m x_0 - x_0\| \underset{m}{\longrightarrow} 0 \ .$$

Consequently, by uniqueness of limits, $T x_0 = x_0$.

<u>Case 2: $x_0 \notin K$.</u>

The limit $x_0 \in W$, and so we may write $x_0 = \sum_{j=1}^{\infty} \beta_j f_j$, where each $\beta_j \geq 0$ and $\sum_{j=1}^{\infty} \beta_j < 1$. We define $\delta := 1 - \sum_{j=1}^{\infty} \beta_j \in (0, 1]$. Let $y_0 := (\beta_1 + \delta) f_1 + \sum_{j=2}^{\infty} \beta_j f_j \in K$. Fix an arbitrary $y := \sum_{j=1}^{\infty} \alpha_j f_j \in K$, so that each $\alpha_j \geq 0$ and $\sum_{j=1}^{\infty} \alpha_j = 1$. Then,

$$\begin{aligned}
\|y - x_0\| = \left\| \sum_{n=1}^{\infty} (\alpha_n - \beta_n) f_n \right\| &= |\alpha_1 - \beta_1| \|f_1\| + \sum_{n=2}^{\infty} |\alpha_n - \beta_n| \|f_n\| \\
&= b\, |\alpha_1 - \beta_1| + \sum_{n=2}^{\infty} |\alpha_n - \beta_n| \\
&= b \left( |\alpha_1 - \beta_1| + \sum_{n=2}^{\infty} |\alpha_n - \beta_n| \right) + (1 - b) \sum_{n=2}^{\infty} |\alpha_n - \beta_n| \\
&\geq b \left| \sum_{n=1}^{\infty} \alpha_n - \sum_{n=1}^{\infty} \beta_n \right| + (1 - b) \sum_{n=2}^{\infty} |\alpha_n - \beta_n| \\
&= b\, |1 - (1 - \delta)| + (1 - b) \sum_{n=2}^{\infty} |\alpha_n - \beta_n| = b\, \delta + (1 - b) \sum_{n=2}^{\infty} |\alpha_n - \beta_n| \\
&\geq b\, \delta \ ;
\end{aligned}$$

with equality if and only if $\alpha_n = \beta_n$, for all $n \geq 2$. Equality implies that

$$\begin{aligned}
\alpha_1 = 1 - \sum_{n=2}^{\infty} \alpha_n &= 1 - \sum_{n=2}^{\infty} \beta_n \\
&= 1 + \beta_1 - \sum_{n=1}^{\infty} \beta_n = 1 + \beta_1 - (1 - \delta) \\
&= \beta_1 + \delta \ .
\end{aligned}$$

Hence we have that for all $y \in K$, $\|y - x_0\| \geq \|y_0 - x_0\|$, with equality if and only if $y = y_0$. It follows that

$$\begin{aligned} r(y_0) &= \|y_0 - x_0\| + r(x_0) \quad , \text{ by } (\dagger) \ ; \\ &\leq \|T^m y_0 - x_0\| + r(x_0) \\ &= r(T^m y_0) \\ &\leq \lambda_m \, r(y_0) \quad , \text{ by } (\ddagger) \ . \end{aligned}$$

But $\lambda_m \underset{m}{\longrightarrow} 1$. Therefore,

$$\|T^m y_0 - x_0\| + r(x_0) \underset{m}{\longrightarrow} \|y_0 - x_0\| + r(x_0) \ ;$$

and so

$$\|T^m y_0 - x_0\| \underset{m}{\longrightarrow} \|y_0 - x_0\|.$$

In this situation, we claim that the previous statement implies

$$\|T^m y_0 - y_0\| \underset{m}{\longrightarrow} 0 \ . \quad (\bigstar)$$

Suppose we have shown $(\bigstar)$. It follows that

$$\begin{aligned} \|Ty_0 - y_0\| &= \|Ty_0 - T^m y_0 + T^m y_0 - y_0\| \\ &\leq \|Ty_0 - T^m y_0\| + \|T^m y_0 - y_0\| \\ &\leq \lambda_1 \|y_0 - T^{m-1} y_0\| + \|T^m y_0 - y_0\| \underset{m}{\longrightarrow} 0 \ ; \end{aligned}$$

and hence $Ty_0 = y_0$.

It remains to show that $(\bigstar)$. We will do this by establishing the following two facts.

1. For all $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that for all $y \in K$,

$$\big| \|y - x_0\| - \|y_0 - x_0\| \big| < \gamma \implies \|y - y_0\| < \varepsilon \ .$$

2. For all sequences $(z_m)_{m \in \mathbb{N}}$ in $K$ such that $\|z_m - x_0\| \underset{m}{\longrightarrow} \|y_0 - x_0\|$, it follows that $\|z_m - y_0\| \underset{m}{\longrightarrow} 0$.

Note that (2) follows immediately from (1), and our desired conclusion follows from (2) with $(z_m)_{m\in\mathbb{N}}$ defined by

$$z_m := T^m y_0 \ , \ \text{for all } m \in \mathbb{N} \ .$$

We will now prove (1). Fix $\varepsilon > 0$ and recall that $0 < b < 1$. Choose

$$\gamma(\varepsilon) := \frac{(1-b)}{(1+b)}\,\varepsilon \ \in (0,\infty) \ .$$

Fix $y = \sum_{j=1}^{\infty} \alpha_j\, f_j \in K$ such that $\big|\|y-x_0\| - \|y_0-x_0\|\big| = \|y-x_0\| - \|y_0-x_0\| < \gamma$. Then,

$$\begin{aligned}\|y-y_0\| &= \left\| \sum_{j=1}^{\infty} \alpha_j\, f_j - (\beta_1+\delta)\, f_1 - \sum_{j=2}^{\infty} \beta_j\, f_j \right\| \\ &= |\alpha_1 - \beta_1 - \delta|\, b + \sum_{j=2}^{\infty} |\alpha_j - \beta_j| \ .\end{aligned}$$

Note that

$$\begin{aligned}|\alpha_1 - \beta_1 - \delta| &= \left|\alpha_1 - \beta_1 - \left(1 - \sum_{j=1}^{\infty}\beta_j\right)\right| = \left|\alpha_1 - 1 + \sum_{j=2}^{\infty}\beta_j\right| \\ &= \left|\sum_{j=2}^{\infty}\beta_j - \sum_{j=2}^{\infty}\alpha_j\right| \le \sum_{j=2}^{\infty}|\alpha_j - \beta_j| \ .\end{aligned}$$

Therefore,

$$\begin{aligned}
\|y - y_0\| &\le b \sum_{j=2}^{\infty} |\alpha_j - \beta_j| + \sum_{j=2}^{\infty} |\alpha_j - \beta_j| \\
&= (1+b) \sum_{j=2}^{\infty} |\alpha_j - \beta_j| = \frac{(1+b)}{(1-b)} (1-b) \sum_{j=2}^{\infty} |\alpha_j - \beta_j| \\
&= \frac{(1+b)}{(1-b)} \left[ b\,|1 - (1-\delta)| + (1-b) \sum_{j=2}^{\infty} |\alpha_j - \beta_j| - b\,\delta \right] \\
&= \frac{(1+b)}{(1-b)} \left[ b \left| \sum_{j=1}^{\infty} \alpha_j - \sum_{j=1}^{\infty} \beta_j \right| + (1-b) \sum_{j=2}^{\infty} |\alpha_j - \beta_j| - b\,\delta \right] \\
&\le \frac{(1+b)}{(1-b)} \left[ b \sum_{j=1}^{\infty} |\alpha_j - \beta_j| + (1-b) \sum_{j=2}^{\infty} |\alpha_j - \beta_j| - b\,\delta \right] \\
&= \frac{(1+b)}{(1-b)} \left[ b\,|\alpha_1 - \beta_1| + \sum_{j=2}^{\infty} |\alpha_j - \beta_j| - b\,\delta \right] \\
&= \frac{(1+b)}{(1-b)} \left[\, \|y - x_0\| - \|y_0 - x_0\| \,\right] \\
&< \frac{(1+b)}{(1-b)}\, \gamma \\
&= \varepsilon\ .
\end{aligned}$$

This proves (1); and therefore the proof of Theorem 3.1.1 is complete. □

We thank Paddy Dowling for pointing out to us that the proof of Case 1 above generalizes, essentially without change, to prove the following corollary.

**Corollary 3.1.2.** *Let $C$ be any nonempty, weak*-compact, convex subset of $\ell^1$. Let $T : C \longrightarrow C$ be a mapping that is asymptotically nonexpansive and has an approximate fixed point sequence. Then $T$ has a fixed point.*

*In particular, all asymptotically nonexpansive affine mappings $T : C \longrightarrow C$ have a fixed point.*

## 3.2 THE RIGHT SHIFT $R$ ON THE C.B.C. SET $K$

Let $b \in (0,1)$. Let $K = K_b$ be the set defined in the previous section. Consider the *right shift map* $R : K \longrightarrow K$ defined by

$$R(x) := \sum_{n=1}^{\infty} t_n \, f_{n+1} \ , \text{ for all } x = \sum_{n=1}^{\infty} t_n \, f_n \in K \ .$$

Note that $R$ is affine and fixed point free. Fix $x = \sum_{n=1}^{\infty} t_n \, f_n$ and $y = \sum_{n=1}^{\infty} s_n \, f_n$ in $K$. Then

$$\|x-y\|_1 = b\,|t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \ .$$

Further, for all $m \in \mathbb{N}$, it is easy to check that

$$\|R^m(x) - R^m(y)\|_1 = |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \ .$$

Fix $m \in \mathbb{N}$. Then

$$\begin{aligned}
\|R^m(x) - R^m(y)\|_1 &= |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&= \left| \left(1 - \sum_{k=2}^{\infty} t_k\right) - \left(1 - \sum_{k=2}^{\infty} s_k\right) \right| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&= \left| \sum_{k=2}^{\infty} (t_k - s_k) \right| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&\leq 2 \sum_{n=2}^{\infty} |t_n - s_n| \leq 2 \left( b\,|t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&= 2\,\|x - y\|_1 \ .
\end{aligned}$$

Further,

$$\begin{aligned}
\|R^m(x) - R^m(y)\|_1 &= |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&= \frac{1}{b} \left( b\,|t_1 - s_1| + b \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&\leq \frac{1}{b} \left( b\,|t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&= \frac{1}{b}\,\|x - y\|_1 \ .
\end{aligned}$$

Therefore,

$$\|R^m(x) - R^m(y)\|_1 \leq \min\left\{2, \frac{1}{b}\right\} \|x - y\|_1 \ .$$

Consider this uniform Lipschitz constant $M_b := \min\left\{2, 1/b\right\}$. Note that for $b \in [1/2, 1)$, $M_b = 1/b \in (1, 2]$. E.g., if $b = 2/3$, then $M_b = 3/2$. Also, for $b \in (0, 1/2]$, $M_b = 2$.

An initial question is: "Is $M_b$ best (i.e., smallest) possible?" The answer is "No", as we will see below.

**Theorem 3.2.1.** *Let $b \in (0,1)$. Let $f_1 := b\, e_1$, and $f_n := e_n$ for all $n \geq 2$. Define*

$$K := K_b := \left\{ \sum_{j=1}^{\infty} t_j f_j : \sum_{j=1}^{\infty} t_j = 1 \ \textit{and each} \ \ t_j \geq 0 \right\} .$$

*Consider the right shift map $R : K \longrightarrow K$ defined by*

$$R(x) := \sum_{n=1}^{\infty} t_n\, f_{n+1} \ , \ \textit{for all}\ x = \sum_{n=1}^{\infty} t_n\, f_n \in K \ .$$

*Let $W_b := 2/(1+b)$. Then, for all $m \in \mathbb{N}$, for every $x, y \in K_b$,*

$$\|R^m(x) - R^m(y)\|_1 \leq W_b\, \|x - y\|_1 \ .$$

*Moreover, $W_b$ is the smallest possible uniform Lipschitz constant for $R$.*

*Proof.* Fix $m \in \mathbb{N}$. Let $x = \sum_{n=1}^{\infty} t_n\, f_n$ and $y = \sum_{n=1}^{\infty} s_n\, f_n$ in $K$. Further, let $\tau \in [0,1]$. We have that

$$\begin{aligned}
\|R^m(x) - R^m(y)\|_1 &= |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&= \tau\, |t_1 - s_1| + (1-\tau)\, |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&= \tau\, |t_1 - s_1| + (1-\tau) \left| \sum_{k=2}^{\infty} t_k - \sum_{k=2}^{\infty} s_k \right| + \sum_{n=2}^{\infty} |t_n - s_n| \\
&\leq \tau\, |t_1 - s_1| + (2-\tau) \sum_{n=2}^{\infty} |t_n - s_n| \\
&\overset{(\bigstar)}{=} \frac{\tau}{b} \left( b\, |t_1 - s_1| + \frac{(2-\tau)\, b}{\tau} \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&\overset{(\bigstar\bigstar)}{=} (2-\tau) \left( \frac{\tau}{2-\tau}\, |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \right) .
\end{aligned}$$

In order to use ($\bigstar$) above to gain a uniform Lipschitz estimate for $R$, we require

$$\frac{(2-\tau)\,b}{\tau} \leq 1 \iff 2\,b - \tau\,b \leq \tau \iff \frac{2\,b}{1+b} \leq \tau \ .$$

Now, to minimize $\tau/b$, we minimize $\tau$; i.e., let

$$\tau := \frac{2\,b}{1+b} \in (0,1) \implies \frac{\tau}{b} = \frac{2}{1+b} \ .$$

On the other hand, in order to use ($\bigstar\bigstar$) above to get a uniform Lipschitz estimate for $R$, we require

$$\frac{\tau}{2-\tau} \leq b \iff \tau \leq 2\,b - b\,\tau \iff \tau \leq \frac{2\,b}{1+b} \ .$$

So, to minimize $2-\tau$, we maximize $\tau$; i.e., let

$$\tau := \frac{2\,b}{1+b} \in (0,1) \implies 2-\tau = \frac{2}{1+b} \ .$$

We see that both ($\bigstar$) and ($\bigstar\bigstar$) lead to our desired uniform Lipschitz constant for $R$; i.e., $W_b := 2/(1+b)$. To see that $W_b$ is best possible, consider $x := f_1$ and $y := f_2 \in K$. Then $\|f_1 - f_2\|_1 = \|b\,e_1 - e_2\|_1 = b+1$. Further, for all $m \in \mathbb{N}$,

$$\begin{aligned} \|R^m(f_1) - R^m(f_2)\|_1 &= \|f_{m+1} - f_{m+2}\|_1 = \|e_{m+1} - e_{m+2}\|_1 = 1+1 = 2 \\ &= W_b\,\|f_1 - f_2\|_1 \ . \end{aligned}$$

□

**Open questions.**

(1) For a given $b \in (0,1)$, are there any fixed point free, affine, uniformly Lipschitzian mappings $U : K \longrightarrow K$ with (best) uniform Lipschitz constant $M \in (1, 2/(1+b))$?

(2) Can we find an explicit example of a mapping $U : K \longrightarrow K$ that is asymptotically nonexpansive and affine, and yet *not* nonexpansive? (Such a $U$ necessarily has a fixed point in $K$, by Theorem 3.1.1 above.)

**An answer for question** (2)**:** "Yes".

Let $r \in \mathbb{R}$ such that $\frac{b+1}{2} < r < 1$. Define $U : K \longrightarrow K$ by

$$U : x = (bt_1, t_2, t_3, t_4 \ldots) \mapsto (b(1-r), rt_1, rt_2, rt_3, \ldots) \ .$$

By induction, we see that for arbitrary $x = \sum_{n=1}^{\infty} t_n f_n$ and $y = \sum_{n=1}^{\infty} s_n f_n \in K$, and for all $m \in \mathbb{N}$,

$$\|U^m x - U^m y\|_1 = r^m \sum_{n=1}^{\infty} |t_n - s_n|.$$

Using the techniques in Theorem 3.2.1, we may write

$$\begin{aligned}
\|U^m(x) - U^m(y)\|_1 \quad &= \quad r^m \left( |t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&= \quad r^m \left( \tau\,|t_1 - s_1| + (1-\tau)\,|t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&= \quad r^m \left( \tau\,|t_1 - s_1| + (1-\tau) \left| \sum_{k=2}^{\infty} t_k - \sum_{k=2}^{\infty} s_k \right| + \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&\leq \quad r^m \left( \tau\,|t_1 - s_1| + (2-\tau) \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&\overset{(\bigstar)}{=} \quad \frac{\tau\, r^m}{b} \left( b\,|t_1 - s_1| + \frac{(2-\tau)\,b}{\tau} \sum_{n=2}^{\infty} |t_n - s_n| \right) \\
&\overset{(\bigstar\bigstar)}{=} \quad (2-\tau)\, r^m \left( \frac{\tau}{2-\tau}\,|t_1 - s_1| + \sum_{n=2}^{\infty} |t_n - s_n| \right) \ .
\end{aligned}$$

Similarly to before, from ($\bigstar$) or ($\bigstar\bigstar$), we obtain a Lipschitz constant for $U^m$ of $W_{b,m} := (2\,r^m)/(1+b)$. To see that $W_{b,m}$ is best possible, consider again $x := f_1$ and $y := f_2 \in K$. Then $\|f_1 - f_2\|_1 = \|b\,e_1 - e_2\|_1 = b + 1$. Further, for all $m \in \mathbb{N}$,

$$\begin{aligned}
\|U^m(f_1) - U^m(f_2)\|_1 \quad &= \quad \|r^m\, f_{m+1} - r^m\, f_{m+2}\|_1 = \|r^m\, e_{m+1} - r^m\, e_{m+2}\|_1 = r^m + r^m = 2\,r^m \\
&= \quad W_{b,m}\, \|f_1 - f_2\|_1 \ .
\end{aligned}$$

In summary, for all $m \in \mathbb{N}$,

$$\|U^m x - U^m y\|_1 \leq \frac{2r^m}{b+1} \|x - y\|_1 \ , \text{ for all } x, y \in K \ ;$$

and the Lipschitz constant for $U^m$ is best possible. Since $r^m \underset{m}{\longrightarrow} 0$, it follows that $U$ is an affine, asymptotically nonexpansive map on $K$. On the other hand, since $r \in \left(\frac{b+1}{2}, 1\right)$, we have that $\frac{2r}{b+1} > 1$. Hence,

$$\|Uf_1 - Uf_2\|_1 > \|f_1 - f_2\|_1 \ ;$$

and consequently $U$ is not nonexpansive.

We note that $U^m$ is a *strict contraction* for all sufficiently large $m$, and $C$ with the $\ell^1$ metric is a complete metric space. Therefore, there is a second way to see that that $U$ has a fixed point in $K$. Indeed, by a corollary of the Banach Contraction Mapping Theorem, we may conclude that $U$ has a unique fixed point in $K$.

**Open Question.** Does there exist an affine, asymptotically nonexpansive map $V$ on $K$ such that $V$ is not nonexpansive, and for which each $V^m$ is not a strict contraction?

## 3.3 A C.B.C. NON-WEAK*-COMPACT SUBSET OF $\ell^1$ THAT HAS THE FIXED POINT PROPERTY FOR NONEXPANSIVE MAPPINGS, BUT FAILS THE F.P.P. FOR AFFINE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Following Section 3 of [7], we define the closed, bounded, and convex subset $G$ of $\ell^1$ by

$$G := \left\{ q + \sum_{n=1}^{\infty} \alpha_n\, u_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

Here,

$$\begin{aligned} q &:= \left( \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots, \frac{1}{3^n}, \ldots \right) ; \\ u_1 &:= \frac{1}{2}\, e_1 \ ; \text{ and} \\ u_n &:= \left( \frac{1}{2} + \frac{1}{3^{2n}} \right) e_{2n-1} - \frac{1}{3^{2n}}\, e_{2n} \ , \text{ for all } n \geq 2 \ . \end{aligned}$$

In [7] it is proven that every nonexpansive map $T$ on $G$ has a fixed point.

Consider the right shift map $R : G \longrightarrow G$ defined by

$$R(x) := q + \sum_{n=1}^{\infty} \alpha_n\, u_{n+1} \ , \text{ for all } x = q + \sum_{n=1}^{\infty} \alpha_n\, u_n \in G \ .$$

The map $R$ is clearly fixed point free; and so $R$ cannot be nonexpansive. It is also affine, and thus has an approximate fixed point sequence.

**Proposition 3.3.1.** *The mapping $R : G \longrightarrow G$ is asymptotically nonexpansive with an approximate fixed point sequence, and $R$ has no fixed point in $G$.*

*Proof.* It remains to show that $R : G \longrightarrow G$ is asymptotically nonexpansive. Note that

$$\|u_1\|_1 = \frac{1}{2} \qquad \text{and} \qquad \|u_n\|_1 = \frac{1}{2} + \frac{2}{3^{2n}} \ , \text{ for all } n \geq 2 \ .$$

Fix $x = q + \sum_{n=1}^{\infty} \alpha_n\, u_n$ and $y = q + \sum_{n=1}^{\infty} \gamma_n\, u_n$ in $G$ and $m \in \mathbb{N}$. Then,

$$\|x - y\|_1 = \frac{1}{2}|\alpha_1 - \gamma_1| + \sum_{n=2}^{\infty} \left( \frac{1}{2} + \frac{2}{3^{2n}} \right) |\alpha_n - \gamma_n|$$

and

$$\|R^m(x) - R^m(y)\|_1 = \left(\frac{1}{2} + \frac{2}{3^{2(m+1)}}\right)|\alpha_1 - \gamma_1| + \sum_{n=2}^{\infty}\left(\frac{1}{2} + \frac{2}{3^{2(m+n)}}\right)|\alpha_n - \gamma_n| \ .$$

It is straightforward to check that

$$\|R^m(x) - R^m(y)\|_1 \leq \lambda_m \|x - y\|_1 \ , \text{ for all } x, y \in G \ ;$$

where

$$\lambda_m := \frac{\left(\frac{1}{2} + \frac{2}{3^{2(m+1)}}\right)}{\left(\frac{1}{2}\right)} = 1 + \frac{4}{3^{2(m+1)}} \ .$$

Note that $(\lambda_m)_{m\in\mathbb{N}}$ is a decreasing sequence that converges to 1. Therefore, $R$ is asymptotically nonexpansive. □

## 3.4 A DISTANCE TO A SIMPLEX THEOREM AND ITS COROLLARY

Let $(a_i)_{i\in\mathbb{N}}$ be any bounded sequence of non-negative real numbers, and set $f^i = (1 + a_i)e_i$. Following Goebel and Kuczumow [9], Example 3.1, define

$$C := \left\{ x = \sum_{i=1}^{\infty} \lambda_i f^i : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{\infty} \lambda_i = 1 \right\}$$

The weak* closure of $C$ is

$$\overline{C} = \left\{ z = \sum_{i=1}^{\infty} \mu_i f^i : \mu_i \geq 0 \text{ and } \sum_{i=1}^{\infty} \mu_i \leq 1 \right\}$$

Since the sequence $(a_i)_{i\in\mathbb{N}}$ is bounded, $\exists B \in \mathbb{R}$ such that $a_i \leq B, \forall i \in \mathbb{N}$. Let $a := \inf_{i\in\mathbb{N}} a_i$ and $N_0 := \{i : a_i = a\}$. Next, $\forall z \in \overline{C}$, set $\delta_z := 1 - \sum_{i=1}^{\infty} \mu_i$. Goebel and Kuczomow showed that $\forall z \in \overline{C}$, $\text{dist}(z, C) = \delta_z(1 + a)$.

For any $z \in \overline{C}$, define

$$\begin{aligned} \text{Proj}(z) :&= \left\{ y \in C : \|z - y\|_1 = \text{dist}(z, C) \right\} \\ &= \overline{co}\left\{ z + \delta_z f^i : i \in N_0 \right\} \end{aligned}$$

Note that in comparison to Theorem 2.0.4, we are using $b_i = 1 + a_i$ and $f_i = f^i$. Also, the set $N_0$ corresponds, as before, to the set $F$ in Theorem 2.3.15 and Theorem 2.3.18.

**Theorem 3.4.1.** *Fix $n \in \mathbb{N}$ $(n \geq 2)$. Let $N_0 \subseteq \mathbb{N}$ with $\#(N_0) = n$. Let $(\sigma_j)_{j\in N_0}$ be such that each $\sigma_j \in \mathbb{R}$.*

*Define*

$$\Phi_n^{(N_0)} = \Phi_n = \left\{ s = (s_j)_{j\in N_0} \in \mathbb{R}^n : \textit{each } s_j \geq 0 \textit{ and } \sum_{j\in N_0} s_j = 1 \right\}$$

$$G_n^{(N_0)} = G_n = \sum_{j\in N_0} |\sigma_j - \alpha_j|, \; \forall \alpha = (\alpha_j)_{j\in N_0} \in \Phi_n$$

1. *If* ($\bigstar_1$) [$\exists j \in N_0$ *such that* $\sigma_j > 1$] *or* ($\bigstar_2$) [$\exists j \neq k \in N_0$ *such that* $\sigma_j + \sigma_k > 1$] *or* $\cdots$ *or* ($\bigstar_{n-1}$) [*there exist distinct integers* $j_1, \ldots, j_{n-1} \in N_0$ *such that* $\sum_{\nu=1}^{n-1} \sigma_{j_\nu} > 1$], *then*

$$\min_{\alpha \in \Phi_n} G_n(\alpha) \leq \sum_{m \in N_0} |\sigma_m| - 1.$$

2. *If not* ($\bigstar_1$) *and not* ($\bigstar_2$) *and* $\cdots$ *and not* ($\bigstar_{n-1}$), *then*

$$\min_{\alpha \in \Phi_n} G_n(\alpha) \leq \left| \sum_{m \in N_0} \sigma_m - 1 \right|.$$

*Proof.* We begin by proving the base case, that is, when $\#(N_0) = 2$. Without loss of generality, assume that $N_0 = \{1, 2\}$.

(1) Assume that there is a $j \in N_0$ such that $\sigma_j > 1$. Without loss of generality, assume that $\sigma_1 > 1$. Then for any $\alpha = (\alpha_1, \alpha_2) \in \Phi_2$,

$$G_2(\alpha_1, \alpha_2) = |\sigma_1 - \alpha_1| + |\sigma_2 - \alpha_2|$$

Hence,

$$\begin{aligned}\min_{\alpha \in \Phi_2} G_2(\alpha) \leq G_2(1, 0) &= |\sigma_1 - 1| + |\sigma_2| \\ &= \sigma_1 - 1 + |\sigma_2| \\ &\leq |\sigma_1| - 1 + |\sigma_2| \\ &= \sum_{j=1}^{2} |\sigma_j| - 1,\end{aligned}$$

as desired.

(2) Assume not ($\bigstar_1$), that is, $\sigma_1, \sigma_2 \in (-\infty, 1]$.

Case 2(a), $\sigma_1, \sigma_2 < 0$.

Note that for all $\alpha = (\alpha_1, \alpha_2) \in \Phi_2$, $\sigma_j - \alpha_j < 0$ for $j = 1, 2$. Fix $\alpha \in \Phi_2$.

$$
\begin{aligned}
G_2(\alpha) &= |\sigma_1 - \alpha_1| + |\sigma_2 - \alpha_2| \\
&= -(\sigma_1 - \alpha_1) + (-(\sigma_2 - \alpha_2)) \\
&= -\sigma_1 + \alpha_1 - \sigma_2 + \alpha_2 \\
&= -(\sigma_1 + \sigma_2) + (\alpha_1 + \alpha_2) \\
&= -(\sigma_1 + \sigma_2) + 1 \\
&= -[(\sigma_1 + \sigma_2) - 1] \\
&= |\sigma_1 + \sigma_2 - 1|
\end{aligned}
$$

Hence,

$$\min_{\alpha \in \Phi_2} G_2(\alpha) \leq \left| \sum_{j=1}^{2} \sigma_j - 1 \right|$$

Case 2(b) $\sigma_1 \in [0,1]$ or $\sigma_2 \in [0,1]$. Without loss of generality, assume $\sigma_1 \in [0,1]$. Note that $1 - \sigma_1 \in [0,1]$. Then, $(\sigma_1, 1 - \sigma_1) \in \Phi_2$, and

$$G_2(\sigma_1, 1 - \sigma_1) = 0 + |\sigma_2 - (1 - \sigma_1)| = |\sigma_1 + \sigma_2 - 1|$$

Hence,

$$\min_{\alpha \in \Phi_2} G_2(\alpha) \leq G_2(\sigma_1, 1 - \sigma_1) = \left| \sum_{j=1}^{2} \sigma_j - 1 \right|$$

Therefore the lemma is true for $\#(N_0) = 2$.

Next, we show that the lemma is true for the general case via induction. Assume that the lemma is true for $\#(N_0) = n - 1$. We show that this implies that the lemma holds for $\#(N_0) = n$.

Without loss of generality, assume $N_0 = \{1, 2, 3, \ldots, n-1, n\}$. Fix $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \sigma_n \in \mathbb{R}$.

(1) ($\bigstar_1$). Assume $\exists j \in N_0$ such that $\sigma_j > 1$. Without loss of generality, $\sigma_1 > 1$. Then,

$$\begin{aligned}\min_{\alpha \in \Phi_n} G_n(\alpha) \leq G(1,0,0,\ldots,0,0) &= |\sigma_1 - 1| + |\sigma_2| + \cdots + |\sigma_n| \\ &= \sigma_1 - 1 + |\sigma_2| + \cdots + |\sigma_n| \\ &= |\sigma_1| - 1 + |\sigma_2| + \cdots + |\sigma_n| \\ &= \sum_{j=1}^{n} |\sigma_j| - 1\end{aligned}$$

Hence,

$$\min_{\alpha \in \Phi_n} G_n(\alpha) \leq \sum_{j=1}^{n} |\sigma_j| - 1$$

($\bigstar_2$) Assume that there exists distinct integers $j, k \in \{1, 2, \ldots, n\}$ such that $\sigma_j + \sigma_k > 1$. Without loss of generality, assume $\sigma_1 + \sigma_2 > 1$.

Fix $\alpha \in \Phi_n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n)$. Set $\alpha_n = 0$. Let

$$\widetilde{N_0} = \{1, 2, \ldots, n-1\}, \quad \Phi_{n-1} = \Phi_{n-1}^{(\widetilde{N_0})}, \quad G_{n-1} = G_{n-1}^{(\widetilde{N_0})}$$

Then,

$$G_n(\alpha) = G_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) + |\sigma_n|$$

Hence,

$$\begin{aligned}\min_{\alpha \in \Phi_n} G_n(\alpha) &\leq \min_{\alpha \in \Phi_{n-1}} \Big[G_{n-1}(\alpha) + |\sigma_n|\Big] \\ &= \Big[\min_{\alpha \in \Phi_{n-1}} G_{n-1}(\alpha)\Big] + |\sigma_n|\end{aligned}$$

By assumption, the lemma is true for $\widetilde{N_0}$. Then by (1)($\bigstar_2$) for $\Phi_{n-1}$,

$$\begin{aligned}\min_{\alpha\in\Phi_n} G_n(\alpha) &\le \left[\min_{\alpha\in\Phi_{n-1}} G_{n-1}(\alpha)\right] + |\sigma_n| \\ &\le \sum_{j=1}^{n-1} |\sigma_j| - 1 + |\sigma_n| \\ &= \sum_{j=1}^{n} |\sigma_j| - 1\end{aligned}$$

($\bigstar_3$) Assume that there exist distinct $i, j, k \in N_0$ such that $\sigma_i + \sigma_j + \sigma_k > 1$. Without loss of generality, $\sigma_1 + \sigma_2 + \sigma_3 > 1$. Again, fix $\alpha = (\alpha_1, \dots, \alpha_n) \in \Phi_n$, and set $\alpha_n = 0$. Then with $\widetilde{N_0}$, $\Phi_{n-1}$, and $G_{n-1}$ as above,

$$\begin{aligned}\min_{\alpha\in\Phi_n} G_n(\alpha) &\le \left[\min_{\alpha\in\Phi_{n-1}} G_{n-1}(\alpha)\right] + |\sigma_n| \\ &\le \sum_{j=1}^{n-1} |\sigma_j| - 1 + |\sigma_n| \quad \text{by (1)}(\bigstar_3) \text{ for } \Phi_{n-1} \\ &= \sum_{j=1}^{n} |\sigma_j| - 1\end{aligned}$$

The proof for cases ($\bigstar_4$) through ($\bigstar_{n-2}$) follow the same method as ($\bigstar_2$) and ($\bigstar_3$), using the corresponding case for $\Phi_{n-1}$.

($\bigstar_{n-1}$). Assume that there are distinct $i_1, i_2, \dots, i_{n-1} \in N_0$ such that $\sigma_{i_1} + \sigma_{i_2} + \cdots + \sigma_{i_{n-1}} > 1$. Without loss of generality, assume that $\sigma_1 + \sigma_2 + \cdots + \sigma_{n-1} > 1$. Also we may assume that ($\bigstar_1$) through ($\bigstar_{n-1}$) do not hold. Then all of $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ must be great than or equal to zero, for if $\sigma_j < 0$, then

$$\begin{aligned}\sigma_1 + \sigma_2 + \cdots + \sigma_{n-1} &= (\sigma_1 + \cdots + \sigma_{j-1} + \sigma_{j+1} + \cdots + \sigma_{n-1}) + \sigma_j \\ &< (\sigma_1 + \cdots + \sigma_{j-1} + \sigma_{j+1} + \cdots + \sigma_{n-1}) + 0 \\ &\le 1 \quad \text{by not}(\bigstar_{n-2})\end{aligned}$$

Then, $(\sigma_1, \sigma_2, \ldots, \sigma_{n-2}, 1-(\sigma_1+\sigma_2+\ldots+\sigma_{n-2}), 0) \in \Phi_n$, and

$$\begin{aligned}
\min_{\alpha\in\Phi_n} G_n(\alpha) &\leq G_n(\sigma_1, \sigma_2, \ldots, \sigma_{n-2}, 1-(\sigma_1+\sigma_2+\ldots+\sigma_{n-2}), 0) \\
&= |0| + \cdots + |0| + |\sigma_{n-1} - 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-2}| + |\sigma_n| \\
&= \sigma_1 + \sigma_2 + \cdots + \sigma_{n-2} + \sigma_{n-1} - 1 + |\sigma_n| \\
&= |\sigma_1| + |\sigma_2| + \cdots + |\sigma_{n-2}| + |\sigma_{n-1}| + |\sigma_n| - 1 \\
&= \sum_{j=1}^{n} |\sigma_j| - 1
\end{aligned}$$

This concludes the proof of part (1).

(2) Assume that $(\bigstar_1)$ through $(\bigstar_{n-1})$ do not hold, i.e, we have that

$$\begin{array}{ll}
\widetilde{(\bigstar_1)} & \forall i \in N_0, \sigma_i \leq 1; \\
\widetilde{(\bigstar_2)} & \forall i_1, i_2 \in N_0, \sigma_{i_1} + \sigma_{i_2} \leq 1; \\
\widetilde{(\bigstar_3)} & \forall i_1, i_2, i_3 \in N_0, \sigma_{i_1} + \sigma_{i_2} + \sigma_{i_3} \leq 1; \\
\ldots & \\
\widetilde{(\bigstar_{n-1})} & \forall i_1, i_2, \ldots, i_{n-1} \in N_0, \sigma_{i_1} + \sigma_{i_2} + \cdots + \sigma_{i_{n-1}} \leq 1.
\end{array}$$

Case 2(0) $\forall i \in \{1, \ldots, n\}$, $\sigma_i < 0$.

$\forall \alpha \in \Phi_n$,

$$
\begin{aligned}
G_n(\alpha) &= \sum_{j=1}^{n} |\sigma_j - \alpha_j| \\
&= \sum_{j=1}^{n} -(\sigma_j - \alpha_j) \\
&= \sum_{j=1}^{n} \alpha_j - \sum_{j=1}^{n} \sigma_j \\
&= 1 - \sum_{j=1}^{n} \sigma_j \\
&= \left| \sum_{j=1}^{n} \sigma_j - 1 \right|
\end{aligned}
$$

<u>Case 2(1) through Case 2($n-2$)</u> where Case 2($\nu$) is:

$\exists i_1, \ldots i_\nu \in \{1, \ldots, n\}$ such that $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_\nu} \geq 0$ and $\sigma_j < 0$ for $j \in \{1, \ldots, n\} \backslash \{i_1, \ldots, i_\nu\}$.

The proof of Case 2($\nu$) proceeds as follows:

Without loss of generality $\sigma_1, \ldots, \sigma_{n-\nu} < 0$ and $\sigma_{n-\nu+1}, \ldots, \sigma_n \geq 0$. Note that

$$\alpha_0 = (0, 0, \ldots, 0, 1 - (\sigma_{n-\nu+1} + \cdots + \sigma_n), \sigma_{n-\nu+1}, \ldots, \sigma_n) \in \Phi_n \quad \text{by } \widetilde{(\bigstar_\nu)}$$

Then,

$$
\begin{aligned}
G_n(\alpha_0) &= |\sigma_1| + \cdots + |\sigma_{n-\nu-1}| + |\sigma_{n-\nu} - 1 + (\sigma_{n-\nu+1} + \cdots + \sigma_n)| \\
&= -\sigma_1 - \cdots - \sigma_{n-\nu-1} - \sigma_{n-\nu} - \sigma_{n-\nu+1} - \cdots - \sigma_n + 1 \quad \text{by } \widetilde{(\bigstar_{\nu+1})} \\
&= \left| \sum_{j=1}^{n} \sigma_j - 1 \right|
\end{aligned}
$$

Hence,

$$\min_{\alpha\in\Phi_n} G_n(\alpha) \leq \left|\sum_{j=1}^{n} \sigma_j - 1\right|$$

Case $2(n-1)$ Either $n-1$ or $n$ of the $\sigma_j$'s are greater than or equal to zero. Without loss of generality, assume that $\sigma_2, \sigma_3, \ldots, \sigma_n \geq 0$. Set

$$\alpha_0 = \left(1 - \sum_{j=2}^{n} \sigma_j, \sigma_2, \ldots, \sigma_n\right)$$

$$G_n(\alpha_0) = \left|\sum_{j=1}^{n} \sigma_j - 1\right|$$

Therefore,

$$\min_{\alpha\in\Phi_n} G_n(\alpha) \leq \left|\sum_{j=1}^{n} \sigma_j - 1\right|$$

This concludes the proof of part (2) and the proof of the theorem. □

In [9], Goebel and Kuczumow showed that the set $C$ above has the fixed point property for nonexpansive maps if and only if $N_0$ is nonempty but finite. We will show that $C$ has the fixed point property for affine asymptotically nonexpansive maps if and only if $N_0$ is nonempty and finite, and a certain extra condition holds.

As mentioned in the Introduction, Corollary 3.4.2 and Remark 3.4.3 are a special case of a theorem of Kaczor and Prus [11], who establish their result by a different proof technique.

**Corollary 3.4.2.** *Let $T : C \longrightarrow C$ be a mapping that is affine and asymptotically nonexpansive. If $N_0$ is nonempty and finite, then $T$ has a fixed point, as long as $\Gamma > a$, where $\Gamma := \inf_{j\neq N_0} a_j > a$.*

*Proof.* Let $C$, $\overline{C}$, $N_0$, and $\Gamma$ be as defined above. Suppose that $T : C \longrightarrow C$ is affine and asymptotically nonexpansive, and that $\Gamma > a$.

Case 1 $\#(N_0) = 1$. Without loss of generality, $N_0 = \{1\}$. Since $T$ is affine, there exists an approximate fixed point sequence $(x_n)_{n\in\mathbb{N}} \subseteq C$ for $T$ and an $x_0 \in \overline{C}$ with $x_n \longrightarrow x_0$ weak* as $n \longrightarrow \infty$.

If $x_0 \in C$, then the corresponding case in the proof of theorem 3.1.1 shows that $T$ has a fixed point.

Assume $x_0 \in \overline{C} \backslash C$ and fix $y \in C$. Note that in this case, $\text{Proj}(x_0) = \{x_0 + \delta_{x_0} f^1\}$. Let $y_0 := x_0 + \delta_{x_0} f^1$.

Note that

$$\|y - y_0\|_1 = |\beta_j - \mu_j - \delta_{x_0}|(1 + a) + \sum_{j=2}^{\infty} |\beta_j - \mu_j|(1 + a_j)$$

$$\begin{aligned}
|\beta_1 - \mu_1 - \delta_{x_0}| &= \left|\beta_1 - \mu_1 - \left(1 - \sum_{j=1}^{\infty} \mu_j\right)\right| \\
&= \left|\left(1 - \sum_{j=2}^{\infty} \beta_j\right) - \mu_1 - \left(1 - \sum_{j=1}^{\infty} \mu_j\right)\right| \\
&= \left|\sum_{j=2}^{\infty} (\mu_j - \beta_j)\right| \\
&\leq \sum_{j=2}^{\infty} |\beta_j - \mu_j|
\end{aligned}$$

$$\begin{aligned}
\delta_{x_0} = 1 - \sum_{j=1}^{\infty} \mu_j &= \sum_{j=1}^{\infty} \beta_j - \sum_{j=1}^{\infty} \mu_j \\
&= \sum_{j=1}^{\infty} (\beta_j - \mu_j) \\
&\leq \sum_{j=1}^{\infty} |\beta_j - \mu_j|
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|y - y_0\|_1 &= |\beta_1 - \mu_1 - \delta_{x_0}|(1+a) + \sum_{j=2}^{\infty} |\beta_j - \mu_j|(1+a_j) \\
&\leq (1+a)\sum_{j=2}^{\infty} |\beta_j - \mu_j| + \sum_{j=2}^{\infty}(1+a_j)|\beta_j - \mu_j| \\
&= \sum_{j=2}^{\infty} \frac{2+a+a_j}{a_j - a}|\beta_j - \mu_j|\,[(1+a_j)-(1+a)] \\
&\leq \sum_{j=2}^{\infty} \left(\frac{2+a+B}{\Gamma - a}\right)|\beta_j - \mu_j|\,[(1+a_j)-(1+a)] \\
&= \left(\frac{2+a+B}{\Gamma - a}\right)\left[(1+a)\delta_{x_0} + \sum_{j=2}^{\infty} |\beta_j - \mu_j|\,[(1+a_j)-(1+a)] - (1+a)\delta_{x_0}\right] \\
&\leq \left(\frac{2+a+B}{\Gamma - a}\right)\left[(1+a)|\beta_1 - \mu_1| + \sum_{j=2}^{\infty} |\beta_j - \mu_j|(1+a_j) - (1+a)\delta_{x_0}\right] \\
&= \left(\frac{2+a+B}{\Gamma - a}\right)\left[\|y - x_0\|_1 - \|y_0 - x_0\|_1\right]
\end{aligned}
$$

From here, the proof of Theorem 3.1.1 shows that $Ty_0 = y_0$ with $\gamma(\varepsilon) = \left(\frac{1-b}{1+b}\right)\varepsilon$ replaced by

$$
\gamma(\varepsilon) = \left(\frac{\Gamma - a}{2+a+B}\right)\varepsilon
$$

<u>Case 2 $\#(N_0) = n$, where $n \geq 2$</u>. Note that in this case,

$$
\mathrm{Proj}(x_0) = \left\{x_0 + \sum_{j\in N_0} \alpha_j \delta_{x_0} f^j : \alpha_j \geq 0 \text{ and } \sum_{j \in N_0} \alpha_j = 1\right\}
$$

and $u \in \mathrm{Proj}(x_0)$ has the form

$$
u = \sum_{j\in \mathbb{N}\setminus N_0} \mu_j f^j + \sum_{j \in N_0} (\mu_j + \alpha_j \delta_{x_0}) f^j
$$

Again, since $T$ is affine, there exists an approximate fixed point sequence $(x_n)_{n\in\mathbb{N}} \subseteq C$ for $T$ and an $x_0 \in \overline{C}$ with $x_n \longrightarrow x_0$ weak$^*$ as $n \longrightarrow \infty$, and if $x_0 \in C$, then the proof of Theorem 3.1.1 shows that $T$ has a fixed point.

Assume $x_0 \in \overline{C} \backslash C$ and fix $y \in C$. We wish to minimize $\|y-u\|_1$ as $u$ varies over $\text{Proj}(x_0)$. Note that for $u \in \text{Proj}(x_0)$,

$$\|y-u\|_1 = \sum_{j \in \mathbb{N} \backslash N_0} |\beta_j - \mu_j|(1+a_j) + \sum_{j \in N_0} |\beta_j - \mu_j - \alpha_j \delta_{x_0}|(1+a)$$

For $\alpha = (\alpha_j)_{j \in N_0}$, define

$$F_n(\alpha) = \sum_{j \in N_0} |\beta_j - \mu_j - \alpha_j \delta_{x_0}| = \delta_{x_0} \sum_{j \in N_0} \left| \frac{\beta_j - \mu_j}{\delta_{x_0}} - \alpha_j \right|$$

From the Lemma, with $\sigma_j = \frac{\beta_j - \mu_j}{\delta_{x_0}}$ and $F_n(\alpha) = \delta_{x_0} G_n(\alpha)$, we have that $\exists \, \widetilde{a} = (\alpha_j)_{j \in N_0}$ with either

$$(\dagger) \quad F_n(\widetilde{\alpha}) \leq \sum_{j \in N_0} |\beta_j - \mu_j| - \delta_{x_0} \quad \text{or}$$

$$(\ddagger) \quad F_n(\widetilde{\alpha}) \leq \left| \sum_{j \in N_0} (\beta_j - \mu_j) - \delta_{x_0} \right|$$

In the case of ($\dagger$), $\exists \, \widetilde{u} \in \text{Proj}(x_0)$ such that $F_n(\widetilde{\alpha}) \leq \sum_{j \in N_0} |\beta_j - \mu_j| - \delta_{x_0}$. Then,

$$\begin{aligned}
\|y - \widetilde{u}\|_1 &= \sum_{j \in \mathbb{N} \backslash N_0} |\beta_j - \mu_j|(1+a_j) + F_n(\widetilde{\alpha})(1+a) \\
&\leq \left( \sum_{j \in \mathbb{N} \backslash N_0} |\beta_j - \mu_j|(1+a_j) + \sum_{j \in N_0} |\beta_j - \mu_j|(1+a) \right) - \delta_{x_0}(1+a) \\
&= \left[ \|y - x_0\|_1 - \|\widetilde{u} - x_0\|_1 \right] (1) \\
&\leq \left( \frac{2+a+B}{\Gamma - a} \right) \left[ \|y - x_0\|_1 - \|\widetilde{u} - x_0\|_1 \right]
\end{aligned}$$

In the case of ($\ddagger$), $\exists \, \widetilde{u} \in \text{Proj}(x_0)$ such that $F_n(\widetilde{\alpha}) \leq \left| \sum_{j \in N_0} (\beta_j - \mu_j) - \delta_{x_0} \right|$.

Note that

$$
\begin{aligned}
F_n(\widetilde{\alpha}) &\leq \left| \sum_{j \in N_0} (\beta_j - \mu_j) - \delta_{x_0} \right| \\
&= \left| \sum_{j \in N_0} \beta_j - \sum_{j \in N_0} \mu_j - \left( 1 - \sum_{j=1}^{\infty} \mu_j \right) \right| \\
&= \left| 1 - \sum_{j \in \mathbb{N} \setminus N_0} \beta_j - \left( 1 - \sum_{j \in \mathbb{N} \setminus N_0} \mu_j \right) \right| \\
&= \left| \sum_{j \in \mathbb{N} \setminus N_0} (\mu_j - \beta_j) \right| \\
&\leq \sum_{j \in \mathbb{N} \setminus N_0} |\beta_j - \mu_j|
\end{aligned}
$$

So that

$$
\begin{aligned}
\|y - \widetilde{u}\|_1 &= \sum_{j \in \mathbb{N} \setminus N_0} |\beta_j - \mu_j|(1 + a_j) + F_n(\widetilde{\alpha})(1 + a) \\
&\leq (1 + a) \sum_{j \in \mathbb{N} \setminus N_0} |\beta_j - \mu_j| + \sum_{j \in \mathbb{N} \setminus N_0} (1 + a_j)|\beta_j - \mu_j| \\
&= \sum_{j \in \mathbb{N} \setminus N_0} \left( \frac{2 + a + a_j}{a_j - a} \right) |\beta_j - \mu_j| \left[ (1 + a_j) - (1 + a) \right] \\
&\leq \sum_{j \in \mathbb{N} \setminus N_0} \left( \frac{2 + a + B}{\Gamma - a} \right) |\beta_j - \mu_j| \left[ (1 + a_j) - (1 + a) \right] \\
&= \left( \frac{2 + a + B}{\Gamma - a} \right) \left[ (1 + a)\delta_{x_0} + \sum_{j \in \mathbb{N} \setminus N_0} |\beta_j - \mu_j| \left[ (1 + a_j) - (1 + a) \right] - (1 + a)\delta_{x_0} \right] \\
&\leq \left( \frac{2 + a + B}{\Gamma - a} \right) \left[ (1 + a) \sum_{j=1}^{\infty} |\beta_j - \mu_j| \right. \\
&\left. + \sum_{j \in \mathbb{N} \setminus N_0} |\beta_j - \mu_j| \left[ (1 + a_j) - (1 + a) \right] - (1 + a)\delta_{x_0} \right] \\
&= \left( \frac{2 + a + B}{\Gamma - a} \right) \left[ (1 + a) \sum_{j \in N_0} |\beta_j - \mu_j| + \sum_{j \in \mathbb{N} \setminus N_0} |\beta_j - \mu_j| \, (1 + a_j) - (1 + a)\delta_{x_0} \right] \\
&= \left( \frac{2 + a + B}{\Gamma - a} \right) \left[ \|y - x_0\|_1 - \|\widetilde{u} - x_0\|_1 \right]
\end{aligned}
$$

Hence, in all cases, $\forall y \in C$, $\exists\, \widetilde{u} \in \operatorname{Proj}(x_0)$ such that

$$\|y-\widetilde{u}\|_1 \leq \left(\frac{2+a+B}{\Gamma-a}\right)\left[\|y-x_0\|_1 - \|\widetilde{u}-x_0\|_1\right]$$

Pick $y_0 \in \mathrm{Proj}(x_0)$ and consider $(T^m y_0)_{m\in\mathbb{N}}$. As in Theorem 3.1.1 we can show that $\|T^m y_0 - x_0\|_1 \longrightarrow \|y_0 - x_0\|_1$ as $m \longrightarrow \infty$. Hence, $\forall m \in \mathbb{N}\cup\{0\}$, $\exists\, u_m \in \mathrm{Proj}(x_0)$ such that

$$\|T^m y_0 - u_m\|_1 \leq \left(\frac{2+a+B}{\Gamma-a}\right)\left[\|T^m y_0 - x_0\|_1 - \|u_m - x_0\|_1\right]$$

and $\|T^m y_0 - u_m\|_1 \longrightarrow 0$ as $m \longrightarrow \infty$.

Define $\nu_m := \frac{u_0+u_1+\cdots+u_m}{m+1} \in \mathrm{Proj}(x_0)$ and $q_m := \frac{y_0+Ty_0+\cdots+T^m y_0}{m+1}$. Since $T$ is affine, $\|Tq_m - q_m\|_1 \longrightarrow 0$ as $m \longrightarrow \infty$. Next, $\mathrm{Proj}(x_0)$ is norm compact, and so there is a subsequence $(\nu_{m_k})_{k\in\mathbb{N}}$ and $\nu_0 \in \mathrm{Proj}(x_0)$ such that $\|\nu_{m_k} - \nu_0\|_1 \longrightarrow 0$ as $k \longrightarrow \infty$.

Observe that

$$\begin{aligned}
\|q_{m_k} - \nu_{m_k}\|_1 &= \left\|\frac{1}{m_k+1}\sum_{j=0}^{m_k} T^j y_0 - \frac{1}{m_k+1}\sum_{j=0}^{m_k} u_j\right\|_1 \\
&\leq \frac{1}{m_k+1}\sum_{j=0}^{m_k} \|T^j y_0 - u_j\|_1
\end{aligned}$$

and therefore, since $\|T^j y_0 - u_j\|_1 \longrightarrow 0$ as $j \longrightarrow \infty$ for $0 \leq j \leq m_k$,

$$\|q_{m_k} - \nu_{m_k}\|_1 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

This implies that

$$\|q_{m_k} - \nu_0\|_1 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

Hence,

$$\begin{aligned}
\|T\nu_0 - \nu_0\|_1 &\leq \|T\nu_0 - Tq_{m_k}\|_1 + \|Tq_{m_k} - q_{m_k}\|_1 + \|q_{m_k} - \nu_0\|_1 \\
&\leq (\lambda_1 + 1)\|q_{m_k} - \nu_0\|_1 + \|Tq_{m_k} - q_{m_k}\|_1 \\
&\longrightarrow 0 + 0 \quad \text{as } k \longrightarrow \infty
\end{aligned}$$

and therefore $T\nu_0 = \nu_0$, i.e. $T$ has a fixed point.

In Goebel and Kuczomow's proof for nonexpansive maps, they provided counterexamples to show that $T$ may not have a fixed point if $N_0$ is empty or infinite. Since nonexpansive maps are also asymptotically nonexpansive (and the counterexamples they gave were affine), the same counterexamples when $N_0$ is empty or infinite work here. □

*Remark* 3.4.3. For the purposes of the following, without loss of generality, $N_0 = \{1, \ldots, n\}$. In our hypothesis, we have made the assumption that $\Gamma := \inf_{j \geq n+1} a_j > a$. What happens when $\Gamma = a$? Making use of a variant of the "Right Shift"operator, we can show that there exists a fixed point free, asymptotically nonexpansive, affine map $T : C \longrightarrow C$.

Assume that $\Gamma = a$. Then there is a subsequence $(a_{j_k})_{k \in \mathbb{N}}$ such that $\lim_{k \longrightarrow \infty} a_{n_k} = a$. We may assume that $j_1 > n$ and that the sequence is decreasing. Define for all $k \in \mathbb{N}$. $\varphi(k) := j_{k+1}$. Define $T : C \longrightarrow C$ by

$$T\left(\sum_{k=1}^{\infty} \beta_k f^k\right) := \sum_{k=1}^{\infty} \beta_k f^{\varphi(k)}$$

Using the notation $\varphi^2(k) = \varphi(\varphi(k))$, $\varphi^3(k) = \varphi(\varphi(\varphi(k)))$, etc, note that

$$T^\nu\left(\sum_{k=1}^{\infty} \beta_k f^k\right) = \sum_{k=1}^{\infty} \beta_j f^{\varphi^\nu(k)}$$

and for $x = \sum_{k=1}^{\infty} \beta_k f^k$, and $y = \sum_{k=1}^{\infty} \gamma_k f^k$ in $C$,

$$\|T^\nu x - T^\nu y\|_1 = \sum_{k=1}^{\infty} |\beta_k - \gamma_k| \left(1 + a_{\varphi^\nu(k)}\right)$$

and

$$\|x - y\|_1 = \sum_{k=1}^{n} |\beta_k - \gamma_k|(1 + a) + \sum_{k=n+1}^{\infty} |\beta_k - \gamma_k| \left(1 + a_k\right)$$

$T$ is an affine, fixed point free mapping from $C$ to $C$. Furthermore, $T$ is asymptotically nonexpansive, as the following calculation demonstrates.

$$
\begin{aligned}
\|T^\nu x - T^\nu y\|_1 &\le \sum_{k=1}^{\infty} |\beta_k - \gamma_k| \left(1 + a_{\varphi^\nu(1)}\right) \\
&= \left(\frac{1 + a_{\varphi^\nu(1)}}{1+a}\right) \sum_{k=1}^{\infty} |\beta_k - \gamma_k|(1+a) \\
&\le \left(\frac{1 + a_{\varphi^\nu(1)}}{1+a}\right) \|x - y\|_1
\end{aligned}
$$

Let $\lambda_\nu = \left(\frac{1+a_{\varphi^\nu(1)}}{1+a}\right)$. Noting that $\lambda_\nu > 1, \forall \nu \in \mathbb{N}$ and $\lambda_\nu \longrightarrow 1$ as $\nu \longrightarrow \infty$, and we see that $T$ is indeed asymptotically nonexpansive.

## 4.0 OPEN QUESTIONS

**Open question (1).** In $(\ell^1, \|\cdot\|_1)$ (or more generally in an arbitrary Banach space $(X, \|\cdot\|)$), can we identify precisely which isomorphic $\ell^1$-basic sequences are such that their closed convex hulls have the FPP(n.e.)? In particular, can we prove a theorem that includes both Theorem 2.2.1 and Theorem 2.3.18 as special cases?

**Open question (2).** In $(\ell^1, \|\cdot\|_1)$ (or more generally in an arbitrary Banach space $(X, \|\cdot\|)$), can we identify precisely which isomorphic $\ell^1$-basic sequences are such that their closed convex hulls have the fixed point property for asymptotically nonexpansive maps (FPP(a.n.e.))?

**Open question (3).** When do the sets discussed in Chapters 3 and 4 have the FPP(a.n.e.)? In the paper of T. Dominguez Benavides, J. Garcia Falset, E. Llorens-Fuster, and P. Lorenzo Ramirez [2], Theorem 4.13 suggests a possible proof strategy in the case of asymptotically nonexpansive mappings with approximate fixed point sequences. (We thank Torrey Gallagher for drawing our attention to this paper).

**Open question (4).** In 2002, Dowling, Lennard, and Turett [6] showed that if $(X, \|\cdot\|)$ is a Banach space and $K$ is a closed, bounded, and convex subset of $X$ that contains an asymptotically isometric $\ell^1$-basic sequence, then $K$ contains a non-empty, closed, bounded, and convex subset $C$ on which there is a nonexpansive mapping $T$ that fails to have a fixed point in $C$.

It is an open question as to whether or not every non-weak$^*$ compact closed, bounded, and convex subset $K$ of $(\ell^1, \|\cdot\|_1)$ contains a further subset $G$ that is non-empty, closed, bounded, convex, and non-weak$^*$ compact such that $G$ has the FPP(n.e.). If we could show this, then the result would imply that inside any non-weak* compact, closed, bounded, and convex subset of $(\ell^1, \|\cdot\|_1)$, there exists a sequence of sets $(K_n)_{n\in\mathbb{N}}$ that are nested and decreasing such that each odd term has the FPP(n.e.) and each even term fails the FPP(n.e.). Note that in Goebel and Kuczumow [9] an example of such a decreasing chain is given.

**Open question (5).** As in Chapter 6, for a given $b \in (0,1)$, are there any fixed point free, affine, uniformly Lipschitzian mappings $U : K \longrightarrow K$ with (best) uniform Lipschitz constant $M \in (1, 2/(1+b))$?

**Open question (6).** As in Chapter 6, does there exist an affine, asymptotically nonexpansive map $V$ on $K$ such that $V$ is not nonexpansive, and for which each $V^m$ is not a strict contraction?

**Open question (7).** Let $(X, \|\cdot\|)$ be a Banach space. Does every asymptotically nonexpansive map $T$ on a closed, bounded, and convex subset $C$ of $X$ have an approximate fixed point sequence?

**Open question (8).** We note that $(\ell^1, \|\cdot\|_1)$ isometrically embeds as diagonal matrices in the space of infinite-by-infinite matrices called the trace class, $\mathcal{C}_1$, with its usual norm $\|\cdot\|_{\mathcal{C}_1}$. Thus, all the Goebel and Kuczumow sets (examples of closed, bounded, and convex sets with the FPP(n.e.)) embed into $(\mathcal{C}_1, \|\cdot\|_{\mathcal{C}_1})$. Are there natural analogues of the Goebel and Kuczumow examples inside $\mathcal{C}_1$ that have off-diagonal non-zero entries?

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