THIN FILM EQUATIONS WITH VAN DER WAALS FORCE

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ABSTRACT

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We are interested in the steady states of thin films in a cylindrical container with van der Waals forces which lead to a singular elliptic equation in a bounded domain with Neumann boundary conditions. Using the prescribed volume of the thin film as a variable parameter we investigated the structure of radial solutions and their associated energies using rigorous asymptotic analysis and numerical computation. Motivated by the existence of rupture solutions for thin film equations, we considered elliptic equations with more general non linearity and obtained sufficient condition for the existence of weak rupture solutions for a class of generalized elliptic equations. Finally such results can be generalized to a class of quasi-linear elliptic partial differential equations.

Keywords: Thin film equations, point rupture solutions, elliptic equations, quasi-linear elliptic equations, asymptotic analysis.

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PREFACE

ACKNOWLEDGMENTS

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1.0 INTRODUCTION

1.1 PHYSICAL BACKGROUND

In fluid dynamics, lubrication theory describes the flow of fluids in a geometry in which one dimension is significantly smaller than the others. In our situation we have a fluid in a solid substrate of bottom a planar region $\Omega \subset \mathbb{R}^2$ and thickness or height usually denoted as u or h is considered smaller than the dimensions of the base region Ω. Interior flows are those where the boundaries of the fluid volume are known. Here a key goal of lubrication theory is to determine the thickness distribution in the fluid volume. The fluid in this case is called a lubricant. Free film lubrication theory is concerned with the case in which one of the fluid's surfaces is a free surface in our case the top one. Surface tension due in particular to the walls of the substrate may become important, additional intermolecular forces such as van der Waalls forces may be significant and issues of wetting and non wetting then arise.

An important and major application area is in the coatings industries including the preparation of thin films, printing, painting and adhesives. Our main interest is in the rupture solutions that occur when the thickness approaches zero. Such problem is motivated by the ruptures in thin films. In the lubrication model of thin films, u will be the thickness of the thin film over a planar region Ω and the dynamic of the thin film can be modeled by the general fourth order partial differential equation.

$$u_t = -\nabla \cdot (u^m \nabla u) - \nabla \cdot (u^n \nabla \Delta u). \qquad (1.1.1)$$

The exponents m, n represent the powers in the destabilizing second-order and the stabilizing fourth-order diffusive terms, respectively. This class of equations occurs in connection with many physical models involving fluid interfaces. Here the fourth-order term in the equation reflects surface tension effects. The second-order term can reflect gravity, van der Waals interactions, or the geometry of the solid substrate.

For instance, the equation describes a thin jet in a Hele-Shaw cell for $n = 1$ and $m = 1$ discussed in detail in [1] [5] [7] [8] [17]. A Hele-Shaw cell is composed by two flat superposed plates that are parallel to each other and separated by a very reduced distance. At least one of the plates is transparent. These cells are used for studying many phenomena, including the behavior of granular materials as they are being poured into the space between the plates.

The equation can also model the fluid droplets hanging from a ceiling, this is the case when both n and m are integers and are equal to three, for this instance a rigorous analysis was done as in [9]. Water molecules attract each other, in fact every molecule of any liquid attracts each other. In general, every neutral atom or molecule attracts one another. The forces of this attraction repulsion are certainly electrical in nature. They are generally known as van der Waals' forces, and are mainly due to forces between two dipoles meaning imbalanced charge distributions whose total charge is zero. However, molecules are not polarized by themselves but they generate a dipole moment when they are brought together. Water in free fall will tend to form itself into a sphere the form of least surface area. The sphere of water into contact with another piece of solid material of a planar surface as the ceiling depending on the strength of the attraction between a water molecule and the molecules of the ceiling if stronger than the attraction between water and water the water will tend to stick to the surface and we say that the water causes the wetting of the surface. If water-water attraction is greater it will not stick to the surface and will cause the non-wetting of the surface. If the water can wet the surface it sticks to it and we say that it has found a way to reduce the tension by reducing the surface area further thus it sticks to the surface of the ceiling.

The equation has the appellation of the modified Kuramoto-Sivashinsky equation, which describes solidification of a hyper-cooled melt when $n = 0$ and $m = 1$ and it was fully studied in [3] and [4]. Hyper-cooled melt is the process of lowering the temperature of a fluid below its freezing point without actually observing solidification. A liquid below its standard freezing point will crystallize in the presence of a seed crystal or nucleus around which a crystal structure can form creating a solid.

The equation models van der Waals force driven thin films and this instance is manifested when both n and m are integers with values 3 and -1 respectively. Numerous rigorous analysis were performed as published in [6] , [12] , [18] , [19] and [20]. When the space dimension is one R. Laugesen and M. Pugh [16] considered positive periodic steady states and touch-down steady states in a more general setting. The dynamics of a special type of thin film equation has been investigated by F. Bernis and A. Friedman [2]. They established the existence of weak solutions and showed that the support of the thin film will expand with time.

1.2 DERIVATION AND FORMULATION OF THE PROBLEM

Let us consider a cylindrical container with base Ω in $\mathbb{R}^2$ inside of which is a viscous fluid. We are interested in studying the behavior of the thin film. Our problem is actually motivated by the ruptures in thin films. In the lubrication model of thin films, u will be the thickness of the thin film over a planar region Ω and the dynamic of the thin film can be controlled by the fourth order partial differential equation

$$u_t = -\nabla \cdot (u^m \nabla u) - \nabla \cdot (u^n \nabla \Delta u) . \tag{1.2.1}$$

Here the fourth-order term in the equation reflects surface tension effects, and the second-order term can reflect gravity, van der Waals interactions, thermocapillary effects or the geometry of the solid substrate. This class of model equation occurs in connection with many physical systems involving fluid interfaces. In our instance, we are dealing with van der Waals force driven thin film, which means that, $n = 3$ and $m = -1$ and we are mainly concerned with steady states solutions of (1.2.1) which is not well understood in the two dimensional case. Since $n - m \neq 1$, we can define the quantity

$$p = -\frac{1}{m-n+1} u^{m-n+1} - \Delta u,$$

which can be viewed as the pressure of the fluid. We can rewrite (1.2.1) as

$$u_t = \nabla \left(u^n \nabla p \right) .$$

Now, we need few assumptions that are physically feasible for the problem. let $\Omega \subset R^2$ be the bottom of a cylindrical container occupied by the thin film fluid, we assume that there is no flux across the boundary, which yields the boundary condition

$$\frac{\partial p}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.2.2}$$

We also ignore the wetting or non-wetting effect, and assume that the fluid surface is orthogonal to the boundary of the container, i.e.,

$$\frac{\partial h}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.2.3}$$

Whenever $m-n+1\neq 0$ and $m-n+2\neq 0$, we may divide by the product of these two terms and associate (1.2.1) with an energy function

$$E(u)=\int_{\Omega}\left(\frac{1}{2}|\nabla u|^{2}-\frac{1}{(m-n+1)(m-n+2)}u^{m-n+2}\right),$$

and formally, using (1.2.2), (1.2.3), we have

$$\frac{d}{dt}E(u)=-\int_{\Omega}u^{n}|\nabla p|^{2}.$$

Hence, for a thin film fluid at rest, p has to be a constant, and u satisfies

$$-\Delta u-\frac{1}{m-n+1}u^{m-n+1}=p\text{ in }\Omega,$$

which is an elliptic equation.

If we further assume $m-n+1<-1$, which includes the van der Waals force case. We can write the equation as

$$\begin{cases}\Delta u=\frac{1}{\alpha}\cdot u^{-\alpha}-p & in\ \ \Omega,\\ \frac{\partial u}{\partial\nu}=0 & on\ \ \Omega,\end{cases}\tag{1.2.4}$$

where p is an unknown constant and

$$\alpha=-(m-n+1)>1.$$

For van der Waals force driven thin film, $\alpha=3$.

In physical experiments, usually the total volume of the fluid is a known parameter identified by the average thickness of the thin film defined as

$$\bar{u}=\frac{1}{|\Omega|}\int_{\Omega}u(x)dx.$$

Therefore for any given $\bar{u}>0$ we need to find a function u and an unknown constant p satisfying,

$$\begin{cases}\Delta u=\frac{1}{\alpha}\cdot u^{-\alpha}-p & in\ \ \Omega,\\ \frac{1}{|\Omega|}\int_{\Omega}u(x)dx=\bar{u}, & \\ \frac{\partial u}{\partial\nu}=0 & on\ \ \partial\Omega.\end{cases}\tag{1.2.5}$$

This singular elliptic problem modeling steady states solutions for van der Waals force driven thin film has been rigorously studied in [15] and a complete description of all continuous, radial and positive solutions was given for problem (1.2.4) . However, the dependence of solutions with a given $\bar{u}$ is still not clear. Using rigorous asymptotic analysis and numerical computation we would like to investigate the structure and behavior of radial solutions and their associated energies for a prescribed volume of the thin film. Many of these results are illustrated in section 2.5, where numerical experiments were performed on the particular equation,

$$u'' + \frac{1}{r}u' - \frac{1}{\alpha}u^{-\alpha} + p = 0, \tag{1.2.6}$$

when $\alpha = 3$ and $p = \frac{1}{3}$. Observe that the smooth and the rupture solutions are global oscillatory, positive, continuous, and approaching 1 at infinity.

1.3 RADIAL STEADY STATES OF VAN DER WAALS FORCE DRIVEN THIN FILMS WITH PRESCRIBED FLUID VOLUME

In chapter two we would like to understand the dependence of radial solutions u and the prescribed volume $\bar{u}$. We consider solving problem (1.2.4) in the unit ball, thus $\Omega = B(0,1)$, and since u is radial then it must satisfy the equation (1.2.6) . Our strategy is to fix $\alpha = 3$, $p = \frac{1}{3}$ and ignore the Neumann boundary condition assuming a given prescribed volume constraint $\bar{u}$. Then for any $\eta > 0$ the unique radial solution, denoted $u_\eta(r)$ to the differential equation (1.2.6) with initial conditions $u(0) = \eta$ and $u'(0) = 0$ was shown in [15] that it is global and oscillates around 1. Also, there exists an increasing sequence of positive radii $r_k^\eta, k = 1, 2, \cdots$ which are the critical points of the solution $u^{\eta,k}(r)$ defined as,

$$u^{\eta,k}(x) = (r_k^\eta)^{-\frac{2}{1+\alpha}} u^\eta(|x|) = (r_k^\eta)^{-\frac{1}{2}} u^\eta(|x|).$$

Now, if we restrict the solution $u_\eta(r)$ to the ball $B_{r_k}(0)$ then the function $u^{\eta,k}$ must satisfy the Neumann boundary problem,

$$\begin{cases} \Delta u = \frac{1}{\alpha} \cdot u^{-\alpha} - p^{\eta,k} \quad in \quad B_1(0), \\ \frac{\partial u}{\partial \nu} = 0 \qquad\qquad on \quad \partial B_1(0), \end{cases} \tag{1.3.1}$$

where the unknown constant $p^{\eta,k}$ is defined as,

$$p^{\eta,k} = \frac{1}{3}((r_k^\eta)^{-\frac{1}{2}}).$$

Now define the constant $\bar{u}(\eta, k)$ by,

$$\bar{u}(\eta, k) = \frac{1}{|B_1(0)|} \int_{B_1(0)} u^{\eta,k}(x)dx = \frac{(r_k^\eta)^{-\frac{1}{2}}}{|B_1(0)|} \int_{B_{r_k}(0)} u^\eta(x)dx.$$

All radial solutions to problem (1.2.5) can be obtained in this fashion. Now, to answer the question of behavior of the solution in terms of $\bar{u}$, all we need to do is to understand the dependance of $\bar{u}$ on η and k. From the theory of ordinary differential equations $\bar{u}$ must be continuous as a function of $\eta \in [0, 1]$ and $\eta \in [1, \infty]$.

In the case when $\eta = 1$ the only radial solution is the constant solution $u \equiv 1$ and the critical point r_1 is not defined. However, using rigorous asymptotic analysis, we are able to show that $\bar{u}$ is still continuous at 1. We can also show that when η approaches zero then the radial solution $u_\eta(r)$ will approach the rupture solution $u_0(r)$ satisfying $u_0(0) = 0$ and $u_0'(0) = \infty$. Hence for each k , $\bar{u}(\eta, k)$ is continuous on $[0, \infty]$. For the behavior of $\bar{u}(\eta, k)$ as η tends to infinity we performed many numerical experiments confirming the asymptotic analysis and we obtained the limiting behavior of $\bar{u}(\eta, k)$ at infinity. Similarly, we studied the energy of the radial solution $u_\eta^k(\mathrm{r})$ denoted by $E(\eta, k)$ and we found that it is also continuous on $[0, \infty]$ and we were also able to derive its limiting profile as η approaches infinity. Combining the numerical computation of both $\bar{u}(\eta, k)$ and $E(\eta, k)$ with their limiting profile we have a better understanding of the dependance of the radial solutions and their energies as η varies.

1.4 RUPTURE SOLUTIONS OF GENERAL ELLIPTIC EQUATION

In chapter three we are interested in rupture solutions of the general elliptic equation for their special properties and would like to understand the mechanism of producing ruptures for elliptic partial differential equations. We also conjecture that the ruptures are discrete for finite energy solutions, and expect that the radial point rupture solutions will serve as the blow up profile of the solution near any point rupture. The general elliptic equation we are interested in has the form

$$\Delta u = f(u)$$

in a region $\Omega \subset \mathbb{R}^2$ and assuming Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. The function f is positive continuous and satisfying

$$\lim_{u \to 0^+} f(u) = \infty.$$

Motivated by the thin film equations, a solution u is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(p) > 0$ in $\Omega \backslash \{p\}$. Our main result is a sufficient condition on f for the existence of radial point rupture solutions. Actually, we are only concerned with the local solutions in a neighborhood of the point rupture. Since the equation has no singularity away from the point rupture, the possible extension of point rupture solution to a global solution could be carried out using similar arguments as in [15] where the case $f(u) = u^{-\alpha} - 1$. In our context, the function f is not required to be monotone decreasing as we do not impose on it to cross the x-axis with negative slope.

In chapter four we apply our results of rupture solutions for general elliptic equations to show existence of point rupture solutions to a class of quasi-linear elliptic equations of the form

$$\operatorname{div}(a(u)\nabla u) = \frac{a'(u)}{2}|\nabla u|^2 + g(u)$$

where $a \in C^1$ and $g \in C^0$ are positive functions of a real variable.

2.0 RADIAL STEADY STATES OF VAN DER WAALS FORCE DRIVEN THIN FILMS WITH PRESCRIBED FLUID VOLUME

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. We consider the thin films in a cylindrical container with base Ω which is governed by the fourth order partial differential equation

$$h_t = \nabla \left(h^n \nabla p\right) \tag{2.0.1}$$

where h is thickness of the thin film and the pressure

$$p = -\Delta h + \frac{1}{\alpha} h^{-\alpha} \tag{2.0.2}$$

is a sum of linearized surface tension and van der Waals force. Here $n > 0$ and $\alpha > 1$ are physical constants and in the van der Waals force driven thin film, we have $n = 3$ and $\alpha = 3$.

We assume that there is no flux across the boundary, which yields the boundary condition

$$\frac{\partial p}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.0.3}$$

We also ignore the wetting or non-wetting effect, and assume that the fluid surface is orthogonal to the boundary of the container, i.e.,

$$\frac{\partial h}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.0.4}$$

We associate (2.0.1) with an energy functional

$$E\left(h\right) = \int_\Omega \left(\frac{1}{2}\left|\nabla h\right|^2 - \frac{1}{\alpha\left(\alpha-1\right)} h^{1-\alpha}\right),$$

and formally, using (2.0.1) and the boundary conditions (2.0.3) , (2.0.4), we have

$$\begin{aligned}\frac{d}{dt}E\left(h\right) &= \int_{\Omega} \nabla h \nabla h_t + \frac{1}{\alpha} h^{-\alpha} h_t \\ &= \int_{\Omega} \left(-\Delta h + \frac{1}{\alpha} h^{-\alpha}\right) h_t \\ &= \int_{\Omega} p \nabla \left(h^n \nabla p\right) \\ &= -\int_{\Omega} h^n \left|\nabla p\right|^2 \leq 0.\end{aligned}$$

Hence, for a thin film fluid at rest, the pressure p has to be a constant, and h satisfies the elliptic equation

$$-\Delta h + \frac{1}{\alpha} h^{-\alpha} = p \text{ in } \Omega$$

with the Neumann boundary condition (2.0.4).

In physical experiments, usually the total volume of the fluid is a known parameter. i.e.,

$$\bar{h} = \frac{1}{|\Omega|} \int_{\Omega} h(x) dx$$

is given. Therefore for any given $\bar{h} > 0$, we need to find a function h and an unknown constant p satisfying

$$\begin{cases} \Delta h = \frac{1}{\alpha} h^{-\alpha} - p \quad \text{in} \quad \Omega, \\ \frac{1}{|\Omega|} \int_{\Omega} h(x) dx = \bar{h}, \\ \frac{\partial u}{\partial \nu} = 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{cases} \tag{2.0.5}$$

Our goal in this chapter is to understand the structure of the radial solutions and their associated energy when the prescribed volume $\bar{h}$ changes.

2.1 SCALING PROPERTY OF GLOBAL RADIAL SOLUTIONS

Let $\Omega = B_1(0)$, a unit disk in $\mathbb{R}^2$. We are interested in the radial solutions of (2.0.5) when $\bar{h}$ is given which leads to

$$\begin{cases} h_{rr} + \frac{1}{r}h_r = \frac{1}{\alpha}h^{-\alpha} - p \ \text{ in } \ B_1(0), \\ 2\int_0^1 rh(r)dr = \bar{h}, \\ h'(1) = 0. \end{cases} \tag{2.1.1}$$

From the elliptic theory, h is smooth whenever it is positive, hence we also require that $h'(0) = 0$ if $h(0) > 0$.

We first ignore the volume constraint and the Neumann boundary condition. Fixing $p = \frac{1}{\alpha}$, we consider the ordinary differential equation

$$h_{rr} + \frac{1}{r}h_r = \frac{1}{\alpha}h^{-\alpha} - \frac{1}{\alpha} \tag{2.1.2}$$

defined on $[0, \infty)$. It has been shown in [15] that for any $\eta > 0$,

$$\begin{cases} h_{rr} + \frac{1}{r}h_r = \frac{1}{\alpha}h^{-\alpha} - \frac{1}{\alpha}, \\ h(0) = \eta, \\ h'(0) = 0 \end{cases} \tag{2.1.3}$$

has a unique positive solution h^η defined on $[0, \infty)$. And when $\eta = 0$, there exists a unique rupture solution h^0 which is continuous on $[0, \infty)$ such that $h^0(0) = 0$ and h^0 is positive and satisfies (2.1.2) on $(0, \infty)$. We remark here that $h_r^0(0) = \infty$. Some plots for the radial rupture solution and smooth radial solutions follow in the next page.

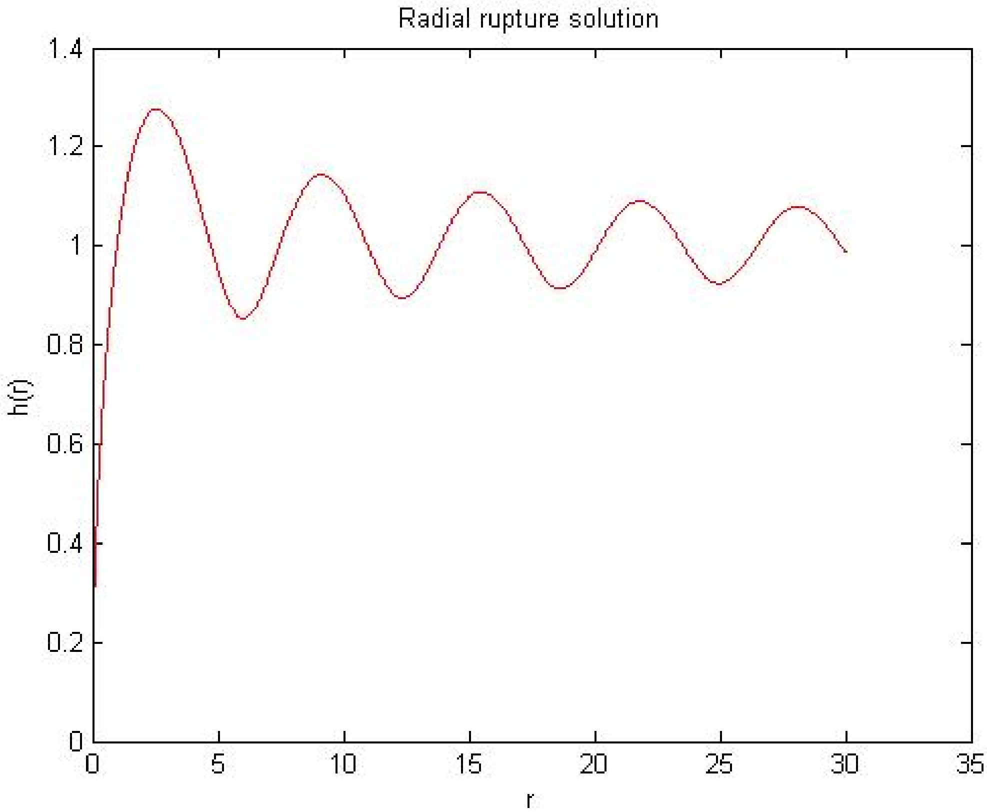

Figure 1: Global rupture solution with 9 oscillations.

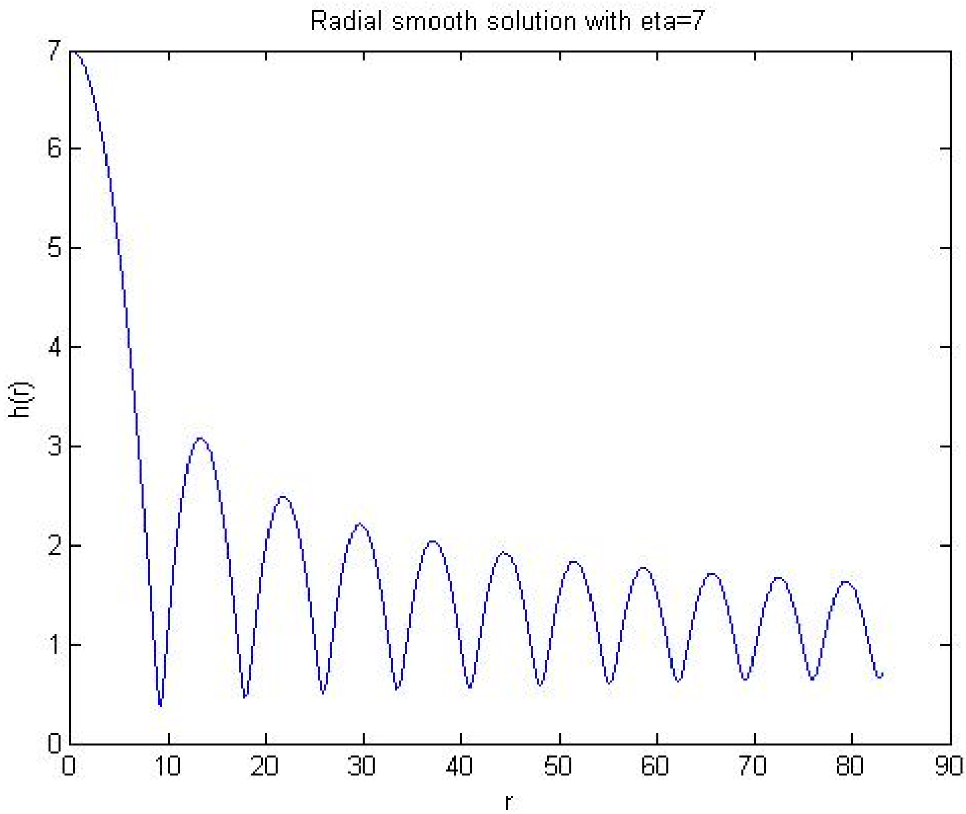

Figure 2: Global radial smooth solution with eta=7.

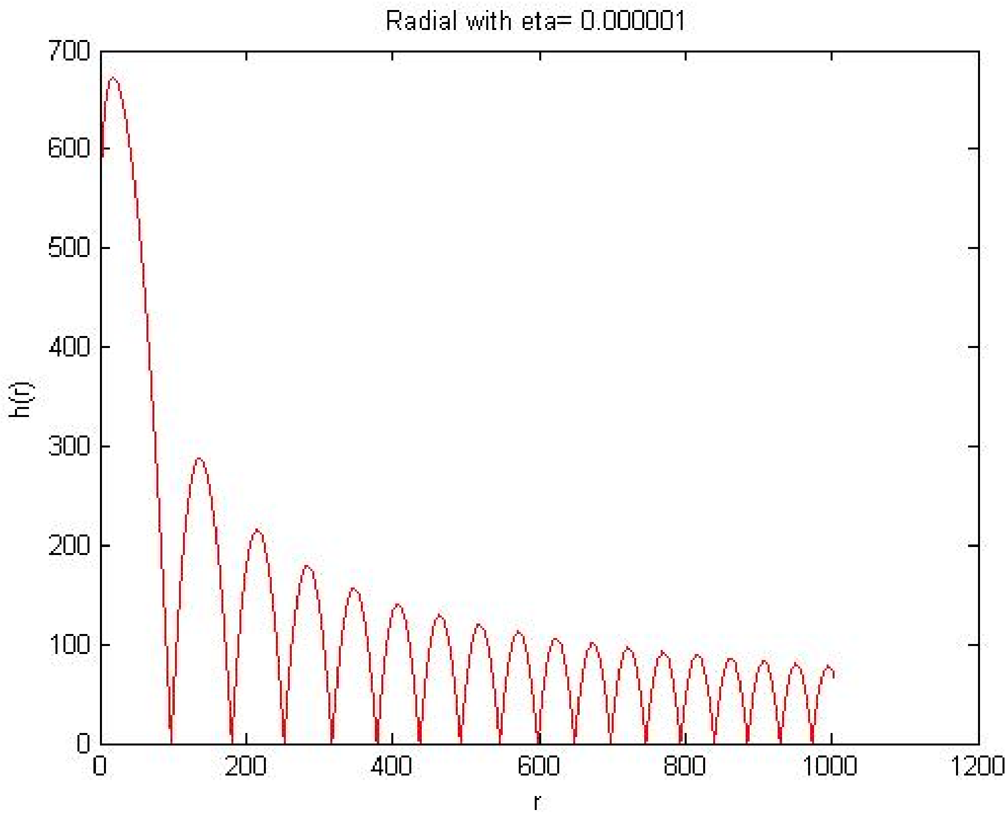

Figure 3: Radial solution with eta = 11.

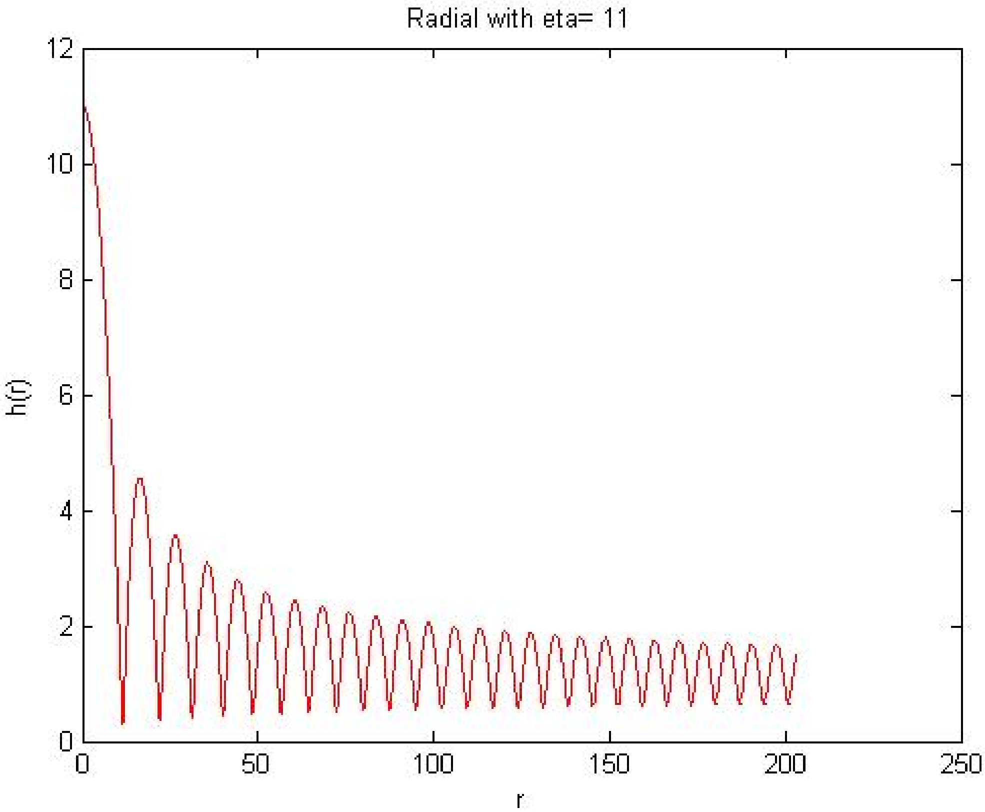

Figure 4: Radial smooth solution with eta = 0.000001

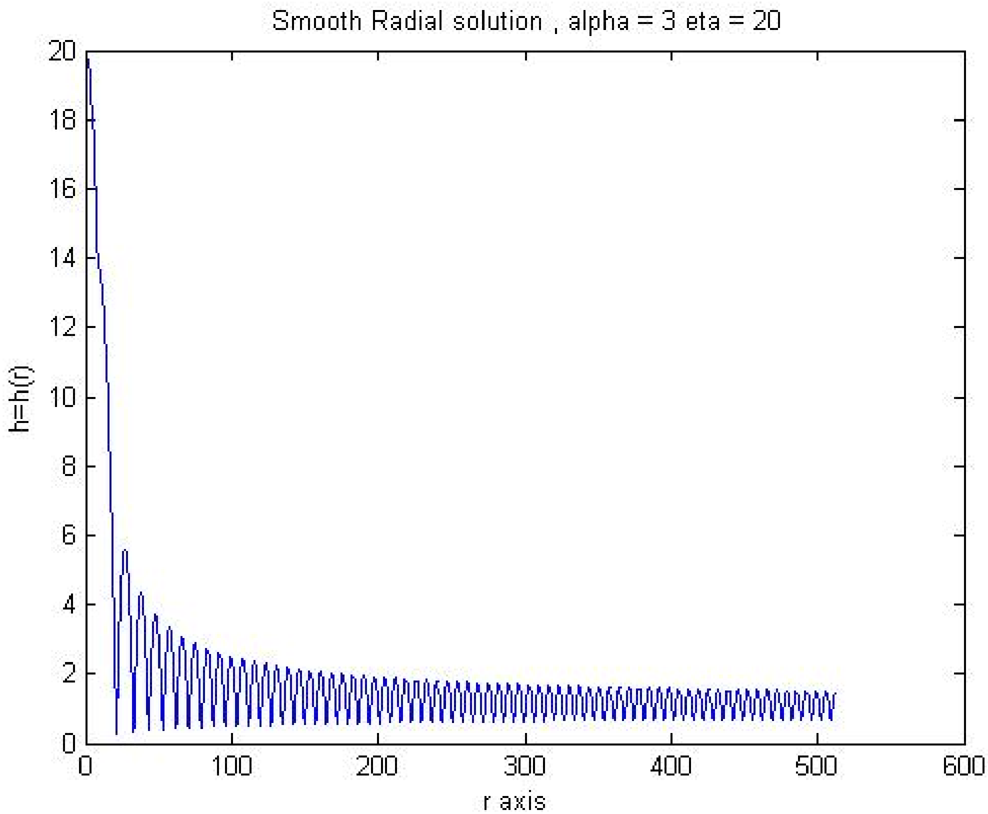

Figure 5: Radial smooth solution with eta = 20.

Obviously $h \equiv 1$ if $\eta = 1$. When $\eta \geq 0$, $\eta \neq 1$, h^{η} oscillates around 1 and there exists an increasing sequence of critical radius $r_k^{\eta} \to \infty$ such that $(h^{\eta})'(r_k^{\eta}) = 0$.

Given $\eta \geq 0$, $\eta \neq 1$ and a positive integer k, $h^{\eta}(r)$ satisfies the Neumann boundary condition at $r = r_k^{\eta}$. We now define

$$h^{\eta,k}(r) = (r_k^{\eta})^{-\frac{2}{1+\alpha}} h^{\eta}(r_k^{\eta} r).$$

One can easily verify that $h^{\eta,k}(x) = h^{\eta,k}(|x|)$ satisfies the elliptic equation

$$\Delta h = \frac{1}{\alpha} \cdot h^{-\alpha} - p^{\eta,k} \quad in \quad B_1(0)$$

with Neumann boundary condition

$$\frac{\partial h}{\partial \nu} = 0 \ on \ \ \partial B_1(0).$$

Here $p^{\eta,k}$ is defined by

$$p^{\eta,k} = \frac{1}{\alpha}(r_k^{\eta})^{\frac{2\alpha}{1+\alpha}}.$$

We can also calculate the average thickness for $h^{\eta,k}$,

$$\begin{aligned}\bar{h}^{\eta,k} &= \frac{1}{|B_1(0)|}\int_{B_1(0)} h^{\eta,k}(x)dx = \frac{(r_k^{\eta})^{-\frac{2}{1+\alpha}}}{|B_{r_k^{\eta}}(0)|}\int_{B_{r_k^{\eta}}(0)} h^{\eta}(r)dr \\ &= 2(r_k^{\eta})^{-\frac{2}{1+\alpha}-2}\int_0^{r_k^{\eta}} r h^{\eta}(r)dr.\end{aligned}$$

Its associated energy is given by

$$\begin{aligned}E^{\eta,k} &= \int_{B_1(0)}\left(\frac{1}{2}\left|\nabla h^{\eta,k}\right|^2 - \frac{1}{\alpha(\alpha-1)}\left(h^{\eta,k}\right)^{1-\alpha}\right) \\ &= (r_k^{\eta})^{-\frac{4}{1+\alpha}}\int_{B_{r_k^{\eta}}(0)}\left(\frac{1}{2}|\nabla h^{\eta}|^2 - \frac{1}{\alpha(\alpha-1)}(h^{\eta})^{1-\alpha}\right) \\ &= 2\pi(r_k^{\eta})^{-\frac{4}{1+\alpha}}\int_0^{r_k^{\eta}}\left(\frac{1}{2}\left(\frac{dh^{\eta}}{dr}\right)^2 - \frac{1}{\alpha(\alpha-1)}(h^{\eta})^{1-\alpha}\right) r dr.\end{aligned}$$

Hence, we constructed a solution to (2.1.1) with

$$\bar{h} = \bar{h}^{\eta,k}.$$

Actually, all nonconstant solutions to (2.1.1) can be obtained in this fashion.

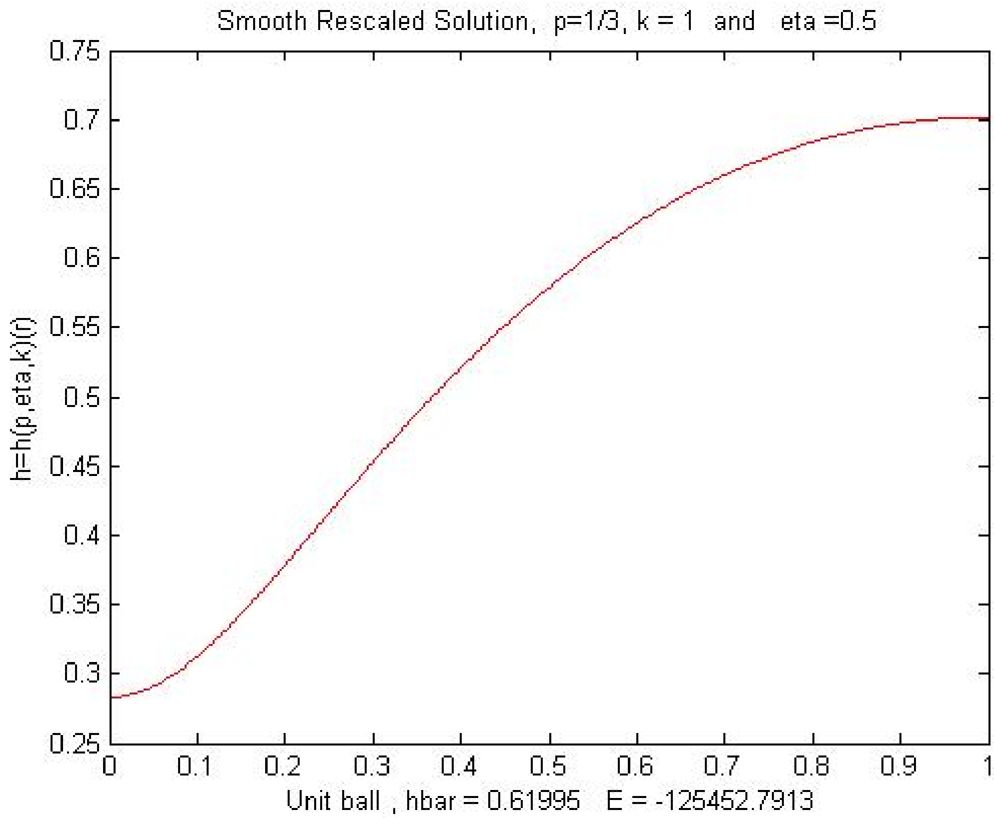

Figure 6: Smooth radial rescaled to the unit ball with eta=.5 and k=1

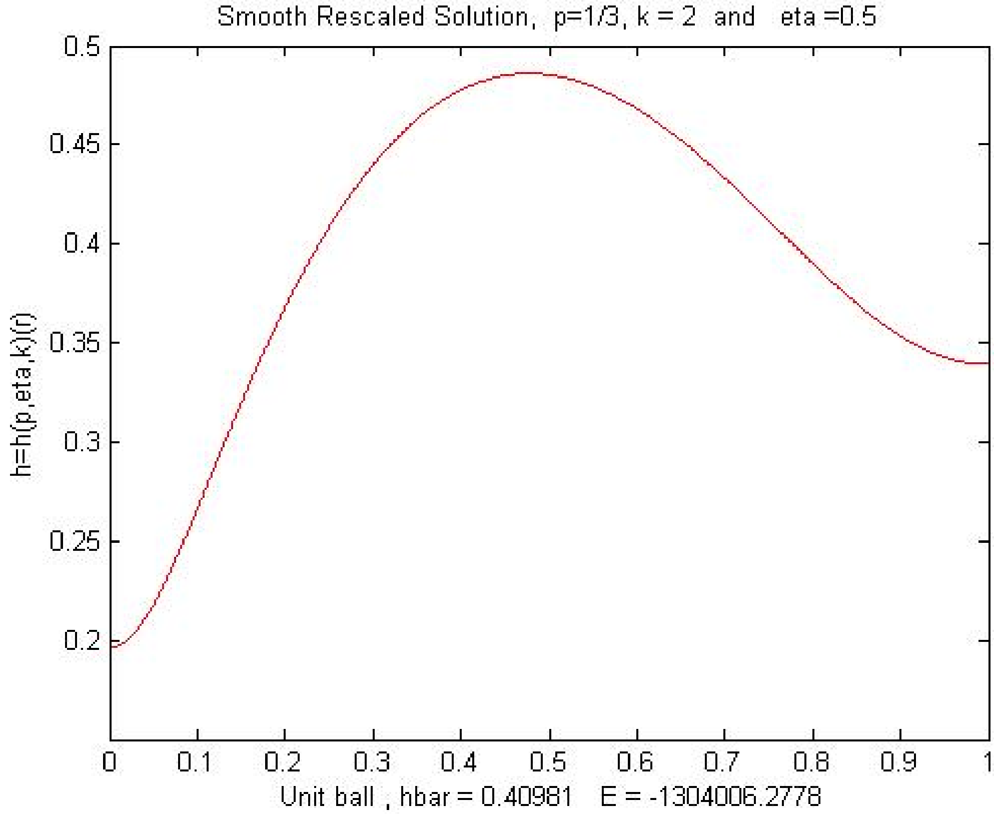

Figure 7: Smooth radial rescaled to the unit ball with eta=.5 and k=2

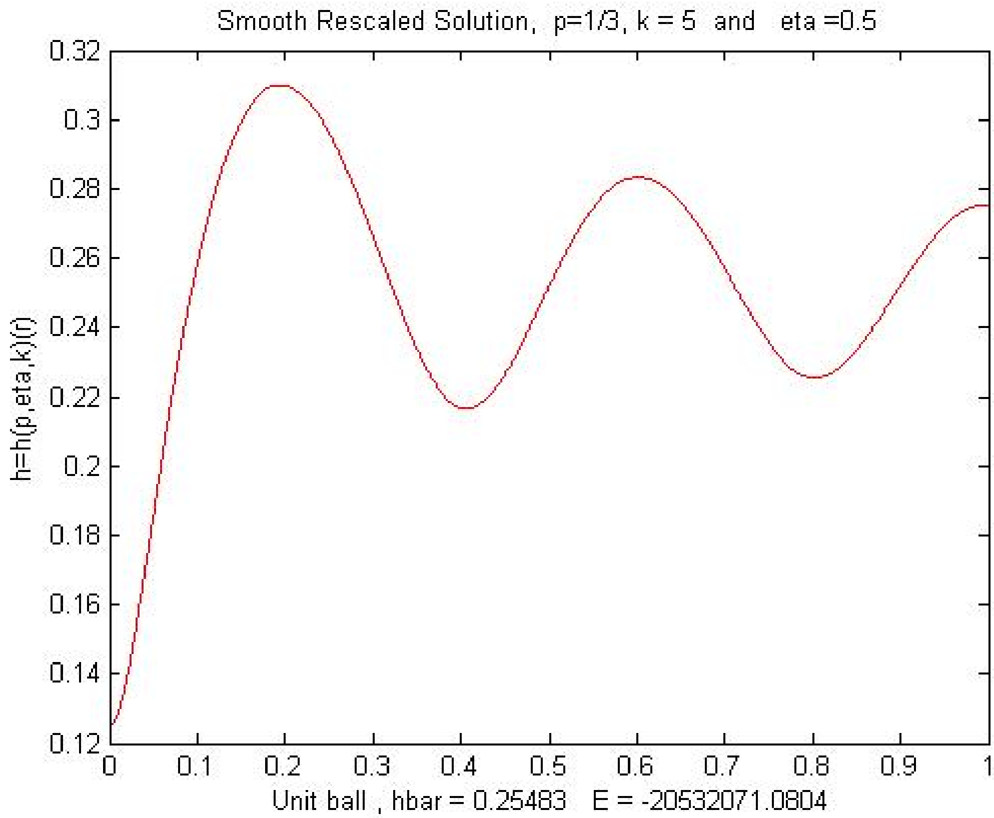

Figure 8: Smooth radial rescaled to the unit ball with eta=.5 and k=5

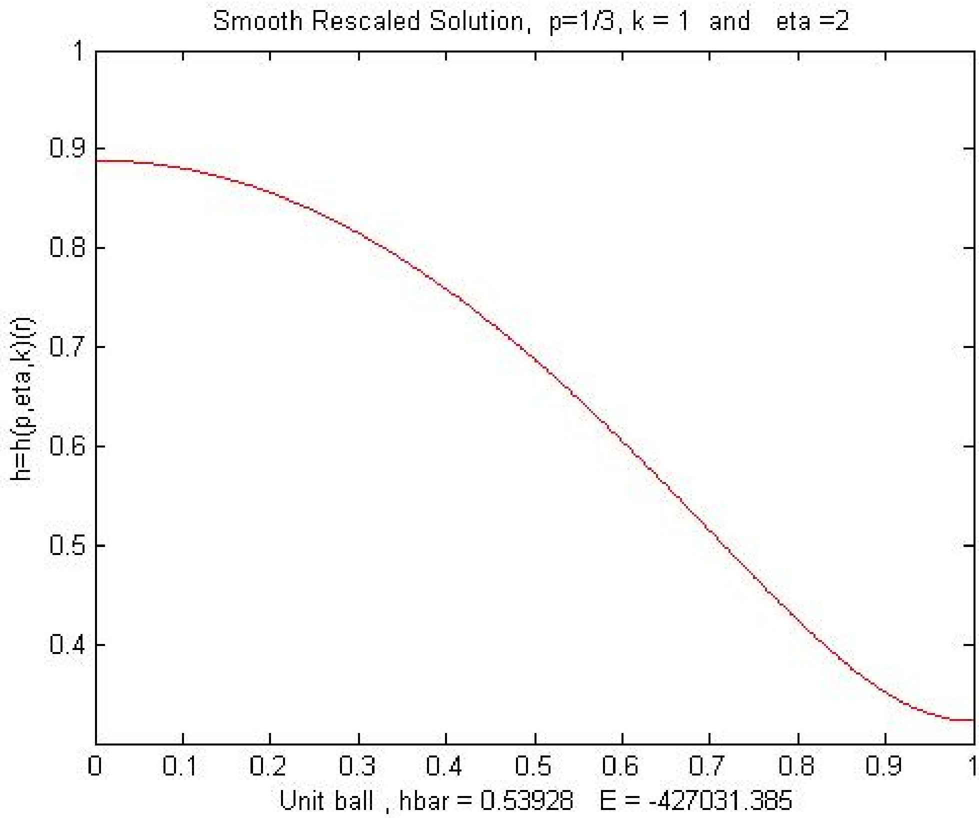

Figure 9: Smooth radial rescaled to the unit ball with eta= 2 and k=1

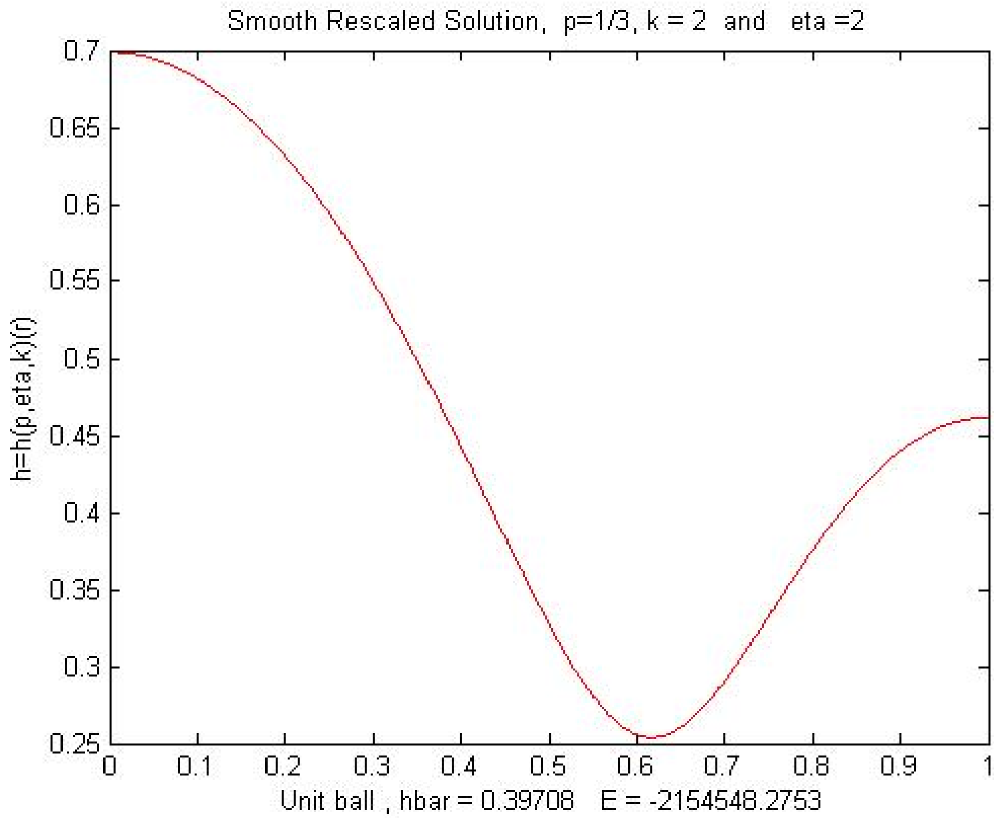

Figure 10: Smooth radial rescaled to the unit ball with eta= 2 and k=2

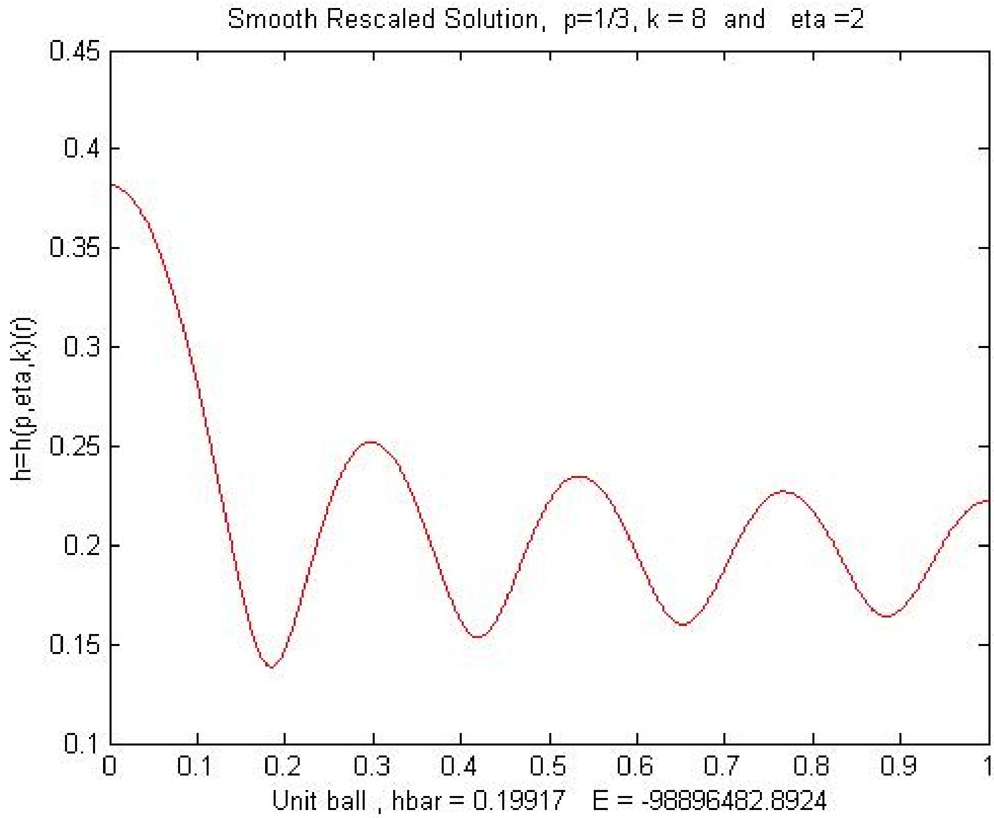

Figure 11: Smooth radial rescaled to the unit ball with eta= 2 and k=8

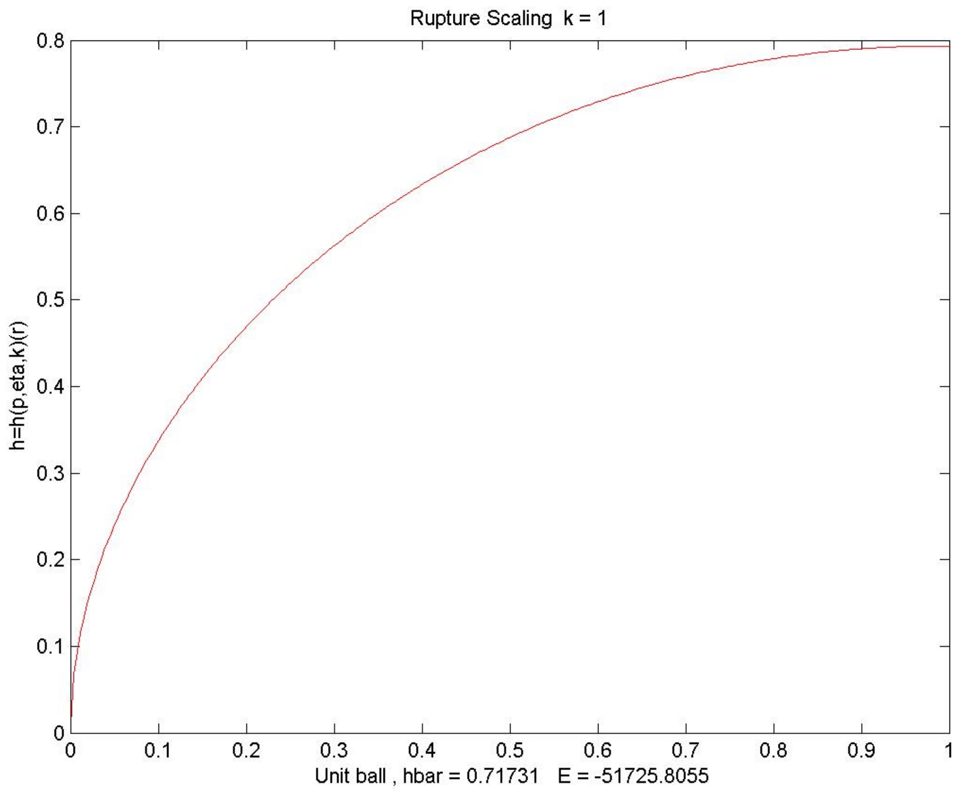

Figure 12: Rupture radial rescaled to the unit ball with k=1

To obtain the structure of nontrivial radial solutions when $\bar{h}$ is given, we need to find η, k so that $\bar{h} = \bar{h}^{\eta,k}$. Hence, we need to completely understand the dependence of $\bar{h}^{\eta,k}$ on η and k.

Fixing a positive integer k, from the continuous dependence of ordinary differential equations on the initial data, $\bar{h}(\eta, k) = \bar{h}^{\eta,k}$ is a continuous function on η on $(0,1) \cup (1,\infty)$. In the next three sections, we want to investigate the behavior of $\bar{h}(\eta, k)$ as $\eta \to 0$, $\eta \to 1$ and $\eta \to \infty$. Also we would like to show that for a fixed η then $\bar{h}(\eta, k)$ approaches zero as $k \to \infty$

2.2 RUPTURE SOLUTION AS A LIMIT OF SMOOTH SOLUTIONS $\eta \to 0$

As $\eta \to 0^+$, h_η converges uniformly to the rupture solution h_0 on $[0,\infty)$. Hence, $\bar{h}(\eta, k)$ is continuous at $\eta = 0$. See the following plots for illustration.

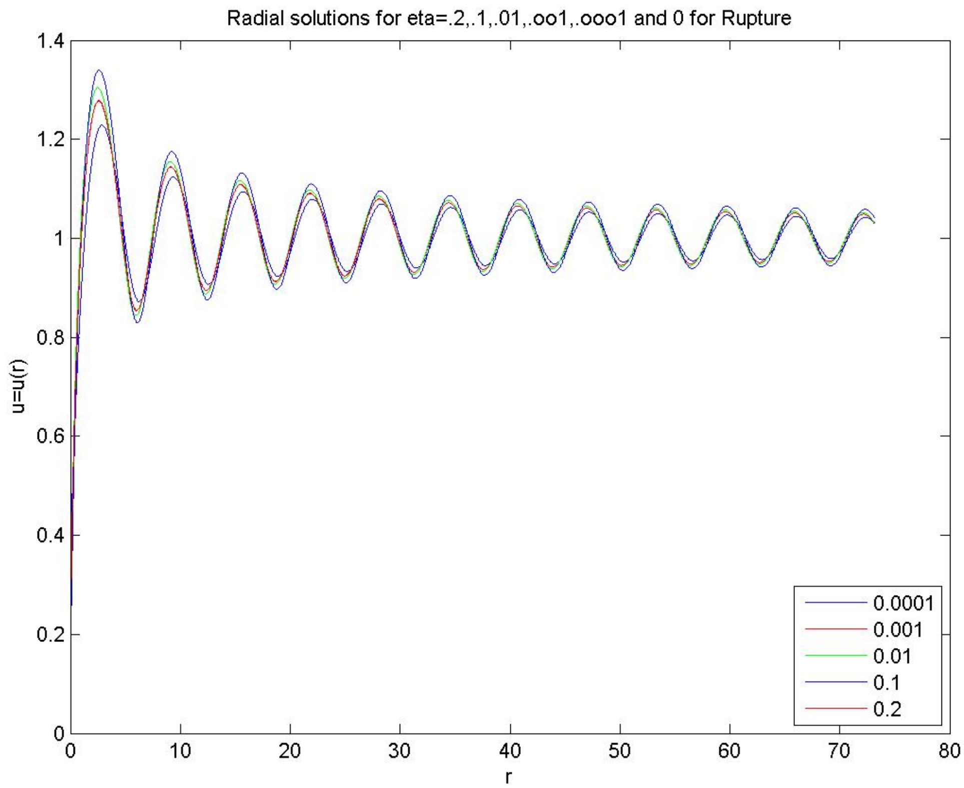

Figure 13: Set of smooth radial solution for eta = 0.2 , 0.1 , 0.01 , 0.001 and 0.0001.

This figure proves that as $\eta \to 0$ the corresponding smooth radial solution approaches the rupture solution.

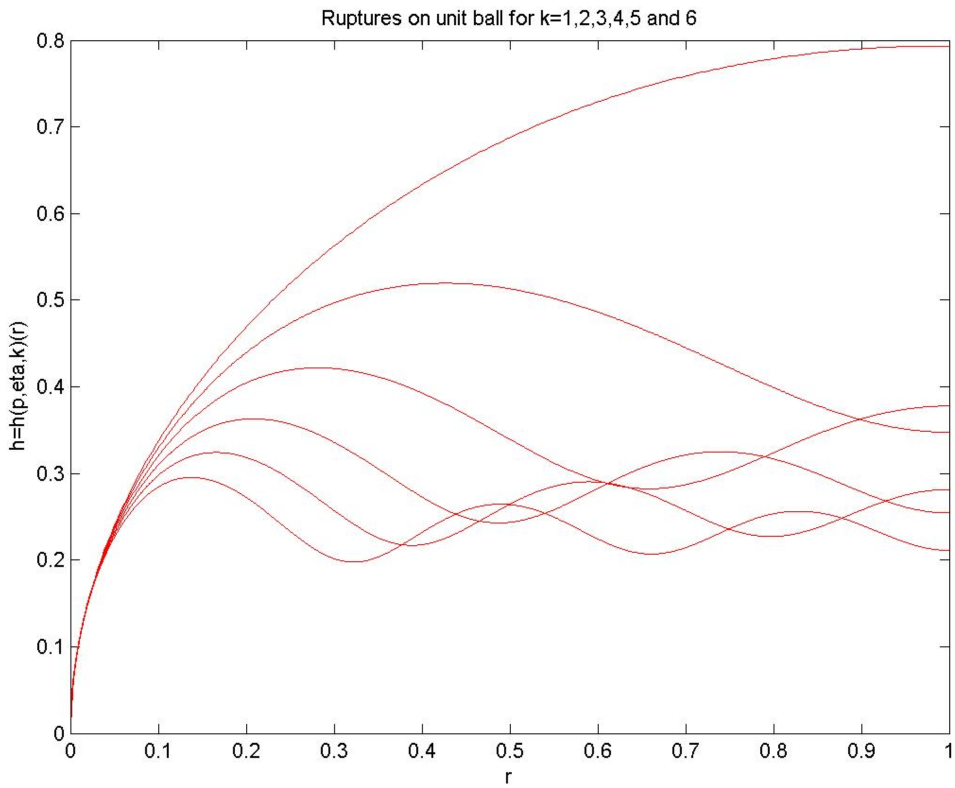

Figure 14: Set of rupture radial solutions rescaled to the unit ball for k=1,2,3,4,5 and 6.

From this plot we see that for fixed η, here it is zero since ruptures are considered only, $\bar{h}^{\eta,k}$ tends to zero as $k \to \infty$ in a decreasing manner. This result is in fact true for any η.

Inspired by the numerical suggestions, we would like to intend to prove that for fixed η the average volume $\bar{h}^{\eta,k}$ tends to zero as $k \to \infty$ in a decreasing manner. We will prove that indeed $\bar{h}^{\eta,k} \to 0$ as $k \to \infty$ however, we are unable to prove monotonicity here.

Proposition 2.1. *For any fixed $\eta \geq 0$ then $\bar{h}^{\eta,k} \to 0$ as $k \to \infty$.*

Proof. We know that as $k \to \infty$ the critical point $r_k^\eta \to \infty$. We also know that as $r \to \infty$ the radial solution $h(r)$ tends to 1. Therefore for any $R > 0$ there exists $N > 0$ such that for $k > N$ then $r_k^\eta > R$ and for some $R > 0$ the function $h(r) < 2$. Thus since we can write,

$$\bar{h}^{\eta,k} = 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\left(\int_0^R rh^\eta(r)dr + \int_R^{r_k^\eta} rh^\eta(r)dr\right),$$

then the result follows. □

2.3 LINEARIZATION WHEN $\eta \to 1$

$\bar{h}(\eta,k)$ is not defined when $\eta = 1$. To understand the behavior of $\bar{h}(\eta,k)$ as $\eta \to 1$, we need to understand the behavior of h^η as $\eta \to 1$. Recall that h^η is a solution to (2.1.2) with $h^\eta(0) = \eta$ and $(h^\eta)'(0) = 0$. We define

$$\varepsilon = \eta - 1,$$

and

$$w^\eta(r) = \frac{h^\eta(r) - 1}{\varepsilon}.$$

Then w^η is a solution to the differential equation

$$w_{rr} + \frac{1}{r}w_r = \frac{1}{\varepsilon}\left[\frac{1}{\alpha}(1+\varepsilon w)^{-\alpha} - \frac{1}{\alpha}\right] \tag{2.3.1}$$

with initial condition

$$w(0) = 1,\ w'(0) = 0.$$

As $\eta \to 1$, $\varepsilon \to 0$, formally, (2.3.1) converges to the Bessel's differential equation with order 0,

$$w_{rr} + \frac{1}{r}w_r + w = 0$$

with the initial date

$$w(0) = 1,\ w'(0) = 0.$$

Such limiting initial value problem has a unique solution which is the Bessel's function of the first kind with order 0 and is given by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

We remark here that $J_0(x)$ is oscillating around 0. See plot of Bessel function of the first kind of order zero in the next page.

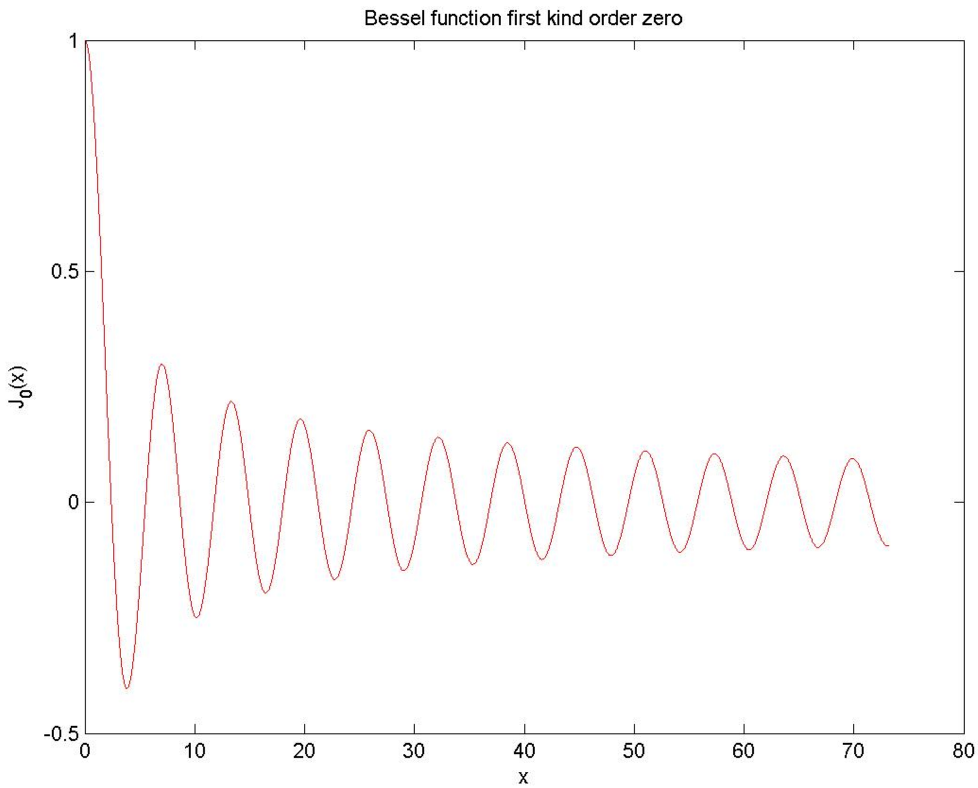

Figure 15: The Bessel function of first kind and of order zero.

This is the plot of the Bessel function of first kind and of order zero ploted over the interval [0,70]. It oscillates around zero. One can see its similarity with the radial solutions.

We can show that as $\eta \to 1$, w^η converges uniformly to J_0. Since both w^η and J_0 are oscillating around 0, $r_k^\eta \to r_k^*$ as $\eta \to 1$ where r_k^*, $k = 1, 2, \cdots$ is an increasing sequence of the critical radius of the Bessel's function $J_0(x)$. Since $h^\eta \to 1$ uniformly as $\eta \to 1$, we have

$$\bar{h}^{\eta,k} = \frac{(r_k^\eta)^{-\frac{2}{1+\alpha}}}{|B_{r_k^\eta}(0)|} \int_{B_{r_k^\eta}(0)} h^\eta(r)dr \to (r_k^*)^{-\frac{2}{1+\alpha}}$$

as $\eta \to 1$ and

$$\begin{aligned} E^{\eta,k} &= (r_k^\eta)^{-\frac{4}{1+\alpha}} \int_{B_{r_k^\eta}(0)} \left(\frac{1}{2} |\nabla h^\eta|^2 - \frac{1}{\alpha(\alpha-1)} (h^\eta)^{1-\alpha} \right) \\ &\to -\frac{1}{\alpha(\alpha-1)} (r_k^*)^{-\frac{4}{1+\alpha}} \left| B_{r_k^*}(0) \right| = -\frac{\pi}{\alpha(\alpha-1)} (r_k^*)^{2-\frac{4}{1+\alpha}}. \end{aligned}$$

as $\eta \to 1$.

Hence, we can define $\bar{h}^{\eta,k}$ and $E^{\eta,k}$ so that they are both continuous functions on $[0, \infty)$.

2.4 LIMITING PROFILE WHEN $\eta \to \infty$

In this section, we want to understand the behavior of $\bar{h}^{\eta,k}$ and $E^{\eta,k}$ as $\eta \to \infty$.

Let $\eta > 1$ and h_η be the solution to (2.1.3). we define the blow down solution z by

$$z(x) = \frac{1}{\eta} h(r)$$

with $r = \sqrt{\alpha\eta}x$. Then we have

$$z_x = \frac{\sqrt{\alpha}}{\sqrt{\eta}} h_r(r),$$

$$z_{xx} = \alpha h_{rr}(r)$$

and hence

$$\begin{aligned} z'' + \frac{1}{x} z' &= \alpha \left(h_{rr} + \frac{1}{r} h_r \right) = h^{-\alpha} - 1 \\ &= \frac{\eta^{-\alpha}}{z^\alpha} - 1. \end{aligned}$$

Denoting $\varepsilon = \frac{1}{\eta}$, we have $\varepsilon \to 0$ as $\eta \to \infty$. The blow down function z is a solution to the initial value problem

$$\begin{cases} z'' + \frac{1}{x} z' = \frac{\varepsilon^\alpha}{z^\alpha} - 1, \\ z(0) = 1, \quad \text{and} \quad z'(0) = 0. \end{cases} \tag{2.4.1}$$

The following plot illustrates the blow down function z for $\varepsilon = 10$.

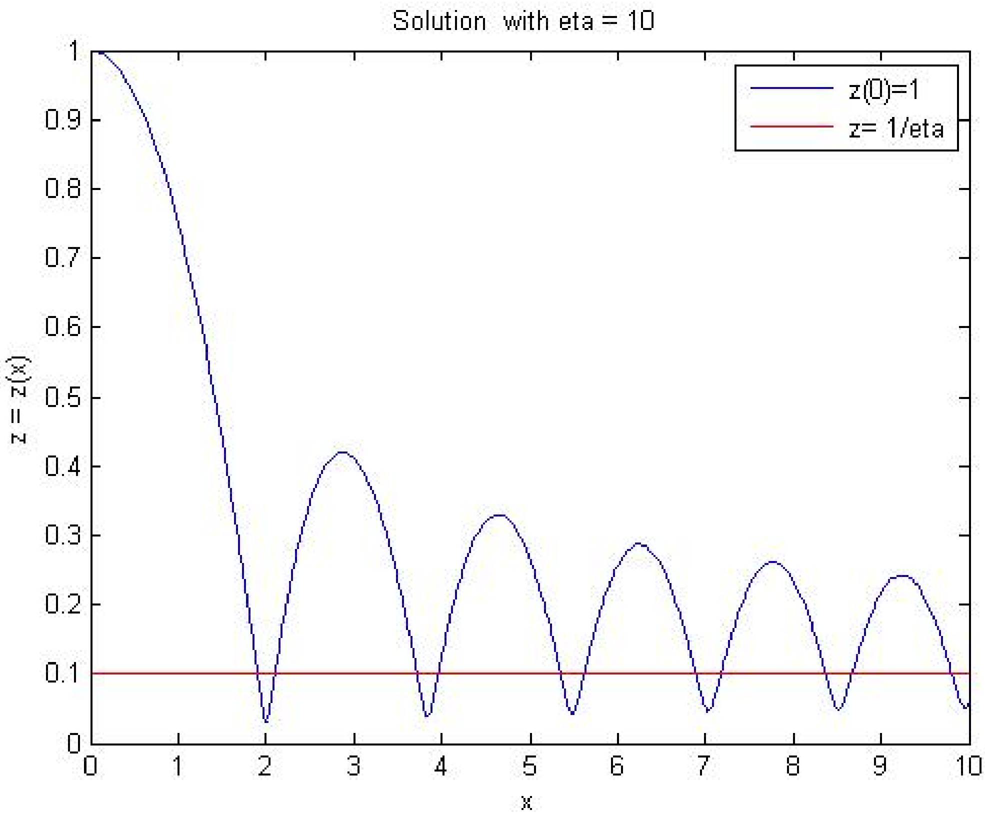

Figure 16: Blow down solution with eta=0.1

This plot show an instance of blow down solution when $\varepsilon = 0.10$.

Formally, as $\varepsilon \to 0$, (2.4.1) converges to the limiting equation

$$\begin{cases} z'' + \frac{1}{x} z' = -1, \\ z(0) = 1, \quad \text{and} \quad z'(0) = 0. \end{cases} \tag{2.4.2}$$

The following plot shows the profile of the asymptotic solution.

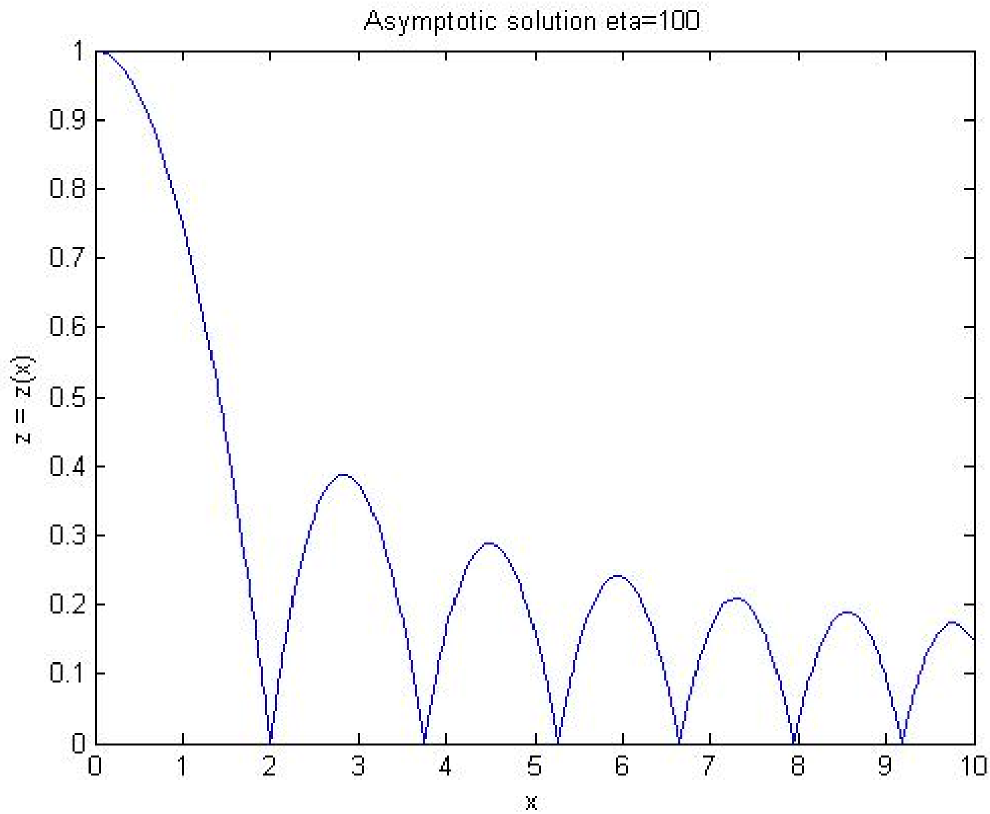

Figure 17: Asymptotic solution eta=100.

This plot illustrates how the asymptotic solution behaves and appears as the parameter $\eta \to \infty$

which has a unique global solution

$$z(x) = 1 - \frac{1}{4}x^2.$$

However, we can't expect

$$\lim_{\varepsilon \to 0} z^{\varepsilon}(x) = 1 - \frac{1}{4}x^2$$

since the function $1 - \frac{1}{4}x^2$ becomes negative when $x > 2$.

Nonetheless, we can establish the following theorem:

Theorem 2.2. *For every $\varepsilon > 0$, let $z^{\varepsilon}(x)$ be the unique solution of the initial value problem (2.4.1).*

Then as ε tends to zero positively, $z^{\varepsilon}(x)$ converges uniformly on $[0, \infty)$ to $z_*(x)$, the solution of the limiting initial value problem

$$\begin{cases} z_*'' + \frac{1}{x} z_*' = -1, \quad z_* > 0 \quad in \bigcup_{j=0}^{\infty} (a_j, a_{j+1}). \\ z_*(0) = 1, \quad and \quad z_*'(0) = 0, \\ z_*(a_j) = 0, \quad z_*'(a_j+) = -z_*'(a_j-) \end{cases} \tag{2.4.3}$$

where $a_0 = 0$, $2 = a_1 < a_2 < \cdots$ are inductively computed by solving the IVP (2.4.3).

Hence, in particular, we have $z^{\varepsilon}(x)$ converges uniformly to $1 - \frac{1}{4}x^2$ on $[0, 2]$ as $\varepsilon \to 0$ and $\frac{r_1^{\eta}}{\sqrt{\alpha\eta}}$ converges to 2 as $\eta \to \infty$. More generally, we have for $k = 1, 2, 3, \cdots$,

$$\lim_{\eta \to \infty} \frac{r_{2k-1}^{\eta}}{\sqrt{\alpha\eta}} = a_k$$

and

$$\lim_{\eta \to \infty} \frac{r_{2k}^{\eta}}{\sqrt{\alpha\eta}} = b_k$$

where b_k is the maximum point of z^* in (a_k, a_{k+1}).

Given a positive integer k and given $\eta > 1$, we have

$$\begin{aligned}
\bar{h}^{\eta,k} &= 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\int_0^{r_k^\eta} r h^\eta(r)dr \\
&= 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\eta\int_0^{r_k^\eta} r z\left(\frac{r}{\sqrt{\alpha\eta}}\right)dr \\
&= 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\alpha\eta^2\int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} s z\left(s\right)ds \\
&= 2\alpha^{-\frac{1}{1+\alpha}}\eta^{\frac{\alpha}{1+\alpha}}\left(\frac{r_k^\eta}{\sqrt{\alpha\eta}}\right)^{-\frac{2}{1+\alpha}-2}\int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} s z\left(s\right)ds
\end{aligned}$$

Hence, we have for $k = 1, 2, 3, \cdots$,

$$\lim_{\eta\to\infty}\frac{\bar{h}^{\eta,2k-1}}{\eta^{\frac{\alpha}{1+\alpha}}} = 2\alpha^{-\frac{1}{1+\alpha}}a_k^{-\frac{2}{1+\alpha}-2}\int_0^{a_k} s z^*\left(s\right)ds$$

and

$$\lim_{\eta\to\infty}\frac{\bar{h}^{\eta,2k}}{\eta^{\frac{\alpha}{1+\alpha}}} = 2\alpha^{-\frac{1}{1+\alpha}}b_k^{-\frac{2}{1+\alpha}-2}\int_0^{b_k} s z^*\left(s\right)ds$$

We remark here that for each positive integer k, $\bar{h}^{\eta,k} \to \infty$ as $\eta \to \infty$.

Next we investigate the energy of radial solutions as $\eta \to \infty$. Since

$$z\left(x\right) = \frac{1}{\eta}h(r),$$

$$\frac{dh^\eta}{dr} = \eta z'\left(x\right)\frac{dx}{dr} = \frac{\sqrt{\eta}}{\sqrt{\alpha}}z'\left(x\right).$$

Hence

$$\begin{aligned}
E^{\eta,k} &= 2\pi(r_k^\eta)^{-\frac{4}{1+\alpha}}\int_0^{r_k^\eta}\left(\frac{1}{2}\left(\frac{dh^\eta}{dr}\right)^2 - \frac{1}{\alpha\left(\alpha-1\right)}\left(h^\eta\right)^{1-\alpha}\right)rdr \\
&= 2\pi\left(\frac{r_k^\eta}{\sqrt{\alpha\eta}}\right)^{-\frac{4}{1+\alpha}}\left(\alpha\eta\right)^{1-\frac{2}{1+\alpha}}\int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}}\left(\frac{1}{2}\frac{\eta}{\alpha}\left(z'\left(x\right)\right)^2 - \frac{1}{\alpha\left(\alpha-1\right)}\left(\eta z\left(x\right)\right)^{1-\alpha}\right)xdx \\
&= \pi\alpha^{-\frac{2}{1+\alpha}}\left(\frac{r_k^\eta}{\sqrt{\alpha\eta}}\right)^{-\frac{4}{1+\alpha}}\eta^{2-\frac{2}{1+\alpha}}\int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} x\left|z'\right|^2dx + O\left(\eta^{2-\alpha-\frac{2}{1+\alpha}}\right).
\end{aligned}$$

Hence, we have for $k = 1, 2, 3, \cdots$,

$$\lim_{\eta\to\infty}\frac{E^{\eta,2k-1}}{\eta^{2-\frac{2}{1+\alpha}}} = \pi\alpha^{-\frac{2}{1+\alpha}}a_k^{-\frac{4}{1+\alpha}}\int_0^{a_k} s\left|(z^*)'\right|^2ds.$$

and

$$\lim_{\eta\to\infty} \frac{E^{\eta,2k}}{\eta^{2-\frac{2}{1+\alpha}}} = \pi\alpha^{-\frac{2}{1+\alpha}} b_k^{-\frac{4}{1+\alpha}} \int_0^{b_k} s \left|(z^*)'\right|^2 ds.$$

The following plots illustrate how the average volume $\bar{h}$ of the fluid changes as η changes from 0 to 100.

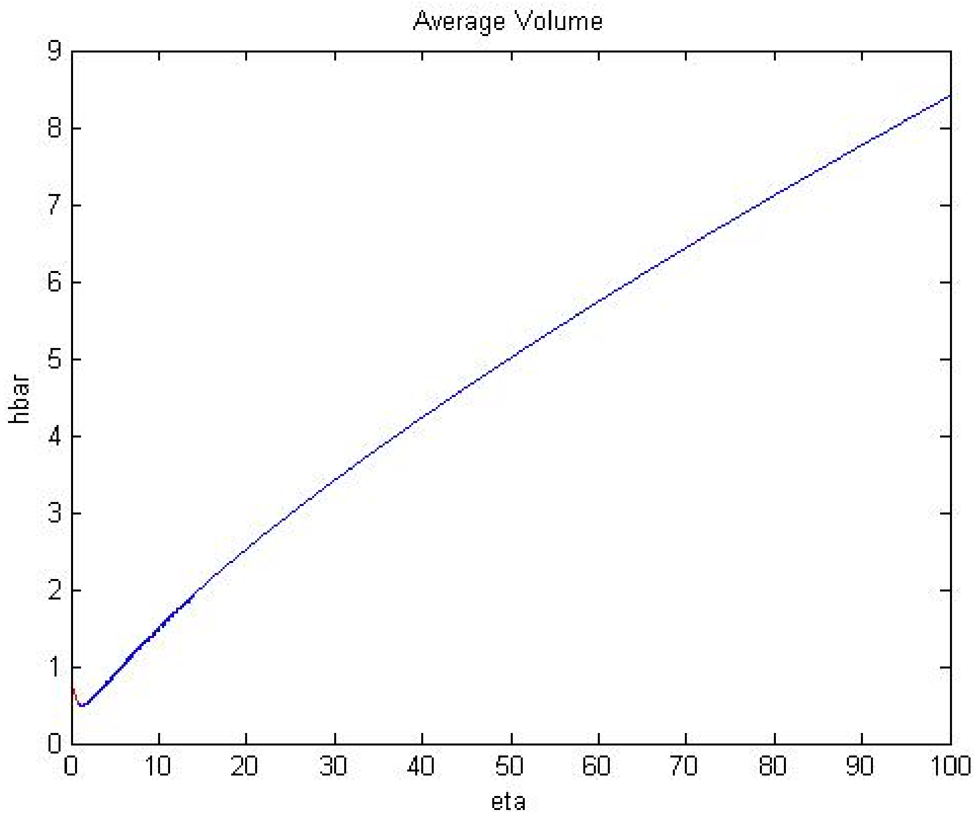

Figure 18: Average volume hbar.

This figure shows how for fixed k the parameter $\bar{h}$ behaves as $\eta \to \infty$. Here we let eta varies from 0 to 100.

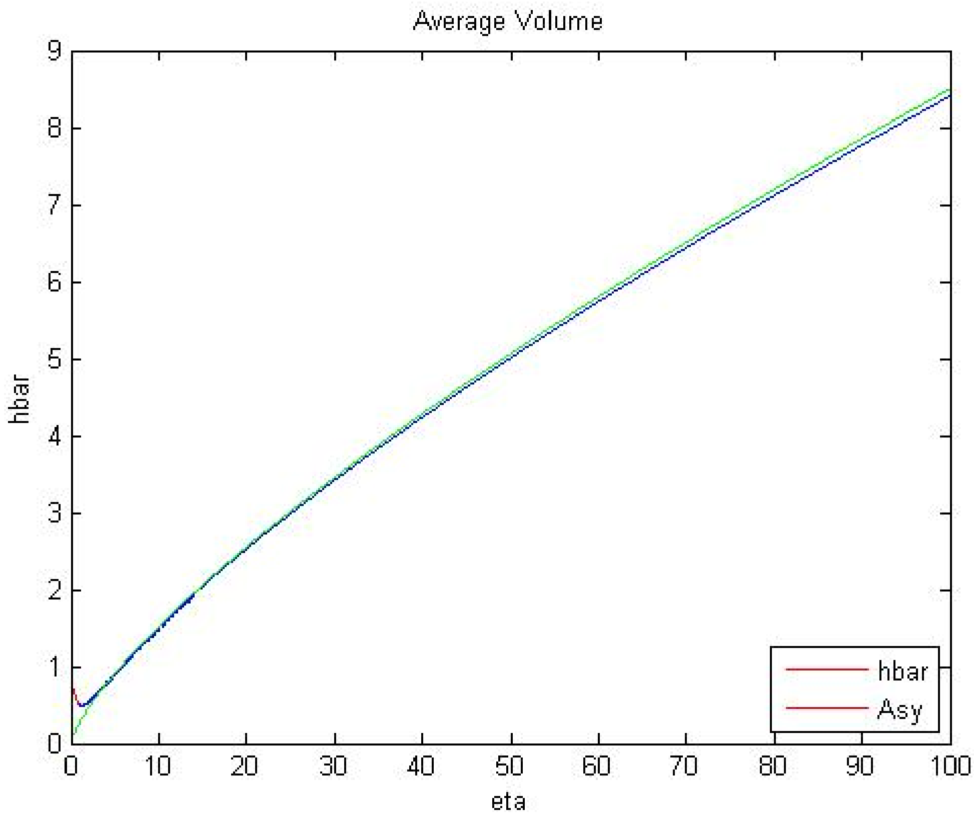

Figure 19: Average volume hbar and its asymptotic approximation.

This plot shows that the plot of $\bar{h}$ coincides with its asymptotic behavior as $\eta \to \infty$. We did the experiment with η changing from 0 to 100. We note the match of the approximation since $\eta \approx 5$.

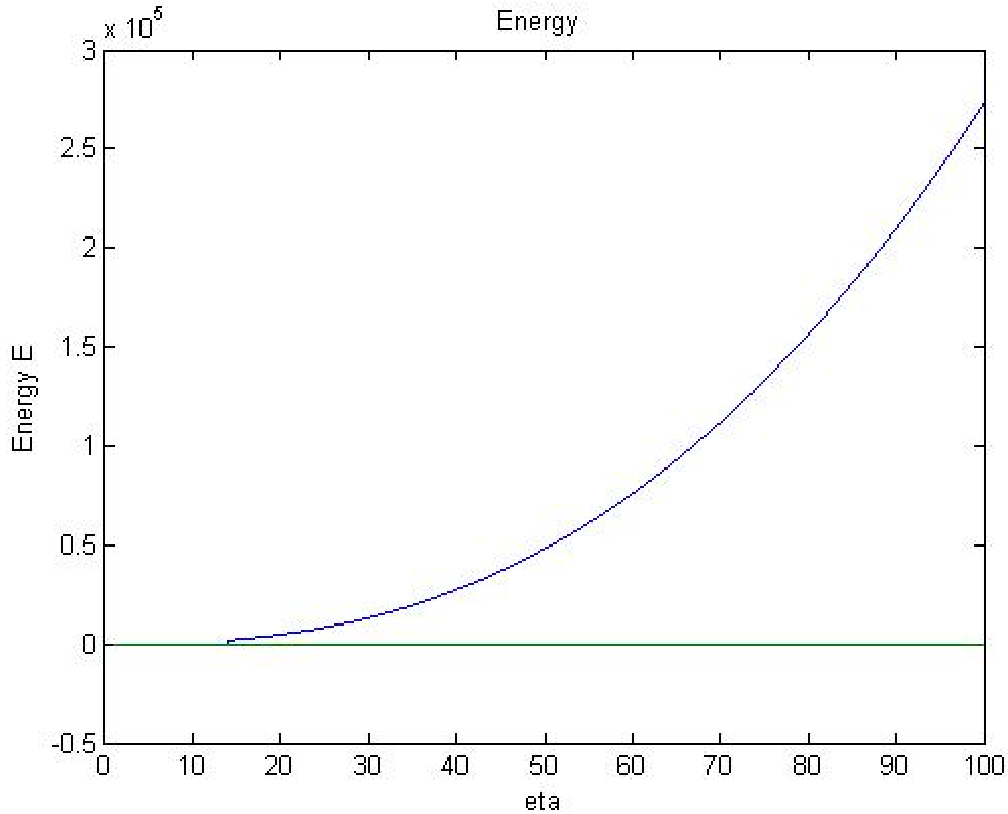

Figure 20: Energy versus eta. Eta changes from 0 to 100.

This plot illustrates the behavior of the energy function as a function of η.

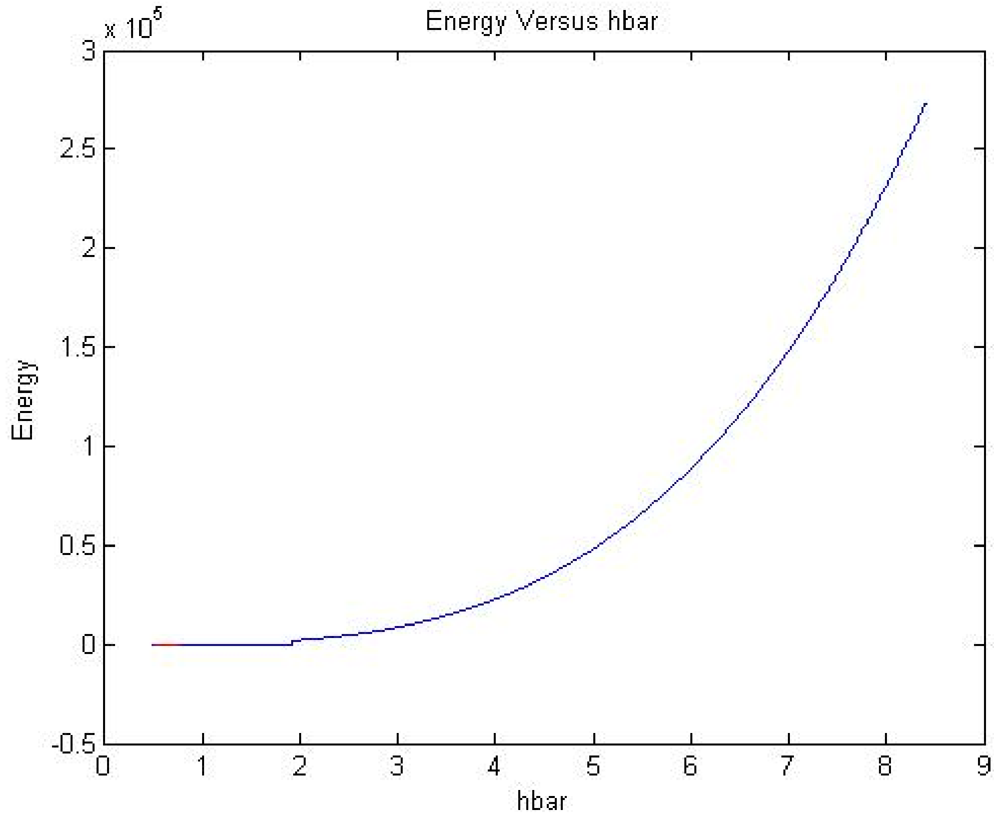

Figure 21: Energy versus hbar.

This plot shows the behavior of the energy in terms of $\bar{h}$.

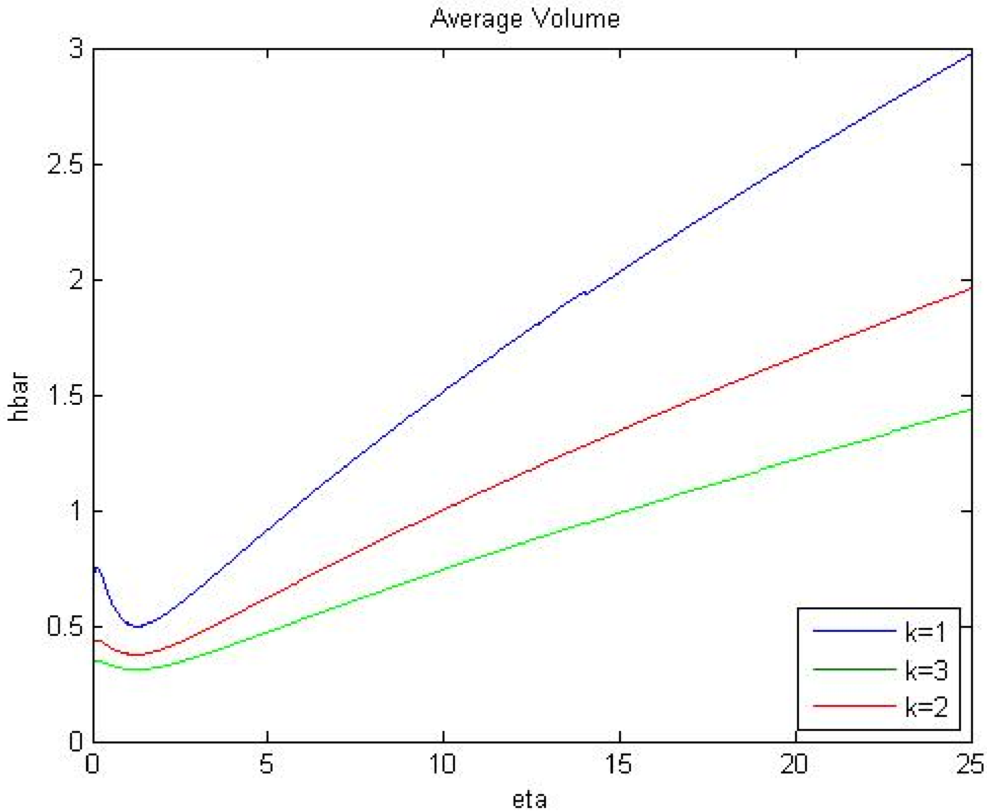

Figure 22: Average volume for k=1,2 and 3.

This plot shows the variation of $\bar{h}$ for $k = 1, 2, 3$ and η in [0, 25]. This graph suggests that we do have bifurcations of solutions for some values of $\bar{h}$.

3.0 RUPTURE SOLUTIONS OF GENERAL ELLIPTIC EQUATION

3.1 INTRODUCTION

In this chapter we will prove that under a light integrability condition the general two dimensional semi linear elliptic equation has a continuous weak radial point rupture solution which is locally monotone increasing near the origin.

$$\Delta u = f(u) \quad \text{in} \quad \Omega \tag{3.1.1}$$

This equation is obtained for steady states solutions of the modeling equation for thin films over a planar region.

$$u_t = -\nabla \cdot (u^m \nabla u) - \nabla \cdot (u^n \nabla \Delta u) . \tag{3.1.2}$$

This equation is only assumed to be valid when $u > 0$ and the set where $u = 0$ is called the rupture set of the thin film. Many rigorous works have been done on the above equation. However, when the space dimension is two, the physically realistic dimension, the dynamics of this equation is still not well understood. For Van der Waals the equation becomes

$$\triangle u = \frac{1}{\alpha} u^{-\alpha} - p \text{ in } \Omega, \tag{3.1.3}$$

where p is an unknown constant to be determined and $\alpha = 3$.

3.2 THE MAIN RESULT

The following is the statement of our main result for sufficient conditions for existence of a rupture solution and its a priori bounds.

Theorem 3.1. *Let $\sigma^* > 0$ and f be a continuous, monotone decreasing positive function on $(0, \sigma^*]$ such that*

$$\lim_{v \to 0^+} f(v) = \infty.$$

Let

$$G(v) = \int_0^v \frac{1}{f(s)} ds. \tag{3.2.1}$$

Assume in addition that

$$\frac{v}{f(v) G(v)} \in L^1 [0, \sigma^*]. \tag{3.2.2}$$

Then there exists $r^* > 0$ and a radial point rupture solution u_0 to (3.1.1) in $B_{r^*}(0)$ such that $u_0 = u_0(r)$ is continuous on $[0, r^*]$,

$$u_0(0) = 0,\ u_0(r) > 0 \text{ for any } r \in (0, r^*],$$

and u is a weak solution to (3.1.1) in $B_{r^*}(0)$. Moreover, u_0 is monotone increasing and

$$G^{-1}\left(\frac{1}{4} r^2\right) \leq u_0(r) \leq \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v) G(v)} ds \text{ for any } r \in [0, r^*].$$

Remark 3.2. Here the technical assumption (3.2.2) is not very strong, for example, if $f(v) = v^{-\alpha}$, for some $\alpha > 0$, we would have

$$\frac{v}{f(v) G(v)} = \frac{v}{v^{-\alpha}\left(\frac{1}{1+\alpha} v^{1+\alpha}\right)} = 1 + \alpha \in L^1 [0, \sigma^*].$$

Such assumption also holds for some singularity of exponential growth, for example, if

$$f(v) = v^{p+1} e^{\frac{1}{v^p}},\ 0 < p < 1,$$

we have

$$\frac{v}{f(v)\,G(v)} = \frac{p}{v^p} \in L^1[0,\sigma^*].$$

Remark 3.3. The assumption that f is monotone decreasing can be replaced by the assumption that f is a product of a monotone decreasing function and a uniformly bounded positive function, one may check how this idea is developed in chapter four in an n dimensional vector space where $n \geq 3$.

3.3 PROOF OF THE MAIN RESULT

For any $\sigma \in (0,\sigma^*)$, we use u_σ to denote the unique solution to the initial value problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r = f(u), \\ u(0) = \sigma, u'(0) = 0. \end{cases} \tag{3.3.1}$$

Lemma 3.4. *There exists $r_\sigma > 0$ such that u_σ is defined on $[0,r_\sigma]$ with $u_\sigma(r_\sigma) = \sigma^*$. Moreover, $u'_\sigma(r) > 0$ on $(0,r_\sigma]$ and*

$$G^{-1}\left(\frac{1}{4}r^2\right) \leq u_\sigma(r) \leq \sigma + \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)\,G(v)} ds \text{ on } [0,r_\sigma]. \tag{3.3.2}$$

Proof. For simplicity, we suppress the σ subscript in this proof. We write

$$u_{rr} + \frac{1}{r}u_r = f(u)$$

in the form of

$$\left(ru_r\right)_r = rf\left(u\right) \geq 0,$$

so we have

$$ru_r = \int_0^r sf\left(u\left(s\right)\right)ds \geq 0.$$

In particular, u is monotone increasing and u can be extended whenever $f\left(u\right)$ is defined and bounded. Hence, there exists $r_\sigma > 0$ such that u_σ is defined on $\left[0, r_\sigma\right]$ with $u_\sigma\left(r_\sigma\right) = \sigma^*$. Since u is monotone increasing and f is monotone decreasing, we have

$$ru_r = \int_0^r sf\left(u\left(s\right)\right)ds \geq f\left(u\left(r\right)\right)\int_0^r sds = \frac{1}{2}r^2 f\left(u\left(r\right)\right),$$

hence,

$$\frac{u_r}{f\left(u\right)} \geq \frac{1}{2}r.$$

Integrating again, we have

$$G\left(u\left(r\right)\right) \geq G\left(\sigma\right) + \frac{1}{4}r^2 \geq \frac{1}{4}r^2.$$

Since G is strictly monotone increasing, we have

$$u\left(r\right) \geq G^{-1}\left(\frac{1}{4}r^2\right).$$

On the other hand,

$$ru_r = \int_0^r sf\left(u\left(s\right)\right)ds \leq \int_0^r f\left(G^{-1}\left(\frac{1}{4}s^2\right)\right)sds.$$

Let $v = G^{-1}\left(\frac{1}{4}s^2\right)$, we have $G\left(v\right) = \frac{1}{4}s^2$, and

$$\frac{1}{f\left(v\right)}dv = \frac{1}{2}sds.$$

Hence,

$$\int_0^r f\left(G^{-1}\left(\frac{1}{4}s^2\right)\right)sds = \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} 2dv = 2G^{-1}\left(\frac{1}{4}r^2\right).$$

Hence,

$$u_r \leq \frac{2}{r} G^{-1} \left(\frac{1}{4} r^2 \right)$$

which yields

$$\begin{aligned} u(r) &\leq \sigma + \int_0^r \frac{2}{s} G^{-1} \left(\frac{1}{4} s^2 \right) ds \\ &= \sigma + \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v) G(v)} dv. \end{aligned}$$

□

The bounds on u_σ can be explored to find a uniform lower bound for all r_σ, more accurately this is stated in the following corollary.

Corollary 3.5. *There exists* $r^* > 0$ *such that for any* $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$,

$$r_\sigma \geq r^*.$$

We can take

$$r^* = 2\sqrt{G\left(H^{-1}\left(\frac{\sigma^*}{2}\right)\right)}$$

where

$$H(u) = \int_0^u \frac{v}{f(v) G(v)} dv.$$

Proof. For any $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$,

$$\sigma^* = u_\sigma(r_\sigma) \leq \sigma + \int_0^{G^{-1}\left(\frac{1}{4}r_\sigma^2\right)} \frac{v}{f(v)\,G(v)}dv$$
$$\leq \frac{\sigma^*}{2} + \int_0^{G^{-1}\left(\frac{1}{4}r_\sigma^2\right)} \frac{v}{f(v)\,G(v)}dv.$$

Hence,

$$\int_0^{G^{-1}\left(\frac{1}{4}r_\sigma^2\right)} \frac{v}{f(v)\,G(v)}dv \geq \frac{\sigma^*}{2}.$$

Since $\frac{v}{f(v)G(v)}$ is integrable, the function

$$H(u) = \int_0^u \frac{v}{f(v)\,G(v)}dv$$

is strictly monotone increasing, so

$$H\left(G^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right) \geq \frac{\sigma^*}{2}$$

implies

$$r_\sigma \geq 2\sqrt{G\left(H^{-1}\left(\frac{\sigma^*}{2}\right)\right)}.$$

□

The point rupture solutions can be constructed as a uniform limit of sequences of smooth solutions of u_{σ_k} as $\sigma_k \to 0$.

Proposition 3.6. *There exists a sequence* $\{\sigma_k\}_{k=1}^\infty \subset \left(0, \frac{\sigma^*}{2}\right]$ *satisfying* $\lim_{k\to\infty} \sigma_k = 0$, *such that* $u_{\sigma_k} \to u_0$ *uniformly in* $\overline{B_{r^*}(0)}$ *as* $k \to \infty$, *for some function* $u_0 \in C^0\left(\overline{B_{r^*}(0)}\right) \cap C^2\left(\overline{B_{r^*}(0)} \setminus \{0\}\right)$. *Moreover,* u_0 *is a classical solution to* (3.1.1) *in* $B_{r^*}(0) \setminus \{0\}$ *and*

$$G^{-1}\left(\frac{1}{4}r^2\right) \leq u_0(r) \leq \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)\,G(v)}ds \;\; on \;\; [0, r^*].$$

Proof. For any $\varepsilon > 0$, u_σ, $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$ is a family of uniformly bounded classical solutions to

$$\Delta u = f(u) \text{ in } \overline{B_{r^*}(0)} \backslash B_\varepsilon(0),$$

hence by a diagonal argument, there exists a sequence $\{\sigma_k\}_{k=1}^{\infty} \subset \left(0, \frac{\sigma^*}{2}\right]$ satisfying $\lim_{k\to\infty} \sigma_k = 0$, such that $u_{\sigma_k} \to u_0$ locally uniformly in $\overline{B_{r^*}(0)} \backslash \{0\}$ as $k \to \infty$. Now (3.3.2) implies

$$G^{-1}\left(\frac{1}{4}r^2\right) \leq u_0(r) \leq \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)\,G(v)} ds \text{ on } [0, r^*].$$

Since

$$\lim_{r\to 0} \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)\,G(v)} ds = 0,$$

it is not difficulty to see, from the bounds of u_σ and u_0, that $u_{\sigma_k} \to u_0$ uniformly in $\overline{B_{r^*}(0)}$ as $k \to \infty$. □

Remark 3.7. The above limit should be independent of the choice of $\{\sigma_k\}_{k=1}^{\infty}$. Actually, we expect that $u_\sigma \to u_0$ uniformly on $[0, r^*]$ as $\sigma \to 0$. Unfortunately, we are unable to provide a proof here and the question remains open.

The following lemma is crucial in the remaining part in order to show that the function u_0 is actually a Sobolev solution to the original problem.

Lemma 3.8.

$$\lim_{r\to 0^+} r u_0'(r) = 0. \tag{3.3.3}$$

Proof. For any $r \in (0, r^*)$, we have

$$(r u_0'(r))' = r f(u_0) > 0.$$

Hence, $ru_0'(r)$ is monotone increasing in $(0, r^*)$. Since $ru_0'(r) \geq 0$ in $(0, r^*)$,

$$\beta = \lim_{r \to 0^+} ru_0'(r) \geq 0$$

is well defined. If $\beta > 0$, we have for r sufficiently small, say $r \in (0, \tilde{r}]$,

$$ru_0'(r) \geq \frac{\beta}{2}$$

hence, for any $r \in (0, \tilde{r}]$,

$$u_0(r) = u_0(\tilde{r}) - \int_r^{\tilde{r}} u_0'(r)\, dr \leq u_0(\tilde{r}) - \int_r^{\tilde{r}} \frac{\beta}{2r} dr.$$

which contradicts to the fact that u_0 is continuous at 0 if we let $r \to 0^+$. Hence $\beta = 0$ and (3.3.3) holds. □

Proposition 3.9. *$f(u_0) \in L^1(B_{r^*}(0))$ and u_0 is a weak solution to (3.1.1) in $B_{r^*}(0)$.*

Proof. For any test function $\varphi \in C_c^\infty(B_{r^*}(0))$, we have

$$\begin{aligned}
&\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon \to 0^+} \int_{B_{r^*}(0) \setminus \overline{B_\varepsilon(0)}} u_0 \Delta\varphi dx \\
&= \lim_{\varepsilon \to 0^+} \left(\int_{B_{r^*}(0) \setminus \overline{B_\varepsilon(0)}} \Delta u_0 \varphi dx - \int_{\partial B_\varepsilon(0)} \left(u_0 \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial u_0}{\partial n} \right) ds_x \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\int_{B_{r^*}(0) \setminus \overline{B_\varepsilon(0)}} f(u_0)\, \varphi dx - \int_{\partial B_\varepsilon(0)} u_0 \frac{\partial\varphi}{\partial n} ds_x + \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial u_0}{\partial n} ds_x \right).
\end{aligned}$$

Now for any $\varepsilon \in (0, r^*)$, since $u_0(\varepsilon) \leq u_0(r^*) \leq \delta^*$, we have

$$\begin{aligned}
\left| \int_{\partial B_\varepsilon(0)} u_0 \frac{\partial\varphi}{\partial n} ds_x \right| &\leq u_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} |\partial B_\varepsilon(0)| \\
&\leq 2\pi\varepsilon u_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$. On the other hand, (3.3.3) implies that

$$\left|\int_{\partial B_\varepsilon(0)} \varphi \frac{\partial u_0}{\partial n} ds_x\right| \leq 2\pi\varepsilon u_0'(\varepsilon) \|\varphi\|_{L^\infty(B_{r^*}(0))} \to 0$$

as $\varepsilon \to 0^+$. Hence, we have for any $\varphi \in C_c^\infty(B_{r^*}(0))$,

$$\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} f(u_0)\varphi dx.$$

Choosing φ such that $\varphi \equiv 1$ near the origin, the above limit implies that $f(u_0)$ is integrable near the origin. Since $f(u_0)$ is a positive continuous function in $B_{r^*}(0)\setminus\{0\}$, we conclude $f(u_0) \in L^1(B_{r^*}(0))$. So we have for any test function $\varphi \in C_c^\infty(B_{r^*}(0))$

$$\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} f(u_0)\varphi dx = \int_{B_{r^*}(0)} f(u_0)\varphi dx,$$

i.e., u_0 is a weak solution to (3.1.1) in $B_{r^*}(0)$. □

3.4 SEVERAL EXAMPLES

In this section, we discuss several examples in order to get a better understanding of the technical assumption.

Example 3.10.

$$f(u) = u^{-\alpha}$$

where $\alpha > 0$ is a constant. Here we have

$$G(v) = \int_0^v \frac{ds}{f(s)} = \int_0^v \frac{ds}{s^{-\alpha}} = \int_0^v s^\alpha ds = \frac{v^{\alpha+1}}{\alpha+1}$$

therefore

$$\frac{v}{G(v)f(v)} = \frac{v}{\frac{v^{\alpha+1}}{\alpha+1}v^{-\alpha}} = \alpha+1 \in L^1[0,\sigma^*].$$

Example 3.11. For some $0 < p < 1$,

$$f(v) = v^{p+1} e^{\frac{1}{v^p}}$$

We have

$$G(v) = \int_0^v \frac{ds}{f(s)} = \int_0^v \frac{ds}{s^{p+1} e^{\frac{1}{s^p}}} = p e^{-\frac{1}{v^p}}$$

therfore,

$$\frac{v}{G(v)f(v)} = \frac{1}{pv^p} \in L^1[0,1] \quad \text{since} \quad \int_0^1 \frac{dv}{pv^p} = \frac{1}{p(1-p)}.$$

Example 3.12. This example shows that our result is not optimal. Let

$$f(u) = 2u^3 e^{\frac{2}{u}}$$

which is monotone decreasing near the origin and

$$\lim_{u \to 0^+} f(u) = \infty.$$

we have

$$\frac{v}{f(v)G(v)} \notin L^1(0, \sigma^*]$$

for any $\sigma^* > 0$. However, let

$$u = \frac{-1}{\ln r},$$

we have

$$u_r = \frac{1}{r \ln^2 r},\; u_{rr} = -\frac{1}{r^2 \ln^2 r} - 2\frac{1}{r^2 \ln^3 r},$$

and so

$$u_{rr} + \frac{1}{r} u_r = -2\frac{1}{r^2 \ln^3 r} = 2u^3 e^{\frac{2}{u}} = f(u)$$

Hence $u = \frac{-1}{\ln r}$, is a rupture solution even if the technical assumption is not satisfied.

3.5 DYNAMICAL SYSTEM FORMULATION

In this section, we give a formulation of our problem in the dynamical system setting which might be helpful in proving the uniqueness of rupture solutions. We are interested in the rupture solutions to

$$u_{rr} + \frac{1}{r}u_r = f(u). \tag{3.5.1}$$

Let $r = e^{-t}$, we have

$$dr = -rdt$$

and

$$u_r = -\frac{1}{r}u_t, u_{rr} = \frac{1}{r^2}u_t + \frac{1}{r^2}u_{tt},$$

hence

$$u_{tt} = r^2 f(u).$$

Let

$$R = r^2, v = -u_t,$$

we obtain an autonomous system

$$\begin{cases} u_t = -v, \\ \\ v_t = -Rf(u), \\ \\ R_t = -2R. \end{cases} \tag{3.5.2}$$

It is not difficult to see that a radial rupture solution to (3.5.1) is equivalent to a trajectory of (3.5.2) in the region $\{(u, v, R) : u, v, R > 0\}$ which approaches the origin as $t \to \infty$. Hence given f, we are interested in the existence and uniqueness of such trajectory.

4.0 POINT RUPTURE SOLUTIONS OF A CLASS OF QUASI-LINEAR ELLIPTIC EQUATIONS

4.1 INTRODUCTION

Let Ω be a region in $\mathbb{R}^2$, and f be a smooth function defined on $(0, \infty)$ satisfying

$$\lim_{s \to 0^+} f(s) = \infty, \tag{4.1.1}$$

we consider the quasi-linear elliptic equations of the form

$$\operatorname{div}(a(u)\nabla u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u) \tag{4.1.2}$$

where the terms depending upon a are formally associated with the functional

$$\int_\Omega a(u)|\nabla u|^2$$

which can be viewed as a minimizing problem in presence of a Riemannian metric tensor depending upon the unknown u itself.

Motivated by the studies of thin film equations, a solution to (4.1.2) is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(x) > 0$ for any $x \in \Omega \backslash \{p\}$. Our main result is the existence of a radial rupture solution:

Theorem 4.1. *Assume that for some* $\sigma^* > 0$, $a \in C^1[0,\sigma^*]$, $f \in C^1(0,\sigma^*]$ *are positive functions such that for some positive constants* $m < M$,

$$m \leq a(u) \leq M$$

holds for any $u \in [0,\sigma^*]$ *and* f *is monotone decreasing function on* $(0,\sigma^*]$ *satisfying*

$$\frac{u}{G(u)\,f(u)} \in L^1[0,\sigma^*] \tag{4.1.3}$$

where

$$G(u) = \int_0^u \frac{1}{f(s)}ds.$$

Then there exists $r^* > 0$ *and a radial point rupture solution* u_0 *to* (4.1.2) *in* $B_{r^*}(0)$ *such that* $u_0 = u_0(r)$ *is continuous on* $[0,r^*]$,

$$u_0(0) = 0,\ u_0(r) > 0 \text{ for any } r \in (0,r^*].$$

Moreover, u_0 *is a weak solution to* (4.1.2) *in the sense that for any* $\varphi \in C_0^\infty(B_1(r^*))$,

$$\int_{B_1(r^*)} A(u_0)\,\Delta\varphi = \int_{B_1(r^*)} \frac{f(u_0)}{\sqrt{a(u_0)}}\varphi$$

where

$$A(u) = \int_0^u \sqrt{a(s)}ds.$$

When $a \equiv 1$, (4.1.2) is reduced to the simpler form

$$\Delta u = f(u)$$

and its rupture solution has been investigated in [13], [15] when $f(u) = u^{-\alpha} - 1$, $\alpha > 1$ which has application to the van der Waals force driven thin films, in [14] with f satisfying the growth condition (4.1.3) and in [11] when the space dimension ≥ 3. (4.1.2) has also been studied by F. Gladiali and M. Squassina [10] where they are interested in the so called explosive solutions.

4.2 PROOF OF THE MAIN RESULT

We consider the quasi-linear equations of the form

$$\operatorname{div}\left(a\left(u\right)\nabla u\right)=\frac{a'\left(u\right)}{2}\left|\nabla u\right|^{2}+f\left(u\right) \tag{4.2.1}$$

in a region $\Omega\subset\mathbb{R}^{N}$ where for some $\delta^{*}>0$, $a\in C^{1}\left[0,\delta^{*}\right]$ and $f\in C^{1}\left(0,\delta^{*}\right]$ are positive functions.

Following [10], let g be the unique solution to the Cauchy problem

$$g'=\frac{1}{\sqrt{a\left(g\right)}},\ g\left(0\right)=0,$$

and let v be a solution to

$$\Delta v=h\left(v\right) \tag{4.2.2}$$

where

$$h\left(v\right)=\frac{f\left(g\left(v\right)\right)}{\sqrt{a\left(g\left(v\right)\right)}}.$$

Define

$$u=g\left(v\right).$$

We have

$$\nabla u=g'\left(v\right)\nabla v=\frac{1}{\sqrt{a\left(g\right)}}\nabla v,$$

hence

$$\nabla v=\sqrt{a\left(u\right)}\nabla u,$$

which leads to

$$\Delta v=\sqrt{a\left(u\right)}\Delta u+\frac{1}{2}\frac{1}{\sqrt{a\left(u\right)}}a'\left(u\right)\left|\nabla u\right|^{2}.$$

Hence (4.2.2) implies

$$\sqrt{a\left(u\right)}\Delta u+\frac{1}{2}\frac{1}{\sqrt{a\left(u\right)}}a'\left(u\right)\left|\nabla u\right|^{2}=\frac{f\left(u\right)}{\sqrt{a\left(u\right)}}$$

which is equivalent to (4.2.1). Hence, (4.2.1) admits a point rupture solution if and only if (4.2.2) has a point rupture solution.

Noticing that $h(v) = \frac{f(g(v))}{\sqrt{a(g(v))}}$ is not necessary monotone decreasing in v, we can't apply the result in [14] directly. However, the boundedness of a and the monotone properties of f and g implies that

$$\frac{1}{\sqrt{M}} f(g(v)) \leq h(v) \leq \frac{1}{\sqrt{m}} f(g(v)),$$

i.e., h is bounded by two monotone decreasing functions.

We need to following theorem:

Theorem 4.2. *Let $\sigma^* > 0$ and $h_1,\ h_2 \in C^1(0, \sigma^*]$ be monotone decreasing functions such that*

$$0 < h_1 \leq h_2 \ \textit{on} \ (0, \sigma^*]$$

and

$$\lim_{v \to 0^+} h_1(v) = \lim_{v \to 0^+} h_2(v) = \infty.$$

Let $h \in C^1(0, \sigma^*]$ satisfy

$$h_1 \leq h \leq h_2 \ \textit{on} \ (0, \sigma^*].$$

Let

$$G_1(v) = \int_0^v \frac{1}{h_1(s)} ds. \tag{4.2.3}$$

Assume in addition that

$$\frac{h_2}{h_1} \in L^1[0, \sigma^*] \ \textit{and} \ \frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u) h_1(u)} \in L^1[0, \sigma^*]. \tag{4.2.4}$$

Then there exists $r^* > 0$ and a radial point rupture solution u_0 to

$$\Delta u = h(u) \tag{4.2.5}$$

in $B_{r^*}(0)$ such that $u_0 = u_0(r)$ is continuous on $[0, r^*]$,

$$u_0(0) = 0,\ u_0(r) > 0 \ \textit{for any} \ r \in (0, r^*],$$

and u is a weak solution to (4.2.5) in $B_{r^*}(0)$. Moreover, u_0 is monotone increasing and

$$G_1^{-1}\left(\frac{1}{4}r^2\right) \leq u_0(r) \leq \int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u) h_1(u)} du \ \textit{for any} \ r \in [0, r^*].$$

We leave its proof to the next section.

Let

$$h_1(v) = \frac{1}{\sqrt{M}} f(g(v)) \text{ and } h_2(v) = \frac{1}{\sqrt{m}} f(g(v)).$$

We have

$$h_1(v) \le h(v) \le h_2(v)$$

on $(0, \sigma^*]$. Moreover, $h_1, h_2 \in C^1(0, \sigma^*]$ are monotone decreasing functions such that

$$0 < h_1 \le h_2 \text{ on } (0, \sigma^*]$$

and

$$\lim_{v \to 0^+} h_1(v) = \lim_{v \to 0^+} h_2(v) = \infty.$$

Next we check the growth condition. We have

$$\frac{h_2}{h_1} = \frac{\sqrt{M}}{\sqrt{m}} \in L^1[0, \sigma^*].$$

Now since

$$g' = \frac{1}{\sqrt{a(g)}},$$

we have

$$\begin{aligned} G_1(v) &= \int_0^v \frac{1}{h_1(s)} ds. = \sqrt{M} \int_0^v \frac{1}{f(g(s))} ds \\ &= \sqrt{M} \int_0^v \frac{\sqrt{a(g)}}{f(g(s))} g'(s) ds \\ &\ge \sqrt{mM} \int_0^v \frac{1}{f(g(s))} g'(s) ds \\ &= \sqrt{mM} \int_0^{g(v)} \frac{1}{f(v)} dv \\ &= \sqrt{mM} G(g(v)) \end{aligned}$$

and

$$\begin{aligned} \frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u) h_1(u)} &= \frac{\sqrt{M}}{\sqrt{m}} \frac{u}{G_1(u) \frac{1}{\sqrt{M}} f(g(u))} \\ &\le \frac{\sqrt{M}}{m} \frac{u}{G(g(u)) f(g(u))} \\ &\le \frac{M}{m} \frac{g(u)}{G(g(u)) f(g(u))} \end{aligned}$$

where we used

$$g(u) \geq \frac{1}{\sqrt{M}} u$$

which follows from $g'(u) \geq \frac{1}{\sqrt{M}}$ and $g(0) = 0$. Hence, for any $\sigma > 0$ sufficiently small,

$$\begin{aligned}
&\int_0^\sigma \frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u) h_1(u)} \\
&\leq \int_0^\sigma \frac{M}{m} \frac{g(u)}{G(g(u)) f(g(u))} du \\
&= \int_0^\sigma \frac{M}{m} \frac{g(u)}{G(g(u)) f(g(u))} \sqrt{a(g)} g'(u) \, du \\
&\leq \frac{M\sqrt{M}}{m} \int_0^{g(\sigma)} \frac{v}{G(v) f(v)} dv.
\end{aligned}$$

So the growth condition in (4.1.3) implies that for some $\tilde{\sigma} > 0$,

$$\frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u) h_1(u)} \in L^1[0, \tilde{\sigma}].$$

Applying Theorem 4.2, we have the existence of rupture solution v_0 to (4.2.2). Then

$$u_0 = g(v_0)$$

is a rupture solution to (4.1.2). Moreover, u_0 is a weak solution to (4.1.2) follows from the fact that v_0 is a weak solution to (4.2.2).

4.3 POINT RUPTURE SOLUTION WITH NON-MONOTONIC H

We prove Theorem 4.2 in this section. For any $\sigma \in (0, \sigma^*)$, we use u_σ to denote the unique solution to the initial value problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r = h(u), \\ u(0) = \sigma, u'(0) = 0. \end{cases} \tag{4.3.1}$$

Lemma 4.3. *There exists $r_\sigma > 0$ such that u_σ is defined on $[0, r_\sigma]$ with $u_\sigma(r_\sigma) = \sigma^*$. Moreover, $u'_\sigma(r) > 0$ on $(0, r_\sigma]$ and*

$$G^{-1}\left(\frac{1}{4}r^2\right) \le u_\sigma(r) \le \sigma + H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right) \text{ on } [0, r_\sigma]. \tag{4.3.2}$$

where

$$H(w) = \int_0^w \frac{\int_0^u \frac{h_2(v)}{h_1(v)}dv}{G_1(u)\,h_1(u)}du.$$

Proof. For simplicity, we suppress the σ subscript in this proof. We write

$$u_{rr} + \frac{1}{r}u_r = h(u)$$

in the form of

$$(ru_r)_r = rh(u) \ge 0,$$

so we have

$$ru_r = \int_0^r sh(u(s))\,ds \ge 0.$$

In particular, u is monotone increasing and u can be extended whenever $f(u)$ is defined and bounded. Hence, there exists $r_\sigma > 0$ such that u_σ is defined on $[0, r_\sigma]$ with $u_\sigma(r_\sigma) = \sigma^*$. Since u is monotone increasing and h_1 is monotone decreasing, we have

$$\begin{aligned} ru_r &= \int_0^r sh(u(s))\,ds \ge \int_0^r sh_1(u(s))\,ds \\ &\ge h_1(u(r)) \int_0^r sds = \frac{1}{2}r^2 h_1(u(r)), \end{aligned}$$

hence,

$$\frac{u_r}{h_1(u)} \ge \frac{1}{2}r.$$

Integrating again, we have

$$G_1\left(u\left(r\right)\right) \geq G_1\left(\sigma\right) + \frac{1}{4}r^2 \geq \frac{1}{4}r^2.$$

where

$$G_1\left(v\right) = \int_0^v \frac{1}{h_1\left(s\right)}ds.$$

Since G_1 is continuous and strictly monotone increasing, G_1^{-1} is well defined and we have

$$u\left(r\right) \geq G_1^{-1}\left(\frac{1}{4}r^2\right).$$

On the other hand, since h_2 is monotone increasing,

$$ru_r = \int_0^r sh\left(u\left(s\right)\right)ds \leq \int_0^r sh_2\left(u\left(s\right)\right)ds \leq \int_0^r h_2\left(G_1^{-1}\left(\frac{1}{4}s^2\right)\right)sds.$$

Let $v = G_1^{-1}\left(\frac{1}{4}s^2\right)$, we have $G_1\left(v\right) = \frac{1}{4}s^2$, and

$$\frac{1}{h_1\left(v\right)}dv = \frac{1}{2}sds.$$

Hence,

$$\int_0^r h_2\left(G_1^{-1}\left(\frac{1}{4}s^2\right)\right)sds = 2\int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{h_2\left(v\right)}{h_1\left(v\right)}dv.$$

Hence,

$$u_r \leq \frac{2}{r}\int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{h_2\left(v\right)}{h_1\left(v\right)}dv$$

which yields

$$\begin{aligned}
u\left(r\right) &\leq \sigma + \int_0^r \frac{2}{s}\left[\int_0^{G_1^{-1}\left(\frac{1}{4}s^2\right)} \frac{h_2\left(v\right)}{h_1\left(v\right)}dv\right]ds \\
&= \sigma + \int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{2}{s}\left[\int_0^w \frac{h_2\left(v\right)}{h_1\left(v\right)}dv\right]\frac{2}{sh_1\left(w\right)}dw \\
&= \sigma + \int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{\int_0^w \frac{h_2(v)}{h_1(v)}dv}{G_1\left(w\right)h_1\left(w\right)}dw \\
&= \sigma + H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right)
\end{aligned}$$

where

$$H\left(w\right) = \int_0^w \frac{\int_0^u \frac{h_2(v)}{h_1(v)}dv}{G_1\left(u\right)h_1\left(u\right)}du$$

and we used substitution

$$w = G_1^{-1}\left(\frac{1}{4}s^2\right).$$

□

The bounds on u_σ implies:

Corollary 4.4. *There exists $r^* > 0$ such that for any $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$,*

$$r_\sigma \geq r^*.$$

We can take

$$r^* = 2\sqrt{G_1\left(H^{-1}\left(\frac{\sigma^*}{2}\right)\right)}.$$

Proof. For any $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$,

$$\begin{aligned} \sigma^* = u_\sigma\left(r_\sigma\right) &\leq \sigma + H\left(G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right) \\ &\leq \frac{\sigma^*}{2} + H\left(G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right). \end{aligned}$$

Hence,

$$H\left(G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right) \geq \frac{\sigma^*}{2}.$$

Since the function H is strictly monotone increasing, we have

$$r_\sigma \geq 2\sqrt{G_1\left(H^{-1}\left(\frac{\sigma^*}{2}\right)\right)}.$$

□

The point rupture solution can be constructed as the limit of u_σ as $\sigma \to 0$.

Proposition 4.5. *There exists a sequence* $\{\sigma_k\}_{k=1}^{\infty} \subset \left(0, \frac{\sigma^*}{2}\right]$ *satisfying*

$$\lim_{k\to\infty} \sigma_k = 0,$$

such that $u_{\sigma_k} \to u_0$ *uniformly in* $\overline{B_{r^*}(0)}$ *as* $k \to \infty$, *for some function*

$$u_0 \in C^0\left(\overline{B_{r^*}(0)}\right) \cap C^2\left(\overline{B_{r^*}(0)} \setminus \{0\}\right).$$

Moreover, u_0 *is a classical solution to* (4.2.5) *in* $B_{r^*}(0) \setminus \{0\}$ *and*

$$G_1^{-1}\left(\frac{1}{4}r^2\right) \le u_0(r) \le H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right) \text{ on } [0, r^*].$$

Proof. For any $\varepsilon > 0$, u_σ, $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$ is a family of uniformly bounded classical solutions to

$$\Delta u = h(u) \text{ in } \overline{B_{r^*}(0)} \setminus B_\varepsilon(0),$$

hence by a diagonal argument, there exists a sequence $\{\sigma_k\}_{k=1}^{\infty} \subset \left(0, \frac{\sigma^*}{2}\right]$ satisfying $\lim_{k\to\infty} \sigma_k = 0$, such that $u_{\sigma_k} \to u_0$ locally uniformly in $\overline{B_{r^*}(0)} \setminus \{0\}$ as $k \to \infty$. Now (4.3.2) implies

$$G_1^{-1}\left(\frac{1}{4}r^2\right) \le u_0(r) \le H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right) \text{ on } [0, r^*].$$

Since

$$\lim_{r\to 0} H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right) = 0,$$

it is not difficulty to see, from the bounds of u_σ and u_0, that $u_{\sigma_k} \to u_0$ uniformly in $\overline{B_{r^*}(0)}$ as $k \to \infty$. □

Remark 4.6. The above limit should be independent of the choice of $\{\sigma_k\}_{k=1}^{\infty}$. Actually, we expect that $u_\sigma \to u_0$ uniformly on $[0, r^*]$ as $\sigma \to 0$. Unfortunately, we are unable to provide a proof here.

In order to show that u_0 is a weak solution. We need the following lemma:

Lemma 4.7.

$$\lim_{r\to 0^+} r u_0'(r) = 0. \tag{4.3.3}$$

Proof. For any $r \in (0, r^*)$, we have

$$(ru_0'(r))' = rf(u_0) > 0.$$

Hence, $ru_0'(r)$ is monotone increasing in $(0, r^*)$. Since $ru_0'(r) \geq 0$ in $(0, r^*)$,

$$\beta = \lim_{r \to 0^+} ru_0'(r) \geq 0$$

is well defined. If $\beta > 0$, we have for r sufficiently small, say $r \in (0, \tilde{r}]$,

$$ru_0'(r) \geq \frac{\beta}{2}$$

hence, for any $r \in (0, \tilde{r}]$,

$$u_0(r) = u_0(\tilde{r}) - \int_r^{\tilde{r}} u_0'(r)\, dr \leq u_0(\tilde{r}) - \int_r^{\tilde{r}} \frac{\beta}{2r} dr.$$

which contradicts to the fact that u_0 is continuous at 0 if we let $r \to 0^+$. Hence $\beta = 0$ and (4.3.3) holds. □

Proposition 4.8. *$h(u_0) \in L^1(B_{r^*}(0))$ and u_0 is a weak solution to (4.2.5) in $B_{r^*}(0)$.*

Proof. For any test function $\varphi \in C_c^\infty(B_{r^*}(0))$, we have

$$\begin{aligned}
&\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon \to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} u_0 \Delta\varphi dx \\
&= \lim_{\varepsilon \to 0^+} \left(\int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} \Delta u_0 \varphi dx - \int_{\partial B_\varepsilon(0)} \left(u_0 \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial u_0}{\partial n} \right) ds_x \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} h(u_0)\varphi dx - \int_{\partial B_\varepsilon(0)} u_0 \frac{\partial\varphi}{\partial n} ds_x + \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial u_0}{\partial n} ds_x \right).
\end{aligned}$$

Now for any $\varepsilon \in (0, r^*)$, since $u_0(\varepsilon) \leq u_0(r^*) \leq \delta^*$, we have

$$\begin{aligned}
\left| \int_{\partial B_\varepsilon(0)} u_0 \frac{\partial\varphi}{\partial n} ds_x \right| &\leq u_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} |\partial B_\varepsilon(0)| \\
&\leq 2\pi\varepsilon u_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$. On the other hand, (4.3.3) implies that

$$\left| \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial u_0}{\partial n} ds_x \right| \leq 2\pi\varepsilon u_0'(\varepsilon) \|\varphi\|_{L^\infty(B_{r^*}(0))} \to 0$$

as $\varepsilon \to 0^+$. Hence, we have for any $\varphi \in C_c^\infty(B_{r^*}(0))$,

$$\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} f(u_0)\varphi dx.$$

Choosing φ such that $\varphi \equiv 1$ near the origin, the above limit implies that $f(u_0)$ is integrable near the origin. Since $f(u_0)$ is a positive continuous function in $B_{r^*}(0)\setminus\{0\}$, we conclude $f(u_0) \in L^1(B_{r^*}(0))$. So we have for any test function $\varphi \in C_c^\infty(B_{r^*}(0))$

$$\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} h(u_0)\varphi dx = \int_{B_{r^*}(0)} h(u_0)\varphi dx,$$

i.e., u_0 is a weak solution to (4.2.5) in $B_{r^*}(0)$. □

Theorem 4.2 is a combination of Proposition 4.5 and Proposition 4.8.

Finally, we discuss several examples in this section to get a better understanding of the technical assumption on the growth rate of h.

Example 4.9.

$$h(u) = b(u)u^{-\alpha}$$

where $\alpha > 0$ is a constant and $b(u)$ satisfies

$$B_1 \le b(u) \le B_2$$

for some constants $0 < B_1 < B_2$. If we take

$$h_1 = B_1 u^{-\alpha} \text{ and } h_2 = B_2 u^{-\alpha},$$

we have

$$\frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u)h_1(u)} = \frac{(1+\alpha)B_2}{B_1} \in L^1[0,1].$$

Example 4.10. For some $0 < p < 1$,

$$h(u) = b(u)\, v^{p+1} e^{\frac{1}{v^p}}$$

and $b(u)$ satisfies

$$B_1 \leq b(u) \leq B_2$$

for some constants $0 < B_1 < B_2$. If we take

$$h_1 = B_1 v^{p+1} e^{\frac{1}{v^p}} \text{ and } h_2 = B_2 v^{p+1} e^{\frac{1}{v^p}},$$

we have

$$\frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u)\, h_1(u)} = \frac{B_2 p}{B_1 v^p} \in L^1[0,1].$$

Example 4.11.

$$f(u) = \frac{1}{2}\left[\left(1 + \sin\frac{1}{u}\right) u^{-\alpha} + \left(1 - \sin\frac{1}{u}\right) u^{-\beta}\right]$$

where

$$0 < \alpha < \beta < \alpha + 1.$$

We take

$$f_1(u) = u^{-\alpha}, f_2(u) = u^{-\beta},$$

then we have for any $u \in (0,1]$,

$$h_1(u) \leq h(u) \leq h_2(u).$$

Hence,

$$\frac{\int_0^u \frac{h_2(v)}{h_1(v)} dv}{G_1(u)\, h_1(u)} = \frac{\int_0^u u^{\alpha-\beta} dv}{\frac{1}{1+\alpha} u} = \frac{1+\alpha}{1+\alpha-\beta} u^{\alpha-\beta} \in L^1[0,1]$$

since $\alpha - \beta > -1$. In this example, h can't be expressed as a product of a bounded function and a monotone function. Interested readers should verify this.

5.0 CONCLUSION

In this thesis, we focused on the dynamics of thin film equation modeling the steady states solutions of van der Waals force driven thin film for a viscous fluid assuming that there is no flux across the boundary and ignoring wetting and non-wetting effect and that the fluid surface is perpendicular to the boundary of the container. The use of radial solutions was the approach to study solutions near the origin, point rupture solutions must be discrete for finite energy solutions and we expect them to be important in the sense that they may serve as model and blow up profile for the solutions near the rupture point.

In chapter two, we performed many numerical experiments to study the most physical case of thin film equation that is the two dimensional type of thin film and for the special equation $\Delta h = \frac{1}{\alpha} \cdot h^{-\alpha} - p$ when $\alpha = 3$. RK4 numerical method was used and most of the properties of the radial solution were clearly revealed in particular, their uniform boundedness and there oscillatory behavior. Smooth radial solutions could be plotted easily for any initial data η, however for larger values then the method starts taking considerable amount of time, therefore we expect to use higher techniques if one needs faster and more accurate numerical approximations to the radial solutions. The rupture solution is computed using RK4, however it starts from zero by the use of an expansion to reach the first step, and then proceeds via RK4 with controllable tolerance. The plots of the energy function versus η and the average volume $\bar{h}$ suggest the existence of bifurcations and their number must increase with k. The behavior of the energy shows that it approaches ∞ as η get larger. Asymptotic analysis revealed properties for the limiting solution as the initial data $\eta \to \infty$ and behavior of the average $\bar{h}$ and the energy were computed in terms of the initial data.

In chapter three, we consider the elliptic equation $\Delta h = f(h)$ as we stated and proved the theorem for a sufficient condition on f for radial solutions as we obtain the bounds for the weak and radial point rupture solution under a light condition on f, also the assumption on f to be monotone decreasing can easily be replaced by assuming that f can be written as the product of two functions f_1 and f_2 where f_1 is uniformly bounded and f_2 is monotone decreasing such hypothesis can be useful in proving a similar result in any space of dimension greater than two. From the proof one sees that the rupture is constructed from an arbitrary uniformly bounded sequence of smooth radial solutions and therefore uniqueness of the point rupture solution may not be guaranteed, thus the question remains open. The rupture set $\Sigma = \{x \in \Omega : u(x) = 0\}$ corresponds to "dry spots" in the thin films, which is of great significance in the coatings industry where non uniformities are very undesirable. In the two dimensional case we proved that the point rupture solution is never zero away from the origin therefore the only point where the solution touches the x-axis is the origin and therefore the measure of the rupture set is indeed zero.

In chapter four, we consider the elliptic equation $\Delta h = f(h)$ in a two dimensional space such that the function f is uniformly bounded near zero by two functions f_1 and f_2 continuous, monotone decreasing and satisfying

$$\lim_{v \to 0^+} f_1(v) = \lim_{v \to 0^+} f_2(v) = \infty$$

and under the conditions,

$$\frac{f_2}{f_1} \in L^1 [0, \sigma^*] \text{ and } \frac{\int_0^u \frac{f_2(v)}{f_1(v)} dv}{G_1(u) f_1(u)} \in L^1 [0, \sigma^*] .$$

The local weak solution u_0 was derived with its appropriate bounds. It is important to note that the solution is actually a global weak solution. Many examples were discussed to show how we can apply these results. In particular, an interesting example of a non monotone function was mentioned. Afterward, we showed how these results can be applied in the plane to a quasi-linear elliptic equation of the form,

$$\text{div}\left(a\left(u\right)\Delta u\right) = \frac{a'\left(u\right)}{2}\left|\nabla u\right|^2 + g\left(u\right)$$

Next, we consider the elliptic equation $\Delta h = f(h)$ in any n dimensional vector space with dimension $n \geq 3$ and using a different approach as in [11] and using change of variable procedure which works in any space dimension, the results of the existence of weak rupture solution still hold with appropriate bounds and can be be proved under the sole condition of f being decreasing. Then we adapted the method to the above quasi-linear equation and obtain similar results.

Furthermore, in the near future I plan and I would like to invest more on thin film equations in general and consider the question of analyzing the dynamic of the fourth order partial differential equation more carefully since it is derived from the very important Navier Stokes Equation under some simplifying assumptions. Radial solutions are very special ones for this partial differential equation and these solutions must be very important in studying and approaching the general solutions and they may serve as a model or a blow up profile. I also would like to consider not only the steady states solutions but solutions depending on time as well. Numerical experiments suggest bifurcations of solutions and the number must vary with the average volume of the container and with k, the order of the critical point. Better tools and numerical solvers could help to give a clearer picture and would help understanding better the behavior of solutions. In the future, I also would like to explore in more detail the question of estimating the Hausdorff dimension of the rupture set.

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