WELL POSEDNESS AND PHYSICAL POSSIBILITY

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There is a sentiment shared among physicists that well posedness is a necessary condition for physical possibility. The arguments usually offered for well posedness have an epistemic flavor and thus they fall short of establishing the metaphysical claim that lack of well posedness implies physical impossibility. In this work we analyze the relationship of well posedness to prediction and confirmation as well as the notion of physical possibility and we devise three novel and independent argumentative strategies that may succeed where the usual epistemic arguments fail.

Keywords: determinism, laws of nature, metaphysics, philosophy of physics, physical possibility, prediction, well posed problem.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

PREFACE

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***

I offer this work to the memory of my mother.

1.0 INTRODUCTION

Si c'est ici le meilleur des mondes possibles,

que sont donc les autres?

(Voltaire: *Candide*)

Better understanding of nature requires better understanding of our theories of it, and better understanding of our theories requires better understanding of the properties of models which are rendered physically possible by them. To achieve this goal we need to probe oddly behaving possibilities besides well behaving ones: the study of properties of an odd model helps understanding a physical theory similarly to how the study of an odd counterexample helps understanding a mathematical theorem. No wonder many transitionary discussions between philosophy and physics are focusing on, what appears to be, unexpected and unintuitive properties of certain odd models of physical theories. Determinism defeating, supertask producing, time traveling, limiting behavior breaking scenarios are among the main targets of investigation by philosophers of physics.

Many of these oddly behaving examples share a common feature: they fail to be solutions of so-called well posed problems. A problem of mathematical physics is called well posed if its solution exists, if its solution is unique, and if its solution depends continuously on the data that is given. It is highly desirable that problems of a physical theory have these properties: Without existence there is no model; without uniqueness there is no determinism; without continuous dependence approximative methods, prediction, and confirmation may be at peril.

There is a sentiment shared widely across the physics community that well posedness is a necessary condition for physical possibility. In spite of being viewed as an important factor in understanding current physical theories as well as in guiding heuristic attempts to arrive at new theories very little if any defense is given by physicists to back this sentiment up. What we typically find are claims to the effect that failure of well posedness would make prediction impossible and impossibility of prediction would render a theory to be physically unviable.

There is an immediate problem with establishing physical impossibility on the basis predictive incapability: it rests on a conflation of ontological and epistemic interests. There is a gap between what there is (what there can be) and what can be known. A theory could still be true of the world even if its models had limited or no use for us. Or, in other words, the world does not need to care about the epistemic needs of its human observers. Hence we can not narrow the set of models deemed physically possible by a theory by a simple surgical removal of some of its predictively defective models. Pointing out the gap between the ontologically grounded and the epistemically accessible is the essence of what we call the *gap objection* against arguments for well posedness.

Familiar philosophical strategies which aim to reduce the ontological-epistemic gap may counter the gap objection and may entail a defense of well posedness. One can imagine that a verificationist-operationalist meaning postulate, combined with an analysis of the relationship between continuous dependence and verifiability, may rule out (some) solutions of non well posed problems as meaningless and leave behind only solutions of well posed problems as meaningful representations of physically possible scenarios. One can also imagine that arguments of Kantian flavor, taking off from the tenet that our physical theories can be confirmed and seeking the necessary conditions which make this confirmation possible, may rule well posedness to be a necessary condition for the possibility of knowledge and as such to be an indispensable foundation for the experienced phenomenal world.

Such philosophical strategies would either require comprehensive and quite restrictive accounts of how scientific statements acquire meaning, or inquiries into the structure and possibility of a priori knowledge and into how the world does after all care about its observers. Albeit these and such investigations may be indispensable in order to convincingly settle the matter it would be reassuring if less philosophically demanding considerations could also underpin the physicist sentiment. In this work we attempt to devise and evaluate such arguments in order to see more clearly what kind of epistemological and metaphysical assumptions may be needed or may be sufficient to defend well posedness.

To meet the gap objection we are going to take a closer look at the notion of physical possibility. A physical theory, broadly speaking, identifies two components in a representation of the world: a component which the theory proclaims to be fixed and a component which the theory may allow to vary. The modal character of the physical theory arises from associating the fixed component with the necessary and the variable component with the accidental. If the variable component is viewed as accidental then it could have been otherwise. The mathematical structure of the physical theory is suggestive of the space of mathematically admissible alternatives to the variable component, and we take these alternatives to represent the physical possibilities. It is in this sense we take solutions of a fundamental differential equation to represent physically possible scenarios: we represent arrangements of facts with a trajectory, notice that the trajectory is a solution of a differential equation, proclaim that the differential equation is the fixed component – the law – and that the solution is the variable component, and proceed to view other compatible variable components – other solutions of the same differential equation – as representations of other physical possibilities.

There are three assumptions behind our usual assessment of physical possibility: that (a) mathematical compatibility of the of the fixed and of the variable components is a sufficient condition for a plausible notion of physical possibility, that (b) differential equations are the appropriate mathematical representations of the fixed component, and that (c) solutions of a differential equation are the appropriate mathematical representations of the variable component. We are going to take a closer look at these assumptions and devise argumentative strategies along which the physicist sentiment that well posedness is necessary for physical possibility may become vindicated. The hope is that these strategies could avoid the gap objection as they do not operate by direct and ad hoc surgical removal of epistemically undesired possibilities.

The first strategy involves tinkering with the notion of physical possibility. We point out a difference between two readings of physical possibility: the first reading merely requires propositions expressing laws of the actual world to be true in the physically possible worlds while the second reading requires propositions expressing laws of the actual world to also be laws of the physically possible worlds. These two readings diverge if one accepts Humean supervenience of laws. In particular for a Best System account a proposition which achieves best balance between informativeness and simplicity in the actual world might fail to achieve such balance in a possible world in which the proposition is nevertheless true, and hence such possible worlds would not count as being physically possible as they wouldn't have the same laws as the actual world. We argue that this may indeed be the case if the proposition at hand is a differential equation whose initial value problems are well posed in the actual world but are not well posed in some other possible worlds; in these latter worlds other propositions achieve a better balance of informativeness and simplicity, hence these worlds are not physically possible according to the second reading.

The second strategy involves tinkering with the mathematical structure of the fixed component of physical theories. We argue that propagator equations can be more directly interpreted as laws than differential equations as long as the main intuition we associate with laws is that they 'evolve,' 'govern,' or 'bring about' physical states. Promoting propagator equations as fixed components comes with a price: well posedness of the corresponding differential equation is necessary for the existence of the propagator. However from the propagator-as-law perspective this restriction can be viewed as a precondition for an appropriate mathematical formulation of the physical theory rather than as a post-hoc condition restricting the number of possibilities.

The third strategy involves tinkering with the mathematical structure of the variable component of physical theories: we replace a solution with an alternative mathematical construction, the 'bolution,' as representation of a physically possible world (see the definition later). As a result the set of physical possibilities becomes a set of all bolutions instead of a set of all solutions. Epistemic concerns are present in the course of the motivation of the mathematical definition of the bolution but once the mathematical definition is given these epistemic concerns can be spared with and the focus can be shifted to the question whether bolutions can indeed serve the role of the variable component in the representation of those physical scenarios which form the basis of our generalization to all possibilities. It turns out that there is a direct relationship between epistemically desired solutions and bolutions and hence we can think of these desired solutions as representational short-hands for the bolutions to which they correspond. However no such relationship exists with undesired solutions and bolutions, hence if we indeed accept bolutions as the appropriate representations of the world these undesired solutions are left without any representational role. In the end this strategy also yields a narrower set of solutions as the set of short-hand representations of physically possible worlds, but without the unwarranted surgical removal procedure problematized by the gap objection.

We briefly overview the technical apparatus and discuss well posedness in Chapter 2 and in Appendix A. Chapter 3 discusses the notion of physical possibility and presents the third argumentative strategy mentioned above; Chapter 4 pursues the first and Chapter 5, supplemented by Appendix B, pursues the second strategy. As we will see results pertaining to the issue of well posedness are sensitive to technical subtleties. Our discussion frequently omits subtle details in order to present the main ideas; whether they survive when adapted to particular circumstances needs further investigation.

2.0 GROUNDWORK

2.1 WELL POSEDNESS: WHAT IT IS AND WHAT IT IS NOT

A mathematical problem used to model a physical process is called *well posed*, *properly posed*, or *correctly set* if it satisfies three conditions[1]:

(1) the solution of the problem *exists*,

(2) the solution is *unique*,

(3) the solution *depends continuously* on the given data.

Well posedness is widely held to be an essential feature of physical theories. Consider the following remarks of Mikhail M. Lavrentiev, Alan Rendall, and Robert M. Wald – leading experts in their respective fields of physics – intended as motivations for the continuous dependence condition:

> One should remember that the main goal of solving mathematical problems is to describe certain physical processes in mathematical terms. In this case the initial data are obtained experimentally; and since measurements cannot be absolutely precise, the data contain measurement errors. For a mathematical model to describe a real physical process, the problem should be supplemented with some additional requirements reflecting, in a physical sense, the fact that the solution should have only small variations under slight changes of initial data or, to put it conventionally, the *stability* of the solution under small perturbations in the data. (Lavrentiev et al.; 2003, p. 6)

> The condition of continuity is sometimes called Cauchy stability. The reason for including it is as follows. If PDE are to be applied to model phenomena in the natural world it

[1] See e.g. (Walter; 1998, p. 142).

must be remembered that measurements are never exact but always associated with some error. As a consequence it is impossible to know initial data for a problem exactly and so if solutions depend on the initial data in an uncontrollable way the model cannot make useful predictions. Cauchy stability guarantees that this does not happen and thus represents a necessary condition for the application of PDE to the real world. (Rendall; 2008, p. 134)

> If a theory can be formulated so that "appropriate initial data" may be specified (possibly subject to constraints) such that the subsequent dynamical evolution of the system is uniquely determined, we say that the theory possesses an initial value formulation. However, even if such a formulation exists, there remain further properties that a physically viable theory should satisfy. First, in an appropriate sense, "small changes" in initial data should produce only correspondingly "small changes" in the solution over any fixed compact region of spacetime. If this property were not satisfied, the theory would lose essentially all predictive power, since initial conditions can be measured only to a finite accuracy. It is generally assumed that the pathological behavior which would result from the failure of this property does not occur in physics. [...][2] (Wald; 1984, p. 224)

These remarks express a sentiment widely shared among physicists: well posedness is a necessary condition for models to describe real physical processes. Lack of well posedness would be pathological and it "does not occur in physics," at least not in describing forward time propagation of physical processes.

To get a better understanding of the physicists' sentiment we would ideally first nail down a precise definition of well posedness. There are many concepts that are left vague by the above characterization. What type of mathematical problems are we considering? What

[2] In the context of general relativity a further condition – that of finite propagation speed – is also sometimes understood to be part of the definition of well posedness. Thus Wald continues (ibid.):

> Second, changes in the initial data in a region, S, of the initial data surface should not produce any changes in the solution outside the causal future, $J^+(S)$, of this region. If such changes occurred, we should be able to use them to propagate signals "faster than the speed of light." This would undermine the entire framework of relativity theory. If a theory possesses an initial value formulation which satisfies both of the above properties, we say that this initial value formulation is *well posed*. Note, however, that we have not attempted to give a mathematically precise definition of "well posed initial value formulation" here since the precise criteria depend on the type of theory considered.

In what follows we do not include the assumption of finite propagation speed in our definition of well posedness.

solution concept are we working with? Do these solutions need to exists and be unique only locally or globally? What is the given data[3]? What do we mean by continuous dependence? Where do we get the topology necessary for the notion of continuity? Whether a certain mathematical problem end up being well posed may clearly depend on how we fix these and such concepts.

Much initial comfort can be drawn from starting with precise definitions as they appear to 'solve,' by fiat, many of the questions posed in the previous paragraph. Starting with definitions does not, however, do complete justice to the way physicists view the general requirement of well posedness. Continuing his remarks Robert Wald states that he did not attempt to give a mathematically precise definition "since the precise criteria depend on the type of theory considered" (ibid.). This suggests that the right motivation should come first and it may be expressed by different mathematical definitions in different theoretical contexts.

We should also acknowledge upfront that the problem setting itself sidesteps several major issues, notably an assumption that we can meaningfully differentiate two sources of knowledge: one that is directly connected to observation or measurement to provide us the 'given data,' and another that allows us to identify structural features of our modeling apparatus. It has proven to be useful to separate two stages in the process of developing a mathematical model adequate to represent a physical process: (1) identification of the general

[3]In what follows we use the term 'given data' as a terminus technicus referring to the quantities on which solutions are supposed to depend continuously according to assumption (3). Typically 'given data' is the data the supposed source of which is empirical observation and/or measurement. 'Given data' should not be immediately equated with the notion of 'initial value' of the theory of differential equations; the latter sometimes does not adequately characterize the empirical data 'that is given' to observers by measurements. For instance in classical mechanics the left-hand side of Newton's law $F = ma$ also incorporates crucial model parameters – i.e. the shape of surfaces on which balls are rolling – the values of which observers establish by measurement. This is a typical case whenever ordinary differential equations are employed and is recognized in the definition of well-posedness of initial value problems of ordinary differential equations (see Appendix A). We take 'state' to be a term of the vocabulary of a physical theory that is supposed to represent a 'configuration' of a 'physical system' (this latter being the entity that is represented by trajectories of states of a true theory).

type and structure of the model – the relevant mathematical framework, the utilized physical theories, differential equations, and general model parameters – and (2) determination of the numerical values of the model parameters. Given data, then, is what we may acquire via measurement during the second stage. This separation also allows us to introduce modal talk: we may regard the structural features obtained in the first stage as fixed components of physical descriptions and ask what are the possible ways the world may be given these fixed components. It is unclear to me how artificial any such separation in the process of our knowledge acquisition may ultimately be, but for our present purposes we assume that the familiar way of drawing a dividing line by fixing a certain well established physical theory is apt.

Some initial clarification is needed regarding the concept of continuous dependence. In line with the separation of the two stages in the modeling process some modern practice- and inverse problem-oriented textbooks[4] characterize well posedness as follows. Many problems in mathematical physics[5] can be cast in the following general form:

$$Ow = d, \tag{2.1}$$

where $w \in W$, $d \in D$, and $O : W \to D$ is an operator. In the context of differential equations W is the set of solutions of a differential equation, D is a space of functions involved in the initial and/or boundary conditions, and O is defined by the differential equation together with some additional conditions. The problem is to find the solution w given O and d, i.e. to find a solution given certain data[6].

The authors offer the following definition of well posedness.

[4]See e.g. Lavrentiev et al. (2003) and other books in the Inverse and Ill-Posed Problems Series of de Gruyter.

[5](Temirbolat; 2003, p. 1) goes as far to say that "Every problem of mathematical physics."

[6]Typically only d is regarded as given data, although there are exceptions. Some authors regard O to represent given data as well and require that continuous dependence (see below) also holds for O in the operator norm. See (Tikhonov and Arsenin; 1977, p. 6), (Kabanikhin; 2011, p. 5); also contrast with perturbation result in Appendix A.

Definition 1 (Well posed problem, initial attempt)**.** *Assume that D and W are normed spaces*[7]*. The solution of problem (2.1)*

(1a) exists *if for all $d \in D$ there is an element $w \in W$ such that (2.1) holds,*

(2a) is unique *if for all $d \in D$ there is at most one $w \in W$ such that (2.1) holds,*

(3a) is stable under small variations of the right side d *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every data $\bar{d} \in D$ satisfying*

$$\|d - \bar{d}\|_D \leq \delta \tag{2.2}$$

the inequality

$$\|w - \bar{w}\|_W \leq \epsilon \tag{2.3}$$

holds for every $Ow = d$ and $O\bar{w} = \bar{d}$.

If conditions (1a)-(3a) are satisfied for all $d \in D$ then the problem (2.1) is well posed*. If one of these conditions fail then the problem is* ill posed.

Simply put, the problem (2.1) is well posed if solutions depend continuously on the given data to which they correspond; it is not assumed that solutions or that given data have any particular structure aside that a notion of distance between different solutions and different given data is available. On the upside Definition 1 is both general and elegant: it rests on an abstract separation of structural features and given data and refers to solutions and continuity as basic concepts. By attaching to it the label 'classical' the authors mean to signal that the concept of well posedness was developed by Hadamard (Lavrentiev et al.; 2003, p. 7). Attributing this particular sense of well posedness to him does not do justice to the horror Hadamard might have felt discovering his well known counterexample (see later) as Definition 1 blurs an important distinction. The case when solutions do not depend

[7]Slightly more general definitions can be based on metric and/or on topological spaces, but for a more manageable discussion we assume that the metric and the topology is induced by a norm. In the context of differential equations used in physics this is not a serious loss of generality.

continuously on the given data is distinct and should be differentiated from the case of chaotic systems whose solutions depend continuously but sensitively on the given data. As the definition above requires continuity in the normed space of entire solutions – which are for dynamical systems also functions of a time parameter – continuous dependence in this sense also fails if the trajectories for nearby initial values diverge. Hence the above definition renders chaotic systems to be ill posed; no wonder that in this sense we may find many physically realistic systems countering the dictum that only well posed problems reflect real phenomena.[8] For numerical approximation chaotic behavior may pose a challenge which could only be overcome only by utilizing further assumptions, but this challenge is different from the one giving physicists like Hadamard a pause.

It is useful to recall Hadamard's example of to illustrate the failure of continuous dependence we want to capture. Consider the Laplace equation in two dimensions:

$$u_{xx} + u_{yy} = 0 \tag{2.4}$$

in the square $\Omega = \{(x, y);\ 0 < x, y < \pi\}$. If the data given for u is

$$u(x, 0) = 0, \quad u(0, y) = u(\pi, y) = 0 \tag{2.5}$$

$$u_y(x, 0) = 0 \tag{2.6}$$

then the problem (2.4)-(2.6) has a unique solution:

$$u(x, y) = 0. \tag{2.7}$$

If we slightly change the given data by replacing the condition (2.6) with

$$u_y(x, 0) = \frac{1}{n}\sin(nx), \tag{2.8}$$

[8]We assume that the norm on the space of solutions is given in the way it is standardly done. With tricky ways of inducing the norms one can trivialize the problem of continuous dependence. I.e. if O is one-to-one, solutions of the equation $Ow = d$ are stable with respect to the normed space with new norm $\|w\|_{W^{new}} \doteq \|Ow\|_D$.

n being a free parameter, the solution of the problem (2.4)-(2.5), (2.8) is again unique:

$$u(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny). \tag{2.9}$$

Note that, due to the exponential increase in the $\sinh(ny)$ term, we can make the new solution (2.9) differ from the old solution (2.7) to an arbitrarily large extent if we choose our parameter n large enough (measuring distances in the natural supremum norm). However, as we increase n, the new given data (2.8) gets closer and closer to the old given data (2.6). And this means that the solution of the problem does not depend continuously on the given data: in any small neighborhood of the old data and in any short span of time we find new data such that the corresponding new solution deviates from the old to an arbitrarily large extent even within that short span of time.

This type of failure of continuous dependence motivated the original distinction between well and ill posed problems (or correctly and incorrectly set problems in Hadamard's terminology). When we adopt our problem setting to handle this distinction suddenly a plethora of options for fixing the notion of continuous dependence surface. To get a sense of the cornucopia consider again the problem

$$Ow = d, \tag{2.10}$$

where $w \in W$, $d \in D$, and $O : W \to D$ is an operator. We now also assume solutions to take the form of a mapping of states to a time parameter: every $w \in W$ is a function $w : T \to S$, where $T \in \hat{\mathcal{T}} \subseteq \mathcal{T}$[9] is potentially different for different solutions. For the Cauchy problem the normed 'state space' S and the 'data space' D are typically the same with the same norm, however there are other problems for which the given data does not have the same

[9] $\mathcal{T}$ is the set of all time intervals (of the form (a,b), $[a,b)$, $(a,b]$, or $[a,b]$, $b=\infty$ or $a=-\infty$ allowed) in $\mathbb{R}$; for simplicity we assume that $0 \in T$ for all $T \in \mathcal{T}$. Let $\mathcal{T}^+ \subset \mathcal{T}$ be the set of all nonnegative $[0,b)$ or $[0,b]$ intervals ($b>0$, $b=\infty$ allowed), and let $\mathcal{T}^-$ be similarly the set of all nonpositive intervals. Also, let $\mathcal{T}_f \subseteq \mathcal{T}$ denote the set of all finite time intervals, $\mathcal{T}_f^+$ the set of nonnegative, and $\mathcal{T}_f^-$ the set of nonpositive finite intervals.

structure as do the states. For simplicity let's assume that solutions are maximally extended in the sense that if $w : T \to S$, $\bar{w} : \bar{T} \to S$, $\bar{T} \subseteq T$, and $w_{|\bar{T}} = \bar{w}$ then from $w, \bar{w} \in W$ it follows that $T = \bar{T}$.

Definition 2 (Well posed problem in d)**.** *Assume that D and S are normed spaces. Let $d \in D$ be a given data. The solution of the problem (2.10)*

(1b) exists at d *if there is an element $w \in W$ such that (2.10) holds,*

(2b) is unique at d *if there is at most one $w \in W$ such that (2.10) holds,*

(3b) depends continuously on the right side d *if*

$$\forall \epsilon > 0 : \ \exists \delta > 0, \ \exists U \subseteq \mathcal{T} \tag{2.11}$$

such that for every data $\bar{d} \in D$ satisfying

$$\|d - \bar{d}\|_D \leq \delta \tag{2.12}$$

the inequality

$$\|w(t) - \bar{w}(t)\|_W \leq \epsilon \tag{2.13}$$

is meaningful and holds for every $Ow = d$ and $O\bar{w} = \bar{d}$ and for all $t \in U$.

If conditions (1b)-(3b) are satisfied then the problem (2.10) is well posed in d (in sense (b)).

According to Definition 2 for every allowed margin of deviation there is a short enough time interval and small enough vicinity of the given data such that any other solution belonging to data within this small vicinity will not deviate, throughout the short time interval, from the original solution belonging to the given data more than the allowed margin of deviation. In this level of generality the definition is permissive regarding existence: initial value problems whose solutions only exist locally may also be well posed.[10] It is frequent, however, to require $\hat{\mathcal{T}} = \mathbb{R}$, meaning that solutions should exists globally.

There are other choices one could make regarding the order and type of quantifiers in (2.11) while still ending up with a version of continuous dependence.

$$\forall U \subseteq \mathcal{T}_f^+,\ \forall \epsilon > 0 :\ \exists \delta > 0 \tag{2.14}$$

would, for instance, require that the deviation of the solutions for $t > 0$ can be arbitrarily constrained throughout any finite time interval by allowing for a sufficiently small vicinity of the given data. Condition (2.14) is stronger than (2.11) in the sense that well posedness of a problem with condition (2.14) implies well posedness with condition (2.11) but the converse is not true. Changing $\mathcal{T}_f^+$ to $\mathcal{T}_f$ or to $\mathcal{T}$ in (2.14) produces stronger and stronger notions of well posedness; the last would actually produce a definition which, after appropriate identifications, is equivalent with Definition 1 of well posedness we introduced before.

The arrangement of quantifiers gets especially messy when we extend the notion of well posedness in a given data d to well posedness in general. Besides the freedom of quantification arrangement we get the additional issue whether existence of solutions should be required for all possible data or for just some (typically dense) subset of them. We showcase the options using the following general definition:

Definition 3 (Well posed problem in $\hat{D}$)**.** *Assume that D and S are normed spaces. Let $\hat{D} \subseteq D$ be a set of data. The solution of the problem (2.10)*

[10] In the sense of Definition 2 Hadamard's example is not well posed but we conjecture that Xia (1992)'s example for non-collision singularity is. Using the notation and results of Xia (1992) the sketch of the argument is as follows. Take an initial condition $x \in \Lambda_0$ of the main Theorem 1.2, $x = (x_q, x_p)$ with x_q being the initial position coordinates. $x_q \notin \Delta$ and there exists a compact ball B_{x_q} of radius r_{x_q} around x_q not intersecting Δ. Let B_x be the (higher dimensional) compact ball centered on x with the same radius r_{x_q}. If for a $y \in B_x$ we denote with σ_y the maximal time until there exists a unique solution of the initial value problem (according to Theorem 0.1) then due to the compactness of B_x we have $\rho \doteq \inf_{y \in B_x}\{\sigma_y\} > 0$. Thus within a small vicinity B_x of an initial condition x of a Xia scenario all solutions exists and are unique at least for a finite time $\rho/2$; that continuous dependence holds should then follow from the fact that throughout $\rho/2$ solutions originating from initial conditions B_x stay within a compact region.

(1c) exists in $\hat{D}$ *if for all* $d \in \hat{D}$ *there is an element* $w \in W$ *such that (2.10) holds,*

(2c) is unique in $\hat{D}$ *if for all* $d \in \hat{D}$ *there is at most one* $w \in W$ *such that (2.10) holds,*

(3c) depends continuously on the right side d *if*

$$\forall d \in \hat{D},\ \forall \epsilon > 0 :\ \exists \delta > 0,\ \exists U \subseteq \mathcal{T} \tag{2.15}$$

such that for every data $\bar{d} \in D$ *satisfying*

$$\|d - \bar{d}\|_D \leq \delta \tag{2.16}$$

the inequality

$$\|w(t) - \bar{w}(t)\|_W \leq \epsilon \tag{2.17}$$

is meaningful and holds for every $Ow = d$ *and* $O\bar{w} = \bar{d}$ *and for all* $t \in U$*.*

If conditions (1c)-(3c) are satisfied then the problem (2.10) is well posed in $\hat{D}$*.*

Table 1 collects a garden variety of definitions we get by changing condition (2.15) in a way that still captures the idea of continuous dependence of solutions on initial data. It is customary to differentiate between continuous dependence of solutions in the future and in the past directions, as solutions of some initial value problems in physics – such as the initial value problem for the heat equation – obey continuous dependence in one but not in the other direction. Pepping up Table 1 with the direction-dependent variants yields an overwhelming list of 36 different definitions of a well posed problem. Allow for *a priori* assumptions on properties of solutions and correspondingly restrict the data space to data that yields a solution satisfying these *a priori* assumptions – which is the notion of conditional well posedness à la Tikhonov to be discussed later – and your count is bumped up to 72!

Among the possibilities listed in Table 1 the weakest condition (1) requires that for each member of a set of possible given data there is a small neighborhood and there is a short time interval so that solutions corresponding to the small neighborhood exist throughout

Table 1: Some versions of continuous dependence.

Alternatives to condition (2.15)	Remark
(1) $\forall d \in \hat{D},\ \forall \epsilon > 0:\ \exists \delta > 0,\ \exists U \subseteq \mathcal{T}$	The original condition (2.15).
(2) $\forall d \in \hat{D}:\ \exists U \subseteq \mathcal{T}:\ \forall \epsilon > 0:\ \exists \delta > 0$	(2) $\to$ (1)
(3) $\exists U \subseteq \mathcal{T}:\ \forall d \in \hat{D},\ \forall \epsilon > 0:\ \exists \delta > 0$	(3) $\to$ (2)
(4) $\forall U \subseteq \mathcal{T}_f:\ \forall d \in \hat{D}:\ \forall \epsilon > 0:\ \exists \delta > 0$	(4) $\to$ (3) Most faithful* to Hadamard's notion.
(5) $\forall d \in \hat{D}:\ \forall \epsilon > 0:\ \exists \delta > 0:\ \forall U \subseteq \mathcal{T}_f$	(5) $\to$ (4), (5) $\equiv$ (7)
(6) $\forall U \subseteq \mathcal{T}:\ \forall d \in \hat{D}:\ \forall \epsilon > 0:\ \exists \delta > 0$	(6) $\to$ (4)
(7) $\forall d \in \hat{D}:\ \forall \epsilon > 0:\ \exists \delta > 0:\ \forall U \subseteq \mathcal{T}$	(7) $\to$ (6), (7) $\equiv$ (5) Corresponds* to Definition 1.
(8) $\forall \epsilon > 0:\ \exists \delta > 0 : \forall d \in \hat{D}:\ \exists U \subseteq \mathcal{T}$	(8) $\to$ (1)
(9) $\forall \epsilon > 0:\ \exists \delta > 0,\ \exists U \subseteq \mathcal{T}:\ \forall d \in \hat{D}$	(9) $\to$ (8)
(10) $\exists U \subseteq \mathcal{T}:\ \forall \epsilon > 0:\ \exists \delta > 0:\ \forall d \in \hat{D}$	(10) $\to$ (9)
(11) $\forall U \subseteq \mathcal{T}_f, \forall \epsilon > 0:\ \exists \delta > 0:\ \forall d \in \hat{D}$	(11) $\to$ (10) Corresponds* to Definition 4.
(12) $\forall \epsilon > 0:\ \exists \delta > 0:\ \forall U \subseteq \mathcal{T}_f, \forall d \in \hat{D}$	(12) $\to$ (11), (12) $\to$ (5), (12) $\equiv$ (14)
(13) $\forall U \subseteq \mathcal{T}, \forall \epsilon > 0:\ \exists \delta > 0:\ \forall d \in \hat{D}$	(13) $\to$ (11)
(14) $\forall \epsilon > 0:\ \exists \delta > 0:\ \forall U \subseteq \mathcal{T}, \forall d \in \hat{D}$	(14) $\to$ (13), (14) $\to$ (7), (14) $\equiv$ (12)

$*$: after appropriate identifications.

this time interval and do not deviate more than a preset value; the neighborhood and the interval may be different for different given data. The strongest condition (14) requires that solutions corresponding to a small neighborhood of data exist for all time and do not deviate more than a preset value, irrespective of the given data on which this small neighborhood is centered. Hadamard's example violates all of these conditions, however his own notion of well posedness most likely corresponds to condition (4)[11]. Probably the most studied are implementations of condition (7) and (11).

Given such an abundance of possible conditions of continuous dependence one could wonder which of these is supposedly necessary for a "physically viable theory." The strongest? The weakest? The one for which we can most straightforwardly offer an epistemic justification i.e. in terms of predictive use? The one which turns out to be mathematically most fruitful? Should there be a single such condition or may the condition vary case by case?

As with religions the mere abundance of possibilities should compel us to take a step back and to try to first get a grasp of the motivations working in the background. Let us now turn to advantages well posedness of some sort supposedly yields, keeping in mind the question whether these advantages could indeed be strong enough to elevate well posedness to a "necessary condition on physical viability / possibility."

2.2 WELL POSEDNESS AND PREDICTION

The story linking well posedness with predictions goes roughly as follows. First, the solution of the mathematical problem describing the physical scenario needs to exists, since without an existing solution there is no prediction we can talk of. Second, the solution needs to be unique, since without a unique solution we do not have a definite prediction, but many

[11] Based the reconstruction (Fattorini; 1983, p. 55-57).

different predictions, and we don't know how to choose among them. Third, the data informing our model comes from measurement, but as measurement precision of physical quantities is finite, the data can always contain some error. The solution needs to depend continuously on the given data, otherwise even the smallest difference in data would yield vastly different solutions and would thus render prediction impossible.

In what follows we are going to pick this story apart, arguing that well posedness is neither sufficient nor necessary to allow for predictions.

2.2.1 Well posedness: not sufficient for prediction

Is well posedness sufficient to allow for predictions? Prediction is an epistemic concept and hence this question can only be answered relative to a set of assumptions about the capabilities of the agents who carry out the predictive tasks. What kind of data do these agents have access to? What kind of limitations do they face in gathering the data? What do we assume about their computational, and in general, mathematical abilities? Only after fixing these and such parameters regarding the capabilities of the agents can we hope to settle the issue of sufficiency. As only imagination bounds the limitations one may impose and here we resort only to a couple of remarks.

We should keep in mind that the notions of *determinism* and *prediction* are distinct. Determinism is a metaphysical concept which, in the context of physical theories, is linked to the issue of uniqueness of solutions. Whether determinism holds – whether a solution is unique given certain data[12] – is independent of the knowledge of the observers of this fact. Prediction on the other hand does depend on whether the observer may access the data on

[12] In a wider sense determinism holds if the mathematical problem determines the solution given certain data, that is if the problem has a unique solution. In philosophy determinism typically signifies a narrower concept – Laplacian determinism – which further requires the given data to be a specification of an instananeous state (or data on a spacelike hypersurface). For an extensive discussion on the status of determinism in physics see Earman (1986), Earman (2007) and their references.

the basis of which she is supposed to predict. Hence even if the problem is well posed given certain data prediction may be rendered impossible by the inaccessibility of this data.

Inaccessibility of data may come in various forms. Physical theories that impose limits on the speed of causal signal propagations in turn impose limits on the extension of observational past of the observers. Such are the theories of special and general relativity; even though determinism prevails in spacetimes that admit a Cauchy surface[13], this Cauchy surface might not be contained in the causal past of the observer[14]. Even in cases when there is a Cauchy surface in the causal past of the observer the observer might not be able to know that there is one: If we require observers to have the resources to know whether they are able to carry out predictions then prediction outside the boundary of one's observational past is not possible in general relativity[15].

There are other types of constraints on accessibility to data implied by the theories within which the mathematical problems are formulated. The most straightforward example is that of quantum mechanics. Consider the interpretation which portrays the quantum world most alike to the familiar dynamical theories, that of Bohm and de Broglie[16]. The time evolution of Bohmian mechanics *is* deterministic: given the guiding equation, the initial wave function

[13] In the context of spacetime theories the *past domain of dependence* $D^-(S)$ of an achronal surface S is the set of spacetime points p such that every future inextendible causal curve through p intersects S; the *causal past* $J^-(q)$ of a spacetime point q is the set of spacetime points p such that there exists a past directed causal curve from q to p; the *chronological past* $I^-(q)$ of q is the set of spacetime points p such that there exists a past directed timelike curve from q to p. The *future domain of dependence* $D^+(S)$ of an achronal surface S, the *causal future* $J^+(q)$ and the *chronological future* $I^+(q)$ of a spacetime point q is defined analogously. The *(total) domain of dependence* $D(S)$ of achronal surface S is the set $D^-(S) \cup D^+(S)$. A closed achronal set S whose total domain of dependence is the entire manifold is called a *Cauchy surface*. See (Wald; 1984, p. 190, pp. 200-201).

[14] In the case of special relativity the Minkowski spacetime $(\mathbb{R}^4, \eta)$ does admit a Cauchy surface; however there is no spacetime point which would contain a Cauchy surface within its causal past. Hence prediction is not possible for the entire spacetime manifold. Moreover observers can't even make local predictions: it is easy to see that for any spacetime point p and any achronal surface S, if $S \subset J^-(q)$ then $D(S) \subset J^-(q)$; see (Earman; 1995, p. 128).

[15] For definitions and results see Manchak (2008).

[16] An accessible introduction to Bohmian mechanics and its philosophical problems is Albert (1992).

and the initial positions of the particles[17] the quantum system has a unique time evolution. The theory however also implies that observers who do not know the initial positions of the particles are unable to reveal them via measurement. Observers can only learn about the evolution of the wave function and about the probabilities of measurement outcomes, but not about the unique time evolution which brings about the measurement outcomes[18].

Besides theory-driven constraints on accessibility to data observers may face pragmatic constraints as well. For instance it seems quite reasonable to assume that only finite number of data can be gathered by an observer. Initial values for a partial differential equation typically consist of infinitely many data points along a surface, i.e. for the heat equation the temperature values at each point, say, along the length of a rod. Solutions may get uniquely determined by these infinitely many data points; they would not, however, be uniquely determined by only finitely many of them. Hence problems which could be well posed for given data consisting of infinitely many data points would not be well posed for given data consisting of only finitely many data points. It seems then that, even though many physically relevant problems are well posed when initial values of the usual sort are given, such well posedness would not be sufficient to allow prediction for observers who are limited to collecting only finitely many data points.

Continuous dependence may come to the rescue. The imposition that observers have finite data collection capability does not necessitate that all data collected by them must be of the same sort, i.e. that all collected data should be an individual temperature reading at a certain place of a rod. Observers may have other sorts of devices with which they can measure aggregate information about an initial state, such as how far this state lies from an appropriately chosen reference state. If observers can perform a finite set of operations which would directly reveal, say, the norm of the measured initial state, then they can carry

[17]For uniqueness results in Bohmian mechanics see Berndl et al. (1995).

[18]See chapter 7 in Albert (1992).

out predictions in case continuous dependence of solutions on the given data holds for the said norm. Continuous dependence guarantees that for sufficiently close initial states the solutions do not deviate more than a desired amount, and hence if the norm of the initial state can be measured with sufficiently small imprecision then the approximate behavior of the (norm of the) solution can be predicted.

Hence assuming that observers can collect only finitely many data does not necessarily threaten prediction as long as a norm which makes the problem well posed is *approximately measurable* (if for any state there is a finite measurement procedure which, up to some error, yields the value of the norm of the state). Whether such connection between the mathematical apparatus and possible measurement procedures exists is crucial in order to appreciate whether a mathematical result regarding the well posedness of a problem has any relevance to the issue of prediction. If the problem is well posed only with norms which are not approximately measurable then well posedness of this sort would not allow prediction to observers handicapped with a finite data handling capability.

Even approximate measurability of the norm only allows to predict the behavior of the *norm* of the solution. This is still a far cry from predicting how the solution itself behaves in terms of its various measurable properties. It may be the case that the norm is not *operationally significant*: sufficient closeness of two states in the norm does not mean that their measurable properties are also sufficiently close to each other[19]. If a problem is well posed only in norms which are not operationally significant then we can at best predict

[19] As an example, the L^2 norm utilized in quantum mechanics is not operationally significant. Assume measurable properties of a quantum mechanical system are the expectation values such as the position and momentum expectation values. The time evolution for free particles is linear and unitary and thus preserves the norm. This means that if two states start close measured in the L^2 norm then they stay close for all time. One can have two wave packets that are arbitrarily close to each other in the L^2 norm such that initially their position expectations are the same and their momentum expectations differ only slightly. Due to Ehrenfest's theorem the expectation values behave as their classical counterparts hence after sufficient amount of time the position expectations can differ to an arbitrary extent, even though the states stay close in the L^2 norm! Thus closeness in the L^2 norm does not imply closeness of measurable properties. See also (Earman; 2007, p. 1403) and Belot and Earman (1999).

how the norm of the solution behaves but not how the measurable world does[20]. As our main interest lies in predicting behavior of measurable properties and not the behavior of a single abstract (albeit frequently informative) quantity, operational significance of the norm is crucial for linking well posedness with prediction even for observers not handicapped with a finite data handling capability[21].

Operational significance or the norm may fail for several reasons; one of its most worrisome aspects is the assumption that measurement precision can be arbitrarily refined. Continuous dependence may only be sufficient for prediction if for any pre-assigned precision observers are capable of building and operating devices which can measure physical quantities up to this said precision. Without such capability observers might run into limits before the deviation of the can be narrowed in any useful way. There are several reasons to think that such measurement limitations exist. Quantum mechanics sheds serious doubts upon the possibility of arbitrary measurement precision as well as on the tacit assumption that measurement of physical quantities can take place without altering the measured quantities themselves. If observers run into hard limits in refining their measurement precision then continuous dependence is again insufficient for prediction[22].

[20]Even such prediction would assume that we have independent basis to know that the initial states indeed fall into the required neighborhood; without additional premises in case the norm is not operationally significant it is unclear how could we experimentally verify that and so even prediction of the behavior of the norm would become problematic.

[21]We drew a distinction between two properties norms of well posed problems should have in order to allow for prediction: norms that are approximately measurable and norms that are operationally significant. I have not seen such distinction being drawn nor have I found any systematic study of how norms used in physical theories correspond to measurement procedures that are used to test the theories. I think understanding this link between the mathematical apparatus and the measurement procedures would be crucial in order to understand what we do when we test our physical theories based on their predictions, and thus this problem should move the forefront of research in the foundations of physics.

[22]Consider the well known Kolmogorov-Arnold-Moser (or KAM) theorem according to which the phase space trajectory of a quasi-integrable Hamiltonian system, for particular choices of initial conditions and for a sufficiently small perturbation parameter, can be confined to a restricted region. Thus as long as observers of such systems can improve their measurement accuracy to get within the small perturbation required by the KAM theorem they can predict that the trajectory stays within a confined (torus-like) region. The size of the perturbation parameter depends on the number of degrees of freedom N of the system, and is typically of

Finally let us mention that observers may also be handicapped with computational limitations which can cause quite serious problems for prediction. Mere existence of a solution does not imply that it can be found or even that it can be approximated by observers who have, say, the computational capabilities of a Turing machine, even in cases when the given data itself is within the computational reach of the observers[23]. Continuous dependence does grant approximative techniques – indeed this is one of its main mathematical advantages, as we will point out later – however there are still cases that show that well posedness is not sufficient for constructively accessible solutions. There is a further obvious issue arising with continuous dependence: the mere assumption that a sufficiently small data neighborhood keeping deviation of solutions within a desired level *exists* does not mean that this neighborhood is also *computable*. Prediction arguably requires that observers are able to tell when they have arrived at their predictions and computability of the error rates is necessary for that[24].

the order $\exp^{-N \log N}$, which means that for large-N systems the region of stability is extremely small. (For a precise statement of the KAM theorem and for a discussion of the dependence on the degrees of freedom see (Pettini; 2007, pp. 59-61).) It had been argued (see Frigg and Werndl (2011), and also Vranas (1998)) that for large non-integrable perturbations of integrable systems the motion is likely epsilon-ergodic (ergodic except for a small phase space region); if we take this to be an indication that outside the small regions of stability prediction becomes impossible then we can conclude that observers who do not have access to sufficiently fine measurements become unable to predict even though they would be able to predict if they were able to measure sufficiently but still finitely finely.

[23] A physically relevant example is provided by (Pour-El and Richards; 1989, p. 116). Pour-El and Richards construct an initial value problem for the relativistic wave equation with computable initial values whose solution, even though it exists and is unique, is not computable. Computability of solutions is a type of approximation and hence it makes reference to a norm with which distances are measured; the norm utilized by their Theorem 6 is the uniform norm on the Banach space of continuous functions of a compact region. If one changes the norm (i.e. to the energy norm) then solutions become computable, see Theorem 7 on p. 118. (ibid). Note that well posedness for the wave equation fails for the space of continuous functions but holds for norm relevant for energy conservation; see Appendix A.

[24] For many interesting counterexamples and for a general overview on computational analysis see Pour-El and Richards (1989). See also Chapter 6 of Earman (1986) for a discussion of the relationship of determinism and computability.

2.2.2 Well posedness: not necessary for prediction

As even the weakest notion of well posedness (Definition 3) may allow observers handicapped with finite measurement precision to predict at least for a brief period of time, failure of stronger notions of well posedness do not automatically entail impossibility of prediction. Besides this general remark it is worthwhile to take a look at other reasons why well posedness does not seem to be necessary for prediction.

2.2.2.1 Existence. Suppose well posedness fails since the solution of the mathematical problem does not exist. Prediction is not possible, the reasoning goes, because without an existing solution we do not have a prediction we can talk of. But is it so? Lack of solution may signal a variety of problems; one such problem can be that we have not utilized the appropriate type of solution concept. Non-existence of a solution is a mathematical claim which have been preceded by definitions of what counts as a mathematical problem and what it is to be a solution of such a problem. The non-existence result is relative to and is dependent on the particular mathematical choices we make in defining our concepts; if we changed some of the requirements the existence result may also change[25].

Such changes in the solution concept have happened during the history of treating the problem of well posedness. For differential equations the concept of a classical solution is defined as a sufficiently many times continuously differentiable function satisfying the differential equation. Many physically relevant differential equations have classical solutions, but many of them do not. Consider the scalar conservation law

$$u_t + F(u)_x = 0$$

that models various phenomena including formation and propagation of shock waves[26]. A

[25] Definitional dependence should be also noted w.r.t. sufficiency of well posedness for prediction.

[26] For an introduction to weak solutions, conservation laws and for a treatment of this equation see Evans (1998).

shock wave is a curve of discontinuity and if we want to study conservation laws we should allow solutions which are not continuously differentiable. The concept of a classical solution does not permit such discontinuities. It is possible, however, to weaken this classical concept; this is where various types of so-called weak solution concepts surface.

The tricky issue is not how to weaken a solution concept, but how to weaken it so that we avoid the trade-off between existence and uniqueness: a problem whose solution is unique with a more stringent solution concept may very well have multiple solutions with a more relaxed solution concept[27]. Appropriate relaxation of the classical solution concept can be done for many conservation laws of the type mentioned above so that their problems become well posed utilizing a weak solution concept. However there is an interpretational price to be paid: it is much less straightforward to see how weak solutions represent physical systems as they are not functions that assign states to a time variable. This may or may not cause problem for prediction.

2.2.2.2 Uniqueness. Let us now turn to the assumption of uniqueness. The existence of multiple solutions is supposed to threaten prediction due to a lack of recipe for choosing among different possibilities. There are couple of questions to be asked about this assessment. Is it necessarily the case that different solutions represent different physical possibilities? Can there indeed be no further criteria on the basis of which we can choose among different solutions?

[27] There are many examples; a simple illustration for calibrating a concept of solution to achieve uniqueness may be the Hamilton-Jacobi equation on an open Ω subset of a Banach space E. The problem is to find solutions $u : E \to \mathbb{R}$ of $\|Du(x)\| = 1$ for all $x \in E$ such that $u(x) = 0$ for all $x \in \partial\Omega$. This problem has no classical solution, it has multiple weak solutions, and among these many weak solutions one can find a 'natural' one, the so called viscosity solution, which is unique (see Deville (1999) for details).
Surprises may also happen. The literature on weak solution is vast and one may easily run into incompatible definitions; even though weak solutions born out of the need to salvage non-existence of classical solutions not all weak solution concepts are strictly weaker than the classical solution concept. For instance the Dirichlet problem for the Laplacian with discontinuous boundary data has a classical solution but appears to not have a sort of weak solution, see Krutitskii (2009).

The negative answer to the first question is a commonplace in the literature of gauge invariance. It may be that several different solutions represent the same physical scenario due to the presence of surplus mathematical structure which do not correspond to any physical property but which mathematically may take different values. The given physical data is insufficient to determine these non-physical values and in turn solutions become non-unique. This non-uniqueness, however, does not threaten prediction as long as we are only supposed to predict the behavior of physical properties. The solutions only differ in some of their mathematical properties, but all of them tell the same story of physical happenings[28].

Thus prediction can also thrive if merely the class of physically equivalent solutions is determined uniquely by the given data. Unfortunately we don't have clearly independent means of establishing physical equivalence; many times we take gauge invariance to be evidence for physical equivalence as opposed to establishing physical equivalence first and judging on this basis that a non-uniqueness of solutions indicates surplus structure. As experimentally we can only confirm observational equivalence, physical equivalence – sameness of physical properties – can only be established by using additional theoretical premises, and the presence of non-uniqueness which only affects non-observable properties is often taken as a basis for such a premise of a Leibnizian flavor.

As physical equivalence is not identical with observational equivalence we may also wonder whether fans of prediction would not be satisfied with being able to predict the course of observable properties. For that it would be sufficient to have the class of observationally equivalent solutions being uniquely determined by the given data. As what solutions count

[28]An example for gauge invariance can be found in electrodynamics. The initial value problem for Maxwell's equation for electromagnetic potentials does not have a unique solution; the solution only becomes unique if one imposes some further condition, such as the Lorentz gauge condition. The different solutions for potentials however tell the same story about the behavior of the electromagnetic field E and B. Only the E and B fields are empirically accessible and hence it may be reasonable to assume that only they represent genuine physical properties and the potentials are merely convenient tools for calculation (albeit the Aharonov-Bohm effect suggests otherwise). See i.e. in (Earman; 2007, p. 1378).

observationally equivalent again depends on the capabilities of the observers, for sufficiently handicapped observers this could be a much less stringent requirement than unique determination of the class of physically equivalent solutions, and a further blow to the assumption that uniqueness of solutions themselves is necessary for prediction[29].

Assume now that our problem has several non-physically and/or non-observationally equivalent solutions given certain data. Under what conditions does this multiplicity threaten the possibility of prediction? Can we indeed not rely on further criteria to choose among these solutions?

When the given data does not include all data that is available to the observer at the time of prediction this further data may be relied upon to get rid of multiplicity – lack of uniqueness may simply be due to not taking into account all available information. Thus the possibility of prediction should only be at peril when the given data includes all data that is available to the observer. However, as we discussed before, different assumptions about capabilities of observers yield different sets of data that is available to them. That this may cause problems can be illustrated by examples. Physicists who emphasize the necessity of well posedness for prediction typically think of initial value problems where the given data is a specification of the properties of an instantaneous state. If we already assume (pace theory of relativity) that an entire instantaneous state can be considered as data that is available for an observer we might as well assume that the states past this instantaneous state also constitute data that is available for the observer – observers, after all, may have memory of states which they lived through. There are, however, examples of problems which are not

[29]When the need for prediction is cited to motivate continuous dependence prediction is understood to take place with some imprecision. This suggests that we may even relax observational equivalence to 'observational similarity' and require only the class of 'observationally similar' solutions to be uniquely determined by the given data. This would be a bit misleading, however, as continuous dependence is also motivated by the possibility to arbitrarily diminish the said prediction imprecision via narrowing the set of possible given data. If only the class of observationally similar solutions were fixed by an individual given data there would be no more ways to improve upon the similarity.

well posed when the given data is an instantaneous state but which become well posed when the given data is a set of (past) states; initial value problems for delay differential equations are often of this sort[30].

Important cases for instantaneous states not furnishing all data that is available to the observer also permeates the literature on inverse problems. Inverse problems deal with retrodiction: they ask what could have been the dynamical evolution in the past that led to a certain present state? An inverse problem which only takes the present state as given data may have multiple solutions. However observers may have additional information on the behavior of the solution that led to the present state i.e. that its norm have not surpassed a certain finite value. Assuming finiteness of the norm of the solution is in many cases sufficient to ensure uniqueness. The assumption that the dynamical evolution leading to the present state was not explosive may in certain circumstances be treated as further data that is given to the observers[31]. Thus one needs to be careful drawing conclusions from the failure of a Laplacian sort of determinism to the failure of prediction; prediction may still be possible even if Laplacian determinism fails.

Assume, then, that the given data does include all data that is available to the observer at the time of prediction and the problem still has several non-observationally equivalent solutions. Prediction of the involved observable quantities with *certainty* may be then at peril but that does not preclude the possibility of prediction with very high *probability*. Even if uniqueness can not be guaranteed, the physical theory in which the problem is formulated may furnish a probability distribution on the space of solutions according to which these solutions get realized. Such is the case is quantum mechanics; in the Bohm–de Broglie formulation the guiding equation uniquely determines the time evolution of the quantum mechanical system given the initial wave function together with the initial position

[30]For examples and physical relevance see (Earman; 2007, p. 1373).

[31]Technical issues surface when we attempt to formulate a requirement for continuous dependence on such type of 'data' but for uniqueness purposes these issues are not present.

of the Bohmian particle. If we make the usual assumption that observers can not know the initial position of the Bohmian particle and thus we assume that only the initial wave function constitutes given data then the solution of the resulting problem of finding the time evolution of the quantum system is not uniquely determined. Still the wave function traces the probability of finding the Bohmian particle in a certain location and hence it provides a probability distribution over the set of possible solutions. Depending on the physical setup the probability of finding the Bohmian particle – and thus the probability to obtain certain measurement outcomes – can get arbitrarily close to one.

We take up further discussion of prediction with uncertainty in the following section. When probabilities or other notions of 'likeliness' for solutions are not supplied by the physical theory[32] there are few options remaining open to eliminate non-uniqueness. One option may be to discard some solutions along the lines that they are 'unphysical.' Such claims are fueled when results show that uniqueness ensues if we can impose suitable boundary conditions or growth conditions or other restrictions on the behavior of the solution. An example could be the initial value problem for the heat equation on the real line[33]. The solution of the initial value problem is not unique without the aid of further conditions, such as a specific exponential growth condition on the solution or the assumption that the solution is non-negative everywhere for all times. Both of these assumptions may be regarded as 'physically reasonable': lack of strong influences coming from infinity may motivate the exponential growth condition, or the physical interpretation of the solution as temperature may moti-

[32]If the physical theory does not supply the probabilities in order to smuggle them in one either needs to resort to some dubious principle such as the principle of indifference or needs to invent extra physical properties. (For a brief discussion of these options in the context of an example in classical physics see (Norton; 2003, p. 10).) Without further empirical justification of the so-introduced probabilities neither of these options seem helpful for prediction.

[33]In general parabolic partial differential equations provide illustrative examples since their initial value problems typically do not have unique solutions (their characteristics coincide with planes of absolute simultaneity). For results for the heat equation and for a discussion of general second order linear parabolic partial differential equations see John (1982).

vate the non-negativity assumption. The crucial issue from the perspective of prediction is whether such 'physically reasonable' assumptions could indeed be taken as available data[34]. As the required assumptions can not be derived from data of any present or past state, observers lacking foresight can not take them to be available data, and hence observers can not rely on such assumptions to pick out a single solution[35] without running into circularity.

There is another way to diminish worry about non-uniqueness arising for problems of a physical theory T if there is another physical theory T' that 'cures' non-uniqueness of T. If T' is in some empirical sense superior to T then we can point fingers towards T' to get rid of some of the solutions of T as 'unphysical.' This elimination in turn may allow us to predict on the basis of the remaining solutions. Such 'curing' relationship between classical and quantum mechanics have recently been explored by Earman (2009) in the context of Laplacian determinism. It remains to be seen to what extent does the 'cure' come from elimination of physical possibilities that results from different mathematical assumptions and to what extent does it get an explanation in terms of new physical knowledge[36]. Albeit

[34]For an extended discussion and criticism of such conditions on the heat equation in the context of Laplacian determinism see (Earman; 1986, pp. 40-45).

[35]Prediction with uncertainty may come again to rescue. We have pointed out the possibility of prediction with uncertainty when we can rely on a notion of 'likeliness' on the space of solutions belonging to a single given data; in the following section we treat the case when a notion of 'likeliness' is available for the space of given data. A third option for prediction with uncertainty could arise if a notion of 'likeliness' were furnished for conditions on the behavior of solutions that ensure uniqueness, i.e. resulting from the 'chance' of strong influences coming from infinity. The required notion of 'likeliness' may be obtained from an additional physical theory or even merely from counting relative frequencies of relevant past observations. To my knowledge motivating uncertain predictions via this third route is unexplored in the philosophy of physics literature.

[36]For the purposes of this footnote let $\mathcal{W}_T$ stand for the set of physically possible worlds according to a theory T. Let $[w_T]$ denote the set of all worlds in $\mathcal{W}_T$ which agree with $w_T \in \mathcal{W}_T$ at some time; let us here refer to $[w_T]$ as the *indeterminism bouquet* of w_T. The world $w_T \in \mathcal{W}_T$ is *(Laplacian) deterministic* if it is the only world which can agree with itself at some time, that is when its indeterminism bouquet has a unique member $[w_T] = \{w_T\}$. (The relation generating $[.]$ is reflexive and symmetric, but not necessarily transitive. After a suitable rescaling of the global time function $[w_T] = \{w_T\}$ whenever $w_T(t^*) = w'_T(t^*)$ for some t^* implies $w_T(t) = w'_T(t)$ for all t. $w_T(t^*)$ for a specific time t^* is called the *state* of the world w_T at time t^*.) Some worlds in $\mathcal{W}_T$ may be deterministic while some other worlds in $\mathcal{W}_T$ may not be deterministic, which is to say they are *indeterministic*. Thus we can partition $\mathcal{W}_T$ as $\mathcal{W}_T^d \cup \mathcal{W}_T^i$ where $\mathcal{W}_T^d$ holds the deterministic worlds and $\mathcal{W}_T^i$ holds the indeterministic worlds. (Note that for $w_T \in \mathcal{W}_T^i$ we have $[w_T] \subseteq \mathcal{W}_T^i$.) The theory

this is a fascinating area of inter-theoretical relationships we can only rely on aid coming from better theories when there are any; unfortunately many of our best physical theories are also plagued by non-uniqueness and thus cure for their non-uniqueness awaits discovery

T is *deterministic* if all worlds in $\mathcal{W}_T$ are deterministic – if $\mathcal{W}_T = \mathcal{W}_T^d$ –, and it is *(partially) indeterministic* otherwise.

Suppose a theory T is indeterministic. What would it take for another theory T' to "cure" the indeterminism of T?

The question clearly assumes that theory T and theory T' can be related to one another in some meaningful way. It seems that addressing the same type of phenomena is necessary for so relating the two theories in a meaningful way: the question whether the theory of business cycles may cure the indeterminism of the theory of quantum mechanics appears moot because these theories do not address the same type of phenomena. Thus there needs to be some type of phenomena to which both some physically possible worlds of T and some physically possible worlds of T' can be related. But if both theories' physically possible worlds can be related to the same phenomena they can also be related to one another. A minimal assumption, then, is that at least some physically possible worlds of T can be related to some physically possible worlds of T'; let's denote the mapping implementing this relation by the partially defined mapping $\phi : \mathcal{W}_T \mapsto \mathcal{W}_{T'}$. For some worlds in $\mathcal{W}_T$ the mapping ϕ may not be defined, and for some worlds it may yield an empty result, meaning that some physically possible worlds of T do not have corresponding physically possible worlds of T'. ϕ may also be one-to-many, as T' may give a more nuanced description of the phenomena than does T. Although specifying ϕ is far from being straightforward let us now assume that it is given.

We say that theory T' *cures* the indeterminism of the physically possible world w_T of T if all physically possible worlds of T' corresponding to w_T are deterministic: if $\phi(w_T) \subseteq \mathcal{W}_{T'}^d$. The theory T' *partially cures* the indeterminism of theory T if it cures the indeterminism of at least some physically possible worlds of T. The theory T' *cures* the indeterminism of theory T if it cures all of T's indeterministic worlds: if $\phi(W_T^i) \subseteq \mathcal{W}_{T'}^d$.

Note that curing the indeterminism of a theory T by T' does not necessarily mean that T' fares better on the front of determinism, as T' may still be indeterministic; perversely some of the indeterministic worlds of T' may even correspond to physically possible worlds of T which were originally deterministic. Those who take indeterminism as a disease for which determinism is the cure strive not merely for curing indeterminism in the sense defined above, but for curing indeterminism by a deterministic theory.

Indeterminism of a physically possible world w_T consists in that $[w_T]$ is not unique. Curing this indeterminism requires reducing the number of corresponding possible worlds in the corresponding indeterminism bouquet to at most a single possible world. Hence curing indeterminism of a possible world can follow a (mix) of three different strategies: elimination of the physically possible world itself, elimination of physically possible worlds in the indeterminism bouquet, and determination of the physically possible worlds (turning indeterministic worlds into deterministic ones).

If successful cure merely means that the therapy leaves no ill patients then the therapist can take care of the illness in a simple but maybe not very satisfying way: by the elimination of all ill patients. According to our definition T' does cure the indeterminism of T in case no physically possible worlds of T' correspond to the indeterministic worlds of T'. We say that T' *eliminates* a physically possible world w_T of T if $\phi(w_T) = \emptyset$. A radical way for a theory T' to cure the indeterminism of T is by elimination of all of T's indeterministic worlds: $\phi(\mathcal{W}_T^i) = \emptyset$.

It is possible to cure the indeterminism of T by elimination of all of its indeterministic worlds, but for an elimination strategy to succeed it is also sufficient to eliminate worlds in indeterminism bouquets. We of yet unknown theories.

Finally we note that even the existence of several non-observationally equivalent solutions, given all available data and all justified restrictions and all likeliness analysis, would not necessarily serve a fatal blow to prediction. As long as our problem narrows down the set of possible solutions in some way it yields us a prediction in the sense that it asserts that only solutions belonging to this set are possible. Non-uniqueness should not be equated with 'everything goes.' It may well be that some observable quantities can not be predicted but some other observable quantities can, or at least that we can predict the range in which the evolution of observable quantities fall. Such predictions would even allow us to empirically test our theories; the tests would arguably be less effective than in cases where uniqueness reigns but they would still make the theory falsifiable[37]. Narrowing the set of observable

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say that T' *eliminates an agreement alternative* of a physically possible world w_T of T if it eliminates at least one world from the indeterminism bouquet $[w_T]$: if there is a $w'_T \in [w_T]\backslash\{w_T\}$ such that $\phi(w'_T) = \emptyset$. Eliminating all agreement alternatives of w_T may be a decisive step in curing the determinism of w_T. As an example consider the case of a theory T' for which $\mathcal{W}_{T'} \doteq \mathcal{W}^d_T \cup V$ where V is a minimal set for which $[V] = \mathcal{W}^i_T$, and set $\phi(w_T) = w_T$ if $w_T \in \mathcal{W}_{T'}$ and $\phi(w_T) = \emptyset$ otherwise. ($[U] \doteq \cup_{w\in U}[w]$. V is minimal in the sense that for any proper subset of V: $[V] \neq \mathcal{W}^i_T$. Such V always exists due to the axiom of choice.) In this example T' eliminates the indeterminism of T and is itself deterministic. Determinism was achieved purely by elimination of some indeterministic worlds, even though it did not require elimination of all of them.

Curing indeterminism may happen without elimination of possible worlds. Here is a simple but useful example of a theory T' curing the indeterminism of T without elimination: for all $w_T \in \mathcal{W}_T$ let $\phi(w_T) = (w_T, \kappa)$, where κ is a different cardinal for each w_T's, and let $\mathcal{W}_{T'} = \phi(\mathcal{W}_T)$. No worlds in $\mathcal{W}_{T'}$ are indeterministic, since due to their different labeling they are never going to agree with each other. Curing happens by 'determination,' by turning indeterministic worlds into deterministic ones. The cure is nevertheless suspicious, as nothing seems to suggest that receiving a different κ label signifies a physical difference between states which would agree otherwise.

In which of these ways do actual physical theories cure the indeterminism of another theory? To my knowledge this is a not yet well understood topic in the foundations of physics. I conjecture that most cases, such as curing the indeterminism of classical physics by quantum mechanics, involves to some degree elimination strategies. From a philosophical point of view curing indeterminism by elimination is less compelling than curing by 'determination': the very same elimination could be achieved without the help of any additional theory as a mere posit in the old, i.e. by an application of a selection principle we discuss later. Elimination by a selection principle may seem more ad hoc than elimination implied by the mathematical structure of another theory but it is unclear to me whether the difference has philosophical significance.

Same distinctions can be made w.r.t. curing non well posedness; to pursue this issue further shall be the topic of another paper.

[37] Take, for instance, the indeterminism produced by Norton (2006)'s Dome. The solution depicting the

possibilities is a valid and pragmatically useful sense of prediction and yet another indication of why issues of determinism should not be equated with issues of prediction.

2.2.2.3 Continuous dependence. There are many tacit assumptions behind the tenet that continuous dependence of solutions on given data is necessary to allow for prediction for observers who are only capable to obtain the given data imprecisely. For one it is tacitly assumed that the given data consists of physical quantities which take continuous values. If the possible values for the quantities were discreet then it would be sufficient if measurement precision could get beyond the distance between successive values, since then it would be possible to reveal by a finite precision measurement the precise value of the physical quantity. We do venture that some fundamental physical quantities are of this sort, i.e. in quantum mechanics the possible energy values are quantized.

In Hadamard's example we have found arbitrarily large deviations of solutions for any small neighborhood of the initial data within any short span of time. The presence of unbounded behavior might seem to be curious: continuous dependence could, in principle, also fail without producing large deviations. It is possible that continuous dependence fails but solutions are still not departing more than a fixed number $\epsilon^* > 0$ after choosing an appropriately small $\delta > 0$ neighborhood for the given data. Even though ϵ^* could not be further decreased by decreasing δ this type of failure of continuous dependence doesn't seem to pose a fatal problem for approximate prediction if ϵ^* is "small," as we could still keep predictions within ϵ^* accuracy by keeping accuracy of the given data within a δ range.[38] We should allow for such failure of continuous dependence i.e. if our continuous models were known to be merely approximations of inherently discreet processes.

ball departing the top of the Dome at $t = 0$ puts an upper bound on the distance the ball may be from the top of the Dome at a given time regardless of which of the multiple non-observationally equivalent solutions gets realized. Claims of the form 'the ball will not hit the pole situated in 100 meters from the top of the Dome under 10 seconds' are valid and testable predictions of the theory.

[38] Compare with remarks on necessity of uniqueness!

In the context of differential equations, given that existence and uniqueness of solutions is established, if continuous dependence fails then arbitrary deviation of the solutions is tacitly assumed[39]. The reason has to do with powerful theorems in functional analysis showing that when solutions fall into a compact set existence and uniqueness entails continuous dependence. These theorems are the main motivations behind Tikhonov's notion of a *conditionally well posed problem* Tikhonov (1943). Tikhonov assumes that the given data may not vary arbitrarily but only within a specific subset of data which yield solutions that belong to a restricted set, i.e. that the norm of the solution stays below a certain upper bound[40]. The challenge of establishing (conditional) well posedness then reduces to the challenge of establishing uniqueness of solutions which significantly decreases mathematical difficulties.

Why should failure of continuous dependence and an ensuing arbitrarily large deviation pose a fatal threat to prediction? Hadamard opines that sans continuous dependence "everything takes place, physically speaking, as if the knowledge of ... [the initial data] would *not* determine the unknown function," and the time development of a system lacking continuous dependence "would appear to us as being governed by pure chance (which, since Poincaré, has been known to consist precisely in such discontinuity in determinism) and not obeying any law whatever" (Hadamard; 1923, p. 38). Thus Hadamard piggybacks failure of continuous dependence to apparent failure of determinism[41] and he equates apparent failure

[39] I am not aware of counterexamples. If uniqueness also fails then solutions might not deviate arbitrarily, i.e. as in the case of Norton (2006)'s Dome.

[40] Using the notation of Definition 1 conditional well posedness requires that (i) the solution w exists and it belongs to M, (ii) the solution is unique in the set M, and (iii) for every $\epsilon > 0$ there exists $\delta > 0$ such that for every data $d, \bar{d} \in OM$ satisfying

$$\|d - \bar{d}\|_D \leq \delta$$

the inequality

$$\|w - \bar{w}\|_W \leq \epsilon$$

holds for $Ow = d$ and $O\bar{w} = \bar{d}$. When the operator O is continuous and M is a compact set existence and uniqueness entail continuous dependence (Lavrentiev et al.; 2003, p. 11). For another significant result connecting existence and uniqueness with continuous dependence, see Appendix A.

[41] Apparent failure of determinism should of course not be confused with actual failure of determinism. Popper (1982) seems to make this sort of mistake as it was pointed out by (Earman; 1986, p. 9).

of determinism with apparent “pure chance” and “not obeying any law whatever.”

We have already commented on the danger of equating non-uniqueness with ‘everything goes.’ The same is true for equating failure of continuous dependence with ‘everything goes:’ “not obeying any law whatever” would require that any function which can represent a time development of a system can be approximated by a solution of the problem that originates from a small pre-assigned neighborhood. In physically relevant problems this is not the case; even if the norm of solutions originating from a small neighborhood varies wildly that does not imply that such solutions are dense in the set of all functions that can represent time development (many of which do not obey the law of the problem). Thus even if continuous dependence failed it would be possible to achieve predictions in the limited sense we described in the previous section. If the norm is not operationally significant then solutions originating from a small neighborhood may even allow prediction of specific values of physical quantities instead of prediction of some set of these quantities.

Continuous dependence favors a notion of prediction with certainty. If observers can sufficiently narrow the set of possible given data they can be certain that the corresponding solutions stay within a desired level of accuracy, no matter which data within this possibility set is actual. Certainty is not the only interesting option from the vantage point of those who would want to predict; failure of (classical) continuous dependence may still allow for predictions with various degrees of certainty. This may be captured by theoretical results in several ways. Suppose, for instance, that the set of possible given data D is endowed with a probability measure and that it has a subset $\hat{D} \subset D$ of measure one such that solutions depend continuously on data that belongs to this subset. If the probability measure reflects on the chance of the occurrence of the given data in observational scenarios then this would mean that continuous dependence holds almost always and observers are almost never mistaken in their predictions. We could even relax the assumption that the measure of this data subset is one to that it is very close to one and still end up with a pragmatically highly justified notion of prediction.

There are rigorous ways other than measure theory to talk about certainty. Solutions that fail to depend continuously on all data D may depend continuously on a set of data $\hat{D} \subset D$ that has the same cardinality as D. If the cardinality of $D \backslash \hat{D}$ is smaller than that of D and $\hat{D}$ then there are 'as many' possible data in $\hat{D}$ as in D while there are relatively 'few' possible data in the omitted set making, in this sense, appearance of continuous dependence 'likely.' A bit more elaborate way of talking of relative sizes is via the notion of a meagre set of descriptive set theory[42]. If $D \backslash \hat{D}$ is a meagre set in D then one could say that the set of exceptions from continuous dependence is negligible compared with all possibilities.

Talking of certainty in terms of measure, cardinality or meagreness may not help criticizing the necessity of continuous dependence for prediction: mathematical results might cut in both ways and it may turn out that the set of data one would need to discard in order to achieve conditional well posedness is 'overwhelming.'[43] If predictions in these cases would actually work out well we would be compelled to conclude that the applied mathemat-

[42] A set is meagre iff it is the countable union of nowhere-dense sets. A set is comeagre iff its complement is meagre. See i.e. Oxtoby (1980).

[43] Consider the case of a point particle constrained to move on a frictionless surface S in the presence of a uniform gravitational field. As it is exemplified by Norton's Dome the initial value problem for Newton's law is not always unique if we only require the surface S to be continuously differentiable (to be $\mathcal{C}^1$). As a consequence of Theorems 1 and 2 in Appendix A the initial value problem does become well posed (in the sense defined there) if the surface S belongs to $\mathcal{C}^2$. One could wonder then about the relationship between $\mathcal{C}^1$ and $\mathcal{C}^2$ in terms of relative 'sizes.' Two claims can be conjectured:
Claim 1. Take an initial value problem of a point particle constrained to move on a frictionless $\mathcal{C}^2$ surface S. In any δ-neighborhood of S (understood in the supremum norm) there are $\mathcal{C}^1$ surfaces for which the initial value problem with the same initial value does not have a unique solution. The cardinality of such $\mathcal{C}^1$ surfaces in the given δ-neighborhood is at least as large as the cardinality of $\mathcal{C}^2$ surfaces in the δ-neighborhood.
Claim 2 $\mathcal{C}^2$ surfaces are megre among the $\mathcal{C}^1$ surfaces.
Claim 1 can be proven by bending the surface S perpendicular to the direction of the particle's velocity using Norton's construction in a sufficiently small area around the particle's initial position. Claim 2 is a consequence of the well known theorem of Banach (1931) according to which the set of continuous functions that have a derivative in at least one point are meagre set in the space of continuous functions. As maybe not all $\mathcal{C}^1 \backslash \mathcal{C}^2$ surfaces lead to indeterminism it is only indicative of the fact that there might be overwhelmingly more indeterminism-yielding surfaces than well behaving ones. If our mathematical notions of "likeliness" had any relationship to the chance of the occurrence of physical systems which are adequately represented by these models then we should expect prediction to fail 'overwhelmingly.'

ical notion of 'likeliness' has no direct relationship with chance of occurrence in predictive scenarios. Without such link between mathematical notions and physical occurrences it is difficult to appreciate any of the results and there is a risk of mistaking elegance of a mighty formalism with relevance for application. When the physical theory itself does not endow a mathematical notion of likeliness with physical interpretation (as it is the case with quantum mechanics) the mathematical notion of likeliness is in grave danger of being purely ad hoc.

Arguing along the lines of *'likeliness'* of some subset $\hat{D}$ of given data D does not exclude the *possibility* of data $D \backslash \hat{D}$: we only assumed that data in $D \backslash \hat{D}$ is not 'likely' and hence its possibility does not threaten prediction. Albeit Tikhonov's restrictions on the set of possible given data seems only to be warranted for inverse problems it does call attention to the strategy of narrowing the set of possible given data to achieve continuous dependence. One may argue that there exists a sort of *selection principle* that supplements the requirement of well posedness: for reason X the set $D \backslash \hat{D}$ contains given data that is not (in some sense) possible and hence we only need to require well posedness conditionally on a restricted subset $\hat{D}$. Reason X many vary; here we briefly concoct some possibilities.

The so-called Past Hypothesis narrows the set of possible initial conditions of our Universe to a specific subset[44]. It has become fashionable to elevate this requirement to the venerable status of a physical law. Riding this tide one could posit the Well Posedness Hypothesis which narrows the set of possible initial conditions of our Universe (for a suitably chosen fundamental physical theory) so that conditioning on this narrower set the initial value problem becomes well posed. If one could supplement this Hypothesis with results showing that well posedness of relevant local direct problems in the future time evolution follows from the Hypothesis then one could get an explanation of why well posedness seems to holds for the typical dynamical systems we are experimentally familiar with.

[44] For an introduction to the Past Hypothesis which assumes that this set comprises states of low entropy see Albert (2000).

For reasons similar to those expressed against the Past Hypothesis the Well Posedness Hypothesis may not even be false: to paraphrase Earman (2006) it is not clear whether it can be properly motivated, whether it can be well defined, and whether its implementation would consist in more than furious hand waving and wishful thinking. It certainly 'solves' the problem of justifying necessity of well posedness by fiat. When no further explanation seems possible such attempt may satisfy someone but we do not pursue this line of thought further here as I'm not aware of any defenders of such Hypothesis.

Another candidate for a selection principle could be selection by measurement. According to this idea both well and not well posed problems may be physically possible but the act of measurement selects a subsample of systems that can be described by well posed problems. One could maintain that measurement always alters the measured system in a way that conditioning of the set of post-measurement systems renders problems conditionally well posed. Such selection by measurement would then explain why do we seem to experience well posedness and why failure of well posedness could nevertheless happen when no-one looks. Although this may be an interesting idea to investigate I'm not aware of any analysis of measurement that would indicate the existence of such a selection by measurement.

The selection principle may be supplied by another theory. We touched upon the possibility of a theory T' 'curing' the non-uniqueness of a theory T by elimination of some solutions of T. A theory T' may similarly 'cure' the failure of continuous dependence of solutions of T by narrowing the set of possible given data. If T' is in some sense superior to T then one may argue that failure of continuous dependence simpliciter is not necessary for prediction; we can eliminate some of the given data on the recommendation of theory T' and it is sufficient for prediction if continuous dependence holds conditionally on the rest.

The idea of cherry-picking from the set of possible given data that is promoted by the above selection principles is rarely cherished outside the literature on inverse problems. The cherry-picking does nevertheless happen albeit it is dressed up more innocuously. Instead of narrowing the set of possible given data D while keeping the norm with which one measures distances between data, curing failure of well posedness typically proceeds by changing the norm $\|.\|_D$ to another norm $\|.\|_{D'}$. As we are going to see many physically relevant problems that are not well posed using one norm becomes well posed using another. What may slip past by the attention of the causal reader is that there is no change of norm without a corresponding change of the underlying space. The mathematical claim we usually read is not that the problem which is not well posed in the space $(D, \|.\|_D)$ becomes well posed in the space $(D, \|.\|_{D'})$, but that the problem becomes well posed in the space $(D', \|.\|_{D'})$ where D and D' typically differs. The difference is induced by the assumption that $\|.\|_{D'}$ is meaningful on the elements of D'; indeed this is the condition through which D' is usually defined. $\|.\|_{D'}$ is frequently not meaningful for all elements of D and hence the pair $(D, \|.\|_{D'})$ would not be a normed space.

Thus changing the norm may implicitly introduce a selection among the possible given data, i.e. when elements of D are states then states that appear possible in one formulation may become not possible in another. Narrowing the set of possible states by means of invoking a selection principle may rightfully leave bad taste in our mouth but then similar bad taste should also accrue seeing changes in norms that also imply elimination of possibilities. Whether the bad taste should persists depends on what do we make of the difference between D and D' as their elements may turn out to be identifiable in some physically relevant sense. The existence of a norm preserving isomorphism between elements of D and D' could be an indication of such identifiability; such is the case when both D and D' are separable infinite dimensional Hilbert spaces[45].

[45]The existence of a norm preserving isomorphism seems as much as one may want for physical identification and it has become a sort of slogan that in physics we do not need to care about the choice of the (infinite dimensional separable) Hilbert space as they are all equivalent. Treating isomorphic Hilbert spaces as representing the same physical possibilities does not seem to me to be the only arguable option. Consider the example $D = L^2(\mathbb{R}^4) = H^0(\mathbb{R}^4)$ and $D' = H^5(\mathbb{R}^4)$, H^k referring to the kth Sobolev space. As both L^2 and H^5 are separable infinite dimensional Hilbert spaces there is a norm preserving isomorphism

As we can see necessity of continuous dependence for prediction is difficult to justify in full generality. We now turn to the issue of connecting pragmatic concerns of predictability with metaphysical claims of possibility.

i according to which all $\psi \in H^5$ corresponds to a $\psi_i = i(\psi) \in L^2$; norm preserving means that we always have $\|\psi\|_{H^5} = \|\psi_i\|_{L^2}$. Every element of H^5 can, however, be also naturally identified with an element of L^2, as every function whose H^5 norm is finite has a finite L^2 norm as well. Loosely speaking every state in H^5 is also a state in L^2. Thus there is a mapping m so that every $\psi \in H^5$ naturally corresponds to a $\psi_m = m(\psi) \in L^2$; furthermore we always have $\|\psi\|_{H^5} = \|\psi_m\|_{H^5}$ (properly understood). The mapping m is, of course, not a bijection, in general $\|\psi\|_{H^5} \neq \|\psi_m\|_{L^2}$, and m and i only agree on the null-vector.

It may be reasonable to argue that it is the mapping m that identifies physically possible states as the 'same' in different mathematical representations and to view a transition from L^2 to H^5 as narrowing the set of possibilities (or a transition from H^5 to L^2 as including more possibilities). L^2 i.e. allows physical states with 'rough edges' while H^5 does not. As some states in L^2 have properties that none of the elements in H^5 have it may be an open debate whether such difference in properties signal physical difference.

Note that the difference between m and i is also carried over to possible time evolutions since i and m identifies different functions $f_i : t \mapsto L^2$ and $f_m : t \mapsto L^2$ with a given time evolution $f : t \mapsto H^5$. Suppose we solve an initial value problem in H^5 and we solve the 'same' initial value problem in L^2 in the sense that the initial values are the same (according to m) and the differential equation takes the 'same' form. If f is the solution of an initial value problem in H^5 then the same function in L^2 (that is, f_m) may not be the solution of the 'same' initial value problem in L^2! Purely from taking a look at a function describing the time evolution of a system we can not tell whether it is a solution of a differential equation until we fix the norm operating on the space of states. (This should be obvious as the definition of a solution of a differential equation invokes a norm according to which derivatives are taken and for different norms the derivatives may differ. This is not emphasized in finite dimension as there all norms are equivalent.) Thus given the time evolution of a system and the form of the differential equation it may be possible to determine the norm (up to equivalence) from the requirement that the time evolution needs to be a solution of the equation. Unfortunately our observers do not have access to the time evolutions, they only have access to the time evolution up to some imprecision, and as imprecision is mathematically represented by a norm the norm needs to be already given. (Some iterative process to find the right norm may be of help but there is no guarantee that such process would not yield multiple potential norms.)

That f_m may not be a solution in L^2 even though f is a solution in H^5 can either be interpreted as a further proof that m should not be taken as physical identification or as an indication that we should be more careful identifying differential equations defined on different spaces as being the 'same' merely on the basis of their formal similarity. On the other hand if we take i to physically identify elements of L^2 and H^5 then we need to make sense of results such as the 'same' initial value problems being well posed in one of the spaces but not in the other. I.e. Hawking and Ellis (1973) famously proved the existence of a well posed initial value formulation of linear diagonal second order hyperbolic systems for globally hyperbolic spacetimes; it is assumed that continuous dependence is understood with norm H^5, not with norm L^2; see (Wald; 1984, pp. 244-267) for details.

2.3 PREDICTION AND PHYSICAL POSSIBILITY – THE GAP OBJECTION

As we have seen quite many assumptions are needed to justify either necessity or sufficiency of well posedness for prediction. Assume now that well posedness is indeed necessary for prediction. Can we then draw conclusions regarding the necessity of well posedness for physical possibility[46]?

Prediction is undoubtedly an important pragmatic virtue; one may even maintain that a theory that is unable to predict is 'not viable' in some pragmatic sense. Pragmatic virtues, however, should not be confused with metaphysical claims. A theory may very well be true of the world even if it does not have certain pragmatic virtues which would allow users of the theory to, say, carry out predictions. Observers' problems with being able to carry out predictions are epistemic problems; claims about physical possibilities entailed by a certain theory are metaphysical claims. There is a gap between epistemic and metaphysical claims – between what may be known and what there may be – and without closing this gap in some way one can not draw inferences from failure of epistemic access to physical impossibility.

One can ask the question: but how do we know whether a certain theory is true of the world? As we typically test our dynamical theories by testing the empirical adequacy of their models well posedness seems instrumental for confirmation. If well posedness were sufficient for prediction then we had a straightforward recipe that allows testing of theories both in inductive and in falsificatory ways: First, decide upon a level of empirical fit with which you want to test your theory. Take a well posed problem, determine the data precision that corresponds to the required fit, and determine the set of possible predictions. Create a

[46]Although the authors quoted above do not directly use the term 'physical possibility' they claim that well posedness is necessary for "a mathematical model to describe a real physical process," and its failure would be "pathological" and "does not occur in physics." I take these quotes to express that models of theories which arise from some relevant set of problems that are not well posed can not (all) be taken to represent physical possibilities.

physical scenario such that its measured physical quantities that are represented by the given data fall within said precision and observe how the system behaves. Contrast the outcome with the predictions. If there is a possible match then the theory passed the particular test and one may refine the level of empirical fit or change the tested problem, rinse, wash and repeat. If there is no possible match and the discrepancy can not attributed to data collection, interference or representational problems then the theory is falsified since if the theory had been true then deviations of solutions should have stayed within the pre-assigned fit.

Lack of well posedness would sabotage several steps in this logic. If well posedness were necessary for prediction (if all objections against its necessity mentioned in the previous section were preempted) then without continuous dependence our theory testers would be in trouble no matter how loose level of empirical fit would they aim at. They would not be able to find a precision such that scenarios whose measured physical quantities fell within said precision could be guaranteed to evolve as to stay within the required fit. Some scenarios may do so, some others might not, and so inductive practices would be in trouble. Even if none of the evolutions stayed within the required fit we could not rely on this fact to falsify the theory as the theory itself allows the existence of arbitrary deviations.

The same worries we raised about sufficiency and necessity of well posedness for prediction apply to its sufficiency and necessity for confirmation; the Reader is urged to revisit the problems from this point of view. To the extent well posedness is not necessary for confirmation we may acquire reasons to believe that solutions of not well posed problems do represent physical possibilities. Furthermore, even if a case can be made for a connection between well posedness and confirmation of theories, arguments connecting physical possibility with confirmation would still suffer from the same problem as arguments connecting physical possibility with prediction do. A theory may well be true of the world even if users of the theory are not able to confirm that it is so. Truth of a theory and consequently truth of a claim about physical possibilities entailed by the theory is a metaphysical claim which should not be conflated with the epistemological question of how observers may confirm the theories they become acquainted with.

In the light of the gap objection the task of establishing well posedness as a necessary condition for physical possibility seems exceedingly difficult. In the rest of this work we invent and evaluate arguments which aim to achieve this feat; the price we need to pay in terms of premises, as we will see, may also be exceeding.

2.4 FURTHER REMARKS

Even if the connection between well posedness and prediction and confirmation is not entirely straightforward well posed problems are important because they are mathematically quite tractable. It is impossible to give here even a brief overview on the mathematical literature; the list of references for the 1983 non-historical survey for linear partial differential equations, Fattorini (1983), is over a hundred pages long, while ill posed inverse problems themselves have their own journal and book series. Thus we are only going to point out few mathematical advantages of assumptions of continuous dependence; the reason why existence and uniqueness of a solution is mathematically desirable is rather clear.

Despite the existence of a garden variety of definitions of continuous dependence for reasons of clarity it is useful to settle with a particular formulation (although we need to apply care to not to over-generalize the particular results we obtain). The archetype case of a mathematical problem used to model a physical process is an initial and/or boundary value problem of a differential equation which expresses a law of a dynamical physical theory[47].

[47] Although there are other relevant and interesting mathematical problems which should also be discussed in the wider context of well posedness – difference equations are among them (see Lax and Richtmyer (1956) or examples e.g. in Lavrent'ev and Savel'ev (2006)) – we are going to restrict our attention to this archetype case.

In the majority of this work we rely on a general and widely used framework developed for abstract differential equations by Peter Lax in the 1950's. This abstract framework is general enough to handle many fundamental dynamical equations in physics – such as the Maxwell, the Schrödinger, the Dirac equation for free particles – and it applies to both initial and initial/boundary value problems (as information about boundary values can be encoded in the domain of the utilized operator A). Appendix A gives a more detailed mathematical background, but the basic ideas can be easily introduced.

Let A be a densely defined operator in an arbitrary Banach space E with a norm $\|.\|$. Consider the equation

$$u'(t) = Au(t) \quad (-\infty < t < \infty) \tag{2.18}$$

A *solution* of (2.18) is a function $t \to u(t)$ such that $u(t)$ is continuously differentiable for $-\infty < t < \infty$, $u(t)$ is in the domain $D(A)$ of A, and (2.18) is satisfied for $-\infty < t < \infty$.

Definition 4. *We say that the Cauchy problem for (2.18) is* well posed in the sense of Lax *(or simply* well posed*) in $-\infty < t < \infty$ if the following two assumptions hold:*

(1) Existence of solutions for sufficiently many initial data: There exists a dense subspace D of E such that, for any $u_0 \in D$, there exists a solution $u(.)$ of (2.18) in $-\infty < t < \infty$ with

$$u(0) = u_0. \tag{2.19}$$

(2) Continuous dependence of solutions on their initial data: There exists a function $C(t)$ defined for $-\infty < t < \infty$ such that $C(t)$ and $C(-t)$ are nondecreasing, nonnegative, and

$$||u(t)|| \leq C(t)||u(0)|| \quad (-\infty < t < \infty) \tag{2.20}$$

for any solution of (2.18).

Condition (2) can be given an equivalent (but more palpable) formulation as:

(2') Let $\{u_n(.)\}$ be a sequence of solutions of (2.18) with $u_n(0) \to 0$. Then $u_n(t) \to 0$ uniformly on compacts of $-\infty < t < \infty$.

One can immediately note that the uniqueness condition in Definition 4 is missing. This is due to the fact that uniqueness is a mathematical consequence of existence 2.19 and continuous dependence 2.20. In a more general setting uniqueness of solution for a studied initial value problem is often established by showing that solutions of suitably taken, tractable initial value problems converge; continuous dependence is then used to infer from convergence of initial data to convergence of solutions.

The relationship between continuous dependence and uniqueness, assuming existence, is even stronger. As one can see from Theorem 6 of Appendix A, if operator A satisfies an additional condition existence and uniqueness of solutions also imply continuous dependence. This tight connection may tempt someone to try to give a non-circular justification for uniqueness – to try to give a non-circular justification for determinism – via providing an independent justification for continuous dependence. To do this was the original motivation for my research; the problem, as we have seen, lies in the inherently epistemic character of the usual justifications that are offered for continuous dependence.

Continuous dependence is also instrumental for various types of approximation results. Solutions of well posed problems can often be approximated by solutions of a series of different, more easily tractable differential equations; the time variable may also be discretized and the equation itself can often be replaced by a finite difference approximation[48]. As it is often indispensable for approximation and discretization continuous dependence is also often needed for the applicability of numerical methods and computer simulation.

[48]For some results see Chapter 5.7 and onwards in Fattorini (1983). Note that results about approximation and the link between continuous dependence and uniqueness do *not* require the norm to be approximately measurable or operationally significant; mathematical results do not require an interpretational link between the formal apparatus and measurement.

Finally, assuming that we are fine with some pre-established fit, continuous dependence allows us to conveniently choose a relatively simple representative from a set of given data that is compatible with measurement results and to predict by calculating the trajectory of a single, and hopefully simple solution. As given data can only be obtained up to a certain imprecision lack of continuous dependence would necessitate tracing the trajectory of every solution in order to arrive at the set of possible trajectories which may not be fitted within bounds. That we may represent time development of systems using a single solution is a major convenience.

What is the status of well posedness in current research? Even though Hadamard seems to have believed that failure of well posedness would not occur in world obeying physical laws, mathematical research unearthed many examples of problems with physically relevant equations that are not well posed; facing the counterexamples in the middle of the 20th century the reference guide of Courant and Hilbert opined that

> Nonlinear phenomena, quantum theory, and the advent of powerful numerical methods have shown that "properly posed" problems are by far not the only ones which appropriately reflect real phenomena. So far, unfortunately, little mathematical progress has been made in the important task of solving or even identifying and formulating such problems that are not "properly posed" but still are important or motivated by realistic situations. (Courant and Hilbert; 1962, p. 230)

Subsequent research has emphasized the importance of the mathematical choices we make in formulating our equations; in particular whether an equation is well posed in the sense of Definition 4 depends on the choice of the Banach space E. Failure of well posedness may then be linked to the inappropriate mathematical choices, as it is emphasized in the mid-80's by the author of *The Cauchy Problem*:

> A look at the vast amount of literature produced during the last two decades on *deterministic* treatment of improperly posed problems (including numerical schemes for the computation of solutions) would appear to prove Hadamard's dictum wrong. However, it may be said to remain true in the sense that, many times, a physical phenomenon appears to be improperly posed not due to its intrinsic character but to unjustified use of the model

> that describes it; for instance the model may have "unphysical " solutions in addition to the ones representing actual trajectories of the system, bounds and constraints implicit in the phenomenon may be ignored in the model, the initial and boundary value problems imposed on the equation may be incorrect translations of physical requirements, and so on. [...] With the possible exception of the reversed Cauchy problem in Section 6.1, none of the problems and examples in this chapter [which is an overview of ill posed problems in physics – B. Gy.] would probably be recognized by specialists as improperly posed, but rather as properly posed problems whose correct formulation is somewhat nonstandard. [...] For this reason such non well posed equations are often referred to as *weakly ill posed*, i.e. they become well posed if we choose another, more suitable space E. (Fattorini; 1983, p. 347,375)

In Appendix A we give examples for physical laws that have a well posed Cauchy problem using one formulation but have a not well posed Cauchy problems using another. We are also going to take this insight as our stepping stone for developing arguments for the necessity of well posedness for physical possibility. In order to proceed first we need to fix the sense of physical possibility that is being invoked in these discussion.

3.0 TWO READINGS OF PHYSICAL POSSIBILITY. BEST SYSTEM AND WELL POSEDNESS

3.1 THE RECEIVED VIEW OF PHYSICAL POSSIBILITY

Modal terms such as ‘possible’ and ‘physically possible’ can be understood in different ways. We do not attempt to overview the enormous philosophical literature: our focus is on the possible worlds approach that understands alethic modal statements that invoke possibility as an existential quantification over possible worlds[1]. According to this characterization if the statement ‘time travel is physically possible’ were true then it should be understood as the statement that ‘there exists a physically possible world in which time travel happens.’ I submit that this approach is not merely predominant among philosophers and philosophers of science but also among physicists when they talk of possibilities offered by our most fundamental theories. When asked whether time travel is physically possible physicists routinely point to the existence of mathematical models that satisfy Einstein’s equation and feature time travel. Taking the existence of such global models as evidence for physical possibility – even though these models do not resemble and may even be incompatible with our actual world – is best explained by the implicit adoption of the possible worlds approach[2].

[1]Thinking in terms of possible worlds goes back at least to Leibniz; for a modern introduction see Kripke (1959, 1963a,b) and Lewis (1973). There are philosophical approaches, such as that of Meinong (1960), Fine (1994), or Jubien (1996) that do not understand possibility in terms of possible worlds; for a general overview of the literature on possibility and ‘possible objects’ see i.e. Yagisawa (2009).

[2]The possible worlds approach have not always been so predominant. Planck and many other nineteenth century physicists understood the term (physically) possible as ‘available in the actual world.’ For them the

Note that physical possibility is not intended as an epistemic notion: physically possible worlds are not to be equated with epistemically possible worlds, that is with possible worlds that are compatible with some state of knowledge of observers. Lack of observers' knowledge may influence their epistemic possibilities but it does not bear upon what systems or interactions are physically possible[3].

The *received view of physical possibility* is the conjunction of the possible worlds approach with the tenet that it is physical laws that determine what is physically possible. The conjunction, albeit it is widely accepted, is not necessary. One may entertain either of these assumptions without the other: one may accept that the notion of physical possibility should be analyzed via a possible world approach but deny that the set of physically possible worlds

truth of the statement that 'a perpetual motion machine is physically possible' depended upon whether we could actually build one given the furniture of our actual world, that is given the systems and interactions that actually are. (For historical notes on different understandings of physical possibility see i.e. Uffink (2001).) Such an actualist account of possibility faces many difficulties, especially if we adopt the view that our actual world is best represented by a distribution of modally thin definite physical quantities on a spacetime structure, which is arguably the way how the actual world is represented by a solution of a differential equation. A system or process then either exists or does not exist in the (modally thin) actual world and hence making room for the possible in the actual is not straightforward (for an alternative research project that represents a possible world as not modally thin see Placek and Belnap (2012) and its references; on modal thinness of physical quantities see also (Belot; 2011, p. 108)). Notions such as 'could actually build' also carry connotations of agency which is difficult to reconcile with modal thinness, definiteness, and the assumption that agents are also subject to the same physical laws as other systems and interactions. Despite these worries I ultimately believe that the correct account of physical possibility is actualist, but an analysis would go beyond the scope of the present work.

[3]The hierarchy of modal notions is convoluted. It is customary to make distinction among logical, epistemic, metaphysical, nomic / natural, and physical possibility, logical possibility being the most and physical possibility being the least permissive. For the purposes of this work we assume that epistemic possibility differs from metaphysical, nomic, and physical possibility; we do not take stance in whether the latter three notions differ but we tend to use them interchangeably. Distinction between physical possibility and physical laws vs. nomological possibility and natural laws becomes important if we contemplate the existence of natural laws which are not (and which do not supervene in an appropriate sense on) physical laws; such distinction is often emphasized by philosophers of mind. Whether nomic and metaphysical possibility coincides is a debate between the so-called necessitarians and contingentist (see Kment (2012)). There are many other modal notions such as practical, medical, moral, legal etc. possibility that do not enter to our discussion, although we note that some nineteenth century physicists, such as Lord Kelvin and Maxwell, were concerned with a notion of possibility that is dependent upon the human condition (see Uffink (2001)).

is determined by physical laws (either because there are no physical laws or because physical laws are not sufficient to determine which possible worlds fall into the set of physically possible worlds)[4]. One may also accept that physical laws determine what is physically possible but deny that what is physically possible should be analyzed in terms of possible worlds[5]. In the rest of this work we concern ourselves with the received view of physical possibility and we explore strategies to defend the idea that only well posed problems give rise to physically possible worlds in the light of the received view.

The received view makes reference to physical laws; we now take a brief detour to discuss the question: what are physical laws?

[4]There is a point of contact here with the ongoing debate about the nature of scientific theories. The so-called *semantic view* maintains that one should think of a scientific theory T as a class of models $\mathcal{M}_T$. The most radical form (SMV5 in the terminology of Earman (2008)) of the semantic view claims that this class of models $\mathcal{M}_T$ in general can not be obtained as a set of L-structures (as a set of models in the logical sense) of formulas Δ of some suitable formal language: $\mathcal{M}_T \neq mod(\Delta)$ for any Δ set of propositions (for such pronouncement see (van Fraassen; 1989, pp. 211-212); also see (Earman; 2008, p. 20)). If laws of a physical theory T can be formulated axiomatically then the radical version of the semantic view denies that the set of models $\mathcal{M}_T$ which a physical theory is supposedly identified with are the L-structures that satisfy these physical laws. Since we can also naturally view $\mathcal{M}_T$ as the set of models that represent physical possibilities according to T the radical version of the semantic view seems to accept the possible world analysis but deny that the set of physically possible worlds are those that satisfy the physical laws. The radical version of the semantic view may then deny the received view while still entertain a possible world analysis of physical possibility. (Some proponents of the radical version of the semantic view such as van Fraassen indeed must deny the received view as he also denies that laws of nature exist; see later.)
I don't find the arguments presently cited in favor of the radical version of the semantic view convincing (mainly in agreement with the challenges formulated by Earman (2008); also see Halvorson (2012) for a different line of attack). I note, however, that while reading (B') of physical possibility (see later) *is* accepting of the received view of physical possibility it is also consistent with the radical claim of the semantic view of scientific theories. In reading (B') the set of physically possible worlds is *determined* by the physical laws but this set may not coincide with the set of models that *satisfy* the laws or satisfy some other set of formulas. Whether this is so depends on the definability of the mapping $\Lambda : \mathcal{L} \to \mathcal{W}$ that assigns to a set of laws the possible worlds in which those are laws. Undefinability of Λ would imply the radical claim of the semantic view.

[5]An actualist approach to physical possibility mentioned in a previous footnote would be such a position.

3.2 LAWS OF NATURE

There is a difference between the question 'what are physical laws' and the question 'what are *the* physical laws.' The latter may be answered by a list of differential equations featured in our most revered physical theories (for a different view, see Chapter 4); the former question asks about the criterion that differentiates laws that make their way to the list from other propositions that are non-laws.

There is not only a plethora of accounts of laws of nature but also many ways to group and label them alongside commonly shared characteristics[6]. Here we follow Cohen and Callender (2009) in distinguishing three varieties.

The *No-Laws* camp believes that there are no laws of nature: in idealized models we may find causes, symmetries, or some general principles but they are not worthy to be elevated to the status of laws. These accounts are frequently driven by a sort of pessimistic meta-induction: van Fraassen[7] draws motivation from the perceived failure of major approaches such as that of Lewis and Armstrong while Giere[8] cites cases from the history of science in which generalizations once thought to be laws were proven to be false. Mumford[9], Cartwright[10], and some projectivist accounts such as that of Ward[11] can also be lumped under the No-Laws heading. If one maintains that there are no laws then one can not appeal to the laws to determine physical possibilities and hence needs to reject the received view of

[6]Thus the main camps for the overview article Carroll (2012) are systems, universals, antirealist, antireductionist and necessitarian approaches, while Swartz (2009) differentiates two main camps, necessitarians and regularity accounts; they differ on whom to count as necessitarian. Belot (2011) focuses on best system, primitivist, and necessitarian approaches (mentioning alongside the Armstrong-Dretske-Tooley and other approaches). Earman (1986) also have a different terminology and analyzes many other accounts as well, such as the naive regularity account.

[7]See i.e. (van Fraassen; 1989, p. 130, pp. 180-181).

[8]See i.e. (Giere; 1999, pp. 86-91).

[9]According to Carroll (2012), Mumford (2004) can be seen as arguing for an antireductionist position.

[10]Cartwright (1983) is interpreted by (Cohen and Callender; 2009, p. 2) as a no-laws position.

[11]Ward (2002) is sometimes viewed as anti-realist (i.e. in Carroll (2012)) and sometimes as promoting the existence of laws (i.e. in Cohen and Callender (2009)) alongside other projectivist accounts, i.e. Goodman (1954) and Ayer (1956).

physical possibility.

The *Governing* camp insists that genuine laws of nature exists and stipulate that these laws govern events and happenings in the world[12]. In the eloquent words of Cohen and Callender, "just as librarians enforce the rules of book borrowing and policemen enforce traffic rules, so some Governing theorists think that necessitarian relations, primitive accessibility relations, or primitive universals enforce certain behaviors upon the events of the world. Other advocates of Governing are silent on how the laws manage these feats, but insist that they do and treat laws with the requisite governing powers as primitive" (ibid. p. 2). Not only do some Governing advocates treat laws as primitive, but they also ascribe the power to the laws to generate, evolve, bring about, or propagate physical states. Thus (Maudlin; 2007, p. 15) writes:

> My own proposal is simple: laws of nature ought to be accepted as ontologically primitive. We may use metaphors to fire the imagination: among the regularities of temporal evolution, some, such as perhaps that described by Schrödingers equation, govern or determine or generate the evolution.

The *Non-Governing* camp also insists that genuine laws of nature exists but deny that these laws govern. Instead of governing the laws merely describe certain aspects of patterns that obtain in the mosaic of events. One of the most prevalent Non-Governing view is the Best System account[13] associated with Mill[14], Ramsey[15], Callender[16], Lewis[17], Loewer[18], and Earman[19]. According to the Best System account the distinguishing feature of laws is

[12]For advocates of the Governing view see Armstrong (1978, 1983, 1991, 1993), Dretske (1977); Tooley (1977); Swoyer (1982); Shoemaker (1998); Maudlin (2007).

[13]After the associated authors the view could be abbreviated as MiRaCLE, although the abbreviation MRL is in current use after Mill, Ramsey and Lewis. We are going to use the BTS abbreviation after 'Best True System'.

[14]See Mill (1947).

[15]See Ramsey (1978).

[16]See Callender (2008); Cohen and Callender (2009).

[17]See Lewis (1973, 1983, 1986, 1994).

[18]See Loewer (1996).

[19]See Earman (1986).

that they are highly informative about the world in a simple way: they are propositions of a deductive system which best balances informativeness and simplicity.

One of the main dividing line between the Governing and the Non-Governing camps is in how they relate to Humean supervenience, “the doctrine that all there is in the world is a vast mosaic of local matters of particular fact, just one little thing and then another”[20]. In general advocates of a Governing view deny Humean supervenience about laws of nature while advocates of Non-Governing views, in particular of the Best System view, embrace it.

All three accounts of laws have been severely criticized. Without laws it is very difficult to explain practice of the physical sciences and to explain why certain regularities and patterns among events hold. While Governing views may be able to explain, in terms of their favored necessetarian relations, primitive accessibility relations, or primitive universals, why certain patterns among the events hold they seem to merely push the explanatory issue one step back as they are unable to explain why the employed particular relations or universals hold[21]. If they provide no explanatory advantage then postulating laws as entities existing in addition to the mosaic of facts seems metaphysically parsimonious. The Best System view is in danger to render laws subject dependent as notions of simplicity, informativeness, and best balance seem to be very difficult to articulate from an objective point of view[22].

In their above-mentioned article Callender and Cohen proposed to relativize the Best System account to specific choices of basic kinds or basic predicates. They claim that even though there is indeed no objective point of view from which simplicity, informativeness and

[20](Lewis; 1986, p. ix). Humean supervenience tries to capture the empiricist constraint that laws should supervene on the mosaic of non-nomic facts; what counts as ‘non-nomic’ and ‘fact’ is a matter of debate. There exists several, more sophisticated statements of Humean supervenience; for an analysis see Earman and Roberts (2005a,b).

[21]Whether by pushing back the explanatory question it also radically shrinks may be up to debate. Advocates of the Governing view tend to think this is so (thanks for Gordon Belot for this remark); I don’t share their impression but this is debate should be pursued elsewhere.

[22]For an overview of the main advantages and the main lines of criticism mounted against these views see i.e. Carroll (2012).

best balance could be judged such vantage point is also not necessary to find a satisfactory account of laws. They hold a true generalization a law relative to a choice of basic kinds or predicates if the generalization appears in all immanently Best Systems relative to these basic kinds or predicates. They take relativity to be an advantage of their account; with its help they claim to be able solve many problems plaguing the traditional Best System view. In addition they see Relativized Best System (RBS) to be capable of giving an account for laws in the special sciences.

This last point will be of interest for us later. In order to accommodate the special sciences the authors concede that RBS needs to allow for exceptions because claims of special sciences "will be in principle defeatable by lower-level physical limitations" (ibid. p. 25). One proposal they make to amend this difficulty is to "relax the requirement that the MRL laws be true and replace it with some other requirement, like pragmatic reliability (as determined by the science of interest)" (ibid. fn 24). Apparently the authors believe that (exact) truth is not a *sine qua non* of laws: it is consistent with an understanding of laws as simple and informative systematizations that under certain circumstances they may happen to be false. Presumably this can only happen if the price of falsehood is paid back in improvement of the overall balance of simplicity and informativeness. As generalizations that are not even approximately true can not be informative this suggestion amounts to relaxing the condition that laws are true generalizations to the condition that laws are approximately true generalizations.

3.3 TWO READINGS OF THE RECEIVED VIEW

There seems to be a not sufficiently recognized split in philosopher's understanding of physical possibility. When philosophers present the view they take to be standard, they usually state either of the following two, non-equivalent formulations:

(A) A possible world is *physically possible* if and only if it `satisfies` the physical laws of the actual world[23].

(B) A possible world is *physically possible* if and only if it `has the same` physical laws as does the actual world[24].[25]

It is difficult to assess what are the physical laws of the actual world, especially given the well known incompatibility of some of our best physical theories. To counter this and other difficulties we frequently rely upon a theory-relativized notion of physical possibility. Thus,

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[23]Examples:

> Letting $\mathcal{W}$ stand for the collection of all physically possible worlds, that is, possible worlds which satisfy the natural laws obtaining in the actual world, [...] (Earman; 1986, p. 13)

> In saying of a certain state of affairs that it is "physically possible," one of the things we might mean is this: that the state of affairs is one such that the statement that it obtains is, by itself, consistent with the laws of nature. (Chisholm; 1967, p. 412)

> Our world seems to be governed by laws, at least around here. When we say that an event or situation is physically possible we mean that its occurrence is consistent with the constraints that derive from the laws. (Maudlin; 2007, p. 18)

[24]Examples:

> A physically possible world is any possible world which has the same natural laws as does the actual world. (R. Bradley; 1979, p. 6)

> [...] There are possible worlds in which Ling-Ling is a plaid panda and in which the laws are exactly the laws of the actual world. Invoking the standard definition of physical possibility, it follows that it is *physically possible* for Ling-Ling to be a plaid panda. (Carroll; 1994, p. 174)

> Let the *physically possible* worlds be those in which all and only the laws of physics of the actual world are laws of physics therein. [...] (Witmer; 2001, p. 62)

[25]I thank Gordon Belot for pointing out that a very similar distinction between two notions of physical possibility appears in Earman (1995).

instead of delivering possible worlds from the actual world, it is customary to say things like "w is physically possible according to theory T," where T is a physical theory, such as the theory of general relativity. We can carry over the distinction between the two readings to this theory-relativized notion:

(A') A possible world is *physically possible according to a theory* T if and only if it `satisfies` the physical laws of T.

(B') A possible world is *physically possible according to a theory* T if and only if it `has the same` physical laws as does T.

If we assume that L is a physical law of T if and only if L is a physical law of the actual world then these theory-relativized notions (A') and (B') reduce to their actual world-based (A) and (B) counterparts.

3.3.1 The two readings produce different physically possible worlds

The set of physically possible worlds under reading (B) may be narrower than under reading (A). Reading (B) not only requires a physically possible world to satisfy the physical laws L of the actual world, but also that L are the physical laws of the physically possible worlds themselves. This latter condition does not follow automatically from the former: depending on our conception of physical laws it may be the case that a possible world w do satisfy L but L is not a physical law in w.

Indeed if Humean supervenience about laws holds we should expect such cases to occur. The Best System account of laws provides an illustrative example. According to the Best System account the distinguishing feature of laws is that they are highly informative about the world in a simple way: they are propositions of a deductive system which best balances informativeness and simplicity. Now suppose that Maxwell's equations are among those propositions which best balance informativeness and simplicity in our actual world. Since the possible world which is empty also satisfies Maxwell's equations it is physically possible according to reading (A). But this empty world may not be physically possible according to reading (B) since the deductive system best balancing informativeness and simplicity in the empty world may simply contain the proposition that 'the world is empty.' Maxwell's equations are true in this empty world but they are not laws since they are not simple enough given the alternatives.

The difference can also be seen without invoking the Best System account of laws. Note again that most of our current physical theories allow for an "empty" world to satisfy its physical laws. Hence the empty world is physically possible according to several different theories under reading (A'). However, this empty world could possess exactly the same laws of at most one of these theories, and hence it can not be physically possible under reading (B') for the other theories[26]. Hence as long as our account of laws of nature is capable to recover at least some of our current physical theories, without which it's arguably without much merit as an account of laws, the two readings will produce different physically possible worlds.

The two readings of physical possibility may only coincide if the recoverable sets of physical laws (the recoverable theories) perfectly partition the possible worlds. If the two readings coincide then there is either only one set of physical laws (one physical theory), regardless of circumstances such as the non-nomic constitution of the actual world, of if there are possibly different sets of physical laws (different physical theories) for different circumstances, then there is no possible world which satisfies more than one of these. Both of these options seems implausible and hence one needs to decide which reading is to be

[26]Slightly more generally: Suppose there is a possible world w_0 which satisfies the laws of two theory T_1 and T_2 which have a different set of physical laws; in other words, w_0 is physically possible for both theories according to reading (A'). Now w_0 can at most have the same physical laws as either those of T_1 or those of T_2, as these latter have different laws. Thus w_0 can be physically possible, according to reading (B'), at most either for T_1 or for T_2. Hence the two readings produce a different set of physically possible worlds for at least one of these two theories.

preferred as a reading of physical possibility.

3.3.2 Side notes

At this point it is reasonable to wonder what is the motivation for the “same” clause in reading (B’). The “same” clause prevents a physically possible world w to have *more* physical laws than does the actual world. But why is it not sufficient to require that all physical laws of the actual world shall also be physical laws of a physically possible world w?

The “sameness” restriction fosters quite unintuitive consequences for an adherent of a Best System account, such as a world w becoming physically impossible merely because it has an additional symmetry (which becomes a law in w) which the actual world does not have[27]. It would then seem that all textbook examples, and indeed exactly those physical models which we can make pragmatic use of by virtue of their additional symmetries, are not examples of physically possible worlds. This is especially counterintuitive given that confirmation of our physical theories typically proceeds through tests of their special models, which models then would not represent a physical possibility if the theory were true. Weird[28].

In this spirit fans of baroque philosophical vogue may find it compelling to combine and generalize the two readings as follows:

(AB’) A possible world is physically possible according to a triple (T_1, T_2, T_3) if and only if

(a) it satisfies the physical laws of theory T_1, and of theory T_2, (b) it has all the physical

[27]Example: w is a swarm of Newtonian particles perfectly aligned in a circle (the Best System laws are Newton’s laws plus a law expressing the circular alignment) while the particles of the actual world are aligned almost exactly like that of w but with slight displacements (hence the laws are merely Newton’s laws). w is then not (B)-physically possible since it also has an additional law which the actual world doesn’t have.

[28]Another weird feature appears when we combine a Best System account with the liberal reading of satisfaction (see below). Then invisible pink unicorns and other random stuff may show up in physically possible worlds but only as long as they don’t behave regularly enough to give rise to an additional law, for then the “only if” clause would break. So according to reading (B) adding random nonsense to our ontology may be permitted until its behavior is not too systematic, which is a quite interesting inversion of Occam’s razor.

laws of T_2, and (c) it might have some additional physical laws, but only if those are included in T_3.

(AB') reduces to reading (A') if we set $T_1 = T$, $T_2 = \emptyset$, and T_3 to include all formulas consistent with T_1. (AB') reduces to reading (B') if we set $T_1 = T$, $T_2 = T$, and $T_3 = \emptyset$. With other choices we get a host of new notions of physical possibility[29].

[29] We should note that there is some ambiguity in how we understand the notion "satisfies the laws." The more liberal understanding is that of model theory: we take the physical laws L to be formulas of an appropriate logic and think of possible worlds as models which satisfy, in the standard logical sense, these formulas. This liberal understanding is relatively precise, however it has a disadvantage in allowing physically possible worlds to be populated by many superfluous entities, such as invisible pink unicorns, whose properties are independent of, and whose behavior does not interact with, the physical laws of the actual world. Given a set theoretic model w satisfying L it is trivial to produce another model w' which also satisfies L but whose universe is much larger that that of w. A suitably expressive logic may also allow w' to satisfy some additional laws L' which also employ predicates not utilized by L.

Although the possibility of superfluous entities is not troublesome when we only want to study the properties of the physical laws of the actual world, the existence of physically possible worlds in which invisible pink unicorns fight a death-match in the immaterial background does bring some uneasiness. So one may opt for a more conservative reading of satisfaction and allow only for models which, say, are minimal models of a logic whose language is in some sense minimally sufficient to formulate the laws L. This reading is more faithful to the intuitive picture of the philosopher of physics who thinks of physically possible worlds as solutions of a differential equation (as the definition of what it is to be a solution of a differential equation brings in required kinds of restrictions), but the details of "minimal models" and "minimally sufficiency language" would needed to be fleshed out and it is unclear to me whether this could be done unambiguously. The logically minded anxious reader is welcomed to fill out the details.

Staying with the liberal understanding of satisfaction we may reformulate the two readings in the context of formal logic (thanks to Gergely Székely for calling my attention to this elegant way of presenting the idea). Let's choose a suitably rich language. If w is (a model representing) a possible world let's write $Laws(w)$ for the set of formulas which express the physical laws of w. We require that all possible worlds w satisfy their own laws $Laws(w)$, that is for all w we have $w \vDash Laws(w)$. Let w_a be (a model representing) the actual world.

(A) A possible world w is physically possible iff $w \vDash Laws(w_a)$.
(B) A possible world w is physically possible iff $Laws(w) = Laws(w_a)$.

Note that $w \vDash Laws(w_a)$ follows from $Laws(w) = Laws(w_a)$ as we require $w \vDash Laws(w)$. With some ambiguity let now $Laws(T)$ be the set of formulas expressing the physical laws of a theory T; the theory-relativized readings can be captured as follows:

(A') A possible world w is physically possible according to T iff $w \vDash Laws(T)$.
(B') A possible world w is physically possible according to T iff $Laws(w) = Laws(T)$.

In this formal context the task of an account of laws of nature is to say something informative about how $Laws(w)$ depend on a possible world w. A few hidden dangers lurk behind this formal approach, primarily

3.3.3 Which reading should be preferred?

It is unclear on what basis we could make up our mind about which reading is the 'right' one. Purely from the perspective of formal simplicity and elegance both readings have advantages and disadvantages. Reading (A) yields a set of physically possible worlds which can be mathematically explored quite effectively: since physical laws are typically formulated as differential equations the set of physically possible worlds is (represented by) the set of solutions of a system of differential equations, a well known object of mathematical study. Handling the set of possible worlds of reading (B) may be a messy business depending on how effectively one can handle the mapping $\Lambda : \mathcal{L} \rightarrow \mathcal{W}$ that assigns to laws the possible worlds in which those are laws; the apparent vagueness of the requirements of the Best System account does not encourage optimism in this regard.

While reading (A) produces a mathematically more elegant set of physically possible worlds, reading (B) produces a more elegant notion of accessibility through physical possibility. If we write $w_1 P_B w_2$ for "w_2 would be physically possible under reading (B) if w_1 were the actual world" then the relation P_B is an equivalence relation: it is reflexive, symmetric and transitive.

The respective relation P_A, however, does not necessarily have any nice structure. P_A is reflexive, however it need to be neither symmetric nor transitive. It might be the case that $w_1 P_A w_2$ but not $w_2 P_A w_1$: w_2 would be physically possible if w_1 were actual but w_1 would not be physically possible if w_2 were actual. And it might be the case that $w_1 P_A w_2$, and $w_2 P_A w_3$, but not $w_1 P_A w_3$: w_2 would be physically possible if w_1 were actual, w_3 would be physically possible if w_2 were actual, but w_3 would not be physically possible if w_1 were

due to the lack of pre-theoretic specification of what counts as a "model" and due to the ambiguities in how much detail these models "represent" the possible worlds they stand for. For instance if the ws only contain the non-nomic facts then the assumption that $Laws(w)$ is always a well defined set of formulas is equivalent with Humean supervenience about laws. It would be unhelpful to sneak in such assumptions from the outset hence we need to keep formalization urges at bay.

actual[30].

As the difference between reading (A) and (B) is a matter of lawhoodness a way to assess the merits of the readings is by regurgitating what laws are supposed to do for us. Do certain roles laws are supposed to play – that they govern, that they lend support for counterfactuals, that they play a role in explanation, etc. – favor one reading over the other[31]?

1) Even though we illuminated differences between reading (A) and (B) by relying on the Best System account of laws proponents of other accounts also face a choice between the two readings. It seems that the Governing view, especially in its more recent formulations, should favor reading (B) of physical possibility. According to some proponents given a state of the world the role of fundamental dynamical laws of nature is to 'generate' future states. Fundamental laws play a role in the becoming of future states akin to how a meat fed grinder plays a role in the becoming of the sausage. Due to their ability to generate the future states the fundamental dynamical laws are necessary constituents of the basic ontology of the world.

If 'generating' future states is a matter of lawhood, for states of physically possible worlds to develop similarly to how they do in the actual world the fundamental laws L of

[30]Take the following plausibility constructions based on the Best System account. Let w_1 be a Newtonian spacetime with a sourceless electromagnetic field and a number of electrically charged massive point particles. We assume that the BTS laws of w_1 are the Maxwell equations, the Lorentz force law and Newton's law. For the failure of symmetry, let w_2 be the completely empty Newtonian spacetime. $w_1 P_A w_2$ since the empty spacetime does satisfy all the mentioned laws. However the simplest and strongest description – the BTS law – of w_2 is the proposition that "w_2 is empty," and hence the Maxwell equations etc. are, albeit true of w_2, not laws of w_2. $w_2 P_A w_1$ is not the case since w_1 is not empty and hence it does not satisfy the BTS law of w_2. For the failure of transitivity let w_2 be a Newtonian spacetime with a number of massive point particles but without any electric charge and field (hence the only BTS law is Newton's law), and let w_3 be a Newtonian spacetime with a number of non-charged massive point particles but with a purportedly electromagnetic field which defies one of Maxwell's equations. We then have $w_1 P_A w_2$, we also have $w_2 P_A w_3$, but we don't have $w_1 P_A w_2$.
Under typical non-Humean approaches P_A may turn out to be an equivalence relation but it is difficult to see what would necessitate that it is; see discussion below.

[31]Proper treatment of this issue should be pursued by another work; we keep the discussion brief. Note that we do not make judgement about what are the laws or compare different conceptions of *laws* here on the basis of the goodies they are supposed to deliver us but we compare different conceptions of *physical possibility* on the basis of some assumed properties of laws. This of course does not make it more clear how valuable reasonings that presuppose that certain goodies need to be delivered can be.

our world would also need to be fundamental laws in the physically possible world. If some fundamental laws of our world were merely true generalizations in a physically possible world w then they would lack the power of generate the states of w. But the set of laws which generate the states are necessary constituents of the basic ontology of a world and hence when we judge physical possibility it seems we ought to keep these generating constituents, which are necessary, fixed. If the successive states of w were generated by another set of fundamental laws L' then w could be physically possible according to L', but that is different from being physically possible according to L. If, on the other hand, the successive states of w were not generated at all, and yet w were sufficiently similar to our world, then the need for laws which govern to account for the unfolding of the world is in serious jeopardy. The argument is similar for other Governing accounts that require laws to enforce certain behavior upon the events of the world.

2) Another feature oft attributed to laws is their ability to support counterfactuals[32]. Suppose I have an apple sitting in my hands. The truth of the counterfactual claim 'had I dropped this apple it would have hit the ground' is said to be supported by the law of gravity. In the language of possible worlds this can be reformulated as the claim that in all gravity-respecting physically possible worlds which are sufficiently close to the actual world and in which this apple is dropped the apple hits the ground.

It is unclear whether a need to support counterfactuals would induce preference of one reading over the other. According to both readings the law of nature which supports the counterfactual is a true generalization in the physically possible worlds, and hence if in some of these worlds the antecedent of the generalization is fulfilled then the consequent follows. Hence even reading (A) allows for evaluation of the truth value of counterfactuals along the lines described in the previous paragraph. Whether we should prefer reading (B) could depend on additional considerations, such as a desire that physically possible worlds should

[32] For an overview see Carroll (2012).

also support the same kind of counterfactuals as does the actual world. One could argue that it is plausible to require that in a physically possible world in which I dropped the apple it should be true that 'had I not dropped this apple it would have stayed in my hands.' After all this is a true proposition in our actual world so it would be weird if it became untrue in a world which is physically possible on the basis of our actual world.

3) Laws are also supposed to be the salt and pepper of scientific explanations. Staying with our previous example, the law of gravity does not merely imply that a dropped apple hits the ground, but on the count of being a law it also explains why does the apple do so. If we cash out this ability of laws to provide explanations in terms of counterfactuals we get a similar analysis as before. But even without such a translation one could wander whether we should let the ability to explain disappear in a physically possible world. If the law of gravity explains falling apples in our world why should it loose its power to explain in a world which, say, differs from our world in some minor detail somewhere far away? Reading (B) would ensure that our physical laws still explain in the physically possible worlds; reading (A) would not.

None of the considerations above are conclusive but in general reading (B) seems preferred when we want laws to deliver the usual goodies. But do we really need to make up our mind? There seems to be at least two ways in which the odium of choice can be shrugged off. Those who maintain that lawhood is independent of the non-nomic constitution of the possible worlds may take this independence to be indicative of the coincidence of the possible worlds of the two readings. After all there is no need to worry that a generalization which is satisfied by a possible world may be prevented from being a law in that possible world by the worlds unfortunate arrangement of non-nomic facts.

Even though under such accounts of laws there may be nothing *preventing* the coextension of the sets of physically possible worlds of reading (A) and (B) it is unclear what would necessitate that they do. A generalization may well be a law in a possible world independently of the worlds non-nomic constitution but this independence does not entail that in all possible worlds in which the generalization is true it is also a law. It is conceivable that the set of possible worlds in which a generalization is a law does not contain some possibilities in which the generalization is merely true. Of course it may be, say, mathematically convenient to work with a set of worlds in which a generalization is true, but mere mathematical convenience is not sufficient to equate this set with the set in which the generalization is also a law; additional philosophical premises would need to entail this coincidence, and without such suitable premises we still need to choose among the readings of physical possibility.

A second way to avoid choice leads through the relativization of the notion of physical possibility and results in conditional philosophical conclusions. That is, instead of making direct claims about what is physically possible, one makes claims of the form "if what you mean by physically possible is (A) then the result is X but if what you mean by physically possible is (B) then the result is Y etc." One can support this attitude by pointing out that the notion of physical possibility is highly technical[33] and hence conditional analysis is not necessarily inappropriate.

While conditionalization is always an option, it is difficult to escape the feeling that it is mere cover for perplexity. We surely intend to mean something substantial when we say that, according to our best understanding of gravitation (general relativity), time traveling spacetimes are physically possible. It may be that we translate this statement in a specific way, but this specific translation often amounts to a choice of a reading of physical possibility. For instance, we may intend this claim to mean that the truth of Einstein's field equations is consistent with the possibility of the existence of closed timelike curves; but this translation amounts to accepting reading (A). With such translations a choice is being made, and we should be able to motivate why are we making this choice and not the other.

[33] Thanks to Barry Loewer for this comment.

3.4 MAY READING (B') SALVAGE WELL POSEDNESS?

It is natural to ask whether there are philosophically interesting problems which hinge on the difference of the two readings. We now turn to a difference which arises if one accepts amenable modifications to the Best System account of laws. Based on this modified Best System account only possible worlds whose subsystems admit a well posed initial value problem may be physically possible under reading (B'). Scenarios of a theory T which feature initial value indeterminism or which feature non-continuous dependence of solutions on initial data are possible worlds which may not be physically possible according to T under reading (B') even though they are physically possible under reading (A'). Since a whole host of issues in philosophy of physics depend on the physical possibility of scenarios featuring indeterminism or essential idealization it is worthwhile to explore how demanding premises are needed to bring about this conclusion. Our tour will also serve as an exploration of the plausibility of Best System approaches to laws.

To unpack the jargon of the previous paragraph recall that a law of a physical theory is typically represented by a differential equation and a possible world is represented by a solution of this differential equation. Differential equations and their solutions have a handy relationship: if the equation is given we may only need some further property of the solution to determine the whole solution. It is especially desirable that we can do this when the needed property of the solution is a so-called initial value. The problem of finding a solution which satisfies both a differential equation and an initial value is called an initial value problem. When an initial value problem has a single solution which depends continuously on the initial values – that is, when sufficiently small variation in the initial value results in small variation in the solution – then we say that the initial value problem is well posed.

The initial value problem is not well posed if one of these mentioned conditions fail: First, the problem might not have a solution at all. Second, the problem might have more than one solution; this is the case of initial value indeterminism. Third, the solution might not depend continuously on the initial data, that is, even the smallest variation in the initial data may lead to infinitely large variation in the solution. For the sake of readability we are going to refer to a well posed initial value problem as a '*graceful problem*' and to a differential equation whose initial value problems are well posed as a '*graceful equation*;' the labels '*ungraceful problem*' and '*ungraceful equation*' will be respectively reserved for not well posed initial value problems and for differential equations whose initial value problems are not well posed. In doing so we bypass many subtle issues we discussed in Chapter 2 for the sake of presenting the main idea.

Graceful problems are highly informative: they not only pick out a single possible world among the many but they also give us approximate truths about this world if the initial value is given only with some approximation. On the contrast ungraceful problems are not very informative: if there is no solution they tell us nothing about the world; if there are many widely differing solutions or when continuous dependence fails and only inexact data is available they may provide truth at best about certain aspects of the world.

The main idea we follow here is to make use of this difference in informativeness of graceful versus ungraceful problems in the context of a Best System account of laws. Consider a physical theory with differential equation L and suppose that w is a solution of an ungraceful problem with equation L. According to reading (A') w represents a physically possible world. According to reading (B') w only represents a physically possible world if L also shows up among the propositions of the deductive system describing w which best balances informativeness and simplicity. If there is another deductive system with a better balance of informativeness and simplicity than what is provided by L then w is not physically possible according to our physical theory under reading (B') for L is not a law of w.

To cook up such an alternative deductive system showing something along the following lines would seem to be sufficient: whenever we have a solution w of an ungraceful problem with differential equation L we can always find a graceful problem with differential equation L' such that w is also a solution of this graceful problem and there is no significant difference in simplicity of L and L'. If informativeness of laws depend significantly on the issue of well posedness and if there is no significant difference in simplicity of their laws then it seems a better balance of informativeness and simplicity is reached by the alternative deductive system featuring L', and hence L is not a law in w and w is not physically possible under reading (B').

John Norton's Dome[34] might serve as an illustration. Imagine a ball resting on the top of a carefully designed Dome-shaped surface. The ball can move frictionlessly but it is restricted to move on the surface and is influenced by a homogeneous gravitational field. Our physical theory is classical mechanics with Newtons' laws, in particular the second law: $F = ma$. In order to find how the ball moves we need to solve the initial value problem where the force F of Newton's law is determined by the shape of the Dome and by the gravitational field and where the initial values are the initial position and momentum of the ball. If this were a commonplace problem in classical mechanics we would get a unique solution telling us how the ball is going to move. However the shape of the Dome was trickily designed so that our initial value problem yields many different solutions: the ball will spontaneously start to roll from the top of the Dome but classical mechanics can't tell us when this starting moment happens. Thus Norton's is a case of an ungraceful problem featuring initial value indeterminism as the initial values and the differential equation do not single out a unique solution.

A possible world w featuring Norton's Dome is physically possible according to reading (A') since it satisfies Newtons' laws. In order for w to be physically possible according to reading (B') we would further need the equation $F = ma$ to also be a law in the world

[34] For an accessible presentation see Norton (2003); for more in-depth analysis see Norton (2006) and Malament (2008).

w. $F = ma$ could fail to be a law according to a Best System type account if we could find another differential equation which is also satisfied by our possible world w, which is as simple as $F = ma$, and which is significantly more informative than $F = ma$ about the happenings in w.

There is a strong intuition that even if another differential equation also satisfied by w existed it would surely be no simpler than $F = ma$. With the possible exception of the case when the ball stays forever on the top of the Dome this intuition is probably right. However we shall argue that we can slightly relax the conditions above so that we may find a candidate for a law that is simpler than $F = ma$. To achieve this we need to alter the Best System account of laws. In particular we shall argue that it is consistent with the spirit of the Best System approach to allow for laws which are approximately true by which we mean that the laws are true up to a certain approximation or coarse graining. As we noted before the relativized Best System approach of Cohen and Callender is an account pointing towards this direction. To articulate these ideas we take a brief detour.

3.5 DETOUR 1: A BEST APPROXIMATELY TRUE SYSTEM?

Should Maxwell's equations be considered laws of our actual world? Consider the following remarks made by Einstein in his seminal 1905 paper on the light quantum:

> The wave theory of light, which operates with continuous spatial functions, has proved itself splendidly in describing purely optical phenomena and will probably never be replaced by another theory. One should keep in mind, however, that optical observations apply to time averages and not to momentary values, and it is conceivable that despite the complete confirmation of the theories of diffraction, reflection, refraction, dispersion, etc., by experiment, the theory of light, which operates with continuous spatial functions, may lead to contradictions with experience when it is applied to the phenomena of production and transformation of light. (Einstein 1905, pp. 132-133.)

Einstein takes Maxwell's as a theory which is true only up to an approximation, the approximation being a result of operating with time averages instead of momentary values of physical quantities. Even though he denies its exactness Einstein still uses a strong language to endorse Maxwell's theory as it "has proved itself splendidly," got "completely confirmed" in several domains of phenomena and "will probably never be replaced by another theory." It may be reasonable to interpret these statements as regarding the Maxwell equations as laws despite of their approximate truth; to the extent this sentiment is shared by other physicists it may point towards an acceptance of merely approximately true generalizations as laws.

To regard generalizations which are only true-to-an-approximation as laws likely does not mesh well with Governing accounts of laws. Approximate truth on the other hand does seem to be consistent with the spirit of the Best System account of laws. To ease discussion let us first highlight the difference between the 'Best True System' and the 'Best Approximately True System' accounts:

(BTS) Laws are propositions of the true deductive systems which best balance simplicity and informativeness.

(BATS) Laws are propositions of the deductive systems which best balance simplicity, approximation to truth, and informativeness.

BTS is a view that balances the virtues of informativeness and of simplicity assuming truth; BATS is a view that balances the virtues of informativeness, of simplicity, and of truth. According to BTS truth is a must; according to BATS, lots of simple informativeness may outweigh a small loss in exact truth. BATS does embrace truth but allows for the introduction of a certain coarse graining in the true description of the world. As BATS proposes a balance between all virtues without singling out one as absolute it seems to follow better the spirit of identifying laws as means to achieve effective organization of information about the mosaic of events than does BTS.

Difference between BTS and BATS may be highlighted by an example. Which deductive system carries more the merit of lawhood: one which implies the exact value of the electromagnetic field in one spacetime point but is completely silent about its values everywhere else, or another which implies the values of the electromagnetic field everywhere but only up to an approximation which lies beyond our measurement capabilities to detect? BTS would force us to choose the first option while BATS also allows for the second.

We do not aim to argue here for the validity of BATS. We merely submit that BATS is an account which should be amenable to defenders of a Best System account of laws. As we have already pointed out the Relativized Best System account implicitly already embraces BATS. Any other account that allows the special sciences to have laws is also likely to implicitly embrace BATS. If the same mosaic of events may be subject to laws of theories located on different levels of the proverbial layer cake then truth with approximation should be sufficient for laws as theories on a higher level typically operate with more coarse-grained descriptions than theories on the lower level. Allowing for approximate truth does come with a major disadvantage, though: it adds approximation to the laundry list of terms such as informativeness and simplicity which we need to make sense of[35].

[35]Without denying the need for clarification I resort to making only one comment. Approximation is best accounted for by appealing to a notion of distance between different initial data, different solutions, and so on: a claim about a physical quantity is approximately true if the true value lies within a certain distance from the claimed value. But this notion of distance is logically independent of the notion of 'distance' referred to in debates about 'closeness' of possible worlds. An example for this latter would be the evaluation of counterfactuals a la Lewis: a counterfactual claim is true if in all sufficiently 'close' worlds in which the antecedent is true the consequent is also true. This 'closeness' is a notion which needs metaphysical grounding; approximation, on the other hand, is primarily tied to epistemic interests, and it relates to issues of measurement precision. A typical physical theory already comes equipped with a notion of distance which reflects the connection of the mathematical apparatus to measurement, and hence this distance notion might be less problematic than the hypothetical closeness-of-possible-worlds relation of Lewis. One may of course connect these two interests, but they do not necessarily need to coincide, and hence we need to keep them separate.

3.6 BATS IN DOMES

Domes do not suit BATS well. To see how BATS change the landscape consider a finite trajectory w of a classical world consisting of a Dome and a ball which starts to spontaneously roll at a certain time. Reading (A') renders this world physically possible according to classical mechanics.

Substituting the force determined by the shape of the Dome and by the gravitational field, Newton's law in this world becomes

$$x^{(2)} = \sqrt{x}, \tag{3.1}$$

where x is the distance of the ball from the top of the Dome on its surface and the $^{(2)}$ superscript denotes the second derivative with respect to the time parameter. Albeit this equation contains some ugly mathematical notation it does strike us as pretty simple. Can we really hope to find an equation which better balances approximation to truth, informativeness, and simplicity than equation (3.1)?

Well, simplicity is in the eye of the beholder. Contrast (3.1) with the following differential equation:

$$x^{(k)} = 0, \tag{3.2}$$

where k is a yet undetermined number. A charitable reader may agree that, at least at a first blush, it is not unreasonable to hold that equation (3.2) is simpler than equation (3.1). There may be issues with the number k not being simple enough[36], but superficially speaking

[36]Merely as a tongue in cheek (not as a serious argument) one can "prove" that there are infinitely many simple natural numbers. A number is simple if it admits a simple definition. If there were finitely many simple natural numbers then there would be a largest among them. But then "the number that follows the largest simple number" would be a definition that is arguably simple, and hence it would define a simple number that would be larger than the largest simple number, a contradiction. Hence there are infinitely many simple numbers. As a corollary k can always chosen to be simple, since if it is not simple we can choose the first larger simple number, as it will also fit our purposes.

(3.2) is homogenous and only makes reference to differentiation, while (3.1) is inhomogenous and involves, besides differentiation, the square root of the variable as well.

The solutions of equation (3.2) are polynomials up to the k-th degree. Polynomials can be used to approximate finite trajectories and thus they can approximately describe the path the ball rolling down on the Dome takes. Mathematically speaking for any finite solution w of equation (3.1) we can find a value for k so that a solution w' of equation (3.2) stays within our desired level of approximation to w. By appropriately choosing k number of initial values we can uniquely determine this w' solution. Thus assuming our Best System only systematizes a finite lifespan of the Dome universe equation (3.2) may count as a proposition of a deductive system which approximates truth, is informative and is simple.

Equation (3.2) is merely approximately true while equation (3.1) is exactly true; if we accept that equation (3.2) is simpler than equation (3.1) then it becomes a question of balance whether (3.2) or (3.1) should count as a law in w according to the Best Approximately True System account. As the loss in truth due to the approximation could be made arbitrarily small, if a gain in simplicity has an effect on the balance of informativeness and simplicity then the overall balance may be improved by a suitable choice of small approximation. In case it turns out that equation (3.2) provides a better balance we get the conclusion that in reading (B') of physical possibility w is not physically possible according to classical mechanics since its laws are not those of Newtons'.

The argument has many shortcomings, notably the assumption that our Best System only systematizes an arbitrarily long, but still finite lifespan of the possible world. We will return to this point later. It is also worthwhile to point out that the argument is somewhat sensitive to the assumption that w merely contains a single Dome scenario. If, instead of a single Dome, w were also populated with many other classically behaving particles in various force fields then the balance between simplicity and approximation to truth could get altered in favor of Newton's law. Note however that even in a larger classical universe we could preserve the simplicity of equation (3.2) by appropriately introducing new dimensions and by increasing the value of k, and the fact that the achieved approximation to truth could still be made arbitrarily small might still keep the balance in favor of (3.2) as a law.

The gain in simplicity by equation (3.2) might very well turn out to be too small to favor one system over the other, and it may even be illusory. One further nagging point is that equation (3.2) requires k number of initial values to determine the solution, and so a large k can make the task of further determination of the world w look quite complicated. This brings up the question whether informativeness or simplicity of laws should depend upon their ability to get combined with additional accessible informative and simple information in order to produce further informative and simple information about the world. In the next section we take a look at arguments to the effect that it should. These arguments turn out to actually strengthen the negative conclusion regarding the physical possibility of the Dome.

3.7 DETOUR 2: WHY WOULD WELL POSEDNESS MATTER FOR INFORMATIVENESS?

Another point in need of clarification is the relationship between informativeness of differential equations and gracefulness of problems. We claimed that graceful problems are more informative about the world than ungraceful problems. Initial value problems are combinations of two components: differential equations and initial values. Best System accounts, however, only require laws to be informative and simple. Indeed this is how a Best System account can distinguish propositions which express laws from propositions which express other information, such as initial values: these latter might not be simple enough to be included in the deductive system which best balances informativeness and simplicity. So are we not comparing apples with oranges when we talk about informativeness of initial value problems in relation to a Best System account of laws?

To answer this question we need to take a closer look at the notion of informativeness. What does it take for a law to be informative? Intuitively we can associate informativeness of a proposition with the extent to which it restricts possibilities. Relative to some background knowledge the narrower the set of possible worlds in which a certain proposition is true the more informative the proposition is. If a proposition can uniquely pick out the actual world it is maximally informative. Tautologies are on the other end of the spectrum, they do not restrict possibilities hence they are not informative at all.

Differential equations are somewhere between maximally informative propositions and tautologies: they do restrict the possibilities to the set of solutions satisfied by them, but in themselves they typically don't single out the actual world. If we solely judge the informativeness of a differential equation by the extent to which it, in itself, restricts possibilities, then there may be no difference between graceful versus ungraceful equations. It is difficult to quantify and compare the size of the set of all solutions of two different equations, but it seems to be likely that, both in the measure-theoretic sense (if a measure over the sets of solutions can be introduced at all) and in the Baire category sense there will typically be no difference between the 'sizes' of respective solution sets.

However the informativeness of a proposition is always relative to a set of other propositions – the background knowledge – which accompanies it to restrict possibilities. For instance in case our proposition logically follows from the background knowledge it does not further restrict possibilities, and hence it is not informative in the sense defined above. Thus informativeness of a proposition is not purely a matter of how well this proposition restricts possibilities, but also how well it restricts possibilities given some other propositions that are present. The interaction between different propositions which are used to characterize the world should also be constitutive of their informativeness.

The sense in which graceful equations are more informative is tied to their interaction with other simple and informative propositions. We can think of differential equations as input-output systems in the sense that given additional information (such as initial data) as input they produce further information about the world (such as a single solution) as output. The advantage of graceful equations is that they are simple and informative input-output systems: they are able to take simple and informative propositions as inputs and produce simple and informative propositions as outputs. Ungraceful equations, however, do not produce simple and informative propositions as outputs even if they are fed simple and informative propositions as inputs.

To see this consider first the case of initial value indeterminism. Suppose we supplement our differential equation – which qualifies as a law under a Best System approach – with an initial value which does not qualify as a law due to not being simple enough. An initial value problem which has a unique solution is clearly superior in informativeness to one which features initial value indeterminism since the deterministic problem is able to uniquely pick out the world while the indeterministic problem is not, and the differences between the multiple different solutions might be large[37]. When presented with exact initial values a deterministic differential equation yields a more informative proposition than one which features initial value indeterminism.

Exact initial values, aside from textbook examples, are rarely simple. However they may be approximated with relatively simple non-exact initial values, and hence a proposition stating that the initial value is within the neighborhood of a simple value can be both simple and informative[38]. If we can supplement a differential equation whose solutions depend continuously on the initial data with such a simple and informative proposition we can end up with another simple and informative proposition about the solution staying within some

[37]We assume here that the indeterminism is not a product of gauge freedom but the multiplicity reflects real physical differences.

[38]We do not assume that an entire Cauchy surface can be approximated by simple and informative initial values, only that some subsystems' initial surfaces can be. If a proposition describing the entire Cauchy surface were simple then it would become a law according to a Best System approach.

time-dependent neighborhood of a simple solution[39]. However in case continuous dependence fails we may not get any informativeness out of the resulting set of solutions as these solutions differ beyond any measure.

To sum up, if the deductive system's interaction with further input and output is taken into consideration, a graceful equation is more informative than an ungraceful equation. Hence ceteris paribus a graceful equation qualify better for lawhood than an ungraceful equation. If in the balance lots of additional informativeness can outweigh a small loss in exact truth then this conclusion also holds if the graceful equation describes the same phenomena merely approximately.

3.8 THE MAIN ARGUMENT

Combining the remarks of the previous section we arrive to the mouthful Best Approximately True Input-Output System account of laws:

(BATIS) Laws are propositions of the deductive systems which best balance simplicity, approximation to truth, and informativeness, especially regarding the ability to entail further informative and simple approximate truths when they are supplemented by accessible informative and simple approximate truths.

We argued that according to BATIS a simple approximately true graceful equation qualifies better for lawhood than a simple true ungraceful equation. This seems to be the differ-

[39] Assuming that relative simplicity of initial data correlates with relative simplicity of the resulting approximate solutions, which seems reasonable at least in the realm of linear differential equations. (If continuous dependence holds typically any solution can be approximated with solutions of a converging set of initial value problems whose initial values, by construction, is relatively simple, and whose solutions are defined by Fourier series whose components involve these simple initial values. If the series converge quickly the first few terms in the series yield an approximation that is relatively simple. Simplicity of the differential equations of which these Fourier functions are solutions of also gives hope that our recipe for finding a simple and informative law can be generalized to physical theories other Newtonian mechanics.)

ence between equation $x^{(k)} = 0$ and Newton's $F = ma$ for possible worlds obeying classical mechanics which contain subsystems whose description under (3.1) feature initial value indeterminism. In such worlds, then, BATIS crowns $x^{(k)} = 0$ as a law instead of Newton's $F = ma$. As a consequence possible worlds which feature initial value indeterminism (or non-continuous dependence on initial data) are not physically possible according to classical mechanics under reading (B') of physical possibility even though $F = ma$ is true of them[40]. Note that this conclusion does not extend to possible worlds obeying classical mechanics which do not contain non well posed systems: in such worlds $F = ma$ is still a law since without the presence of non well posed systems $F = ma$ is as informative as $x^{(k)} = 0$ and has the further advantage of being true instead of being merely approximately true. Hence possible worlds which only contain subsystems with graceful problems remain physically possible according to classical mechanics.

Although we used Norton's Dome and classical mechanics as an illustration our plausibility argument can be generalized to cover cases of non well posedness showing up in other physical theories as well. Given a possible world w the task is to come up with a simple graceful equation which has solutions approximating w. For field theories we can keep relying on the homogenous differential equation whose solutions are the (now four dimensional) polynomials to approximate finite solutions. For other cases and for non-finite solutions we would need to come up with alternative mathematical constructions.

Despite its many shortcomings the argument presented here has one advantage. We have seen in Chapter 2 that prediction-based arguments for well posedness manifestly conflate epistemic and ontological interests: pragmatic difficulties with prediction does not justify

[40]There is another potential outcome of the balance struggle which, however, would yield the same conclusion. Consider again a possible world w which features a Dome. Even with BATIS the best balance may be reached by a deductive system which has $F = ma$ *and* a proposition setting the value of the time T at which the ball starts to spontaneously roll as laws. If this were the case w would still not be physically possible according to Newtonian mechanics under reading (B') for its laws would not exactly be those of Newtonian mechanics.

exclusion of solutions of ungraceful problems from the set representing physically possible worlds. The argument presented here against physical possibility of worlds describable by non well posed problems, however, does not promote removal of possible worlds from the set of physically possible worlds on the basis of alleged epistemic defects. Once we accept the outlined account of laws and reading (B') of physical possibility they jointly determine the whole set of of physically possible worlds. This whole set happens to not contain worlds describable by non well posed problems; no further surgical removal of possible worlds or further epistemic arguments are needed to achieve the conclusion that only worlds describable by well posed problems represent physical possibilities. The argument presented here thus seems to avoid the gap objection.

3.9 OBJECTIONS

The conclusion of an argument can only be as strong as the premises on which it is based. As the intended conclusion narrows the gap between 'what can be known' and 'what can there be' assumptions with epistemic implications needed to have been snuggled in at some point. The entry point for such assumptions here is the account of laws of nature. To the extent BATIS is acceptable as an account of laws we massaged the part of our formal apparatus which is relied on to generate modalities – the part of the formal apparatus on the basis of which the 'what can there be' question is answered – and as a result the gap between accessible and possible got narrowed without an apparent conflation of the epistemic and the ontological.

It is questionable whether BATIS is not overly epistemically laden to qualify as an account of laws of nature. The most obvious epistemic entry point in is conditioning on the accessibility of the given data. This accessibility condition was added to single out initial values as the data on which solutions need to depend continuously. The deductive system's desirable interaction with further simple and informative propositions may be used as a justification for continuous dependence on some data (and hence for well posedness in general) but without additional requirements such as accessibility nothing seems to necessitate that the data should consist of initial values. One could argue the same way for why continuous dependence on boundary values or on any other type of data is crucial for informativeness. So why would an adherent of a Best System account of laws single out initial values as the type of data on which solutions should continuously depend? The same point can be raised regarding uniqueness: even if uniqueness of solutions given some data matters for informativeness why should this data be initial data as opposed to some other type of data[41]? Initial data may indeed be accessible for observers in a way boundary data or other data from the future is not, but it is unclear why should laws care about being compliant with data that is accessible for observers as, after all, they are supposedly determined by the whole mosaic of events and not merely by events that are accessible for observers.

There is hope that the requirement of accessibility can be dropped altogether from BATIS. Based on an analysis of differential equations in physics Callender (2008) argues that the direction of time – and thus the direction from which data is judged to be initial – is determined by the informativeness requirement of a Best System. Under this analysis 'initial' turns out to be mere labeling that signifies the direction from which a given input, coupled with the differential equations that qualify as laws, most increases informativeness. Requiring data to be 'initial' data then is not an epistemic requirement but a further elaboration on a notion of informativeness that is not based on observers' limitations but on

[41] For instance, in the case of Norton's Dome the initial data consisting of the position and of the momentum of the ball is insufficient to determine a single solution, but the initial position and momentum joined by one further data – the time at which the ball starts to spontaneously roll – does single out a unique solution. Hence for the Dome Newton's (3.1) merely requires three values to determine a solution, while (3.2) requires k values to determine a solution. As k is typically large, the only real advantage of the latter equation is that all k values are initial values, while the time at which the ball starts to roll is not an initial data.

objective features of the Humean mosaic of events. This result would in turn allow us to drop the 'accessibility' requirement from BATIS while keeping the presented argument for well posedness intact. Many details in Callender's analysis are left open for future research[42]; we do not pursue this matter further here either.

Hailing $x^{(k)} = 0$ as a law for worlds in which Newton's $F = ma$ nevertheless holds is going to strike some as a reductio refuting either reading (B') of physical possibility, or BATIS, or some other premises[43]. This is a licit move, but one should not dismiss the conclusion without identifying the premise that is the target of the modus tollens. We have contrasted the two readings of physical possibility earlier; some considerations seem to favor reading (B') over reading (A') but it is unclear where to find higher grounds on which we could stand to reject either of the readings. Before we quickly dismiss BATIS note again that BATIS does not depart too far from the widely accepted BTS account. We argued that allowing for truth with an approximation is consistent with the spirit of BTS and in fact appears in its recently defended version. Introducing a preference for well posedness is, on the other hand, merely an elaboration on the notion of informativeness which is integral part of BTS itself[44]. Hence if the modus tollens is directed against BATIS the hit should likely be taken by its core: the idea that laws are propositions which allow for a best balance between informativeness and

[42]For a similar view see Skow (2007).

[43]It may be argued that the premises could all be accepted individually but their combination is inconsistent. If one accepts Humean supervenience about laws as well as that only the actual Humean mosaic of events exists in a meaningful sense then talk of physical possibility becomes problematic. Here we only point out that all previously mentioned defendants of the Best System approach do talk in their works about physical possibility in the same unproblematic way as most other philosophers do. Exploring possible inconsistencies of this approach is out of the scope of our present treatment.

[44]Callender (2008) seems to embrace the very same elaboration on the notion of informativeness as BATIS albeit he does not make the difference between his account of laws and the standard BTS account explicit as we do here. Callender also argues for the importance of well posedness as an element of informativeness and he also assumes, without arguments, that laws need not only be simple and informative themselves but also need to interact well with further data. See i.e. (Callender; 2008, p. 3) : "[a best system] will instead contain a way to generate some pieces of the domain of events given other pieces. In other words, it will favor algorithms, and short ones at that. The more of what happens that is generated by small input the better."

simplicity[45].

One of the most serious defects of our 'wonderlaw' (3.2) is that it can only guarantee approximate truth if we assume that our Best System systematizes the happenings of a world with a finite lifespan. There is no guarantee that our wonderlaw can do the same for a world whose happenings go on forever. Worse, there is no guarantee that our wonderlaw can systematize happenings of a world which contain subsystems which disappear to infinity in a finite time, such as Xia's example of a non-collision singularity. The reason is that polynomials can be used to approximate continuous functions on a compact domain but the same is not true for continuous functions of an unbounded domain or on an open domain.

While this is a serious objection against (3.2) being applicable for all interesting scenarios in classical mechanics it is not a refutation of the existence of another simple approximately true[46] graceful equation which could be relied on to carry the same argumentation through. To render the objection conclusive one would need to find a solution of a physically relevant

[45] In my view the main problem with BATIS (and with BTS), apart from the inherent vagueness of concepts such as simplicity, is not that it is too epistemically laden but that it is not sufficiently epistemically motivated. Examples such as our wonderlaw 3.2 become possible because there is no requirement built in to the account that would force discoverability of laws. How such requirement could be built in would hinge upon a way to solve the problem of induction which we are not touching upon here.

[46] A further note should be made about approximate truth of deductive systems. When we talk of truth in the context of formal languages we mean truth relative to a model w: a deductive system is true relative to w if w satisfies the formulas of the deductive system. We handled approximate truth similarly: we held a deductive system to be approximately true relative to w if there is a model w' satisfying the formulas of the deductive system which is 'close' to w in some sense in which distance of models can be judged. Such comparison of models yielded our conclusions in particular about the approximate truth of some differential equations: we held a differential equation to be approximately true relative to w if the differential equation had a solution which was 'close' to w.
However there are other ways in which approximate truth of deductive systems could be cached out. Comparison of deductive systems may proceed on the basis of more than a pair of their models, or it may proceed without making any reference to their models at all. The former strategy can be naturally exemplified in the context of linear differential equations: one could say that two differential equations are 'close' to each other if the differential operators – or, alternatively, if the propagators – of the two equations are 'close' to each other in the operator norm. The latter strategy could proceed by introducing a distance among formulas, for instance on the basis of the number of steps it would take to get from one formula to the other according to some well defined method of formula construction and deconstruction. If these alternative methods of comparison can be philosophically motivated different conclusions may be reached along the same line of argumentation.

ungraceful problem which can not be approximated by a solution of any other simple graceful problem. It is a question open for further investigation whether for specific scenarios such suitable differential equations can or can not always be found. Without some further elaboration on the notions of simplicity and informativeness this question can not be conclusively decided.

Even though no unambiguous elaboration on the notions of simplicity and informativeness is forthcoming I believe that the investigation of this mathematical issue should be pursued further as it is likely going to provide valuable insights. To bring this project to fruition, instead of attempting to find some abstract scheme which allows approximation of an arbitrary differential equation, we should get our hands dirty and overview the relationship of various concrete partial differential equations in physics. Elliptic and parabolic differential equations are the typical sources of failure of well posedness in physics and there is a general sense in which such equations can be approximated by quasilinear first order hyperbolic equations whose initial value problems are well posed. Robert Geroch, one of the main authorities on partial differential equations in physics opines that

> [...] A case could be made that, at least on a fundamental level, all the "partial differential equations of physics" are hyperbolic – that, e.g. elliptic and parabolic systems arise in all cases as mere approximations of hyperbolic systems. Thus, Poisson's equation for the electric potential is just a facet of a hyperbolic system, Maxwells equations. (Geroch; 2008, pp. 2-3)

Geroch then proceeds to show that a general symmetrization procedure is available for quasilinear first order hyperbolic systems; for symmetric systems general theorems on existence and uniqueness of solutions are available. This is a strong indication that in a relevant sense solutions of non well posed problems in physics can be approximated by solutions of well posed problems. As these latter are problems of some fundamental differential equation required simplicity of the graceful equation may get automatically resolved via the oft-argued simplicity of laws of our fundamental physical theories.

These issues are not pursued further here; we hope to have shown that it is not implausible that the required kind of approximating graceful problems exist and hence that the presented argument for well posedness may succeed even in infinite cases. We should nevertheless emphasize again that many important technical subtleties got suppressed for the sake of presenting the main idea. We introduced the notion of graceful vs. ungraceful problems and graceful vs. ungraceful equations to improve the readability of the text but these notions carry the danger of oversimplification. For instance depending on our particular definition of well posedness graceful and ungraceful equations might not be exhaustive categories, as there might exist differential equations which have both well and not well posed initial value problems. It is straightforward to modify the argument to allow for such mixed cases, but for the sake of readability we leave this task to the Reader.

4.0 LAWS AND WELL POSEDNESS

4.1 INTRODUCTION

In Chapter 3 we pointed out that the theory-relativized received view of physical possibility has two non-equivalent formulations:

(A') A possible world is physically possible according to a theory T if and only if it *satisfies* the physical laws of T.

(B') A possible world is physically possible according to a theory T if and only if it *has the same* physical laws as does T.

Depending on our conception of laws of nature reading (B') may yield a narrower set of physically possible worlds than does reading (A'). Utilizing reading (B') we constructed an argument for well posedness by adopting a suitable variant of the Best System account of laws. The argument avoided the gap objection as it did not operate by a selective removal of physical possibilities on the basis of epistemic motivations: the whole set of physically possible worlds happens to not contain worlds that are represented by solutions of non well posed problems.

Non-philosophers likely remain unimpressed by this argument. Even if all philosophically satisfying considerations preferred reading (B') mathematical pragmatism would trump these considerations in the mind of a practicing physicist. By choosing a particular mathematical formulation of a physical theory reading (A') allows discussion of possibilities in a mathematically clear and unambiguous way while reading (B'), sans any clearly and effectively formulated account of laws, does not. As the sentiment that well posedness is a necessary condition for physical possibility is widespread among practicing physicists it would be more satisfying to back up their sentiment by arguments relying on reading (A')[1].

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[1] It is instructive to take a look at a (highly idealized) genesis of reading (A') of physical possibility. The story goes as follows. We observe a number of 'similar' physical systems and collect data about their behavior. Based on the data we construct mathematical representations of the observed systems. During this process we find commonalities among the representations; if we get lucky we can capture some of these commonalities mathematically. The mathematical structure of the commonalities is then suggestive of mathematically admissible representations of yet unobserved physical systems that also obey the commonalities. We may then conduct experiments to verify whether yet unobserved physical systems indeed behave as the mathematically generated representations, to which they correspond to, suggest they should. If we are lucky they do. After substantial effort to verify a variety of mathematically admissible representations our confidence may reach a point to make two sort of generalizations about the commonalities.

The first generalization asserts that (G1) all representations of 'similar' physical systems obey the identified commonalities. This generalization is a claim about physical systems that appear in our actual world. As an example if mathematical representations were functions of the form $f : t \mapsto \mathbb{R}^{3N}$ and if identified commonalities were Newton's Laws then the first generalization would assert that all physical systems in our actual world that are 'similar' to the one we studied and can be represented by a function f obeys Newton's Laws.

The second generalization asserts that (G2) all representations that obey the identified commonalities are representations of physical systems that are possible. This generalization is a claim about modalities and it is not tied to physical systems that appear in our actual world. As an example if mathematical representations were functions of the form $f : t \mapsto \mathbb{R}^{3N}$ and if identified commonalities were Newton's Laws then the second generalization would assert that all functions f that obey Newton's Laws represent physical systems that are possible.

These two generalizations are independent from one another. The first generalization appears more humble but it suffers from vagueness resulting from the notion of 'similar' systems. Unless the identified commonalities happen to hold for *all* actual physical systems (which would be characteristic of a fundamental true physical theory that is yet to be found) the first generalization requires a way of identifying systems as similar in a non-circular way, that is without relying on the success of the commonalities to hold for the representations. The first generalization is also merely an assertion about what there actually is and does not imply claims about what is possible. The second generalization appears to suffer less from vagueness but it operates with a notion of possibility that outstretches the resources of the actual world. The second generalization is also ridden with semantic problems as generalization from a small set of mathematical representations to all mathematically admissible representations does not automatically imply that the semantical link between members of the small set of mathematical representations and actual physical systems also gets generalized to a semantical link between the set of all mathematically admissible representations and physical systems they are supposedly representing.

Reading (A') of physical possibility is an expression of the permissive attitude of the second generalization: physical possibility is essentially mathematical compatibility with certain constraints imposed by the laws. Until proven otherwise any representation is taken as physically possible if it takes the right mathematical form and if it meets the constraints imposed by the laws of a theory. A physical theory thus identifies two

Adopting reading (A') leaves us with fewer options to back up the physicists' sentiment. There are examples of physically relevant differential equations with non well posed problems. If the differential equation represents the law and if the solutions of the differential equation represent the physical possibilities then, according to reading (A'), there are physically possible worlds represented by solutions of non well posed problems. Whether reading (A') leaves room for arguments for well posedness depends on whether there is room left for said differential equations not to represent the laws of the physical theory or for solutions of these differential equations not to represent possible worlds of the physical theory[2].

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components in a representation of the world: a component which the theory proclaims to be fixed and a component which the theory may allow to vary. The modal character of the physical theory arises from associating the fixed component with the necessary and the variable component with the accidental. It is in this sense we take solutions of a fundamental differential equation to represent physically possible scenarios: we construct a mathematical representation of an arrangements of facts, we realize that this mathematical representation can be viewed as a solution of a differential equation, proclaim that the differential equation is the fixed component – the law – and that the solution is the variable component, and proceed to view other compatible variable components – other solutions of the same differential equation – as representations of other physical possibilities.

[2]This question in turn depends on how narrowly we construe a 'physical theory.' Do we allow a 'physical theory' to potentially have different mathematical formulations? Do we allow a 'physical theory' to potentially identify different mathematical structures as representations of laws or as representations of possible worlds?

Mathematical logic is suggestive of answering both questions negatively and construing 'theories' narrowly. A fixed mathematical formulation becomes constitutive of a 'theory' if we identify a 'theory' with a specific set of formulas F of a fixed formal language. Identification of a fixed representation of laws and of possible worlds becomes constitutive of a 'theory' if we identify physical laws of a 'theory' with this set of formulas F and if we identify possible worlds with L-structures. With these choices the set of physically possible worlds of the 'theory' according to reading (A') are the set of L-structures that satisfy F. If for these structures the question of well posedness can be meaningfully asked then it is either the case that there are possible worlds which are solutions of non well posed problems or it is the case that there are no such possible worlds. Aside from tinkering with the definition of well posedness there is little room left to influence the outcome.

'Physical theories' can rarely be found as a set of formulas of a fixed formal language. Treatises on classical and quantum mechanics, electrodynamics and relativity theory present many formal and semi-formal assumptions and methods but they do not present the theory as an axiomatic system of a specific formal language. Although for pragmatic purposes the treatises are sufficiently mathematically precise there seems to be a leeway in how to ground their concepts in formal systems. Shall the underlying language be first order or second order? Shall the logic be classical or intuitionistic? Shall the language contain countably many or continuously many of the various logical constants? With each choice we can get a different formal system. (There are ways to compare and identify formal systems that are formulated in different languages. The problem is that there are multiple ways to do so; for instance there exist multiple notions of definitional equivalence of theories (see Szczerba (1977)). Identification of a formal systems thus depends on a choice of

If there are several ways to identify laws of a physical theory and to identify the possible worlds that are supposed to satisfy these laws then reading (A') may allow tinkering with the set of physically possible worlds. In this Chapter we focus on the laws; in Chapter 5 we

a notion of definitional equivalence and with different choices different formal systems may count being the same.). If we construe 'theory' narrowly then different formal systems become different 'theories.' However, as long as these different formal systems all respect some important features (such as they contain formulas expressing the laws) we tend to naturally identify them as different axiomatizations of the *same physical theory*. This identification of different formal systems as being axiomatizations of the same physical theory only makes sense if a fixed mathematical formulation is not constitutive of a 'physical theory' but a 'physical theory' can have alternative axiomatizations.

A 'physical theory,' as presented by standard treatises, may also be consistent with different choices for the mathematical objects that are supposed to be representations of laws and of possible worlds. Let us briefly recall one among the many examples we already mentioned in another context in Chapter 2. (Identification of solutions in the presence of gauge freedom by the 'same physical theory' could be another example.) It is typical to take a solution of a differential equation to represent a physically possible world. There exist, however, several non-equivalent solution concepts for differential equations and thus the set of physically possible worlds depends on which of these solutions concepts we choose to represent possibilities. For instance if we formulate Maxwell's equations using college analysis and we rely on the notion of a classical solution of a differential equation then, strictly speaking, there are no physically possible worlds with point charges as the Dirac delta is not a function in the required sense. However if we consider distributional solutions for differential equations to be representations of physical possibilities then there are physically possible worlds with point charges as distribution theory is capable of handling Dirac-delta-like entities. Understanding 'theories' narrowly would imply that these representations belong to different 'theories' but there is a natural sense in which the representations, albeit being different, belong to the same 'physical theory,' namely to Maxwell's theory.

Suppose it is reasonable to follow these linguistic clues and we take a 'physical theory' to be an entity that in itself lacks complete specification of the mathematical objects that represent laws or represent possible worlds. Then in order to determine what are the physically possible worlds we need to add such a specification. Thus instead of talking of

physically possible worlds of physical theory T

we need to talk of

physically possible worlds of physical theory T with specification S.

As highlighted by the examples above for a different specifications S we may end up with a different set of physically possible worlds.

What is the point of stripping off specification S from the 'physical theory' T if we add S back right when we need to determine the set of physically possible worlds? Separating T and S may illuminate what an argument for well posedness aims to achieve. For some choices of S physically possible words of $T + S$ may respect the requirement of well posedness, for other choices of S they may not. Choice of S may not be completely arbitrary: there might be good reasons to prefer some choices of S to others. If these good reasons prefer a choice of S so that physically possible worlds of $T + S$ respect the requirement of well posedness and if they are non-circular then we have an argument for well posedness. This logic behind choosing a particular mathematical formulation of a physical theory remains hidden if a 'theory' is understood in a narrow way.

treat the possible worlds.

It is commonplace to assume that the mathematical structure which represents the laws of dynamical physical theories are differential equations. Philosophers often debate the question what makes laws of nature laws[3] but when the question becomes what are *the* laws of nature they point in unison with the physicists to a list of well known differential equations, such as Newton's $F = ma$, Maxwell's equations, Schrödinger's equation, the Dirac equation, Einstein's field equations etc.

Differential equations may arise naturally from empirical studies. When the dominant physical quantities affecting a given physical system are known one can often make educated guesses about the relationship of these quantities. Talking of dynamical theories these relationships typically involve rates of change and displacement of the quantities. As the number of independently observable quantities and the number of independently entertained connections among these quantities is often limited the relationship between these rates of-

How wide shall be the reach of freedom for choosing a specification S? A typical way to narrow the set of physically possible worlds of a theory T_L is by imposing additional constraints. An example could be the imposition of an energy condition C which is not implied by the dynamical laws L of the theory T_L. By imposition of a constraint the set of possible worlds which satisfy laws L gets narrowed to the set of possible worlds which satisfy laws L and also satisfy the further constraints C. This narrowed set of worlds then might have desired properties that the wider set of worlds lacks i.e. requirement of well posedness may be respected in the narrower set even though it is not respected in the wider set.

The ambiguity in the notion of a 'physical theory' (the freedom for choosing different specifications of S) can not be as wide as to allow ambiguity regarding the inclusion of additional *laws* like an energy condition C. An energy condition is a statement that evidently alters the physical content of the theory. A theory T_{L+C} whose laws are $L + C$ is a different 'physical theory' than T_L whose laws are merely L, and thus discovery of an additional law C can not be taken as an argument for well posedness for the physical theory T_L even if it produces a different physical theory T_{L+C} which respects the requirement of well posedness. (If we add C to T_L not as a law but merely as a condition that have pragmatic use for calculation for specific scenarios then T_L may remain intact as a theory but then we again do not have an argument for well posedness for T_L. According to reading (A') the set of possible worlds that satisfy $L + C$ is the same as the set of physically possible worlds of a theory that has *laws* $L + C$; if only L are the laws then the physically possible worlds are those that satisfy L only and thus C is of no help for narrowing this set.)

I can offer here no principled way to draw a line for deciding what freedom should or should not be allowed in what a 'physical theory' consists of; in general empirical distinguishability seems to be a good guideline but further elaboration on this issue is needed.

[3] See our discussion in Chapter 3.

ten takes the form of a differential equation. These differential equations compactly and informatively summarize the relationship among the physical quantities and they may often be generalized to describe other similar physical phenomena as well. Hence from a heuristic - epistemological standpoint differential equations seem natural candidates when we seek a mathematical object that is to be interpreted as the modality-generating fixed component – as the 'law' – of a theory.

Differential equations, however, may not be natural choices as objects of interpretation when we consider one of the strongest intuitions we tend to associate with laws: that laws 'govern,' 'evolve,' 'propagate' or 'bring about' the states. A differential equation typically connects time and spacial derivatives of state variables but it is not straightforward to interpret this connection as an expression of one thing evolving another thing. There exists another mathematical object which fits these intuitions better: the so-called propagator.

To fix ideas consider the following abstract differential equation:

$$u'(t) = Au(t) \tag{4.1}$$

where A is a densely defined operator in a Banach space E (see Appendix A for a precise treatment). We can think of E as the set containing the possible states and of a solution $u(t)$ satisfying (4.1) as a (representation of a) physically possible world according to this differential equation. Many fundamental differential equations in physics fit this abstract scheme[4]. Under certain circumstances – soon to be addressed – the solutions of this differential equation can be expressed in the following form:

$$u(t) \doteq S(t)u_0 \tag{4.2}$$

where u_0 is an arbitrary element of E. $S(t)$ is called the *time evolution operator* or *propagator* of equation (4.1).

[4]Examples include the Maxwell equation, the Schrödinger equation, and the Dirac equation for free particles. See Appendix A for some details.

The propagator does what its name suggests: it takes a state and a time t as an input and evolves the state to a new state after t amount of time passes. Thus as long as we think about laws as entities which 'govern,' 'evolve,' 'propagate' or 'bring about' states the propagator may be a more natural choice for the mathematical object we intend to interpret as a law.

To find the propagator one needs to solve the differential equation. This is the sweaty part, the bread and butter of the pragmatics of physics. It can be quite difficult to find the solution of particular initial value problems – hefty cash prizes await those who solve some of the outstanding open problems –, let alone expressing a propagator which captures the time evolution for all initial states. But these problems we face in the course of finding the propagator are 'epistemic-pragmatic' in their origin: we humans have difficulties in finding solutions or expressing solutions in a compact form which is easy for us to comprehend and work with (i.e. in an analytic form). The existence of the propagator as a mathematical object is independent from these epistemic-pragmatic difficulties.

We typically rely on differential equations to dissect descriptions of a particular type of dynamical system to fixed and variable components: the differential equation is identified as the fixed component and its solutions are identified as the variable components. A similar dissection can be achieved via propagators: the propagator is to be regarded as the fixed component and the solutions generated by the propagator equation (4.2) as the variable components. Thus we arrive at different notions of of physical possibility depending on whether we take the differential equation (4.1) or the propagator equation (4.2) to express the law of our physical theory T:

($\mathbf{A}_d$') A physically possible world of a theory T is represented by a solution of T's fundamental differential equation.

(A_p') A physically possible world of a theory T is represented by a solution[5] generated by T's fundamental propagator equation.

Both formulations are variants of reading (A') of physical possibility but they regard a different mathematical object as the appropriate representation of the dynamical law of a theory: a differential equation versus a propagator equation. Both formulations rely on a mathematically sound and unambiguous separation of fixed and variable components of given physical descriptions. Each formulation has its own advantage and disadvantage: the differential equation may be easier to find and may be more cogently expressed than the propagator, which gives an advantage to variant (A_d') over (A_p'). On the other hand given the propagator and the state space E the set $\mathcal{W}_{Tp}$ of (representations of) physically possible worlds of (A_p') is straightforwardly generated, while generating the respective set $\mathcal{W}_{Td}$ of (A_d') requires finding all solutions of the differential equation and to do so we need to face the same difficulties we had in finding the propagator. In the end the same difficulties arise in generating $\mathcal{W}_{Tp}$ and $\mathcal{W}_{Td}$, but the two variants locate the difficulties in different stages of the generation of the set of physically possible worlds. As we noted these difficulties are essentially 'epistemic' in their origin.

4.2 PROPAGATORS AND WELL POSEDNESS

The choice between these formulation gets interesting when we realize that the sets of physically possible worlds $\mathcal{W}_{Td}$ and $\mathcal{W}_{Tp}$ may not coincide. The first apparent difference is that we only required the operator A to be densely defined while the propagator S is defined on all elements of the underlying space E. Thus apparently $\mathcal{W}_{Tp}$ contains more worlds than $\mathcal{W}_{Td}$; this should immediately raise suspicion. This difference, however, can be regarded as a tech-

[5] The appropriate concept here is that of a *weak solution*, as explained below.

nicality, for $\mathcal{W}_{Tp}$ only contains, besides solutions of (4.1), the so-called weak solutions of (4.1). Weak solutions are familiar from distribution theory, they are generalizations of the notion of a solution of a differential equation. The interpretational importance of this difference should be discussed elsewhere; here we only note that if we wish to take common textbook examples involving point masses and point charges of field theories as representations of physically possible scenarios we need to resort to such generalization. Thus broadening our scope to include weak solutions of (4.1) can probably be regarded unproblematic; allowing for weak solutions the difference between $\mathcal{W}_{Td}$ and $\mathcal{W}_{Tp}$ seems to disappear. When it does not lead to confusion we are going to refer to weak solutions and functions generated by the propagator equation simply as solutions.

We need to address the issue whether the propagator exists at all. Note that the solution generated by S is unique for all initial states from E. We know, however, that certain initial value problems have multiple solutions. Are we not facing a contradiction here?

Albeit there is no contradiction there is a tension which leads us to subtle issues in the theory of differential equations. The existence of a propagator is closely linked with well posedness of the differential equation. We call the equation (4.1) *well posed* if a solution exists for a dense set of initial values and if solutions depend continuously on the initial values. These conditions for (4.1) entail the uniqueness of solutions; on a dense set one can then define an operator valued time evolution function via the existing unique solutions, and extend the operator to the entire space E as due to continuous dependence the operator is bounded. This everywhere defined operator valued function is the so-called propagator $S(.)$ we introduced earlier. Such propagator of a well posed problem is a strongly continuous group. Conversely, if $S(.)$ is a strongly continuous group then there exists a unique closed, densely defined operator A for which the equation (4.1) is well posed such that S is the propagator of (4.1). For precise definitions and treatment see Appendix A.

Consequently if the equation is well posed then (allowing for weak solutions) the physically possible worlds of (A_{Td}') and (A_{Tp}') coincide. A not well posed differential equation, however, does not have a propagator in the sense defined above. If there is no propagator there are no physically possible worlds generated by the propagator equation and we can't take it to represent the fix component of our physical theory. As non well posedness seems to plague our best physical theories the non-existence of the propagator may seem as a fatal blow to the (A_{Tp}') view.

Before jumping to conclusions it is worthwhile to contemplate about reasons of failure of well posedness. Well posedness is sensitive to the mathematical choices we make in formulating the differential equation in the first place. For a mathematically rigorous treatment of differential equations one needs to make sense of the differential operators involved; this requires specifying the type of functions on which the differential operators as operators act; specifying these functions in turn requires specifying the space of states to which the ostensive solutions map to. Differentiation with respect to the time parameter also requires that we make sense distance between states; distance of states is typically introduced by invoking the notion of a norm. The abstract framework presented above makes this logic quite clear: first we need to specify the space E to which solutions map and then need to define the differential operator A as one acting on some subdomain of E; E also needs to be endowed with a norm (implicit in the notion of a Banach space) to allow for differentiation[6].

Treatment of differential equations in physics and engineering textbooks often obscures this logic and leaves the state space E vague and unspecified. Strictly speaking for different choices of the Banach space E equation (4.1) is a different differential equation; we tend to lump these different equations together on the basis of similarity of form. This attitude may be well justified when we only care about heuristically finding individual solutions of a type of differential equation and we are content with finding post hoc a Banach space to host the

[6]Typically the norm also plays a role in defining the dense domain on which the operator acts. Requiring $(E, \|.\|)$ to be a Banach space adds the additional requirement of closure whose justification is primarily technical.

solution. $\mathcal{W}_{Td}$, however, is sensitive to the choice of E and hence if we are interested in the physical possibilities we do need to nail down the appropriate state space E[7].

Well posedness of an equation depends crucially on the choice of the state space E. If E is too narrow solutions may fail to exist; if E is too wide then uniqueness or continuous dependence of solutions might fail[8]. Well posedness can only be achieved when E strikes the right balance between the two ends of this spectrum.

One can often choose a Banach space E with which equation (4.1) becomes non well posed. Around the middle of the last century this and other observations led to the view that non well posedness plagues our best physical theories. Mathematical research since the 1950's has also progressively shown, however, that with the exception of certain inverse problems most dynamical equations in physics do have a well posed formulation if the space E is chosen wisely[9]. For this reason such non well posed equations are often referred to as *weakly ill posed*, i.e. they become well posed if we choose another, more suitable space E.

Choice of E can often be motivated on independent grounds, i.e. by additional interpretational premises or by additional information about the investigated physical systems. As

[7] (A'_{Td}) and (A'_{Tp}) can be understood both in a *narrow* and in a *vague* way. As we pointed out fixing the space E is a precondition for making precise sense of an equation and of a solution of this equation, concepts to which both (A'_{Td}) and (A'_{Tp}) makes reference to. The *narrow* understanding, accordingly, assumes that the space E is fixed.

The *vague* understanding does not assume the space E to be fixed but appeals to our sense of identifying equations which are defined on different spaces on the basis of their formulaic similarity. If there is a Banach space E with which our equation is meaningful and gives rise to a solution then we shall regard this solution as a representation of a physical possibility.

Heuristically it is more tempting to adhere to a vague understanding (see also footnote 2 on what a 'physical theory' consists of) but quantifying over possible mathematical structures can be a recipe for philosophical disaster. The concept of similar differential equations is inherently vague. It is also easy to underestimate the weirdness of the mathematical structures which may fit our bill. If similarity of equations is the only limitation there is no reason to exclude, for instance, Tsirelson spaces as candidates for physical possibility producing Banach spaces. Neither C_0 nor L^p spaces can be embedded into a Tsirelson space so interpreting the resulting solutions in physical terms becomes very problematic.

[8] This parlance is an oversimplification as choice of norm also has a defining influence.

[9] For brief historical notes and references see Chapter 2.

an example consider the familiar Schrödinger equation for a free particle:

$$-\frac{h}{i}\frac{\partial\psi}{\partial t} = -\frac{h^2}{2M}\Delta\psi \tag{4.3}$$

This differential equation can be cast in the abstract form (4.1). Is this equation well posed? As careful readers surely point out this question can not be answered until we specify the space E on which the differential operator and the ostensive $\psi : t \mapsto E$ solutions are defined. As it turns out there are many choices of E which make perfect mathematical sense; for some such choices the Schrödinger equation is well posed, for some other choices it is not. The Schrödinger equation is not well posed for $E = L^1$ while it is well posed for $E = L^2$; indeed for $E = L^p$ the Schrödinger equation is well posed if and only if $p = 2$[10]. The choice of $E = L^2$, of course, also has a motivation independent from the desirable result that it implies well posedness: the probabilistic interpretation of the mathematical apparatus and the Born rule suggests that $\psi(t)$ needs to be square integrable and thus needs to belong to L^2. This motivation, however, does not follow merely from the mathematics of the Schrödinger equation but from additional physical premises regarding the intended interpretation of ψ.

If it is possible to find a sensible space E that makes our differential equation well posed then with this space E the equation does have a propagator. If there is a propagator then there are possible worlds generated by the propagator equation and we can take it to represent the fix component of the physical theory. Hence for at least weakly ill posed equations the (A'_{Tp}) view remains viable.

Thus weakly ill posed differential equations in physics can be viewed through different looking glasses depending on whether we uphold differential equations or propagator equations as the modality-generating fixed components (as the laws) of the physical theory. An adherent of the (A'_{Td}) view says it might very well be the case that the proper mathematical

[10]For the result see Proposition 2 of Appendix A. That fact that the differential form of the Schrödinger equation may allow for indeterminism while the integral form does not have already been noted in the philosophy of physics literature, see Norton (1999), esp. Proposition 1 and Proposition 6.

formulation of the physical theory is one in which its dynamical equation is not well posed. In such cases non well posedness – such as the presence of initial value indeterminism – genuinely reflect physical possibilities regardless of the ensuing epistemic concerns.

The looking glass of (A'_{Tp}) shows a different view: as a propagator only exists when the corresponding equation is well posed the presence of non well posedness does not genuinely reflect on physical possibilities but on our failure to arrive at a proper mathematical formulation of the physical theory. Differential equations are but epistemically useful tools to arrive at histories representing physical possibilities but they need to be used wisely to serve well in this role, that is they need to be formulated in a way which permits the existence of the propagator. If, given some mathematical choices, our equation turns out to be non well posed we need to scrutinize our mathematical choices instead of fancying the results as genuine reflections on physical possibility. In this view well posedness can and should be taken as a guide to the proper mathematical formulation of differential equations for it is a necessary condition for identifying the solutions of these equations as representations of physically possible worlds.

Thus the (A'_{Tp}) variant of physical possibility supplies an argument for well posedness in the sense that it limits the choice of the space E to Banach spaces that allow for the existence of the propagator. The propagator-as-law point of view also has a further philosophical advantage in allowing for a direct correspondence between elements of the state space E and physically possible worlds. This is especially desirable since it also allows us to take the notion of a physically possible 'state' (configuration) to be mereologically prior to the notion of a physically possible world: as it is tempting to think of physically possible worlds as entities made up from a succession of physically possible 'states' it is also tempting to think of the modality of possible worlds being dependent on the modality of possible 'states.' (A'_{Tp}) reflects this intuition but (A'_{Td}) needs to get the relationship backwards. With (A'_{Td}) in order to know which elements of the space E represent physically possible states we first need to know what are the physically possible worlds. This difference shows when, as (A'_{Td}) suggests, we allow ill posed equations to generate physical possible worlds. When ill posedness shows in the failure of the existence condition for some elements of E solution of the initial value problem does exist. Such elements of E can not represent physically possible states for they are not states of any physically possible world: under the assumption that physically possible worlds are represented by *solutions* of a differential equation we automagically disqualify worlds which would only have instantaneous existence from being physically possible. Hence with (A'_{Td}) elements of E do not necessarily correspond to physically possible states while E receives a natural interpretation as the set of physically possible states under (A'_{Tp}) .

4.3 PEACE BETWEEN GOVERNING AND NON-GOVERNING INTUITIONS?

When the differential equation is well posed the physically possible worlds of the two variants of reading (A') of physical possibility coincide. From a modal perspective there is little gain in splitting hairs whether we should regard the differential equation or the propagator equation as the fixed component of the theory: the theory has the same modal character either way since the variable components end up being the same.

From a philosophical perspective the variants may be taken to express different intuitions about laws of nature[11]. According to some Non-Governing intuitions (notably those of the Best System account) laws of nature are simple and informative systematizers. Differential equations fit this bill nicely: they provide surprisingly simple descriptions which can be true of many, seemingly widely different possibilities. This simplicity of the differential equations may not be present in the propagators. Hence its seems more natural to associate the

[11] For an overview of the different positions and references to literature see Chapter 3.

Non-Governing account of laws with the (A'_{T_d}) view.

Even though simplicity of differential equations may not show up in the propagators they express more palpably the intuition of some Governing theorists that laws 'govern,' or that they 'produce,' 'evolve,' or 'bring about' the physical states. If so then it seems more natural to associate the Governing account of laws with the (A'_{T_p}) view[12].

The Governing and the Non-Governing intuitions are often thought to be in conflict with each other. Assumption of well posedness may soothe the conflict between Governing and Non-Governing intuitions: even though they allocate the modality-generating fixed component of the physical theory to distinct mathematical structures this distinction does not show up in any modal difference. Different intuitions about laws match up with emphasizing different mathematical aspects of the same physical theory.

It this a truce or a détente? We motivated associating Governing intuitions with (A'_{T_p}) and Non-Governing intuitions with (A'_{T_d}) but this association is not necessary; defenders of the accounts would need to speak for themselves about their allegiances. Even if the association is apt well posedness is necessary for soothing the conflict. If well posedness fails then, as we have seen, modal difference between the two variants resurface and starting with different intuitions about laws we end up with different judgements about physical possibilities. Note also that the Governing and Non-Governing theorists intend to give account not only of laws of our actual world but also of laws of other possible worlds. Soothing may only touch upon importance of systematizing versus governing aspects of laws for worlds whose laws can be expressed as differential or propagator equations; it does not touch upon the general issue of Humean supervenience, a front line between those who attach metaphysical significance to the methodological separation of laws and initial conditions and

[12]A Governing theorist who maintains (A_d') may quickly run into problems when well posedness fails. For instance the presence of initial value indeterminism poses a challenge for the Governing theorist for in such cases there is nothing in the differential equation which would set definite path for the states to evolve. It becomes difficult to maintain the intuition that the differential equation generates the states if it produces multiple possibilities. Thanks to Barry Loewer for a discussion about this point.

those who would rather not. Thus in other possible worlds cease-fire between the Governing and the Non-Governing theorist may quickly be abandoned. We wish them a good fight.

Facing a weakly ill posed equation we need to choose between the two variants: we either keep our mathematical formulation and rely on (A'_{Td}) as a guide to physical possibility or we revamp the mathematical formulation in the spirit of (A'_{Tp}). Are there any empirical means to decide in the favor of one variant or the other?

4.4 ARE THE TWO VARIANTS EMPIRICALLY DISTINGUISHABLE?

So far not much had been said about connection to experimental evidence. Does difference of the (A'_{Td}) and the (A'_{Tp}) variants of reading (A') of physical possibility allow to distinguish them empirically? For ill posed problems (A'_{Td}) allows physical possibilities which (A'_{Tp}) disavows; could we not simply decide between the variants by experimental means?

Testing the difference may seem straightforward. Let's call s a *treacherous state* if s belongs to a state space E which renders the initial value problem ill posed, the initial value problem for s in E has at least one solution, and s does not belong to any other Banach space that renders the initial value problem well posed. Successfully engineering a physical system whose initial configuration is represented by a treacherous state s could be evidence for the (A'_{Td}) variant as this variant allows s to represent a physically possible state while (A'_{Td}) does not. Impossibility of engineering a physical system described by any of the treacherous states could similarly be taken as evidence for the (A'_{Tp}) variant.

The proposed test faces several problems. First we would need to make sense of the condition that s does not 'belong' to any other Banach space that renders the initial value problem well posed. This condition presumes that an identification of states of different Banach spaces on the count that they represent the same physical configurations is available.

If this physical identification of states is achieved, as it is frequently assumed, by a norm-preserving isomorphism then in the physically most relevant cases (when the Banach spaces are separable infinite dimensional Hilbert spaces) such identification always exists for all elements. Hence there are no examples of states s that 'belong' to some of the state spaces but do not belong to another and hence there are no treacherous states. The test then can not even get off the ground[13].

Even if a treacherous state s existed in its state space E we would still face serious difficulties in the engineering part of the test. Many worries we raised in Chapter 2 about the necessity and sufficiency of well posedness for prediction and confirmation could be reiterated; we only bring up here the issue of measurement inaccuracy. Recall that one of the empirical limitations typically attributed to observers is that they are only able to prepare and measure physical systems inaccurately. This inaccuracy gets a mathematical representation by means of closeness in the norm. The norm of space E is either operationally significant or it is not operationally significant[14]. If the norm of E is not operationally significant then it does not allow to correlate a set of close states with the prepared systems. If the norm E is operationally significant we still have the problem that any actual measurement has finite precision and hence our handicapped observers are never in the position to verify whether the prepared configuration is indeed appropriately represented by the treacherous state or by some other non-treacherous state close to the treacherous state[15]. The same problem would surface if we wanted to empirically test whether initial value indeterminism genuinely reflects a property of our world: such test would presumably require observers to repeatedly prepare a treacherous state s that gives rise to multiple solutions in E and check whether

[13] Physical identification of states of different Banach spaces on the basis of norm preserving isomorphism may however be problematic as we pointed out in Chapter 2.

[14] A norm is operationally significant if states are sufficiently close in the norm iff their measurable properties are also sufficiently close. See Chapter 2 for a discussion of operational significance and approximate measurability of the norm.

[15] If non-treacherous states are dense in E this problem can not be avoided.

indeed more than one of the solutions can get realized. Non-uniqueness implies failure of continuous dependence and without continuous dependence it would be difficult to discern whether multiple solutions arise from the same treacherous state s or from different states nearby s[16].

Testing failure of continuous dependence seems more promising. One could prepare many systems whose initial configurations do not differ more than a pre-set measurement precision and track the behavior of these systems. For an operationally significant norm such a close set of configurations can be mapped to a close set of states; the theory then could be used to estimate the maximal deviation of the solutions which (in case the norm is also approximately measurable) can be compared with the behavior of the system. A system that exhibits a deviation falling beyond the upper bound for the deviation could be taken as evidence for the failure of the (A'_{Tp}) variant.

One problem with such test could be that estimations of maximal deviation are tied to particular choices for the space E; typically there are multiple choices of E that can make the initial value problems well posed and it may not be the case that a universal upper bound exists that would be valid for all such possible E's[17]. If there is no universal upper bound then the existence of a large deviation would not evidence failure of (A'_{Tp}) as it may only be an indication that we have not chosen the right space E to begin with[18].

A more serious problem with this test, and with any other similar proposals, is that in order to draw any conclusion from experiments first we would need to be able to identify the norm or norms that are operationally significant and/or approximately measurable. Without a link connecting observed deviations with distances of states relating any experimental data

[16] Albeit see our comments in Chapter 2 about failure of continuous dependence not implying 'everything goes.'

[17] This problem may be moot but I'm not aware of mathematical results settling this question either way.

[18] The existence of several Banach spaces E that can make the problem well posed also points to a difficulty shared by both variants: even though (A'_{Tp}) helps in reducing the number of possible candidates it also faces the problem of choosing among the rest.

to the mathematical models is problematic. To my knowledge no systematic study have been made in correlating norms to available experimental techniques. Sans establishing such correspondence it is not possible to do justice to the issue of continuous dependence by means of observation and we need to apply great care when we intend to interpret mathematical examples of failure of continuous dependence empirically[19]. This remark also applies to other empirical means that have less of an operationalist flavor. Although it would be interesting to see a detailed overview of the possible ways to empirically assess the difference of the two variants, and without such a detailed overview it would be hasty to conclude that no empirical means are available to tell them apart, we leave this discussion with the sense that finding such means is beset by many yet-to-be resolved problems.

4.5 OBJECTIONS

Variant (A'_{Tp}) of physical possibility supplies an argument for well posedness in limiting the choice of the state space E to Banach spaces that allow for the existence of the propagator. This is a formal requirement: it does not imply approximate measurability or operational significance of the norm of E. Albeit severing ties with empirical justification may seem as an advantage it also makes the approach suspicious. It may be possible, for instance, that the equation is well posed in a norm that entails the existence of the modality generating propagator equation but it is not well posed in the operationally significant norm. This would then pose difficulties for prediction based on and confirmation of the underlying physical theory, which is the complaint originally raised against ill posed equations.

In Chapter 2 we argued that well posedness is neither necessary nor sufficient for predic-

[19] To avoid circularity we should also not base judgement of operational significance on whether a given norm renders the differential equations well posed. It is difficult to overcome such temptation given the prejudice for well posedness in the practice of mathematical physics.

tion and confirmation. Among other objections we pointed out that even though a theory may not be able to supply prediction with certainty it may still be able to supply probabilities with which certain outcomes get realized. Quantum mechanics is a straightforward example: it is well confirmed in its probabilistic assertions but it is unable to supply definite predictions. Quantum mechanics then may also be an example that supposedly should have raised our suspicion according to the previous paragraph. The Schrödinger equation is well posed in the operationally not significant L^2 norm[20]; if it were possible to define an alternative notion of 'state' and an operationally significant norm in terms of the measurable quantities alone then, since no measurable quantities fix the time evolution of these quantities uniquely, the initial value problems for this alternative notion of 'state' would not be well posed. Then quantum mechanics would be a theory for which there is an argument for well posedness even though the argument does not establish that it is the operationally significant norm in which the theory should be well posed.

The argument presented in this chapter for well posedness has many limitations. Albeit many fundamental differential equations in physics can be cast in the 4.1 form – i.e. the Maxwell equation, the Schrödinger equation, and the Dirac equation for free particles (see Appendix A for some details) – there are others, such as Einstein field equations, that can not; it remains to be seen whether similar ideas can be pursued for nonlinear cases where no such straightforward relationship between the existence of propagators and well posedness of the differential equation is available.

Another contentious issue is that equations of the form 4.1 handle the time parameter in a distinguished way which is then inherited by the propagator equation. Partial differential equations, at least prima facie, do not need to distinguish the time parameter from other physical variables. Thus the preferential treatment of time cries out for justification and so

[20]See Chapter 2 for an argument why the L^2 norm is not operationally significant; for the well posedness result see Appendix A.

does the implied assumption of instantaneous states. In special relativity we don't have a preferred coordinate system; general relativity even have models not admitting a global time function[21].

The preferential treatment of the time variable may be less problematic than it appears; an increasing literature on the nature of time points out that many differential equations of physics do treat the time parameter distinctly and indeed some philosophers, such as Callender (2008) or Skow (2007), make use of this feature of differential equations to explicate what makes time special. Coordinate system dependence and abandonment of instantaneous states may also be less of an issue, although in order to preserve well posedness additional requirements, such as finite signal propagation speed, might also need to be motivated[22].

The main weakness of the approach lies in the arbitrariness of the applied mathematical concepts. Mathematical physics treats propagators as natural mathematical objects; from a pragmatic or an aesthetic point of view propagators can be viewed as natural candidates for representing a fixed component of a theory. But if we make propagators responsible for the modal character of physical theories and if the issue of well posedness turns on the propagators then it is natural to ask for a motivation for their mathematical properties. A propagator is a strongly continuous group[23]. Both strong continuity (which asserts that a solution does not have 'jumps' and is not to be confused with continuous dependence on the data) and the group property are, in the end, assumptions about how trajectories generated by the propagator behave. Any justification that can be given for such assumptions could likely directly act as a justification for well posedness of the differential equations.

I don't know how this objection of *petitio principii* could be conclusively preempted. We

[21]Note that models of general relativity that do not admit a global time function do not admit a well posed initial value formulation either. The mathematical framework we use here for abstract differential equations is not directly capable to handle differential equations of general relativity.

[22]The assumption of finite propagation speed is typically added to the notion of well posedness in general relativistic contexts, see i.e. (Wald; 1984, p. 224)).

[23]See definitions in Appendix A

have seen above that treating propagators as the modality generating fixed component brings some philosophical advantages – allowing for a truce between Governing and Non-Governing intuitions about laws, allowing interpreting the space E as the space of physically possible 'states,' etc. – apart from entailing well posedness. These may or may not be sufficiently compelling to accept the propagators-as-laws point of view. Although it is not a recourse for a philosopher, history of physics is also a history of well chosen definitions that cleverly hide conceptual problems, and choosing propagators as representations of laws might be one such choice to be made.

5.0 POSSIBLE WORLDS AND WELL POSEDNESS

5.1 INTRODUCTION

The sentiment that well posedness is necessary for physical possibility can be understood in different ways. Chapter 4 argued that in the propagators-as-laws perspective well posedness is necessary for physical possibility in the sense that it is a necessary condition for the appropriate mathematical formulation of the physical theory. Here we are going to take a look at another way of understanding well posedness as a necessary condition for physical possibility: we investigate the

Desired Solution Thesis: Solutions of a well posed initial value problem represent physically possible worlds but solutions of a non well posed initial value problem do not represent physically possible worlds.

If we accept reading (A') of physical possibility and if we accept that differential equations express the laws then the Thesis seems unsalvageable. There exists non well posed initial value problems whose differential equations are physically relevant: Chapter 2 discusses the Cauchy problem for the Laplace equation and Appendix A shows that many other physically relevant differential equations (the Maxwell equations, the Schrödinger equation, etc.) have formulations that render them ill posed. If such a differential equation represents the law of a physical theory T and if trajectories represent possible worlds then according to our best theories and according to reading (A') of physical possibility, stating that

(A') A possible world is physically possible according to a theory T if and only if it satisfies the physical laws of T,

there are physically possible worlds represented by solutions (trajectories that satisfy the differential equation) of non well posed problems.

Does it follow from this conclusion that the Thesis must be false? In this Chapter we argue that it does not. The premise that physically possible worlds are represented by solutions can be challenged. We argue that this premise may also be challenged in a way that leads to the opposite conclusion and vindicates the Thesis. We outline a general argument that would achieve this goal; we also construct an example that may serve as an alternative to solutions as representations of physically possible worlds. The Chapter proceeds informally; a mathematically rigorous presentation is relegated to Appendix B.

5.2 ALTERNATIVE REPRESENTATIONS OF PHYSICALLY POSSIBLE WORLDS

Choice of the mathematical construction via which possible worlds get represented is crucial for reading (A'). In order to find the set of all physically possible worlds first we need to specify this mathematical construction. For dynamical physical theories the customary mathematical construction C used to represent a possible world is a trajectory and the set of all trajectories that satisfy the fundamental differential equation (the law) of the theory, namely the set of solutions of the differential equation, represent the physically possible worlds. Trajectories are, however, not the only mathematical constructions that may serve as representations of possible worlds. If we propose another mathematical construction C' – for instance a certain equivalence class of trajectories – as a representation of a possible world, and if satisfaction of the fundamental differential equation by C' can be meaningfully formulated, then the set of all physically possible worlds of reading (A') becomes represented by the set of all mathematically possible constructions C' that satisfies the differential equation.

Challenging the assumption that solutions of differential equations represent physical possibilities is neither novel nor uncommon. We emphasized in Chapter 2 that what counts as a 'solution' of a differential equation is subject to definition and many different solution concepts can be relied upon. The so-called classical solution is neither the only nor the most commonly invoked solution concept in the literature on partial differential equations. Indeed one can trace a tendency to prefer various weak solution concepts, and one of the main motivations for the shift towards weak solutions is that they are more successful in averting failure of well posedness that would result from non-existence of classical solutions.

Lumping together different states or different solutions to form a single representation is also quite common in physics; states and solutions frequently contain 'surplus' mathematical structure that is washed out by identification of different states or different solutions as representations of the same physical system. Solutions tied to particular coordinate systems get identified on the basis of symmetry under Galilean or Lorentz transformation. Field theories identify functions that differ in a measure zero set of points even though particular calculations are frequently carried out using a single function of this equivalence class. Elements of the L^2 space are further identified in quantum mechanics when they only differ in a phase factor. Gauge invariance is also taken as a basis for identifying different solutions as representations of the same physical scenario. These identifications keep eroding the natural intuition that representations of possible worlds are straightforward assignments of values of physical quantities to points in space and time.

Even if some physical theories employ mathematical constructions that are not straightforward assignments of values of physical quantities to spacetime points these mathematical constructions are nevertheless representationally successful in describing observationally accessible phenomena. Loosely speaking representational success of a mathematical con-

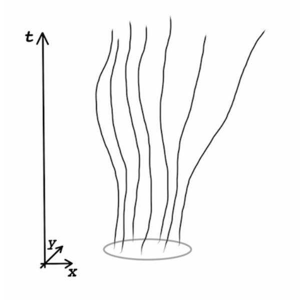

Figure 1: Solutions of a differential equation.

struction minimally requires that observations or experiments that fit within the descriptive scope of the theory can be related to said mathematical construction. Solutions of differential equations of dynamical physical theories are for the most part representationally successful; we know how to describe appropriately designed experiments so that these descriptions can be compared with solutions and thus empirical adequacy of solutions can be judged upon. With other commonly used mathematical constructions – weak solutions, equivalence classes of solutions – the question of how can they can be interpreted so to make them representationally successful is more tricky and would deserve its own treatment but for the time being we assume that their physical application evidences that they can be physically interpreted.

5.3 THE MAIN ARGUMENT

How could the Desired Solution Thesis be argued for if we accept reading (A') of physical possibility and if we accept a differential equation to represent the law of a sufficiently confirmed physical theory? Consider first the following reconstruction of the reasoning that had been considered in Chapter 2:

(i) Assume that solutions of the differential equation are representationally successful and empirically adequate in experimental cases we have tested so far.

(ii) Argue that well posedness of initial value problems is a necessary and sufficient condition for success of experimental testing.

(iii) Conclude that solutions of well posed initial value problems represent physically possible worlds.

(iv) Conclude that solutions of non well posed initial value problems do not represent physically possible worlds.

Reasoning (i)-(iv) is not sound. The conclusion of (iii) could only go through if we accepted that trajectories that satisfy the differential equation represent possible worlds. According to reading (A') the set of physically possible worlds is the set of all possible worlds that satisfy the law. If trajectories represent possible worlds then the set of physically possible worlds is the set of all trajectories that satisfy the law, that is the set of all solutions. Since whether a trajectory is a solution does not depend on whether initial values determine it uniquely or whether it continuously depends on initial values (iv) can not follow from the premises. Epistemic considerations such as (i) do influence initial choice of the mathematical construction that has a representational role but after the choice is being made the set of all possible constructions is determined by mathematical constraints; further epistemic considerations, such as (ii), do not post facto influence the set of all physically possible worlds[1].

The only way getting around this objection seems to lead through tinkering with the mathematical construction that represents possible worlds. If we accept assumption (i) this task is not easy but it may be feasible. Consider the following attempt at replacing solutions as representations with a hypothetical mathematical construction, the '**w**olution:'

[1]This objection lines up with the separation of two stages in the process of knowledge acquisition we alluded to in Chapter 2. In the first stage we assume that confirmation of physical theories is an empirical matter and hence epistemic considerations, such as limitations human observers may face, do play a role in selection of the right theory. The epistemic ladder is thrown away when we ask, in the second stage, what would be the physical possibilities if the theory were true. Conceptual clarity favors sticking to this modus operandi and we do so here; how reasonable maintaining such a strict separation of the two stages can be from a broader perspective is up for another debate.

(1) Assume that solutions of the differential equation are representationally successful and empirically adequate in experimental cases we have tested so far.
(2) Introduce another mathematical construction which meaningfully satisfies the differential equation and thus is another candidate for representing of a physically possible world. For the time being let us refer to this alternative mathematical construction as a '**w**olution.'
(3) Formulate the **w**olution counterpart of an initial value, an initial value problem, continuous dependence of solutions on initial values, and well posed initial value problem: '**w**initial value,' '**w**initial value problem,' 'continuous dependence of **w**olutions on **w**initial values,' and 'well posed **w**initial value problem.'
(4) Show that by construction a well posed **w**initial value problem have a unique **w**olution that depends continuously on **w**initial values.
(5) Show that by construction there are either no non well posed **w**initial value problems or that a non well posed **w**initial value problem do not have **w**olutions (that a **w**olution only exists if it is unique and depends continuously on **w**initial values).

From (4) and (5) we arrive at the conclusion

(C1) If **w**olutions represent physically possible worlds then only **w**olutions of a well posed **w**initial value problem represent physically possible worlds.

Conclusion (C1) bears similarity to the Thesis yet (2)-(5) are not be sufficient to motivate replacing solutions with **w**olutions as representations of physically possible worlds. Why would we do so? According to (1) solutions are representationally successful; **w**olutions are, so far, merely mathematical constructions. We would further need to

(6) Argue that **w**olutions share the representational success and empirical adequacy of solutions in the experimental cases we have tested so far.

Given (1)-(6) we have an alternative mathematical construction, the **w**olution, with which we could replace solutions as representations of physically possible worlds. Assuming (6) in itself would be puzzling; if (6) were true then solutions and **w**olutions would need to be related to each other some way since otherwise they could not be both representationally successful and empirically adequate. Let us choose a particular ways solutions and **w**olutions could be related that will serve our purposes. So let us further assume that we can

(7) Show that by construction a **w**olution, a **w**initial value, and a **w**initial value problem reduces, in an appropriate sense, to a solution, an initial value, and an initial value problem, respectively.

Thus if **w**olutions represent physically possible worlds then a solution s that has a **w**olution reducing to s can be regarded as an *indirect representation* of a physically possible world. If however there is no **w**olution that reduces to s then s does not represent a physical possible world neither directly nor indirectly.

(8) Show that by construction if an initial value problem is well posed then the corresponding **w**initial value problem is also well posed.

(9) Show that by construction if an initial value problem is not well posed then there is either no corresponding **w**initial value problem or the corresponding **w**initial value problem has no **w**olution.

From (1)-(9) we can then conclude that

(C2) If **w**olutions represent physically possible worlds then a solution of a well posed initial value problem does (indirectly) represent a physically possible world.

(C3) If **w**olutions represent physically possible worlds then a solution of a non well posed initial value problem does not represent a physically possible world neither directly nor indirectly.

Given (1)-(9) **w**olutions are alternative mathematical constructions that could represent physically possible worlds; furthermore if they did then we would have an argument for the Desired Solution Thesis. Albeit in this case solutions would only represent physically possible worlds indirectly – a solution of a well posed problem would only stand in as a representational short-hand for the **w**olution that do represents the physically possible world – (C2) and (C3) would still conform to the spirit of the Desired Solution Thesis.

To argue for the Thesis we need to supplement (1)-(9) with an argument for choosing **w**olutions to be the mathematical constructions that represent physically possible worlds. On the basis of (1)-(9) alone one could still decide to stick to solutions as constructions that represent physical possibilities. One would further need to

(10) Argue that **w**olutions are better motivated as mathematical representations of the experimental cases we have tested so far than solutions.

(11) Argue that (6), (7), and (10) provides sufficient reason to prefer **w**olutions as representations of physically possible worlds.

Given (C2), (C3), and (11) it then follows that

(C4) A solution of a well posed initial value problem does (indirectly) represent a physically possible world.

(C5) A solution of a non well posed initial value problem does not represent a physically possible world.

Conclusions (C4) and (C5) vindicate the Desired Solution Thesis in a slightly modified yet faithful form. Premise (11) is the tricky bit and to establish (11) a lot depends on how strong the arguments for (10) are. Yet if we assume that theories are confirmed on the basis of experimental tests (of the sort (10) argues **w**olutions fare better in representing) then it should be possible to motivate change of representation from solutions to **w**olutions on the basis of (10).

The existence of a **w**olution with the properties required by (1)-(11) would then provide a general argument that could vindicate the Desired Solution Thesis. In the rest of this paper we attempt to give an example for a **w**olution : the **b**olution (named after a **b**undle of solutions). The **b**olution is a particular construction for abstract differential equations; other constructions may also serve in the role of a **w**olution.

5.4 BOLUTION-CHUNKS, BOLUTION-PATHS, AND BOLUTIONS

To keep an eye on (10) we invoke certain assumptions about experiments that have been conducted so far to confirm the physical theory. We assume that the obtained experimental results do not falsify the differential equation given the limitations that observers who conduct the experiments face. The basic idea is to take into account some of these limitations in the construction of our mathematical representation. If certain limitations characterize the observation of experiments then it would be prudent if our mathematical construction that represents the experiments took these limitations into account. Building in observational limitations to the mathematical representation may protect us from surplus mathematical structure that reflect unobservable features or properties of the world.

Table 2: An argument for the Desired Solution Thesis.

(1) Assume that solutions of the differential equation are representationally successful and empirically adequate in experimental cases we have tested so far.

(2) Introduce another mathematical construction which meaningfully satisfies the differential equation and thus is another candidate for representing of a physically possible world. For the time being let us refer to this alternative mathematical construction as a '**w**olution.'

(3) Formulate the **w**olution counterpart of an initial value, an initial value problem, continuous dependence of solutions on initial values, and well posed initial value problem: '**w**initial value,' '**w**initial value problem,' 'continuous dependence of **w**olutions on **w**initial values,' and 'well posed **w**initial value problem.'

(4) Show that by construction a well posed **w**initial value problem have a unique **w**olution that depends continuously on **w**initial values.

(5) Show that by construction there are either no non well posed **w**initial value problems or that a non well posed **w**initial value problem do not have **w**olutions (that a **w**olution only exists if it is unique and depends continuously on **w**initial values).

(C1) If **w**olutions represent physically possible worlds then only **w**olutions of a well posed **w**initial value problem represent physically possible worlds.

(6) Argue that **w**olutions share the representational success and empirical adequacy of solutions in the experimental cases we have tested so far.

(7) Show that by construction a **w**olution, a **w**initial value, and a **w**initial value problem reduces, in an appropriate sense, to a solution, an initial value, and an initial value problem, respectively. Thus if **w**olutions represent physically possible worlds then a solution s that has a **w**olution reducing to s can be regarded as an *indirect representation* of a physically possible world. If however there is no **w**olution that reduces to s then s does not represent a physical possible world neither directly nor indirectly.

(8) Show that by construction if an initial value problem is well posed then the corresponding **w**initial value problem is also well posed.

(9) Show that by construction if an initial value problem is not well posed then there is either no corresponding **w**initial value problem or the corresponding **w**initial value problem has no **w**olution.

(C2) If **w**olutions represent physically possible worlds then a solution of a well posed initial value problem does (indirectly) represent a physically possible world.

(C3) If **w**olutions represent physically possible worlds then a solution of a non well posed initial value problem does not represent a physically possible world neither directly nor indirectly.

(10) Argue that **w**olutions are better motivated as mathematical representations of the experimental cases we have tested so far than solutions.

(11) Argue that (6), (7), and (10) provides sufficient reason to prefer **w**olutions as representations of physically possible worlds.

(C4) A solution of a well posed initial value problem does (indirectly) represent a physically possible world.

(C5) A solution of a non well posed initial value problem does not represent a physically possible world.

On the other hand we should also strive to disengage a representation of the world from the particular viewpoint of the potential observers since we want the representation to be a representation of the world and not a representation of the world-as-seen-by-a-particular-observer. A way to achieve a balance may be to construct our representation on the basis of the following recipe: in the first step we create a mathematical representation that respects the limitations of particular observers and in the second step we disengage particular observers from the representation by removing the limitations we imposed in the first step.

What are the limitations faced by observers of the experiments that have been conducted so far? Recall from Chapter 2 that the main motivation cited for the requirement of continuous dependence is the assumption that an observation is limited to finite measurement precision; another such limitation is that an observation only lasts for a finite time. Accordingly we are going to assume that any potential observer (any observer who is or who could potentially be in the position to confirm the physical theory) faces two kinds of limitations: that her observation lasts for a finite time and that her measurements have a finite precision. Our mathematical construction then implements the two-step recipe sketched above for local dynamical theories of classical physics. First we define the notion of a **b**olution-chunk that takes finite time and finite precision limitations into account; then in two steps we remove these limitations to arrive at the notion of a **b**olution.

5.4.1 Bolution-chunks

Assume that all observations of experiments share a common feature, namely that they last for some finite time and that data is available with finite measurement accuracy. Since local dynamical theories of classical physics rely on continuous representation of at least some physical quantities it is typically the case that measurement data is compatible with more than one solution of the governing differential equation – that measurement data is compatible with all solutions which are so close to each other that we can't discern them within a given measurement precision. Hence it seems that any particular experiment should be represented by not one solution of a differential equation, but by what we are going to call a **b***olution-chunk*: a maximal set of solutions of a differential equation defined on a finite time interval which stay within some finite distance from each other throughout this time interval[2]. The greater the measurement precision the narrower the **b**olution-chunk's *width*; some **b**olution-chunks are proper subsets of other **b**olution-chunks, some pairs of **b**olution-chunks have non-zero intersections, and yet some other pairs are disjoint or can have different *time intervals* with different *length*s.

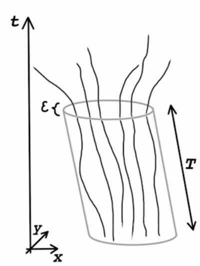

Figure 2: **B**olution-chunk: a maximal set of ϵ close solutions throughout time interval T.

A **b**olution-chunk thus represents the viewpoint of a particular observer who conducts her observation for a given finite time and with a given finite precision. On the other hand a **b**olution-chunk is laden with the limitations of the particular observer; the observation could have lasted for a longer period of time and that the measurement precision could have

[2]We carry out the construction using the framework developed for abstract differential equations in Appendix A. Appendix B also recalls the basic details. Thus the differential equation is assumed to take the form B.0.1. For definition of a bolution-chunk and length and width of bolution-chunks see Definition 13. We assume that the distance between solutions is given by the norm $\|.\|$ of the Banach space E. See discussion later.

been more refined[3]. We now thus proceed to remove these limitations.

5.4.2 Bolution-path

Assume that our observer extended the length of her observation. As she would still face observational limitations her longer observation would still be compatible with a set of solutions that stayed close to each other. Thus her longer observation would still be adequately represented by a **b**olution-chunk of a particular width. This new, longer **b**olution-chunk needs to be consistent with the older, shorter **b**olution-chunk in the sense that the longer **b**olution-chunk can only consist of solutions whose segments were already contained in the shorter **b**olution-chunk.

The observation could be continued even further, potentially up to any given point in time; the observation could also have been conducted for a shorter time. Hence the adequate representation for a potential observer who has a limited measurement capability but who lives long enough to see how the world unfolds is an assignment, to any given length of time, of a **b**olution-chunk of this length, with the restriction that a longer **b**olution-chunk can only consist of continuation of solutions which are already present in a shorter **b**olution-chunk. Such an assignment of **b**olution-chunks to time-lengths will be called a **b***olution-path*[4].

Given mild technical conditions **b**olution-chunks exist and any **b**olution-chunk can be extended to a **b**olution-path[5]. Whether this extension is unique depends on the differential equation. If the time evolution preserves distance between the solutions, e.g. when the time evolution is unitary, then extension of a **b**olution-chunk to a **b**olution-path is unique. If the time evolution does not preserve distance between solutions, then extension of a

[3]Chapter 2 calls attention to the dangers of this assumption. Here we go along with intuitions from classical physics and assume that an ideal observer's measurement precision could be arbitrarily refined; adding a lower boundary to our construction is feasible but it would change the results we end up with esp. regarding conclusion (C4).

[4]See Definition 14.

[5]See Proposition 6 and Proposition 7.

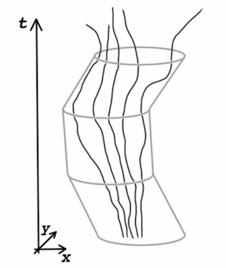

Figure 3: **B**olution-path: an assignment of **b**olution-chunks to all points in time.

bolution-chunk to a **b**olution-path may not be unique (if solutions diverge it is not). In case a **b**olution-chunk does not determine the **b**olution-path then having represented the experiment only up to a certain point in time puts limits on how the observed system can behave in the future but it could still behave in discernibly different ways.

5.4.3 Bolution

The **b**olution-path removes the limitation that particular observations are only conducted for a finite length of time. Observations could have been made with better precision as well. Had our observer made her observations with a better precision her adequate representation of the experiment would still be a **b**olution-path but it would be a **b**olution-path with a narrower width. This new, narrower **b**olution-path needs to be consistent with the older, wider **b**olution-path in the sense that the narrower can only contain solutions which were contained in the wider.

We may assume that measurement precision could have been even better, potentially reaching any finite accuracy. Hence a representation that removes the assumption that the measurement precision is a given finite value is an assignment, to any given positive

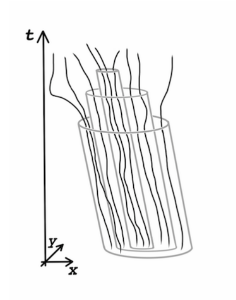

Figure 4: **B**olution: an assignment of **b**olution-paths to all levels of precision.

measurement precision, of a **b**olution-path with a width defined by this precision, with the restriction that a **b**olution-path with better precision is contained in a **b**olution-path with worse precision. Such an assignment of **b**olution-paths to levels of measurement precision is a **b***olution*[6].

Again, given some mild technical conditions, any **b**olution-path can be extended to a **b**olution[7]. The extension is not unique; a series of **b**olution-paths can "zoom on" many solutions in the **b**olution-path, leading to different **b**olutions[8].

What is a **b**olution, then? We can think of a **b**olutions as a more elaborate mathematical representation of a possible world than a solution; a **b**olution also tells, for any measurement precision and for any length of time, how observations would be represented if we took into account the limitations of the observer who conducts her measurement with said precision for the said length of time.

We offer the considerations leading to the definition of the bolution as evidence that (6) **b**olutions share the representational success and empirical adequacy of solutions and (10) **b**olutions are better motivated as representations of the experiments through which we tested

[6]See Definition 15.
[7]See Proposition 8.
[8]For a more precise statement see Proposition 9.

the theory. If we then generalize and (11) take **b**olutions to represent physically possible worlds then we should not let ourselves be led astray by the fact that **b**olutions are defined via solutions. Logical primacy should not be conflated with representational primacy; if **b**olutions are the mathematical constructions that represent physical possibilities then the representational role some solutions might have is only derivative from the representational role of **b**olutions. As the next section shows some solutions do indeed have such indirect representational role as in certain circumstances a **b**olution reduces to a solution.

5.5 THE RELATIONSHIP BETWEEN SOLUTIONS AND BOLUTIONS

Prima facia there seems to be little difference between generalizing from a few solutions to all solutions versus generalizing from a few **b**olutions to all **b**olutions; it may seem that for all practical purposes solutions could be identified with **b**olutions that "zoom on" them. Reducing **b**olutions to solutions by means of the "zooming in" property of **b**olutions may suggest that **b**olutions are only an unnecessary complication which can be entirely dealt away with.

It turns out that this identification depends on whether the solution is explosive, that is, whether the norm of the solution approaches infinity in a finite time[9]. Any non-explosive solution has a **b**olution counterpart which zooms on it[10]. Hence, for all practical purposes, we can rely on non-explosive solutions as *indirect representations* for their corresponding **b**olutions. There is no harm in thinking about a non-explosive solution as a (short-hand) representation of a physically possible world.

But there is a twist: there are no **b**olutions zooming on explosive solutions[11]. This means

[9]See Definition 10.
[10]See Proposition 9.
[11]See Proposition 10.

that an explosive solution can not be thought of as a short-hand for a **b**olution. Thus if we accept **b**olutions as representations of physically possible worlds then we can not think of an explosive solution as an indirect representation of a physically possible world.

This doesn't mean that explosive solutions are entirely worthless; they still have use for prediction. Any explosive solution, for any time t before its norm becomes infinite, and for any desired level of measurement precision, belongs to a **b**olution-chunk with named precision[12]. Hence an explosive solution may be used for prediction until time t by an observer with observation capabilities limited by named precision. However, as there is no **b**olution to which the entire explosive solution belongs an explosive solution can not stand as an indirect representation for a physically possible world.

In short not all solutions are equal; if **b**olutions represent physically possible worlds then some solutions can be though of as indirect representations of a physically possible world but some others can only be thought of as tools for prediction up until a certain point in time.

5.6 STATES, INITIAL VALUE PROBLEMS

We motivated the introduction of **b**olution-paths by assuming that an observation could potentially be continued; we also assumed that an observation could have potentially been conducted for a shorter length of time. Hence we required a **b**olution-path to be an assignment of a **b**olution-chunk to any length of time. Consequently, since a **b**olution is a collection of narrowing **b**olution-paths, a **b**olution is also an assignment of a narrowing set of **b**olution-chunks to any length of time. By taking this time length limit to zero we arrive at what can be naturally regarded as the corresponding notion of an initial value in the

[12]See Corollary 2.

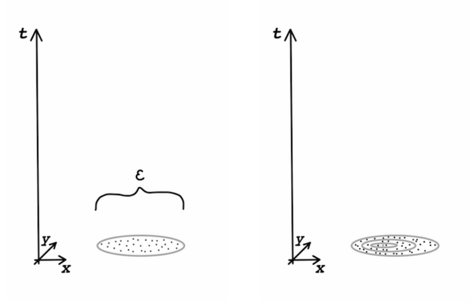

Figure 5: A set of ϵ-close initial values and a **b**initial value.

bolution-terminology: an assignment of a maximal set of close initial values to every level of precision, with the condition that the set belonging to a better precision needs to be a subset of the set belonging to a worse precision. Let us call such an assignment a **b***initial value*[13]. A state of the world thus gets represented, instead of a single initial value, by a **b**initial value. As it can be readily suspected, a **b**initial value "zooms on" a single initial value, similarly to how some **b**olutions zoom on a single solution[14].

By making use of **b**initial values we can formulate the analog an initial value problem in the language of **b**olutions. Recall that a solution of an initial value problem is a solution of the differential equation which satisfies the given initial value. Similarly, we want to say that a **b**olution of a **b**initial value problem is a **b**olution which satisfies the given **b**initial value. **B**olutions by definition satisfy the differential equation (as all solutions contained in them do so). However it is not yet clear what would satisfaction of a given **b**initial value by a **b**olution mean.

To massage intuitions let's recap how a **b**olution-path behaves. Recall that a **b**olution-path tells, for any given time, what is the **b**olution-chunk – what is the maximal set of close solutions – compatible with the observations of an observer. As this time increases, the

[13]See Definition 17.
[14]See Proposition 11.

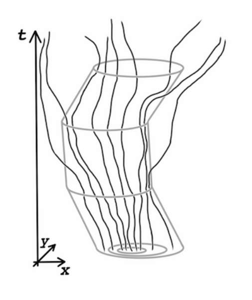

Figure 6: As $t \to 0$ the set of initial values from which solutions in the **b**olution-path originate gets larger and larger.

assigned **b**olution-chunk contains (not strictly) fewer and fewer solutions, and in turn the set of initial values from which these contained solutions originate gets smaller and smaller. We can think of this behavior in informational terms: learning more about how a physical system evolves, if its governing laws are known, may entail learning more about its initial state. Conversely if we shorten the time elapsing after the initial state the assigned **b**olution-chunk contains more and more solutions, and in turn the set of initial values from which these solutions originate gets larger and larger. Again, we can think of this limiting behavior in informational terms: assuming that the **b**olution-chunk is given what are the initial values from which the solutions within the **b**olution-chunk originate from? In the zero limit the set of originating initial values becomes the whole set of initial values which are within the distance set by the **b**olution-path's width and which give rise to an existing solution[15].

Thus the set of solution-originating initial values starts with the whole set of close initial values defined by the **b**olution-path's width and, as time moves forward, it narrows down to a smaller and smaller subset of these initial values. We want to grasp this behavior in the definition of satisfaction. Reversing the limiting process and thinking about it in causal,

[15] See Proposition 12.

rather than informational terms, we can ask the question: what happens with solutions originating from a certain set of initial values? In particular, as we are given a **b**initial value, what happens with solutions originating from a maximal set of close initial values? Are these solutions going to be compatible with the measurements of the observer, at least for a while? Translating this question to technical terms: suppose we are given a **b**initial value. Is it the case that, for any **b**olution-chunk (assigned to a certain length of time by the **b**olution-path) there is a maximal set of close initial values (assigned to a level of precision by the **b**initial value) so that all solutions originating from this set of close initial values stay within the **b**olution-chunk? If the answer is affirmative, then we say that the **b**olution-path satisfies the **b**initial value[16]. And if in turn all **b**olution-paths of a **b**olution satisfy the **b**initial value then we can say that the **b***olution satisfies the* **b***initial value*, or that the **b**olution is a **b***olution of the* **b***initial value problem* with the given **b**initial value[17].

Assuming this definition of a **b**olution of a **b**initial value problem, we can now think of **b**initial value problems as producing representations of the world similarly to how we ordinarily think of initial value problems producing representations of the world. If a **b**initial value problem has no **b**olution, then it does not produce a possible world; if it has exactly one **b**olution, then it produces one possible world, and we say that the world is deterministic; if it has multiple **b**olutions, then it can produce many possible worlds, and we say that the world is not deterministic, according to the physical theory under consideration.

5.7 WELL POSEDNESS RESULTS AND DISCUSSION

There is a close relationship between properties of initial value problems and properties of the corresponding **b**initial value problems. One can formulate the natural counterpart

[16] See Definition 18.

[17] See Definitions 19 and 20.

of the requirement of continuous dependence of solutions on initial data for **b**olutions and hence the natural counterpart of a **b**olution well posed problem[18]. Suppose that the initial value problems are well posed[19]. Then the corresponding **b**initial value problems also turn out to be well posed[20]. This then entails that a solution of an initial value problem can be identified with the **b**olution solution of the corresponding **b**initial value problem which establishes premise (8).

The situation is different when the initial value problem is not well posed. If an initial value problem has an explosive solution then the corresponding **b**initial value problem does not have a **b**olution[21]. Failure of continuous dependence also yields similar result: if a solution of an initial value problem exists but it does not depend continuously on its initial value then the corresponding **b**initial value problem again does not have a **b**olution[22]. These two results come close to establish premise (9) that if an initial value problem is not well posed then there is either no corresponding **b**initial value problem or the corresponding **b**initial value problem has no **b**olution. (Premise (9) is not established in its full generality because we operate with a strict definition of well posedness that requires uniform continuity on the initial data. Hence in this sense an initial value problem may fail to be well posed due to failure of uniform continuity and yet it may still be the case that all solutions depend continuously on their initial values. Relaxing the definition of well posedness may help establishing (9) in its full generality but this have yet to be worked out.)

With this minor remark we can sum up the results by saying that **b**olutions seem to validate premises (1)-(10) and thus if we accept them as representations of physically possible worlds then a close kin of the Desired Solution Thesis becomes vindicated: solutions of well posed initial value problems do represent physically possible worlds while explosive solutions

[18]See Definition 21.
[19]See Definition 11.
[20]See Proposition 13.
[21]See Proposition 14.
[22]See Proposition 15

or solutions that fail to depend continuously on their initial value do not.

Let us consider an example. As it is shown in Appendix A the Maxwell equation is not well posed in the supremum norm. The reason is that in the supremum norm explosive solutions of the Maxwell equation exist: for a particular choice of initial values the electromagnetic field at a certain spatial point may approach infinity in finite time. A physically intuitive example is a specially aligned spherical electromagnetic wave that focuses on a point in finite time[23]. If we take solutions to represent physical possibilities then, according to the Maxwell equation formulated with the supremum norm, such converging spherical waves are physically possible. As the solution only exists for a finite time we would accordingly need to draw the conclusion that some physically possible worlds of electrodynamics cease to exist after a finite period of time. If however we accept **b**olutions to represent physical possibilities then these specially designed spherical wave solutions do not represent physical possibilities for they have no **b**olution counterparts. **B**olutions do not blow up and so physically possible worlds of electrodynamics do not cease to exist after a finite period of time.

To add to the puzzle recall that the Maxwell equation *is* well posed in the L^2 norm (again see Appendix A for details). Thus as long as we take the state space to consist of elements of the L^2 space there are no explosive solutions in the L^2 norm[24]. Thus if we take solutions to represent physical possibilities then according to the Maxwell equation formulated with the L^2 norm all solutions exists globally and hence there are no physically possible worlds of electrodynamics that would cease to exist after a finite period of time. All of these solutions then have a **b**olution counterpart and thus if we take **b**olutions to represent physically possible worlds all familiar solutions would be indirect representations of physically possible worlds.

[23] For an example of focusing of waves for the wave equation see (Bers et al.; 1964, p. 13).

[24] Note that while the supremum norm blows up due to divergence of the electromagnetic field at a single point the L^2 norm is not determined by field values at singular points: states are equivalence classes of functions that may disagree at a set of measure zero points. Hence the L^2 norm of a solution may stay finite even if a representative of the equivalence class assigns infinite value to the electromagnetic field at a point.

This example calls attention to the importance of the choice of the norm. As the example of the Maxwell equation shows if we take solutions to represent physically possible worlds then different choices of norms produce different sets of physically possible worlds, some of which contain physically possible worlds with only finite lifespan while some others don't. **B**olutions are not immune from norm relativity either. As we have seen if a solution of an initial value problem does not depend continuously on its initial value then the corresponding **b**initial value problem have no **b**olution. In the extreme case when there are no initial value problems with solutions that depend continuously on their initial values this entails that none of the **b**initial value problems have **b**olutions. A change of norm that renders the initial value problems well posed would however entail that **b**olutions of **b**initial value problems exist. Thus the existence of **b**olutions of **b**initial value problems does depend on the choice of the norm.

It is not apparent what would motivate a choice of norm if we take solutions to represent physically possible worlds. Without some principled choice the best we can do is to give a conditional analysis of the form 'if we understand electrodynamics to be a theory with norm $\|.\|_X$ then the physically possible worlds are $\mathcal{W}_X$, but if we understand electrodynamics to be a theory with norm $\|.|_Y$ then the physically possible worlds are $\mathcal{W}_Y$' etc.

Bolutions, however, do motivate a specific choice of the norm. **B**olution-chunks were motivated by the limitation observers face in telling different solutions apart: we assumed that observers can only measure the values of physical quantities imprecisely. This motivation only makes sense if the norm which measures closeness of states is operationally significant[25]: if closeness of states in the norm does not imply closeness of observable properties then a maximal set of solutions that are close in the norm does not represent the possibilities that are consistent with observers' measurements. Thus **b**olutions do offer a motivation for choosing a specific norm for the theory: the chosen norm needs to be operationally significant.

[25] See Chapter 2 for a discussion.

The question then becomes whether for specific differential equations an observationally significant norm exists and whether an operationally significant norm would render the equation well posed. If so then taking **b**olutions to represent physical possibilities would offer a motivation for choosing a mathematical formulation of the theory that renders its equation well posed. This would then be a justification of well posedness that is more akin to the one given in Chapter 4. Unfortunately we know little about which norms of which theories are operationally significant and thus whether **b**olutions or constructions similar to **b**olutions would be of help in motivating well posedness is unclear.

5.8 OBJECTIONS

Consider first the claim that there is no **b**olution zooming on an explosive solution, and that a **b**initial value problem with **b**initial value zooming on an initial value which would produce such an explosive solution has no **b**olution. One can point out that this feature follows from defining **b**olution-paths as assignments of **b**olution-chunks to *all* lengths of time and explosive solutions evidently can not stay in a set of solutions which have close norms for a time after the explosion takes place. Thus one can argue that we exclude explosive solutions by simply requiring that a solution should not be explosive, which seems as a tight petitio principii.

There is, however, a difference between the cases when initial value problems vs. when **b**initial value problems are taken to produce representations of possible worlds. When an initial value problem gives rise to an explosive solution there seems little reason to doubt the physical possibility of its originating initial value in the sense that it could be the case that the world is such that at a given moment of time its state is represented by this originating initial value. There exists a local solution of this initial value problem, which tells a story about how the world unfolds for a finite period of time in perfect agreement with the physical laws, which could even be consistent with our limited experiences. A blow up would be uncomfortable but raising blow up as an evidence against physical possibility would come with the heavy baggage of retrocausality, an arguably high price to salvage global determinism. Hence the aversion against an ad hoc requirement on future behavior of the world in order to save the philosophical doctrine of determinism seems apt.

In the case of a **b**initial value problem, whose **b**initial value zooms on an initial value which originates an explosive solution, we do not need to face such objections from retrocausality, since the **b**initial value problem simply does not have a **b**olution. Albeit it is true that this non-existence of a **b**olution is entailed by a future behavior of a solution which would need to be a constituent of the **b**olution, if we accept **b**olutions as the mathematical construction that have representational role then, even though solutions as mathematical objects are used in the construction of a **b**olution and hence are logically prior to the **b**olution, they are representationally merely derivative. Retrocausal considerations could apply to the **b**olutions, but they don't touch mere mathematical constituents of the **b**olution representation, which are in themselves devoid of meaning. (Some of the solutions may function as an indirect representation but this role would be derived from the representational role of the **b**olution.)

In defense of the definition a more technical consideration can also be brought up. Given mild conditions any **b**olution-chunk can be extended to a **b**olution-path[26] and any **b**olution-path can be extended to a **b**olution[27], hence we can always find a **b**olution which gives rise to the particular **b**olution-path which is experienced by a given observer. Since for any observation length and precision the world looks to an observer as a **b**olution-chunk this means that forever-continuation of the world is always going to be compatible with

[26] See Proposition 7.
[27] See Proposition 8.

the experience of all of its potential observers. Thus, as long as we rely on **b**olutions as representations of the world, there is no reason why any of the observers should doubt global existence. This is in sharp contrast with relying on solutions as representations.

The original objection can be sharpened by pointing out that excluding explosive solutions as short-hand representations follows from the particular way we arrived at the definition of a **b**olution. The notion of a **b**olution-chunk came with two limitations: that experiments can only be conducted for a given length of time, and that they can only be conducted at a given level of precision. We removed the time limitation *first* and removed the precision limitation *second*. However, if we switched this order of removing the limitations we would end up with a notion of a **b**olution which may zoom on explosive solutions but which may only be defined until the explosion takes place, thereby mirroring the behavior of the explosive solution. **B**initial value problems based on this alternative definition then could have **b**olution solutions which are not globally extendable, similarly to initial value problems. And so the objection is that the order in which we removed limitations is unmotivated.

At this point it is worthwhile to ask which limitation removal has better inductive support: that an observation can be continued, or that the same observation could have been conducted with a better precision? Every observation conducted so far seems to support the continuability of observations. However the idea that the very same observation could have been conducted with a better precision is at best supported indirectly by conducting other similar experiments with better precision, for the exact same circumstances can never be repeated. And, as we know from quantum mechanics, even this indirect support is on shaky grounds. On this basis one may argue that we have firmer motivation to remove the time limitation first and hence to adopt the **b**olution definition we gave above.

We should also take a critical look at the definitions leading to the non-existence of a **b**olution of a **b**initial value problem corresponding to non well posed initial value problems. Arguably the main issue arises with the notion of satisfaction (see Definition 18 in Appendix

B). The definition of satisfaction was motivated above by a conjunctive-objective interpretation of the possibilities present in a maximal set of close initial values: all initial values in this set are taken as objectively possible for the observer with the corresponding capabilities and hence all solutions stemming from this set should be, at least for a while, present in the **b**olution-path produced by this set of close values. However one could have a different, disjunctive-subjective interpretation of the possibilities present in a maximal close set of initial values, namely that these initial values should only be taken as subjectively possible, and it is only due to the ignorance of the observer that she can't tell, at the given moment of time, which one of these possibilities is actually real.

This latter interpretation, which is closer to our original way of thinking about initial values and solutions as adequate representations would suggest another definition of satisfaction, one which only requires that at least one of the solutions stemming from the close set of initial values should stay in the **b**olution-path. Although results regarding the non-existence of **b**olutions in the explosive case would stay intact, with this alternative definition some not well posed **b**initial value problems would end up having **b**olutions. For instance in the important case when the initial value problems do have an existing and unique solution which nevertheless fails to depend continuously on the initial value the corresponding **b**initial value problems would also have an existing **b**olution. Interestingly, however, this **b**olution might not be unique, despite that the solution to which the **b**olution zooms on was unique. (One can easily give sufficient conditions which entail such non-uniqueness, although it is unknown to me what portion of not well posed initial value problems in physics would satisfy such conditions.) Thus, although with this alternative definition of satisfaction one would again arrive at the conclusion that some not well posed initial value problems produce indirect representations of physically possible worlds, the epistemic problem of prediction due to failure of continuous dependence in the solution-based representation would translate to a non-epistemic problem of indeterminism in the **b**olution representation. Hence the continuous dependence property of the underlying initial value problem could then be understood as a condition necessary for maintaining determinism.

We end this discussion with a similar tone to that of Chapter 4. In the end of the day definitions are what they are: definitions. We gave at least a partial defense to motivate them, but the crucial question is whether they can deliver the job, that is whether they lead to representations which are able to account for the experimental support we have for our dynamical theories. **B**olutions may be able to do so and hence they are candidate replacements of solutions as representations of physically possible worlds; there might be other such constructions. Even if we ultimately reject shifting representation from solutions to **b**olutions we can think of **b**olutions as a proof-of-concept: it is possible to change representation in such a way that, even though we remain faithful to all possible and actual experimental results, the set of physically possible worlds becomes more narrow, leaving out some of the craziest examples produced by philosophy of physics.

6.0 CONCLUSIONS

There is a sentiment shared widely across the physics community that well posedness is a necessary condition for physical possibility; evidence offered for this sentiment typically takes the form of a claim that we would not be able to predict or to confirm physical theories based on non well posed problems. In Chapter 2 we argued that well posedness is neither necessary nor sufficient for prediction and confirmation, and even if it were arguments based on pragmatic difficulties would still not succeed in establishing physical impossibility as they conflate epistemic and metaphysical concerns.

The question remained open whether the physicist sentiment could be vindicated without conflation of epistemic and metaphysical concerns. In order to talk clearly about modality we summoned the received view of physical possibility: according to the received view statements invoking physical possibility are to be understood as an existential quantification over a set of possible worlds determined by the laws of a physical theory T. With the aid of the received view a strong argument against the physicist sentiment can be formulated as follows. Examples reviewed in Chapter 2 and in Appendix A show that our best physical theories feature differential equations whose problems are not well posed. If such a differential equation represents the law of theory T and if solutions of differential equations represent possible worlds and if

(A') A possible world is physically possible according to a theory T if and only if it satisfies the physical laws of T

then, according to our best physical theories, there are physically possible worlds represented by solutions of non well posed problems.

To vindicate the physicist sentiment it would not be sufficient to challenge this conclusion by challenging one of its three premises; one would also need to replace the premises with credible alternatives that allow reaching the opposite conclusion. The dissertation undertook this task and attempted to replace each of the premises with a suitable alternative. Chapter 3 pointed out that reading (A') is not the only way how the received view of physical possibility can and have been understood and argued that coupling an alternative reading with a certain Non-Governing account of laws could validate the physicist sentiment. Chapter 4 argued that Governing intuitions about laws may prefer a different mathematical object, the propagator, as representation of laws, and contended that from a propagator-as-law perspective well posedness becomes a condition for appropriate mathematical formulation of a theory. Chapter 5 advocated replacing solutions as representations of possible worlds with another mathematical object, the **b**olution, which only happens to exists when the corresponding initial value problems are well posed. These three are independent and novel attempts to defend the physicist sentiment. Along the way we introduced distinctions which so far have escaped philosophers and showed that these distinctions matter for attempts at justifying well posedness.

Why do we need three separate arguments when a single good one should suffice? Arguments are only as strong as their premises and as we have pointed out all three arguments have premises that are problematic. These problems are also opportunities for further research. Better appreciation of the main argument of Chapter 3 requires a better understanding of the approximative relationship of different physical theories and a further analysis of the special status of initial conditions in a Best System account and of issues of simplicity and strength. Chapter 4 calls for a further investigation of dynamical systems, in particular of non-linear differential equations for which the relationship between existence of a propagator and well posedness of the equation is less straightforward. Chapter 5 poses the task of creating mathematical constructions that serve better in the role of defending the Desired Solution Thesis than does the **b**olution.

In general this dissertation calls for a more careful understanding of physical theories that does not neglect the role norms play in their mathematical formulation, advocates a systematic study of relating these norms to available experimental procedures, and points out that understanding how well posed problems of one theory approximates non well posed problems of another theory does not merely improve our understanding of inter-theoretical relationships but may also shed light on physical possibilities offered by these theories on their own. We hope that our investigation helped appreciating the potentials and limitations of vindicating the physicist sentiment and showed that there are many interesting puzzles to tackle and lessons to be learned along the way.

APPENDIX A

MATHEMATICAL PRELIMINARIES

A.1 WELL POSEDNESS FOR ORDINARY DIFFERENTIAL EQUATIONS

A.1.1 Basic definitions

We are going to denote a system of simultaneous first order differential equations of the form

$$\begin{aligned} x_1' &= f_1(t, x_1, ..., x_n) \\ \vdots \quad & \quad \vdots \\ x_n' &= f_n(t, x_1, ..., x_n) \end{aligned} \tag{A.1.1}$$

using the vector notation

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \ . \tag{A.1.2}$$

Here the functions $f_1, ..., f_n$ are defined on $D \subseteq \mathbb{R}^{n+1}$. $\mathbf{x}(t)$ is a *solution* of (A.1.2) in the interval J if its component functions $x_i(t)$ are differentiable in J and if they satisfy equations (A.1.1) identically; naturally we require that $(t, \mathbf{x}) \in D$ for $t \in J$.

An *initial value problem* for (A.1.2) asks for a solution that passes through a given point $(t_0, \mathbf{x}_0) \in D$:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \ . \tag{A.1.3}$$

Definition 5. *A vector function* $\mathbf{f}(t, \mathbf{x})$ *satisfies a* Lipschitz condition *with respect to* $\mathbf{x}$ *in* D *with Lipschitz constant* L *if*

$$|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| \leq L|\mathbf{x}_1 - \mathbf{x}_2| \tag{A.1.4}$$

for all $(t, \mathbf{x}_1), (t, \mathbf{x}_2) \in D$. *We say that* $\mathbf{f}(t, \mathbf{x})$ *satisfies a* local Lipschitz condition *with respect to* $\mathbf{x}$ *if for every point* $(t, \mathbf{x}) \in D$ *there exists an open neighborhood* U *such that* $\mathbf{f}$ *satisfies a Lipschitz condition in* $D \cap U$.

Note: As all norms in $\mathbb{R}^{n+1}$ are equivalent the question whether $\mathbf{f}$ satisfies a Lipschitz condition is independent from the chosen norm.

Theorem 1. Existence and uniqueness. *Let* $\mathbf{f}(t, \mathbf{x})$ *be continuous in a domain* $D \subseteq \mathbb{R}^{n+1}$ *and satisfy a local Lipschitz condition with respect to* $\mathbf{x}$ *in* D. *For any* $(t_0, \mathbf{x}_0) \in D$ *the initial value problem (A.1.3) has exactly one solution. The solution can be extended to the left and right up to the boundary of* D.

Example: when $\mathbf{f}$ and $\partial \mathbf{f}/\partial \mathbf{x}$ are continuous in D then $\mathbf{f}$ satisfies a local Lipschitz condition and hence the conditions of theorem (1).

We call a problem of mathematical physics *well posed* when its solution exists, when its solution is unique, and when its solution depends continuously on the data that is given. To formulate the third requirement we need to decide which properties of the problem we take to vary with data. For partial differential equations of physics we typically take initial (and perhaps boundary) values to represent given data. For ordinary differential equations in physics the right hand side $\mathbf{f}$ typically also expresses empirically determined information regarding the modeled physical system (such as the shape of a surface on which a ball is rolling). Hence besides initial values we also require the solution to depend continuously on $\mathbf{f}$. We have the following sufficient condition for the well posedness of a system of first order ordinary differential equations:

Let $S_\alpha^{\hat{\mathbf{x}}}$ denote the α-neighborhood of the graph of a vector function $\hat{\mathbf{x}}(t)$, that is the set of all points $(t, \mathbf{x})$ with $t \in J$, $|\mathbf{x} - \hat{\mathbf{x}}(t)| \leq \alpha$.

Theorem 2. Continuous dependence. *Let J be a compact interval with $t_0 \in J$ and let the function $\hat{\mathbf{x}}(t)$ be a solution of the initial value problem (A.1.3). Suppose there exists $\alpha > 0$ such that $\mathbf{f}(t, \mathbf{x})$ is continuous and satisfies the Lipschitz condition (A.1.4) in $S_\alpha^{\hat{\mathbf{x}}}$. Then the solution $\hat{\mathbf{x}}(t)$ depends continuously on the initial values and on the right-hand side* $\mathbf{f}$.

That is, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{g}$ is continuous in $S_\alpha^{\hat{\mathbf{x}}}$ and the inequalities

$$|\mathbf{g}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{x})| < \delta \;\; \textit{in} \;\; S_\alpha^{\hat{\mathbf{x}}}, \qquad |\mathbf{y}_0 - \mathbf{x}_0| < \delta \tag{A.1.5}$$

are satisfied, then every solution $\mathbf{y}$ of the "perturbed" initial value problem

$$\mathbf{y}' = \mathbf{g}(t, \mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{A.1.6}$$

exists in all of J and satisfies the inequality

$$|\mathbf{y}(t) - \hat{\mathbf{x}}(t)| < \epsilon \;\; \textit{in} \;\; J \; . \tag{A.1.7}$$

Example: when D is open and $\mathbf{f}$ and $\partial \mathbf{f}/\partial \mathbf{x}$ are both in $C^0(D)$ the conditions of theorem (2) (and of theorem (1)) are satisfied.

The Lipschitz condition in theorem (2) can be replaced with significantly weaker conditions; discussion and proof of theorem (1) and (2) can be found e.g. in Walter (1998).

A.2 BASIC DEFINITIONS FOR ABSTRACT DIFFERENTIAL EQUATIONS

A.2.1 L^p space, L^p norm

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, Ω be a Borel subset of $\mathbb{R}^m$ and μ a positive Borel measure in Ω. Given a $1 \leq p < \infty$ real number $L^p(\Omega, \mu)$ is the space of all μ-measurable funtions u in Ω with norm

$$\|u\|_p = \|u\|_{L^p(\Omega,\mu)} = \left(\int_\Omega |u(x)|^p \mu(dx)\right)^{1/p}; \tag{A.2.1}$$

for $p = \infty$ the Banach space $L^\infty(\Omega, \mu)$ consists of all μ-measurable, μ-essentially bounded functions u with the norm

$$\|u\|_\infty = \|u\|_{L^\infty(\Omega,\mu)} = \mu \operatorname{ess\,sup}\{|u(x)|;\ x \in \Omega\}. \tag{A.2.2}$$

Strictly speaking the elements of $L^p(\Omega, \mu)$ are not functions but equivalence classes of functions, the equivalence relation being equality almost everywhere with respect to μ. When μ is the Lebesgue measure we simply write $L^p(\Omega, \mu) = L^p(\Omega)$; when Ω is clear we further simplify to $L^p(\Omega, \mu) = L^p$. When it is unambiguous we also write the norm $\|u\|_{L^p(\Omega,\mu)}$ simply as $\|u\|_p$.

For all $1 \leq p \leq \infty$ the space $L^p(\Omega, \mu)$ is a Banach space, which is also separable if $p \neq \infty$. For $p = 2$ the space $L^2(\Omega, \mu)$ is also a Hilbert space with the scalar product

$$(u, v)_2 = \int_\Omega \bar{u}(x) v(x) \mu(dx). \tag{A.2.3}$$

The $L^p(\Omega, \mu)^\nu$ space of vector-valued functions for finite integer ν consists of all vector-valued functions $u(x) = (u_1(x), ..., u_\nu(x))$ where each of the components is μ-measurable and

$$\|u\|_p = \left(\int_\Omega \sum_{j=1}^{\nu} |u_j(x)|^p \mu(dx)\right)^{1/p} < \infty$$

in the case $1 \leq p < \infty$. Again $L^p(\Omega, \mu)^\nu$ is a separable Banach-space and for $p = 2$ the scalar product can be defined similarly to (A.2.3).

A.2.2 $C(K)$ space of continuous functions

Another important space is that of continuous functions. Let K be a compact subset of $\mathbb{R}^m$. The space $C(K)$ consists of all continuous functions u in K endowed with the supremum norm

$$\|u\|_C = \sup\{|u(x)|;\ x \in K\}. \tag{A.2.4}$$

The so-defined $C(K)$ is a Banach space. When K is closed but not compact we define $C_0(K)$ to be the space of all continuous functions in K with

$$\lim_{|x|\to\infty} u(x) = 0,$$

again endowed with the supremum norm

$$\|u\|_{C_0} = \sup\{|u(x)|;\ x \in K\}. \tag{A.2.5}$$

Both $C(K)$ and $C_0(K)$ are separable.

For the Banach space $C_0(K)^\nu$ of all vector functions $u = (u_1, ..., u_\nu)$ where all components u_j belong to $C_0(K)$ we define the supremum norm as

$$\|u\|_{C_0} = \sup_{1\le j\le\nu} \|u_j\|_{C_0}. \tag{A.2.6}$$

A.2.3 Sobolev spaces, Sobolev norm

A multi-index α is an m-tuple $\alpha = (\alpha_1, ..., \alpha_m)$ of non-negative integers α_i; for multi-indices we write $|\alpha| = \alpha_1 + ... + \alpha_m$. In what follows let $D^\alpha = (D^1)^{\alpha_1} \cdots (D^m)^{\alpha_m}$ be an arbitrary differentiation monomial with $D^j = \partial/\partial x_j$.

Let $\Omega \subseteq \mathbb{R}^m$ be an arbitrary domain, $j \geq 0$ integer and $1 \leq p < \infty$ real. The *Sobolev space* $W^{j,p}(\Omega)$ is defined as the space of all functions $u \in L^p(\Omega)$ such that

$$D^\alpha u \in L^p(\Omega)$$

for all $|\alpha| = \alpha_1 + ... + \alpha_m \leq j$ (the derivatives understood in the sense of distributions). The space $W^{j,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{j,p} = \|u\|_{W^{j,p}(\Omega)} = \left(\sum_{|\alpha| \leq j} \int_\Omega |D^\alpha u(x)|^p dx \right)^{1/p}. \tag{A.2.7}$$

$H^j(\Omega) = W^{j,2}(\Omega)$ is also a Hilbert space for any j, the scalar product defined by

$$(u, v)_{j,2} = \sum_{|\alpha| \leq j} \int_\Omega D^\alpha \bar{u}(x) D^\alpha v(x) \mu(dx). \tag{A.2.8}$$

For $\Omega = \mathbb{R}^m$ and alternative characterization of $H^j(\mathbb{R}^m)$ can be given using the *Fourier-Plancherel transform* in $L^2(\mathbb{R}^m)$,

$$\mathcal{F}u(\sigma) = \tilde{u}(\sigma) = \lim_{a \to \infty} \frac{1}{(2\pi)^{m/2}} \int_{|x| \leq a} e^{i(\sigma,x)} u(x) dx \tag{A.2.9}$$

where $\sigma = (\sigma_1, ..., \sigma_m)$, $(\sigma, x) = \sigma_1 x_1 + ... + \sigma_m x_m$, and the limit in (A.2.9) is understood in the $L^2(\mathbb{R}^m)$ norm. The operator $\mathcal{F}$ is an isometric isomorphism from $L^2_x(\mathbb{R}^m)$ to $L^2_\sigma(\mathbb{R}^m)$ (we use the subindices to signal the change in variables). The inverse of $\mathcal{F}$ is given by

$$u(x) = \mathcal{F}^{-1}\tilde{u}(x) = \lim_{a \to \infty} \frac{1}{(2\pi)^{m/2}} \int_{|\sigma| \leq a} e^{-i(\sigma,x)} \tilde{u}(\sigma) d\sigma. \tag{A.2.10}$$

A function $u \in L^2$ satisfies $D^\alpha u \in L^2_x$ if and only if $\sigma^\alpha \mathcal{F}u \in L^2_\sigma$ (where $\sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_m^{\alpha_m}$). We also have $\mathcal{F}(D^\alpha u)(\sigma) = (-i\sigma)^\alpha \mathcal{F}u(\sigma)$. It follows that $u \in H^j(\mathbb{R}^m)$ if and only if $(1+|\sigma|^2)^{j/2}\tilde{u} \in L^2$ and the norm $\|u\|'_{j,2} = \|(1+|\sigma|^2)^{j/2}\tilde{u}\|_2$ is equivalent to the norm of $H^j(\mathbb{R}^m)$ (where $|\sigma|^2 = \sigma_1^2 + ... + \sigma_m^2$). These remarks make natural the introduction of the spaces $H^s(\mathbb{R}^m)$ – s now being an arbitrary nonnegative parameter – that consists of all $u \in L_x^2$ such that $(1+|\sigma|^2)^{s/2}\tilde{u}$ belongs to L_σ^2. These are Hilbert spaces under the scalar products

$$(u,v)_{s,2} = \int_{\mathbb{R}^m} (1+|\sigma|^2)^s \bar{\tilde{u}}(\sigma)\tilde{v}(\sigma)d\sigma \tag{A.2.11}$$

corresponding to the norm

$$\|u\|_{s,2} = \|(1+|\sigma|^2)^{s/2}\tilde{u}\|_{L^2(\mathbb{R}^m)}. \tag{A.2.12}$$

The $H^s(\mathbb{R}^m)^\nu$ spaces of vector-valued functions can be introduced with the scalar product $(u(\sigma), v(\sigma))$ being in the integrand of (A.2.11).

A.2.4 Operators

An operator A is *closed* if and only if whenever $\{u_n\}$ is a sequence in $D(A)$ such that $u_n \to u$ and $Au_n \to v$ for some $u, v \in E$, it follows that $u \in D(A)$ and $Au = v$.

Theorem 3. Closed Graph Theorem. *Let A be closed and everywhere defined. Then A is bounded.*

A.2.5 Well posedness in the sense of Lax

Let A be a densely defined operator in an arbitrary Banach space E with a norm $\|.\|$. Consider the equation

$$u'(t) = Au(t) \quad (-\infty < t < \infty) \tag{A.2.13}$$

A *solution* of (A.2.13) is a function $t \to u(t)$ such that $u(t)$ is continuously differentiable for $-\infty < t < \infty$, $u(t)$ is in the domain $D(A)$ of A, and (A.2.13) is satisfied for $-\infty < t < \infty$.

Definition 6. *We say that the Cauchy problem for (A.2.13) is* well posed in the sense of Lax *(or simply* well posed*) in $-\infty < t < \infty$ if the following two assumptions hold:*

(1) Existence of solutions for sufficiently many initial data: There exists a dense subspace D of E such that, for any $u_0 \in D$, there exists a solution $u(.)$ of (A.2.13) in $-\infty < t < \infty$ with

$$u(0) = u_0. \tag{A.2.14}$$

(2) Continuous dependence of solutions on their initial data: There exists a function $C(t)$ defined for $-\infty < t < \infty$ such that $C(t)$ and $C(-t)$ are nondecreasing, nonnegative, and

$$||u(t)|| \leq C(t)||u(0)|| \quad (-\infty < t < \infty) \tag{A.2.15}$$

for any solution of (A.2.13).

Condition (2) can be given an equivalent (but more palpable) formulation as:

(2') Let $\{u_n(.)\}$ be a sequence of solutions of (A.2.13) with $u_n(0) \to 0$. Then $u_n(t) \to 0$ uniformly on compacts of $-\infty < t < \infty$.

Note that we treat initial value problems and mixed initial-boundary value problems within the same abstract framework. Boundary values, if given, are incorporated as restrictions on E or on the domain $D(A)$ of the operator A.

In this framework the distribution-theoretic idea of a weak solution of a differential equation can be formulated as follows. (Motivation: an otherwise appropriate $u(t)$ might not belong to the domain of A.) Let A be closed and let $u(t)$ be a locally integrable function in $-\infty < t < \infty$. Denote the adjoint of A by A^*. We say that $u(t)$ is a *weak solution* of (A.2.13) and (A.2.14) if and only if, for every $u^* \in D(A^*)$ and for every Schwartz test function $\varphi \in \mathcal{D}$ (consisting of all infinitely differentiable functions whose support is compact and contained in $\mathbb{R}^m$) we have

$$\int_0^\infty \langle A^* u^*, u(t)\rangle \varphi(t) dt = -\int_0^\infty \langle u^*, u(t)\rangle \varphi'(t) dt - \langle u^*, u_0\rangle \varphi(0) \ . \tag{A.2.16}$$

A.2.6 Propagators

A useful consequence of the definition of well posedness is the existence of the so-called propagator. Let us assume that equation (A.2.13) is well posed in $-\infty < t < \infty$, let $u_0 \in D$, and define the operator valued function $S(.)$, for all such t, by

$$S(t)u_0 \doteq u(t), \tag{A.2.17}$$

where $u(.)$ is the only solution of (A.2.13) with $u(0) = u_0$. Due to condition (A.2.15) $S(t)$ is a bounded operator in D, and since D is dense in E we can extend $S(t)$ to a bounded operator $\bar{S}(t)$ in E. This extended function $\bar{S}(.)$ is called the *propagator* of equation (A.2.13).

Note that due to the extension to E the E-valued function $\bar{S}(t)v_0$ makes sense for all $v_0 \in E$; writing now

$$v(t) \doteq \bar{S}(t)v(0) \quad (-\infty < t < \infty) \tag{A.2.18}$$

where $v(0) = v_0$ we arrive at the notion of a *generalized solution* of equation (A.2.13). A generalized solution is identical to a solution when the latter exists, but it does not need to be a genuine solution of (A.2.13) i.e. when v_0 does not belong to the dense subset D on which we assumed the existence of a solution.

The notion of a generalized solution is equivalent with the notion of a weak solution in the sense defined above, see (Fattorini; 1983, p. 30) and onwards.

Definition 7. *We call a* group *any functions $S(.)$ with values in the set of bounded operators over E defined in $-\infty < t < \infty$ which satisfy the equations*

$$S(0) = I \tag{A.2.19}$$

$$S(s+t) = S(s)S(t) \tag{A.2.20}$$

for $-\infty < s, t < \infty$.

Definition 8. *A group $S(.)$ is* strongly continuous *if*

$$\forall u_0 \in E: \quad \|S(t)u_0 - u_0\| \to 0 \quad as \quad t \to 0\ . \tag{A.2.21}$$

The propagator of equation (A.2.13) for which the Cauchy problem is well posed is a strongly continuous group (see (Fattorini; 1983, p. 81)), and in general any strongly continuous group is called a *propagator*. The converse relationship between well posedness and the existence of a propagator also holds:

Theorem 4. *Let $S(.)$ be a strongly continuous group. Then there exists a (unique) closed, densely defined operator A for which the Cauchy problem of (A.2.13) is well posed in the sense of Lax in $-\infty < t < \infty$ such that S is the propagator of this equation (A.2.13).*

A.3 BASIC RESULTS FOR THE CAUCHY PROBLEM

A.3.1 The general case

Let A be a densely defined operator in an arbitrary Banach space E with a norm $\|.\|$; we denote the domain of A as $D(A)$. The *resolvent set* $\sigma(A)$ of A is defined as the set of all complex λ such that $\lambda I - A$ has a bounded inverse $R(\lambda)$; the *spectrum* of A is the complement of $\sigma(A)$.

Consider again the equation

$$u'(t) = Au(t) \quad (-\infty < t < \infty). \tag{A.3.1}$$

We have the following general

Theorem 5. *Let A be closed. The Cauchy problem for (A.3.1) is well posed in the sense of Lax in $(-\infty, \infty)$ and its propagator S satisfies*

$$\|S(t)\| \leq Ce^{\omega|t|} \quad (-\infty < t < \infty) \tag{A.3.2}$$

if and only if $\sigma(A)$ is contained in the strip $|Re\lambda| \leq \omega$ and

$$\|R(\lambda)^n\| \leq C(|Re\lambda| - \omega)^{-n} \quad (|Re\lambda| > \omega, \ n \geq 1). \tag{A.3.3}$$

Existence and uniqueness of solutions, coupled with an additional technical condition, as the following result shows, ensures continuous dependence:

The *resolvent set* $\rho(A)$ of a densely defined operator A is the set of all complex λ such that $\lambda I - A$ has a bounded inverse $R(\lambda)$; the *spectrum* of A is the complement of $\rho(A)$.

Theorem 6. *Let A be a densely defined operator in the Banach space E such that $\rho(A) \neq \emptyset$. Assume that for every $u \in D(A)$ there exists a unique solution of*

$$u'(t) = Au(t) \quad (t \geq 0) \tag{A.3.4}$$

with $u(0) = u$. Then the Cauchy problem for (A.3.4) is well posed in the sense of Lax in $t \geq 0$.

A.3.2 The case of symmetric hyperbolic systems

More can be said about *symmetric hyperbolic systems*

$$D_t u = \sum_{j=1}^{m} A_j D^j u + Bu, \tag{A.3.5}$$

where $u = (u_1, ..., u_\nu)$ and $A_1, ..., A_m, B$ are complex constant matrices, $A_1, ..., A_m$ self-adjoint, B skew-adjoint.

Let's define the operator A as

$$Au = \sum_{j=1}^{m} A_j D^j u + Bu \tag{A.3.6}$$

where the domain $D(A)$ is the space of all $u \in H^s(\mathbb{R}^m)^\nu$ such that Au (understood in the sense of distributions) belongs to $H^s(\mathbb{R}^m)^\nu$. Consider now the Cauchy problem for

$$u'(t) = Au(t) \quad (-\infty < t < \infty) \tag{A.3.7}$$

with the so-defined operator A. We have:

Theorem 7. *The Cauchy problem for (A.3.7) is well posed in the sense of Lax in $-\infty < t < \infty$ in all spaces $E = H^s(\mathbb{R}^m)^\nu$, $s \geq 0$ (in particular in $E = L^2(\mathbb{R}^m)^\nu$); if u is a solution in L^2 such that $u(0) \in H^s$, then $u(t) \in H^s$ for all t and*

$$\|u(t)\|_{s,2} = \|u(0)\|_{s,2} \quad (-\infty < t < \infty). \tag{A.3.8}$$

For $s > m/2$, $u(t)$ (modified in a null set if necessary) belongs to $C_0(\mathbb{R}^m)$, and

$$\|u(t)\|_{C_0} \leq C\|u(0)\|_{s,2} \quad (-\infty < t < \infty) \tag{A.3.9}$$

holds with $C = (2\pi)^{-m/2}K_s^{1/2}$, $K_s = \int_{\mathbb{R}^m}(1 + |\sigma|^2)^{-s}d\sigma$.

If $s \leq m/2$ and p satisfies

$$2 < p < \frac{2m}{m - 2s} \leq \infty$$

then $u(t)$ belongs to L^p, and

$$\|u(t)\|_p \leq C(p)\|u(0)\|_{s,2} \quad (-\infty < t < \infty) \tag{A.3.10}$$

holds with $C(p) = (2\pi)^{-d}K(p)$, $K(p) = (K_{ps/(p-2)})^{(p-2)/2p}$.

The condition that B is skew-adjoint can be relaxed to the case where B is an arbitrary complex $\nu \times \nu$ matrix. Let ω be the maximum of the absolute value of the eigenvalues of $\frac{1}{2}(B + B^*)$; instead of the equality (A.3.8) we end up with the estimate

$$\|u(t)\| \leq e^{\omega|t|}\|u(0)\| \quad (-\infty < t < \infty), \tag{A.3.11}$$

which still guarantees continuous dependence on the initial data, etc.

A.3.3 Perturbation results

Theorem 8. *If the Cauchy problem for*

$$u'(t) = Au(t) \quad (-\infty < t < \infty). \tag{A.3.12}$$

is well posed in the sense of Lax then the Cauchy problem for

$$u'(t) = (A + P)u(t) \quad (-\infty < t < \infty). \tag{A.3.13}$$

is also well posed in the sense of Lax for any bounded operator P.

The condition of boundedness of P can be significantly weakened. For proof of theorem (8) and other results see (Fattorini; 1983, p. 268-272).

A.4 EXAMPLES

A.4.1 The Schrödinger equation

The Schrödinger equation for a particle of mass M under the influence of an external potential U is

$$-\frac{h}{i}\frac{\partial \psi}{\partial t} = -\frac{h^2}{2M}\Delta\psi + U\psi \tag{A.4.1}$$

where h is Planck's constant.

We consider here the free particle case in $m \geq 1$ dimension. Then equation (A.4.1) can be abstractly written as

$$u'(t) = Au(t) \quad (-\infty < t < \infty) \tag{A.4.2}$$

with operator

$$Au = i\kappa\Delta u = i\kappa\left((D^1)^2 u + \ldots + (D^m)^2 u\right) \tag{A.4.3}$$

where $\kappa \neq 0$ is a real number. We ask how we need to choose the Banach space E and the domain $D(A)$ of the operator A to make the Cauchy problem for (A.4.2) well posed.

We know from Theorem 7 that

Proposition 1. *The Cauchy problem for the Schrödinger equation (A.4.2)* is *well posed in the sense of Lax when $E = H^s(\mathbb{R}^m)$ and when the domain of A consists of all $u \in H^s(\mathbb{R}^m)$ such that Au (understood in the sense of distributions) belongs to $H^s(\mathbb{R}^m)$. In particular this is true for $E = L^2(\mathbb{R}^m)$.*

However this is not the case when we consider other Banach spaces and other norms. Consider first L^p spaces in general. Let $\mathcal{S}(\mathbb{R}^m)$ consists of all infinitely differentiable functions on $\mathbb{R}^m$ dying down at infinity faster than any power of $|x|$ together with all their derivatives. We have

Proposition 2. *The Cauchy problem for the Schrödinger equation (A.4.2)* is not *well posed in the sense of Lax when $E = L^p(\mathbb{R}^m)$ $(1 \leq p < \infty,\ p \neq 2)$ and when the domain of A is $D(A) = \mathcal{S}(\mathbb{R}^m)$.*

Let's consider now the case of the supremum norm. We have

Proposition 3. *The Cauchy problem for the Schrödinger equation (A.4.2)* is not *well posed in the sense of Lax when $E = C_0(\mathbb{R}^m)$ and when the domain of A is $D(A) = \mathcal{S}(\mathbb{R}^m)$.*

Proposition 2 is a consequence of a result of Hörmander ((Hörmander; 1960, p. 109), see also in Fattorini (1983)); Proposition 3 can be found in (Fattorini; 1983, pp. 45-49).

Remark: the choice of $L^2(\mathbb{R}^m)$ norm has a physical motivation, as the function $|\psi|^2$ is interpreted in quantum mechanics as the probability density of the particle. This interpretation imposes

$$\int_{\mathbb{R}^m} |\psi(x,t)|^2 dx = 1 \tag{A.4.4}$$

for all t and suggests that $L^2(\mathbb{R}^m)_{\mathbb{C}}$ is the proper space to study (A.4.1). However the differential equation itself has solutions for other function spaces as well, and in some contexts it is applied to classical fields as well (e.g. for studying a coherent matter wave of a Bose condensate or of a superfluid).

A.4.2 The Maxwell equation

The case for the Maxwell equation is similar.

Let $\mathcal{E}, \mathcal{B}$ be three dimensional vector functions of $x = (x_1, x_2, x_3)$ and t. We consider the case of propagating electromagnetic field in a homogeneous, isotropic medium in the absence of charges and currents:

$$\frac{\partial \mathcal{E}}{\partial t} = c \, \nabla \times \mathcal{B} \tag{A.4.5}$$

$$\frac{\partial \mathcal{B}}{\partial t} = -c \, \nabla \times \mathcal{E} \tag{A.4.6}$$

$$\nabla \mathcal{E} = 0 \tag{A.4.7}$$

$$\nabla \mathcal{B} = 0, \tag{A.4.8}$$

where c is a constant.

Let's define the operator A as

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -cD^3 & cD^2 \\ 0 & 0 & 0 & cD^3 & 0 & -cD^1 \\ 0 & 0 & 0 & -cD^2 & cD^1 & 0 \\ 0 & cD^3 & -cD^2 & 0 & 0 & 0 \\ -cD^3 & 0 & cD^1 & 0 & 0 & 0 \\ cD^2 & -cD^1 & 0 & 0 & 0 & 0 \end{bmatrix} = A_1 D^1 + A_2 D^2 + A_3 D^3, \tag{A.4.9}$$

where $D^j = \partial / \partial x_j$; A_1, A_2, A_3 are 6×6 symmetric matrices. The abstract formulation of Maxwell's equations (A.4.5)-(A.4.6) becomes the familiar

$$u'(t) = Au(t) \quad (-\infty < t < \infty). \tag{A.4.10}$$

Again, we know from Theorem 7 that

Corollary 1. *The Cauchy problem for the Maxwell equation (A.4.10)* is *well posed in the sense of Lax when* $E = H^s(\mathbb{R}^3)^6$ *and when the domain of* A *consists of all* $u \in H^s(\mathbb{R}^3)^6$ *such that* Au *(understood in the sense of distributions) belongs to* $H^s(\mathbb{R}^3)^6$*. In particular this is true for* $E = L^2(\mathbb{R}^3)^6$.

We neglected conditions (A.4.7)-(A.4.8); note however that they are satisfied for all t if they are satisfied by the initial conditions $\mathcal{E}(x, 0)$, $\mathcal{B}(x, 0)$.

However the situation changes when we consider other Banach spaces and other norms. Consider first L^p spaces in general. Again let $\mathcal{S}(\mathbb{R}^3)$ consists of all infinitely differentiable functions on $\mathbb{R}^3$ dying down at infinity faster than any power of $|x|$ together with all their derivatives, and let $\mathcal{S}(\mathbb{R}^3)^6$ be all 6-vectors with components in $\mathcal{S}(\mathbb{R}^3)$. We have:

Proposition 4. *The Cauchy problem for the Maxwell equation (A.4.10)* is not *well posed in the sense of Lax when* $E = L^p(\mathbb{R}^3)^6$ $(1 \leq p < \infty,\ p \neq 2)$ *and when the domain of* A *is* $D(A) = \mathcal{S}(\mathbb{R}^3)^6$.

Remark: the solution exists; the problem is the failure of continuous dependence. Proposition 4 follows from a result of Brenner (1966). One might be tempted to think that taking into account the further constraints (A.4.7) and (A.4.8) may improve the situation, but this is not so, see (Fattorini; 1983, pp. 46-48).

Let's consider again the case of the supremum norm. Let $\mathcal{F}_s$ be the subspace of $C_0(\mathbb{R}^3)^6$ consisting of all vectors u which satisfy the constraints $div(u_1, u_2, u_3) = div(u_4, u_5, u_6) = 0$. We have

Proposition 5. *The Cauchy problem for the Maxwell equation (A.4.10)* is not *well posed in the sense of Lax when* $E = \mathcal{F}_s$ *and when the domain of* A *is* $D(A) = \mathcal{S}(\mathbb{R}^3)^6$.

Remark: For a proof of Proposition 5 see (Fattorini; 1983, pp. 46-48).

Remark: the choice of $L^2(\mathbb{R}^m)$ norm has a physical motivation, since the function

$$\frac{1}{8\pi}\|u(t)\|^2 = \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|\mathcal{E}(x,t)|^2 + |\mathcal{B}(x,t)|^2\right)dx \tag{A.4.11}$$

is interpreted as the total energy of the electromagnetic field, which one would hope to remain constant in time in the absence of external influences. This is thus also a motivation for application of an energy norm. One would however hope to prove the constancy of the total energy on the basis of Cauchy data rather than assuming it.

A.5 A REMARK ON THE CHOICE OF THE APPROPRIATE BANACH SPACE

Consider again the Maxwell equation. Suppose our measurement devices can directly measure the strength of the $\mathcal{E}$ and $\mathcal{B}$ fields at spatial points[1] but we can't directly measure the values of their derivatives.

Then the operationally significant norm is the supremum norm and Proposition 5 shows that the initial value problem for the Maxwell equation is not well posed for the space of continuous functions that die off at infinity. This poses then a threat to prediction and confirmation (but: see our discussion of necessity of well posedness for prediction in Chapter 2.).

We may change the set of possible initial values to the functions space $L^2(\mathbb{R}^3)^6$, and change the norm to the L^2 norm. Then, according to Proposition 4 the initial value problem becomes well posed. However closeness in the L^2 norm is not helpful in carrying out the prediction task of establishing value of the field in a certain point, for it only provides mean

[1]This is questionable as arguably we never measure values of fields at exact single points but we rather measure average values in open domains. However this is interpretation-dependent and it *is* customary to ask what are the values of fields in given spatial points.

square estimate for the outcome. In order to predict values at points we need bounds in the supremum norm; the technical result that the initial value problem is well posed in the L^2 norm is of no pragmatic use.

This is where turning to Sobolev norm results could become useful. As Theorem 7 shows (see Proposition 1) the initial value problem for the Maxwell equation is well posed in the Sobolev space $H^s(\mathbb{R}^3)^6$. Furthermore, when $s > m/2$ (in our case: $s > 3/2$, say, $s = 2$) then estimates in the Sobolev space $H^s(\mathbb{R}^3)^6$ become estimates in the $C_0(\mathbb{R}^3)^6$ space as well (see (A.3.9)). Hence if we take our 'physically reasonable' space to be $H^2(\mathbb{R}^3)^6$ then we could both get well posedness in the norm of the space as well as the ability predict values of fields in given spatial locations.

APPENDIX B

BOLUTIONS AND SOLUTIONS

This Appendix is a supplement to Chapter 5. We recap and slightly modify some of the definitions of Appendix A – for reasons of convenience we only deal with the case when $t \geq 0$ but with suitable changes discussion could be extended to the $t \leq 0$ or to the $-\infty < t < \infty$ case.

Thus let $\mathcal{T}^+$ be the set of all nonnegative $[0, b)$ or $[0, b]$ time intervals ($b > 0$, $b = \infty$ allowed) and let $\mathcal{T}_f^+ \subseteq \mathcal{T}^+$ denote the set of all finite time intervals ($b = \infty$ is not allowed). Let A be a densely defined operator in an arbitrary topological vector space F, let $E \subseteq F$ be a Banach space with norm $\|.\|$, and let $T \subseteq [0, \infty)$ be an interval. Consider the equation

$$u'(t) = Au(t) \quad (t \geq 0) \tag{B.0.1}$$

A *solution of (B.0.1) in* $T \in \mathcal{T}^+$ is a function $t \to u(t)$ such that $u(t)$ is continuously differentiable, $u(t)$ is in the domain $D(A)$ of A, and (B.0.1) is satisfied for all $t \in T$. When $T = [0, \infty)$ we will simply call a solution of (B.0.1) in T a *solution* of (B.0.1).

Definition 9. *Let $u_1(t)$ be a solution of (B.0.1) in T_1, $T_1 \subseteq T_2 \in \mathcal{T}^+$, and let $u_2(t)$ be a solution on T_2 which agrees with u_1 everywhere in T_1. Then we say that the solution u_2* extends *u_1 and that u_1 is* extendable *to T_2. When a solution can not be extended to $t \geq 0$ we say it is* not globally extendable*.*

Definition 10. *Suppose $T \in \mathcal{T}_f^+$; we call a solution $u(t)$ in T* explosive *if $\|u(t)\| \to \infty$ when t approaches the boundary of T. Solutions which are not explosive for any $T \in \mathcal{T}_f^+$ are called* non-explosive. *(Note: $u(t)$ may be non-explosive even though $\|u(t)\| \to \infty$ as $t \to \infty$.)*

We now define the notion of a well posed problem (this is the same as definition 6 for $t \geq 0$).

Definition 11. *We say that the equation (B.0.1) is* well posed in $t \geq 0$ *(in the sense of Lax) if the following two assumptions hold:*
(1) Existence of solutions for sufficiently many initial data: There exists a dense subspace D of E such that, for any $u_0 \in D$, there exists a solution $u(.)$ of (B.0.1) (in $t \geq 0$) with

$$u(0) = u_0. \tag{B.0.2}$$

(2) Continuous dependence of solutions on their initial data: There exists a function $C(t)$ defined for $t \geq 0$ such that $C(t)$ is nondecreasing, nonnegative, and

$$||u(t)|| \leq C(t)||u(0)|| \quad (0 \leq t < \infty) \tag{B.0.3}$$

for any solution $u(t)$ of (B.0.1).

Condition (2) can be given an equivalent (but more palpable) formulation as:
(2') Let $\{u_n(.)\}$ be a sequence of solutions of (B.0.1) with $u_n(0) \to 0$. Then $u_n(t) \to 0$ uniformly on compacts of $0 \leq t < \infty$.

A useful consequence of well posedness is the existence of the so-called propagator. Let us assume that equation (B.0.1) is well posed in $t \geq 0$, let $u_0 \in D$, and define the operator valued function $S(.)$, for all $t \geq 0$, by

$$S(t)u_0 \doteq u(t), \tag{B.0.4}$$

where $u(.)$ is the only solution of (B.0.1) with $u(0) = u_0$. Due to condition (B.0.3) $S(t)$ is a bounded operator in D, and since D is dense in E we can extend $S(t)$ to a bounded operator $\bar{S}(t)$ in E. This extended function $\bar{S}(.)$ is called the *propagator* of equation (B.0.1).

Note that due to the extension to E the E-valued function $\bar{S}(t)v_0$ makes sense for all $v_0 \in E$; writing now

$$v(t) \doteq \bar{S}(t)v(0) \quad (t \geq 0) \tag{B.0.5}$$

where $v(0) = v_0$ we arrive at the notion of a *generalized solution* of equation (B.0.1). A generalized solution is identical to a solution when the latter exists but it does not need to be a genuine solution of (B.0.1) i.e. when v_0 does not belong to the dense subset D on which we assumed the existence of a solution. The notion of a generalized solution is equivalent with the notion of a weak solution (in the sense of distributions); for a further discussion see Appendix A.

We now define the notion of a bolution friendly differential equation. Bolution friendliness is a significant weakening of the notion of well posedness; it already implies basic results regarding bolutions but it is not as strong as to make the argumentation circular by presupposing well posedness.

Definition 12. *We say that the equation (B.0.1) is* bolution friendly *in $t \geq 0$ if the following two assumptions hold:*

(1) Existence of solutions for sufficiently many initial data: There exists a dense subspace D of E such that, for any $u_0 \in D$, there exists a solution $u(.)$ of (B.0.1) (in $t \geq 0$) with

$$u(0) = u_0. \tag{B.0.6}$$

(2) Approximation of solutions in a finite time interval: for every solution $u(.)$ of (B.0.1) defined in a closed interval $T \in \mathcal{T}_f^+$ there exists a sequence of solutions $\{u_n(.)\}$ of (B.0.1) (in $t \geq 0$) with $u_n \to u$ uniformly in T.

Bolution friendliness is a weak property of differential equations; it only requires that for any solution there are *some* non-explosive solutions of the differential equation which stay close to it in a closed time interval. It's easy to see that every well posed differential equation is bolution friendly (but not the converse). I'm not aware of examples of differential equations appearing in non-relativistic physics which satisfy condition (1) but not (2).

We now define the notion of a bolution-chunk, bolution-path, and bolution:

Definition 13. *Let $T \in \mathcal{T}^+$ and let $\epsilon > 0$. A* bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,T}$ of (B.0.1) *is a maximal set of solutions of (B.0.1) in T which stay within ϵ distance throughout T. That is $\breve{\boldsymbol{B}}^{\epsilon,T}$ is a bolution-chunk if*

(a) for all $u_1, u_2 \in \breve{\boldsymbol{B}}^{\epsilon,T}$: $\|u_1(t) - u_2(t)\| < \epsilon$ for all $t \in T$ and

(b) whenever we have a solution u_1 of (B.0.1) in T for which for all $u_2 \in \breve{\boldsymbol{B}}^{\epsilon,T}$: $\|u_1(t) - u_2(t)\| < \epsilon$ for all $t \in T$ then $u_1 \in \breve{\boldsymbol{B}}^{\epsilon,T}$.

We refer to T as the time interval*, $|T|$ as the* (time) length*, and ϵ as the* width *of the bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,T}$.*

Let's denote the set of solutions in the bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,[0,t']}$ restricted on the $[0,t]$ interval as $\breve{\boldsymbol{B}}^{\epsilon,[0,t']}_{|[0,t]}$.

Definition 14. *A* bolution-path $\hat{\boldsymbol{B}}^{\epsilon}(t)$ *is an assignment $t \mapsto \breve{\boldsymbol{B}}^{\epsilon,[0,t]}$ of a bolution-chunk with time interval $[0,t]$ to a length of time $t \geq 0$ such that $\hat{\boldsymbol{B}}^{\epsilon}(t')_{|[0,t]} \subseteq \hat{\boldsymbol{B}}^{\epsilon}(t)$ whenever $t' \geq t$.*

Definition 15. *Let $\epsilon > 0$. A* bolution $\tilde{\boldsymbol{B}}(\epsilon, t)$ *is an assignment $\epsilon \mapsto \hat{\boldsymbol{B}}^{\epsilon}(t)$ of a bolution-path to an ϵ level of precision such that for all $t > 0$: $\tilde{\boldsymbol{B}}(\epsilon, t) \subseteq \tilde{\boldsymbol{B}}(\epsilon', t)$ whenever $\epsilon \leq \epsilon'$.*

We also want to refer to the values taken by the solutions in a bolution at time t: we write $\boldsymbol{B}(\epsilon, t)$ for the set $\{u(t) : u \in \tilde{\boldsymbol{B}}(\epsilon, t)\}$. When it is not confusing we also refer to $\boldsymbol{B}(\epsilon, t)$ as a bolution*.*

We say that the bolution satisfies *the equation (B.0.1) with respect to which it is defined.*

The proofs of the following Propositions are self evident.

Proposition 6. *For every solution u of (B.0.1) in $[0, t^*]$ there exists at least one bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,[0,t^*]}$ which contains u.*

A bolution-chunk can be extended to a bolution-path:

Proposition 7. *Let equation (B.0.1) be bolution friendly, $\epsilon > 0$. For any bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,[0,t^*]}$ there exists a bolution-path $\hat{\boldsymbol{B}}^{\epsilon}(.)$ such that $\hat{\boldsymbol{B}}^{\epsilon}(t^*) = \breve{\boldsymbol{B}}^{\epsilon,[0,t^*]}$.*
$\hat{\boldsymbol{B}}^{\epsilon}(.)$ is unique if the time evolution preserves the norm.

A bolution-path can be extended to a bolution:

Proposition 8. *Let equation (B.0.1) be bolution friendly, $\epsilon > 0$. For any bolution-path $\hat{\boldsymbol{B}}^{\epsilon}(.)$ there exists a bolution $\tilde{\boldsymbol{B}}(.)$ such that $\tilde{\boldsymbol{B}}(\epsilon, .) = \hat{\boldsymbol{B}}^{\epsilon}(.)$.*

Every (non-explosive) solution has a bolution "zooming on" it:

Proposition 9. *Let $u(t)$ be a non-explosive solution of a bolution friendly differential equation (B.0.1) in $t \geq 0$. There exists a bolution $\boldsymbol{B}(\epsilon, t)$ such that for all t: $\lim_{\epsilon \to 0} \boldsymbol{B}(\epsilon, t) = \{u(t)\}$.*

If it exists, we denote the solution $u(t)$ on which a bolution $\boldsymbol{B}$ zooms – in the sense of Proposition 9 – as B.

Corollary 2. *Let $u(t)$ be an explosive solution or a not globally extendable solution of a bolution friendly differential equation (B.0.1) in $[0, T)$. For any $\epsilon > 0$ and any $0 < \delta < T$ there exists bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,[0,T-\delta]}$ such that $u_{|[0,T-\delta]} \in \breve{\boldsymbol{B}}^{\epsilon,[0,T-\delta]}$.*

Note: according to Proposition 7 this bolution-chunk $\breve{\boldsymbol{B}}^{\epsilon,[0,T-\delta]}$ *can* be extended to a bolution-path. No bolution zooms however on an explosive solution:

Proposition 10. *Let $u(t)$ be an explosive solution of a bolution friendly differential equation (B.0.1) in $[0, T)$ with $lim_{t \to T} \|u(t)\| = \infty$, and let $\epsilon > 0$. There exists no bolution-path $\hat{\boldsymbol{B}}^{\epsilon}(.)$ such that $u_{|[0,T-\delta]} \in \hat{\boldsymbol{B}}^{\epsilon}(T - \delta)$ for all $\delta > 0$. Consequently there exists no bolution $\boldsymbol{B}(\epsilon, t)$ such that for any $t \in [0, T]$: $\lim_{\epsilon \to 0} \boldsymbol{B}(\epsilon, t) = u(t)$.*

The interpretation of Proposition 10 is that even though explosive solutions are not short-hand representations of physically possible worlds, up until any moment before the explosion happens there is a possible world and a possible observer who may use the solution for purposes of non-precise calculation until that moment. However for different moments the world and the observer for whom the solution up until this point has pragmatic use may differ!

Let us now define the notion of a binitial value-chunk and of a binitial value:

Definition 16. *A binitial value-chunk is a maximal set of close initial values. That is, a* ${\boldsymbol{B_0}}^{\epsilon} \subseteq E$ *is a* binitial value-chunk *if*

(a) for all $u_1, u_2 \in {\boldsymbol{B_0}}^{\epsilon}$*:* $\|u_1 - u_2\| < \epsilon$*, and*

(b) whenever for a $u_1 \in E$ *we have* $\|u_1 - u_2\| < \epsilon$ *for all* $u_2 \in {\boldsymbol{B_0}}^{\epsilon}$ *then* $u_1 \in {\boldsymbol{B_0}}^{\epsilon}$*.*

Definition 17. *A binitial value is an assignment of a narrowing set of binitial value-chunks to a level of precision. That is, a* $\boldsymbol{B_0}(\epsilon) : \epsilon \mapsto {\boldsymbol{B_0}}^{\epsilon}$ *is a* binitial value *if* $\boldsymbol{B_0}(\epsilon)$ *is a binitial value-chunk for all* $\epsilon > 0$ *and if* $\boldsymbol{B_0}(\epsilon) \subseteq \boldsymbol{B_0}(\epsilon')$ *whenever* $\epsilon \leq \epsilon'$*.*

As can readily be suspected, a binitial value "zooms on" an initial value:

Proposition 11. *Let* $\boldsymbol{B_0}(\epsilon)$ *be a binitial value. There exists a unique* $u_0 \in E$ *such that* $u_0 \in \boldsymbol{B_0}(\epsilon)$ *for all* $\epsilon > 0$*.*

Conversely, for any $u_0 \in E$ *there exists a binitial value such that* $u_0 \in \boldsymbol{B_0}(\epsilon)$ *for all* $\epsilon > 0$*.*

On the basis of Proposition 11, we denote the unique u_0 on which the binitial value $\boldsymbol{B_0}$ zooms as B_0.

The zero time limit of a bolution path is a (subset of a) binitial value-chunk:

Proposition 12. *Let* $\overline{\epsilon} > 0$ *be fixed and let* $\boldsymbol{B}(\overline{\epsilon}, t)$ *be a bolution.* $\lim_{t \to 0} \boldsymbol{B}(\overline{\epsilon}, t) = {\boldsymbol{B_0}}^{\overline{\epsilon}} \backslash X$*, where* ${\boldsymbol{B_0}}^{\overline{\epsilon}}$ *is a binitial value-chunk and* $X \subseteq E$ *is a set of* u_0 *initial values for which the initial value problem (B.0.1), (B.0.2) has no solution.* ${\boldsymbol{B_0}}^{\overline{\epsilon}}$ *is unique.*

Similarly, we can regard a binitial value $\boldsymbol{B_0}(.)$ as a zero time limit of a bolution $\boldsymbol{B}(., t)$ if, for all $\epsilon > 0$, the binitial value-chunk $\boldsymbol{B_0}(\epsilon)$ is the zero time limit of $\boldsymbol{B}(\epsilon, t)$ in the sense of Proposition 12.

The inverse of this limiting process motivates the following definition:

Definition 18. *A bolution-path* $\hat{\boldsymbol{B}}$ satisfies *the binitial value* $\boldsymbol{B_0}$ *if for all* $T > 0$ *there is an* $\epsilon > 0$ *such that all solutions of the initial value problems with initial values in* $\boldsymbol{B_0}(\epsilon)$ *stay in* $\hat{\boldsymbol{B}}(T)$.

Definition 19. *A bolution* $\boldsymbol{B}$ satisfies *the binitial value* $\boldsymbol{B_0}$ *if for all* $\epsilon > 0$ *the bolution-path* $\tilde{\boldsymbol{B}}(\epsilon)$ *satisfies the binitial value* $\boldsymbol{B_0}$.

Again, let A be a densely defined operator in an arbitrary topological vector space F, let $E \subseteq F$ be a Banach space with norm $\|.\|$. Consider the equation

$$u'(t) = Au(t) \quad (t \geq 0) \tag{B.0.7}$$

Definition 20. *Let* $\boldsymbol{B}$ *be a bolution of (B.0.7). If*

$$\boldsymbol{B} \; satisfies \; bolution \; initial \; value \; \boldsymbol{B_0} \tag{B.0.8}$$

then we say that the bolution is a bolution of the binitial value problem (B.0.7)-(B.0.8) with binitial value $\boldsymbol{B_0}$.

We can also define the natural counterpart of the notion of well posedness for binitial value problems.

Definition 21. *We say that the equation (B.0.7) is* bolution well posed in $t \geq 0$ *if the following two assumptions hold:*
(1) Existence of bolutions for sufficiently many binitial data: There exists a dense subspace D *of* E *such that every initial value* $B_0 \in D$ *has a corresponding binital value* $\boldsymbol{B_0}$*, and for all such* $\boldsymbol{B_0}$ *there exists a bolution* $\boldsymbol{B}$ *of (B.0.7) satisfying the binitial value* $\boldsymbol{B_0}$.

(2) Continuous dependence of bolutions on the binitial data: For every $\epsilon > 0$ there exists a function $C_\epsilon(t)$ defined for $t \geq 0$ such that $C_\epsilon(t)$ is nondecreasing, nonnegative, and

$$||u(t)|| \leq C_\epsilon(t)||u(0)|| \quad (0 \leq t < \infty)$$

for any solution $u(.)$ in any bolution $\tilde{\boldsymbol{B}}(\epsilon, .)$ of (B.0.7) when $0 < \epsilon < \epsilon^*$ is suitably small.

Proposition 13. *Let (B.0.1) be an equation whose initial value problems are well posed in $t \geq 0$. Then the binitial value problems of (B.0.7) are bolution well posed in $t \geq 0$.*

Proposition 14. *Let $u_0 \in E$ be an initial value such that the initial value problem (B.0.1)-(B.0.2) has an explosive solution $u(t)$. Then the binitial value problem (B.0.7)-(B.0.8) has no bolution.*

Definition 22. *Let $u(.)$ be a solution of (B.0.1), and let $u_0 = u(0)$. We say that the solution of equation (B.0.1)* depends continuously on its initial value u_0 in $t \geq 0$ *if there exists a neigborhood $E' \subseteq E$ of u_0 such that*

(1) There exists a dense subset D of E' such that, for any $v_0 \in D$, there exists a solution $v(.)$ of (B.0.1) (in $t \geq 0$) with

$$v(0) = v_0.$$

(2) For all sequence $\{u_n(.)\}$ of solutions of (B.0.1) for which $u_n(0) \to u(0)$, $u_n(0) \in E'$ we have $u_n(t) \to u(t)$ uniformly on compacts of $0 \leq t < \infty$.

Note that if equation (B.0.1) is well posed in the sense of Definition 11 then all of its solutions depend continuously on their initial values. The converse is not necessarily true as the continuous dependence might not be uniform.

Proposition 15. *Let $\boldsymbol{B_0}$ be a binitial value, let $u_0 = B_0$. Suppose that a solution of equation (B.0.1) does not depend continuously on its initial value u_0 in $t \geq 0$. Then the binitial value* *problem (B.0.7)-(B.0.8) has no bolution for the binitial value* $\boldsymbol{B_0}$ *even if the initial value problem (B.0.1)-(B.0.2) had a solution.*

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