HIGHER-ORDER, STRONGLY STABLE METHODS FOR UNCOUPLING GROUNDWATER-SURFACE WATER FLOW

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Many environmental problems today involve the prediction of the migration of contaminants in groundwater-surface water flow. Sources of contaminated groundwater-surface water flow include: landfill leachate, radioactive waste from underground storage containers, and chemical run-off from pesticide usage in agriculture, to name a few. Before we can track the transport of pollutants in environmental flow, we must first model the flow itself, which takes place in a variety of physical settings. This necessitates the development of accurate numerical models describing coupled fluid (surface water) and porous media (groundwater) flow, which we assume to be described by the fully evolutionary Stokes-Darcy equations. Difficulties include finding methods that converge within a reasonable amount of time, are stable when the physical parameters of the flow are small, and maintain stability and accuracy along the interface. Ideally, because there exist a wide variety of physical scenarios for this coupled flow, we desire numerical methods that are versatile in terms of stability and practical in terms of computational cost and time.

The approach to model this flow studied herein seeks to take advantage of existing efficient solvers for the separate sub-flows by uncoupling the flow so that at each time level we may solve a separate surface and groundwater problem. This approach requires only one (SPD) Stokes and one (SPD) Darcy sub-physics and sub-domain solve per time level for the time-dependent Stokes-Darcy problem. In this dissertation, we investigate several different methods that uncouple groundwater-surface water flow, and provide thorough analysis of the stability and convergence of each method along with numerical experiments.

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PREFACE

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1.0 INTRODUCTION

Throughout history, the survival of humankind has depended largely on the accessibility of sufficient quantities of clean freshwater for agricultural, industrial, and domestic purposes. Of all the water on planet Earth, only 2.5% of it is freshwater, with the majority of it being frozen and inaccessible (see, for example, [11] for more details on the scarcity of freshwater). Groundwater contained in aquifers makes up 90% of the world's available freshwater. Unfortunately, this valuable resource frequently becomes contaminated by both human and natural processes. For example, in hydro-fracturing, a mixture of water with sand and chemicals is injected at high pressure into a well to create fractures to allow for the collection of shale gas. The majority of the chemicals in this mixture are not recovered and eventually leave the well to contaminate local groundwater supply. In another example, pesticide application in agriculture can have devastating effects on surrounding freshwater resources due to chemical run-off into nearby rivers, lakes, and streams, and seepage deep into the soil. Also, many storage facilities for radioactive materials exist underground. Over time as storage containers become compromised, nuclear waste can migrate into nearby freshwater aquifers. Even natural processes may result in contaminated freshwater, such as salt-water intrusion in coastal aquifers.

Before we can track the movement of contaminants, we must first develop numerical models which accurately describe and predict this coupled flow. Separate groundwater and surface water flows have been studied by many scientists. See, for example, Pinder and Celia [58], Watson and Burnett [70], or Bear [8] for an extensive study on subsurface flows. An in-depth description of surface flow can be found, for example, in Kundu and Cohan [43]. There exist many accurate and efficient solvers for the independent flows. Difficulty arises, however, when we consider the interaction of groundwater with surface water, resulting in a challenging coupled problem. To model such a flow, we need to preserve the physics of the sub-flows in each region, yet still accurately describe the interaction of fluids between the two media. One way to approach this problem is to start from scratch and develop new codes and solvers. Instead, this research seeks to make use of the existing solvers for separate fluid and porous media flow by investigating methods that uncouple the flow equations in time so that the individual flow problems may be solved separately. Called partitioned methods, these methods allow us to utilize, in a black-box manner, solvers already optimized for the separate flow problems.

It is important that the partitioned methods maintain stability and accuracy along the interface where the two flows meet. Also, potentially small physical parameters create an additional challenge for stability. Because groundwater moves slowly, we are concerned with methods that are stable over long-time intervals, since numerical simulations may span long-time periods. Along these same lines, we want methods that converge within a reasonable amount of time to be of practical use, making higher-order convergence a desirable property. In this work, we will present and analyze several partitioned methods applied to the groundwater-surface water flow problem.

The modeling of this coupled fluid-porous media flow begins with the coupling of the Stokes or Navier-Stokes equations describing the flow in the fluid region, along with the Darcy or Brinkman equations for the flow in the aquifer, or porous media region containing the groundwater. This research focuses on the Stokes-Darcy coupling which is suitable for slow moving flows over large domains.

1.1 THE STOKES-DARCY EQUATIONS

Consider the equations describing the motion of an incompressible viscous fluid in a coupled fluid-porous media domain of two or three dimensions ($d = 2, 3$). Denote the fluid region by Ω_f and the porous media region by Ω_p. Assume both domains are bounded and regular. Let I represent the interface between the two domains. An example of a coupled domain for $d = 2$ can be seen in Figure 1.1. A brief description of the derivation of the equations of

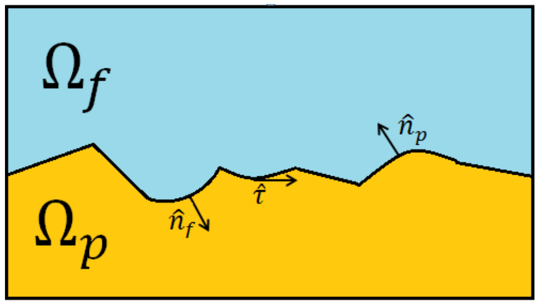

Figure 1.1: Coupled Flow Domain

fluid motion in each domain ensues. For more details on the derivation of the surface and groundwater flow equations, see, for example [7, 8, 70, 58].

1.1.1 The Stokes Equations

Begin with the equations of fluid motion in the fluid region. These equations arise from conservation laws, namely conservation of mass and conservation of momentum. Let $x_0 \in \Omega_f$, $\varepsilon > 0$. Consider a small ε-ball about x_0, denoted by $B_\varepsilon := \{x \in \Omega_f : |x - x_0| \leq \varepsilon\}$. Conservation of mass in B_ε states that the rate of change with respect to time of the net amount of fluid mass in B_ε is balanced by the net flux through the boundary of the ball plus the sources multiplied by the volume. As an integral expression, assuming no sources of mass, this becomes

$$\frac{d}{dt}\left[\int_{B_\varepsilon} \rho \, dx\right] = -\int_{\partial B_\varepsilon} \rho(u \cdot \hat{n}) \, ds,$$

where $\rho = \rho(x,t)$ represents the fluid density and $u = u(x,t)$ the fluid velocity in Ω_f, with $\hat{n}$ being the outward-oriented unit normal. After applying the Divergence Theorem, this becomes

$$\frac{d}{dt}\left[\int_{B_\varepsilon} \rho\, dx\right] = -\int_{B_\varepsilon} \nabla \cdot (\rho u)\, dx.$$

Assuming continuity in all variables, if we shrink this ball to a point, then, since x_0 was arbitrary, we obtain the partial differential equation for conservation of mass in a fluid:

$$\rho_t + \nabla \cdot (\rho u) = 0 \text{ in } \Omega_f \times [0,T]. \tag{1.1}$$

Because this research specifically focuses on groundwater-surface water flow, we may assume homogeneous incompressibility, meaning that the fluid density, ρ, does not vary in space and is not sensitive to changes in fluid pressure, p. This means that $\rho \equiv$ constant, and so (1.1) becomes

$$\nabla \cdot u = 0 \text{ in } \Omega_f \times [0,T], \tag{1.2}$$

meaning the fluid velocity, u, is "divergence free".

Next we discuss the second governing equation in fluid flow, conservation of momentum. Because a fluid convects its own momentum, the rate of change of linear momentum is balanced by the net forces acting on a collection of fluid particles. In relation to the collection of fluid particles in B_ε, the net forces acting on these particles are a combination of internal (surface) forces and external forces. In integral form, this implies

$$\int_{B_\varepsilon} \rho\left(\frac{Du}{dt}\right)\, dx = \int_{\partial B_\varepsilon} \overrightarrow{t}\, ds + \int_{B_\varepsilon} f_f\, dx,$$

where $\frac{Du}{dt} = u_t + u \cdot \nabla u$ is the material derivative of u, describing the evolution of velocity, $\overrightarrow{t}$ represents the Cauchy stress vector, or internal forces, and f_f represents the external forces. The Cauchy stress vector, $\overrightarrow{t}$, obeys a linear relationship with the shear stress tensor, Π. Namely, $\overrightarrow{t} = \hat{n} \cdot \Pi$. After implementing the Divergence Theorem, the surface integral of the Cauchy stress vector becomes

$$\int_{\partial B_\varepsilon} \vec{t}\, ds = \int_{B_\varepsilon} \nabla \cdot \Pi \, dx.$$

Therefore, after letting $\varepsilon \to 0$, assuming continuity in all variables, we obtain the partial differential equation describing conservation of momentum in a fluid

$$\rho(u_t + u \cdot \nabla u) = \nabla \cdot \Pi + f_f \text{ in } \Omega_f \times [0, T]. \tag{1.3}$$

For this research, we focus on slow moving, creeping flow, or flow at low Reynolds' numbers. This means that the convective term, $u \cdot \nabla u$, may be omitted from the equation because this quadratic nonlinear term is negligible compared to other terms in the equation. Thus, the equation for conservation of momentum becomes

$$\rho u_t - \nabla \cdot \Pi = f_f \text{ in } \Omega_f \times [0, T]. \tag{1.4}$$

The shear stress tensor, Π, represents the tangential (viscous) and normal (pressure) forces in the fluid, with $\Pi = 2\mu\mathbb{D} - p\mathbb{I}$. μ represents the dynamic viscosity of the fluid, $\mathbb{D} := \frac{1}{2}(\nabla u + \nabla u^T)$ the deformation tensor, and $\mathbb{I}$ the identity tensor. Since $\nabla \cdot (\nabla u^T) = \hat{0}$, (1.4) is equivalent to

$$\rho u_t - \nu \Delta u + \nabla p = f_f \text{ in } \Omega_f \times [0, T]. \tag{1.5}$$

Going forward, assume that the fluid pressure, p and forcing term, f_f have been rescaled by the fluid density, ρ, which is constant ($p \equiv \frac{1}{\rho}p$, $f_f \equiv \frac{1}{\rho}f_f$). Also, in lieu of the dynamic viscosity, μ, we will refer to the kinematic viscosity, $\nu = \frac{\mu}{\rho}$. For the boundary condition, assume $u = \hat{0}$ on the external boundary, $\partial\Omega_f \setminus I$. We also assume an initial condition. Thus, the evolutionary Stokes equations governing the flow in the fluid region become

$$\begin{aligned}
&u_t - \nu \Delta u + \nabla p = f_f \text{ in } \Omega_f \times [0, T],\\
&\nabla \cdot u = 0 \text{ in } \Omega_f \times [0, T],\\
&u = 0 \text{ on } \partial\Omega_f \backslash I \times [0, T],\\
&u(x, 0) = u_0(x) \text{ in } \Omega_f.
\end{aligned} \tag{Stokes}$$

Table 1.1: Sample values (percentages) of volumetric porosity for various materials (Bear [8] p. 46.)

Material	Porosity Value (percent)	Material	Porosity Value (percent)
Peat soil	$60-80$	Gravel and sand	$30-35$
Soil	$50-60$	Gravel	$30-40$
Clay	$45-55$	Sandstone	$10-20$
Silt	$40-50$	Shale	$1-10$
Uniform sand	$30-40$	Limestone	$1-10$

1.1.2 The Darcy Equation

Next we discuss the equations of fluid motion in the aquifer, or porous media region. Let n represent the volumetric porosity of the porous media. A dimensionless percentage, n represents the ratio of the volume of void space to the bulk volume. See Table 1.1 for sample values of porosity percentages from Bear [8] p. 46. Let q represent the specific discharge and f_p the sources in Ω_p. Then the conservation of mass equation in the porous media region reads

$$\frac{\partial(\rho n)}{\partial t} + \nabla \cdot (\rho q) = f_p \text{ in } \Omega_p \times [0, T]. \tag{1.6}$$

Other variables of interest in the aquifer flow problem include the hydraulic (piezometric) head, or Darcy pressure, ϕ, which satisfies

$$\begin{aligned} \phi &= z + \tfrac{1}{\rho g} p_p \\ &= \text{elevation head} + \text{pressure head} \end{aligned}$$

where p_p represents the fluid pressure in Ω_p. By the chain rule, the first term in (1.6), $\frac{\partial(\rho n)}{\partial t}$

can be written as

$$\frac{\partial(\rho n)}{\partial t} = \frac{\partial \rho}{\partial t} n + \rho \frac{\partial n}{\partial t}.$$

Consider the term, $\frac{\partial \rho}{\partial t}$. Water is slightly compressible, meaning ρ depends *weakly* on the pressure head, p_p, and in turn on ϕ. By chain rule, $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \phi}\frac{\partial \phi}{\partial t}$. Let β represent the compressibility of water, or the measure of volume changes when water is subjected to changes in normal pressures. By definition

$$\beta = \frac{1}{\rho}\frac{\partial \rho}{\partial p_p} = \text{constant},$$

By chain rule, $\frac{\partial \rho}{\partial p_p} = \frac{\partial \rho}{\partial \phi}\frac{\partial \phi}{\partial p_p}$. Thus

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\partial \rho}{\partial \phi}\frac{\partial \phi}{\partial t} \\
&= \left(\frac{\partial \rho}{\partial p_p} \left(\frac{\partial \phi}{\partial p_p} \right)^{-1} \right) \frac{\partial \phi}{\partial t} \\
&= (\rho^2 g \beta) \frac{\partial \phi}{\partial t}.
\end{aligned}$$

Aquifers are poroelastic media, meaning that the spaces between the pores expand and contract in response to changes in pressure. In Bear [7] pp.204-206 it is shown that

$$\begin{aligned}
\frac{\partial n}{\partial t} &= \frac{\partial n}{\partial p_p}\frac{\partial p_p}{\partial t} \\
&= \frac{\partial n}{\partial p_p} \rho g \frac{\partial \phi}{\partial t} \\
&= \alpha'(1-n)\rho g \frac{\partial \phi}{\partial t},
\end{aligned}$$

where α' represents the coefficient of compressibility for a fixed mass of moving solids. Therefore, our conservation of mass equation now reads

$$\rho^2 g(\alpha'(1-n) + \beta n)\phi_t - \nabla \cdot (\rho q) = f_p \text{ in } \Omega_p \times [0, T]. \tag{1.7}$$

The specific (volumetric) storativity of an aquifer, S_0, is defined as $S_0 := \rho g(\alpha'(1-n) + \beta n)$ (see [7], p. 207). It refers to the volume of water an aquifer releases per unit volume due to a unit decrease in the hydraulic head, ϕ, while remaining fully saturated. In simple cases when an aquifer is homogeneous and non-deformable, S_0 is a constant. More realistically, it is bounded function dependent on space. The research presented herein will assume that S_0 is constant, but may be extended naturally to $S_0 = S_0(x)$ being a bounded function. See Table 1.2 for sample values of S_0 for various types of porous media gathered from Domenico and Mifflin [28] and Johnson [42]. Using this specific storage parameter, S_0, (1.7) becomes

$$\rho S_0 \phi_t + \nabla \cdot (\rho q) = f_p \text{ in } \Omega_p \times [0, T]. \tag{1.8}$$

Next, we implement Darcy's Law. Darcy's law relates the specific discharge, q, to the hydraulic head by

$$q = -\mathcal{K} \nabla \phi \text{ in } \Omega_p \times [0, T],$$

where $\mathcal{K}$ represents the hydraulic conductivity tensor. This tensor represents the ease with which water moves through pore spaces or fractures. The tensor, $\mathcal{K}$, is symmetric positive definite (SPD) and unless the aquifer is homogeneous, depends on space. Let k_{min} represent the smallest eigenvector of $\mathcal{K}$. Depending on the composition of the aquifer, this eigenvalue could be very small. See Table 1.3 for sample values of k_{min} taken from Bear in [8].

Consider the term $\nabla \cdot (\rho q) = q \nabla \rho + \rho \nabla \cdot q$. Since $\nabla \rho$ is negligible in comparison to the other terms, we omit it from the equation. Thus (1.8), representing the conservation of mass in the aquifer, becomes

$$S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p \text{ in } \Omega_p \times [0, T], \tag{1.9}$$

where we have rescaled the source function, f_p, by the density, ρ.

In addition to the conservation of mass in the aquifer, we assume $\phi = 0$ on the external boundary as well as an initial condition. This completes the Darcy equation governing flow in the porous media.

$$\begin{aligned} S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) &= f_p \text{ in } \Omega_p \times [0, T], \\ \phi &= 0 \text{ on } \partial \Omega_p \setminus I \times [0, T], \\ \phi(x, 0) &= \phi_0(x) \text{ in } \Omega_p. \end{aligned} \tag{Darcy}$$

Table 1.2: Values of specific (volumetric) storativity, S_0, for different materials in a confined aquifer (see Domenico and Mifflin [28] and Johnson [42]).

Material	Specific Storage S_0 (m^{-1})
Plastic clay	$2.6 \times 10^{-3} - 2.0 \times 10^{-2}$
Stiff clay	$1.3 \times 10^{-3} - 2.6 \times 10^{-3}$
Medium hard clay	$9.2 \times 10^{-4} - 1.3 \times 10^{-3}$
Loose sand	$4.9 \times 10^{-4} - 1.0 \times 10^{-3}$
Dense sand	$1.3 \times 10^{-4} - 2.0 \times 10^{-4}$
Dense sandy gravel	$4.9 \times 10^{-5} - 1.0 \times 10^{-4}$
Rock, sound	less than 3.3×10^{-6}

1.1.3 Coupling Conditions Along the Interface

The (Stokes) and (Darcy) equations describe the fluid motion in the separate domains. To study the coupled flow necessitates the addition of coupling conditions describing the flow along the interface, I.

The first coupling condition represents conservation of mass along the interface. In Ω_p, the (averaged) fluid velocity, u_p, satisfies, $u_p = \frac{q}{n}$. Conservation of mass along the interface implies

$$\begin{aligned} u \cdot \widehat{n}_f + u_p \cdot \widehat{n}_p = 0 \text{ on } I, \text{ or using Darcy's Law,} \\ nu \cdot \widehat{n}_f - \mathcal{K}\nabla\phi \cdot \widehat{n}_p = 0 \text{ on } I, \end{aligned} \tag{Coupling 1}$$

where $\widehat{n}_{f/p}$ represent the outward oriented unit normal vectors on the interface in each subdomain. The second coupling condition describes the balance of normal forces along the interface. Recall that in a fluid body the Cauchy stress vector, $\overrightarrow{t}$, represents the internal forces exerted on a fluid volume by the fluid outside this volume. It satisfies a linear relationship with the stress tensor, Π,

Table 1.3: Values of k_{min}, the smallest eigenvector of the hydraulic conductivity tensor, $\mathcal{K}$, for various materials (see Bear [8]).

Material	Hydraulic Conductivity k_{min} (m/s)
Well sorted gravel	$10^{-1} - 10^{0}$
Highly fractured rocks	$10^{-3} - 10^{0}$
Well sorted sand or sand & gravel	$10^{-4} - 10^{-2}$
Oil reservoir rocks	$10^{-6} - 10^{-4}$
Very fine sand, silt, loess, loam	$10^{-8} - 10^{-5}$
Layered Clay	$10^{-8} - 10^{-6}$
Sandstone, limestone, dolomite, granite	$10^{-12} - 10^{-7}$
Fat/Unweathered Clay	$10^{-12} - 10^{-9}$

$$\overrightarrow{t} = \widehat{n}_f \cdot \Pi.$$

Thus the normal forces on I emanating from Ω_f are given by $-\overrightarrow{t} \cdot \widehat{n}_f$. The normal forces on I exerted by Ω_p are $\rho g \phi$. This yields the second coupling condition:

$$-\overrightarrow{t} \cdot \widehat{n}_f = \rho g \phi. \qquad \text{(Coupling 2)}$$

The third and final condition accounts for viscosity on the interface by enforcing a condition on the tangential fluid velocity. The exact mathematical formulation of this condition is not completely understood. In part this may be due to trying to match the point-wise velocity, u, in the fluid region, Ω_f to the average velocity, u_p, in the porous media region, Ω_p. Let $\{\widehat{\tau}_i\}_{i=1}^{d-1}$ be an orthonormal set of tangent vectors on I. The most natural assumption would be "no-slip" on the boundary, i.e.

$$u \cdot \widehat{\tau}_j = u_p \cdot \widehat{\tau}_j = 0 \text{ for } j = 1, ..., d-1.$$

However, this did not reflect experimental data discovered by Beavers and Joseph in 1967

[9]. Through several experiments, they noted that the mass flux through Ω_f is larger than that predicted by the no-slip boundary conditions. Their experiments led them to derive the following slip-flow condition, expressing that slip velocity along I is proportional to the shear stresses along I.

$$(u - u_p) \cdot \hat{\tau}_j = \left(\frac{\sqrt{\hat{\tau}_j \cdot \mathcal{K} \cdot \hat{\tau}_j}}{\alpha_{\text{BJ}}} \right) (-\overrightarrow{t} \cdot \hat{\tau}_j) \text{ for } j = 1, ..., d-1$$

Here α_{BJ} represents the experimentally determined slip coefficient. It depends solely on the porous media properties and ranges from .01 to 5 (see [9]). In 1971, Saffman [63] proposed a modification to the Beavers-Joseph coupling condition. This proposal dropped the porous media averaged velocity based on observations showing that the term $u_p \cdot \widehat{\tau}_j$ is negligible compared to the fluid velocity $u \cdot \widehat{\tau}_j$. Referred to as the Beavers-Joseph-Saffman (-Jones) coupling condition, this is the third and final coupling condition used in this research:

$$u \cdot \hat{\tau}_j = -\frac{\sqrt{\hat{\tau}_j \cdot \mathcal{K} \cdot \hat{\tau}_j}}{\alpha_{\text{BJ}}} \overrightarrow{t} \cdot \hat{\tau}_j. \qquad \text{(Coupling 3)}$$

1.1.4 The Evolutionary Stokes-Darcy Equations

We review the fully evolutionary, Stokes-Darcy system. Let Ω_f represent the fluid region and Ω_p the porous media region, or aquifer. Let $0 < T \leq \infty$. The time-dependent Stokes-Darcy problem reads: Find u, p, ϕ satisfying

$$\left.\begin{aligned} & u_t - \nu \Delta u + \nabla p = f_f \text{ in } \Omega_f \times [0, T], \\ & \nabla \cdot u = 0 \text{ in } \Omega_f \times [0, T], \end{aligned}\right\} \qquad \text{(Stokes)}$$

$$\left. S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p \text{ in } \Omega_p \times [0, T], \right\} \qquad \text{(Darcy)}$$

$$\left.\begin{aligned} & n u \cdot \hat{n}_f - \mathcal{K} \nabla \phi \cdot \hat{n}_p = 0 \text{ on } I \times [0, T], \\ & \nu \hat{n}_f \cdot \nabla u \cdot \hat{n}_f - p = -g\phi \text{ on } I \times [0, T], \\ & u \cdot \hat{\tau}_j = -\frac{\rho \sqrt{\hat{\tau}_j \cdot \mathcal{K} \cdot \hat{\tau}_j}}{\alpha_{\text{BJ}}} \nu \hat{n}_f \cdot \nabla u \cdot \hat{\tau}_j \text{ for any tangent, } \hat{\tau}_j \text{ on } I \times [0, T], \end{aligned}\right\} \qquad \text{(Coupling)}$$

$$\left.\begin{aligned} u = 0 \text{ on } \partial\Omega_f \setminus I \times [0,T], \\ \phi = 0 \text{ on } \partial\Omega_p \setminus I \times [0,T], \end{aligned}\right\} \qquad \text{(Boundary Conditions)}$$

$$\left. u(x,0) = u_0(x) \text{ in } \Omega_f, \phi(x,0) = \phi_0(x) \text{ in } \Omega_f, \right\} \qquad \text{(Initial Conditions)}$$

where the variables and parameters are as follows:

$u =$ fluid velocity in Ω_f (Stokes velocity),

$p =$ kinematic fluid pressure in Ω_f (Stokes pressure),

$\phi =$ hydraulic head in Ω_p(Darcy pressure) $=$ elevation head $+$ pressure head,

$f_f =$ body forces in Ω_f,

$f_p =$ sources in Ω_p,

$\nu =$ kinematic viscosity of fluid,

$\mathcal{K} =$ hydraulic conductivity tensor,

$S_0 =$ specific (volumetric) storage,

$g =$ gravitational acceleration constant,

$n =$ volumetric porosity percentage,

$\alpha_{\text{BJ}} =$ measured slip coefficient.

1.2 VARIATIONAL FORMULATION AND PRELIMINARIES

Having derived the fully evolutionary Stokes-Darcy problem, the next step towards numerical approximation of solutions to this coupled problem is to derive the variational formulation. Let $(.,.)_{f/p}$ represent the L^2-inner product over the regions Ω_f and Ω_p respectively. Let $< .,. >_I$ represent the L^2-inner product over I. Denote the L^2 and H^1 norms induced by these inner products by $\|.\|_{f/p/I}$ and $\|.\|_{1,f/p/I}$.

Consider the conservation of momentum equation in the Stokes equations (Stokes). Choose the test function space $X_f = \{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \backslash I\}$. Multiply by

$v \in X_f$ and integrate over Ω_f to obtain

$$(u_t, v)_f - (\nu \Delta u, v)_f + (\nabla p, v)_f = (f_f, v)_f.$$

Consider the term $(\nu \Delta u, v)_f$. After integrating by parts and applying the Divergence Theorem, since $v = 0$ on $\Omega_f \setminus I$, this becomes

$$(\nu \Delta u, v)_f =< \nu \hat{n}_f \cdot \nabla u, v >_I -(\nu \nabla u, \nabla v)_f.$$

Similarly, the Stokes pressure term is equivalent to

$$(\nabla p, v)_f =< p, v \cdot \hat{n}_f >_I -(p, \nabla \cdot v)_f.$$

Thus, the conservation of momentum equation becomes

$$(u_t, v)_f + (\nu \nabla u, \nabla v)_f - (p, \nabla \cdot v)_f - < \nu \hat{n}_f \cdot \nabla u, v >_I - < p, v \cdot \hat{n}_f >_I = (f_f, v)_f \tag{1.10}$$

Rewrite the first interface term, $- < \nu \hat{n}_f \cdot \nabla u, v >_I$, in terms of normal and tangential components: $\hat{n}_f \cdot \nabla u = (\hat{n}_f \cdot \nabla u \cdot \hat{n}_f)\hat{n}_f + (\hat{n}_f \cdot \nabla \cdot \hat{\tau}_j)\hat{\tau}_j$, $v = (v \cdot \hat{n}_f)\hat{n}_f + \sum_{j=1}^{d-1}(v \cdot \hat{\tau}_j)\hat{\tau}_j$. Utilizing that $\hat{n}_f \cdot \hat{\tau}_j = 0$, this first interface term may be written as

$$- < \nu \hat{n}_f \cdot \nabla u \cdot \hat{n}_f, v \cdot \hat{n}_f >_I - \sum_{j=1}^{d-1} \langle \nu \hat{n}_f \cdot \nabla u \cdot \hat{\tau}_j, v \cdot \hat{\tau}_j \rangle_I$$

Together the interface terms in (1.10) equal

$$- < \nu \hat{n}_f \cdot \nabla u \cdot \hat{n}_f - p, v \cdot \hat{n}_f >_I - \sum_{j=1}^{d-1} \langle \nu \hat{n}_f \cdot \nabla u \cdot \hat{\tau}_j, v \cdot \hat{\tau}_j \rangle_I .$$

Therefore, by the second and third coupling conditions (Coupling), (1.10) becomes

$$\begin{aligned}(u_t, v)_f + (\nu \nabla u, \nabla v)_f + \sum_{j=1}^{d-1} \int_I \frac{\alpha_{\mathrm{BJ}}}{\rho \sqrt{\widehat{\tau}_j \cdot \mathcal{K} \cdot \widehat{\tau}_j}} \langle u \cdot \widehat{\tau}_j, v \cdot \widehat{\tau}_j \rangle_I \\ -(p, \nabla \cdot v)_f + g < \phi, v \cdot \widehat{n}_f >_I = (f_f, v)_f\end{aligned} \tag{1.11}$$

We next derive the variational formulation of the Darcy equation in the coupled system. Begin with the conservation of mass equation in (Darcy) in the porous media region:

$$S_0\phi_t - \nabla \cdot (\mathcal{K}\nabla\phi) = f_p \text{ in } \Omega_p \times [0, T].$$

Let $\psi \in X_p = \{\psi \in H^1(\Omega_p) : \phi = 0 \text{ on } \partial\Omega_p \backslash I\}$. Multiply both sides by ψ and integrate over the porous media domain. Integrate by parts once and apply the Divergence Theorem to obtain

$$\begin{aligned}(f_p, \psi)_p &= (S_0\phi_t, \psi)_p - (\nabla \cdot (\mathcal{K}\nabla\phi, \psi)_p \\ &= (S_0\phi_t, \psi)_p - < \mathcal{K}\nabla\phi \cdot \widehat{n}_p, \psi >_I + (\mathcal{K}\nabla\phi, \nabla\psi)_p\end{aligned}$$

Multiply through by g and apply the first coupling condition in (Coupling).

$$(gf_p, \psi)_p = (gS_0\phi_t, \psi)_p - gn < u \cdot \widehat{n}_f, \psi >_I + g(\mathcal{K}\nabla\phi, \nabla\psi)_p$$

To simplify notation, define the following bilinear forms. Let $a_f(.,.) : X_f \times X_f \to \mathbb{R}$, $b(.,.) : Q_f \times Q_f \to \mathbb{R}$, $a_p(.,.) : X_p \to \mathbb{R}$, and $c_I(.,.) : X_p \times X_f \to \mathbb{R}$. Define

$$a_f(u, v) = n\nu \int_{\Omega_f} \nabla u : \nabla v \, dx + \sum_{j=1}^{d-1} \int_I \frac{\alpha_{\mathrm{BJ}} n}{\rho\sqrt{\widehat{\tau}_j \cdot \kappa \cdot \widehat{\tau}_j}} (u \cdot \widehat{\tau}_j)(v \cdot \widehat{\tau}_j) \, ds, \tag{1.12}$$

$$b(v, q) = n \int_{\Omega_f} q\nabla \cdot v \, dx, \tag{1.13}$$

$$a_p(\phi, \psi) = g \int_{\Omega_p} \nabla\psi \cdot \mathcal{K} \cdot \nabla\phi \, dx, \tag{1.14}$$

$$c_I(v, \psi) = gn \int_I \psi v \cdot \widehat{n}_f . \, ds. \tag{1.15}$$

Thus, the variational formulation for the coupled Stokes-Darcy problem is given by

$$\begin{aligned}
\text{For every } t>0 \text{ find } u(.,t)\in X_f, p(.,t)\in Q_f, \phi(.,t)\in X_p \\
\text{satisfying for all } v\in X_f, q\in Q_p, \psi\in X_p: \\
n(u_t(t),v)_f + a_f(u(t),v) - b(v,p(t)) + c_I(v,\phi(t)) = n(f_f(t),v)_f, \\
b(u(t),q)=0, \\
g(S_0\phi_t(t),\psi)_p + a_p(\phi(t),\psi) - c_I(u(t),\psi) = g(f_p(t),\psi)_p.
\end{aligned} \tag{1.16}$$

Notice that this is an exactly skew-symmetric coupling. Studies on the existence and uniqueness of solution to this continuous problem in stationary form can be found in [24, 45]. Using the Trace and Poincaré inequalities given below, one can show that the bilinear forms $a_f(.,.)$ in (1.12) and $a_p(.,.)$ in (1.14) are both continuous and coercive on their respective domains, as given in Lemma 3.

Lemma 1. *(A Trace Inequality) Let Ω be a bounded regular domain, $u\in H^1(\Omega)$. Then there exists a constant $C_\Omega>0$ depending on the domain Ω such that the following inequality holds.*

$$\|u\|_{L^2(\partial\Omega)} \le C_\Omega \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

Proof. See, for example Brenner and Scott in [10], Ch. 1.6 p.36-38. □

Lemma 2. *(Poincaré Inequality) Let $v\in X_f$, $\psi\in X_p$. Then there exists a constant $C_P>0$ such that the following holds for $w=v$ or ψ.*

$$\|w\| \le C_P\|\nabla w\|.$$

Lemma 3. *(Continuity and Coercivity of the Bilinear Forms) The following inequalities hold:*

$$\begin{aligned}
a_f(u,v) &\le M_f\|\nabla u\|_f\|\nabla v\|_f, \\
a_p(\phi,\psi) &\le gk_{max}\|\nabla\phi\|_p\|\nabla\psi\|_p, \\
a_f(u,u) &\ge n\nu\|\nabla u\|_f^2, \\
a_p(\phi,\phi) &\ge gk_{min}\|\nabla\phi\|_p^2,
\end{aligned}$$

where $M_f = n\left(\nu + \dfrac{\alpha_{\mathrm{BJ}} C_{P,f} C_{\Omega_f}}{\rho\sqrt{k_{min}}}\right) > 0.$

Proof. We show the proof for the continuity of $a_f(.,.)$. Let $u, v \in X_f$. Recall that because $\mathcal{K}$ is (SPD), $k_{min} > 0$. By the Cauchy-Schwarz, Trace, and Poincaré inequalities,

$$\begin{aligned} a_f(u,v) &\le n\nu\|\nabla u\|_f\|\nabla v\|_f + \sum_{i=1}^{d-1} \tfrac{n\alpha_{\mathrm{BJ}}}{\rho\sqrt{\widehat{\tau}_i\cdot\mathcal{K}\cdot\widehat{\tau}_i}}\|u\cdot\widehat{\tau}_i\|_I\|v\cdot\widehat{\tau}_i\|_I \\ &\le n\nu\|\nabla u\|_f\|\nabla v\|_f + \tfrac{n\alpha_{\mathrm{BJ}}}{\rho\sqrt{k_{min}}}\sum_{i=1}^{d-1}\|u\cdot\widehat{\tau}_i\|_I\|v\cdot\widehat{\tau}_i\|_I \\ &\le n\nu\|\nabla u\|_f\|\nabla v\|_f + \tfrac{n\alpha_{\mathrm{BJ}} C_{P,f} C_{\Omega_f}}{\rho\sqrt{k_{min}}}\|\nabla u\|_f\|\nabla v\|_f. \end{aligned}$$

Coercivity of $a_f(.,.)$ and $a_p(.,.)$ follows immediately from calculating $a_f(u,u)$ and $a_p(\phi,\phi)$ and the fact that $\mathcal{K}$ is SPD. Likewise, continuity of $a_p(.,.)$ follows by the Cauchy-Schwarz inequality and boundedness of $\mathcal{K}$. □

Next we derive an energy estimate for the solutions to (1.16). Define the following norms on the dual spaces, $(X_f)^*$ and $(X_p)^*$.

$$\|f\|_{-1,f/p} = \sup_{0\neq w\in X_{f/p}} \frac{(f,w)_{f/p}}{\|\nabla w\|_{f/p}}$$

Lemma 4. *(Energy Estimate for Stokes-Darcy) Let* $(u(t),p(t),\phi(t)) \in (X_f,Q_f,X_p)\times[0,T]$ *be solutions to* (1.16)*. Then, for any* $0 < t \le T$*,*

$$\begin{aligned} n\|u(t)\|_f^2 + gS_0\|\phi(t)\|_p^2 + n\nu\int_0^t \|\nabla u(s)\|_f^2\,ds + gk_{min}\int_0^2 \|\nabla\phi(s)\|_p^2\,ds \\ \le \frac{m}{\nu}\int_0^t \|f_f(s)\|_{-1,f}^2\,ds + \frac{g}{k_{min}}\int_0^t \|f_p(s)\|_{-1,p}^2\,ds + n\|u_0\|_f^2 + gS_0\|\phi_0\|_p^2. \end{aligned}$$

Proof. Set $v = u$ and $\psi = \phi$ in (1.16) and add the first and third equations. By skew-symmetry, the coupling terms cancel, leaving

$$n(u_t,u)_f + (gS_0\phi_t,\phi)_p + a_f(u,u) + a_p(\phi,\phi) = n(f_f,u)_f + (gf_p,\phi)_p$$

Note that $(u_t, u)_f = \frac{1}{2}\frac{d}{dt}\|u\|^2$ and similar for $(\phi_t, \phi)_p$. Apply coercivity of the two bilinear forms, utilize the dual norm, and implement Young's inequality on the right-hand side to obtain

$$\frac{1}{2}\frac{d}{dt}\left(n\|u\|^2 + gS_0\|\phi\|^2\right) + \frac{n\nu}{2}\|\nabla u\|_f^2 + \frac{gk_{min}}{2}\|\nabla \phi\|_p^2 \leq \frac{n}{2\nu}\|f_f\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_p\|_{-1,p}^2.$$

Integrating in time produces the energy estimate for the coupled system. □

1.3 SEMI-DISCRETE APPROXIMATION USING THE FINITE ELEMENT METHOD

Having established the variational formulation of the fully evolutionary Stokes-Darcy problem (1.16), we next discretize the problem in space using the Finite Element Method (FEM). Select quasi-uniform meshes for Ω_f and Ω_p, $\mathcal{T}_{h_1}^f$ and $\mathcal{T}_{h_2}^p$ respectively. Let $\mathcal{T}_h = \mathcal{T}_{h_1}^f \bigcup \mathcal{T}_{h_2}^p$, with the maximum triangle diameter over the combined meshes denoted by h. Select finite element spaces,

$$\text{Stokes velocity: } X_f^h \subset X_f,$$
$$\text{Darcy pressure: } X_p^h \subset X_p,$$
$$\text{Stokes pressure: } Q_f^h \subset Q_f,$$

based on a conforming FEM triangulation. We assume that the Stokes velocity-pressure FEM spaces, X_f^h and Q_f^h, satisfy the usual discrete inf-sup condition for stability of the discrete pressure, denoted by (LBB^h) (see, for example, [34]), and stated below.

$$\exists\, \beta_h > 0 \text{ such that } \inf_{q_h \in Q_f^h,\, q_h \neq 0} \sup_{v_h \in X_f^h,\, v_h \neq 0} \frac{(q_h, \nabla \cdot v_h)_f}{\|\nabla v_h\|_f \|q_h\|_f} > \beta_h \qquad (LBB^h)$$

No assumption is made on the mesh compatibility or inter-domain continuity on the interface, I, between the two FEM meshes in the two sub-domains. Assume that X_f^h, X_p^h, and Q_f^h satisfy approximation properties of piecewise polynomials on quasi-uniform meshes of local degrees $r-1, r$, and $r+1$. That is,

$$\begin{aligned}
\inf_{x_h \in X_f^h} \|u - x_h\|_f &\leq Ch^{r+1}\|u\|_{H^{r+1}(\Omega_f)}, \\
\inf_{x_h \in X_f^h} \|u - x_h\|_{H^1(\Omega_f)} &\leq Ch^{r}\|u\|_{H^{r+1}(\Omega_f)}, \\
\inf_{y_h \in X_p^h} \|\phi - y_h\|_p &\leq Ch^{r+1}\|\phi\|_{H^{r+1}(\Omega_p)}, \\
\inf_{y_h \in X_p^h} \|\phi - y_h\|_{H^1(\Omega_p)} &\leq Ch^{r}\|\phi\|_{H^{r+1}(\Omega_p)}, \\
\inf_{z_h \in Q_f^h} \|p - z_h\|_f &\leq Ch^{r+1}\|p\|_{H^{r+1}(\Omega_f)}.
\end{aligned} \tag{1.17}$$

Analysis of some of the methods considered in this research requires an inverse inequality given below in Lemma 5. Note that this assumption implies a minimum angle condition. See Brenner and Scott, [10], chapter 4 for more on inverse inequalities. The inverse inequality constant, $C_{(inv)}$, depends on the angles in the mesh, but not the domain size or shape.

Lemma 5. *(An Inverse Inequality) Let $w_h \in X_f^h$ or X_p^h, then*

$$h\|\nabla w_h\|_{f/p} \leq C_{(inv)}\|w_h\|_{f/p}.$$

The semi-discretization for the coupled Stokes-Darcy problem is as follows.

$$\begin{aligned}
\text{Find } (u_h(\cdot,t), p_h(\cdot,t), \phi_h(\cdot,t)) : [0,\infty) &\to (X_f^h, Q_f^h, X_p^h) \\
\text{satisfying for all } (v_h, q_h, \psi_h) &\in (X_f^h, Q_f^h, X_p^h), \\
n(u_{h,t}, v_h)_f + a_f(u_h, v_h) - b(v_h, p_h) + c_I(v_h, \phi_h) &= n(f_f, v_h)_f, \\
b(u_h, q_h) &= 0, \\
gS_0(\phi_{h,t}, \psi_h)_p + a_p(\phi_h, \psi_h) - c_I(u_h, \psi_h) &= g(f_p, \psi_h)_p.
\end{aligned} \tag{FEM-SD}$$

Note in particular the preservation of the skew-symmetric coupling between the Stokes and the Darcy sub-problems.

After applying the Finite Element Method to the Stokes-Darcy problem, the system can be further reduced to a coupled evolution equation of the form

$$\begin{aligned} u_t + A_f u + C\phi &= F_f, \\ \phi_t + A_p\phi - Cu &= F_p, \end{aligned} \tag{1.18}$$

where A_f and A_p are SPD and $C = C^T$. Like the Stokes-Darcy semi-discretization in (FEM-SD), the above coupling is exactly skew-symmetric. The equations may then be written as a 2-block ODE system:

$$\frac{d}{dt}\begin{bmatrix} u \\ \phi \end{bmatrix} + \begin{bmatrix} A_f & 0 \\ 0 & A_p \end{bmatrix}\begin{bmatrix} u \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix}\begin{bmatrix} u \\ \phi \end{bmatrix} = \begin{bmatrix} F_f \\ F_p \end{bmatrix}. \tag{1.19}$$

This, in turn may be further simplified to an evolution equation of the form

$$w_t + Aw + \Lambda w = F, \tag{1.20}$$

where A is SPD, and Λ is exactly skew symmetric ($\Lambda^T = -\Lambda$). While most of the methods presented in this research will be applied directly to (FEM-SD) so that we may study effects of small parameters in the Stokes-Darcy problem, there will be times when it is helpful to refer to one of the aforementioned simpler systems.

1.3.1 Partitioned Methods

As mentioned previously, the challenges of the Stokes-Darcy problem include (i) capturing with accuracy the different physical process happening in each sub-domain, (ii) computing solutions over large domains and long-time intervals, and finally, (iii) maintaining stability when faced with small parameters such as specific storage, S_0, and hydraulic conductivity, k_{min}, as discussed in Section 1.1.2. Stability regardless of small parameters is a key role in developing methods for groundwater-surface water flow, since the singular limits $S_0 \to 0$ and $k_{min} \to 0$ have physical relevance given confined aquifers and impermeable porous media.

There has been considerable growth in the development and study of numerical methods for Stokes-Darcy coupled problems. A recent summary of methods and analyses of the Stokes-Darcy coupling can be found in [25]. Studies on the continuum model have been performed in [37, 9, 15, 57, 63]. The analysis of the equilibrium problem is advanced, see for example, [45, 24, 12, 26, 59, 61]. A more recent survey of the Stokes-Darcy domain decomposition for the equilibrium problem has been studied in [23, 27]. Analysis and development of methods for the time-dependent problem using the Beavers-Joseph interface conditions can be found in [16, 15, 14, 13]. Studies on extensions to the Navier-Stokes-Darcy coupling were performed in [6, 1, 25, 35].

The methods developed and analyzed in this research uncouple the Stokes-Darcy equations so that at each time step one can solve a separate Stokes and Darcy problem. Called partitioned methods, these algorithms allow us to utilize existing legacy codes already optimized for the sub-flows, thus minimizing computational cost and time. Partitioned methods for the Stokes-Darcy problem were first proposed by Mu and Zhu in [56]. The methods they proposed were first-order convergent. These partitioned methods uncouple the equations by utilizing implicit-explicit (IMEX) time-discretization, in particular by treating the coupling terms explicitly. Some general theory on (IMEX) discretizations can be found in [4, 33, 3, 69]. There are three main types of partitioned methods: parallel, splitting, and asynchronous. Parallel partitioned methods uncouple the equations so that the separate sub-problems may be solved in parallel, whereas splitting methods require you to solve them sequentially at each time step. Asynchronous methods allow one to take different time step sizes in each domain. More studies on partitioned methods for two domain problems have been performed in [17, 18, 19, 67]. In particular, in [49], Layton and Trenchea studied the stability of two (IMEX) partitioned methods applied to the related coupled evolution equations in (3.1).

Partitioned methods specifically for the Stokes-Darcy method have been studied in [56, 47, 48]. Asynchronous partitioned methods for Stokes-Darcy have been proposed and studied in [46, 73]. In this research, we will focus on error analysis for some the splitting partitioned methods presented in [47], presented in Chapter 2. The primary focus of this research, however, is to study higher-order methods, thus making computations over long-time intervals more efficient. The methods studied in Chapters 3-5 are both summaries and expansions on ideas presented by the author in [65, 66, 41].

2.0 SPLITTING METHODS FOR THE STOKES-DARCY PROBLEM

In this chapter we study the convergence properties of splitting methods applied to the Stokes-Darcy problem. As mentioned previously, the goal of a partitioned method is to decompose this complicated dual-physics problem into two, simpler, sub-physics problems. A splitting method accomplishes this goal by uncoupling the Stokes-Darcy equations in such a way that at each time level, we may solve the Stokes (Darcy) problem first and then use that solution to solve the Darcy (Stokes) problem. This idea of "splitting-up" a complicated problem in mathematical physics so that it may be solved sequentially has been well-studied, see for example [50, 51, 53, 52, 74].

The splitting methods studied in this chapter were proposed by Layton, Tran, and Xiong in [48]. In their work, they analyzed the stability properties of four splitting methods and performed several numerical tests on stability and convergence. In this chapter, the author expands on [48] by performing convergence analysis on three of the splitting methods, (BEsplit1-SD), (BEsplit2-SD), and (CNsplit-SD). The first two splitting methods, (BEsplit1-SD) and (BEsplit2-SD) are first-order convergent, but exhibit good stability properties when faced with either small k_{min} or small S_0 (but not both). The third splitting method studied, herein, (CNsplit-SD), has better convergence properties, but exhibits very restrictive stability properties.

The convergence analyses of the aforementioned methods presented in this chapter will utilize the equilibrium projection operator for the Stokes-Darcy problem, defined as follows.

Let $T \in (0, \infty)$. Define the equilibrium projection operator, P_h by

$$P_h : (X_f, Q_f, X_p) \to \left(X_f^h, Q_f^h, X_p^h\right),$$
$$P_h : (u(t), p(t), \phi(t)) \to (P_h u(t), P_h p(t), P_h \phi(t)), \forall t \in [0, T]$$
where $(P_h u(t), P_h p(t), P_h \phi(t))$ satisfies

$$\begin{aligned} a_f(u(t), v_h) - b(v_h, p(t)) + c_I(v_h, \phi(t)) \\ = a_f(P_h u(t), v_h) - b(v_h, P_h p(t)) + c_I(v_h, P_h \phi(t)), \\ b(P_h u(t), q_h) = 0, \\ a_p(\phi(t), \psi_h) - c_I(u(t), \phi_h) = a_p(P_h \phi(t), \psi_h) - c_I(P_h u(t), \psi_h). \end{aligned} \tag{SD-PROJ}$$

The operator, P_h, exists and is well defined, see for example [23, 24]. Under certain regularity assumptions, there holds

$$\begin{aligned} \|P_h w(t) - w(t)\|_{f/p} \leq Ch^2 \|w(t)\|_{f/p}, \\ \|\nabla (P_h w(t) - w(t))\|_{f/p} \leq Ch \|\nabla w(t)\|_{f/p}, \\ \|P_h p(t) - p(t)\|_f \leq Ch \|p(t)\|_f, \end{aligned} \tag{2.1}$$

for $w = u, \phi$. The first inequality above is derived in [55], and the second two inequalities are derived in [23].

Denote the true solutions at time t^k to the Stokes-Darcy equation (1.16) by (u^k, p^k, ϕ^k), whereas solutions to a numerical approximation to (1.16) at time t^k are given by (u_h^k, p_h^k, ϕ_h^k). The true solution satisfies the equation below.

$$\begin{aligned} n(u_t^{k+1}, v_h)_f + gS_0(\phi_t^{k+1}, \psi_h)_p + a_f(u^{k+1}, v_h) + a_p(\phi^{k+1}, \psi_h) \\ + c_I(v_h, \phi^{k+1}) - c_I(u^{k+1}, \psi_h) - b(v_h, p^{k+1}) \\ = n(f_f^{k+1}, v_h)_f + g(f_p^{k+1}, \psi_h)_p \end{aligned}$$

If we apply the projection property (SD-PROJ) for t^{k+1} to the above equation, we obtain

$$\begin{aligned} n(u_t^{k+1}, v_h)_f + gS_0(\phi_t^{k+1}, \psi_h)_p + a_f(P_h u^{k+1}, v_h) + a_p(P_h \phi^{k+1}, \psi_h) \\ + c_I(v_h, P_h \phi^{k+1}) - c_I(P_h u^k, \psi_h) - b(v_h, P_h p^{k+1}) = n(f_f^{k+1}, v_h)_f + g(f_p^{k+1}, \psi_h)_p \end{aligned} \tag{2.2}$$

In the following analyses, we utilize a special treatment of the coupling terms, $c_I(.,.)$ that requires an additional inequality, proven by Moraiti in [54]. The inequality holds under conditions on the domains Ω_f, Ω_p. The constant $C_{f,p}$ depends on $\Omega_{f/p}$ and in the special case of a flat interface I, with Ω_f and Ω_p being arbitrary domains, $C_{f,p}$ equals 1 (we will assume that our domain fits this case). See Moraiti in [54] Section 3 Lemmas 3.1 and 3.2 for further details on the derivation of this inequality.

$$|c_I(u,\phi)| \leq ng\|u\|_{DIV,f}\|\phi\|_{1,p}. \qquad \text{(HDIV-TRACE)}$$

Furthermore, we will need the following bounds for consistency errors produced by the (BEsplit1-SD),(BEsplit2-SD),(CNsplitA-SD),and (CNsplitB-SD) methods.

Lemma 6.

$$\Delta t \sum_{k=0}^{N-1} \|u^{k+1} - u^k - \Delta t u_t^{k+1}\|_f^2 \leq \tfrac{\Delta t^4}{3}\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{2.3}$$

$$\Delta t \sum_{k=0}^{N-1} \|\phi^{k+1} - \phi^k - \Delta t \phi_t^{k+1}\|_p^2 \leq \tfrac{\Delta t^4}{3}\|\phi_{tt}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{2.4}$$

$$\Delta t \sum_{k=0}^{N-1} \|u^{k+1} - u^k\|_f^2 \leq \Delta t\|u_t\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{2.5}$$

$$\Delta t \sum_{k=0}^{N-1} \|\phi^{k+1} - \phi^k\|_p^2 \leq \Delta t\|\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{2.6}$$

$$\Delta t \sum_{k=0}^{N-1} \|(2u^{k+1} - 2u^k) - \Delta t(u_t^{k+1} + u_t^k)\|_f^2 \leq \tfrac{2\Delta t^4}{3}\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{2.7}$$

$$\Delta t \sum_{k=0}^{N-1} \|(2\phi^{k+1} - 2\phi^k) - \Delta t(\phi_t^{k+1} + \phi_t^k)\|_p^2 \leq \tfrac{2\Delta t^4}{3}\|\phi_{tt}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{2.8}$$

$$\Delta t \sum_{k=0}^{N-1} \|f_f^{k+1} + f_f^k - 2f_f^{k+1/2}\|_f^2 \leq \tfrac{\Delta t^4}{12}\|(f_f)_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{2.9}$$

$$\Delta t \sum_{k=0}^{N-1} \|f_p^{k+1} + f_p^k - 2f_p^{k+1/2}\|_p^2 \leq \tfrac{\Delta t^4}{12}\|(f_p)_{tt}\|^2_{L^2(0,T;L^2(\Omega_p))},. \tag{2.10}$$

Proof. All inequalities can be shown using integration by parts along with the Cauchy-

Schwarz inequality. For the first inequality,

$$\begin{aligned}
\Delta t\sum_{k=0}^{N-1}\left\|\left(u^{k+1}-u^{k}\right)-\Delta t u_t^{k+1}\right\|_f^2 &= \Delta t\sum_{k=0}^{N-1}\int_{\Omega_f}\left\|\left(u^{k+1}-u^{k}\right)-\Delta t u_t^{k+1}\right\|_2^2\,dx\\
&= \Delta t\sum_{k=0}^{N-1}\int_{\Omega_f}\left\|\int_{t^k}^{t^{k+1}}(t^k-t)u_{tt}\,dt\right\|_2^2\,dx\\
&\le \Delta t\sum_{k=0}^{N-1}\int_{\Omega_f}\left(\int_{t^k}^{t^{k+1}}(t^k-t)^2\,dt\int_{t^k}^{t^{k+1}}\|u_{tt}\|_2^2\,dt\right)dx\\
&= \tfrac{\Delta t^4}{3}\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))},
\end{aligned}$$

and similar for ϕ in (2.4).

Inequalities (2.6)-(2.7) are similar, so we only show the proof for ϕ:

$$\begin{aligned}
\Delta t\sum_{k=0}^{N-1}\|\phi^{k+1}-\phi^k\|_p^2 &= \Delta t\int_{\Omega_p}\sum_{k=0}^{N-1}\left(\int_{t^k}^{t^{k+1}}\phi_t\,dt\right)^2 dx\\
&\le \Delta t\sum_{k=0}^{N-1}\int_{\Omega_p}\left(\int_{t^k}^{t^{k+1}}1^2\,dt\int_{t^k}^{t^{k+1}}|\phi_t|_2^2\,dt\right)dx\\
&= \Delta t^2\|\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}.
\end{aligned}$$

Inequality (2.7) follows below.

$$\begin{aligned}
\Delta t\sum_{k=0}^{N-1}\|(2u^{k+1}-2u^k)-\Delta t(u_t^{k+1}+u_t^k)\|_f^2 &= \Delta t\sum_{k=0}^{N-1}\int_{\Omega_f}\left\|2\left(u^{k+1}-u^k\right)-\Delta t(u_t^{k+1}+u_t^k)\right\|_2^2 dx\\
&= \Delta t\sum_{k=0}^{N-1}\int_{\Omega_f}\left\|\int_{t^k}^{t^{k+1}}(t^{k+1}+t^k-2t)u_{tt}\,dt\right\|_2^2 dx\\
&\le \Delta t\sum_{k=0}^{N-1}\int_{\Omega_f}\left(\int_{t^k}^{t^{k+1}}(t^{k+1}+t^k-2t)^2\,dt\int_{t^k}^{t^{k+1}}\|u_{tt}\|_2^2\,dt\right)dx\\
&= \tfrac{2\Delta t^4}{3}\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}
\end{aligned}$$

Proof of inequality (2.8) is omitted due to its similarity to the proof of inequality (2.7).

Finally, we prove the last two inequalities.

$$\Delta t \sum_{k=0}^{N-1} \|f_{f/p}^{k+1} + f_{f/p}^{k} - 2f_{f/p}^{k+1/2}\|_{f/p}^2 = \int_{\Omega_{f/p}} \left[\int_{t^{k+1/2}}^{t^{n+1}} (f_{f/p})_t \, dt + \int_{t^{k+1/2}}^{t^k} (f_{f/p})_t \, dt \right]^2 dx$$

$$= \Delta t \sum_{k=0}^{N-1} \int_{\Omega_{f/p}} \left[\int_{t^{k+1/2}} (t - t^{n+1})(f_{f/p})_{tt} \, dt + \int_{t^{k+1/2}}^{t^k} (t - t^k)() f_{f,p,tt} \, dt \right]^2 dx$$

$$\leq 2\Delta t \sum_{k=0}^{N-1} \int_{\Omega_{f/p}} \left[\left(\int_{t^{k+1/2}}^{t^{k+1}} (t - t^{k+1}(f_{f/p})_{tt} \, dt) \right)^2 + \left(\int_{t^{k+1/2}}^{t^k} (t - t^n)(f_{f/p})_{tt} \, dt \right)^2 \right] dx$$

$$\leq 2\Delta t \sum_{k=0}^{N-1} \int_{\Omega_{f/p}} \left[\int_{t^{k+1/2}}^{t^{k+1}} (t - t^{k+1})^2 dt \int_{t^{k+1/2}}^{t^{k+1}} |(f_{f/p})_{tt}|_2^2 dt + \int_{t^{k+1/2}}^{t^k} (t - t^k)^2 dt \int_{t^{k+1/2}}^{t^k} |f_{f/p})_{tt}|_2^2 dt \right] dx$$

$$= \tfrac{\Delta t^4}{12} \|(f_{f/p})_{tt}\|_{L^2(0,T;L^2(\Omega_{f/p}))}^2.$$

□

Additionally, we will need the following bounds involving the projection operator.

Lemma 7.

$$\sum_{k=0}^{N-1} \|(P_h - I)(u^{k+1} - u^k)\|_f^2 \leq C\Delta t h^4 \|u_t\|_{L^2(0,T;L^2(\Omega_f))}^2, \tag{2.11}$$

$$\sum_{k=0}^{N-1} \|(P_h - I)(\phi^{k+1} - \phi^k)\|_p^2 \leq C\Delta t h^4 \|\phi_t\|_{L^2(0,T;L^2(\Omega_p))}^2, , \tag{2.12}$$

$$\sum_{k=0}^{N-1} \|\nabla(P_h u^{k+1} - P_h u^k)\|_f^2 \leq C\Delta t(h^2 + 1)\|\nabla u_t\|_{L^2(0,T;L^2(\Omega_f))}^2, , \tag{2.13}$$

$$\sum_{k=0}^{N-1} \|\nabla(P_h \phi^{k+1} - P_h \phi^k)\|_p^2 \leq C\Delta t(h^2 + 1)\|\nabla \phi_t\|_{L^2(0,T;L^2(\Omega_p))}^2, , \tag{2.14}$$

Proof. We prove the first two using integration by parts and the Cauchy-Schwarz inequality, followed by the projection error bounds given in (2.1). We show the proof for (2.12). The proof for (2.11) is similar.

$$\sum_{k=0}^{N-1} \|(P_h - I)(\phi^{k+1} - \phi^k)\|_p^2 = \sum_{k=0}^{N-1} \int_{\Omega_p} \left((P_h - I) \int_{t^k}^{t^{k+1}} \phi_t \, dt \right)^2 dx$$
$$\leq \sum_{k=0}^{N-1} \int_{\Omega_p} \left\{ \Delta t \int_{t^k}^{t^{k+1}} |(P_h - I)\phi_t|^2 \, dt \right\} dx$$
$$\leq C\Delta t h^4 \|\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}.$$

To prove inequalities (2.13)-(2.14), first note that by triangle inequality

$$\|\nabla(P_h w^{k+1} - P_h w^k)\|_{f/p} \leq \|\nabla((P_h - I)(w^{k+1} - w^k))\|_{f/p} + \|\nabla(w^{k+1} - w^k)\|_{f/p},$$

for $w = u, \phi$.

Therefore, similar to (2.11) and (2.12),

$$\sum_{k=0}^{N-1} \|\nabla(P_h(w^{k+1} - w^k))\|^2_{f/p} \leq 2 \sum_{k=0}^{N-1} \left(\|\nabla((P_h - I)(w^{k+1} - w^k))\|^2_{f/p} + \|\nabla(w^{k+1} - w^k)\|^2_{f/p} \right)$$
$$\leq C\Delta t(h^2 + 1)\|\nabla w_t\|^2_{L^2(0,T;L^2(\Omega_{f/p}))},$$

for $w = u, \phi$.

□

2.1 CONVERGENCE OF THE BACKWARD-EULER SPLITTING METHODS FOR STOKES-DARCY

The Backward-Euler Splitting Methods discretize the Stokes-Darcy problem, (1.16), in time using the fully implicit Backward-Euler (BE) method, but lag the coupling term in either the Stokes equation (BEsplit1-SD) or the Darcy equation (BEsplit2-SD) at the previous time step. In doing so, these methods uncouple the equations so that at each time step one solves a separate Stokes and Darcy problem sequentially, making this a splitting partitioned method. The algorithms for (BEsplit1-SD) and (BEsplit2-SD) follow below.

Definition 8. *(Backward-Euler Splitting Method 1 for Stokes-Darcy (BEsplit1-SD)) Given (u_h^k, p_h^k, ϕ_h^k) in (X_f^h, Q_f^h, X_p^h), find $(u_h^{k+1}, p^{k+1}, \phi_h^{k+1})$ satisfying for all (v_h, q_h, ψ_h) in (X_f^h, Q_f^h, X_p^h):*

$$
\begin{aligned}
&1.\ n\left(\tfrac{u_h^{k+1}-u_h^k}{\Delta t}, v_h\right)_f + a_f\left(u_h^{k+1}, v_h\right) - b(v_h, p_h^{k+1}) \\
&\qquad\qquad\qquad\qquad + c_I\left(v_h, \phi_h^k\right) = n(f_f^{k+1}, v_h)_f, \\
&\quad b(u_h^{k+1}, q_h) = 0. \\
&2.\ gS_0\left(\tfrac{\phi_h^{k+1}-\phi_h^k}{\Delta t}, \psi_h\right)_p + a_p\left(\phi_h^{k+1}, \psi_h\right) - c_I\left(u_h^{k+1}, \psi_h\right) = g\left(f_p^{k+1}, \psi_h\right)_p.
\end{aligned}
\tag{BEsplit1-SD}
$$

Definition 9. *(Backward-Euler Splitting Method 2 for Stokes-Darcy (BEsplit2-SD)) Given (u_h^k, p_h^k, ϕ_h^k) in (X_f^h, Q_f^h, X_p^h), find $(u_h^{k+1}, p^{k+1}, \phi_h^{k+1})$ satisfying for all (v_h, q_h, ψ_h) in (X_f^h, Q_f^h, X_p^h):*

$$
\begin{aligned}
&1.\ gS_0\left(\tfrac{\phi_h^{k+1}-\phi_h^k}{\Delta t}, \psi_h\right)_p + a_p\left(\phi_h^{k+1}, \psi_h\right) - c_I\left(u_h^{k}, \psi_h\right) = g\left(f_p^{k+1}, \psi_h\right)_p. \\
&2.\ n\left(\tfrac{u_h^{k+1}-u_h^k}{\Delta t}, v_h\right)_f + a_f\left(u_h^{k+1}, v_h\right) - b(v_h, p_h^{k+1}) \\
&\qquad\qquad\qquad\qquad + c_I\left(v_h, \phi_h^{k+1}\right) = n(f_f^{k+1}, v_h)_f, \\
&\quad b(u_h^{k+1}, q_h) = 0.
\end{aligned}
\tag{BEsplit2-SD}
$$

In [48], they analyzed stability properties of the BEsplit-SD methods. The restrictions derived for stability are key in the convergence analysis. Define the following:

$$
\begin{aligned}
\Delta T_1 &:= 2\min\left\{\nu k_{min} S_0 \tfrac{16}{(C_{\Omega_f} C_{\Omega_p})^4 g^2}, 1\right\}, \\
\Delta T_2 &:= \tfrac{2\min\{1, gS_0\}}{gC_{\Omega_f} C_{\Omega_p} C_{(inv)}} h, \\
\Delta T_3 &:= 2gS_0 \nu h \left[gC_{\Omega_f} C_{\Omega_p}\right]^{-2} (C_{(inv)} C_{P,f})^{-1}, \\
\Delta T_4 &:= \tfrac{2\min\{1,\rho\}}{\rho g(1+C_{P,p}^2)} k_{min}, \\
\Delta T_5 &:= \tfrac{h k_{min}}{ng(C_{\Omega_f} C_{\Omega_p})^2 C_{(inv)}},
\end{aligned}
$$

$$\Delta T_6 := \frac{k_{min}}{ng(1+C_{P,p}^2)},$$

$$Parameters := (1 + C_{P,p}^2)(C_{P,f}^2 + d)\frac{gn}{k_{min}\nu}.$$

Implications of the time-step and parameter restrictions for stability of the two methods will be further discussed in the conclusion of this chapter.

2.1.1 Convergence of (BEsplit1-SD)

We begin by proving first-order convergence of the method (BEsplit1-SD). In [48], they showed the method is uniformly stable in time. For reference, the result is summarized below.

Theorem 10 (Stability of (BEsplit1-SD))**.** *Suppose that either the problem parameters satisfy*

$$Parameters < 1,$$

or, Δt satisfies the time-step restriction

$$\Delta t < \max\{\Delta T_1, \Delta T_2, \Delta T_3, \Delta T_4\}.$$

Then (BEsplit1-SD) is stable uniformly in time. In particular, if one of the time-step restrictions $\Delta T_{1,2,4}$ or $Parameters$ holds, then there is $\alpha \in (0,1)$ such that for $N > 0$,

$$\alpha\left(n\|u_h^N\|_f^2 + gS_0\|\phi_h^N\|_p^2\right) + \frac{\Delta t}{2}\sum_{k=0}^{N-1}\left\{a_f(u_h^{k+1}+u_h^k, u_h^{k+1}, u_h^k) + a_p(\phi_h^{k+1}+\phi_h^k, \phi_h^{k+1}+\phi_h^k)\right\}$$
$$\leq \alpha\left(n\|u_h^0\|_f^2 + gS_0\|\phi_h^0\|_p^2\right) + \Delta t\sum_{k=0}^{N-1} n\left\{(f_f^{k+1}, u_h^{k+1}+u_h^k)_f + g(f_p^{k+1}, \phi_h^{k+1}+\phi_h^k)_p\right\}.$$

Proof. See [48]. □

The convergence analysis of (BEsplit1-SD) assumes the condition $Parameters < 1$ holds. Analysis for the other cases is similar in nature but omitted from this research due to length.

Theorem 11 (Convergence of (BEsplit1-SD))**.** *Suppose that u, ϕ, p satisfy the regularity conditions required for the projection error inequalities in (2.1) for $w = u, \phi, u_t, \phi_t$. Assume* *that the following time-step condition holds:*

$$Parameters := (1 + C_{P,p}^2)(C_{P,f}^2 + d)\frac{g}{k_{min}\nu} < 1.$$

Then the errors between the true solutions to the Stokes-Darcy problem (1.16) *and the* (BEsplit1-SD) *method satisfy* $\mathcal{O}(\Delta t(h+1) + h^2)$.

Proof. Add the projections of the discrete time derivatives for u and ϕ to both sides of (2.2).

$$\begin{aligned} n\left(\tfrac{P_h u^{k+1} - P_h u^k}{\Delta t}, v_h\right)_f + gS_0\left(\tfrac{P_h \phi^{k+1} - P_h \phi^k}{\Delta t}, \psi_h\right)_p + a_f(P_h u^{k+1}, v_h) \\ + a_p(P_h \phi^{k+1}, \psi_h) + c_I(v_h, P_h \phi^{k+1}) - c_I(P_h u^{k+1}, \psi_h) - b(v_h, P_h p^{k+1}) \\ = n\left(\tfrac{P_h u^{k+1} - P_h u^k}{\Delta t}, v_h\right)_f + gS_0\left(\tfrac{P_h \phi^{k+1} - P_h \phi^k}{\Delta t}, \psi_h\right)_p \\ - n(u_t^{k+1}, v_h)_f - gS_0(\phi_t^{k+1}, \psi_h)_p + n(f_f^{k+1}, v_h)_f + g(f_p^{k+1}, \psi_h)_p \end{aligned} \tag{2.15}$$

Consider (BEsplit1-SD). Define the errors between the projections of the true solutions and the solutions to the method (BEsplit1-SD) as

$$e_u^k = P_h u^k - u_h^k,\ e_p^k = P_h p^k - p_h^k,\ e_\phi^k = P_h \phi^k - \phi_h^k.$$

Note that these errors are in our finite element spaces, X_f^h, Q_f^h, and X_p^h. Subtract (BEsplit1-SD) from (2.15). Using the prescribed error notation yields

$$\begin{aligned} n\left(\tfrac{e_u^{k+1} - e_u^k}{\Delta t}, v_h\right)_f + gS_0\left(\tfrac{e_\phi^{k+1} - e_\phi^k}{\Delta t}\right)_p + a_f(e_u^{k+1}, v_h) + a_p(e_\phi^{k+1}, \psi_h) \\ + c_I(v_h, e_\phi^k) - c_I(e_u^{k+1}, \psi_h) - b(v_h, e_p^{k+1}) \\ = n\left(\tfrac{P_h u^{k+1} - P_h u^k}{\Delta t}, v_h\right)_f + gS_0\left(\tfrac{P_h \phi^{k+1} - P_h \phi^k}{\Delta t}, \psi_h\right)_p \\ - c_I(v_h, P_h \phi^{k+1} - P_h \phi^k) - n(u_t^{k+1}, v_h)_f - gS_0(\phi_t^{k+1}, \psi_h)_p. \end{aligned} \tag{2.16}$$

In the above equation we added $c_I(v_h, P_h \phi^k)$ to both sides to make the term $c_I(v_h, e_\phi^k)$ appear on the left. Choose $v_h = \Delta t(e_u^{k+1} + e_u^k)$, $\psi_h = \Delta t(e_\phi^{k+1} + e_\phi^k)$. Then the equation becomes

$$
\begin{aligned}
&\left(n\|e_u^{k+1}\|_f^2 + gS_0\|e_\phi^{k+1}\|_p^2\right) - \left(n\|e_u^k\|_f^2 + gS_0\|e_\phi^k\|_p^2\right) \\
&+\Delta t a_f(e_u^{k+1}, e_u^{k+1} + e_u^k) + \Delta t a_p(e_\phi^{k+1}, e_\phi^{k+1} + e_\phi^k) \\
&+\Delta t c_I(e_u^{k+1} + e_u^k, e_\phi^k) - \Delta t c_I(e_u^{k+1}, e_\phi^{k+1} + e_\phi^k) \\
&= n\left(P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}, e_u^{k+1} + e_u^k\right)_f \\
&+gS_0\left(P_h \phi^{k+1} - P_h \phi^k - \Delta t \phi_t^{k+1}, e_\phi^{k+1} + e_\phi^k\right)_p \\
&-\Delta t c_I(e_u^{k+1} + e_u^k, P_h\phi^{k+1} - P_h\phi^k)
\end{aligned}
\tag{2.17}
$$

Define the energy, diffusive, and coupled terms as follows. Note that by coercivity of $a_f(.,.)$ and $a_p(.,.)$, $\mathcal{D}^{k+1/2} \leq a_f(e_u^{k+1} + e_u^k, e_u^{k+1} + e_u^k) + a_p(e_\phi^{k+1} + e_\phi^k, e_\phi^{k+1} + e_\phi^k)$.

$$
\begin{aligned}
E^k &= n\|e_u^k\|_f^2 + gS_0\|e_\phi^k\|_p^2 + \tfrac{\Delta t}{2}\left(a_f(e_u^k, e_u^k) + a_p(e_\phi^k, e_\phi^k)\right), \\
\mathcal{D}^{k+1/2} &= n\nu\|\nabla(e_u^{k+1} + e_u^k)\|_f^2 + gk_{min}\|\nabla(e_\phi^{k+1} + e_\phi^k)\|_p^2, \\
\mathcal{C}^k &= c_I(e_u^k, e_\phi^k).
\end{aligned}
$$

After incorporating the above notation, (2.17) becomes the following inequality:

$$
\begin{aligned}
&(E^{k+1} + \Delta t\mathcal{C}^{k+1}) - (E^k + \Delta t\mathcal{C}^k) + \tfrac{\Delta t}{2}\mathcal{D}^{k+1/2} \\
&\leq n\left(P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}, e_u^{k+1} + e_u^k\right)_f \\
&+gS_0\left(P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}, e_\phi^{k+1} + e_\phi^k\right)_p \\
&-\Delta t c_I(e_u^{k+1} + e_u^k, P_h\phi^{k+1} - P_h\phi^k).
\end{aligned}
\tag{2.18}
$$

Consider the coupling term on the right-hand side. Using the Cauchy-Schwarz, Trace (Lemma 1), Poincaré (Lemma 2), and Young inequalities we find

$$
\begin{aligned}
&\Delta t c_I(e_u^{k+1} + e_u^k, P_h\phi^{k+1} - P_h\phi^k) = \Delta t n g \int_I \left[(e_u^{k+1} + e_u^k)\cdot \widehat{n}_f (P_h\phi^{k+1} - P_h\phi^k)\right] ds \\
&\leq \Delta t n g \|e_u^{k+1} + e_u^k\|_I \|P_h\phi^{k+1} - P_h\phi^k\|_I \\
&\leq \Delta t n g C_{\Omega_f} C_{\Omega_p} \|e_u^{k+1} + e_u^k\|_f^{1/2} \|\nabla(e_u^{k+1} + e_u^k)\|_f^{1/2} \|P_h\phi^{k+1} - P_h\phi^k\|_p^{1/2} \|\nabla(P_h\phi^{k+1} - P_h\phi^k)\|_p^{1/2} \\
&\leq \Delta t n g C_{\Omega_f} C_{\Omega_p} C_{P,f}^{1/2} C_{P,p}^{1/2} \|\nabla(e_u^{k+1} + e_u^k)\|_f \|\nabla(P_h\phi^{k+1} - P_h\phi^k)\|_p \\
&\leq \tfrac{\Delta t n\nu}{8}\|\nabla(e_u^{k+1} + e_u^k)\|_f^2 + \frac{2\Delta t n g^2 C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f} C_{P,p}}{\nu}\|\nabla(P_h\phi^{k+1} - P_h\phi^k)\|_p^2.
\end{aligned}
$$

Next we bound the consistency errors on the right-hand side.

$$n\left(P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}, e_u^{k+1} + e_u^k\right)_f \le n\|P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}\|_f \|e_u^{k+1} + e_u^k\|_f$$
$$\le \tfrac{\Delta t n\nu}{8}\|\nabla(e_u^{k+1} + e_u^k)\|_f^2 + \tfrac{8C_{P,f}^2 n}{\Delta t\nu}\|P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}\|_f^2,$$

$$gS_0\left(P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}, e_\phi^{k+1} + e_\phi^k\right)_p \le gS_0\|P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}\|_p \|e_\phi^{k+1} + e_\phi^k\|_p$$
$$\le \tfrac{\Delta t g k_{min}}{4}\|\nabla(e_\phi^{k+1} + e_\phi^k)\|_p^2 + \tfrac{C_{P,p}^2 g S_0^2}{\Delta t k_{min}}\|P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}\|_p^2.$$

After subsuming terms into the diffusive terms on the left-hand side, (2.18) becomes

$$\begin{aligned} &(E^{k+1} + \Delta t\mathcal{C}^{k+1}) - (E^k + \Delta t\mathcal{C}^k) + \tfrac{\Delta t}{4}\mathcal{D}^{k+1/2} \\ \le \tfrac{8C_{P,f}^2 n}{\Delta t\nu}\|P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}\|_f^2 &+ \tfrac{C_{P,p}^2 g^2 g S_0^2}{\Delta t k_{min}}\|P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}\|_p^2 \\ &+ \tfrac{2\Delta t n g^2 C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f} C_{P,p}}{\nu}\|\nabla(P_h\phi^{k+1} - P_h\phi^k)\|_p^2. \end{aligned} \tag{2.19}$$

Split the consistency errors on the right-hand side as follows using triangle inequality:

$$\|P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}\|_f \le \|\left(P_h - I\right)\left(u^{k+1} - u^k\right)\|_f + \|\left(u^{k+1} - u^k\right) - \Delta t u_t^{k+1}\|_f,$$
$$\|P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}\|_p \le \|\left(P_h - I\right)\left(\phi^{k+1} - \phi^k\right)\|_p + \|\left(\phi^{k+1} - \phi^k\right) - \Delta t\phi_t^{k+1}\|_p.$$

Sum inequality (2.19) from $k = 0$ to $N - 1$. Absorb all constants on the right-hand side into one constant, $C_0 > 0$, independent of mesh width, h, and time-step size, Δt. This produces

$$\begin{aligned} &(E^{k+1} + \Delta t\mathcal{C}^{k+1}) + (E^k + \Delta t\mathcal{C}^k) + \tfrac{\Delta t}{4}\mathcal{D}^{k+1/2} \\ &\le C_0\sum_{k=0}^{N-1}\{\tfrac{1}{\Delta t}\left[\|(P_h - I)(u^{k+1} - u^k)\|_f^2 + \|(u^{k+1} - u^k) - \Delta t u_t^{k+1}\|_f^2\right] \\ &\quad + \tfrac{1}{\Delta t}\left[\|(P_h - I)(\phi^{k+1} - \phi^k)\|_p^2 + \|(\phi^{k+1} - \phi^k) - \Delta t\phi_t^{k+1}\|_p^2\right] \\ &\quad + \Delta t\left\|\nabla\left(P_h(\phi^{k+1} - \phi^k)\right)\right\|_p^2\}. \end{aligned} \tag{2.20}$$

Apply Lemmas 6 and 7 to the terms on the right-hand side of (2.20) to obtain

$$E^N + \Delta t\mathcal{C}^N + \tfrac{\Delta t}{4}\sum_{k=0}^{N-1}\mathcal{D}^{k+1/2} \leq E^0 + \Delta t\mathcal{C}^0 + C_1\{h^4\left(\|u_t\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}\right)$$
$$+ \Delta t^2\left(\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;L^2(\Omega_p))}\right) + \Delta t^2\left(h^2+1\right)\|\nabla\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))}\},$$

for some $C_1 > 0$. This implies that the errors between the projections of the true solutions of Stokes-Darcy and the solutions of the method (BESPLIT1-SD) satisfy

$$\|e_w^N\|_{f/p} \approx \mathcal{O}(\Delta t(h+1) + h^2),$$

for $w = u, \phi$ provided $E^N - \Delta t\mathcal{C}^N > 0$. We will show $E^N + \Delta t\mathcal{C}^N > 0$ holds momentarily. Bound $\Delta t\mathcal{C}^N$ using inequality (HDIV-TRACE), along with Poincaré (Lemma 2), and $\|\nabla\cdot u\|_f \leq \sqrt{d}\|\nabla u\|_f$ to find

$$\begin{aligned}\Delta t\mathcal{C}^N &\leq \Delta tgn\|e_\phi^N\|_p^2\|e_u^N\|^2_{DIV,f}\\ &\Delta tgn\sqrt{(1+C_(P,p)^2)(C^2_{P,f}+d)}\|\nabla e_\phi^N\|_p\|\nabla e_u^N\|_f\\ &\leq \Delta tgk_{min}\|\nabla e_\phi^N\|_p^2 + \tfrac{\Delta t(1+C^2_{P,p})(C^2_{P,f}+d)gn^2}{k_{min}}\|\nabla e_u^N\|_f^2.\end{aligned}$$

Since $E^N \geq n\|e_u^N\|_f^2 + gS_0\|e_\phi^N\|2_p + \Delta t\left(n\nu\|\nabla e_u^N\|_f^2 + gk_{min}\|\nabla e_\phi^N\|_p^2\right)$, by the assumptions that $Parameters < 1$, we have $E^N - \Delta\mathcal{C}^N > 0$. To complete the proof, notice that by triangle inequality, the errors between the true solutions of the Stokes-Darcy equations and the solutions of (BESPLIT1-SD) satisfy

$$\|w^N - w_h^N\|_{f/p} \leq \|(P_h - I)w^N\|_{f/p} + \|e_w^N\|_{f/p},$$

for $w = u, \phi$. By the bounds for the projection errors given in (2.1), we have $\|(P_h - I)w^N\|_{f/p} \leq Ch^2\|w^N\|_{f/p}$. Therefore, the errors in the (BESPLIT1-SD) method are $\mathcal{O}(\Delta t(h+1) + h^2)$, implying this method first-order convergent in time.

□

2.1.2 Convergence of (BESPLIT2-SD)

We recall the stability result for the (BESPLIT2-SD) method proven in [48].

Theorem 12 (Stability of (BEsplit2-SD)). *Suppose that the problem parameters satisfy either*

$$Parameters < 1,$$

or Δt satisfies

$$\Delta t < \max\{\Delta T_1, \Delta T_2, \Delta T_5, \Delta T_6\}.$$

Then (BEsplit2-SD) is uniformly stable in time and in S_0. In particular, there exists $\alpha > 0$ such that for $N > 0$,

$$\begin{aligned} &\tfrac{1}{2}\left(n\|u_h^N\|_f^2 + gS_0\|\phi_h^N\|_p^2\right) \\ &+\Delta t \sum_{k=0}^{N-1}\left\{\tfrac{\Delta t}{2} gS_0 \left\|\tfrac{\phi_h^{k+1}-\phi_h^n}{\Delta t}\right\|_p^2 + a_f(u_h^{k+1} u_h^{k+1}) + \alpha a_p(\phi_h^{k+1}, \phi_h^{k+1})\right\} \\ &\leq \tfrac{1}{2}\left(n\|u_h^0\|_f^2 + gS_0\|\phi_h^0\|_p^2\right) + \Delta t \sum_{k=0}^{N-1}\left\{(f_f^{k+1}, u_h^{k+1})_f + g(f_p^{k+1}, \phi_h^{k+1})_p\right\}. \end{aligned}$$

Proof. See [48]. □

Next, we prove first-order in time convergence of the method when the time-step size, Δt, satisfies $\Delta t < \Delta T_5$. Proof of convergence for the other possible scenarios are similar in procedure and omitted due to length.

Theorem 13 (Convergence of (BEsplit2-SD)). *Suppose the following time-step condition holds*

$$\Delta t \leq \frac{hk_{min}}{(C_{\Omega_f} C_{\Omega_p})^2 gnC_{(inv)}} \tag{2.21}$$

Then the errors between the true solutions to the Stokes-Darcy problem (1.16) and the solutions to the (BEsplit2-SD) method satisfy $\mathcal{O}(\Delta t(h+1) + h^2)$.

Proof. Subtract (BEsplit2-SD) from (2.2). This yields

$$\begin{aligned} n\left(\tfrac{e_u^{k+1}-e_u^k}{\Delta t}, v_h\right)_f + gS_0\left(\tfrac{e_\phi^{k+1}-e_\phi^k}{\Delta t}\right)_p + a_f(e_u^{k+1}, v_h) + a_p(e_\phi^{k+1}, \psi_h) \\ +c_I(v_h, e_\phi^{k+1}) - c_I(e_u^k, \psi_h) - b(v_h, e_p^{k+1}) \\ = n\left(\tfrac{P_h u^{k+1} - P_h u^k}{\Delta t}, v_h\right)_f + gS_0\left(\tfrac{P_h\phi^{k+1} - P_h\phi^k}{\Delta t}, \psi_h\right)_p \\ -c_I(v_h, P_h\phi^{k+1} - P_h\phi^k) - n(u_t^{k+1}, v_h)_f - gS_0(\phi_t^{k+1}, \psi_h)_p, \end{aligned} \tag{2.22}$$

where, as before, $e_w^k = P_h w^k - w_h^k$ for $w = u, \phi$. In the above equation we added $c_I(P_h u^k, \psi_h)$ to both sides to make the term $c_I(e_u^k, \psi_h)$ appear on the left. Choose $v_h = 2\Delta t e_u^{k+1}$, $\psi_h = 2\Delta t e_\phi^{k+1}$. Then (2.22) becomes

$$\begin{aligned} \left(n\|e_u^{k+1}\|_f^2 + gS_0\|e_\phi^{k+1}\|_p^2\right) - \left(n\|e_u^k\|_f^2 + gS_0\|e_\phi^k\|_p^2\right) \\ n\|e_u^{k+1} - e_u^k\|_f^2 + gS_0\|e_\phi^{k+1} - e_\phi^k\|_p^2 + 2\Delta t a_f(e_u^{k+1}, e_u^{k+1}) + 2\Delta t a_p(e_\phi^{k+1}, e_\phi^{k+1}) \\ +2\Delta t c_I(e_u^{k+1} - e_u^k, e_\phi^{k+1}) = 2n\left(P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}, e_u^{k+1}\right)_f \\ +2gS_0\left(P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}, e_\phi^{k+1}\right)_p + 2\Delta t c_I(P_h u^{k+1} - P_h u^k, e_\phi^{k+1}). \end{aligned} \tag{2.23}$$

Define the new energy and diffusive terms as follows:

$$\begin{aligned} E^k &= \|e_u^k\|_f^2 + gS_0\|e_\phi^k\|_p^2, \\ \mathcal{D}^k &= n\nu\|\nabla e_u^{k+1}\|_f^2 + gk_{min}\|\nabla e_\phi^{k+1}\|_p^2. \end{aligned}$$

By coercivity, $a_f(e_u^k, e_u^k) + a_p(e_\phi^k, e_\phi^k) \geq \mathcal{D}^k$. Therefore, after incorporating the above notation, (2.23) becomes the following inequality:

$$\begin{aligned} E^{k+1} - E^k + 2\Delta t\mathcal{D}^{k+1} + n\|e_u^{k+1} - e_u^k\|_f^2 + gS_0\|e_\phi^{k+1} - e_\phi^k\|_p^2 + 2\Delta t c_I(e_u^{k+1} - e_u^k, e_\phi^{k+1}) \\ \leq 2n\left(P_h u^{k+1} - P_h u^k - \Delta t u_t^{k+1}, e_u^{k+1}\right)_f + 2gS_0\left(P_h\phi^{k+1} - P_h\phi^k - \Delta t\phi_t^{k+1}, e_\phi^{k+1}\right)_p \\ +2\Delta t c_I(P_h u^{k+1} - P_h u^k, e_\phi^{k+1}). \end{aligned} \tag{2.24}$$

Consider the coupling term on the right-hand side. Using the Cauchy-Schwarz, Trace (Lemma 1), and Young inequalities we find

$$\begin{aligned} &2\Delta t c_I(P_h u^{k+1} - P_h u^k, e_\phi^{k+1}) \\ &\quad \leq 2\Delta t n g C_{\Omega_f} C_{\Omega_p}\|P_h u^{k+1} - P_h u^k\|_f^{1/2}\|\nabla(P_h u^{k+1} - P_h u^k)\|_f^{1/2}\|e_\phi^{k+1}\|_p^{1/2}\|\nabla e_\phi^{k+1}\|_p^{1/2} \\ &\quad \leq \tfrac{\Delta t(C_{\Omega_f} C_{\Omega_p})^2 n^2 g C_{P,f} C_{P,p}}{k_{min}}\|\nabla(P_h u^{k+1} - P_h u^k)\|_f^2 + \tfrac{\Delta t g k_{min}}{2}\|\nabla e_\phi^{k+1}\|_p^2. \end{aligned}$$

Bound the coupling term on the left-hand side using Cauchy-Schwarz, Trace (Lemma 1), Inverse (Lemma 5), and Young inequalities to obtain

$$2\Delta t c_I(e_u^{k+1} - e_u^k, e_\phi^{k+1}) \leq \tfrac{\Delta t(C_{\Omega_f} C_{\Omega_p})^2 n^2 g C_{(inv)}}{h k_{min}}\|e_u^{k+1} - e_u^k\|_f^2 + \Delta t g k_{min}\|\nabla e_\phi^{k+1}\|_p^2.$$

Next, bound the consistency terms on the right-hand side.

$$2n\left(P_hu^{k+1}-P_hu^k-\Delta tu_t^{k+1},e_u^{k+1}\right)_f \le \|P_hu^{k+1}-P_hu^k-\Delta tu_t^{k+1}\|_f\|e_u^{k+1}\|_f$$
$$\le \Delta tn\nu\|\nabla e_u^{k+1}\|_f^2+\tfrac{nC_{P,f}^2}{\Delta t\nu}\|P_hu^{k+1}-P_hu^k-\Delta tu_t^{k+1}\|_f^2$$

$$2gS_0\left(P_h\phi^{k+1}-P_h\phi^k-\Delta t\phi_t^{k+1},e_\phi^{k+1}\right)_p \le gS_0\|P_h\phi^{k+1}-P_h\phi^k-\Delta t\phi_t^{k+1}\|_p\|e_\phi^{k+1}\|_p$$
$$\le \tfrac{\Delta tgk_{min}}{2}\|\nabla e_\phi^{k+1}\|_p^2+\tfrac{C_{P,p}^2gS_0^2}{\Delta tk_{min}}\|P_h\phi^{k+1}-P_h\phi^k-\Delta t\phi_t^{k+1}\|_p^2$$

Subsume terms on the left-hand side and sum (2.24) from $k=0$ to $N-1$. After using the consistency error bounds from Lemmas 6 and 7, the equation becomes

$$E^N-E^0+\sum_{k=0}^{N-1}\left\{\left(n-\tfrac{\Delta t(C_{\Omega_f}C_{\Omega_p})^2n^2gC_{(inv)}}{hk_{min}}\right)\|e_u^{k+1}-e_u^k\|_f^2+gS_0\|e_\phi^{k+1}-e_\phi^k\|_p^2\right\}$$
$$\le C_0\{h^4(\|u_t\|^2_{L^2(0,T;L^2(\Omega_f))}+\|\phi_t\|^2_{L^2(0,T;L^2(\Omega_p))})$$
$$+\Delta t^2\left(\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}+\|\phi_{tt}\|L^2(0,T;L^2(\Omega_p))+(h^2+1)\|\nabla u_t\|^2_{L^2(0,T;L^2(\Omega_f))}\right)\} \tag{2.25}$$

First-order convergence in time of (BEsplit2-SD) follows since (2.21) implies that $\left(1-\tfrac{\Delta t(C_{\Omega_f}C_{\Omega_p})^2ngC_{(inv)}}{hk_{min}}\right)\ge 0$. □

2.2 CONVERGENCE OF CRANK-NICOLSON (CN) SPLITTING METHOD FOR STOKES-DARCY

In this section we prove convergence rates for the Crank-Nicolson (CN) Splitting Method applied to the Stokes-Darcy system. The definition of the method follows.

Definition 14. *(Crank Nicolson Splitting Method for Stokes-Darcy (*CNsplit-SD*))* *Given* $(\widehat{u}_h^k,\widehat{p}_h^k,\widehat{\phi}_h^k)$, $(\tilde{u}_h^k,\tilde{p}_h^k,\tilde{\phi}_h^k)$ *in* (X_f^h,Q_f^h,X_p^h), *find* $(\widehat{u}_h^{k+1},\widehat{p}_h^{k+1},\widehat{\phi}_h^{k+1})$, $(\tilde{u}_h^{k+1},\tilde{p}_h^{k+1},\tilde{\phi}_h^{k+1})$ *in*

(X_f^h, Q_f^h, X_p^h) *satisfying for all* (v_h, q_h, ψ_h) *in* (X_f^h, Q_f^h, X_p^h)*:*

$$\begin{aligned} &1.\ n\left(\frac{\widehat{u}_h^{k+1}-\widehat{u}_h^k}{\Delta t}, v_h\right)_f + a_f\left(\frac{\widehat{u}^{k+1}+\widehat{u}^k}{2}, v_h\right) - b\left(v_h, \frac{\widehat{p}_h^{k+1}+\widehat{p}_h^k}{2}\right) \\ &\qquad\qquad + c_I(v_h, \widehat{\phi}_h^k) = n(f^{k+1/2}, v_h)_f, \\ &\quad b(\widehat{u}_h^{k+1}, q_h) = 0, \\ &2.\ gS_0\left(\frac{\widehat{\phi}_h^{k+1}-\widehat{\phi}_h^k}{\Delta t}, \psi_h\right)_p + a_p\left(\frac{\widehat{\phi}_h^{k+1}+\widehat{\phi}_h^k}{2}, \psi_h\right) - c_I(\widehat{u}_h^{k+1}, \psi_h) = g(f^{k+1/2}, \psi_h). \end{aligned} \tag{CNsplitA-SD}$$

as well as

$$\begin{aligned} &1.\ gS_0\left(\frac{\tilde{\phi}_h^{k+1}-\tilde{\phi}_h^k}{\Delta t}, \psi_h\right)_p + a_p\left(\frac{\tilde{\phi}_h^{k+1}+\tilde{\phi}_h^k}{2}, \psi_h\right) - c_I(\tilde{u}_h^k, \psi_h) = g(f^{k+1/2}, \psi_h), \\ &2.\ n\left(\frac{\tilde{u}_h^{k+1}-\tilde{u}_h^k}{\Delta t}, v_h\right)_f + a_f\left(\frac{\tilde{u}^{k+1}+\tilde{u}^k}{2}, v_h\right) - b\left(v_h, \frac{\tilde{p}_h^{k+1}+\tilde{p}_h^k}{2}\right) \\ &\qquad\qquad + c_I(v_h, \tilde{\phi}_h^{k+1}) = n(f^{k+1/2}, v_h)_f, \\ &\quad b(\tilde{u}_h^{k+1}, q_h) = 0. \end{aligned} \tag{CNsplitB-SD}$$

Then $(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1})$ *is defined by* $w_h^{k+1} = \frac{\widehat{w}_h^{k+1}+\tilde{w}_h^{k+1}}{2}$*, for* $w = u, p,$ *and* ϕ

To implement this method, first compute $(\widehat{u}_h^{k+1}, \widehat{p}_h^{k+1}, \tilde{\phi}_h^{k+1})$ in parallel by completing the first steps in the (CNsplitA-SD) and (CNsplitB-SD) methods. Then, in parallel, compute $(\tilde{u}_h^{k+1}, \tilde{p}_h^{k+1}, \widehat{\phi}_h^{k+1})$ in the second steps of the method. Finally, compute $(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1})$ by averaging the two solutions, $w_h^{k+1} = \frac{\widehat{w}_h^{k+1}+\tilde{w}_h^{k+1}}{2}$, for $w = u, p,$ and ϕ.

Results of the stability analysis performed in [48] are summarized below for reference.

Theorem 15 (Stability of CNsplit-SD)**.** *Suppose* Δt *satisfies*

$$\Delta t < \frac{\sqrt{2S_0}h}{\sqrt{gn}C_{\Omega_f}C_{\Omega_p}C_{(inv)}}.$$

Then both (CNsplitA-SD) *and* (CNsplitB-SD) *are stable uniformly in time over long-time intervals. That is, there exists* $\alpha > 0$ *such that for every* $N \geq 1$

$$\begin{aligned} \alpha\left[n\|\widehat{u}_h^N\|_f^2 + gS_0\|\widehat{\phi}_h^N\|_p^2\right] + \frac{\Delta t}{2}\sum_{k=0}^{N-1}\left[a_f(\widehat{u}_h^{k+1}+\widehat{u}_h^k, \widehat{u}_h^{k+1}+\widehat{u}_h^k) + a_p(\widehat{\phi}_h^{k+1}+\widehat{\phi}_h^k, \widehat{\phi}_h^{k+1}+\widehat{\phi}_h^k)\right] \\ \leq n\|\widehat{u}_h^0\|_f^2 + gS_0\|\widehat{\phi}_h^0\|_p^2 - \Delta t c_I(\widehat{u}_h^0, \widehat{\phi}_h^0) \\ + \Delta t\sum_{k=0}^{N-1}\left[n(f_f^{k+1/2}, \widehat{u}_h^{k+1}+\widehat{u}_h^k)_f + g(f_p^{k+1/2}, \widehat{\phi}_h^{k+1}+\widehat{\phi}_h^k)_p\right], \end{aligned}$$

for (CNsplitA-SD) *and*

$$\alpha\left[n\|\tilde{u}_h^N\|_f^2+gS_0\|\tilde{\phi}_h^N\|_p^2\right]+\frac{\Delta t}{2}\sum_{k=0}^{N-1}\left[a_f(\tilde{u}_h^{k+1}+\tilde{u}_h^k,\tilde{u}_h^{k+1}+\tilde{u}_h^k)+a_p(\tilde{\phi}_h^{k+1}+\tilde{\phi}_h^k,\tilde{\phi}_h^{k+1}+\tilde{\phi}_h^k)\right]$$
$$\le n\|\tilde{u}_h^0\|_f^2+gS_0\|\tilde{\phi}_h^0\|_p^2-\Delta t c_I(\tilde{u}_h^0,\tilde{\phi}_h^0)$$
$$+\Delta t\sum_{k=0}^{N-1}\left[n(f_f^{k+1/2},\tilde{u}_h^{k+1}+\tilde{u}_h^k)_f+g(f_p^{k+1/2},\tilde{\phi}_h^{k+1}+\tilde{\phi}_h^k)_p\right],$$

for (CNsplitB-SD).

Proof. See [48]. □

To study the convergence of the CNsplit-SD method, first average the methods (CNsplitA-SD) and (CNsplitB-SD). This produces

$$\begin{aligned}
n\left(\frac{u_h^{k+1}-u_h^k}{\Delta t},v_h\right)_f+a_f\left(\frac{u_h^{k+1}+u_h^k}{2},v_h\right)-b\left(v_h,\frac{p_h^{k+1}+p_h^k}{2}\right)&\\
+\tfrac{1}{2}c_I(v_h,\widehat{\phi}_h^k+\tilde{\phi}_h^{k+1})=n(f_f^{k+1/2},v_h)_f&,\\
b(u_h^{k+1},q_h)=0&, \qquad \text{(CNsplit-SD)}\\
gS_0\left(\frac{\phi_h^{k+1}-\phi_h^k}{\Delta t},\psi_h\right)_p+a_p\left(\frac{\phi_h^{k+1}+\phi_h^k}{2},\psi_h\right)-\tfrac{1}{2}c_I(\widehat{u}_h^{k+1}+\tilde{u}_h^k,\psi_h)&\\
=g(f_p^{k+1/2},\psi_h)&.
\end{aligned}$$

Begin by computing the consistency errors in the method. To determine the consistency errors, plug the true solutions of the Stokes-Darcy problem into the (CNsplit-SD) method above. The coupling terms exactly cancel, leaving the following consistency errors in the fluid and porous media domains:

$$\tau_f^k=\left(\frac{u^{k+1}-u^k}{\Delta t}-\frac{u_t^{k+1}+u_t^k}{2},v_h\right)_f+n\left(\frac{f_f^{k+1}+f_f^k}{2}-f^{k+1/2},v_h\right)_f,$$
$$\tau_p^k=gS_0\left(\frac{\phi^{k+1}-\phi^k}{\Delta t}-\frac{\phi_t^{k+1}+\phi_t^k}{2},\psi_h\right)_p+\left(\frac{f_p^{k+1}+f_p^k}{2}-f_p^{k+1/2},\psi_h\right)_p).$$

By Lemma 6,

$$\begin{aligned}
\Delta t\sum_{k=0}^{N-1}\|\tau_f^k\|_f^2 &\le C\Delta t^4\left(\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}+\|(f_f)_{tt}\|^2_{L^2(0,T;L^2(\Omega_f))}\right),\\
\Delta t\sum_{k=0}^{N-1}\|\tau_p^k\|_p^2 &\le C\Delta t^4\left(\|\phi_{tt}\|^2_{L^2(0,T;L^2(\Omega_p))}+\|(f_p)_{tt}\|^2_{L^2(0,T;L^2(\Omega_p))}\right),
\end{aligned} \tag{2.26}$$

implying that the consistency error is second-order in time.

Apply the projection property (SD-PROJ) to the variational form, (FEM-SD), with the true solutions and average. This yields

$$\begin{aligned}
n\left(\tfrac{u_t^{k+1}+u_t^k}{2},v_h\right)_f + gS_0\left(\tfrac{\phi_t^{k+1}+\phi_t^k}{2},\psi_h\right)_p + a_f\left(\tfrac{P_h(u^{k+1}+u^k)}{2},v_h\right)+a_p\left(\tfrac{P_h(\phi^{k+1}+\phi^k)}{2},\psi_h\right)&\\
+c_I(v_h,\tfrac{P_h(\phi^{k+1}+\phi^k)}{2})-c_I(\tfrac{P_h(u^{k+1}+u^k)}{2},\psi_h)-b(v_h,\tfrac{P_h(p^{k+1}+p^k)}{2})&\\
=n(\tfrac{f_f^{k+1}+f_f^k}{2},v_h)_f+g(\tfrac{f_p^{k+1}+f_p^k}{2},\psi_h)_p,&
\end{aligned} \tag{2.27}$$

which is used in the following proof of convergence for the (CNsplitA-SD) and (CNsplitB-SD) methods.

Theorem 16 (Convergence of (CNsplitA-SD) and (CNsplitB-SD))**.** *Assume that the following time-step condition holds:*

$$\Delta t < \tfrac{\sqrt{2S_0}h}{\sqrt{gn}C_{\Omega_f}C_{\Omega_p}C_{(inv)}}.$$

Then the errors in (CNsplitA-SD) and (CNsplitB-SD) are $\mathcal{O}(h^2+\Delta t^2+\Delta t(h^2+1))$.

Proof. We present the proof for (CNsplitA-SD) only, as the proof for (CNsplitB-SD) is similar. Add the projections of the discrete time derivatives for u and ϕ to both sides of (2.27) and subtract (CNsplitA-SD). Assume $v_h\in V^h$, so that we omit the term $b(.,.)$. This produces

$$\begin{aligned}
n\left(\tfrac{\widehat{e}_u^{k+1}-\widehat{e}_u^k}{\Delta t},v_h\right)_f+gS_0\left(\tfrac{\widehat{e}_\phi^{k+1}-\widehat{e}_\phi^k}{\Delta t},\psi_h\right)_p+a_f(\tfrac{\widehat{e}_u^{k+1}+\widehat{e}_u^k}{2},v_h)+a_p(\tfrac{\widehat{e}_\phi^{k+1}+\widehat{e}_\phi^k}{2},\psi_h)&\\
+c_I(v_h,\widehat{e}_\phi^k)-c_I(\widehat{e}_u^{k+1},\psi_h)=n\left(\tfrac{(P_h-I)(u^{k+1}-u^k)}{\Delta t}-\tfrac{u_t^{k+1}+uk_t}{2},v_h\right)_f&\\
+gS_0\left(\tfrac{(P_h-I)(\phi^{k+1}-\phi^k)}{\Delta t}-\tfrac{\phi_t^{k+1}+\phi_t^k}{2},\psi_h\right)_p+(\tau_f^k,v_h)_f+(\tau_p^k,\psi_h)_p&\\
-\tfrac{1}{2}c_I(v_h,P_h(\phi^{k+1}-\phi^k))-\tfrac{1}{2}c_I(P_h(u^{k+1}-u^k),\psi_h),&
\end{aligned} \tag{2.28}$$

where $\widehat{e}_u^k = P_h u^k - \widehat{u}_h^k$.

Choose $v_h = \Delta t(\widehat{e}_u^{k+1} + \widehat{e}_u^k)$, and $\psi_h = \Delta t(\widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)$. After simplifying, (2.28) becomes

$$\begin{aligned}
\left(n\|\widehat{e}_u^{k+1}\|_f^2 + gS_0\|\widehat{e}_\phi^{k+1}\|_p^2\right) - \left(n\|\widehat{e}_u^{k}\|_f^2 + gS_0\|\widehat{e}_\phi^{k}\|_p^2\right)\\
+\Delta t\left[a_f(\widehat{e}_u^{k+1} + \widehat{e}_u^k, \widehat{e}_u^{k+1} + \widehat{e}_u^k) + a_p(\widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k, \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)\right]\\
+\Delta t\left[c_I(\widehat{e}_u^k, \widehat{e}_\phi^k) - c_I(\widehat{e}_u^{k+1}, \widehat{e}_\phi^{k+1})\right]\\
= n\left((P_h - I)(u^{k+1} - u^k), \widehat{e}_u^{k+1} + \widehat{e}_u^k\right)_f\\
+gS_0\left((P_h - I)(\phi^{k+1} - \phi^k), \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k\right)_p\\
+\Delta t(\tau_f^k, \widehat{e}_u^{k+1} + \widehat{e}_u^k)_f + \Delta t(\tau_p^k, \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)_p\\
-\tfrac{\Delta t}{2} c_I(\widehat{e}_u^{k+1} + \widehat{e}_u^k, P_h(\phi^{k+1} - \phi^k)) - \tfrac{\Delta t}{2} c_I(P_h(u^{k+1} - u^k), \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k).
\end{aligned} \tag{2.29}$$

Define the energy, diffusive, and coupled terms as follows. The inequality in the diffusive term follows by the coercivity of a_f and a_p.

$$\begin{aligned}
\widehat{E}^k &= n\|\widehat{e}_u^k\|_f^2 + gS_0\|\widehat{e}_\phi^k\|_p^2\\
\widehat{D}^{k+1/2} &= \tfrac{1}{2}\Delta t n\nu\|\nabla(\widehat{e}_u^{k+1} + \widehat{e}_u^k)\|_f^2 + gk_{min}\|\nabla(\widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)\|_p^2\\
&\le \tfrac{1}{2}\Delta t\left[a_f(\widehat{e}_u^{k+1} + \widehat{e}_u^k, \widehat{e}_u^{k+1} + \widehat{e}_u^k) + a_p(\widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k, \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)\right]\\
\widehat{C}^k &= c_I(\widehat{e}_u^k, \widehat{e}_\phi^k)
\end{aligned}$$

After incorporating the above notation, (2.29) becomes the following inequality:

$$\begin{aligned}
(\widehat{E}^{k+1} - \Delta t\widehat{C}^{k+1}) - (\widehat{E}^k - \Delta t\widehat{C}^k) + \tfrac{\Delta t}{2}\widehat{D}^{k+1/2}\\
= n\left((P_h - I)(u^{k+1} - u^k) - \Delta t\left(\tfrac{u_t^{k+1} + uk_t}{2}\right), \widehat{e}_u^{k+1} + \widehat{e}_u^k\right)_f\\
+gS_0\left((P_h - I)(\phi^{k+1} - \phi^k) - \Delta t\left(\tfrac{\phi_t^{k+1} + \phi_t^k}{2}\right), \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k\right)_p\\
+\Delta t(\tau_f^k, \widehat{e}_u^{k+1} + \widehat{e}_u^k)_f + \Delta t(\tau_p^k, \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)_p\\
-\tfrac{\Delta t}{2} c_I(\widehat{e}_u^{k+1} + \widehat{e}_u^k, P_h(\phi^{k+1} - \phi^k)) - \tfrac{\Delta t}{2} c_I(P_h(u^{k+1} - u^k), \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k)
\end{aligned} \tag{2.30}$$

Next bound the consistency error terms on the right-hand side by the Poincaré (Lemma 2), and Young's inequalities:

$$
\begin{aligned}
n\left((P_h-I)(u^{k+1}-u^k), e_u^{k+1}+e_u^k\right)_f &\leq nC_{P,f}\|(P_h-I)(u^{k+1}-u^k)\|_f\|\nabla(e_u^{k+1}+e_u^k)\|_f \\
&\leq \tfrac{\Delta t n\nu}{6}\|\nabla(e_u^{k+1}+e_u^k)\|_f^2+\tfrac{3nC_{P,f}^2}{2\Delta t\nu}\|(P_h-I)(u^{k+1}-u^k)\|_f^2, \\
gS_0\left((P_h-I)(\phi^{k+1}-\phi^k), e_\phi^{k+1}+e_\phi^k\right)_p &\leq gS_0\|(P_h-I)(\phi^{k+1}-\phi^k)\|_p\|\nabla(e_\phi^{k+1}+e_\phi^k)\|_p \\
&\leq \tfrac{\Delta t g k_{min}}{6}\|\nabla(e_\phi^{k+1}+e_\phi^k)\|_p^2+\tfrac{3gS_0^2C_{P,p}^2}{2\Delta t k_{min}}\|\|(P_h-I)(\phi^{k+1}-\phi^k)\|_p^2, \\
\Delta t(\tau_f^k, e_u^{k+1}+e_u^k)_f &\leq \tfrac{\Delta t n\nu}{6}\|\nabla(e_u^{k+1}+e_u^k)\|_f^2+\tfrac{3\Delta t}{2n\nu}\|\tau_f^k\|_f^2, \\
\Delta t(\tau_p^k, e_\phi^{k+1}+e_\phi^k)_p &\leq \tfrac{\Delta t g k_{min}}{6}\|\nabla(e_\phi^{k+1}+e_\phi^k)\|_p^2+\tfrac{3\Delta t}{2gk_{min}}\|\tau_p^k\|_p^2.
\end{aligned}
$$

Thus, after subsuming terms into the right-hand side of (2.30), and absorbing the constants into one constant, $C_0>0$, (2.30) becomes

$$
\begin{aligned}
&(\widehat{E}^{k+1}-\Delta t\widehat{C}^{k+1})-(\widehat{E}^k-\Delta t\widehat{C}^k)+\tfrac{\Delta t}{2}\widehat{D}^{k+1/2} \\
&\leq C_0\{\tfrac{1}{\Delta t}\left(\|(P_h-I)(u^{k+1}-u^k)\|_f^2+\|(P_h-I)(\phi^{k+1}-\phi^k)\|_p^2\right)+\Delta t\left(\|\tau_f^k\|_f^2+\|\tau_p^k\|_p^2\right) \\
&\quad -\Delta t c_I(\widehat{e}_u^{k+1}+\widehat{e}_u^k, P_h(\phi^{k+1}-\phi^k))-\Delta t c_I(P_h(u^{k+1}-u^k), \widehat{e}_\phi^{k+1}+\widehat{e}_\phi^k)\}.
\end{aligned}
\tag{2.31}
$$

Sum (2.31) from $k=0$ to $k=N-1$:

$$
\begin{aligned}
&\widehat{E}^N-\Delta t\widehat{C}^N+\tfrac{\Delta t}{2}\sum_{k=0}^{N-1}\widehat{D}^{k+1/2}\leq \widehat{E}^0-\Delta t\widehat{C}^0 \\
&+C_0\sum_{k=0}^{N-1}\{\tfrac{1}{\Delta t}\|(P_h-I)(u^{k+1}-u^k)\|_f^2+\tfrac{1}{\Delta t}\|(P_h-I)(\phi^{k+1}-\phi^k)\|_p^2 \\
&-\Delta t c_I(\widehat{e}_u^{k+1}+\widehat{e}_u^k, P_h(\phi^{k+1}-\phi^k))-\Delta t c_I(P_h(u^{k+1}-u^k), \widehat{e}_\phi^{k+1}+\widehat{e}_\phi^k)+\Delta t\left(\|\tau_f^k\|_f^2+\|\tau_p^k\|_p^2\right)\}.
\end{aligned}
$$

By the projection error bounds in Lemma 7 and consistency error bounds, (2.26), we have

$$
\begin{aligned}
&\widehat{E}^N-\Delta t\widehat{C}^N+\tfrac{\Delta t}{2}\sum_{k=0}^{N-1}\widehat{D}^{k+1/2}\leq \widehat{E}^0-\Delta t\widehat{C}^0 \\
&+C_1\{h^4\left(\|u_{tt}\|_{L^2(0,T;L^2(\Omega_f))}^2+\|\phi_{tt}\|_{L^2(0,T;L^2(\Omega_p))}^2\right)+\Delta t^4(\|u_{tt}\|_{L^2(0,T;L^2(\Omega_f))}^2 \\
&+\|(f_f)_{tt}\|_{L^2(0,T;L^2(\Omega_f))}^2+\|\phi_{tt}\|_{L^2(0,T;L^2(\Omega_p))}^2+\|(f_p)_{tt}\|_{L^2(0,T;L^2(\Omega_p))}^2)\} \\
&-\sum_{k=0}^{N-1}\left(\Delta t c_I(\widehat{e}_u^{k+1}+\widehat{e}_u^k, P_h(\phi^{k+1}-\phi^k))+\Delta t c_I(P_h(u^{k+1}-u^k), \widehat{e}_\phi^{k+1}+\widehat{e}_\phi^k)\right).
\end{aligned}
\tag{2.32}
$$

Consider the remaining coupling term on the right-hand side. Using the Cauchy-Schwarz, Trace (Lemma 1), Poincaré (Lemma 2), and Young inequalities, as well as implementing the bound on $\|\nabla(P_h(w^{k+1}-w^k))\|$ from Lemma 7 we find

$$\begin{aligned}
&\sum_{k=0}^{N-1} \tfrac{\Delta t}{2} c_I(\widehat{e}_u^{k+1}+\widehat{e}_u^k, P_h(\phi^{k+1}-\phi^k)) \\
&\leq \sum_{k=0}^{N-1}\left(\tfrac{\Delta t n\nu}{6}\|\nabla(\widehat{e}_u^{k+1}+\widehat{e}_u^k)\|_f^2 + \tfrac{3\Delta t n g^2 C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f}C_{P,p}}{8\nu}\|\nabla(P_h(\phi^{k+1}-\phi^k))\|_p^2\right) \\
&\leq \tfrac{\Delta t n\nu}{6}\sum_{k=0}^{N-1}\|\nabla(\widehat{e}_u^{k+1}+\widehat{e}_u^k)\|_f^2 + C\Delta t^2(h^2+1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=0}^{N-1} \Delta t c_I(P_h(u^{k+1}-u^k), e_\phi^{k+1}+e_\phi^k) \\
&\leq \sum_{k=0}^{N-1}\left(\tfrac{\Delta t g k_{min}}{6}\|\nabla(e_\phi^{k+1}+e_\phi^k)\|_p^2 + \tfrac{3\Delta t n^2 C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f}C_{P,p}}{8k_{min}}\|\nabla(P_h(u^{k+1}-u^k))\|_f^2\right) \\
&\leq \sum_{k=0}^{N-1}\tfrac{\Delta t g k_{min}}{6}\|\nabla(e_\phi^{k+1}+e_\phi^k)\|_p^2 + C\Delta t^2(h^2+1)\|\nabla u_t\|_{L^2(0,T;L^2(\Omega_f))}^2.
\end{aligned}$$

Using the above bounds implies that the errors between the projections of the true solutions of Stokes-Darcy and the solutions to the method (CNsplitA-SD) obey

$$\|\widehat{e}_w^N\|_{f/p} \approx \mathcal{O}(h^2+\Delta t^2+\Delta t(h+1)),$$

for $w=u,\phi$ provided $\widehat{E}^N - \Delta t\widehat{C}^N > 0$. To show $\widehat{E}^N - \Delta t\widehat{C}^N > 0$ holds, apply the Cauchy-Schwarz, Trace (Lemma 1), Inverse (Lemma 5) and Young's inequality to $\widehat{C}^N$ to obtain the following bound:

$$\Delta t\widehat{C}^N \leq \tfrac{g\Delta t^2}{2S_0h^2}(nC_{\Omega_f}C_{\Omega_p}C_{(inv)})^2\|u^N\|_f^2 + \tfrac{gS_0}{2}\|\phi^N\|_p^2.$$

Hence, $\widehat{E}^N - \Delta t\widehat{C}^N > 0$ since by assumption Δt satisfies

$$\Delta t < \tfrac{\sqrt{2S_0}h}{\sqrt{gn}C_{\Omega_f}C_{\Omega_p}C_{(inv)}}.$$

By triangle inequality, the errors between the true solutions of Stokes-Darcy and the solutions of (CNsplitA-SD) obey

$$\|w^N - w_h^N\|_{f/p} \le \|(I - P_h)w^N\|_{f/p} + \|e_w^N\|_{f/p},$$

Therefore, by the bounds for the projection errors (2.1), $\|(I - P_h)w^N\|_{f/p} \le Ch^2\|w^N\|_{f/p}$ for $w = u, \phi$, this implies that the errors of the (CNsplitA-SD) method satisfy
$\mathcal{O}(h^2 + \Delta t^2 + \Delta t(h + 1))$. □

Remark 17 (*(On the Convergence Rate of* (CNsplit-SD)*)*)**.** *The convergence rate proven in Theorem 16 is not second-order in time, as is normally expected with the Crank-Nicolson discretization. This is related to the explicit treatment of the coupling term in the first step. While the coupling terms in the consistency errors of* (CNsplitA-SD) *and* (CNsplitB-SD) *cancel due to exact opposite signs, the coupling terms on the right-hand side in the proof of convergence during step* (2.32)*, given by*

$$-c_I(\widehat{e}_u^{k+1} + \widehat{e}_u^k, P_h(\phi^{k+1} - \phi^k)) - c_I(P_h(u^{k+1} - u^k), \widehat{e}_\phi^{k+1} + \widehat{e}_\phi^k) \text{ for (CNsplitA-SD), and}$$
$$+c_I(\tilde{e}_u^{k+1} + \tilde{e}_u^k, P_h(\phi^{k+1} - \phi^k)) + c_I(P_h(u^{k+1} - u^k), \tilde{e}_\phi^{k+1} + \tilde{e}_\phi^k) \text{ for (CNsplitB-SD)}$$

do not cancel. In fact, when averaged, they simplify to

$$c_I((\tilde{u}_h^{k+1} - \widehat{u}_h^{k+1}) + (\tilde{u}_h^k - \widehat{u}_h^k), P_h(\phi^{k+1} + \phi^k))$$
$$+c_I(P_h(u^{k+1} + u^k), (\tilde{\phi}_h^{k+1} - \widehat{\phi}_h^{k+1}) + (\tilde{\phi}_h^{k+1} - \widehat{\phi}_h^{k+1})).$$

Averaging the convergence rates of (CNsplitA-SD) *and* (CNsplitB-SD) *to obtain a rate for* (CNsplit-SD) *implies at least second-order convergence in space and first-order in time. Numerical experiments for* (CNsplit-SD)*, however, showed second-order convergence in time and space, suggesting that the proven convergence rate is not optimal. An optimal proof of second-order in time convergence remains an open problem.*

2.3 NUMERICAL EXPERIMENTS

Numerical experiments verify the predicted rates of convergence of (BEsplit1-SD), (BEsplit2-SD), and (CNsplit-SD). All tests use the same domain and exact solutions chosen to satisfy the interface conditions and introduced by Mu and Zhu in [56]. Calculations were made using FreeFem++ software [36]. The code for the experiments is included in the appendix. We use Taylor-Hood elements (P2-P1) for the Stokes problem, thereby satisfying the (LBB^h) condition. For the Darcy problem, we use piecewise quadratics (P2). The initial and forcing terms are chosen to correspond with the exact solutions given below.

$$\begin{aligned} \Omega_f = (0,1)\times(1,2), \qquad \Omega_p = (0,1)\times(0,1), \qquad I = \{(x,1) : x \in (0,1)\} \\ u(x,y,t) = \left((x^2(y-1)^2+y)\cos(t), (\frac{2}{3}x(1-y)^3+2-\pi\sin(\pi x))\cos(t)\right), \\ p(x,y,t) = (2-\pi\sin(\pi x))\sin(\frac{\pi}{2}y)\cos(t), \\ \phi(x,y,t) = (2-\pi\sin(\pi x))(1-y-\cos(\pi y))\cos(t). \end{aligned} \tag{Test}$$

All parameters, $n, \rho, g, \alpha_{\mathrm{BJ}}, \rho, S_0$, and k_{min} are set to one unless otherwise indicated.

2.3.1 Convergence Rates

To test the predicted rates of convergence for the (BEsplit1-SD), (BEsplit2-SD), and (CNsplit-SD) methods, we set $h = \Delta t$ and enforce inhomogeneous Dirichlet boundary conditions: $u_h - u$ on $\partial\Omega_f \setminus I$ and $\phi_h = \phi$ on $\partial\Omega_p \setminus I$. With $T_{final} = 1.0$, we measure errors ($\mathcal{E}_w^k = w^k - w_h^k$ for $w = u, p, \phi$) in the norm $L^\infty(0, T_{final}; L^2(\Omega_{f/p}))$ for u, p, ϕ in each method. As indicated in previous sections, analysis dictates that the (BEsplit-SD) methods are first-order convergent in time. The experiments for (CNsplit-SD) imply second-order convergence in both time and space. See tables 2.1-2.3 for experiment results.

2.3.2 Stability Experiments

We include a few tests on the stability of the studied methods when faced with small parameters. While the stability properties of these methods were studied in [48] and are not the focus of this research, understanding the behavior of these splitting methods in regards to small values of k_{min} and S_0 provides motivation for the methods developed and studied in the subsequent chapters of this dissertation. To test the stability of the methods we set the forcing terms equal to zero and enforce homogeneous Dirichlet boundary conditions on the external boundaries. All parameters are set to one, with the exception of k_{min} and S_0. We calculate the final system energy, $E(N) = n\|u_h^N\|_f^2 + gS_0\|\phi_h^N\|_p^2$ over the time interval $[0, 10]$. In the absence of external forcing terms and under homogeneous Dirichlet boundary conditions, the solution decays to zero as $t \to \infty$ when the system is stable. We test the stability of the method in four scenarios: (1) small S_0 and k_{min} ($S_0 = k_{min} = 10^{-6}$), (2) small k_{min} ($S_0 = 1$ and $k_{min} = 10^{-6}$), (3) small S_0 ($S_0 = 10^{-6}$ and $k_{min} = 1$), and (4) $S_0 = k_{min} = 1$. In all graphs, please note the logarithmic scale.

The stability behavior of the two (BEsplit) methods are very similar, see Figures 2.1 and 2.2. As predicted by the CFL-type stability conditions outlined in Section 2.1, these splitting methods perform well for either small values of hydraulic conductivity, k_{min}, or small of specific storage, S_0. However, the methods become highly unstable given both small k_{min} and S_0. As for (CNsplit-SD), the method is stable for small k_{min} (with moderate S_0) and unstable for small S_0, as seen in Figure 2.3.

Table 2.1: Convergence Rates for (BEsplit1-SD)

$h = \Delta t$	$\|\mathcal{E}_u\|$	rate	$\|\mathcal{E}_p\|$	rate	$\|\mathcal{E}_\phi\|$	rate
1/10	1.657e-3		2.995e-2		1.161e-3	
1/20	8.405e-4	0.9798	1.521e-2	0.9774	5.409e-4	1.102
1/40	4.239e-4	0.9875	7.675e-3	0.9871	2.706e-4	0.9994
1/80	2.129e-4	0.9939	3.855e-3	0.9933	1.356e-4	0.9962

2.4 CONCLUSIONS FOR SPLITTING METHODS FOR STOKES-DARCY

Splitting methods for the Stokes-Darcy problem have several advantages. First, they uncouple the equations allowing one to utilize optimize solvers for each sub-problem. Consider the (BEsplit1-SD) and (BEsplit2-SD) methods. If we simplify the CFL-type conditions for stability of these methods in terms of $\Delta t, h, k_{min}$ and S_0, the conditions are equivalent to

$$\Delta t \lesssim C \max\{k_{min}, S_0 k_{min}, S_0 h\} \text{ or } \sqrt{k_{min}} > 1, \qquad \text{(for (BEsplit1-SD))}$$

$$\Delta t \lesssim C \max\{k_{min}, S_0 k_{min}, k_{min} h\} \text{ or } \sqrt{k_{min}} > 1. \qquad \text{(for (BEsplit2-SD))}$$

As suggested by the above conditions, and evidenced in numerical experiments, these methods perform well given one small parameter: either k_{min} or S_0. However, neither method is stable when both of these parameters are small, a possibility in groundwater-surface water flow, especially when involving confined aquifers. Also, the method is only first-order convergent in time, and the focus of the research is higher-order, strongly stable methods for this coupled flow problem.

The (CNsplit-SD) method, comprised of (CNsplitA-SD) and (CNsplitB-SD), is an interesting splitting method that consists of two methods solved sequentially in parallel, and

Table 2.2: Convergence Rates for (BEsplit2-SD)

$h = \Delta t$	$\|\mathcal{E}_u\|$	rate	$\|\mathcal{E}_p\|$	rate	$\|\mathcal{E}_\phi\|$	rate
1/10	9.213e-4		2.743e-2		4.834e-3	
1/20	4.391e-4	1.069	1.336e-2	1.038	2.447e-3	0.9820
1/40	2.195e-4	0.9999	6.645e-3	1.007	1.233e-3	0.9891
1/80	1.100e-4	0.9971	3.319e-3	1.001	6.188e-4	0.9945

Table 2.3: Convergence Rates for (CNsplit-SD)

$h = \Delta t$	$\|\mathcal{E}_u\|$	rate	$\|\mathcal{E}_p\|$	rate	$\|\mathcal{E}_\phi\|$	rate
1/10	3.894e-4		4.211e-2		1.521e-3	
1/20	5.035e-5	2.95	1.020e-2	2.046	3.654e-4	2.057
1/40	7.713e-6	2.707	2.530e-3	2.011	9.080e-5	2.009
1/80	1.564e-6	2.302	6.253e-4	2.017	2.266e-5	2.003

then averaged. However, simplified in terms of $\Delta t, h, k_{min}$, and S_0, the CFL-type time-step condition for stability becomes

$$\Delta t \lesssim C\sqrt{S_0}h,$$

which is very restrictive given small values of specific storage, S_0. Therefore, while numerical tests for method imply higher-order convergence, its stability properties make the method impractical when faced with physical situations that involve small values of specific storage.

In conclusion, while splitting methods exhibit many desirable properties, none of these methods satisfy the goals for this research: higher-order convergent numerical methods that exhibit stability in a wide variety of physical situations. In the next several chapters, we develop, analyze, and test a method that satisfies both of these criteria.

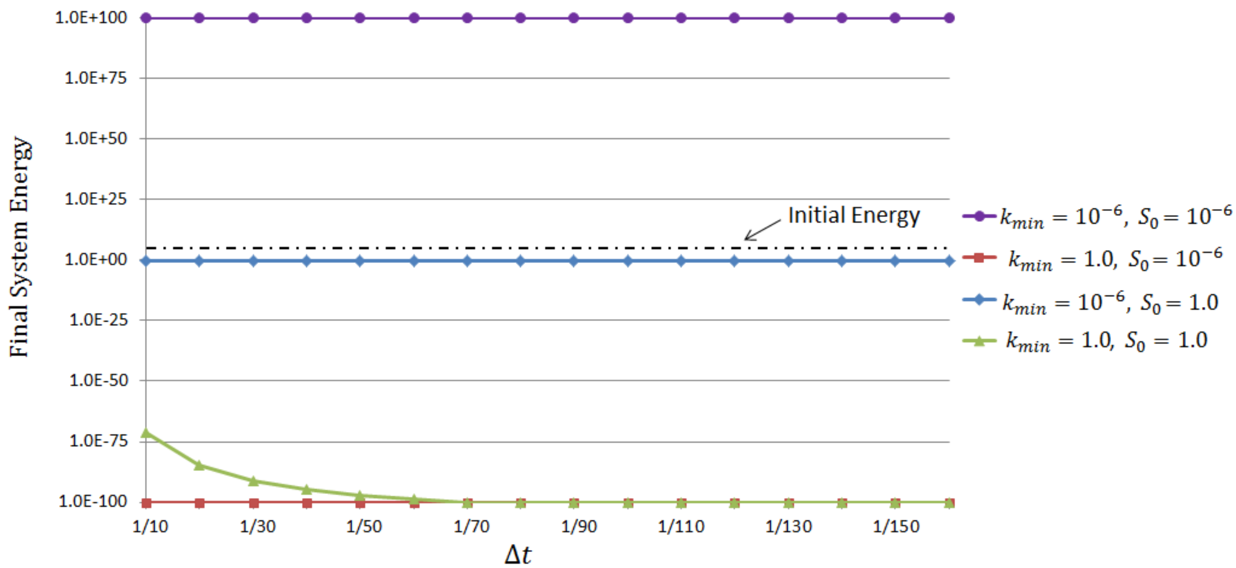

Figure 2.1: Final System Energy ($E(N)$) versus time-step size (Δt) for (BEsplit1-SD). The minimum is truncated at 1.0E-100 and the maximum at 1.0E+100.

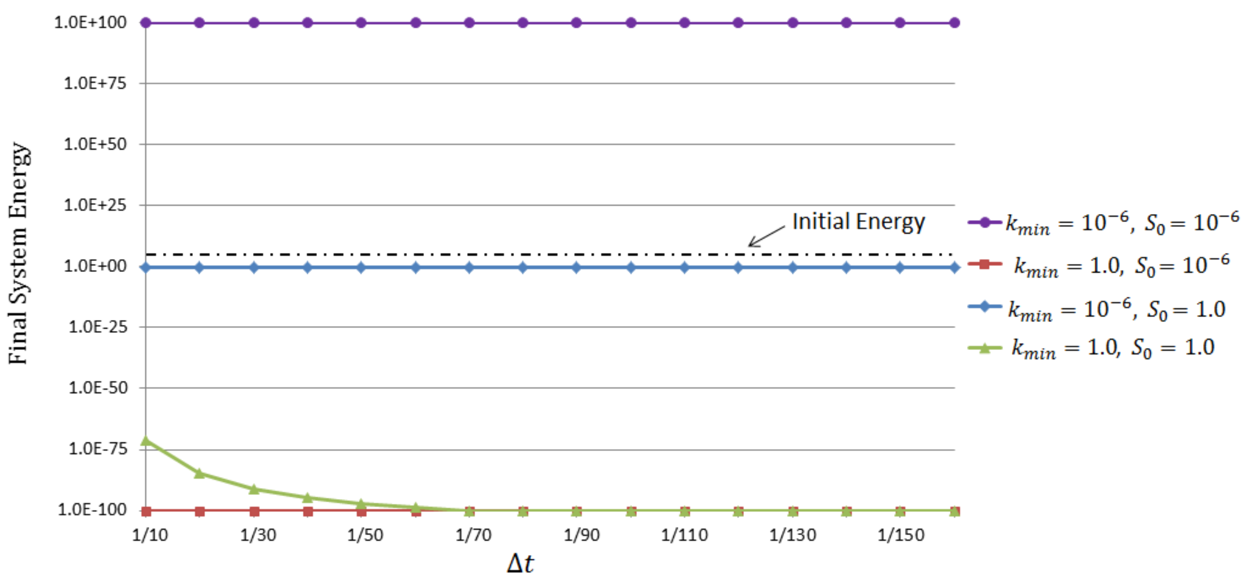

Figure 2.2: Final System Energy ($E(N)$) versus time-step size (Δt) for (BEsplit2-SD). The minimum is truncated at 1.0E-100 and the maximum at 1.0E+100.

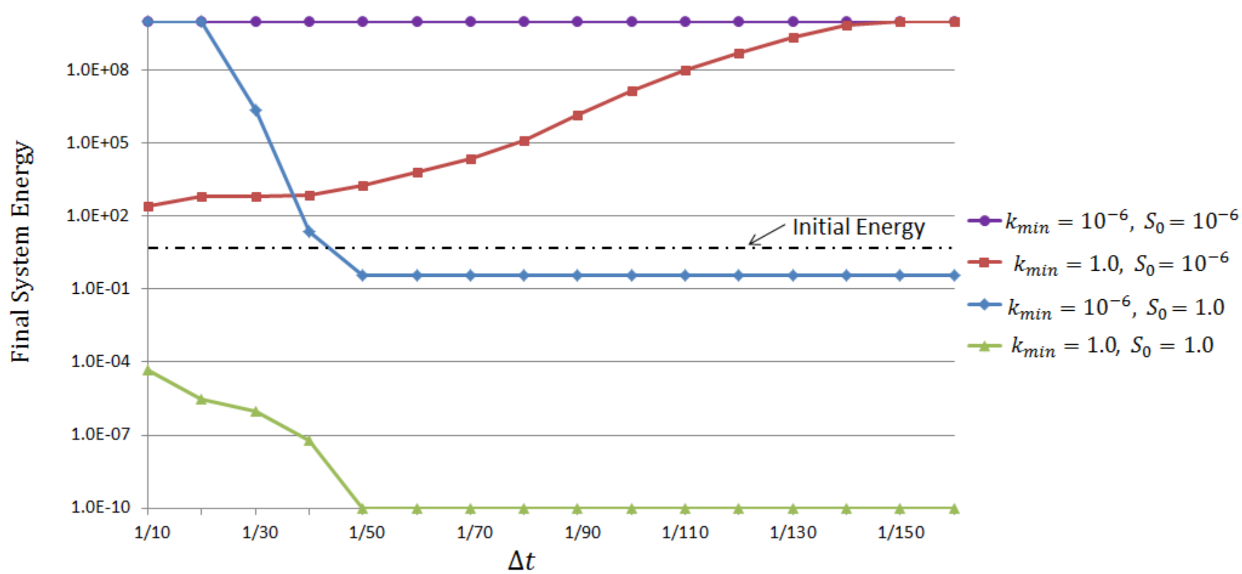

Figure 2.3: Final System Energy ($E(N)$) versus time-step size (Δt) for (CNsplit-SD). The minimum is truncated at 1.0E-10 and the maximum at 1.0E+10.

3.0 CRANK-NICOLSON LEAPFROG (CNLF) METHOD FOR STOKES-DARCY (CNLF-SD)

It remains to develop a higher-order convergent method for the groundwater-surface water flow problem that exhibits stable behavior given small problem parameters. In this chapter, we consider the stability and convergence properties of the Crank-Nicolson Leapfrog (CNLF) time discretization applied to the (FEM-SD) formulation.

Recall that, after applying the Finite Element Method to the Stokes-Darcy problem, the system further reduces to a coupled evolution equation:

$$\begin{aligned} u_t + A_f u + C\phi &= f_f, \\ \phi_t + A_p \phi - Cu &= f_p, \end{aligned} \tag{3.1}$$

where A_f, A_p are SPD and $C = C^T$. The Crank-Nicolson Leapfrog (CNLF) time discretization is an implicit-explicit (IMEX) method that successfully uncouples these equations by treating the coupling term explicitly with Leapfrog. This allows one to solve the two equations separately, in parallel, at each time step, making it a parallel partitioned method. The (CNLF) method applied to general evolution equations of the form (3.1) was first presented by Layton and Trenchea in [49]. In [49], they proved stability for general coupled evolution equations under the necessary and sufficient time-step condition, $\Delta t\sqrt{\lambda_{max}(C^T C)} < 1$, which is related to the stability theory from the Leapfrog method.

$$\begin{aligned} \tfrac{u^{k+1}-u^{k-1}}{2\Delta t} + A_f\left(\tfrac{u^{k+1}+u^{k-1}}{2}\right) + C\phi^k &= f_f^k, \\ \tfrac{\phi^{k+1}-\phi^{k-1}}{2\Delta t} + A_p\left(\tfrac{\phi^{k+1}+\phi^{k-1}}{2}\right) - Cu^k &= f_p^k. \end{aligned} \tag{CNLF}$$

We now consider this time discretization applied specifically to the evolutionary Stokes-Darcy problem, with special attention to the affects of the potentially small parameters:

S_0 and k_{min}. When applied to the semi-discrete, evolutionary Stokes-Darcy system in (FEM-SD), (CNLF-SD) uncouples the Stokes-Darcy equations by lagging the coupling terms, $c_I(.,.)$, at the previous time step, similar to (CNLF). Hence, each time step only requires a separate fluid flow and porous media flow solve performed in parallel, thus minimizing computational cost and time. The use of Crank-Nicolson on the diffusive terms, $a_f(.,.)$ and $a_p(.,.)$, adds additional numerical dissipation into the system. However, as expected by the stability condition for (CNLF) applied to coupled evolution equations, (CNLF-SD) is only conditionally long-time stable, and the CFL-type condition sufficient for stability, (Δt_{CNLF}), derived shortly herein, is sensitive to small values of specific storage, S_0.

On the other hand, the method is second-order convergent in time and space, giving it an added advantage over other first-order methods studied in [56, 47] and the splitting methods presented in Chapter 2. This chapter presents an analysis of the stability and convergence properties of the (CNLF-SD) method followed by numerical experiments.

Let $t^k := k\Delta t$ and $w^k := w(x, t^k)$ for any function $w(x,t)$. Let $N \in \mathbb{N}$ and denote by $T := N\Delta t$. The (CNLF-SD) algorithm follows below.

Definition 18. *(Crank-Nicolson Leapfrog Method for Stokes-Darcy (CNLF-SD))*
Given $\left(u_h^k, p_h^k, \phi_h^k\right)$ *and* $\left(u_h^{k-1}, p_h^{k-1}, \phi_h^{k-1}\right)$ *in* $\left(X_f^h, Q_f^h, X_p^h\right)$, *find* $\left(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}\right)$ *in* $\left(X_f^h, Q_f^h, X_p^h\right)$ *satisfying for all* (v_h, q_h, ψ_h) *in* $\left(X_f^h, Q_f^h, X_p^h\right)$:

$$
\begin{aligned}
&n\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) - b\left(v_h, \frac{p_h^{k+1}+p_h^{k-1}}{2}\right) \\
&\qquad\qquad\qquad\qquad\qquad + c_I(v_h, \phi_h^k) = n(f_f^k, v_h)_f, \\
&b\left(u_h^{k+1}, q_h\right)_f = 0, \\
&gS_0\left(\frac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - c_I(u_h^k, \psi_h) = g(f_p^k, \psi_h)_p.
\end{aligned}
\tag{CNLF-SD}
$$

Because (CNLF-SD) is a three-level method, it requires two terms to initiate. The terms (u_h^0, p_h^0, ϕ_h^0) arise from the initial conditions of the problem. To obtain (u_h^1, p_h^1, ϕ_h^1) one must use another numerical method. Errors in this first step will affect the overall convergence rate of the method.

Energy stability analysis of the method requires special treatment of the coupling terms, presented in Lemma 19 below.

Lemma 19. *(Coupling Inequalities for (CNLF-SD)) For all $v_h \in X_f^h$ and $\psi_h \in X_p^h$, with $\mathcal{C} = C_{\Omega_f} C_{\Omega_p} C_{(inv)} g$ we have*

$$c_I(v_h, \psi_h) \le \frac{1}{2}\mathcal{C} h^{-2} n \|v_h\|_f^2 + \frac{1}{2}\mathcal{C} n \|\psi_h\|_p^2, \tag{CNLF Coupling 1}$$

$$c_I(v_h, \psi_h) \le \frac{1}{2}\mathcal{C} h^{-2} n^2 \|v_h\|_f^2 + \frac{1}{2}\mathcal{C} \|\psi_h\|_p^2, \tag{CNLF Coupling 2}$$

$$c_I(v_h, \psi_h) \le \frac{1}{2}\mathcal{C} h^{-2} \|v_h\|_f^2 + \frac{1}{2}\mathcal{C} n^2 \|\psi_h\|_p^2, \tag{CNLF Coupling 3}$$

$$c_I\left(v_h, \psi_h\right) \le \frac{1}{2}\mathcal{C} h^{-1} n \|v_h\|_f^2 + \frac{1}{2}\mathcal{C} h^{-1} n \|\psi_h\|_p^2, \tag{CNLF Coupling 4}$$

$$c_I\left(v_h, \psi_h\right) \le \frac{1}{2}\mathcal{C} h^{-1} n^2 \|v_h\|_f^2 + \frac{1}{2}\mathcal{C} h^{-1} \|\psi_h\|_p^2, \tag{CNLF Coupling 5}$$

$$c_I\left(v_h, \psi_h\right) \le \frac{1}{2}\mathcal{C} h^{-1} \|v_h\|_f^2 + \frac{1}{2}\mathcal{C} h^{-1} n^2 \|\psi_h\|_p^2. \tag{CNLF Coupling 6}$$

Proof. Use the Cauchy-Schwarz, Lemma 1 (Trace), Lemma 5 (Inverse), and Young inequalities in that order, picking up the corresponding constants which depend on the geometry of the spaces Ω_f or Ω_p:

$$\begin{aligned}
c_I(v_h, \psi_h) &= ng \int_I \psi_h v_h \cdot \hat{n}_f ds \le \left| ng \int_I \psi_h v_h \cdot \hat{n}_f ds \right| \\
&\le ng \|\psi_h\|_I \|v_h\|_I \\
&\le C_{\Omega_f} C_{\Omega_p} ng \|\psi_h\|_p^{\frac{1}{2}} \|\nabla \psi_h\|_p^{\frac{1}{2}} \|v_h\|_f^{\frac{1}{2}} \|\nabla v_h\|_f^{\frac{1}{2}} \\
&\le C_{\Omega_f} C_{\Omega_p} C_{(inv)} h^{-1} ng \|\psi_h\|_p \|v_h\|_f \\
&\le \frac{1}{2} C_{\Omega_f} C_{\Omega_p} C_{(inv)} h^{-2} ng \|v_h\|_f^2 + \frac{1}{2} C_{\Omega_f} C_{\Omega_p} C_{(inv)} ng \|\psi_h\|_p^2.
\end{aligned}$$

This proves (CNLF Coupling 1). (CNLF Coupling 2)-(CNLF Coupling 3) follow similarly with different treatment of the porosity parameter, n. Replace the last line for the proof of (CNLF Coupling 1) with

$$c_I\left(v_h, \psi_h\right) \le \frac{1}{2} C_{\Omega_f} C_{\Omega_p} C_{(inv)} h^{-1} ng \|v_h\|_f^2 + \frac{1}{2} C_{\Omega_f} C_{\Omega_p} C_{(inv)} h^{-1} ng \|\psi_h\|_p^2,$$

to obtain (CNLF Coupling 4). (CNLF Coupling 5)-(CNLF Coupling 6) follow likewise, with different treatment of the porosity parameter, n. □

3.1 STABILITY OF CNLF-SD

Energy stability analysis of (CNLF-SD) leads to a CFL-type time-step condition. Under this condition, approximate solutions to the Stokes-Darcy coupled problem are uniform in time stable and convergent.

Theorem 20. *(Stability for (CNLF-SD) Method) Suppose Δt satisfies*

$$\begin{aligned}\Delta t < \mathcal{C}^{-1} \max \{ & \min\left\{h^2, gS_0 n^{-1}\right\}, \min\left\{n^{-1}h^2, gS_0\right\}, \min\left\{nh^2, gS_0 n^{-2}\right\} \\ & \min\left\{h, gS_0 n^{-1}h\right\}, \min\left\{n^{-1}h, gS_0 h\right\}, \min\left\{nh, gS_0 n^{-2}h\right\} \},\end{aligned} \tag{Δt_{CNLF}}$$

where $\mathcal{C} = C_{\Omega_f} C_{\Omega_p} C_{(inv)} g$. Then there exist $\alpha, \beta > 0$ such that, for $N = 1, 2, 3, \ldots$, (CNLF-SD) is stable over long-time intervals:

$$\begin{aligned}&\alpha(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) + \beta(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2) \\ &\le n(\|u_h^1\|_f^2 + \|u_h^0\|_f^2) + gS_0(\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2) + 2\Delta t\left(c_I(u_h^1, \phi_h^0) - c_I(u_h^0, \phi_h^1)\right) \\ &\qquad + 2\Delta t \sum_{k=1}^{N-1} \left\{n(\nu)^{-1}\|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2\right\}.\end{aligned} \tag{3.2}$$

Proof. Choose $v_h = u_h^{k+1} + u_h^{k-1}$ and $\psi_h = \phi_h^{k+1} + \phi_h^{k-1}$. Then the second equation of (CNLF-SD) drops out and the first and third equations of the method added together become

$$\begin{aligned}&\tfrac{1}{2\Delta t}\left(n\|u_h^{k+1}\|_f^2 + gS_0\|\phi_h^{k+1}\|_g^2 - n\|u_h^{k-1}\|_f^2 - gS_0\|\phi_h^{k-1}\|_g^2\right) \\ &\qquad + a_f(\tfrac{u_h^{k+1}+u_h^{k-1}}{2}, u_h^{k+1} + u_h^{k-1}) + a_p(\tfrac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \phi_h^{k+1} + \phi_h^{k-1}) \\ &\qquad + c_I(u_h^{k+1} + u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1} + \phi_h^{k-1}) \\ &\qquad\qquad = n(f_f^k, u_h^{k+1} + u_h^{k-1})_f + g(f_p^k, \phi_h^{k+1} + \phi_h^{k-1})_p.\end{aligned}$$

Consider the right-hand-side of the above equation. Use the Cauchy-Schwarz and Young inequalities to obtain the following bound

$$\begin{aligned}&n(f_f^k, u_h^{k+1} + u_h^{k-1})_f + g(f_p^k, \phi_h^{k+1} + \phi_h^{k-1})_p \\ &\quad\le \|f_f^k\|_{-1,f}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f + g\|f_p^k\|_{-1,p}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p \\ &\quad\le \tfrac{\nu}{4}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2 + (\nu)^{-1}\|f_f^k\|_{-1,f}^2 + \tfrac{gk_{min}}{4}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2.\end{aligned}$$

This bound, along with coercivity of $a_f(.,.)$ and $a_p(.,.)$, gives

$$\frac{1}{2\Delta t}\left(n\|u_h^{k+1}\|_f^2 + gS_0\|\phi_h^{k+1}\|_p^2 - n\|u_h^{k-1}\|_f^2 - gS_0\|\phi_h^{k-1}\|_p^2\right) + \frac{n\nu}{4}\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2$$
$$+\frac{gk_{min}}{4}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p + c_I(u_h^{k+1}+u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}+\phi_h^{k-1})$$
$$\leq n(\nu)^{-1}\|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2.$$

Consider the coupling terms $c_I(u_h^{k+1}+u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}+\phi_h^{k-1})$. Define

$$C^{k+\frac{1}{2}} = c_I(u_h^{k+1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}).$$

The coupling terms equal $C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}$. Define the energy terms:

$$E^{k+1/2} = n\|u_h^{k+1}\|_f^2 + n\|u_h^k\|_f^2 + gS_0\|\phi_h^{k+1}\|_p^2 + gS_0\|\phi_h^k\|_p^2.$$

Using this notation and multiplying by $2\Delta t$ produces

$$\begin{aligned} E^{k+1/2} + 2\Delta t C^{k+1/2} - E^{k-1/2} - 2\Delta t C^{k-1/2} \\ +\Delta t\left\{\frac{n\nu}{2}\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p\right\} \\ \leq 2\Delta t\left\{n(\nu)^{-1}\|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2\right\}. \end{aligned} \tag{3.3}$$

Sum the above inequality from $k = 1$ to $N-1$.

$$E^{N-1/2} + 2\Delta t C^{N-1/2} + \Delta t\sum_{k=1}^{N-1}\left\{\frac{n\nu}{2}\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}$$
$$\leq 2\Delta t\sum_{k=1}^{N-1}\left\{n(\nu)^{-1}\|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2\right\} + E^{1/2} + 2\Delta t C^{1/2}.$$

The above inequality implies the stability of the CNLF-method provided that $E^{N-1/2} + 2\Delta t C^{N-1/2} > 0$. By Lemma 19,

$$C^{N-1/2} \geq -\frac{1}{2}\mathcal{C}h^{-2}n\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}n\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right),$$
$$C^{N-1/2} \geq -\frac{1}{2}\mathcal{C}h^{-1}n\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}h^{-1}n\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right),$$
$$C^{N-1/2} \geq -\frac{1}{2}\mathcal{C}h^{-2}n^2\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right),$$
$$C^{N-1/2} \geq -\frac{1}{2}\mathcal{C}h^{-1}n^2\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}h^{-1}\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right),$$

$$C^{N-1/2} \geq -\frac{1}{2}\mathcal{C}h^{-2}\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}n^2\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right),$$
$$C^{N-1/2} \geq -\frac{1}{2}\mathcal{C}h^{-1}\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}h^{-1}n^2\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right).$$

Apply these bounds separately to the energy term $E^{N-1/2} + 2\Delta t C^{N-1/2}$ to find

$$\begin{aligned}
E^{N-1/2} + 2\Delta t C^{N-1/2} &\geq (n - \Delta t\mathcal{C}nh^{-2})(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\quad + (gS_0 - \Delta t\mathcal{C})n(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2), \\
E^{N-1/2} + 2\Delta t C^{N-1/2} &\geq (n - \Delta t\mathcal{C}h^{-1}n)(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\quad + (gS_0 - \Delta t\mathcal{C}h^{-1}n)(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2), \\
E^{N-1/2} + 2\Delta t C^{N-1/2} &\geq (n - \Delta t\mathcal{C}h^{-2}n^2)(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\quad + (gS_0 - \Delta t\mathcal{C})(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2), \\
E^{N-1/2} + 2\Delta t C^{N-1/2} &\geq (n - \Delta t\mathcal{C}h^{-1}n^2)(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\quad + (gS_0 - \Delta t\mathcal{C}h^{-1})(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2), \\
E^{N-1/2} + 2\Delta t C^{N-1/2} &\geq (n - \Delta t\mathcal{C}h^{-2})(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\quad + (gS_0 - \Delta t\mathcal{C}n^2)(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2), \\
E^{N-1/2} + 2\Delta t C^{N-1/2} &\geq (n - \Delta t\mathcal{C}h^{-1})(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\quad + (gS_0 - \Delta t\mathcal{C}h^{-1}n^2)(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2),
\end{aligned}$$

Therefore $E^{N-1/2} + 2\Delta t C^{N-1/2} > 0$ provided that at least one of the inequalities above has positive coefficients on the right-hand side. Hence, we must have

$$\begin{aligned}
\Delta t < \mathcal{C}^{-1}\max\{\, &\min\left\{h^2, gS_0n^{-1}\right\}, \min\left\{n^{-1}h^2, gS_0\right\}, \min\left\{nh^2, gS_0n^{-2}\right\} \\
&\min\left\{h, gS_0n^{-1}h\right\}, \min\left\{n^{-1}h, gS_0h\right\}, \min\left\{nh, gS_0n^{-2}h\right\}\,\}.
\end{aligned}$$

□

This time-step condition, (Δt_{CNLF}), is independent of k_{min} but sensitive to small values S_0. In fact, if $h = 1/10$, $n = 0.10$, and $S_0 = 10^{-6}$, then the time-step restriction implies (assuming $\mathcal{C}$ is $O(1)$) that one must take $\Delta t < 10^{-3}$ for stability.

3.1.1 Control over the Stable and Unstable Modes of CNLF-SD

The use of the Leapfrog time discretization produces the numerical phenomenon of stable and unstable modes. These modes have no physical meaning with respect to the actual fluid-porous media flow. Nevertheless, understanding the behavior of these modes is crucial for predicting the overall behavior of solutions to (CNLF-SD) since accumulation of numerical noise in the unstable mode has been known to correspond to energy blow-up in finite time [30, 31]. In geophysics, techniques such as time-filtering [40, 5, 71, 72] are often applied to counteract growth in the unstable mode that leads to system instability. However, we show that provided satisfaction of the CFL-type condition, (Δt_{CNLF}), (3.2) controls both the stable and unstable modes.

We examine stability of the stable and unstable modes, represented by $w_h^{k+1} + w_h^{k-1}$ and $w_h^{k+1} - w_h^{k-1}$ for $w = u, \phi$ respectively, inherently present in the (CNLF-SD) algorithm. We show that provided (Δt_{CNLF}) is satisfied, (CNLF-SD) effectively damps both the stable and unstable modes. This implies that spurious oscillations or growth in the unstable mode may be attributed to accumulation of round-off error or to an incorrect implementation of the method, such as a violation of (Δt_{CNLF}). Further, control over modes of the Stokes velocity and Darcy pressure approximations leads to a stability estimate for the averages of the Stokes pressure.

To prove stability of the modes, we will utilize of the following corollary to Theorem 20 regarding the stable modes of the Stokes velocity and Darcy pressure.

Corollary 21. *(Control of the Stable Mode of* (CNLF-SD)*) Suppose* Δt *satisfies* (Δt_{CNLF}). *Then the following inequality for the stable modes holds.*

$$\begin{aligned} &\Delta t \sum_{k=1}^{N-1} \left\{ \tfrac{n\nu}{2} \|\nabla \left(u_h^{k+1} + u_h^{k-1}\right)\|_f^2 + \tfrac{gk_{min}}{2} \|\nabla \left(\phi_h^{k+1} + \phi_h^{k-1}\right)\|_p^2 \right\} \\ &\leq 2\Delta t \sum_{k=1}^{N-1} \left\{ n(\nu)^{-1} \|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1} \|f_p^k\|_{-1,p}^2 \right\} + E^{1/2} + 2\Delta t C^{1/2}. \end{aligned} \tag{3.4}$$

Proof. Drop the positive term $E^{N-1/2} + 2\Delta t C^{N-1/2}$ from the left hand side of (3.3) in the proof of Theorem 20. □

Theorem 22 (Control of the Stable and Unstable Modes of (CNLF-SD))**.** *Suppose Δt satisfies (Δt_{CNLF}), recalled below*

$$\begin{aligned}\Delta t < \mathcal{C}^{-1} \max \{ & \min\left\{h^2, gS_0 n^{-1}\right\}, \min\left\{n^{-1}h^2, gS_0\right\}, \min\left\{nh^2, gS_0 n^{-2}\right\} \\ & \min\left\{h, gS_0 n^{-1}h\right\}, \min\left\{n^{-1}h, gS_0 h\right\}, \min\left\{nh, gS_0 n^{-2}h\right\} \}, \end{aligned} \tag{Δt_{CNLF}}$$

where $\mathcal{C} = C_{\Omega_f} C_{\Omega_p} C_{(inv)} g$. Then (CNLF-SD) *controls both the stable and unstable modes. That is, there exists a positive constant, $\mathcal{M}$, satisfying for any $N \geq 2$,*

$$\begin{aligned} \mathcal{M} \Bigg\{ \Delta t \sum_{k=1}^{N-1} & \left\{ \|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2 + \|\nabla(\phi_h^{k+1} + \phi_h^{k+1})\|_p^2 \right\} \\ & + \sum_{k=1}^{N-1} \left\{ \|u_h^{k+1} - u_h^{k-1}\|_f^2 + \|\phi_h^{k+1} - \phi_h^{k-1}\|_p^2 \right\} \Bigg\} \\ \leq \Delta t \sum_{k=1}^{N-1} & \left\{ \|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t \left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2 \right) \right\} \\ & + \|u_h^1\|_f^2 + \|u_h^0\|_f^2 + \|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2 \\ & + \Delta t \left\{ \|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla \phi_h^1\|_p^2 + \|\nabla \phi_h^0\|_p^2 \right\} \\ + \Delta t & \left\{ c_I(u_h^1, \phi_h^0) - c_I(u_h^0, \phi_h^1) + c_I(u_h^2 - u_h^0, \phi_h^1) - c_I(u_h^1, \phi_h^2 - \phi_h^0) \right\}. \end{aligned} \tag{3.5}$$

Proof. Choose $v_h = 2\delta\Delta t(u_h^{k+1} - u_h^{k-1})$ and $\psi_h = 2\delta\Delta t(\phi_h^{k+1} - \phi_h^{k-1})$ in (CNLF-SD) where $\delta > 0$ and add the equations. This produces

$$\begin{aligned} \delta\left\{n\|u_h^{k+1} - u_h^{k-1}\|_f^2 + gS_0\|\phi_h^{k+1} - \phi_h^{k+1}\|_p^2\right\} + \delta\Delta t\left\{\mathcal{A}^{k+1/2} - \mathcal{A}^{k-1/2}\right\} \\ + 2\delta\Delta t\left\{c_I(u_h^{k+1} - u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1} - \phi_h^{k-1})\right\} \\ = 2\delta\Delta t\left\{n(f_f^k, u_h^{k+1} - u_h^{k-1})_f + g(f_p^k, \phi_h^{k+1} - \phi_h^{k-1})_p\right\}, \end{aligned}$$

where $\mathcal{A}^{k+1/2} = a_f(u_h^{k+1}, u_h^{k+1}) + a_p(\phi_h^{k+1}, \phi_h^{k+1}) + a_f(u_h^k, u_h^k) + a_p(\phi_h^k, \phi_h^k)$. Apply Cauchy-Schwarz and Young's inequality to the right hand side to obtain

$$\begin{aligned} \delta(1-\varepsilon)\left\{n\|u_h^{k+1} - u_h^{k-1}\|_f^2 + gS_0\|\phi_h^{k+1} - \phi_h^{k+1}\|_p^2\right\} + \delta\Delta t\left\{\mathcal{A}^{k+1/2} - \mathcal{A}^{k-1/2}\right\} \\ + 2\delta\Delta t\left\{c_I(u_h^{k+1} - u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1} - \phi_h^{k-1})\right\} \leq \tfrac{\delta\Delta t^2}{\varepsilon}\left\{n\|f_f^k\|_f^2 + \tfrac{g}{S_0}\|f_p^k\|_p^2\right\}, \end{aligned} \tag{3.6}$$

where $\varepsilon \in (0,1)$. Define the new energy:

$$\tilde{E}^{k+1/2} = E^{k+1/2} + 2\Delta t C^{k+1/2} + \delta \Delta t \mathcal{A}^{k+1/2}.$$

Note that under the time-step restriction given in (Δt_{CNLF}), $\tilde{E}^{k+1/2} > 0$. Next, add (3.6) to (3.3) in Corollary 21. Using the new notation yields

$$\begin{aligned}
\tilde{E}^{k+1/2} - \tilde{E}^{k-1/2} + \tfrac{\Delta t}{2}\left\{n\nu\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2 + gk_{min}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}&\\
+\delta(1-\varepsilon)\left\{n\|u_h^{k+1}-u_h^{k-1}\|_f^2 + gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\}&\\
+2\delta\Delta t\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\}&\\
\leq 2\Delta t\left\{\tfrac{n}{\nu}\|f_f^k\|_{-1,f}^2 + \tfrac{g}{k_{min}}\|f_p^k\|_{-1,p}^2\right\} + \tfrac{\delta\Delta t^2}{\varepsilon}\left\{n\|f_f^k\|_f^2 + \tfrac{g}{S_0}\|f_p^k\|_p^2\right\}&
\end{aligned}$$

Sum the above from $k=1$ to $N-1$. This produces

$$\begin{aligned}
\tilde{E}^{N-1/2} + \tfrac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{n\nu\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2 + gk_{min}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}&\\
+\delta(1-\varepsilon)\sum_{k=1}^{N-1}\left\{n\|u_h^{k+1}-u_h^{k-1}\|_f^2 + gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\} + \mathcal{Q}& \qquad (3.7)\\
\leq \sum_{k=1}^{N-1}\left\{2\Delta t\left(\tfrac{n}{\nu}\|f_f^k\|_{-1,f}^2 + \tfrac{g}{k_{min}}\|f_p^k\|_{-1,p}^2\right) + \tfrac{\delta\Delta t^2}{\varepsilon}\left(n\|f_f^k\|_f^2 + \tfrac{g}{S_0}\|f_p^k\|_p^2\right)\right\} + \tilde{E}^{1/2},&
\end{aligned}$$

where

$$\mathcal{Q} = 2\delta\Delta t\sum_{k=1}^{N-1}\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\}.$$

Next, bound $\mathcal{Q}$ using techniques from Lemma 19. Begin by rewriting the interface integrals in terms of the stable and unstable modes. Notice that for $k \geq 2$,

$$\begin{aligned}
c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1}) = \tfrac{1}{2}c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k-\phi_h^{k-2})&\\
+\tfrac{1}{2}c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k+\phi_h^{k-2}) - \tfrac{1}{2}c_I(u_h^k-u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}) - \tfrac{1}{2}c_I(u_h^k+u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}).&
\end{aligned}$$

Bound each term as follows using Cauchy-Schwarz, Trace (Lemma 1), Poincaré (Lemma 2), Inverse (Lemma 5) and Young's inequalities as necessary.

$$\begin{aligned}
c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k-\phi_h^{k-2}) &\le \tfrac{\mathcal{C}n}{2h}\left\{\|u_h^{k+1}-u_h^{k-1}\|_f^2+\|\phi_h^k-\phi_h^{k-2}\|_f^2\right\},\\
c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k+\phi_h^{k-2}) &\le \frac{ngC_{\Omega_f}C_{\Omega_p}\sqrt{C_{(inv)}C_{P,p}}}{\sqrt{h}}\|u_h^{k+1}-u_h^{k-1}\|_f\|\nabla(\phi_h^k+\phi_h^{k-2})\|_p\\
&\le \tfrac{n\mathcal{C}\epsilon_1}{h}\|u_h^{k+1}-u_h^{k-1}\|_f+\frac{ngC_{\Omega_f}C_{\Omega_p}C_{P,p}}{2\epsilon_1}\|\nabla(\phi_h^k+\phi_h^{k-2})\|_p,\\
c_I(u_h^{k}-u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}) &\le \tfrac{n\mathcal{C}}{2h}\left\{\|u_h^k-u_h^{k-2}\|_f^2+\|\phi_h^{k+1}-\phi_h^{k-1}\|_f^2\right\},\\
c_I(u_h^{k}+u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}) &\le \frac{ngC_{\Omega_f}C_{\Omega_p}\sqrt{C_{(inv)}C_{P,f}}}{\sqrt{h}}\|\nabla(u_h^k+u_h^{k-2})\|_f\|\phi_h^{k+1}-\phi_h^{k-1}\|_p\\
&\le \frac{ngC_{\Omega_f}C_{\Omega_p}C_{P,f}}{2\epsilon_2}\|\nabla(u_h^k+u_h^{k-2})\|_f^2+\tfrac{n\mathcal{C}\epsilon_2}{h}\|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2,
\end{aligned} \tag{3.8}$$

where, as before, $\mathcal{C}=gC_{\Omega_f}C_{\Omega_p}C_{(inv)}$, and $\epsilon_{1,2}>0$ are from Young's inequality. Notice that the first and third terms above were bounded using (CNLF Coupling 4) from Lemma 19.

Combine terms and simplify to obtain the following bound for $\mathcal{Q}$:

$$\begin{aligned}
\mathcal{Q} \le \tfrac{\delta\Delta t\mathcal{C}}{h}\sum_{k=1}^{N-1}\left\{n(1+\tfrac{\epsilon_1}{2})\|u_h^{k+1}-u_h^{k-1}\|_f^2+n(1+\tfrac{\epsilon_2}{2})\|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2\right\}\\
+\delta\Delta t\sum_{k=1}^{N-1}\left\{\frac{ngC_{\Omega_f}C_{\Omega_p}C_{P,f}}{2\epsilon_2}\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2+\frac{ngC_{\Omega_f}C_{\Omega_p}C_{P,p}}{2\epsilon_1}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}\\
+2\delta\Delta t\left[c_I(u_h^2-u_h^0,\phi_h^1)-c_I(u_h^1,\phi_h^2-\phi_h^0)\right].
\end{aligned}$$

Since $\tilde{E}^{N-1/2}>0$ by (Δt_{CNLF}), our energy inequality in (3.7) becomes

$$\begin{aligned}
&\tfrac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\left(n\nu-\frac{\delta ngC_{\Omega_f}C_{\Omega_p}C_{P,f}}{\epsilon_2}\right)\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2\right\}\\
&+\tfrac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\left(gk_{min}-\frac{\delta ngC_{\Omega_f}C_{\Omega_p}C_{P,p}}{\epsilon_1}\right)\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\}\\
&+\delta\sum_{k=1}^{N-1}\left\{n\left((1-\varepsilon)-(1+\tfrac{\epsilon_1}{2})\tfrac{\Delta t\mathcal{C}}{h}\right)\|u_h^{k+1}-u_h^{k-1}\|_f^2\right\}\\
&+\delta\sum_{k=1}^{N-1}\left\{\left(gS_0(1-\varepsilon)-n(1+\tfrac{\epsilon_2}{2})\tfrac{\Delta t\mathcal{C}}{h}\right)\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\}\\
&\le\sum_{k=1}^{N-1}\left\{2\Delta t\left(\tfrac{n}{\nu}\|f_f^k\|_{-1,f}^2+\tfrac{g}{k_{min}}\|f_p^k\|_{-1,p}^2\right)+\tfrac{\delta\Delta t^2}{\varepsilon}\left(n\|f_f^k\|_f^2+\tfrac{g}{S_0}\|f_p^k\|_p^2\right)\right\}\\
&+\tilde{E}^{1/2}+2\delta\Delta t\left[c_I(u_h^2-u_h^0,\phi_h^1)-c_I(u_h^1,\phi_h^2-\phi_h^0)\right],
\end{aligned}$$

This implies control over the stable and unstable modes provided the coefficients of the sums on the left hand side are positive, i.e.

$$\begin{aligned} \nu - \tfrac{\delta g C_{\Omega_f} C_{\Omega_p} C_{P,f}}{\epsilon_2} &> 0, \\ g k_{min} - \tfrac{\delta g C_{\Omega_f} C_{\Omega_p} C_{P,p}}{\epsilon_1} &> 0, \\ (1-\varepsilon) - (1+\tfrac{\epsilon_1}{2})\tfrac{\Delta t \mathcal{C}}{h} &> 0, \\ g S_0 (1-\varepsilon) - n(1+\tfrac{\epsilon_2}{2})\tfrac{\Delta t \mathcal{C}}{h} &> 0. \end{aligned}$$

The last two inequalities are equivalent to

$$\tfrac{1+\frac{\epsilon_1}{2}}{1-\varepsilon} < \tfrac{h}{\Delta t \mathcal{C}}, \qquad \tfrac{1+\frac{\epsilon_2}{2}}{1-\varepsilon} < \tfrac{g S_0 h}{n \Delta t \mathcal{C}}.$$

Since $\epsilon_{1,2}$ are arbitrary positive numbers and $\varepsilon \in (0,1)$, we have $\frac{1+\frac{\epsilon_{1,2}}{2}}{1-\varepsilon} > 1$. If the maximum in ($\Delta t_{\text{CNLF}}$) is $\min\{h, g S_0 n^{-1} h\}$, then the inequalities hold since $\Delta t < \mathcal{C}^{-1} \min\{h, g S_0 n^{-1} h\}$. To complete the proof in this case, choose

$$\delta = \min\left\{ \tfrac{g k_{min} \epsilon_1}{n g C_{\Omega_f} C_{\Omega_p} C_{P,p}}, \tfrac{\nu \epsilon_2}{g C_{\Omega_f} C_{\Omega_p} C_{P,f}} \right\}.$$

To complete the proof for the other cases in (Δt_{CNLF}), go back to (3.8) and apply one of (CNLF Coupling 1)-(CNLF Coupling 3), (CNLF Coupling 5), or (CNLF Coupling 6) from Lemma 19 to obtain the coefficients for these cases. □

3.1.2 Stability of the Stokes Pressure

Theorem 23 (Stability of the Stokes Pressure)**.** *Suppose Δt satisfies the CFL-condition for stability given in (Δt_{CNLF}). Then there exists a constant, $\mathcal{H} = C(1+\Delta t)(1+h)$ for $C > 0$, such that*

$$\begin{aligned} \Delta t^2 \sum_{k=1}^{N-1} \left\| \tfrac{p_h^{k+1} + p_h^{k-1}}{2} \right\|_f^2 \leq \mathcal{H} \{ \ \Delta t \sum_{k=1}^{N-1} \left[\|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t \left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2 \right) \right] \\ + \|u_h^1\|_f^2 + \|u_h^0\|_f^2 + \|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2 \\ + \Delta t \left[\ \|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla \phi_h^1\|_p^2 + \|\nabla \phi_h^0\|_p^2 \right] \\ + \Delta t \left(c_I(u_h^1, \phi_h^0) - c_I(u_h^0, \phi_h^1) + c_I(u_h^2 - u_h^0, \phi_h^1) - c_I(u_h^1, \phi_h^2 - \phi_h^0) \right) \ \}. \end{aligned} \tag{3.9}$$

Proof. By (LBB^h),

$$\beta \left\| \frac{p_h^{k+1}+p_h^{k-1}}{2} \right\|_f \le \sup_{v_h \in X_f^h} \frac{b\left(v_h, \frac{p_h^{k+1}+p_h^{k-1}}{2}\right)}{\|\nabla v_h\|_f}$$
$$= \sup_{v_h \in X_f^h} \frac{n\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) + c_I(v_h, \phi_h^k) - n(f_f^k, v_h)_f}{\|\nabla v_h\|_f}.$$

Bound the terms on the right hand side using the Cauchy-Schwarz inequality, Lemma 2 (Poincaré), and the continuity of $a_f(.,.)$ (Lemma 3).

$$\beta \left\| \frac{p_h^{k+1}+p_h^{k-1}}{2} \right\|_f \le \sup_{v_h \in X_f^h} \{\, nC_{P,f} \left\| \frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t} \right\|_f + M_f \left\| \nabla \left(\frac{u_h^{k+1}+u_h^{k-1}}{2} \right) \|_f$$
$$+ \frac{c_I(v_h, \phi_h^k)}{\|\nabla v_h\|_f} + n\|f_f^k\|_{-1,f} \,\}.$$

Next, bound the coupling term by the Cauchy-Schwarz, Trace (Lemma 1), Poincaré (Lemma 2), and Inverse (Lemma 5) inequalities.

$$\beta \left\| \frac{p_h^{k+1}+p_h^{k-1}}{2} \right\|_f \le nC_{P,f} \left\| \frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t} \right\|_f + M_f \left\| \nabla \left(\frac{u_h^{k+1}+u_h^{k-1}}{2} \right) \right\|_f$$
$$+ \frac{ngC_{\Omega_f} C_{\Omega_p} \sqrt{C_{P,f} C_{(inv)}}}{\sqrt{h}} \|\phi_h^k\|_p + n\|f_f^k\|_{-1,f}.$$

Square both sides, sum from $k=1$ to $k=N-1$ and apply Cauchy-Schwarz again to obtain

$$\beta^2 \sum_{k=1}^{N-1} \left\| \frac{p_h^{k+1}+p_h^{k-1}}{2} \right\|_f^2 \le \sum_{k=1}^{N-1} \Bigg\{ nC_{P,f} \left\| \frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t} \right\|_f + M_f \left\| \nabla \left(\frac{u_h^{k+1}+u_h^{k-1}}{2} \right) \right\|_f$$
$$+ \frac{ngC_{\Omega_f} C_{\Omega_p} \sqrt{C_{P,f} C_{(inv)}}}{\sqrt{h}} \|\phi_h^k\|_p + n\|f_f^k\|_{-1,f} \Bigg\}^2$$
$$\le H \sum_{k=1}^{N-1} \left\{ \left\| \frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t} \right\|_f^2 + \left\| \nabla \left(u_h^{k+1} + u_h^{k-1} \right) \right\|_f^2 + \frac{\mathcal{C}}{h} \|\phi_h^k\|_p^2 + \|f_f^k\|_{-1,f}^2 \right\},$$

where $H = n^2 C_{P,f}^2 + \frac{M_f^2}{4} + n^2 g C_{\Omega_f} C_{\Omega_p} C_{P,f} + n^2$. Next, we bound the summation containing $\|\phi_h^k\|_p^2$ similar by rewriting it in terms of the stable and unstable modes via the parallelogram law.

$$\sum_{k=1}^{N-1}\|\phi_h^k\|_p^2 = \sum_{k=1}^{N-2}\|\phi_h^{k+1}\|_p^2 + \|\phi_h^1\|_p^2$$

$$= \tfrac{1}{2}\sum_{k=1}^{N-2}\left\{\|(\phi_h^{k+1}-\phi_h^{k-1})\|_p^2 + C_{P,p}^2\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2 - 2\|\phi_h^{k-1}\|_p^2\right\} + \|\nabla\phi_h^1\|_p^2$$

$$\leq \tfrac{1}{2}\sum_{k=1}^{N-1}\left\{\|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2 + C_{P,p}^2\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\} + \|\nabla\phi_h^1\|_p^2.$$

Thus we have

$$\beta^2\sum_{k=1}^{N-1}\left\|\tfrac{p_h^{k+1}+p_h^{k-1}}{2}\right\|_f^2 \leq H\,\Big\{\,\sum_{k=1}^{N-1}\Big[\,\left\|\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right\|_f^2 + \left\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\right\|_f^2$$

$$+\tfrac{\mathcal{C}}{2h}\left(\|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2 + C_{P,p}^2\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right) + \|f_f^k\|_{-1,f}^2\,\Big] + \|\nabla\phi_h^1\|_p^2\,\Big\}\,.$$

Bound each term on the right hand side above using the inequality in Theorem 22. That is,

$$\sum_{k=1}^{N-1}\|\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\|_f^2 \leq \tfrac{1}{4\mathcal{M}\Delta t^2}\,\Big\{\,\Delta t\sum_{k=1}^{N-1}\left[\|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t\left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2\right)\right]$$

$$+\|u_h^1\|_f^2 + \|u_h^0\|_f^2 + \|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2$$

$$+\Delta t\,\left\{\,\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla\phi_h^1\|_p^2 + \|\nabla\phi_h^0\|_p^2\right\}$$

$$+\Delta t\left(c_I(u_h^1,\phi_h^0) - c_I(u_h^0,\phi_h^1) + c_I(u_h^2-u_h^0,\phi_h^1) - c_I(u_h^1,\phi_h^2-\phi_h^0)\right)\,\Big\}\,,$$

$$\tfrac{\mathcal{C}}{2h}\sum_{k=1}^{N-1}\|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2 \leq \tfrac{\mathcal{C}}{2\mathcal{M}h}\,\Big\{\,\Delta t\sum_{k=1}^{N-1}\left[\|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t\left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2\right)\right]$$

$$+\|u_h^1\|_f^2 + \|u_h^0\|_f^2 + \|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2$$

$$+\Delta t\,\left\{\,\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla\phi_h^1\|_p^2 + \|\nabla\phi_h^0\|_p^2\right\}$$

$$+\Delta t\left(c_I(u_h^1,\phi_h^0) - c_I(u_h^0,\phi_h^1) + c_I(u_h^2-u_h^0,\phi_h^1) - c_I(u_h^1,\phi_h^2-\phi_h^0)\right)\,\Big\}\,,$$

$$\tfrac{\mathcal{C}C_{P,p}^2}{2h}\sum_{k=1}^{N-1}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2 \leq \tfrac{\mathcal{C}C_{P,p}^2}{2\mathcal{M}h}\,\Big\{\,\sum_{k=1}^{N-1}\left[\|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t\left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2\right)\right]$$

$$+\tfrac{1}{\Delta t}\left(\|u_h^1\|_f^2 + \|u_h^0\|_f^2 + \|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2\right)$$

$$\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla\phi_h^1\|_p^2 + \|\nabla\phi_h^0\|_p^2$$

$$+\left(c_I(u_h^1,\phi_h^0) - c_I(u_h^0,\phi_h^1) + c_I(u_h^2-u_h^0,\phi_h^1) - c_I(u_h^1,\phi_h^2-\phi_h^0)\right)\,\Big\}\,,$$

$$\sum_{k=1}^{N-1} \|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 \leq \tfrac{1}{\mathcal{M}}\Big\{ \sum_{k=1}^{N-1} \left[\|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t\left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2\right)\right]$$
$$+\tfrac{1}{\Delta t}\left(\|u_h^1\|_1^2 + \|u_h^0\|_1^2 + \|\phi_h^1\|_1^2 + \|\phi_h^0\|_1^2\right)$$
$$\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla \phi_h^1\|_p^2 + \|\nabla \phi_h^0\|_p^2$$
$$+\left(c_I(u_h^1,\phi_h^0) - c_I(u_h^0,\phi_h^1) + c_I(u_h^2 - u_h^0,\phi_h^1) - c_I(u_h^1,\phi_h^2-\phi_h^0)\right)\Big\}.$$

Use the above bounds to obtain the final result. Multiply through by Δt^2, and utilize that since the porosity, n, satisfies, $0 < n < 1$, by (Δt_{CNLF}), $\frac{\mathcal{C}\Delta t}{h} < n^{-1}(1+h)$. □

3.1.3 Special Case: Convergence to the Equilibrium Stokes-Darcy Problem

The special case of convergence to the equilibrium Stokes-Darcy problem, with forcing terms f_f^∞ and f_p^∞ independent of time, follows. The equilibrium solution, denoted by $(u_\infty, p_\infty, \phi_\infty)$, obeys

$$\begin{aligned} a_f(u_\infty, v_h) - b(v_h, p_\infty) + c_I(v_h, \phi_\infty) &= n(f_f^\infty, v_h)_f, \\ b(u_\infty, q_h) &= 0, \\ a_p(\phi_\infty, \psi_h) - c_I(u_\infty, \psi_h) &= (f_p^\infty, \psi_h)_p. \end{aligned} \tag{SD-Equi}$$

Using the equilibrium projection operator, P_h, discussed in Chapter 2, we may project the true solutions to the equilibrium problem onto the finite element spaces, X_f^h, Q_f^h, and X_p^h. Let $\tilde{w}_h^k = w_h^k - P_h w_\infty$ for $w = u, \phi$, and $\tilde{f}_{f,p}^k = f_{f,p}^k - f_{f,p}^\infty$. By the projection property (SD-PROJ) we have

$$\begin{aligned} a_f(P_h u_\infty, v_h) - b(v_h, P_h p_\infty) + c_I(v_h, P_h\phi_\infty) &= (f_f^\infty, v_h)_f, \\ a_f(P_h\phi_\infty, \psi_h) - c_I(P_h u_\infty) &= (f_p^\infty, \psi_h)_p. \end{aligned}$$

Thus, by linearity, the terms $\tilde{u}, \tilde{p}$, and $\tilde{\phi}$ satisfy

$$
\begin{aligned}
& n\left(\frac{\tilde{u}_h^{k+1}-\tilde{u}_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{\tilde{u}_h^{k+1}+\tilde{u}_h^{k-1}}{2}, v_h\right) - b\left(v_h, \frac{\tilde{p}_h^{k+1}+\tilde{p}_h^{k-1}}{2}\right)_f + c_I(v_h, \tilde{\phi}_h^k) = n(\tilde{f}_f^k, v_h)_f, \\
& b\left(\tilde{u}_h^{k+1}, q_h\right) = 0, \\
& gS_0\left(\frac{\tilde{\phi}_h^{k+1}-\tilde{\phi}_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\tilde{\phi}_h^{k+1}+\tilde{\phi}_h^{k-1}}{2}, \psi_h\right) - c_I(\tilde{u}_h^k, \psi_h) = g(\tilde{f}_p^k, \psi_h)_p.
\end{aligned} \tag{3.10}
$$

Corollary 24 (Convergence of (CNLF-SD) to Equilibrium)**.** *Suppose* (Δt_{CNLF}) *holds and both* $f_f^k \to f_f^\infty$ *and* $f_p^k \to f_p^\infty$ *as* $k \to \infty$ *in the sense that the series* $\sum_{k=0}^{\infty} \|f_f^k - f_f^\infty\|_{-1,f}^2$*,* $\sum_{k=0}^{\infty} \|f_p^k - f_p^\infty\|_{-1,p}^2$*,* $\sum_{k=0}^{\infty} \|f_f^k - f_f^\infty\|_f^2$*, and* $\sum_{k=0}^{\infty} \|f_p^k - f_p^\infty\|_p^2$ *all converge. Then,* $u_h^k \to P_h u_\infty$*,* $\frac{p_h^{k+1}+p_h^{k-1}}{2} \to P_h p_\infty$*, and* $\phi_h^k \to P_h \phi_\infty$ *as* $k \to \infty$*.*

Proof. Note that the terms $\tilde{u}$ and $\tilde{\phi}$ lie in our finite element solutions spaces and obey (CNLF-SD) by linearity (3.10). Insert $\tilde{u}$ and $\tilde{\phi}$ into the inequality in Theorem 22. Let $N \to \infty$. The resulting inequality implies that $\|\nabla(w_h^{k+1}+w_h^{k-1})\| \to 0$ and $\|w_h^{k+1}-w_h^{k-1}\| \to 0$ for $w = \tilde{u}, \tilde{\phi}$ as $k \to \infty$. By the triangle and Poincaré's inequality,

$$
\begin{aligned}
\|w_h^{k+1}\| &= \tfrac{1}{2}\|w_h^{k+1} + w_h^{k-1} - w_h^{k-1} + w_h^{k+1}\| \\
&\leq \tfrac{1}{2}\|w_h^{k+1} + w_h^{k-1}\| + \tfrac{1}{2}\|w_h^{k+1} - w_h^{k-1}\| \\
&\leq \tfrac{1}{2}\left(C_P\|\nabla(w_h^{k+1} + w_h^{k-1})\| + \|w_h^{k+1} - w_h^{k-1}\|\right),
\end{aligned}
$$

for $w = \tilde{u}, \tilde{\phi}$. This implies both $\|\tilde{u}_h^{k+1}\|_f$ and $\|\tilde{\phi}_h^{k+1}\|_p$ converge to zero, meaning the Stokes velocity and Darcy pressure converge to the finite element projections of the equilibrium solutions. As for the Stokes pressure, since $\frac{\tilde{p}_h^{k+1}+\tilde{p}_h^{k-1}}{2} \in Q_f^h$, by ($LBB^h$), we have

$$
\begin{aligned}
\beta_h\left\|\frac{\tilde{p}_h^{k+1}+\tilde{p}_h^{k-1}}{2}\right\|_f &\leq \sup_{v_h \in X_f^h} \frac{b\left(v_h, \frac{\tilde{p}_h^{k+1}+\tilde{p}_h^{k-1}}{2}\right)}{\|\nabla v_h\|_f} \\
&= \sup_{v_h \in X_f^h} \frac{n\left(\frac{\tilde{u}_h^{k+1}-\tilde{u}_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{\tilde{u}_h^{k+1}+\tilde{u}_h^{k-1}}{2}, v_h\right) + c_I(v_h, \tilde{\phi}_h^k) - (\tilde{f}_f^\infty, v_h)_f}{\|\nabla v_h\|_f} \\
&\leq C\{\tfrac{1}{\Delta t}\|\tilde{u}_h^{k+1} - \tilde{u}_h^{k-1}\|_f + \|\nabla(\tilde{u}_h^{k+1} + \tilde{u}_h^{k-1})\|_f + \|\nabla\tilde{\phi}_h^k\|_p + \|\tilde{f}_f^k\|_{-1,f}\}.
\end{aligned}
$$

Let $k \to \infty$. Since $\lim_{k\to\infty} \tilde{w}_h^k = 0$ for $w = u, \phi$, and the forcing term $\|\tilde{f}_f^\infty\|_{-1,f} \to 0$, we have

$$\lim_{k\to\infty} \left\| \frac{\tilde{p}_h^{k+1}+\tilde{p}_h^{k-1}}{2} \right\|_f = 0.$$

□

3.2 ANALYSIS OF THE CONVERGENCE OF CNLF-SD

Error analysis of the (CNLF-SD) method over long-time intervals ensues. Recall that the FEM spaces, X_f^h, X_p^h and Q_f^h were chosen to satisfy approximation properties of piecewise polynomials of degree $r-1$, r, and $r+1$ as stated previously in (1.17). Since we assumed that X_f^h and Q_f^h satisfy (LBB^h), there exists some constant C such that if $u \in V$, where $V := \{v \in X_f : \nabla \cdot v = 0\}$, then

$$\inf_{v_h \in V_h} \|u - v_h\|_{1,f} \leq C \inf_{x_h \in X_f^h} \|u - x_h\|_{1,f}, \tag{3.11}$$

(see, for example, Girault and Raviart [34]). Let $N \in \mathbb{N}$ be given. Denote $t^n = n\Delta t$ and $T = N\Delta t$. If $T = \infty$ then $N = \infty$. In order to conduct error analysis for (CNLF-SD), define the following discrete-in-time norms:

$$|||v|||_{L^2(0,T;H^s(\Omega_{f,p}))} := \left(\sum_{k=1}^{N} \|v^k\|^2_{H^s(\Omega_{f,p})} \Delta t \right)^{1/2},$$

$$|||v|||_{L^\infty(0,T;H^s(\Omega_{f,p}))} := \max_{0 \leq k \leq N} \|v^k\|_{H^s(\Omega_{f,p})}.$$

To bound the consistency errors of (CNLF-SD) we will use the lemma below.

Lemma 25. *(Consistency Errors for (CNLF-SD)) The following inequalities hold:*

$$\Delta t \sum_{k=1}^{N-1} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 \leq \frac{(\Delta t)^4}{20} \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))},$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}\|_p^2 \leq \frac{(\Delta t)^4}{20} \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))},$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \leq \frac{7(\Delta t)^4}{6} \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))},$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \leq \frac{7(\Delta t)^4}{6} \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}.$$

Proof. Proofs for the second and fourth inequalities follow. The proofs for the other inequalities are similar. We prove the first inequality by integrating by parts twice and the Cauchy-Schwarz inequality.

$$\begin{aligned}
&\sum_{k=1}^{N-1} \left\| \phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p^2 \\
&= \frac{1}{4(\Delta t)^2} \int_{\Omega_p} \sum_{k=1}^{N-1} \left(\int_{t^k}^{t^{k+1}} (t - t^{k+1}) \phi_{tt} \, dt + \int_{t^{k-1}}^{t^k} (t - t^{k-1}) \phi_{tt} \, dt \right)^2 dx \\
&= \frac{1}{4(\Delta t)^2} \int_{\Omega_p} \sum_{k=1}^{N-1} \left(\int_{t^k}^{t^{k+1}} \frac{(t - t^{k+1})^2}{2} \phi_{ttt} \, dt + \int_{t^{k-1}}^{t^k} \frac{(t - t^{k-1})^2}{2} \phi_{ttt} \, dt \right)^2 dx \\
&\leq \frac{1}{4(\Delta t)^2} \int_{\Omega_p} \sum_{k=1}^{N-1} \frac{(\Delta t)^5}{20} \left(\int_{t^{k-1}}^{t^{k+1}} |\phi_{ttt}|^2 \, dt \right) dx \\
&\leq \frac{(\Delta t)^3}{20} \int_{\Omega_p} \int_0^T |\phi_{ttt}|^2 \, dt \, dx \leq \frac{(\Delta t)^3}{20} \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))}.
\end{aligned}$$

This next inequality is proved similarly.

$$\begin{aligned}
&\sum_{k=1}^{N-1} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 = \int_{\Omega_p} \sum_{k=1}^{N-1} \left| \nabla \left(\frac{\phi^k - \phi^{k+1}}{2} + \frac{\phi^k - \phi^{k-1}}{2} \right) \right|^2 dx \\
&= \tfrac{1}{4} \int_{\Omega_p} \sum_{k=1}^{N-1} |\nabla(\phi^k - \phi^{k+1}) + \nabla(\phi^k - \phi^{k-1})|^2 \, dx \\
&= \frac{1}{4} \int_{\Omega_p} \sum_{k=1}^{N-1} \left| \int_{t^{k+1}}^{t^k} \nabla \phi_t \, dt + \int_{t^{k-1}}^{t^k} \nabla \phi_t \, dt \right|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{4}\int_{\Omega_p}\sum_{k=1}^{N-1}\left|\int_{t^{k+1}}^{t^k}(t-t^k)'\nabla\phi_t dt+\int_{t^{k-1}}^{t^k}(t-t^k)'\nabla\phi_t dt\right|^2 dx\\
&= \tfrac{1}{4}\int_{\Omega_p}\sum_{k=1}^{N-1}\left|-\Delta t\int_{t^{k-1}}^{t^{k+1}}\nabla\phi_{tt}\,dt+\int_{t^k}^{t^{k+1}}(t-t^k)\nabla\phi_{tt}\,dt+\int_{t^{k-1}}^{t^k}(t^k-t)\nabla\phi_{tt}\,dt\right|^2 dx\\
&\le \tfrac{1}{2}\int_{\Omega_p}\sum_{k=1}^{N-1}\left((\Delta t)^3\int_{t^{k-1}}^{t^{k+1}}|\nabla\phi_{tt}|^2\,dt+\frac{(\Delta t)^3}{3}\int_{t^{k-1}}^{t^{k+1}}|\nabla\phi_{tt}|^2\,dt\right)dx\\
&\le \frac{7(\Delta t)^3}{6}\|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}.
\end{aligned}
$$

□

3.2.1 Convergence of the Stokes Velocity and Darcy Pressure

Proof of convergence of the Stokes velocity, u, and Darcy pressure, ϕ, with optimal rates over long-time intervals under condition (Δt_{CNLF}) follows. Denote the errors in the Stokes velocity and Darcy pressure variables by $\mathcal{E}_u^k = u^n - u_h^k$ and $\mathcal{E}_\phi^k = \phi^k - \phi_h^k$. Note that in the analysis of the convergence of this method we are not employing the projection operator, P_h, utilized in the convergence analysis of the splitting methods in Chapter 2.

Theorem 26. *(Convergence of (CNLF-SD)) Consider (CNLF-SD). Suppose that the time-step condition (Δt_{CNLF}) holds and u, p, ϕ satisfy the following regularity conditions:*

$$
\begin{aligned}
&u \in L^2(0,T;H^{r+2}(\Omega_f)) \cap L^\infty(0,T;H^{r+1}(\Omega_f)) \cap H^3(0,T;H^1(\Omega_f)),\\
&p \in L^2(0,T;L^2(\Omega_f)),\\
&\phi \in L^2(0,T;H^{r+2}(\Omega_p)) \cap L^\infty(0,T;H^{r+1}(\Omega_p)) \cap H^3(0,T;H^1(\Omega_p)).
\end{aligned}
$$

Then, for any $0 \le t_N \le \infty$, there is a positive constant, $\widehat{C}$, independent of the mesh width, h, and time-step size, Δt, such that for some $\alpha, \beta > 0$ there holds

$$\frac{\alpha}{2}(\|\mathcal{E}_u^N\|_f^2 + \|\mathcal{E}_u^{N-1}\|_f^2 + \frac{\beta}{2}(\|\mathcal{E}_\phi^N\|_p^2 + \|\mathcal{E}_\phi^{N-1}\|_p^2)$$
$$+ \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{4}\|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|_p^2\right)$$

$$\leq \widehat{C}\Big\{ \quad h^{2r}\Big\{ \ |||u|||^2_{L^2(0,T;H^{r+1}(\Omega_f))} + |||\phi|||^2_{L^2(0,T;H^{r+1}(\Omega_p))}\Big\}$$
$$+ h^{2r+2}\Big\{ \ \|u_t\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + \|\phi_t\|_{L^2(0,T;H^{r+1}(\Omega_p))} + |||u|||^2_{L^\infty(0,T;H^{r+1}(\Omega_f))}$$
$$+ |||\phi|||^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} + |||p|||^2_{L^2(0,T;H^{r+1}(\Omega_f))}\Big\}$$
$$+ (\Delta t)^4\Big\{ \ \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}$$
$$+ \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big\} \ + \Delta t(\|\nabla\mathcal{E}_u^1\|_f^2 + \|\nabla\mathcal{E}_u^0\|_f^2 + \|\nabla\mathcal{E}_\phi^1\|_p^2 + \|\nabla\mathcal{E}_\phi^0\|_p^2)$$
$$+ \|\mathcal{E}_u^1\|_f^2 + \|\mathcal{E}_u^0\|_f^2 + \|\mathcal{E}_\phi^1\|_p^2 + \|\mathcal{E}_\phi^0\|_p^2 \quad \Big\} \ .$$

Proof. Recall that solution $u^k = u(t^k)$ where $t^k = k\Delta t$, satisfies (FEM-SD). Consider (CNLF-SD) over the discretely divergence free space $V^h := \{v_h \in X_f^h : b(v_h, q_h) = 0 \,\forall q_h \in Q_f^h\}$ instead of X_f^h. Subtract (CNLF-SD) from (FEM-SD) evaluated at time t^k. Note that since $v_h \in V^h$, the term $b\left(v_h, \frac{p_h^{k+1}+p_h^{k-1}}{2}\right)$ equals zero, and can therefore be omitted from the equation. We have

$$n\left(u_t^k - \frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(u^k - \frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right)$$
$$-b\left(v_h, p^k\right) + c_I\left(v_h, \phi^k - \phi_h^k\right) = 0,$$
$$gS_0\left(\phi_t^k - \frac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\phi^k - \frac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - c_I\left(u^k - u_h^k, \psi_h\right) = 0.$$

Since v_h is discretely divergence free, $b\left(v_h, p^k\right) = b(v_h, p^k - \lambda_h^k)$, for any $\lambda_h \in Q_f^h$. After rearranging terms, the error equations become

$$n\left(\frac{\mathcal{E}_u^{k+1}-\mathcal{E}_u^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{\mathcal{E}_u^{k+1}+\mathcal{E}_u^{k-1}}{2}, v_h\right) + c_I\left(v_h, \mathcal{E}_\phi^k\right)$$
$$= n\left(\frac{u^{k+1}-u^{k-1}}{2\Delta t}, v_h\right)_f - n(u_t^k, v_h)_f - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}, v_h\right) + b\left(v_h, p^k - \lambda_h^k\right),$$

$$gS_0\left(\frac{\mathcal{E}_\phi^{k+1}-\mathcal{E}_\phi^{k-1}}{2\Delta t},\psi_h\right)_p + a_p\left(\frac{\mathcal{E}_\phi^{k+1}+\mathcal{E}_\phi^{k-1}}{2},\psi_h\right) - c_I\left(\mathcal{E}_u^k,\psi_h\right)$$
$$= gS_0\left(\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\psi_h\right)_p - gS_0(\phi_t^k,\psi_h)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2},\psi_h\right).$$

The consistency errors are:

$$\tau_f^k(v_h) = n\left(\frac{u^{k+1}-u^{k-1}}{2\Delta t},v_h\right)_f - n(u_t^k,v_h)_f - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2},v_h\right),$$
$$\tau_p^k(\psi_h) = gS_0\left(\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\psi_h\right)_p - gS_0(\phi_t^k,\psi_h)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2},\psi_h\right).$$

Split the error terms into:

$$\mathcal{E}_u^{k+1} = u^{k+1} - u_h^{k+1} = (u^{k+1} - \tilde{u}^{k+1}) + (\tilde{u}^{k+1} - u_h^{k+1}) = \eta_u^{k+1} + \xi_u^{k+1},$$
$$\mathcal{E}_\phi^{k+1} = \phi^{k+1} - \phi_h^{k+1} = (\phi^{k+1} - \tilde{\phi}^{k+1}) + (\tilde{\phi}^{k+1} - \phi_h^{k+1}) = \eta_\phi^{k+1} + \xi_\phi^{k+1}.$$

Take $\tilde{u}^{k+1} \in V^h$ and $\tilde{\phi}^{k+1} \in X_p^h$ so that $\xi_u^{k+1} \in V^h$. Rearranging error equations produces

$$n\left(\frac{\xi_u^{k+1}-\xi_u^{k-1}}{2\Delta t},v_h\right)_f + a_f\left(\frac{\xi_u^{k+1}+\xi_u^{k-1}}{2},v_h\right) + c_I(v_h,\xi_\phi^k)$$
$$= -n\left(\frac{\eta_u^{k+1}-\eta_u^{k-1}}{2\Delta t},v_h\right)_f - a_f\left(\frac{\eta_u^{k+1}+\eta_u^{k-1}}{2},v_h\right)$$
$$- c_I(v_h,\eta_\phi^k) + \tau_f^k(v_h) + b\left(v_h,p^k - \lambda_h^k\right),$$
$$gS_0\left(\frac{\xi_\phi^{k+1}-\xi_\phi^{k-1}}{2\Delta t},\psi_h\right)_p + a_p\left(\frac{\xi_\phi^{k+1}+\xi_\phi^{k-1}}{2},\psi_h\right) - c_I(\xi_u^k,\psi_h)$$
$$= -gS_0\left(\frac{\eta_\phi^{k+1}-\eta_\phi^{k-1}}{2\Delta t},\psi_h\right)_p - a_p\left(\frac{\eta_\phi^{k+1}+\eta_\phi^{k-1}}{2},\psi_h\right)$$
$$+ c_I(\eta_u^k,\psi_h) + \tau_p^k(\psi_h).$$

Choose $v_h = \xi_u^{k+1} + \xi_u^{k-1} \in V^h$ and $\psi_h = \xi_\phi^{k+1} + \xi_\phi^{k-1} \in X_p^h$ and add both error equations to find

$$\frac{1}{2\Delta t}\left(n\|\xi_u^{k+1}\|_f^2 + gS_0\|\xi_\phi^{k+1}\|_p^2 - n\|\xi_u^{k-1}\|_f^2 - gS_0\|\xi_\phi^{k-1}\|_p^2\right)$$
$$+ \left[c_I(\xi_u^{k+1} + \xi_u^{k-1},\xi_\phi^k) - c_I(\xi_u^k,\xi_\phi^{k+1} + \xi_\phi^{k-1})\right]$$
$$+\frac{1}{2}\left[a_f(\xi_u^{k+1} + \xi_u^{k-1},\xi_u^{k+1} + \xi_u^{k-1}) + a_p(\xi_\phi^{k+1} + \xi_\phi^{k-1},\xi_\phi^{k+1} + \xi_\phi^{k-1})\right]$$

$$= -\tfrac{1}{2\Delta t}\left[n\left(\eta_u^{k+1}-\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}\right)_f + gS_0\left(\eta_\phi^{k+1}-\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1}\right)_p\right]$$
$$-\tfrac{1}{2}\left[a_f\left(\eta_u^{k+1}+\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}\right)+a_p\left(\eta_\phi^{k+1}+\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1}\right)\right]$$
$$-\left[c_I(\xi_u^{k+1}+\xi_u^{k-1},\eta_\phi^k)-c_I(\eta_u^k,\xi_\phi^{k+1}+\xi_\phi^{k-1})\right]$$
$$+\tau_f^k(\xi_u^{k+1}+\xi_u^{k-1})+b\left(\xi_u^{k+1}+\xi_u^{k-1},p^k-\lambda_h^k\right)+\tau_p^k(\xi_\phi^{k+1}+\xi_\phi^{k-1}).$$

Split the coupled terms on the left hand side in the following way:

$$\begin{aligned}
c_I(\xi_u^{k+1}+\xi_u^{k-1},\xi_\phi^k)-&c_I(\xi_u^k,\xi_\phi^{k+1}+\xi_\phi^{k-1})\\
&=\left(c_I(\xi_u^{k+1},\xi_\phi^k)-c_I(\xi_u^k,\xi_\phi^{k+1})\right)-\left(c_I(\xi_u^k,\xi_\phi^{k-1})-c_I(\xi_u^{k-1},\xi_\phi^k)\right)\\
&=C_\xi^{k+\frac{1}{2}}-C_\xi^{k-\frac{1}{2}}.
\end{aligned}$$

Denote the ξ energy terms by

$$E_\xi^{k+1/2} := n\|\xi_u^{k+1}\|_f^2+gS_0\|\xi_\phi^{k+1}\|_p^2+n\|\xi_u^k\|_f^2+gS_0\|\xi_\phi^k\|_p^2.$$

Applying the coercivity of $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ we have

$$\begin{aligned}
E_\xi^{k+1/2}&+2\Delta t C_\xi^{k+\frac{1}{2}}-E_\xi^{k-1/2}-2\Delta t C_\xi^{k-\frac{1}{2}}\\
&\qquad+\Delta t\left(n\nu\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2+gk_{min}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1}\|_p^2\right)\\
&\le-\left[n\left(\eta_u^{k+1}-\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}\right)_f+gS_0\left(\eta_\phi^{k+1}-\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1}\right)_p\right]\\
&\quad-\Delta t\left[a_f\left(\eta_u^{k+1}+\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}\right)+a_p\left(\eta_\phi^{k+1}+\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1}\right)\right]\\
&\quad-2\Delta t\left[c_I(\xi_u^{k+1}+\xi_u^{k-1},\eta_\phi^k)-c_I(\eta_u^k,\xi_\phi^{k+1}+\xi_\phi^{k-1})\right]\\
&\quad+2\Delta t\left(\tau_f^k(\xi_u^{k+1}+\xi_u^{k-1})+b(\xi_u^{k+1}+\xi_u^{k-1},p^k-\lambda_h^k)+\tau_p^k(\xi_\phi^{k+1}+\xi_\phi^{k-1})\right).
\end{aligned}$$

Now bound the right hand side of the inequality from above. Begin by bounding the first term on the right using the standard Cauchy-Schwarz, Poincaré (Lemma 2), and Young inequalities.

$$\begin{aligned}
n(\eta_u^{k+1}-\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1})_f&+gS_0(\eta_\phi^{k+1}-\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1})_p\\
&\le\tfrac{3nC_{P,f}^2}{\nu\Delta t}\|\eta_u^{k+1}-\eta_u^{k-1}\|_f^2+\tfrac{5gS_0^2C_{P,p}^2}{2k_{min}\Delta t}\|\eta_\phi^{k+1}-\eta_\phi^{k-1}\|_p^2\\
&\quad+\Delta t\tfrac{n\nu}{12}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2+\Delta t\tfrac{gk_{min}}{10}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2.
\end{aligned}$$

Next, we apply the continuity of the bilinear forms $a_f(.,.)$ and $a_p(.,.)$ to bound the second term on the right.. Using the notation, M_f, from Lemma 3, we obtain

$$
\begin{aligned}
a_f(\eta_u^{k+1} + \eta_u^{k-1}\xi_u^{k+1} + \xi_u^{k-1}) &+ a_p(\eta_\phi^{k+1} + \eta_\phi^{k-1}, \xi_\phi^{k+1} + \xi_\phi^{k-1}) \\
&\leq M_f \|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f \\
&\quad + g k_{max} \|\nabla(\eta_\phi^{k+1} + \eta_\phi^{k-1})\|_p \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p \\
&\leq \tfrac{3M_f^2}{n\nu} \|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f^2 + \tfrac{5 g k_{max}^2}{2k_{min}} \|\nabla(\eta_\phi^{k+1} + \eta_\phi^{k-1})\|_p^2 \\
&\quad + \tfrac{n\nu}{12} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2 + \tfrac{g k_{min}}{10} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^2.
\end{aligned}
$$

We bound the coupled terms on the right hand side using Lemmas 1 (Trace), 2 (Poincaré), and Young's inequalities. Let $C = C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f} C_{P,p} g^2$. Then

$$
\begin{aligned}
&c_I(\xi_u^{k+1} + \xi_u^{k-1}, \eta_\phi^k) - c_I(\eta_u^k, \xi_\phi^{k+1} + \xi_\phi^{k-1}) \\
&\quad \leq ng \left(\|(\xi_u^{k+1} + \xi_u^{k-1}) \cdot \hat{n}_f\|_I \|\eta_\phi^k\|_I + \|\eta_u^k \cdot \hat{n}_f\|_I \|\xi_\phi^{k+1} + \xi_\phi^{k-1}\|_I \right) \\
&\quad \leq C_{\Omega_f} C_{\Omega_p} ng \big(\|\xi_u^{k+1} + \xi_u^{k-1}\|_f^{1/2} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^{1/2} \|\eta_\phi^k\|_p^{1/2} \|\nabla \eta_\phi^k\|_p^{1/2} \\
&\qquad + \|\xi_\phi^{k+1} + \xi_\phi^{k-1}\|_p^{1/2} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^{1/2} \|\eta_u^k\|_f^{1/2} \|\nabla \eta_u^k\|_f^{1/2} \big) \\
&\quad \leq n\sqrt{C} \left(\|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f \|\nabla \eta_\phi^k\|_p + \|\nabla \eta_u^k\|_f \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p \right) \\
&\quad \leq \tfrac{6nC}{\nu} \|\nabla \eta_u^k\|_f^2 + \tfrac{5n^2 C}{g k_{min}} \|\nabla \eta_\phi^k\|_p^2 \\
&\qquad + \tfrac{n\nu}{24} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2 + \tfrac{g k_{min}}{20} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^2.
\end{aligned}
$$

Finally, we bound the consistency errors, τ_f^k and τ_p^k, and the pressure term as follows.

$$
\begin{aligned}
\tau_f^k(\xi_u^{k+1} + \xi_u^{k-1}) &= n \left(\tfrac{u^{k+1} - u^{k-1}}{2\Delta t}, \xi_u^{k+1} + \xi_u^{k-1} \right) - n(u_t^k, \xi_u^{k+1} + \xi_u^{k-1})_u \\
&\quad - a_f \left(u^k - \tfrac{u^{k+1} + u^{k-1}}{2}, \xi_u^{k+1} + \xi_u^{k-1} \right) \\
&\leq n \left\| u_t^k - \tfrac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f \|\xi_u^{k+1} + \xi_u^{k-1}\|_f \\
&\quad + M_f \left\| \nabla \left(u^k - \tfrac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f \\
&\leq \tfrac{6 C_{P,f}^2 n}{\nu} \left\| u_t^k - \tfrac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 + \tfrac{6 n M_f^2}{\nu} \left\| \nabla \left(u^k - \tfrac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \\
&\quad + \tfrac{n\nu}{12} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2,
\end{aligned}
$$

$$
\begin{aligned}
\tau_p^k(\xi_\phi^{k+1}+\xi_\phi^{k-1}) &= gS_0\left(\tfrac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}, \xi_\phi^{k+1}+\xi_\phi^{k-1}\right)_p - gS_0(\phi_t^k, \xi_\phi^{k+1}+\xi_\phi^{k-1})_p \\
&\qquad - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}, \xi_\phi^{k+1}+\xi_\phi^{k-1}\right) \\
&\le gS_0\left\|\phi_t^k - \tfrac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p \|\xi_\phi^{k+1}+\xi_\phi^{k-1}\|_p \\
&\qquad + gk_{max}\left\|\nabla\left(\phi^k - \tfrac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p \|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \\
&\le \tfrac{5gS_0^2C_{P,p}^2}{k_{min}}\left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \tfrac{5gk_{max}}{k_{min}}\left\|\nabla\left(\phi^k - \tfrac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2 \\
&\qquad + \tfrac{gk_{min}}{10}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2,
\end{aligned}
$$

$$
\begin{aligned}
b\left(\xi_u^{k+1}+\xi_u^{k-1}, p^k-\lambda_h^k\right) &\le n\|p^k-\lambda_h^k\|_f \|\nabla\cdot(\xi_u^{k+1}+\xi_u^{k-1})\|_f \\
&\le \tfrac{6nd}{\nu}\|p^k-\lambda_h^k\|_f^2 + \tfrac{n\nu}{24}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2.
\end{aligned}
$$

Having bounded each term on the right hand side from above, we now subsume the ξ terms on the right into the left hand side of the inequality to obtain, for some $\mathcal{C}_0 > 0$,

$$
\begin{aligned}
&E_\xi^{k+\frac{1}{2}} + 2\Delta t C_\xi^{k+\frac{1}{2}} - E_\xi^{k-\frac{1}{2}} - 2\Delta t C_\xi^{k-\frac{1}{2}} \\
&\qquad\qquad + \Delta t\left(\tfrac{n\nu}{2}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2 + \tfrac{gk_{min}}{2}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2\right) \\
&\le \mathcal{C}_0\Big\{(\Delta t)^{-1}\left\{\|\eta_u^{k+1}-\eta_u^{k-1}\|_f^2 + \|\eta_\phi^{k+1}-\eta_\phi^{k-1}\|_p^2\right\} \\
&\quad + \Delta t\Big\{\|\nabla(\eta_u^{k+1}+\eta_u^{k-1})\|_f^2 + \|\nabla(\eta_\phi^{k+1}+\eta_\phi^{k-1})\|_p^2 + \|\nabla\eta_u^k\|_f^2 \\
&\quad + \|\nabla\eta_\phi^k\|_p^2 + \left\|u_t^k - \tfrac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 \\
&\quad + \left\|\nabla\left(u^k - \tfrac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 + \|p^k-\lambda_h^k\|_f^2 \\
&\quad + \left\|\phi_t^k - \tfrac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \left\|\nabla\left(\phi^k - \tfrac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2\Big\}\Big\}.
\end{aligned}
$$

Sum the previous inequality from $k = 1, ..., N-1$. Then

$$
\begin{aligned}
&E_\xi^{N-\frac{1}{2}} + 2\Delta t C_\xi^{N-\frac{1}{2}} - E_\xi^{\frac{1}{2}} - 2\Delta t C_\xi^{\frac{1}{2}} \\
&\qquad\qquad + \Delta t \sum_{k=1}^{N-1} \left(\tfrac{n\nu}{2} \|\nabla \left(\xi_u^{k+1} + \xi_u^{k-1}\right)\|_f^2 + \tfrac{g k_{min}}{2} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^2 \right) \\
&\leq \mathcal{C}_0 \Bigg\{ (\Delta t)^{-1} \sum_{k=1}^{N-1} \left[\|\eta_u^{k+1} - \eta_u^{k-1}\|_f^2 + \|\eta_\phi^{k+1} - \eta_\phi^{k-1}\|_p^2 \right] \\
&+ \Delta t \sum_{k=1}^{N-1} \Bigg[\|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f^2 + \|\nabla(\eta_\phi^{k+1} + \eta_\phi^{k-1})\|_p^2 \\
&+ \|\nabla \eta_u^k\|_f^2 + \|\nabla \eta_\phi^k\|_p^2 + \tfrac{12 C_{P,f}^2}{\nu} \left\| u_t^k - \tfrac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 \\
&+ \left\| \nabla \left(u^k - \tfrac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 + \|p^k - \lambda_h\|_f^2 \\
&+ \left\| \phi_t^k - \tfrac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \right\|_p^2 + \left\| \nabla \left(\phi^k - \tfrac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \Bigg] \Bigg\}.
\end{aligned}
$$

Next bound this in terms of norms instead of summations. Using Cauchy-Schwarz and other basic inequalities, we bound the first term on the right hand side as follows.

$$
\begin{aligned}
\sum_{k=1}^{N-1} \|\eta_u^{k+1} - \eta_u^{k-1}\|_f^2 &= \sum_{k=1}^{N-1} \left\| \int_{t^{k-1}}^{t^{k+1}} \eta_{u,t} dt \right\|_f^2 \\
&\leq \sum_{k=1}^{N-1} \int_{\Omega_f} (2\Delta t) \int_{t^{k-1}}^{t^{k+1}} |\eta_{u,t}|^2 dt \quad dx \\
&\leq 4\Delta t \|\eta_{u,t}\|_{L^2(0,T;L^2(\Omega_f))}^2.
\end{aligned} \tag{3.12}
$$

We treat the second term similarly.

$$
\sum_{k=1}^{N-1} \|\eta_\phi^{k+1} - \eta_\phi^{k-1}\|_f^2 \leq 4\Delta t \|\eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2. \tag{3.13}
$$

We bound the remaining η terms using Cauchy-Schwartz and the discrete norms.

$$
\begin{aligned}
\sum_{k=1}^{N-1} \|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f^2 &\leq 2 \sum_{k=1}^{N-1} \left(\|\nabla \eta_u^{k+1}\|_f^2 + \|\nabla \eta_u^{k-1}\|_f^2 \right) \\
&\leq 4 \sum_{k=0}^{N} \|\nabla \eta_u^k\|_f^2 \leq 4(\Delta t)^{-1} \||\nabla \eta_u|\|_{L^2(0,T;L^2(\Omega_f))}^2,
\end{aligned} \tag{3.14}
$$

$$\sum_{k=1}^{N-1} \|\nabla(\eta_{\phi}^{k+1} + \eta_{\phi}^{k-1})\|_f^2 \leq 4(\Delta t)^{-1} \||\nabla \eta_{\phi}|\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{3.15}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_u^k\|_f^2 \leq (\Delta t)^{-1} \||\nabla \eta_u|\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{3.16}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_{\phi}^k\|_p^2 \leq (\Delta t)^{-1} \||\nabla \eta_{\phi}|\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{3.17}$$

$$\sum_{k=1}^{N-1} \|p^k - \lambda_h^k\|_f^2 \leq (\Delta t)^{-1} \||p - \lambda_h|\|^2_{L^2(0,T;L^2(\Omega_f))}. \tag{3.18}$$

Recall from the proof of stability that since (Δt_{CNLF}) holds, we have the following lower bound for the energy terms:

$$E_{\xi}^{N-1/2} + 2\Delta t C_{\xi}^{N-\frac{1}{2}} \geq \alpha(\|\xi_u^N\|_f^2 + \|\xi_u^{N-1}\|_f^2) + \beta(\|\xi_{\phi}^N\|_p^2 + \|\xi_{\phi}^{N-1}\|_p^2) > 0,$$

for some $\alpha, \beta > 0$. After applying bounds (3.12)-(3.18), along with Lemma 25, and absorbing all the constants into one constant, $\widehat{C}_1$, the inequality becomes

$$\begin{aligned}
&\alpha(\|\xi_u^N\|_f^2 + \|\xi_u^{N-1}\|_f^2) + \beta(\|\xi_{\phi}^N\|_p^2 + \|\xi_{\phi}^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{n\nu}{2} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2 + \frac{g k_{min}}{2} \|\nabla(\xi_{\phi}^{k+1} + \xi_{\phi}^{k-1})\|_p^2 \right) \\
&\quad \leq \widehat{C}_1 \Big\{ \|\eta_{u,t}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + \||\nabla \eta_u|\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad + \||\nabla \eta_{\phi}|\|^2_{L^2(0,T;L^2(\Omega_p))} + \||p - \lambda_h|\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad + (\Delta t)^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad\quad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) \Big\} + E_{\xi}^{1/2} + 2\Delta t C_{\xi}^{\frac{1}{2}}
\end{aligned} \tag{3.19}$$

Recall that $\mathcal{E}_u^N = u^N - u_h^N$ and $\mathcal{E}_{\phi}^N = \phi^N - \phi_h^N$. Use the triangle inequality on the error equation to split the error terms into terms of η and ξ.

$$\begin{aligned}
&\frac{\alpha}{2}(\|\mathcal{E}^N{}_u\|_f^2 + \|\mathcal{E}_u^{N-1}\|_f^2 + \frac{\beta}{2}(\|\mathcal{E}_\phi^N\|_p^2 + \|\mathcal{E}_\phi^{N-1}\|_p^2) \\
&\qquad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{4} \|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|_f^2 + \frac{g k_{min}}{4} \|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|_p^2 \right) \\
&\qquad \le \alpha(\|\xi_u^N\|_f^2 + \|\xi_u^{N-1}\|_f^2) + \beta(\|\xi_\phi^N\|_p^2 + \|\xi_\phi^{N-1}\|_p^2) \\
&\qquad\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2 + \frac{g k_{min}}{2} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^2 \right) \\
&\qquad\quad + \alpha(\|\eta_u^N\|_f^2 + \|\eta_u^{N-1}\|_f^2) + \beta(\|\eta_\phi^N\|_p^2 + \|\eta_\phi^{N-1}\|_p^2) \\
&\qquad\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2} \|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f^2 + \frac{g k_{min}}{2} \|\nabla(\eta_\phi^{k+1} + \eta_\phi^{k-1})\|_p^2 \right)
\end{aligned}$$

Note that $\|\eta_{u,\phi}^N\|_{f,p}^2, \|\eta_{u,\phi}^{N-1}\|_{f,p}^2 \le \||\eta_{u,\phi}|\|_{L^\infty(0,T;L^2(\Omega_{f,p}))}^2$. Using this, the previous bounds for η terms, applying inequality (3.19), and absorbing constants into a new constant, $\widehat{C}_2$ produces

$$\begin{aligned}
&\frac{\alpha}{2}(\|\mathcal{E}^N{}_u\|_f^2 + \|\mathcal{E}_u^{N-1}\|_f^2 + \frac{\beta}{2}(\|\mathcal{E}_\phi^N\|_p^2 + \|\mathcal{E}_\phi^{N-1}\|_p^2) \\
&\qquad + \Delta t \sum_{k=1}^{N-1} \left(\frac{n\nu}{4} \|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|_f^2 + \frac{g k_{min}}{4} \|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|_p^2 \right) \\
&\qquad \le \widehat{C}_2 \Big\{ \ \|\eta_{u,t}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2 + \||\nabla \eta_u|\|_{L^2(0,T;L^2(\Omega_f))}^2 \\
&\qquad\quad + \||\nabla \eta_\phi|\|_{L^2(0,T;L^2(\Omega_p))}^2 + \||p - \lambda_h|\|_{L^2(0,T;L^2(\Omega_f))}^2 \\
&\qquad + (\Delta t)^4 (\|u_{ttt}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\phi_{ttt}\|_{L^2(0,T;L^2(\Omega_p))}^2 \\
&\qquad\qquad + \|u_{tt}\|_{L^2(0,T;H^1(\Omega_f))}^2 + \|\phi_{tt}\|_{L^2(0,T;H^1(\Omega_p))}^2) \\
&\qquad\quad + \||\eta_u|\|_{L^\infty(0,T;L^2(\Omega_f))}^2 + \||\eta_\phi|\|_{L^\infty(0,T;L^2(\Omega_p))}^2 \Big\} \\
&\qquad\quad + \|\xi_f^1\|_f^2 + \|\xi_p^1\|_p^2 + \|\xi_f^0\|_f^2 + \|\xi_p^0\|_p^2 + 2\Delta t C_\xi^{1/2}.
\end{aligned} \tag{3.20}$$

Bound the coupled terms on the right hand side by

$$C_\xi^{1/2} \le \frac{C}{2} \left(\|\nabla \xi_\phi^0\|_p^2 + \|\nabla \xi_\phi^1\|_p^2 + \|\nabla \xi_u^0\|_f^2 + \|\nabla \xi_u^1\|_f^2 \right).$$

Since (3.20) holds for any $\tilde{u} \in V^h$, $\lambda_h \in Q_f^h$, and $\tilde{\phi} \in X_p^h$, we may take the infimum over V^h,

Q_f^h, and X_p^h. By (3.11), we may bound the infimum over V^h by the infimum over X_f^h so the following holds for some positive constant $\widehat{C}_3$:

$$
\begin{aligned}
&\frac{\alpha}{2}(\|\mathcal{E}_u^N\|_f^2 + \|\mathcal{E}_u^{N-1}\|_f^2 + \frac{\beta}{2}(\|\mathcal{E}_\phi^N\|_p^2 + \|\mathcal{E}_\phi^{N-1}\|_p^2) \\
&+ \Delta t \sum_{k=1}^{N-1} \left(\frac{n\nu}{4} \|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|_f^2 + \frac{gk_{min}}{4} \|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|_p^2 \right) \\
&\leq \widehat{C}_3 \Big\{ \inf_{\tilde{u}\in X_f^h} \Big\{ \|\eta_{u,t}\|^2_{L^2(0,T;L^2(\Omega_f))} + |||\eta_u|||^2_{L^2(0,T;H^1(\Omega_f))} + |||\eta_u|||^2_{L^\infty(0,T;L^2(\Omega_f))} \\
&\quad + \|\xi_u^1\|_f^2 + \|\xi_u^0\|_f^2 + \Delta t(\|\nabla \xi_u^1\|_f^2 + \|\nabla \xi_u^0\|_f^2) \Big\} + \inf_{\lambda_h \in Q_f^h} |||p - \lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\quad + \inf_{\tilde{\phi}\in X_p^h} \Big\{ \|\eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\eta_\phi|||^2_{L^2(0,T;H^1(\Omega_p))} + |||\eta_\phi|||^2_{L^\infty(0,T;L^2(\Omega_p))} \\
&\quad + (\|\xi_\phi^1\|_p^2 + \|\xi_\phi^0\|_p^2) + \Delta t(\|\nabla \xi_\phi^1\|_p^2 + \|\nabla \xi_\phi^0\|_p^2) \Big\} + (\Delta t)^4 \Big\{ \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&\quad + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big\} \Big\} .
\end{aligned}
$$

After applying the approximation assumptions (1.17) we get the final result. □

Corollary 27. *(Rates of Convergence) Let (X_f^h, Q_f^h) be the finite element spaces associated with Taylor-Hood elements and X_p^h be continuous piecewise quadratics. Suppose also that the assumptions of Theorem 26 hold. Then,*

$$
\begin{aligned}
&\tfrac{\alpha}{2}(\|\mathcal{E}_u^N\|_f^2 + \|\mathcal{E}_u^{N-1}\|_f^2 + \tfrac{\beta}{2}(\|\mathcal{E}_\phi^N\|_p^2 + \|\mathcal{E}_\phi^{N-1}\|_p^2) \\
&+ \Delta t \sum_{k=1}^{N-1} \left(\tfrac{n\nu}{4} \|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|_f^2 + \tfrac{gk_{min}}{4} \|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|_p^2 \right) \\
&\leq \widehat{C} \Big\{ h^4 \Big\{ |||u|||^2_{L^2(0,T;H^{r+1}(\Omega_f))} + |||\phi|||^2_{L^2(0,T;H^{r+1}(\Omega_p))} + |||p|||^2_{L^2(0,T;H^{r+1}(\Omega_f))} \\
&\quad + h^2 \|u_t\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + h^2 \|\phi_t\|_{L^2(0,T;H^{r+1}(\Omega_p))} + h^2 |||u|||^2_{L^\infty(0,T;H^{r+1}(\Omega_f))} \\
&\quad + h^2 |||\phi|||^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} \Big\} + (\Delta t)^4 \Big\{ \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\quad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big\} + \Delta t(\|\nabla \mathcal{E}_u^1\|_f^2 + \|\nabla \mathcal{E}_u^0\|_f^2 \\
&\quad + \|\nabla \mathcal{E}_\phi^1\|_p^2 + \|\nabla \mathcal{E}_\phi^0\|_p^2) + \|\mathcal{E}_u^1\|_f^2 + \|\mathcal{E}_u^0\|_f^2 + \|\mathcal{E}_\phi^1\|_p^2 + \|\mathcal{E}_\phi^0\|_p^2 \Big\} .
\end{aligned}
$$

3.3 NUMERICAL EXPERIMENTS FOR CNLF-SD

Numerical experiments verify the stability properties and predicted rates of convergence of (CNLF-SD). All tests use the same domain and exact solutions used in Chapter 2, chosen to satisfy the interface conditions and introduced by Mu and Zhu in [56]. Calculations were made using FreeFem++ software [36]. The code for the experiments is included in the appendix. We use Taylor-Hood elements (P2-P1) for the Stokes problem and piecewise quadratics (P2) for the Darcy problem. The choice of Taylor-Hood elements in the Stokes problem satisfies the (LBB^h) requirement for stability of the discrete pressure. The initial and first terms are chosen to correspond with the exact solutions, recalled below.

$$\begin{aligned}
&\Omega_f = (0,1)\times(1,2), \qquad \Omega_p = (0,1)\times(0,1), \qquad I = \{(x,1) : x \in (0,1)\}\\
&u(x,y,t) = \left((x^2(y-1)^2 + y)\cos(t), (\frac{2}{3}x(1-y)^3 + 2 - \pi\sin(\pi x))\cos(t)\right),\\
&\qquad p(x,y,t) = (2-\pi\sin(\pi x))\sin(\frac{\pi}{2}y)\cos(t),\\
&\qquad \phi(x,y,t) = (2-\pi\sin(\pi x))(1-y-\cos(\pi y))\cos(t).
\end{aligned} \tag{TEST}$$

3.3.1 Stability Experiments

We examine the stability region of CNLF applied to Stokes-Darcy (CNLF-SD) to determine if the CFL-type condition, (Δt_{CNLF}), derived in Theorem 20, is sharp. To begin, we must first determine the size of the constant, $\mathcal{C} = C_{\Omega_f} C_{\Omega_p} C_{(inv)} g$, in the stability condition, recalled below.

$$\begin{aligned}
\Delta t < \mathcal{C}^{-1}\max\{\, &\min\left\{h^2, gS_0 n^{-1}\right\}, \min\left\{n^{-1}h^2, gS_0\right\}, \min\left\{nh^2, gS_0 n^{-2}\right\}\\
&\min\left\{h, gS_0 n^{-1}h\right\}, \min\left\{n^{-1}h, gS_0 h\right\}, \min\left\{nh, gS_0 n^{-2}h\right\}\,\}.
\end{aligned} \tag{Δt_{CNLF}}$$

Recall that as a consequence to Theorem 22, when Δt satisfies (Δt_{CNLF}), both the stable and unstable modes are damped by (CNLF-SD). This means that any instabilities are due to

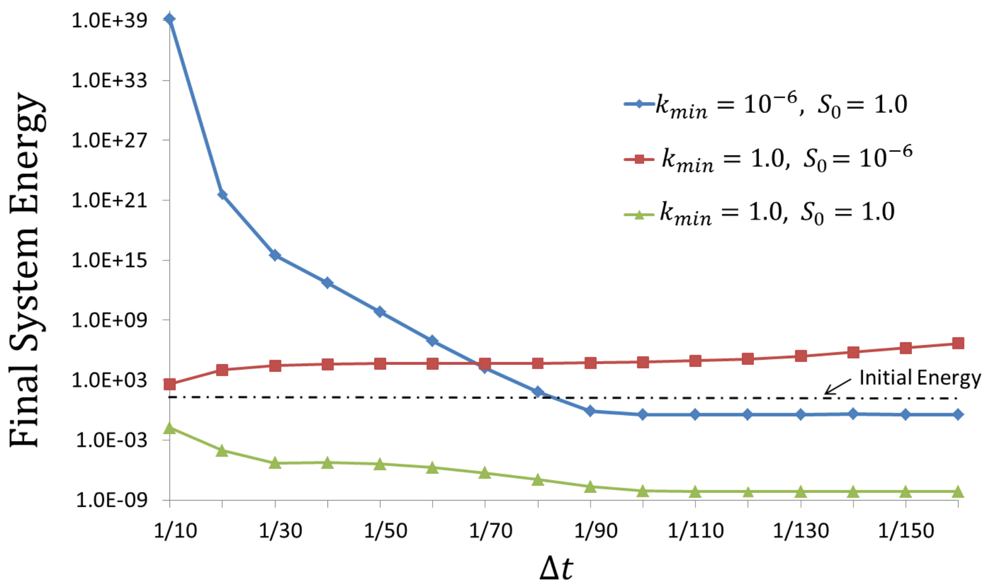

Figure 3.1: Final System Energy $(E(N))$ versus time-step size (Δt) for (CNLF-SD).

improper implementation, such as accumulation of round-off errors or violation of (Δt_{CNLF}). Therefore, to determine the size stability region (in particular, the size of $\mathcal{C}$ in (Δt_{CNLF})), we set the forcing terms equal to zero, enforce homogeneous Dirichlet boundary conditions, and eliminate the Crank-Nicolson terms which add numerical dissipation into the system. When the stability condition is met, the solution should decay to zero over time. Set the parameters n, g, α_{BJ}, ρ, and S_0 equal to 1. To eliminate the Crank-Nicolson terms, we set and $\nu = k_{min} = 0$, with the exception of $k_{min} = 10^{-6}$ in $a_f(.,.)$. We set $h = 0.1$ and calculate the system energy, $E^{N+1/2} = \|u_h^{N+1}\|_f^2 + \|u_h^N\|_h^2 + gS_0\|\phi_h^{N+1}\|_p^2 + gS_0\|\phi_h^N\|_p^2$, at the final time step, $T_{final} = 10$. Note that the initial energy of the system, $E^{1/2} \approx \mathcal{O}(10)$. Results of this series of experiments are summarized in Table 3.1. (CNLF-SD) becomes stable when $\Delta t < 1/110$. Using these results we estimate that $\mathcal{C} \approx 111/10 = \mathcal{O}(10)$.

The CFL-type condition (Δt_{CNLF}) implies sensitivity of CNLF to small values of S_0 but not k_{min}. We compute the final system energy over the time interval $[0, 10]$ for fixed $h = 0.1$ and successively smaller time-step sizes in four different situations: (1) $S_0 = k_{min} = 10^{-6}$,

Table 3.1: Stability Region CNLF

Δt	E_{final}	Δt	E_{final}	Δt	E_{final}
1/10	∞	1/60	∞	1/110	$2.7666E093$
1/20	∞	1/80	∞	1/111	1.60628
1/40	∞	1/100	∞	1/112	1.59992
				1/120	1.59636

(2) $S_0 = 1$ and $k_{min} = 10^{-6}$, (3) $S_0 = 10^{-6}$ and $k_{min} = 1$, and (4) $S_0 = k_{min} = 1$. Results are summarized in Figure 3.1 in a plot of final system energy, $E(N)$ against time-step size, Δt. Note the logarithmic scale. All tests for situation (1) resulted in energy blow-up before the conclusion of the time interval. Note that when $S_0 = 1$, we are guaranteed stability once $\Delta t < 1/110$. However, when $S_0 = 10^{-6}$, the stability condition requires $\Delta t < \mathcal{O}(10^{-6})$ as well, therefore the lack of stability in situation (2) is as predicted. For the case of small k_{min} (situation (2)), (CNLF-SD) becomes stable before we reach the boundary of the stability region, $\Delta t_{crit} \approx 1/110$. Recall that k_{min} affects the strength of the numerical diffusion in the system.

To illustrate with further details, see the break-down of the system energy and modes in Figures 3.2, 3.3, and 3.4. In Figure 3.2, $k_{min} = 10^{-6}$ and $S_0 = 1.0$. The small value of k_{min} greatly weakens the numerical dissipation in the system, and violation of (Δt_{CNLF}) leads to spurious oscillations in the unstable mode and a drastic blow-up in system energy. In Figure 3.3, $k_{min} = 1.0$ and $S_0 = 10^{-6}$. (Δt_{CNLF}) is violated and both modes and the energy exhibit growth as time progresses. Finally, in Figure 3.4, when $S_0 = k_{min} = 1.0$, the CFL-type condition holds and both modes and the system energy converge to zero as predicted.

3.3.2 CNLF-SD and Time Filtering

One popular technique in geophysics used to counteract the accumulation of numerical noise in the unstable mode induced by Leapfrog is to implement time-filters. We use the Robert-

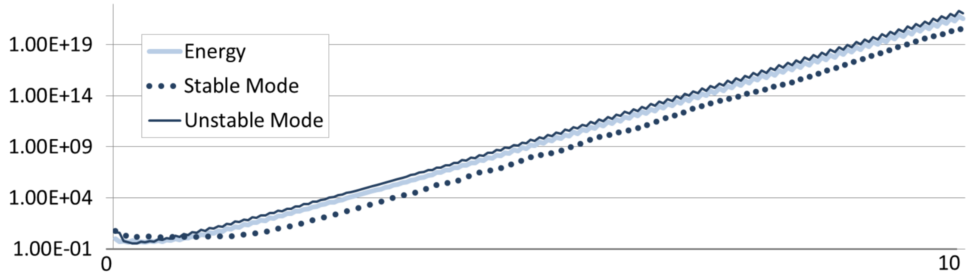

Figure 3.2: Break-down of the evolution of energy and modes of (CNLF-SD) over $[0, 10]$ for situation (2) ($k_{min} = 10^{-6}$ and $S_0 = 1.0$) when $\Delta t = 1/20$. (Δt_{CNLF}) is VIOLATED. Spurious oscillations in the unstable mode correspond to a drastic blow-up in energy.

Asselin Filter, or RA-filter ([62], [5]). At every time step, after computing $u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}$, we update the previous k^{th} values and replace them with filtered values, given below.

$$\overline{w}_h^k = w_h^k + \alpha(\overline{w}_h^{k-1} - 2w_h^k + w_h^{k+1}), \text{ where } w = u, p, \text{ or } \phi, 0 \leq \alpha \leq 1.$$

The RA-filter damps the computational mode in Leapfrog (see e.g. Durran [30]). Analysis of the RA-filter is still an interesting open problem. Some of the analytical theory of the related Robert-Asselin-Williams (RAW) time filter applied to (CNLF) is discussed in [39]. For this test set $\alpha = 0.10$. For more discussion on the choice of the parameter α see, for example [40] p. 437. To see if the addition of an RA-filter step mitigates the (CNLF-SD)

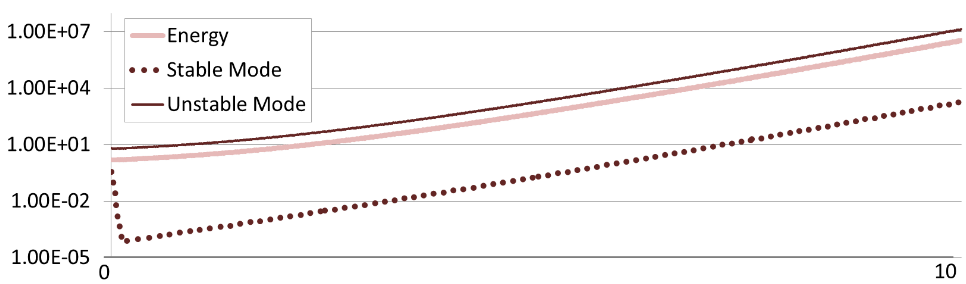

Figure 3.3: Break-down of the evolution of energy and modes of (CNLF-SD) over $[0, 10]$ for situation (3) ($k_{min} = 1.0$ and $S_0 = 10^{-6}$) when $\Delta t = 1/80$. (Δt_{CNLF}) is VIOLATED and the energy along with both modes increases as time progresses, leading to an unstable system.

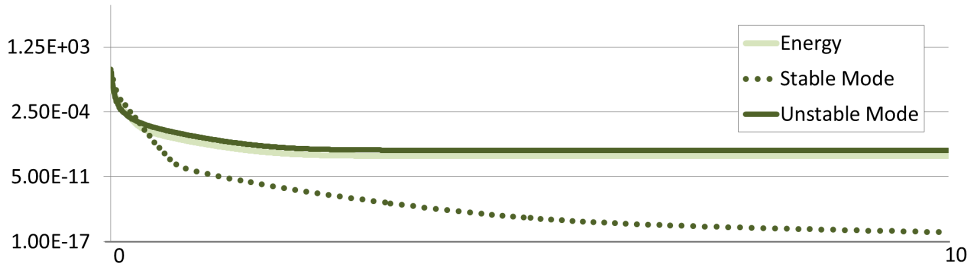

Figure 3.4: Break-down of the evolution of energy and modes of (CNLF-SD) over $[0, 10]$ for situation (4) ($k_{min} = S_0 = 1.0$) when $\Delta t = 1/120$. (Δt_{CNLF}) HOLDS and the energy along with both modes decay to zero.

method's sensitivity to small values of specific storage, we perform the same stability tests for the four situations as previously described. Results are shown in Figure. For further illustration we match the situation illustrated in Figure 3.3 by setting $h = 0.1$, $\Delta t = 1/80$, $S_0 = 10^{-6}$, and $k_{min} = 1.0$. In contrast to (CNLF-SD), with the addition of the RA-filter, the energy, stable, and unstable modes decay rapidly to zero (note that the time interval is $[0, 1]$ instead of $[0, 10]$), as seen in Figure 3.6. While the RA-filter appears to greatly enlarge that stability region of (CNLF-SD), because the RA-filter step is only first-order, we lose second-order accuracy in time. In the following chapters we will investigate an alternate route to gain stability and still preserve higher-order accuracy.

3.3.3 Convergence Rate Experiments

For the convergence rate experiments all parameters, n, α_{BJ}, ν, S_0, κ, ρ and $g = 1.0$. We set the boundary condition on the problem to be inhomogeneous Dirichlet: $u_h = u$ on $\partial\Omega_f / I$, and similar for the Darcy pressure, ϕ. We set the mesh size, h, equal to the time step, Δt. While this violates the CFL-type condition for long-time stability, (CNLF-SD) is (short-term) stable over $[0, 1]$ for these choices of parameters due to the numerical dissipation. The errors for various values of h are given in Table 3.2. We denote $L^\infty(0, 1; L^2(\Omega_{f,p}))$ by $L^\infty_{f,p}$. The rates of convergence in the table exhibit second order convergence for u and ϕ. This agrees with the error analysis for the Taylor-Hood elements as evidenced in Corollary 27.

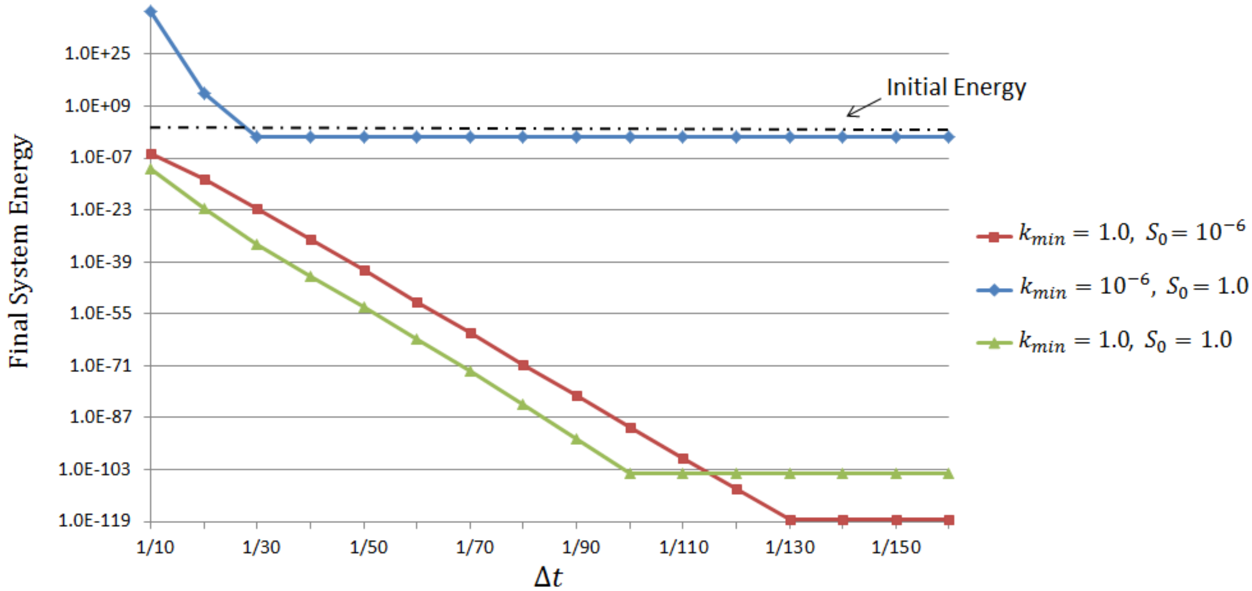

Figure 3.5: (CNLF-SD) + **RA-Filter**. Final System Energy ($E(N)$) versus time-step size (Δt) for (CNLF-SD) with the RA-filter step.

Remark 28. *(Convergence of the Stokes Pressure) The expected rate of convergence for the (average) pressure is* $O(\Delta t^2 + h^2)$*. We omit the proof of this estimate due to length. However, even though the problem is linear, the proof has a few unexpected points. One has to first bound the error in the time differences. Then, the average pressure error is bounded in terms of the errors in* u*,* ϕ*, and their time differences using the discrete inf-sup condition (*LBB^h*). In order to complete the first step one must use a discrete Gronwall inequality. As a result, the predicted errors in the time differences and thus the pressure contain a multiplier of the form* $exp(aT)$*.*

3.4 CONCLUSION FOR CNLF-SD

(CNLF-SD) is a parallel partitioned method that allows one to implement existing black box solvers optimized for surface and groundwater flow, thus preserving the physics of the coupled problem and allowing us to utilize existing computational tools. Analysis of the

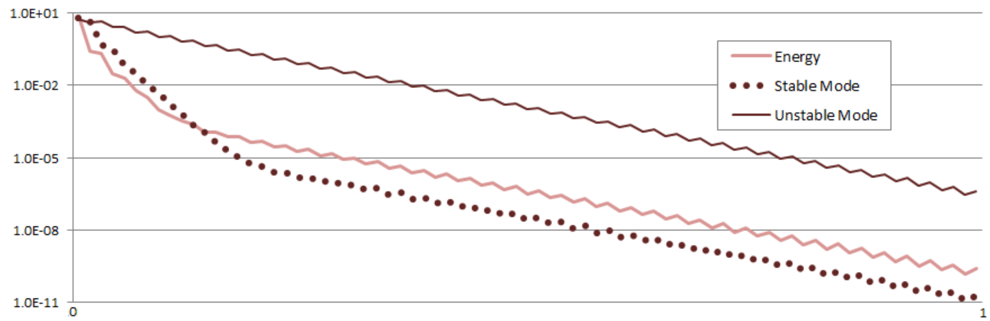

Figure 3.6: CNLF-SD with the RA-filter. Compare to Figure 3.3, with $\Delta t = 1/80$, $k_{min} = 1.0$, and $S_0 = 10^{-6}$. While (Δt_{CNLF}) is violated, the energy, unstable, and stable modes decay rapidly to zero in the presence of the RA-filter.

Table 3.2: Rates of Convergence for the Stokes velocity, pressure, and Darcy pressure in (CNLF-SD).

$h = \Delta t$	$\|\|u - u^h\|\|_{L_f^\infty}$	rate	$\|\|p - p^h\|\|_{L_f^\infty}$	rate	$\|\|\phi - \phi^h\|\|_{L_p^\infty}$	rate
$\frac{1}{10}$	8.62671e-4		1.56045e-1		6.54407e-3	
$\frac{1}{20}$	1.77135e-4	2.28	3.77064e-2	2.05	1.46515e-3	2.16
$\frac{1}{40}$	3.54644e-5	2.32	8.9672e-3	2.07	3.4904e-4	2.07
$\frac{1}{80}$	6.72106e-6	2.40	2.15951e-3	2.05	8.70886e-5	2.00

Crank-Nicolson Leapfrog method applied to the Stokes-Darcy equations lead to a CFL-type condition, (Δt_{CNLF}), sufficient for stability and convergence. Under this time-step restriction, both the stable and unstable modes arising from Leapfrog are controlled.

However, the sensitivity of this condition to small values of S_0, is restrictive in cases of confined aquifers since in such cases S_0 is very small (as small as 10^{-6} as seen in Table 1.2). Numerical experiments confirmed sensitivity of (CNLF-SD) to small values S_0. Additional dissipation from the use of Crank-Nicolson on the diffusive terms adds (temporary) stability to the system, but over time if the CFL-type condition is violated, the system will eventually destabilize. Implementing a time-filter such as the RA-filter enlarges the stability region, although it will decrease the order of convergence since the RA-filter step is only first-order. The convergence analysis and numerical experiments confirms that this method is second-order in time and space.

The method, (CNLF-SD), is higher-order convergent but does not exhibit the desired strong stability properties. In the next chapters we develop and analyze an adaptation of this method, (CNLFstab-SD), which maintains the second-order convergence of (CNLF-SD), yet is unconditionally stable.

4.0 STABILIZED CNLF FOR EVOLUTION EQUATIONS (CNLFSTAB)

In this chapter, we present a stabilized version of the Crank-Nicolson Leapfrog, (CNLF), method for a general evolution equation of the form

$$w_t + Aw + \Lambda w = 0, \tag{4.1}$$

where A is symmetric-positive-definite (SPD) and Λ is linear, skew-symmetric ($\Lambda^T = -\Lambda$) and bounded. This system is similar to the system obtained by applying the Finite Element Method to the Stokes-Darcy variational equations (FEM-SD) as described in the end of Chapter 1. Applying the Crank-Nicolson Leapfrog (CNLF) method to the general evolution equation in (4.1) produces

$$\frac{w^{k+1} - w^{k-1}}{2\Delta t} + A\left(\frac{w^{k+1} + w^{k-1}}{2}\right) + \Lambda w^k = 0. \tag{CNLF}$$

In [49, 38], stability analysis for Crank-Nicolson Leapfrog (CNLF) for general and coupled evolution equations produced a stability condition of the form $\Delta t\|\Lambda\| < 1$. This chapter presents a stabilized version of (CNLF) for the general evolution equation that is unconditionally stable as well as second-order convergent. It is both a summary and expansion of results obtained in [41, 66].

The stabilized version of (CNLF), denoted by (CNLFSTAB), is

$$\frac{w^{k+1} - w^{k-1}}{2\Delta t} + \mathbf{\Delta t \Lambda^* \Lambda (w^{k+1} - w^{k-1})} + A\left(\frac{w^{k+1} - w^{k-1}}{2}\right) + \Lambda w^k = 0. \tag{CNLFSTAB}$$

The stabilization term, $\mathbf{\Delta t \Lambda^* \Lambda (w^{k+1} - w^{k-1})}$, is both linear and SPD. It contributes an added consistency error of $2\Delta t^2 \Lambda^* \Lambda w_t = \mathcal{O}(\Delta t^2)$, the same size as the consistency error in (CNLF); therefore it preserves second-order convergence. The motivation of the stabilization term arises from the energy stability analysis of (CNLF), given in Theorem 29. This stabilized version of (CNLF) is similar to methods studied in [3, 44, 29, 20].

Let $< .,. >$ denote the Euclidean inner product and $||.||$ the Euclidean norm. Since A is SPD, the norm given by $\|w\|_A = \sqrt{< w, Aw >}$ is well-defined.

Theorem 29 (Stability of (CNLF) for General Evolution Equation)**.** *Suppose Δt satisfies $\Delta t\|\Lambda\| < 1$, then (CNLF) is stable. That is, for $N \geq 1$*

$$(1 - \|\Lambda\|\Delta t)\left(\|w^{N+1}\|^2 + \|w^N\|^2\right) + \sum_{k=1}^{N-1} \|w^{k+1} + w^{k-1}\|_A^2 = 0$$

Proof. The following proof was first presented in [49], and has been adapted to fit the notation of this research. As in Theorem 20, take the inner product of (CNLF) with its stable mode, $w^{k+1} + w^{k-1}$ and multiply by $2\Delta t$. This produces

$$\|w^{k+1}\|^2 - \|w^{k-1}\|^2 + \Delta t\|w^{k+1} + w^{k-1}\|_A^2 + 2\Delta t < w^{k+1} + w^{k-1}, \Lambda w^k >= 0.$$

Similar to the proof of stability for (CNLF-SD), we define the following:

$$E^{k+1/2} = \|w^{k+1}\|^2 + \|w^k\|^2,$$
$$C^{k+1/2} =< w^{k+1}, \Lambda w^k > .$$

Simplify and use the skew-symmetry of Λ to obtain

$$E^{k+1/2} + 2\Delta t C^{k+1/2} - E^{k-1/2} - 2\Delta t C^{k-1/2} + \Delta t\|w^{k+1} + w^{k-1}\|_A^2 = 0.$$

Next, sum from $k = 1$ to $N - 1$:

$$E^{N-1/2} + 2\Delta t C^{N-1/2} + \sum_{k=1}^{N-1} \|w^{k+1} + w^{k-1}\|_A^2 = E^{1/2} + 2\Delta t C^{1/2}.$$

Stability thus holds if $E^{N-1/2}+2\Delta t C^{N-1/2} > 0$. By Cauchy-Schwarz and Young's inequality,

$$C^{N+1/2} =< w^{N+1}, \Lambda w^N > \leq \|w^{N+1}\|\|\Lambda\|\|w^N\| \leq \tfrac{\|\Lambda\|}{2}(\|w^{N+1}\|^2 + \|w^N\|^2),$$

and thus $E^{N-1/2} + 2\Delta t C^{N-1/2} > 0$ provided $1 - \Delta t\|\Lambda\| > 0$, or $\Delta t\|\Lambda\| < 1$. □

Notice that the stability condition arises out of the need to subsume the skew-symmetric term, $(w^{k+1} + w^{k-1})^T \Lambda w^k$, into the discrete system energy. The added stability term, $\mathbf{\Delta t \Lambda^* \Lambda (w^{k+1} - w^{k-1})}$, provides an alternative for the absorption of the skew-symmetric term by providing anadditional SPD term in the stability equation specifically suited for this purpose.

4.1 STABILITY OF (CNLFSTAB)

Theorem 30. *The method* (CNLFSTAB) *is unconditionally stable. That is, for every* $N \geq 1$

$$\begin{aligned} \tfrac{1}{2}||w^{N+1}||^2 + ||w^N||^2 + 2\Delta t^2 ||\Lambda w^{N+1}||^2 \\ \leq \|w^1\|^2 + \|w^0\|^2 + 2\Delta t \left\langle \Lambda w^0, w^1 \right\rangle + 2\Delta t^2 (\|\Lambda w^1\|^2 + \|\Lambda w^2\|^2). \end{aligned} \tag{4.2}$$

Proof. Begin like the proof of Theorem 29. Take the inner product of (CNLFSTAB) with its stable mode, $(w^{k+1} + w^{k-1})$, and multiply through by $2\Delta t$. This produces

$$\begin{aligned} (||w^{k+1}||^2 + ||w^k||^2) - (||w^k||^2 + ||w^{k-1}||^2) + \\ +2\Delta t^2 \left\langle \Lambda^* \Lambda \left(w^{k+1} - w^{k-1} \right), w^{k+1} + w^{k-1} \right\rangle + \\ +\Delta t \left\langle A(w^{k+1} + w^{k-1}), w^{k+1} + w^{k-1} \right\rangle + 2\Delta t \left\langle \Lambda w^k, w^{k+1} + w^{k-1} \right\rangle = 0. \end{aligned}$$

Rewrite the added stability term as follows

$$\begin{aligned} 2\Delta t^2 \left\langle \Lambda^* \Lambda \left(w^{k+1} - w^{k-1} \right), w^{k+1} + w^{k-1} \right\rangle &= 2\Delta t^2 \langle \Lambda \left(w^{k+1} - w^{k-1} \right), \Lambda (w^{k+1} + w^{k-1}) \rangle \\ &= 2\Delta t^2 \left(||\Lambda w^{k+1}||^2 - ||\Lambda w^{k-1}||^2 \right) \\ = 2\Delta t^2 \left[\left(||\Lambda w^{k+1}||^2 \right. \right. &+ ||\Lambda w^k||^2 - \left(||\Lambda w^k||^2 + ||\Lambda w^{k-1}||^2 \right). \end{aligned}$$

Denote the stabilized system energy by

$$E_{\text{STAB}}^{k+1/2} := ||w^{k+1}||^2 + ||w^k||^2 + 2\Delta t^2 \left(||\Lambda w^{k+1}||^2 + ||\Lambda w^k||^2 \right).$$

Thus the stability equation becomes

$$E_{\text{STAB}}^{k+1/2} - E_{\text{STAB}}^{k-1/2} + \Delta t \left\langle A(w^{k+1} + w^{k-1}), w^{k+1} + w^{k-1} \right\rangle$$
$$+ 2\Delta t \left\langle \Lambda w^k, w^{k+1} + w^{k-1} \right\rangle = 0.$$

As in 29, let $C^{k+1/2} := \left\langle \Lambda w^k, w^{k+1} \right\rangle$, so that, by skew-symmetry of Λ,

$$\left\langle \Lambda w^k, w^{k+1} + w^{k-1} \right\rangle = C^{k+1/2} - C^{k-1/2}.$$

Simplify the stability equation to obtain

$$E_{\text{STAB}}^{k+1/2} - E_{\text{STAB}}^{k-1/2} + \Delta t \left\langle A(w^{k+1} + w^{k-1}), w^{k+1} + w^{k-1} \right\rangle$$
$$+ 2\Delta t \left(C^{k+1/2} - C^{k-1/2} \right) = 0.$$

Sum the above from $k = 1, \ldots, N$.

$$E_{\text{STAB}}^{N+1/2} + 2\Delta t C^{N+1/2} + \Delta t \sum_{k=1}^{N} \left\langle A(w^{k+1} + w^{k-1}), w^{k+1} + w^{k-1} \right\rangle = E_{\text{STAB}}^{1/2} + 2\Delta t C^{1/2}.$$

It remains to prove that $E_{\text{STAB}}^{N+1/2} + 2\Delta t C^{N+1/2} > 0$. Apply the Cauchy-Schwarz and Young inequality to show

$$2\Delta t C^{N+1/2} \leq 2\Delta t^2 ||\Lambda w^N||^2 + \frac{1}{2}||w^{N+1}||^2,$$

thus implying

$$E_{\text{STAB}}^{N+1/2} + 2\Delta t C^{N+1/2} \geq \tfrac{1}{2}||w^{N+1}||^2 + ||w^N||^2 + 2\Delta t^2 ||\Lambda w^{N+1}||^2 > 0.$$

Hence,

$$\begin{aligned} 0 &\leq \tfrac{1}{2}||w^{N+1}||^2 + ||w^N||^2 + 2\Delta t^2 ||\Lambda w^{N+1}||^2 \\ &\quad + \Delta t \sum_{k=1}^{N} \left\langle A(w^{k+1} + w^{k-1}), w^{k+1} + w^{k-1} \right\rangle \leq E_{\text{STAB}}^{1/2} + 2\Delta t C^{1/2}, \end{aligned} \tag{4.3}$$

This implies (4.2), since A is SPD. □

It remains to prove that (CNLFSTAB) is unconditionally, asymptotically stable over long-time intervals. To do this, we show that (CNLFSTAB) effectively controls both the stable and unstable modes, $w^{k+1}+w^{k-1}$ and $w^{k-1}-w^{k-1}$, for all time-step sizes, Δt.

Theorem 31 (Unconditional, asymptotic stability of (CNLFstab))**.** *Consider the method (CNLFstab). Both the stable mode, $(w^{k+1}+w^{k-1})$, and the unstable mode,$(w^{k-1}-w^{k-1})$, are unconditionally, asymptotically stable. That is,*

$$w^{k+1}+w^{k-1} \longrightarrow 0 \quad \text{and} \quad w^{k+1}-w^{k-1} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty,$$

and hence $w^k \to 0$ as $k \to \infty$.

Proof. Take the inner product of (CNLFstab) with the unstable mode, $(w^{k+1}-w^{k-1})$, and multiply through by $2\delta\Delta t$ for some $\delta>0$ (to be specified subsequently). This yields

$$\begin{aligned} &\delta\|w^{k+1}-w^{k-1}\|^2 + 2\delta\Delta t^2 \left\langle \Lambda^*\Lambda(w^{k+1}-w^{k-1}), w^{k+1}-w^{k-1}\right\rangle \\ +&\delta\Delta t \left\langle A(w^{k+1}+w^{k-1}), w^{k+1}-w^{k-1}\right\rangle + 2\delta\Delta t \left\langle \Lambda w^k, w^{k+1}-w^{k-1}\right\rangle = 0. \end{aligned} \tag{4.4}$$

The term, $\delta\Delta t\langle A(w^{k+1}+w^{k-1}), w^{k+1}-w^{k-1}\rangle$, can be written as

$$\delta\Delta t \left\langle A(w^{k+1}+w^{k-1}, w^{k+1}-w^{k-1})\right\rangle = \delta\Delta t \left[(\|w^{k+1}\|_A^2 + \|w^k\|_A^2) - (\|w^k\|_A^2 + \|w^{k-1}\|_A^2)\right].$$

Define $\mathcal{A}^{k+1/2} := \|w^{k+1}\|_A^2 + \|w^k\|_A^2 \geq 0$. Simplify and sum (4.4) from $k=1,\ldots,N$:

$$\begin{aligned} &\delta\sum_{k=1}^{N}\left[\|w^{k+1}-w^{k-1}\|^2 + 2\Delta t^2\|\Lambda(w^{k+1}-w^{k-1})\|^2\right] \\ &+ 2\delta\Delta t\sum_{k=1}^{N}\left\langle \Lambda w^k, w^{k+1}-w^{k-1}\right\rangle + \delta\Delta t\mathcal{A}^{N+1/2} = \delta\Delta t\mathcal{A}^{1/2}. \end{aligned} \tag{4.5}$$

Adding (4.3) to (4.5) gives

$$\begin{aligned} &\tfrac{1}{2}||w^{N+1}||^2 + ||w^N||^2 + 2\Delta t^2||\Lambda w^{N+1}||^2 + \delta\Delta t\mathcal{A}^{N+1/2} + F^N \\ &\quad + \sum_{k=1}^{N}\left[\Delta t\|w^{k+1}+w^{k-1}\|_A^2 + \delta\|w^{k+1}-w^{k-1}\|^2 + 2\delta\Delta t^2\|\Lambda(w^{k+1}-w^{k-1})\|^2\right] \\ &\leq E_{\text{STAB}}^{1/2} + 2\Delta t C^{1/2} + \delta\Delta t\mathcal{A}^{1/2}, \end{aligned} \tag{4.6}$$

where

$$F^N = \sum_{k=1}^{N} 2\delta\Delta t \left\langle \Lambda w^k, w^{k+1}-w^{k-1}\right\rangle.$$

It remains to subsume F^N into the positive terms on the left-hand side of (4.6). This requires an upper bound for F^N. Begin by applying Young's inequality which implies, for or any ϵ, $0 < \epsilon < 1$, there holds:

$$|F^N| \leq \delta\epsilon \sum_{k=1}^{N} \|w^{k+1} - w^{k-1}\|^2 + \frac{\delta}{\epsilon} \sum_{k=1}^{N} \Delta t^2 \|\Lambda w^k\|^2.$$

Rewrite the second term on the right above in terms of the stable and unstable modes:

$$\|\Lambda w^k\|^2 = \|\Lambda(\frac{w^k + w^{k-2}}{2}) + \Lambda(\frac{w^k - w^{k-2}}{2})\|^2$$
$$= 2\|\Lambda(\frac{w^k + w^{k-2}}{2})\|^2 + 2\|\Lambda(\frac{w^k - w^{k-2}}{2})\|^2 - \|\Lambda(w^{k-2})\|^2,$$

which holds for all $k \geq 2$. The upper bound on F_N now becomes

$$|F^N| \leq \delta\epsilon \sum_{k=1}^{N} \|w^{k+1} - w^{k-1}\|^2 + \frac{\delta}{\epsilon}\Delta t^2 \|\Lambda w^1\|^2 - \frac{\delta}{\epsilon} \sum_{k=2}^{N} \Delta t^2 \|\Lambda(w^{k-2})\|^2$$
$$+ \frac{\delta}{2\epsilon} \sum_{k=2}^{N} \Delta t^2 \left(\|\Lambda(w^k + w^{k-2})\|^2 + \|\Lambda(w^k - w^{k-2})\|^2 \right).$$

Shift the index of the third sum above and drop the negative term $-\frac{\delta}{\epsilon}\sum_{k=2}^{N} \Delta t^2 \|\Lambda(w^{k-2})\|^2$. The bound for F^N is thus:

$$|F^N| \leq \delta\epsilon \sum_{k=1}^{N} \|w^{k+1} - w^{k-1}\|^2 + \frac{\delta}{\epsilon}\Delta t^2 \|\Lambda w^1\|^2 - \frac{\delta}{\epsilon} \sum_{k=2}^{N} \Delta t^2 \|\Lambda(w^{k-2})\|^2$$
$$+ \frac{\delta}{2\epsilon} \sum_{k=1}^{N-1} \Delta t^2 \left(\|\Lambda(w^{k+1} + w^{k-1})\|^2 + \|\Lambda(w^{k+1} - w^{k-1})\|^2 \right). \tag{4.7}$$

Apply (4.7) to (4.6) to show

$$\begin{aligned}
&\tfrac{1}{2}\|w^{N+1}\|^2 + \|w^N\|^2 + 2\Delta t^2 \|\Lambda w^{N+1}\|^2 + \delta \Delta t \mathcal{A}^{N+1/2} \\
&+ \sum_{k=1}^{N} \left[\Delta t \|w^{k+1} + w^{k-1}\|_A^2 - \tfrac{\delta \Delta t^2}{2\epsilon} \|\Lambda(w^{k+1} + w^{k-1})\|^2 \right] \\
&\sum_{k=1}^{N} \delta \left[(1-\epsilon)\|w^{k+1} - w^{k-1}\|^2 + \Delta t^2 \left(2 - \tfrac{1}{2\epsilon}\right) \|\Lambda(w^{k+1} - w^{k-1})\|^2 \right] \\
&\leq E_{\text{STAB}}^{1/2} + 2\Delta t C^{1/2} + \delta \Delta t \mathcal{A}^{1/2} + \tfrac{\delta \Delta t^2}{\epsilon} \|\Lambda w^1\|^2.
\end{aligned} \tag{4.8}$$

Recall that A is SPD, so $\lambda_{min}(A)$, the smallest eigenvalue of A, is positive. This implies

$\|w^{k+1}+w^{k-1}\|_A^2 \geq \lambda_{min}(A)\|w^{k+1}+w^{k-1}\|^2$. Choose $\epsilon = \frac{1}{4}$ and $\delta = \frac{\lambda_{min}(A)}{4\Delta t\|\Lambda\|^2}$. Since $\frac{1}{2}\|w^{N+1}\|^2 + \|w^N\|^2 + 2\Delta t^2\|\Lambda w^{N+1}\|^2 > 0$ by Theorem 30, it can be omitted from the above equation. Therefore (4.8) simplifies to

$$\begin{aligned} 0 &< \sum_{k=1}^{N}\left[\tfrac{\Delta t\lambda_{min}(A)}{2}\|w^{k+1}+w^{k-1}\|^2 + \tfrac{3\lambda_{min}(A)}{16\Delta t\|\Lambda\|^2}\|w^{k+1}-w^{k-1}\|^2\right] \\ &\leq E_{\text{STAB}}^{1/2} + 2\Delta t C^{1/2} + \tfrac{\lambda_{min}(A)}{4\|\Lambda\|^2}\left(\mathcal{A}^{1/2} + 4\Delta t\|\Lambda w^1\|^2\right), \end{aligned}$$

which further reduces to

$$\sum_{k=1}^{N}\left[\|w^{k+1}+w^{k-1}\|^2 + \|w^{k+1}-w^{k-1}\|^2\right] \leq \mathcal{C}(w^1, w^0).$$

The constant, $\mathcal{C}(w^1, w^0)$, above depends on w^1 and w^0 but is independent of N. This implies both $\|w^{k+1}+w^{k-1}\|^2 \to 0$ and $\|w^{k+1}-w^{k-1}\|^2 \to 0$ as $k \to \infty$. □

Remark 32. *Theorem 31 implies asymptotic stability about zero. These results may be extended to include nonzero forcing terms on the right hand side, $F^k = F(t^k)$, by linearity. Let F_∞ represent the nonzero forcing term in the equilibrium problem:*

$$Aw_\infty + \Lambda w_\infty = F_\infty.$$

Then, if $F^k \to F_\infty$ as $k \to \infty$ in the sense that the series, $\sum_{k=1}^{\infty}\|F^k - F_\infty\|_*^2$ converges, by following the steps of Theorems 30 and 31 one concludes that $w^{k+1} + w^{k-1} \to 2w_\infty$, $w^{k+1} - w^{k-1} \to 0$, and $w^k \to w_\infty$.

4.2 NUMERICAL EXPERIMENTS

To illustrate the enhanced stability properties of (CNLFstab) we perform a simple numerical experiment using MATLAB. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Set $w^0 = [1,1]^T$ and $w^1 = [1,-1]^T$. If the system is stable, then $\|w^k\| \to 0$ as $k \to \infty$. According to Theorem 29, (CNLF) is stable when $\Delta t < 1$. In Figure 4.1, the size of the approximate solution, $\|w^k\|$, is computed at each time step and plotted against time. In the first experiment, $\Delta t = 0.99$, which is within the stability region of (CNLF). Note how once $\Delta t \geq 1$, (CNLF) becomes unstable, as seen in the middle and bottom graphs of Figure 4.1, while (CNLFSTAB) remains stable.

A detail of the behavior of the modes is given in Figure 4.2 for the case when $\Delta t = 1.0$. In this case, the norm of the unstable mode of (CNLF) is a positive constant whereas the norm of the stable mode equals zero. In contrast, the norms of the energy, stable, and unstable modes of (CNLFSTAB) decay to zero as expected.

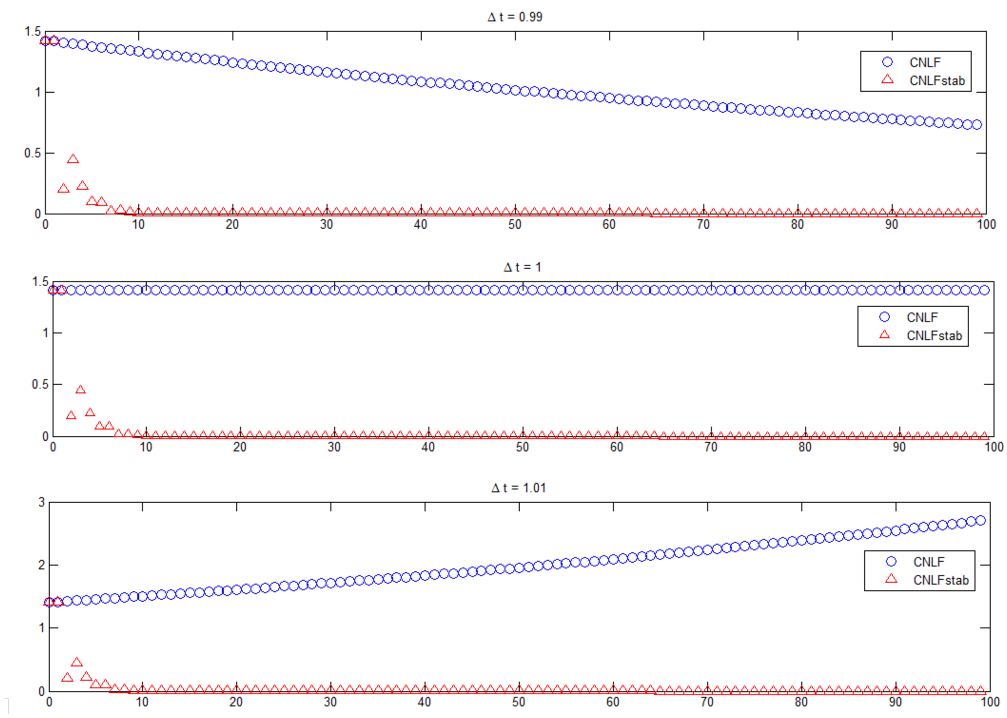

Figure 4.1: $\|w^k\|$ vs. time, $k = 1, \ldots, 100$, for $\Delta t = 0.99$ (above), $\Delta t = 1.0$ (middle), and $\Delta t = 1.01$ (below). Note the loss of stability for (CNLF) once $\Delta t \geq 1$.

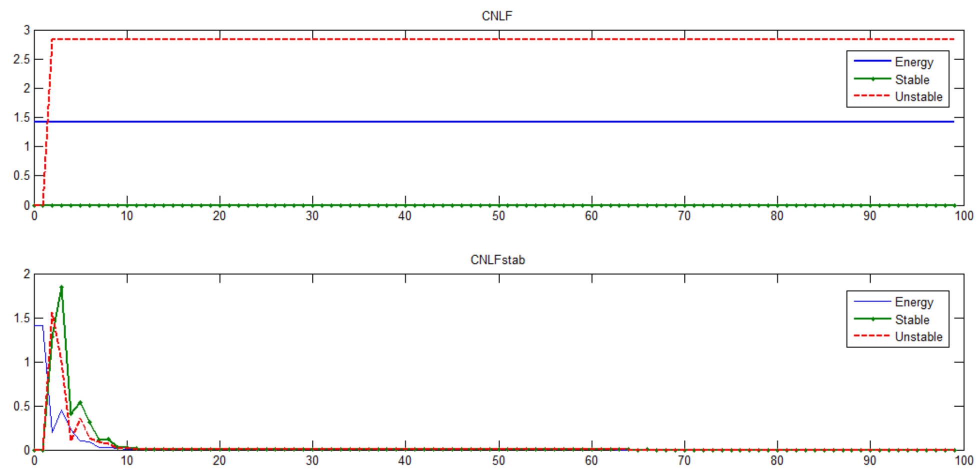

Figure 4.2: Energy and modes versus time for (CNLF) (top) and (CNLFSTAB) (bottom) for $\Delta t = 1.0$ over $[0, 1]$.

5.0 STABILIZED CNLF FOR STOKES-DARCY (CNLFSTAB-SD)

In this chapter, we present an adaptation of the method, (CNLFSTAB), from Chapter 4 for the Stokes-Darcy system. Recall from Chapter 3 that (CNLF-SD) is a conditionally stable, second-order convergent, parallel partitioned method. While the higher-order convergence of this method is desirable, the time-step condition derived for stability in Theorem 20,

$$\begin{aligned}\Delta t < \mathcal{C}^{-1} \max\{ & \min\left\{h^2, gS_0 n^{-1}\right\}, \min\left\{n^{-1}h^2, gS_0\right\}, \min\left\{nh^2, gS_0 n^{-2}\right\} \\ & \min\left\{h, gS_0 n^{-1}h\right\}, \min\left\{n^{-1}h, gS_0 h\right\}, \min\left\{nh, gS_0 n^{-2}h\right\} \}, \end{aligned} \tag{Δt_{CNLF}}$$

is impractical in physical situations that correspond to small values of specific storage, S_0. In this chapter, we addrss this potential issue by adapting the unconditionally stable, second-order convergent (CNLFSTAB)-method for the Stokes-Darcy problem.

To review, the (CNLF) method for the Stokes-Darcy system is as follows.

$$\begin{aligned} & n\left(\frac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) - b\left(v_h, \frac{p_h^{k+1}+p_h^{k-1}}{2}\right) \\ & \qquad\qquad + c_I(v_h, \phi_h^k) = n(f_f^k, v_h)_f, \\ & b\left(u_h^{k+1}, q_h\right)_f = 0, \\ & gS_0\left(\frac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - c_I(u_h^k, \psi_h) = g(f_p^k, \psi_h)_p. \end{aligned} \tag{CNLF-SD}$$

Recall the stabilization term in (CNLFSTAB):

$$2\Delta t \Lambda^* \Lambda(w^{k+1} - w^{k-1}). \tag{5.1}$$

To translate the above exactly for the Stokes-Darcy problem, first define a linear operator

$\Lambda = (\Lambda_f, \Lambda_p) : X_f^h \times X_p^h \to X_f^h \times X_p^h$ via the Riesz Representation Theorem by

$$(\Lambda_f(u,\phi), v)_f + (\Lambda_p(u,\phi), \psi)_p = \int_I \psi u \cdot \widehat{n} ds - \int_I \phi v \cdot \widehat{n} ds.$$

Then, the natural implementation of (5.1) would be to add

$$\Delta t g^2 \int_I (\phi_h^{k+1} - \phi_h^{k-1})\psi_h \, ds \text{ and } \Delta t n^2 \int_I \left(u_h^{k+1} - u_h^{k-1}\right) \cdot \hat{n}_f v_h \cdot \hat{n}_f \, ds,$$

into the (CNLF-SD)-method. However, numerical tests using this exact implementation of the (CNLFstab)-method for Stokes-Darcy suggest it is insufficient (see discussion in Section 5.3).

We present an adaptation of the (CNLFstab)-method specifically for the Stokes-Darcy problem. Recall the derivation of the CFL-type stability condition for (CNLF-SD). In Theorem 20, the stability condition, (Δt_{CNLF}), for (CNLF-SD) arose out of bounding the coupling terms, $c_I(.,.)$, by Lemma 19, which utilized an inverse inequality. Then the effects of the coupling terms were absorbed into the discrete energy, $E^{k+1/2} = \|u_h^{k+1}\|_f^2 + \|u_h^k\|_f^2 + gS_0(\|\phi_h^{k+1}\|_p^2 + \|\phi_h^k\|_p^2)$, of (CNLF-SD).

In this chapter, we treat the coupling term differently. Recall the (HDIV-TRACE) inequality, proven by Moraiti in [54] and used in Chapter 2:

$$|c_I(u,\phi)| \le ngC_{f,p}\|u\|_{DIV,f}\|\phi\|_{1,p}. \quad \text{(HDIV-TRACE)}$$

This inequality holds under conditions on the domains Ω_f, Ω_p, with the constant, $C_{f,p}$, depending on $\Omega_{f/p}$. The above inequality suggests the following adaptation of (CNLFstab) for Stokes-Darcy, referred to as (CNLFstab-SD), in which the added stability terms are

$$n\left(\nabla \cdot \left(\frac{u_h^{k+1} - u_h^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f, \text{ in Stokes, and}$$

$$\Delta t n g^2 C_{f,p}^2 \left(\nabla(\phi_h^{k+1} - \phi_h^{k-1}), \nabla\psi_h\right)_p, \Delta t n g^2 C_{f,p}^2 \left(\phi_h^{k+1} - \phi_h^{k-1}, \psi_h\right)_p, \text{ in Darcy.}$$

Similar to (CNLFstab), the added stability terms in (CNLFstab-SD), these terms are both SPD and $\mathcal{O}(\Delta t^2)$, thus preserving the second-order convergence of (CNLF-SD). However, in contrast to (CNLF-SD), (CNLFstab-SD), is unconditionally stable and therefore not sensitive to small values of specific storage, S_0. A full definition of the (CNLFstab-SD) method, followed by an analysis of its stability and convergence properties ensues.

Definition 33 (Stabilized Crank-Nicolson Leapfrog for Stokes-Darcy (CNLFSTAB-SD))**.** *Given* $\left(u_h^k, p_h^k, \phi_h^k\right)$ *and* $\left(u_h^{k-1}, p_h^{k-1}, \phi_h^{k-1}\right)$ *in* $\left(X_f^h, Q_f^h, X_p^h\right)$*, find* $\left(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}\right)$ *in* $\left(X_f^h, Q_f^h, X_p^h\right)$ *satisfying for all* (v_h, q_h, ψ_h) *in* $\left(X_f^h, Q_f^h, X_p^h\right)$ *:*

$$\begin{aligned} n\left(\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + n\left(\nabla\cdot\left(\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla\cdot v_h\right)_f + a_f\left(\tfrac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) & \\ -b\left(v_h, \tfrac{p_h^{k+1}+p_h^{k-1}}{2}\right)_f + c_I(v_h, \phi_h^k) = n(f_f^k, v_h)_f, & \\ b\left(u_h^{k+1}, q_h\right)_f = 0, & \qquad \text{(CNLFSTAB-SD)} \\ gS_0\left(\tfrac{\phi_h^{n+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\tfrac{\phi_h^{k+1}-\phi_h^{k-1}}{2}, \psi_h\right) - c_I(u_h^k, \psi_h) & \\ +\Delta t n g^2 C_{f,p}^2\left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_{H^1(\Omega_p)} = g(f_p^k, \psi_h)_p, & \end{aligned}$$

where $C_{f,p}$ *is the constant from inequality (HDIV-TRACE), and* $\left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_{H^1(\Omega_p)} = \left(\nabla\left(\phi_h^{k+1}-\phi_h^{k-1}\right), \nabla\psi_h\right)_p + \left(\phi_h^{k+1}-\phi_h^{k-1}, \psi_h\right)_p.$

5.1 UNCONDITIONAL STABILITY OF CNLFSTAB-SD METHOD

Theorem 34 (Unconditional Stability of (CNLFSTAB-SD))**.** (CNLFSTAB-SD) *is unconditionally stable over long-time intervals. That is, for* $N = 1, 2, 3, \ldots,$ *there holds*

$$\begin{aligned} &\frac{n}{2}\left(\|u_h^N\|_{DIV,f}^2 + \|u_h^{N-1}\|_{DIV,f}^2\right) + gS_0\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right) \\ &\quad + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{n\nu\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2 + gk_{min}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2\right\} \\ &\quad \le n(\|u_h^1\|_{DIV,f}^2 + \|u_h^0\|_{DIV,f}^2) + gS_0\left(\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2\right) \\ &\quad + 2\Delta t^2 n g^2 C_{f,p}^2\left(\|\phi_h^1\|_{1,p}^2 + \|\phi_h^0\|_{1,p}^2\right) + 2\Delta t\left\{c_I(\phi_h^0, u_h^1) - c_I(\phi_h^1, u_h^0)\right\} \\ &\quad + \frac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{\frac{g}{k_{min}}\|f_p^k\|_{-1,p}^2 + \frac{n}{\nu}\|f_f^k\|_{-1,f}^2\right\}. \end{aligned} \tag{5.2}$$

Proof. In (CNLFSTAB-SD), set $v_h = u_h^{k+1} + u_h^{k-1}$, $\psi_h = \phi_h^{k+1} + \phi_h^{k-1}$, add the equations, and multiply by $2\Delta t$ to obtain

$$
\begin{aligned}
&n(\|u_h^{k+1}\|^2_{DIV,f} - \|u_h^{k-1}\|^2_{DIV,f}) + gS_0\left(\|\phi_h^{k+1}\|_p^2 - \|\phi_h^{k-1}\|_p^2\right)\\
&+ 2\Delta t^2 ng^2C^2_{f,p}\left(\|\phi_h^{k+1}\|^2_{1,p} - \|\phi_h^{k-1}\|^2_{1,p}\right)\\
&+ \Delta t\left\{a_p\left(\phi_h^{k+1}+\phi_h^{k-1}, \phi_h^{k+1}+\phi_h^{k-1}\right) + a_f\left(u_h^{k+1}+u_h^{k-1}, u_h^{k+1}+u_h^{k-1}\right)\right\}\\
&+ 2\Delta t\left(c_I(u_h^{k+1}+u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}+\phi_h^{k-1})\right)\\
&\quad = 2\Delta t\left\{g\left(f_p^k, \phi_h^{k+1}+\phi_h^{k-1}\right)_p + n\left(f_f^k, u_h^{k+1}+u_h^{k-1}\right)_f\right\}.
\end{aligned}
$$

As before, let $C^{k+1/2} = c_I(\phi_h^k, u_h^{k+1}) - c_I(\phi_h^{k+1}, u_h^k)$. Then the coupling terms may be rewritten as $c_I(u_h^{k+1}+u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}+\phi_h^{k-1}) = C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}$. Using this notation, by coercivity of the bilinear forms, $a_{f/p}(.,.)$, and Young's inequality, the equation becomes,

$$
\begin{aligned}
&n(\|u_h^{k+1}\|^2_{DIV,f} - \|u_h^{k-1}\|^2_{DIV,f}) + gS_0\left(\|\phi_h^{k+1}\|_p^2 - \|\phi_h^{k-1}\|_p^2\right)\\
&+ 2\Delta t^2 ng^2C^2_{f,p}\left(\|\phi_h^{k+1}\|^2_{1,p} - \|\phi_h^{k-1}\|^2_{1,p}\right) + 2\Delta t\left\{C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}\right\}\\
&+ \Delta t\left\{\tfrac{gk_{min}}{2}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2 + \tfrac{n\nu}{2}\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2\right\}\\
&\quad \le \Delta t\frac{n}{2\nu}\|f_f^k\|^2_{-1,f} + \Delta t\frac{g}{2k_{min}}\|f_p^k\|^2_{-1,p}.
\end{aligned}
$$

Denote the stabilized energy terms by

$$
\begin{aligned}
E_{\text{STAB}}^{k+1/2} = n(\|u_h^{k+1}\|^2_{DIV,f} + \|u_h^k\|^2_{DIV,f}) &+ gS_0\left(\|\phi_h^{k+1}\|_p^2 + \|\phi_h^k\|_p^2\right)\\
&+2\Delta t^2 ng^2C^2_{f,p}\left(\|\phi_h^{k+1}\|^2_{1,p} + \|\phi_h^k\|^2_{1,p}\right).
\end{aligned}
$$

Then the inequality becomes

$$
\begin{aligned}
E_{\text{STAB}}^{k+1/2} - E_{\text{STAB}}^{k-1/2} &+ \Delta t\left\{\tfrac{gk_{min}}{2}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2 + \tfrac{n\nu}{2}\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2\right\}\\
&+2\Delta t\left\{C^{k+1/2} - C^{k-1/2}\right\} \le \Delta t\frac{n}{2\nu}\|f_f^k\|^2_{-1,f} + \Delta t\frac{g}{2k_{min}}\|f_p^k\|^2_{-1,p}.
\end{aligned}
$$

Sum the inequality from $k = 1$ to $N-1$. This produces

$$
\begin{aligned}
E_{\text{STAB}}^{N-1/2} &+ \Delta t\sum_{k=1}^{N-1}\left\{\tfrac{gk_{min}}{2}\|\nabla\left(\phi_h^{k+1}+\phi_h^{k-1}\right)\|_p^2 + \tfrac{n\nu}{2}\|\nabla\left(u_h^{k+1}+u_h^{k-1}\right)\|_f^2\right\}\\
&+2\Delta t C^{N-1/2} \le E_{\text{STAB}}^{1/2} + C^{1/2} + \Delta t\frac{g}{2k_{min}}\|f_p^k\|^2_{-1,p} + \Delta t\frac{n}{2\nu}\|f_f^k\|^2_{-1,f}.
\end{aligned}
\tag{5.3}
$$

Apply inequality (HDIV-TRACE) to the terms in $C^{N-1/2}$ to show

$$|c_I(u_h^N, \phi_h^{N-1})| \leq ngC_{f,p}\|u_h^N\|_{DIV,f}\|\phi_h^{N-1}\|_{1,p} \text{ and}$$
$$|c_I(u_h^{N-1}, \phi_h^N)| \leq ngC_{f,p}\|u_h^{N-1}\|_{DIV,f}\|\phi_h^N\|_{1,p}.$$

Next, bound $C^{N-1/2}$ by the Cauchy-Schwarz and Young inequalities:

$$\begin{aligned}|2\Delta t C^{N-1/2}| \leq \frac{n}{2}\left(\|u_h^N\|^2_{DIV,f} + \|u_h^{N-1}\|^2_{DIV,f}\right)\\ + 2\Delta t^2 ng^2C^2_{f,p}\left(\|\phi_h^{N-1}\|^2_{1,p} + \|\phi_h^N\|^2_{1,p}\right).\end{aligned}$$

Hence,

$$E_{\text{STAB}}^{N-1/2} + 2\Delta t C^{N-1/2} \geq \frac{n}{2}\left(\|u_h^N\|^2_{DIV,f} + \|u_h^{N-1}\|^2_{DIV,f}\right) + gS_0\left(\|\phi_h^N\|^2_p + \|\phi_h^{N-1}\|^2_p\right) > 0.$$

This inequality, along with (5.3) imply the unconditional stability bound given in (5.2). □

5.1.1 Control over the Stable and Unstable Modes of CNLFstab-SD

In addition to being unconditionally stable, the (CNLFSTAB-SD)-method controls both the stable and unstable modes linked to Leapfrog for all choices of time-step size, Δt, and mesh width, h.

Corollary 35 (Control of the Stable Mode of (CNLFSTAB-SD))**.** *The following inequality for the stable mode of* (CNLFSTAB-SD) *holds for all* $N = 2, 3, ...$

$$\begin{aligned}\Delta t\sum_{k=1}^{N-1}\left\{\tfrac{n\nu}{2}\|\nabla\left(u_h^{k+1} + u_h^{k-1}\right)\|^2_f + \tfrac{gk_{min}}{2}\|\nabla\left(\phi_h^{k+1} + \phi_h^{k-1}\right)\|^2_p\right\}\\ \leq E^{1/2}_{\text{STAB}} + C^{1/2} + \Delta t\frac{g}{2k_{min}}\|f_p^k\|^2_{-1,p} + \Delta t\frac{n}{2\nu}\|f_f^k\|^2_{-1,f}.\end{aligned}$$

Proof. Drop the positive term $E_{\text{STAB}}^{N-1/2} + 2\Delta t C^{N-1/2}$ from the left-hand side of (5.3) in the proof of Theorem 34. □

Therefore, (CNLFSTAB-SD) controls the stable mode for all time-step sizes. Next, we show the same applies for the unstable mode. The proof will be similar to Theorem 22 for (CNLF-SD) given in Chapter 3, with carefully modified treatment of the coupling terms.

Theorem 36 (Control of the Modes of (CNLFSTAB-SD)). *The method, (CNLFSTAB-SD), controls both the stable and unstable modes for all choices of time-step size, $\Delta t > 0$, and mesh width, h. That is, there exists a positive constant $\mathcal{M}$, satisfying for any $N \geq 2$,*

$$\begin{aligned}\mathcal{M}\Bigg\{\Delta t \sum_{k=1}^{N-1} \{\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2 + \|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\} \\ + \sum_{k=1}^{N-1}\{\|u_h^{k+1}-u_h^{k-1}\|_{DIV,f}^2 + \|\phi_h^{k+1}-\phi_h^{k-1}\|_p^2 + \Delta t^2\|\phi_h^{k+1}-\phi_h^{k-1}\|_{1,p}^2\}\Bigg\} \\ \leq \Delta t \sum_{k=1}^{N-1}\{\|f_f^k\|_{-1,f}^2 + \|f_p^k\|_{-1,p}^2 + \Delta t\left(\|f_f^k\|_f^2 + \|f_p^k\|_p^2\right)\} \\ +\|u_h^1\|_{DIV,f}^2 + \|u_h^0\|_{DIV,f}^2 + \|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2 + \Delta t^2(\|\phi_h^1\|_{1,p}^2 + \|\phi_h^0\|_{1,p}^2) \\ +\Delta t\left(\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2 + \|\nabla\phi_h^1\|_p^2 + \|\nabla\phi_h^0\|_p^2\right) \\ +\Delta t\left(c_I(u_h^1,\phi_h^0) - c_I(u_h^0,\phi_h^1) + c_I(u_h^2-u_h^0,\phi_h^1) - c_I(u_h^1,\phi_h^2-\phi_h^0)\right).\end{aligned} \tag{5.4}$$

Proof. As in the proof of Theorem 22, begin by choosing $v_h = 2\delta\Delta t(u_h^{k+1} - u_h^{k-1})$ and $\psi_h = 2\delta\Delta t(\phi_h^{k+1} - \phi_h^{k-1})$ in (CNLFSTAB-SD) where $\delta > 0$. Add the equations to obtain

$$\begin{aligned}\delta\left\{n\|u_h^{k+1}-u_h^{k-1}\|_{DIV,f}^2 + gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\} + 2\delta\Delta t^2 ng^2C_{f,p}^2\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2 \\ +\delta\Delta t\left\{\mathcal{A}^{k+1/2}-\mathcal{A}^{k-1/2}\right\} + 2\delta\Delta t\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\} \\ = 2\delta\Delta t\left\{n(f_f^k, u_h^{k+1}-u_h^{k-1})_f + g(f_p^k,\phi_h^{k+1}-\phi_h^{k-1})_p\right\},\end{aligned}$$

where, as before, $\mathcal{A}^{k+1/2} = a_f(u_h^{k+1},u_h^{k+1}) + a_p(\phi_h^{k+1},\phi_h^{k+1}) + a_f(u_h^k,u_h^k) + a_p(\phi_h^k,\phi_h^k) \geq 0$. Applying Cauchy-Schwarz and Young's inequality on the right-hand side and summing from $k = 1$ to $N-1$ produces

$$\begin{aligned}\delta(1-\varepsilon)\sum_{k=1}^{N-1}\left\{n\|u_h^{k+1}-u_h^{k-1}\|_{DIV,f}^2 + gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\} + \delta\varepsilon n\sum_{k=1}^{N-1}\|\nabla\cdot(u_h^{k+1}-u_h^{k-1})\|_f^2 \\ +2\delta\Delta t^2 ng^2C_{f,p}^2\sum_{k=1}^{N-1}\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2 + 2\delta\Delta t\sum_{k=1}^{N-1}\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\} \\ \leq \delta\Delta t\mathcal{A}^{1/2} + \tfrac{\delta\Delta t^2}{\varepsilon}\sum_{k=1}^{N-1}\left\{n\|f_f^k\|_f^2 + \tfrac{g}{S_0}\|f_p^k\|_p^2\right\},\end{aligned}$$

where $\varepsilon \in (0,1)$. Add this inequality to the result from Corollary 35:

$$\begin{aligned}
\delta\Delta t\mathcal{A}^{N-1/2} + \delta(1-\varepsilon)\sum_{k=1}^{N-1}\left\{n\|u_h^{k+1}-u_h^{k-1}\|_{DIV,f}^2 + gS_0\|\phi_h^{k+1}-\phi_h^{k+1}\|_p^2\right\} \\
+\delta\varepsilon n\sum_{k=1}^{N-1}\|\nabla\cdot(u_h^{k+1}-u_h^{k-1})\|_f^2 + 2\delta\Delta t^2 ng^2C_{f,p}^2\sum_{k=1}^{N-1}\|\phi_h^{k+1}-\phi_h^{k+1}\|_{1,p}^2 \\
+2\delta\Delta t\sum_{k=1}^{N-1}\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\} \\
+\tfrac{\Delta t}{2}\sum_{k=1}^{N-1}\left\{n\nu\|\nabla(u_h^{k+1}+u_h^{k-1})\|_f^2 + gk_{min}\|\nabla(\phi_h^{k+1}+\phi_h^{k-1})\|_p^2\right\} \\
\leq \delta\Delta t\mathcal{A}^{1/2} + E_{\text{STAB}}^{1/2} + 2\Delta tC^{1/2} \\
+2\Delta t\sum_{k=1}^{N-1}\left\{\tfrac{n}{\nu}\|f_f^k\|_{-1,f}^2 + \tfrac{g}{k_{min}}\|f_p^k\|_{-1,p}^2\right\} + \tfrac{\delta\Delta t^2}{\varepsilon}\sum_{k=1}^{N-1}\left\{n\|f_f^k\|_f^2 + \tfrac{g}{S_0}\|f_p^k\|_p^2\right\}.
\end{aligned} \tag{5.5}$$

Let $\mathcal{Q} = 2\delta\Delta t\sum_{k=1}^{N-1}\left\{c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1})\right\}$. Similar to before, rewrite the interface integrals in terms of the stable and unstable modes. For $k \geq 2$,

$$\begin{aligned}
c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k) - c_I(u_h^k,\phi_h^{k+1}-\phi_h^{k-1}) = \tfrac{1}{2}c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k-\phi_h^{k-2}) \\
+\tfrac{1}{2}c_I(u_h^{k+1}-u_h^{k-1},\phi_h^k+\phi_h^{k-2}) - \tfrac{1}{2}c_I(u_h^k-u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}) - \tfrac{1}{2}c_I(u_h^k+u_h^{k-2},\phi_h^{k+1}-\phi_h^{k-1}).
\end{aligned}$$

By (HDIV-TRACE):

$$\begin{aligned}
\mathcal{Q} \leq \delta\Delta tngC_{f,p}\sum_{k=2}^{N-1}\left\{\|u_h^{k+1}-u_h^{k-1}\|_{DIV,f}\left(\|\phi_h^k-\phi_h^{k-2}\|_{1,p} + \|\phi_h^k+\phi_h^{k-2}\|_{1,p}\right)\right. \\
\left.+\left(\|u_h^k-u_h^{k-2}\|_{DIV,f} + \|u_h^k+u_h^{k-2}\|_{DIV,f}\right)\|\phi_h^{k+1}-\phi_h^{k-1}\|_p\right\} \\
+2\delta\Delta t\left[c_I(u_h^2-u_h^0,\phi_h^1) - c_I(u_h^1,\phi_h^2-\phi_h^0)\right].
\end{aligned} \tag{5.6}$$

We bound the stable modes above using the Poincaré inequality (Lemma 2) as follows:

$$\begin{aligned}
\|u_h^k+u_h^{k-2}\|_{DIV,f} \leq \sqrt{C_{P,f}^2+d}\,\|\nabla(u_h^k+u_h^{k-2})\|_f, \\
\|\phi_h^k+\phi_h^{k-2}\|_{1,p} \leq \sqrt{1+C_{P,p}^2}\,\|\nabla(\phi_h^k+\phi_h^{k-2})\|_p.
\end{aligned}$$

Next, we apply Young's inequality to each term in (5.6). Let $\mathcal{B} = \Delta tgC_{f,p}$. By Young's inequality, for any $\epsilon_{1,2,3} > 0$, there holds

$$n\mathcal{B}\|u_h^{k+1} - u_h^{k-1}\|_{DIV,f}\|\phi_h^k - \phi_h^{k-2}\|_{1,p} \le \tfrac{n\epsilon_1}{2}\|u_h^{k+1} - u_h^{k-1}\|^2_{DIV,f} + \tfrac{n\mathcal{B}^2}{2\epsilon_1}\|\phi_h^k - \phi_h^{k-2}\|^2_{1,p},$$

$$n\mathcal{B}\|u_h^{k+1} - u_h^{k-1}\|_{DIV,f}\|\phi_h^k + \phi_h^{k-2}\|_{1,p} \le \tfrac{n\epsilon_2}{2}\|u_h^{k+1} - u_h^{k-1}\|^2_{DIV,f} + \tfrac{n\mathcal{B}^2(1+C_{P,p}^2)}{2\epsilon_2}\|\nabla(\phi_h^k + \phi_h^{k-2})\|_p,$$

$$n\mathcal{B}\|u_h^{k} - u_h^{k-2}\|_{DIV,f}\|\phi_h^{k+1} - \phi_h^{k-1}\|_{p} \le \tfrac{\epsilon_1}{2}\|u_h^{k} - u_h^{k-2}\|^2_{DIV,f} + \tfrac{n\mathcal{B}^2}{2\epsilon_1}\|\phi_h^{k+1} - \phi_h^{k-1}\|^2_{1,p},$$

$$n\mathcal{B}\|u_h^{k} + u_h^{k-2}\|_{DIV,f}\|\phi_h^{k+1} - \phi_h^{k-1}\|_{p} \le \tfrac{n\epsilon_3(C_{P,f}^2+d)}{2}\|\nabla(u_h^{k} + u_h^{k-2})\|^2_{f} + \tfrac{n\mathcal{B}^2}{2\epsilon_3}\|\phi_h^{k+1} - \phi_h^{k-1}\|^2_{1,p}.$$

Simplify (5.6) by utilizing the above bounds and shifting the index of the sums:

$$\begin{aligned}
\mathcal{Q} \le \delta \sum_{k=1}^{N-1} &\Big\{(\varepsilon_1 + \tfrac{\varepsilon_2}{2})n\|u_h^{k+1} - u_h^{k-1}\|^2_{DIV,f} + ng^2\Delta t^2 C_{f,p}^2(\tfrac{1}{\varepsilon_1} + \tfrac{1}{2\varepsilon_3})\|\phi_h^{k+1} - \phi_h^{k-1}\|^2_{1,p}\Big\} \\
+\tfrac{\delta}{2}\sum_{k=1}^{N-1} &\Big\{n(C_{P,f}^2 + d)^2\varepsilon_3\|\nabla(u_h^{k+1} - u_h^{k-1})\|_f^2 + \tfrac{ng^2\Delta t^2 C_{f,p}^2(1+C_{P,p}^2)}{\varepsilon_2}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p^2\Big\} \\
&+2\delta\Delta t\left[c_I(u_h^2 - u_h^0, \phi_h^1) - c_I(u_h^1, \phi_h^2 - \phi_h^0)\right].
\end{aligned}$$

After using this bound for $\mathcal{Q}$, our inequality (5.5) becomes

$$\begin{aligned}
&\tfrac{1}{2}\sum_{k=1}^{N-1}\left\{\left(n\nu\Delta t - \delta n(d + C_{P,f}^2)\varepsilon_3\right)\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2\right\} \\
&+\tfrac{1}{2}\sum_{k=1}^{N-1}\left\{\left(gk_{min}\Delta t - \tfrac{\delta ng^2\Delta t^2 C_{f,p}^2(1+C_{P,p}^2)}{\epsilon_2}\right)\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p^2\right\} \\
&+\delta n\sum_{k=1}^{N-1}\left\{\left((1-\varepsilon) - (\varepsilon_1 + \tfrac{\epsilon_2}{2})\right)\|u_h^{k+1} - u_h^{k-1}\|^2_{DIV,f}\right\} \\
&+\delta\sum_{k=1}^{N-1}\left\{(gS_0(1-\varepsilon)))\|\phi_h^{k+1} - \phi_h^{k+1}\|_p^2\right\} \\
&+\delta\varepsilon n\sum_{k=1}^{N-1}\left\{\|\nabla\cdot(u_h^{k+1} - u_h^{k-1})\|_f^2\right\} \\
&+\delta g^2 n\Delta t^2 C_{f,p}^2\sum_{k=1}^{N-1}\left\{\left(2 - (\tfrac{1}{\varepsilon_1} + \tfrac{1}{2\varepsilon_3})\right)\|\phi_h^{k+1} - \phi_h^{k+1}\|^2_{1,p}\right\} \\
\le &\sum_{k=1}^{N-1}\left\{2\Delta t\left(\tfrac{n}{\nu}\|f_f^k\|^2_{-1,f} + \tfrac{g}{k_{min}}\|f_p^k\|^2_{-1,p}\right) + \tfrac{\delta\Delta t^2}{\varepsilon}\left(n\|f_f^k\|_f^2 + \tfrac{g}{S_0}\|f_p^k\|_p^2\right)\right\} \\
&+\delta\Delta t\mathcal{A}^{1/2} + E_{\text{STAB}}^{1/2} + 2\Delta t C^{1/2} + 2\delta\Delta t\left[c_I(u_h^2 - u_h^0, \phi_h^1) - c_I(u_h^1, \phi_h^2 - \phi_h^0)\right].
\end{aligned} \tag{5.7}$$

This implies control over the stable, $\|\nabla(w_h^{k+1} + w_h^{k-1})\|_{f,p}$ for $w = u, \phi$, and unstable,

$\|w_h^{k+1} - w_h^{k-1}\|_{f,p}$ for $w = u, \phi$, modes as given in (5.4) provided the coefficients of the sums on the left hand side are positive, i.e.

$$\begin{aligned}
\nu\Delta t - \delta(C_{P,f}^2 + d)\varepsilon_3 &> 0,\\
gk_{min}\Delta t - \tfrac{\delta n g^2 \Delta t^2 C_{f,p}^2(1+C_{P,p}^2)}{\epsilon_2} &> 0,\\
(1-\varepsilon) - (\varepsilon_1 + \tfrac{\epsilon_2}{2}) &> 0,\\
2 - (\tfrac{1}{\varepsilon_1} + \tfrac{1}{2\varepsilon_3}) &> 0,
\end{aligned}$$

It remains to choose $\delta, \varepsilon_{1,2,3} > 0$ and $\epsilon \in (0,1)$ so that the above inequalities hold. The last two inequalities may be rearranged as

$$\frac{\varepsilon_1 + \frac{\varepsilon_2}{2}}{1-\varepsilon} < 1, \quad \frac{1}{\varepsilon_1} + \frac{1}{2\varepsilon_3} < 2.$$

Many choices of ϵ and $\varepsilon_{1,2,3}$ will satisfy the above. For example, choose $\varepsilon = \varepsilon_2 = \frac{1}{8}$, $\varepsilon_1 = \frac{3}{4}$, and $\varepsilon_3 = \frac{3}{2}$. Then

$$\frac{\varepsilon_1 + \frac{\varepsilon_2}{2}}{1-\varepsilon} = \frac{\frac{3}{4} + \frac{1}{16}}{\frac{7}{8}} = \frac{13}{14} < 1, \quad \frac{1}{\varepsilon_1} + \frac{1}{2\varepsilon_3} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3} < 2.$$

As for δ, choose

$$\delta = \min\left\{ \frac{\nu\Delta t}{(d + C_{P,f}^2)\varepsilon_3}, \frac{k_{min}\varepsilon_2}{ng\Delta t C_{f,p}^2(1+C_{P,p}^2)} \right\} > 0.$$

□

5.2 CONVERGENCE OF CNLFSTAB-SD

This section establishes the method's *unconditional*, second-order convergence over long-time intervals. An essential feature of the error analysis is that no form of Gronwall's inequality is used.

Recall that the FEM spaces, X_f^h, X_p^h and Q_f^h, satisfy approximation properties of piecewise polynomials of degree $r-1$, r, and $r+1$, stated previously in (1.17). Also, recall that since we assumed the spaces X_f^h and Q_f^h satisfied the (LBB^h) condition, there exists some constant C such that

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega_f)} \leq C \inf_{x_h \in X_f^h} \|u - x_h\|_{H^1(\Omega_f)}, \tag{5.8}$$

(see, for example, Girault and Raviart [34], Chapter II, Proof of Theorem 1.1, Equation (1.12)).

As a reminder, we will use the following notation for discrete-in-time norms.

$$|||v|||^2_{L^2(0,T;H^s(\Omega_{f,p}))} := \sum_{k=1}^{N} \|v^k\|^2_{H^s(\Omega_{f,p})} \Delta t,$$

$$|||v|||_{L^\infty(0,T;H^s(\Omega_{f,p}))} := \max_{0 \leq k \leq N} \|v^k\|_{H^s(\Omega_{f,p})}.$$

Recall the consistency error bounds, derived in Lemma 25 and used in the convergence analysis of (CNLF-SD):

$$\Delta t \sum_{k=1}^{N-1} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 \leq \frac{(\Delta t)^4}{20} \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{5.9}$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}\|_p^2 \leq \frac{(\Delta t)^4}{20} \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{5.10}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \leq \frac{7(\Delta t)^4}{6} \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}, \tag{5.11}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \leq \frac{7(\Delta t)^4}{6} \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}. \tag{5.12}$$

Convergence analysis of (CNLFstab-SD), requires additional consistency error bounds, given in the following lemma.

Lemma 37. *(Additional Consistency Errors for (CNLFstab-SD)) The following inequalities hold:*

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right) \right\|_f^2 \leq \frac{(\Delta t)^4}{20} \|\nabla u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{5.13}$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi^{k+1} - \phi^{k-1}\|^2_{1,p} \leq 4\Delta t^2 \|\phi_t\|^2_{L^2(0,T,H^1(\Omega_p))}. \tag{5.14}$$

Proof. For (5.13), follow the proof for (5.9) in Lemma 25, replacing u with ∇u. For the proof of (5.14), we have

$$\begin{aligned}\Delta t\sum_{k=1}^{N-1}\|\phi^{k+1}-\phi^{k-1}\|_p^2 &= \Delta t\sum_{k=1}^{N-1}\int_{\Omega_f}\left(\int_{t_{k-1}}^{t^{k+1}}\phi_t\ dt\right)\ dx\\ &\le \Delta t\int_{\Omega_f}\sum_{k=1}^{N-1}\int_{t^{k-1}}^{t^{k+1}}\ dt\int_{t^{k-1}}^{t^{k+1}}\phi_t^2\ dt\ dx\\ &= \Delta t\int_{\Omega_f}\sum_{k=1}^{N-1}2\Delta t\int_{t^{k-1}}^{t^{k+1}}\phi_t^2\ dt\ dx\\ &\le 2\Delta t^2\int_{\Omega_f}2\sum_{k=1}^{N}\int_{t^{k-1}}^{t^k}\phi_t^2\ dt\ dx\\ &= 4\Delta t^2\|\phi_t\|^2_{L^2(0,T,L^2(\Omega_p))}.\end{aligned}$$

Similarly,

$$\Delta t\sum_{k=1}^{N-1}\|\nabla\left(\phi^{k+1}-\phi^{k-1}\right)\|_p^2\le 4\Delta t^2\|\nabla\phi_t\|^2_{L^2(0,T,L^2(\Omega_p))}. \tag{5.15}$$

Combining the above two inequalities produces (5.14). □

5.2.1 Convergence of the Stokes Velocity and Darcy Pressure

Denote the errors by $\mathcal{E}_u^k = u^k - u_h^k$, $\mathcal{E}_p^k = p^k - p_h^k$, and $\mathcal{E}_p^k = \phi^k - \phi_h^k$. Convergence analysis of the (CNLFstab-SD)-method follows.

Theorem 38. *(Convergence of (CNLFstab-SD)) Consider the CNLF-stab method (CNLFstab-SD). For any $0 < t_N = T \le \infty$, if u, p, ϕ satisfy the regularity conditions*

$$\begin{aligned}&u\in L^2(0,T;H^{r+2}(\Omega_f))\cap L^\infty(0,T;H^{r+1}(\Omega_f))\cap H^3(0,T;H^1(\Omega_f)),\\ &p\in L^2(0,T;H^{r+1}(\Omega_f)),\\ &\phi\in L^2(0,T;H^{r+2}(\Omega_p))\cap L^\infty(0,T;H^{r+1}(\Omega_p))\cap H^3(0,T;H^1(\Omega_p)),\end{aligned}$$

then there exists a constant $\widehat{C} > 0$, independent of the mesh width, h, time step, Δt, and final time, $t_N = T$, such that

$$
\begin{aligned}
&\frac{n}{2}(\|\mathcal{E}_u^N\|^2_{DIV,f} + \|\mathcal{E}_u^{N-1}\|^2_{DIV,f}) + gS_0(\|\mathcal{E}_\phi^N\|^2_p + \|\mathcal{E}_\phi^{N-1}\|^2_p) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{n\nu}{2} \|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|^2_f + \frac{gk_{min}}{2} \|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|^2_p \right) \\
&\quad \leq \widehat{C} \Big\{ h^{2r} \{ \|u_t\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + \||u|\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + \||u|\|^2_{L^\infty(0,T;H^{r+1}(\Omega_f))} \\
&\qquad + \Delta t^4 \|\phi_t\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \||\phi|\|^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} \} \\
&\qquad + h^{2r+2} \{ \||p|\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T;H^{r+1}(\Omega_p))} + \||\phi|\|^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} \} \\
&\qquad + \Delta t^4 \{ \|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \} + \|\mathcal{E}_u^1\|^2_{DIV,f} + \|\mathcal{E}_\phi^1\|^2_{1,p} \Big\}.
\end{aligned}
\tag{5.16}
$$

Proof. Consider (CNLFSTAB-SD) over the discretely divergence free space $V^h := \{v_h \in X_f^h : (q_h, \nabla \cdot v_h)_f = 0 \, \forall q_h \in Q_f^h\}$, instead of X_f^h, so that the pressure term $b\left(v_h, \frac{p_h^{k+1}+p_h^{k-1}}{2}\right)$ equals zero. Subtract (CNLFSTAB-SD) from the semi-discrete variational form, (FEM-SD), evaluated at time t^k to obtain

$$
\begin{aligned}
n\left(u_t^k - \tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f - n\left(\nabla \cdot \left(\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f + a_f\left(u^k - \tfrac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) \\
-b\left(v_n, p^k\right) + c_I\left(v_h, \phi^k - \phi_h^k\right) = 0, \\
gS_0\left(\phi_t^k - \tfrac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\phi^k - \tfrac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - \Delta t n g^2 C_{f,p}^2 (\nabla(\phi_h^{k+1} - \phi_h^{k-1}), \nabla \psi_h)_p \\
-\Delta t n g^2 C_{f,p}^2 (\phi_h^{k+1} - \phi_h^{k-1}, \psi_h)_p - c_I\left(u^k - u_h^k, \psi_h\right) = 0.
\end{aligned}
$$

Because v_h is discretely divergence free, $b\left(v_h, p^k\right) = b\left(v_h, p^k - \lambda_h^k\right)$, for any $\lambda_h \in Q_f^h$. Further, $\left(\nabla \cdot u_t^k, v_h\right)_f = 0$. Thus, after rearranging we find

$$
\begin{aligned}
&n\left(\tfrac{\mathcal{E}_u^{k+1}-\mathcal{E}_u^{k-1}}{2\Delta t}, v_h\right)_f + n\left(\nabla \cdot \left(\tfrac{\mathcal{E}_u^{k+1}-\mathcal{E}_u^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f \\
&\quad + a_f\left(\tfrac{\mathcal{E}_u^{k+1}+\mathcal{E}_u^{k+1}}{2}, v_h\right) + c_I\left(v_h, \mathcal{E}_\phi^k\right) \\
&\quad = -n\left(u_t^k - \tfrac{u^{k+1}-u^{k-1}}{2\Delta t}, v_h\right)_f - n\left(\nabla \cdot \left(u_t^k - \tfrac{u^{k+1}-u^{k-1}}{2\Delta t}\right), \nabla \cdot v_h\right)_f \\
&\qquad - a_f\left(u^k - \tfrac{u^{k+1}+u^{k-1}}{2}, v_h\right) + b\left(v_h, p^k - \lambda_h^k\right),
\end{aligned}
$$

$$
\begin{aligned}
&gS_0\left(\frac{\mathcal{E}_\phi^{k+1}-\mathcal{E}_\phi^{k-1}}{2\Delta t},\psi_h\right)_p + a_p\left(\frac{\mathcal{E}_\phi^{k+1}+\mathcal{E}_\phi^{k-1}}{2},\psi_h\right) + \Delta t n g^2 C_{f,p}^2(\nabla(\mathcal{E}_\phi^{k+1}-\mathcal{E}_\phi^{k-1}),\nabla\psi_h)_p \\
&\quad + \Delta t n g^2 C_{f,p}^2(\mathcal{E}_\phi^{k+1}-\mathcal{E}_\phi^{k-1},\psi_h)_p - c_I\left(\mathcal{E}_u^k,\psi_h\right) \\
&\quad\quad = -gS_0\left(\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\psi_h\right)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2},\psi_h\right) \\
&\quad + \Delta t n g^2 C_{f,p}^2(\nabla(\phi^{k+1}-\phi^{k-1}),\nabla\psi_h)_p + \Delta t n g^2 C_{f,p}^2(\phi^{k+1}-\phi^{k-1},\psi_h)_p.
\end{aligned}
$$

Denote the consistency errors by:

$$
\begin{aligned}
\tau_f^k(v_h) &= -n\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}, v_h\right)_f - n\left(\nabla\cdot\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right),\nabla\cdot v_h\right)_f \\
&\quad - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}, v_h\right), \\
\tau_p^k(\psi_h) &= -gS_0\left(\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\psi_h\right)_p + \Delta t n g^2 C_{f,p}^2(\nabla(\phi^{k+1}-\phi^{k-1}),\nabla\psi_h)_p \\
&\quad + \Delta t n g^2 C_{f,p}^2(\phi^{k+1}-\phi^{k-1},\psi_h)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2},\psi_h\right).
\end{aligned}
$$

Decompose the error terms into

$$
\begin{aligned}
\mathcal{E}_u^{k+1} &= u^{k+1}-u_h^{k+1} = (u^{k+1}-\tilde{u}^{k+1}) + (\tilde{u}^{k+1}-u_h^{k+1}) = \eta_u^{k+1}+\xi_u^{k+1}, \\
\mathcal{E}_\phi^{k+1} &= \phi^{k+1}-\phi_h^{k+1} = (\phi^{k+1}-\tilde{\phi}^{k+1}) + (\tilde{\phi}^{k+1}-\phi_h^{k+1}) = \eta_\phi^{k-1}+\xi_\phi^{k+1}.
\end{aligned}
$$

Take $\tilde{u}^{k+1}\in V^h$, so that $\xi_u^{k+1}\in V^h$ and let $\tilde{\phi}^{k+1}\in X_p^h$. Then the error equations become:

$$
\begin{aligned}
&n\left(\frac{\xi_u^{k+1}-\xi_u^{k-1}}{2\Delta t},v_h\right)_f + n\left(\nabla\cdot\left(\frac{\xi_u^{k+1}-\xi_u^{k-1}}{2\Delta t}\right),v_h\right)_f + a_f\left(\frac{\xi_u^{k+1}+\xi_u^{k-1}}{2},v_h\right) + c_I(v_h,\xi_\phi^k) \\
&\quad = -n\left(\frac{\eta_f^{k+1}-\eta_u^{k-1}}{2\Delta t},v_h\right)_f - n\left(\nabla\cdot\left(\frac{\eta_u^{k+1}-\eta_u^{k-1}}{2\Delta t}\right),\nabla\cdot v_h\right)_f - a_f\left(\frac{\eta_u^{k+1}+\eta_u^{k-1}}{2},v_h\right) \\
&\quad - c_I(v_h,\eta_\phi^k) + \tau_f^k(v_h) + b\left(v_h,p^k-\lambda_h^k\right),
\end{aligned}
$$

$$
\begin{aligned}
&gS_0\left(\frac{\xi_\phi^{k+1}-\xi_\phi^{k-1}}{2\Delta t},\psi_h\right)_p + a_p\left(\frac{\xi_\phi^{k+1}+\xi_\phi^{k-1}}{2},\psi_h\right) + \Delta t n g^2 C_{f,p}^2(\nabla(\xi_\phi^{k+1}-\xi_\phi^{k-1}),\nabla\psi_h)_p \\
&+ \Delta t n g^2 C_{f,p}^2(\xi_\phi^{k+1}-\xi_\phi^{k-1},\psi_h)_p - c_I(\xi_u^k,\psi_h) \\
&\quad = -gS_0\left(\frac{\eta_\phi^{k-1}-\eta_\phi^{k-1}}{2\Delta t},\psi_h\right)_p - a_p\left(\frac{\eta_\phi^{k+1}+\eta_\phi^{k-1}}{2},\psi_h\right) + c_I(\eta_u^k,\psi_h) \\
&\quad - \Delta t n g^2 C_{f,p}^2(\nabla(\eta_\phi^{k+1}-\eta_\phi^{k-1}),\nabla\psi_h)_p - \Delta t n g^2 C_{f,p}^2(\eta_\phi^{k+1}-\eta_\phi^{k-1},\psi_h)_p + \tau_p^k(\psi_h).
\end{aligned}
$$

Pick $v_h = \xi_u^{k+1} + \xi_u^{k-1} \in V^h$ and $\psi_h = \xi_\phi^{k+1} + \xi_\phi^{k-1} \in X_p^h$ in the equations above, multiply by $2\Delta t$, and add. This produces

$$
\begin{aligned}
&\left(n\|\xi_u^{k+1}\|_{DIV,f}^2 + gS_0\|\xi_\phi^{k+1}\|_p^2 + \Delta t^2 ng^2 C_{f,p}^2\|\xi_\phi^{k+1}\|_{H^1(\Omega_p)}^2\right)\\
&-\left(n\|\xi_u^{k-1}\|_{DIV,f}^2 + gS_0\|\xi_\phi^{k-1}\|_p^2 + \Delta t^2 ng^2 C_{f,p}^2\|\xi_\phi^{k-1}\|_{H^1(\Omega_p)}^2\right)\\
&+2\Delta t\left[c_I(\xi_u^{k+1}+\xi_u^{k-1},\xi_\phi^k) - c_I(\xi_u^k,\xi_\phi^{k+1}+\xi_\phi^{k-1})\right]\\
&+\Delta t\left[a_f(\xi_u^{k+1}+\xi_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}) + a_p(\xi_\phi^{k+1}+\xi_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1})\right]\\
=&n\left[\left(\eta_u^{k+1}-\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}\right)_f + n\left(\nabla\cdot(\eta_u^{k+1}-\eta_u^{k-1}),\nabla\cdot(\xi_u^{k+1}-\xi_u^{k+1})\right)_f\right]\\
&-[gS_0\left(\eta_\phi^{k+1}-\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1}\right)_p\\
&+2\Delta t^2 ng^2C_{f,p}^2(\nabla(\eta_\phi^{k+1}-\eta_\phi^{k-1}),\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1}))_p\\
&+2\Delta t^2 ng^2C_{f,p}^2(\eta_\phi^{k+1}-\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1})_p]\\
&-\Delta t\left[a_f\left(\eta_u^{k+1}+\eta_u^{k-1},\xi_u^{k+1}+\xi_u^{k-1}\right) + a_p\left(\eta_\phi^{k+1}+\eta_\phi^{k-1},\xi_\phi^{k+1}+\xi_\phi^{k-1}\right)\right]\\
&-2\Delta t\left[c_I(\xi_u^{k+1}+\xi_u^{k-1},\eta_\phi^k) - c_I(\eta_f^k,\xi_\phi^{k+1}+\xi_\phi^{k-1})\right]\\
&+2\Delta t\left[\tau_f^k(\xi_u^{k+1}+\xi_u^{k-1}) + b\left(\xi_u^{k+1}+\xi_u^{k-1},p^k-\lambda_h^k\right) + \tau_p^k(\xi_\phi^{k+1}+\xi_\phi^{k-1})\right].
\end{aligned}
$$

Similar to prior analyses, rewrite the coupling terms on the left hand side equivalently as

$$
\begin{aligned}
c_I(\xi_u^{k+1}&+\xi_u^{k-1},\xi_\phi^k) - c_I(\xi_u^k,\xi_\phi^{k+1}+\xi_\phi^{k-1})\\
&=\left(c_I(\xi_u^{k+1},\xi_\phi^k) - c_I(\xi_u^k,\xi_\phi^{k+1})\right) - \left(c_I(\xi_u^k,\xi_\phi^{k-1}) - c_I(\xi_u^{k-1},\xi_\phi^k)\right)\\
&= C_\xi^{k+\frac{1}{2}} - C_\xi^{k-\frac{1}{2}}.
\end{aligned}
$$

If we denote the ξ energy terms by

$$
\begin{aligned}
E_{\xi,\text{STAB}}^{k+1/2} := &\, n\|\xi_u^{k+1}\|_{DIV,f}^2 + gS_0\|\xi_\phi^{k+1}\|_p^2 + \Delta t^2 ng^2C_{f,p}^2\|\xi_\phi^{k+1}\|_{H^1(\Omega_p)}^2\\
&+n\|\xi_u^k\|_{DIV,f}^2 + gS_0\|\xi_\phi^k\|_p^2 + \Delta t^2 ng^2C_{f,p}^2\|\xi_\phi^k\|_{H^1(\Omega_p)}^2,
\end{aligned}
$$

and apply the coercivity of the forms $a_f(.,.)$ and $a_p(.,.)$, the inequality becomes

$$
\begin{aligned}
&E^{k+1/2}_{\xi,\text{STAB}} + 2\Delta t C^{k+\frac{1}{2}}_{\xi} - E^{k-1/2}_{\xi,\text{STAB}} - 2\Delta t C^{k-\frac{1}{2}}_{\xi} \\
&\qquad + \Delta t \left(n\nu \|\nabla(\xi^{k+1}_u + \xi^{k-1}_u)\|^2_f + g k_{min} \|\nabla(\xi^{k+1}_\phi + \xi^{k-1}_\phi\|^2_p \right) \\
&\leq - \Big[n \left(\eta^{k+1}_u - \eta^{k-1}_u, \xi^{k+1}_u + \xi^{k-1}_u\right)_f + n \left(\nabla\cdot\left(\eta^{k+1}_u - \eta^{k-1}_u\right), \nabla\cdot\left(\xi^{k+1}_u + \xi^{k-1}_u\right)\right)_f \Big] \\
&- [gS_0 \left(\eta^{k+1}_\phi - \eta^{k-1}_\phi, \xi^{k+1}_\phi + \xi^{k-1}_\phi\right)_p + 2\Delta t^2 n g^2 C^2_{f,p} (\nabla(\eta^{k+1}_\phi - \eta^{k-1}_\phi), \nabla(\xi^{k+1}_\phi + \xi^{k-1}_\phi))_p \\
&\quad + 2\Delta t^2 n g^2 C^2_{f,p} (\eta^{k+1}_\phi - \eta^{k-1}_\phi, \xi^{k+1}_\phi + \xi^{k-1}_\phi)_p] \\
&\quad - \Delta t \left[a_f \left(\eta^{k+1}_u + \eta^{k-1}_u, \xi^{k+1}_u + \xi^{k-1}_u\right) + a_p \left(\eta^{k+1}_\phi + \eta^{k-1}_\phi, \xi^{k+1}_\phi + \xi^{k-1}_\phi\right) \right] \\
&\quad - 2\Delta t \left[c_I(\xi^{k+1}_u + \xi^{k-1}_u, \eta^k_\phi) - c_I(\eta^k_f, \xi^{k+1}_\phi + \xi^{k-1}_\phi) \right] \\
&\quad + 2\Delta t \left[\tau^k_f(\xi^{k+1}_u + \xi^{k-1}_u) + b(\xi^{k+1}_u + \xi^{k-1}_u, p^k - \lambda^k_h) + \tau^k_p(\xi^{k+1}_\phi + \xi^{k-1}_\phi) \right],
\end{aligned}
\tag{5.17}
$$

Next, we bound each term on the right hand side of the above inequality. We bound the first two terms by the standard Cauchy-Schwarz and Young inequalities along with the Poincaré inequality (Lemma 2).

$$
\begin{aligned}
n\Big(\eta^{k+1}_u - \eta^{k-1}_u, \xi^{k+1}_u + \xi^{k-1}_u\Big)_f &+ n \left(\nabla\cdot\left(\eta^{k+1}_u - \eta^{k-1}_u\right), \nabla\cdot\left(\xi^{k+1}_u + \xi^{k-1}_u\right)\right)_f \\
&\leq \frac{6nC^2_{P,f}}{\nu\Delta t} \|\eta^{k+1}_u - \eta^{k-1}_u\|^2_f + \frac{6nd^2}{\nu\Delta t} \|\nabla\left(\eta^{k+1}_u - \eta^{k-1}_u\right)\|^2_f \\
&\quad + \Delta t \frac{n\nu}{12} \|\nabla(\xi^{k+1}_u + \xi^{k-1}_u)\|^2_f ,
\end{aligned}
$$

$$
\begin{aligned}
gS_0(\eta^{k+1}_\phi - \eta^{k-1}_\phi, \xi^{k+1}_\phi + \xi^{k-1}_\phi)_p &+ 2\Delta t^2 n g^2 C^2_{f,p} \left(\nabla\left(\eta^{k+1}_\phi - \eta^{k-1}_\phi\right), \nabla\left(\xi_\phi^{k+1} + \xi_\phi^{k-1}\right)\right)_p \\
&+ 2\Delta t^2 n g^2 C^2_{f,p} \left(\eta^{k+1}_\phi - \eta^{k-1}_\phi, \xi_\phi^{k+1} + \xi_\phi^{k-1}\right)_p \\
&\leq \frac{15gC^2_{P,p}}{2k_{min}\Delta t} \left(S^2_0 + 4\Delta t^4 g^2 C^4_{f,p}\right) \|\eta^{k+1}_\phi - \eta^{k-1}_\phi\|^2_p \\
&\quad + \frac{30\Delta t^3 g^3 C^4_{f,p}}{k_{min}} \|\nabla\left(\eta^{k+1}_\phi - \eta^{k-1}_\phi\right)\|^2_p + \Delta t \frac{g k_{min}}{10} \|\nabla(\xi^{k+1}_\phi + \xi^{k-1}_\phi)\|^2_p.
\end{aligned}
$$

To bound the second term, we apply the continuity of the bilinear forms, $a_f(.,.)$ and $a_p(.,.)$, proven in Lemma 3. This yields

$$
\begin{aligned}
a_f(\eta_u^{k+1}+&\eta_u^{k-1}, \xi_u^{k+1}+\xi_u^{k-1}) + a_p(\eta_\phi^{k+1}+\eta_\phi^{k-1}, \xi_\phi^{k+1}+\xi_\phi^{k-1}) \\
&\leq M_f \|\nabla(\eta_u^{k+1}+\eta_u^{k-1})\|_f \|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f \\
&\qquad + gk_{max}\|\nabla(\eta_\phi^{k+1}+\eta_\phi^{k-1})\|_p \|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \\
&\leq \frac{3M_f^2}{n\nu}\|\nabla(\eta_u^{k+1}+\eta_u^{k-1})\|_f^2 + \frac{5gk_{max}^2}{2k_{min}}\|\nabla(\eta_\phi^{k+1}+\eta_\phi^{k-1})\|_p^2 \\
&\quad + \frac{n\nu}{12}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2 + \frac{gk_{min}}{10}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2.
\end{aligned}
$$

We bound the coupling terms on the right hand side using the (HDIV-TRACE), Poincaré (Lemma 2) and Young inequalities. Letting $C = C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f} C_{P,p} g^2 n$, this produces

$$
\begin{aligned}
&c_I(\xi_u^{k+1}+\xi_u^{k-1}, \eta_\phi^k) - c_I(\eta_f^k, \xi_\phi^{k+1}+\xi_\phi^{k-1}) \\
&\quad \leq ng\left(\|(\xi_u^{k+1}+\xi_u^{k-1})\cdot\hat{n}_f\|_I \|\eta_\phi^k\|_I + \|\eta_f^k\cdot\hat{n}_u\|_I \|\xi_\phi^{k+1}+\xi_\phi^{k-1}\|_I\right) \\
&\quad \leq C_{\Omega_f} C_{\Omega_p} ng \Big\{ \|\xi_u^{k+1}+\xi_u^{k-1}\|_f^{1/2} \|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^{1/2} \|\eta_\phi^k\|_p^{1/2} \|\nabla\eta_\phi^k\|_p^{1/2} \\
&\qquad + \Big\|\xi_\phi^{k+1}+\xi_\phi^{k-1}\|_p^{1/2} \|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^{1/2} \|\eta_u^k\|_f^{1/2} \|\nabla\eta_u^k\|_f^{1/2} \Big\} \\
&\quad \leq \sqrt{C}\left\{ \|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f \|\nabla\eta_\phi^k\|_p + \|\nabla\eta_u^k\|_f \|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \right\{ \\
&\quad \leq \frac{6C}{\nu}\|\nabla\eta_\phi^k\|_f^2 + \frac{5Cn}{gk_{min}}\|\nabla\eta_u^k\|_p^2 + \frac{n\nu}{24}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2 + \frac{gk_{min}}{20}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2.
\end{aligned}
$$

Finally, we bound the pressure term, as well as the consistency errors, τ_f^k and τ_p^k, as follows. By the Cauchy-Schwarz, Young and Poincaré (Lemma 2) inequalities, these boundes are:

$$
\begin{aligned}
b\left(\xi_u^{k+1}+\xi_u^{k-1}, p^k-\lambda_h^k\right) &\leq n\|p^k-\lambda_h^k\|_f \|\nabla\cdot(\xi_u^{k+1}+\xi_u^{k-1})\|_f \\
&\leq \frac{6dn}{\nu}\|p^k-\lambda_h^k\|_f^2 + \frac{n\nu}{24}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2,
\end{aligned}
$$

$$
\begin{aligned}
\tau_f^k(\xi_u^{k+1}+\xi_u^{k-1}) &= -n\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}, \xi_u^{k+1}+\xi_u^{k-1}\right)_f \\
&\quad - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}, \xi_u^{k+1}+\xi_u^{k-1}\right) \\
&\quad - n\left(\nabla\cdot\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right), \nabla\cdot\left(\xi_u^{k+1}+\xi_u^{k-1}\right)\right)_f
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_{P,f}n\left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f \|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f \\
&\quad + dn\left\|\nabla\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right)\right\|_f \|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f \\
&\quad + M_f\left\|\nabla\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f \|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f \\
&\leq \frac{9C_{P,f}^2 n}{\nu}\left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 + \frac{9M_f^2 n}{\nu}\left\|\nabla\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 \\
&\quad + \frac{9d^2 n}{\nu}\left\|\nabla\left(u_t^k - \frac{u^{k+1}+u^{k-1}}{2\Delta t}\right)\right\|_f^2 + \frac{n\nu}{12}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2,
\end{aligned}
$$

$$
\begin{aligned}
\tau_p^k(\xi_\phi^{k+1}+\xi_\phi^{k-1}) &= -gS_0\left(\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}, \xi_\phi^{k+1}+\xi_\phi^{k-1}\right)_p \\
&\quad + \Delta t n g^2 C_{f,p}^2(\nabla(\phi^{k+1}-\phi^{k-1}), \nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1}))_p \\
&+ \Delta t n g^2 C_{f,p}^2(\phi^{k+1}-\phi^{k-1}, \xi_\phi^{k+1}+\xi_\phi^{k-1})_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}, \xi_\phi^{k+1}+\xi_\phi^{k-1}\right) \\
&\leq gS_0C_{P,p}\left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p \|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \\
&\quad + \Delta t n g^2 C_{f,p}^2\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \\
&\quad + \Delta t n g^2 C_{f,p}^2 C_{P,p}\|\phi^{k+1}-\phi^{k-1}\|_p\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \\
&\quad + gk_{max}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p \|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p \\
&\leq \frac{10gS_0^2C_{P,p}^2}{k_{min}}\left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \frac{10\Delta t^2 n^2 g^3 C_{f,p}^4}{k_{min}}\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p^2 \\
&\quad + \frac{10\Delta t^2 n^2 g^3 C_{f,p}^4 C_{P,p}^2}{k_{min}}\|\phi^{k+1}-\phi^{k-1}\|_p^2 + \frac{10gk_{max}}{k_{min}}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2 \\
&\quad + \frac{gk_{min}}{10}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2.
\end{aligned}
$$

After subsuming all the resulting ξ terms into the left hand side of inequality, absorb all constants into some $\hat{C}_0 > 0$, independent of time-step size, Δt, and mesh width, h. Group together the remaining terms, so that inequality (5.17) becomes

$$
\begin{aligned}
&E^{k+\frac{1}{2}}_{\xi,\text{STAB}} + 2\Delta t C^{k+\frac{1}{2}}_{\xi} - E^{k-\frac{1}{2}}_{\xi,\text{STAB}} - 2\Delta t C^{k-\frac{1}{2}}_{\xi} \\
&\quad + \Delta t\left(\frac{n\nu}{2}\|\nabla(\xi_u^{k+1}+\xi_u^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2\right) \\
&\le \hat{C}_0\Bigg\{(\Delta t)^{-1}\left\{\|\eta_u^{k+1}-\eta_u^{k-1}\|_f^2 + \|\eta_\phi^{k+1}-\eta_\phi^{k-1}\|_p^2 + \left\|\nabla\left(\eta_u^{k+1}-\eta_u^{k-1}\right)\|_f^2\right\}\right. \\
&+ \Delta t\Big\{\Delta t^2\left\|\eta_\phi^{k+1}-\eta_\phi^{k-1}\right\|_{1,p}^2 + \left\|\nabla(\eta_u^{k+1}+\eta_u^{k-1})\right\|_f^2 + \|\nabla(\eta_\phi^{k+1}+\eta_\phi^{k-1})\|_p^2 \\
&+ \|\nabla\eta_u^k\|_f^2 + \|\nabla\eta_\phi^k\|_p^2 + \left\|u_t^k - \tfrac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 + \left\|\nabla\left(u^k - \tfrac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 \\
&+ \left\|\nabla\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right)\right\|_f^2 + \|p^k-\lambda_h^k\|_f^2 + \left\|\phi_t^k - \tfrac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 \\
&+ \left\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p^2 + \Delta t^2\|\phi^{k+1}-\phi^{k-1}\|_p^2 + \left\|\nabla\left(\phi^k - \tfrac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2\Big\}\Bigg\}\right. .
\end{aligned}
$$

Now, sum this inequality from $k = 1, ..., N-1$. This yields

$$
\begin{aligned}
&E^{N-\frac{1}{2}}_{\xi,\text{STAB}} + 2\Delta t C^{N-\frac{1}{2}}_{\xi} - E^{\frac{1}{2}}_{\xi,\text{STAB}} - 2\Delta t C^{\frac{1}{2}}_{\xi} \\
&\quad + \Delta t\sum_{k=1}^{N-1}\left(\frac{n\nu}{2}\|\nabla\left(\xi_u^{k+1}+\xi_u^{k-1}\right)\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\xi_\phi^{k+1}+\xi_\phi^{k-1})\|_p^2\right) \\
&\le \hat{C}_0\Bigg\{(\Delta t)^{-1}\sum_{k=1}^{N-1}\left\{\|\eta_u^{k+1}-\eta_u^{k-1}\|_f^2 + \|\eta_\phi^{k+1}-\eta_\phi^{k-1}\|_p^2 + \|\nabla\left(\eta_u^{k+1}-\eta_u^{k-1}\right)\|_f^2\right\} \\
&\quad + \Delta t\sum_{k=1}^{N-1}\Big\{\Delta t^2\|\nabla\left(\eta_\phi^{k+1}-\eta_\phi^{k-1}\right)\|_{1,p}^2 + \|\nabla(\eta_u^{k+1}+\eta_u^{k-1})\|_f^2 \\
&\qquad\qquad + \|\nabla(\eta_\phi^{k+1}+\eta_\phi^{k-1})\|_p^2 + \frac{12C}{\nu}\|\nabla\eta_u^k\|_f^2 + \frac{10C}{gk_{min}}\|\nabla\eta_\phi^k\|_p^2 \\
&\quad + \left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 + \left\|\nabla\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 + \left\|\nabla\left(u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right)\right\|_f^2 \\
&\quad + \|p^k-\lambda_h^k\|_f^2 + \left\|\phi_t^k - \frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \Delta t^2\|\nabla(\phi^{k+1}-\phi^{k-1})\|_p^2 \\
&\quad + \Delta t^2\|\phi^{k+1}-\phi^{k-1}\|_p^2 + \left\|\nabla\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2\Big\}\Bigg\}.
\end{aligned}
$$

To obtain a bound involving norms instead of summations, we use the following bounds derived in the proof of Theorem 26 in Chapter 3:

$$\sum_{k=1}^{N-1} \|\eta_u^{k+1} - \eta_u^{k-1}\|_f^2 \leq 4\Delta t \|\eta_{u,t}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{5.18}$$

$$\sum_{k=1}^{N-1} \|\eta_\phi^{k+1} - \eta_\phi^{k-1}\|_f^2 \leq 4\Delta t \|\eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{5.19}$$

$$\sum_{k=1}^{N-1} \|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f^2 \leq 4(\Delta t)^{-1} \||\nabla \eta_f|\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{5.20}$$

$$\sum_{k=1}^{N-1} \|\nabla(\eta_\phi^{k+1} + \eta_\phi^{k-1})\|_f^2 \leq 4(\Delta t)^{-1} \||\nabla \eta_p|\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{5.21}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_u^k\|_f^2 \leq (\Delta t)^{-1} \||\nabla \eta_f|\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{5.22}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_\phi^k\|_p^2 \leq (\Delta t)^{-1} \||\nabla \eta_p|\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{5.23}$$

$$\sum_{k=1}^{N-1} \|p^k - \lambda_h^k\|_f^2 \leq (\Delta t)^{-1} \||p - \lambda_h|\|^2_{L^2(0,T;L^2(\Omega_f))}. \tag{5.24}$$

Substitute $\nabla\eta$ for η in the proofs of (5.18)-(5.19) to obtain

$$\sum_{k=1}^{N-1} \|\nabla\left(\eta_u^{k+1} - \eta_u^{k-1}\right)\|_f^2 \leq 4\Delta t \|\nabla \eta_{u,t}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{5.25}$$

$$\sum_{k=1}^{N-1} \|\nabla\left(\eta_\phi^{k+1} - \eta_\phi^{k-1}\right)\|_p^2 \leq 4\Delta t \|\nabla \eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))}. \tag{5.26}$$

Inequalities (5.25)-(5.26) imply

$$\sum_{k=1}^{N-1} \left\{\|\eta_u^{k+1} - \eta_u^{k-1}\|_f^2 + \|\nabla\left(\eta_u^{k+1} - \eta_u^{k-1}\right)\|_f^2\right\} \leq 4\Delta t \|\eta_{u,t}\|^2_{L^2(0,T;H^1(\Omega_f))}, \tag{5.27}$$

$$\sum_{k=1}^{N-1} \left\{\|\eta_\phi^{k+1} - \eta_\phi^{k-1}\|_p^2 + \|\nabla\left(\eta_\phi^{k+1} - \eta_\phi^{k-1}\right)\|_p^2\right\} \leq 4\Delta t \|\eta_{\phi,t}\|^2_{L^2(0,T;H^1(\Omega_p))}. \tag{5.28}$$

After applying bounds (5.18)-(5.28), along with the consistency error bounds from Lemmas 25 and 37, and the bound (3.3) from the stability proof, there holds, for some $\hat{C}_1$:

$$
\begin{aligned}
&\frac{1}{2}(\|\xi_u^N\|^2_{DIV,f} + \|\xi_u^{N-1}\|^2_{DIV,f}) + gS_0(\|\xi_\phi^N\|_p^2 + \|\xi_\phi^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{n\nu}{2} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2 + \frac{gk_{min}}{2} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^2 \right) \\
&\quad \leq \widehat{C}_1 \Big\{ \|\eta_{u,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + \Delta t^4 \|\eta_{\phi,t}\|^2_{L^2(0,T;H^1(\Omega_p))} \\
&\quad + \||\nabla \eta_u|\|^2_{L^2(0,T;L^2(\Omega_f))} + \||\nabla \eta_\phi|\|^2_{L^2(0,T;L^2(\Omega_p))} + \Delta t^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} \\
&\quad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T,H^1(\Omega_p))} \\
&\quad + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) + \||p - \lambda_h|\|^2_{L^2(0,T;L^2(\Omega_f))} \Big\} + E^{1/2}_{\xi,\text{STAB}} + 2\Delta t C_\xi^{1/2}.
\end{aligned} \tag{5.29}
$$

Recall that the error terms equal $\mathcal{E}_u^N = u^N - u_h^N = \eta_u^N + \xi_u^N$ and $\mathcal{E}_\phi^N = \phi^N - \phi_h^N = \eta_\phi^N + \xi_\phi^N$. Applying the triangle inequality produces

$$
\begin{aligned}
&\tfrac{1}{4}(\|\mathcal{E}_u^N\|^2_{DIV,f} + \|\mathcal{E}_u^{N-1}\|^2_{DIV,f}) + \tfrac{gS_0}{2}(\|\mathcal{E}_\phi^N\|_p^2 + \|\mathcal{E}_\phi^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\tfrac{n\nu}{4} \|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|_f^2 + \tfrac{gk_{min}}{4} \|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|_p^2 \right) \\
&\quad \leq \tfrac{1}{2}(\|\xi_u^N\|^2_{DIV,f} + \|\xi_u^{N-1}\|^2_{DIV,f}) + gS_0(\|\xi_\phi^N\|_p^2 + \|\xi_\phi^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\tfrac{n\nu}{2} \|\nabla(\xi_u^{k+1} + \xi_u^{k-1})\|_f^2 + \tfrac{gk_{min}}{2} \|\nabla(\xi_\phi^{k+1} + \xi_\phi^{k-1})\|_p^2 \right) \\
&\quad + \tfrac{1}{2}(\|\eta_u^N\|^2_{DIV,f} + \|\eta_u^{N-1}\|^2_{DIV,f}) + gS_0(\|\eta_\phi^N\|_p^2 + \|\eta_\phi^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\tfrac{n\nu}{2} \|\nabla(\eta_u^{k+1} + \eta_u^{k-1})\|_f^2 + \tfrac{gk_{min}}{2} \|\nabla(\eta_\phi^{k+1} + \eta_\phi^{k-1})\|_p^2 \right).
\end{aligned}
$$

Because $\|\eta_{f,p}^N\|^2_{f,p}, \|\eta_{f,p}^{N-1}\|^2_{f,p} \leq \||\eta_{f,p}|\|^2_{L^\infty(0,T;L^2(\Omega_{f,p}))}$, $\|\eta_f^N\|^2_{DIV,f} \leq d\||\eta_f|\|^2_{L^\infty(0,T;H^1(\Omega_f))}$. This, along with the previous bounds for η terms and inequality (5.29) implies

$$
\begin{aligned}
&\tfrac{1}{4}(\|\mathcal{E}_u^N\|^2_{DIV,f} + \|\mathcal{E}_u^{N-1}\|^2_{DIV,f}) + \tfrac{gS_0}{2}(\|\mathcal{E}_\phi^N\|^2_p + \|\mathcal{E}_\phi^{N-1}\|^2_p) \\
&+ \Delta t \sum_{k=1}^{N-1} \left(\tfrac{n\nu}{4}\|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|^2_f + \tfrac{gk_{min}}{4}\|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|^2_p \right) \\
&\leq \widehat{C}_2 \Big\{ \|\eta_{u,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + \Delta t^4 \|\eta_{\phi,t}\|^2_{L^2(0,T;H^1(\Omega_p))} \\
&+ |||\nabla \eta_u|||^2_{L^2(0,T;L^2(\Omega_f))} + |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} + |||p - \lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&+ \Delta t^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&+ \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) + |||\eta_f|||^2_{L^\infty(0,T;H^1(\Omega_f))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))} \\
&+ \|\xi_u^1\|^2_{DIV,f} + \|\xi_u^0\|^2_{DIV,f} + \|\xi_\phi^1\|^2_\phi + \|\xi_\phi^0\|^2_p + \Delta t^2 (\|\xi_\phi^1\|^2_{1,p} + \|\xi_\phi^0\|^2_{1,p}) + 2\Delta t C_\xi^{1/2} \Big\},
\end{aligned} \tag{5.30}
$$

where we absorbed all constants into a new constant, $\widehat{C}_2 > 0$.

Now, we bound the coupling terms on the right hand side as follows:

$$
C_\xi^{1/2} \leq \frac{C}{2} \left(\|\xi_\phi^0\|^2_{1,p} + \|\xi_\phi^1\|^2_{1,p} + \|\xi_u^0\|^2_{DIV,f} + \|\xi_u^1\|^2_{DIV,f} \right). \tag{5.31}
$$

Inequality (5.30) holds for any $\tilde{u} \in V^h$, $\lambda_h \in Q_f^h$, and $\tilde{\phi} \in X_p^h$. Taking the infimum over the spaces V^h, Q_f^h, and X_p^h, using (5.8) to bound the infimum over V^h by the infimum over X_f^h, and finally, using bound (5.31), the following holds for some positive constant $\widehat{C}_3$:

$$
\begin{aligned}
&\frac{1}{2}(\|\mathcal{E}_u^N\|^2_{DIV,f} + \|\mathcal{E}_u^{N-1}\|^2_{DIV,f}) + gS_0(\|\mathcal{E}_\phi^N\|^2_p + \|\mathcal{E}_\phi^{N-1}\|^2_p) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2}\|\nabla(\mathcal{E}_u^{k+1} + \mathcal{E}_u^{k-1})\|^2_f + \frac{gk_{min}}{2}\|\nabla(\mathcal{E}_\phi^{k+1} + \mathcal{E}_\phi^{k-1})\|^2_p \right) \\
&\quad \leq \widehat{C}_3 \Big\{ \inf_{\tilde{u} \in X_f^h} \{ \|\eta_{u,t}\|^2_{L^2(0,T;H^1(\Omega_f))} + |||\nabla \eta_u|||^2_{L^2(0,T;L^2(\Omega_f))} + |||\eta_u|||^2_{L^\infty(0,T;H^1(\Omega_f))} \\
&\qquad + \|\xi_u^1\|^2_{DIV,f} + \|\xi_u^0\|^2_{DIV,f} \} + \inf_{\lambda_h \in Q_f^h} |||p - \lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad + \inf_{\tilde{\phi} \in X_p^h} \{ \|\eta_{\phi,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + \Delta t^4 \|\eta_{\phi,t}\|^2_{L^2(0,T;H^1(\Omega_p))} + |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))} + \|\xi_\phi^1\|^2_{1,p} + \|\xi_\phi^0\|^2_{1,p} \} + \Delta t^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega_f))} \\
&\qquad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) \Big\}.
\end{aligned}
$$

The final result, (5.16), immediately follows by applying the approximation assumptions given in (1.17). □

5.3 NUMERICAL EXPERIMENTS FOR CNLFSTAB-SD

Using the same exact solutions given in (Test) from [56], we verify the unconditional stability and second-order convergence of the (CNLFstab-SD)-method. All experiments were conducted using FreeFEM++ [36]. The code for the experiments is included in the appendix. In the Stokes region, we utilize Taylor-Hood elements (P2-P1) and piecewise quadratics (P2) in the Darcy region. Like the numerical tests performed in Chapter 3 for (CNLF-SD), we set the initial and first terms equal to the exact solutions.

5.3.1 Stability Experiments

To confirm unconditional stability of (CNLFstab-SD), set the body force and source functions, f_f and f_p equal to zero. Enforce homogeneous Dirichlet boundary conditions on the external boundary. Set the parameters n, g, α_{BJ},ν, and ρ equal to one. We test the stability of the method in four situations, (1) small k_{min} and S_0, (2) small S_0, (3) small k_{min}, and (4) $k_{min} = S_0 = 1.0$ over the time interval $[0, 10]$. With $h = 0.1$, we plot the final system energy for each choice of time-step size, Δt in Figure 5.1. We see that, in all situations, (CNLFstab-SD) is stable for large time steps, regardless of the size of k_{min} and S_0, as expected. This is a vast improvement over (CNLF-SD), which for $\Delta t \leq 1/80$ was only stable when $k_{min} = S_0 = 1.0$, and unstable in all other cases (see Figure 5.2).

We run the same stability test for the more exact implementation of (CNLFstab) for the Stokes-Darcy problem, with the added stability terms on the interface only, given below.

$$\Delta t g^n \int_I (\phi_h^{k+1} - \phi_h^{k-1})\psi_h \, ds \text{ and } \Delta t n^2 \int_I \left(u_h^{k+1} - u_h^{k-1}\right) \cdot \hat{n}_f v_h \cdot \hat{n}_f \, ds.$$

In Figure 5.3, it is clear that only adding stabilizing terms on the interface, I, is not sufficient for gaining unconditional stability. The method is unstable for all tested time-step sizes when

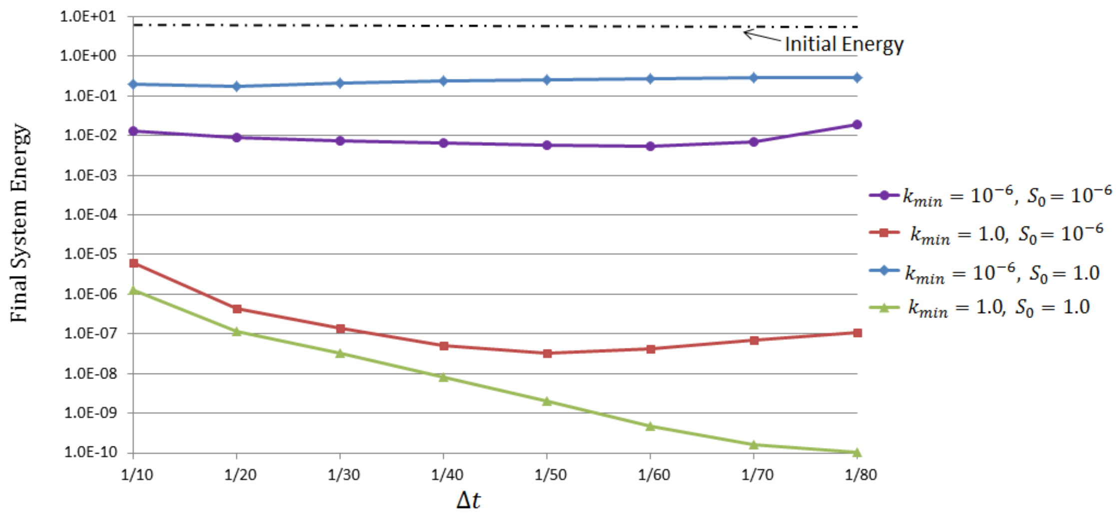

Figure 5.1: Final System Energy versus time-step size (Δt) for (CNLFSTAB-SD). The method is stable for all choice of time-step sizes.

both k_{min} and S_0 are small, does not become stable until $\Delta t < 1/70$ for small k_{min}, and becomes unstable for small S_0 as Δt decreases. Analysis of exactly why this method fails is still an open problem.

5.3.2 Convergence Rate Experiments

We next test the convergence rate of the (CNLFSTAB-SD)-method. Set the parameters n, ρ, α_{BJ}, ν, S_0, $\mathcal{K}$, and g equal to 1 and apply inhomogeneous Dirichlet external boundary conditions: $u_h = u$ on $\Omega_f \setminus I$, $\phi_h = \phi$ on $\Omega_p \setminus I$. As before, set the initial conditions, as well as the first terms in the method, to match the exact solutions. We set $h = \Delta t$ and calculate the errors and convergence rates for the variables u, p, and ϕ in Table 5.1 over the time interval $[0, 1]$. Define the norms for the errors, $E(u)$, $E(p)$, and $E(\phi)$, as follows.

$$E(u) = |||u - u_h|||_{L^\infty(0,T;DIV(\Omega_f))},$$

$$E(p) = |||p - p_h|||_{L^\infty(0,T;L^2(\Omega_f))},$$

$$E(\phi) = |||\phi - \phi_h|||_{L^\infty(0,T;L^2(\Omega_p))}.$$

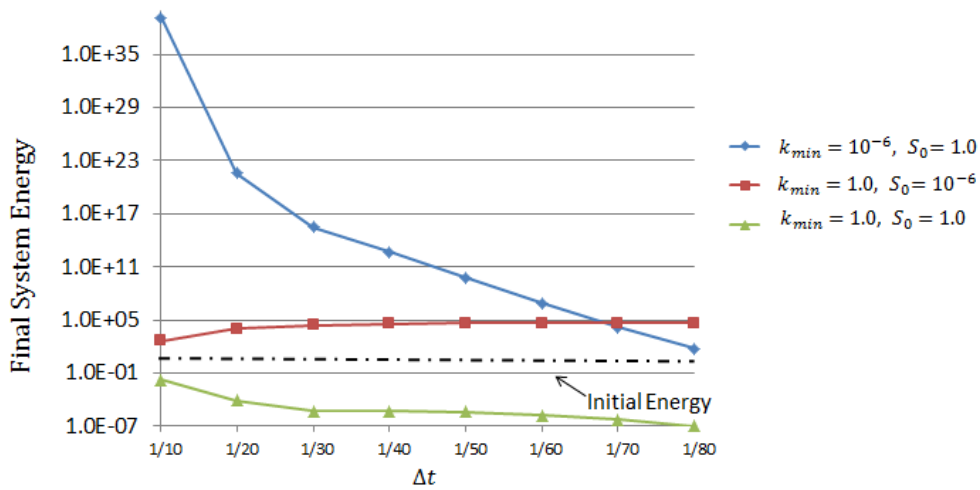

Figure 5.2: Final System Energy versus time-step size (Δt) for (CNLF-SD). The method is unstable for all $0 < \Delta t < 1/80$ for cases of small parameters.

As predicted, we have second-order convergence for the Stokes velocity, u, Stokes pressure, p, and Darcy pressure, ϕ. Convergence of the Stokes pressure, p, should follow a similar path as convergence of p in (CNLF-SD), discussed briefly in Remark 28.

5.4 CONCLUSIONS FOR CNLFSTAB-SD

As discussed in Chapter 3, (CNLF-SD), while second-order convergent, has less than desirable stability properties when faced with small problem parameters. This is due to the sensitivity of the CFL-type time-step condition, (Δt_{CNLF}), to small values of specific storage, S_0. The adaptation of the method, (CNLFSTAB), for the Stokes-Darcy problem presented in this chapter addresses this sensitivity and is an unconditionally stable, second-order, parallel partitioned method. In addition, the (CNLFSTAB-SD)-method effectively controls the unstable mode arising from the use of Leapfrog. It is interesting to see numerical experiments imply that the natural implementation of the (CNLFSTAB)-method, which involves adding

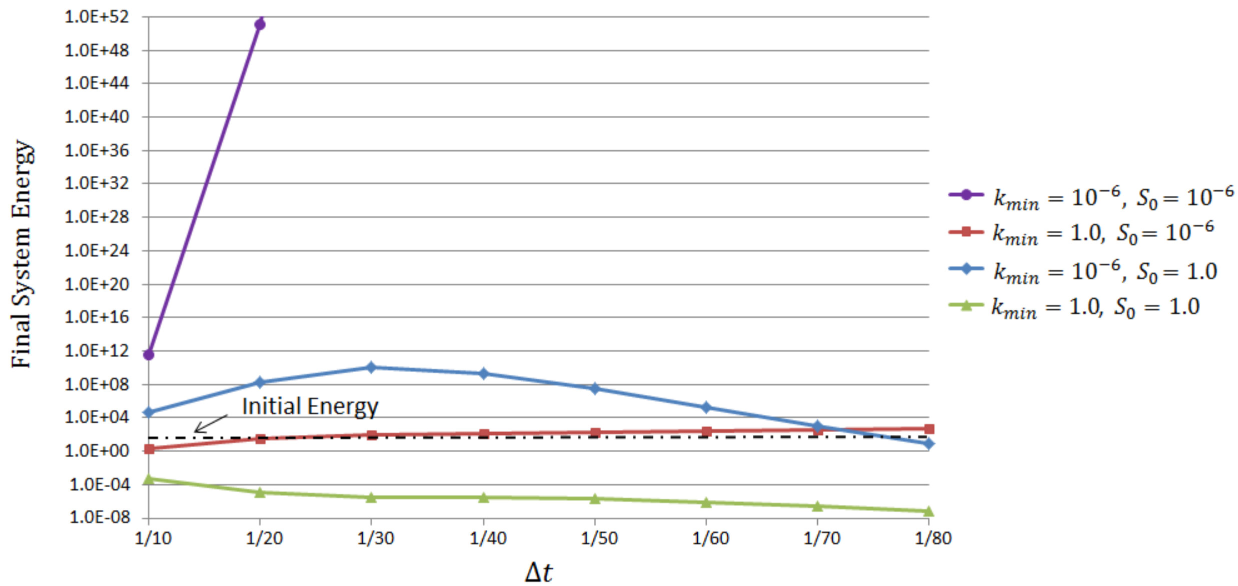

Figure 5.3: Final System Energy versus time-step size (Δt) for the natural adaptation of (CNLFSTAB) for Stokes-Darcy in which we only add stability terms on the interface, I.

stabilizing terms only on the interface, I, is not sufficient to produce an unconditionally stable method. The analysis of the insufficiency of this natural adaptation remains an open problem.

Table 5.1: Rates of convergence for (CNLFSTAB-SD)

$h = \Delta t$	$E(u)$	rate	$E(p)$	rate	$E(\phi)$	rate
1/10	7.30706e-3		1.0229		1.46307e-1	
1/20	1.43894e-3	2.34	2.56199e-1	1.99	3.65586e-2	2.00
1/40	3.02353e-4	2.25	6.09220e-2	2.07	9.01390e-3	2.02
1/80	6.02521e-5	2.33	1.45354e-2	2.07	2.223117e-3	2.01

6.0 CONCLUSIONS AND FUTURE WORK

6.1 SUMMARY OF RESULTS

With freshwater being a necessary component for survival of humankind, it is imperative that we take measures to protect this vital resource. Hence, it is important to develop tools for tracking pollutants that enter groundwater and surface water flow so that we may act accordingly to prevent and react to contamination. Because these flows occur on such a large scale, accurate numerical models are a great asset to this problem. The first step is to develop numerical methods for describing groundwater-surface water flow. To be of practical use, these methods must be adaptable to a wide variety of geographical situations, capture both the separate fluid and porous media fluid movement as well as the interactions between the flows, stay stable over long-time intervals, and converge within a reasonable amount of time.

In this research, we focused on the development and study of higher-order convergent methods for the fully evolutionary Stokes-Darcy problem with strong stability properties. Because groundwater flow moves slowly, calculations must be made over longer time intervals, making higher-order convergence a great practical advantage. In this problem, strong stability properties refers to being long-time stable as well as stable in a variety of physical situations, especially given the small parameters, S_0 (specific storage), and k_{min} (smallest eigenvalue of hydraulic conductivity). The singular limits of these parameters have physical meaning, with $S_0 \to 0$ corresponding to the case of confined aquifers, and $k_{min} \to 0$ corresponding to impermeable porous media. Table 6.1 gives a summary of the methods studied herein, along with their stability and convergence properties.

The splitting methods, (BEsplit1-SD) and (BEsplit2-SD) are both long-time stable

Table 6.1: Summary of convergence rates and stability properties of partitioned methods applied to Stokes-Darcy. C represents a positive constant (different for each condition) that is independent of h and Δt.

Method	Convergence Rate	Stability Condition
(BEsplit1-SD)	$\mathcal{O}(\Delta t(h+1)+h^2)$	$\Delta t < C\max\{k_{min}, S_0 k_{min}, S_0 h\}$ or $C\sqrt{k_{min}} \geq 1$
(BEsplit2-SD)	$\mathcal{O}(\Delta t(h+1)+h^2)$	$\Delta t < Ck_{min}\max\{1, S_0, h\}$ or $C\sqrt{k_{min}} \geq 1$
(CNsplit-SD)	$\mathcal{O}(\Delta t^2 + h^2 + \Delta t(h+1))$	$\Delta t < C\sqrt{S_0}h$
(CNLF-SD)	$\mathcal{O}(\Delta t^2 + h^2)$	$\Delta t < C\max\{\min\{h^2, S_0\}, \min\{h, S_0 h\}\}$
(CNLFstab-SD)	$\mathcal{O}(\Delta t^2 + h^2)$	none

and have desirable stability properties when faced with either small S_0 or small k_{min}. However, these methods become highly unstable when both parameters are small. Both the (CNsplit-SD) and (CNLF-SD) methods have stronger convergence properties, but demonstrate sensitivity to small values of S_0, making these methods impractical in those scenarios. Finally, the adaptation of the (CNLFstab)-method for the Stokes-Darcy problem, (CNLFstab-SD), exhibits the best stability and convergence properties of all the methods studied in this research. Its unconditional stability and second-order convergence make it a highly desirable method for modeling groundwater-surface flow over long-time periods in a variety of physical situations.

6.2 FUTURE WORK

6.2.1 Asynchronous Stokes-Darcy Methods

This dissertation culminated in the development and analysis of a second-order, unconditionally stable numerical method for modeling groundwater-surface water flow. Development and analysis of methods for groundwater-surface water flow, while well-studied by many mathematicians (see, for example, [24, 27, 45, 15], to name a few), still remains an active area of research. The current frontier in the area lies with higher-order, multi-rate methods. A multi-rate method can take advantage of the reality that groundwater flow moves at a much slower rate than surface water. Thus, it is practical to take different time-step sizes in each domain, relative to the rates of the flows.

Partitioned methods for uncoupling groundwater-surface water flow extend naturally to multi-rate methods, in which for each large time-step in the Darcy region, we solve for the flow in the Stokes region several times at a smaller time step. Multi-rate methods are especially advantageous for studying flow in large aquifers with low conductivity, requiring accurate calculations over long-time periods. Two multi-rate partitioned methods were studied in [73, 46]. However, these multi-rate methods are first-order convergent with stability restrictions dependent on the potentially small parameters, S_0 and k_{min}. Thus, future research in the area should involve:

- *Developing and analyzing higher-order, strongly stable multi-rate partitioned methods for groundwater-surface water flow.*
- *Exploring the behavior of these methods in the physically relevant singular limits:* $S_0 \to 0$ *(quasistatic Stokes-Darcy) and* $k_{min} \to 0$ *(impermeable porous media).*
- *Developing a systematic theory of the methods and separate the particular from the universal principles so that these methods may be applied to other physically important coupled problems.*

6.2.2 Stokes-Darcy + Transport

The primary motivation for this research on numerical methods for the fully evolutionary Stokes-Darcy problem originated from several modern environmental problems that require tracking contaminants in groundwater-surface water flow. While we have developed one higher-order, unconditionally stable method, (CNLFstab-SD), it remains to couple this method with the convection-diffusion equation (6.1) to model the path of pollutants in coupled flow.

In time-dependent Stokes-Darcy flow, the inclusion of transport adds a hidden reaction term since the fluid velocity in the evolutionary porous media flow equation is not divergence free. In coupled flow, the transport of contaminants is given by

$$\beta c_t + \nabla \cdot (u_\Omega c - D\nabla c) = \beta s, \tag{6.1}$$

where c represents some concentration, β the porosity, D the diffusion/dispersion tensor, and s any sources or sinks. Recall that, when applied to groundwater-surface water flow, $u_\Omega = u$ in Ω_f and u_p in Ω_p. A system nonlinearity arises from the term $\nabla \cdot (u_\Omega c)$. If $\nabla \cdot u_\Omega$ equals zero or is bounded, as is true in the fluid region ($\nabla \cdot u = 0$), then standard tools suffice. However, in the porous media region, after multiplying by c and integrating in space, the energy estimate produces the extra term

$$\int_{\Omega_p} (\nabla \cdot u_p) c^2 \, dx = \int_{\Omega_p} \nabla \cdot (-\mathcal{K}\nabla\phi) c^2 \, dx.$$

Recall that in evolutionary porous media flow, ϕ satisfies

$$S_0 \phi_t - \nabla \cdot (\mathcal{K}\nabla\phi) = 0 \text{ (or some source)},$$

Slow moving flow over large domains necessitates predictions over long-time intervals, during which this nonlinear reaction term dominates the physical behavior of the convective flux. Prior work on coupled flow with transport has assumed quasi-static ($S_0 = 0$) or steady fluid flow (see, for example, [22, 21, 68, 60, 2]), in which the reaction term satisfies $-\nabla \cdot (\mathcal{K}\nabla\phi) \equiv 0$ (or some source). Thus it remains to develop and analyze numerical methods for the fully evolutionary coupled flow with transport problem.

There is also the case of reactive transport for biological, chemical, or nuclear contaminants, in which the source function, s, satisfies, $s = s(c)$. This case is physically relevant for many environmental problems, such as tracking radioactive contaminants or landfill leachate. In [32] and [64], *stationary* porous media flow and coupled flow with evolutionary reactive transport are presented and studied, but research on the fully evolutionary case remains an open question.

In connection to the research on higher-order, strongly stable partitioned methods for coupled flow presented in this dissertation, the next steps towards modeling pollution in groundwater-surface water flow should involve:

- *Developing and analyzing methods for contaminant transport compatible with partitioned and multi-rate methods for fully evolutionary Stokes-Darcy flow.*
- *Developing and analyzing space-time discontinuous Galerkin methods for the fully time-dependent Stokes-Darcy-Transport problem.*
- *Adapting these methods to the case of reactive transport, in which $s = s(c)$.*
- *Developing a systematic theory of the methods by separating the particular from the universal principles so that these methods may be applied to other physically important problems involving coupled flow with transport.*

APPENDIX

STOKES-DARCY CODE

All numerical experiments for the Stokes-Darcy problem in this research were performed using FreeFEM++ [36]. The code for the experiments follows below.

```
/*
Michaela Kubacki
Code for Stokes-Darcy problem. Includes splitting methods
    (BEsplit1,BEsplit2,CNsplitA,CNsplitB,CNsplit) CNLF, CNLFstab, CNLFstab on I.
    Stability and Convergence tests.
*/

verbosity=0;
int plotson=0
;

int j;
//start time-loop for different choices of dt, h
for(j=1;j< 17; j++){
string fnse = "SD_DATE_METHOD_TEST_CONDITION";
int M=10 * j; // =10 * 2 ^ (j-1); for error tests, = 10 * j; for stability tests
real dt=1.0/M; //time step size
int numsteps = 10*M; //number of time steps = 10*M; for stability, = M; for
    convergence
int i;

ofstream report(fnse + "_j_" + j + "_dt_" + dt + "_M_" + M + ".txt");

//**********************THE MESH*************************
mesh Omegaf; // mesh in fluid region
mesh Omegap; // mesh in porous media region

border C1(t=0.0,1.0){ x=1.0-t; y=2.0;   label=1; }   //Stokes top
```

```
border C2(t=0.0,1.0){ x=0.0; y=2.0-t;   label=2; }  //Stokes left
border C3(t=0.0,1.0){ x=t;   y=1.0;     label=3; }  //Stokes bottom --> Interface
border C4(t=0.0,1.0){ x=1.0; y=1.0+t;   label=4; }  //Stokes right

border B1(t=0.0,1.0){ x=1.0-t; y=1.0;    label=5; } //Darcy top --> Interface
border B2(t=0.0,1.0){ x=0.0;  y=1.0-t;   label=6; } //Darcy left
border B3(t=0.0,1.0){ x=t;    y=0.0;     label=7; } //Darcy bottom
border B4(t=0.0,1.0){ x=1.0;  y=t;       label=8; } //Darcy right

//for fixed meshwidth (stability tests)
Omegaf=buildmesh(C1(10) + C2(10) + C3(10) + C4(10));
Omegap=buildmesh(B1(10) + B2(10) + B3(10) + B4(10));

//for varied meshwidth (h = \Delta t)
Omegaf=buildmesh(C1(M) + C2(M) + C3(M) + C4(M));
Omegap=buildmesh(B1(M) + B2(M) + B3(M) + B4(M));

//plot mesh
if (plotson){
plot(Omegaf,Omegap,wait=1);
}

//***************SPACES, FUNCTIONS, VARIABLES*******************

//finite element spaces ------>Taylor Hood in Stokes, P2 in Darcy
fespace Xf(Omegaf,P2); //Stokes velocity space
fespace Qf(Omegaf,P1); // Stokes pressure space
fespace Xp(Omegap,P2); // Darcy pressure space



//Stokes velocity, Stokes pressure, Darcy pressure functions
Xf u1, u2, u1old, u2old, u1old2, u2old2, u1soln, u2soln, divuold, uA1, uA2,
   uA1old, uA2old, uB1, uB2, uB1old, uB2old, divu, v1, v2;
Qf p, pold, pold2, pA, pB, pAold, pBold, psoln, q;
Xp phi, phiold, phiold2, phisoln, up1soln, up2soln, phiA, phiAold, phiB,
   phiBold, psi, w1, w2;

//constants for the flow problem
real nu = 1.0; //kinematic viscosity of fluid
real S0= 0.000001; //specific storage coefficient
real g = 1.0; //gravitational acceleration constant
real alpha = 1.0; //measured slip coefficient
real PRESSUREPENALTY=0.000001;
real Kappa1 = 0.000001; // first term of the hydraulic conductivity tensor
real Kappa2 = 0.0;
real Kappa3 = 0.0;
```

```
real Kappa4 = 1.0;
real gamma = 0.0; // RAW filter constant
real chi = 0.1; // RA filter constant

macro contract(v1,v2,v3,v4) ( dx(v1)*dx(v3) + dy(v1)*dy(v3) + dx(v2)*dx(v4) +
    dy(v2)*dy(v4) ) //tensor contraction for grad u : grad v
macro div(v1,v2) ( dx(v1) + dy(v2) ) // for div(u)
macro dot(v1,v2,v3,v4) (v1*v3 + v2*v4) // dot product of 2 vectors


//*********************DEFINE TRUE SOLUTIONS*******************
//from Mu Zhu paper

//Stokes velocity, pressure and Darcy velocity, pressure
func real u1true(real t) {return ( (x^2.0*(y-1.0)^2.0 + y) * cos(t) ); }
func real u2true(real t) {return ( (-2.0/3.0 * x * (y-1.0)^3.0) * cos(t) +
    (2.0-pi*sin(pi*x))* cos(t) ); }
func real phitrue(real t) {return ( ( 2.0-pi*sin(pi*x) )*( 1.0-y-cos(pi*y)
    )*cos(t) ); }
func real ptrue(real t) {return ( 2.0 - pi*sin(pi*x) )*(sin(.5*pi*y) )*cos(t); }
func real divutrue(real t) {return (2.0*x*(y-1.0)^2.0*cos(t) -2.0 * x*
    (y-1.0)^2.0*cos(t)); }
func real up1true(real t) {return (
    Kappa1*cos(t)*pi^2.0*cos(pi*x)*(1.0-y-cos(pi*y)) ); }
func real up2true(real t) {return ( -Kappa1*cos(t)*(2.0-pi*sin(pi*x))*(-1.0 +
    pi*sin(pi*y))); }

// forcing in fluid region f_f=(f1,f2)
func real f1(real t) {return ( -(x^2.0*(y-1.0)^2.0 + y)*sin(t)- nu*(
    2.0*(y-1.0)^2.0*cos(t) + 2.0*x^2.0*cos(t) ) -
    pi^2.0*cos(pi*x)*sin(pi/2.0*y)*cos(t) ); }
func real f2(real t) {return ( -(-2.0/3.0*x*(y-1.0)^3.0)*sin(t) -
    (2.0-pi*sin(pi*x))*sin(t)- nu*( pi^3.0*sin(pi*x)*cos(t) -
    4.0*x*(y-1.0)*cos(t) ) + (2.0-pi*sin(pi*x))*(pi/2.0)*cos(pi/2.0*y)*cos(t) ); }
// source in porous region (Darcy)
func real fp(real t) {return ( S0*( 2.0-pi*sin(pi*x) )*( 1.0-y-cos(pi*y)
    )*(-1.0)*sin(t)
                             - Kappa1*( pi^3.0*sin(pi*x) )*( 1.0-y-cos(pi*y)
                               )*cos(t)
                             - Kappa1*( 2.0-pi*sin(pi*x) )*( pi^2.0*cos(pi*y)
                               )*cos(t) ); }

//assign values for the first two time steps (for CNLF, CNLFstab)
//For 3-level methods: CNLF and CNLFstab
real tnminus1=0.0;
real tn=dt;
real tnplus1=2*dt;
```

```
phiold2=phitrue(0.0);
 phiold=phitrue(dt);
u1old2=u1true(0.0);
 u1old=u1true(dt);
u2old2=u2true(0.0);
 u2old=u2true(dt);
pold2=ptrue(0.0);
 pold=ptrue(dt);
 divuold=divutrue(0.0);
 divu=divutrue(dt);

 //For 2-level methods: BE and CN split
 real tn=0.0;
 real tnplus1=dt;
 real thalf = 0.5 * (tn + tnplus1);
 phiold=phitrue(0.0);
 u1old=u1true(0.0);
 u2old=u2true(0.0);
 pold=ptrue(0.0);
 uA1old=u1true(0.0);
 uA2old=u2true(0.0);
 phiAold=phitrue(0.0);
 pAold=ptrue(0.0);
 uB1old=u1true(0.0);
 uB2old=u2true(0.0);
 phiBold=phitrue(0.0);
 pBold=ptrue(0.0);

report << " *** BEGIN REPORT, Michaela Kubacki ***" << endl;
report << " # of boundary nodes per side is " << M << endl;
report << " Time Step = " << dt << endl;
report << " # of Time Steps= " << numsteps << endl;
report << " S0 = " << S0 << endl;
report << " kmin = " << Kappa1 << endl;


// define initial norms and errors
real errorUstabLdivL2 = 0.0;
real errorULinfL2 = 0.0;
real errorPhiLinfL2 = 0.0;
real errorPLinfL2 = 0.0;
real errorUoldLinfL2 = 0.0;
real errorPhioldLinfL2 = 0.0;
real errorPoldLinfL2 = 0.0;
real errorPavgLinfL2 = 0.0;
real errorUtLdivL2 = 0.0;
real L2normPSquared = 0.0;
```

```
real L2normDivUSquared = 0.0;
real L2normDivUtSquared = 0.0;
real L2normUold = 0.0;
real L2normPhiold = 0.0;
real L2normUSquared = 0.0;
real L2normPhiSquared = 0.0;
real initialenergySD;
real L2normPtrueSquared = 0.0;
real L2normUtrueSquared = 0.0;
real L2normPhitrueSquared = 0.0;
real temperrorUstab = 0.0;
real temperrorU = 0.0;
real temperrorPhi = 0.0;
real temperrorPavg = 0.0;
real temperrorP = 0.0;
real temperrorUold = 0.0;
real temperrorPhiold = 0.0;
real temperrorPold = 0.0;
real L2normPCurrent;
real L2normDivUCurrent;
real L2normDivUtCurrent;
real L2normUCurrent;
real L2normPhiCurrent;
real L2normPtrueCurrent;
real L2normUtrueCurrent;
real L2normPhitrueCurrent;


//************BEGIN TIME LOOP for Partitioned Method**********

for(i=0;i<numsteps; i++){

// For convergence tests for all CNLF-type methods
       func F1=f1(tn);
       func F2=f2(tn);
       func Fp=g*fp(tn);
       func Uf1=u1true(tnplus1);
       func Uf2=u2true(tnplus1);
       func PHI=phitrue(tnplus1);

// For stability tests for all CNLF-type methods
       func F1=0.0;
       func F2=0.0;
       func Fp=0.0;
       func Uf1=0.0;
       func Uf2=0.0;
       func PHI=0.0;
```

```
//*******************CNLF Method **********************

problem CNLFStokes([u1,u2,p],[v1,v2,q],solver=UMFPACK) =
   int2d(Omegaf)(
      (0.5 / dt) * (dot(u1,u2,v1,v2) ) //u_t
    + p * q * PRESSUREPENALTY) //pressure penalty term
        + int2d(Omegaf)(
    - (0.5 / dt) * (dot(u1old2,u2old2,v1,v2) ) //u_t
    -dot(F1,F2,v1,v2) //forcing
    )
       + int2d(Omegaf)( 0.5 * nu * contract(u1,u2,v1,v2)) + int1d(Omegaf,3)(0.5
         * alpha / sqrt(Kappa1) * u1 * v1) //.5*a_f(u_{k+1},v)
       + int2d(Omegaf)(0.5 * nu * contract(u1old2,u2old2,v1,v2)) +
          int1d(Omegaf,3)(0.5 * alpha / sqrt(Kappa1) * u1old2 * v1)
          //.5*a_f(u_{k-1},v)
       - int2d(Omegaf)(0.5 * p * div(v1,v2) ) //-.5*b(v, p_{k+1})
       - int2d(Omegaf)(0.5 * pold2 * div(v1,v2) ) //-.5*b(v,p_{k-1})
       + int2d(Omegaf)(q * div(u1,u2)) // incompressibility condition

 + int1d(Omegaf,3)(-g * phiold * v2) //c_I(v, phi_k) ---> v \cdot n_f = -v2
  + on ( 1,2,4, u1=Uf1, u2=Uf2) ;//external BCs

problem CNLFDarcy(phi,psi,solver=LU) =
   int2d(Omegap)((0.5 * S0 * g / dt) * phi * psi) //g*S_0*phi_t
 + int2d(Omegap)(- (0.5 * g * S0 / dt) * phiold2 * psi //g*S_0*phi_t
        -Fp*psi //source
       )
 + int2d(Omegap)(0.5* g * Kappa1 * dot(dx(phi),dy(phi),dx(psi),dy(psi) ))
    //.5*a_p(phi_{k+1},psi)
 + int2d(Omegap)(0.5* g * Kappa1 * dot(dx(phiold2),dy(phiold2),dx(psi),dy(psi)
    )) //.5*a_p(phi_{k-1},psi)
 - int1d(Omegap,5)(-g * psi * u2old) //c_I(u_k, psi) ---> u_k \cdot n_f = -u2_k
 + on( 6,7,8, phi = PHI) ; //external BCs

//Implement CNLF method
CNLFStokes;
CNLFDarcy;

//*********************CNLFStab Method *********************
 problem CNLFStabStokes([u1,u2,p],[v1,v2,q],solver=LU) =
   int2d(Omegaf)(
      (0.5 / dt) * (dot(u1,u2,v1,v2) + gamma * div(u1,u2)*div(v1,v2) )
                //u_t + grad-div(u_t) for k+1
    + p * q * PRESSUREPENALTY)              //pressure penalty term
        + int2d(Omegaf)(
    - (0.5 / dt) * (dot(u1old2,u2old2,v1,v2) - gamma *
```

```
      div(u1old2,u2old2)*div(v1,v2) )   //u_t + grad-div(u_t) for k-1
   -dot(F1,F2,v1,v2) //forcing
  )
      + int2d(Omegaf)( 0.5 * nu * contract(u1,u2,v1,v2)) + int1d(Omegaf,3)(0.5
         * alpha / sqrt(Kappa1) * u1 * v1)  //.5*a_f(u_{k+1},v)
      + int2d(Omegaf)(0.5 * nu * contract(u1old2,u2old2,v1,v2)) +
         int1d(Omegaf,3)(0.5 * alpha / sqrt(Kappa1) * u1old2 * v1)
         //.5*a_f(u_{k-1},v)
      - int2d(Omegaf)(0.5 * p * div(v1,v2) ) //-.5*b(v, p_{k+1})
      - int2d(Omegaf)(0.5 * pold2 * div(v1,v2) ) //-.5*b(v,p_{k-1})
      + int2d(Omegaf)(q * div(u1,u2)) //incompressibility condition
 + int1d(Omegaf,3)(-g * phiold * v2) //c_I(v, phi_k) ---> v \cdot n_f = -v2
  + on ( 1,2,4, u1=Uf1, u2=Uf2); //external BCs

 problem CNLFStabDarcy(phi,psi,solver=LU) =
   int2d(Omegap)((0.5 * S0 * g / dt) * phi * psi + g^2 * dt *(phi * psi +
      dot(dx(phi),dy(phi),dx(psi),dy(psi))) )//g*S_0*phi_t + added stab for k+1
 + int2d(Omegap)(- (0.5 * g * S0 / dt) * phiold2 * psi - g^2 * dt *( phiold2 *
    psi + dot(dx(phiold2),dy(phiold2),dx(psi),dy(psi)))//g*S_0*phi_t + added
    stab for k-1
        -Fp*psi //source
       )
 + int2d(Omegap)(0.5* g * Kappa1 * dot(dx(phi),dy(phi),dx(psi),dy(psi) ))
    //.5*a_p(phi_{k+1},psi)
 + int2d(Omegap)(0.5* g * Kappa1 * dot(dx(phiold2),dy(phiold2),dx(psi),dy(psi)
    )) //.5*a_p(phi_{k-1},psi)
 - int1d(Omegap,5)(-g * psi * u2old) //c_I(u_k, psi) ---> u_k \cdot n_f = -u2_k
 + on( 6,7,8, phi = PHI) ; //external BCs

//Implement CNLFstab method
CNLFStabStokes;
CNLFStabDarcy;

//*****************CNLFstab on I Method **********************
problem CNLFstabIStokes([u1,u2,p],[v1,v2,q],solver=UMFPACK) =
   int2d(Omegaf)(
      (0.5 / dt) * (dot(u1,u2,v1,v2) ) //u_t
    + p * q * PRESSUREPENALTY) //pressure penalty term
        + int2d(Omegaf)(
    - (0.5 / dt) * (dot(u1old2,u2old2,v1,v2) ) //u_t
    -dot(F1,F2,v1,v2) //forcing
    )
      + int2d(Omegaf)( 0.5 * nu * contract(u1,u2,v1,v2)) + int1d(Omegaf,3)(0.5
         * alpha / sqrt(Kappa1) * u1 * v1) //.5*a_f(u_{k+1},v)
      + int2d(Omegaf)(0.5 * nu * contract(u1old2,u2old2,v1,v2)) +
         int1d(Omegaf,3)(0.5 * alpha / sqrt(Kappa1) * u1old2 * v1)
         //.5*a_f(u_{k-1},v)
```

```
       - int2d(Omegaf)(0.5 * p * div(v1,v2) ) //-.5*b(v, p_{k+1})
       - int2d(Omegaf)(0.5 * pold2 * div(v1,v2) ) //-.5*b(v,p_{k-1})
       + int2d(Omegaf)(q * div(u1,u2)) // incompressibility condition

 + int1d(Omegaf,3)(-g * phiold * v2) //c_I(v, phi_k) ---> v \cdot n_f = -v2
 +int1d(Omegaf,3)(dt * u2 * v2) //stab on I term
 -int1d(Omegaf,3)(dt * u2old2 * v2) // stab on I term
  + on ( 1,2,4, u1=Uf1, u2=Uf2) ;//external BCs

problem CNLFstabIDarcy(phi,psi,solver=LU) =
   int2d(Omegap)((0.5 * S0 * g / dt) * phi * psi) //g*S_0*phi_t
 + int2d(Omegap)(- (0.5 * g * S0 / dt) * phiold2 * psi //g*S_0*phi_t
        -Fp*psi //source
       )
 + int2d(Omegap)(0.5* g * Kappa1 * dot(dx(phi),dy(phi),dx(psi),dy(psi) ))
    //.5*a_p(phi_{k+1},psi)
 + int2d(Omegap)(0.5* g * Kappa1 * dot(dx(phiold2),dy(phiold2),dx(psi),dy(psi)
    )) //.5*a_p(phi_{k-1},psi)
 - int1d(Omegap,5)(-g * psi * u2old) //c_I(u_k, psi) ---> u_k \cdot n_f = -u2_k
 + int1d(Omegap,5)(dt * g^2 * phi * psi) //stab on I term
 -int1d(Omegap,5)(dt * g^2 * phiold2 * psi) //stab on I term
 + on( 6,7,8, phi = PHI) ; //external BCs

//Implement CNLFstab on I method
 CNLFstabIStokes;
 CNLFstabIDarcy;

//*************BE-split Methods*****************

 //For convergence tests
       func F1plus1=f1(tnplus1);
       func F2plus1=f2(tnplus1);
       func Fpplus1=g*fp(tnplus1);
       func Uf1=u1true(tnplus1);
       func Uf2=u2true(tnplus1);
       func PHI=phitrue(tnplus1);

 //For stability tests
       func F1plus1=0.0;
       func F2plus1=0.0;
       func Fpplus1=0.0;
       func Uf1=0.0;
       func Uf2=0.0;
       func PHI=0.0;

 //**************BE-Split 1 Method **************
```

```
 problem BEsplit1Stokes([u1,u2,p],[v1,v2,q],solver=UMFPACK) =
   int2d(Omegaf)(
      (1.0 / dt) * (dot(u1,u2,v1,v2) ) //u_t
    + p * q * PRESSUREPENALTY) //pressure penalty term
         + int2d(Omegaf)(
    - (1.0 / dt) * (dot(u1old,u2old,v1,v2) ) //u_t
    -dot(F1plus1,F2plus1,v1,v2) //forcing
    )
        + int2d(Omegaf)( nu * contract(u1,u2,v1,v2)) + int1d(Omegaf,3)(alpha /
           sqrt(Kappa1) * u1 * v1) //*a_f(u_{k+1},v)
        - int2d(Omegaf)(p * div(v1,v2) ) //*b(v, p_{k+1})
        + int2d(Omegaf)(q * div(u1,u2)) // incompressibility condition

 + int1d(Omegaf,3)(-g * phiold * v2) //c_I(v, phi_k) ---> v \cdot n_f = -v2
  + on ( 1,2,4, u1=Uf1, u2=Uf2) ;//external BCs

problem BEsplit1Darcy(phi,psi,solver=LU) =
   int2d(Omegap)((S0 * g / dt) * phi * psi) //g*S_0*phi_t
 + int2d(Omegap)(- (g * S0 / dt) * phiold * psi //g*S_0*phi_t
         -Fpplus1*psi //source
        )
 + int2d(Omegap)(g * Kappa1 * dot(dx(phi),dy(phi),dx(psi),dy(psi) ))
    //a_p(phi_{k+1},psi)
 - int1d(Omegap,5)(-g * psi * u2) //c_I(u_k+1, psi)
 + on( 6,7,8, phi = PHI) ; //external BCs

 //Implement BE-split1 Method
 BEsplit1Stokes;
 BEsplit1Darcy;

  //**********************BE-Split 2 Method ********************
 problem BEsplit2Stokes([u1,u2,p],[v1,v2,q],solver=UMFPACK) =
   int2d(Omegaf)(
      (1.0 / dt) * (dot(u1,u2,v1,v2) ) //u_t
    + p * q * PRESSUREPENALTY) //pressure penalty term
         + int2d(Omegaf)(
    - (1.0 / dt) * (dot(u1old,u2old,v1,v2) ) //u_t
    -dot(F1plus1,F2plus1,v1,v2) //forcing
    )
        + int2d(Omegaf)( nu * contract(u1,u2,v1,v2)) + int1d(Omegaf,3)(alpha /
           sqrt(Kappa1) * u1 * v1) //*a_f(u_{k+1},v)
        - int2d(Omegaf)(p * div(v1,v2) ) //*b(v, p_{k+1})
        + int2d(Omegaf)(q * div(u1,u2)) // incompressibility condition

 + int1d(Omegaf,3)(-g * phi * v2) //c_I(v, phi_k) ---> v \cdot n_f = -v2
  + on ( 1,2,4, u1=Uf1, u2=Uf2) ;//external BCs
```

```
problem BEsplit2Darcy(phi,psi,solver=LU) =
   int2d(Omegap)((S0 * g / dt) * phi * psi) //g*S_0*phi_t
 + int2d(Omegap)(- (g * S0 / dt) * phiold * psi //g*S_0*phi_t
        -Fpplus1*psi //source
       )
 + int2d(Omegap)(g * Kappa1 * dot(dx(phi),dy(phi),dx(psi),dy(psi) ))
    //a_p(phi_{k+1},psi)
 - int1d(Omegap,5)(-g * psi * u2old) //c_I(u_k, psi)
 + on( 6,7,8, phi = PHI) ; //external BCs

 //Implement BE-split 2 Method
 BEsplit2Darcy;
 BEsplit2Stokes;

//************CN-Split Method**********************

//For convergence tests
       func F1half = f1(thalf);
       func F2half = f2(thalf);
       func Fphalf = g*fp( thalf);
       func Uf1=u1true(tnplus1);
       func Uf2=u2true(tnplus1);
       func PHI=phitrue(tnplus1);

//For stability tests
       func F1half = 0.0;
       func F2half = 0.0;
       func Fphalf = 0.0;
       func Uf1=0.0;
       func Uf2=0.0;
       func PHI=0.0;

  problem CNsplitAStokes([uA1,uA2,pA],[v1,v2,q],solver=UMFPACK) =
   int2d(Omegaf)(
      (1.0 / dt) * (dot(uA1,uA2,v1,v2) ) //u_t
    + 0.5 * pA * q * PRESSUREPENALTY) //pressure penalty term
        + int2d(Omegaf)(
    - (1.0 / dt) * (dot(uA1old,uA2old,v1,v2) ) //u_t
       + 0.5* (pAold * q * PRESSUREPENALTY) )//pressure penalty term
    + int2d(Omegaf)(-dot(F1half,F2half,v1,v2)) //forcing
       + int2d(Omegaf)(0.5 * nu * contract(uA1,uA2,v1,v2))
       +  int1d(Omegaf,3)(0.5 * alpha / sqrt(Kappa1) * uA1 * v1)
          //*a_f(u_{k+1},v)
       + int2d(Omegaf)(0.5 * nu * contract(uA1old,uA2old,v1,v2)) +
          int1d(Omegaf,3)(0.5 * alpha / sqrt(Kappa1) * uA1old * v1)
          //*a_f(u_{k},v)
```

```
      + int2d(Omegaf)(-0.5 * pA * div(v1,v2) ) //*b(v, p_{k+1})
      + int2d(Omegaf)(-0.5 * pAold * div(v1,v2) ) //*b(v, p_{k+1})
      + int2d(Omegaf)(q * div(uA1,uA2)) // incompressibility condition

 + int1d(Omegaf,3)(-g * phiAold * v2) //c_I(v, phi_k) ---> v \cdot n_f = -v2
  + on ( 1,2,4, uA1=Uf1, uA2=Uf2) ;//external BCs

problem CNsplitADarcy(phiA,psi,solver=LU) =
   int2d(Omegap)((S0 * g / dt) * phiA * psi) //g*S_0*phi_t
 + int2d(Omegap)(- (g * S0 / dt) * phiAold * psi) //g*S_0*phi_t
        -int2d(Omegap)(Fphalf*psi) //source
 + int2d(Omegap)(0.5 * g * Kappa1 * dot(dx(phiA),dy(phiA),dx(psi),dy(psi) ))
    //a_p(phi_{k+1},psi)
  +  int2d(Omegap)(0.5 * g * Kappa1 * dot(dx(phiAold),dy(phiAold),dx(psi),dy(psi)
     )) //a_p(phi_{k},psi)
 - int1d(Omegap,5)(-g * psi * uA2) //c_I(u_k+1, psi)
 + on( 6,7,8, phiA = PHI) ; //external BCs

 problem CNsplitBStokes([uB1,uB2,pB],[v1,v2,q],solver=UMFPACK) =
   int2d(Omegaf)(
      (1.0 / dt) * (dot(uB1,uB2,v1,v2) ) //u_t
    + 0.5* pB * q * PRESSUREPENALTY) //pressure penalty term
        + int2d(Omegaf)(
    - (1.0 / dt) * (dot(uB1old,uB2old,v1,v2) ) //u_t
       + 0.5*(pBold * q * PRESSUREPENALTY) //pressure penalty term
    -dot(F1half,F2half,v1,v2) //forcing
    )
      + int2d(Omegaf)(0.5 * nu * contract(uB1,uB2,v1,v2)) + int1d(Omegaf,3)(0.5
         * alpha / sqrt(Kappa1) * uB1 * v1) //*a_f(u_{k+1},v)
      + int2d(Omegaf)(0.5 * nu * contract(uB1old,uB2old,v1,v2)) +
         int1d(Omegaf,3)(0.5 * alpha / sqrt(Kappa1) * uB1old * v1)
         //*a_f(u_{k},v)
      - int2d(Omegaf)(0.5 * pB * div(v1,v2) ) //*b(v, p_{k+1})
      - int2d(Omegaf)(0.5 * pBold * div(v1,v2) ) //*b(v, p_{k+1})
      + int2d(Omegaf)(q * div(uB1,uB2)) // incompressibility condition

 + int1d(Omegaf,3)(-g * phiB * v2) //c_I(v, phi_k+1) ---> v \cdot n_f = -v2
  + on ( 1,2,4, uB1=Uf1, uB2=Uf2) ;//external BCs

problem CNsplitBDarcy(phiB,psi,solver=LU) =
   int2d(Omegap)((S0 * g / dt) * phiB * psi) //g*S_0*phi_t
 + int2d(Omegap)(- (g * S0 / dt) * phiBold * psi //g*S_0*phi_t
        -Fphalf*psi //source
       )
 + int2d(Omegap)( 0.5 * g * Kappa1 * dot(dx(phiB),dy(phiB),dx(psi),dy(psi) ))
    //a_p(phi_{k+1},psi)
  + int2d(Omegap)(0.5 * g * Kappa1 * dot(dx(phiBold),dy(phiBold),dx(psi),dy(psi)
```

```
     )) //a_p(phi_{k},psi)
 - int1d(Omegap,5)(-g * psi *uB2old) //c_I(u_k+1, psi)
 + on( 6,7,8, phiB = PHI) ; //external BCs

 //Implement CN-split Method
 CNsplitAStokes;
 CNsplitBDarcy;
 CNsplitADarcy;
 CNsplitBStokes;

 u1 = 0.5 * (uA1 + uB1);
 u2 = 0.5 * (uA2 + uB2);
 p = 0.5 * (pA + pB);
 phi = 0.5 * (phiA + phiB);

//***************************Time Filters*****************************

//RA Filter
u1old = u1old + chi * (u1old2 - 2 * u1old + u1);
u2old = u2old + chi * (u2old2 - 2 * u2old + u2);
phiold = phiold + chi * (phiold2 - 2 * phiold + phi);
pold = pold + chi * (pold2 - 2 * pold + p);

//RAW Filter
u1 = u1 + gamma * (chi-1) / 2 * (u1 - 2 * u1old + u1old2);
u1old = u1old + chi *gamma / 2 * (u1old2 - 2 * u1old + u1);
u2 = u2 + gamma * (chi-1) / 2 * (u2 - 2 * u2old + u2old2);
u2old = u2old + chi *gamma / 2 * (u2old2 - 2 * u2old + u2);
phi = phi + gamma * (chi-1) / 2 * (phi - 2 * phiold + phiold2);
phiold = phiold + chi *gamma / 2 * (phiold2 - 2 * phiold + phi);
p = p + gamma * (chi-1) / 2 * (p - 2 * pold + pold2);
pold = pold + chi *gamma / 2 * (pold2 - 2 * pold + p);

//**********************Plots****************************
//solve for u_p
//w1=-Kappa1*dx(phi);
//w2=-Kappa1*dy(phi);

//plot solution
u1soln=u1true(tnplus1);
u2soln=u2true(tnplus1);
up1soln=up1true(tnplus1);
up2soln=up2true(tnplus1);
if(plotson){
plot([u1soln,u2soln],[up1soln,up2soln], value=1, wait=1);
}
```

```
//************Calculate Errors for Convergence Tests*****************

temperrorU = sqrt( int2d(Omegaf)( (u1-u1true(tnplus1))^2 +
    (u2-u2true(tnplus1))^2));
  if ( temperrorU> errorULinfL2 ){
    errorULinfL2 = temperrorU;
  }  //for CNLF,BE-split,CN-split

temperrorUstab = sqrt( int2d(Omegaf)( (u1-u1true(tnplus1))^2 +
    (u2-u2true(tnplus1))^2 + (div(u1,u2)-divutrue(tnplus1))^2) );
  if ( temperrorUstab > errorUstabLdivL2 ){
    errorUstabLdivL2 = temperrorUstab;
  }  //for CNLFstab (div-norm of error in u)

temperrorPhi = sqrt( int2d(Omegap)( (phi-phitrue(tnplus1))^2 ));
  if ( temperrorPhi > errorPhiLinfL2 ){
    errorPhiLinfL2 = temperrorPhi;
  }

temperrorPavg = sqrt( int2d(Omegaf)( (0.5*(p-ptrue(tnplus1)) +
    0.5*(pold2-ptrue(tnminus1)))^2) ); //pressure average error
  if ( temperrorPavg > errorPavgLinfL2 ){
    errorPavgLinfL2 = temperrorPavg;
  }

temperrorP = sqrt( int2d(Omegaf)( (p-ptrue(tnplus1))^2) ); //pressure error
  if ( temperrorP > errorPLinfL2 ){
    errorPLinfL2 = temperrorP;
  }


//******************For Stability Tests*****************************
L2normUCurrent = int2d(Omegaf)( (u1)^2 + (u2)^2 );
L2normUSquared = L2normUSquared + L2normUCurrent * dt;
L2normUtrueCurrent = int2d(Omegaf)( (u1true(tnplus1))^2 + (u2true(tnplus1))^2 );
L2normUtrueSquared = L2normUtrueSquared + L2normUtrueCurrent * dt;

L2normPhiCurrent = int2d(Omegap)( (phi)^2 );
L2normPhiSquared = L2normPhiSquared + L2normPhiCurrent * dt;
L2normPhitrueCurrent = int2d(Omegap)( (phitrue(tnplus1))^2 );
L2normPhitrueSquared = L2normPhitrueSquared + L2normPhitrueCurrent * dt;


//Stability test (energy, stable, and unstable modes)

//For CNLF, CNLFstab, CNLFstabI
```

```
real [int] Energy(numsteps);
real [int] E1U(numsteps); // Unstable U mode
real [int] E2U(numsteps); // Stable U mode
real [int] E1Phi(numsteps); //Unstable Phi mode
real [int] E2Phi(numsteps); //Stable Phi mode
real [int] Unstable(numsteps);
real [int] Stable(numsteps);

Energy[i] = L2normUCurrent+ g*S0*L2normPhiCurrent + L2normUold +
    g*S0*L2normPhiold;
E1U[i]=int2d(Omegaf)( (u1-u1old2)^2 + (u2-u2old2)^2 );
E2U[i]=int2d(Omegaf)( (u1+u1old2)^2 + (u2+u2old2)^2 );
E1Phi[i]=int2d(Omegap)( (phi-phiold2)^2 );
E2Phi[i]=int2d(Omegap)( (phi+phiold2)^2 );
Unstable[i] = E1U[i] + E1Phi[i]; // Unstable Mode
Stable[i] = E2U[i] + E2Phi[i]; // Stable Mode

//For splitting methods
real [int] Energy(numsteps);
Energy[i] = L2normUCurrent + g * S0 * L2normPhiCurrent;


report<< "Energy " << Energy[i] << endl;
//report << "Unstable " << Unstable[i] << endl;
//report << "Stable " << Stable[i] << endl;

//***************Update and end time stepping loop****************

//For CNLF, CNLFstab
u1old2=u1old;
u2old2=u2old;
phiold2=phiold;
pold2=pold;
L2normUCurrent=L2normUold;
L2normPhiCurrent=L2normPhiold;

//For CNLF, CNLFstab, BEsplit1-2
u1old=u1;
u2old=u2;
phiold=phi;
pold=p;


//For CNsplit method
uA1old=uA1;
uA2old=uA2;
phiAold=phiA;
```

```
pAold=pA;
uB1old=uB1;
uB2old=uB2;
phiBold=phiB;
pBold=pB;

tnminus1 = tnminus1 + dt; //for CNLF, CNLFstab
tn = tn + dt;
tnplus1 = tnplus1 + dt;

}  //end of time loop for partitioned method

// Print largest error over time interval
 report << "Error Stokes Velocity & " << errorULinfL2<< endl ; //for CNLF,
    BE-split, CN-split
 report << "Error Stokes Velocity & " << errorULdivL2<< endl ; //for CNLFstab
 report << "Error Stokes AvgPressure & " << errorPavgLinfL2<< endl;
 report << "Error Stokes Pressure & " << errorPLinfL2<< endl;
 report << "Error Darcy Pressure & "<< errorPhiLinfL2<< endl;

} //end outer loop (for j)
```

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