JOYAL'S CONJECTURE IN HOMOTOPY TYPE THEORY

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Joyal's Conjecture asserts, in a mathematically precise way, that Martin-Löf dependent type theory gives rise to locally cartesian closed quasicategory. We prove this conjecture.

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1.0 INTRODUCTION

Let me ask you the following: when boarding an airplane or saving money in a bank account for retirement, would you like to know that the—often fairly sophisticated—software used on this airplane or by your bank is bug-free? Or will you settle for knowing that the programmer who wrote it promised—or maybe even double-promised—that it is bug-free? (Although, frankly, the buggy moral code of the bankers can probably do you more harm then the buggy software they may be using.)

Formal verification of software typically amounts to the verification of the mathematics that underlies it [Hal08]. This mathematics is often very intricate, common examples being elliptic curve cryptography and sheaf-theoretic network sensoring.

Unfortunately, many mathematicians tend to think that formal verification is not worth their time and many hope to delegate this task to computer scientists. This is due to the fact that working from the very first axioms of mathematics towards a theorem in, say, algebraic geometry would require one to redevelop hundreds of years of mathematics, but in more detailed way. I have some empathy for the people who, faced with this problem, say "whatever."

So is the formal verification of ongoing, cutting edge research only an unrealistic dream? While it is not quite yet a reality, it is not far away. To understand its recent advances we need to delve into the area of *foundations of mathematics*.

The development of foundations of mathematics was the project of many leading mathematicians at the beginning of the 20th century. Of the several foundations proposed at that time, set theory (or, more precisely, Zermelo–Fraenkel set theory with the axiom of choice, also known as *ZFC*) became and remains the most prominent foundation used. Its simple language consisting of only one primitive relation $\in$ turned out to serve well as a common basis for all of mathematics.

In hindsight, however, bare set theory may not be the best system in which to formally develop mathematics. Indeed, in set theory, *everything is a set*, and therefore most statements of ZFC are mathematically meaningless. It does, for example, make sense to ask whether the number π is a group with 5 elements. On a more serious note, rebuilding mathematics from just one relation and a small set of axioms is highly impractical! The quest for a practical and widely implementable proof assistant calls for the need to change the system from set theory to *type theory*. Whereas the basic object of study in set theory is a set, type theory is concerned with *types* and their elements, called *terms*. The key change is that in type theory, every term has exactly one type, and hence π, a term of the type of real numbers, cannot even conceivably be considered as a group.

It is important to point out that set theory and type theory are not necessarily competing choices for the foundation of mathematics. In fact, type theory can be seen as a layer of notation on top of set theory, as suggested in the previous paragraph. This is indeed a commonly shared view best exemplified by the proof assistant Mizar which is based on set theory, but implements aspects of type theory on top of this base for practical and notational convenience.

Historically, the first usage of type theory goes back to Russell [Rus08, WR62] who saw it as a way to block certain paradoxes of set theory (such as the existence of a set of all sets). It was, however, not until Church [Chu33, Chu40, Chu41] that type theory started to be seen as a unifying foundation of mathematics and computation.

Martin-Löf dependent type theory (see, for example, [ML72]; more references will be provided in Chapter 2, where we discuss a specific type theory) goes a step further by enriching the language of type theory to allow types to depend on other types. It gives rise to several other similar systems (for example, Calculus of Inductive Constructions) that underlie many now-prominent proof assistants, including Coq and AGDA. Indeed, the suitability for large-scale implementation is now a major motivation for considering systems that build upon Martin-Löf type theory. The current frontiers in the usage of such systems include the verification of the Four-Color Theorem [Gon08] and the very recently formally checked proof of the Feit–Thompson (Odd Order) Theorem, due to a large team of mathematicians and computer scientists working under the leadership of Gonthier [GAA+13].

The way in which Gonthier's team approached this formal proof (and the choice of the theorem itself) deserves a mention in our context. It built a large repository of formally verified mathematics ranging over several areas that other mathematicians can later use and contribute to. And besides formal verification as a goal in itself, creating such repositories is one of the main goals of formal verification.

1.1 HOMOTOPY TYPE THEORY

The program of Univalent Foundations of Mathematics extends these ideas further. Syntactically, it is a further extension of Martin-Löf type theory; however, it drastically differs in its intended semantics. That is to say, it gives a different answer to the question of *what the meaning of the word "type" is.*

The program was proposed by Voevodsky, a Fields medalist working in the areas of algebraic geometry and homotopy theory. Voevodsky often mentions that what drove him to propose this new program was the fact that some of his already published papers were later found to contain mistakes. A particular example of this is his paper with Kapranov [KV91]; indeed, seven(!) years after publication, Simpson [Sim98] found a counterexample to its main theorem. Interestingly enough, in his paper, Simpson says that he cannot identify exactly where the mistake occurred. Reading the proof of, say, Lemma 3.4 of [KV91] one can see why: the statements are vague and the details are omitted. This is by no means criticism of Kapranov and Voevodsky's paper; it is just the style that the majority of research papers are required to obey.

Returning to the question at hand, Voevodsky suggested to interpret types not as bare sets, but as more highly structured entities, *homotopy types*. He emphasized its treatment of equality, which at the time was a comparatively unexploited feature of Martin-Löf type theory. In mathematics based on set theory, equality does not carry any information beyond its truth value: two sets are either equal or not. In type theory, equality can carry more information; this resembles the notion of *homotopy equivalence* familiar from algebraic topology: two spaces can be homotopy equivalent in many different ways, and indeed, the homotopy equivalences between two given spaces form a space in their own right.

Voevodsky's advancement of the program of Univalent Foundations was in parallel with the work of Awodey and Warren [AW09] and their collaborators (see [Awo12] for a survey of these results). In an attempt to understand the mysterious nature of equality in Martin-Löf type theory, they constructed models of equality types in model categories, categories equipped with a (cloven) weak factorization systems and so on; they also used a higher-categorical perspective to study structures arising from the syntax of type theory. We will survey this work in Chapter 2.

Voevodsky started building a library of results in type theory with this interpretation in mind [V+] and indeed his naming conventions (weak equivalences, homotopy fiber products, ...) mirror the vocabulary of homotopy theory. However, without any further axioms, type theory merely allows types to behave like higher-dimensional categories, but it still allows the types-as-sets interpretation. Voevodsky therefore decided to add an axiom to type theory, called the *Univalence Axiom*.

In a nutshell, the Univalence Axiom identifies the type of equality proofs between two given types with the type of equivalences between them. While satisfied by the model of type theory in simplicial sets [KLV12], it is fundamentally not true under the types-as-sets interpretation, as it would imply that any two sets of a given cardinality must be equal. It also bears a resemblance to Rezk's completeness condition for Segal spaces [Rez01] and the descent condition from higher topos theory [Lur09a]. We will discuss both of these later on (in Chapters 3 and 4, respectively).

A further extension, by Higher Inductive Types, was later proposed by Lumsdaine and Shulman. It adds more general schemes for inductive definitions, allowing one to define several homotopy-theoretic objects and constructions directly in type theory. Using these results, a large number of results from homotopy theory were formalized during the special year 2012-13 on Univalent Foundations at the Institute for Advanced Study in Princeton. These included: Freudenthal's Suspension Theorem, Blakers–Massey Theorem, computations of $\pi_k(S^n)$ for $k \leq n$, basic results on covering spaces, and many more. It is worth mentioning that some of these proofs are new and suggest techniques that may be of interest not only for people interested in formalization, but for active researchers in the area of homotopy theory. It promises a new approach to homotopy theory that can be called *synthetic homotopy theory*. An excellent survey of these results is given in [Uni13]; a survey of other results obtained during the special year at the IAS can be found in [APW13].

1.2 HIGHER CATEGORY THEORY AND JOYAL'S CONJECTURE

The connection between type theory and higher category theory was mentioned, although not emphasized, in the previous section. As the present thesis is concerned with some aspects of this connection, let us briefly review the main ideas of higher category theory. (One reference worth mentioning here is a survey article [Lur08].)

Higher category theory, in its present form, arose from the structures appearing in homotopy theory. In algebraic topology one defines the fundamental groupoid $\Pi_1(X)$ of a space X by declaring the objects of $\Pi_1(X)$ to be the points of X and the morphisms to be homotopy classes of paths between these points. One is forced to take homotopy classes of paths, rather than the paths themselves, since the concatenation of paths is associative only up to homotopy, and hence the resulting structure would not otherwise obey the axioms of category theory. The passage from paths to homotopy classes thereof is, however, highly unsatisfying, and conceals much information about the space at hand.

The solution is to allow *higher morphisms* (that is, 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on) and require that composition be associative and unital only up to a higher morphism. The question of how best to implement these ideas in a mathematically precise way remains to be answered. Partial answers were given in the case of so-called (∞, n)-categories, that is higher categories that have morphisms in arbitrary degree and morphisms in degrees $k > n$ are invertible in the appropriate up-to-homotopy sense. Surveys of such definitions can be found in [Ber10] for $n = 1$ and [BR13] for arbitrary n.

Higher categories therefore provide a convenient language for describing homotopy-universal properties, that is properties that define objects up to homotopy equivalence. This renders higher categories a very useful tool in topology (see, for instance, [CJ13]). Since notions of equivalence weaker than isomorphism are present in many other areas of mathematics, the use of higher categories has become widespread. For example, in geometry and mathematical physics, one is interested in the notion of cobordism between manifolds, and the language of higher categories allows one to address some resulting fundamental questions [FHLT10, Lur09b].

Among higher categories, ∞-groupoids play a special role. They are $(\infty, 0)$-categories, that is ∞-categories in which all morphisms in all dimensions are (weakly) invertible. For Grothendieck [Gro83], ∞-groupoids were the true object of study in homotopy theory. This statement was known as the *Homotopy Hypothesis*, and has been used as a benchmark for the definition of a higher category.

In this thesis, we will be concerned only with $(\infty, 1)$-categories. Among many possible (but equivalent) definitions, one has in some sense become prominent, namely the definition asserting that an $(\infty, 1)$-category is a *quasicategory*. (A quasicategory is a simplicial set having the inner horn filling property.) Even though it was defined back in 1973 by Boardman and Vogt [BV73], it is only in the last 10–15 years that its great applications have been developed by Joyal [Joy02, Joy09] and Lurie [Lur09a, Lur12].

Dwyer and Kan showed [DK80c, DK80a] that, given a category with a notion of a weak equivalence, one can extract from it a quasicategory. Later work of Barwick and Kan [BK12a] shows that this assignment (called *localization*) is itself an equivalence of categories equipped with the notion of a weak equivalence.

The connections between higher category theory and homotopy type theory are manifold. Indeed, higher categories with appropriate structure should provide models for type theory: as we will discuss in Chapter 4, every locally cartesian closed quasicategory can be turned into a model of a fragment of Martin-Löf type theory. However this is by no means immediate: a model for type theory is a category with certain, very strict, extra structure, which is not immediately possessed by a category extracted from a quasicategory. Thus more work is needed.

One may also ask the opposite question: is every model of type theory a locally cartesian closed quasicategory? The model itself would be a (1-)category equipped with certain extra structure sufficient to endow it with a notion of weak equivalence. One can therefore apply the localization construction to obtain a quasicategory. The question then becomes: is the resulting quasicategory locally cartesian closed?

An affirmative answer to this question is known as *Joyal's Conjecture*, formulated in 2011 during the *Oberwolfach MiniWorkshop 1109a: Homotopy Interpretation of Constructive Type Theory* [AGMLV11]. Joyal's conjecture begins to unwind the variety of higher-categorical structures present in type theory (as suggested in [Awo12, Sec. 3]). The proof of this conjecture is the main result of the present thesis.

It should be said, however, that the fact that type theory should in some way give rise to locally cartesian closed quasicategories occurred to other people before Joyal formulated his conjecture. For example, the author of this thesis heard it mentioned in a conversation between Steve Awodey and Peter LeFanu Lumsdaine back in 2009. Joyal's conjecture is therefore a way of making this anticipated connection precise.

1.3 ORGANIZATION OF THE THESIS

After this rather high-level overview, let us now return to planet Earth and discuss the organization of the thesis.

In Chapter 2 we will review type-theoretic preliminaries. We will carefully describe the syntax of the type theory under consideration (Section 2.1). It is a fragment of Martin-Löf type theory with a limited number of logical constructors. We will also discuss possible extensions of the theory considered there. In Section 2.2, we describe categorical semantics of type theory. In particular, we say precisely what we mean by a model of type theory and what some properties of such models are. Finally, in Section 2.3, we will review the basics of homotopy type theory. The content of this chapter is not original and parts of the texts are taken almost verbatim from [KLV12] and [AKL13].

Chapter 3 is devoted to an introduction to abstract homotopy theory. In Section 3.1 we describe various *models* of higher categories. That is, we give four possible definitions of what a higher category could be. These are: homotopical categories (i.e. categories equipped with a suitable class of weak equivalences), simplicial categories, quasicategories, and complete Segal spaces. Besides giving basic definitions, we also put these notions in a broader context, describing their properties and applications. In Section 3.2 we show that all these notions are equivalent in a suitable sense and the ways of translating between them are essentially equivalent. Finally, in Section 3.3, we review basic quasicategory theory, showing how to lift certain categorical notions (slices, limits, adjoints, ...) to quasicategories. The content of this chapter is not original.

Building on the developments of the previous chapter, in Chapter 4 we define locally cartesian closed quasicategories and study their basic properties (Section 4.1); we, in precise mathematical terms, explain the already existing connections between type theory and locally cartesian closed quasicategories (Section 4.2); and we state Joyal's conjecture, and discuss our proof strategy (Section 4.3). The content of this chapter is not original; however, some of the results presented there exist only as comments on some mathematical blogs and some of the proofs are new.

Our proof strategy is based on an observation from [AKL13] that models of type theory carry more structure than just that of a homotopical category: they are fibration categories. While localizations of arbitrary homotopical categories can be difficult to work with, the situation simplifies when the category in question is known to possess the structure of a fibration category. Szumiło [Szu14] defined a functor associating to a fibration category the quasicategory of frames in it, and showed that this quasicategory possesses finite limits.

The use of this construction plays a fundamental role in our proof. We identify the structure that one has to equip a fibration category with in order for the resulting quasicategory to be locally cartesian closed. We furthermore prove that the construction of Szumiło is equivalent to the standard localization functor, as used in the formulation of Joyal's conjecture.

To this end, in Chapter 5 we review the theory of fibration categories and Szumiło's construction. Fibration categories of diagrams play an important role in this development and therefore, after reviewing the basic definitions in Section 5.1, we turn in Section 5.2 towards the rich theory of fibration categories of diagrams. Next, we review the results of Szumiło (Section 5.3), in particular the construction of the quasicategory of frames. Section

5.4 serves as a repository of technical results on extending Reedy fibrant diagrams along sieve inclusions. This section can easily be skipped at first reading and referred to when necessary. The results of this chapter are mostly not original. We believe that the statements and proofs of the last section exist in folklore, but we could not find them in the existing literature, thus we gave our own proofs.

Chapter 6 is devoted to a technical result regarding the preservation of equivalences under a certain operation on fibration categories. This result will later be used in Chapter 8, but we dedicate a separate chapter to it in order to emphasize a rather interesting and intricate technique used to prove it. Indeed, in Section 6.2 we introduce and study *partial Reedy structures*; while our use of them is limited to one proof, we present them in detail as we believe they may in the future lead to more powerful applications. The main result of Section 6.2 depends also on the basic theory of homotopy pullbacks; we review this theory in the framework of fibration categories in Section 6.1. The results of this and subsequent chapters contain original research.

The convenience of working with the quasicategories of frames is best demonstrated in Chapter 7. This is where we establish the main properties of the quasicategory of frames. Recall that a locally cartesian closed category is a category with a terminal object all of whose slices are cartesian closed (i.e. the product functor has a right adjoint). We therefore show that slices of quasicategories of frames are equivalent to quasicategories of frames in the corresponding slices of fibration categories (Section 7.2); and moreover, adjunctions between fibration categories preserving enough structure (that is, some or all of the structure of a fibration category) are carried to adjunctions between quasicategories (Section 7.3). The first of these results uses a lemma (proven in Section 7.1) simplifying the criterion for a map to be an equivalence of quasicategories, when the quasicategories in question are known to arise from fibration categories.

Chapter 8 contains a technical result used later (Section 9.1) to establish an equivalence between the quasicategory of frames and the standard version of localization. Even though the result is not of independent interest, a particular lemma used to prove it (from Section 8.1) may have applications going beyond the scope of this thesis. In Chapter 9, after establishing the aforementioned equivalence, we introduce the notion of a locally cartesian closed fibration category and prove, using the results of Chapter 7, that the quasicategory of frames in such a category is a locally cartesian closed quasicategory (Section 9.2). Finally, in Section 9.3, we verify that every categorical model of type theory carries the structure of a locally cartesian closed fibration category, thus proving Joyal's conjecture. (In particular, the verification that each such model is a fibration category is taken almost verbatim from [AKL13].)

1.4 PREREQUISITES

This thesis contains a mixture of logic and abstract homotopy theory, with the vast majority being the latter. Thus in order to understand all the statements, some background in both of these areas is required. It was my intention to keep the logical and homotopy-theoretic aspects separate. Therefore, Chapters 3 and 5–8 require no knowledge of logic and type theory, while Chapter 2 requires no knowledge of homotopy theory. The two areas mix, however—and they genuinely have to—in two chapters: the one giving the statement of the conjecture (Chapter 4) and the one containing its proof (Chapter 9).

It was also my intention for this thesis to be as self-contained as possible, but making it fully self-contained was simply not possible. I therefore assume that the reader is familiar with basic category theory as presented in [ML98a], basic notions of type theory as presented in [Uni13, Ch. 1], and basic definitions and properties of model categories, including Quillen model structure on simplicial sets (for Kan complexes), for which the reference [Hov99, Ch. 1–3] is sufficient.

Finally, in the parts of thesis that were crucial for the proof of Joyal's conjecture each statement is given in full and is either proven in detail, or a specific reference is given (that is, a reference that includes either the number of a specific theorem or of the page that the statement can be found on). The only departure from this occurs in Chapter 4 when providing the motivation for Joyal's conjecture. In these cases we do not give the statements of some theorems, but only specific references where both the statement and the proof can be found. We afford ourselves this liberty as these issues have no impact on the proof of

Joyal's conjecture, but only the mathematics surrounding it.

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Last, but not least, I would like to thank my mother for 28 years of continuing support and everything she did for me. Mum, it is my great honor to dedicate this thesis to you.

2.0 HOMOTOPY TYPE THEORY

2.1 MARTIN-LÖF TYPE THEORY

2.1.1. Martin-Löf Dependent Type Theory is a formal system of logic, designed as an alternative foundation of mathematics. In this section, we will discuss the specific theory under consideration and its relation to other systems, in particular Calculus of Inductive Constructions. We will therefore introduce briefly the syntax of type theory. (We note however that this is not a comprehensive introduction to type theory; for this we refer the reader to [SU06] for general type theory, and to [NPS90], [Uni13, Ch. 1] for the dependent type theory.)

2.1.2. First, one constructs the *raw syntax*—the set of formulas that are at least parsable, if not necessarily meaningful—as certain strings of symbols, or alternatively, certain labeled trees. On this, one then defines *α-equivalence*, i.e. equality up to suitable renaming of bound variables, and the operation of *capture-free substitution*. This first stage is well standardized in the literature; see e.g. [Hof97] for details.

Second, one defines on the raw syntax several multi-place relations—the *judgements* of the theory. These relations are defined by mutual induction, as the smallest family of relations satisfying a bevy of specified closure conditions, the *inference rules* of the theory. The details of the judgements and inference rules used vary somewhat and in fact Martin-Löf could not quite settle on a single formulation of the theory, making multiple changes over time [ML72], [ML75], [ML84], [ML82], [ML98b]; we therefore set our choice out in full in Appendix A. For the structural rules, our presentation is based largely on [Hof97]; as for the logical rules, we present only those mentioned explicitly in Joyal's Conjecture. Their statements are taken from [ML84].

2.1.3. Let us mention, however, that Joyal's Conjecture will be formulated for any extension of the theory presented here; it is the extensions that make the conjecture difficult and interesting at the same time. We will return to this point in Paragraph 4.3.3.

2.1.4. We take as basic the judgement forms

$$\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash A = A' \ \mathsf{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash a = a' : A.$$

We treat contexts as a derived judgement: $\vdash \Gamma \ \mathsf{cxt}$ means that Γ is a list $(x_i{:}A_i)_{i<n}$, with x_i distinct variables, and such that for each $i < n$, $(x_j{:}A_j)_{j<i} \vdash A_i \ \mathsf{type}$.

2.1.5. Let us finally mention some possible extensions that we have in mind. The most natural one is Martin-Löf Type Theory as presented in [ML84]; it adds to the rules of Appendix A.1 and A.2 more logical constructors (W-types, the unit type $\mathsf{1}$, the empty type $\mathsf{0}$, and coproduct types $+$) and a (sequence of) universe(s). Another extension is Calculus of Inductive Constructions [CH88, PPM90, PM93, Wer94], which the Coq proof assistant is based on. CIC differs from Martin-Löf Type Theory, most notably in its very general scheme for inductive definitions and its treatment of universes. In the case of extensional type theory, the inductive definitions of CIC are known to reduce to the aforementioned logical constructors of MLTT (see e.g. [PM96] or [Bar12]). For intensional type theory, this only exists in folklore, but some discussion is present in [Voe10b, Sec. 6.2].[1]

2.1.6. Besides these, we may also want to consider Higher Inductive Types [Uni13, Ch. 6] and the Univalence Axiom [Voe10a]. And indeed, these possible extensions are what stands behind Joyal's Conjecture. It is really only the first step in unwinding the variety of homotopy-theoretic structures behind the new foundations, coming from the above extensions.

2.2 CATEGORICAL MODELS OF TYPE THEORY

2.2.1. In this section, we will review the basics of categorical semantics of type theory. Before delving into the definitions, a few comments are in order. Instead of interpreting type theory

[1] The last reference seems to reduce all inductive constructions of CIC to Σ, Id, $\mathsf{1}$, $\mathsf{0}$, $+$, and a dependent version of W-types, as studied, from the semantics viewpoint in [GH04].

in categories directly, we introduce an intermediate notion of a categorical model. In fact, there is no *we* as different authors use different notions of a categorical model; and so, there are: **comprehension categories** [Jac93, Jac99], **categories with families** [Dyb96, Hof97], **categories with attributes** [Car78, Mog91] (also referred to as **type categories** [Pit00]). We choose to work with yet another notion of **contextual categories** (also referred to as **C-systems** in [Voe10b]), originally introduced by Cartmell [Car78, Sec. 2.2] and studied extensively in [Car86] and [Str91].

2.2.2. It should be said, however, that all of these notions are essentially equivalent and the only differences lie in the *adjectives*. For example, contextual categories are the same as *full, split comprehension categories with unit*. Our choice of contextual categories as the framework to work with arose primarily due to the convenient fact that they are the only notion of a model not requiring any further adjectives. Moreover, the structure of contextual categories appears to resemble the notion of a fibration category that will be used in the proof.

Definition 2.2.3 (cf. [Str91, Def. 1.2])**.** A **contextual category** consists of the following data:

1. a category $\mathcal{C}$;
2. a grading of objects as $\operatorname{Ob}\mathcal{C} = \coprod_{n\in\mathbb{N}} \operatorname{Ob}_n \mathcal{C}$;
3. an object $\mathbf{1} \in \operatorname{Ob}_0 \mathcal{C}$;
4. maps $\mathrm{ft}_n \colon \operatorname{Ob}_{n+1} \mathcal{C} \to \operatorname{Ob}_n \mathcal{C}$ (whose subscripts we usually suppress);
5. for each $X \in \operatorname{Ob}_{n+1} \mathcal{C}$, a map $p_X \colon X \to \mathrm{ft}X$ (the *canonical projection* from X);
6. for each $X \in \operatorname{Ob}_{n+1} \mathcal{C}$ and $f \colon Y \to \mathrm{ft}(X)$, an object $f^*(X)$ and a map $q(f, X) \colon f^*(X) \to X$;

such that:

1. $\mathbf{1}$ is the unique object in $\operatorname{Ob}_0(\mathcal{C})$;
2. $\mathbf{1}$ is a terminal object in $\mathcal{C}$;

3. for each $X \in \mathrm{Ob}\,\mathcal{C}$ and $f\colon Y \to \mathrm{ft}(X)$, we have $\mathrm{ft}(f^*X) = Y$, and the square

$$\begin{tikzcd} f^*X \arrow[r, "q(f,X)"] \arrow[d, "p_{f^*X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p_x"] \\ Y \arrow[r, "f"] & \mathrm{ft}(X) \end{tikzcd}$$

is a pullback (the *canonical pullback* of X along f); and

4. these canonical pullbacks are strictly functorial: that is, for $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$, $1_{\mathrm{ft}X}^*X = X$ and $q(1_{\mathrm{ft}X}, X) = 1_X$; and for $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$, $f\colon Y \to \mathrm{ft}X$ and $g\colon Z \to Y$, we have $(fg)^*(X) = g^*(f^*(X))$ and $q(fg, x) = q(f, X)q(g, f^*X)$.

2.2.4. Note that these may be seen as models of a multi-sorted essentially algebraic theory [AR94, 3.34], with sorts indexed by $\mathbb{N} + \mathbb{N} \times \mathbb{N}$. This definition is best understood via the following example.

Example 2.2.5. Let $\mathbb{T}$ be any type theory. Then there is a contextual category $\mathcal{C}\ell(\mathbb{T})$, described as follows:

- $\mathrm{Ob}_n\,\mathcal{C}\ell(\mathbb{T})$ consists of the contexts $[x_1{:}A_1,\ \ldots,\ x_n{:}A_n]$ of length n, up to definitional equality and renaming of free variables;
- maps of $\mathcal{C}\ell(\mathbb{T})$ are *context morphisms*, or *substitutions*, considered up to definitional equality and renaming of free variables. That is, a map

  $$f\colon [x_1{:}A_1,\ \ldots,\ x_n{:}A_n] \to [y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1, \ldots, y_{m-1})]$$

  is represented by a sequence of terms

  $$\begin{gathered} x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_1 : B_1 \\ \vdots \\ x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_m : B_m(f_1,\ \ldots,\ f_{m-1}) \end{gathered}$$

  and two such maps $[f_i]$, $[g_i]$ are equal just if for each i,

  $$x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_i = g_i \colon B_i;$$

- composition is given by substitution, and the identity $\Gamma \to \Gamma$ by the variables of Γ, considered as terms;

- $\mathbf{1}$ is the empty context $[\cdot]$;
- $\mathrm{ft}[x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}] = [x_1{:}A_1,\ \ldots,\ x_n{:}A_n]$;
- for $\Gamma = [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}]$, the map $p_\Gamma \colon \Gamma \to \mathrm{ft}\Gamma$ is the *dependent projection* context morphism

$$(x_1, \ldots, x_n)\colon [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}] \to [x_1{:}A_1,\ \ldots,\ x_n{:}A_n],$$

  simply forgetting the last variable of Γ;
- for contexts

$$\Gamma = [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}(x_1, \ldots, x_n)],$$

$$\Gamma' = [y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1, \ldots, y_{m-1})],$$

  and a map $f = [f_i(\vec{y})]_{i \leq n} \colon \Gamma' \to \mathrm{ft}\Gamma$, the pullback $f^*\Gamma$ is the context

$$[y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1, \ldots, y_{m-1}),\ y_{m+1}{:}A_{n+1}(f_1(\vec{y}), \ldots, f_n(\vec{y}))],$$

  and $q(\Gamma, f)\colon f^*\Gamma \to \Gamma$ is the map

$$[f_1,\ \ldots, f_n,\ y_{n+1}].$$

2.2.6. For an object Γ, we will write e.g. (Γ, A) to denote an arbitrary object with $\mathrm{ft}(\Gamma, A) = \Gamma$, and will then write the dependent projection $p_{(\Gamma,A)}$ simply as p_A; similarly, (Γ, A, B), and so on.

2.2.7. The plain definition of a contextual category corresponds precisely to the basic judgements and structural rules of dependent type theory (Appendix A.1). Similarly, each logical rule or type- or term-constructor corresponds to certain extra structure on a contextual category. We make this correspondence precise in Theorem 2.2.18 below, after we have set up the appropriate definitions.

Definition 2.2.8. A Π-*type structure* on a contextual category $\mathcal{C}$ consists of:

1. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, an object $(\Gamma, \Pi(A, B)) \in \mathrm{Ob}_{n+1}\,\mathcal{C}$;
2. for each such (Γ, A, B) and section $b\colon (\Gamma, A) \to (\Gamma, A, B)$, a section $\lambda(b)\colon \Gamma \to (\Gamma, \Pi(A, B))$;
3. for each pair of sections $k\colon \Gamma \to (\Gamma, \Pi(A, B))$ and $a\colon \Gamma \to (\Gamma, A)$, a section $\mathsf{app}(k, a)\colon \Gamma \to (\Gamma, A, B)$ such that $p_B \cdot \mathsf{app}(k, a) = a$,

4. and such that for $a\colon \Gamma \to (\Gamma, A)$ and $b\colon (\Gamma, A) \to (\Gamma, A, B)$, we have $\mathsf{app}(\lambda(b), a) = b \cdot a$;
5. and for $f\colon \Gamma' \to \Gamma$, and all appropriate arguments as above,

$$f^*(\Gamma, \Pi(A, B)) = (\Gamma', \Pi(f^*A, f^*B)),$$
$$f^*\lambda(b) = \lambda(f^*b), \qquad f^*(\mathsf{app}(k, a)) = \mathsf{app}(f^*k, f^*a).$$

2.2.9. Given a Π-structure on $\mathcal{C}$, and (Γ, A, B) as above, write $\mathsf{app}_{A,B}$ for the morphism

$$\begin{aligned} & q(q(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A}, A), B) \cdot \\ & \qquad \mathsf{app}_{(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A})^*A,\, (p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A})^*B} \big((1, p_{p^*_{\Pi(A,B)}A}), (1, q(p_{\Pi(A,B)}, A))\big) \\ & \qquad\qquad : (\Gamma, \Pi(A, B), p^*_{\Pi(A,B)}A) \to (\Gamma, A, B); \end{aligned}$$

the general form $\mathsf{app}_{A,B}(k, a)$ can be re-derived from these instances. Also, for objects (Γ, A), (Γ, B) in $\mathcal{C}$, write $(\Gamma, [A, B])$ for $(\Gamma, \Pi(A, p^*_A B))$.

Definition 2.2.10. A Σ-*type structure* on a contextual category $\mathcal{C}$ consists of:

1. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, an object $(\Gamma, \Sigma(A, B)) \in \mathrm{Ob}_{n+1}\,\mathcal{C}$;
2. for each such (Γ, A, B) a morphism $\mathsf{pair}_{A,B}\colon (\Gamma, A, B) \to (\Gamma, \Sigma(A, B))$ over Γ;
3. for each such (Γ, A, B), each object $(\Gamma, \Sigma(A, B), C)$, and each morphism $d\colon (\Gamma, A, B) \to (\Gamma, \Sigma(A, B), C)$ with $p_C \cdot d = \mathsf{pair}_{A,B}$, a section $\mathsf{split}_d\colon (\Gamma, \Sigma(A, B)) \to (\Gamma, \Sigma(A, B), C)$, with $\mathsf{split}_d \cdot \mathsf{pair}_{A,B} = d$;
4. such that for $f\colon \Gamma' \to \Gamma$, and all appropriate arguments as above,

$$f^*(\Gamma, \Sigma(A, B)) = (\Gamma', \Sigma(f^*A, f^*B)),$$
$$f^*\mathsf{pair}_{A,B} = \mathsf{pair}_{f^*A, f^*B}, \qquad f^*\mathsf{split}_d = \mathsf{split}_{f^*d}.$$

Definition 2.2.11. An Id-*type structure* on a contextual category $\mathcal{C}$ consists of:

1. for each (Γ, A), an object $(\Gamma, A, p^*_A A, \mathsf{Id}_A)$;
2. for each (Γ, A), a morphism $\mathsf{refl}_A\colon (\Gamma, A) \to (\Gamma, A, p^*_A A, \mathsf{Id}_A)$, such that $p_{\mathsf{Id}_A} \cdot \mathsf{refl}_A = (1_A, 1_A)\colon (\Gamma, A) \to (\Gamma, A, p^*_A A)$;
3. for each $(\Gamma, A, p^*_A A, \mathsf{Id}_A, C)$ and $d\colon (\Gamma, A) \to (\Gamma, A, p^*_A A, \mathsf{Id}_A, C)$ with $p_C \cdot d = \mathsf{refl}_A$, a section $\mathsf{J}_{C,d}\colon (\Gamma, A, p^*_A A, \mathsf{Id}_A) \to (\Gamma, A, p^*_A A, \mathsf{Id}_A, C)$, such that $\mathsf{J}_{C,d} \cdot \mathsf{refl}_A = d$;

4. such that for $f\colon \Gamma' \to \Gamma$, and all appropriate arguments as above,

$$f^*(\Gamma, A, p_A^* A, \mathsf{Id}_A) = (\Gamma', f^*A, (p_{f^*A})^*(f^*A), \mathsf{Id}_{f^*A}),$$
$$f^*\mathsf{refl}_A = \mathsf{refl}_{f^*A}, \qquad f^*\mathsf{J}_{C,d} = \mathsf{J}_{f^*C, f^*d}.$$

Definition 2.2.12. Say that $\mathcal{C}$ satisfies the Π-η *rule* if for any (Γ, A, B), the "η-expansion" map

$$q(p_{\Pi(A,B)}, \Pi(A,B)) \cdot \lambda(1_{p^*_{\Pi(A,B)}A}, \mathsf{app}_{A,B}) \colon (\Gamma, \Pi(A,B)) \to (\Gamma, \Pi(A,B))$$

is equal to $1_{(\Gamma, \Pi(A,B))}$.

A Π-EXT *structure* on $\mathcal{C}$ is an operation giving for each (Γ, A, B) a map

$$\mathsf{ext}_{A,B} \colon (\Gamma, \Pi(A,B), p^*_{\Pi(A,B)}\Pi(A,B), \mathsf{Htp}_{A,B}) \to \\ (\Gamma, \Pi(A,B), p^*_{\Pi(A,B)}\Pi(A,B), \mathsf{Id}_{A,B})$$

over $(\Gamma, \Pi(A,B), p^*_{\Pi(A,B)}\Pi(A,B))$, stably in Γ, where $\mathsf{Htp}_{A,B}$ is the object

$$\Big(\Gamma,\ \Pi(A,B),\ p^*_{\Pi(A,B)}\Pi(A,B),\ \Pi\Big(\big(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}\Pi(A,B)}\big)^* A, \\ \big(\mathsf{app}_{A,B} \cdot q(p_{\Pi(A,B)}, A), \mathsf{app}_{A,B} \cdot q(q(p_{\Pi(A,B)}, \Pi(A,B)), A)\big)^* \mathsf{Id}_B\Big)\Big).$$

Given a Π-EXT structure on $\mathcal{C}$, a Π-EXT-COMP-PROP *structure* for it is an operation giving, for each (Γ, A, B) and section $f\colon \Gamma \to (\Gamma, \Pi(A,B))$, a map

$$\mathsf{ext\text{-}comp}(f) \colon \Gamma \to \\ (\Gamma,\ \Pi(A,B),\ p^*_{\Pi(A,B)}\Pi(A,B),\ \mathsf{Id}_{\Pi(A,B)},\ p^*_{\mathsf{Id}_{\Pi(A,B)}}\mathsf{Id}_{\Pi(A,B)},\ \mathsf{Id}_{\mathsf{Id}_{\Pi(A,B)}})$$

over the pair of maps

$$\mathsf{ext}_{A,B}(f,g) \cdot \lambda(1_A, \mathsf{refl}_B \cdot p_{p^*_A B} \cdot \mathsf{app}((1_A, f), (1_A, 1_A))) \ \ , \ \ \mathsf{refl}_{\Pi(A,B)} \cdot f \\ \colon \Gamma \to (\Gamma,\ \Pi(A,B),\ p^*_{\Pi(A,B)}\Pi(A,B),\ \mathsf{Id}_{\Pi(A,B)}),$$

stably as ever in Γ.

Example 2.2.13. If $\mathbb{T}$ is a type theory with Π-types, then $\mathcal{C}\ell(\mathbb{T})$ carries an evident Π-type structure; similarly for Σ-types and Id-types.

2.2.14. Note that all of these structures, like the definition of contextual categories themselves, are again essentially algebraic.

Definition 2.2.15. A map $F\colon \mathcal{C} \to \mathcal{D}$ of contextual categories, or **contextual functor**, consists of a functor $\mathcal{C} \to \mathcal{D}$ between underlying categories, respecting the gradings, and preserving (on the nose!) all the structure of a contextual category.

Similarly, a map of contextual categories with Π-type structure, Σ-type structure, etc., is a contextual functor preserving the additional structure.

2.2.16. These are exactly the maps given by considering contextual categories as essentially algebraic structures.

2.2.17. We are now equipped to state precisely the sense in which the structures defined above correspond to the appropriate syntactic rules:

Theorem 2.2.18. *Let $\mathbb{T}$ be the type theory given by just the structural rules of Section A.1. Then $\mathcal{C}\ell(\mathbb{T})$ is the initial contextual category.*

Similarly, let $\mathbb{T}$ be the type theory given by the structural rules, plus the logical rules of Sections A.2, A.3. Then $\mathcal{C}\ell(\mathbb{T})$ is initial among contextual categories with the appropriate extra structure.

2.2.19. This is essentially the Correctness Theorem of [Str91, Ch. 3, p. 181], with a slightly different selection of logical constructors.

In other words, Theorem 2.2.18 says that if $\mathcal{C}$ is a contextual category with structure corresponding to the logical rules of a type theory $\mathbb{T}$, then there is a functor $\mathcal{C}\ell(\mathbb{T}) \to \mathcal{C}$ interpreting the syntax of $\mathbb{T}$ in $\mathcal{C}$. This justifies the definition:

Definition 2.2.20. A **model of dependent type theory** with any selection of the logical rules of Sections A.2 and A.3 is a contextual category equipped with the structure corresponding to the chosen rules.

Examples 2.2.21.

- The category $\mathcal{S}\mathrm{et}$ of sets and functions is a contextual category (with an arbitrary grading on objects and arbitrary choice of ft maps) equipped with all the structures discussed in Appendix A. Given a map $f\colon B \to A$ of sets, we may view it as an A-indexed family

$(B_a \mid a \in A)$, where $B_a = f^{-1}(a)$. In this presentation, we have:

$$\begin{aligned} f^*(X_a \mid a \in A) &= (X_{f(b)} \mid b \in B) \\ \Sigma_f(Y_b \mid b \in B) &= (\textstyle\sum_{b \in B_a} Y_b \mid a \in A) \\ \Pi_f(Y_b \mid b \in B) &= (\textstyle\prod_{b \in B_a} Y_b \mid a \in A) \end{aligned}$$

This yields the interpretations of the canonical pullback, $\mathsf{\Sigma}$-, and $\mathsf{\Pi}$-structures. The Id-structure is given by the diagonal $\Delta\colon A \to A \times A$.

- One may observe that the structures described above generalize to all locally cartesian closed categories. Recall that a category $\mathcal{C}$ is **locally cartesian closed** if it has a terminal object, and for any map $f\colon B \to A$, the pullback functor $f^*\colon \mathcal{C}/A \to \mathcal{C}/B$ admits a right adjoint (typically denoted f_* or Π_f). This is equivalent to asking for $\mathcal{C}$ to have a terminal object and each of its slices $\mathcal{C}/A$ to be cartesian closed. In particular, a locally cartesian closed category has all finite limits. For the details of the interpretation, see [See84] and [Hof95b].

2.3 HOMOTOPY TYPE THEORY

2.3.1. It can be shown [AW09, Prop. 2.1] that every model of a type theory $\mathbb{T}$ in the style of [See84] will satisfy an additional rule, called **reflection rule**:

$$\frac{\Gamma \vdash p : \mathsf{Id}_A(a,b)}{\Gamma \vdash a = b : A}\ \mathsf{Id\text{-}reflection}$$

This rule is highly undesirable from the proof-theoretic point of view as it destroys several good properties of the system, such as *decidability of type-checking*. The quest for models that do not satisfy this rule gave rise to Homotopy Type Theory. Let us then turn now our attention to this program, describing its origins and main milestones.

2.3.2. There are two meanings of the term *Homotopy Type Theory*. One definition is as the underlying type theory of the Univalent Foundations, which is Calculus of Inductive Constructions together with the Univalence Axiom and Higher Inductive Types. The other definition refers to the interpretation of Martin-Löf Type Theory into various categorical structures arising from homotopy theory, as well as the study of the homotopy theory of type theory. Since the type theories considered here are fairly minimalistic, it is the latter definition that we will use.

2.3.3. In this section, we will review some basic results from the program of Homotopy Type Theory, as they are relevant to our main result. To this end, let $\mathbb{T}$ be any type theory satisfying the rules of Appendix A.

2.3.4. The development of Homotopy Type Theory derives from the work of Hofmann and Streicher [HS98b] on the groupoid model of type theory. The groupoid model introduced the structure of a contextual category with all the structures present in $\mathbb{T}$ on the category $\mathcal{G}$pd of groupoids. It was the first model in which the reflection rule was not validated. This was accomplished by interpreting the Id-type not as a the diagonal $A \to A \times A$, but as the arrow groupoid $A^{\to} \to A \times A$. It hence made use of two notions of equality: the equality of objects of groupoids as set-theoretic entities, and their isomorphisms as objects of a category. And since not all isomorphisms are identities, this violated the reflection rule.

2.3.5. Awodey and Warren [AW09] noticed that the rules of Id-types correspond to the axioms of a weak factorization system, forcing types to be interpreted as *fibrations* (or, more generally, maps belonging to the right class of the factorization system in question). Their paper was a genuine breakthrough and opened the gate for others to work on these topics.

2.3.6. Models of type theories with Id-types, based on model categories and weak factorization systems, were found later by Warren [War08] and by Garner and van den Berg [vdBG12].

2.3.7. Gambino and Garner [GG08, Thm. 10] identified a weak factorization system on the classifying category $\mathcal{C}\ell(\mathbb{T})$, thus showing that every categorical model of dependent type theory is in fact equipped with such a factorization system. Their construction uses Garner's Identity Contexts [Gar09b, Prop. 3.3.1].

2.3.8. For completeness, we also mention the work of Lumsdaine [Lum09, Lum10] on connecting the structures present on $\mathcal{C}\ell(\mathbb{T})$ to Leinster's definition of a weak ω-category [Lei04], and that of Garner and van den Berg [vdBG11] on the same topic. From a slightly different perspective, this connection was studied by Awodey, Hofstra, and Warren [AHW13, HW13]

2.3.9. At around the same time, Voevodsky suggested his Univalent Foundations Program [Voe10c], which suggests using the system Coq, together with an additional axiom, called the Univalence Axiom as foundations for mathematics. In this proposal, Voevodsky already had the homotopy-theoretic interpretation in mind (see his earlier work, e.g. [Voe06]), but he also managed to formally develop (in the system of Coq) sizeable portions of classical homotopy theory [V^+]. An excellent introduction to formalization in Homotopy Type Theory can be found [PW12].

2.3.10. The ideas of Voevodsky planted the seed from which much research has grown. The HoTT group now has its own repository of formally verified results [HoTa] and a blog [HoTb], where a full list of papers can be found. Voevodsky identified a model of the Univalence Axiom in the category of simplicial sets [KLV12] and this result was later extended by Shulman [Shu14] to a larger class of models.

2.3.11. Let us now recall a few basic definitions from the Univalent Foundations. These definitions will be necessary later, when formulating Joyal's Conjecture. We adopt here the informal style of presentation developed in [Uni13]. Thus it is important to point out that for the next couple of definitions and theorems, we are working *inside* type theory.

Definition 2.3.12. A type X is **contractible** if there is some $x_0 : X$, and a function giving for each $x : X$ a path in $\mathsf{Id}(x, x_0)$. Formally, the proposition "X is contractible" is defined as follows:[2]

$$\mathsf{isContr}(X) := \sum_{x_0 : X} \prod_{x : X} \mathsf{Id}(x, x_0).$$

Definition 2.3.13. The *homotopy fiber* of a map $f\colon X \to Y$ over an element $y : Y$ is defined

[2]One might at first read this as a definition of connectedness—for each x, there exists some path from x to x_0—but remember that one should think of the function sending x to the path as *continuous*, so as giving a contraction of X to x_0. Precisely, in the simplicial and similar interpretations, the Π-type becomes a space of continuous functions, and so $\mathsf{isContr}$ gets interpreted as the property of contractibility; and moreover, working within the theory, the logic forces $\mathsf{isContr}$ to behave like contractibility, not like connectedness.

by:

$$\mathsf{hfib}(f, y) := \sum_{x:X} \mathsf{Id}_Y(f(x), y).$$

Definition 2.3.14. A map $f\colon X \to Y$ is an *equivalence* if for all $y : Y$ the homotopy fiber of f over y is contractible, i.e.:

$$\mathsf{isEquiv}(f) := \prod_{y:Y} \mathsf{isContr}(\mathsf{hfib}(f, y)).$$

Theorem 2.3.15. *The following are equivalent for a map f:*

1. *f is an equivalence.*
2. *there exists $g\colon B \to A$ together with a homotopy $\eta\colon \prod_{x:A} \mathsf{Id}_A(x, gfx)$ and $\varepsilon\colon \prod_{y:B} \mathsf{Id}_B(fgy, y)$.*
3. *there exists $g_1\colon B \to A$ together with $\eta\colon \prod_{x:A} \mathsf{Id}_A(x, g_1fx)$ and $g_2\colon B \to A$ with $\varepsilon\colon \prod_{y:B} \mathsf{Id}_B(fg_2y, y)$.*

2.3.16. We will now move back to classical theory and study type theory $\mathbb{T}$ externally. Given $c\colon \sum_{x:A} B(x)$, using Σ-ELIM, we may obtain terms: $\pi_1(c){:}A$ and $\pi_2(c){:}B(\pi_1(c))$. We may then add the following rule to type theory $\mathbb{T}$:

$$\frac{\Gamma \vdash c : \Sigma_{x:A}B(x)}{\Gamma \vdash c = \mathsf{pair}(\pi_1(c), \pi_2(c)) : \Sigma_{x:A}B(x)}\ \Sigma\text{-}\eta$$

2.3.17. When working with the category $\mathcal{C}\ell(\mathbb{T})$ of contexts, it is often convenient to use the internal reasoning. Thus, for convenience of exposition later on, we also assume the above η-rules for Σ-types, so that every context is isomorphic to (a context consisting of just) a single iterated Σ-type: for instance,

$$[x{:}A,\ y{:}B(x)] \cong [p : \Sigma_{x:A}B(x)].$$

Indeed, the maps $\mathsf{pair}\colon [x{:}A,\ y{:}B(x)] \to [p : \Sigma_{x:A}B(x)]$ and $[\pi_1, \pi_2]\colon [p : \Sigma_{x:A}B(x)] \to [x{:}A,\ y{:}B(x)]$ can easily be seen as each other inverses, using the Σ-η rule. Thus, the inclusion of the subcategory of $\mathcal{C}\ell(\mathbb{T})$ consisting only of single type extensions (a.k.a. types, a.k.a. objects of grading 1 in $\mathcal{C}\ell(\mathbb{T})$) is an equivalence of categories. Therefore, any categorical properties of $\mathcal{C}\ell(\mathbb{T})$ can be detected on the level of types, without a reference to more general contexts.

2.3.18. Nothing here depends on that assumption, however; one may simply replace types with contexts and Σ-types with context extensions.

2.3.19. To demonstrate the advantage of working with types, rather than contexts, let us show that that the category $\mathcal{C}\ell(\mathbb{T})$ is a homotopical category (in the sense of Definition 3.1.3). We defined the notion of a weak equivalence internally to type theory, so we may now say that a morphism $f\colon \Gamma \to \Delta$ of contexts is a **weak equivalence** if the corresponding morphism of iterated Σ-types is provably a weak equivalence (that is, the type $\mathsf{isEquiv}(f)$ is inhabited). Moreover, one can easily show—working internally to type theory—that weak equivalences are closed under composition and that every identity morphism is a weak equivalence. This implies the desired external statement that these weak equivalences form a subcategory.

3.0 ABSTRACT HOMOTOPY THEORY

Abstract homotopy theory traditionally has two incarnations: homotopical algebra and higher category theory. The former deals with categories equipped with a class of weak equivalences (e.g. model categories, Waldhausen categories), while the latter makes the information about higher-categorical structure explicit. However, both of these approaches capture the same information. We will therefore discuss different models of what can be called a *homotopy theory* (Section 3.1) and show that all these models are equivalent in a suitable sense (Section 3.2). Finally in Section 3.3, we shall review, following [Joy09] and [Lur09a], the basics of how to lift category theory to higher category theory.

3.1 MODELS OF $(\infty, 1)$-CATEGORIES

3.1.1. In this section, we will briefly review the basics of four different models for $(\infty, 1)$-categories. They are: homotopical categories (Sec. 3.1.1), simplicial categories (Sec. 3.1.2), quasicategories (Sec. 3.1.3), and complete Segal spaces (Sec. 3.1.4).

3.1.1 Homotopical categories

3.1.2. Recall that a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ of a category $\mathcal{C}$ is called **wide** if $\mathcal{C}'$ contains all objects of $\mathcal{C}$.

Definition 3.1.3. A **homotopical category** consists of a category $\mathcal{C}$ together with a wide subcategory $\mathcal{W} \subseteq \mathcal{C}$, whose morphisms are called **weak equivalences**.

3.1.4. Clearly, we could alternatively define a homotopical category as a category $\mathcal{C}$ together with a class of maps $\mathcal{W}$ such that:

- all identity morphisms belong to $\mathcal{W}$ and
- the class $\mathcal{W}$ is closed under composition.

We shall mention that some other authors prefer calling what we call homotopical categories, **relative categories** (see, for example, [BK12b]). However, we find this terminology to be somewhat unilluminating.

3.1.5. The significance of homotopical categories lies in the fact that every homotopical category $\mathcal{C}$ admits the homotopy category $\mathrm{Ho}(\mathcal{C})$.

Definition 3.1.6 (cf. [GZ67, Ch. 1.1.1])**.** The **homotopy category** of a homotopical category $(\mathcal{C}, \mathcal{W})$ is a category $\mathrm{Ho}(\mathcal{C}, \mathcal{W})$ (or just $\mathrm{Ho}\mathcal{C}$, if no confusion possible) together with a identity-on-objects functor $\gamma\colon \mathcal{C} \to \mathrm{Ho}\mathcal{C}$, with $\gamma(w)$ an isomorphism for all $w \in \mathcal{W}$, satisfying the following universal property: given any category $\mathcal{D}$ with a functor $\delta\colon \mathcal{C} \to \mathcal{D}$ such that $\delta(w)$ is an isomorphism for all $w \in \mathcal{W}$, there exists a unique functor $\mathrm{Ho}\mathcal{C} \to \mathcal{D}$ making the following triangle commute:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\delta"] \arrow[d, "\gamma"'] & \mathcal{D} \\
\mathrm{Ho}\mathcal{C} \arrow[ur, dotted] &
\end{tikzcd}$$

3.1.7. Explicitly, one may construct the homotopy category of $(\mathcal{C}, \mathcal{W})$ as follows. The objects of $\mathrm{Ho}\mathcal{C}$ are the objects of $\mathcal{C}$ and the morphisms of $\mathrm{Ho}\mathcal{C}$ are equivalence classes of zigzags built from:

1. morphisms of $\mathcal{C}$
2. inverted morphisms from $\mathcal{W}$ i.e. morphisms $\overline{w}\colon X \to Y$ such that $w\colon Y \to X \in \mathcal{W}$

quotiented by the relations:

1. $X \xrightarrow{f} Y \xrightarrow{g} Z \ \sim\ X \xrightarrow{g\cdot f} Z$;
2. $1_X^{\mathcal{C}} \sim 1_X^{\mathrm{Ho}\mathcal{C}}$;
3. for any $w \in \mathcal{W}$,
   a. $X \xrightarrow{w} Y \xrightarrow{\overline{w}} X \ \sim 1_X^{\mathrm{Ho}\mathcal{C}}$;
   b. $Y \xrightarrow{\overline{w}} X \xrightarrow{w} Y \ \sim 1_Y^{\mathrm{Ho}\mathcal{C}}$.

One then defines $\gamma\colon \mathcal{C} \to \mathrm{Ho}\mathcal{C}$ as the identity-on-objects functor taking a morphism f to the class $[f]_\sim$, where we consider f as a zigzag of length 1. The verification that $\mathrm{Ho}\mathcal{C}$ satisfies the required universal property is straightforward (see [GZ67, Ch. 1] for details).

Definition 3.1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories. A **homotopical functor** $F\colon \mathcal{C} \to \mathcal{D}$ is a functor taking weak equivalences to weak equivalences.

3.1.9. Notice that if $\mathcal{C}$ is a homotopical category and J is any small category, then the functor category $\mathcal{C}^J$ is again a homotopical category, whose weak equivalences are natural weak equivalences i.e., natural transformations $\alpha\colon F \to G$ such that $\alpha_j\colon F(j) \to G(j)$ is a weak equivalence for all $j \in J$.

Proposition 3.1.10. *A homotopical functor* $F\colon \mathcal{C} \to \mathcal{D}$ *induces a functor* $\mathrm{Ho}F\colon \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$.

Proof. Since F preserves weak equivalences and the canonical map $\delta\colon \mathcal{D} \to \mathrm{Ho}\mathcal{D}$ inverts them, by the universal property of $\mathrm{Ho}\mathcal{C}$ we obtain the induced map:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \arrow[r, "\delta"] & \mathrm{Ho}\mathcal{D} \\
\mathrm{Ho}\mathcal{C} \arrow[urr, dotted] & &
\end{tikzcd}$$

□

Definition 3.1.11. Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories. A homotopical functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **homotopy equivalence** if there exists a homotopical functor $G\colon \mathcal{D} \to \mathcal{C}$ and two zigzags of natural weak equivalences connecting $F \cdot G$ with $1_\mathcal{D}$ and $G \cdot F$ with $1_\mathcal{C}$.

3.1.2 Simplicial categories

3.1.12. Simplicial categories were among the first models used to handle in a modular way the infinite amount of coherence data required in many constructions in homotopy theory [CP86, CP88]. Unlike homotopical categories, simplicial categories and all following models belong to the higher-categorical approach.

Definition 3.1.13. A **simplicial category** is a category enriched over the cartesian monoidal category $s\mathcal{S}et$ of simplicial sets.

3.1.14. The category of simplicial categories (and simplicially-enriched functors) will be denoted by $s\mathcal{C}at$.

3.1.15. The term *simplicial category* is also sometimes used to describe a simplicial object in the category $\mathcal{C}at$ of small categories. As it turns out, the simplicial categories in the sense of the above definition form a subcategory of the category $\mathcal{C}at^{\Delta^{\mathrm{op}}}$ (of simplicial objects in $\mathcal{C}at$). Indeed, there is a natural correspondence between simplicial categories and the simplicial objects $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathcal{C}at$ with a fixed set of objects (i.e. $\mathrm{Ob}\,\mathcal{C}_n = O$ for some set O and the face and degeneracy maps are identity-on-objects).

3.1.16. Let $\mathcal{C}$ be a simplicial category. We will typically write $\mathrm{Ob}\,\mathcal{C}$ for the collection of objects of $\mathcal{C}$ and $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(X, Y)$ for the simplicial sets of morphisms from X to Y. We say that a simplicial category is **locally Kan** if for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\mathcal{M}\mathrm{ap}(X, Y)$ is a Kan complex (see Paragraph 3.1.32).

3.1.17. The **homotopy category** $\mathrm{Ho}\mathcal{C}$ of a simplicial category $\mathcal{C}$ is defined as the category whose:

- objects are the objects of $\mathcal{C}$;
- hom-sets are given by:

$$\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(X, Y) = \pi_0(\mathcal{M}\mathrm{ap}(X, Y)'),$$

  where $\mathcal{M}\mathrm{ap}(X, Y)'$ is the fibrant replacement of $\mathcal{M}\mathrm{ap}(X, Y)$ in the Quillen model structure on $s\mathcal{S}et$.

By functoriality of π_0, it is immediate that a simplicial functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{Ho}(F)\colon \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$.

Definition 3.1.18. A simplicial functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **Dwyer–Kan equivalence** if $\mathrm{Ho}(F)$ is an equivalence and for all $X, Y \in \mathcal{C}$, the induced map of simplicial sets

$$\mathcal{M}\mathrm{ap}_{\mathcal{C}}(X, Y) \to \mathcal{M}\mathrm{ap}_{\mathcal{D}}(FX, FY)$$

is a weak equivalence in the Quillen model structure on $s\mathcal{S}et$ (see [Hov99, Sec. 3.2]).

3.1.19. We next describe a way to assign to each homotopical category, a simplicial category. The construction and the in-depth study of its properties is due to Dwyer and Kan [DK80c, DK80a, DK80b].

Definition 3.1.20. Let $(\mathcal{C}, \mathcal{W})$ be a homotopical category. The **hammock localization** of $(\mathcal{C}, \mathcal{W})$ is the simplicial category $\mathrm{L^H}(\mathcal{C}, \mathcal{W})$ defined as follows:

- The objects of $\mathrm{L^H}\mathcal{C}$ are the objects of $\mathcal{C}$;
- Given $X, Y \in \mathcal{C}$, the set of n-simplices of the simplicial set $\mathcal{M}\mathrm{ap}(X, Y)$ is given by commutative diagrams of the form:

$$\begin{tikzcd}
 & C_{1,1} \arrow[r, dash] \arrow[d] & C_{1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{1,k} \arrow[d] \arrow[ddr, dash] & \\
 & C_{2,1} \arrow[r, dash] \arrow[d] & C_{2,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{2,k} \arrow[d] \arrow[dr, dash] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \arrow[dl, dash] \\
 & C_{n,1} \arrow[r, dash] & C_{n,2} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{n,k} &
\end{tikzcd}$$

  subject to the following conditions:

  1. $k \geq 0$.
  2. all vertical maps are weak equivalences.
  3. in each column all horizontal maps go in the same direction; if they all go to the left, then they are weak equivalences.
  4. the maps in the adjacent columns go in the opposite directions.
  5. no column contains only the identity maps.

  The i-th face map is obtained by omitting the i-th row and composing the vertical arrows; and the i-th degeneracy is given by repeating the i-th row and inserting a vertical row of identities.

3.1.21. One can easily check that this assignment extends to a functor $\mathrm{L^H}\colon \mathrm{h}\mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{C}\mathrm{at}$. We can therefore define the equivalences of homotopical categories.

Definition 3.1.22. A homotopical functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **Dwyer–Kan equivalence** (or simply, **DK-equivalence**) if the induced simplicial functor $\mathrm{L^H}(F)$ is a Dwyer–Kan equivalence.

Explicitly, $F\colon \mathcal{C} \to \mathcal{D}$ is a Dwyer–Kan equivalence if $\mathrm{Ho}F\colon \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ is an equivalence of categories and for any $X, Y \in \mathcal{C}$, the induced map:

$$\mathrm{L^H}(F)_{X,Y}\colon \mathcal{M}\mathrm{ap}_{\mathrm{L^H}\mathcal{C}}(X,Y) \to \mathcal{M}\mathrm{ap}_{\mathrm{L^H}\mathcal{D}}(FX,FY)$$

is an equivalence in Quillen's model structure on simplicial sets.

Examples 3.1.23.

- Let $\mathcal{M}$ and $\mathcal{M}'$ be model categories and $F\colon \mathcal{M} \to \mathcal{M}'$ a right Quillen functor which is a Quillen equivalence. Then F does not have to be a Dwyer–Kan equivalence of the underlying homotopical categories, but its restriction to the fibrant objects $F_{\mathrm{f}}\colon \mathcal{M}_{\mathrm{f}} \to \mathcal{M}'_{\mathrm{f}}$ is.
- For any model category $\mathcal{M}$, the inclusion of the full subcategory of fibrant objects $\mathcal{M}_{\mathrm{f}} \hookrightarrow \mathcal{M}$ is a Dwyer–Kan equivalence.

Theorem 3.1.24 (Barwick–Kan, [BK12a, Sec. 1])**.** *The hammock localization functor* $\mathrm{L^H}\colon \mathrm{h}\mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{C}\mathrm{at}$ *is a Dwyer-Kan equivalence, where the weak equivalences in both* $\mathrm{h}\mathcal{C}\mathrm{at}$ *and* $\mathrm{s}\mathcal{C}\mathrm{at}$ *are taken to be Dwyer-Kan equivalences.*

3.1.25. The category $\mathrm{s}\mathcal{C}\mathrm{at}$ of simplicial categories carries more structure than just this of a homotopical category; it is in fact a model category. In order to define its cofibrations and weak equivalences, we need however a preliminary notion of a homotopy equivalence in a simplicial category.

Definition 3.1.26. A map $f\colon X \to Y$ in a simplicial category $\mathcal{C}$ is a **homotopy equivalence** if it becomes an isomorphism in $\mathrm{Ho}\mathcal{C}$.

3.1.27 (cf. [Ber07, Thm. 1.1])**.** The category $\mathrm{s}\mathcal{C}\mathrm{at}$ carries a model structure, that we will refer to as **Bergner model structure**, in which:

- weak equivalences are Dwyer-Kan equivalences.
- fibrations are simplicial functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(X,Y) \to \mathcal{M}\mathrm{ap}_{\mathcal{D}}(FX,FY)$ is a Kan fibration of simplicial sets, for any $X, Y \in \mathcal{C}$;
2. given any $X \in \mathcal{C}$ and a homotopy equivalence $f\colon FX \to A \in \mathcal{D}$, there exists a homotopy equivalence $\overline{f}\colon X \to Y \in \mathcal{C}$ such that $F(\overline{f}) = f$.

3.1.28. Using the Rezk model structure on bisimplicial sets, Barwick and Kan [BK12b] were able to create a model structure on the category $\mathrm{h}\mathcal{C}\mathrm{at}$ and show [BK12a] that its weak equivalences coincide with the Dwyer-Kan equivalences. The hammock localization is not a right or left Quillen functor with respect to this structure, but it is equivalent to one.

3.1.3 Quasicategories

3.1.29. In this section, we will review the basic definitions and examples. It is worth mentioning that it is in quasicategories where the theory of higher categories is developed. That is, they are the nicest model for the theory of $(\infty,1)$-categories. In a separate section 3.3, we will discuss in more detail this internal development.

Definition 3.1.30. A **quasicategory** is a simplicial set $\mathcal{C}$ satisfying the following **inner horn filling condition**: for each $n \in \mathbb{N}$ and any $0 < i < n$, given a map $\Lambda^i[n] \to \mathcal{C}$, there exists an extension $\Delta[n] \to \mathcal{C}$ making the following triangle commute:

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

3.1.31. We shall next discuss two large classes of examples of quasicategories. They are: Kan complexes and (nerves of) categories. The former can be seen as a way of embedding *weak* ∞-groupoids (a.k.a. spaces, a.k.a. homotopy types) into the world of ∞-categories and the latter as an embedding of *strict* 1-categories.

3.1.32 (Kan complexes)**.** Recall that a simplicial set K is a **Kan complex**, if it satisfies the (general) **horn filling condition**: for each $n \in \mathbb{N}$ and any $0 \leq i \leq n$, given a map

$\Lambda^i[n] \to \mathcal{C}$, there exists an extension $\Delta[n] \to \mathcal{C}$ making the following triangle commute:

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

Thus, every Kan complex is a a quasicategory. In fact, quasicategories were first introduced under the name of *weak Kan complex* in [BV73].

3.1.33 (Nerve of a category)**.** Let $\mathcal{C}$ be a category. Define a simplicial set $\mathrm{N}(\mathcal{C})$ called the **nerve** of $\mathcal{C}$ by:

$$\mathrm{N}(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$$

where $[n]$ denotes the linear order $\{0 \leq 1 \leq \ldots \leq n\}$, regarded as a category. It is easy to see that $\mathrm{N}(\mathcal{C})$ is a quasicategory and moreover, such quasicategories are characterized by satisfying the **unique inner horn filling condition**, which asserts that for each $n \in \mathbb{N}$ and any $0 < i < n$, given a map $\Lambda^i[n] \to \mathcal{C}$, there exists a unique extension $\Delta[n] \to \mathcal{C}$ making the following triangle commute:

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

3.1.34 (cf. [Joy09, p. 158])**.** We recall moreover that the nerve functor is fully faithful and has a left adjoint denoted $\tau_1 \colon \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$. Explicitly, $\tau_1(\mathcal{C})$ is the category whose objects are 0-simplices of $\mathcal{C}$ and whose morphisms are freely generated by composites of 1-simplices modulo, for each 2-simplex α, the relation $s_0(\alpha) \cdot s_2(\alpha) = s_1(\alpha)$, where $s_0, s_1, s_2 \colon \mathcal{C}_2 \to \mathcal{C}_1$ are the face maps. For a quasicategory $\mathcal{C}$, the category $\tau_1(\mathcal{C})$ is called the **homotopy category** of $\mathcal{C}$, and we will sometimes write $\mathrm{Ho}\mathcal{C}$ instead. Similarly as in the case of simplicial categories (Def. 3.1.26), we define a 1-simplex $f \colon x \to y$ in $\mathcal{C}$ to be an **equivalence** if it becomes an isomorphism in the homotopy category.

One can also verify that the nerve of a category $\mathrm{N}(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid. Similarly, a quasicategory $\mathcal{C}$ is a Kan complex if and only if $\mathrm{Ho}\mathcal{C}$ is a groupoid.

3.1.35 (cf. [Joy09, Ch. 6])**.** On a related note, recall that Kan complexes are fibrant objects in the Quillen model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$. There is an analogous model structure, which we will refer to as **Joyal model structure**, in which:

- cofibrations are monomorphisms;
- weak equivalences are **weak categorical equivalences**, i.e. maps $w\colon X \to Y$ such for all quasicategories $\mathcal{C}$, the induced map

  $$\tau_1(\mathcal{C}^w)\colon \tau_1(\mathcal{C}^Y) \to \tau_1(\mathcal{C}^X)$$

  is an equivalence of categories.

The fibrant objects of this model structure are quasicategories; we will call weak categorical equivalences between quasicategories, **categorical equivalences**. Notice however that when restricted to Kan complexes, categorical equivalences and weak equivalences (from Quillen's model structure) coincide.

3.1.36. The adjunction $\tau_1\colon \mathrm{s}\mathcal{S}\mathrm{et} \leftrightarrows \mathcal{C}\mathrm{at} : \mathrm{N}$ is a Quillen adjunction, where $\mathcal{C}\mathrm{at}$ is considered with its natural model structure; that is, the one in which cofibrations are functors injective on objects, and weak equivalences are equivalences of categories (cf. [Joy09, p. 162]).

3.1.37. Let $\mathcal{K}\mathrm{an}$ and $\mathrm{q}\mathcal{C}\mathrm{at}$ denote the full subcategories of $\mathrm{s}\mathcal{S}\mathrm{et}$ consisting of Kan complexes and quasicategories, respectively. There is an obvious inclusion $\mathcal{K}\mathrm{an} \hookrightarrow \mathrm{q}\mathcal{C}\mathrm{at}$.

Proposition 3.1.38 ([JT07, Prop. 1.16])**.** *The inclusion* $\mathcal{K}\mathrm{an} \hookrightarrow \mathrm{q}\mathcal{C}\mathrm{at}$ *admits a right adjoint*

$$\mathrm{J}\colon \mathrm{q}\mathcal{C}\mathrm{at} \to \mathcal{K}\mathrm{an}.$$

Moreover, J *takes categorical equivalences between quasicategories to weak homotopy equivalences of Kan complexes.*

3.1.39. Intuitively, J simply picks out the maximal subgroupoid of a quasicategory.

We now turn towards establishing the connection between quasicategories and two previously defined models of $(\infty, 1)$-categories.

3.1.40 (Homotopy coherent nerve)**.** We will define a right Quillen functor $\mathrm{N}_\Delta\colon \mathrm{s}\mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$ (where the model structures on them are the ones defined by Bergner and Joyal, respectively). Our definition will follow a similar idea as the one developed for the nerve functor $\mathrm{N}\colon \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$. That is, we first construct a cosimplicial object $\mathfrak{C}\colon \Delta \to \mathrm{s}\mathcal{C}\mathrm{at}$ by:

- $\mathrm{Ob}\,\mathfrak{C}[n] = \{0, 1, \ldots, n\}$;
- $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[n]}(i,j) = \mathrm{N}(\{I \subseteq \{i, i+1, \ldots, j\} \mid i, j \in I\})$ (i.e. the nerve of the poset of subsets of $\{i, i+1, \ldots, j\}$ that contain both i and j).

Now, we can define the **homotopy coherent nerve** $\mathrm{N}_\Delta\colon \mathrm{s}\mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$ by setting:

$$\mathrm{N}_\Delta(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{s}\mathcal{C}\mathrm{at}}(\mathfrak{C}[n], \mathcal{C}).$$

3.1.41. Since the category $\mathrm{s}\mathcal{C}\mathrm{at}$ of simplicial categories possesses all colimits, we may define the left adjoint $\mathfrak{C}\colon \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{C}\mathrm{at}$ of N_Δ as a left Kan extension:

$$\begin{tikzcd}
\mathrm{s}\mathcal{S}\mathrm{et} \arrow[rr, bend left, "\mathfrak{C} := \mathrm{Lan}_{\mathbf{y}}\mathfrak{C}"] \arrow[rr, phantom, "\perp"] & & \mathrm{s}\mathcal{C}\mathrm{at} \arrow[ll, "\mathrm{N}_\Delta"] \\
& \Delta \arrow[ul, "\mathbf{y}"] \arrow[ur, "\mathfrak{C}"'] &
\end{tikzcd}$$

The adjunction $\mathfrak{C} \vdash \mathrm{N}_\Delta$ is a Quillen equivalence between the category of simplicial categories with Bergner's model structure and the category of simplicial sets with Joyal's model structure (see, for example, [Lur09a, Thm. 2.2.5.1] or [DS11, Cor. 8.2] for a proof).

3.1.42. Composing the hammock localization with the coherent nerve functor (and taking fibrant replacement in the category of simplicial categories inbetween) assigns to each homotopical category $(\mathcal{C}, \mathcal{W})$ a quasicategory. Notice that the fibrancy assumption is important since N_Δ as a right Quillen functor returns the *homotopically correct* object only on fibrant objects.

We will next give a more direct construction realizing this specification, but first we have to define a simplicial set that we will use in our construction.

3.1.43. Define a simplicial set $\mathcal{I}$ as the pushout:

$$\begin{tikzcd} \Delta[1] \sqcup \Delta[1] \arrow[r] \arrow[d, hook, "{[02,13]}"'] & \Delta[0] \sqcup \Delta[0] \arrow[d] \\ \Delta[3] \arrow[r] & \mathcal{I} \end{tikzcd}$$

Precomposing the $\Delta[3] \to \mathcal{I}$ with the inclusion $\Delta[1] \xhookrightarrow{[12]} \Delta[3]$, yields a map $\Delta[1] \hookrightarrow \mathcal{I}$.

3.1.44. From the definition, it is easy to see that maps $\mathcal{I} \to K$ *classify* morphisms in K together with a choice of left and right quasi-inverses. Thus, for a quasicategory $\mathcal{C}$, asking whether a 1-simplex $f\colon x \to y \in \mathcal{C}$ is an equivalence is the same as asking whether the map $\Delta[1] \xrightarrow{f} \mathcal{C}$ factors through $\mathcal{I}$.

Equivalently, we may give another description of equivalences. Let $E[1]$ denote the nerve of the contractible groupoid with two objects. A choice of equivalence in $\mathcal{C}$ is equivalent to a map $E[1] \to \mathcal{C}$.

3.1.45. Let $(\mathcal{C}, \mathcal{W})$ be a homotopical category. We will define a quasicategory $\mathrm{L}(\mathcal{C}, \mathcal{W})$, which will be called the **standard localization** of a homotopical category $(\mathcal{C}, \mathcal{W})$.

Since 1-simplices of the nerve $\mathrm{N}(\mathcal{C})$ are morphisms of $\mathcal{C}$, we have $\mathcal{W} \subseteq \mathrm{N}(\mathcal{C})_1$. By the Yoneda Lemma, for each $w \in \mathcal{W}$, there is a simplicial map $\Delta[1] \to \mathrm{N}(\mathcal{C})$, picking out w. This gives a map

$$\coprod_{w \in \mathcal{W}} \Delta[1] \to \mathrm{N}(\mathcal{C}).$$

Taking the pushout of this map along the $\mathcal{W}$-indexed coproduct of inclusions $\Delta[1] \hookrightarrow \mathcal{I}$, we obtain:

$$\begin{tikzcd} \coprod_{w \in \mathcal{W}} \Delta[1] \arrow[r] \arrow[d, hook] & \mathrm{N}(\mathcal{C}) \arrow[d] \\ \coprod_{w \in \mathcal{W}} \mathcal{I} \arrow[r] & \mathrm{N}(\mathcal{C})[\mathcal{W}^{-1}] \end{tikzcd}$$

and we define $\mathrm{L}(\mathcal{C}, \mathcal{W})$ as the fibrant replacement (in Joyal model structure) of $\mathrm{N}(\mathcal{C})[\mathcal{W}^{-1}]$.

Proposition 3.1.46 (cf. [HS98a, Prop. 8.7])**.** *The functor* $\mathrm{L}\colon \mathrm{h}\mathcal{C}\mathrm{at} \to \mathrm{q}\mathcal{C}\mathrm{at}$ *is a Dwyer-Kan equivalence, and for any homotopical category* $\mathcal{C}$*, there is an equivalence:*

$$\mathrm{L}(\mathcal{C}) \simeq \mathrm{N}_{\Delta}(\mathrm{L}^{\mathrm{H}}(\mathcal{C})'),$$

where $\mathrm{L}^{\mathrm{H}}(\mathcal{C})'$ *denotes the fibrant replacement of* $\mathrm{L}^{\mathrm{H}}(\mathcal{C})$ *in Bergner's model structure on* $\mathrm{s}\mathcal{C}\mathrm{at}$ *(Paragraph 3.1.27).*

3.1.4 Complete Segal spaces

3.1.47. The last model for the theory of $(\infty, 1)$-categories that we will consider is the complete Segal spaces. While working inside a complete Segal space may not be as convenient as working inside a quasicategory, a particular model category structure on the category $\mathrm{ss}\mathcal{S}\mathrm{et}$ of bisimplicial sets is much better behaved.

3.1.48. Recall that a **simplicial model category** is a model category $\mathcal{M}$ together with a tensor $\otimes\colon \mathrm{s}\mathcal{S}\mathrm{et} \times \mathcal{M} \to \mathcal{M}$, which is required to be a left Quillen bifunctor (with respect to Quillen model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$). A canonical example of such a simplicial model structure is Quillen model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$. As simplicial model categories are in many respects better behaved, we may ask whether the Joyal model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$ is also simplicial. (Here, we wish to consider the natural tensor $\mathrm{s}\mathcal{S}\mathrm{et} \times \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ given by the cartesian product.) We shall see that this is not the case.

Let us write $\mathrm{s}\mathcal{S}\mathrm{et}_Q$ for the category of simplicial sets with Quillen model structure and $\mathrm{s}\mathcal{S}\mathrm{et}_J$ for the same category, but with Joyal model structure. In order for a model category to be simplicial, it would have to satisfy the pushout-product axiom, asserting that for any pair of cofibrations ($i \in \mathrm{s}\mathcal{S}\mathrm{et}_Q, j \in \mathrm{s}\mathcal{S}\mathrm{et}_J$), their pushout-product $i \widehat{\otimes} j$ is again a cofibration, and is acyclic if either i or j is. However, we have the following:

$$\begin{array}{c} \Lambda^0[2] \\ \big\downarrow \\ \Delta[2] \end{array} \;\widehat{\otimes}\; \begin{array}{c} \emptyset \\ \big\downarrow \\ \Delta[0] \end{array} \;=\; \begin{array}{c} \Lambda^0[2] \\ \big\downarrow \\ \Delta[2] \end{array}$$

but since $\Lambda^0[2] \hookrightarrow \Delta[2]$ is an acyclic cofibration only in Quillen model structure, $\mathrm{s}\mathcal{S}\mathrm{et}_J$ fails to satisfy the pushout-product axiom, and hence is not a simplicial model category.

3.1.49. A remedy for this problem is the Rezk model structure on the category of bisimplicial sets, that is on the functor category $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ or equivalently $\Delta^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}$. It is Quillen equivalent to the Joyal model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$, but enjoys several categorical properties (such as being simplicial) that Joyal model structure lacks.

In this section, we will briefly review the basic properties of complete Segal spaces as they will play a crucial role later in the proof. Our notation and terminology follows [Rez01] for the first part and [JT07] for the second.

3.1.50. In our review, we will make use of the notion of Reedy fibrations and fibrant objects for diagrams in model categories. The reader who is unfamiliar with these notions may want to consult either [Hov99, Ch. 5] or Section 5.2 of the present thesis, where we discuss Reedy fibrations in a more general framework that subsumes the model-categorical case.

Definition 3.1.51. A **Segal space** is a Reedy fibrant functor $W\colon \Delta^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}$ such that for each $n \in \mathbb{N}$, the canonical map:

$$W_n \to \underbrace{W_1 \times_{W_0} W_1 \times_{W_0} \ldots \times_{W_0} W_1}_{n \text{ times}},$$

induced by n distance-preserving inclusions $[1] \hookrightarrow [n]$, is a weak equivalence.

3.1.52. A Segal space $W\colon \Delta^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}$ gives a functor $W^{\mathrm{op}}\colon \Delta \to \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}}$. Let $\widetilde{W}$ denote the left Kan extension of W^{op} along the Yoneda embedding:

$$\begin{tikzcd}
\Delta \arrow[r, "W^{\mathrm{op}}"] \arrow[d, hook, "\mathbf{y}"'] & \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}} \\
\mathrm{s}\mathcal{S}\mathrm{et} \arrow[ur, dotted, bend right, "\widetilde{W} := \mathrm{Lan}_{\mathbf{y}} W^{\mathrm{op}}"']
\end{tikzcd}$$

Application of $\widetilde{W}$ to the unique map $\mathcal{I} \to \Delta[0]$ yields a map of simplicial sets:

$$W_0 = \widetilde{W}(\Delta[0]) \to \widetilde{W}(\mathcal{I}) =: \mathrm{hoequiv}(W),$$

where the equality $W_0 = \widetilde{W}(\Delta[0])$ follows by the Yoneda Lemma.

Definition 3.1.53. A Segal space $W\colon \Delta^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is **complete** if the canonical map $W_0 \to \mathrm{hoequiv}(W)$ is a weak equivalence.

3.1.54 (cf. [Rez01, Thm. 7.2])**.** There is a model structure on the category $\mathrm{ss}\mathcal{S}\mathrm{et}$ of bisimplicial sets, that we will refer to as the **Rezk model structure**, in which:

- cofibrations are the monomorphisms;
- fibrant objects are complete Segal spaces;

- weak equivalences are morphisms $s\colon X \to Y$ such that for any complete Segal space W the induced map

$$W^s\colon W^Y \to W^X$$

is a weak equivalence of simplicial sets, where W^X denotes the simplicial set given by $(W^X)_n = W_n^{X_n}$.

3.1.55. Having described complete Segal spaces, we will now relate them to all three preceding models.

Theorem 3.1.56 ([JT07, Thm. 4.11])**.** *The functor $(-)_0\colon \mathrm{ss\mathcal{S}et} \to \mathrm{s\mathcal{S}et}$ taking a bisimplicial set $(X_{m,n})$ to the simplicial set $(X_{m,0})$ is a right Quillen functor, and part of a Quillen equivalence.*

3.1.57. A key lemma in the proof of the above theorem is the following result that establishes an important property of the maximal subgroupoid functor J. We mention it here as we shall need it in one of our later proofs.

Proposition 3.1.58 ([JT07, Thm. 4.10])**.** *Let $\mathcal{C} \in \mathrm{q\mathcal{C}at}$. Then the bisimplicial set $\mathrm{J}(\mathcal{C}^{\Delta[-]})$ is a complete Segal space.*

3.1.59. Let $\mathcal{C}$ be a simplicial category. In light of the discussion of Paragraph 3.1.15, we can view it as a simplicial object $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ with a constant set of objects. We define the bisimplicial set $\mathrm{L_C}(\mathcal{C})$ by setting $\mathrm{L_C}(\mathcal{C})_n = \mathrm{N}(\mathcal{C}_n)$. After possibly taking Reedy fibrant replacement, this functor gives an DK-equivalence $\mathrm{s\mathcal{C}at} \to \mathrm{ss\mathcal{S}et}$, where $\mathrm{s\mathcal{C}at}$ is considered with Bergner model structure and $\mathrm{ss\mathcal{S}et}$ with Rezk model structure (cf. [Ber09, Sec. 6])

3.1.60. Let $(\mathcal{C}, \mathcal{W})$ be a homotopical category and let $[m] = \{0 \leq 1 \leq \ldots \leq m\}$ be the ordinal regarded as a category. The functor category $\mathcal{C}^{[m]}$ is again a homotopical category where weak equivalences are natural weak equivalences. We will denote it by $(\mathcal{C}^{[m]}, \mathcal{W}_{\mathcal{C}^{[m]}})$.

The **classification diagram** $\mathcal{N}(\mathcal{C}, \mathcal{W})$ of a homotopical category $(\mathcal{C}, \mathcal{W})$ is a bisimplicial set $\Delta^{\mathrm{op}} \to \mathrm{s\mathcal{S}et}$, whose n-simplices are given by:

$$\mathcal{N}(\mathcal{C}, \mathcal{W})_n = \mathrm{N}(\mathcal{W}_{\mathcal{C}^{[m]}}).$$

3.1.61. In particular, we have:

- $\mathcal{N}(\mathcal{C},\mathcal{W})_0 = \mathrm{N}\mathcal{W}$;
- $\mathcal{N}(\mathcal{C},\mathcal{W})_1 = \mathrm{N}\left(\text{diagrams } \begin{tikzcd} \bullet \arrow[r] \arrow[d, "\sim"'] & \bullet \arrow[d, "\sim"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \text{ in } \mathcal{C}\right)$
- $\mathcal{N}(\mathcal{C},\mathcal{W})_1 = \mathrm{N}\left(\text{diagrams } \begin{tikzcd} \bullet \arrow[r] \arrow[d, "\sim"'] & \bullet \arrow[r] \arrow[d, "\sim"'] & \bullet \arrow[d, "\sim"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \text{ in } \mathcal{C}\right)$

3.1.62 (cf. [BK12b, Sec. 1])**.** Finally, we shall note that the functor $\mathcal{N}\colon \mathrm{h}\mathcal{C}\mathrm{at} \to \mathrm{ss}\mathcal{S}\mathrm{et}$ is a Dwyer-Kan equivalence and it is equivalent to a Quillen functor.

3.1.63. The category $\mathcal{C}\mathrm{at}$ is a full subcategory of the category $\mathrm{h}\mathcal{C}\mathrm{at}$ of homotopical categories, whose weak equivalences are exactly isomorphisms. The restriction $\mathcal{N}|\mathcal{C}\mathrm{at}\colon \mathcal{C}\mathrm{at} \hookrightarrow \mathrm{h}\mathcal{C}\mathrm{at} \to \mathrm{ss}\mathcal{S}\mathrm{et}$ is easily seen to be fully faithful. The Reedy fibrant replacement of $\mathcal{N}\mathcal{C}$ is how the category $\mathcal{C}$ was encoded in the formalization of category theory in the Univalent Foundations [AKS13] after interpretation of type theory in the simplicial mode of [KLV12].

3.2 COMPARISON PROBLEM

3.2.1. In the previous section, we defined several models of $(\infty,1)$-categories: homotopical categories, simplicial categories, quasicategories, and complete Segal spaces. Other models, that were not discussed, are topological categories (categories enriched over $\mathcal{T}\mathrm{op}$) and Segal categories (Segal spaces, in which W_0 is discrete). We have also defined functors between some of these categories. More precisely, the picture is:

$$\begin{tikzcd} \mathrm{h}\mathcal{C}\mathrm{at} \arrow[r, "\mathrm{L^H}"] \arrow[d, "\mathcal{N}"'] \arrow[dr, "\mathrm{L}"] & \mathrm{s}\mathcal{C}\mathrm{at} \arrow[d, "\mathrm{N}_\Delta"] \arrow[dl, dashed, "\mathrm{L_C}"] \\ \mathrm{ss}\mathcal{S}\mathrm{et} \arrow[r, "(-)_0"'] & \mathrm{s}\mathcal{S}\mathrm{et} \end{tikzcd}$$

3.2.2. As we have verified, all of these maps are equivalences of *homotopy theories*, i.e. Quillen equivalences between model categories or DK-equivalences of homotopical categories. Joyal's Conjecture will be a statement asserting something about the category $\mathrm{L}(\mathcal{C}\ell(\mathbb{T}))$, the standard localization (Paragraph 3.1.45) of the classifying category of type theory, regarded as a homotopical category. We will be able to show the corresponding result about the classification diagram (or more precisely, the functor $(-)_0$ applied to its fibrant replacement). Thus not only is it important to be able to compare these models of $(\infty, 1)$-categories, but it is also crucial to relate different localizations.

3.2.3. Toën showed [Toë05, Thm. 6.3] that the homotopy theory of homotopy theories (that is, the homotopical category of homotopical categories with DK-equivalences) admits, up to equivalence, only two autoequivalences: identity and the operation of taking the opposite category. All of the localizations in the diagram of Paragraph 3.2.1 therefore induce automorphisms of this homotopy theory. But examining them on the full subcategory of $\mathrm{h}\mathcal{C}\mathrm{at}$ consisting of the terminal category and the category $[1]$ (with two objects and one non-identity arrow), we can tell whether they are equivalent to the identity or to the opposite category functor.

- the simplicial category corresponding via L^{H} to $[1] \in \mathrm{h}\mathcal{C}\mathrm{at}$, will consist of two objects $\{0, 1\}$ and the only non-trivial mapping space will be $\mathcal{M}\mathrm{ap}(0, 1) = \Delta[0]$; the other ones will contain either only identities or be empty;
- the corresponding quasicategory will be $\Delta[1]$;
- the corresponding bisimplicial set will consist of $\partial\Delta[1]$ in dimension 0, $\Delta[1]$ in dimension 1; and higher simplicial sets will be degenerate (in the sense of diagram of simplicial sets).

3.2.4. Given the above descriptions, it is immediate to see that the functors in the diagram of Paragraph 3.2.1 preserve the orientation of the arrow in the aforementioned objects.[1] Hence, for any homotopical category $\mathcal{C}$ and any two composites of functors in the diagram of Paragraph 3.2.1 applied to it, the resulting objects will be weakly equivalent. In particular, we obtain the following theorem:

Theorem 3.2.5. *For every homotopical category* $(\mathcal{C}, \mathcal{W})$*, the quasicategories* $\mathrm{L}(\mathcal{C}, \mathcal{W})$ *and* $(\mathcal{N}(\mathcal{C}, \mathcal{W})')_0$ *are weakly equivalent in the Joyal model structure on* $\mathrm{s}\mathcal{S}\mathrm{et}$*. (Here,* $\mathcal{N}(\mathcal{C}, \mathcal{W})'$ *denotes the fibrant replacement of* $\mathcal{N}(\mathcal{C}, \mathcal{W})$ *in the Rezk model structure on* $\mathrm{ss}\mathcal{S}\mathrm{et}$*.)*

[1] I learned this argument from Chris Schommer–Pries on MathOverflow `http://mathoverflow.net/questions/92916/does-the-classification-diagram-localize-a-category-with-weak-equivalences`.

3.3 THEORY OF QUASICATEGORIES

3.3.1. As we mentioned before, quasicategories give an extremely nice model of $(\infty, 1)$-categories for the development of the theory. The profound work of Joyal [Joy09] and Lurie [Lur09a, Lur12] lifted the existing category theory to higher category theory. In this section, we will review the basics of the theory of quasicategories. Before doing that, let us recall the following contractibility criterion.

3.3.2 (Contractibility criterion)**.** Let $X \in \mathrm{s}\mathcal{S}\mathrm{et}$. Since the monomorphisms of simplicial sets are generated under (possibly transfinite) composition and pushouts from the boundary inclusions, we see immediately that X is contractible if and only if for all $n \geq 0$ and all maps $\partial\Delta[n] \to X$, there exists an extension:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

Definition 3.3.3. Let $\mathcal{C}$ be a quasicategory and $x, y \in \mathcal{C}$. The **mapping space** $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)$ (or $\mathcal{C}(x, y)$) is a simplicial set defined as a pullback:

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta[1]} \arrow[d] \\ \Delta[0] \arrow[r, "{(x,y)}"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Proposition 3.3.4 ([Lur09a, Cor. 4.2.1.8])**.** *For any* 0*-simplices* x *and* y *in a quasicategory* $\mathcal{C}$*, the mapping space* $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)$ *is a Kan complex.*

3.3.5. A quasicategory $\mathcal{C}$ is **locally small** if it is weakly equivalent to a small simplicial set.

3.3.6 (Join of categories)**.** The first notion that we wish to generalize to quasicategories is the notion of a join. Recall that given two categories $\mathcal{C}$ and $\mathcal{D}$, their $\mathcal{C} \star \mathcal{D}$ is defined as a category whose objects are $\operatorname{Ob}\mathcal{C} \amalg \operatorname{Ob}\mathcal{D}$ and whose hom-sets are given by:

$$\operatorname{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y) = \begin{cases} \operatorname{Hom}_{\mathcal{C}}(X,Y) & \text{if } X, Y \in \mathcal{C} \\ \operatorname{Hom}_{\mathcal{D}}(X,Y) & \text{if } X, Y \in \mathcal{D} \\ \{*\} & \text{if } X \in \mathcal{C}, Y \in \mathcal{D} \\ \emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C} \end{cases}$$

As we shall see later, a big part of categorical notions can be defined in terms of joins. Thus, since a generalization of joins from categories to simplicial sets is rather straightforward, this is where we will begin.

Definition 3.3.7. Let $K, L \in \mathrm{s}\mathcal{S}\mathrm{et}$. Define the **join** of K and L by:

$$(K \star L)_n = \coprod_{\substack{i,j \geq -1 \\ i+j=n-1}} K_i \times L_j$$

where we assume that $K_{-1} = L_{-1} = \{*\}$.

Examples 3.3.8.

- The join of the representables is given by $\Delta[n] \star \Delta[m] = \Delta[m+n+1]$.
- The unit of the join operation is given by the empty simplicial set, i.e. $K \star \emptyset = K$.
- The join of nerves of categories reduces to the nerve of their join as categories (see Paragraph 3.3.6), i.e. $\mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D}) \cong \mathrm{N}(\mathcal{C} \star \mathcal{D})$ [Joy09, Cor. 3.3].

3.3.9. Given $K, L \in \mathrm{s}\mathcal{S}\mathrm{et}$, there is a canonical morphism $K \to K \star L$ given by $x \mapsto (x, *)$. Thus the operation of taking the join with a fixed simplicial set K defines a functor

$$K \star -\colon \mathrm{s}\mathcal{S}\mathrm{et} \to K/\mathrm{s}\mathcal{S}\mathrm{et}$$

to the coslice category of simplicial sets.

This functor admits a right adjoint $K/\mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ taking a simplicial morphism $X\colon K \to L$ to the simplicial set $X \downarrow L$ defined as:

$$(X \downarrow L)_n = \left\{ Y\colon K \star \Delta[n] \to L \;\middle|\; Y|K = X \right\}.$$

3.3.10 (Coslice of quasicategory)**.** Let $\mathcal{C}$ be a quasicategory and $x \in \mathcal{C}$ be a 0-simplex. We may view x as a map $\Delta[0] \to \mathcal{C}$. Instantiating the construction $X \downarrow \mathcal{C}$ of Paragraph 3.3.9 with $X := x \colon \Delta[0] \to \mathcal{C}$, we obtain the **coslice** quasicategory $\mathcal{C}/x$ as:

$$(x/\mathcal{C})_n = \left\{ X \colon \Delta[n+1] \to \mathcal{C} \;\middle|\; X|\Delta^{\{0\}} = x \right\}.$$

3.3.11. Dually, given $K, L \in \mathrm{s}\mathcal{S}\mathrm{et}$, there is also a canonical morphism $L \to K \star L$ given by $y \mapsto (*, y)$. We can therefore form another adjunction:

$$- \star L \colon \mathrm{s}\mathcal{S}\mathrm{et} \to L/\mathrm{s}\mathcal{S}\mathrm{et}$$

to the coslice category of simplicial sets.

This functor admits a right adjoint $L/\mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ taking a simplicial morphism $X \colon L \to K$ to the simplicial set $L \downarrow X$ defined as:

$$(L \downarrow X)_n = \left\{ Y \colon \Delta[n] \star L \to K \;\middle|\; Y|L = X \right\}.$$

Again, instantiating it with the inclusion of a 0-simplex, we obtain the notion of a **slice quasicategory**

$$(\mathcal{C}/x)_n = \left\{ X \colon \Delta[n+1] \to \mathcal{C} \;\middle|\; X|\Delta^{\{n+1\}} = x \right\}.$$

3.3.12. Let $\mathcal{C}$ be a quasicategory and let $x \in \mathcal{C}$ be a 0-simplex. There is an obvious projection map:

$$\mathcal{C}/x \longrightarrow \mathcal{C}$$

taking an n-simplex $X \colon \Delta[n+1] \to \mathcal{C}$ of $\mathcal{C}/x$ to its restriction $X|\Delta^{\{0,1,\ldots,n\}}$.

Proposition 3.3.13 ([Joy09, Thm. 3.19])**.** *The projection map $\mathcal{C}/x \to \mathcal{C}$ has the right lifting property (see [Hov99, Def. 1.1.2]) with respect to the horn inclusions $\Lambda^i[n] \hookrightarrow \Delta[n]$ for $0 < i \leq n$. In particular, the slice of a quasicategory is again a quasicategory.*

3.3.14. Given a functor $X\colon J \to \mathcal{C}$ of 1-categories, one defines a cone over X as a object $\widetilde{X}$ (vertex of the cone) together with morphisms $\pi_i\colon \widetilde{X} \to X(i)$ such that for all $\alpha\colon i \to j$ in J, we have $\pi_j = X(\alpha)\cdot \pi_i$.

Alternatively, we can define a cone over X as an extension $\widetilde{X}\colon \{*\} \star J \to \mathcal{C}$, where the vertex of the cone is now the value $\widetilde{X}(*)$. We will see that definition will generalize easily to higher-categorical setting. To simplify the notation later on, we will write $K^{\triangleleft} = \Delta[0] \star K$, where K is any simplicial set.

Definition 3.3.15.

- Let $\mathcal{C}$ be a quasicategory and $X\colon K \to \mathcal{C}$ a map of simplicial sets. A **cone** over X is a simplicial map $Y\colon K^{\triangleleft} \to \mathcal{C}$ such that $Y|K = X$ i.e.

$$\begin{tikzcd} K \arrow[r, "X"] \arrow[d, hook] & \mathcal{C} \\ K^{\triangleleft} \arrow[ur, dotted, "Y"'] & \end{tikzcd}$$

- A cone $\widetilde{X}\colon K^{\triangleleft} \to \mathcal{C}$ is **universal** (or a **limit**) if for all $n > 0$ and all $Z\colon \partial\Delta[n] \star K \to \mathcal{C}$ such that $Z|K^{\triangleleft} = \widetilde{X}$, there exists an extension:

$$\begin{tikzcd} \partial\Delta[n] \star K \arrow[r, "Z"] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \star K \arrow[ur, dotted] & \end{tikzcd}$$

3.3.16. Using the adjunction of Paragraph 3.3.11, we can alternatively describe limits using slices. More precisely, a cone over X corresponds to a diagram $Y\colon \Delta[0] \to \mathcal{C} \downarrow X$. Such a cone $\widetilde{X}\colon \Delta[0] \to \mathcal{C} \downarrow X$ is universal if for all $n > 0$ and all $Z\colon \partial\Delta[n] \to \mathcal{C} \downarrow X$ such that $Z|\Delta^{\{n\}} = \widetilde{X}$, there exists an extension:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "Z"] \arrow[d, hook] & \mathcal{C} \downarrow X \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

Example 3.3.17. The **terminal object** 1 in a quasicategory $\mathcal{C}$ is the limit of $\emptyset \to \mathcal{C}$. Combining the discussion of Paragraph 3.3.16 with the Contractibility Criterion 3.3.2, we see that for any 0-simplex $x \in \mathcal{C}$, the mapping space $\mathcal{M}\mathrm{ap}(x, 1)$ is a contractible Kan complex.

Example 3.3.18. Let $\mathcal{C}$ be a quasicategory and $x, y \in \mathcal{C}$ be 0-simplices. The **(binary) product** $x \times y$ in $\mathcal{C}$ is given as a limit of the functor $[x, y]\colon \partial\Delta[1] \to \mathcal{C}$.

Example 3.3.19. A **pullback** in a quasicategory $\mathcal{C}$ is given as the limit of the diagram $\Lambda^2[2] \to \mathcal{C}$.

3.3.20 (Colimits)**.** All the notions above admit dual versions. One defines a **cone under** $X\colon K \to \mathcal{C}$ as an extension $Y\colon K^{\triangleright} \to \mathcal{C}$, where $K^{\triangleright} := K \star \Delta[0]$, with $Y|K = X$. A cone $\widetilde{X}$ under X is **universal** if for all $n > 0$ and all $Z\colon K \star \partial\Delta[n] \to \mathcal{C}$ such that $Z|(K \star \Delta^{\{0\}})$, there exists an extension $\widetilde{Z}\colon K \star \Delta[n] \to \mathcal{C}$.

Since join is not a commutative operation, we do not have $K \star \Delta[0] \cong \Delta[0] \star K$, and hence $K^{\triangleleft} \ncong K^{\triangleright}$. (Notice that the triangles are facing opposite directions.)

3.3.21 (Uniqueness of limits)**.** In (1-)category theory one proves that (co)limits, if they exist, are unique up to a unique isomorphism. A similar statement is also true, although appropriately refined, for quasicategories. Let $X\colon K \to \mathcal{C}$ be a diagram that admits a universal cone $\widetilde{X}$. Then the simplicial subset of such universal cones forms a contractible Kan complex. This follows easily by the Contractibility Criterion 3.3.2 and the equivalent description of limits of Paragraph 3.3.16.

3.3.22. We will call a simplicial set **finite** if it has only finitely many non-degenerate cells. Thus, for example, standard n-simplices are finite, but the nerve of any non-trivial group is not.

We say that a quasicategory $\mathcal{C}$ has **finite limits** if for all finite simplicial sets K and all functors $K \to \mathcal{C}$, there exists a universal cone.

A simplicial map (or a functor of quasicategories) $F\colon \mathcal{C} \to \mathcal{D}$ is said to **preserve (finite) limits**, if F takes universal cones (of finite diagrams) to universal cones.

3.3.23. We next turn towards adjunctions between quasicategories. The first definition that we will give is perhaps the best-known one as it has a great advantage of being concise. It is however difficult to work with in practice. Thus following Definition 3.3.24, in Proposition 3.3.26 we will give an equivalent characterization. Since in the subsequent chapters, we will work only with the characterization and not Definition 3.3.24, we can treat this definition as a black box. In particular, the definition uses the notions of cartesian fibration and opfibration that we have not introduced, but that can be found in [Lur09a, Ch. 2].

Definition 3.3.24. Let $\mathcal{C}$, $\mathcal{D}$ be quasicategories. An **adjunction** between $\mathcal{C}$ and $\mathcal{D}$ consists of a morphism $p\colon \mathcal{M} \to \Delta[1]$ of simplicial sets, which is both a cartesian fibration and cartesian opfibration, together with equivalences $\mathcal{C} \to p^{-1}(0)$ and $\mathcal{D} \to p^{-1}(1)$.

3.3.25. Extracting from it the simplicial maps forming the adjunction involves using the lifting properties of a cartesian (op)fibration. Thus we would like to have an explicit and easily verifiable criterion whether a simplicial map $F\colon \mathcal{C} \to \mathcal{D}$ between quasicategories admits a left or right adjoint.

Proposition 3.3.26 (cf. [RV13, Obs. 4.3.8])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories and $G\colon \mathcal{D} \to \mathcal{C}$ be a simplicial map. Then G has a left adjoint if and only if for each $x \in \mathcal{C}$ the slice category $(x \downarrow G)$ has an initial object.*

3.3.27. Notice that Proposition 3.3.26 is exactly the most brute force generalization of the standard 1-categorical characterization of adjunction to the world of higher category theory. Indeed, in category theory, we say that $F \dashv G$ if there is a natural transformation $\eta\colon 1 \to G \cdot F$ such that for any $X \in \mathcal{C}$ and any map $f\colon X \to GA \in \mathcal{C}$, there exists a unique map $\overline{f}\colon FX \to A \in \mathcal{D}$, making the following triangle commute:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & GFX \arrow[d, dotted, "G(\overline{f})"] \\ & GA \end{tikzcd}$$

3.3.28. Let J be a small category and $\mathcal{C}$ any category. Suppose that $\mathcal{C}$ has all limits of shape J; that is, for any $X\colon J \to \mathcal{C}$, the limit of X exists. This yields a well-defined functor $\lim\colon \mathcal{C}^J \to \mathcal{C}$ taking a diagram to its limit. We would like to establish a similar result for quasicategories.

3.3.29. Now, let K be a simplicial set and $\mathcal{C}$ a quasicategory; and suppose that for any $X\colon K \to \mathcal{C}$, the limit of X exists. Since the limits in quasicategories are unique only up to a contractible ambiguity (more precisely, as explained in Paragraph 3.3.21, the subquasicategory of universal cones over X is a contractible Kan complex), an arbitrary choice of limits may not lead to a simplicial map $\mathcal{C}^K \to \mathcal{C}$.

This issue can be resolved using [Lur09a, Prop. 4.2.2.7]. Suppose that each diagram $X\colon K \to \mathcal{C}$ admits a limit. Then applying Proposition 4.2.2.7 to the obvious maps:

$$K \times \mathcal{C}^K \longrightarrow \mathcal{C} \times \mathcal{C}^K \longrightarrow \mathcal{C}^K$$

we obtain a map $\mathcal{C}^K \to \mathcal{C}^{K^\triangleleft}$ assigning to each X a universal cone over X.

3.3.30. The last notion discussed in this section will be an extremely useful criterion for verifying that a map is a categorical equivalence. It is based on the following standard fact from classical algebraic topology. A map $f\colon X \to Y$ of CW-complexes is a weak equivalence if and only if for every $n \in \mathbb{N}$ and every commutative square:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ D^{n+1} \arrow[r, "e"] & Y \end{tikzcd}$$

there exists a map $\overline{e}\colon D^{n+1} \to X$ such that in the diagram:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ D^{n+1} \arrow[r, "e"] \arrow[ur, dotted, "\overline{e}"] & Y \end{tikzcd}$$

the upper triangle commutes and the lower triangle commutes up to homotopy relative to the boundary inclusion (that is, $f \cdot \overline{e} \sim e$ rel S^n).

Thus in order to translate the above result to quasicategories, we need to introduce an appropriate notion of homotopy between quasicategories.

3.3.31. Let $f, g\colon K \to L$ be simplicial maps.

- An $E[1]$**-homotopy** from f to g is a map $H\colon K \times E[1] \to L$ such that

$$H|K \times \{0\} = f \text{ and } H|K \times \{1\} = g.$$

- Suppose $K' \hookrightarrow K$. An $E[1]$-homotopy H from f to g is **relative** to K', if the composite

$$K' \times E[1] \hookrightarrow K \times E[1] \xrightarrow{H} L$$

  factors through the projection $K' \times E[1] \to K'$.

Lemma 3.3.32 ([Szu14, Lem. 4.5])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a map of quasicategories such that for any $n \in \mathbb{N}$ and any square of the form:*

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\ \Delta[n] \arrow[r] & \mathcal{D} \end{tikzcd}$$

there exists a diagonal filler

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \mathcal{D} \end{tikzcd}$$

making the upper triangle commute and the lower triangle commute up to $E[1]$-homotopy relative to $\partial\Delta[n]$. Then F is a categorical equivalence.

4.0 LOCALLY CARTESIAN CLOSED QUASICATEGORIES AND JOYAL'S CONJECTURE

This chapter will combine large parts of the previous two chapters. Our goal now is to formulate Joyal's Conjecture, explain its importance, and outline our proof strategy. To this end, we first introduce and study in some detail the notion of a locally cartesian closed quasicategory (Section 4.1) and afterwards explain the importance of locally cartesian closed quasicategories in Homotopy Type Theory (Section 4.2). Finally, in Section 4.3, we formulate the conjecture.

4.1 LOCALLY CARTESIAN CLOSED QUASICATEGORIES

Definition 4.1.1. A quasicategory $\mathcal{C}$ is

- **cartesian closed** if it has finite products and for all $x \in \mathcal{C}$, the product functor

  $$x \times - : \mathcal{C} \to \mathcal{C}$$

  has a right adjoint.
- **locally cartesian closed** if it has a terminal object and for every 0-simplex $x \in \mathcal{C}$, the slice quasicategory $\mathcal{C}/x$ is cartesian closed.

Example 4.1.2. If $\mathcal{C}$ is a (1-)category, then $\mathcal{C}$ is (locally) cartesian closed in 1-categorical sense if and only if its nerve $\mathrm{N}\mathcal{C}$ is (locally) cartesian closed in the quasicategorical sense.

4.1.3. There seems to be a general disagreement in the literature about whether or not to include the existence of a terminal object in the definition of a locally cartesian closed category. There is, to the best of the author's knowledge, only one natural example of a category with no terminal object whose slices are cartesian closed, namely the category of topological spaces with local homeomorphisms.

4.1.4. Before we can proceed with our investigation of locally cartesian closed quasicategories, let us recall a basic fact from model category theory. Suppose that $p\colon B \to A$ is an acyclic fibration in a model category $\mathcal{M}$ whose codomain is cofibrant. Then p admits a section. Indeed, since the unique map $0 \to A$ is a cofibration, by lifting:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

we obtain the desired section. In particular, every acyclic fibration in either Quillen's or Joyal's model structure on $\mathcal{sSet}$ admits a section since all objects there are cofibrant.

4.1.5. As in the case of 1-categories, we have the following characterization of locally cartesian closed quasicategories.

Proposition 4.1.6. *A quasicategory $\mathcal{C}$ is locally cartesian closed if and only if for any 1-simplex $f\colon x \to y \in \mathcal{C}$, the pullback functor $f^*\colon \mathcal{C}/y \to \mathcal{C}/x$ has a right adjoint.*

4.1.7. The proof of Proposition 4.1.6 follows verbatim the proof of the very same statement from ordinary category theory [Awo10, Prop. 9.20] and will be given after Paragraph 4.1.10. However, the **if** direction in 1-category theory requires knowing that *a slice of a slice is again a slice*. Formally, one has that for any morphism $f\colon X \to Y$ in a category $\mathcal{C}$ there is an equivalence of categories:

$$(\mathcal{C}/Y)/f \simeq \mathcal{C}/X.$$

Lemma 4.1.8. *Let $\mathcal{C}$ be a quasicategory and $f\colon x \to y$ a 1-simplex in $\mathcal{C}$. Then the canonical projection $(\mathcal{C}/y)/f \to \mathcal{C}/x$ is an acyclic fibration, thus in particular a categorical equivalence.*

Proof. Since the boundary inclusions generate all monomorphisms, it suffices to show that for all $n \in \mathbb{N}$, every commutative square of the form:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & (\mathcal{C}/y)/f \arrow[d] \\ \Delta[n] \arrow[r] & \mathcal{C}/x \end{tikzcd}$$

admits a diagonal filler. By adjunction of Paragraph 3.3.11, this is equivalent to asking for a filler in the square:

$$\begin{tikzcd} \partial\Delta[n] \star \Delta[0] \arrow[r] \arrow[d, hook] & \mathcal{C}/y \arrow[d] \\ \Delta[n] \star \Delta[0] \arrow[r] & \mathcal{C} \end{tikzcd}$$

but since $\partial\Delta[n] \star \Delta[0] \cong \Lambda^{n+1}[n+1]$, this follows by Proposition 3.3.13. $\square$

4.1.9. In Paragraph 3.3.29, we showed that for any diagram K, one can always choose a functor $\mathcal{C}^K \to \mathcal{C}^{K^{\triangleleft}}$ assigning to each diagram $K \to \mathcal{C}$ a universal cone on it. In the case of a pullback we get a simplicial map $\mathcal{C}^{\Lambda^2[2]} \to \mathcal{C}^{\Delta[1]\times\Delta[1]}$, picking out universal cones. Proposition 4.1.6 ask however for a functor $\mathcal{C}/y \to \mathcal{C}/x$. Let $\mathcal{C}^{\Lambda^2[2]_f}$ and $\mathcal{C}^{\Delta[1]\times\Delta[1]_f}$ denote the subobjects of $\mathcal{C}^{\Lambda^2[2]}$ and $\mathcal{C}^{\Delta[1]\times\Delta[1]}$, respectively, spanned by diagrams in which bottom 1-simplex is mapped to f. The composite $\mathcal{C}^{\Lambda^2[2]_f} \hookrightarrow \mathcal{C}^{\Lambda^2[2]} \to \mathcal{C}^{\Delta[1]\times\Delta[1]}$ factors through $\mathcal{C}^{(\Delta[1]\times\Delta[1])_f}$:

$$\begin{tikzcd} \mathcal{C}^{\Lambda^2[2]_f} \arrow[r] \arrow[d, hook] & \mathcal{C}^{(\Delta[1]\times\Delta[1])_f} \arrow[d, hook] \\ \mathcal{C}^{\Lambda^2[2]} \arrow[r] & \mathcal{C}^{\Delta[1]\times\Delta[1]} \end{tikzcd}$$

The canonical morphism $\mathcal{C}^{\Lambda^2[2]_f} \to \mathcal{C}/y$ is an isomorphism of simplicial sets, hence admits an inverse. The desired pullback functor is then given by the composite:

$$\mathcal{C}/y \longrightarrow \mathcal{C}^{\Lambda^2[2]_f} \longrightarrow \mathcal{C}^{(\Delta[1]\times\Delta[1])_f} \longrightarrow \mathcal{C}/x$$

4.1.10. Also, similarly as in the 1-categorical case, the pullback functor $f^*\colon \mathcal{C}/y \to \mathcal{C}/x$ admits a left adjoint $f_!\colon \mathcal{C}/x \to \mathcal{C}/y$, given essentially by composition with f. Since composition in a quasicategory is defined only up to homotopy one has to *choose* the composites in order to define a simplicial map. This can be done by first choosing a section of the acyclic fibration $(\mathcal{C}/y)/f \to \mathcal{C}/x$ of Lemma 4.1.8, and composing it with the projection $(\mathcal{C}/y)/f \to \mathcal{C}/y$. (A slightly different description can also be found in [Lur09a, Prop. 6.1.1.1].)

Proof of Prop. 4.1.6. First, let us assume that all pullback functors $f^*\colon \mathcal{C}/y \to \mathcal{C}/x$ have right adjoints. We need to show that the product with a fixed object in every slice $\mathcal{C}/y$ admits a right adjoint. Let $f\colon x \to y$ be an object in $\mathcal{C}/y$. Unwinding the definitions, we see that the product functor in $\mathcal{C}/y$ is given by:

$$\mathcal{C}/y \xrightarrow{f^*} \mathcal{C}/x \xrightarrow{f_!} \mathcal{C}/y.$$

Thus its right adjoint is given by the composite:

$$\mathcal{C}/y \xleftarrow{f_*} \mathcal{C}/x \xleftarrow{f^*} \mathcal{C}/y.$$

The converse repeats verbatim the standard 1-category theoretic proof, using Lemma 4.1.8. □

4.1.11. The notion of a locally cartesian closed category plays an important role in higher topos theory. In the remainder of the section, we will try to gather a few important facts about locally presentable locally cartesian closed quasicategories. Some preliminary definitions are in order.

4.1.12 (Filtered categories)**.** Recall that a non-empty category J is called **filtered**, if it satisfies the following conditions:

1. for any pair of objects $j, j' \in J$, there exists an object $i \in J$ with maps $j \to i$ and $j' \to i$;
2. for any pair of parallel morphisms $f, g\colon j \to j'$, there exists a morphism $h\colon j' \to i$ such that $h \cdot f = h \cdot g$.

Equivalently, we may say that every diagram $K \to J$ admits a cocone. This equivalent formulation can easily be generalized to the case of quasicategories.

4.1.13. Let κ be a cardinal. A quasicategory K is **κ-filtered**, if for every κ-small S and every map $S \to K$ admits an extension:

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & K \\ S^{\triangleright} \arrow[ur, dotted] & \end{tikzcd}$$

The colimit of a functor $K \to \mathcal{C}$ is called **κ-filtered colimit**, if K is a κ-filtered quasicategory. An object $x \in \mathcal{C}$ is called **κ-compact**, if the functor $\mathcal{M}\mathrm{ap}(x, -) \colon \mathcal{C} \to \mathcal{K}\mathrm{an}$ commutes with κ-filtered colimits.

4.1.14. A quasicategory $\mathcal{C}$ is **κ-accessible**, if:

1. it is locally small;
2. has all κ-filtered colimits;
3. the subquasicategory of κ-compact objects is essentially small ;
4. every object of $\mathcal{C}$ is a κ-filtered colimit of κ-compact objects.

A quasicategory is **accessible**, if it is κ-accessible for some cardinal κ.

Definition 4.1.15. A quasicategory is **locally presentable** if it has all small colimits and is accessible.

4.1.16. The reader should be aware that what we call a locally presentable quasicategory is sometimes called just a **presentable** quasicategory, most notably by Lurie [Lur09a]. We choose the name 'locally presentable' as it is aligned with its 1-categorical part [AR94].

The importance of locally presentable quasicategories lies in the fact that the Adjoint Functor Theorem holds in them.

Theorem 4.1.17 (Adjoint Functor Theorem, [Lur09a, Cor. 5.5.2.9])**.** *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a morphism between locally presentable quasicategories. Then F has a right adjoint if and only if F preserves (small) colimits.*

4.1.18 (Universality of colimits)**.** Let $\mathcal{C}$ be a quasicategory. We say that colimits in $\mathcal{C}$ are **universal** if, for any diagram $X \colon K \to \mathcal{C}$ and any map $f \colon \overline{y} \to \mathrm{colim} X$, we have:

$$\mathrm{colim}(f^* X) = \overline{y}.$$

In other words, colimits in $\mathcal{C}$ are stable under pullback.

Theorem 4.1.19. *A locally presentable quasicategory $\mathcal{C}$ is locally cartesian closed if and only if the colimits in $\mathcal{C}$ are universal.*

Proof. If $\mathcal{C}$ is locally cartesian closed, then each pullback functor is a left adjoint, hence it preserves colimits. Conversely, if $\mathcal{C}$ is locally presentable, then so is each of its slices, and the result now follows by the universality of colimits. □

4.1.20. Notice that for the implication $\Rightarrow$, we did not have to use the presentability assumption.

4.1.21. In the remainder of this section we will give equivalent characterizations of locally presentable locally cartesian closed quasicategories. These presentations will be then used later to show that, in a suitable sense, locally presentable locally cartesian closed quasicategories are models of type theory.

I do not want to claim much originality here. In the remainder of this section, I gathered a few facts about locally presentable locally cartesian closed quasicategories that are available on-line: as comments on blogs or on wikis. The main theorem of the remainder of this section (Theorem 4.1.25) was—at least its hardest part—proven in the comments under Shulman's post *The mysterious nature of right properness* on The n-Category Café. This is why I give credit to Cisinski and Shulman, who were the most active participants of the discussion.

4.1.22. Recall that a model category is said to be **combinatorial** if it is locally presentable (as a category) and the model structure is cofibrantly generated. A **Cisinski model category** is a cofibrantly generated model structure on a (1-)topos $\mathcal{E}$, in which cofibrations are monomorphisms (cf. [Cis02] and [Cis06, Thm. 1.3.22]).

4.1.23. A **locally cartesian closed model category** is a model category $\mathcal{M}$ which—perhaps unsurprisingly—is locally cartesian closed, and in addition, for any fibration $p\colon B \to A$ between fibrant objects, the adjunction

$$\begin{tikzcd} \mathcal{M}/A \arrow[r, bend left, "p^*"] \arrow[r, phantom, "\perp"] & \mathcal{M}/B \arrow[l, bend left, "p_*"] \end{tikzcd}$$

is a Quillen adjunction between corresponding induced model structures on slice categories.

4.1.24. In Section 2.2, we discussed categorical models of type theory. Such models are 1-categories with certain additional structure. Nevertheless, we would like to talk about models of type theory in higher categories. In order to do so, we develop some more convenient presentations of locally cartesian closed quasicategories. That is to say, we will show that to each locally cartesian closed quasicategory $\mathcal{C}$, one may associate a homotopical or simplicial category that is taken to a quasicategory equivalent to $\mathcal{C}$ by the coherent nerve functor. Any such category is called a **presentation** of $\mathcal{C}$. Of course, we would like to establish rather highly structured presentations, so that we can grant ourselves that the presentation will be a model of type theory.

Theorem 4.1.25 (Cisinski–Shulman)**.** *The following conditions are equivalent for a quasicategory $\mathcal{C}$:*

1. *$\mathcal{C}$ is locally presentable locally cartesian closed quasicategory.*
2. *$\mathcal{C}$ admits a presentation as a right proper left Bousfield localization of the injective model structure on simplicial presheaves.*
3. *$\mathcal{C}$ admits a presentation as a right proper Cisinski model category.*
4. *$\mathcal{C}$ admits a presentation as combinatorial locally cartesian closed model category.*

Proof. **1 ⇒ 2.** This is exactly [GK12, Thm. 6.7].

2 ⇒ 3. This is immediate from the definitions.

3 ⇒ 4. Every topos is a locally cartesian closed category, so it suffices to verify that for each fibration $p\colon B \to A$, the adjunction $p^* \dashv p_*$ is Quillen. The pullback of a monomorphism is a monomorphism, which combined with right properness shows that p^* is a left Quillen functor, completing the proof.

4 ⇒ 1. The localization of a combinatorial model category is locally presentable by [Lur12, Prop. 1.3.3.9]. By [Lur12, Cor. 1.3.3.13], the functor induced on the localization by the pullback functor admits a right adjoint. □

4.2 LOCALLY CARTESIAN CLOSED CATEGORIES AND TYPE THEORY

4.2.1. We will now briefly discuss the importance of locally cartesian closed categories in establishing models of type theory.

Theorem 4.2.2 (Shulman)**.** *Every locally presentable locally cartesian closed quasicategory admits a presentation as a model of type theory.*

Proof. By 4.1.25, every such quasicategory admits a presentation as a right proper left Bousfield localization of the injective model structure on simplicial presheaves and the indexing category can always be chosen to be inverse. The category of simplicial sets admits a model of type theory by [KLV12, Cor. 2.3.5]; and [Shu14, Thm. 11.11] shows how to lift it to simplicial presheaves over an arbitrary inverse category. □

4.2.3. One may therefore ask whether all models of type theory are presentations of locally presentable locally cartesian closed quasicategories. By initiality of $\mathcal{C}\ell(\mathbb{T})$ that would mean that a localization of $\mathcal{C}\ell(\mathbb{T})$ would have to be a locally presentable locally cartesian closed quasicategory. Part of that is not true since $\mathcal{C}\ell(\mathbb{T})$ has only those homotopy colimits that can be defined internally, so it is unlikely that the localization of $\mathcal{C}\ell(\mathbb{T})$ will have arbitrary small colimits as required in the definition of a locally presentable quasicategory. It does make sense, however, to ask whether the localization of $\mathcal{C}\ell(\mathbb{T})$ is a locally cartesian closed quasicategory. This is precisely the statement of Joyal's Conjecture and a positive answer to this question is the content of the main theorem of this thesis.

4.2.4. Before stating Joyal's Conjecture formally, let us briefly mention a further possible extension that we will not address here. It is a deep and interesting connection with higher topos theory. The subject of higher topos theory has been developed by multiple authors [Lur09a, Rez05, TV05] and has found applications in areas such as algebraic topology and algebraic geometry. It is a belief of many, including the author of this thesis, that understanding the connection between higher topos theory and type theory will benefit both areas and allow them to interact in a similar way that 1-topos theory leads to a fruitful interaction between the logic and geometry (see, for instance, [MLM94] or [Joh02b, Joh02a]).

4.2.5. The addition to type theory of the rules corresponding to having a univalent universe is what allows the **encode-decode** proofs to work (see [Uni13, Sec. 8.9]). In terms of higher categorical semantics, that corresponds to the universal property of a small object classifier, which is part of the definition of a higher topos. Indeed, one of the equivalent definitions of a higher topos says that it is a locally presentable locally cartesian closed quasicategory with a small object classifier.

4.3 STATEMENT OF JOYAL'S CONJECTURE AND PROOF STRATEGY

4.3.1. Joyal's Conjecture is a statement about the classifying category of any sufficiently rich type theory; more precisely, any type theory admitting the rules of Appendix A. It was formulated by André Joyal during the Oberwolfach Mini-Wokshop 1109a: *Homotopy Interpretation of Constructive Type Theory* in 2011 [Joy11].

Conjecture 4.3.2 (Joyal)**.** *For any dependent type theory $\mathbb{T}$ that admits the rules described in Appendix A, the standard localization (see Paragraph* **??***)* $\mathrm{L}(\mathcal{C}\ell(\mathbb{T}))$ *of its classifying category is a locally cartesian closed quasicategory.*

4.3.3. Notice that if $\mathbb{T}$ is the theory described *only* by the rules of Appendix A, then $\mathcal{C}\ell(\mathbb{T})$ is the terminal category, and hence its localization is the terminal quasicategory $\Delta[0]$. Of course, it is trivially locally cartesian closed.

It is therefore really crucial that we allow extensions of this theory to allow e.g. universes, inductive and higher inductive constructions, and axioms. This is where the significance of the conjecture can be fully demonstrated.

4.3.4. The positive answer to Joyal's Conjecture opens the door to the study of internal languages of higher categories, as outlined in [Shu12]. Indeed, Shulman [Shu14, Sec. 4.2] began this program by defining an internal type theory of a type-theoretic fibration category. It follows from our proof of Joyal's Conjecture that each such category presents a locally cartesian closed quasicategory. This opens the possibility of lifting the results related to the internal languages of categories (see e.g. [LS86]) to the world of higher category theory.

4.3.5. We next explain our proof strategy. The main difficulty in proving this conjecture lies in the fact that the construction of $\mathrm{L}(\mathcal{C}\ell(\mathbb{T}))$ involves taking fibrant replacement. Because of that, it is typically difficult to work with the quasicategory in question directly.

Instead, we need to look for a different construction of the standard localization. Unfortunately, to the best of the author's knowledge, for a general homotopical category $\mathcal{C}$, there does not exist a functorial construction assigning to $\mathcal{C}$ a quasicategory equivalent to $\mathrm{L}(\mathcal{C})$. The situation changes, however, if $\mathcal{C}$ is known to possess more structure. For example, if $\mathcal{C}$ is a *fibration category*, there is a functorial construction, due to Szumiło, $\mathcal{C} \mapsto \mathrm{N_f}(\mathcal{C})$ assigning to it, its quasicategory of frames [Szu14].

4.3.6. We will use Szumiło's construction and show that:

1. the slices of the quasicategory of frames in a fibration category $\mathcal{C}$ can be expressed using slices of the fibration category $\mathcal{C}$;
2. adjoint functors between fibration categories are mapped by $\mathrm{N_f}$ to adjunctions between corresponding quasicategories, provided that they preserve enough structure;
3. the quasicategory of frames is equivalent to the standard localization in Joyal's model structure;
4. the classifying category of a type theory admitting the rules described in Appendix A carries the structure sufficient for it to be mapped to a locally cartesian closed category by $\mathrm{N_f}$ in light of what we will have proven.

5.0 FIBRATION CATEGORIES

In this chapter, we will review several results from the theory of fibration categories. We begin by introducing fibration categories in Section 5.1 and then move to the study of fibration category structure on the diagram categories (Section 5.2). We then review the work of Szumiło and his results involving the quasicategory of frames in Section 5.3. We end by gathering several lemmas regarding Reedy structures on the categories of diagrams (Section 5.4).

5.1 DEFINITION AND FIRST PROPERTIES

Definition 5.1.1. A **fibration category** consists of a category $\mathcal{C}$ together with two wide subcategories $\mathcal{F}$ (called **fibrations**) and $\mathcal{W}$ (called **weak equivalences**) such that (in what follows, an **acyclic fibration** is a map that is both a fibration and a weak equivalence):

F1. $\mathcal{W}$ satisfies two-out-of-six property; that is, given a composable triple of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} Z$$

if $hg, gf \in \mathcal{W}$, then all $f, g, h \in \mathcal{W}$.

F2. all isomorphisms are acyclic fibrations.

F3. pullbacks along fibrations exist; fibrations and acyclic fibrations are stable under pullback.

F4. $\mathcal{C}$ has a terminal object $\mathbf{1}$; the canonical map $X \to \mathbf{1}$ is a fibration for any object $X \in \mathcal{C}$ (that is, all objects are **fibrant**).

F5. every map can be factored as a weak equivalence followed by a fibration.

5.1.2. There is, of course, the dual notion of a cofibration category. A **cofibration category** is a category equipped with two classes of maps: cofibrations and weak equivalences, satisfying the dual versions of the above axioms

Example 5.1.3. Let $\mathcal{M}$ be a model category. Then the full subcategory $\mathcal{M}_{\mathrm{f}} \subseteq \mathcal{M}$ of fibrant objects in $\mathcal{M}$ with fibrations and weak equivalence from the model structure on $\mathcal{M}$ is a fibration category. In particular, the category $\mathrm{q}\mathcal{C}\mathrm{at}$ of quasicategories is a fibration category, as the full subcategory of fibrant objects in Joyal's model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$.

Perhaps unsurprisingly, typical examples of cofibration categories are full subcategories of model categories, consisting of cofibrant objects.

Example 5.1.4 (cf. [Sch84])**.** The category of C^*-algebras is naturally a fibration, but there is no model category structure underlying it.

5.1.5. This definition differs from the one given by Brown [Bro73] in two ways. First, Brown required the class of weak equivalences $\mathcal{W}$ to satisfy only 2-out-of-3 axiom, rather than 2-out-of-6. The latter implies the former, but not the other way round. In fact, Cisinski showed [RB09, Thm. 7.2.7] that, in the presence of the Axioms F2–F5, the following conditions are equivalent:

1. $\mathcal{W}$ satisfies 2-out-of-3 and is saturated (i.e. under passage to $\mathrm{Ho}(\mathcal{C})$, weak equivalences are the only morphisms that become isomorphisms);
2. $\mathcal{W}$ satisfies 2-out-of-6.

5.1.6. In his axioms, Brown required only the existence of the factorization of the diagonal maps $\Delta\colon A \to A \times A$, not all maps as in our F5. Our requirement is not any stronger than Brown's. He proves existence of factorizations for diagonals implies existence of factorizations for all maps (see [Bro73, Factorization Lemma]).

5.1.7. The axioms of a fibration category imply several desirable properties. For example, one can show that projections from the product $A \times B \to A$ are fibrations, since the product can be defined as a pullback:

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

5.1.8. As all objects in fibration categories are fibrant, it is easy to show that fibration categories are *right proper.*

Lemma 5.1.9 (Right properness, [Bro73, Lem. 2])**.** *In a fibration category, the pullback of a weak equivalence along a fibration is again a weak equivalence.*

5.1.10 (Homotopy in a fibration category)**.** A **path object** for A in a fibration category $\mathcal{C}$ is any factorization of the diagonal map $\Delta_A \colon A \to A \times A$ into a weak equivalence followed by a fibration $A \xrightarrow{\sim} \widetilde{A} \to A \times A$.

Let $f, g \colon A \to B$ be a pair of maps in $\mathcal{C}$. A **right homotopy** between f and g is a commutative square:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\mathcal{W}\cap\mathcal{F}\ni"'] & \widetilde{B} \arrow[d] \\ A \arrow[r, "{(f,g)}"] & B \times B \end{tikzcd}$$

We say that f and g are **right-homotopic** if there is a right homotopy between f and g.

Theorem 5.1.11 (Brown, [Bro73, Thm. 1])**.**

1. *The relation of being right-homotopic is an equivalence relation on* $\operatorname{Hom}(A, B)$ *and is respected by pre- and postcomposition.*
2. *If two morphisms $f, g \colon A \to B$ are right-homotopic, then they are homotopic (i.e. equal in* $\operatorname{Ho}\mathcal{C}$*).*

Definition 5.1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be fibration categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **exact** if it preserves fibrations, acyclic fibrations, pullbacks along fibrations, and the terminal object.

The category of fibration categories and exact functors will be denoted $\mathcal{F}\mathrm{ib}$.

Lemma 5.1.13 (Ken Brown, [Bro73, Lem. 4.1])**.** *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between fibration categories that takes acyclic fibrations to weak equivalences. Then F takes all weak equivalences to weak equivalences.*

Proof. The well-known proof is typically given in the context of model categories [Hov99, Lem. 1.1.12], but it works for fibration categories as well. Let $f\colon A \to B$ be a weak equivalence and consider a factorization of $(1_A, f)\colon A \to A \times B$ into a weak equivalence followed by a fibration:

$$A \xrightarrow{w} \widetilde{A} \xrightarrow{(p_1,p_2)} A \times B$$

Both p_1 and p_2 must be acyclic fibrations and hence by assumption $F(p_1)$ and $F(p_2)$ are weak equivalences. By 2-out-of-3 for $1_{FA} = F(1_A) = F(p_1) \cdot F(w)$, $F(w)$ must be a weak equivalence, and so must the composite $F(f) = F(p_2) \cdot F(w)$. □

Corollary 5.1.14. *Every exact functor is homotopical.*

Corollary 5.1.15. *There is an obvious forgetful functor* $\mathcal{F}\mathrm{ib} \to \mathrm{h}\mathcal{C}\mathrm{at}$.

Definition 5.1.16. An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ is an **equivalence** of fibration categories, if it induces a categorical equivalence $\mathrm{Ho}(F)\colon \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ of homotopy categories.

Theorem 5.1.17 (Waldhausen's Approximation Criteria, [Cis10, Thm. 3.19])**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be fibration categories. An exact functor* $F\colon \mathcal{C} \to \mathcal{D}$ *is an equivalence if and only if the following two conditions are satisfied:*

(App1) F *reflects weak equivalences (that is,* Ff *being a weak equivalence in* $\mathcal{D}$ *implies that* f *is a weak equivalence in* $\mathcal{C}$*).*

(App2) for any morphism $f\colon Y \to FA$ *there exists a map* $p\colon B \to A$ *in* $\mathcal{C}$ *and a commutative square:*

$$\begin{tikzcd} Z \arrow[r, "\sim"] \arrow[d, "\sim"'] & FB \arrow[d, "Fp"] \\ Y \arrow[r, "f"] & FA \end{tikzcd}$$

Let us try to demystify the condition (App2) appearing in the theorem above. Substituting $A = \mathbf{1}$, we get that it is equivalent to the induced functor $\mathrm{Ho}(F)$ being essentially surjective. Similarly, we can show that $\mathrm{Ho}(F)$ has to be full and faithful.

5.2 FIBRATION CATEGORIES OF DIAGRAMS

5.2.1. Suppose $\mathcal{C}$ is a fibration category and J a small category. One can then ask: can the functor category $\mathcal{C}^J$ be equipped with the structure of a fibration category? The answer to this question is affirmative, provided that J is sufficiently nice. More precisely, J has to be an inverse category (see Definition 5.2.4 below).

5.2.2. In this section, we will introduce two fibration category structures on the functor category $\mathcal{C}^J$. In brief, they will be:

- the **levelwise structure**, in which a morphism (i.e. a natural transformation) is a weak equivalence (resp. a fibration) if each of its components is a weak equivalence (resp. a fibration);
- the **Reedy structure**, in which weak equivalences are still levelwise weak equivalences, but fibrations are required to satisfy a slightly stronger condition;

5.2.3. The first structure that we will introduce is the Reedy structure. The reason for delaying the introduction of the levelwise structure is that one uses the Reedy structure to deduce the existence of factorizations (Axiom F5) in the levelwise structure.

Definition 5.2.4.

1. A category J is **direct** if there is a function, called **degree**, $\deg\colon \mathrm{Ob}(J) \to \mathbb{N}$ such that for every non-identity map $j \to j'$ in J we have $\deg(j) < \deg(j')$.
2. A category J is **inverse** if J^{op} is direct.

5.2.5. Two remarks are in order.

First, we shall notice that existence of the function deg is the above definition is a *property, not structure*. Indeed, most maps between direct (and inverse) categories will *not* preserve the degree.

Second, from the point of view of fibration categories, inverse categories are more natural to consider. Indeed, it is the assumption that J is an inverse category that allows us to equip $\mathcal{C}^J$ with the fibration category structure. However, most of our indexing categories will have a natural description as direct categories and we will in some sense artificially force them to be inverse by passing to the opposite category.

Definition 5.2.6. Let J be an inverse category.

1. Let $j \in J$. The **matching category** $\partial(j/J)$ of j is the full subcategory of the slice category j/J consisting of all objects except 1_j. There is a canonical functor $\mathrm{cod}\colon \partial(j/J) \to J$, assigning to a morphism (regarded as object of $\partial(j/J)$) its codomain.

2. Let $X\colon J \to \mathcal{C}$ and $j \in J$. The **matching object** of X at j is defined as a limit of the composite
$$M_j(X) := \lim(\partial(j/J) \longrightarrow J \xrightarrow{X} \mathcal{C}),$$
where the first map sends an arrow in the slice to its codomain and the second is X. The canonical map $X(j) \to M_j(X)$ is called the **matching map**.

3. Let $\mathcal{C}$ be a fibration category. A diagram $X\colon J \to \mathcal{C}$ is called **Reedy fibrant**, if for all $j \in J$, the matching object $M_j(X)$ exists and the matching map $X(j) \to M_j(X)$ is a fibration.

4. Let $\mathcal{C}$ be a fibration category and let $X, Y\colon J \to \mathcal{C}$ be Reedy fibrant diagrams in $\mathcal{C}$. A map $f\colon X \to Y$ of diagrams is a **Reedy fibration**, if for all $j \in J$ the induced map $X(j) \to Y(j) \times_{M_j Y} M_j X$ is a fibration.

5.2.7. The notions of matching category, matching object, and Reedy fibrations and fibrant objects is specific to diagrams defined on an inverse category J. All of these notions admit dual formulations, when J is direct and $\mathcal{C}$ is a cofibration category. For example, if $j \in J$ is an object of direct category, we define its **latching category** $\partial(J/j)$ as the full subcategory of the slice category J/j consisting of all objects except the identity 1_j.

Similarly, one defines the **latching object**, **Reedy cofibrant objects**, and **Reedy cofibrations**. In this thesis, we will be dealing only with fibration categories, hence we will not make much use of latching objects, Reedy cofibrant objects and cofibrations. The notion of the latching category will be however useful to us; indeed, a since a lot of our examples of inverse can be most naturally described as duals of some direct categories, we may prefer to talk about latching categories, instead of matching caegories.

Examples 5.2.8.

- The constant diagram $J \to \mathcal{C}$ taking the value 1 (terminal object) is Reedy.

- If $Y\colon J \to \mathcal{C}$ is Reedy fibrant and $f\colon X \to Y$ is a Reedy fibration, then X is also Reedy fibrant.
- A diagram $X\colon \{1 \rightrightarrows 0\} \to \mathcal{C}$ is Reedy fibrant if and only if the induced map $X(1) \to X(0) \times X(0)$ is a fibration.

5.2.9. The advantage of Reedy fibrant diagrams over arbitrary diagrams lies in the fact that one can always take the limit over a Reedy fibrant diagram.

5.2.10. Let J be a finite inverse category and choose the degree function $\deg\colon \operatorname{Ob} J \to \mathbb{N}$. Then for some $d \in \mathbb{N}$, we have $J = J_{\le d}$, where $J_{\le d}$ is the full subcategory of J consisting of those $j \in J$ for which $\deg j \le d$. The minimal d such that $J = J_{\le d}$ will be called the **degree** of J. By induction with respect to the degree of J, we will construct the limit of a diagram $X\colon J \to \mathcal{C}$ (simultaneously for all fibration categories $\mathcal{C}$, inverse categories J of degree, and Reedy fibrant diagrams $X\colon J \to \mathcal{C}$).

Base case. If all objects are of degree 0, then the limit is given by:

$$\lim X = \prod_{j \in J} X(j).$$

Induction step. Suppose now that we have constructed the limit of $X|J_{\le d}$ and J is of degree $d+1$. Then the limit of X is given by the pullback:

$$\begin{tikzcd}
\lim X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod\limits_{\deg j = d+1} X(j) \arrow[d] \\
\lim(X|J_{\le d}) \arrow[r] & \prod\limits_{\deg j = d+1} M_j(X)
\end{tikzcd}$$

Definition 5.2.11. Let $\mathcal{C}$ be a fibration category and $X, Y\colon J \to \mathcal{C}$ functors. A natural transformation $f\colon X \to Y$ is called a **levelwise fibration** (resp. **weak equivalence**) if for all $j \in J$, the map $f_j\colon X(j) \to Y(j)$ is a fibration (resp. weak equivalence).

Proposition 5.2.12. *Let $\mathcal{C}$ be a fibration category and J and inverse category. Let $f\colon X \to Y$ be a morphism in the category of diagrams $\mathcal{C}^J$, whose codomain is Reedy fibrant. Then f admits a factorization $f = p \cdot w$, where p is a Reedy fibration and w is a levelwise weak equivalence.*

Proof. We will proceed by induction on degree of $j \in J$. In each step we will define a factorization $X(j) \xrightarrow{w_j} \widetilde{X}(j) \xrightarrow{p_j} Y(j)$ with $\widetilde{X}(j)$ Reedy fibrant. Suppose such factorizations are already defined for all objects of degree smaller than $\deg(j)$ for some $j \in J$. Thus $M_j(\widetilde{X})$ exists. (Here, by $M_j(\widetilde{X})$ we mean the limit of the composite $\partial(j/J) \to J \to \mathcal{C}$, which is well-defined even though $\widetilde{X}$ is not yet defined on the whole J.) By the induction hypothesis there is a map $X(j) \to M_j(\widetilde{X})$ making the following square commute:

$$\begin{tikzcd}
X(j) \arrow[r, "f_j"] \arrow[d] & Y(j) \arrow[d] \\
M_j(\widetilde{X}) \arrow[r] & M_j(Y)
\end{tikzcd}$$

Since Y is Reedy fibrant and thus the map $Y(j) \to M_j(Y)$ is a fibration, we can form the pullback $Y(j) \times_{M(Y)} M_j(\widetilde{X})$ and by the commutativity of the above square, we obtain a map $X(j) \to Y(j) \times_{M_j(Y)} M_j(\widetilde{X})$. We define $\widetilde{X}(j)$ by choosing a factorization of $X(j) \to \widetilde{X}(j) \to Y(j) \times_{M_j(Y)} M_j(\widetilde{X})$ into a weak equivalence followed by a fibration. $\square$

5.2.13. We are now ready to prove the existence of **Reedy structure** on the subcategory $\mathcal{C}^J_{\mathrm{R}}$ of Reedy fibrant diagrams in the functor category $\mathcal{C}^J$. This restriction is required since not every functor $J \to \mathcal{C}$ is Reedy fibrant and hence $\mathcal{C}^J$ would not satisfy Axiom F4 of Definition 5.1.1.

Theorem 5.2.14. *Let J be an inverse category and $\mathcal{C}$ a fibration category. Then the category $\mathcal{C}^J_{\mathrm{R}}$ of Reedy fibrant diagrams $J \to \mathcal{C}$ with Reedy fibrations as fibrations and levelwise weak equivalences as weak equivalences is a fibration category.*

Proof. The verification of Axioms F1–F4 is routine (see [RB09, Thm. 9.3.8.(2a)]). Axiom F5 is exactly the statement of Proposition 5.2.12. $\square$

Proposition 5.2.15. *Let $\mathcal{C}$ be a fibration category and I and J inverse categories. Then there is a natural equivalence of fibration categories:*

$$\mathcal{C}^{I \times J}_{\mathrm{R}} \simeq (\mathcal{C}^I_{\mathrm{R}})^J_{\mathrm{R}}$$

Proof. It suffices to show that a diagram $I \times J \to \mathcal{C}$ is Reedy fibrant if and only if its transpose $I \to \mathcal{C}^J_{\mathrm{R}}$ is.

This follows readily by the following description of matching categories in the product, which can be regarded as yet another incarnation of Leibniz rule:

$$\partial((i,j)/I \times J) \cong \Big(\partial(i/I) \times j/J\Big) \cup \Big((i/I) \times \partial(j/J)\Big)$$

$\square$

5.2.16. We can turn attention to the **levelwise structure** on the functor category.

Theorem 5.2.17. *Let J be an inverse category and $\mathcal{C}$ a fibration category. Then the functor $\mathcal{C}^J$ with levelwise fibrations and weak equivalences is a fibration category.*

Proof. As in Theorem 5.2.14, F1–F4 are clear (see [RB09, Thm. 9.3.8.(2b)]). For F5, let $f\colon X \to Y$ be the map to be factored. Let $w_Y\colon Y \xrightarrow{\sim} \widetilde{Y}$ be a Reedy fibrant replacement of Y (i.e. factorization of $Y \to 1$ of Proposition 5.2.12). By Proposition 5.2.12, we can factor $w_Y \cdot f$ as a weak equivalence followed by a fibration:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "w"'] & Y \arrow[d, "w_Y"] \\
\widetilde{X} \arrow[r, "p"] & \widetilde{Y}
\end{tikzcd}$$

Since $p\colon \widetilde{X} \to \widetilde{Y}$ is a fibration, the pullback $\widetilde{X} \times_{\widetilde{Y}} Y$ exists and by the commutativity of the above square we obtain the induced map $X \to \widetilde{X} \times_{\widetilde{Y}} Y$. We claim that $X \to \widetilde{X} \times_{\widetilde{Y}} Y \to Y$ is the desired factorization of f. Indeed, the map $X \to \widetilde{X} \times_{\widetilde{Y}} Y$ by a 2-out-of-3 argument and right properness (Lemma 5.1.9) of the Reedy fibration category structure and $\widetilde{X} \times_{\widetilde{Y}} Y \to Y$ is a Reedy fibration (as a pullback of p), hence also a levelwise fibration. $\square$

Theorem 5.2.18. *Given an inverse category J and a fibration category $\mathcal{C}$, the inclusion $\mathcal{C}^J_{\mathrm{R}} \hookrightarrow \mathcal{C}^J$ is an equivalence of fibration categories.*

Proof. We will verify Waldhausen's approximation criteria 5.1.17. (App1) is automatic since weak equivalences are levelwise in both Reedy and levelwise structure. For (App2), by Proposition 5.2.12, we have a factorization of every morphism $f\colon X \to A \in \mathcal{C}^J$ whose codomain is Reedy fibrant into a levelwise weak equivalence followed by a Reedy fibration $X \xrightarrow{\sim} B \to A$. Then the diagram:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "1_X"'] & B \arrow[d] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

fulfils the requirement of (App2). □

5.2.19. Next, we will investigate under what conditions a functor $f\colon I \to J$ between inverse categories induces by precomposition an exact functor

$$f^*\colon \mathcal{C}^J \to \mathcal{C}^I \qquad \text{and} \qquad f^*\colon \mathcal{C}^J_{\mathrm{R}} \to \mathcal{C}^I_{\mathrm{R}}.$$

5.2.20. For the levelwise structures the answer is immediate. The functor $f^*\colon \mathcal{C}^J \to \mathcal{C}^I$ given by $f^*(X) := X \cdot f$ is always exact.

5.2.21 (Sieves)**.** Sieves play an important role in verifying exactness of functors between fibration categories. Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **sieve** if it is fully faithful and for any object $c \in \mathcal{C}$ and a morphism $f\colon d \to Fc$, there exists a unique $h\colon c' \to c$ such that $Fh = f$.

Proposition 5.2.22 (Exactness Criterion)**.** *Let $f\colon I \to J$ be a homotopical functor between homotopical direct categories with finite latching categories such that for all $i \in I$, the induced map on the matching categories factors as:*

$$\begin{tikzcd} & \bullet \arrow[dr, hook, "\textit{sieve}"] & \\ I/i \arrow[ur, "\textit{cofinal}"] \arrow[rr] & & J/f(i) \end{tikzcd}$$

Then the induced functor $f^*\colon \mathcal{C}^{J^{\mathrm{op}}}_{\mathrm{R}} \to \mathcal{C}^{I^{\mathrm{op}}}_{\mathrm{R}}$ is an exact functor of fibration categories.

Proof. Preservation of weak equivalences, pullbacks, and the terminal object is clear. Since the induced map on matching objects factors as cofinal followed by sieve, it will preserve fibrancy of the diagram. □

5.2.23. Here, by a **cofinal** functor we mean a functor F such that precomposition with F induces isomorphism on the colimits. That is, $\alpha\colon I \to J$ is cofinal if for any $\mathcal{C}$ and any $F\colon J \to \mathcal{C}$, $\operatorname{colim} F = \operatorname{colim}(F \cdot \alpha)$.

5.2.24 (Cofinality Criterion)**.** As we shall employ the above criterion frequently, we will have to verify that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is cofinal. Thus we give here a useful criterion establishing cofinality of a functor (cf. [ML98a, Thm. IX.3.1]): $F\colon \mathcal{C} \to \mathcal{D}$ is cofinal if and only if for every $d \in \mathcal{D}$, the slice category $d \downarrow F$ is connected (that is, any two objects of $d \downarrow F$ can be connected by a zigzag of maps).

Lemma 5.2.25. *Let $\mathcal{C}$ be a fibration category and $f\colon I \to J$ be a homotopy equivalence (see Definition 3.1.11) of homotopical direct categories. Then $f^*\colon \mathcal{C}^{J^{\mathrm{op}}} \to \mathcal{C}^{I^{\mathrm{op}}}$ is an equivalence of fibration categories.*

Proof. This follows easily by 2-out-of-3. □

Proposition 5.2.26. *Let $\mathcal{C}$ be a fibration category and $f\colon I \to J$ a map of homotopical direct categories such that the induced map $f^*\colon \mathcal{C}_{\mathrm{R}}^{J^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{R}}^{I^{\mathrm{op}}}$ is exact. If f is a homotopy equivalence, then $f^*\colon \mathcal{C}_{\mathrm{R}}^{J^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{R}}^{I^{\mathrm{op}}}$ is an equivalence.*

Proof. By Lemma 5.2.25, $f^*\colon \mathcal{C}^{J^{\mathrm{op}}} \to \mathcal{C}^{I^{\mathrm{op}}}$ is an equivalence. Since, by assumption, $f^*\colon \mathcal{C}_{\mathrm{R}}^{J^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{R}}^{I^{\mathrm{op}}}$ is exact, we have a square of fibration categories and exact functors:

$$\begin{tikzcd} \mathcal{C}_{\mathrm{R}}^{J^{\mathrm{op}}} \arrow[r, "f^*"] \arrow[d, hook] & \mathcal{C}_{\mathrm{R}}^{I^{\mathrm{op}}} \arrow[d, hook] \\ \mathcal{C}^{J^{\mathrm{op}}} \arrow[r, "f^*"] & \mathcal{C}^{I^{\mathrm{op}}} \end{tikzcd}$$

in which both vertical arrows are equivalences by Theorem 5.2.18, and the bottom arrow is an equivalence by Lemma 5.2.25. Hence, by 2-out-of-3, all functors are equivalences. □

5.3 THE QUASICATEGORY OF FRAMES IN A FIBRATION CATEGORY

5.3.1. In this section, we will review relevant parts of the work of Szumiło, who has recently established a direct (i.e. not involving fibrant replacement) construction assigning to a fibration category $\mathcal{C}$ a quasicategory $\mathrm{N_f}\mathcal{C}$, and has managed to prove several properties of it. Unless explicitly stated otherwise, all the results of this section are from [Szu14]. Let us just mention that Szumiło worked with cofibration categories and hence some of the results that we state are dual to his, but the translations are in each case straightforward.

5.3.2. Szumiło noticed that the category $\mathcal{F}\mathrm{ib}$ of fibration categories and exact functors can be equipped with the structure of a fibration category, in which weak equivalences are the equivalences of fibration categories.

5.3.3. For convenience of exposition later on, let us introduce the following notations. For $n \in \mathbb{N}$, let:

- $[n]$ be the linearly ordered set $\{0 \leq 1 \leq \ldots \leq n\}$ that, when regarded as a homotopical category, has only trivial weak equivalences;
- $\mathcal{P}_{\mathrm{n}}([n])$ be the set of proper non-empty subsets of $[n]$;
- $(n]$ be the linearly ordered set $\{1 \leq 2 \leq \ldots \leq n\}$;
- $[n)$ be the linearly ordered set $\{0 \leq 1 \leq \ldots \leq n-1\}$;
- $\widehat{[n]}$ be the linearly ordered set $[n]$, regarded as a homotopical category, in which all maps are weak equivalences.

Definition 5.3.4. An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories is said to have the **lifting property with respect to**:

- **isofibrations** if for any isomorphism $f\colon FA \to X$ in $\mathcal{D}$, there exists an isomorphism $\overline{f}\colon A \to B$ in $\mathcal{C}$ such that $F(\overline{f}) = f$.
- **factorizations** if for any $f\colon A \to B$ in $\mathcal{C}$ and a factorization $Ff = p \cdot w$ as weak equivalence followed by a fibration in $\mathcal{D}$, there exists a factorization $f = p' \cdot w'$ in $\mathcal{C}$ such that $F(w') = w$ and $F(p') = p$.

- **pseudofactorizations** if for any $f\colon A \to B$ in $\mathcal{C}$ and any commutative square in $\mathcal{D}$ of the form:

$$\begin{tikzcd} Z \arrow[r, "w"] \arrow[d, "v"'] & Y \arrow[d, "p"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

where p is a fibration and w and v are weak equivalences, there exists a square:

$$\begin{tikzcd} Z' \arrow[r, "w'"] \arrow[d, "v'"'] & Y' \arrow[d, "p'"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where p' is a fibration, w' and v' are weak equivalences, and $F(p') = p$, $F(w') = w$, and $F(v') = v$.

Theorem 5.3.5 ([Szu14, Thm. 1.16])**.** *The category $\mathcal{F}\mathrm{ib}$ of fibration categories and exact functors has itself the structure of a fibration category, in which:*

- *weak equivalences are equivalences of fibration categories (i.e. exact functors $F\colon \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Ho}(F)$ is an equivalence of categories);*
- *fibrations are functors having the lifting property with respect to factorizations, pseudo-factorizations, and isofibrations.*

5.3.6. The next theorem gives a useful characterization of acyclic fibrations between fibration categories. For that, we need however a preliminary definition. An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ has a **Reedy lifting property** with respect to the map $f\colon I \to J$ of inverse categories if every commutative square of the form:

$$\begin{tikzcd} I \arrow[r] \arrow[d, "f"'] & \mathcal{C} \arrow[d, "F"] \\ J \arrow[r] & \mathcal{D} \end{tikzcd}$$

in which the horizontal maps are Reedy fibrant, admits a diagonal filler $J \to \mathcal{C}$ which is also required to be Reedy fibrant.

Theorem 5.3.7 ([Szu14, Lem. 1.25])**.** *A fibration $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories is acyclic if and only F has the Reedy lifting property with respect to the inclusions $[0] \hookrightarrow [1]$ (picking out 1) and $[1] \hookrightarrow \widehat{[1]}$.*

5.3.8. Given a homotopical category J, we will construct a direct homotopical category DJ, together with a homotopical functor $p\colon DJ \to J$. The objects of DJ are pairs $([n], \varphi\colon [n] \to J)$, where $n \in \mathbb{N}$ and φ is an arbitrary functor. A map

$$f\colon ([n], \varphi) \to ([m], \psi)$$

is an injective, order preserving map $f\colon [n] \hookrightarrow [m]$ making the following square commute:

$$\begin{tikzcd} {[n]} \arrow[r, "\varphi"] \arrow[d, hook, "f"'] & J \arrow[d, equal] \\ {[m]} \arrow[r, "\psi"] & J \end{tikzcd}$$

It is clear that DJ is a direct category (with $\deg([n], \varphi) = n$). To define $p\colon DJ \to J$ we put $p([n], \varphi) = \varphi(n)$. This gives a contravariant functor and we can define $\mathcal{W}_{DJ} := p^{-1}(\mathcal{W}_J)$. So $(DJ, \mathcal{W}_{DJ})$ is a homotopical category and p is a homotopical functor.

We will sometimes refer to DJ as the **fat barycentric subdivision** of J.

Definition 5.3.9. Let $\mathcal{C}$ be a fibration category. We define the simplicial set $\mathrm{N_f}(\mathcal{C})$, called the **quasicategory of frames** in $\mathcal{C}$, by setting:

$$\mathrm{N_f}(\mathcal{C})_n := \mathrm{Fun}^{\mathrm{hR}}(D[n]^{\mathrm{op}}, \mathcal{C}).$$

That is, the n-simplices of $\mathrm{N_f}(\mathcal{C})$ are homotopical, Reedy fibrant diagrams $D[n]^{\mathrm{op}} \to \mathcal{C}$.

5.3.10. The name *quasicategory of frames* is motivated by the fact that the category $D[0]$ is equivalent to the semisimplex category Δ_{inj} of finite non-empty ordinals and injective monotone functions. A homotopical, Reedy fibrant diagram $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ is called a frame (see also [Hov99, Ch. 5] for the discussion in the context of model categories). Thus the 0-simplices of $\mathrm{N_f}(\mathcal{C})$ are exactly the frames in $\mathcal{C}$.

5.3.11. Throughout the remainder of this thesis, we will be working with fibration categories of homotopical, Reedy fibrant diagrams $J^{\mathrm{op}} \to \mathcal{C}$, where J is a direct homotopical category. The category of such diagrams $\mathcal{C}_{\mathrm{hR}}^{J^{\mathrm{op}}}$ is again a fibration category and all the results that we had for $\mathcal{C}_{\mathrm{R}}^{J^{\mathrm{op}}}$ translate in a straightforward manner under the addition of the homotopicality assumption.

Theorem 5.3.12 ([Szu14, Thm. 3.2])**.** *The functor* $\mathrm{N_f}\colon \mathcal{F}\mathrm{ib} \to \mathrm{s}\mathcal{S}\mathrm{et}$ *takes values in finitely complete quasicategories and is exact as a functor between fibration categories* $\mathcal{F}\mathrm{ib} \to \mathrm{q}\mathcal{C}\mathrm{at}$.

5.3.13. A slightly more familiar version of the D-construction of Paragraph 5.3.8 is the (ordinary) **barycentric subdivision** construction. Unlike D, however, it is only available for posets. Given a homotopical poset P, define a homotopical direct category $\mathrm{Sd}(P)$ as follows:

- objects of $\mathrm{Sd}(P)$ are injective monotone functions $\varphi\colon [n] \hookrightarrow P$, i.e. linearly ordered non-empty subsets of P;
- a morphism from $\varphi\colon [n] \hookrightarrow P$ to $\psi\colon [m] \hookrightarrow P$ is an injective function $f\colon [n] \to [m]$ making the following square commute:

$$\begin{tikzcd} {[n]} \arrow[r, "\varphi"] \arrow[d, hook, "f"'] & P \arrow[d, equal] \\ {[m]} \arrow[r, "\psi"] & P \end{tikzcd}$$

Finally, we set $\mathcal{W}_{\mathrm{Sd}(P)} := p^{-1}(\mathcal{W}_P)$, where $p(\varphi\colon [n] \to P) = \varphi(n)$.

5.3.14. Similarly, we may define D and Sd of simplicial sets, rather than of categories. Let $K \in \mathrm{s}\mathcal{S}\mathrm{et}$ and define the underlying category of DK to be the category of elements of K, considered as a semisimplicial set (i.e. without degeneracy maps). The weak equivalences in DK are generated under composition from equivalences in K (i.e. maps $\Delta[1] \to K$ that factor through $\mathcal{I}$ or $E[1]$). One defines Sd of a simplicial set in the analogous manner, but this construction is only available for those $K \in \mathrm{s}\mathcal{S}\mathrm{et}$ for which there exists an inclusion into the nerve of a poset $K \hookrightarrow \mathrm{N}(P)$. (This still captures the crucial examples of simplices $\Delta[n]$, boundaries $\partial\Delta[n]$, and horns $\Lambda^i[n]$.)

The usefulness of the Sd-construction is best demonstrated by the following theorem.

Theorem 5.3.15 ([Szu14, Lem. 3.19])**.** *Let P be a poset, $K \hookrightarrow L \hookrightarrow \mathrm{N}(P)$ inclusion of simplicial sets, and $\mathcal{C}$ a fibration category. Then the inclusion $DK \cup \mathrm{Sd}L \hookrightarrow DL$ induces an acyclic fibration $\mathcal{C}_{\mathrm{hR}}^{DL^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{hR}}^{(DK \cup \mathrm{Sd}L)^{\mathrm{op}}}$ of fibration categories.*

5.3.16. This theorem is of particular importance when solving lifting problems (e.g. when verifying that a map is a categorical equivalence using Lemma 3.3.32) involving $\mathrm{N_f}(\mathcal{C})$. Notice that for $L = \Delta[n]$ and $K = \partial\Delta[n], \Lambda^i[n]$, the canonical map

$$\mathcal{C}_{\mathrm{hR}}^{D\Delta[n]^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{hR}}^{(DK \cup \mathrm{Sd}\Delta[n])^{\mathrm{op}}}$$

is an acyclic fibration of fibration categories and hence admits a section. We will revisit this situation frequently in subsequent chapters.

5.3.17. The last outstanding piece of Szumiło's work, that we will employ in the present thesis, is the following *fake adjunction*. Indeed, it does look like an adjunction at first, but after a closer inspection it is clear that there are no categories that can serve as its domain and codomain.

Theorem 5.3.18 ([Szu14, Prop. 3.5])**.** *Let $\mathcal{C}$ be a fibration category and K a simplicial set. Then there is a natural bijection:*

$$\left\{ \begin{array}{c} \text{simplicial maps} \\ K \to \mathrm{N_f}\mathcal{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{homotopical Reedy fibrant functors} \\ DK^{\mathrm{op}} \to \mathcal{C} \end{array} \right\}$$

In other words, the contravariant functor $\mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ *taking a simplicial set to the set of homotopical Reedy fibrant diagrams $DK^{\mathrm{op}} \to \mathcal{C}$ is representable, represented by* $\mathrm{N_f}\mathcal{C}$*.*

5.4 REEDY FIBRANCY AND SIEVES

5.4.1. In this last section of this introductory chapter to fibration categories, we gather several lemmas about extending Reedy fibrant diagrams along inclusions of sieves. Most or all of these exist in folklore, but are presented here for reference later in the thesis.

Lemma 5.4.2. *Let $\mathcal{C}$ be a fibration category and $I \hookrightarrow J$ be a sieve of direct categories with finite latching categories. Suppose given $X \colon J^{\mathrm{op}} \to \mathcal{C}$ such that $X|I^{\mathrm{op}}$ is Reedy fibrant. Then there exists a Reedy fibrant diagram $\widetilde{X} \colon J^{\mathrm{op}} \to \mathcal{C}$ together with a natural weak equivalence $w \colon X \to \widetilde{X}$ such that $\widetilde{X}|I^{\mathrm{op}} = X|I^{\mathrm{op}}$ and $w|I^{\mathrm{op}} = 1_{X|I^{\mathrm{op}}}$.*

Proof. Let $j \in J \setminus I$ be of minimal degree. Then the matching object $M_j(X)$ exists and we can factor $X(j) \to M_j(X)$ as a weak equivalence followed by a fibration: $X(j) \to \widetilde{X}(j) \to M_j(X)$, obtaining the extension of $\widetilde{X}$ to $j \in J$. We then repeat this procedure for the sieve $I \cup \{j\} \hookrightarrow J$. □

Lemma 5.4.3. *Let $I \hookrightarrow J$ be a cosieve of inverse categories and let $X \colon J \to \mathcal{C}$ and $Y \colon I \to \mathcal{C}$ be Reedy fibrant diagrams together with a natural weak equivalence $w \colon Y \to X|I$. Then there exists a Reedy fibrant diagram $\widetilde{Y} \colon J \to \mathcal{C}$ such that $\widetilde{Y}|I = Y$ and natural weak equivalence $w \colon \widetilde{Y} \to X$.*

Proof. We will build both $\widetilde{Y}(j)$ and $w_j \colon \widetilde{Y}(j) \to X(j)$ by induction with respect to the degree of $j \in J$. Let j be of minimal degree such that $\widetilde{Y}(j)$ and w_j have not yet been defined. Then $M_j(\widetilde{Y})$ exists and the map $M_j(\widetilde{Y}) \to M_j(X)$ is a weak equivalence. We can define $\widetilde{Y}$ as the pullback:

$$\begin{tikzcd}
\widetilde{Y}(j) \arrow[r, "w_j"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(j) \arrow[d] \\
M_j(\widetilde{Y}) \arrow[r] & M_j(X)
\end{tikzcd}$$

The matching map $\widetilde{Y}(j) \to M_j(\widetilde{Y})$ is a fibration as a pullback of a fibration and w_j is a weak equivalence by right properness 5.1.9. □

Lemma 5.4.4. *Let $I \hookrightarrow J$ be a sieve of direct categories with finite latching categories. Suppose given a diagram $X \colon J^{\mathrm{op}} \to \mathcal{C}$ and a Reedy fibrant diagram $Y \colon I^{\mathrm{op}} \to \mathcal{C}$ together with a natural weak equivalence $w \colon X|I^{\mathrm{op}} \to Y$. Then there exists a Reedy fibrant diagram $\widetilde{Y} \colon J^{\mathrm{op}} \to \mathcal{C}$ such that $\widetilde{Y}|I^{\mathrm{op}} = Y$ and a natural weak equivalence $\widetilde{w} \colon X \to \widetilde{Y}$ such that $\widetilde{w}|I^{\mathrm{op}} = w$.*

Proof. We will construct $\widetilde{Y}(j)$ and $\widetilde{w}_j$ simultaneously by induction on the degree of $j \in J$. Let j be of minimal degree such that $\widetilde{Y}(j)$ and $\widetilde{w}_j$ have not yet been constructed. The matching object $M_j(\widetilde{Y})$ exists (as formal limit) and there is a canonical map $X(j) \to M_j(\widetilde{Y})$. We factor this map as a weak equivalence followed by a fibration obtaining the desired $\widetilde{Y}(j)$ and $\widetilde{w}_j$:

$$X(j) \xrightarrow{w_j} \widetilde{Y}(j) \longrightarrow M_j(Y).$$

$\square$

Lemma 5.4.5. *Let $\mathcal{C}$ be a fibration category and I be a finite homotopical inverse category with a terminal object 1 such that every morphism of I is a weak equivalence. Then for any homotopical, Reedy fibrant functor $X\colon I \to \mathcal{C}$, the limit $\lim X$ is weakly equivalent to $X(1)$ (and hence to every object in the image of X).*

Proof. This follows directly by the construction of limits of Reedy fibrant diagrams, described in Paragraph 5.2.10. $\square$

6.0 PARTIAL REEDY STRUCTURES

The goal of this chapter is to prove Corollary 6.2.11. While this corollary is not of particular interest outside of this thesis, the techniques used to prove it may find applications going beyond the scope of this thesis. Indeed, in Section 6.2 we introduce a new fibration category structure on the category of diagrams $\mathcal{C}^J$, which depends on an additional parameter, namely a map $f\colon I \to J$ of inverse categories. Before that, in Section 6.1 we review the basic theory of homotopy pullbacks (the results of this section are to some extent known in folklore, but there are no comprehensive references on the level of generality that we require).

6.1 HOMOTOPY PULLBACKS

6.1.1. Our treatment of homotopy pullbacks requires us to first establish the right notion of slice of fibration categories.

6.1.2. Recall that, given a model category $\mathcal{M}$ and an object $A \in \mathcal{M}$, one can easily show that the slice category $\mathcal{M}/A$ is again a model category, in which a morphism is a fibration/cofibration/weak equivalence, if it is fibration/cofibration/weak equivalence, regarded as a morphism of $\mathcal{M}$ (cf. [Hov99, Prop. 1.1.8]).

6.1.3. Let $\mathcal{C}$ be a fibration category and $A \in \mathcal{C}$. The slice category $\mathcal{C}/A$ does not have to be a fibration category since not every object of $\mathcal{C}/A$ (i.e. a morphism $B \to A$ in $\mathcal{C}$) has to be fibrant in $\mathcal{C}/A$ (i.e. a fibration in $\mathcal{C}$). To resolve this issue, let us define $\mathcal{C}(A)$ as the full subcategory of the slice category $\mathcal{C}/A$ whose objects are fibrations $B \to A$.

Proposition 6.1.4. *Declare a map in $\mathcal{C}(A)$ to be a fibration (respectively, a weak equivalence)* *if it is a fibration (resp. a weak equivalence) regarded as map in $\mathcal{C}$. Then the category $\mathcal{C}(A)$ with this choice of fibrations and weak equivalences is a fibration category.*

Proof. Straightforward verification. $\square$

Lemma 6.1.5. *Let $f\colon X \to Y$ be a morphism in a fibration category $\mathcal{C}$. Then the functor $f^*\colon \mathcal{C}(Y) \to \mathcal{C}(X)$ given by pullback is exact.*

Proof. Preservation of pullbacks and the terminal object is clear. Preservation of fibrations and acyclic fibrations follows by the two-pullback lemma. $\square$

Definition 6.1.6. A square of the form:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

in a fibration category $\mathcal{C}$ is a **homotopy pullback** if given any factorization $X \to \widetilde{X} \to Y$, the induced map $U \to \widetilde{X} \times_Y V$ is a weak equivalence.

6.1.7. In order to show that the notion of homotopy pullback is well-defined, we need to show that Definition 6.1.6 does not depend on the choice of factorization and the choice of map to be factored (we could have chosen $V \to Y$ instead). We will address these issues in Lemmas 6.1.8 and 6.1.9, respectively.

Lemma 6.1.8. *Definition 6.1.6 does not depend on the choice of factorization.*

Proof. Any two factorizations of $X \to Y$ can be connected by a zigzag of weak equivalences. Indeed, given:

$$\begin{tikzcd} & X \arrow[dl, "\sim"'] \arrow[dr, "\sim"] \arrow[dd] & \\ \widetilde{X} \arrow[dr] & & \widetilde{X}' \arrow[dl] \\ & Y & \end{tikzcd}$$

we can factor the induced map $X \to \widetilde{X} \times_Y \widetilde{X}'$ into a weak equivalence followed by a fibration. Thus, we have reduced the problem to showing that for any factorizations connected by a weak equivalence:

$$\begin{tikzcd}
& X \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
\widetilde{X} \arrow[rr, "\sim"] \arrow[dr] & & \widetilde{X}' \arrow[dl] \\
& Y &
\end{tikzcd}$$

the induced map $U \to \widetilde{X} \times_Y V$ is a weak equivalence if and only if the induced map $U \to \widetilde{X}' \times_Y V$ is. This follows by exactness of the pullback functor 6.1.5, Ken Brown's lemma, and 2-out-of-3. □

Lemma 6.1.9. *Definition 6.1.6 does not depend on the choice of morphism for factorization. That is, the square of Definition 6.1.6 is a homotopy pullback if and only if for any factorization $V \to \widetilde{V} \to Y$ of $V \to Y$ into a weak equivalence followed by a fibration, the induced map $U \to \widetilde{V} \times_Y X$ is a weak equivalence.*

Proof. It suffices to show that in the diagram:

$$\begin{tikzcd}
U \arrow[r, dotted] \arrow[d, dotted] & \bullet \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\sim"] \\
\bullet \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d] \\
V \arrow[r, "\sim"] & \bullet \arrow[r] & Y
\end{tikzcd}$$

one of the dotted arrows is a weak equivalence if and only if the other one is. This is however an immediate consequence of 2-out-of-3. □

Lemma 6.1.10. *Given a commutative cube of the form:*

$$\begin{tikzcd}
& U_0 \arrow[rr] \arrow[dl, "\sim"'] \arrow[dd, dotted] & & X_0 \arrow[dl, "\sim"'] \arrow[dd] \\
U_1 \arrow[rr] \arrow[dd] & & X_1 & \\
& V_0 \arrow[rr, dotted] \arrow[dl, dotted, "\sim"'] & & Y_0 \arrow[dl, "\sim"'] \\
V_1 \arrow[rr] & & Y_1 \arrow[from=uu, crossing over] &
\end{tikzcd}$$

the front square is a homotopy pullback if and only if the back square is a homotopy pullback.

Proof. Let $X_1 \to \widetilde{X}_1 \to Y_1$ be any factorization of $X_1 \to Y_1$ and let $X_0 \to \widetilde{X}_0 \to \widetilde{X}_1 \times_{Y_1} Y_0$ be a factorization of the induced map $X_0 \to \widetilde{X}_1 \times_{Y_1} Y_0$. By Lemma 6.1.8, we can choose the factorizations as we please, thus it suffices to show that in the diagram:

$$\begin{tikzcd}
& U_0 \arrow[rr] \arrow[dl, "\sim"'] \arrow[dd, dotted] & & X_0 \arrow[dl, "\sim"'] \arrow[dd] \\
U_1 \arrow[rr] \arrow[dd] & & X_1 & \\
& V_0 \times_{Y_0} \widetilde{X}_0 \arrow[rr, dotted] \arrow[dl, dotted] & & \widetilde{X}_0 \arrow[dl, "\sim"'] \\
V_1 \times_{Y_1} \widetilde{X}_1 \arrow[rr] & & \widetilde{X}_1 \arrow[from=uu, crossing over] &
\end{tikzcd}$$

the map $U_0 \to V_0 \times_{Y_0} \widetilde{X}_0$ is a weak equivalence if and only if the map $U_1 \to V_1 \times_{Y_1} \widetilde{X}_1$ is. However, by the gluing lemma, the map $V_0 \times_{Y_0} \widetilde{X}_0 \to V_1 \times_{Y_1} \widetilde{X}_1$ is a weak equivalence and the result now follows by 2-out-of-3. □

Corollary 6.1.11. *A square of Definition 6.1.6 in a fibration category $\mathcal{C}$ is a homotopy pullback if and only if it can be connected by a zigzag of natural weak equivalences (in the category of commutative squares in $\mathcal{C}$) with a pullback along a fibration.*

Proof. The **only if** part is obvious and the **if** part follows from Lemma 6.1.10. □

Lemma 6.1.12 (Two-hopullback lemma). *Suppose given a commutative diagram:*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Z \arrow[d] \\ V \arrow[r] & Y \arrow[r] & W \end{tikzcd}$$

in a fibration category such that the right hand side square is a homotopy pullback. Then the left hand side square is a homotopy pullback if and only if the outer rectangle is a homotopy pullback.

Proof. This follows immediately by the standard two-pullbacks lemma. □

6.2 PARTIAL REEDY STRUCTURES (IN ACTION)

6.2.1. Partial Reedy structures are fibration category structures on a (subcategory of a) diagram category $\mathcal{C}^J$, but they depend not only on J, but also on a functor $f\colon I \to J$ of inverse categories. These structures are not necessarily of independent interest, but are rather a technical device introduced to prove Corollary 6.2.11.

6.2.2 (Reedy f-fibrant diagrams). Given a map $s\colon I \to J$ of inverse categories and a fibration category $\mathcal{C}$, define $\mathcal{C}^J_{f\text{-R}}$ as the full subcategory of $\mathcal{C}^J$ consisting of diagrams $X\colon J \to \mathcal{C}$ such that $f^*X\colon I \to \mathcal{C}$ is Reedy fibrant. We will call such diagrams **Reedy f-fibrant**.

6.2.3. If $f\colon I \hookrightarrow J$ is an inclusion, then a Reedy f-fibrant diagram $X\colon J \to \mathcal{C}$ is a diagram whose restriction $X|I$ is Reedy fibrant.

6.2.4 (Reedy f-fibrations). Let $f\colon I \to J$ and $\mathcal{C}$ be as above, a natural transformation $F\colon X \to Y$ between two Reedy f-fibrant diagrams $X, Y\colon J \to \mathcal{C}$ is a **Reedy f-fibration** if it is a fibration in the levelwise structure on $\mathcal{C}^J$ and the induced map $f^*(F)\colon f^*(X) \to f^*(Y)$ is a Reedy fibration in $\mathcal{C}^I_{\mathrm{R}}$.

Theorem 6.2.5. *Let $\mathcal{C}$ be a fibration category and $f\colon I \to J$ a functor between inverse categories. Then the category $\mathcal{C}^J_{f\text{-R}}$ with Reedy f-fibrations as fibrations and levelwise weak equivalences as weak equivalences is a fibration category.*

Proof. Analogous to the proof of Theorem 5.2.17. □

Examples 6.2.6.

1. If $I = \emptyset$, that is, f is the inclusion of the empty category, then $\mathcal{C}^J_{f\text{-R}}$ is simply the levelwise structure $\mathcal{C}^J$.
2. If $f = 1_J \colon J \to J$, then $\mathcal{C}^J_{1_J\text{-R}}$ is the Reedy fibration category structure $\mathcal{C}^J_{\mathrm{R}}$.

Theorem 6.2.7. *Let $f \colon I \to J$ be a map of homotopical inverse categories. Then the inclusions*

$$\mathcal{C}^J_{\mathrm{R}} \hookrightarrow \mathcal{C}^J_{f\text{-R}} \hookrightarrow \mathcal{C}^J$$

are equivalences of fibration categories.

Proof. The composite of these maps is an equivalence by Theorem 5.2.18, thus, by 2-out-of-3, it suffices to show that one of the two maps is an equivalence. The proof of Theorem 5.2.18 shows that $\mathcal{C}^J_{\mathrm{R}} \hookrightarrow \mathcal{C}^J_{f\text{-R}}$ is so. □

6.2.8. Let us once again emphasize that partial Reedy structures are of no independent interest; they are, however, a useful tool, allowing us to prove the following results.

Lemma 6.2.9. *Let $\mathcal{C}$ be a fibration category and let*

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "f"'] & J \arrow[d, "g"] \\ K \arrow[r, hook] & L \end{tikzcd}$$

be a pushout square of direct categories such that such that the maps $I \hookrightarrow J$ and $K \hookrightarrow L$ are sieves, and $I \to K$ and $J \to L$ satisfy the assumptions of the Exactness Criterion 5.2.22. Then there is an induced square homotopy pullback square of fibration categories and exact functors:

$$\begin{tikzcd} \mathcal{C}^{L^{\mathrm{op}}}_{\mathrm{R}} \arrow[r, "g^*"] \arrow[d] & \mathcal{C}^{J^{\mathrm{op}}}_{\mathrm{R}} \arrow[d] \\ \mathcal{C}^{K^{\mathrm{op}}}_{\mathrm{R}} \arrow[r, "f^*"] & \mathcal{C}^{I^{\mathrm{op}}}_{\mathrm{R}} \end{tikzcd}$$

Proof. A morphism $\varphi\colon X \to Y$ of Reedy g-fibrant diagrams $X, Y\colon L^{\mathrm{op}} \to \mathcal{C}$ is a Reedy g-fibration, then its restriction $\varphi|K^{\mathrm{op}}\colon X|K^{\mathrm{op}} \to Y|K^{\mathrm{op}}$ is a Reedy f-fibration. For it suffices to show that the restriction $\varphi|I^{\mathrm{op}}$ is Reedy fibrant, but that follows from the fact that $\varphi|J^{\mathrm{op}}$ is Reedy fibrant and the map $I \hookrightarrow J$ is a sieve. Similarly, if a diagram $X\colon L^{\mathrm{op}} \to \mathcal{C}$ is Reedy g-fibrant, then $X|K^{\mathrm{op}}$ is Reedy f-fibrant.

It follows that the square:

$$\begin{tikzcd} \mathcal{C}^{L^{\mathrm{op}}}_{g\text{-R}} \arrow[r, "g^*"] \arrow[d] & \mathcal{C}^{J^{\mathrm{op}}}_{\mathrm{R}} \arrow[d] \\ \mathcal{C}^{K^{\mathrm{op}}}_{f\text{-R}} \arrow[r, "f^*"] & \mathcal{C}^{I^{\mathrm{op}}}_{\mathrm{R}} \end{tikzcd}$$

is a pullback of fibration categories and exact functors. Thus, by Theorem 6.2.7, the outer square

$$\begin{tikzcd} \mathcal{C}^{L^{\mathrm{op}}}_{\mathrm{R}} \arrow[r, hook, "\sim"] \arrow[d] & \mathcal{C}^{L^{\mathrm{op}}}_{g\text{-R}} \arrow[r, "g^*"] \arrow[d] & \mathcal{C}^{J^{\mathrm{op}}}_{\mathrm{R}} \arrow[d] \\ \mathcal{C}^{K^{\mathrm{op}}}_{\mathrm{R}} \arrow[r, hook, "\sim"] & \mathcal{C}^{K^{\mathrm{op}}}_{f\text{-R}} \arrow[r, "f^*"] & \mathcal{C}^{I^{\mathrm{op}}}_{\mathrm{R}} \end{tikzcd}$$

is a homotopy pullback. $\square$

Corollary 6.2.10. *In the situation of Lemma 6.2.9, if $I \hookrightarrow K$ is a sieve, then the resulting square is a pullback (not just a homotopy pullback).*

Proof. We need to show that a morphism $f\colon X \to Y$ of diagrams $X, Y\colon L^{\mathrm{op}} \to \mathcal{C}$ is a Reedy fibration if and only if it is a Reedy fibration when restricted to both J and K. Given $j \in J$, we have that the canonical map $\partial(J/j) \to \partial(L/j)$ is an isomorphism, since $J \hookrightarrow L$ is a sieve. Similarly, for any $k \in K$, the map $\partial(K/k) \longrightarrow \partial(L/k)$ is an isomorphism. This proves the **only if** part. For the **if** part, we simply use the fact that L is a pushout and hence for each $l \in L$, we have either $l \in J$ or $l \in K$. $\square$

Corollary 6.2.11. *Let $\mathcal{C}$ be a fibration category and consider a pushout square of direct categories:*

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "f"'] & J \arrow[d, "g"] \\ K \arrow[r] & L \end{tikzcd}$$

in which $I \hookrightarrow K$ is a sieve and f satisfies the assumptions of the Exactness Criterion 5.2.22 and suppose that $f^* \colon \mathcal{C}_{\mathrm{R}}^{K^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{R}}^{I^{\mathrm{op}}}$ is an equivalence. Then $g^* \colon \mathcal{C}_{R}^{L^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{R}}^{J^{\mathrm{op}}}$ is an equivalence of fibration categories.

Proof. By Lemma 6.2.9, g^* is a homotopy pullback of f^*. But a homotopy pullback of a weak equivalence is again a weak equivalence (Corollary 6.1.11), which completes the proof. $\square$

7.0 PROPERTIES OF THE QUASICATEGORY OF FRAMES

We will now investigate the properties of the quasicategory of frames that are crucial from the viewpoint of the proof of Joyal's conjecture. In Section 7.1, we will prove an important technical lemma establishing a criterion for a simplicial map to be a categorical equivalence between quasicategories of frames. Then, we will show that the assignment $\mathcal{C} \mapsto \mathrm{N_f}\mathcal{C}$, in a suitable sense and under suitable assumptions, preserves slices (Section 7.2) and adjoints (Section 7.3).

7.1 EQUIVALENCES INVOLVING QUASICATEGORIES OF FRAMES

7.1.1. In this section, we will prove only one lemma, but an extremely important one from the point of view of the results that we are aiming to prove. Indeed, this lemma explains how to combine the Criterion 3.3.32 with Theorem 5.3.18 to prove that a map of quasicategories is an equivalence, by working on the level of fibration categories.

Almost every major theorem in the remainder of the thesis will in some way rely on this lemma.

Lemma 7.1.2.

1. *Let $\mathcal{C}$ be a fibration category, P a poset, and $K \hookrightarrow \mathrm{N}P$ an inclusion of simplicial sets. Let $X, Y \colon K \to \mathrm{N_f}\mathcal{C}$ be maps of simplicial sets and $f \colon X \to Y$ a natural weak equivalence between corresponding functors $DK^{\mathrm{op}} \to \mathcal{C}$. Then f induces a $E[1]$-homotopy (see Paragraph 3.3.31) between X and Y.*
2. *Moreover, if there is $L \hookrightarrow K$ such that $X|L = Y|L$ and for all $\varphi \in DL^{\mathrm{op}}$, $f_\varphi = \mathrm{id} \in \mathcal{C}_{\mathrm{h}}^{\widehat{[1]}}$,*

then this $E[1]$-homotopy can be chosen relative to L.

7.1.3. The proof of Lemma 7.1.2 will be preceded by a short discussion. Let $\mathcal{C}$ be a homotopical category and J a small category. A natural transformation f between functors $X, Y\colon J \to \mathcal{C}$ can be seen as a functor

$$f\colon J \times [1] \to \mathcal{C}$$

such that $f(j,0) = X(j)$ and $f(j,1) = Y(j)$. Such a natural transformation is a *natural weak equivalence* if it homotopical, regarded as functor $f\colon J \times \widehat{[1]} \to \mathcal{C}$, where J has only trivial weak equivalences.

Alternatively, we may view a natural transformation $X \to Y$ as a functor:

$$f\colon J \to \mathcal{C}^{[1]}$$

such that $f(j)(0) = X(0)$ and $f(j)(1) = Y(j)$. Again, if we replace $[1]$ with $\widehat{[1]}$ and consider only the subcategory $\mathcal{C}_{\mathrm{h}}^{\widehat{[1]}} \subseteq \mathcal{C}^{[1]}$ of homotopical diagrams, we get an equivalent description of a natural weak equivalence i.e. as a functor $f\colon J \to \mathcal{C}_{\mathrm{h}}^{\widehat{[1]}}$.

In light of the above discussion, the additional condition in (2) of the Lemma 7.1.2, asserts that f, regarded as a functor $f\colon J \to \mathcal{C}^{\widehat{[1]}}$, for each $j \in J$, takes value $f_j = 1_{X(j)}$.

Proof. The composite

$$H\colon D(K \times \widehat{[1]})^{\mathrm{op}} \hookrightarrow DK^{\mathrm{op}} \times \widehat{[1]} \xrightarrow{f} \mathcal{C}$$

is a homotopical diagram and the restrictions $H|D(K\times\{0\})^{\mathrm{op}} = X$ and $H|D(K\times\{1\})^{\mathrm{op}} = Y$ are Reedy fibrant. Moreover, the inclusions: $D(K \times \{i\}) \hookrightarrow D(K \times \widehat{[1]})$ are jointly sieve for $i = 0, 1$. Thus by Lemma 5.4.2, we may extend it to a homotopical Reedy fibrant diagram $\overline{H}\colon D(K \times \widehat{[1]})^{\mathrm{op}} \to \mathcal{C}$ such that $\overline{H}|D(K \times \{0\})^{\mathrm{op}} = X$ and $\overline{H}|D(K \times \{1\})^{\mathrm{op}} = Y$.

By Theorem 5.3.18, this gives a simplicial map $\overline{H}\colon K \times \Delta[1] \to \mathrm{N_f}(\mathcal{C})$ such that for each $x \in K$, $\overline{H}|\{x\} \times \Delta[1]$ picks out an equivalence. By Paragraph 3.3.31 this gives the desired $E[1]$-homotopy and hence proves (1).

For (2), we notice that the corresponding restriction $D(L \times \Delta[1])^{\mathrm{op}} \to \mathcal{C}$ must factor as:

$$D(L \times \Delta[1])^{\mathrm{op}} \to DL^{\mathrm{op}} \to \mathcal{C}.$$

But by assumption $f|(DL^{\mathrm{op}} \times \widehat{[1]})$ factors through:

$$DL^{\mathrm{op}} \times \widehat{[1]} \to DL^{\mathrm{op}} \to \mathcal{C}.$$

The result thus follows by commutativity of the following triangle:

$$\begin{tikzcd} D(L \times \Delta[1]) \arrow[rr] \arrow[dr] & & DL \times [1] \arrow[dl] \\ & DL & \end{tikzcd}$$

□

7.2 SLICES OF THE QUASICATEGORY OF FRAMES

7.2.1. Our main goal in this section is to show the following theorem, describing the slices of the quasicategory $\mathrm{N_f}(\mathcal{C})$ in terms of the *fibered* slices $\mathcal{C}(A)$ as described in Proposition 6.1.4.

Theorem 7.2.2. *Let $\mathcal{C}$ be a fibration category and let $A \colon D[0]^{\mathrm{op}} \to \mathcal{C}$ be a 0-simplex in $\mathrm{N_f}(\mathcal{C})$. Then there is an equivalence of quasicategories:*

$$\mathrm{N_f}(\mathcal{C})/A \simeq \mathrm{N_f}(\mathcal{C}(A_0))$$

Proof. We will begin by defining a map $F \colon \mathrm{N_f}(\mathcal{C})/A \to \mathrm{N_f}(\mathcal{C}(A_0))$ that will be an equivalence.

By Theorem 5.3.18, an n-simplex in $\mathrm{N_f}(\mathcal{C})/A$ is given by a homotopical Reedy fibrant diagram $B \colon D[n+1]^{\mathrm{op}} \to \mathcal{C}$ such that

$$B_{\underbrace{n \ldots n}_{k \text{ times}}} = A_{k-1},$$

where $B_{\underbrace{n \ldots n}_{k \text{ times}}}$ denotes the value of B on a function $[k] \to [n]$ constant equal to n. In particular, $B_n = A_0$.

On the other hand, an n-simplex in $\mathrm{N_f}(\mathcal{C}(A_0))$ is a homotopical, Reedy fibrant diagram $B \colon D[n]^{\mathrm{op}} \to \mathcal{C}(A_0)$. Equivalently, it is a homotopical, Reedy fibrant diagram $B \colon D[n]^{\mathrm{op}} \to$

$\mathcal{C}$, together with, for each $([k], \varphi\colon [k] \to [n])$, a fibration $B_\varphi \to A_0$ such that for any $\varphi'\colon [k'] \to [n]$ injective, order-preserving function $f\colon [k] \to [k']$ the following triangle commutes:

$$\begin{tikzcd} B_{\varphi'} \arrow[rr, "B_f"] \arrow[dr] & & B_{\varphi'} \arrow[dl] \\ & A_0 & \end{tikzcd}$$

For an n-simplex $B\colon D[n+1]^{\mathrm{op}} \to \mathcal{C}$ in $\mathrm{N_f}(\mathcal{C})/A$, define $F(B)\colon D[n]^{\mathrm{op}} \to \mathcal{C}$ by setting:

$$F(B)_{a_1 a_2 \dots a_i} := B_{a_1 a_2 \dots a_i n}$$

For every sequence of the form $(a_1 a_2 \dots a_i n)$, there is a canonical map $B_{a_1 a_2 \dots a_i n} \to B_n = A_0$. Since B was Reedy fibrant, this map is a fibration and the map F is therefore well-defined.

Our next goal is to prove that $F\colon \mathrm{N_f}(\mathcal{C})/A \to \mathrm{N_f}(\mathcal{C}(A_0))$ is an equivalence. By Lemma 3.3.32, we need to find a filler for the following diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "X"] \arrow[d, hook] & \mathrm{N_f}(\mathcal{C})/A \arrow[d, "F"] \\ \Delta[n] \arrow[r, "X'"] & \mathrm{N_f}(\mathcal{C}(A_0)) \end{tikzcd}$$

As explained in Paragraph 3.3.9, a simplicial map $X\colon \partial\Delta[n] \to \mathrm{N_f}(\mathcal{C})/A$ corresponds naturally to a map $\partial\Delta[n] \star \Delta[0] = \Lambda^{n+1}[n+1] \to \mathrm{N_f}(\mathcal{C})$, whose restriction to the $(n+1)$-st vertex is A. By Theorem 5.3.18, this in turn corresponds naturally to a diagram:

$$X\colon D(\Lambda^{n+1}[n+1])^{\mathrm{op}} \to \mathcal{C}.$$

We may view $D(\Lambda^{n+1}[n+1])$ as a subcategory of $D[n+1]$, consisting of those monotone functions $\varphi\colon [k] \to [n+1]$ that skip some $i \in [n] \subseteq [n+1]$. That is,

$$D(\Lambda^{n+1}[n+1]) = \{[k] \xrightarrow{\varphi} [n+1] \mid \text{there exists } i \leq n \text{ s.th. } i \notin \mathrm{im}(\varphi)\}$$

Via this representation:

$$X_{\underbrace{(n+1)\dots(n+1)}_{k \text{ times}}} = A_{\underbrace{0\dots0}_{k \text{ times}}}.$$

By Theorem 5.3.18, $X'\colon \Delta[n] \to \mathrm{N_f}(\mathcal{C}(A_0))$ corresponds naturally to a diagram $D[n]^{\mathrm{op}} \to \mathcal{C}(A_0)$. This in turn, corresponds naturally to a diagram $X'\colon D^a[n]^{\mathrm{op}} \to \mathcal{C}$ with $X'_\emptyset = A_0$.

As above, we may view $D^a[n]$ as a full subcategory of $D[n+1]$ via the inclusion sending $\varphi\colon [k] \to [n] \in D[n]$ to $\varphi'\colon [k+1] \to [n+1]$ defined by:

$$\varphi'(i) = \begin{cases} \varphi(i) & \text{if } i \leq k \\ n+1 & \text{if } i = k+1 \end{cases}$$

That is,

$$D^a[n] = \{[k] \xrightarrow{\varphi} [n+1] \mid \varphi(k) = n+1 \text{ and } \varphi(k-1) \leq n\} \cup \{\emptyset \to [n+1]\}.$$

Thus, viewing X' as defined on this subcategory of $D[n+1]^{\mathrm{op}}$, we know that $X'_{n+1} = A_0$.

We seek an extension:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "X"] \arrow[d, hook] & \mathrm{N_f}(\mathcal{C})/A \arrow[d, "F"] \\ \Delta[n] \arrow[r, "X'"'] \arrow[ur, dotted, "\widetilde{X}"] & \mathrm{N_f}(\mathcal{C}(A_0)) \end{tikzcd}$$

hence, by Theorem 5.3.18, a diagram $\widetilde{X}\colon D[n+1]^{\mathrm{op}} \to \mathcal{C}$ such that:

- $\widetilde{X}|D(\Lambda^{n+1}[n+1]) = X'$ and
- $\widetilde{X} \sim_{E[1]} X$ relative to the boundary.

By Theorem 5.3.15, it suffices to find an extension $\widetilde{X}\colon D(\partial\Delta[n])^{\mathrm{op}} \cup \mathrm{Sd}[n+1]^{\mathrm{op}} \to \mathcal{C}$.

To define $\widetilde{X}$, let us first consider the case when $\varphi\colon [k] \to [n+1] \in D(\partial\Delta[n])$ i.e. φ is not surjective. Then we set

$$\widetilde{X}_\varphi = \begin{cases} X_\varphi & \text{if } \varphi(k) = n+1 \\ X'_{\varphi'} & \text{if } \varphi(k) < n+1 \end{cases}$$

where $\varphi'\colon [k+1] \to [n+1]$ is given by:

$$\varphi'(i) = \begin{cases} \varphi(i) & \text{if } i \leq k \\ n+1 & \text{if } i = k+1 \end{cases}$$

It remains to define $\widetilde{X}_{1_{[n+1]}}$. In order to obtain it, take the factorization of the induced map:

$$X'_{1_{[n+1]}} \to \lim(\mathcal{P}_{\mathrm{n}}([n+1])^{\mathrm{op}} \xrightarrow{\widetilde{X}} \mathcal{C}).$$

Indeed, by construction of $\widetilde{X}$ on the boundary, $X'_{1_{[n+1]}}$ admits a map to each component of the diagram and hence to the limit. This gives an extension of $\widetilde{X}$ to $\mathrm{Sd}[n+1]^{\mathrm{op}}$ and consequently to $D[n+1]^{\mathrm{op}}$.

It is clear that the upper triangle in the diagram above commutes. The lower triangle commutes up to $E[1]$-homotopy relative to the boundary by Proposition 5.4.3. $\square$

7.3 PRESERVATION OF ADJOINTS

7.3.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a homotopical functor between fibration categories. We define a simplicial set $\mathcal{C}_F$ by:

$$(\mathcal{C}_F)_m := \{(X \in (\mathrm{N_f}\mathcal{C})_m, Y \in (\mathrm{N_f}\mathcal{D})_m, w\colon FX \xrightarrow{\sim} Y)\}$$

Intuitively, an m-simplex of $\mathcal{C}_F$ consists of an m-simplex of $X \in (\mathrm{N_f}\mathcal{C})$, together with a fibrant replacement of FX.

Lemma 7.3.2. *For any homotopical functor $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories, the canonical projection $\mathcal{C}_F \to \mathrm{N_f}(\mathcal{C})$ is an acyclic fibration.*

Proof. We need to find a lift for the following square:

$$\begin{tikzcd} \partial\Delta[m] \arrow[r] \arrow[d, hook] & \mathcal{C}_F \arrow[d] \\ \Delta[m] \arrow[r] & \mathrm{N_f}(\mathcal{C}) \end{tikzcd}$$

Thus by Theorem 5.3.18, we are given:

- homotopical and Reedy fibrant $X\colon D[m]^{\mathrm{op}} \to \mathcal{C}$ (bottom map)
- homotopical and Reedy fibrant $Y\colon D(\partial\Delta[m])^{\mathrm{op}} \to \mathcal{D}$, together with a natural weak equivalence $w\colon F \cdot X|D(\partial\Delta[m])^{\mathrm{op}} \to Y$ (top map).

We need to find extensions of Y and w to $D[m]^{\mathrm{op}}$. This follows from Lemma 5.4.4. $\square$

Corollary 7.3.3. *$\mathcal{C}_F$ is a quasicategory.*

7.3.4. This allows us to define the value of $\mathrm{N_f}$ on a homotopical functor $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories that is not necessarily exact. Indeed, by Lemma 7.3.2, the canonical projection $\mathcal{C}_F \to \mathrm{N_f}(\mathcal{C})$ is an acyclic fibration, and hence admits a section. Postcomposing this section with the projection $\mathcal{C}_F \to \mathrm{N_f}(\mathcal{D})$ yields a map $\mathrm{N_f}(\mathcal{C}) \to \mathrm{N_f}(\mathcal{D})$ that we may take as $\mathrm{N_f}(F)$.

7.3.5. Of course, since the definition of $\mathrm{N_f}(F)$ involved choosing some sections, it is not reasonable to expect that this assignment is strictly functorial. However, it is functorial in an appropriate up-to-homotopy sense. More precisely, for any composable pair of homotopical functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$, there exists a quasicategory $\mathcal{H}$ fitting into the following Reedy fibrant fraction:

$$\begin{tikzcd}
 & & \mathcal{D} & & \\
 & \mathcal{C}_F \arrow[ur] \arrow[dl] & & \mathcal{D}_G \arrow[ul] \arrow[dr] & \\
 & & \mathcal{H} \arrow[ul] \arrow[ur] \arrow[d] & & \\
\mathcal{C} & & \mathcal{C}_{GF} \arrow[ll] \arrow[rr] & & \mathcal{E}
\end{tikzcd}$$

Indeed, we may define $\mathcal{H}$ as the pullback:

$$\begin{tikzcd}
\mathcal{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_F \arrow[d] \\
\mathcal{D}_G \arrow[r] & \mathrm{N_f}(\mathcal{D})
\end{tikzcd}$$

so, explicitly, m-simplices of $\mathcal{H}$ are quintuples:

$$(X \in \mathrm{N_f}\mathcal{C}, Y \in \mathrm{N_f}\mathcal{D}, Z \in \mathrm{N_f}\mathcal{E}, w\colon FX \xrightarrow{\sim} Y, u\colon GY \xrightarrow{\sim} Z).$$

Lemma 7.3.6. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be adjoint functors between fibration categories in which all objects are cofibrant such that F is exact and G is homotopical. Then a pair of morphism $f, g\colon X \to GA$ is homotopic if and only if their adjoint transposes $\overline{f}, \overline{g}\colon FX \to A$ are homotopic.*

7.3.7. Note that by Theorem 5.1.11, the easiest way to establish that two maps are homotopic is to find a right-homotopy between them. If we knew that the right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ is exact, our proof would follow this idea. Unfortunately, we only know that G preserves weak equivalences, hence a more refined proof is needed.

Proof. First, assume that $\overline{f}, \overline{g}\colon FX \to A$ are homotopic and let choose a path object for A in $\mathcal{C}$:

$$\begin{tikzcd} & \widetilde{A} \arrow[d] \\ A \arrow[ur, bend left] \arrow[r, "\Delta_A"'] & A \times A \end{tikzcd}$$

Then we have

$$\begin{tikzcd} & \widetilde{A} \arrow[d] \\ FX \arrow[ur, bend left] \arrow[r, "(\overline{f},\overline{g})"'] & A \times A \end{tikzcd}$$

Taking the adjoint transpose of this diagram, we get:

$$\begin{tikzcd} & G\widetilde{A} \arrow[d] \\ X \arrow[ur, bend left] \arrow[r, "(f,g)"'] & GA \times GA \end{tikzcd}$$

since G preserves products. This establishes f and g as homotopic.

Conversely, assume that $f, g\colon X \to GA$ are homotopic. Factor the map $G(\widetilde{A}) \to GA \times GA$ as a weak equivalence followed by a fibration $G(\widetilde{A}) \xrightarrow{\sim} \widetilde{G(\widetilde{A})} \to GA \times GA$. By assumption there is a homotopy $H\colon X \to \widetilde{G(\widetilde{A})}$ that we can factor into a weak equivalence followed by a fibration:

$$\begin{tikzcd} \widetilde{X} \arrow[r] & \widetilde{G(\widetilde{A})} \arrow[d] \\ X \arrow[u] \arrow[ur, "H"] \arrow[r, "(f,g)"'] & GA \times GA \end{tikzcd}$$

Taking the pullback we obtain:

$$\begin{tikzcd}
\widetilde{X} \times_{\widetilde{G(A)}} G(\widetilde{A}) \arrow[r] \arrow[d] & G(\widetilde{A}) \arrow[d] \\
\widetilde{X} \arrow[r] & \widetilde{G(A)} \arrow[d] \\
X \arrow[u] \arrow[ur] \arrow[r, "{(f,g)}"'] & GA \times GA
\end{tikzcd}$$

Finally, composing the right hand side map gives us a square:

$$\begin{tikzcd}
\widetilde{X} \times_{\widetilde{G(A)}} G(\widetilde{A}) \arrow[r] \arrow[d] & G(\widetilde{A}) \arrow[dd] \\
\widetilde{X} \arrow[d, leftrightarrow] \arrow[dr] & \\
X \arrow[r, "{(f,g)}"'] & GA \times GA
\end{tikzcd}$$

which we can take the adjoint transpose of:

$$\begin{tikzcd}
F(\widetilde{X} \times_{\widetilde{G(A)}} G(\widetilde{A})) \arrow[r] \arrow[d] & \widetilde{A} \arrow[dd] \\
F(\widetilde{X}) \arrow[d, leftrightarrow] \arrow[dr] & \\
FX \arrow[r, "{(\overline{f},\overline{g})}"'] & A \times A
\end{tikzcd}$$

$\square$

7.3.8. If in Lemma 7.3.6, we assume that G is exact, we could drop the assumption that all objects of $\mathcal{C}$ and $\mathcal{D}$ are cofibrant and instead proceed as follows. We begin by choosing a path object for A in $\mathcal{C}$:

$$\begin{tikzcd}
& P(A) \arrow[d] \\
A \arrow[ur, bend left] \arrow[r, "\Delta_A"'] & A \times A
\end{tikzcd}$$

By assumption and exactness of R, there exists a commutative square of the form:

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d, two heads, "\sim"'] & R(P(A)) \arrow[d] & \\
X \arrow[r, "(f,g)"'] & R(A) \times R(A) \arrow[r, phantom, "\cong"] & R(A \times A)
\end{tikzcd}$$

By adjointness and exactness of L, we get:

$$\begin{tikzcd}
L(\widetilde{X}) \arrow[r] \arrow[d, two heads, "\sim"'] & P(A) \arrow[d] \\
LX \arrow[r, "(\bar{f},\bar{g})"'] & A \times A
\end{tikzcd}$$

Theorem 7.3.9. *Let $F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors between fibration categories with all object cofibrant. Assume that G is homotopical and F is exact. Then $\mathrm{N_f}(G) \colon \mathrm{N_f}(\mathcal{D}) \to \mathrm{N_f}(\mathcal{C})$ is a right adjoint in $\mathrm{q}\mathcal{C}\mathrm{at}$ (with left adjoint $\mathrm{N_f}(F)$).*

7.3.10. The proof of this theorem will be preceded by a discussion that will explain the proof strategy and will address a special case. By Proposition 3.3.26, a functor $G \colon \mathcal{D} \to \mathcal{C}$ is a right adjoint if for all $x \in \mathcal{C}$ the slice quasi-category $x \downarrow G$ has an initial object (the unit of the adjunction). Thus a 1-simplex $x \to G(a)$ of $\mathcal{C}$ is an initial object of $x \downarrow G$ if for all $n > 0$ and any map $\Delta[0] \star \partial\Delta[n] \to x \downarrow \mathrm{N_f}(G)$ there exists an extension:

$$\begin{tikzcd}
\Delta[0] \star \partial\Delta[n] \arrow[r] \arrow[d, hook] & x \downarrow \mathrm{N_f}(G) \\
\Delta[0] \star \Delta[n] \arrow[ur, dotted] &
\end{tikzcd}$$

Let us start by describing the initial object. Since $F \dashv G$, there is a natural transformation $\eta \colon 1_{\mathcal{C}} \to G \cdot F$. We will use it in order to produce the desired 1-simplex.

7.3.11. Let $X \in \mathrm{N_f}(\mathcal{C})$ be a 0-simplex (that is, a frame). To construct the lift in the diagram of Paragraph 7.3.10, by Theorem 5.3.15 it suffices to find a Reedy fibrant fraction $X_0 \leftarrow X_{01} \rightarrow GF(X_0) =: X_1$. Define a factorization of the map $X_0 \to X_0 \times X_1$ as a weak equivalence followed by a fibration:

$$\begin{tikzcd}
X_0 \arrow[rr, "\langle 1_{X_0}, \eta_{X_0} \rangle"] \arrow[dr, "w"'] & & X_0 \times X_1 \\
& X_{01} \arrow[ur, "\langle p_0, p_1 \rangle"'] &
\end{tikzcd}$$

Notice that since G was not required to be exact, we may need to fibrantly replace the frame, however without changing the value GFX_0. Thus, thanks to Theorem 5.3.15, we will be able to use the universal property of η in the proof.

7.3.12. We shall verify the universal property of the unit constructed in Paragraph 7.3.11. Before giving the proof in full generality, we consider the case $n = 1$ separately as our working example. This proof will span Paragraphs 7.3.12–7.3.16. Unwinding the definition, we see that, by Theorem 5.3.15, given:

$$\begin{tikzcd}
& & X_1 & & \\
& X_{01} \arrow[ur] \arrow[dl] & & & \\
X_0 & & X_{02} \arrow[ll] \arrow[rr] & & X_2
\end{tikzcd}$$

where we have $X_1 = G(B_1)$ (and, in turn, $B_1 = F(X_0)$) and $X_2 = G(B_2)$, we have to find a Reedy fibrant fraction $F(X_0) = B_1 \leftarrow B_{12} \rightarrow B_2$ and X_{012} with the appropriate maps, completing the above diagram to a homotopical Reedy fibrant diagram on $\mathrm{Sd}[2]^{\mathrm{op}}$:

$$\begin{tikzcd}
& & X_1 & & \\
& X_{01} \arrow[ur] \arrow[dl] & & X_{12} \arrow[ul] \arrow[dr] & \\
& & X_{012} \arrow[ul] \arrow[ur] \arrow[d] & & \\
X_0 & & X_{02} \arrow[ll] \arrow[rr] & & X_2
\end{tikzcd}$$

where $X_{12} = G(B_{12})$.

7.3.13. First, we define $\widetilde{X}_{012}$ as the pullback:

$$\begin{tikzcd}
\widetilde{X}_{012} \arrow[r] \arrow[d] & X_{01} \arrow[d] \\
X_{02} \arrow[r] & X_0
\end{tikzcd}$$

Note that all maps in the above diagram are acyclic fibrations.

7.3.14. Next, by the universal property of η, there exists a unique map $\widetilde{B}_{12} := F(\widetilde{X}_{012}) \to B_1 \times B_2$ making the following square commute:

$$\begin{tikzcd}
\widetilde{X}_{012} \arrow[r, "\eta_{\widetilde{X}_{012}}"] \arrow[d] & GF(\widetilde{X}_{012}) \arrow[d, dotted] \\
X_{01} \times X_{02} \arrow[r] & G(B_1) \times G(B_2)
\end{tikzcd}$$

We claim that up to Reedy fibrancy of the resulting diagrams, the objects $\widetilde{X}_{012}$ and $\widetilde{B}_{12}$ with the maps defined above give the extension required in the universal property. To see this, it remains to verify that the map $\widetilde{B}_{01} \to B_1$ is a weak equivalence. Notice that then we could simply fibrantly replace the diagrams, obtaining the desired ones.

Thus, let α be the map $\widetilde{B}_{01} \to B_1$.

7.3.15. Note that the map $\beta \colon \widetilde{X}_{012} \to X_0$ in the diagram of Paragraph 7.3.13 is a weak equivalence and hence so is $F\beta \colon F(\widetilde{X}_{012}) \to F(X_0)$. By Lemma 7.3.6, since a map homotopic to a weak equivalence is itself a weak equivalence, it suffices to show that the adjoint transposes of α and $F\beta$ are homotopic.

We will construct these transposes explicitly. By the universal property of η, there exists a unique morphism $F(\widetilde{X}_{012}) \to F(X_0)$ making the diagram

$$\begin{tikzcd}
\widetilde{X}_{012} \arrow[rr, "\eta_{\widetilde{X}_{012}}"] \arrow[d] & & GF(\widetilde{X}_{012}) \arrow[dddd, dotted] \\
X_{01} \times X_{02} \arrow[d] & & \\
X_{01} \arrow[dr, "\sim"] & & \\
& X_0 \arrow[dr, "\eta_{X_0}"] & \\
& & GF(X_0)
\end{tikzcd}$$

commute.

But by naturality of η, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{X}_{012} \arrow[r, "\eta_{\widetilde{X}_{012}}"] \arrow[d, "\beta"'] & GF(\widetilde{X}_{012}) \arrow[d, "GF\beta"] \\
X_0 \arrow[r, "\eta_{X_0}"] & GF(X_0)
\end{tikzcd}$$

and hence $F\beta$ is the adjoint transpose of the composite:

$$\widetilde{X}_{012} \longrightarrow X_{01} \times X_{02} \longrightarrow X_{01} \longrightarrow X_0 \longrightarrow GF(X_0)$$

On the other hand, by the definition of $\widetilde{B}_{12}$ and commutativity of the following diagram:

$$\begin{tikzcd}
\widetilde{X}_{012} \arrow[r, "\eta_{X_{012}}"] \arrow[d] & GF(\widetilde{X}_{012}) \arrow[d] \\
X_{01} \times X_{02} \arrow[r] \arrow[d] & G(B_1) \times G(B_2) \arrow[d] \\
X_{01} \arrow[r] & GF(X_0)
\end{tikzcd}$$

we get that the adjoint transpose of $\alpha\colon \widetilde{B}_{12} \to B_1$ is given by the composite:

$$\widetilde{X}_{012} \longrightarrow X_{01} \times X_{02} \longrightarrow X_{01} \longrightarrow GF(X_0)$$

in the above diagram.

7.3.16. Thus, in order to verify that α and $F\beta$ are homotopic, and hence complete the proof in case $n = 1$, it remains to verify that the following triangle commutes up to homotopy:

$$
\begin{tikzcd}
X_{01} \arrow[rr, "p_2"] \arrow[dr, "p_1"'] & & GF(X_0) \arrow[dd, equal] \\
& X_0 \arrow[dr, "\eta_{X_0}"'] & \\
& & GF(X_0)
\end{tikzcd}
$$

For this, it is enough to show that these two maps become equal after precomposition with some weak equivalence. Let $w\colon X_0 \to X_{01}$ be the weak equivalence obtained in the factorization defining X_{01}. Then we have:

$$
\begin{aligned}
\eta_{X_0} \cdot p_1 \cdot w &= \eta_{X_0} \\
&= p_2 \cdot w
\end{aligned}
$$

that yields the desired conclusion.

Proof of Theorem 7.3.9. Suppose given $X\colon \Delta[0] \star \partial\Delta[n] \to \mathrm{N_f}(\mathcal{C}) \downarrow G$. We wish to extend it to $\widetilde{X}\colon \Delta[0] \star \Delta[n] \to \mathrm{N_f}(\mathcal{C}) \downarrow G$. By Theorem 5.3.15, it suffices to show that given any diagram

$$X\colon \mathcal{P}_{\mathrm{n}}([n])^{\mathrm{op}} \to \mathcal{C}$$

with the property that for any $A \subsetneq (n]$, we have $X_A = G(B_A)$, where $B_A \in \mathcal{D}$ and the maps connecting such $G(B_A)$'s are also of the form $G(-)$, we need to find $X_{[1+n]} \in \mathcal{C}$ and $B_{(1+n]} \in \mathcal{D}$ together with maps extending X to $\widetilde{X}\colon \mathrm{Sd}[1+n]^{\mathrm{op}} \to \mathcal{C}$.

To simplify the notation, we will write $G(B_A)$ as X_A throughout. We start by defining $\widetilde{X}_{[1+n]}$:

$$\widetilde{X}_{[1+n]} = \lim_{0 \in A} X_A.$$

By construction, the canonical map $\widetilde{X}_{[1+n]} \to X_A$ is a fibration for any A such that $0 \in A$. By Lemma 5.4.5, it is moreover an acyclic fibration.

Given $A \subseteq [n]$ such that $0 \notin A$, there is a map $X_{A\cup\{0\}} \to X_A = G(B_A)$. Thus we obtain an induced map:

$$\lim_{0 \notin A} X_{A\cup\{0\}} \longrightarrow \lim_{0 \notin A} G(B_A) \cong G(\lim_{0 \notin A} B_A)$$

since G is a right adjoint.

By the universal property of η, there is a unique morphism

$$\widetilde{B}_{(1+n]} := F(\widetilde{X}) \longrightarrow \lim_{0 \notin A} B_A$$

making the following square commute:

$$\begin{tikzcd}
\widetilde{X}_{[1+n]} \arrow[r, "\eta_{\widetilde{X}_{[1+n]}}"] \arrow[d] & GF(\widetilde{X}_{[1+n]}) \arrow[d] \\
\lim\limits_{0 \notin A} X_{A \cup \{0\}} \arrow[r] & G(\lim\limits_{0 \notin A} B_A)
\end{tikzcd}$$

We claim that $\widetilde{X}_{[1+n]}$ and $\widetilde{B}_{(1+n]}$ as constructed above, together with the above maps, give the desired extension. It remains to verify that the canonical map $\widetilde{B}_{(1+n]} \to B_1$ is a weak equivalence.

The proof follows verbatim the proof given in the case $n = 1$ in the Paragraphs 7.3.15–7.3.16. $\square$

8.0 SIMPLICES OF THE QUASICATEGORY OF FRAMES

This chapter is devoted to the proof of the following result: given any fibration category $\mathcal{C}$ and a natural number m, there is a categorical equivalence:

$$\mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \simeq \mathrm{N_f}(\mathcal{C})^{\Delta[m]}.$$

To this end, we first prove (Section 8.1) a lemma that resembles the familiar Dold's Lemma from algebraic topology, and then using it, we show the desired result (Section 8.2). The arguments used in Section 8.2 are very combinatorial in nature and nicely demonstrate the main advantages of working with Szumiło's construction.

8.1 DOLD'S LEMMA FOR FIBRATION CATEGORIES

8.1.1 (Dold's Lemma for topological spaces)**.** Let B be a topological space consider the following commutative triangle:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

where both $X \to B$ and $Y \to B$ are Hurewicz fibrations and the map $X \to Y$ is a homotopy equivalence. Then $X \to Y$ is a fiberwise homotopy equivalence; that is, its quasi-inverse $Y \to X$ as well as the homotopies can be chosen to be fiberwise.

8.1.2. Notice that in the special case $Y = B$, Dold's Lemma asserts that an acyclic Hurewicz fibration admits a deformation section (both the section and the homotopies can be chosen fiberwise).

8.1.3. Dold's Lemma is just a special case of **Whitehead's Theorem** for model categories that says that weak equivalences between objects that are both fibrant and cofibrant are homotopy equivalences. Indeed, Dold's Lemma is exactly an instance of Whitehead's Theorem for Hurewicz model structure on the slice category $\mathcal{T}\mathrm{op}/B$.

8.1.4. Since one can talk about fibrant and cofibrant in a fibration category, there is an analogous statement of Whitehead's Theorem for an arbitrary fibration category.

We wish to apply it however to the fibration category $\mathcal{F}\mathrm{ib}$ of fibration category. In this case the statement is very interesting, because there are not *enough* cofibrant fibration categories. The word "enough" in the previous sentence does not have a precise mathematical meaning; it simply refers to the fact that we cannot use this statement.

Proposition 8.1.5. *Consider the following commutative triangle of fibration categories and exact functors:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "P"'] & & \mathcal{D} \arrow[dl, "Q"] \\ & \mathcal{B} & \end{tikzcd}$$

in which P and Q are fibrations and F is an equivalence. Suppose we have a frame $Y\colon D[0]^{\mathrm{op}} \to \mathcal{D}$. Then there exists an object $X \in \mathcal{C}$ such that $P(X) = Q(Y_0)$ and there is a fraction $F(X) \leftarrow Z \to Y_0$, in which both maps are acyclic fibrations and that under Q becomes the initial part of the frame $Q \cdot Y\colon D[0]^{\mathrm{op}} \to \mathcal{B}$ (i.e. $Q(Z) = Q(Y_{00})$ and the two maps from the fraction become $Q(Y_{00}) \rightrightarrows Q(Y_0)$).

8.1.6. Notice that it is indeed a generalization of Dold's Lemma to the case when the fibration categories $\mathcal{C}$ and $\mathcal{D}$ are not cofibrant in $\mathcal{F}\mathrm{ib}/\mathcal{B}$. If they were, we could simply choose a fiberwise quasi-inverse and take $X := G(Y_0)$.

Proof. Factor the map $(1_{\mathcal{C}}, F)\colon \mathcal{C} \to \mathcal{C} \times_{\mathcal{B}} \mathcal{D}$ as an equivalence followed by a fibration:

$$\begin{tikzcd} & \widetilde{\mathcal{C}} \arrow[dr, "{(S,T)}"] & \\ \mathcal{C} \arrow[ur, "J"] \arrow[rr, "{(1_{\mathcal{C}},F)}"] & & \mathcal{C} \times_{\mathcal{B}} \mathcal{D} \end{tikzcd}$$

Functors S and T are easily seen to be fibrations (as composites of fibrations). They are moreover acyclic fibrations. Indeed, we a commutative triangle:

$$\begin{tikzcd} & \widetilde{\mathcal{C}} \arrow[d, "T"] \\ \mathcal{C} \arrow[ur, "J"] \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

in which both J and F are equivalences, hence by 2-out-of-3 so is T. One proves similarly that S is an equivalence.

Since $T\colon \widetilde{\mathcal{C}} \to \mathcal{D}$ is an acyclic fibration and $Y \to 1$ is a fibration in $\mathcal{D}$, by Theorem 5.3.7, there exists $\widetilde{X} \in \widetilde{\mathcal{C}}$ such that $T(\widetilde{X}) = Y_0$. Define $X := S(\widetilde{X})$. By commutativity of the square:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "S"] \arrow[d, "T"'] & \mathcal{C} \arrow[d, "P"] \\ \mathcal{D} \arrow[r, "Q"] & \mathcal{B} \end{tikzcd}$$

we obtain $P(X) = Q(Y_0)$.

Since $P\colon \mathcal{C} \to \mathcal{B}$ is a fibration and we have $P(X) = Y_0$, we can find a Reedy fibrant diagram $\overline{X}_{00} \rightrightarrows X$ (in which both maps are weak equivalences) such that

$$P(\overline{X}_{00} \rightrightarrows X) = Q(Y_{00}) \rightrightarrows Q(Y_0).$$

Since $S\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is an acyclic fibration and we have $S(\widetilde{X} \times J(X)) = X \times X$, by Theorem 5.3.7, we can lift the fibration $\overline{X}_{00} \to X \times X \in \mathcal{C}$ to a fibration $\widehat{X}_{00} \to \widetilde{X} \times J(X) \in \widetilde{\mathcal{C}}$, hence a fraction $\widetilde{X} \leftarrow \widehat{X}_{00} \to J(X)$. Applying T to it we obtain the desired fraction since $T(\widetilde{X}) = Y$ and $TJ(X) = F(X)$. $\square$

8.2 SIMPLICES OF THE QUASICATEGORY OF FRAMES

8.2.1. Our main goal in this section is to establish an equivalence of quasicategories:

$$\mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \simeq \mathrm{N_f}(\mathcal{C})^{\Delta[n]}.$$

Before constructing an explicit map and proving it is an equivalence, we need a few lemmas.

8.2.2. Let P be a homotopical poset. In Paragraph 5.3.13, we defined the barycentric subdivision of P as the the homotopical poset $\mathrm{Sd}P$ of non-empty subsets of P, whose weak equivalences are created by the functor $\max\colon \mathrm{Sd}P \to P$. In the special case $P = [m]$, this functor admits a left adjoint:

$$[m] \hookrightarrow \mathrm{Sd}[m]$$

taking $i \in [m]$ to the canonical inclusion $[i] \hookrightarrow [m]$. Indeed, for a subset $S \subseteq P$ and any $i \in [m]$, we have:

$$\max S \leq i \qquad \text{iff} \qquad S \subseteq [i].$$

8.2.3. There is also a pair of adjoint functors between $\mathrm{Sd}P$ and DP, for any homotopical poset P. The obvious inclusion $\mathrm{Sd}P \hookrightarrow DP$ taking a subset $S \subseteq P$ to the inclusion $[|S|] \to P$ picking out S, admits a left adjoint $\mathrm{im}\colon DP \to \mathrm{Sd}P$ that takes a monotone function $\varphi\colon [k] \to P$ to its image $\mathrm{im}(\varphi) \subseteq P$.

8.2.4. In order to show the main theorem of this section (Theorem 8.2.25), we shall show that the following functors:

- $\mathcal{C}^{D[m]^{\mathrm{op}}} \to \mathcal{C}^{[m]^{\mathrm{op}}}$
- $\mathcal{C}^{(\mathrm{Sd}P)^{\mathrm{op}}} \hookrightarrow \mathcal{C}^{DP^{\mathrm{op}}}$
- $\mathcal{C}^{(\mathrm{Sd}P)^{\mathrm{op}}} \to \mathcal{C}^{P^{\mathrm{op}}}$

are equivalences of fibration categories.

Lemma 8.2.5. *Let $\mathcal{C}$ be a fibration category and $m \in \mathbb{N}$. Then the canonical inclusion $[m] \hookrightarrow D[m]$ given by $i \mapsto ([i] \hookrightarrow [m])$ induces an equivalence of fibration categories:*

$$\mathcal{C}^{D[m]^{\mathrm{op}}} \longrightarrow \mathcal{C}^{[m]^{\mathrm{op}}}.$$

8.2.6. Before giving the proof for an arbitrary $m \in \mathbb{N}$, we will treat the case $m = 0$ separately to explain the intuition behind the general case. (An alternative proof for the case $m = 0$ can also be found in [Sch13, Thm. 3.10].) By Lemma 5.2.25, it suffices to show that the inclusion $[0] \hookrightarrow D[0]$ is a homotopy equivalence. Since $[0]$ is the terminal category, there is only one choice of the quasi-inverse, that is, the canonical map $D[0] \to [0]$. Clearly, the composite $[0] \hookrightarrow D[0] \to [0]$ is equal to $1_{[0]}$. We have to construct a zigzag between $1_{D[0]}$ and the other composite (by construction, it is $\mathrm{const}_{[0]} \colon D[0] \to D[0]$).

To do that, define $S \colon D[0] \to D[0]$ by:

$$S[k] := [k+1]$$

(here we are using the fact that $D[0] \cong \Delta_{\mathrm{inj}}$) and $S(\varphi \colon [k+1] \to [l+1])$ is given by:

$$S\varphi(i) = \begin{cases} \varphi(i) & \text{if } i \leq k \\ l+1 & \text{for } i = k+1. \end{cases}$$

Since $[0]$ is the value of the composite $D[0] \to [0] \hookrightarrow D[0]$ on an arbitrary $[k]$ and for any $[k]$, there are obvious inclusions:

$$[k] \xhookrightarrow{\sim} S[k] \xhookleftarrow{\sim} [0]$$

Thus we obtain natural weak equivalences:

$$1_{D[0]} \xhookrightarrow{\sim} S \xhookleftarrow{\sim} \mathrm{const}_{[0]}$$

yielding the desired zigzag.

Proof of Lemma 8.2.5. Let $\iota\colon [m] \to D[m]$ be the inclusion under consideration. By Lemma 5.2.25, it suffices to show that it is a homotopy equivalence. Let $p\colon D[m] \to [m]$ be the map evaluating $\varphi\colon [k] \to [m]$ at k i.e.

$$p([k] \xrightarrow{\varphi} [m]) = \varphi(k).$$

The composite $p \cdot \iota\colon [m] \to D[m] \to [m]$ is equal to the identity. Thus to fulfill the assumptions of Lemma 5.2.25, it suffices to construct a zigzag of natural weak equivalences $\iota \cdot p \sim 1_{D[m]}$.

As in the case $m = 0$, we will construct an intermediate endofunctor S on $D[m]$ along with natural weak equivalences $1_{D[m]} \to S \leftarrow \iota \cdot p$. We can view $\varphi\colon [k] \to [m] \in D[m]$ as a non-decreasing sequence of length $k+1$ of elements of $[m]$. Define the sequence $S(\varphi)\colon [k+\varphi(k)+1] \to [m]$ as follows. In the sequence that represents φ, insert an additional occurrence of each of the numbers: $0, 1, \ldots, \varphi(k)$. Thus the resulting sequence $S(\varphi)$ will be of length $k+1+\varphi(k)+1$, hence a function $[k+\varphi(k)+1] \to [m]$. It is easy to see that S is functorial; indeed, given a commutative triangle:

$$\begin{tikzcd}
{[k]} \arrow[rr, hook, "\theta"] \arrow[dr, "\varphi"'] & & {[l]} \arrow[dl, "\psi"] \\
& {[m]} &
\end{tikzcd}$$

we may define $S(\theta)\colon [k+\varphi(k)+1] \to [l+\psi(l)+1]$ by saying that $S(\theta)$ does on what θ did on the existing occurrences and naturally extends to the additional occurrences.

Finally, notice that by definition of p and ι, the value of the composite $\iota \cdot p$ is given by the canonical inclusion:

$$\iota \cdot p([k] \xrightarrow{\varphi} [m]) = ([\varphi(k)] \hookrightarrow [m]).$$

Thus we have a commutative diagram:

$$\begin{tikzcd}
{[k]} \arrow[r, hook] \arrow[dr, "\varphi"'] & {[k+\varphi(k)+1]} \arrow[d, "S(\varphi)"] & {[\varphi(k)]} \arrow[l, hook'] \arrow[dl, "\iota \cdot p(\varphi)"] \\
& {[m]} &
\end{tikzcd}$$

in which the horizontal arrows are weak equivalences. Hence, we obtain natural weak equivalences:

$$1_{D[m]} \xhookrightarrow{\sim} S \xleftarrow{\sim} \iota \cdot p$$

what completes the proof. $\square$

8.2.7. It is easy to check that the map $[m] \hookrightarrow D[m]$ satisfies the assumptions of the Exactness Criterion 5.2.22 and hence the induced map:

$$\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{hR}}^{[m]^{\mathrm{op}}}$$

is also an equivalence of fibration categories of Reedy fibrant diagrams. We mention this result only as a side remark; it will not be used further in the proof.

Moreover, one quickly deduces the following from Lemma 8.2.5.

Corollary 8.2.8. *The inclusion $[m] \times [n] \hookrightarrow D[m] \times D[n]$ induces an equivalence of fibration categories*

$$\mathcal{C}^{(D[m]\times D[n])^{\mathrm{op}}} \longrightarrow \mathcal{C}^{([m]\times[n])^{\mathrm{op}}}.$$

8.2.9. Combining the above Corollary with Proposition 5.2.15 and the Exactness Criterion 5.2.22, one obtains that the induced map:

$$\mathcal{C}_{\mathrm{hR}}^{(D[m]\times D[n])^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{hR}}^{([m]\times[n])^{\mathrm{op}}}$$

is an equivalence of fibration categories (hence, in particular, exact).

8.2.10. We will now turn to the second of the outstanding equivalences.

Lemma 8.2.11. *Let P be a homotopical poset and $\mathcal{C}$ a fibration category. The canonical inclusion map $\mathrm{Sd}(P) \hookrightarrow DP$ induces an equivalence of fibration categories:*

$$\mathcal{C}^{DP^{\mathrm{op}}} \longrightarrow \mathcal{C}^{\mathrm{Sd}(P)^{\mathrm{op}}}.$$

Proof. As discussed in Paragraph 8.2.3, there is a pair of adjoint functors:

$$\begin{tikzcd} \mathrm{Sd}(P) \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & DP \arrow[l, bend left, "\mathrm{im}"] \end{tikzcd}$$

We claim that they are quasi-inverses as required in Lemma 5.2.25. For future reference, we will call the inclusion map $\mathrm{Sd}(P) \hookrightarrow DP$, ι. It is immediate to see that $\mathrm{im} \cdot \iota 1_{\mathrm{Sd}(P)}$. Thus it suffices to construct a zigzag of natural weak equivalences $\iota \cdot \mathrm{im} \sim 1_{DP}$.

We will mimic the proof of Lemma 8.2.5. Thus, we begin by defining a functor $S\colon DP \to DP$ as follows. Given $\varphi\colon [m] \to P$, regarded is an ordered and non-decreasing sequence of elements of p and let $S(\varphi)$ be a map

$$[m + |\mathrm{im}(\varphi)|] \longrightarrow P$$

that inserts a new occurrence of each $p \in P$ already appearing in the image of φ. The functoriality of S is verified as follows. Given a map θ between two such sequences, $S(\theta)$ acts like θ on the old occurrences and does the only possible thing it could do on the old ones.

For each $\varphi\colon [k] \to [m]$, there is a commutative diagram:

$$\begin{tikzcd} {[k]} \arrow[r, hook] \arrow[dr, "\varphi"'] & {[k + \mathrm{im}(\varphi)]} \arrow[d, "S(\varphi)"] & {[|\mathrm{im}(\varphi)|]} \arrow[l, hook'] \arrow[dl, "\iota\cdot\mathrm{im}(\varphi)"] \\ & P & \end{tikzcd}$$

in which the horizontal arrows are weak equivalences. Hence, we obtain natural weak equivalences:

$$1_{D[m]} \overset{\sim}{\hookrightarrow} S \overset{\sim}{\hookleftarrow} \iota \cdot \mathrm{im}$$

what completes the proof. $\square$

Proposition 8.2.12. *Let P be a poset and $\mathcal{C}$ a fibration category. The functor* $\max\colon \mathrm{Sd}(P) \to P$ *induces an equivalence of fibration categories:*

$$\max{}^*\colon \mathcal{C}^{P^{\mathrm{op}}} \to \mathcal{C}^{\mathrm{Sd}(P)^{\mathrm{op}}}.$$

8.2.13. The proof of this proposition will be split over several lemmas.

Lemma 8.2.14. *Let P and $\mathcal{C}$ be as in Proposition 8.2.12 and let $X\colon \mathrm{Sd}(P)^{\mathrm{op}} \to \mathcal{C}$ be a homotopical Reedy fibrant diagram. Then the restriction $X|\max^{-1}\{p\}^{\mathrm{op}}$ is again a Reedy fibrant diagram.*

Proof. Let $\downarrow p := \{q \in P | q \leq p\}$ denote the downward set of $p \in P$. We have a pair of inclusions $\max^{-1}\{p\} \hookrightarrow \mathrm{Sd}(\downarrow p) \hookrightarrow \mathrm{Sd}(P)$. The latter is a sieve, hence it suffices to show that the inclusion $\max^{-1}\{p\} \hookrightarrow \mathrm{Sd}(\downarrow p)$ satisfies the assumption of the Exactness Criterion 5.2.22.

For let $A \in \max^{-1}\{p\}$ i.e. $A \subseteq P$ satisfies $\max A = p$. We have:

$$\partial(\max{}^{-1}\{p\}/A) = \{B \subsetneq A \mid B \neq \emptyset, A \text{ and } \max B = p\}$$

and

$$\partial(\mathrm{Sd}(\downarrow p)/A) = \{B \subsetneq A \mid B \neq \emptyset\}$$

The map $\partial(\max{}^{-1}\{p\}/A) \hookrightarrow \partial(\mathrm{Sd}(\downarrow p)/A)$ factors through:

$$L := \{B \subseteq A \mid B \neq \emptyset \text{ and there exists } C \supseteq B \text{ such that } C \neq \emptyset \text{ and } \max C = p\}.$$

The inclusion $L \hookrightarrow \partial(\mathrm{Sd}(\downarrow p)/A)$ is clearly a sieve. Thus it remains to show that $\partial(\max{}^{-1}\{p\}/A) \hookrightarrow L$ is cofinal.

We will use the Cofinality Criterion 5.2.24. For let us choose $B \in L$; we need to show that the slice category $B/\partial(\max{}^{-1}\{p\}/A)$ is connected. (Notice that we slightly abused the notation here by identifying the inclusion with its image.) Explicitly, we have:

$$B/\partial(\max{}^{-1}\{p\}/A) = \{C \supseteq B \mid \max C = p\}.$$

This category has the least element, namely $B \cup \{p\}$, and hence is connected. $\square$

Lemma 8.2.15. *Let $X\colon \mathrm{Sd}(P)^{\mathrm{op}} \to \mathcal{C}$ be a homotopical, Reedy fibrant diagram. Then the right Kan extension $\mathrm{Ran}_{\max}(X)\colon P^{\mathrm{op}} \to \mathcal{C}$ exists:*

$$\begin{tikzcd}
\mathrm{Sd}(P)^{\mathrm{op}} \arrow[r, "X"] \arrow[d, "\max"'] & \mathcal{C} \\
P^{\mathrm{op}} \arrow[ur, dotted, "\mathrm{Ran}_{\max}(X)"'] &
\end{tikzcd}$$

and is given by $\mathrm{Ran}_{\max}(X)_p = \lim(X|\max{}^{-1}\{p\}^{\mathrm{op}})$.

Proof. For $p \in P$, the obvious inclusions:

$$\max{}^{-1}\{p\} \hookrightarrow \mathrm{Sd}(\downarrow p) \hookrightarrow (\max \downarrow p)^{\mathrm{op}}$$

are both cofinal (and the latter is, in fact, an ismorphism). By Lemma 8.2.14, $X|\max^{-1}\{p\}^{\mathrm{op}}$ is Reedy and hence the limit:

$$\lim(X|\max{}^{-1}\{p\}^{\mathrm{op}})$$

exists. Thus, both $\lim(X|\mathrm{Sd}(\downarrow p)^{\mathrm{op}})$ and $\lim(X|(\max \downarrow p))$ must exist as well and, by cofinality be equal.

Hence by the pointwise formula for Kan extensions [ML98a, Thm. X.5.1], we have:

$$\begin{aligned}
\mathrm{Ran}_{\max}(X)_p &= \lim(X|(\max \downarrow p)) \\
&= \lim(X|\mathrm{Sd}(\downarrow p)^{\mathrm{op}}) \\
&= \lim(X|\max{}^{-1}\{p\}^{\mathrm{op}}).
\end{aligned}$$

□

8.2.16. By Lemma 8.2.15, we have the following diagram:

$$\begin{tikzcd}
& \mathcal{C}^{P^{\mathrm{op}}} \arrow[d, "\max^*"] \\
\mathcal{C}^{\mathrm{Sd}(P)^{\mathrm{op}}}_{\mathrm{hR}} \arrow[ur, dotted, bend left, "\mathrm{Ran}_{\max}(-)"] \arrow[ur, phantom, shift left=4ex, "\Downarrow"] \arrow[r, hook, "\sim"] & \mathcal{C}^{\mathrm{Sd}(P)^{\mathrm{op}}}
\end{tikzcd}$$

Lemma 8.2.17. *For any $p \in P$, the canonical projection:*

$$\lim(X|\max{}^{-1}\{p\}^{\mathrm{op}}) \longrightarrow X_{\{p\}}$$

is a weak equivalence.

Proof. Poset $\max^{-1}\{p\}$ is a direct category with an initial object, namely $\{p\}$. Since weak equivalences in $\mathrm{Sd}(P)$ are created by $\max\colon \mathrm{Sd}(P) \to P$ and X is homotopical, Lemma 5.4.5 applies yielding the desired conclusion. □

Lemma 8.2.18. *Let $A\colon P^{\mathrm{op}} \to \mathcal{C}$ and $X\colon \mathrm{Sd}(P)^{\mathrm{op}} \to \mathcal{C}$ be Reedy fibrant. Then a map $A \to \mathrm{Ran}_{\max}(X)$ is an equivalence if and only if its transpose $\max^* A \to X$ is an equivalence.*

Proof. We need to show that the following conditions are equivalent:

1. $A_p \to \mathrm{Ran}_{\max}(X)_p$ is a weak equivalence for all $p \in P$.
2. $\max^* A_S \to X_S$ is a weak equivalence for all $S \in \mathrm{Sd}(P)$.

By Lemma 8.2.17 and 2-out-of-3, 1. is equivalent to:

1'. the composite $A_p \to \mathrm{Ran}_{\max}(X)_p \to X_{\{p\}}$ is a weak equivalence for all $p \in P$.

We will then show that $1'. \Leftrightarrow 2.$.

For $2. \Rightarrow 1'.$, simply take $S = \{p\}$. For $1'. \Rightarrow 2.$, consider the following commutative square:

$$\begin{tikzcd}
(\max{}^*A)_S \arrow[r] \arrow[d, equal] & X_S \arrow[d] \\
A_{\{\max S\}} \arrow[r] & X_{\{\max S\}}
\end{tikzcd}$$

Since X is homotopical and weak equivalences in $\mathrm{Sd}(P)$ are created by max, the vertical right-hand arrow is a weak equivalence. By assumption the bottom arrow is a weak equivalence, hence by 2-out-of-3 so is the top one. $\square$

Proof of Prop. 8.2.12. Putting $A := \mathrm{Ran}_{\max}(X)$ in Lemma 8.2.18, we get that the counit in the diagram of Paragraph 8.2.16 is a natural weak equivalence and hence the composite $\max^* \cdot \mathrm{Ran}_{\max}$ is homotopic to a weak equivalence, thus is itself a weak equivalence.

So by 2-out-of-3, it suffices to show that $\mathrm{Ran}_{\max}$ is a weak equivalence. For that we verify the approximation criteria of Theorem 5.1.17.

(App1). Let $X \to Y$ be a map in $\mathcal{C}_{\mathrm{hR}}^{\mathrm{Sd}(P)^{\mathrm{op}}}$ whose image $\mathrm{Ran}_{\max}(X) \to \mathrm{Ran}_{\max}(Y)$ in $\mathcal{C}^{P^{\mathrm{op}}}$ is a weak equivalence. We need to show that $X \to Y$ is a weak equivalence, that is, for all $S \in \mathrm{Sd}(P)$, $X_S \to Y_S$ is a weak equivalence. Since both X and Y are homotopical and weak equivalences in $\mathrm{Sd}(P)$ are created by max, we have a commutative diagram:

$$\begin{tikzcd}
X_S \arrow[r] \arrow[d] & Y_S \arrow[d] \\
X_{\max S} \arrow[r] & Y_{\max S}
\end{tikzcd}$$

in which both vertical arrows are weak equivalences. Combining Lemmas 8.2.17, 8.2.15, and the assumption that for all $p \in P$, $\mathrm{Ran}_{\max}(X)_p \to \mathrm{Ran}_{\max}(Y)_p$ is an equivalence, we see that the bottom map is a weak equivalence as well. Hence, by 2-out-of-3 so is the top map.

(App2). Let $f\colon A \to \mathrm{Ran}_{\max}(X)$. Factor the transpose $\overline{f}\colon \max^* A \to X$ as weak equivalence followed by a fibration:

$$\begin{tikzcd}
 & \widetilde{A} \arrow[dr, "p"] & \\
\max^* A \arrow[ur, "w"] \arrow[rr, "f"] & & X
\end{tikzcd}$$

Then we have a commutative square:

$$\begin{tikzcd}
A \arrow[r, "\overline{w}"] \arrow[d, "1_A"'] & \mathrm{Ran}_{\max}(\widetilde{A}) \arrow[d, "\mathrm{Ran}_{\max}(p)"] \\
A \arrow[r, "f"] & \mathrm{Ran}_{\max}(X)
\end{tikzcd}$$

where $\overline{w}$ is the transpose of w and hence, by Lemma 8.2.18, a weak equivalence. Thus (App2) is satisfied. ☐

8.2.19. Let $m, n \in \mathbb{N}$. By postcomposition with the projections, a monotone function $[k] \to [m] \times [n]$ (i.e. an object in $D([m] \times [n])$) gives a pair of functions $([k] \to [m], [k] \to [n])$ (i.e. an object in $D[m] \times D[n]$). This defines a functor:

$$D([m] \times [n]) \longrightarrow D[m] \times D[n].$$

Proposition 8.2.20. *The canonical map $D([m] \times [n]) \to D[m] \times D[n]$ of Paragraph 8.2.19 induces an equivalence of fibration categories:*

$$\mathcal{C}^{(D[m]\times D[n])^{\mathrm{op}}} \longrightarrow \mathcal{C}^{D([m]\times[n])^{\mathrm{op}}}.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Sd}([m] \times [n]) \arrow[r, "①"] \arrow[dr, "②"'] & D([m] \times [n]) \arrow[r, "⑤"] \arrow[d, "③"] & D[m] \times D[n] \arrow[dl, "④"] \\
 & [m] \times [n] &
\end{tikzcd}$$

By Lemma 8.2.11, ① induces an equivalence; by Proposition 8.2.12 so does ②. Hence, by 2-out-of-3, so does ③. By Corollary 8.2.8, ④ induces an equivalence, and hence, by 2-out-of-3, so does ⑤. ☐

Corollary 8.2.21. *The canonical map $D([m]\times[n]) \to D[m]\times D[n]$ of Paragraph 8.2.19, induces an equivalence*

$$\mathcal{C}_{\mathrm{hR}}^{(D[m]\times D[n])^{\mathrm{op}}} \longrightarrow \mathcal{C}_{\mathrm{hR}}^{D([m]\times[n])^{\mathrm{op}}}.$$

Proof. By Propositions 8.2.20 and 5.2.26, it suffices to show that the map $D([m]\times[n]) \to D[m]\times D[n]$ induces an exact functor on fibration categories of homotopical and Reedy fibrant diagrams. By the Exactness Criterion 5.2.22, it suffices to show that for any $(\varphi,\psi) \in D([m]\times[n])$, the induced map on latching categories factors as: (sieve) $\circ$ (cofinal).

Let $(\varphi,\psi)\colon [k] \to [m]\times[n]$. We may the latching categories as follows:

$$\partial\big(D([m]\times[n])/(\varphi,\psi)\big) = \{A \subsetneq [k] \mid A \neq \emptyset\}$$

and

$$\partial\big(D[m]\times D[n]/(\varphi\times\psi)\big) = \{A\times B \subsetneq [k]\times[k] \mid A, B \neq \emptyset\}$$

and the map is given by $A \mapsto A\times A$. It factors through:

$$L := \{A\times B \subseteq [k]\times[k] \mid A, B \neq \emptyset \text{ and } A\cup B \neq [k]\}.$$

The inclusion $L \hookrightarrow \partial\big(D[m]\times D[n]/(\varphi\times\psi)\big)$ is easily seen to be a sieve; thus, it remains to show that $\partial\big(D([m]\times[n])/(\varphi,\psi)\big) \hookrightarrow L$ is cofinal.

We will use the Cofinality Criterion 5.2.24. Given $A\times B \in L$, we need to show that the slice category

$$A\times B/\partial\big(D([m]\times[n])/(\varphi,\psi)\big)$$

is connected. But this is clear since it has the initial object given by $A\times B \hookrightarrow (A\cup B)\times(A\cup B)$. $\square$

8.2.22. In fact, given $K, L \in \mathrm{sSet}$, the canonical map $D(K\times L) \to DK\times DL$ induces an exact functor:

$$\mathcal{C}_{\mathrm{hR}}^{DK^{\mathrm{op}}\times DL^{\mathrm{op}}} \longrightarrow \mathcal{C}_{\mathrm{hR}}^{D(K\times L)^{\mathrm{op}}}.$$

This is the case because the argument for exactness in the proof of Corollary 8.2.21 depends only on latching categories and these are the same for $D[n]$ and DK.

8.2.23. Our next goal is to prove the equivalence $\mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \simeq \mathrm{N_f}(\mathcal{C})^{\Delta[n]}$. We start by constructing a map $\mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \to \mathrm{N_f}(\mathcal{C})^{\Delta[n]}$.

8.2.24. A k-simplex $\Delta[k] \to \mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}})$ is, by definition, a homotopical Reedy fibrant diagram $D[k]^{\mathrm{op}} \to \mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}$. By Proposition 5.2.15, this corresponds to a homotopical Reedy fibrant functor $D[k]^{\mathrm{op}} \times D[m]^{\mathrm{op}} \to \mathcal{C}$. Precomposing with the canonical inclusion $D([k] \times [m]) \hookrightarrow D[k] \times D[m]$, we obtain a homotopical Reedy fibrant functor $D([k] \times [m])^{\mathrm{op}} \to \mathcal{C}$. By Theorem 5.3.18, this corresponds naturally to a simplicial map $\Delta[k] \times \Delta[m] \to \mathrm{N_f}\mathcal{C}$ and hence a k-simplex $\Delta[k] \to \mathrm{N_f}(\mathcal{C})^{\Delta[m]}$.

Since every step of the above construction was natural, we obtain a well-defined map $\mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \to \mathrm{N_f}(\mathcal{C})^{\Delta[n]}$.

Theorem 8.2.25. *Let $\mathcal{C}$ be a fibration category and $m \in \mathbb{N}$. Then the map*

$$F \colon \mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \to \mathrm{N_f}(\mathcal{C})^{\Delta[m]}$$

constructed in Paragraph 8.2.24 is a categorical equivalence of quasicategories.

Proof. In order to show that F is an equivalence, we may use Lemma 3.3.32. Consider a commuting square:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "U"] \arrow[d, hook] & \mathrm{N_f}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \arrow[d, "F"] \\ \Delta[n] \arrow[r, "V"] & \mathrm{N_f}(\mathcal{C})^{\Delta[m]} \end{tikzcd}$$

for we need to find a diagonal filler, making the upper triangle commute strictly and the lower triangle commute up to $E[1]$-homotopy relative to the boundary.

By Theorem 5.3.18, we may rephrase the problem as follows. Given:

- homotopical Reedy fibrant diagram $U \colon D[m]^{\mathrm{op}} \times D(\partial\Delta[n])^{\mathrm{op}} \to \mathcal{C}$ and
- homotopical Reedy fibrant diagram $V \colon D([m] \times [n])^{\mathrm{op}} \to \mathcal{C}$

that agree on $D(\Delta[m] \times \partial\Delta[n])^{\mathrm{op}}$, we need an extension $X \colon D[m]^{\mathrm{op}} \times D[n]^{\mathrm{op}} \to \mathcal{C}$ such that:

- $X|(D[m]^{\mathrm{op}} \times D(\partial\Delta[n])^{\mathrm{op}}) = U$;
- $FX \sim_{E[1]} V$ rel $\partial\Delta[n]$.

Consider now the following pushout diagram of homotopical direct categories and homotopical functors:

$$\begin{tikzcd}
D(\Delta[m] \times \partial\Delta[n]) \arrow[r, hook, "①"] \arrow[d, "②"'] & D([m] \times [n]) \arrow[d, "④"] \arrow[ddr, bend left, "⑤"] & \\
D[m] \times D(\partial\Delta[n]) \arrow[r, "③"] \arrow[drr, bend right, "⑥"'] & \bullet \arrow[dr, dotted, "⑦"] & \\
 & & D[m] \times D[n]
\end{tikzcd}$$

Notice that F is the map induced by ⑦. We claim that all maps in the above diagram induce exact functors of fibration categories under $\mathcal{C}^{(-)^{\mathrm{op}}}$. ① is a sieve; as explained in Paragraph 8.2.22, ② satisfies the assumptions of the Exactness Criterion 5.2.22; the map ③ is easily seen to be a sieve and ④ satisfies the assumptions of the Exactness Criterion 5.2.22; ⑤ induces an exact functor by Corollary 8.2.21; ⑥ and ⑦ are sieves.

For any fibration category $\mathcal{C}$, the map $D(\Delta[m] \times \partial\Delta[n]) \to D[m] \times D(\partial\Delta[n])$ induces an equivalence on diagram categories:

$$\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}} \times D(\partial\Delta[n])^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{hR}}^{D(\Delta[m] \times \partial\Delta[n])^{\mathrm{op}}}.$$

To see it, we first $\partial\Delta[n] = \operatorname*{colim}_{\int \partial\Delta[n]} \Delta[i]$ and use the fact that both D and cartesian product commute with colimits (the latter by the fact that in the category of simplicial sets, it is a left adjoint to exponential).

Thus ② induces an equivalence. So, by Corollary 6.2.11, so does ④. Moreover, ⑤ induces an equivalence by Corollary 8.2.21. By 2-out-of-3, so does ⑦. Finally, both ③ and ⑤ induce fibrations of fibration categories.

Let $\mathcal{J}$ be the pushout i.e. $\mathcal{J} = D[m] \times D(\partial\Delta[n]) \cup D([m] \times [n])$. The diagram:

$$\begin{tikzcd}
\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}} \times D[n]^{\mathrm{op}}} \arrow[rr, "F"] \arrow[dr] & & \mathcal{C}_{\mathrm{hR}}^{\mathcal{J}^{\mathrm{op}}} \arrow[dl] \\
 & \mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}} \times D(\partial\Delta[n])^{\mathrm{op}}} &
\end{tikzcd}$$

and $[U, V] \in \mathcal{C}_{\mathrm{hR}}^{\mathrm{J}^{\mathrm{op}}}$ with its natural frame.

By Dold's Lemma, we obtain $X \in \mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}} \times D[n]^{\mathrm{op}}}$ such that $X|(D[m] \times D(\partial\Delta[n]))^{\mathrm{op}} = U$ together with a fraction:

$$FX \overset{\sim}{\leftarrow} \widetilde{X} \overset{\sim}{\rightarrow} [U, V].$$

We claim that this is the right X. It remains to produce the required $E[1]$-homotopy out of the fraction $\widetilde{X}$. Restricting this fraction to $D([m] \times [n])^{\mathrm{op}}$ we get:

$$X|D([m] \times [n])^{\mathrm{op}} \overset{\sim}{\leftarrow} \widetilde{X}|D([m] \times [n])^{\mathrm{op}} \overset{\sim}{\rightarrow} V.$$

But $X|D([m] \times [n])^{\mathrm{op}} = FX$, by definition of F.

Thus we found a diagram:

$$D([m] \times [n])^{\mathrm{op}} \times \mathrm{Sd}\widehat{[1]} \xrightarrow{\widetilde{X}} \mathcal{C}$$

The composite:

$$D([m] \times [n] \times \widehat{[1]})^{\mathrm{op}} \longrightarrow D([m] \times [n])^{\mathrm{op}} \times \mathrm{Sd}\widehat{[1]} \xrightarrow{\widetilde{X}} \mathcal{C}$$

is a homotopical diagram, whose restrictions to $D([m] \times [n] \times \{0\})^{\mathrm{op}}$ and $D([m] \times [n] \times \{1\})^{\mathrm{op}}$ (equal to FX and V, respectively) are Reedy fibrant and jointly sieve. By Lemma 5.4.2, we may replace it with a homotopical Reedy fibrant diagram without changing it on the boundary.

By Theorem 5.3.18 and Lemma 7.1.2, this corresponds to a diagram $\Delta[m] \times \Delta[n] \times \Delta[1] \to \mathrm{N_f}(\mathcal{C})$ giving the desired homotopy. $\square$

9.0 PROOF OF JOYAL'S CONJECTURE

We finally return to Joyal's conjecture. Having established several results, we now put them to good use. In Section 9.1 we show that the quasicategory of frames in a fibration category is equivalent to the standard localization of this fibration category, regarded as a homotopical category. In Section 9.2, we will introduce the notion of a locally cartesian closed fibration category, and show that if $\mathcal{C}$ is a locally cartesian closed fibration category, then $\mathrm{N_f}\mathcal{C}$ is a locally cartesian closed quasicategory. Finally, in Section 9.3, we verify that for any type theory $\mathbb{T}$ admitting the rules of Appendix A, $\mathcal{C}\ell(\mathbb{T})$ is a locally cartesian closed fibration category, which allows us to deduce Joyal's conjecture.

9.1 QUASICATEGORY OF FRAMES AS SIMPLICIAL LOCALIZATION

9.1.1. This section is devoted to the proof of the following theorem.

Theorem 9.1.2. *For any fibration category $\mathcal{C}$, the quasicategories $\mathrm{L}(\mathcal{C})$ and $\mathrm{N_f}(\mathcal{C})$ are weakly equivalent.*

9.1.3. In particular, it then follows that $\mathrm{L}(\mathcal{C})$ is locally cartesian closed if and only if $\mathrm{N_f}(\mathcal{C})$ is. The proof of Theorem will be given at the very end of the section as we need to establish several intermediate results. Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a fibration category. For each $[m] \in \Delta$, the simplicial set $\mathrm{N_f}(\mathcal{C})^{\Delta[m]}$ is a quasicategory, and hence by Proposition 3.1.58, the bisimplicial set $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})$ is a complete Segal space. Explicitly, it can be described via:

$$\begin{aligned} \mathrm{J}(\mathrm{N_f}(C)^{\Delta[-]})_{m,n} &= \mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[m]})_n \\ &\cong \mathrm{Fun}^{\mathrm{hR}}(D(\widehat{[n]} \times [m])^{\mathrm{op}}, \mathcal{C}). \end{aligned}$$

9.1.4. Let us now explain our proof strategy. We will show that the bisimplicial sets $\mathcal{N}(\mathcal{C}, \mathcal{W})$ and $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})$ are levelwise weakly equivalent and thus the latter is a model for the fibrant replacement $\mathcal{N}(\mathcal{C}, \mathcal{W})'$ of the former. Thus by Theorem 3.2.5, $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})_0$ is equivalent with $\mathrm{L}(\mathcal{C})$. But calculating $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})_0$ explicitly by instantiating the calculation of Paragraph 9.1.3 with $m = 0$ we get $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})_0 \simeq \mathrm{N_f}(\mathcal{C})$.

Proposition 9.1.5. *For each $[m] \in \Delta$, the simplicial sets $\mathrm{N}(\mathcal{W}_{\mathcal{C}^{[m]}})$ and $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[m]})$ are weakly equivalent in $\mathrm{s}\mathcal{S}\mathrm{et}_Q$.*

9.1.6. Notice that since Quillen's model structure is a localization of Joyal's, it is easier to be an equivalence in Quillen's model structure. This is the key idea standing behind our proof.

Furthermore, Rezk model structure is a localization of the Reedy model structure on $\mathrm{ss}\mathcal{S}\mathrm{et}$. It thus follow from Proposition 9.1.5 that $\mathrm{N}(\mathcal{C}, \mathcal{W})$ and $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})$ are weakly equivalent in Rezk model structure.

9.1.7. To prove Proposition 9.1.5, we need a few intermediate notions and lemmas. The first of these notions is a variation on Kan's functor Ex (see [Kan57]). Recall that $\mathrm{Ex}\colon \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ was defined as

$$\mathrm{Ex}(X)_m = \mathrm{Hom}(\mathrm{NSd}[m], X)$$

For our purposes certain modifications are required. First, we work with *fat* subdivisions D, rather than with the ordinary subdivision functor Sd. The second modification requires taking $D[m]^{\mathrm{op}}$, rather simply $D[m]$ since we are working in the framework of fibration categories.

Definition 9.1.8. Given $X \in \mathrm{s}\mathcal{S}\mathrm{et}$, define the simplicial set $\mathbf{E}\mathrm{x}(X)$ by

$$\mathbf{E}\mathrm{x}(X)_m := \mathrm{Hom}(\mathrm{N}D[m]^{\mathrm{op}}, X).$$

9.1.9. Given $[m] \in \Delta$, there is a map $\mathrm{ev}_0\colon D[m]^{\mathrm{op}} \to [m]$ taking a monotone function $\varphi\colon [k] \to [m] \in D[m]$ to $\varphi(0) \in [m]$. Moreover, the family $\{\mathrm{ev}_0\colon D[m]^{\mathrm{op}} \to [m]\}_{[m]\in\Delta}$ is easily seen to be natural and hence, by Yoneda, there is an induced map:

$$\mathrm{ev}_0^*\colon X \to \mathbf{E}\mathrm{x}(X).$$

9.1.10. Each of the maps $\mathrm{ev}_0\colon D[m]^{\mathrm{op}} \to [m]$ admits a section $\iota\colon [m] \to D[m]^{\mathrm{op}}$ taking $k \in [m]$ to the inclusion $\varphi\colon [m-k] \hookrightarrow [m]$, given by $\varphi(i) = i + k$. Indeed, one easily sees that $\mathrm{ev}_0 \cdot \iota = 1_{[m]}$. Moreover, there is a natural transformation $\iota \cdot \mathrm{ev}_0 \to 1_{D[n]^{\mathrm{op}}}$ coming from commutativity of the following triangle:

$$\begin{tikzcd}
{[k]} \arrow[rr, "j \mapsto \varphi(j)-\varphi(0)"] \arrow[dr, "\varphi"'] & & {[n - \varphi(0)]} \arrow[dl, hook] \\
& {[n]} &
\end{tikzcd}$$

Thus, since the nerve functor N takes natural transformations to homotopies, the maps $\mathrm{N}(\mathrm{ev}_0)$ and $\mathrm{N}(\iota)$ form a homotopy equivalence $\mathrm{N}(D[m]^{\mathrm{op}}) \simeq \Delta[m]$.

Lemma 9.1.11. *Let $f, g\colon K \to L$ be homotopic maps in sSet_Q. Then $\mathbf{E}\mathrm{x}(f)$ and $\mathbf{E}\mathrm{x}(g)$ are also homotopic.*

Proof. Choose a homotopy $H\colon K \times \Delta[1] \to L$ between f and g. The following composite gives a homotopy between $\mathbf{E}\mathrm{x}(f)$ and $\mathbf{E}\mathrm{x}(g)$:

$$\begin{aligned}
\mathbf{E}\mathrm{x}(K) \times \Delta[1] &\xrightarrow{1\times \mathrm{ev}_0^*} \mathbf{E}\mathrm{x}(K) \times \mathbf{E}\mathrm{x}(\Delta[1]) \\
&\xrightarrow{\cong} \mathbf{E}\mathrm{x}(K \times \Delta[1]) \\
&\xrightarrow{\mathbf{E}\mathrm{x}(H)} \mathbf{E}\mathrm{x}(L)
\end{aligned}$$

□

9.1.12. The unique map $!\colon [n] \to [0]$ admits a section that includes $\mathrm{const}_0\colon [0] \to [n]$ and we have a natural transformation $1_{[n]} \to \mathrm{const}_0 \cdot !$. Thus $\mathrm{N}(!)$ and $\mathrm{N}(\mathrm{const}_0)$ form a homotopy equivalence $\Delta[n] \simeq \Delta[0]$.

Lemma 9.1.13. *The map $\mathrm{ev}_0^*\colon X \to \mathbf{E}\mathrm{x}(X)$ is a weak equivalence of simplicial sets.*

9.1.14. The main idea of the proof lies in a clever application of the Diagonal Lemma (adapted from [BK12b]). Before giving it, let us recall the following lemma:

Lemma 9.1.15 (Diagonal Lemma, [GJ09, Thm. 4.1.9])**.** *Let $X \to Y$ be a map of bisimplicial sets such that for any n, the induced map $X_n \to Y_n$ is a weak equivalence. Then the induced map $\mathrm{diag}X \to \mathrm{diag}Y$ is also a weak equivalence of simplicial sets, where $\mathrm{diag}\colon \mathrm{ss}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is the functor taking a simplicial set X to $\mathrm{diag}(X)_n := X_{n,n}$.*

Proof of Lemma 9.1.13. The proof will span Paragraphs 9.1.16–9.1.19.

9.1.16. Consider the following commutative square:

$$\begin{tikzcd}
\mathrm{Hom}(\Delta[m] \times \Delta[0], X) \arrow[r, "(1\times !)^*"] \arrow[d, "(\mathrm{Nev}_0 \times 1)^*"'] & \mathrm{Hom}(\Delta[m] \times \Delta[n], X) \arrow[d, "(\mathrm{Nev}_0 \times 1)^*"] \\
\mathrm{Hom}(\mathrm{N}D[m]^{\mathrm{op}} \times \Delta[0], X) \arrow[r, "(1\times !)^*"] & \mathrm{Hom}(\mathrm{N}D[m]^{\mathrm{op}} \times \Delta[n], X)
\end{tikzcd}$$

As m and n vary each of the objects becomes a (possibly constant) bisimplicial set.

9.1.17. First, fix $n \in \mathbb{N}$. Then the diagram of Paragraph 9.1.16 becomes:

$$\begin{tikzcd}
X^{\Delta[0]} \arrow[r] \arrow[d] & X^{\Delta[n]} \arrow[d] \\
\mathbf{Ex}(X^{\Delta[0]}) \arrow[r] & \mathbf{Ex}(X^{\Delta[n]})
\end{tikzcd}$$

in which:

- the top map $X^{\Delta[0]} \to X^{\Delta[n]}$ is a homotopy equivalence as the image of homotopy equivalence of Paragraph 9.1.12 under $X^{(-)}$;
- the bottom map $\mathbf{Ex}(X^{\Delta[0]}) \to \mathbf{Ex}(X^{\Delta[n]})$ is homotopy equivalence since $X^{\Delta[0]} \to X^{\Delta[n]}$ is a homotopy equivalence and, by Lemma 9.1.11 $\mathbf{Ex}$ preserves homotopy equivalences.

9.1.18. Second, fix $m \in \mathbb{N}$. Then the diagram of Paragraph 9.1.16 becomes:

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & X^{\Delta[m]} \arrow[d] \\
? \arrow[r] & X^{\mathrm{N}(D[m]^{\mathrm{op}})}
\end{tikzcd}$$

and the right hand side vertical map $X^{\Delta[m]} \to X^{\mathrm{N}(D[m]^{\mathrm{op}})}$ is a homotopy equivalence as the image of homotopy equivalence of Paragraph 9.1.10 under $X^{(-)}$.

9.1.19. Finally, applying the diagonal functor diag: ss$\mathcal{S}$et $\to$ s$\mathcal{S}$et to diagram of Paragraph 9.1.16, we obtain:

$$\begin{tikzcd} X \arrow[r, "①"] \arrow[d, "\mathrm{ev}_0^*"'] & ? \arrow[d, "③"] \\ \mathbf{E}\mathrm{x}(X) \arrow[r, "②"] & ? \end{tikzcd}$$

in which maps ① and ② are weak equivalences by the Diagonal Lemma 9.1.15 and Paragraph 9.1.17 and ③ is a weak equivalence by the Diagonal Lemma 9.1.15 and Paragraph 9.1.18. Thus, by 2-out-of-3, ev_0^* is also a weak equivalence. □

9.1.20. Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a fibration category. The set of n-simplices of the Kan complex $\mathrm{J} \cdot \mathrm{N_f}(\mathcal{C})$ can be described explicitly:

$$\mathrm{J} \cdot \mathrm{N_f}(\mathcal{C})_n = \mathrm{Fun}^{\mathrm{hR}}(\widehat{D[n]}^{\mathrm{op}}, \mathcal{C}).$$

The n-simplices of the simplicial set $\mathbf{E}\mathrm{x}(\mathrm{N}\mathcal{W})$ are on the other hand given by:

$$\mathbf{E}\mathrm{x}(\mathrm{N}\mathcal{W})_n = \mathrm{Fun}^{\mathrm{h}}(\widehat{D[n]}^{\mathrm{op}}, \mathcal{C}).$$

Thus, there is an evident inclusion

$$\mathrm{J} \cdot \mathrm{N_f}(\mathcal{C}) \hookrightarrow \mathbf{E}\mathrm{x}(\mathrm{N}\mathcal{W}).$$

Lemma 9.1.21. *The inclusion* $\mathrm{J} \cdot \mathrm{N_f}(\mathcal{C}) \hookrightarrow \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_\mathcal{C})$ *of Paragraph 9.1.20 is a categorical equivalence.*

Proof. By Lemma 3.3.32, it suffices to find a solution to the following lifting problem:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathrm{J} \cdot \mathrm{N_f}(\mathcal{C}) \arrow[d, hook] \\ \Delta[n] \arrow[r, "X"] \arrow[ur, dotted, "\widetilde{X}"] & \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_\mathcal{C}) \end{tikzcd}$$

making the upper triangle commute and lower commute up to $E[1]$-homotopy relative to $\partial\Delta[n]$.

By Theorem 5.3.18, the diagram $X\colon \Delta[n] \to \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_{\mathcal{C}})$ corresponds to a homotopical functor $\widehat{D[n]}^{\mathrm{op}} \to \mathcal{C}$. Moreover, by commutativity of the square above, the restriction of X to the boundary $\partial\Delta[n]$ is a Reedy fibrant and homotopical functor:

$$D(\widehat{\partial\Delta[n]})^{\mathrm{op}} \xrightarrow{X} \mathrm{J} \cdot \mathrm{N}_{\mathrm{f}}(\mathcal{C}) \subseteq \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_{\mathcal{C}}).$$

(Here, $D(\widehat{\partial\Delta[n]})$ indicates that all maps of $D(\partial\Delta[n])$ are weak equivalences.) Since $D(\widehat{\partial\Delta[n]}) \hookrightarrow D[n]$ is a sieve, by Lemma 5.4.2, we may find an extension $\widetilde{X}$ and natural weak equivalence $X \to \widetilde{X}$ that by Lemma 7.1.2 gives the required homotopy. $\square$

Proof of Proposition 9.1.5. Consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{N}(\mathcal{W}_{\mathcal{C}}) \arrow[r] & \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_{\mathcal{C}}) & \mathrm{J} \cdot \mathrm{N}_{\mathrm{f}}(\mathcal{C}) \arrow[l] \\
\mathcal{C}_{\mathrm{h}}^{[m]} \arrow[d] & \mathrm{N}(\mathcal{W}_{(\mathcal{C}_{\mathrm{h}}^{[m]})}) \arrow[r] \arrow[d] & \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_{(\mathcal{C}_{\mathrm{h}}^{[m]})}) \arrow[d] & \mathrm{J} \cdot \mathrm{N}_{\mathrm{f}}(\mathcal{C}_{\mathrm{h}}^{[m]}) \arrow[l] \arrow[d] \\
\mathcal{C}_{\mathrm{h}}^{D[m]^{\mathrm{op}}} & \mathrm{N}(\mathcal{W}_{(\mathcal{C}_{\mathrm{h}}^{D[m]^{\mathrm{op}}})}) \arrow[r] & \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_{(\mathcal{C}_{\mathrm{h}}^{D[m]^{\mathrm{op}}})}) & \mathrm{J} \cdot \mathrm{N}_{\mathrm{f}}(\mathcal{C}_{\mathrm{h}}^{D[m]^{\mathrm{op}}}) \arrow[l] \\
\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}} \arrow[u] & \mathrm{N}(\mathcal{W}_{(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}})}) \arrow[r] \arrow[u] & \mathbf{E}\mathrm{x} \cdot \mathrm{N}(\mathcal{W}_{(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}})}) \arrow[u] & \mathrm{J} \cdot \mathrm{N}_{\mathrm{f}}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}}) \arrow[l] \arrow[u]
\end{tikzcd}$$

By Lemma 9.1.13 the left arrow in the top row is a weak equivalence; and by Lemma 9.1.21 the right arrow in the top row is a weak equivalence. Hence all horizontal arrows in the diagram above are weak equivalences (as instances of the top row).

Since N_{f} is exact (Theorem 5.3.12)and J takes categorical equivalences to weak equivalences (Proposition 3.1.38) the arrows in the right most column are weak equivalences. Hence by 2-out-of-3 all maps in this diagram are weak equivalences.

This gives a categorical equivalence $\mathrm{N}(\mathcal{W}_{(\mathcal{C}_h^{[m]})}) \simeq \mathrm{J} \cdot \mathrm{N}_{\mathrm{f}}(\mathcal{C}_{\mathrm{hR}}^{D[m]^{\mathrm{op}}})$, which combined with 8.2.25 gives

$$\mathrm{N}(\mathcal{W}_{(\mathcal{C}_h^{[m]})}) \simeq \mathrm{J}(\mathrm{N}_{\mathrm{f}}(\mathcal{C})^{\Delta[n]})$$

completing the proof. $\square$

Proof of Theorem 9.1.3. By Proposition 9.1.5, $\mathrm{J}(\mathrm{N_f}(\mathcal{C})^{\Delta[-]})$ is a levelwise fibrant replacement of the classification diagram $\mathrm{N}(\mathcal{C}, \mathcal{W})$. Hence it is also its fibrant replacement in the Rezk model structure. The discussion of Paragraph 9.1.4 yields the desired conclusion. $\square$

9.2 LOCALLY CARTESIAN CLOSED FIBRATION CATEGORIES

Definition 9.2.1. A fibration category $\mathcal{C}$ is a **locally cartesian closed fibration category** if all objects of $\mathcal{C}$ are cofibrant and, for any fibration $p\colon B \to A$ in $\mathcal{C}$, the pullback functor

$$p^*\colon \mathcal{C}(A) \to \mathcal{C}(B)$$

is exact and has a right adjoint, which in turn is a homotopical functor.

9.2.2. Notice that a locally cartesian closed fibration category is not simply a fibration category that is locally cartesian closed. We only request the right adjoint to the pullback functor p^* to exist when p is a fibration. Moreover, this adjoint is not to be defined on slices of $\mathcal{C}$, but rather on their subcategories consisting only of fibrations as objects.

Proposition 9.2.3. *If $\mathcal{C}$ is a locally cartesian closed fibration category, then for each $A \in \mathcal{C}$, the category $\mathcal{C}(A)$ is cartesian closed. Moreover, for any fibration $p\colon B \to A$, the product functor $p \times -\colon \mathcal{C}(A) \to \mathcal{C}(A)$ is exact and its right adjoint is homotopical.*

Proof. This immediate by the construction of the exponentials in the slices from the right adjoint to the pullback functor. $\square$

9.2.4. Let $\mathcal{C}$ be a fibration category and let $X\colon \Lambda^0[2] \to \mathrm{N_f}(\mathcal{C})$ be a simplicial map. Recall that $\Lambda^0[2] \cong \Delta[0] \star \partial\Delta[1]$, i.e. X is a cone over a diagram $\partial\Delta[1] \to \mathrm{N_f}(\mathcal{C})$. We wish to give a criterion that assures that X is a universal cone i.e. a product diagram.

By Theorem 5.3.18, this simplicial map corresponds to a diagram $X\colon D(\Lambda^0[2])^{\mathrm{op}} \to \mathcal{C}$. Let

$$\begin{tikzcd}
 & & X_1 & \\
 & X_{01} \arrow[ur] \arrow[dl] & & \\
X_0 & & X_{02} \arrow[ll] \arrow[r] & X_2
\end{tikzcd}$$

be its restriction to $\mathrm{Sd}(\Lambda^0[2])^{\mathrm{op}}$. It is easy to see that X is a product diagram if and only if the induced map:

$$X_{01} \times_{X_0} X_{02} \to X_1 \times X_2$$

is a weak equivalence.

9.2.5. Let $\mathcal{C}$ be a fibration category and let $B\colon D[0]^{\mathrm{op}} \to \mathcal{C}$ be a frame in $\mathcal{C}$. Taking the product with B is a simplicial map (see Paragraph 3.3.29):

$$B \times -\colon \mathrm{N_f}\mathcal{C} \longrightarrow \mathrm{N_f}\mathcal{C}.$$

On the other hand, taking the product with B_0 gives an exact functor:

$$B_0 \times -\colon \mathcal{C} \longrightarrow \mathcal{C}.$$

Proposition 9.2.6. *The simplicial maps $B \times -$ and $\mathrm{N_f}(B_0 \times -)$ are homotopic.*

Proof. We need to show that given any 0-simplex $A \in \mathrm{N_f}\mathcal{C}$, $B_0 \times A$ has the universal property of $B \times A$. Thus we begin by first equipping it with cone structure; that is, we need two *projections* whose restriction to $\mathrm{Sd}(\Lambda^0[2])^{\mathrm{op}}$ looks like:

$$\begin{tikzcd}
 & & B_0 & \\
 & X_{01} \arrow[ur] \arrow[dl] & & \\
B_0 \times A_0 & & X_{02} \arrow[ll] \arrow[r] & A_0
\end{tikzcd}$$

As ever, by Theorem 5.3.15, it suffices to find the extension to $\mathrm{Sd}(\Lambda^0[2])^{\mathrm{op}}$. In order to define X_{01} take a factorization of $(1_{B_0 \times A_0}, \pi_1) \colon B_0 \times A_0 \to (B_0 \times A_0) \times B_0$ into a weak equivalence followed by a fibration:

$$\begin{tikzcd}
 & X_{01} \arrow[d] \\
B_0 \times A_0 \arrow[ur, bend left] \arrow[r, "{(1,\pi_2)}"'] & (B_0 \times A_0) \times B_0
\end{tikzcd}$$

and similarly, for the definition of X_{02} take such a factorization of $(1_{B_0 \times A_0}, \pi_2) \colon B_0 \times A_0 \to (B_0 \times A_0) \times A_0$. This gives a homotopical Reedy fibrant diagram on $\mathrm{Sd}(\Lambda^0[2])^{\mathrm{op}}$. By construction, the map

$$X_{01} \times_{B_0 \times A_0} X_{02} \longrightarrow B_0 \times A_0$$

is a weak equivalence and hence the two functors must be homotopic. $\square$

9.2.7. A small comment about the above proof is due. Typically, when constructing a homotopy between two simplicial maps, we need to establish a map commuting with all the face and degeneracy operators. The reason that in this case we could only compare these two functors "locally" lies in the fact that they are given by certain universal properties.

Theorem 9.2.8. *Let $\mathcal{C}$ be a locally cartesian closed fibration category. Then $\mathrm{N_f}(\mathcal{C})$ is a locally cartesian closed quasicategory.*

Proof. By Theorem 5.3.12, $\mathrm{N_f}(\mathcal{C})$ has a terminal object, so it remains to show that each slice $\mathrm{N_f}(\mathcal{C})/A$ is a cartesian closed category. By Theorem 7.2.2, it suffices to check that $\mathrm{N_f}(\mathcal{C}(A_0))$ is cartesian closed. By Proposition 9.2.3, the (product $\dashv$ exponential) adjunction in each $\mathcal{C}(A_0)$ satisfies the assumptions of Theorem 7.3.9. The result then follows by the characterization of products 9.2.4. $\square$

9.3 LOCALLY CARTESIAN CLOSED FIBRATION CATEGORIES AND JOYAL'S CONJECTURE

9.3.1. We begin by proving that $\mathcal{C}\ell(\mathbb{T})$ is a fibration category for $\mathbb{T}$ admitting the rules of Appendix A. The weak equivalences in $\mathcal{C}\ell(\mathbb{T})$ are those defined in Paragraph 2.3.19 (and used in the formulation of Joyal's conjecture).

9.3.2. Following the ideas of [GG08, Thm. 10], we define **fibrations** in $\mathcal{C}\ell(\mathbb{T})$ as maps isomorphic to (in $\mathcal{C}\ell(\mathbb{T})^{\to}$) composites of the canonical projections p_Γ. We stress, however, that our definition deviates from theirs in that we do not require the class of fibrations to be closed under arbitrary retracts.

Theorem 9.3.3 (Avigad, Kapulkin, Lumsdaine, [AKL13, Thm. 2.2.5])**.** *Let $\mathbb{T}$ be a type theory admitting rules for the Π, Σ, and Id-types. Then $\mathcal{C}\ell(\mathbb{T})$ with the classes of fibrations and weak equivalences described above is a fibration category.*

9.3.4. For convenience of exposition, we will work with types, rather than context, as discussed in Paragraph 2.3.17. The proof of Theorem 9.3.3 will be preceded by two lemmas.

Lemma 9.3.5. *Let $\pi_1 \colon \sum_{x:A} B(x) \to A$ be a fibration. Then for any $a : A$, we have $B(a) \simeq \mathsf{hfib}(\pi_1, a)$.*

Proof. Take any $a : A$. For the map $B(a) \to \mathsf{hfib}(\pi_1, a)$, send $b : B(a)$ to $((a, b), \mathsf{refl}(a))$. Conversely, send $((a', b), p) : \mathsf{hfib}(\pi_1, a)$ (where $b : B(a')$ and $p : \mathsf{Id}a'a$) to the transported element $p_!(b) : B(a)$. The verification that these are mutually inverse is straightforward. □

Lemma 9.3.6. *Pullbacks of fibrations exist.*

Proof. The pullback of a dependent projection is given by substituting into the corresponding dependent type; that is, the following square is a pullback:

$$\begin{tikzcd}
\sum\limits_{x:A'} B(fx) \arrow[r] \arrow[d] & \sum\limits_{x:A} B(x) \arrow[d] \\
A' \arrow[r, "f"] & A.
\end{tikzcd}$$

(See also [Pit00, Lem. 6.3.2].) □

Proof of Thm. 9.3.3. The proof will span Paragraphs 9.3.7–9.3.11. Each paragraph will contain the verification of a separate axiom of Definition 5.1.1.

9.3.7 (Axiom F1.)**.** If f, g, h form a composable triple of maps such that $g \cdot f$ and $h \cdot g$ are weak equivalences, then we have their quasi-inverses: $\overline{g \cdot f}$ and $\overline{h \cdot g}$, respectively. We form the quasi-inverses of $h \cdot g \cdot f$, f, g, and h as follows:

- $\overline{(g \cdot f)} \cdot g \cdot \overline{(h \cdot g)}$ is the quasi-inverse of $h \cdot g \cdot f$;
- $\overline{(h \cdot g)} \cdot h$ and $f \cdot \overline{(g \cdot f)}$ give left and right quasi-inverses for g, respectively;
- $\overline{(g \cdot f)} \cdot g$ gives a quasi-inverse for f;
- $g \cdot \overline{(h \cdot g)}$ gives a quasi-inverse for h.

9.3.8 (Axiom F2.)**.** Clear.

9.3.9 (Axiom F3.)**.** The pullback of a dependent projection is given by substituting into the corresponding dependent type; that is, the following square is a pullback:

$$\begin{tikzcd} \sum\limits_{x:A'} B(fx) \arrow[r] \arrow[d] & \sum\limits_{x:A} B(x) \arrow[d] \\ A' \arrow[r, "f"] & A. \end{tikzcd}$$

The two pullbacks lemma implies that pullbacks of their composites then also exist.

Preservation of fibrations is clear by construction from the proof of Lemma 9.3.6. For acyclicity, suppose $\pi = \pi_1 \colon \sum_{x:A} B(x) \to A$ is an acyclic fibration, and $f \colon A' \to A$ is a map. Write $f^*\pi$ for the pullback fibration $\pi_1 \colon \sum_{x:A'} B(f(x)) \to A'$. Then for any $x : A'$,

$$\mathsf{hfib}(f^*\pi, x) \simeq B(f(x)) \simeq \mathsf{hfib}(\pi, f(x))$$

by Lemma 9.3.5; and $\mathsf{hfib}(\pi, f(x))$ is contractible by hypothesis, so since equivalence preserves contractibility, $\mathsf{hfib}(f^*\pi, x)$ is again contractible. So $f^*\pi$ is again acyclic, as required.

9.3.10 (Axiom F4.)**.** The empty context (or unit type) is the terminal object.

9.3.11 (Axiom F5.)**.** The factorization of $f\colon A \to B$ is given by the *mapping path-space*

$$\mathrm{P}(f) := \sum_{y:B} \sum_{x:A} \mathsf{Id}_B(fx, y).$$

The fibration $\mathrm{P}(f) \to B$ is given by the obvious projection. The weak equivalence $w\colon A \to \mathrm{P}(f)$ is given by

$$\lambda x : A.(fx, x, \mathsf{refl}_B(fx)).$$

It is clearly a weak equivalence with quasi-inverse given by the second projection. □

9.3.12. It makes sense to ask if the fibration category structure on $\mathcal{C}\ell(\mathbb{T})$ is a part of the full model structure on $\mathcal{C}\ell(\mathbb{T})$. Lumsdaine [Lum11] showed that, if $\mathbb{T}$ admits also certain higher inductive types (specifically, mapping cylinders), then $\mathcal{C}\ell(\mathbb{T})$ carries a *pre-model structure*; that is, a model structure, but without limits and colimits. However, one may hope that there are enough colimits for $\mathcal{C}\ell(\mathbb{T})$ to be a cofibration category. This is, unfortunately, not the case. One can show that for a constructor of (possibly higher) inductive type to admit pushouts (that Lumsdaine identified as generating cofibrations), it has to satisfy an appropriate η-rule. Requesting the constructor $\mathsf{refl}\colon A \to \mathsf{Id}_A$ to satisfy it is stronger than enforcing the reflection rule. Hence $\mathcal{C}\ell(\mathbb{T})$ is not a cofibration category.

9.3.13. Nevertheless, we have the following lemma:

Lemma 9.3.14 ([AKL13, Lem. 2.2.14])**.** *All objects of $\mathcal{C}\ell(\mathbb{T})$ are cofibrant.*

Proof. Lemma 9.3.5 implies that every acyclic fibration $\pi_1\colon \sum_{x:A} B(x) \to A$ admits some section: take some family of contractions of the fibers $\mathsf{hfib}(\pi_1, x)$, and send $x : A$ to the image of the center of contraction $*_x : \mathsf{hfib}(\pi_1, x)$ under the equivalence $\mathsf{hfib}(\pi_1, x) \simeq B(x)$. Now, given f as above, take $\bar{f}$ to be the composite of f with this section. □

Theorem 9.3.15. *Let $\mathbb{T}$ be any type theory admitting the rules of Appendix A. Then $\mathcal{C}\ell(\mathbb{T})$ is a locally cartesian closed fibration category.*

Proof. By Theorem 9.3.3, $\mathcal{C}\ell(\mathbb{T})$ is a fibration category. By Paragraph 9.3.9 and the two-pullback lemma, the pullback functor preserves fibrations and acyclic fibrations. As a right adjoint, it also preserves pullbacks and the terminal object, hence it is exact.

Its right adjoint is given by Π. That is, consider:

$$\sum_{x:A} \sum_{y:B(x)} C(x,y) \xrightarrow{p_C} \sum_{x:A} B(x) \xrightarrow{p_B} A.$$

Then the right adjoint p_B is given by

$$(p_B)_*(p_C) = p_{\Pi_B C} \colon \sum_{x:A} \left(\prod_{y:B(x)} C(x,y) \right) \to A,$$

and it is homotopical by function extensionality (Appendix A.3). ☐

9.3.16. Finally, putting it all together, we obtain:

Theorem 9.3.17 (Joyal's Conjecture 4.3.2, restated)**.** *For any dependent type theory* $\mathbb{T}$ *that admits the rules described in Appendix A, the standard localization* $\mathrm{L}(\mathcal{C}\ell(\mathbb{T}))$ *of its classifying category is a locally cartesian closed quasicategory.*

Proof. By Theorem 9.3.15, $\mathcal{C}\ell(\mathbb{T})$ is a locally cartesian closed fibration category, and hence by Theorem 9.2.8, $\mathrm{N_f}(\mathcal{C}\ell(\mathbb{T}))$ is a locally cartesian closed quasicategory. But by Theorem 9.1.3, $\mathrm{L}(\mathcal{C}\ell(\mathbb{T})) \simeq \mathrm{N_f}(\mathcal{C}\ell(\mathbb{T}))$, so it is also locally cartesian closed. ☐

APPENDIX A

RULES OF TYPE THEORY

A.1 STRUCTURAL RULES

The structural rules of the type theory are (where $\mathcal{J}$ may be the conclusion of any of the judgement forms):

$$\frac{\vdash \Gamma,\ x{:}A,\ \Delta\ \mathsf{cxt}}{\Gamma,\ x{:}A,\ \Delta \vdash x : A}\,\mathsf{Vble} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma,\ x{:}A,\ \Delta \vdash \mathcal{J}}{\Gamma,\ \Delta[a/x] \vdash \mathcal{J}[a/x]}\,\mathsf{Subst}$$

$$\frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma,\ \Delta \vdash \mathcal{J}}{\Gamma,\ x{:}A,\ \Delta \vdash \mathcal{J}}\,\mathsf{Wkg}$$

Definitional equality (also known as syntactic or judgemental equality):

$$\frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma \vdash A = A\ \mathsf{type}} \qquad \frac{\Gamma \vdash A = B\ \mathsf{type}}{\Gamma \vdash B = A\ \mathsf{type}} \qquad \frac{\Gamma \vdash A = B\ \mathsf{type} \qquad \Gamma \vdash B = C\ \mathsf{type}}{\Gamma \vdash A = C\ \mathsf{type}}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a = a : A} \qquad \frac{\Gamma \vdash a = b : A}{\Gamma \vdash b = a : A} \qquad \frac{\Gamma \vdash a = b : A \qquad \Gamma \vdash b = c : A}{\Gamma \vdash a = c : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash A = B\ \mathsf{type}}{\Gamma \vdash a : B} \qquad \frac{\Gamma \vdash a = b : A \qquad \Gamma \vdash A = B\ \mathsf{type}}{\Gamma \vdash a = b : B}$$

A.2 LOGICAL CONSTRUCTORS

In this section, we present rules introducing various type- and term-constructors. For each such constructor, we assume (besides the explicit rules introducing and governing it) a rule stating that it preserves definitional equality in each of its arguments; for instance, along with the Π-INTRO rule introducing the constructor λ, we assume the rule

$$\frac{\Gamma \vdash A = A' \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B(x) = B'(x) \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash b(x) = b'(x) : B(x)}{\Gamma \vdash \lambda x{:}A.b(x) = \lambda x{:}A'.b'(x) : \Pi_{x:A} B(x)} \ \lambda\text{-EQ}$$

Π-types (Dependent products; dependent function types).

$$\frac{\Gamma,\ x{:}A \vdash B(x) \ \mathsf{type}}{\Gamma \vdash \Pi_{x:A} B(x) \ \mathsf{type}} \ \Pi\text{-FORM} \qquad \frac{\Gamma,\ x{:}A \vdash B(x) \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x{:}A.b(x) : \Pi_{x:A} B(x)} \ \Pi\text{-INTRO}$$

$$\frac{\Gamma \vdash f{:}\Pi_{x:A} B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(f, a) : B(a)} \ \Pi\text{-APP}$$

$$\frac{\Gamma,\ x{:}A \vdash B(x) \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash b(x) : B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(\lambda x{:}A.b(x), a) = b(a) : B(a)} \ \Pi\text{-COMP}$$

As a special case of this, when B does not depend on x, we obtain the ordinary function type $[A, B] := \Pi_{x:A} B$.

Σ-types (Dependent sums; disjoint sums.)

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B(x) \ \mathsf{type}}{\Gamma \vdash \Sigma_{x:A} B(x) \ \mathsf{type}} \ \Sigma\text{-FORM}$$

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B(x) \ \mathsf{type}}{\Gamma,\ x{:}A,\ y{:}B(x) \vdash \mathsf{pair}(x, y) : \Sigma_{x:A} B(x)} \ \Sigma\text{-INTRO}$$

$$\frac{\Gamma,\ z{:}\Sigma_{x:A}B(x) \vdash C(z)\ \mathsf{type} \qquad \Gamma,\ x{:}A,\ y{:}B(x) \vdash d(x,y) : C(\mathsf{pair}(x,y))}{\Gamma,\ z{:}\Sigma_{x:A}B(x) \vdash \mathsf{split}_d(z) : C(z)}\ \Sigma\text{-}\textsc{elim}$$

$$\frac{\Gamma,\ z{:}\Sigma_{x:A}B(x) \vdash C(z)\ \mathsf{type} \qquad \Gamma,\ x{:}A,\ y{:}B(x) \vdash d(x,y) : C(\mathsf{pair}(x,y))}{\Gamma,\ x{:}A,\ y{:}B(x) \vdash \mathsf{split}_d(\mathsf{pair}(x,y)) = d(x,y) : C(\mathsf{pair}(x,y))}\ \Sigma\text{-}\textsc{comp}$$

Again, the special case where B does not depend on x is of particular interest: this gives the cartesian product $A \times B := \Sigma_{x:A}B$.

Id-types. (Identity types, equality types.)

$$\frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma,\ x,y{:}A \vdash \mathsf{Id}_A(x,y)\ \mathsf{type}}\ \mathsf{Id}\text{-}\textsc{form} \qquad \frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma,\ x{:}A \vdash \mathsf{refl}_A(x) : \mathsf{Id}_A(x,x)}\ \mathsf{Id}\text{-}\textsc{intro}$$

$$\frac{\Gamma,\ x,y{:}A,\ u{:}\mathsf{Id}_A(x,y) \vdash C(x,y,u)\ \mathsf{type} \qquad \Gamma,\ z{:}A \vdash d(z) : C(z,z,\mathsf{refl}_A(z))}{\Gamma,\ x,y{:}A,\ u{:}\mathsf{Id}_A(x,y) \vdash J_{z.d}(x,y,u) : C(x,y,u)}\ \mathsf{Id}\text{-}\textsc{elim}$$

$$\frac{\Gamma,\ x,y{:}A,\ u{:}\mathsf{Id}_A(x,y) \vdash C(x,y,u)\ \mathsf{type} \qquad \Gamma,\ z{:}A \vdash d(z) : C(z,z,r(z))}{\Gamma,\ x{:}A \vdash J_{z.d}(x,x,\mathsf{refl}_A(x)) = d(x) : C(x,x,\mathsf{refl}_A(x))}\ \mathsf{Id}\text{-}\textsc{comp}$$

A.3 FURTHER RULES

The rules above are somewhat weak in their implications for equality of functions. To this end, some further rules are often adopted: the *η-rule* for Π-types, and the *functional extensionality* rule(s). Our formulation of the latter is taken from [Gar09a]; see also [Hof95a].

$$\frac{\Gamma \vdash f : \mathsf{\Pi}_{x:A} B(x)}{\Gamma \vdash \eta(f) : f = \lambda x{:}A.\mathsf{app}(f,x) : \mathsf{\Pi}_{x:A} B(x)}\ \mathsf{\Pi}\text{-}\eta$$

$$\frac{\Gamma \vdash f, g : \mathsf{\Pi}_{x:A} B(x) \qquad \Gamma \vdash h : \mathsf{\Pi}_{x:A} \mathsf{Id}_{B(x)}(\mathsf{app}(f,x), \mathsf{app}(g,x))}{\Gamma \vdash \mathsf{ext}(f,g,h) : \mathsf{Id}_{\mathsf{\Pi}_{x:A} B(x)}(f,g)}\ \mathsf{\Pi}\text{-}\textsc{ext}$$

$$\frac{\Gamma, x{:}A \vdash b : B(x)}{\begin{array}{l}\Gamma \vdash \mathsf{ext\text{-}comp}(x.b) : \mathsf{Id}_{\mathsf{\Pi}_{x:A} B(x)} \\ \qquad\qquad (\mathsf{ext}(\lambda x{:}A.b, \lambda x{:}A.b, \lambda x{:}A.\mathsf{refl} b), \mathsf{refl}(\lambda x{:}A.b))\end{array}}\ \mathsf{\Pi}\text{-}\textsc{ext-comp-prop}$$

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