SOME ANALYTICAL ISSUES FOR THE SELECTED COMPLEX FLUIDS MODELS

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SOME ANALYTICAL ISSUES FOR THE SELECTED COMPLEX FLUIDS MODELS

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In this dissertation, we study the selected models from complex fluids: compressible flow of liquid crystals and the incompressible fluid-particles flow. On the compressible flow of liquid crystals, we establish the global existence of renormalized weak solutions when $\gamma > \frac{3}{2}$ through a three-level approximation, energy estimates, and weak convergence methods in the spirit of the so-called Lions-Feireisl method. On the incompressible fluid-particles flow, we establish the global existence of Leray weak solutions which was constructed by the Galerkin methods, fixed point arguments, and convergence analysis with the large initial data. The uniqueness was established by the classical theory of Stokes equations and a bootstrap argument in the two dimensional space.

Keywords: Global weak solutions, existence, uniqueness, liquid crystal, Navier-Stokes equations, Vlasov equation.

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1.0 INTRODUCTION

In this dissertation, our interests include the rigorous mathematical study of fluid models derived from the Navier-Stokes theory. In particular, we have been working on the compressible flow of nematic liquid crystals and the incompressible fluid-particles flow. Understanding these is a fundamental challenge in both mathematics and science. Our principal goal is to develop new analytic methods to tackle the mathematical issues and to gain new physical insights into the above flows and related applications. More precisely, the focus of this dissertation includes:

- Renormalized weak solutions to the compressible flow of nematic liquid crystals.
- Leray weak solutions to the incompressible fluid-particles flow, including global existence in three dimensions, and uniqueness in two dimensions.

1.1 COMPRESSIBLE FLOW OF LIQUID CRYSTALS

The various applications of liquid crystals motivate us to investigate the related mathematical problems. The motion of nematic liquid crystals is governed by the forced Navier-Stokes equations and a parabolic type equation.

The hydrodynamic equations for the three-dimensional flow of nematic liquid crystals ([10, 27, 34])has the following form:

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{1.1.1a}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mu \Delta \mathbf{u} - \lambda \mathrm{div}\left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2}|\nabla \mathbf{d}|^2 + F(\mathbf{d})) I_3 \right), \tag{1.1.1b}$$

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \theta(\Delta \mathbf{d} - f(\mathbf{d})). \tag{1.1.1c}$$

The system (1.1.1) is subject to the following initial-boundary conditions:

$$(\rho, \rho\mathbf{u}, \mathbf{d})|_{t=0} = (\rho_0(x), \mathbf{m}_0(x), \mathbf{d}_0(x)), \quad x \in \Omega, \tag{1.1.2}$$

and

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{d}_0(x), \tag{1.1.3}$$

where

$$\rho_0 \in L^\gamma(\Omega), \quad \rho_0 \geq 0; \quad \mathbf{d}_0 \in L^\infty(\Omega) \cap H^1(\Omega);$$
$$\mathbf{m}_0 \in L^1(\Omega), \quad \mathbf{m}_0 = 0 \text{ if } \rho_0 = 0; \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega).$$

Here $\Omega \subset \mathbb{R}^3$ is a smooth boundary domain, $\rho \geq 0$ is the density of fluid, $\mathbf{u} \in \mathbb{R}^3$ is the velocity of fluid, $\mathbf{d} \in \mathbb{R}^3$ is the direction field for the averaged macroscopic molecular orientations, and $P = a\rho^\gamma$ is the pressure with constants $a > 0$ and $\gamma \geq 1$. The constants $\mu > 0, \lambda > 0, \theta > 0$ denote the viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relation time for the molecular orientation field, respectively. The notation $\otimes$ denotes the Kronecker tensor product, I_3 is the 3×3 identity matrix, and $\nabla\mathbf{d} \odot \nabla\mathbf{d}$ denotes the 3×3 matrix whose ij-th entry is $< \partial_{x_i}\mathbf{d}, \partial_{x_j}\mathbf{d} >$. The penalty function $f(\mathbf{d})$ is the vector-valued smooth function and has the following form:

$$f(\mathbf{d}) = \nabla_{\mathbf{d}} F(\mathbf{d}),$$

where the scalar function $F(\mathbf{d})$ denotes the bulk part of the elastic energy. Typically, we choose $F(\mathbf{d})$ as the Ginzburg-Landau penalization thus yielding the penalty function $f(\mathbf{d})$ as:

$$F(\mathbf{d}) = \frac{1}{4\sigma_0^2}(|\mathbf{d}|^2 - 1)^2, \quad f(\mathbf{d}) = \frac{1}{2\sigma_0^2}(|\mathbf{d}|^2 - 1)\mathbf{d},$$

where $\sigma_0 > 0$ is a constant. We refer the readers to [7, 10, 16, 27, 33, 34] for more mathematical models and physical background of liquid crystals.

The first objective of this dissertation is to establish the existence of global weak solutions to (1.1.1)-(1.1.3). There have been extensive mathematical results on the incompressible flows of liquid crystals, for example, the existence of global weak solutions with large data, the global existence of strong solutions, and the partial regularity of the weak solutions similar to the classical theorem of Caffarelli-Kohn-Nirenberg [5], see [30, 34, 35, 36, 37, 51] and the references cited therein. The existence of weak solutions to the density-dependent incompressible flow of liquid crystals was proved in [32]. The three dimensional compressible flow (1.1.1)-(1.1.3) of liquid crystals is much more complicated and difficult to establish the global existence due to strong nonlinearity. In the one-dimensional case the global existence of smooth and weak solutions to the compressible flow of liquid crystals was obtained in [11].

When the direction field $\mathbf{d}$ is absent in the system (1.1.1), the system reduces to the compressible Navier-Stokes equations. For the multidimensional compressible Navier-Stokes equations, Lions in [39] proved the global existence of finite energy weak solutions for $\gamma > 9/5$ by pioneering the concept of renormalized solutions to overcome the difficulties of large oscillations, and then Feireisl, *et al*, in [21, 18, 19] developed this method and extended the existence results to $\gamma > 3/2$. We shall study the initial-boundary value problem (1.1.1)-(1.1.3) for liquid crystals with large initial data in certain functional spaces with $\gamma > 3/2$. To achieve our goal, We shall employ a three-level approximation scheme similar to that in [21, 18] to prove the global existence, which consists of Faedo-Galerkin approximation, artificial viscosity, and artificial pressure. Then, in sprite of the work of [18], we prove that the uniform estimate of the density $\rho^{\gamma+\alpha}$ in L^1 for some $\alpha > 0$ guarantees the vanishing of artificial pressure and the strong compactness of the density. We adopt the methods of Lions and Feireisl in [21, 18, 36] to build the weak continuity of the effective viscous flux for the compressible flow of liquid crystals similar to that for compressible Navier-Stokes equations to remove the difficulty of possible large oscillation of the density. To obtain the related estimates on effective viscous flux, we need to establish some estimates to deal with the direction field and its coupling and interaction with the fluid variables.

1.2 INCOMPRESSIBLE FLUID-PARTICLES FLOW

On physical grounds, the motivation of our study of the incompressible fluid-particle flow is of primary importance in the modeling of sprays. There are many relevant applications, such as combustion theory, pollutant transport, and many more. The flow of the continuous phase is modeled by the forced Navier-Stokes equations, and the flow of the particles is governed by the kinetic equation. The fluid-particle interactions are described by a friction force exerted from the fluid onto the particles.

The second objective of this dissertation is to establish the global existence of weak solutions for the following partial differential equations, namely Navier-Stokes-Vlasov equations:

$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\Delta\mathbf{u} = -\int_{\mathbb{R}^d}(\mathbf{u}-\mathbf{v})f\,d\mathbf{v},\\
&\mathrm{div}\mathbf{u} = 0, \\
&f_t + \mathbf{v}\cdot\nabla_x f + \mathrm{div}_{\mathbf{v}}((\mathbf{u}-\mathbf{v})f) = 0,
\end{aligned} \tag{1.2.1}$$

in $\Omega\times\mathbb{R}^d\times(0,T)$, where $\Omega\subset\mathbb{R}^d$ is a bounded domain, $d=2,3$, $\mathbf{u}$ is the velocity of the fluid, and p is the pressure. Without loss of generality, we take kinematic viscosity of fluid $\mu = 1$ throughout the paper. The distribution function $f(t,x,\mathbf{v})$ depends on the time $t\in[0,T]$, the physical position $x\in\Omega$, and the velocity of particle $\mathbf{v}\in\mathbb{R}^d$. The notation $f(t,x,\mathbf{v})\,d\mathbf{v}$ is the number of particles enclosed at $t\geq 0$ and location $x\in\Omega$ in the volume element $d\mathbf{v}$.

The system is completed by the initial data

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad f(0,x,\mathbf{v}) = f_0(x,\mathbf{v}), \tag{1.2.2}$$

and with the following boundary conditions:

$$\mathbf{u} = 0 \text{ on } \partial\Omega, \quad \text{and} \quad f(t,x,\mathbf{v}) = f(t,x,\mathbf{v}^*) \text{ for } x\in\partial\Omega,\ \mathbf{v}\cdot\nu(x)<0 \tag{1.2.3}$$

where $\mathbf{v}^* = \mathbf{v} - 2(\mathbf{v}\cdot\nu(x))\nu(x)$ is the specular velocity, $\nu(x)$ is the outward normal to Ω.

In general, the mathematical analysis of fluid-particle flow is challenging because the distribution function f depends on more variables than the fluid density ρ and velocity $\mathbf{u}$. The rigorous mathematical study to such coupled systems is far from being complete but recently has received much attention. The global existence of weak solutions to Stokes-Vlasov system with boundary was established in 1990s, see [26]. The existence theorem for weak solutions has been extended to Navier-Stokes-Vlasov equations within a periodic domain in [4]. The global existence of smooth solutions for Navier-Stokes-Vlasov-Fokker-Planck equations with small data was proved in [23]. More Recently, the existence of global weak solutions of Navier-Stokes-Vlasov-Poisson system with corresponding boundary value problem was established in [1]. Meanwhile, there are many works in the direction of hydrodynamic limits, we refer the reader to [6, 24, 25, 46]. In works [6, 24, 25, 46], the authors used some scaling issues and convergence methods to investigate the hydrodynamic limits. A key idea in [24, 25] is to control the dissipation rate of a certain free energy associated with the whole space. For the compressible version, local strong solutions of Euler-Vlasov equations was established in [2]. Global existence of weak solutions for compressible Navier-Stokes equations coupled to Vlasov-Fokker-Planck equations was established in [45].

In Section 3, we shall establish the global existence of weak solutions to the initial-boundary value problem (1.2.1)-(1.2.3) for Navier-Stokes-Vlasov equations with large data in three dimensional space. To this end, we construct a new approximation scheme motivated by the works of [13, 44, 45]. The key idea of this approximation is to control the modified force term of regularized Navier-Stokes equations. The existence and uniqueness of the modified Vlasov equation is classically obtained, for example, see [3, 12, 26]. The controls of $\int_{\mathbb{R}^d} f d\mathbf{v}$ and $\int_{\mathbb{R}^d} \mathbf{v} f\, d\mathbf{v}$ ensure that the modified Navier-Stokes equations could be solved. The compactness properties of the system will allow us to pass the limit to recover the original system. We shall also establish the uniqueness of the weak solutions in the two dimensional space.

1.3 DENSITY-DEPENDENT INCOMPRESSIBLE FLUID-PARTICLE FLOW

Now, let us move to the Navier-Stokes-Vlasov equations for particles dispersed in a density-dependent incompressible viscous fluid:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.3.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p - \mu\Delta\mathbf{u} = -\int_{\mathbb{R}^3} m_p F f d\mathbf{v}, \tag{1.3.2}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.3.3}$$

$$f_t + \mathbf{v}\cdot\nabla_x f + \operatorname{div}_{\mathbf{v}}(Ff) = 0, \tag{1.3.4}$$

for $(x, \mathbf{v}, t)$ in $\Omega \times \mathbb{R}^3 \times (0, \infty)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain, ρ is the density of the fluid, $\mathbf{u}$ is the velocity of the fluid, p is the pressure, μ is kinematic viscosity of fluid. The density distribution function $f(t, x, \mathbf{v})$ of particles depends on the time $t \in [0, T]$, the physical position $x \in \Omega$ and the velocity of particle $\mathbf{v} \in \mathbb{R}^3$. In (1.3.2), m_p is the mass of the particle and F is the drag force. The interaction of the fluid and particles is through the drag force exerted by the fluid onto the particles. Typically, the drag force F depends on the relative velocity $\mathbf{u} - \mathbf{v}$ and on the density of fluid ρ (e.g. [46]), such as

$$F = F_0 \rho(\mathbf{u} - \mathbf{v}), \tag{1.3.5}$$

where F_0 is a positive constant. Without loss of generality we take $\mu = F_0 = m_p = 1$ throughout the paper.

The final objective is to establish the global existence of weak solutions to the initial-boundary value problem for the system (1.3.1)-(1.3.5) subject to the following initial data:

$$\rho|_{t=0} = \rho_0(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \Omega, \tag{1.3.6}$$

$$(\rho \mathbf{u})|_{t=0} = \mathbf{m}_0(x), \quad \mathbf{x} \in \Omega, \tag{1.3.7}$$

$$f|_{t=0} = f_0(x, \mathbf{v}), \quad \mathbf{x} \in \Omega, \quad \mathbf{v} \in \mathbb{R}^N, \tag{1.3.8}$$

and the following boundary conditions:

$$\begin{aligned} &\mathbf{u}(t, x) = 0 \quad \text{on } \partial\Omega, \\ &f(t, x, \mathbf{v}) = f(t, x, \mathbf{v}^*) \;\; \text{for } x \in \partial\Omega, \; \mathbf{v} \cdot \nu(x) < 0, \end{aligned} \tag{1.3.9}$$

where

$$\mathbf{v}^* = \mathbf{v} - 2(\mathbf{v} \cdot \nu(x))\nu(x)$$

is the specular velocity, and $\nu(x)$ is the outward normal vector to Ω. When the drag force is assumed independent on density in (1.3.5), hydrodynamic limits and the global existence of weak solutions to the Navier-Stokes and Vlasov-Fokker-Planck equations were studied in [24, 25, 45, 46]. When the drag force depends on the density as in (1.3.5), a relaxation of the kinetic regime toward a hydrodynamic regime with velocity $\mathbf{u}$ on the vacuum $\{\rho = 0\}$ can not be excepted. It is difficult to establish a priori lower estimates on the density from the mathematics view point.

2.0 COMPRESSIBLE FLOW OF LIQUID CRYSTALS

The global existence of weak solutions with large initial data to the compressible flows is always a basic and interesting problem of the mathematical study. The goal of this chapter is to study the global existence of weak solutions to the three dimensional compressible flow of liquid crystals with bounded domain.

The remaining part of this chapter is organized as follows. In Section 2.1, after deduce the basic energy law, we state the main existence result of this chapter. In the following Sections, we use the three-level approximations, namely Faedo-Galerkin, vanishing viscosity, and artificial pressure, respectively, to prove our main result.

2.1 ENERGY ESTIMATES AND MAIN RESULTS

In this section, we derive some basic energy estimates for the initial-boundary problem (1.1.1)-(1.1.3), introduce the notion of finite energy weak solutions in the spirit of Feireisl [21, 18], and state the main results.

Without loss of generality, we take $\theta = a = 1$. First we formally derive the energy equality and some a priori estimates, which will play a very important role in our paper. Multiplying (1.1.1b) by $\mathbf{u}$, integrating over Ω, and using the boundary condition (1.1.3), we obtain

$$\begin{aligned}&\partial_t \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1}\right) dx + \mu \int_\Omega |\nabla \mathbf{u}|^2 dx \\ &= -\lambda \int_\Omega \operatorname{div}\left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2}|\nabla \mathbf{d}|^2 + F(\mathbf{d}))I_3\right) \mathbf{u} dx.\end{aligned}$$

Using the equality

$$\mathrm{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d}) = \nabla(\frac{1}{2}|\nabla \mathbf{d}|^2) + (\nabla \mathbf{d})^\top \cdot \Delta \mathbf{d},$$

we have

$$\begin{aligned}
&\int_\Omega \mathrm{div}\left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2}|\nabla \mathbf{d}|^2 + F(\mathbf{d}))I_3\right)\mathbf{u}dx \\
&= \int_\Omega (\nabla \mathbf{d})^\top \cdot \Delta \mathbf{d} \cdot \mathbf{u}dx - \int_\Omega \nabla_{\mathbf{d}} F(\mathbf{d})\mathbf{u}dx.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\partial_t \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1}\right)dx + \int_\Omega |\nabla \mathbf{u}|^2 dx \\
&= -\lambda \int_\Omega (\nabla \mathbf{d})^\top \cdot \Delta \mathbf{d} \cdot \mathbf{u}dx + \lambda \int_\Omega \nabla_{\mathbf{d}} F(\mathbf{d})\mathbf{u}dx.
\end{aligned} \tag{2.1.1}$$

Multiplying by $\lambda(\Delta \mathbf{d} - f(\mathbf{d}))$ on the both sides of (1.1.1c) and integrating over Ω, we get

$$\begin{aligned}
&-\partial_t \int_\Omega \lambda \frac{|\nabla \mathbf{d}|^2}{2}dx - \partial_t \int_\Omega \lambda F(\mathbf{d})dx - \int_\Omega \lambda \nabla_{\mathbf{d}} F(\mathbf{d})\mathbf{u}dx + \lambda \int_\Omega (\nabla \mathbf{d})^\top \cdot \Delta \mathbf{d} \cdot \mathbf{u}dx \\
&= \lambda \int_\Omega |\Delta \mathbf{d} - f(\mathbf{d})|^2 dx.
\end{aligned}$$

Then, from (2.1.1), we have the following energy equality to the system (1.1.1),

$$\begin{aligned}
&\partial_t \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{\lambda}{2}|\nabla \mathbf{d}|^2 + \lambda F(\mathbf{d})\right)dx \\
&\qquad + \int_\Omega \left(\mu|\nabla \mathbf{u}|^2 dx + \lambda|\Delta \mathbf{d} - f(\mathbf{d})|^2\right)dx \\
&= 0.
\end{aligned} \tag{2.1.2}$$

Set

$$E(t) = \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{\lambda}{2}|\nabla \mathbf{d}|^2 + \lambda F(\mathbf{d})\right)(t,x)dx,$$

and assume that $E(0) < \infty$. From (2.1.2), we have the following a priori estimates:

$$\rho|\mathbf{u}|^2 \in L^\infty([0,T]; L^1(\Omega));$$

$$\rho \in L^\infty([0,T]; L^\gamma(\Omega));$$

$$\nabla \mathbf{d} \in L^\infty([0,T]; L^2(\Omega));$$

$$F(\mathbf{d}) \in L^\infty([0,T]; L^1(\Omega));$$

$$\nabla \mathbf{u} \in L^2([0,T]; L^2(\Omega));$$

and also

$$\Delta \mathbf{d} - f(\mathbf{d}) \in L^2([0,T]; L^2(\Omega)). \tag{2.1.3}$$

Although the above estimates will play very important roles in proving of our main existence theorem, they cannot provide sufficient regularity for the direction field $\mathbf{d}$ to control the strongly nonlinear terms containing $\nabla \mathbf{d}$.

Remark 2.1.1. We infer from Gagliardo-Nirenberg inequality that

$$\begin{aligned} &\|\mathbf{d}\|_{L^{10}} \le C\|\mathbf{d}\|_{L^6}^{\frac{4}{5}} \|\Delta \mathbf{d}\|_{L^2}^{\frac{1}{5}} + C\|\mathbf{d}\|_{L^6} \le C\|\mathbf{d}\|_{H^1}^{\frac{4}{5}} \|\Delta \mathbf{d}\|_{L^2}^{\frac{1}{5}} + C\|\mathbf{d}\|_{H^1}, \\ &\|\nabla \mathbf{d}\|_{L^{\frac{10}{3}}} \le C\|\mathbf{d}\|_{L^2}^{\frac{2}{5}} \|\Delta \mathbf{d}\|_{L^2}^{\frac{3}{5}} + C\|\nabla \mathbf{d}\|_{L^2}. \end{aligned}$$

Using $\mathbf{d} \in L^\infty(0,T; H^1(\Omega))$ and $\Delta \mathbf{d} \in L^2(0,T; L^2(\Omega))$, we will have

$$\mathbf{d} \in L^{10}(0,T;\Omega) \quad \text{and } \Delta \mathbf{d} \in L^{\frac{10}{3}}(0,T;\Omega).$$

Through our paper, we will use C to denote a generic positive constant, $\mathcal{D}$ to denote C_0^∞, and $\mathcal{D}'$ to denote the sense of distributions. To introduce the finite energy weak solution $(\rho, \mathbf{u}, \mathbf{d})$, we also need to take a differentiable function b, and multiply (1.1.1a) by $b'(\rho)$ to get the renormalized form:

$$b(\rho)_t + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0. \tag{2.1.4}$$

We define the finite energy weak solution $(\rho, \mathbf{u}, \mathbf{d})$ to the initial-boundary value problem (1.1.1)-(1.1.3) in the following sense: for any $T > 0$,

- $\rho \ge 0, \quad \rho \in L^\infty([0,T]; L^\gamma(\Omega)), \quad \mathbf{u} \in L^2([0,T]; W_0^{1,2}(\Omega)),$

  $$\mathbf{d} \in L^\infty((0,T)\times\Omega) \cap L^\infty([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)),$$

  with $(\rho, \rho\mathbf{u}, \mathbf{d})(0,x) = (\rho_0(x), \mathbf{m}_0(x), \mathbf{d}_0(x))$ for $x \in \Omega$;
- The equations (1.1.1) hold in $\mathcal{D}'((0,T)\times\Omega)$, and (1.1.1a) holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ provided $\rho, \mathbf{u}$ are prolonged to be zero on $\mathbb{R}^3 \setminus \Omega$;

- (2.1.4) holds in $\mathcal{D}'((0,T)\times\Omega)$, for any $b\in C^1(\mathbb{R}^+)$ such that

$$b^{'}(z)=0 \text{ for all } z\in\mathbb{R}^+ \text{ large enough, say } z\geq M, \tag{2.1.5}$$

  where the constant M may vary for different function b;
- The energy inequality

$$E(t)+\int_0^t\int_\Omega \left(\mu|\nabla\mathbf{u}|^2dx+\lambda|\Delta\mathbf{d}-f(\mathbf{d})|^2\right)dxds\leq E(0)$$

  holds for almost every $t\in[0,T]$.

Remark 2.1.2. It's possible to deduce that (2.1.4) will hold for any $b\in C^1(0,\infty)\cap C[0,\infty)$ satisfying the following conditions

$$|b^{'}(z)|\leq c(z^\alpha+z^{\frac{\gamma}{2}}) \text{ for all } z>0 \text{ and a certain } \alpha\in(0,\frac{\gamma}{2}) \tag{2.1.6}$$

provided $(\rho,\mathbf{u},\mathbf{d})$ is a finite energy weak solution in the sense of the above definition (see details in [18]).

Now, our main result on the existence of finite energy weak solutions reads as follows:

Theorem 2.1.1. *Assume that $\Omega\subset\mathbb{R}^3$ is a bounded domain of the class $C^{2+\nu}$, $\nu>0$, and $\gamma>\frac{3}{2}$. then for any given $T>0$ the initial-boundary value problem (1.1.1)-(1.1.3) has a finite weak energy solution $(\rho,\mathbf{u},\mathbf{d})$ on $(0,T)\times\Omega$.*

The proof of Theorem 2.1.1 is based on the following approximation scheme:

$$\rho_t+\operatorname{div}(\rho\mathbf{u})=\varepsilon\Delta\rho, \tag{2.1.7a}$$

$$(\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla P(\rho)+\delta\nabla\rho^\beta+\varepsilon\nabla\mathbf{u}\cdot\nabla\rho$$
$$=\mu\Delta\mathbf{u}-\lambda\operatorname{div}\left(\nabla\mathbf{d}\odot\nabla\mathbf{d}-(\frac{1}{2}|\nabla\mathbf{d}|^2+F(\mathbf{d}))I_3\right), \tag{2.1.7b}$$

$$\mathbf{d}_t+\mathbf{u}\cdot\nabla\mathbf{d}=\Delta\mathbf{d}-f(\mathbf{d}), \tag{2.1.7c}$$

with appropriate initial-boundary conditions. Following the approach of Feireisl [21, 18], we shall obtain the solution of (1.1.1) when $\varepsilon\to0$ and $\delta\to0$ in (2.1.7). We can solve equation (2.1.7a) provided $\mathbf{u}$ is given. Indeed, we can obtain the existence by using classical theory of parabolic equation and overcome the difficulty of vacuum. Next we can also solve equation (2.1.7c) when $\mathbf{u}$ is fixed. By a direct application of the Schauder fixed point theorem, we can establish the local existence of $\mathbf{u}$, and then extend this local solution to the whole time interval. Note that the addition of the extra term $\varepsilon\nabla\mathbf{u}\cdot\nabla\rho$ is necessary for keeping the energy conservation. The last step is to let $\varepsilon \to 0$ and $\delta \to 0$ to recover the original system. We remark that the strongly nonlinear terms containing $\nabla\mathbf{d}$ can be controlled by the sufficiently strong estimate about $\nabla\mathbf{d}$ obtained from the Gagliardo-Nirenberg inequality. In order to control the possible oscillations of the density ρ, we adopt the methods in Lions [39] and Feireisl [21, 18] which is based on the celebrated weak continuity of the effective viscous flux $P - \mu\mathrm{div}\mathbf{u}$. We refer the readers to Lions [39], Feireisl [21, 18], and Hu-Wang [29] for discussions on the effective viscous flux.

2.2 THE SOLVABILITY OF THE DIRECTION VECTOR

To solve the approximation system (2.1.7) by the Faedo-Galerkin method, we need to show that the following system can be uniquely solved in terms of $\mathbf{u}$:

$$\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} - f(\mathbf{d}), \tag{2.2.1a}$$

$$\mathbf{d}|_{t=0} = \mathbf{d}_0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{d}_0, \tag{2.2.1b}$$

which can be achieved by the two lemmas below.

Lemma 2.2.1. *If $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}, \mathbb{R}^3))$, then there exists at most one function*

$$\mathbf{d} \in L^2(0,T; H^2(\Omega)) \cap L^\infty([0,T]; H^1(\Omega))$$

which solves (2.2.1) in the weak sense on $\Omega \times (0,T)$, and satisfies the initial and boundary conditions in the sense of traces.

Proof. Let $\mathbf{d}_1, \mathbf{d}_2$ be two solutions of (2.2.1) with the same data, then we have

$$(\mathbf{d}_1-\mathbf{d}_2)_t+\mathbf{u}\cdot\nabla(\mathbf{d}_1-\mathbf{d}_2)=\Delta(\mathbf{d}_1-\mathbf{d}_2)-(f(\mathbf{d}_1)-f(\mathbf{d}_2)). \tag{2.2.2}$$

Multiplying (2.2.2) by $\Delta(\mathbf{d}_1-\mathbf{d}_2)$, integrating it over Ω, and using integration by parts and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\partial_t\int_\Omega|\nabla(\mathbf{d}_1-\mathbf{d}_2)|^2dx+2\int_\Omega|\Delta(\mathbf{d}_1-\mathbf{d}_2)|^2dx\\
&=2\int_\Omega(\nabla(\mathbf{d}_1-\mathbf{d}_2))^\top\cdot(\Delta(\mathbf{d}_1-\mathbf{d}_2))\cdot\mathbf{u}dx+2\int_\Omega(f(\mathbf{d}_1)-f(\mathbf{d}_2))(\Delta(\mathbf{d}_1-\mathbf{d}_2))dx\\
&\le C\int_\Omega|\nabla(\mathbf{d}_1-\mathbf{d}_2)|^2dx+\int_\Omega|\Delta(\mathbf{d}_1-\mathbf{d}_2)|^2dx,
\end{aligned}$$

where we used the fact that f is smooth, then

$$\partial_t\int_\Omega|\nabla(\mathbf{d}_1-\mathbf{d}_2)|^2dx+\int_\Omega|\Delta(\mathbf{d}_1-\mathbf{d}_2)|^2dx\le C\int_\Omega|\nabla(\mathbf{d}_1-\mathbf{d}_2)|^2dx, \tag{2.2.3}$$

and Lemma 2.2.1 follows from Grönwall's inequality, and the above inequality. □

Lemma 2.2.2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$, $\nu>0$. Assume that $\mathbf{u}\in C([0,T];C_0^2(\bar\Omega,\mathbb{R}^3))$ is a given velocity field. Then the solution operator*

$$\mathbf{u}\longmapsto\mathbf{d}[\mathbf{u}]$$

assigns to $\mathbf{u}\in C([0,T];C_0^2(\bar\Omega;\mathbb{R}^3))$ the unique solution $\mathbf{d}$ of (2.2.1). Moreover, the operator $\mathbf{u}\longmapsto\mathbf{d}[\mathbf{u}]$ maps bounded sets in $C([0,T];C_0^2(\bar\Omega;\mathbb{R}^3))$ into bounded subsets of

$$Y:=L^2([0,T];H^2(\Omega))\cap L^\infty([0,T];H^1(\Omega)),$$

and the mapping

$$\mathbf{u}\in C([0,T];C_0^2(\bar\Omega;\mathbb{R}^3))\longmapsto\mathbf{d}\in Y$$

is continuous on any bounded subsets of $C([0,T];C_0^2(\bar\Omega;\mathbb{R}^3))$.

Proof. The uniqueness of the solution to (2.2.1) is a consequence of Lemma 2.2.2, and the existence of a solution can be guaranteed by the standard parabolic equation theory. By (2.2.3), we can conclude that the solution operator $\mathbf{u} \longmapsto \mathbf{d}(\mathbf{u})$ maps bounded sets in $C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$ into bounded subsets of the set Y. Our next step is to show that the solution operator is continuous from any bounded subset of $C([0,T]; C_0^2(\bar{\Omega}))$ to Y. Let $\{\mathbf{u}_n\}_{n=1}^{\infty}$ be a bounded sequence in $C([0,T]; C_0^2(\bar{\Omega}))$, that is to say, $\mathbf{u}_n \in B(0,R) \subset C([0,T]; C_0^2(\bar{\Omega}))$ for some $R > 0$, and

$$\mathbf{u}_n \to \mathbf{u} \text{ in } C([0,T]; C_0^2(\bar{\Omega})) \quad \text{as } n \to \infty.$$

Here, we denote $\mathbf{d}[\mathbf{u}] = \mathbf{d}$, and $\mathbf{d}[\mathbf{u}_n] = \mathbf{d}_n$, so we have

$$\begin{aligned}
&\partial_t \int_\Omega \frac{1}{2} |\nabla(\mathbf{d}_n - \mathbf{d})|^2 dx + \int_\Omega |\Delta(\mathbf{d}_n - \mathbf{d})|^2 dx \\
&= \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{d} - \mathbf{u}_n \cdot \nabla \mathbf{d}_n)(\nabla(\mathbf{d}_n - \mathbf{d})) dx + \int_\Omega (f(\mathbf{d}) - f(\mathbf{d}_n)) \cdot (\Delta(\mathbf{d}_n - \mathbf{d})) dx \\
&\le \int_\Omega (|\mathbf{u} - \mathbf{u}_n| \cdot |\nabla \mathbf{d}| + |\mathbf{u}_n||\nabla(\mathbf{d} - \mathbf{d}_n)|)|\Delta(\mathbf{d}_n - \mathbf{d})| dx + C \int_\Omega |\nabla(\mathbf{d}_n - \mathbf{d})|^2 dx \\
&\le \|\mathbf{u}_n - \mathbf{u}\|_{L^\infty} \|\nabla \mathbf{d}\|_{L^2}^2 + C\|\nabla(\mathbf{d} - \mathbf{d}_n)\|_{L^2}^2 + \frac{1}{2} \int_\Omega |\Delta(\mathbf{d}_n - \mathbf{d})|^2 dx \\
&\le C\|\mathbf{u}_n - \mathbf{u}\|_{L^\infty} + \frac{1}{2} \|\nabla(\mathbf{d} - \mathbf{d}_n)\|_{L^2}^2,
\end{aligned} \tag{2.2.4}$$

where we used facts that $\mathbf{d}_n$ is bounded in Y and f is smooth. This implies that

$$\begin{aligned}
&\frac{1}{2} \partial_t \int_\Omega |\nabla(\mathbf{d}_n - \mathbf{d})|^2 dx + \frac{1}{2} \int_\Omega |\Delta(\mathbf{d}_n - \mathbf{d})|^2 dx \\
&\le C\|\mathbf{u}_n - \mathbf{u}\|_{L^\infty} + C\|\nabla(\mathbf{d}_n - \mathbf{d})\|_{L^2}^2.
\end{aligned} \tag{2.2.5}$$

Integrating (2.2.5) over time $t \in (0,T)$, and then taking the upper limit over n on the both sides, we get, noting that $\mathbf{u}_n \to \mathbf{u}$ in $C([0,T]; C_0^2(\bar{\Omega}); \mathbb{R}^3)$,

$$\begin{aligned}
&\frac{1}{2} \limsup_n \int_\Omega |\nabla(\mathbf{d}_n - \mathbf{d})|^2 dx + \frac{1}{2} \limsup_n \int_0^T \int_\Omega |\Delta(\mathbf{d}_n - \mathbf{d})|^2 dx dt \\
&\le C \limsup_n \int_0^T \|\nabla(\mathbf{d}_n - \mathbf{d})\|_{L^2}^2 dt \\
&\le C \int_0^T \limsup_n \|\nabla(\mathbf{d}_n - \mathbf{d})\|_{L^2}^2 dt,
\end{aligned} \tag{2.2.6}$$

thus, using Grönwall's inequality to (2.2.6) and noting that $\mathbf{d}_n$, $\mathbf{d}$ share the same initial data, we have

$$\limsup_n \int_\Omega |\nabla(\mathbf{d}_n - \mathbf{d})|^2 dx = 0,$$

which means, from (2.2.6) again,

$$\limsup_n \int_0^T \int_\Omega |\Delta(\mathbf{d}_n - \mathbf{d})|^2 dxdt = 0.$$

Thus, we obtain

$$\mathbf{d}_n \to \mathbf{d} \text{ in } Y.$$

This completes the proof of the continuity of the solution operator. □

2.3 THE FAEDO-GALERKIN APPROXIMATION SCHEME

In this section, we establish the existence of solution to the following approximation scheme:

$$\rho_t + \text{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho, \tag{2.3.1a}$$

$$\begin{aligned}(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) + \delta \nabla \rho^\beta + \varepsilon \nabla \mathbf{u} \cdot \nabla \rho \\ = \mu \Delta \mathbf{u} - \lambda \text{div}\left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2}|\nabla \mathbf{d}|^2 + F(\mathbf{d}))I_3 \right),\end{aligned} \tag{2.3.1b}$$

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} - f(\mathbf{d}), \tag{2.3.1c}$$

with boundary conditions

$$\nabla \rho \cdot \nu|_{\partial\Omega} = 0, \tag{2.3.2a}$$

$$\mathbf{d}|_{\partial\Omega} = \mathbf{d}_0, \tag{2.3.2b}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{2.3.2c}$$

together with modified initial data

$$\rho|_{t=0} = \rho_{0,\delta}(x), \tag{2.3.3a}$$

$$\rho\mathbf{u}|_{t=0} = \mathbf{m}_{0,\delta}(x), \tag{2.3.3b}$$

$$\mathbf{d}|_{t=0} = \mathbf{d}_0(x). \tag{2.3.3c}$$

Here the initial data $\rho_{0,\delta}(x) \in C^3(\overline{\Omega})$ satisfies the following conditions:

$$0 < \delta \leq \rho_{0,\delta}(x) \leq \delta^{-\frac{1}{2\beta}}, \tag{2.3.4}$$

and

$$\rho_{0,\delta}(x) \to \rho_0 \text{ in } L^\gamma(\Omega), \quad |\{\rho_{0,\delta} < \rho_0\}| \to 0 \quad \text{ as } \delta \to 0. \tag{2.3.5}$$

Moreover,

$$\mathbf{m}_{0,\delta}(x) = \begin{cases} \mathbf{m}_0 & \text{if } \rho_{0,\delta}(x) \geq \rho_0(x), \\ 0 & \text{if } \rho_{0,\delta}(x) < \rho_0(x). \end{cases} \tag{2.3.6}$$

The density $\rho = \rho[\mathbf{u}]$ is determined uniquely as the solution of the following Neumann initial-boundary value problem (see Lemmas 2.1 and 2.2 of [18]):

$$\rho_t + \text{div}(\rho\mathbf{u}) = \varepsilon\Delta\rho, \tag{2.3.7a}$$

$$\nabla\rho \cdot \nu|_{\partial\Omega} = 0, \tag{2.3.7b}$$

$$\rho|_{t=0} = \rho_{0,\delta}(x), \tag{2.3.7c}$$

To solve (2.3.1b) by a modified Faedo-Galerkin method, we need to introduce the finite-dimensional space endowed with the L^2 Hilbert space structure:

$$X_n = span(\eta_i)_{i=1}^n, \quad n \in \{1, 2, 3, \cdots\},$$

where the linearly independent functions $\eta_i \in \mathcal{D}(\Omega)^3$, $i = 1, 2, \ldots,$ form a dense subset in $C_0^2(\overline{\Omega}, \mathbb{R}^3)$. The approximate solution $\mathbf{u}_n$ should be given by the following form:

$$\begin{aligned} &\int_\Omega \rho\mathbf{u}_n(\tau) \cdot \eta dx - \int_\Omega m_{0,\delta} \cdot \eta dx \\ &= \int_0^\tau \int_\Omega \big((\mu\Delta\mathbf{u}_n - \text{div}(\rho\mathbf{u}_n \otimes \mathbf{u}_n)) - \nabla(\rho^\gamma + \delta\rho^\beta) - \varepsilon\nabla\rho \cdot \nabla\mathbf{u}_n\big) \cdot \eta dxdt \\ &- \int_0^\tau \int_\Omega \lambda\text{div}(\nabla\mathbf{d} \odot \nabla\mathbf{d} - (\frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d}))I_3) \cdot \eta dxdt \end{aligned} \tag{2.3.8}$$

for any $t \in [0,T]$ and any $\eta \in X_n$, where $\varepsilon, \delta, \beta$ are fixed. Due to Lemmas 2.1 and 2.2 of [18] and our Lemmas 2.2.1 and 2.2.2, the problem (2.2.1), (2.3.7) and (2.3.8) can be solved at least on a short time interval $(0, T_n)$ with $T_n \leq T$ by a standard fixed point theorem on the Banach space $C([0,T], X_n)$. We refer the readers to [18] for more details. Thus we obtain a local solution $(\rho_n, \mathbf{u}_n, \mathbf{d}_n)$ in time.

To obtain uniform bounds on $\mathbf{u}_n$, we derive an energy inequality similar to (2.1.2) as follows. Taking $\eta = \mathbf{u}_n(t,x)$ with fixed t in (2.3.1) and repeating the procedure for a priori estimates in Section 2, we deduce a "Kinetic energy equality":

$$\begin{aligned} &\partial_t \int_\Omega \left(\frac{1}{2}\rho_n |\mathbf{u}_n|^2 + \frac{1}{\gamma - 1}\rho_n^\gamma + \frac{\delta}{\beta - 1}\rho_n^\beta + \frac{\lambda}{2}|\nabla \mathbf{d}_n|^2 + \lambda F(\mathbf{d}_n) \right) dx + \mu \int_\Omega |\nabla \mathbf{u}_n|^2 dx \\ &+ \lambda \int_\Omega |\Delta \mathbf{d}_n - f(\mathbf{d}_n)|^2 dx + \varepsilon \int_\Omega (\gamma \rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2})|\nabla \rho_n|^2 dx = 0. \end{aligned} \tag{2.3.9}$$

The uniform estimates obtained from (2.3.9) furnish the possibility of repeating the above fixed point argument to extend the local solution $\mathbf{u}_n$ to the whole time interval $[0,T]$. Then, by the solvability of equation (2.3.7) and (2.2.1), we obtain the functions $(\rho_n, \mathbf{d}_n)$ on the whole time interval $[0,T]$.

The next step in the proof of Theorem 2.1.1 is to pass the limit as $n \to \infty$ in the sequence of approximate solutions $\{\rho_n, \mathbf{u}_n, \mathbf{d}_n\}$ obtained above. We observe that the terms related to $\mathbf{u}_n$ and ρ_n can be treated similarly to [18]. It remains to show the convergence of the terms related to $\mathbf{d}_n$.

By (2.3.9), smoothness of f, and elliptic estimates, we conclude

$$\nabla \mathbf{u}_n \in L^2([0,T]; L^2(\Omega)), \tag{2.3.10}$$

$$\Delta \mathbf{d}_n - f(\mathbf{d}_n) \text{ is bounded in } L^2([0,T]; L^2(\Omega)), \tag{2.3.11}$$

and

$$\mathbf{d}_n \in L^\infty([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)).$$

This yields that

$$\Delta \mathbf{d}_n - f(\mathbf{d}_n) \to \Delta \mathbf{d} - f(\mathbf{d}) \text{ weakly in } L^2([0,T]; L^2(\Omega)),$$

and

$$\mathbf{d}_n \to \mathbf{d} \text{ weakly in } L^\infty([0,T];H^1(\Omega)) \cap L^2([0,T];H^2(\Omega)). \tag{2.3.12}$$

Using corollary 2.1 in [21] and (2.3.1c), we can improve (2.3.12) as follows:

$$\mathbf{d}_n \to \mathbf{d} \quad \text{ in } C([0,T];L^2_{weak}(\Omega)).$$

Next we need to rely on the following Aubin-Lions compactness lemma (see [41]):

Lemma 2.3.1. *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1; Suppose also that X_0 and X_1 are reflexive spaces. For $1 < p, q < \infty$, let*

$$W = \{u \in L^p([0,T];X_0) | \frac{du}{dt} \in L^q([0,T];X_1)\}.$$

Then the embedding of W into $L^p([0,T];X)$ is also compact.

We are now applying the Aubin-Lions lemma to obtain the convergence of $\mathbf{d}_n$ and $\nabla \mathbf{d}_n$. From Remark 2.1.1, we have

$$\mathbf{d}_n \in L^{10}((0,T) \times \Omega),$$

and

$$\nabla \mathbf{d}_n \in L^{\frac{10}{3}}((0,T) \times \Omega). \tag{2.3.13}$$

Using (2.3.1c), we have

$$\begin{aligned}
\|\partial_t \mathbf{d}_n\|_{L^{\frac{3}{2}}(\Omega)} &\le C\|\mathbf{u}_n \cdot \nabla \mathbf{d}_n\|_{L^{\frac{3}{2}}(\Omega)} + C\|\Delta \mathbf{d}_n - f(\mathbf{d}_n)\|_{L^{\frac{3}{2}}(\Omega)} \\
&\le C\|\mathbf{u}_n\|_{L^6(\Omega)}\|\nabla \mathbf{d}_n\|_{L^2(\Omega)} + C\|\Delta \mathbf{d}_n - f(\mathbf{d}_n)\|_{L^2(\Omega)}, \\
&\le C\|\nabla \mathbf{u}_n\|_{L^2(\Omega)} + C\|\Delta \mathbf{d}_n - f(\mathbf{d}_n)\|_{L^2(\Omega)},
\end{aligned}$$

where we used embedding inequality, the values of C are variant. Thus, (2.3.10), (2.3.11) and (2.3.13) yield

$$\|\partial_t \mathbf{d}_n\|_{L^2([0,T];L^{\frac{3}{2}}(\Omega))} \le C.$$

Notice that $H^2 \subset H^1 \subset L^{\frac{3}{2}}$ and the injection $H^2 \hookrightarrow H^1$ is compact, applying Lemma 2.3.1 we deduce that the sequence $\{\mathbf{d}_n\}_{n=1}^\infty$ is precompact in $L^2([0,T];H^1(\Omega))$.

Summing up the previous results, by taking a subsequence if necessary, we can assume that:

$$\mathbf{d}_n \to \mathbf{d} \quad \text{ in } C([0,T]; L^2_{weak}(\Omega)),$$

$$\mathbf{d}_n \to \mathbf{d} \text{ weakly in } L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)),$$

$$\mathbf{d}_n \to \mathbf{d} \text{ strongly in } L^2(0,T; H^1(\Omega)),$$

$$\nabla \mathbf{d}_n \to \nabla \mathbf{d} \text{ weakly in } L^{\frac{10}{3}}((0,T) \times \Omega),$$

$$\Delta \mathbf{d}_n - f(\mathbf{d}_n) \to \Delta \mathbf{d} - f(\mathbf{d}) \text{ weakly in } L^2(0,T; L^2(\Omega)),$$

$$F(\mathbf{d}_n) \to F(\mathbf{d}) \text{ strongly in } L^2(0,T; H^1(\Omega)).$$

Now, we consider the convergence of the terms related to $\mathbf{d}_n$ and $\nabla \mathbf{d}_n$. Let φ be a test function, then

$$\begin{aligned} &\int_\Omega (\nabla \mathbf{d}_n \odot \nabla \mathbf{d}_n - \nabla \mathbf{d} \odot \nabla \mathbf{d}) \cdot \nabla \varphi dx dt \\ &\leq \int_\Omega (\nabla \mathbf{d}_n \odot \nabla \mathbf{d}_n - \nabla \mathbf{d}_n \odot \nabla \mathbf{d}) \nabla \varphi dx dt + \int_\Omega (\nabla \mathbf{d}_n \odot \nabla \mathbf{d} - \nabla \mathbf{d} \odot \nabla \mathbf{d}) \nabla \varphi dx dt \\ &\leq C \|\nabla \mathbf{d}_n\|_{L^2(\Omega)} \|\nabla \mathbf{d}_n - \nabla \mathbf{d}\|_{L^2(\Omega)} + C \|\nabla \mathbf{d}\|_{L^2(\Omega)} \|\nabla \mathbf{d}_n - \nabla \mathbf{d}\|_{L^2(\Omega)} \end{aligned} \tag{2.3.14}$$

By the strong convergence of $\nabla \mathbf{d}_n$ in $L^2(\Omega)$ and (2.3.14), we conclude that

$$\nabla \mathbf{d}_n \odot \nabla \mathbf{d}_n \to \nabla \mathbf{d} \odot \nabla \mathbf{d} \text{ in } \mathcal{D}'(\Omega \times (0,T)).$$

Similarly,

$$\frac{1}{2} |\nabla \mathbf{d}_n|^2 I_3 \to \frac{1}{2} |\nabla \mathbf{d}|^2 I_3 \text{ in } \mathcal{D}'(\Omega \times (0,T)),$$

and

$$\mathbf{u}_n \nabla \mathbf{d}_n \to \mathbf{u} \nabla \mathbf{d} \text{ in } \mathcal{D}'(\Omega \times (0,T)),$$

where we used

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2([0,T]; H^1_0(\Omega)).$$

Therefore, (2.2.1) and (2.3.8) hold at least in the sense of distribution. Moreover, by the uniform estimates on $\mathbf{u}, \mathbf{d}$ and (1.1.1c), we know that the map

$$t \to \int_\Omega \mathbf{d}_n(x,t) \varphi(x) dx \quad \text{ for any } \varphi \in \mathcal{D}(\Omega),$$

is equi-continuous on $[0, T]$. By the Ascoli-Arzela Theorem, we know that

$$t \to \int_\Omega \mathbf{d}(x,t)\varphi(x)dx$$

is continuous for any $\varphi \in \mathcal{D}(\Omega)$. Thus, $\mathbf{d}$ satisfies the initial condition in (2.2.1).

Now we have the existence of a global solution to (2.3.1) as follows:

Proposition 2.3.1. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of the class $C^{2+\nu}$, $\nu > 0$; Let $\varepsilon > 0, \delta > 0$, and $\beta > max\{4, \gamma\}$ be fixed. Then for any given $T > 0$, there is a solution $(\rho, \mathbf{u}, \mathbf{d})$ to the initial-boundary value problem of (2.3.1) in the following sense:*

(1) The density ρ is a nonnegative function such that

$$\rho \in L^\gamma([0,T]; W^{2,r}(\Omega)), \quad \partial_t \rho \in L^r((0,T)\times\Omega),$$

for some $r > 1$, the velocity $\mathbf{u} \in L^2([0,T]; H_0^1(\Omega))$, and (2.3.1a) holds almost everywhere on $(0,T)\times\Omega$, and the initial and boundary data on ρ are satisfied in the sense of traces. Moreover, the total mass is conserved, i.e.

$$\int_\Omega \rho(x,t)dx = \int_\Omega \rho_{\delta,0}dx,$$

for all $t \in [0,T]$; and the following inequalities hold

$$\delta \int_0^T \int_\Omega \rho^{\beta+1}dxdt \le C(\varepsilon),$$

$$\varepsilon \int_0^T \int_\Omega |\nabla\rho|^2 dxdt \le C \text{ with } C \text{ independent of } \varepsilon.$$

(2) All quantities appearing in equation (2.3.1b) are locally integrable, and the equation is satisfied in $\mathcal{D}'(\Omega\times(0,T))$. Moreover,

$$\rho\mathbf{u} \in C([0,T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$

and $\rho\mathbf{u}$ satisfies the initial data.

(3) All terms in (2.3.1c) are locally integrable on $\Omega\times(0,T)$. The direction $\mathbf{d}$ satisfies the equation (2.2.1a) and the initial data (2.2.1b) in the sense of distribution.

(4) The energy inequality

$$
\begin{aligned}
&\partial_t \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{\delta}{\beta-1}\rho^\beta + \frac{\lambda}{2}|\nabla \mathbf{d}|^2 + \lambda F(\mathbf{d}) \right) dx \\
&+ \mu \int_\Omega |\nabla \mathbf{u}|^2 dx + \lambda \int_\Omega |\Delta \mathbf{d} - f(\mathbf{d})|^2 dx \\
&\leq 0
\end{aligned}
$$

holds almost everywhere for $t \in [0, T]$.

To complete our proof of the main theorem, we will take vanishing artificial viscosity and vanishing artificial pressure in the following sections.

2.4 VANISHING VISCOSITY LIMIT

In this section, we will pass the limit as $\varepsilon \to 0$ in the family of approximate solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$ obtained in Proposition 2.3.1. The estimates in Proposition 2.3.1 are independent of n, and those estimates are still valid for $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$. But, we need to remark that ρ_ε will lose some regularity when $\varepsilon \to 0$ because the term $\varepsilon \Delta \rho_\varepsilon$ goes away. The space $L^\infty(0, T; L^1(\Omega))$ is a non-reflexive space, and the artificial pressure is bounded only in space $L^\infty(0, T; L^1(\Omega))$ from the estimates of Proposition 2.3.1. It is crucial to establish the strong compactness of the density ρ_ε for passing the limits. To this end, we need to obtain better estimates on the artificial pressure.

2.4.1 Uniform estimates of the density

We first introduce an operator

$$B : \left\{ f \in L^p(\Omega) : \int_\Omega f dx = 0 \right\} \longmapsto [W_0^{1,p}(\Omega)]^3$$

which is a bounded linear operator satisfying

$$\|B[f]\|_{W_0^{1,p}(\Omega)} \leq c(p)\|f\|_{L^p(\Omega)} \quad \text{for any } 1 < p < \infty, \tag{2.4.1}$$

where the function $W = B[f] \in \mathbb{R}^3$ solves the following equation:

$$\mathrm{div} W = f \text{ in } \Omega, \quad W|_{\partial\Omega} = 0.$$

Moreover, if the function f can be written in the form $f = \mathrm{div} g$ for some $g \in L^r$, and $g \cdot \nu|_{\partial\Omega} = 0$, then

$$\|B[f]\|_{L^r(\Omega)} \leq c(r)\|g\|_{L^r(\Omega)}$$

for any $1 < r < \infty$. We refer the readers to [21, 18] for more background and discussion of the operator B. Define the function:

$$\varphi(t,x) = \psi(t)B[\rho_\varepsilon - \widehat{\rho}], \quad \psi \in \mathcal{D}(0,T), \quad 0 \leq \psi \leq 1,$$

where

$$\widehat{\rho} = \frac{1}{|\Omega|}\int_\Omega \rho(t)dx.$$

Since ρ_ε is a solution to (2.3.1a), by Proposition 2.3.1 and $\beta > 4$, we have

$$\rho_\varepsilon - \widehat{\rho} \in C([0,T], L^4(\Omega)).$$

Therefore, from (2.4.1), we have $\varphi(t,x) \in C([0,T], W^{1,4}(\Omega))$. In particular, $\varphi(t,x) \in C([0,T]\times\Omega)$ by the Sobolev embedding theorem. Consequently, φ can be used as a test function for (2.3.1b).

After a little bit lengthy but straightforward computation, we obtain:

$$\begin{aligned}
&\int_0^T \int_\Omega \psi(\rho_\varepsilon^{\gamma+1} + \delta\rho_\varepsilon^{\delta+1}) dxdt \\
&= \widehat{\rho} \int_0^T \int_\Omega \psi(\rho_\varepsilon^{\gamma} + \delta\rho_\varepsilon^{\beta}) dxdt + \int_0^T \int_\Omega \psi\rho_\varepsilon \mathbf{u}_\varepsilon B[\rho_\varepsilon - \widehat{\rho}] dxdt \\
&\quad + \mu \int_0^T \int_\Omega \psi \nabla \mathbf{u}_\varepsilon \nabla B[\rho_\varepsilon - \widehat{\rho}] dxdt \\
&\quad - \int_0^T \int_\Omega \psi\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \nabla B[\rho_\varepsilon - \widehat{\rho}] dxdt \\
&\quad - \varepsilon \int_0^T \int_\Omega \psi\rho_\varepsilon \mathbf{u}_\varepsilon B[\Delta\rho_\varepsilon] dxdt \\
&\quad - \int_0^T \int_\Omega \psi\rho_\varepsilon \mathbf{u}_\varepsilon B[\operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon)] dxdt \\
&\quad + \varepsilon \int_0^T \int_\Omega \nabla \mathbf{u}_\varepsilon \nabla \rho_\varepsilon B[\rho_\varepsilon - \widehat{\rho}] dxdt \\
&\quad - \lambda \int_0^T \int_\Omega \left(\nabla \mathbf{d}_\varepsilon \otimes \nabla \mathbf{d}_\varepsilon - (\frac{|\nabla \mathbf{d}_\varepsilon|^2}{2} + F(\mathbf{d})) I_3 \right) \psi \nabla B[\rho_\varepsilon - \widehat{\rho}] dxdt \\
&= \sum_{j=1}^{7} I_j.
\end{aligned} \tag{2.4.2}$$

To achieve our lemma below, we need to estimate that the terms $I_1 - I_7$ are bounded. We can treat the terms related to $\rho_\varepsilon, \mathbf{u}_\varepsilon$ similar to [18]. It remains to estimate the term I_7. Indeed,

$$\begin{aligned}
|I_7| &= \left| \lambda \int_0^T \int_\Omega \left(\nabla \mathbf{d}_\varepsilon \otimes \nabla \mathbf{d}_\varepsilon - (\frac{|\nabla \mathbf{d}_\varepsilon|^2}{2} + F(\mathbf{d})) I_3 \right) \psi \nabla B[\rho_\varepsilon - \widehat{\rho}] dxdt \right| \\
&\leq C\lambda \int_0^T \|\nabla \mathbf{d}_\varepsilon\|^2_{L^{\frac{10}{3}}(\Omega)} \|B[\rho_\varepsilon - \widehat{\rho}]\|_{W^{1,\frac{5}{2}}(\Omega)} dt + C \int_0^T \|B[\rho_\varepsilon - \widehat{\rho}]\|_{W^{1,\frac{5}{2}}(\Omega)} dt \\
&\leq C,
\end{aligned} \tag{2.4.3}$$

where we used

$$\|B[\rho_\varepsilon - \widehat{\rho}]\|_{W^{1,\frac{5}{2}}(\Omega)} \leq C_0 \|\rho_\varepsilon - \widehat{\rho}\|_{L^{\frac{5}{2}}(\Omega)},$$

and $\beta \geq 4$. Consequently, we have proved the following result:

Lemma 2.4.1. *Let $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$ be the solutions of the problem (2.3.1) constructed in Proposition 2.3.1, then*

$$\|\rho_\varepsilon\|_{L^{\gamma+1}((0,T)\times\Omega)} + \|\rho_\varepsilon\|_{L^{\beta+1}((0,T)\times\Omega)} \leq C,$$

where C is independent of ε.

2.4.2 The vanishing viscosity limit passage

From the previous energy estimates, we have

$$\varepsilon\Delta\rho_\varepsilon \to 0 \quad \text{in } L^2(0,T;W^{-1,2}(\Omega))$$

and

$$\varepsilon\nabla\mathbf{u}_\varepsilon\nabla\rho_\varepsilon \to 0 \quad \text{in } L^1(0,T;L^1(\Omega))$$

as $\varepsilon \to 0$.

Due to the above estimates so far, we may now assume that

$$\rho_\varepsilon \to \rho \text{ in } C([0,T], L^\gamma_{weak}(\Omega)), \tag{2.4.4a}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}_0(\Omega)), \tag{2.4.4b}$$

$$\rho_\varepsilon\mathbf{u}_\varepsilon \to \rho\mathbf{u} \text{ in } C([0,T], L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)). \tag{2.4.4c}$$

Then we can pass the limits of the terms related to $\rho_\varepsilon, \mathbf{u}_\varepsilon$ similarly to [18]. It remains to show the convergence of $\mathbf{d}_\varepsilon$. Following the same arguments of Section 4, by taking a subsequence if necessary, we can assume that:

$$\mathbf{d}_\varepsilon \to \mathbf{d} \quad \text{in } C([0,T];L^2_{weak}(\Omega)) \tag{2.4.5a}$$

$$\mathbf{d}_\varepsilon \to \mathbf{d} \text{ weakly in } L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{2.4.5b}$$

$$\mathbf{d}_\varepsilon \to \mathbf{d} \text{ strongly in } L^2(0,T;H^1(\Omega)), \tag{2.4.5c}$$

$$\nabla\mathbf{d}_\varepsilon \to \nabla\mathbf{d} \text{ weakly in } L^{\frac{10}{3}}((0,T)\times\Omega), \tag{2.4.5d}$$

$$\Delta\mathbf{d}_\varepsilon - f(\mathbf{d}_\varepsilon) \to \Delta\mathbf{d} - f(\mathbf{d}) \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{2.4.5e}$$

$$F(\mathbf{d}_\varepsilon) \to F(\mathbf{d}) \text{ strongly in } L^2(0,T;H^1(\Omega)). \tag{2.4.5f}$$

Consequently, letting $\varepsilon \to 0$ and making use of (2.4.4) and (2.4.5), we conclude that the limit of $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon)$ satisfies the following system:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{2.4.6a}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\bar{P} = \mu\Delta\mathbf{u} - \lambda\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d} - (\frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d}))I_3), \tag{2.4.6b}$$

$$\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} - f(\mathbf{d}) \tag{2.4.6c}$$

where $\bar{P} = \overline{a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta}$, here $\overline{K}(x)$ stands for a weak limit of $\{K_\varepsilon\}$.

2.4.3 The strong convergence of the density

We observe that $\rho_\varepsilon, \mathbf{u}_\varepsilon$ is a strong solution of parabolic equation (2.3.1a), then the renormalized form can be written as

$$\begin{aligned} &\partial_t b(\rho_\varepsilon) + \mathrm{div}(b(\rho_\varepsilon)\mathbf{u}_\varepsilon) + (b^{'}(\rho_\varepsilon)\rho_\varepsilon - b(\rho_\varepsilon))\mathrm{div}\mathbf{u}_\varepsilon \\ &= \varepsilon \mathrm{div}(\chi_\Omega \nabla b(\rho_\varepsilon)) - \varepsilon\chi_\Omega b^{''}(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 \end{aligned} \tag{2.4.7}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, with $b \in C^2[0,\infty)$, $\quad b(0) = 0$, and $b^{'}$, $\quad b^{''}$ bounded functions and b convex, where χ_Ω is the characteristics function of Ω. By the virtue of (2.4.7) and the convexity of b, we have

$$\int_0^T\int_\Omega \psi(b^{'}(\rho_\varepsilon)\rho_\varepsilon - b(\rho_\varepsilon)))\mathrm{div}\mathbf{u}_\varepsilon dxdt \le \int_\Omega b(\rho_{0,\delta})dx + \int_0^T\int_\Omega \partial_t\psi b(\rho_\varepsilon)dxdt$$

for any $\psi \in C^\infty[0,T]$, $\quad 0 \le \psi \le 1$, $\quad \psi(0) = 1$, $\quad \psi(T) = 0$. Taking $b(z) = z\log z$ gives us the following estimate:

$$\int_0^T\int_\Omega \psi\rho_\varepsilon \mathrm{div}\mathbf{u}_\varepsilon dxdt \le \int_\Omega \rho_{0,\delta}\log(\rho_{0,\delta})dx + \int_0^T\int_\Omega \partial_t\psi\rho_\varepsilon\log\rho_\varepsilon dxdt,$$

and letting $\varepsilon \to 0$ yields

$$\int_0^T\int_\Omega \psi\overline{\rho\mathrm{div}\mathbf{u}}dxdt \le \int_\Omega \rho_{0,\delta}\log\rho_{0,\delta}dx + \int_0^T\int_\Omega \partial_t\psi\overline{\rho\log\rho}dxdt,$$

that is,

$$\int_0^T\int_\Omega \overline{\rho\mathrm{div}\mathbf{u}}dxdt \le \int_\Omega \rho_{0,\delta}\log\rho_{0,\delta}dx - \int_\Omega \overline{\rho\log\rho}(t)dx. \tag{2.4.8}$$

Meanwhile, $(\rho, \mathbf{u})$ satisfies

$$\partial_t b(\rho) + \mathrm{div}(b(\rho)\mathbf{u}) + (b^{'}(\rho)\rho - b(\rho))\mathrm{div}\mathbf{u} = 0. \tag{2.4.9}$$

Using (2.4.9) and $b(z) = z\log z$, we deduce the following inequality:

$$\int_0^T\int_\Omega \rho\mathrm{div}u dxdt \le \int_\Omega \rho_{0,\delta}\log\rho_{0,\delta}dx - \int_\Omega \rho\log\rho(t)dx. \tag{2.4.10}$$

From (2.4.10) and (2.4.8), we deduce that

$$\int_\Omega \overline{\rho\log\rho} - \rho\log(\rho)(\tau)dx \le \int_0^T\int_\Omega \rho\mathrm{div}\mathbf{u} - \overline{\rho\mathrm{div}\mathbf{u}}dxdt \tag{2.4.11}$$

for a.e. $\tau \in [0,T]$.

To obtain the strong convergence of density ρ_ε, the crucial point is to get the weak continuity of the viscous pressure, namely:

Lemma 2.4.2. *Let $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ be the sequence of approximate solutions constructed in Proposition 2.3.1, then*

$$\lim_{\varepsilon\to 0^+}\int_0^T\int_\Omega \psi\eta(a\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta-\mu\mathrm{div}\mathbf{u}_\varepsilon)\rho_\varepsilon dxdt$$
$$=\int_0^T\int_\Omega \psi\eta(\bar{P}-\mu\mathrm{div}\mathbf{u})\rho dxdt \quad \text{for any } \psi\in\mathcal{D}(0,T), \quad \eta\in\mathcal{D}(\Omega),$$

where $\bar{P}=\overline{a\rho^\gamma+\delta\rho^\beta}$.

Proof. We need to introduce a new operator

$$A_i=\Delta^{-1}(\partial_{x_i}v),\ i=1,2,3,$$

where Δ^{-1} stands for the inverse of the Laplace operator on $\mathbb{R}^3$. To be more specific, A_i can be expressed by their Fourier symbol

$$A_i(\cdot)=\mathcal{F}^{-1}(\frac{-i\xi_i}{|\xi|^2}\mathcal{F}(\cdot)), \quad i=1,2,3,$$

with the following properties (see [18]):

$$\|A_iv\|_{W^{1,s}(\Omega)}\le c(s,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, \quad 1<s<\infty,$$

$$\|A_iv\|_{L^q(\Omega)}\le c(q,s,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, \quad q<\infty, \quad \text{provided } \frac{1}{q}\ge\frac{1}{s}-\frac{1}{3},$$

and

$$\|A_iv\|_{L^\infty(\Omega)}\le c(s,\Omega)\|v\|_{L^s(\mathbb{R}^3)} \quad \text{if } s>3.$$

Next, we use the quantities

$$\varphi(t,x)=\psi(t)\eta(x)A_i[\rho_\varepsilon], \quad \psi\in\mathcal{D}(0,T), \quad \eta\in\mathcal{D}(\Omega), \quad i=1,2,3,$$

as a test function for (2.3.1b) to obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega \varphi\eta((\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta) - \mu \mathrm{div}\mathbf{u}_\varepsilon)\rho_\varepsilon dxdt \\
&= \mu \int_0^T \int_\Omega \psi \nabla \mathbf{u}_\varepsilon \nabla\eta A[\rho_\varepsilon] dxdt - \int_0^T \int_\Omega \psi(\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\nabla\eta A[\rho_\varepsilon] dxdt \\
&\quad - \int_0^T \int_\Omega \psi\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \nabla\eta A[\rho_\varepsilon] dxdt - \int_0^T \int_\Omega \psi_t \eta\rho_\varepsilon \mathbf{u}_\varepsilon A[\rho_\varepsilon] dxdt \\
&\quad - \varepsilon \int_0^T \int_\Omega \psi\eta\rho_\varepsilon \mathbf{u}_\varepsilon A[\mathrm{div}(\chi_\Omega \nabla\rho_\varepsilon)] dxdt \\
&\quad + \varepsilon \int_0^T \int_\Omega \psi\eta\nabla\rho_\varepsilon \nabla\mathbf{u}_\varepsilon A[\rho_\varepsilon] dxdt + \mu \int_0^T \int_\Omega \psi\mathbf{u}_\varepsilon \nabla\eta\rho_\varepsilon dxdt \\
&\quad - \mu \int_0^T \int_\Omega \psi\mathbf{u}_\varepsilon \nabla\eta\nabla A[\rho_\varepsilon] dxdt + \int_0^T \int_\Omega \psi\mathbf{u}_\varepsilon(\rho_\varepsilon R[\rho_\varepsilon \mathbf{u}_\varepsilon] - \rho_\varepsilon \mathbf{u}_\varepsilon R[\rho_\varepsilon]) dxdt \\
&\quad - \lambda \int_0^T \int_\Omega \left(\nabla\mathbf{d}_\varepsilon \odot \nabla\mathbf{d}_\varepsilon - (\frac{1}{2}|\nabla\mathbf{d}_\varepsilon|^2 + F(\mathbf{d}_\varepsilon))I_3 \right) \psi\nabla\eta A[\rho_\varepsilon] dxdt \\
&\quad - \lambda \int_0^T \int_\Omega \left(\nabla\mathbf{d}_\varepsilon \odot \nabla\mathbf{d}_\varepsilon - (\frac{1}{2}|\nabla\mathbf{d}_\varepsilon|^2 + F(\mathbf{d}_\varepsilon))I_3 \right) \psi\eta\nabla A[\rho_\varepsilon] dxdt
\end{aligned}
\tag{2.4.12}
$$

where χ_Ω is the characteristics function of Ω, $A[x] = \nabla\Delta^{-1}[x]$.

Meanwhile, we can use

$$
\varphi(t,x) = \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho], \quad \psi \in \mathcal{D}(0,T), \quad \eta \in \mathcal{D}(\Omega),
$$

as a test function for (2.4.6b) to obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega \varphi\eta(\bar{P} - \mu\mathrm{div}\mathbf{u})\rho dxdt \\
&= \mu \int_0^T \int_\Omega \psi\nabla\mathbf{u}\nabla\eta A[\rho] dxdt - \int_0^T \int_\Omega \psi P \nabla\eta A[\rho] dxdt \\
&\quad - \int_0^T \int_\Omega \psi\rho\mathbf{u} \otimes \mathbf{u}\nabla\eta A[\rho] dxdt - \int_0^T \int_\Omega \psi_t \eta\rho\mathbf{u} A[\rho] dxdt \\
&\quad + \mu \int_0^T \int_\Omega \psi\mathbf{u}\nabla\eta\rho dxdt - \mu \int_0^T \int_\Omega \psi\mathbf{u}\nabla\eta\nabla A[\rho] dxdt \\
&\quad + \int_0^T \int_\Omega \psi\mathbf{u}(\rho R[\rho\mathbf{u}] - \rho\mathbf{u}R[\rho]) dxdt \\
&\quad - \lambda \int_0^T \int_\Omega \left(\nabla\mathbf{d} \odot \nabla\mathbf{d} - (\frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d}))I_3 \right) \psi\nabla\eta A[\rho] dxdt \\
&\quad - \lambda \int_0^T \int_\Omega \left(\nabla\mathbf{d} \odot \nabla\mathbf{d} - (\frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d}))I_3 \right) \psi\eta\nabla A[\rho] dxdt.
\end{aligned}
\tag{2.4.13}
$$

For the related terms of $\rho_\varepsilon, \mathbf{u}_\varepsilon$, following the same line in [18] we can show that these terms in (2.4.12) converge to their counterparts in (2.4.13). It remains to handle the terms related to $\mathbf{d}_\varepsilon$ in (2.4.12). By virtue of the classical Mikhlin multiplier theorem (see [18]), we have

$$\nabla A[\rho_\varepsilon] \to \nabla A[\rho] \text{ in } C([0,T]; L^\beta_{weak}(\Omega)) \quad \text{as } \varepsilon \to 0, \tag{2.4.14}$$

and

$$A[\rho_\varepsilon] \to A[\rho] \text{ in } C(\overline{(0,T) \times \Omega)}) \quad \text{as } \varepsilon \to 0. \tag{2.4.15}$$

Since

$$\begin{aligned} &\int_\Omega |\nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon A[\rho_\varepsilon] - \nabla \mathbf{d} \odot \nabla \mathbf{d} A[\rho]| \, dx \\ &\leq \int_\Omega |\nabla \mathbf{d}_\varepsilon|^2 \, |A[\rho_\varepsilon] - A[\rho]| \, dx + \int_\Omega |\nabla \mathbf{d}_\varepsilon| \, |\nabla \mathbf{d}_\varepsilon - \nabla \mathbf{d}| \, |A[\rho]| dx \\ &\quad + \int_\Omega |\nabla \mathbf{d}| \, |\nabla \mathbf{d}_\varepsilon - \nabla \mathbf{d}| \, |A[\rho]| dx, \end{aligned} \tag{2.4.16}$$

using Hölder's inequality to (2.4.16), by (2.4.14), (2.4.15), and (2.4.5c) we have

$$\int_0^T \int_\Omega (\nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon) \psi \nabla \eta A[\rho_\varepsilon] dx dt \to \int_0^T \int_\Omega (\nabla \mathbf{d} \odot \nabla \mathbf{d}) \psi \nabla \eta A[\rho] dx dt \quad \text{as } \varepsilon \to 0.$$

Similarly,

$$\int_0^T \int_\Omega (\frac{1}{2} |\nabla \mathbf{d}_\varepsilon|^2 I_3) \psi \nabla \eta A[\rho_\varepsilon] dx dt \to \int_0^T \int_\Omega (\frac{1}{2} |\nabla \mathbf{d}|^2 I_3) \psi \nabla \eta A[\rho] dx dt \quad \text{as } \varepsilon \to 0.$$

Using the strong convergence of $F(\mathbf{d}_\varepsilon)$, we conclude that,

$$\begin{aligned} &\lambda \int_0^T \int_\Omega \left(\nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon - (\frac{1}{2} |\nabla \mathbf{d}_\varepsilon|^2 + F(\mathbf{d}_\varepsilon)) I_3 \right) \psi \nabla \eta A[\rho_\varepsilon] dx dt \\ &\to \lambda \int_0^T \int_\Omega \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2} |\nabla \mathbf{d}|^2 + F(\mathbf{d})) I_3 \right) \psi \nabla \eta A[\rho] dx dt \quad \text{as } \varepsilon \to 0. \end{aligned}$$

And similarly,

$$\begin{aligned} &\lambda \int_0^T \int_\Omega \left(\nabla \mathbf{d}_\varepsilon \odot \nabla \mathbf{d}_\varepsilon - (\frac{1}{2} |\nabla \mathbf{d}_\varepsilon|^2 + F(\mathbf{d}_\varepsilon)) I_3 \right) \psi \eta \nabla A[\rho_\varepsilon] dx dt \\ &\to \lambda \int_0^T \int_\Omega \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2} |\nabla \mathbf{d}|^2 + F(\mathbf{d})) I_3 \right) \psi \eta \nabla A[\rho] dx dt \quad \text{as } \varepsilon \to 0. \end{aligned}$$

So we deduce that

$$\lim_{\varepsilon\to 0^+}\int_0^T\int_\Omega \psi\eta(\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta-\mu\mathrm{div}\mathbf{u}_\varepsilon)\rho_\varepsilon dxdt$$
$$=\int_0^T\int_\Omega \psi\eta(\bar{P}-\mu\mathrm{div}\mathbf{u})\rho dxdt \quad \text{for any } \psi\in\mathcal{D}(0,T), \quad \eta\in\mathcal{D}(\Omega),$$

where $\bar{P}=\overline{\rho^\gamma+\delta\rho^\beta}$. The proof of Lemma 2.4.2 is complete. □

From Lemma 2.4.2, we have

$$\int_0^T\int_\Omega \rho\mathrm{div}\mathbf{u}-\overline{\rho\mathrm{div}\mathbf{u}}dxdt\le\frac{1}{\mu}\int_0^T\int_\Omega(\bar{P}\rho-\overline{a\rho^{\gamma+1}+\delta\rho^{\beta+1}})dxdt. \tag{2.4.17}$$

By(2.4.11) and (2.4.17), we deduce that

$$\int_\Omega \overline{\rho\log(\rho)}-\rho\log(\rho)(\tau)dx\le\frac{1}{\mu}\int_0^T\int_\Omega(\bar{P}\rho-\overline{a\rho^{\gamma+1}+\delta\rho^{\beta+1}})dxdt,$$

and

$$\bar{P}\rho-\overline{\rho^{\gamma+1}+\delta\rho^{\beta+1}}\le 0$$

due to the convexity of $\rho^\gamma+\delta\rho^\beta$. So

$$\int_\Omega \overline{\rho\log(\rho)}-\rho\log(\rho)(t)dx\le 0.$$

On the other hand,

$$\overline{\rho\log(\rho)}-\rho\log(\rho)\ge 0.$$

Consequently $\overline{\rho\log(\rho)}=\rho\log(\rho)$ that means

$$\rho_\varepsilon\to\rho \text{ in } L^1((0,T)\times\Omega).$$

Thus, we can pass to the limit as $\varepsilon\to 0$ to obtain the following result:

Proposition 2.4.1. *Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $C^{2+\vartheta}, \vartheta > 0$. let $\delta > 0$, and*

$$\beta > \max\left\{4, \frac{6\gamma}{2\gamma - 3}\right\}$$

be fixed. Then, for any given $T > 0$, there exists a finite energy weak solution $(\rho, \mathbf{u}, \mathbf{d})$ of the problem:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{2.4.18a}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla(\rho^\gamma + \delta\rho^\beta)$$
$$= \mu\Delta\mathbf{u} - \lambda\operatorname{div}\left(\nabla\mathbf{d} \odot \nabla\mathbf{d} - (\frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d}))I_3\right), \tag{2.4.18b}$$

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla\mathbf{d} = \Delta\mathbf{d} - f(\mathbf{d}) \tag{2.4.18c}$$

with the boundary condition $\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{d}_0$ and initial condition (1.1.2). Moreover, $\rho \in L^{\beta+1}((0,T) \times \Omega)$ and the equation (2.4.18a) holds in the sense of renormalized solutions on $D'((0,T) \times \mathbb{R}^3)$ provided $\rho, \mathbf{u}$ were prolonged to be zero on $\mathbb{R}^3 \setminus \Omega$. Furthermore, $(\rho, \mathbf{u}, \mathbf{d})$ satisfies the following uniform estimates:

$$\sup_{t\in[0,T]} \|\rho(t)\|^\gamma_{L^\gamma(\Omega)} \le CE_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0], \tag{2.4.19}$$

$$\delta \sup_{t\in[0,T]} \|\rho(t)\|^\beta_{L^\beta(\Omega)} \le CE_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0], \tag{2.4.20}$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho(t)}\mathbf{u}(t)\|^2_{L^2(\Omega)} \le CE_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0], \tag{2.4.21}$$

$$\|\mathbf{u}(t)\|_{L^2([0,T];H^1_0(\Omega))} \le CE_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0], \tag{2.4.22}$$

$$\sup_{t\in[0,T]} \|\nabla\mathbf{d}\|^2_{L^2(\Omega)} \le CE_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0], \tag{2.4.23}$$

$$\|\mathbf{d}\|_{L^2([0,T];H^2(\Omega))} \le CE_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0], \tag{2.4.24}$$

where C is independent of $\delta > 0$ and

$$E_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0] = \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}} + \frac{1}{\gamma - 1}\rho^\gamma_{0,\delta} + \frac{\delta}{\beta - 1}\rho^\beta_{0,\delta} + \frac{\lambda}{2}|\nabla\mathbf{d}_0|^2 + \lambda F(\mathbf{d}_0)\right) dx.$$

Remark 2.4.1. Recalling the modified initial data (2.3.3)-(2.3.6), we conclude that the modified energy $E_\delta[\rho_0, \mathbf{m}_0, \mathbf{d}_0]$ is bounded, and consequently the estimates in Proposition 2.4.1 hold independently of δ.

2.5 PASSING TO THE LIMIT IN THE ARTIFICIAL PRESSURE TERM

The objective of this section is to recover the original system by vanishing the artificial pressure term. Again in this part the crucial issue is to recover the strong convergence for ρ_δ in L^1 space.

2.5.1 Better estimate of density

Let us begin with a renormalized continuity equation

$$b(\rho_\delta)_t + \mathrm{div}(b(\rho_\delta)\mathbf{u}_\delta) + (b'(\rho_\delta)\rho_\delta - b(\rho_\delta))\mathrm{div}\mathbf{u}_\delta = 0 \quad \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3)$$

for any uniformly bounded function $b \in C^1[0,\infty)$. We can regularize the above equation as

$$\partial_t S_m[b(\rho)] + \mathrm{div}(S_m[b(\rho)]\mathbf{u}) + S_m[(b'(\rho)\rho - b(\rho))\mathrm{div}\mathbf{u}] = q_m \quad \text{ on } (0,T)\times\mathbb{R}^3, \tag{2.5.1}$$

where $S_m(v)$ denotes a spatial convolution with a family of regularizing kernels, and

$$q_m \to 0 \text{ in } L^2(0,T;L^2(\mathbb{R}^3)) \text{ as } m \to \infty,$$

provided b is uniformly bounded (see details in [18]).

We use the operator B to construct multipliers of the form

$$\varphi(t,x) = \psi(t)B[S_m[b(\rho_\delta)] - \frac{1}{|\Omega|}\int_\Omega S_m[b(\rho_\delta)]dx], \quad \psi \in \mathcal{D}(0,T), \quad 0 \le \psi \le 1,$$

where the operator B was defined in Section 2.4. Taking $b(\rho_\delta) = \rho_\delta^\sigma$, using (2.5.1) and (2.4.19), with σ small enough, we see that

$$S_m[\rho_\delta^\sigma] - \frac{1}{|\Omega|}\int_\Omega S_m[\rho_\delta^\sigma]dx$$

is in the space $C([0,T];L^p(\Omega))$ for any finite $p>1$. By (2.4.1) and the embedding theorem, we have $\varphi(t,x)\in C([0,T]\times\Omega)$. Consequently, $\varphi(t,x)$ can be used as a test function for (2.4.18b), then one arrives at the following formula:

$$\begin{aligned}
&\int_0^T\int_\Omega \psi(\rho_\delta^\gamma+\delta\rho_\delta^\beta)S_m[\rho_\delta^\sigma]dxdt\\
&=\int_0^T\psi(t)\left(\int_\Omega(\rho_\delta^\gamma+\delta\rho_\delta^\beta)dx)(\frac{1}{|\Omega|}\int_\Omega S_m[\rho_\delta^\sigma]dx\right)dt\\
&\quad-\int_0^T\int_\Omega\psi_t\rho_\delta\mathbf{u}_\delta B[S_m[\rho_\delta^\sigma]-\frac{1}{|\Omega|}\int_\Omega S_m[\rho_\delta^\sigma]dx]dxdt\\
&\quad+\int_0^T\int_\Omega\psi(\mu\psi\nabla\mathbf{u}_\delta-\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta)\nabla B[S_m[\rho_\delta^\sigma]-\frac{1}{|\Omega|}\int_\Omega S_m[\rho_\delta^\sigma]dx]dxdt\\
&\quad+\int_0^T\int_\Omega\psi\rho_\delta\mathbf{u}_\delta B[S_m(\rho_\delta^\sigma-\sigma\rho_\delta^\sigma)\mathrm{div}\mathbf{u}-\frac{1}{|\Omega|}\int_\Omega S_m[(\rho_\delta^\sigma-\sigma\rho_\delta^\sigma)\mathrm{div}\mathbf{u}_\delta]dx]dxdt\\
&\quad-\int_0^T\int_\Omega\psi\rho_\delta\mathbf{u}_\delta B[\mathrm{div}S_m[(\rho_\delta^\sigma\mathbf{u}_\delta)]]dxdt+\int_0^T\int_\Omega\psi\rho_\delta\mathbf{u}_\delta B[q_m-\frac{1}{|\Omega|}\int_\Omega q_m dx]dxdt\\
&\quad+\lambda\int_0^T\int_\Omega\psi\left(\nabla\mathbf{d}_\delta\odot\nabla\mathbf{d}_\delta-(\frac{1}{2}|\nabla\mathbf{d}_\delta|^2+F(\mathbf{d}_\delta))I_3\right)\nabla B[S_m[\rho_\delta^\sigma]-\frac{1}{|\Omega|}\int_\Omega S_m[\rho_\delta^\sigma]dx]dxdt\\
&=\sum_{i=1}^6 I_i+\int_0^T\int_\Omega\psi\rho_\delta\mathbf{u}_\delta B[q_m-\frac{1}{|\Omega|}\int_\Omega q_m dx]dxdt.
\end{aligned}$$

Noting that $q_m\to 0$ in $L^2(0,T;L^2(\mathbb{R}^3))$ as $m\to\infty$, we can pass to the limit for $m\to\infty$ in the above equality to get the following:

$$\int_0^T\int_\Omega\psi(\rho_\delta^{\gamma+\sigma}+\delta\rho_\delta^{\beta+\sigma})dxdt\le\sum_{i=1}^6|I_i|.$$

Now, we can estimate the integrals I_1-I_6 as follows.

(1) We see that

$$I_1=\int_0^T\psi(t)\left(\int_\Omega(a\rho_\delta^\gamma+\delta\rho_\delta^\beta)dx)(\frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma)dx\right)dt$$

is bounded uniformly in δ provided $\sigma\le\gamma$ by (2.4.19) and (2.4.20).

(2) As for the second term, by (2.4.19), (2.4.21), (2.4.22) and together with the embedding $W^{1,p}(\Omega)\hookrightarrow L^\infty(\Omega)$ for $p>3$, we have

$$\begin{aligned}
|I_2|&=\left|\int_0^T\int_\Omega\psi_t\rho_\delta\mathbf{u}_\delta B[S_m(\rho_\delta^\sigma)-\frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma)dx]dxdt\right|\\
&\le c\int_0^T|\psi_t|dt\le C
\end{aligned}$$

provided $\sigma \le \frac{\gamma}{3}$.

(3) Similarly, for the third term, we have

$$\begin{aligned} |I_3| &= \left| \int_0^T \int_\Omega \psi(\mu\psi\nabla \mathbf{u}_\delta - \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta)\nabla B[S_m(\rho_\delta^\sigma) - \frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma)dx]dxdt \right| \\ &\le C \end{aligned}$$

if we choose $\sigma \le \frac{\gamma}{2}$;

(4) For I_4, by Hölder inequality, we have

$$\begin{aligned} |I_4| &= \left| \int_0^T \int_\Omega \psi\rho_\delta \mathbf{u}_\delta B[S_m(\rho_\delta^\sigma - \sigma\rho_\delta^\sigma)\mathrm{div}\mathbf{u} - \frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma - \sigma\rho_\delta^\sigma)\mathrm{div}\mathbf{u}_\delta dx]dxdt \right| \\ &\le C\int_0^T \|\rho_\delta\|_{L^\gamma(\Omega)}\|\mathbf{u}_\delta\|_{L^6(\Omega)}\|\rho_\delta^\theta \mathrm{div}\mathbf{u}_\delta\|_{L^q(\Omega)}dt, \end{aligned}$$

where

$$p = \frac{6\gamma}{5\gamma - 6}, \qquad q = \frac{6\gamma}{7\gamma - 6}.$$

If we choose $\sigma \le \frac{2\gamma}{3} - 1$, and use (2.4.19), (2.4.20) and (2.4.22), we conclude that I_4 is uniformly bounded.

(5) Using the embedding inequality, we have

$$\begin{aligned} |I_5| &= \left| \int_0^T \int_\Omega \psi\rho_\delta \mathbf{u}_\delta B[\mathrm{div} S_m(\rho_\delta^\sigma \mathbf{u}_\delta)]dxdt \right| \\ &\le \int_0^T \|\rho_\delta\|_{L^\gamma}\|\mathbf{u}_\delta\|_{L^6}\|\rho_\delta^\sigma \mathbf{u}_\delta\|_{L^p}dt \\ &\le C\int_0^T \|\rho_\delta\|_{L^\gamma}\|\mathbf{u}_\delta\|_{L^6}^2\|\rho_\delta^\sigma\|_{L^r}dt, \end{aligned}$$

where $r = \frac{3\gamma}{2\gamma - 3}$. If we choose $\sigma \le \frac{2\gamma}{3} - 1$, and use (2.4.19), (2.4.20) and (2.4.22), then I_5 is bounded.

(6) Finally, we estimate term I_6, let $\sigma \le \frac{\gamma}{2}$, then

$$\begin{aligned} &|I_6| \\ &= \left| \lambda\int_0^T \int_\Omega \psi\left(\nabla \mathbf{d}_\delta \odot \nabla \mathbf{d}_\delta - (\frac{1}{2}|\nabla \mathbf{d}_\delta|^2 + F(\mathbf{d}_\delta))I_3\right)\nabla B[S_m(\rho_\delta^\sigma) - \frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma)dx]dxdt \right| \\ &\le C\int_0^T \|\nabla \mathbf{d}_\delta\|_{L^{\frac{10}{3}}(\Omega)}^2 \|\nabla B[S_m(\rho_\delta^\sigma) - \frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma)dx]\|_{L^{\frac{5}{2}}(\Omega)}dt \\ &\quad + C\int_0^T \|\nabla B[S_m(\rho_\delta^\sigma) - \frac{1}{|\Omega|}\int_\Omega S_m(\rho_\delta^\sigma)dx]\|_{L^{\frac{5}{2}}(\Omega)}dt \\ &\le C, \end{aligned}$$

where we used the smoothness of F, (2.4.1), (2.4.19), (2.4.20) and

$$\nabla \mathbf{d}_\delta \in L^{\frac{10}{3}}((0,T)\times\Omega).$$

All those above estimates together yield the following lemma:

Lemma 2.5.1. *Let $\gamma > \frac{3}{2}$. There exists $\sigma > 0$ depending only on γ, such that*

$$\rho_\delta^{\gamma+\sigma} + \delta\rho_\delta^{\beta+\sigma} \textit{ is bounded in } L^1((0,T)\times\Omega).$$

2.5.2 The limit passage

By virtue of the estimates in Proposition 2.4.1 and Remark 2.4.1, we can assume that, up to a subsequence if necessary,

$$\rho_\delta \to \rho \text{ in } C([0,T], L^\gamma_{weak}(\Omega)), \tag{2.5.2}$$

$$\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^2([0,T]; H^1_0(\Omega)), \tag{2.5.3}$$

$$\mathbf{d}_\delta \to \mathbf{d} \text{ weakly in } L^2([0,T]; H^2(\Omega)) \cap L^\infty([0,T]; H^1(\Omega)). \tag{2.5.4}$$

$$\mathbf{d}_\delta \to \mathbf{d} \text{ strongly in } L^2(0,T; H^1(\Omega)), \tag{2.5.5}$$

$$\nabla\mathbf{d}_\delta \to \nabla\mathbf{d} \text{ weakly in } L^{\frac{10}{3}}((0,T)\times\Omega), \tag{2.5.6}$$

$$\Delta\mathbf{d}_\delta - f(\mathbf{d}_\delta) \to \Delta\mathbf{d} - f(\mathbf{d}) \text{ weakly in } L^2(0,T; L^2(\Omega)), \tag{2.5.7}$$

$$F(\mathbf{d}_\delta) \to F(\mathbf{d}) \text{ strongly in } L^2(0,T; H^1(\Omega)). \tag{2.5.8}$$

Letting $\delta \to 0$, we have,

$$\rho_\delta^\gamma \to \overline{\rho^\gamma} \text{ weakly in } L^1((0,T)\times(\Omega)), \tag{2.5.9}$$

subject to a subsequence.

From (2.5.5) and (2.5.8), we have, as $\delta \to 0$,

$$\begin{aligned} &\nabla \mathbf{d}_\delta \odot \nabla \mathbf{d}_\delta - (\frac{1}{2}|\nabla \mathbf{d}_\delta|^2 + F(\mathbf{d}_\delta))I_3 \\ &\to \nabla \mathbf{d} \odot \nabla \mathbf{d} - (\frac{1}{2}|\nabla \mathbf{d}|^2 + F(\mathbf{d}))I_3 \quad \text{in } \mathcal{D}'(\Omega \times (0,T)), \end{aligned} \tag{2.5.10}$$

and

$$\mathbf{u}_\delta \cdot \nabla \mathbf{d}_\delta \to \mathbf{u} \cdot \nabla \mathbf{d} \quad \text{in } \mathcal{D}'(\Omega \times (0,T)), \tag{2.5.11}$$

as $\delta \to 0$.

On the other hand, by virtue of (2.4.18b), (2.4.19)-(2.4.22), we obtain

$$\rho_\delta \mathbf{u}_\delta \to \rho \mathbf{u} \text{ in } C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)). \tag{2.5.12}$$

Similarly, we have, as $\delta \to 0$,

$$\mathbf{d}_\delta \to \mathbf{d} \text{ in } C([0,T]; L^2_{weak}(\Omega)).$$

By Lemma 2.5.1, we get

$$\delta \rho_\delta^\beta \to 0 \text{ in } L^1((0,T)\times\Omega) \text{ as } \delta \to 0.$$

Thus, the limit of $(\rho, \rho\mathbf{u}, \mathbf{d})$ satisfies the initial and boundary conditions of (1.1.2) and (1.1.3).

Since $\gamma > \frac{3}{2}$, (2.5.3) and (2.5.12) combined with the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$ imply, as $\delta \to 0$,

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \rho \mathbf{u} \otimes \mathbf{u} \qquad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

Consequently, letting $\delta \to 0$ in (2.4.18) and making use of (2.5.2)-(2.5.12), the limit of $(\rho_\delta, \mathbf{u}_\delta, \mathbf{d}_\delta)$ satisfies the following system:

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{2.5.13a}$$

$$(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\overline{\rho^\gamma} = \mu\Delta\mathbf{u} - \lambda \mathrm{div}\left(\nabla\mathbf{d}\odot\nabla\mathbf{d} - (\frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d}))I_3\right) \tag{2.5.13b}$$

$$\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} - f(\mathbf{d}) \tag{2.5.13c}$$

in $\mathcal{D}'(\Omega \times (0,T))$.

2.5.3 The strong convergence of density

In order to complete the proof of Theorem 2.1.1, we still need to show the strong convergence of ρ_δ in $L^1(\Omega)$, or, equivalently $\bar{\rho^\gamma} = \rho^\gamma$.

Since $\rho_\delta, \mathbf{u}_\delta$ is a renormalized solution of the equation (2.5.13a) in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, we have

$$T_k(\rho_\delta)_t + \mathrm{div}(T_k(\rho_\delta \mathbf{u}_\delta)) + (T_k(\rho_\delta)\rho_\delta - T_k(\rho_\delta))\mathrm{div}(\mathbf{u}_\delta) = 0 \text{ in } \mathcal{D}'((0,T)\times R^3),$$

where $T_k(z) = kT(\frac{z}{k})$ for $z \in \mathbb{R}$, $\quad k = 1,2,3\cdots$ and $T \in C^\infty(\mathbb{R})$ is chosen so that

$$T(z) = z \text{ for } z \le 1, \quad T(z) = 2 \text{ for } z \ge 3, \quad T \text{ convex}.$$

Passing to the limit for $\delta \to 0$ we deduce that

$$\partial_t \overline{T_k(\rho)} + \mathrm{div}((\overline{T_k(\rho)})\mathbf{u}) + \overline{(T_k'(\rho)\rho - T_k(\rho))\mathrm{div}\mathbf{u}} = 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3)),$$

where

$$T_k'(\rho\delta)\rho_\delta - T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta \to \overline{(T_k'(\rho)\rho - T_k(\rho))\mathrm{div}\mathbf{u}} \quad \text{weakly in } L^2((0,T)\times\Omega),$$

and

$$T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0,T]; L^p_{weak}(\Omega)) \text{ for all } 1 \le p < \infty.$$

Using the function

$$\varphi(t,x) = \psi(t)\eta(x)A_i[T_k(\rho_\delta)], \quad \psi \in \mathcal{D}[0,T], \quad \eta \in \mathcal{D}(\Omega),$$

as a test function for (2.4.18b), by a similar calculation to the previous sections, we can deduce the following result:

Lemma 2.5.2. *Let $(\rho_\delta, \mathbf{u}_\delta)$ be the sequence of approximate solutions constructed in Proposition 2.4.1, then*

$$\lim_{\delta\to 0}\int_0^T \int_\Omega \psi\eta(\rho_\delta^\gamma - \mu\mathrm{div}\mathbf{u}_\delta)T_k(\rho_\delta)dxdt = \int_0^T\int_\Omega \psi\eta(\overline{\rho^\gamma} - \mu\mathrm{div}\mathbf{u})\overline{T_k(\rho)}dxdt$$

for any $\psi \in \mathcal{D}(0,T), \eta \in \mathcal{D}(\Omega)$.

In order to get the strong convergence of ρ_δ, we need to define the oscillation defect measure as follows:

$$OSC_{\gamma+1}[\rho_\delta \to \rho]((0,T)\times\Omega) = \sup_{k\geq 1}\lim_{\delta\to 0}\sup \int_0^T\int_\Omega |T_k(\rho_\delta) - T_k(\rho))|^{\gamma+1}dxdt.$$

Here we state a lemma about the oscillation defect measure:

Lemma 2.5.3. *There exists a constant C independent of k such that*

$$OSC_{\gamma+1}[\rho_\delta \to \rho]((0,T)\times\Omega) \leq C$$

for any $k \geq 1$.

Proof. Following the line of argument presented in [18], and by Lemma 2.5.2, we obtain

$$OSC_{\gamma+1}[\rho_\delta \to \rho]((0,T)\times\Omega) \leq \lim_{\delta\to 0}\int_0^T\int_\Omega \operatorname{div}\mathbf{u}_\delta T_k(\rho_\delta) - \operatorname{div}\mathbf{u}\overline{T_k(\rho)}dxdt.$$

On the other hand,

$$\begin{aligned}
&\lim_{\delta\to 0}\int_0^T\int_\Omega \operatorname{div}\mathbf{u}_\delta T_k(\rho_\delta) - \operatorname{div}\mathbf{u}\overline{T_k(\rho)}dxdt \\
&= \lim_{\delta\to 0}\int_0^T\int_\Omega (T_k(\rho_\delta) - T_k(\rho) + T_k(\rho) - \overline{T_k(\rho)})\operatorname{div}\mathbf{u}_\delta dxdt \\
&\leq 2\sup_\delta \|\nabla\mathbf{u}_\delta\|_{L^2((0,T)\times\Omega)} \lim_{\delta\to 0}\sup \|T_k(\rho_\delta) - T_k(\rho)\|_{L^2((0,T)\times\Omega)}.
\end{aligned}$$

So we can conclude the Lemma. □

We are now ready to show the strong convergence of the density. To this end, we introduce a sequence of functions $L_k \in C^1(\mathbb{R})$:

$$L_k(z) = \begin{cases} zlnz, & 0 \leq z < k \\ zln(k) + z\int_k^z \frac{T_k(s)}{s^2}ds, & z \geq k. \end{cases}$$

Noting that L_k can be written as

$$L_k(z) = \beta_k z + b_k z,$$

where b_k satisfy (2.1.6), we deduce that

$$\partial_t L_k(\rho_\delta) + \operatorname{div}(L_k(\rho_\delta)\mathbf{u}_\delta) + T_k(\rho_\delta)\operatorname{div}\mathbf{u}_\delta = 0, \tag{2.5.14}$$

and

$$\partial_t L_k(\rho) + \mathrm{div}(L_k(\rho)\mathbf{u}) + T_k(\rho)\mathrm{div}\mathbf{u} = 0 \tag{2.5.15}$$

in $\mathcal{D}'((0,T)\times\Omega)$. Letting $\delta \to 0$, we can assume that

$$L_k(\rho_\delta) \to \overline{L_k(\rho)} \text{ in } C([0,T]; L^\gamma_{weak}(\Omega)).$$

Taking the difference of (2.5.14) and (2.5.15), and integrating with respect to time t, we obtain

$$\begin{aligned}
&\int_\Omega (L_k(\rho_\delta) - L_k(\rho))\phi dx \\
&= \int_0^T \int_\Omega \Big((L_k(\rho_\delta)\mathbf{u}_\delta - L_k(\rho)\mathbf{u}) \cdot \nabla\phi + (T_k(\rho)\mathrm{div}\mathbf{u} - T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta)\phi \Big) dxdt,
\end{aligned}$$

for any $\phi \in \mathcal{D}(\Omega)$. Following the line of argument in [18], we get

$$\begin{aligned}
&\int_\Omega \left(\overline{L_k(\rho)} - L_k(\rho) \right)(t) dx \\
&= \int_0^T \int_\Omega T_k(\rho)\mathrm{div}\mathbf{u} dxdt - \lim_{\delta\to 0^+} \int_0^T \int_\Omega T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta dxdt.
\end{aligned} \tag{2.5.16}$$

We observe that the term $\overline{L_k(\rho)} - L_k(\rho)$ is bounded by its definition. Using Lemma 2.5.3 and the monotonicity of the pressure, we can estimate the right-hand side of (2.5.16):

$$\begin{aligned}
&\int_0^T \int_\Omega T_k(\rho)\mathrm{div}\mathbf{u} dxdt - \lim_{\delta\to 0^+} \int_0^T \int_\Omega T_k(\rho_\delta)\mathrm{div}\mathbf{u}_\delta dxdt \\
&\le \int_0^T \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}\mathbf{u} dxdt.
\end{aligned} \tag{2.5.17}$$

By virtue of Lemma 2.5.3, the right-hand side of (2.5.17) tends to zero as $k \to \infty$. So we conclude that

$$\overline{\rho \log(\rho)(t)} = \rho \log(\rho)(t)$$

as $k \to \infty$. Thus we obtain the strong convergence of ρ_δ in $L^1((0,T)\times\Omega)$.

Therefore we complete the proof of Theorem 2.1.1.

3.0 INCOMPRESSIBLE FLUID-PARTICLE FLOWS

On physical grounds, the motivation of our study of the incompressible fluid-particle flow is of primary importance in the modeling of sprays. There are many relevant applications, such as combustion theory, pollutant transport, and many more. The flow of the continuous phase is modeled by the forced Navier-Stokes equations, and the flow of the particles is governed by the kinetic equation. The fluid-particle interactions are described by a friction force exerted from the fluid onto the particles. From the mathematical viewpoint, it is challenging because the systems always couple nonlinear evolution equations for unknowns that depend on the different sets of variables, that is, one of the unknowns, f, depends on more variables than the other, $\vec{u}$. There are many mathematical works regarding the Vlasov-Poisson or Vlasov-Maxwell system. Recently other complicated couplings have received more attention. One typical example is the Navier-Stokes-Vlasov system. We aim to establish the existence of weak solutions to the incompressible Navier-Stokes-Vlasov equations in three dimensions and the uniqueness in two dimensions.

we proved the existence of global weak solutions in three dimensions and the uniqueness in two dimensions with certain boundary conditions (no-slip is imposed for the velocity of the fluid, and specular reflection for the particles subject to the Vlasov equation). The global existence was constructed by the Galerkin methods, fixed point arguments, and convergence analysis with the large initial data. The uniqueness was established by the classical theory of Stokes equations and a bootstrap argument. In [58], the global well-posedness to the Cauchy problem of the two dimensional incompressible Navier-Stokes-Vlasov equations was established. We set up an iteration for the velocity of fluid and the distribution function of the particles using a characteristic method and semigroup analysis. Higher regularity can be obtained by the energy method. Applying the a *priori* estimates, we can show that the iteration has a fixed point on a small time interval. Finally, the global well-posedness of classical solutions follows from the bootstrap argument.

3.1 A PRIORI ESTIMATES AND MAIN RESULTS

Here we define the energy functional of the particles density:

$$F(f) := \int_\Omega \int_{\mathbb{R}^d} f(1 + |\mathbf{v}|^2)\, d\mathbf{v}\, dx.$$

If $\mathbf{u} = \text{constant}$, $F(f)$ is an energy functional to the third equation in (1.2.1). When $\mathbf{u} \neq \text{constant}$, we will have the following energy inequality, more precisely:

Lemma 3.1.1. *The system* (1.2.1) *has an energy functional:*

$$E(t) := \left(\int_\Omega \frac{1}{2} |\mathbf{u}|^2\, dx + F(f) \right)(t).$$

If $d = 2, 3$ and $(\mathbf{u}, f)$ is a smooth solution to system (1.2.1) *such that*

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)); \tag{3.1.1}$$

$$f(1 + |\mathbf{v}|^2) \in L^\infty(0, T; L^1(\Omega \times \mathbb{R}^d)), \tag{3.1.2}$$

then, for all $t < T$, we have:

$$\frac{d}{dt} E(t) = -\left(\int_\Omega |\nabla \mathbf{u}|^2\, dx + \int_\Omega \int_{\mathbb{R}^d} f |\mathbf{u} - \mathbf{v}|^2\, d\mathbf{v}\, dx \right) \leq 0. \tag{3.1.3}$$

Proof. Multiplying by $\mathbf{u}$ the both sides of the first equation in (1.2.1), and integrating over Ω and by parts, we have

$$\frac{d}{dt} \int_\Omega \frac{1}{2} |\mathbf{u}|^2\, dx + \int_\Omega |\nabla \mathbf{u}|^2\, dx = -\int_\Omega \int_{\mathbb{R}^d} f(\mathbf{u} - \mathbf{v}) \cdot \mathbf{u}\, d\mathbf{v}\, dx. \tag{3.1.4}$$

Multiplying by $(1 + \frac{1}{2}|\mathbf{v}|^2)$ the both sides of the third equation in (1.2.1), integrating over Ω, and using integration by parts, one obtains that

$$\frac{d}{dt} F(f)(t) + \int_\Omega \int_{\mathbb{R}^d} f |\mathbf{u} - \mathbf{v}|^2\, d\mathbf{v}\, dx = \int_\Omega \int_{\mathbb{R}^d} f(\mathbf{u} - \mathbf{v}) \cdot \mathbf{u}\, d\mathbf{v}\, dx. \tag{3.1.5}$$

Using (3.1.4)-(3.1.5), one obtains (3.1.3), and (3.1.1)-(3.1.2) are the consequences of (3.1.3).

The proof is complete.

□

In what follows, we denote

$$m_k f = \int_{\mathbb{R}^d} |\mathbf{v}|^k f \, d\mathbf{v}, \quad \text{and} \quad M_k f = \int_\Omega \int_{\mathbb{R}^d} |\mathbf{v}|^k f \, d\mathbf{v} dx.$$

Clearly,

$$M_k f = \int_\Omega m_k f \, dx$$

Here we state the following lemmas which are due to [26]:

Lemma 3.1.2. *Suppose that* $(\mathbf{u}, f)$ *is a smooth solution to* (1.2.1)*. If* $f_0 \in L^p$ *for any* $p > 1$*, we have*

$$\|f(t, x; \mathbf{v})\|_{L^p} \leq e^{dT} \|f_0\|_{L^p}, \text{ for any } t \geq 0;$$

and if $|\mathbf{v}|^k f_0 \in L^1(\Omega \times \mathbb{R}^d)$*, then we have*

$$\int_{\Omega \times \mathbb{R}^d} |\mathbf{v}|^k f d\mathbf{v} dx \leq C(d, T) \left(\left(\int_{\Omega \times \mathbb{R}^d} |\mathbf{v}|^k f_0 \, d\mathbf{v} dx \right)^{\frac{1}{d+k}} + (||f_0||_{L^\infty} + 1) \|\mathbf{u}\|_{L^r(0,T;L^{d+k})} \right)^{d+k}$$

for all $o \leq t \leq T$ *where the constant* $C(d, T) > 0$ *depends only on* d *and* T*.*

Lemma 3.1.3. *Under hypotheses of Lemma 3.1.2 and* $d = 3$*, the density* $m_0 f$ *and the mean velocity* $m_1 f$ *have the following estimates for all* $0 \leq t \leq T$*,*

$$\|m_0 f\|_{L^2(\Omega)} \leq C(1 + \|f_0\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3)})A^3$$
$$\|m_1 f\|_{L^{3/2}(\Omega)} \leq C(1 + \|f_0\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3)})A^4,$$

where $A = (\int_\Omega \int_{\mathbb{R}^3} |\mathbf{v}|^3 f_0 \, dx d\mathbf{v})^{\frac{1}{6}} + (\|f_0\|_{L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^3))} + 1)\|\mathbf{u}\|_{L^2(0,T;L^6(\Omega))}$*.*

Remark 3.1.1. Similar estimates hold in the two dimensional space.

Our main result reads as follows.

Theorem 3.1.1. *For* $d = 2, 3$*, if* $E_0 < \infty$*,* $m_5 f_0 < \infty$*, and* $f_0 \in L^\infty(\Omega \times \mathbb{R}^3) \cap L^1(\Omega \times \mathbb{R}^3)$*, then there is a weak solution* $(\mathbf{u}, f)$ *to the system* (1.2.1) *with the initial data* (1.2.2) *and boundary condition* (1.2.3) *for any* $T > 0$*.*

The weak solution to system (1.2.1)-(1.2.3) is defined as follows.

Definition 3.1.1. A pair $(\mathbf{u}, f)$ is called a weak solution to the system (1.2.1)-(1.2.3) in the sense of distribution if

- $\mathbf{u} \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)$;
- $f(t,x,\mathbf{v}) \geq 0$, for any $(t,x,\mathbf{v}) \in (0,T) \times \Omega \times \mathbb{R}^d$;
- $f \in L^\infty(0,T;L^\infty(\Omega \times \mathbb{R}^d) \cap L^1(\Omega \times \mathbb{R}^d))$;
- $f|\mathbf{v}|^2 \in L^\infty(0,T;L^1(\Omega \times \mathbb{R}^d))$;
- for all $\varphi \in C^\infty([0,\infty) \times \Omega)$ with $\mathrm{div}\varphi = 0$ we have

$$\int_0^\infty \int_\Omega (-\mathbf{u}\varphi_t + \mathbf{u} \cdot \nabla \mathbf{u} \varphi + \nabla \mathbf{u} \cdot \nabla \varphi)\, dx\, dt$$
$$= -\int_0^\infty \int_{\Omega \times \mathbb{R}^d} f(\mathbf{u} - \mathbf{v}) \cdot \varphi\, dx\, d\mathbf{v}\, ds + \int_\Omega \mathbf{u}_0 \cdot \varphi(0,x)\, dx;$$

- for all $\phi \in C^\infty([0,\infty) \times \Omega \times \mathbb{R}^d)$ with compact support in $\mathbf{v}$, such that $\phi(T,x,\mathbf{v}) = 0$, we have

$$-\int_0^T \int_{\Omega \times \mathbb{R}^d} f(\phi_t + \mathbf{v} \cdot \nabla_x \phi + (\mathbf{u} - \mathbf{v}) \cdot \nabla_{\mathbf{v}} \phi)\, dx\, d\mathbf{v}\, dt$$
$$= \int_{\Omega \times \mathbb{R}^d} f_0 \phi(0,x,\mathbf{v})\, dx\, d\mathbf{v}.$$

- the energy inequality

$$\int_\Omega \frac{1}{2}|\mathbf{u}|^2\, dx + \int_\Omega \int_{\mathbb{R}^d} f(1 + |\mathbf{v}|^2)\, dxd\mathbf{v}$$
$$+ \int_0^T \int_\Omega |\nabla \mathbf{u}|^2\, dxdt + \int_0^T \int_{\Omega \times \mathbb{R}^d} f|\mathbf{u} - \mathbf{v}|^2\, d\mathbf{v}dxdt$$
$$\leq \int_\Omega \frac{1}{2}|\mathbf{u}_0|^2\, dx + \int_\Omega \int_{\mathbb{R}^d} f_0(1 + |\mathbf{v}|^2)\, dx$$

  holds for $t \in [0,T]$ a.e.

In the two dimensional space, we can obtain more regularity and the uniqueness of global weak solution. More precisely, we have:

Theorem 3.1.2. *If $\mathbf{u}_0 \in H^1(\Omega)$, $f_0 \in L^\infty(\Omega \times \mathbb{R}^2) \cap L^1(\Omega \times \mathbb{R}^2)$, and $\int_{\mathbb{R}^2} |\mathbf{v}|^6 f_0\, d\mathbf{v} < \infty$, there exists a unique global solution $(\mathbf{u}, f)$ to the system (1.2.1) with the initial data (1.2.2) and boundary condition (1.2.3), such that*

$$\mathbf{u} \in L^2(0,T;H_0^2(\Omega)) \cap C([0,T];H_0^1(\Omega)), \quad \frac{\partial \mathbf{u}}{\partial t} \in L^2((0,T) \times \Omega),$$
$$f \in C([0,T];L^\infty(\Omega \times \mathbb{R}^2))$$

for any $T > 0$.

3.2 THE EXISTENCE OF WEAK SOLUTIONS

The goal of this section is to show the existence of global weak solutions to (1.2.1) with initial data (1.2.2) and boundary condition (1.2.3). The key idea is to construct an approximation scheme, establish its existence for the global weak solutions, and pass to the limit for recovering the original system. In this section, we shall prove our main result Theorem 3.1.1 in the case $d = 3$. All arguments do work in the case $d = 2$.

3.2.1 Approximation Scheme

We regularize the equations (1.2.1) and construct a solution of the regularized system of equations. We view the first two equations in (1.2.1) as Navier-Stokes equations with a source term $\int_{\mathbb{R}^3}(\mathbf{u}-\mathbf{v})f\,d\mathbf{v}$. The key idea is to control $\int_{\mathbb{R}^3}(\mathbf{u}-\mathbf{v})f\,d\mathbf{v}$ in $L^2((0,T)\times\Omega)$ so that we can solve the Navier-Stokes equations directly. For that purpose, we follow the spirit in [45] to modify the Vlasov equation by truncating the velocity field $\mathbf{u}$: we consider

$$\partial_t f + \mathbf{v}\cdot\nabla_x f + \mathrm{div}_{\mathbf{v}}\left((\chi_\lambda(\mathbf{u})-\mathbf{v})f\right) = 0 \tag{3.2.1}$$

where

$$\chi_\lambda(\mathbf{u}) = \mathbf{u}1_{\{|\mathbf{u}|\le\lambda\}}.$$

To preserve the similar energy inequality, we need to modify Navier-Stokes equations accordingly. This can be done by substituting the right hand side of the first equation in (1.2.1) by

$$-\int_{\mathbb{R}^3}(\mathbf{u}-\mathbf{v})f\,d\mathbf{v}1_{\{|\mathbf{u}|\le\lambda\}}.$$

To establish the global weak solutions, we find a modified Galerkin method particularly convenient. We define the space H as the closure of the space $C_0^\infty(\Omega,\mathbb{R}^3)\cap\{\mathbf{u}:\mathrm{div}\mathbf{u}=0\}$ in $L^2(\Omega,\mathbb{R}^3)$. We let $\{\phi_i\}_{i=1}^\infty$ be an orthogonal basis of the functional space H and such that

$$\Delta\phi_i + \nabla P_i = -e_i\phi_i \quad \text{in } \Omega,$$
$$\phi_i = 0 \quad \text{on } \partial\Omega$$

for $i = 1, 2, 3....$ Here $0 \leq e_1 \leq e_2 \leq e_3 \leq ... \leq e_n \leq ...$ with $e_n \to \infty$ as $n \to \infty$. Define $P_m : H \mapsto H_m = \text{span}\{\phi_1, \phi_2, ..., \phi_m\}$ as the orthogonal projection. We propose the following regularized Navier-Stokes equations

$$\begin{aligned}
&\partial_t \mathbf{u}_m = P_m \left(\Delta \mathbf{u}_m - \mathbf{u}_m \cdot \nabla \mathbf{u}_m - \int_{\mathbb{R}^3} (\tilde{\mathbf{u}} - \mathbf{v}) f_m \, d\mathbf{v} 1_{\{|\tilde{\mathbf{u}}| \leq \lambda\}} \right), \\
&\mathbf{u}_m(x,t) \in H_m, \quad \text{and } \operatorname{div} \mathbf{u}_m = 0.
\end{aligned}$$

Thus the approximation scheme for the Navier-Stokes-Vlasov equations (1.2.1) is given by the following system

$$\begin{aligned}
&\partial_t \mathbf{u}_m = P_m \left(\Delta \mathbf{u}_m - \mathbf{u}_m \cdot \nabla \mathbf{u}_m - G_m \right), \\
&\mathbf{u}_m(x,t) \in H_m, \quad \text{and} \quad \operatorname{div} \mathbf{u}_m = 0. \\
&\partial_t f_m + \mathbf{v} \cdot \nabla_x f_m + \operatorname{div}_{\mathbf{v}}((\chi_\lambda(\tilde{\mathbf{u}}) - \mathbf{v}) f_m) = 0,
\end{aligned} \tag{3.2.2}$$

with the initial data,

$$\mathbf{u}_m|_{t=0} = P_m \mathbf{u}_0, \qquad f_m|_{t=0} = f_0,$$

where

$$G_m = \int_{\mathbb{R}^3} (\tilde{\mathbf{u}} - \mathbf{v}) f_m \, d\mathbf{v} 1_{\{|\tilde{\mathbf{u}}| \leq \lambda\}}$$

and

$$\tilde{\mathbf{u}} \text{ is given in } L^2(0,T; L^2(\Omega)).$$

The rest of this subsection is devoted to show the global existence to (3.2.2) with its initial data. For given $\tilde{\mathbf{u}}$, we can get enough regularity of $-\int_{\mathbb{R}^3} (\tilde{\mathbf{u}} - \mathbf{v}) f \, d\mathbf{v} 1_{\{|\tilde{\mathbf{u}}| \leq \lambda\}}$ to solve the modified Navier-Stokes equations.

Indeed, we have

$$\chi_\lambda(\tilde{\mathbf{u}}) \in L^\infty((0,T) \times \Omega),$$

due to

$$\tilde{\mathbf{u}} \in L^2(0,T; L^2(\Omega)).$$

Considering the following equation

$$\begin{aligned}
&\partial_t f + \mathbf{v} \cdot \nabla_x f + \operatorname{div}_{\mathbf{v}}((\chi_\lambda(\tilde{\mathbf{u}}) - \mathbf{v}) f) = 0; \\
&f(x, \mathbf{v}, 0) = f_0(x, \mathbf{v}), \quad f(t, x, \mathbf{v}) = f(t, x, \mathbf{v}^*) \;\; \text{for } x \in \partial\Omega, \; \mathbf{v} \cdot \nu(x) < 0,
\end{aligned}$$

where $\mathbf{v}^* = \mathbf{v} - 2(\mathbf{v}\cdot\nu(x))\nu(x)$, the existence and uniqueness of the solution can be obtained as in [3, 12, 26].

Applying the maximal principle to the above equation, we have

$$\|f(t,x,\mathbf{v})\|_{L^p} \leq C(T)\|f_0\|_{L^p}, \text{ for any } p > 1. \tag{3.2.3}$$

Thanks to Lemma 3.1.2,

$$\chi_\lambda(\tilde{\mathbf{u}}) \in L^\infty((0,T)\times\Omega),$$

and

$$\int_\Omega\int_{\mathbb{R}^3} |\mathbf{v}|^5 f_0 \, d\mathbf{v}dx < +\infty,$$

we have

$$\int_\Omega\int_{\mathbb{R}^3} |\mathbf{v}|^5 f d\mathbf{v}dx < +\infty.$$

This, together with Lemma 1 in [4], yields

$$\int_{\mathbb{R}^3} f \, d\mathbf{v} \in L^2(0,T;L^2(\Omega)), \text{ and } \int_{\mathbb{R}^3} \mathbf{v}f \, d\mathbf{v} \in L^2(0,T;L^2(\Omega)). \tag{3.2.4}$$

By (3.2.4), we get, for all $t > 0$, that $G_m \in L^2(0,T;L^2(\Omega))$.

For each m we define an approximate solution $\mathbf{u}_m$ of (3.2.2) as follows:

$$\mathbf{u}_m = \sum_{i=1}^m g_{im}(t)\phi_i(x),$$

and hence (3.2.2) is equivalent to

$$\frac{dg_m^i(t)}{dt} = -e_i g_m^i(t) - \int_\Omega (\phi_j(x)\cdot\nabla\phi_k(x))\phi_i(x)\,dx\, g_m^j(t)g_m^k(t) + \int_\Omega G_m\phi_i(x)dx. \tag{3.2.5}$$

The initial data becomes

$$\sum_{i=1}^m g_m^i(0)\phi_i(x) = P_m\mathbf{u}_0(x),$$

which is equivalent to saying that

$$g_m^i(0) = (\mathbf{u}_0,\phi_i) \quad \text{for } i = 1,2,3,...,m. \tag{3.2.6}$$

So the system (3.2.5) with its initial data (3.2.6) can be viewed as an ordinary differential equations in L^2 verifying the conditions of the Cauchy-Lipschitz theorem. Thus it admits a unique maximal solution

$$\mathbf{u}_m \in C^1([0,T_m];L^2(\Omega)).$$

It is easy to find the energy inequality to regularize Navier-Stokes equations as follows

$$\int_\Omega |\mathbf{u}_m|^2\,dx + 2\int_0^t\int_\Omega |\nabla \mathbf{u}_m|^2\,dxdt \le \int_\Omega |\mathbf{u}_{m0}|^2\,dx + \int_0^t\int_\Omega G_m\mathbf{u}_m\,dxdt,$$

which, together with $G_m \in Ł^2(0,T;L^2(\Omega))$, allows us to take $T_m = T$.

We define an operator

$$S: \quad L^2((0,T)\times\Omega) \mapsto L^2((0,T)\times\Omega)$$
$$\tilde{\mathbf{u}} \mapsto \mathbf{u}_m.$$

Here we need to rely on the following lemma:

Lemma 3.2.1. *The operator S has a fixed point in $L^2((0,T)\times\Omega)$, that is, there is a point $\mathbf{u}_m \in L^2((0,T)\times\Omega)$ such that $S\mathbf{u}_m = \mathbf{u}_m$.*

Proof. Multiplying by $\mathbf{u}_m$ the both sides of (3.2.2), and using integration by parts, one obtains that

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\mathbf{u}_m|^2\,dx + \int_\Omega |\nabla\mathbf{u}_m|^2\,dx \le \int_\Omega\left(\int_{\mathbb{R}^3}(\tilde{\mathbf{u}}-\mathbf{v})f_m\,d\mathbf{v}1_{\{|\tilde{\mathbf{u}}|\le\lambda\}}\right)\mathbf{u}_m\,dx. \tag{3.2.7}$$

Considering the force term of the modified Navier-Stokes equations, we have

$$|\int_{\Omega\times\mathbb{R}^3}(\tilde{\mathbf{u}}-\mathbf{v})f_m 1_{\{|\tilde{\mathbf{u}}|\le\lambda\}}\mathbf{u}_m\,d\mathbf{v}\,dx| \le (\|\int_{\mathbb{R}^3}\mathbf{v}f\,d\mathbf{v}\|_{L^2} + \lambda\|\int_{\mathbb{R}^3}f\,d\mathbf{v}\|_{L^2})^2 + \|\mathbf{u}_m\|_{L^2}^2.$$

This, together with (3.2.7), implies that

$$\partial_t\int_\Omega\frac{1}{2}|\mathbf{u}_m|^2\,dx + \int_\Omega|\nabla\mathbf{u}_m|^2\,dx \le \int_{\mathbb{R}^3}|\mathbf{u}_m|^2dx + C(m).$$

By Gronwall's inequality, we have

$$\sup_{t\in(0,T)}\int_\Omega |\mathbf{u}_m|^2\,dx \le C(m),$$

which means that

$$\|S\tilde{\mathbf{u}}_m\|_{L^2((0,T);H_0^1(\Omega))} \le C(m). \tag{3.2.8}$$

By the first equation in (3.2.2), one obtains that

$$\|\partial_t S\tilde{\mathbf{u}}_m\|_{L^2(0,T;H_0^{-1}(\Omega))} \le C(m). \tag{3.2.9}$$

By (3.2.8) and (3.2.9), we conclude that the operator S is compact in $L^2(0,T;L^2(\Omega))$ and the image of the operator S is bounded in $L^2(0,T;\Omega)$. So Schauder's fixed point theorem will give us that the operator S has a fixed point $\mathbf{u}_m$ in $L^2(0,T;\Omega)$. □

Applying Lemma 3.2.1, for any $T>0$, there exists a solution $(\mathbf{u}_m, f_m)$ to the following system

$$\begin{aligned}
&\partial_t \mathbf{u}_m = P_m\left(\Delta\mathbf{u}_m - \mathbf{u}_m\cdot\nabla\mathbf{u}_m - \int_{\mathbb{R}^3}(\mathbf{u}_m-\mathbf{v})f_m\,d\mathbf{v}1_{\{|\mathbf{u}_m|\le\lambda\}}\right),\\
&\mathbf{u}_m(x,t)\in H_m, \quad \text{and} \quad \mathrm{div}\mathbf{u}_m = 0.\\
&\partial_t f_m + \mathbf{v}\cdot\nabla_x f_m + \mathrm{div}_{\mathbf{v}}((\chi_\lambda(\mathbf{u}_m)-\mathbf{v})f_m) = 0,
\end{aligned} \tag{3.2.10}$$

with its initial data

$$\mathbf{u}_m(0,x) = P_m\mathbf{u}_0, \quad f(0,x,\mathbf{v}) = f_0(x,\mathbf{v}),$$

and boundary conditions

$$\mathbf{u}_m|_{\partial\Omega} = 0, \quad \text{and} \quad f(t,x,\mathbf{v}) = f(t,x,\mathbf{v}^*) \quad \text{for any } x\in\partial\Omega,\ \mathbf{v}\cdot\nu(x)<0.$$

Concerning the system (3.2.10) with the initial-boundary data, we have the following energy inequality

$$\begin{aligned}
&\int_\Omega \frac{1}{2}|\mathbf{u}_m|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f_m(1+|\mathbf{v}|^2)\,d\mathbf{v}dx\\
&+\int_0^T\int_\Omega |\nabla\mathbf{u}_m|^2\,dxdt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f_m|\chi_\lambda(\mathbf{u}_m)-\mathbf{v}|^2\,d\mathbf{v}dxdt\\
&\le \int_\Omega\frac{1}{2}|\mathbf{u}_0|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx,
\end{aligned} \tag{3.2.11}$$

due to the fact

$$\int_{\Omega\times\mathbb{R}^3}(\mathbf{u}_m-\mathbf{v})f_m 1_{\{|\mathbf{u}_m|\le\lambda\}}\mathbf{u}_m\,d\mathbf{v}\,dx = \int_{\Omega\times\mathbb{R}^3}\chi_\lambda(\mathbf{u}_m)(\chi_\lambda(\mathbf{u}_m)-\mathbf{v})f_m\,dx\,d\mathbf{v}.$$

Then we have the following result:

Proposition 3.2.1. *For any $T > 0$, there is a weak solution $(\mathbf{u}^m, f^m)$ to the following system*

$$\partial_t \mathbf{u}_m = P_m \left(\Delta \mathbf{u}_m - \mathbf{u}_m \cdot \nabla \mathbf{u}_m - \int_{\mathbb{R}^3} (\mathbf{u}_m - \mathbf{v}) f_m \, d\mathbf{v} 1_{\{|\mathbf{u}_m| \le \lambda\}} \right),$$
$$\mathbf{u}_m(x,t) \in H_m, \quad \textit{and} \quad \mathrm{div}\mathbf{u}_m = 0.$$
$$\partial_t f_m + \mathbf{v} \cdot \nabla_x f_m + \mathrm{div}_{\mathbf{v}}((\chi_\lambda(\mathbf{u}_m) - \mathbf{v}) f_m) = 0,$$

with its initial data

$$\mathbf{u}_m(0,x) = P_m \mathbf{u}_0, \quad f(0,x,\mathbf{v}) = f_0(x,\mathbf{v}),$$

and boundary conditions

$$\mathbf{u}_m|_{\partial\Omega} = 0, \quad \textit{and} \quad f(t,x,\mathbf{v}) = f(t,x,\mathbf{v}^*) \quad \textit{for any } x \in \partial\Omega, \ \mathbf{v} \cdot \nu(x) < 0.$$

In additional, the solution satisfies the following energy inequality:

$$\begin{aligned} &\int_\Omega \frac{1}{2} |\mathbf{u}_m|^2 \, dx + \int_{\Omega \times \mathbb{R}^3} f_m (1 + |\mathbf{v}|^2) \, d\mathbf{v} dx \\ &+ \int_0^T \int_\Omega |\nabla \mathbf{u}_m|^2 \, dx dt + \int_0^T \int_{\Omega \times \mathbb{R}^3} f_m |\chi_\lambda(\mathbf{u}_m) - \mathbf{v}|^2 \, d\mathbf{v} dx dt \\ &\le \int_\Omega \frac{1}{2} |\mathbf{u}_0|^2 \, dx + \int_{\Omega \times \mathbb{R}^3} f_0 (1 + |\mathbf{v}|^2) \, d\mathbf{v} \, dx, \end{aligned}$$

3.2.2 Passing to the Limit as $m \to \infty$

In this section, we will pass the limit as m goes to infinity in the family of approximate solutions $(\mathbf{u}^m, f^m)$ obtained in Proposition 3.2.1. The estimates in Proposition 3.2.1 are independent of m, λ, and those estimates still hold for any m. By (3.2.3), we have

$$\|f^m\|_{L^\infty(0,T;L^p(\Omega \times \mathbb{R}^3))} \le C$$

for all $1 \le p \le \infty$. Proposition 3.2.1 yields the following estimates

$$\|\mathbf{u}^m\|_{L^\infty(0,T;L^2(\Omega))} \le C,$$
$$\|\nabla \mathbf{u}^m\|_{L^2(0,T;L^2(\Omega))} \le C.$$

From the above a priori estimates, we conclude that there exists a function f such that

$$f^m \rightharpoonup f \quad \text{weak star in } L^\infty(0,T;L^p(\Omega \times \mathbb{R}^3))$$

for all $p \in (1, \infty)$.

This weak convergence cannot provide us enough information for passing the limit. For our purpose, we rely on the following average compactness results for the Vlasov equation due to DiPerna-Lions-Meyer [14]:

Lemma 3.2.2.

$$\frac{\partial f^n}{\partial t} + \mathbf{v} \cdot \nabla_x f^n = \mathrm{div}_{\mathbf{v}}(F^n f^n) \quad \text{in } \mathcal{D}'(\mathbb{R}^3_x + \mathbb{R}^3_{\mathbf{v}} \times (0, \infty))$$

where f^n *is bounded in* $L^\infty(0, \infty; L^2_{x,\mathbf{v}} \bigcap L^1_{x,\mathbf{v}}(1+|\mathbf{v}|^2))$, $\frac{F^n}{1+|\mathbf{v}|}$ *is bounded in* $L^\infty((0,\infty)\times\mathbb{R}^3_{\mathbf{v}}; L^2(\mathbb{R}^3_x))$. *Then* $\int_{\mathbb{R}^3} f^n \eta(\mathbf{v})\, d\mathbf{v}$ *is relatively compact in* $L^q(0, T; L^p(B_R))$ *for all* $R, T < \infty$, $1 \le q < \infty, 1 \le p < 2$ *and for* η *such that* $\frac{\eta}{(1+|\mathbf{v}|)^\sigma} \in L^1 + L^\infty, \sigma \in [0, 2)$.

Remark 3.2.1. It is crucial to use this lemma to get the strong convergence of $m_0 f^n$ and $m_1 f^n$.

Let $m_0 f$ and $m_1 f$ be the density and mean velocity associate with f. Applying Lemma 3.2.2 to the Vlasov equation of (3.2.2), one obtains that

$$m_0 f^m(t, x) \to m_0 f(t, x), \quad m_1 f^m(t, x) \to m_1 f(t, x) \tag{3.2.12}$$

in $L^q(0, T; L^p(B_R))$ for any positive number R and $1 \le q < \infty, 1 \le p < 2$.

Noticing that the right side of Navier-Stokes equations (3.2.2)

$$\int_\Omega (\mathbf{u}^m - \mathbf{v}) f^m \, d\mathbf{v} 1_{\{|\mathbf{u}^m| \le \lambda\}}$$

is bounded in $L^\infty(0, T; L^2(\Omega))$ when λ is fixed, one obtains that

$$\|\partial_t \mathbf{u}^n\|_{L^2(0,T;H^{-1})} \le C < \infty. \tag{3.2.13}$$

By (3.2.11)-(3.2.13), applying the Aubin-Lions Lemma, (see [52]), there exist a $\mathbf{u} \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1_0)$, such that

$$\begin{aligned} &\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ weak star in } L^\infty(0, T; L^2) \quad \text{and} \quad \mathbf{u}^m \rightharpoonup \mathbf{u} \text{ in } L^2(0, T; H^1_0) \\ &\mathbf{u}^m \to \mathbf{u} \text{ strongly in } L^2(0, T; H^1_0). \end{aligned} \tag{3.2.14}$$

The next step is to show the convergence of $(\int_{\mathbb{R}^3} f^m \, d\mathbf{v})\mathbf{u}^m 1_{\{|\mathbf{u}^m| \le \lambda\}}$ in the sense of distributions.

Note that $\nabla \mathbf{u}^m$ is bounded in $L^2(0,T,L^2(\Omega))$ and (3.2.12), we have

$$\left(\int_{\mathbb{R}^3} f^m \, d\mathbf{v}\right) \mathbf{u}^m 1_{\{|\mathbf{u}^m|\le\lambda\}} \to \left(\int_{\mathbb{R}^3} f \; d\mathbf{v}\right) \mathbf{u} 1_{\{|\mathbf{u}|\le\lambda\}}$$

in the sense of distributions. Therefore,

$$\left(\int_{\mathbb{R}^3} f^n \, d\mathbf{v}\right) \chi_\lambda(\mathbf{u}^n) \to \left(\int_{\mathbb{R}^3} f \, d\mathbf{v}\right) \chi_\lambda(\mathbf{u})$$

in the sense of distributions. Applying these convergence results, one concludes that $(\mathbf{u}, f)$ is a weak solution to the following system

$$\begin{aligned} &\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \Delta \mathbf{u} = -\int_{\mathbb{R}^3} (\mathbf{u}-\mathbf{v}) f \, d\mathbf{v} 1_{\{|\mathbf{u}|\le\lambda\}}, \\ &\mathrm{div}\mathbf{u} = 0, \\ &\partial_t f + \mathbf{v}\cdot\nabla_x f + \mathrm{div}_{\mathbf{v}}((\chi_\lambda(\mathbf{u}) - \mathbf{v}) f) = 0 \end{aligned} \tag{3.2.15}$$

with its initial data

$$u(0,x) = \mathbf{u}_0(x), \;\; f(0,x,\mathbf{v}) = f(0,x,\mathbf{v}),$$

and boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \text{and} \quad f(t,x,\mathbf{v}) = f(t,x,\mathbf{v}^*) \quad \text{for} \;\; x\in\partial\Omega, \;\mathbf{v}\cdot\nu(x) < 0.$$

Next, we show that this solution satisfies a particular energy inequality.

Because the solution $(\mathbf{u}^m, f^m)$ satisfies the energy inequality in Proposition 3.2.1, we have

$$\begin{aligned} &\int_\Omega \frac{1}{2}|\mathbf{u}^m|^2 \, dx + \int_{\Omega\times\mathbb{R}^3} f^m(1+|\mathbf{v}|^2) \, d\mathbf{v} \, dx \\ &+ \int_0^T \int_\Omega |\nabla \mathbf{u}^m|^2 \, dx \, dt + \int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\chi_\lambda(\mathbf{u}^m) - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt \\ &\le \int_\Omega \frac{1}{2}|\mathbf{u}_0|^2 \, dx + \int_{\Omega\times\mathbb{R}^3} f_0(1+|\mathbf{v}|^2) \, d\mathbf{v} \, dx. \end{aligned}$$

The difficulty of passing the limit for the energy inequality is the convergence of the term $\int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\chi_\lambda(\mathbf{u}^m) - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt$. Here we write the term as:

$$\begin{aligned} &\int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\chi_\lambda(\mathbf{u}^m) - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt \\ &\qquad = \int_0^T \int_{\Omega\times\mathbb{R}^3} \left(f^m |\chi_\lambda(\mathbf{u}^m)|^2 - 2 f^m \chi_\lambda(\mathbf{u}^m)\mathbf{v} + f^m|\mathbf{v}|^2\right) \, dx d\mathbf{v} dt. \end{aligned} \tag{3.2.16}$$

By (3.2.14), we have

$$\chi_\lambda(\mathbf{u}_m) \to \chi_\lambda(\mathbf{u}) \quad \text{in } L^2(0,T;L^6(\Omega)). \tag{3.2.17}$$

Let us look at

$$\begin{aligned} &\left| \int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\chi_\lambda(\mathbf{u}^m)|^2 \, d\mathbf{v} dx dt - \int_0^T \int_{\Omega\times\mathbb{R}^3} f |\chi_\lambda(\mathbf{u})|^2 \, d\mathbf{v} dx dt \right| \\ &\le \int_0^T \int_\Omega \left(\int_{\mathbb{R}^3} (f^m - f) \, d\mathbf{v} \right) |\chi_\lambda(\mathbf{u}^m)|^2 d \, dx + \int_0^T \int_\Omega \left(\int_{\mathbb{R}^3} f \, d\mathbf{v} \right) \left(|\chi_\lambda(\mathbf{u}^m)|^2 - |\chi_\lambda(\mathbf{u})|^2 \right) \, dx. \end{aligned} \tag{3.2.18}$$

Applying (3.2.12) and (3.2.17) to (3.2.18), we deduce that

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\chi_\lambda(\mathbf{u}^m)|^2 \, d\mathbf{v} dx dt \to \int_0^T \int_{\Omega\times\mathbb{R}^3} f |\chi_\lambda(\mathbf{u})|^2 \, d\mathbf{v} dx dt$$

as $m \to \infty$. Similarly,

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} \mathbf{v} f^m \chi_\lambda(\mathbf{u}^m) \, d\mathbf{v} dx dt \to \int_0^T \int_{\Omega\times\mathbb{R}^3} \mathbf{v} f \chi_\lambda(\mathbf{u}) \, d\mathbf{v} dx dt$$

for all $t > 0$.

Finally, because

$$f^m \rightharpoonup f \qquad \text{weak star in } L^\infty(0,T;L^p(\Omega\times\mathbb{R}^3))$$

for all $p \in (1,\infty]$ and $m_2 f^m$ is bounded in $L^\infty(0,T;L^1(\Omega))$, then for any fixed $R > 0$, we have

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\mathbf{v}|^2 \, dx d\mathbf{v} dt = \int_0^T \int_{\Omega\times\mathbb{R}^3} \chi(|\mathbf{v}| < R) |\mathbf{v}|^2 f^m \, dx d\mathbf{v} dt + O(\frac{1}{R})$$

uniformly in m, where χ is the characteristic function of the ball of $\mathbb{R}^3$ of radius R.

Letting $m \to \infty$, then $R \to \infty$, we find

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\mathbf{v}|^2 \, dx d\mathbf{v} dt \to \int_0^T \int_{\Omega\times\mathbb{R}^3} f |\mathbf{v}|^2 \, dx d\mathbf{v} dt.$$

Thus, we have proved

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} f^m |\chi_\lambda(\mathbf{u}^m) - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt \to \int_0^T \int_{\Omega\times\mathbb{R}^3} f |\chi_\lambda(\mathbf{u}) - \mathbf{v}|^2 \, d\mathbf{v} \, dx \, dt \tag{3.2.19}$$

as $m \to \infty$.

Letting m go to infinity, using the convexity of the energy, the weak convergence of f^m and $\mathbf{u}^m$, and (3.2.19), we deduce

$$\begin{aligned}
&\int_\Omega \frac{1}{2}|\mathbf{u}|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx \\
&+ \int_0^T\int_\Omega |\nabla\mathbf{u}|^2\,dx\,dt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f|\chi_\lambda(\mathbf{u})-\mathbf{v}|^2 d\mathbf{v}\,dx\,dt \\
&\le \int_\Omega \frac{1}{2}|\mathbf{u}_0|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx.
\end{aligned}$$

Thus, we have proved the following result:

Proposition 3.2.2. *For any $T>0$, there is a weak solution $(\mathbf{u}^\lambda, f^\lambda)$ to (3.2.15) with the initial data*

$$\mathbf{u}^\lambda(0,x)=\mathbf{u}_0(x),\;\; f^\lambda(0,x,\mathbf{v})=f_0(x,\mathbf{v}),$$

and boundary condition

$$\mathbf{u}^\lambda|_{\partial\Omega}=0,\;\; \textit{and}\;\; f^\lambda(t,x,\mathbf{v})=f^\lambda(t,x,\mathbf{v}^*) \quad \textit{for}\;\; x\in\partial\Omega,\, \mathbf{v}\cdot\nu(x)<0.$$

In additional, the solution satisfies the following energy inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{2}|\mathbf{u}^\lambda|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f^\lambda(1+|\mathbf{v}|^2)d\mathbf{v}dx \\
&+ \int_0^T\int_\Omega |\nabla\mathbf{u}^\lambda|^2\,dx\,dt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f^\lambda|\chi_\lambda(\mathbf{u}^\lambda)-\mathbf{v}|^2\,d\mathbf{v}\,dx\,dt \\
&\le \int_\Omega \frac{1}{2}|\mathbf{u}_0|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx.
\end{aligned}$$

3.2.3 Passing the limit as $\lambda \to \infty$

The last step of showing the global weak solution is to pass the limit as λ goes to infinity. First, we let $(\mathbf{u}^\lambda, f^\lambda)$ be a solution constructed by Proposition 3.2.2. It is easy to find that all estimates for $(\mathbf{u}^m, f^m)$ still hold for $(\mathbf{u}^\lambda, f^\lambda)$. So we can treat these terms as before.

It only remains to show that we can pass the limit in the coupling terms $\chi_\lambda(\mathbf{u}^\lambda)f^\lambda$ and $\int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v}1_{\{|\mathbf{u}^\lambda|\le\lambda\}} = \int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v}\chi_\lambda(\mathbf{u}^\lambda)$. Here, we treat these terms as follows

$$\int_{\mathbb{R}^3} f^\lambda \mathbf{u}^\lambda \, d\mathbf{v}1_{\{|\mathbf{u}^\lambda|\le\lambda\}} = \int_{\mathbb{R}^3} f^\lambda \mathbf{u}^\lambda \, d\mathbf{v} - \int_{\mathbb{R}^3} f^\lambda \mathbf{u}^\lambda \, d\mathbf{v}1_{\{|\mathbf{u}^\lambda|>\lambda\}}, \tag{3.2.20}$$

and for the second term in (3.2.20),

$$\begin{aligned}
&\| \int_{\mathbb{R}^3} f^\lambda \mathbf{u}^\lambda \, d\mathbf{v}1_{\{|\mathbf{u}^\lambda|>\lambda\}}\|_{L^1(0,T;\Omega)} \\
&\le \| \int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v}\|_{L^\infty(0,T;L^2(\Omega)}\|\mathbf{u}^\lambda\|_{L^2(0,T;L^6(\Omega))}\|1_{\{|\mathbf{u}^\lambda|>\lambda\}}\|_{L^2(0,T;L^6(\Omega))} \\
&\le \| \int_{\mathbb{R}^3} f^\lambda d\mathbf{v}\|_{L^\infty(0,T;L^2(\Omega)}\|\mathbf{u}^\lambda\|_{L^2(0,T;L^6(\Omega))}(\frac{\|\mathbf{u}^\lambda\|_{L^2(0,T;L^6(\Omega))}}{\lambda}) \\
&\le \frac{C}{\lambda} \to 0
\end{aligned} \tag{3.2.21}$$

as $\lambda \to \infty$, where we used Sobolev embedding theorem.

On the other hand, we have

$$\partial_t(\int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v}) + \mathrm{div}_x(\int_{\mathbb{R}^3} \mathbf{v} f^\lambda \, d\mathbf{v}) = 0,$$

which implies that $\partial_t(\int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v})$ is bounded in $L^2(0,T;H^{-1})$. This, with the help of $\nabla\mathbf{u}^\lambda$ bounded in $L^2((0,T)\times\Omega)$, yields that

$$\mathbf{u}^\lambda \left(\int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v}\right) \to \mathbf{u}\left(\int_{\mathbb{R}^3} f \, d\mathbf{v}\right) \quad \text{as } \lambda \to \infty \tag{3.2.22}$$

in the sense of distributions.

By (3.2.20)-(3.2.22), one deduces that

$$\mathbf{u}^\lambda \left(\int_{\mathbb{R}^3} f^\lambda \, d\mathbf{v}1_{\{|\mathbf{u}^\lambda|\le\lambda\}}\right) \to \mathbf{u}\left(\int_{\mathbb{R}^3} f \, d\mathbf{v}\right) \quad \text{as } \lambda \to \infty$$

in the sense of distributions. Thus we can pass the limit in the weak solutions of (1.2.1) as $\lambda \to \infty$. We remark that the solution $(\mathbf{u}^\lambda, f^\lambda)$ satisfies the energy inequality in Proposition 3.2.2:

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}|\mathbf{u}^\lambda|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f^\lambda(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx \\
&+ \int_0^T\int_\Omega |\nabla \mathbf{u}^\lambda|^2\,dx\,dt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f^\lambda|\chi_\lambda(\mathbf{u}^\lambda)-\mathbf{v}|^2 d\mathbf{v}\,dx\,dt \\
&\le \int_\Omega \frac{1}{2}|\mathbf{u}_0|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx.
\end{aligned}
$$

Using the same approach as in last subsection, letting λ go to infinity, using the convexity of the energy and the weak convergence of f^λ and $\mathbf{u}^\lambda$, we deduce

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}|\mathbf{u}|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx \\
&+ \int_0^T\int_\Omega |\nabla \mathbf{u}|^2\,dx\,dt + \int_0^T\int_{\Omega\times\mathbb{R}^3} f|\mathbf{u}-\mathbf{v}|^2\,d\mathbf{v}\,dx\,dt \\
&\le \int_\Omega \frac{1}{2}|\mathbf{u}_0|^2\,dx + \int_{\Omega\times\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}\,dx.
\end{aligned}
$$

So we have proved Theorem 3.1.1.

3.3 UNIQUENESS IN THE TWO DIMENSIONAL SPACE

The goal of this section is to establish the uniqueness of global solutions in the two dimensional space. For that purpose, we shall study the regularity first.

3.3.1 Regularity

The existence of global weak solution to (1.2.1) was obtained by Theorem 3.1.1. We multiply the first equation of (1.2.1) by $\mathbf{u}_t$ and use integration by parts over Ω to obtain,

$$\begin{aligned}&\int_\Omega |\mathbf{u}_t|^2 dx + \partial_t \int_\Omega |\nabla \mathbf{u}|^2\, dx \\ &\le \int_\Omega |m_0 f \cdot \mathbf{u} \cdot \mathbf{u}_t|\, dx + \int_\Omega |\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{u}_t|\, dx + \int_\Omega |m_1 f||\mathbf{u}_t|\, dx.\end{aligned}$$

By the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}&\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} + \frac{d}{dt}\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \\ &\le \|m_0 f\|_{L^4(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)}\|\mathbf{u}_t\|_{L^2(\Omega)} + \|\mathbf{u}_t\|_{L^2(\Omega)}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(\Omega)} \\ &+ \|m_1 f\|_{L^2(\Omega)}\|\mathbf{u}_t\|_{L^2(\Omega)}.\end{aligned} \tag{3.3.1}$$

By Theorem 3.1.1, we have

$$\mathbf{u} \in L^\infty(0,T;L^2(\Omega)), \nabla \mathbf{u} \in L^2(0,T;L^2(\Omega)).$$

Using the Gagliardo-Nirenberg inequality

$$\|v\|_{L^4} \le C\|v\|^{1/2}_{L^2}\|\nabla v\|^{1/2}_{L^2}, \tag{3.3.2}$$

one obtains that

$$\int_0^T \int_\Omega |\mathbf{u}|^4\, dx\, dt \le C\|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\|\nabla \mathbf{u}\|^2_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.3.3}$$

Since $\mathbf{u} \in L^2(0,T;H^1_0(\Omega))$, by the Sobolev imbedding inequality, we obtain $\mathbf{u} \in L^2(0,T;L^p(\Omega))$ for any $1 \le p < \infty$.

Thanks to Lemma 3.1.2 with $d = 2$, we have

$$M_6 f < \infty \quad \text{for any } 0 \le t \le T,$$

if $m_6 f_0 < \infty$.

Let us estimate $m_0 f$ in the two dimensional space:

$$m_0 f = \int_{\mathbb{R}^2} f\, d\mathbf{v} = \int_{|\mathbf{v}|<r} f\, d\mathbf{v} + \int_{|\mathbf{v}|\geq r} f\, d\mathbf{v}$$
$$\leq C\|f\|_{L^\infty} r^2 + \frac{1}{r^k}\int_{|\mathbf{v}|\geq r} |\mathbf{v}|^k f\, d\mathbf{v}$$

for all $k \geq 0$. Letting $r = (\int_{\mathbb{R}^2} |\mathbf{v}|^k f d\mathbf{v})^{\frac{1}{k+2}}$, then

$$m_0 f \leq C(\|f\|_{L^\infty} + 1)(\int_{\mathbb{R}^2} |\mathbf{v}|^k f\, d\mathbf{v})^{\frac{2}{k+2}}$$

for all $k \geq 0$. Taking $k = 6$, then $m_0 f \leq C(m_6 f)^{1/4}$, which means

$$\|m_0 f\|_{L^\infty(0,T;L^4(\Omega))} < \infty. \tag{3.3.4}$$

Similarly, we have

$$\|m_1 f\|_{L^\infty(0,T;L^2(\Omega))} < \infty. \tag{3.3.5}$$

By (3.3.1)-(3.3.5), we have for all $\varepsilon > 0$

$$\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} + \frac{d}{dt}\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \leq \frac{C(t)}{\varepsilon}(1 + \|\nabla \mathbf{u}\|^2_{L^2(\Omega)}) + \varepsilon\|D^2\mathbf{u}\|^2_{L^2(\Omega)} \tag{3.3.6}$$

where $C(t) \geq 0$, and $\int_0^T C(t)dt \leq C$ for all $T > 0$.

Next, observe that for all $t \geq 0$ in view of (1.2.1), we have

$$\begin{aligned}&\| - \Delta \mathbf{u} + \nabla p\|_{L^2(\Omega)}\\ &\leq C\left(\|m_1 f\|_{L^2(\Omega)} + \|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(\Omega)} + \||\mathbf{u}||\nabla \mathbf{u}|\|_{L^2(\Omega)} + \|m_0 f \cdot \mathbf{u}\|_{L^2(\Omega)}\right).\end{aligned}$$

Due to $\mathrm{div}\mathbf{u} = 0$, by the classical regularity on Stokes equations, we obtain

$$\begin{aligned}&\|\mathbf{u}\|_{H^2(\Omega)}\\ &\leq C\left(\|m_1 f\|_{L^2(\Omega)} + \|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(\Omega)} + \||\mathbf{u}||\nabla \mathbf{u}|\|_{L^2(\Omega)} + \|m_0 f \cdot \mathbf{u}\|_{L^2(\Omega)}\right).\end{aligned}$$

Following the same argument of (3.3.6), we have for all $\varepsilon' > 0$,

$$\|\mathbf{u}\|_{H^2(\Omega)} \leq \frac{1}{\varepsilon'}C_1(t) + C\|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(\Omega)} + \varepsilon'\|\mathbf{u}\|_{H^2(\Omega)} \tag{3.3.7}$$

where $C_1(t) \geq 0$, and $\int_0^T C_1^2(t)dt \leq C$. Choosing $\varepsilon' = \frac{1}{2}$, we obtain

$$\|\mathbf{u}\|_{H^2(\Omega)}^2 \leq C_2(t) + C\|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(\Omega)} \tag{3.3.8}$$

where $C_2(t) \geq 0$, and $\int_0^T C_2^2(t)dt \leq C$. Inserting (3.3.8) in (3.3.6) and choose $\varepsilon = 1/2C$, we obtain for all $t \geq 0$

$$\|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(\Omega)}^2 + \frac{d}{dt}\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 \leq C_3(t)(1 + \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2) \tag{3.3.9}$$

where $C_3(t) \geq 0$, and $\int_0^T C_3^2(t)dt \leq C$.

Applying Gronwall's inequality to (3.3.9), we obtain

$$\frac{\partial \mathbf{u}}{\partial t} \quad \text{is bounded in } L^2(\Omega \times (0, T)),$$

and

$$\mathbf{u} \quad \text{is bounded in } L^\infty(0, T; H_0^1(\Omega)).$$

This, with the help (3.3.8), implies that

$$\mathbf{u} \quad \text{is bounded in } L^2(0, T; H^2(\Omega)).$$

Here, we need to rely on the following Lemma which is a very special case of interpolation theorem of Lions-Magenes. We refer the readers to [52] for the proof of this lemma.

Lemma 3.3.1. *Let $V \subset H \subset V'$ be three Hilbert spaces, V' is a dual space of V. If a function $\mathbf{u}$ belong to $L^2(0, T; V)$ and its derivative $\mathbf{u}'$ belongs to $L^2(0, T; V')$ then $\mathbf{u}$ is almost everywhere equal to a function continuous from $[0, T]$ into H.*

Thanks to

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(\Omega \times (0, T)), \text{ and } \mathbf{u} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

we conclude that $\mathbf{u} \in C([0, T], H^1(\Omega))$ by Lemma 3.3.1, consequently $f \in C^1([0, T]; L^\infty(\mathbb{R}^2 \times \Omega))$.

3.3.2 Uniqueness of solutions

To show the uniqueness, we rely on the following parabolic regularity due to [16, 49, 52]:

Lemma 3.3.2. *If* $\mathbf{u}$ *solves*

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p = F,$$
$$\mathbf{u}(t=0) = \mathbf{u}_0, \quad \mathbf{u}|_{\partial\Omega} = 0,$$

on some time interval $(0,T)$*, then we have*

$$\|\mathbf{u}\|_{L^\infty([0,T);H_0^1)\cap L^2(0,T;H^2)} \le C\left(\|F\|_{L^2((0,T)\times\Omega)} + \|\mathbf{u}_0\|_{H_0^1}\right).$$

Now we are ready to show the uniqueness. Let $(\mathbf{u}_1, f_1)$ and $(\mathbf{u}_2, f_2)$ be two different solutions to (1.2.1)-(1.2.3). Let $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, and $\bar{f} = f_1 - f_2$, then we have the following equations:

$$\begin{aligned} &\bar{\mathbf{u}}_t + \nabla p - \Delta\bar{\mathbf{u}} = -\int_{\mathbb{R}^2} (\bar{\mathbf{u}} f_1 + \mathbf{u}_2 \bar{f} - \mathbf{v}\bar{f})\, d\mathbf{v} - (\bar{\mathbf{u}}\cdot\nabla\mathbf{u}_1 + \mathbf{u}_2\cdot\nabla\bar{\mathbf{u}}) \\ &\mathrm{div}\bar{\mathbf{u}} = 0, \\ &\bar{f}_t + \mathbf{v}\cdot\nabla_x \bar{f} + \mathrm{div}_{\mathbf{v}}(\bar{\mathbf{u}} f_1 + \mathbf{u}_2\bar{f} - \mathbf{v}\bar{f}) = 0 \end{aligned} \tag{3.3.10}$$

in $\Omega \times \mathbb{R}^2 \times (0,T)$, subject to the following initial data

$$\bar{\mathbf{u}}(0,x) = 0, \qquad \bar{f}(0,x,\mathbf{v}) = 0,$$

and boundary condition

$$\bar{\mathbf{u}}|_{\partial\Omega} = 0, \quad \bar{f}(t,x,\mathbf{v}^*) = \bar{f}(t,x,\mathbf{v}) \quad \text{for } x\in\partial\Omega,\ \mathbf{v}\cdot\nu(x) < 0.$$

Here, we denote the space $X = L^\infty(0,T;H_0^1) \cap L^2(0,T;H_0^2)$. Applying Lemma 3.3.2 with $\mathbf{u}_0 = 0$, we have the following regularity:

$$\begin{aligned} \|\bar{\mathbf{u}}\|_X &\le C\|\int_{\mathbb{R}^2} (\bar{\mathbf{u}} f_1 + \mathbf{u}_2\bar{f} - \mathbf{v}\bar{f}) d\mathbf{v} + (\bar{\mathbf{u}}\cdot\nabla\mathbf{u}_1 + \mathbf{u}_2\cdot\nabla\bar{\mathbf{u}})\|_{L^2((0,T)\times\Omega)} \\ &= J_1 + J_2. \end{aligned} \tag{3.3.11}$$

For J_1:

$$
\begin{aligned}
&\| \int_{\mathbb{R}^2} (\bar{\mathbf{u}} f_1 + \mathbf{u}_2 \bar{f} - \mathbf{v}\bar{f}) d\mathbf{v} \|_{L^2((0,T)\times\Omega)} \\
&= \|\bar{\mathbf{u}} m_0 f_1 + \mathbf{u}_2 m_0 \bar{f} - m_1 \bar{f}\|_{L^2((0,T)\times\Omega)} \\
&\leq \|\bar{\mathbf{u}}\|_{L^\infty(0,T;L^{p_1}(\Omega))} \|m_0 f_1\|_{L^2(0,T;L^3(\Omega))} + \|1\|_{L^6(0,T;L^x(\Omega))} \|\mathbf{u}_2\|_{L^\infty(0,T;L^{p_2}(\Omega))} \|m_0 \bar{f}\|_{L^3((0,T)\times\Omega)} \\
&+ \|m_1 \bar{f}\|_{L^2((0,T)\times\Omega)} \\
&\leq \sup_{t\in[0,T]} \|m_0 f_1\|_{L^3(\Omega)} T^{\frac{1}{2}} \|\bar{\mathbf{u}}\|_X + CT^{\frac{1}{6}} \|\mathbf{u}_2\|_{L^\infty(0,T;L^p(\Omega))} \|m_0 \bar{f}\|_{L^\infty(0,T;L^3(\Omega))} \\
&+ \|m_1 \bar{f}\|_{L^2((0,T)\times\Omega)},
\end{aligned}
\tag{3.3.12}
$$

where $p_1 = 6$, $x = \frac{6p_2}{p_2-6}$, for any $p_2 > 6$, C depends on the domain Ω. And for J_2:

$$
\begin{aligned}
&\|\bar{\mathbf{u}} \cdot \nabla \mathbf{u}_1 + \mathbf{u}_2 \cdot \nabla \bar{\mathbf{u}}\|_{L^2((0,T)\times\Omega)} \\
&\leq \|1\|_{L^4(0,T;L^y(\Omega))} \|\bar{\mathbf{u}}\|_{L^\infty(0,T;L^p(\Omega))} \|\nabla \mathbf{u}_1\|_{L^4((0,T)\times\Omega)} \\
&+ \|1\|_{L^4(0,T;L^\infty(\Omega))} \|\mathbf{u}_2\|_{L^4(0,T;L^\infty(\Omega))} \|\nabla \bar{\mathbf{u}}\|_{L^\infty(0,T;L^2(\Omega))} \\
&\leq CT^{\frac{1}{4}} \|\nabla \mathbf{u}_1\|_{L^4((0,T)\times\Omega)} \|\mathbf{u}\|_X + CT^{\frac{1}{4}} \|\mathbf{u}_2\|_{L^4(0,T;L^\infty(\Omega))} \|\mathbf{u}\|_X,
\end{aligned}
\tag{3.3.13}
$$

where $y = \frac{4p}{p-4}$ for any $p > 4$, C only depends on the domain Ω, and we used the Gagliardo-Nirenberg inequality for $\nabla \mathbf{u}_1$, the Gagliardo-Nirenberg inequality and embedding inequality for $\mathbf{u}_2$. By (3.3.11)-(3.3.13), we can choose small T such that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|m_0 f_1\|_{L^3(\Omega)} T^{\frac{1}{2}} + C\|\mathbf{u}_2\|_{L^\infty(0,T;L^p(\Omega))} T^{\frac{1}{6}} \\
&+ CT^{\frac{1}{4}} \|\nabla \mathbf{u}_1\|_{L^4((0,T)\times\Omega)} + CT^{\frac{1}{4}} \|\mathbf{u}_2\|_{L^4(0,T;L^\infty(\Omega))} \leq \frac{1}{2},
\end{aligned}
\tag{3.3.14}
$$

then

$$
\|\bar{\mathbf{u}}\|_X \leq \|m_0 \bar{f}\|_{L^\infty(0,T;L^3(\Omega))} + \|m_1 \bar{f}\|_{L^2((0,T)\times\Omega)}. \tag{3.3.15}
$$

The next step is to show $\|m_0 \bar{f}\|_{L^3((0,T)\times\Omega)} + \|m_1 \bar{f}\|_{L^2((0,T)\times\Omega)}$ can be controlled by $\|\mathbf{u}\|_X$.

In the two dimensional space, we have

$$
\int_0^T \|m_0 \bar{f}\|^3_{L^3(\Omega)}\, dt \leq C \int_0^T \int_\Omega \int_{\mathbb{R}^2} |\mathbf{v}|^4 \bar{f}\, d\mathbf{v}\, dx\, dt,
$$

and

$$\int_0^T \|m_1\bar{f}\|^2_{L^2(\Omega)}\,dt \le C\int_0^T\int_\Omega\int_{\mathbb{R}^2}|\mathbf{v}|^4\bar{f}\,d\mathbf{v}\,dx\,dt.$$

Pluging them into (3.3.15), and choosing T small enough again, one deduces that

$$\|\bar{\mathbf{u}}\|_X \le C\int_0^T\int_\Omega\int_{\mathbb{R}^2}|\mathbf{v}|^4\bar{f}d\mathbf{v}\,dx\,dt. \tag{3.3.16}$$

We multiply the second equation of (3.3.10) by $|\mathbf{v}|^k$ for $k\ge 1$, and use integration by parts over $\Omega\times\mathbb{R}^2$:

$$\begin{aligned}\partial_t\int_\Omega\int_{\mathbb{R}^2}\bar{f}|\mathbf{v}|^k\,d\mathbf{v}\,dx + k\int_\Omega\int_{\mathbb{R}^2}\bar{f}|\mathbf{v}|^k\,d\mathbf{v}\,dx\\ = k\int_\Omega\int_{\mathbb{R}^2}\bar{\mathbf{u}}f_1|\mathbf{v}|^{k-1}\,d\mathbf{v}\,dx + k\int_\Omega\int_{\mathbb{R}^2}\mathbf{u}_2\bar{f}|\mathbf{v}|^{k-1}\,d\mathbf{v}\,dx.\end{aligned} \tag{3.3.17}$$

We estimate the right hand side terms of (3.3.17):

$$\begin{aligned}&k\int_\Omega\int_{\mathbb{R}^2}\bar{\mathbf{u}}f_1|\mathbf{v}|^{k-1}d\mathbf{v}dx + k\int_\Omega\int_{\mathbb{R}^2}\mathbf{u}_2\bar{f}|\mathbf{v}|^{k-1}\,d\mathbf{v}\,dx\\ &\le C\int_\Omega|\bar{\mathbf{u}}||m_{k-1}f_1|\,dx + C\int_\Omega|\mathbf{u}_2||m_{k-1}\bar{f}|\,dx\\ &\le C\|\bar{\mathbf{u}}\|_{L^2(\Omega)}\|m_{k-1}f_1\|_{L^2(\Omega)} + C\|\mathbf{u}_2\|_{H^2_0(\Omega)}\|m_{k-1}\bar{f}\|_{L^1(\Omega)}.\end{aligned} \tag{3.3.18}$$

By (3.3.17) and (3.3.18), we have

$$\begin{aligned}&\sup_{t\in(0,T)}\int_\Omega\int_{\mathbb{R}^2}\bar{f}|\mathbf{v}|^k\,d\mathbf{v}\,dx + k\int_0^T\int_\Omega\int_{\mathbb{R}^2}\bar{f}|\mathbf{v}|^k\,d\mathbf{v}\,dx\,dt\\ &\le\int_0^T\|\bar{\mathbf{u}}\|_{L^2(\Omega)}\|m_{k-1}f_1\|_{L^2(\Omega)}dt + \int_0^T\|\mathbf{u}_2\|_{H^2_0(\Omega)}\|m_{k-1}\bar{f}\|_{L^1(\Omega)}dt\\ &\le\|\bar{\mathbf{u}}\|_{L^2((0,T)\times\Omega)}\|m_{k-1}f_1\|_{L^2(\Omega)}T^{\frac12} + \|\mathbf{u}_2\|_{L^2(0,T;H^2_0(\Omega))}T^{\frac12}\sup_{t\in[0,T]}\int_\Omega\int_{\mathbb{R}^2}|\mathbf{v}|^{k-1}\bar{f}\,d\mathbf{v}\,dx.\end{aligned} \tag{3.3.19}$$

for all $k\ge 1$. Meanwhile, we integrate the third equation in (3.3.10) over $\Omega\times\mathbb{R}^2$ and use integration by parts:

$$\int_\Omega\int_{\mathbb{R}^2}\bar{f}\,d\mathbf{v}\,dx = \int_\Omega\int_{\mathbb{R}^2}\bar{f}_0\,d\mathbf{v}\,dx = 0. \tag{3.3.20}$$

Using (3.3.19)-(3.3.20) and by induction, we deduce

$$\begin{aligned}&\int_0^T\int_\Omega\int_{\mathbb{R}^2}\bar{f}|\mathbf{v}|^4\,d\mathbf{v}\,dx\,dt \le (\sup_{t\in[0,T]}\|m_3f_1\|_{L^2(\Omega)}T^{\frac12} + \|\mathbf{u}_2\|_X\sup_{t\in[0,T]}\|m_2f_1\|_{L^2(\Omega)}T\\ &+\|\mathbf{u}_2\|_X^2\sup_{t\in[0,T]}\|m_1f_1\|_{L^2(\Omega)}T^{\frac32} + \|\mathbf{u}_2\|_X^3\sup_{t\in[0,T]}\|m_0f_1\|_{L^2(\Omega)}T^2)\|\bar{\mathbf{u}}\|_X.\end{aligned} \tag{3.3.21}$$

Thanks to (3.3.21) and (3.3.16), choosing $T > 0$ small enough, we obtain:

$$\|\bar{\mathbf{u}}\|_X \leq \frac{1}{2}\|\bar{\mathbf{u}}\|_X.$$

Thus, $\bar{\mathbf{u}} = 0$, and hence $\mathbf{u}_1 = \mathbf{u}_2$ on a small time interval. Thus, we have proved the uniqueness of $\mathbf{u}$ on a small time interval.

On the other hand, we have the following equation on the same time interval,

$$\bar{f}_t + \mathbf{v}\nabla\bar{f} + \mathrm{div}_{\mathbf{v}}((\mathbf{u}_2 - \mathbf{v})\bar{f}) = 0 \quad \text{in } \Omega \times \mathbb{R}^2 \times (0, T], \tag{3.3.22}$$

with its initial data

$$\bar{f}_0 = 0,$$

and boundary condition

$$\bar{f}(t, x, \mathbf{v}) = \bar{f}(t, x, \mathbf{v}^*) \quad \text{for } x \in \partial\Omega,\, \mathbf{v}\cdot\nu(x) < 0.$$

By (3.3.22), we have

$$\|\bar{f}\|_{L^\infty((0,T]\times\Omega\times\mathbb{R}^2)} \leq C\|\bar{f}_0\|_{L^\infty((0,T]\times\Omega\times\mathbb{R}^2)},$$

which yields $f_1 = f_2$. So we have proved the uniqueness of solution $(\mathbf{u}, f)$ on a small time interval $[0, T_0]$. For any given $T > 0$, we consider the maximal interval of the uniqueness, $T_1 = \sup T_0 \leq T$, such that the solution is unique on $[0, T_0]$. The main goal is to prove that T_1 can be taken to be equal to ∞. For any given $T_0 > 0$, we use

$$\bar{\mathbf{u}}(T_0, x) = 0, \;\; \bar{f}(T_0, x, \mathbf{v}) = 0,$$

as the new initial data to (3.3.10). Applying the same argument to equation (3.3.10) with the new data, the uniqueness of solution can be extended to $[0, T_0 + T^*]$ for a small number $T^* > 0$.

By (3.3.14) and (3.3.21), T^* can be chosen only depending on the upper bounds of

$$\sup_{t\in[0,T]} \|m_0 f_1\|_{L^3(\Omega)}, \;\; \|\mathbf{u}_1\|_X, \;\; \|\mathbf{u}_2\|_X, \;\; \sup_{t\in[0,T]} \|m_i f_1\|_{L^2(\Omega)} \;\text{ for } i = 0, 1, 2, 3.$$

By the regularity of $\mathbf{u}$ in Section 4.1, $\mathbf{u}_1$, $\mathbf{u}_2$ are uniformly bounded in the space X. Applying the same argument of (3.3.4) and Lemma 3.1.2, we can show that the other terms are uniformly bounded for all time $t \geq 0$. All such terms are uniformly bounded for all time $t \geq 0$. Thus, T^*

can be chosen not depending on the initial data at time T_0. In fact, we can choose $T^* = T_0$. One can then repeat the argument many times and obtain the uniqueness of $(\mathbf{u}, f)$ on the whole time.

4.0 DENSITY-DEPENDENT FLUID-PARTICLE FLOW

It is very hard to carry out the mathematical analysis for the system with density-dependent interactions because a relaxation of the kinetic regime toward a hydrodynamic regime with velocity in the vacuum can not be excepted. In [55], we established the existence of global weak solutions to the Navier-Stokes-Vlasov equations with large data when the density appears in the interactions. In [55], to overcome this difficulty, we decomposed the interaction of the Navier-Stokes equations into two components. One component is viewed as the work of internal force appearing in the left side of the Navier-Stokes equations, and the other component is viewed as an external force. Thus, we can construct an approximation scheme and use a compactness argument to obtain the solutions.

4.1 A PRIORI ESTIMATES AND MAIN RESULTS

In this section, we shall derive some fundamental a priori estimates and then state our main results. These estimates will play an important role in the compactness analysis later since they will allow us to deduce the global existence upon passing to the limit in the regularized approximation scheme. We shall develop these a priori estimates in the three-dimensional space, but they all hold in the two-dimensional space.

First, roughly speaking, (1.3.1) and the incompressibility condition mean that the density $\rho(t,x)$ is independent of time t. In fact, we take any function $\beta \in C^1([0,\infty);\Omega)$, multiply (1.3.1) by $\beta'(\rho)$, use the incompressibility condition, and integration by parts over Ω, then we have

$$\frac{d}{dt}\int_\Omega \beta(\rho)dx = 0.$$

Applying the maximum principle to the transport equations (1.3.1) and (1.3.3), one deduces that

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty},$$

and also $\rho \ge 0$, so we have

$$0 \le \rho(t,x) \le \|\rho_0\|_{L^\infty} \tag{4.1.1}$$

for almost every t.

We now multiply (1.3.2) by $\mathbf{u}$ and integrate over Ω, and use (1.3.1), (1.3.3), and (1.3.5) to deduce that

$$\frac{d}{dt}\int_\Omega \rho|\mathbf{u}|^2\,dx + 2\int_\Omega |\nabla \mathbf{u}|^2\,dx = -2\int_\Omega\int_{\mathbb{R}^3} \rho f(\mathbf{u}-\mathbf{v})\cdot \mathbf{u}\,d\mathbf{v}\,dx. \tag{4.1.2}$$

On the other hand, we multiply the Vlasov equation (1.3.4) by $\frac{|\mathbf{v}|^2}{2}$, integrate over $\Omega \times \mathbb{R}^3$, and use integration by parts to obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} f|\mathbf{v}|^2\,d\mathbf{v}dx\\ &= -2\int_\Omega\int_{\mathbb{R}^3} \rho f|\mathbf{u}-\mathbf{v}|^2\,d\mathbf{v}dx + 2\int_\Omega\int_{\mathbb{R}^3} \rho f(\mathbf{u}-\mathbf{v})\mathbf{u}\,d\mathbf{v}dx.\end{aligned} \tag{4.1.3}$$

Using (4.1.2)-(4.1.3) and the conservation of mass:

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} f\,d\mathbf{v}dx = 0, \tag{4.1.4}$$

we obtain the following energy equality for the system (1.3.1)-(1.3.5):

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \rho|\mathbf{u}|^2\,dx + \frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3} f(1+|\mathbf{v}|^2)\,d\mathbf{v}dx + 2\int_\Omega\int_{\mathbb{R}^3} \rho f|\mathbf{u}-\mathbf{v}|^2\,d\mathbf{v}dx\\ &+2\int_\Omega |\nabla\mathbf{u}|^2\,dx = 0.\end{aligned} \tag{4.1.5}$$

Integrating (4.1.5) with respect to t, we obtain for all t,

$$\begin{aligned}&\int_\Omega \rho|\mathbf{u}|^2\,dx + \int_\Omega\int_{\mathbb{R}^3} f(1+|\mathbf{v}|^2)\,d\mathbf{v}dx + 2\int_0^t\int_\Omega\int_{\mathbb{R}^3} \rho f|\mathbf{u}-\mathbf{v}|^2\,d\mathbf{v}dxdt\\ &+2\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,dxdt\\ &=\int_\Omega \frac{|\mathbf{m}_0|^2}{\rho_0}\,dx + \int_\Omega\int_{\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}dx.\end{aligned} \tag{4.1.6}$$

By (4.1.5), it is easy to find that the global energy is non-increasing with respect to t:

$$\frac{d}{dt}\left(\int_\Omega \rho|\mathbf{u}|^2\,dx + \int_\Omega\int_{\mathbb{R}^3} f(1+|\mathbf{v}|^2)\,d\mathbf{v}dx\right) \le 0.$$

Assume

$$\int_\Omega \frac{|\mathbf{m}_0|^2}{\rho_0}\,dx + \int_\Omega\int_{\mathbb{R}^3} f_0(1+|\mathbf{v}|^2)\,d\mathbf{v}dx < \infty,$$

then

$$\int_0^t\int_\Omega\int_{\mathbb{R}^3} \rho f|\mathbf{u}-\mathbf{v}|^2\,d\mathbf{v}dxdt \le C,$$

and

$$\|\nabla\mathbf{u}\|_{L^2((0,T)\times\Omega)} \le C, \tag{4.1.7}$$

$$\sup_{0\le t\le T}\|\rho|\mathbf{u}|^2\|_{L^1(\Omega)} \le C, \tag{4.1.8}$$

for any given $T>0$ and some generic positive constant C. Moreover, by the Poincaré inequality we obtain

$$\|\mathbf{u}\|_{L^2(0,T;H_0^1(\Omega))} \le C. \tag{4.1.9}$$

The maximum principle applied to (1.3.4) implies that

$$\|f\|_{L^\infty} \le C\|f_0\|_{L^\infty} \tag{4.1.10}$$

for all $t\in[0,T]$. Moreover, $f_0\ge 0$ implies $f\ge 0$ for almost every $(t,x,\mathbf{v})$. Then, by the conservation of mass (4.1.4) and (4.1.10), one has the following estimate:

$$\begin{aligned}&\|f\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} + \|f\|_{L^\infty(0,T;L^1(\Omega\times\mathbb{R}^3))}\\ &\le C\left(\|f_0\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} + \|f_0\|_{L^\infty(0,T;L^1(\Omega\times\mathbb{R}^3))}\right).\end{aligned} \tag{4.1.11}$$

Let $w(t,x)$ be a smooth vector field in $\mathbb{R}^3$ and let f be a solution to the following kinetic equation:

$$\begin{aligned}&\partial_t f + \mathbf{v}\cdot\nabla_x f + \mathrm{div}_{\mathbf{v}}((w-\mathbf{v})f) = 0,\\ &f(0,x,\mathbf{v}) = f_0(x,\mathbf{v}),\quad f(t,x,\mathbf{v}) = f(t,x,\mathbf{v}^*)\ \text{ for } x\in\partial\Omega,\ \ \mathbf{v}\cdot\nu(x)<0\end{aligned} \tag{4.1.12}$$

in $\Omega \times \mathbb{R}^3$. DiPerna-Lions [12] obtained the existence and uniqueness of solution to (4.1.12) when w is not smooth. Denote the moments of f by

$$m_k f(t,x) = \int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^k \, d\mathbf{v},$$
$$M_k f(t) = \int_\Omega \int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^k \, d\mathbf{v} dx,$$

for any $t \in [0,T]$, $x \in \Omega$, and integer $k \geq 0$. It is clear that

$$M_k f(t) = \int_\Omega m_k f(t,x) dx.$$

We first recall the following lemma [26]:

Lemma 4.1.1. *Let $w \in L^p(0,T; L^{N+k}(\Omega))$ with $1 \leq p \leq \infty$ and $k \geq 1$. Assume that $f_0 \in (L^\infty \cap L^1)(\Omega \times \mathbb{R}^3)$ and $m_k f_0 \in L^1(\Omega \times \mathbb{R}^3)$. Then, the solution f of (4.1.12) should have the following estimates*

$$M_k f \leq C \left((M_k f_0)^{1/(N+k)} + (|f_0|_{L^\infty} + 1) \|w\|_{L^p(0,T;L^{N+k}(\Omega))} \right)^{N+k}$$

for all $0 \leq t \leq T$ where the constant C depends only on T.

We also recall the average compactness result for the Vlasov equation due to Di Perna-Lions-Meyer [14]:

Lemma 4.1.2. *Suppose*

$$\frac{\partial f^n}{\partial t} + \mathbf{v} \cdot \nabla_x f^n = \operatorname{div}_{\mathbf{v}}(F^n f^n) \qquad \text{in } \mathcal{D}'(\Omega + \mathbb{R}^3 \times (0,\infty))$$

where f^n is bounded in $L^\infty(0,\infty; L^2(\Omega \times \mathbb{R}^3))$ and $f^n(1+|\mathbf{v}|^2)$ is bounded in $L^\infty(0,\infty; L^1(\Omega \times \mathbb{R}^3)$, $\frac{F^n}{1+|\mathbf{v}|}$ is bounded in $L^\infty((0,\infty) \times \mathbb{R}^3; L^2(\Omega))$. Then $\int_{\mathbb{R}^3} f^n \eta(\mathbf{v}) d\mathbf{v}$ is relatively compact in $L^q(0,T; L^p(\Omega))$ for $1 \leq q < \infty, 1 \leq p < 2$ and for η such that $\frac{\eta}{(1+|\mathbf{v}|)^\sigma} \in L^1 + L^\infty, \sigma \in [0,2)$.

Remark 4.1.1. We shall use this lemma for the Vlasov equation to obtain the compactness of $m_0 f$ and $m_1 f$, which will allow us to pass the limit when ε and δ go to zero in the approximation.

In this paper, we assume that

$$
\begin{aligned}
&\rho_0 \geq 0 \ \text{ almost everywhere in } \Omega, \ \ \rho_0 \in L^\infty(\Omega), \\
&\mathbf{m}_0 \in L^2(\Omega), \quad \mathbf{m}_0 = 0 \ \text{ almost everywhere on } \{\rho_0 = 0\}, \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega), \\
&f_0 \in L^\infty(\Omega \times \mathbb{R}^3), \quad m_3 f_0 \in L^\infty((0,T) \times \Omega).
\end{aligned}
\tag{4.1.13}
$$

Definition 4.1.1. We say that $(\rho, \mathbf{u}, f)$ is a global weak solution to problem (1.3.1)-(1.3.8) if the following conditions are satisfied: for any $T > 0$,

- $\rho \geq 0, \quad \rho \in L^\infty([0,T] \times \Omega), \quad \rho \in C([0,T]; L^p(\Omega)), \quad 1 \leq p < \infty;$
- $\mathbf{u} \in L^2(0,T; H_0^1(\Omega));$
- $\rho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega));$
- $f(t,x,\mathbf{v}) \geq 0,$ for any $(t,x,\mathbf{v}) \in (0,T) \times \Omega \times \mathbb{R}^3;$
- $f \in L^\infty(0,T; L^\infty(\Omega \times \mathbb{R}^3) \cap L^1(\Omega \times \mathbb{R}^3));$
- $m_3 f \in L^\infty(0,T; L^1(\Omega \times \mathbb{R}^3));$
- For any $\varphi \in C^1([0,T] \times \Omega)$, such that $\mathrm{div}_x \varphi = 0$, for almost everywhere t,

$$
\begin{aligned}
-\int_\Omega \mathbf{m}_0 \cdot \varphi(0,x)\, dx + \int_0^t \int_\Omega \Big(&-\rho\mathbf{u} \cdot \partial_t \varphi - (\rho\mathbf{u} \otimes \mathbf{u}) \cdot \nabla\varphi \\
&+ \mu \nabla\mathbf{u} \cdot \nabla\varphi + \mu\rho \int_{\mathbb{R}^3} f(\mathbf{u} - \mathbf{v}) \cdot \varphi \, d\mathbf{v} \Big)\, dxdt = 0;
\end{aligned}
\tag{4.1.14}
$$

- For any $\phi \in C^1([0,T] \times \Omega \times \mathbb{R}^3)$ with compact support in $\mathbf{v}$, such that $\phi(T,\cdot,\cdot) = 0$,

$$
-\int_0^T \int_\Omega \int_{\mathbb{R}^3} f\left(\partial_t \phi + \mathbf{v} \cdot \nabla_x \phi + \rho(\mathbf{u} - \mathbf{v}) \cdot \nabla_{\mathbf{v}} \phi\right)\, dxd\mathbf{v}ds = \int_\Omega \int_{\mathbb{R}^3} f_0 \phi(0,\cdot,\cdot)\, dxd\mathbf{v};
\tag{4.1.15}
$$

- The energy inequality

$$
\begin{aligned}
&\int_\Omega \rho|\mathbf{u}|^2 dx + \int_\Omega \int_{\mathbb{R}^3} f(1 + |\mathbf{v}|^2)\, d\mathbf{v}dx + 2\int_0^T \int_\Omega \int_{\mathbb{R}^3} f|\mathbf{u} - \mathbf{v}|^2\, d\mathbf{v}dxdt \\
&\qquad + 2\int_0^T \int_\Omega |\nabla\mathbf{u}|^2\, dxdt \\
&\leq \int_\Omega \frac{|\mathbf{m}_0|^2}{\rho_0}\, dx + \int_\Omega \int_{\mathbb{R}^3} (1 + |\mathbf{v}|^2) f_0\, d\mathbf{v}dx
\end{aligned}
\tag{4.1.16}
$$

  holds for almost everywhere $t \in [0,T]$.

Our main result on the global weak solutions reads as follows.

Theorem 4.1.1. *Under the assumption* (4.1.13), *there exists a global weak solution* $(\rho, \mathbf{u}, f)$ *to the initial-boundary value problem* (1.3.1)-(1.3.9) *for any* $T > 0$.

Remark 4.1.2. The same existence of global weak solutions holds also in two-dimensional spaces.

4.2 EXISTENCE OF GLOBAL WEAK SOLUTIONS

In this section, we are going to prove Theorem 4.1.1 in two steps. First, we build a regularized approximation system for the original system, and solve this approximation system. Then, we recover the original system from the approximation scheme by passing to the limit of the sequence of solutions obtained in the first step.

4.2.1 Construction of approximation solutions

For each $\varepsilon > 0$, we define

$$\theta_\varepsilon := \varepsilon^3 \theta\left(\frac{x}{\varepsilon}\right)$$

and denote

$$\mathbf{u}_\varepsilon := \mathbf{u} * \theta_\varepsilon,$$

where θ is the the standard mollifier satisfying

$$\theta \in C^\infty(\mathbb{R}^3), \theta \geq 0, \int_{\mathbb{R}^3} \theta \, dx = 1.$$

By (4.1.1), all values of the solution ρ are bounded uniformly. The regularity of the term $-\int_{\mathbb{R}^3}(\mathbf{u} - \mathbf{v})\rho f d\mathbf{v}$ is not enough to solve the Navier-Stokes equation directly. Inspired by the work of [26], we introduce the following regularization function

$$R_\delta(m_0 f, m_1 f) = \frac{1}{1 + \delta \int_{\mathbb{R}^3} f \; d\mathbf{v} + \delta \left| \int_{\mathbb{R}^3} f \mathbf{v} d\mathbf{v} \right|}, \qquad \text{for any fixed } \delta > 0.$$

Clearly

$$0 < R_\delta(m_0 f, m_1 f) < 1$$

for any $\delta > 0$, and

$$R_\delta(m_0 f, m_1 f) \to 1$$

as $\delta \to 0$. For any fixed $\delta > 0$, as mentioned in the introduction, the regularized force term

$$\rho R_\delta \int_{\mathbb{R}^3} (\mathbf{u} - \mathbf{v}) f \, d\mathbf{v}$$

consists of two terms:

$$\rho \left(R_\delta \int_{\mathbb{R}^3} f \, d\mathbf{v} \right) \mathbf{u} \quad \text{and} \quad \rho \left(R_\delta \int_{\mathbb{R}^3} \mathbf{v} f \, d\mathbf{v} \right)$$

the first one is viewed as the work of internal force, and the second one is viewed as the external force. The regularized external force is in $L^2((0,T) \times \Omega)$, which ensures that the regularized Navier-Stokes equations with the work of internal force have a smooth solution. To keep a similar energy inequality for the approximation scheme, we need to regularize the acceleration term as

$$R_\delta(\mathbf{u} - \mathbf{v})\rho f$$

in the Vlasov equation. Thus, we consider the following approximation problem:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}_\varepsilon) = 0, \tag{4.2.1}$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \operatorname{div}(\rho \mathbf{u}_\varepsilon \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p + \rho \left(R_\delta \int_{\mathbb{R}^3} f \, d\mathbf{v} \right) \mathbf{u} = \rho \left(R_\delta \int_{\mathbb{R}^3} \mathbf{v} f \, d\mathbf{v} \right), \tag{4.2.2}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{4.2.3}$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \operatorname{div}_{\mathbf{v}}(R_\delta(\mathbf{u} - \mathbf{v})\rho f) = 0. \tag{4.2.4}$$

To define $\mathbf{u}_\varepsilon$ well, we need to set

$$\Omega_\varepsilon = \{x \in \Omega, \ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

for any $\varepsilon > 0$ if Ω is smooth. Otherwise, we can choose a smooth connected domain Ω_ε such that

$$\{x \in \Omega, \ \operatorname{dist}(x, \partial\Omega) > \varepsilon\} \subset \Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega.$$

We let $\hat{\mathbf{u}}^\varepsilon$ to be the truncation in Ω_ε of $\mathbf{u}$, and we extended it by 0 to Ω. We define $\mathbf{u}_\varepsilon = \hat{\mathbf{u}}^\varepsilon * \theta_{\frac{\varepsilon}{2}}$. It is easy to find that $\mathbf{u}_\varepsilon$ is a smooth function with respect to x, and

$$\mathbf{u}_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \text{div}\mathbf{u}_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^d.$$

To impose the initial value for our approximate system, we need the following elementary variant of Hodge-de Rham decomposition (see [39]):

Lemma 4.2.1. *Let $N \geq 2$, $\rho \in L^\infty(\mathbb{R}^N)$ such that $\rho \geq \underline{\rho} \geq 0$ almost everywhere on $\mathbb{R}^N$ for some $\underline{\rho} \in (0, \infty)$. Then there exists two bounded operators P_δ, Q_δ on $L^2(\mathbb{R}^N)$ such that for all $\mathbf{m} \in L^2(\mathbb{R}^N)$, $(\mathbf{m}_p, \mathbf{m}_q) = (P_\rho \mathbf{m}, Q_\rho \mathbf{m})$ is the unique solution in $L^2(\mathbb{R}^N)$ of*

$$\mathbf{m} = \mathbf{m}_p + \mathbf{m}_q, \qquad (-\Delta)^{-1/2}\text{div}(\rho^{-1}\mathbf{m}_p) = 0, \quad (-\Delta)^{-1/2} rot(\mathbf{m}_q) = 0.$$

Furthermore, if $\rho_n \in L^\infty(\mathbb{R}^N)$, $\underline{\rho} \leq \rho_n \leq \bar{\rho}$ almost everywhere on $\mathbb{R}^N$ for some $0 < \underline{\rho} \leq \bar{\rho} < \infty$ and ρ_n converges almost everywhere to ρ, then $(P_{\rho_n}\mathbf{m}_n, Q_{\rho_n}\mathbf{m}_n)$ converges weakly in $L^2(\mathbb{R}^N)$ to $(P_\rho \mathbf{m}, Q_\rho \mathbf{m})$ whenever $\mathbf{m}_n$ converges weakly to $\mathbf{m}$.

We are ready to discuss the initial conditions for the approximation scheme (4.2.1)-(4.2.4). Before imposing initial data, we have to point out that the initial density may be vanish in a domain: an initial vacuum may exist, and then in this case we cannot directly impose initial data on the velocity $\mathbf{u}$. To remove this difficulty, we adopt the idea from [39] to define

$$\hat{\rho_0} = \begin{cases} \rho_0, & \text{if } x \text{ is in } \Omega \\ 1, & \text{if } x \text{ is in } \Omega^c, \end{cases}$$

define

$$\begin{aligned} &(\rho_0)_\varepsilon = \hat{\rho_0} * \theta\varepsilon|_\Omega, \\ &(\rho_0^{\frac{1}{2}})_\varepsilon = \hat{\rho_0}^{\frac{1}{2}} * \theta_\varepsilon|_\Omega, \\ &(m_0 \rho_0^{-\frac{1}{2}})_\varepsilon = \left(m_0 \rho_0^{-\frac{1}{2}} 1_{\{d > 2\varepsilon\}}\right) * \theta_\varepsilon, \end{aligned}$$

where $d = \text{dist}(x, \partial\Omega)$.

Now we define

$$\rho|_{t=0} = \rho_0^\varepsilon = (\rho_0)_\varepsilon + \varepsilon, \tag{4.2.5}$$

which implies

$$\varepsilon \leq \rho_0^\varepsilon \leq C_0, \tag{4.2.6}$$

where C_0 is independent on ε, and

$$(\rho_0)_\varepsilon = \rho_0 * \theta_\varepsilon.$$

Clearly, $\rho_0^\varepsilon \in C^\infty(\Omega)$, and

$$\rho_0^\varepsilon \to \rho_0 \ \text{ in } L^p(\Omega) \quad \text{for all } \ 1 \leq p < \infty.$$

We define

$$\rho\mathbf{u}|_{t=0} = \mathbf{m}_0^\varepsilon,$$

and

$$\bar{\mathbf{m}}_0^\varepsilon = (\mathbf{m}_0 {\rho_0}^{-1/2})_\varepsilon (\rho_0^{1/2})_\varepsilon \in C_0^\infty(\Omega).$$

It is easy to see

$$\bar{\mathbf{m}}_0^\varepsilon \to \mathbf{m}_0 \ \text{ in } L^2(\Omega), \qquad \bar{\mathbf{m}}_0^\varepsilon (\rho_0^\varepsilon)^{-1/2} \to \mathbf{m}_0 {\rho_0}^{-1/2} \ \text{ in } L^2(\Omega).$$

Relying on Lemma 4.2.1, we decompose $\bar{\mathbf{m}}_0^\varepsilon$ as

$$\begin{aligned} &\bar{\mathbf{m}}_0^\varepsilon = \rho_0^\varepsilon \bar{\mathbf{u}}_0^\varepsilon + \nabla q_0^\varepsilon, \quad \text{where} \ \ \bar{\mathbf{u}}_0^\varepsilon, \ q_0^\varepsilon \in C^\infty(\Omega), \\ &\mathrm{div} \bar{\mathbf{u}}_0^\varepsilon = 0 \quad \text{in } \Omega, \end{aligned}$$

and then

$$\mathrm{div} \left(\frac{1}{\rho_0^\varepsilon} (\nabla q_0^\varepsilon - \bar{\mathbf{m}}_0^\varepsilon) \right) = 0 \quad \text{ in } \Omega.$$

Letting

$$\begin{aligned} &\mathbf{m}_0^\varepsilon = \rho_0^\varepsilon \mathbf{u}_0^\varepsilon + \nabla q_0^\varepsilon, \quad \text{where } \mathbf{u}_0^\varepsilon \in C_0^\infty(\Omega), \\ &\|\mathbf{u}_0^\varepsilon - \bar{\mathbf{u}}_0^\varepsilon\| \leq \varepsilon, \quad \mathrm{div} \mathbf{u}_0^\varepsilon = 0 \qquad \text{in } \Omega. \end{aligned} \tag{4.2.7}$$

We have

$$\mathbf{m}_0^\varepsilon \to \mathbf{m}_0 \quad \text{in } L^2(\Omega), \quad \mathbf{m}_0^\varepsilon (\rho_0^\varepsilon)^{-1/2} \to \mathbf{m}_0 {\rho_0}^{-1/2} \quad \text{in } L^2(\Omega). \tag{4.2.8}$$

Thus

$$\rho\mathbf{u}|_{t=0} = \mathbf{m}_0^\varepsilon = \bar{\mathbf{m}}_0^\varepsilon + \rho_0^\varepsilon (\mathbf{u}_0^\varepsilon - \bar{\mathbf{u}}_0^\varepsilon),$$

and we can impose the initial condition of $\mathbf{u}$ as

$$\mathbf{u}|_{t=0} = \mathbf{u}_0^\varepsilon. \tag{4.2.9}$$

Finally, we impose the initial condition for f as

$$f|_{t=0} = f_0. \tag{4.2.10}$$

We now state and prove the following existence result.

Theorem 4.2.1. *With the above notations and assumptions, there exists a solution $(\rho, \mathbf{u}, f)$ of (4.2.1)-(4.2.4) with the initial conditions (4.2.5), (4.2.9) and (4.2.10), and the boundary conditions (1.3.9), such that $\rho \in C^\infty(\Omega \times [0, \infty))$, $\mathbf{u} \in C^\infty(\Omega \times [0, \infty))$ and $f \in L^\infty(\Omega \times \mathbb{R}^3 \times [0, \infty))$.*

Remark 4.2.1. Our approximation scheme is inspired by Lions' work on the density-dependent Navier-Stokes equations [39] and Hamdache's work on the Vlasov-Stokes equations [26].

Remark 4.2.2. If the initial data f_0 is smooth enough, we can show that the solutions are classical solutions. In fact, we can also show the uniqueness of such solutions.

Proof of Theorem 4.2.1. We define M as the convex set in

$$C([0,T] \times \Omega) \times L^2(0,T; H_0^1(\Omega))$$

by

$$\begin{aligned} M = \Big\{ &(\bar\rho, \bar{\mathbf{u}}) \in C([0,T] \times \Omega) \times L^2(0,T; H_0^1(\Omega)), \\ &\varepsilon \le \bar\rho \le C_0 \text{ in } [0,T] \times \Omega,\ \operatorname{div}\bar{\mathbf{u}} = 0 \quad \text{almost everywhere on } (0,T) \times \Omega, \\ &\|\bar{\mathbf{u}}\|_{L^2(0,T;H_0^1(\Omega))} \le K \Big\}, \end{aligned}$$

where $K > 0$ is to be determined. Here we define a map T from M into itself as

$$T(\bar\rho, \bar{\mathbf{u}}) = (\rho, \mathbf{u}).$$

As a first step, we consider the following initial-value problem:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\bar{\mathbf{u}}_\varepsilon \rho) = 0, \qquad \rho|_{t=0} = \rho_0^\varepsilon, \tag{4.2.11}$$

in $(0,T)\times\Omega$, where $\bar{\mathbf{u}}_\varepsilon = \bar{\mathbf{u}} * \theta_\varepsilon$. The construction of $\bar{\mathbf{u}}_\varepsilon$ implies that $\bar{\mathbf{u}}_\varepsilon \in L^2(0,T;C^\infty(\Omega))$, and $\mathrm{div}\bar{\mathbf{u}}_\varepsilon = 0$ in $(0,T)\times\Omega$. The solution of (4.2.11) can be written in terms of characteristics:

$$\frac{dX}{ds} = \bar{\mathbf{u}}_\varepsilon(X,s), \qquad X(x;x,t) = x, \quad x\in\Omega, \quad t\in[0,T]. \tag{4.2.12}$$

By the properties of $\bar{\mathbf{u}}_\varepsilon \in L^2(0,T;C^\infty(\Omega))$, and the basic theory of ordinary differential equations, we know that there exists a unique solution X of (4.2.12). Therefore, we have

$$\rho(t,x) = \rho_0^\varepsilon(X(0;t,x)), \quad \text{for all } t\in[0,T], \quad x\in\Omega.$$

It is clear that $\varepsilon \le \rho \le C_0$ in $[0,T]\times\Omega$. Since $\bar{\mathbf{u}}_\varepsilon \in L^2(0,T;C^\infty(\Omega))$, then $\rho(t,x)$ lies in $C([0,T];C^\infty(\Omega))$. By (4.2.11) and the properties of $\bar{\mathbf{u}}_\varepsilon$, we have $\frac{\partial\rho}{\partial t} \in L^2(0,T;C^\infty(\Omega))$. Thus, $\rho, \frac{\partial\rho}{\partial t}$ are bounded in these spaces uniformly in $(\bar{\rho},\bar{\mathbf{u}}) \in M$. In particular, by the Aubin-Lions lemma, the set of ρ built in this way is clearly compact in $C([0,T]\times\Omega)$.

The second step is to build $\mathbf{u}$ by solving the following problem:

$$\begin{aligned} &\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\bar{\mathbf{u}}_\varepsilon\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p + \rho\left(R_\delta\int_{\mathbb{R}^3} f\,d\mathbf{v}\right)\mathbf{u} = \rho\left(R_\delta\int_{\mathbb{R}^3}\mathbf{v}f\,d\mathbf{v}\right), \\ &\mathrm{div}\mathbf{u} = 0, \qquad \mathbf{u}|_{t=0} = \mathbf{u}_0^\varepsilon, \qquad \mathrm{div}\mathbf{u}_0^\varepsilon = 0, \end{aligned} \tag{4.2.13}$$

in$(0,T)\times\Omega$. Let

$$e = R_\delta\int_{\mathbb{R}^3} f\,d\mathbf{v} \ge 0, \quad g = R_\delta\int_{\mathbb{R}^3}\mathbf{v}f\,d\mathbf{v}.$$

Multiplying $\mathbf{u}$ on both sides of (4.2.13), one obtains the following energy equality related to (4.2.13):

$$\partial_t\int_\Omega\frac{1}{2}\rho|\mathbf{u}|^2\,dx + \int_\Omega|\nabla\mathbf{u}|^2\,dx + \int_\Omega e\rho|\mathbf{u}|^2\,dx = \int_\Omega\rho g\mathbf{u}\,dx.$$

The right-hand side of above energy equality is bounded by

$$\begin{aligned} \int_\Omega\rho g\mathbf{u}\,dx &\le \left(\int_\Omega\rho|g|^2\,dx\right)^{\frac{1}{2}}\cdot\left(\int_\Omega\rho|\mathbf{u}|^2\,dx\right)^{\frac{1}{2}} \\ &\le \|\rho_0\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|g\|_{L^2(\Omega)}\|\sqrt{\rho}\mathbf{u}\|_{L^2(\Omega)}. \end{aligned}$$

In conclusion, we obtain for all $t\in(0,T)$,

$$\begin{aligned} &\frac{1}{2}\int_\Omega\rho|\mathbf{u}|^2dx + \int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,dxdt + \int_0^t\int_\Omega e\rho|\mathbf{u}|^2\,dxdt \\ &\le C\int_0^t\int_\Omega|g|^2\,dxdt + C\int_0^t\int_\Omega\rho|\mathbf{u}|^2\,dxdt + \frac{1}{2}\int_\Omega\frac{|\mathbf{m}_0|^2}{\rho_0}\,dx. \end{aligned}$$

Applying Gronwall inequality, we obtain

$$
\begin{aligned}
&\sup_{t\in(0,T)} \|\rho|\mathbf{u}|^2\|_{L^1(\Omega)} \le C,\\
&\|\sqrt{e\rho}\mathbf{u}\|_{L^2(0,T;\Omega)} \le C, \qquad (4.2.14)\\
&\|\mathbf{u}\|_{L^2(0,T;H_0^1(\Omega))} \le C,
\end{aligned}
$$

where C denotes various constant which depend only on $T, \Omega, \varepsilon, \delta$ and bounds on $\|\rho_0\|_{L^\infty(\Omega)}$, $\|\rho_0|\mathbf{u}|^2\|_{L^1(\Omega)}$.

Rewriting (4.2.13) as follows

$$
\begin{aligned}
&c\frac{\partial \mathbf{u}}{\partial t} + b\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p + a\mathbf{u} = h,\\
&\text{div}\mathbf{u} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0^\varepsilon, \quad \text{div}\mathbf{u}_0^\varepsilon = 0,
\end{aligned} \qquad (4.2.15)
$$

in $(0,T)\times\Omega$, where

$$
\begin{aligned}
&c \in L^\infty((0,T)\times\Omega), \quad b\in L^2(0,T;L^\infty(\Omega)), \quad a\in L^\infty((0,T)\times\Omega),\\
&h\in L^\infty((0,T)\times\Omega), \quad c\ge\delta>0.
\end{aligned}
$$

To continue our proof, we need the following lemma:

Lemma 4.2.2. *There exists a unique solution* $\mathbf{u}$ *of* (4.2.15) *with the following regularity:*

$$
\mathbf{u}\in L^2(0,T;H^2(\Omega))\cap C([0,T];H_0^1(\Omega)); \quad \nabla p, \frac{\partial\mathbf{u}}{\partial t}\in L^2((0,T)\times\Omega). \qquad (4.2.16)
$$

Proof. First, we multiply (4.2.15) by $\frac{\partial\mathbf{u}}{\partial t}$ and use integration by parts over Ω to obtain:

$$
\begin{aligned}
&\delta\int_\Omega\left|\frac{\partial\mathbf{u}}{\partial t}\right|^2 dx + \frac{d}{dt}\int_\Omega\frac{1}{2}|\nabla\mathbf{u}|^2dx\\
&\le \int_\Omega\left(|h||\mathbf{u}_t| + |b||\nabla\mathbf{u}||\mathbf{u}_t| + |a||\mathbf{u}||\mathbf{u}_t|\right)dx
\end{aligned}
$$

Using the Cauchy-Schwarz inequality and embedding inequality, one deduces that

$$
\begin{aligned}
&\frac{\delta}{2}\int_\Omega\left|\frac{\partial\mathbf{u}}{\partial t}\right|^2 dx + \frac{d}{dt}\int_\Omega\frac{1}{2}|\nabla\mathbf{u}|^2dx\\
&\le C\left(1+\|b\|^2_{L^\infty(\Omega)} + \|a\|^2_{L^\infty(\Omega)}\lambda_0\right)\int_\Omega|\nabla\mathbf{u}|^2dx,
\end{aligned} \qquad (4.2.17)
$$

where λ_0 is a constant from the Sobloev inequality. By the regularity of a, b and Gronwall's inequality, we deduce that

$$\mathbf{u} \in L^\infty(0, T; H_0^1(\Omega)), \ \ \frac{\partial \mathbf{u}}{\partial t} \in L^2((0, T) \times \Omega).$$

We rewrite (4.2.15) as follows

$$\begin{aligned} & -\Delta \mathbf{u} + \nabla p = h - c\mathbf{u}_t - b \cdot \nabla \mathbf{u} - a\mathbf{u}, \\ & \text{div}\mathbf{u} = 0, \end{aligned} \tag{4.2.18}$$

in $\Omega \times (0, T)$, and $\mathbf{u} \in H_0^1(\Omega)$. Let $\widetilde{h} = h - c\mathbf{u}_t - b \cdot \nabla \mathbf{u} - a\mathbf{u}$, and

$$\widetilde{h} \in L^2(0, T; \Omega),$$

thus we have

$$\begin{aligned} & -\Delta \mathbf{u} + \nabla p = \widetilde{h}, \\ & \text{div}\mathbf{u} = 0. \end{aligned}$$

By the regularity of $\mathbf{u}$ and $\widetilde{h}$, we conclude that p is bounded in $L^2((0, T); H^{-1}(\Omega))$. We deduce that p is bounded in $L^2((0, T) \times \Omega)$ if we normalize p by imposing

$$\int_\Omega p \, dx = 0, \quad \text{almost everywhere } t \in (0, T).$$

To normalize p, we refer the readers to [39, 52] for more details. Also we conclude that $\mathbf{u}$ is bounded in $L^2(0, T; H^2(\Omega))$ by the classical regularity on Stokes equation. Thus, we proved the regularity of (4.2.16). The existence and uniqueness of (4.2.15) follows from the Lax-Milgram theorem, see for example [9]. □

By Lemma 4.2.2, there exists a unique solution to (4.2.13) with the regularity of (4.2.16). By the Aubin-Lions Lemma, $\mathbf{u}$ is compact in $L^2(0,T;H_0^1(\Omega))$. This, with the help of compactness of ρ in M, implies that the mapping T is compact in M.

To find the fixed point of map T by the Schauder theorem, it remains to find $K > 0$ such that

$$\|\mathbf{u}\|_{L^2(0,T;H_0^1(\Omega))} \leq K.$$

By (4.2.14), we have

$$\|\mathbf{u}\|_{L^2(0,T;H_0^1(\Omega))} \leq K',$$

this K' only depends on initial data. Thus, we can choose $K = K' + 1$.

Following the same argument of the proof of Lemma 4.2.2, we deduce that

$$\mathbf{u} \in L^p(0,T;W^{2,p}(\Omega)), \quad \frac{\partial \mathbf{u}}{\partial t} \in L^p((0,T)\times\Omega)$$

for all $1 < p < \infty$. With such regularity of $\mathbf{u}$, we can bootstrap and obtain more time regularity on $\mathbf{u}_\varepsilon$ and then on ρ and thus more regularity on $\mathbf{u}$.

In the third step, we would like to find the solutions to the following nonlinear Vlasov equation:

$$\begin{aligned} &\frac{\partial f}{\partial t} + \mathbf{v}\nabla_x f + \mathrm{div}_{\mathbf{v}}(R_\delta(\mathbf{u}-\mathbf{v})\rho f) = 0, \\ &f(0,x,\mathbf{v}) = f_0(x,\mathbf{v}), \;\; f(t,x,\mathbf{v}) = f(t,x,\mathbf{v}^*), \;\; \text{for } x \in \partial\Omega, \; \mathbf{v}\cdot\nu(x) < 0. \end{aligned} \tag{4.2.19}$$

where $\mathbf{u}, \rho$ are smooth functions obtained in step 2. The existence and uniqueness for the above Vlasov equation can be obtained as in [3, 12].

Thus we have proved Theorem 4.2.1. □

Remark 4.2.3. The solutions $(\rho, \mathbf{u}, f)$ obtained in Theorem 4.2.1 satisfy the following energy inequality

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2\,dx + \int_\Omega\int_{\mathbb{R}^3} \frac{1}{2} f(1+|\mathbf{v}|^2)\,dxd\mathbf{v}\right) \\ &+ \int_\Omega |\nabla\mathbf{u}|^2\,dx + \int_\Omega\int_{\mathbb{R}^3} R_\delta \rho f(\mathbf{u}-\mathbf{v})^2\,dxd\mathbf{v} \leq 0. \end{aligned} \tag{4.2.20}$$

The energy inequality will be crucial in deriving a prior estimates on the solutions $(\rho, \mathbf{u}, f)$ of the approximate system of equations.

4.2.2 Pass to the limit as $\varepsilon \to 0$.

The objective of this section is to recover the original system from the approximation scheme (4.2.1)-(4.2.4) upon letting ε goes to 0. Here and below, we denote by $(\rho^\varepsilon, \mathbf{u}^\varepsilon, f^\varepsilon)$ the solution constructed in Theorem 4.2.1.

We take $\beta \in C(\Omega, \mathbb{R}^3)$, use (4.2.1) and (4.2.3) to find that $\int_\Omega \beta(\rho^\varepsilon)dx$ is independent of time t, that is,

$$\int_\Omega \beta(\rho^\varepsilon)\, dx = \int_\Omega \beta(\rho_0^\varepsilon)\, dx \quad \text{for all } t \in (0, \infty). \tag{4.2.21}$$

Observing that $(\rho^\varepsilon, \mathbf{u}^\varepsilon, f^\varepsilon)$ satisfies (4.2.20), one obtains

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{1}{2}\rho^\varepsilon |\mathbf{u}^\varepsilon|^2\, dx + \int_\Omega\int_{\mathbb{R}^3} \frac{1}{2} f^\varepsilon(1+|\mathbf{v}|^2)\, dxd\mathbf{v}\right)\\
&+ \int_\Omega |\nabla \mathbf{u}^\varepsilon|^2\, dx + \int_\Omega\int_{\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon (\mathbf{u}^\varepsilon - \mathbf{v})^2\, dxd\mathbf{v} \le 0.
\end{aligned}$$

Integrating it from 0 to t, we have

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho^\varepsilon |\mathbf{u}^\varepsilon|^2\, dx + \int_\Omega\int_{\mathbb{R}^3} \frac{1}{2} f^\varepsilon(1+|\mathbf{v}|^2)\, dxd\mathbf{v}\\
&+ \int_0^t \int_\Omega |\nabla \mathbf{u}^\varepsilon|^2\, dxdt + \int_0^t\int_\Omega\int_{\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{u}^\varepsilon - \mathbf{v}|^2\, dxd\mathbf{v}dt\\
&\le \frac{1}{2}\int_\Omega \rho_0^\varepsilon |\mathbf{u}_0^\varepsilon|^2\, dx + \frac{1}{2}\int_\Omega\int_{\mathbb{R}^3} (1+|\mathbf{v}|^2) f_0\, dxd\mathbf{v}
\end{aligned} \tag{4.2.22}$$

for all $t > 0$. By (4.2.22), one obtains the following estimates:

$$\begin{aligned}
&\|\mathbf{u}^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le C,\\
&\sup_{0\le t\le T} \|\rho^\varepsilon |\mathbf{u}^\varepsilon|^2\|_{L^1(\Omega)} \le C,
\end{aligned} \tag{4.2.23}$$

where C denotes a generic positive constant independent of ε.

By (4.2.5) and (4.2.6), we assume that, up to the extraction of subsequences,

$$\rho^\varepsilon \to \rho \quad \text{in } C([0,T]; L^p(\Omega)) \text{ for any } 1 \le p < \infty. \tag{4.2.24}$$

We denote by $\mathbf{u}$ the weak limit of $\mathbf{u}^\varepsilon$ in $L^2(0,T;H_0^1(\Omega))$ due to (4.2.23). By the compactness of the embedding $L^p(\Omega) \hookrightarrow W_0^{-1,2}(\Omega)$ for any $p > 6/5$, one deduces from (4.2.24):

$$\rho^\varepsilon \to \rho \quad \text{in } C([0,T]; W_0^{-1,2}(\Omega)). \tag{4.2.25}$$

This, together with (4.2.23), yields

$$\rho^\varepsilon \mathbf{u}^\varepsilon \to \rho \mathbf{u} \quad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

Let a function $g \in C([0,T]; L^p(\Omega))$ for any $1 < p < \infty$ satisfy $g(0) = 0$ on Ω and

$$\frac{\partial g}{\partial t} + \operatorname{div}(g\mathbf{u}) = 0 \quad \text{in } \mathcal{D}'((0,T)\times\Omega),$$

then $g \equiv 0$, which implies the uniqueness of the density ρ when $\mathbf{u}$ is fixed. Thus we have proved that ρ is the solution to (1.3.1).

We now estimate $m_0 f^\varepsilon$:

$$\begin{aligned} m_0 f^\varepsilon &= \int_{\mathbb{R}^3} f^\varepsilon \, d\mathbf{v} = \int_{|\mathbf{v}|<r} f^\varepsilon \, d\mathbf{v} + \int_{|\mathbf{v}|\ge r} f^\varepsilon \, d\mathbf{v} \\ &\le C\|f^\varepsilon\|_{L^\infty} r^3 + \frac{1}{r^k}\int_{|\mathbf{v}|\ge r} |\mathbf{v}|^k f^\varepsilon \, d\mathbf{v} \end{aligned}$$

for all $k \ge 0$. Taking $r = (\int_{\mathbb{R}^3} |\mathbf{v}|^k f^\varepsilon d\mathbf{v})^{\frac{1}{k+3}}$, we have

$$m_0 f^\varepsilon \le C\|f^\varepsilon\|_{L^\infty}\left(\int_{\mathbb{R}^3} |\mathbf{v}|^k f^\varepsilon d\mathbf{v}\right)^{\frac{3}{k+3}} + \left(\int_{\mathbb{R}^3} |\mathbf{v}|^k f^\varepsilon d\mathbf{v}\right)^{\frac{3}{k+3}}.$$

Letting $k = 3$, then

$$\|m_0 f^\varepsilon\|_{L^2(\Omega)} \le C(\|f^\varepsilon\|_{L^\infty} + 1)\left(\int_\Omega\int_{\mathbb{R}^3} |\mathbf{v}|^3 f^\varepsilon d\mathbf{v}\right)^{1/2}.$$

Thanks to Lemma 4.1.1, we conclude that $m_3 f^\varepsilon$ is bounded in $L^\infty(0,T;L^1(\Omega))$. This yields

$$\|m_0 f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{4.2.26}$$

Following the same argument, one deduces that

$$\|m_1 f^\varepsilon\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))} \le C. \tag{4.2.27}$$

Using the fact $R_\delta \le 1$, we see that

$$\begin{aligned} &\|\rho R_\delta m_0 f^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \\ &\le C\|\rho_0\|_{L^\infty((0,T)\times\Omega)}\|m_0 f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\cdot\|\mathbf{u}^\varepsilon\|_{L^2(0,T;L^6(\Omega))}, \end{aligned} \tag{4.2.28}$$

and

$$\|\rho R_\delta m_1 f^\varepsilon\|_{L^\infty(0,T;L^{3/2}(\Omega))} \leq C\|\rho_0\|_{L^\infty((0,T)\times\Omega)}\|m_1 f^\varepsilon\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))}. \tag{4.2.29}$$

Observing

$$\rho^\varepsilon R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon d\mathbf{v} = \rho^\varepsilon R_\delta m_0 f^\varepsilon \mathbf{u}^\varepsilon - \rho R_\delta m_1 f^\varepsilon,$$

and using (4.2.28) and (4.2.29), we obtain that

$$\rho^\varepsilon R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon \, d\mathbf{v} \quad \text{is bounded in } L^2(0,T;L^{3/2}(\Omega)).$$

Since

$$\frac{\partial(\rho^\varepsilon \mathbf{u}^\varepsilon)}{\partial t} = -\mathrm{div}(\rho^\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}^\varepsilon) + \Delta \mathbf{u}^\varepsilon + \nabla p + \rho R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon d\mathbf{v},$$

and in particular, $\nabla \mathbf{u}^\varepsilon$ is bounded in $L^2((0,T)\times\Omega)$ and

$$\rho^\varepsilon R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v}) f^\varepsilon d\mathbf{v} \quad \text{is bounded in } L^2(0,T;L^{3/2}(\Omega))$$

while $\rho^\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}^\varepsilon$ is bounded in $L^2(0,T;L^{\frac{3}{2}}(\Omega))$, one obtains that

$$\frac{\partial(\rho^\varepsilon \mathbf{u}^\varepsilon)}{\partial t} \quad \text{is bounded in } L^2(0,T;H^{-1}(\Omega)).$$

By Theorem 2.4 of [39], we obtain

$$\sqrt{\rho^\varepsilon}\mathbf{u}^\varepsilon \to \sqrt{\rho}\mathbf{u} \quad \text{in } L^p(0,T;L^r(\Omega))$$

for $2 < p < \infty$ and $1 \leq r < \frac{6p}{3p-4}$, and thus

$$\rho^\varepsilon \mathbf{u}^\varepsilon \to \rho \mathbf{u} \quad \text{in } L^p(0,T;L^r(\Omega))$$

for the same values of p, r.

Applying Lemma 4.1.2 to (4.2.4), we obtain

$$m_0 f^\varepsilon \to m_0 f, \quad m_1 f^\varepsilon \to m_1 f \quad \text{for almost everywhere } (t,x). \tag{4.2.30}$$

By (4.2.26) and (4.2.27), the relation (4.2.30) can be strengthened to the following statements:

$$\begin{aligned} m_0 f^\varepsilon &\to m_0 f \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)), \\ m_1 f^\varepsilon &\to m_1 f \quad \text{strongly in } L^\infty(0,T;L^{3/2}(\Omega)). \end{aligned} \tag{4.2.31}$$

By (4.2.24), we have

$$\rho^\varepsilon m_0 f^\varepsilon \to \rho m_0 f \quad \text{strongly in } L^\infty(0,T; L^{\frac{2p}{2+p}}(\Omega)), \tag{4.2.32}$$

and

$$\rho^\varepsilon m_1 f^\varepsilon \to \rho m_1 f \quad \text{strongly in } L^\infty(0,T; L^{\frac{3p}{2p+3}}(\Omega)). \tag{4.2.33}$$

Thanks to (4.2.32)-(4.2.33) and the weak convergence of $\mathbf{u}^\varepsilon$ in $L^2(0,T;H_0^1(\Omega))$, one has

$$R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v})\rho^\varepsilon f^\varepsilon \, d\mathbf{v} \to R_\delta \int_{\mathbb{R}^3} (\mathbf{u} - \mathbf{v})\rho f \, d\mathbf{v} \quad \text{in the sense of distributions.} \tag{4.2.34}$$

The next step is to deal with the convergence of $\mathrm{div}_{\mathbf{v}}(R_\delta \rho^\varepsilon(\mathbf{u}^\varepsilon - \mathbf{v})f^\varepsilon)$. Let $\phi(\mathbf{v}) \in \mathcal{D}(\mathbb{R}^3)$ be a test function, we want to show

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\int_\Omega \int_{\mathbb{R}^3} (R_\delta \rho^\varepsilon(\mathbf{u}^\varepsilon - \mathbf{v})f^\varepsilon)\nabla_{\mathbf{v}}\phi \, d\mathbf{v}dx\right) = \\ &\int_\Omega \int_{\mathbb{R}^3} (R_\delta \rho(\mathbf{u} - \mathbf{v})f)\nabla_{\mathbf{v}}\phi \, d\mathbf{v}dx,\end{aligned} \tag{4.2.35}$$

which can be reached by (4.2.34).

We consider a test function $\varphi \in C_0^1([0,T]\times\Omega)$ such that $\mathrm{div}\varphi = 0$, and a test function $\phi \in C^1([0,T]\times\Omega\times\mathbb{R}^3)$ with compact support in $\mathbf{v}$, such that $\phi(T,\cdot,\cdot) = 0$. The weak formulation associated with the approximation scheme (4.2.1)-(4.2.4) should be

$$\begin{aligned}&-\int_\Omega \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \cdot \varphi(0,x)\, dx + \int_0^t \int_\Omega \{-\rho^\varepsilon \mathbf{u}^\varepsilon \cdot \partial_t \varphi - (\rho \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon)\cdot \nabla\varphi \\ &+ \nabla \mathbf{u}^\varepsilon \cdot \nabla\varphi + \varphi \cdot R_\delta \int_{\mathbb{R}^3} (\mathbf{u}^\varepsilon - \mathbf{v})\rho^\varepsilon f^\varepsilon d\mathbf{v}\}\, dxdt = 0;\end{aligned} \tag{4.2.36}$$

and

$$\begin{aligned}&-\int_0^T \int_\Omega \int_{\mathbb{R}^3} f^\varepsilon \left(\partial_t\phi + \mathbf{v}\cdot\nabla_x\phi + R_\delta(\mathbf{u}^\varepsilon - \mathbf{v})\rho^\varepsilon \cdot \nabla_{\mathbf{v}}\phi\right)\, dxd\mathbf{v}ds \\ &= \int_\Omega \int_{\mathbb{R}^3} f_0\phi(0,\cdot,\cdot)\, dxd\mathbf{v}.\end{aligned} \tag{4.2.37}$$

By (4.2.7)-(4.2.8), we have

$$\int_\Omega \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \cdot \varphi \, dx = \int_\Omega \mathbf{m}_0^\varepsilon \cdot \varphi dx \to \int_\Omega \mathbf{m}_0 \cdot \varphi \, dx \quad \text{ as } \varepsilon \to 0,$$

for all test functions φ.

All the above convergence results in this subsection allow us to recover (4.1.14)-(4.1.15) by passing to the limits in (4.2.36) and (4.2.37) as $\varepsilon \to 0$.

From (4.2.22), the solution $(\rho^\varepsilon, \mathbf{u}^\varepsilon, f^\varepsilon)$ satisfies the following:

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho^\varepsilon|\mathbf{u}^\varepsilon|^2\,dx + \int_\Omega\int_{\mathbb{R}^3}\frac{1}{2}f^\varepsilon(1+|\mathbf{v}|^2)\,dxd\mathbf{v}\\
&+\int_0^t\int_\Omega |\nabla\mathbf{u}^\varepsilon|^2\,dxdt + \int_0^t\int_\Omega\int_{\mathbb{R}^3} R_\delta\rho^\varepsilon f^\varepsilon|\mathbf{u}^\varepsilon-\mathbf{v}|^2\,dxd\mathbf{v}dt\\
&\le \frac{1}{2}\int_\Omega \rho_0^\varepsilon|\mathbf{u}_0^\varepsilon|^2\,dx + \frac{1}{2}\int_\Omega\int_{\mathbb{R}^3}(1+|\mathbf{v}|^2)f_0\,dxd\mathbf{v}.
\end{aligned}$$

The difficulty of passing the limit for the energy inequality is the convergence of the term $\int_0^t\int_{\Omega\times\mathbb{R}^3} R_\delta\rho^\varepsilon f^\varepsilon|\mathbf{u}^\varepsilon-\mathbf{v}|^2\,d\mathbf{v}dxdt$. We follow the same way as in [26, 57] to treat the term as follows

$$\begin{aligned}
&\int_0^T\int_{\Omega\times\mathbb{R}^3} R_\delta\rho^\varepsilon f^\varepsilon|\mathbf{u}^\varepsilon-\mathbf{v}|^2\,d\mathbf{v}\,dx\,dt\\
&\qquad = \int_0^T\int_{\Omega\times\mathbb{R}^3}\left(R_\delta\rho^\varepsilon f^\varepsilon|\mathbf{u}^\varepsilon|^2 - 2R_\delta\rho^\varepsilon f^\varepsilon\mathbf{u}^\varepsilon\mathbf{v} + R_\delta\rho^\varepsilon f^\varepsilon|\mathbf{v}|^2\right)\,dxd\mathbf{v}dt.
\end{aligned} \tag{4.2.38}$$

By the embedding inequality, we have

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{in } L^2(0,T;L^6(\Omega)). \tag{4.2.39}$$

By (4.2.32), (4.2.39), we deduce that

$$R_\delta\rho^\varepsilon m_0 f^\varepsilon|\mathbf{u}^\varepsilon|^2 \to R_\delta\rho m_0 f|\mathbf{u}|^2 \quad \text{weakly in } L^1(0,T;\Omega)$$

as $\varepsilon \to \infty$. Similarly,

$$R_\delta\rho^\varepsilon m_1 f^\varepsilon\mathbf{u}^\varepsilon \to R_\delta\rho m_1 f\mathbf{u} \quad \text{weakly in } L^1(0,T;\Omega)$$

as $\varepsilon \to 0$.

Finally, let us look at the terms:

$$\begin{aligned}
&\left|\int_0^t\int_\Omega\int_{\mathbb{R}^3} R_\delta\rho^\varepsilon f^\varepsilon|\mathbf{v}|^2\,d\mathbf{v}dxdt - \int_0^t\int_\Omega\int_{\mathbb{R}^3} R_\delta\rho f|\mathbf{v}|^2\,d\mathbf{v}dxdt\right|\\
&\le C\,|\rho^\varepsilon-\rho||_{L^\infty}\int_0^T\int_\Omega m_2 f^\varepsilon dxdt + C\|\rho\|_{L^\infty}\int_0^t\int_\Omega(m_2 f - m_2 f^\varepsilon)dxdt\\
&= I_1 + I_2.
\end{aligned}$$

It is clear to see that $I_1 \to 0$ as $\varepsilon \to 0$. For the term I_2, because

$$f^\varepsilon \rightharpoonup f \qquad \text{weak star in } L^\infty(0,T;L^p(\Omega\times\mathbb{R}^3))$$

for all $p \in (1,\infty]$ and $m_3 f^\varepsilon$ is bounded in $L^\infty(0,T;L^1(\Omega))$, then for any fixed $r > 0$, we have

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} f^\varepsilon |\mathbf{v}|^2 \, dx d\mathbf{v} dt = \int_0^T \int_{\Omega\times\mathbb{R}^3} \chi(|\mathbf{v}| < r)|\mathbf{v}|^2 f^\varepsilon \, dx d\mathbf{v} dt + O(\frac{1}{r})$$

uniformly in ε where χ is the characteristic function of the ball of $\mathbb{R}^3$ of radius r. Letting $\varepsilon \to 0$, then $r \to \infty$, we find

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} f^\varepsilon |\mathbf{v}|^2 \, dx d\mathbf{v} dt \to \int_0^T \int_{\Omega\times\mathbb{R}^3} f |\mathbf{v}|^2 \, dx d\mathbf{v} dt,$$

which means $I_2 \to 0$ as $\varepsilon \to 0$. Thus, we have proved

$$\int_0^T \int_{\Omega\times\mathbb{R}^3} R_\delta \rho^\varepsilon f^\varepsilon |\mathbf{u}^\varepsilon - \mathbf{v}|^2 \, d\mathbf{v} dx dt \to \int_0^T \int_{\Omega\times\mathbb{R}^3} R_\delta \rho f |\mathbf{u} - \mathbf{v}|^2 \, d\mathbf{v} dx dt \tag{4.2.40}$$

as $\varepsilon \to \infty$.

We observe that

$$\begin{aligned}
&\int_\Omega \rho_0^\varepsilon |\mathbf{u}_0^\varepsilon|^2 dx = \int_\Omega \frac{1}{\rho_0^\varepsilon} |\mathbf{m}_0^\varepsilon - \nabla q_0^\varepsilon|^2 \, dx \\
&= \int_\Omega \left(\frac{|\mathbf{m}_0^\varepsilon|^2}{\rho_0^\varepsilon} + \frac{|\nabla q_0^\varepsilon|^2}{\rho_0^\varepsilon} - \frac{2}{\rho_0^\varepsilon}(\rho_0^\varepsilon \mathbf{u}_0^\varepsilon + \nabla q_0^\varepsilon)\cdot \nabla q_0^\varepsilon \right) \, dx \\
&= \int_\Omega \left(\frac{|\mathbf{m}_0^\varepsilon|^2}{\rho_0^\varepsilon} - 2\mathbf{u}_0^\varepsilon \cdot \nabla q_0^\varepsilon - \frac{|\nabla q_0^\varepsilon|^2}{\rho_0^\varepsilon} \right) \, dx,
\end{aligned} \tag{4.2.41}$$

where we used Lemma 4.2.1.

Using $\operatorname{div}\mathbf{u}_0^\varepsilon = 0$, one obtains

$$\int_\Omega \rho_0^\varepsilon |\mathbf{u}_0^\varepsilon|^2 \, dx + \int_\Omega \frac{|\nabla q_0^\varepsilon|^2}{\rho_0^\varepsilon} dx = \int_\Omega \frac{|\mathbf{m}_0^\varepsilon|^2}{\rho_0^\varepsilon} \, dx. \tag{4.2.42}$$

Letting $\varepsilon \to 0$, using (4.2.8), (4.2.22), (4.2.40), (4.2.42), and the weak convergence of $(\rho^\varepsilon, \mathbf{u}^\varepsilon, f^\varepsilon)$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2 \, dx + \int_\Omega \int_{\mathbb{R}^3} \frac{1}{2} f(1+|\mathbf{v}|^2) \, dx d\mathbf{v} \\
&+ \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \, dx dt + \int_0^t \int_\Omega \int_{\mathbb{R}^3} R_\delta \rho f |\mathbf{u} - \mathbf{v}|^2 \, dx d\mathbf{v} dt \\
&\le \frac{1}{2} \int_\Omega \frac{|m_0|^2}{\rho_0} \, dx + \frac{1}{2} \int_\Omega \int_{\mathbb{R}^3} (1+|\mathbf{v}|^2) f_0 \, dx d\mathbf{v}.
\end{aligned}$$

So far, we have proved the following result:

Proposition 4.2.1. *For any $T > 0$, there is a weak solution $(\rho^\delta, \mathbf{u}^\delta, f^\delta)$ to the following system:*

$$\begin{aligned}
&\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0,\\
&\frac{\partial(\rho \mathbf{u})}{\partial t} + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p = \rho R_\delta \int_{\mathbb{R}^3} (\mathbf{u} - \mathbf{v}) f \, d\mathbf{v},\\
&\mathrm{div}\mathbf{u} = 0,\\
&\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathrm{div}_{\mathbf{v}}(R_\delta(\mathbf{u} - \mathbf{v})\rho f) = 0.
\end{aligned}$$

with the initial data $\mathbf{u}(0, x) = \mathbf{u}_0$ and $f(0, x, \mathbf{v}) = f_0(x, \mathbf{v})$, and boundary conditions

$$\begin{aligned}
&\mathbf{u}(t, x) = 0 \quad \text{on } \partial\Omega,\\
&f(t, x, \mathbf{v}) = f(t, x, \mathbf{v}^*) \quad \text{for } x \in \partial\Omega, \mathbf{v} \cdot \nu(x) < 0.
\end{aligned}$$

In additional, the solution satisfies the following energy inequality:

$$\begin{aligned}
&\int_\Omega \frac{1}{2} \rho |\mathbf{u}|^2 \, dx + \int_\Omega \int_{\mathbb{R}^3} \frac{1}{2} f(1 + |\mathbf{v}|^2) \, dx d\mathbf{v}\\
&+ \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \, dx dt + \int_0^t \int_\Omega \int_{\mathbb{R}^3} R_\delta \rho f |\mathbf{u} - \mathbf{v}|^2 \, dx d\mathbf{v} dt\\
&\leq \frac{1}{2} \int_\Omega \frac{|m_0|^2}{\rho_0} \, dx + \frac{1}{2} \int_\Omega \int_{\mathbb{R}^3} (1 + |\mathbf{v}|^2) f_0 \, dx d\mathbf{v}.
\end{aligned}$$

4.2.3 Pass the limit as $\delta \to 0$

The last step of showing the global weak solution is to pass the limit as δ goes to zero. First, we let $(\rho^\delta, f^\delta, \mathbf{u}^\delta)$ be a solution constructed by Proposition 4.2.1. It is easy to find that all estimates for $(\rho^\varepsilon, f^\varepsilon, \mathbf{u}^\varepsilon)$ still hold for $(\rho^\delta, f^\delta, \mathbf{u}^\delta)$, thus we can treat these terms as before.

It only remains to show the convergence of the terms

$$\int_{\mathbb{R}^3} R_\delta \rho^\delta f^\delta (\mathbf{u}^\delta - \mathbf{v}) \, d\mathbf{v}, \quad \text{and} \quad \mathrm{div}(R_\delta \rho^\delta (\mathbf{u}^\delta - \mathbf{v})).$$

The next step is to deal with the convergence of $\mathrm{div}_{\mathbf{v}}(R_\delta(\mathbf{u}^\delta - \mathbf{v})\rho^\delta f^\delta)$. Let $\phi(\mathbf{v}) \in \mathcal{D}(\mathbb{R}^3)$ to be a test function, we want to show

$$\begin{aligned}
&\lim_{\delta \to 0} \left(\int_\Omega R_\delta \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi d\mathbf{v} \right) - \int_\Omega \int_{\mathbb{R}^3} R_\delta \rho^\delta f^\delta \mathbf{v} \nabla_{\mathbf{v}} \phi \right)\\
&= \int_\Omega \rho \mathbf{u} \left(\int_{\mathbb{R}^3} f \nabla_{\mathbf{v}} \phi d\mathbf{v} \right) \, dx - \int_\Omega \int_{\mathbb{R}^3} \rho f \mathbf{v} \nabla_{\mathbf{v}} \phi \, d\mathbf{v} \, dx.
\end{aligned} \tag{4.2.43}$$

To prove (4.2.43), we introduce a new function $Q_\delta = 1 - R_\delta$ (see [26]), it is easy to see that

$$Q_\delta \to 0 \quad \text{as } \delta \to 0.$$

Writing

$$\begin{aligned}\int_\Omega R_\delta \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi \, d\mathbf{v} \right) dx = &\int_\Omega \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi \, d\mathbf{v} \right) dx \\ &- \int_\Omega Q_\delta \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi \, d\mathbf{v} \right) dx.\end{aligned} \tag{4.2.44}$$

On one hand, applying Lemma 4.1.2 to (4.2.4), we see that

$$\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi d\mathbf{v} \to \int_{\mathbb{R}^3} f \nabla_{\mathbf{v}} \phi d\mathbf{v} \quad \text{almost everywhere } (t, x). \tag{4.2.45}$$

It is easy to see

$$\left| \int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi d\mathbf{v} \right| \le C |m_0 f^\delta|. \tag{4.2.46}$$

This, combined with (4.2.31), strengthens (4.2.45) as follows:

$$\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi d\mathbf{v} \to \int_{\mathbb{R}^3} f \nabla_{\mathbf{v}} \phi d\mathbf{v} \quad \text{strongly in } L^\infty(0, T; L^2(\Omega)). \tag{4.2.47}$$

By the convergence of ρ^δ, (4.2.47) and the weak convergence of $\mathbf{u}^\delta$ in $L^2(0, T; H_0^1(\Omega))$, one deduces

$$\int_\Omega \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi \, d\mathbf{v} \right) dx \to \int_\Omega \rho \mathbf{u} \left(\int_{\mathbb{R}^3} f \nabla_{\mathbf{v}} \phi \, d\mathbf{v} \right) dx.$$

On the other hand,

$$\begin{aligned}&\left| \int_\Omega Q_\delta \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi \ d\mathbf{v} \right) dx \right| \\ &\le C \int_\Omega Q_\delta m_0 f |\mathbf{u}^\delta| \ dxdt \\ &\le C \|m_0 f^\delta\|_{L^\infty(0,T;L^2(\Omega))} \|\mathbf{u}^\delta\|_{L^2(0,T;L^6(\Omega))} \|Q_\delta\|_{L^2(0,T;L^3(\Omega))} \\ &\le C \|Q_\delta\|_{L^2(0,T;L^3(\Omega))},\end{aligned} \tag{4.2.48}$$

which yields

$$\left| \int_\Omega Q_\delta \rho^\delta \mathbf{u}^\delta \left(\int_{\mathbb{R}^3} f^\delta \nabla_{\mathbf{v}} \phi d\mathbf{v} \right) \right| \to 0 \quad \text{as } \delta \to 0,$$

where we used that $m_0 f^\delta$ is bounded in$L^\infty(0, T; L^2(\Omega))$ and $\mathbf{u}^\delta$ is bounded in $L^2(0, T; L^6(\Omega))$, and $Q_\delta \to 0$ strongly in $L^2(0, T; L^3(\Omega))$.

So we have proved the convergence of the first integral on the left of (4.2.43). We can treat similarly the convergence of the second integral of (4.2.43). Thus, we finish the proof of (4.2.43).

To complete the proof of Theorem 4.1.1, it only remains to check that $(\rho, \mathbf{u}, f)$ satisfies the energy inequality (4.1.16). In order to verify the energy inequality (4.1.16), we need to show

$$\int_0^t \int_\Omega \int_{\mathbb{R}^3} R_\delta \rho^\delta f^\delta |\mathbf{u}^\delta - \mathbf{v}|^2 \, dxd\mathbf{v}dt \to \int_0^t \int_\Omega \int_{\mathbb{R}^3} \rho f |\mathbf{u} - \mathbf{v}|^2 \, dxd\mathbf{v}dt \tag{4.2.49}$$

as $\delta \to 0$.

Denote

$$\begin{aligned} E^\delta &= \int_\Omega \int_{\mathbb{R}^3} \rho^\delta f^\delta |\mathbf{u}^\delta - \mathbf{v}|^2 \, d\mathbf{v}dx, \\ E^\delta &= E_1^\delta - 2E_2^\delta + E_3^\delta, \end{aligned}$$

where

$$\begin{aligned} E_1^\delta &= \int_\Omega \int_{\mathbb{R}^3} \rho^\delta f^\delta |\mathbf{u}^\delta|^2 \, d\mathbf{v}dx = \int_\Omega \rho^\delta m_0 f^\delta |\mathbf{u}^\delta|^2 \, dx, \\ E_2^\delta &= \int_\Omega \int_{\mathbb{R}^3} \rho^\delta f^\delta \mathbf{u}^\delta \mathbf{v} \, d\mathbf{v}dx = \int_\Omega \rho^\delta m_1 f^\delta \mathbf{u}^\delta \, dx, \end{aligned}$$

and

$$E_3^\delta = \int_\Omega \int_{\mathbb{R}^3} \rho^\delta f^\delta |\mathbf{v}|^2 \, d\mathbf{v}dx = \int_\Omega \rho^\delta m_2 f^\delta \, dx.$$

Write $R_\delta E^\delta = E^\delta - Q_\delta E^\delta$, we consider the convergence of E^δ first.

Since

$$\begin{aligned} &\left| \int_0^T \int_{\Omega\times\mathbb{R}^3} \rho^\delta f^\delta |\mathbf{u}^\delta|^2 \, d\mathbf{v}dxdt - \int_0^T \int_{\Omega\times\mathbb{R}^3} \rho f |\mathbf{u}|^2 \, d\mathbf{v}dxdt \right| \\ &\leq \int_0^T \int_\Omega \left(\rho^\delta - \rho\right) m_0 f^\delta |\mathbf{u}^\delta|^2 \, dxdt + \int_0^T \int_\Omega \rho (m_0 f^\delta - m_0 f) |\mathbf{u}^\delta|^2 \, dx\, dt \\ &+ \int_0^T \int_\Omega \rho m_0 f \left(|\mathbf{u}^\delta|^2 - |\mathbf{u}|^2\right) \, dxdt, \end{aligned}$$

then

$$\int_0^t E_1^\delta \, dt \to \int_0^t \int_\Omega \int_{\mathbb{R}^3} \rho f |\mathbf{u}|^2 d\mathbf{v}dxdt \qquad \text{as } \delta \to 0$$

for all $t > 0$. Similarly, we obtain

$$\int_0^t E_2^\delta \, dt \to \int_0^t \int_\Omega \int_{\mathbb{R}^3} \rho f \mathbf{u}\mathbf{v} \, d\mathbf{v}dxdt \qquad \text{as } \delta \to 0$$

for all $t > 0$.

Finally, let us examine

$$\begin{aligned}
&\left|\int_0^t\int_\Omega\int_{\mathbb{R}^3}\rho^\delta f^\delta|\mathbf{v}|^2\,d\mathbf{v}dxdt-\int_0^t\int_\Omega\int_{\mathbb{R}^3}\rho f|\mathbf{v}|^2\,d\mathbf{v}dxdt\right|\\
&\le\|\rho^\delta-\rho\|_{L^\infty}\int_0^T\int_\Omega m_2f^\delta dxdt+C\|\rho\|_{L^\infty}\int_0^t\int_\Omega(m_2f-m_2f^\delta)dxdt\\
&=I_1+I_2.
\end{aligned}$$

It is clear that $I_1\to 0$ as $\delta\to 0$. For the term I_2, because

$$f^\delta\rightharpoonup f\quad\text{weak star in } L^\infty(0,T;L^p(\Omega\times\mathbb{R}^3))$$

for all $p\in(1,\infty]$ and m_3f^δ is bounded in $L^\infty(0,T;L^1(\Omega))$, then for any fixed $r>0$, we have

$$\int_0^T\int_{\Omega\times\mathbb{R}^3}f^\delta|\mathbf{v}|^2\,dxd\mathbf{v}dt=\int_0^T\int_{\Omega\times\mathbb{R}^3}\chi(|\mathbf{v}|<r)|\mathbf{v}|^2f^\delta\,dxd\mathbf{v}dt+O(\frac{1}{r})$$

uniformly in δ where χ is the characteristic function of the ball of $\mathbb{R}^3$ of radius r. Letting $\delta\to 0$, then $r\to\infty$, we find

$$\int_0^T\int_{\Omega\times\mathbb{R}^3}f^\delta|\mathbf{v}|^2\,dxd\mathbf{v}dt\to\int_0^T\int_{\Omega\times\mathbb{R}^3}f|\mathbf{v}|^2\,dxd\mathbf{v}dt,$$

which means $I_2\to 0$ as $\delta\to 0$. Thus, we have proved

$$\int_0^t E^\delta dt\to\int_0^t\int_\Omega\int_{\mathbb{R}^3}\rho f|\mathbf{u}-\mathbf{v}|^2\,d\mathbf{v}dxdt\quad\text{ as }\delta\to 0.$$

In order to show (4.2.49), it remains to show that

$$\int_0^t Q_\delta E^\delta dt\to 0\quad\text{ as }\delta\to 0. \tag{4.2.50}$$

By the Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega Q_\delta\rho^\delta m_0f^\delta|\mathbf{u}^\delta|^2\,dxdt\\
&\le C\|Q_\delta\|_{L^2(0,T;L^6(\Omega))}\|m_0f^\delta\|_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}^\delta\|_{L^2(0,T;L^6(\Omega))}.
\end{aligned}$$

This, together with the definition of Q_δ, implies that

$$\int_0^t\int_\Omega Q_\delta\rho^\delta m_0f^\delta|\mathbf{u}^\delta|^2dxdt\to 0\quad\text{as }\delta\to 0$$

for all $t > 0$. Following the same argument, it is easy to see

$$\int_0^t \int_\Omega Q_\delta \rho^\delta m_1 f^\delta \mathbf{u}^\delta dxdt \to 0 \quad \text{as } \delta \to 0.$$

We write

$$\int_0^t \int_\Omega \int_{\mathbb{R}^3} Q_\delta \rho^\delta |\mathbf{v}|^2 f^\delta d\mathbf{v}dxdt = \int_0^t \int_\Omega \int_{|\mathbf{v}|\le r} Q_\delta \rho^\delta |\mathbf{v}|^2 f^\delta d\mathbf{v}dxdt + Q_\delta \frac{C}{r},$$
$$= \int_0^t \int_\Omega \int_{\mathbb{R}^3} \chi(|\mathbf{v}| < r) Q_\delta \rho^\delta |\mathbf{v}|^2 f^\delta d\mathbf{v}dxdt + \frac{C}{r} Q_\delta$$

uniformly in δ, where $\chi(x)$ is a characterized function. We have

$$\rho^\delta \to \rho \quad \text{in } C([0,T]; L^p(\Omega)) \text{ for any } 1 \le p < \infty,$$

and by the definition of Q^δ, we have

$$\chi(|\mathbf{v}| < r) Q_\delta \rho^\delta \to 0 \quad \text{strongly in } L^p(0,T; L^q(\Omega)) \quad \text{for any } 1 \le p, q < \infty.$$

It follows

$$\int_0^t \int_\Omega \int_{\mathbb{R}^3} Q_\delta \rho^\delta |\mathbf{v}|^2 f^\delta d\mathbf{v}dxdt \to 0$$

when letting $\delta \to 0$ and $r \to \infty$. Thus, we have proved that (4.2.50), and hence have proved (4.2.49).

Thanks to the convergence facts and the convexity of the energy inequality, we deduce (4.1.16) from energy inequality in Proposition 4.2.1.

The proof of Theorem 4.1.1 is complete.

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