TUKEY ORDER ON SETS OF COMPACT SUBSETS OF TOPOLOGICAL SPACES

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ABSTRACT

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A partially ordered set (poset), Q, is a *Tukey quotient* of a poset, P, written $P \geq_T Q$, if there exists a map, a *Tukey quotient*, $\phi : P \to Q$ such that for any cofinal subset C of P the image, $\phi(C)$, is cofinal in Q. Two posets are *Tukey equivalent* if they are Tukey quotients of each other. Given a collection of posets, $\mathcal{P}$, the relation $\leq_T$ is a partial order. The Tukey structure of $\mathcal{P}$ has been intensively studied for various instances of $\mathcal{P}$ [13, 14, 48, 53, 58]. Here we investigate the Tukey structure of collections of posets naturally arising in Topology.

For a space X, let $\mathcal{K}(X)$ be the poset of all compact subsets of X, ordered by inclusion, and let $Sub(X)$ be the set of all homeomorphism classes of subsets of X. Let $\mathcal{K}(Sub(X))$ be the set of all Tukey classes of the form $[\mathcal{K}(Y)]_T$, where $Y \in Sub(X)$. The main purpose of this work is to study order properties of $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ and $(\mathcal{K}(Sub(\omega_1)), \leq_T)$.

We attack this problem using two approaches. The first approach is to study internal order properties of elements of $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$ that respect the Tukey order — calibres and spectra. The second approach is more direct and studies the Tukey relation between the elements of $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ and $(\mathcal{K}(Sub(\omega_1)), \leq_T)$.

As a result we show that $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ has size $2^{\mathfrak{c}}$, has no largest element, contains an antichain of maximal size, $2^{\mathfrak{c}}$, its additivity is $\mathfrak{c}^+$, its cofinality is $2^{\mathfrak{c}}$, $\mathcal{K}(Sub(\mathbb{R}))$ has calibre (κ, λ, μ) if and only if $\mu \leq \mathfrak{c}$ and $\mathfrak{c}^+$ is the largest cardinal that embeds in $\mathcal{K}(Sub(\mathbb{R}))$. While the size and the existence of large antichains of $\mathcal{K}(Sub(\omega_1))$ have already been established in [58], we determine special classes of $\mathcal{K}(Sub(\omega_1))$ and the relation between these classes and

the elements of $\mathcal{K}(Sub(\mathbb{R}))$.

Finally, we explore connections of the Tukey order with function spaces and the Lindelöf Σ property, which require giving the Tukey order more flexibility and larger scope. Hence we develop the *relative* Tukey order and present applications of relative versions of results on $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ and $(\mathcal{K}(Sub(\omega_1)), \leq_T)$ to function spaces.

Keywords: antichain, calibre, cofinal, compact, continuum, embedding, function space, graph, Lindelöf Σ, metrizable, n-arc connected, n-strongly arc connected, partial order, relative Tukey order, separable, stationary, Tukey order, unbounded.

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0.0 INTRODUCTION

0.1 OVERVIEW

The primary objective of this work is to study the order structure of the family of compact subsets, $\mathcal{K}(X)$, of a topological space, X, ordered by set inclusion. For much of this thesis we focus on the case where X is a subspace of one of the two most fundamental topological spaces — the real line, $\mathbb{R}$, and the first uncountable ordinal ω_1. Analysis is founded on the topological properties of the real line; while the ordinals form the back-bone of the set theoretic universe and ω_1 is the boundary point between the countable and uncountable, and so is one of the richest sources of fundamental questions in set theory.

We compute invariants which measure the 'width', 'height' and cofinality of most $\mathcal{K}(M)$'s, where M is a subspace of the real line, and all $\mathcal{K}(S)$'s where S is a subspace of ω_1. We also investigate the relationships between two $\mathcal{K}(M)$'s, or two $\mathcal{K}(S)$'s, or between a $\mathcal{K}(M)$ and a $\mathcal{K}(S)$. The primary order-theoretic tool used to understand both the internal properties of individual $\mathcal{K}(X)$'s and the relationship between $\mathcal{K}(X)$'s is the *Tukey order*. Our results on the Tukey order applied to $\mathcal{K}(X)$'s have applications to the general theory of the Tukey order, including the solution to a fundamental problem on the number of distinct continuum sized directed sets up to Tukey equivalence.

Additional applications of the main results are given to function spaces with the pointwise or compact-open topology. In particular it is shown that there is a $2^{\mathfrak{c}}$-sized family of separable metrizable spaces with pairwise non-linearly homeomorphic function spaces. This is the optimal sized such family, and improves on a previous best of a $\mathfrak{c}$-sized family due to Marciszewski [32]. These applications required the generalization of the Tukey order to the *relative Tukey order*, and all our results are obtained in this more general context.

The Tukey order [60] was originally introduced, early in the 20th century, as a tool to understand convergence in general topological spaces. But it was quickly seen to have broad applicability in comparing partial orders, for example by Isbell [35]. The seeds for the work presented here were sown in the late 1980s by three researchers — Christensen, Todorčević and Fremlin — each working in rather different fields (descriptive set theory, set theory and real analysis, respectively).

Fremlin went on to use the Tukey order as the main organizing theme for his work in category and measure theory (see his five volume series [22]). The last few years have seen a surge in activity [44, 50, 55, 56] in this area. Todorčević, with his co-authors, Dobrinen and Raghavan, has a major project underway classifying ultrafilters under the Tukey order [13, 14, 53].

Christensen's work, in turn, was embraced, developed, and applied, by a significant group of analysts (see [40] for a recent (2011) overview of this work). This school does *not* use the Tukey order. Perhaps because the original result of Christensen was not expressed in terms of the Tukey order. But also perhaps because the basic Tukey order concept is not sufficient to cover all necessary cases. (For example these researchers are interested in the order structure of compact covers, and not just cofinal collections of compact sets.) Our introduction of the relative Tukey order provides a uniform tool to deal with these cases (and much more). We consider this another important, if hidden and under-developed, consequence of the work presented here.

0.2 CONTEXT AND MOTIVATION — THREE SEEDS

0.2.1 Topological and Order Basics

Throughout this work all spaces are assumed to be Tychonoff. For a space X, let $Sub(X)$ be the set of all homeomorphism classes of subspaces of X, and let $\mathcal{K}(X)$ be the set of all compact subsets of X. Then $\mathcal{K}(X)$ is a partially ordered set ordered by the set inclusion. We will concentrate on $Sub(\mathbb{R})$, $Sub(\omega_1)$ and $\mathcal{K}(M)$, $\mathcal{K}(S)$ where $M \in Sub(\mathbb{R})$ and $S \in Sub(\omega_1)$, respectively. (The letters M and N will be reserved for separable metrizable spaces, while the letters S and T will be reserved for subsets of ω_1.) Otherwise our topological notation and definitions are as in [16].

We adopt standard set-theoretic notation such as in [42]. Ordinals such as 1, ω, ω_1 and all cardinals receive their usual order. By $[A]^{<\omega}$ we mean all finite subsets of a set A ordered by inclusion. Products of partially ordered sets receive the pointwise product order. When standard topological notation and set-theory notation clash we give precedence to set-theory. For example we use ω to denote the (topological space of) natural numbers, and not $\mathbb{N}$. Consequently ω^ω is both an ordered set (the countable power of the ordinal ω) but also an important space (the Baire space — countable power of the countable discrete space, with the Tychonoff product topology). Recall that the Baire space is homeomorphic to the irrationals.

Calibres, which measure the width of a partially ordered set, will play a central role in our study of $\mathcal{K}(X)$. Recall that a partially ordered set (poset), P, has *calibre* (κ, λ, μ) if for any κ-sized $P_0 \subseteq P$ there is a λ-sized $P_1 \subseteq P_0$ such that each μ-sized $P_2 \subseteq P_1$ is bounded in P. Calibre $(\kappa, \lambda, \lambda)$ is abbreviated to calibre (κ, λ), and calibre (κ, κ) to calibre κ.

The Tukey ordering compares partially ordered sets. All posets, P, considered here, including $\mathcal{K}(X)$, are *directed*: if p and q are in P then there is an r in P such that $p \leq r$ and $q \leq r$. One directed poset, Q, is a *Tukey quotient* of another, P, denoted $P \geq_T Q$, if there is a map $\phi : P \to Q$, called a *Tukey quotient*, that takes cofinal subsets of P to cofinal subsets of Q. If two posets, P and Q, are Tukey quotients of each other, we call them *Tukey equivalent* and write $P =_T Q$. We note that $\mathcal{K}(\omega^\omega)$ and ω^ω are Tukey equivalent.

For a space X, let $\mathcal{K}(Sub(X))$ be the set of all Tukey equivalence classes $[\mathcal{K}(Y)]_T$, where $Y \in Sub(X)$. The Tukey order is a partial order on $\mathcal{K}(Sub(X))$. The main goal of this work is to study the structure of the two posets $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ and $(\mathcal{K}(Sub(\omega_1)), \leq_T)$. This was motivated by the following 'three seeds'.

0.2.2 Seed 1: Todorčević's Work on the Tukey Order

Isbell in [35] presented five directed sets — 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ — of size $\leq \omega_1$, and two of size $\mathfrak{c}$, which are pairwise Tukey inequivalent. He asked how many Tukey inequivalent directed sets there are of size $\leq \omega_1$. Evidently an upper bound is 2^{ω_1}.

In [58] Todorčević gave a wonderful two part answer. Consistently (under PFA) Isbell's list with the five directed sets is a *complete* list of all directed sets of size $\leq \omega_1$, and so the answer to Isbell's question is, '5'. However, Todorčević also showed, in ZFC, that there is a 2^{ω_1}-sized family, $\mathcal{A}$, of subsets of ω_1 such that for distinct elements S and S' from $\mathcal{A}$ the directed sets $\mathcal{K}(S)$ and $\mathcal{K}(S')$ (which have size $\mathfrak{c}$) are Tukey incomparable, $\mathcal{K}(S) \not\geq_T \mathcal{K}(S')$ and $\mathcal{K}(S') \not\geq_T \mathcal{K}(S)$, and so definitely Tukey inequivalent. Hence consistently (under the continuum hypothesis, CH, $\mathfrak{c} = \omega_1$) the answer to Isbell's question is, '2^{ω_1}'.

The gulf between these two answers — '5' *versus* '$2^{\mathfrak{c}}$' — is amazing.

Actually Todorčević denoted his directed sets by $D(S)$, for an arbitrary subset S of ω_1, and defined them to be the set of all countable subsets C of S such that $\sup(C \cap \alpha) \in C$ for all α, but this is precisely $\mathcal{K}(S)$.

0.2.3 Seed 2: Christensen's Characterization of Polish Spaces

In his book, [12], Christensen proved (without the Tukey order notation) that:

> If M is a separable metrizable space, then $\omega^\omega \geq_T \mathcal{K}(M)$ if and only if M is Polish (in other words, *completely* metrizable).

It seems surprising, even mysterious, that the existence, or otherwise, of a compatible complete metric on a separable metrizable M should be connected to the cofinal structure of the compact subsets of M. The definition, 'Cauchy sequences converge', and other characterizations of completeness (in terms of sieves, for example) seem far away. Recalling that ω^ω is Tukey equivalent to $\mathcal{K}(\omega^\omega)$, and ω^ω is the archetypal Polish space, we can loosely interpret the theorem as saying, if the compact subsets of M are 'organized' like the compact subsets of a Polish space then it is also Polish. This, at least, explains the significance of the order ω^ω.

Since completeness is a key concept in functional analysis — Banach spaces are *complete* normed vector spaces — it is not surprising that analysts took Christensen's alternative view of completeness and developed it intensively. These results (see [40] for a recent survey) are not used here, so we do not go into details. However we will briefly explain, through another result of Christensen's, how the standard Tukey order fails to capture all scenarios that we might wish.

Christensen proved that: if M is a separable metrizable space, then there is a compact cover $\mathcal{K} = \{K_\sigma : \sigma \in \omega^\omega\}$ of M such that $K_\sigma \subseteq K_\tau$ when $\sigma \leq \tau$ if and only if M is analytic (so, the continuous image of a Polish space). If the cover $\mathcal{K}$ were cofinal in $\mathcal{K}(M)$ we would be back to Christensen's first theorem, but without the Tukey notation. But since $\mathcal{K}$ is, necessarily, *not* cofinal, but merely a cover, we cannot use the Tukey order.

Note that saying that '$\mathcal{K}$ is a cover of M' is precisely the same as saying, 'for every compact subset of M of the form $\{x\}$ there is a K in $\mathcal{K}$ such that $\{x\} \subseteq K$'. Making the natural identification of x in M with $\{x\}$ in $\mathcal{K}(M)$, we see that a compact cover $\mathcal{K}$ is essentially a subset of $\mathcal{K}(M)$ cofinal for M in $\mathcal{K}(M)$ — a *relative* cofinal set.

0.2.4 Seed 3: Fremlin's Use of the Tukey Order in Analysis

Fremlin took Christensen's result and in [21] used it to investigate the initial part of $\mathcal{K}(Sub(\mathbb{R}))$. But this was a side branch from his bigger project which was to use the Tukey order to investigate the many natural partially ordered sets arising in the study of measure and category. We give here just one example.

Write $\mathcal{N}$ for the subsets of $[0, 1]$ with measure zero (the null sets), and $\mathcal{M}$ for the meagre subsets of $[0, 1]$. Both are directed sets under inclusion. Fremlin showed that $\mathcal{N} \geq_T \mathcal{M}$, but the converse is not constructively provable.

It follows that, in ZFC, $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$ and $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M})$. Here the $\mathrm{cof}(P)$ denotes the minimal size of cofinal subset of the poset P (cofinality of P), and $\mathrm{add}(P)$ stands for the minimal size of an unbounded subset of P (additivity of P). These two inequalities had been proven earler by Bartoszynski, but Fremlin's proof via the Tukey relation is more natural, and explains *why* the inequalities hold.

Further, since $\mathcal{M} \geq_T \mathcal{N}$ is not constructively provable, there is revealed, contrary to expectations, a fundamental asymmetry between measure and category.

0.2.5 Objectives and Problems

To summarize the key points from the three seeds:

- in answering a question of Isbell on the number of Tukey equivalence classes of directed sets of size ω_1, Todorčević constructed a 2^{ω_1}-sized antichain in $\mathcal{K}(Sub(\omega_1))$, while
- Christensen and Fremlin investigated the initial structure of $\mathcal{K}(Sub(\mathbb{R}))$, which led them and others, to
- applications in real analysis (measure and category) and functional analysis, although
- some of the work in functional analysis is difficult to express in terms of the basic Tukey theory.

Our objectives, and motivation, should now be clear:

- investigate the order structure, especially additivity, cofinality and calibres, of individual $\mathcal{K}(M)$ from $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(S)$ from $\mathcal{K}(Sub(\omega_1))$,
- investigate the order structure of $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$, extending Christensen/Fremlin on the initial structure of the former, and Todorčević's antichain in the latter, and in particular,
- show that $\mathcal{K}(Sub(\mathbb{R}))$ contains an antichain of size $2^{\mathfrak{c}}$, thereby answering the natural variant of Isbell's problem, 'what is the number of Tukey equivalence classes of directed sets of size $\mathfrak{c}$?',
- introduce and study *relative* cofinal sets, and *relative* Tukey maps, in preparation for,
- applications to function spaces.

0.3 THESIS STRUCTURE

This thesis is organized in four chapters containing the main results, concluded by a chapter on related open problems. There are also two appendices. The first, Appendix A, summarizes some results on strengthenings of arc connectedness in continua contained in the two papers: *n-Arc Connected Spaces*, by Benjamin Espinoza, Paul Gartside and Ana Mamatelashvili, [17], published in Colloquium Mathematicum and *Strong Arcwise Connectedness*, by Benjamin Espinoza, Paul Gartside, Merve Kovan-Bakan and Ana Mamatelashvili, [18], accepted by the Houston Journal of Mathematics. The second appendix contains a summary of work on two projects, one on special subsets of function spaces, separators and generators, and the other on the connections between elementary submodels of set theory and function spaces.

In Chapter 1 we establish a series of preliminary but essential lemmas in a more general setting than that of the Tukey order defined above. These preliminaries establish a close connection between the Tukey ordering and calibre properties. So we take a closer look at posets $\mathcal{K}(M)$, $\mathcal{K}(S)$ and investigate their calibre-related properties (Chapter 2). At this point enough ground has been laid to pursue the main goal of the research presented here, and we discuss the structures of $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ and $(\mathcal{K}(Sub(\omega_1)), \leq_T)$ (Chapter 3). This enables us to compare posets $\mathcal{K}(M)$ and $\mathcal{K}(S)$ (Chapter 3), and lastly, we consider various applications of the Tukey ordering to function spaces (Chapter 4).

0.4 SUMMARY OF RESULTS

Relative Tukey order and preliminaries. Our study of posets of the form $\mathcal{K}(X)$, especially when considering applications, revealed the need for a more general version of the Tukey order – a relative Tukey order on pairs of posets (P', P) and (Q', Q), where P and Q are directed, P' is a subset of P and Q' is a subset of Q (see Section 1.1). We establish a relation between relative Tukey order and cofinality. We show that the relative Tukey order preserves calibres: if (P', P) has a given calibre property and $(P', P) \geq_T (Q', Q)$, then (Q', Q)

also has that calibre property (see Section 1.2). Also, (P', P) fails to have calibre κ if and only if $(P', P) \geq_T \kappa$. If $P = \bigcup_{\alpha<\kappa} P_\alpha$ and each (P_α, P) is a Tukey quotient of Q, then $Q \times [\kappa]^{<\omega} \geq_T P$ (see Section 1.2).

The cardinal $\text{add}(P)$ order-embeds in P as an unbounded subset. Whenever $\kappa < \text{add}(Q)$, $(Q, P) =_T (Q^\kappa, P^\kappa)$ (see Section 1.3). Next we define the spectrum of a poset P, denoted $\text{spec}(P)$, to be the set of all regular cardinals κ such that $P \geq_T \kappa$ (or equivalently, P does not have calibre κ). We establish the relationships between the spectrum, the additivity and the cofinality of a poset (see Section 1.4).

The most interesting relative pair, for the purposes of this work, is $(X, \mathcal{K}(X))$, where X can be thought of as a subspace of $\mathcal{K}(X)$ by identifying each $x \in X$ with $\{x\} \in \mathcal{K}(X)$. We investigate the relationship between the relative Tukey order and the standard topological operations (continuous images, perfect images, closed subsets, products and et cetera). We also show that for every separable metrizable M, there is a subset, M_0, of the Cantor set, $\{0,1\}^\omega$, such that $\mathcal{K}(M) =_T \mathcal{K}(M_0)$. Hence studying Tukey classes arising from subsets of $\mathbb{R}$ is the same as studying Tukey classes arising from arbitrary separable metrizable spaces (see Section 1.5).

The last and the most important lemma of these preliminaries gives a condition equivalent to existence of relative Tukey quotient maps. The most useful instances of this Key Lemma are: suppose X is compact and metrizable and $M, N \subseteq X$, then $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ if and only if there is a closed subset, C, of $\mathcal{K}(X)^2$ such that $C[\mathcal{K}(M)] = \mathcal{K}(N)$; and $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$ if and only if there is a closed subset, C, of $\mathcal{K}(X)^2$ such that $\bigcup C[\mathcal{K}(M)] = N$ (see Section 1.6).

Spectra and calibres of $\mathcal{K}(M)$ and $\mathcal{K}(S)$. The underlying fact for the majority of arguments on this topic is that in most cases ω^ω is a Tukey quotient of $\mathcal{K}(M)$ and $\mathcal{K}(S)$. In particular, whenever M is non-locally compact, $\omega^\omega \leq_T \mathcal{K}(M)$ and whenever $\overline{S}\setminus S$ is not closed (or, equivalently, when S is not locally compact) $\omega^\omega \leq_T \mathcal{K}(S)$. These include all the interesting cases for $\mathcal{K}(M)$ since for locally compact M, $\mathcal{K}(M) \leq_T \omega$. Also, when $\overline{S}\setminus S$ is closed $\mathcal{K}(S) \in \{\mathbf{1}, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^\omega\}$ and calibres and spectra of these posets are known.

Whenever ω^ω is a Tukey quotient of P, $\text{spec}(\omega^\omega)$ is a subset of $\text{spec}(P)$. By calculating additional Tukey-bounds for $\mathcal{K}(M)$ and $\mathcal{K}(S)$, as well as their additivities, cofinalities and sizes, we establish spectra for some $\mathcal{K}(M)$ and every $\mathcal{K}(S)$, in terms of the spectrum of ω^ω. Namely, $\operatorname{spec}(\mathcal{K}(\mathbb{Q})) = \{\omega_1\} \cup \operatorname{spec}(\omega^\omega)$, and when M is *totally imperfect* (contains no Cantor set, or equivalently, all compact subsets are countable), $\operatorname{spec}(\mathcal{K}(M)) = \operatorname{spec}(\mathbb{N}^\mathbb{N})$ together with all regular cardinals between ω_1 and $|M|$. On the other hand, if $\overline{S} \setminus S$ is not closed, we have exactly two possibilities. When S is bounded, $\mathcal{K}(S) =_T \omega^\omega$ and $\operatorname{spec}(\mathcal{K}(S)) = \operatorname{spec}(\omega^\omega)$. When S is unbounded, $\operatorname{spec}(\mathcal{K}(S)) = \{\omega_1\} \cup \operatorname{spec}(\omega^\omega)$.

It remains to determine $\operatorname{spec}(\omega^\omega)$. From the preliminary lemmas, we know $\omega, \mathfrak{b}, \operatorname{cof}(\mathfrak{d}) \in \operatorname{spec}(\mathbb{N}^\mathbb{N}) \subseteq \{\omega\} \cup [\mathfrak{b}, \mathfrak{d}]$. We show that for any finite set, F, of regular uncountable cardinals, it is consistent that $\operatorname{spec}(\mathbb{N}^\mathbb{N}) = \{\omega\} \cup F$. For an infinite set, I, of regular uncountable cardinals, we know that it is consistent that I is a subset of $\operatorname{spec}(\omega^\omega)$.

Since $\mathcal{K}(M)$ and $\mathcal{K}(S)$ have size at most $\mathfrak{c}$, we focus on calibres ω_1, $(\omega_1, \omega_1, \omega)$ and (ω_1, ω). Every $\mathcal{K}(M)$ has calibre (ω_1, ω). For every M, $\mathcal{K}(M)$ has calibre ω_1 if and only if it has calibre $(\omega_1, \omega_1, \omega)$. The above spectra result, $\mathfrak{b} \in \operatorname{spec}(\omega^\omega) \subseteq \{\omega\} \cup [\mathfrak{b}, \mathfrak{d}]$, implies that $\omega^\omega =_T \mathcal{K}(\omega^\omega)$ has calibre ω_1 if and only if $\omega_1 < \mathfrak{b}$. Whenever M is totally imperfect (or when $M = \mathbb{Q}$), $\mathcal{K}(M)$ does not have calibre ω_1. However, if $\omega_1 < \mathfrak{p}$, there exists M such that $\mathcal{K}(\mathbb{N}^\mathbb{N}) <_T \mathcal{K}(M)$ and $\mathcal{K}(M)$ has calibre ω_1.

Calibres of $\mathcal{K}(S)$ are completely resolved. If S is bounded then $\mathcal{K}(S) \in \{\mathbf{1}, \omega, \omega^\omega\}$ and we know its calibres. If S is unbounded, then $\mathcal{K}(S)$ fails to have calibre ω_1. It was proven in [58], that $\mathcal{K}(S)$ has calibre (ω_1, ω) if and only if S is stationary. We show that $K(S)$ has calibre $(\omega_1, \omega_1, \omega)$ if and only if $\overline{S} \setminus S$ is bounded and either $\overline{S} \setminus S$ is closed or $\omega_1 < \mathfrak{b}$.

Structure of $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ *and* $(\mathcal{K}(Sub(\omega_1)), \leq_T)$. The initial structure of $\mathcal{K}(Sub(\mathbb{R}))$ was presented in [21]. The smallest elements of $\mathcal{K}(Sub(R))$ form a chain $\mathbf{1} <_T \omega <_T \omega^\omega$. A poset $\mathcal{K}(M)$ is in the Tukey class of $\mathbf{1}$ if and only if M is compact, $\mathcal{K}(M)$ is in the Tukey class of ω if and only if M is locally compact, non-compact, while $\mathcal{K}(M) =_T \omega^\omega$ if and only if M is Polish, non-locally compact. Then we conclude that for every non-Polish M, $\omega^\omega <_T \mathcal{K}(M)$. We determine the corresponding initial structure of the relative Tukey classes of pairs, $(M, \mathcal{K}(M))$.

Using the Key Lemma, we establish that a subset of $\mathcal{K}(Sub(\mathbb{R}))$ is bounded if and only if it has size $\leq \mathfrak{c}$. Since $|Sub(\mathbb{R})| = 2^{\mathfrak{c}}$, we deduce that $\text{add}(\mathcal{K}(Sub(\mathbb{R}))) = \mathfrak{c}^+$, $\text{cof}(\mathcal{K}(Sub(\mathbb{R}))) = 2^{\mathfrak{c}}$, $\mathcal{K}(Sub(\mathbb{R}))$ has calibre (κ, λ, μ) if and only if $\mu \leq \mathfrak{c}$, $\mathfrak{c}^+$ is the largest cardinal that embeds in $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\mathbb{R}))$ has no largest element. Working inside a $\mathfrak{c}$-sized totally imperfect subset $\mathbb{R}$ and using the Key Lemma we construct a $2^{\mathfrak{c}}$-sized collection, $\mathcal{A}$, in $Sub(\mathbb{R})$ with the property that for any two elements M, N of $\mathcal{A}$, $\mathcal{K}(M) \not\geq_T (N, \mathcal{K}(N))$ and $\mathcal{K}(N) \not\geq_T (M, \mathcal{K}(M))$. A similar construction allows us to embed certain $\mathfrak{c}$-sized subposets into $\mathcal{K}(Sub(\mathbb{R}))$. For example, the poset $[0,1]^\omega$ embeds into $\mathcal{K}(Sub(\mathbb{R}))$. Therefore $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$ embed in $\mathcal{K}(Sub(\mathbb{R}))$ as well. Also $(\mathcal{P}(\omega), \subseteq)$ embeds into $\mathcal{K}(Sub(\mathbb{R}))$. It follows that every countable poset embeds.

The poset $\mathcal{K}(Sub(\omega_1))$ has a somewhat different structure than $\mathcal{K}(Sub(\mathbb{R}))$. It has the largest element, $[\omega_1]^{<\omega} \times \omega^\omega$, and most of its elements fall into one of the finite number of classes. Each bounded subset S of ω_1 is Polish and therefore the corresponding $\mathcal{K}(S)$ falls into the equivalence class of one of $\mathbf{1}$, ω or ω^ω. For all closed unbounded S, $\mathcal{K}(S) =_T \omega_1$. For all non-stationary S, $\mathcal{K}(S)$ falls into the class of either $[\omega_1]^{<\omega}$ if $\overline{S} \backslash S$ is closed or $[\omega_1]^{<\omega} \times \omega^\omega$ if $\overline{S} \backslash S$ is not closed. For every S that contains a closed unbounded set, $\mathcal{K}(S) =_T \omega_1 \times \omega$ if $\overline{S} \backslash S$ is non-empty, closed and bounded; $\mathcal{K}(S) =_T \omega_1 \times \omega^\omega$ if $\overline{S} \backslash S$ is bounded but not closed; and $\mathcal{K}(S) =_T \Sigma(\omega^{\omega_1})$ if $\overline{S} \backslash S$ is unbounded.

Elements of $\mathcal{K}(Sub(\omega_1))$ that do not fall into any of the classes mentioned above are associated with subsets of ω_1 that are stationary and co-stationary. We know that for all such S, $\mathcal{K}(S)$ lies strictly between $\omega_1 \times \omega^\omega$ and $[\omega_1]^{<\omega} \times \omega^\omega$ and that it is not possible to have $\mathcal{K}(S) \leq \Sigma(\omega^{\omega_1})$. By a theorem in [58], we know there are 2^{ω_1}-many pairwise incomparable Tukey classes $\mathcal{K}(S)$ where S is stationary and co-stationary. As in the case of the antichain in $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$, this collection is an antichain in a stronger relative Tukey sense.

Comparing $\mathcal{K}(M)$ and $\mathcal{K}(S)$. We investigate under what conditions we get $\mathcal{K}(M) \geq_T \mathcal{K}(S)$ and $\mathcal{K}(S) \geq_T \mathcal{K}(M)$. The answer in the case that M is Polish and $\mathcal{K}(S) \leq \omega_1 \times \omega^\omega$ is trivial from work done already. We show that $\omega_1 \times \omega^\omega <_T \mathcal{K}(\mathbb{Q}) <_T [\omega_1]^{<\omega} \times \omega^\omega$, and $\omega_1 \times \omega^\omega <_T \mathcal{K}(M) <_T [\omega_1]^{<\omega} \times \omega^\omega$ for any totally imperfect M of size ω_1. It was proven in [21] that for any non-Polish M, $\mathcal{K}(M) \not\leq_T \omega_1 \times \omega^\omega$. For unbounded S, there is M with $\mathcal{K}(M) \geq_T (S, \mathcal{K}(S))$ if and only if S contains a closed unbounded set, and there is M with

$\mathcal{K}(M) \geq_T \mathcal{K}(S)$ if and only if $\overline{S} \backslash S$ is bounded.

Applications. Recall that $C_k(X)$ and $C_p(X)$ stand for the set of all continuous, real-valued functions on X with the compact-open topology and topology of pointwise convergence, respectively. There is a very strong connection between the function space $C_k(X)$ and the Tukey ordering. In particular, suppose X and Y are non-compact subsets of $\mathbb{R}$ or ω_1. If there is a continuous open surjection from $C_k(X)$ to $C_k(Y)$ or if $C_k(Y)$ embeds in $C_k(X)$ then we have $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$. Therefore the collection $\mathcal{A}$ that gives a $2^{\mathfrak{c}}$-sized antichain in $\mathcal{K}(Sub(\mathbb{R}))$ also satisfies the property that whenever $M, N \in \mathcal{A}$, $C_k(M)$ and $C_k(N)$ are not homeomorphic. Antichain of size 2^{ω_1} in $\mathcal{K}(Sub(\omega_1))$ returns a similar family of subsets of ω_1.

On the there hand, the function space $C_p(X)$, considered as a locally convex topological vector space, is connected with the relative Tukey class $(X, \mathcal{K}(X))$. Suppose X and Y are non-compact subsets of $\mathbb{R}$ or ω_1. If there is a continuous linear surjection from $C_p(X)$ to $C_p(Y)$ or if $C_p(Y)$ linearly embeds in $C_p(X)$ then we have $\mathcal{K}(X) \geq_T (Y, \mathcal{K}(Y))$. Therefore the collection $\mathcal{A}$ that gives a $2^{\mathfrak{c}}$-sized antichain in $\mathcal{K}(Sub(\mathbb{R}))$ also satisfies the property that whenever $M, N \in \mathcal{A}$, $C_p(M)$ and $C_p(N)$ are not homeomorphic. The antichain in $\mathcal{K}(Sub(\omega_1))$ yields a family of subsets of ω_1 with analogous properties.

For the last two applications we consider the Tukey relation between $\mathcal{K}(M)$ and $\mathcal{K}(X)$, where X is an arbitrary Tychonoff space. It was proven in [11] that this condition is closely related to X being Lindelöf Σ. Recall that a space is Lindelöf Σ if there is a countable collection $\mathcal{W}$ and a compact cover $\mathcal{C}$ such that for every $C \in \mathcal{C}$ and an open set U with $C \subseteq U$, there is $W \in \mathcal{W}$ such that $C \subseteq W \subseteq U$. We construct a subset of $\omega_2 + 1$ to show that in Baturov's theorem the condition 'X is Lindelöf Σ' cannot be substituted by the condition 'there exists separable metrizable M such that $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$'. In [11] it was proven that, under the Continuum Hypothesis, if X is compact and there exists separable metrizable M such that $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$, then X must be countable. We use the fact that every $\mathcal{K}(M)$ has calibre (ω_1, ω) to show that this is true in ZFC as well.

1.0 TUKEY ORDER

In this chapter we introduce the relative Tukey order and present preliminary results. In Section 1.1 we present general results about the relative Tukey order. In sections 1.2, 1.3 and 1.4 we review connections of the relative Tukey order with cofinality, additivity, calibres, powers, embeddings of well-orders and spectrum. In Section 1.5 we single out the poset $\mathcal{K}(X)$ and focus on related relative Tukey pairs. In this section we prove that the Tukey classes arising from subsets of $\mathbb{R}$ include Tukey classes arising from arbitrary separable metrizable spaces. Lastly, in Section 1.6, we prove the Key Lemma that underlies most results in Chapter 3.

1.1 RELATIVE TUKEY ORDER

Let P be a partially ordered set (poset) and let P' be a subset of P. A subset C of P is called *cofinal for P' in P* if for every $p \in P'$ there is $c \in C$ such that $p \leq c$. Suppose also that Q' is a subset of Q. Then (Q', Q) *is a relative Tukey quotient of* (P', P), denoted $(P', P) \geq_T (Q', Q)$, if there is a map $\phi : P \to Q$, a *relative Tukey quotient*, such that whenever C is cofinal for P' in P, $\phi(C)$ is cofinal for Q' in Q. If $P' = P$ and $Q' = Q$ then $(P', P) \geq_T (Q', Q)$ just means $P \geq_T Q$. When $P' = P$, we write $P \geq_T (Q', Q)$ instead of $(P', P) \geq_T (Q', Q)$ and when $Q' = Q$, we write $(P', P) \geq_T Q$ instead of $(P', P) \geq_T (Q', Q)$. It is clear from the definition that a composition of two relative Tukey quotients is a relative Tukey quotient, so, relative Tukey order is transitive.

Assumption about directedness: all posets in this text will be directed with one set of exceptions. In a relative Tukey pair, (P', P), the first poset, P', does not have to be directed and in many cases it will not be.

Tukey quotients have dual counterparts, *Tukey maps*. A map $\psi : Q \to P$ is called a Tukey map if for every unbounded subset U of Q, $\psi(U)$ is unbounded in P. It is known that there exists a Tukey quotient from P to Q if and only if there exists a Tukey map from Q to P [54, 58, 60]. Similar correspondence holds for relative Tukey quotients. Call $\psi : Q' \to P'$ a relative Tukey map from (Q', Q) to (P', P) if and only if for any $U \subseteq Q'$ unbounded in Q, $\psi(U) \subseteq P'$ is unbounded in P.

Taking the contra-positive, $\psi : Q' \to P'$ is a relative Tukey map from (Q', Q) to (P', P) if and only if for any subset B of P' bounded in P, $\psi^{-1}(B) \subseteq Q'$ is bounded in Q.

Lemma 1. *There exists a relative Tukey quotient ϕ from (P', P) to (Q', Q) if and only if there exists a relative Tukey map ψ from (Q', Q) to (P', P).*

Proof. We modify the proof of the non-relative version. Suppose a relative Tukey quotient $\phi : P \to Q$ is given and let $q \in Q'$. Then there is $p_q \in P'$ such that whenever $p \geq p_q$, we have $\phi(p) \geq q$. Otherwise, for each $p \in P'$, there is $c_p \geq p$ such that $\phi(c_p) \not\geq q$. Then $C = \{c_p : \ p \in P'\}$ is a subset of P cofinal for P' and no element of $\phi(C)$ is above $q \in Q'$. Thus $\phi(C) \subseteq Q$ is not cofinal for Q', which is a contradiction. Now define $\psi : Q' \to P'$ by setting $\psi(q) = p_q$.

To show that ψ is a relative Tukey map, let $B \subseteq Q'$ and let $\psi(B)$ be bounded by some $p \in P$. For any $q \in B$, $\psi(q) = p_q \leq p$ and by definition $\phi(p) \geq q$. So $\phi(p)$ bounds B.

Now suppose a relative Tukey map $\psi : Q' \to P'$ is given. For each $p \in P$, let $Q_p = \{q \in Q' : \ \psi(q) \leq p\}$. Then $\psi(Q_p)$ is bounded in P by p and therefore Q_p must be bounded by some $q_p \in Q$. Let $\phi(p) = q_p$. Suppose $C \subseteq P$ is cofinal for P' and $q \in Q'$. Then there is $p \in C$ with $p \geq \psi(q)$ and therefore $q \in Q_p$ but then, since $q_p = \phi(p)$ bounds Q_p, we have $q \leq \phi(p) \in \phi(C)$. So $\phi(C) \subseteq Q$ is cofinal for Q'. □

Recall that a poset P is *Dedekind complete* if and only if every subset of P with an upper bound has the least upper bound.

Lemma 2. *If $(P', P) \geq_T (Q', Q)$ and Q is Dedekind complete then there is a Tukey quotient witnessing $(P', P) \geq_T (Q', Q)$ that is order-preserving.*

Conversely, if $\phi : P \to Q$ is order-preserving and $\phi(P')$ is cofinal for Q' in Q, then ϕ is a relative Tukey quotient.

Proof. Assume Q is Dedekind complete. Then in the second part of the proof of Lemma 1 $\phi(p) = q_p$ can be taken to be the least upper bound of the set Q_p, which ensures that ϕ is order-preserving. This gives the first part of the lemma.

The second part follows easily from the fact that P' is cofinal for itself in P. Let $C \subseteq P$ be cofinal for P' and let $q \in Q'$. Since $\phi(P')$ is cofinal for Q', there exists $p \in P'$ such that $\phi(p) \geq q$. Now there is $c \in C$ such that $c \geq p$ and since ϕ is order-preserving, we get $\phi(c) \geq \phi(p) \geq q$. So $\phi(C)$ is cofinal for Q' and ϕ is a relative Tukey quotient. □

The following result when combined with Lemma 2 above is highly convenient.

Lemma 3. *If C is a cofinal set of a poset P then C and P are Tukey equivalent.*

Proof. Let $\phi : C \to P$ be defined by $\phi(c) = c$. Clearly, ϕ is order-preserving and $\phi(C) = C$ is cofinal in P. So ϕ is a Tukey quotient. But ϕ is also a Tukey map. Suppose B is a bounded subset of P. Then, since C is cofinal in P, B is bounded by an element of C. Now, since $\phi^{-1}(B) \subseteq B$, $\phi^{-1}(B)$ is also bounded in C which proves that ϕ is a Tukey map. □

The following lemma is straightforward from definitions:

Lemma 4. *(1) Suppose $P_1' \subseteq P_2' \subseteq P_2 \subseteq P_1$ and $Q_2' \subseteq Q_1' \subseteq Q_1 \subseteq Q_2$. Then $(P_1', P_1) \geq_T (Q_1', Q_1)$ implies $(P_2', P_2) \geq_T (Q_2', Q_2)$.*

(2) If P' is directed and Q and R are Dedekind complete then $(P', P) \geq_T (Q', Q)$ and $(P', P) \geq_T (R', R)$ imply $(P', P) \geq_T (Q' \times R', Q \times R)$.

(3) Whenever $(P', P) \geq_T (Q' \times R', Q \times R)$ we have $(P', P) \geq_T (Q', Q)$ and $(P', P) \geq_T (R', R)$.

Proof. For the first part, suppose a relative Tukey quotient, ϕ, witnesses $(P_1', P_1) \geq_T (Q_1', Q_1)$. Any $C \subseteq P_2$ cofinal for P_2' in P_2 is also cofinal for P_1' in P_1 and therefore $\phi(C)$ is cofinal for Q_1' in Q_1. So, $\phi \restriction_{P_2}: P_2 \to Q_1$ witnesses $(P_2', P_2) \geq_T (Q_1', Q_1)$. Now let $\psi : Q_1' \to P_2'$ be a relative Tukey map witnessing $(P_2', P_2) \geq_T (Q_1', Q_1)$. Then any $U \subseteq Q_2'$ unbounded in Q_2 is also a subset of Q'_1 unbounded in Q_1, so $\psi(U) \subseteq P'_2$ will be unbounded in P_2 and $\psi \restriction_{Q'_2}: Q'_2 \to P'_2$ witnesses $(P'_2, P_2) \geq_T (Q'_2, Q_2)$.

For the second part, suppose ϕ_1 and ϕ_2 are order-preserving Tukey quotients from (P', P) to (Q', Q) and to (R', R) respectively. Then $\phi : P \to Q \times R$ defined by $\phi(p) = \phi_1(p) \times \phi_2(p)$ is also order-preserving. To show $\phi(P')$ is cofinal for $Q' \times R'$, pick arbitrary $(q, r) \in Q' \times R'$. Then there is $p_1, p_2 \in P'$ such that $\phi_1(p_1) \geq q$ and $\phi_2(p_2) \geq r$. Pick $p \in P'$ with $p \geq p_1, p_2$. Then $\phi(p) \geq (q, r)$.

Lastly, given a relative Tukey quotient ϕ from (P', P) to $(Q' \times R', Q \times R)$, the restriction maps to each coordinate give the desired relative Tukey quotients. □

1.2 COFINALITY, ADDITIVITY AND CALIBRES

Define the *cofinality* of P' in P to be $\mathrm{cof}(P', P) = \min\{|C| : C \text{ is cofinal for } P' \text{ in } P\}$. Define the *additivity* of P' in P to be $\mathrm{add}(P, P') = \min\{|S| : S \subseteq P' \text{ and } S \text{ has no upper bound in } P\}$. Then $\mathrm{cof}(P) = \mathrm{cof}(P, P)$ and $\mathrm{add}(P) = \mathrm{add}(P, P)$ coincide with the usual notions of cofinality and additivity of a poset.

Lemma 5. *If* $(P', P) \geq_T (Q', Q)$*, then* $\mathrm{cof}(P', P) \geq \mathrm{cof}(Q', Q)$ *and* $\mathrm{add}(P', P) \leq \mathrm{add}(Q', Q)$.

Proof. If $\phi : P \to Q$ is a relative Tukey quotient, then for any $C \subseteq P$ cofinal for P', $|\phi(C)| \leq |C|$ and $\phi(C)$ is cofinal for Q'. So we get $\mathrm{cof}(P', P) \geq \mathrm{cof}(Q', Q)$.

If $\psi : Q' \to P'$ is a relative Tukey map, then for any $B \subseteq Q'$ unbounded in Q, $|\psi(B)| \leq |B|$ and $\psi(B) \subseteq P'$ is unbounded in P. So we get $\mathrm{add}(P', P) \leq \mathrm{add}(Q', Q)$. □

Lemma 6. *The cofinality of* P' *in* P *is* $\leq \kappa$ *if and only if* $[\kappa]^{<\omega} \geq_T (P', P)$.

Proof. Suppose cofinality of P' in P is $\leq \kappa$. Then we can pick $C \subseteq P$ that is cofinal for P' with $|C| \leq \kappa$. Let $j : \kappa \to C$ be a surjection. Define $\phi : [\kappa]^{<\omega} \to P$ by $\phi(F) =$ an upper bound of $\{j(\alpha) : \alpha \in F\}$. Since F is finite and P is directed, ϕ is well-defined. Even though, ϕ might not be order-preserving, it is still a relative Tukey quotient. Let A be a cofinal subset of $[\kappa]^{<\omega}$ and let $c \in C$. Then since j is surjective, there is $\alpha < \kappa$ with $j(\alpha) = c$. Since

A is cofinal in $[\kappa]^{<\omega}$, there is $F \in A$ such that $\{x\} \in F$ and by definition $\phi(F) \geq j(\alpha) = c$. So, $\phi(A)$ is cofinal for C, which implies $\phi(A)$ is cofinal for P' as well.

On the other hand, since $\mathrm{cof}([\kappa]^{<\omega}) = \kappa$, $[\kappa]^{<\omega} \geq_T (P', P)$ and Lemma 5 imply $\mathrm{cof}((P', P)) \leq \kappa$. □

Lemma 7. *For a Dedekind complete poset P, suppose $P = \bigcup_{\alpha \in \kappa} P_\alpha$ and for each α we have $Q \geq_T (P_\alpha, P)$. Then $Q \times [\kappa]^{<\omega} \geq_T P$.*

Proof. As P is Dedekind complete, for each $\alpha < \kappa$, fix an order-preserving $\phi_\alpha : Q \to P$ such that $\phi_\alpha(Q)$ is cofinal for P_α in P. Define $\phi : Q \times [\kappa]^{<\omega} \to P$ by $\phi(q, F) = \sup\{\phi_\alpha(q) : \alpha \in F\}$, which is well-defined since P is directed and Dedekind complete.

Then ϕ is order-preserving. If p is any element of P, then p is in P_α, for some α. Pick q from Q such that $\phi_\alpha(q) \geq p$. Then $\phi(q, \{\alpha\}) = \phi_\alpha(q) \geq p$, and thus ϕ has cofinal image. □

We notice that the posets of the form $[\kappa]^{<\omega}$ have a special role. The next corollary describes how they interact with each other. Since $\mathrm{cof}([\kappa]^{<\omega}) = \kappa$, Lemma 6 immediately gives the following result.

Corollary 8. *Let κ and λ be cardinals. Then (1) $[\kappa]^{<\omega} \leq_T [\lambda]^{<\omega}$ if and only if $\kappa \leq \lambda$ and (2) $\kappa \leq_T [\lambda]^{<\omega}$ if and only if* $\mathrm{cof}(\kappa) \leq \lambda$.

The following lemma is useful in spectrum calculations in Chapter 2.

Proposition 9. *If $n \in \omega$, then $\omega^\omega \times [\omega_n]^{<\omega} =_T ([\omega_n]^{<\omega})^\omega$.*

Proof. One direction is clear: $([\omega_n]^{<\omega})^\omega = ([\omega_n]^{<\omega})^\omega \times [\omega_n]^{<\omega} \geq_T \omega^\omega \times [\omega_n]^{<\omega}$. For the other direction we use induction on n.

When $n = 0$, since $\omega =_T [\omega]^{<\omega}$, we have $\omega^\omega \times [\omega]^{<\omega} =_T ([\omega]^{<\omega})^\omega$. So assume $([\omega_{n-1}]^{<\omega})^\omega =_T \omega^\omega \times [\omega_{n-1}]^{<\omega}$, for some $n \geq 1$. Then, $([\omega_n]^{<\omega})^\omega = \bigcup_{\omega_{n-1} \leq \alpha < \omega_n} ([[0, \alpha]]^{<\omega})^\omega =_T \omega^\omega \times [\omega_{n-1}]^{<\omega} \times [\omega_n]^{<\omega} =_T \omega^\omega \times [\omega_n]^{<\omega}$, using the inductive hypothesis and Lemma 7. □

Let $\kappa \geq \lambda \geq \mu$ be cardinals. We say that P' has *calibre* (κ, λ, μ) in P if for every κ-sized subset P_0 of P' there is a λ-sized $P_1 \subseteq P_0$ such that every μ-sized subset P_2 of P_1 has an upper bound in P. Sometimes we say '(P', P) has calibre (κ, λ, μ)' instead of 'P' has calibre

(κ, λ, μ) in P'. When $P' = P$ this definition coincides with the standard definition of a calibre of a poset.

Lemma 10. *If $(P', P) \geq_T (Q', Q)$, P' has calibre (κ, λ, μ) in P and κ is regular, then Q' has calibre (κ, λ, μ) in Q.*

Proof. Suppose Q' does not have calibre (κ, λ, μ) in Q. Then there is κ-sized $Q_0 \subseteq Q'$ that satisfies (*): each λ-sized subset of Q_0 has a μ-sized subset that is unbounded in Q. Then we have following facts:

(1) Every $\geq \lambda$-sized subset of Q_0 satisfies (*).

(2) Every $\geq \lambda$-sized subset of Q_0 is unbounded in Q.

Let $\psi : Q' \to P'$ be a relative Tukey map. Then $|\psi(Q_0)| = \kappa$. Otherwise, since κ is regular, there is $p \in P$ such that $|\psi^{-1}(p) \bigcap Q_0| = \kappa$. By (2), $\psi^{-1}(p) \bigcap Q_0$ is unbounded but is mapped to $\{p\}$, which is a contradiction.

We may shrink Q_0 without changing its size, so that $\psi \restriction_{Q_0}$ is injective and by (1) it will keep property (*). Then $|\psi(Q_0)| = \kappa$ implies that there is λ-sized $Q_1 \subseteq Q_0$ such that every μ-sized subset of $\psi(Q_1)$ is bounded in P. But Q_1 has an unbounded μ-sized subset, which contradicts the assumption that ψ is a relative Tukey map. □

Lemma 11. *Suppose κ is regular. Then (1) P' fails to have calibre κ in P if and only if $(P', P) \geq_T \kappa$, (2) If $(P', P) \geq_T [\kappa]^{<\lambda}$ then P' fails to have calibre (κ, λ) and the converse is true if* $\mathrm{add}(P') \geq \lambda$ *(equivalently, all subsets of P' of size $< \lambda$ are bounded in P').*

Proof. Clearly, κ does not have calibre κ. So, by Lemma 10, $(P', P) \geq_T \kappa$ implies that P' does not have calibre κ in P.

Similarly, $[\kappa]^{<\lambda}$ does not have calibre (κ, λ) ($\{\{\alpha\} : \alpha < \kappa\}$ is a κ-sized collection in $[\kappa]^{<\lambda}$ but none of its λ-sized subcollections is bounded in $[\kappa]^{<\lambda}$) and therefore $(P', P) \geq_T [\kappa]^{<\lambda}$ implies that P' fails to have calibre (κ, λ) in P.

On the other hand, suppose P' fails to have calibre κ in P. Then there exists κ-sized $P_0 \subseteq P'$ such that all κ-sized subsets of P_0 are unbounded. Let $\psi : \kappa \to P_0 \subseteq P$ be a bijection. Since κ is regular, all unbounded subsets of κ are κ-sized and their images are unbounded as well. Therefore ϕ is a relative Tukey map.

Similarly, suppose P' fails to have calibre (κ, λ) in P. Then there exists κ-sized $P_1 \subseteq P'$ such that all λ-sized subsets of P_1 are unbounded in P. Let $j : \kappa \to P_1 \subseteq P$ be a bijection. Since $\mathrm{add}(P') \geq \lambda$ we can define $\psi : [\kappa]^{<\lambda} \to P'$ by $\psi(F) =$ an upper bound of $\{j(\alpha) : \alpha \in F\}$ in P'. Suppose U is an unbounded subset of $[\kappa]^{<\lambda}$. This means that $\bigcup U$ has size $\geq \lambda$ and therefore $\{j(\alpha) : \alpha \in \bigcup U\}$, a subset of P_1 of size $\geq \lambda$, is also unbounded in P. Since any bound of $\{\psi(F) : F \in U\}$ is also a bound of $\{j(\alpha) : \alpha \in \bigcup U\}$ in P, we get that $\{\psi(F) : F \in U\}$ is unbounded and ψ is a relative Tukey map. $\square$

The next two lemmas give relative versions of known facts on productivity of calibres.

Lemma 12. *If P' (or Q') fails to have calibre (κ, λ, μ) in P (respectively, in Q) then $P' \times Q'$ also fails to have calibre (κ, λ, μ) in $P \times Q$.*

Proof. Suppose P' does not have calibre (κ, λ, μ) in P and $P_0 \subseteq P'$ witnesses this. Pick $q \in Q'$. Then $\{(p, q) : p \in P_0\}$ witnesses $P' \times Q'$ not having calibre (κ, λ, μ) in $P \times Q$. The case for Q' is similar. $\square$

Lemma 13. *Both (P', P) and (Q', Q) have calibre (κ, κ, μ) if and only if $(P' \times Q', P \times Q)$ has calibre (κ, κ, μ).*

Proof. One direction follows from Lemma 12. For the other direction, suppose (P', P) and (Q', Q) have calibre (κ, κ, μ) and let $A \subseteq P' \times Q'$ has size κ. By symmetry we may assume that $P_0 = \{p : (p, q) \in A \text{ for some } q \in Q'\}$ also has size κ. Then there exists κ-sized $P_1 \subseteq P_0$ such that all μ-sized subsets of P_1 are bounded. For each $p \in P_1$ pick $q_p \in Q'$ such that $(p, q_p) \in A$. Define $Q_1 = \{q_p : p \in P_1\}$. If $|Q_1| < \kappa$ then there exists κ-sized $P_2 \in P_1$ such that $\{q_p : p \in P_2\} = \{q\}$. Then $A_2 = \{(p, q) : p \in P_2\}$ is a κ-sized subset of A with all μ-sized subsets bounded.

If Q_1 is a κ-sized subset of Q', then it contains a κ-sized subset $Q_2 \subseteq Q_1$ such that all μ-sized subsets of Q_2 are bounded in Q. Now let $A_3 = \{(p, q_p) : q_p \in Q_2\}$. Then A_3 is a κ-sized and every μ-sized subset of A_3 is bounded in $P \times Q$. $\square$

Lemma 14. *Suppose $\kappa < add(Q)$ and $(Q, P) \geq_T (Q_\alpha, P_\alpha)$ for each $\alpha \in \kappa$. Also assume each P_α is Dedekind complete. Then $(Q, P) \geq_T (\prod_{\alpha\in\kappa} Q_\alpha, \prod_{\alpha\in\kappa} P_\alpha)$.*

Proof. For each $\alpha \in \kappa$, let ϕ_α be an order-preserving relative Tukey quotient witnessing $(Q, P) \geq_T (Q_\alpha, P_\alpha)$. Define $\phi(x) = \mathbf{x}$ where $\mathbf{x}(\alpha) = \phi_\alpha(x)$ for all $\alpha < \kappa$. Evidently ϕ is an order-preserving map from P to $\prod_{\alpha\in\kappa} P_\alpha$. Take any $(x_\alpha)_{\alpha<\kappa}$ in $\prod_{\alpha\in\kappa} Q_\alpha$. For $\alpha \in \kappa$, $\phi_\alpha(Q)$ is cofinal in Q_α and we can pick $y_\alpha \in Q$ such that $\phi_\alpha(y_\alpha) \geq x_\alpha$. Then $\{y_\alpha : \alpha < \kappa\}$ has an upper bound in Q, say y. Now we see that $\phi(y) \geq (x_\alpha)_{\alpha<\kappa}$, and thus $\phi(Q)$ is cofinal for $\prod_{\alpha\in\kappa} Q_\alpha$. By Lemma 2, ϕ is a relative Tukey quotient and $(Q, P) \geq_T (\prod_{\alpha\in\kappa} Q_\alpha, \prod_{\alpha\in\kappa} P_\alpha)$. □

The following special case of Lemma 14 is particularly useful.

Corollary 15. *Suppose $Q \subseteq P$. If $\kappa < add(Q)$, then $(Q, P) =_T (Q^\kappa, P^\kappa)$.*

Proof. We do not need to assume that P is Dedekind complete as we may choose each ϕ_α from the proof of Lemma 14 to be the identity map on P. Then the argument from Lemma 14 works in this case as well. Note that, by Lemma 4, we always have $(Q^\kappa, P^\kappa) \geq_T (Q, P)$. □

Lemma 16. *Suppose $Q \subseteq P$ and Q is directed. Then the following are equivalent:*

(1) $(Q, P) \geq_T \omega$, (2) $(Q, P) \geq_T (Q \times \omega, P \times \omega)$, and (3) the additivity of (Q, P) is $\aleph_0$.

Proof. If (1) holds, and $(Q, P) \geq_T \omega$, then $(Q, P) \geq_T (\omega, \omega)$, and since $(Q, P) \geq_T (Q, P)$, (2) follows from the proof of Lemma 4, part (2) (ω is Dedekind complete, the identity map $i : P \to P$ is order-preserving and thus we do not need P to be Dedekind complete). (2) $\to$ (1) follows from Lemma 4, part (3). Hence (1) and (2) are equivalent.

Suppose $(Q, P) \geq_T \omega$ and let $\psi : \omega \to Q$ be a relative Tukey map. Then $\psi(\omega)$ is a countably infinite subset of Q with no upper bound in P. Conversely, suppose $A = \{x_n : n \in \omega\}$ is a countably infinite subset of Q with no upper bound in P. As Q is directed we can assume $x_m < x_n$ if $m < n$. Then $\psi : \omega \to Q$ defined by $\psi(n) = x_n$ is a relative Tukey map. Consequently statements (1) and (3) are equivalent. □

If we set $Q = P$ in Corollary 15 and Lemma 16, the next corollary follows immediately.

Corollary 17. *If $\kappa < add(P)$, then $P =_T P^\kappa$.*

Further, the following are equivalent:

(1) $P \geq_T \omega$, (2) $P \geq_T P \times \omega$, and (3) the additivity of P is $\aleph_0$.

Lemma 18. *For a directed poset P without the largest element, the ordinal* $\mathrm{add}(P)$ *order-embeds in P as an unbounded subset.*

Proof. Let $\kappa = \mathrm{add}(P)$ and $\{u_\alpha : \alpha < \kappa\}$ be some unbounded subset of P. We will construct $\{p_\alpha : \alpha \in \kappa\}$ such that $\beta < \alpha$ implies $p_\beta < p_\alpha$ and $p_\alpha \geq u_\alpha$ for each $\alpha < \kappa$.

Pick any $p_0 \in P$ with $p_0 \geq u_0$. Let $\alpha < \kappa$ and suppose $\{p_\beta : \beta < \alpha\}$ have been constructed such that $\beta < \beta'$ implies $p_\beta < p_{\beta'}$ and $p_\beta \geq u_\beta$ for each $\beta < \alpha$. Since $\{p_\beta : \beta < \alpha\}$ has size less than κ, it is bounded, say, by $s \in P$. Since P has no largest element, there is $t \in P$ such that $s \ngeq t$. Since P is directed, there is $p_\alpha \in P$ with $p_\alpha \geq s, t, u_\alpha$. Since $p_\alpha \geq t$ we get that $p_\alpha \neq s$. So $p_\alpha > p_\beta$ for each $\beta < \alpha$. $\square$

1.4 SPECTRUM

From Lemma 3 we know that $\kappa =_T \mathrm{cof}(\kappa)$ and from Section 1.2 we know that $P \geq_T \kappa$ is related to calibres of P whenever κ is regular. Hence regular cardinals are special and we will devote this section to studying when regular cardinals are Tukey-quotients of a poset P. For a poset, P, define a *spectrum* of P, $\mathrm{spec}(P)$, to be the collection of all regular cardinals κ with the property that $P \geq_T \kappa$. If κ_1 and κ_2 are (regular) cardinals let us write $[\kappa_1, \kappa_2]^r$ for the set of all *regular* cardinals τ such that $\kappa_1 \leq \tau \leq \kappa_2$. The following lemmas will be particularly useful in Chapter 2.

Lemma 19. *Let P_1, P_2 be posets and κ be a regular cardinal. Then $P_1 \times P_2 \geq_T \kappa$ if and only if $P_1 \geq_T \kappa$ or $P_2 \geq_T \kappa$.*

Proof. Since P has calibre κ if and only if $P \ngeq_T \kappa$, Lemma 13 gives the desired conclusion. $\square$

Corollary 20. $\mathrm{spec}(P_1 \times P_2) = \mathrm{spec}(P_1) \cup \mathrm{spec}(P_2)$.

Lemma 21. *If $Q \leq_T P$ then* $\mathrm{spec}(Q) \subseteq \mathrm{spec}(P)$.

Proof. Immediately follows from $\kappa \leq_T Q \leq_T P$. $\square$

In the light of Lemma 18, additivity of a poset is always a regular cardinal. However, cofinality might not be regular. Since every cofinal subset is also unbounded, $\mathrm{add}(P) \leq \mathrm{cof}(P)$. The following lemma establishes a close relationship between the spectrum of a poset and its additivity and cofinality.

Lemma 22. *Let P be a directed poset without the largest element. Then:*

(1) $P \geq_T \mathrm{add}(P)$;

(2) $P \geq_T \mathrm{cof}(P)$;

(3) If $P \geq_T \kappa$, then $\mathrm{add}(P) \leq \mathrm{cof}(\kappa) \leq \mathrm{cof}(P)$.

In short we have, $\mathrm{add}(P), \mathrm{cof}(\mathrm{cof}(P)) \in \mathrm{spec}(P) \subseteq [\mathrm{add}(P), \mathrm{cof}(P)]^r$.

Proof. To show (1), let $\psi : \mathrm{add}(P) \to P$ be the order-embedding constructed in the proof of Lemma 18. Since the image of ψ is unbounded, this is a Tukey map and we have $\mathrm{add}(P) \leq_T P$.

For (2), let $\{p_\alpha : \alpha < \mathrm{cof}(P)\}$ be a cofinal subset of P and define $\phi : P \to \mathrm{cof}(P)$ by setting $\phi(p)$ to equal some α such that $p_\alpha \geq p$. Then whenever C is a cofinal subset of P, the set $\{p_\alpha : \alpha \in \phi(C)\}$ is also cofinal in P and therefore has size at least $\mathrm{cof}(P)$. This implies that $\phi(C)$ has size $\mathrm{cof}(P)$ and therefore must be cofinal in $\mathrm{cof}(P)$.

For (3), suppose $P \geq_T \kappa$. Then Lemma 5 implies that $\mathrm{add}(P) \leq \mathrm{add}(\kappa)$ and $\mathrm{cof}(\kappa) \leq \mathrm{cof}(P)$. Then the fact that $\mathrm{add}(\kappa) = \mathrm{cof}(\kappa)$ finishes the proof. $\square$

Even when we do not know what $\mathrm{add}(P)$ and $\mathrm{cof}(P)$ are, we still have the following corollary.

Corollary 23. *For a poset P, $\mathrm{spec}(P) \subseteq [\omega, |P|]^r$.*

On the other hand, we may not restrict the spectrum further to $[\mathrm{add}(P), \mathrm{cof}(\mathrm{cof}(P))]^r$ in general. For example, consider $P = \Sigma_*(\Pi_{n\in\omega}\omega_n) = \{s \in \Pi_{n\in\omega}\omega_n : |\{n : s(n) \neq 0\}| < \omega\}$. Then $\mathrm{cof}(P) = \aleph_\omega$ but $\mathrm{spec}(P) = \{\aleph_n : n \in \omega\}$.

1.5 THE POSET $\mathcal{K}(X)$ AND THE TUKEY RELATION OF X IN $\mathcal{K}(X)$

Let X be any topological space, and $\mathcal{K}(X)$ the set of compact subsets of X (including the empty set) ordered by inclusion. Since union of two compact subsets is compact, each $K(X)$ is directed. We will investigate pairs $(D, \mathcal{K}(X))$, where $X \subseteq D \subseteq \mathcal{K}(X)$, and pay particular attention to the cases when $D = X$, $D = \mathcal{F}(X) =$ all finite subsets of X, and $D = \mathcal{K}(X)$.

In addition to being a subposet of $\mathcal{K}(X)$, X also has a strong topological connection with $\mathcal{K}(X)$. Recall that $\mathcal{K}(X)$ has a natural topology, the Vietoris topology, which is generated by the sets of the form

$$B(U_0, U_1, \cdots, U_n) = \{K \in \mathcal{K}(X) : K \subseteq \bigcup_{i \leq n} U_i \text{ and } K \cap U_i \neq \emptyset, \text{ for all } i \leq n\},$$

where $U_0, U_1, \cdots, U_n$ are open subsets of X. Clearly, if X is second countable, so is $\mathcal{K}(X)$. On the other hand, the topology of X coincides with the subspace topology of X in $\mathcal{K}(X)$, and therefore, the converse is also true. The following lemma is known [16, 36].

Lemma 24. *(1) If X is Tychonoff, then $\mathcal{K}(X)$ is also Tychonoff. (2) If X is Hausdorff, then X embeds as a closed subspace of $\mathcal{K}(X)$.*

Proof. For (1) let $K \in B(U_0, U_1, \cdots, U_n)$. Then $K \subseteq \bigcup_{i \leq n} U_i = U$ and we can pick $x_i \in U_i \cap K$ for each $i \leq n$. For each $i \leq n$, there exists continuous $f_i : X \to [0, 1]$ such that $f_i(x_i) = 1$ and $f_i(X \backslash U_i) = \{0\}$. It is easy to show, using the assumption that X is Tychonoff, that there exists continuous $f : X \to [0, 1]$ such that $f(K) = \{1\}$ and $f(X \backslash U) = \{0\}$.

Define $F : \mathcal{K}(X) \to [0, 1]$ by $F(L) = \min\{\max\{f_i(L)\} : i \leq n\} \cdot \min\{f(L)\}$, which is well-defined and continuous since L is compact. Since $f(K) = \{1\}$ and $\max\{f_i(K)\} = f_i(x_i) = 1$ for each $i \leq n$, $F(K) = 1$. If $L \notin B(U_0, U_1, \cdots, U_n)$ then either $L \nsubseteq U$, in which case $\min\{f(L)\} = 0$, or there is $i \leq n$ such that $L \cap U_i = \emptyset$, in which case $\max\{f_i(L)\} = 0$. So, in either case $F(L) = 0$ and we are done.

For (2) suppose $K \in \mathcal{K}(X) \backslash X$. Then there exist $x, y \in K$ with $x \neq y$. Since X is Hausdorff, there exist disjoint open subsets of X, U and V, with $x \in U$ and $y \in V$. Then $K \in B(X, U, V) \subseteq \mathcal{K}(X) \backslash X$ and therefore $\mathcal{K}(X) \backslash X$ is open. $\square$

Lemma 24 implies that X is separable metrizable if and only if $\mathcal{K}(X)$ is separable metrizable. It is also known that when X is metrizable the Vietoris topology on $\mathcal{K}(X)$ is compatible with the Hausdorff metric; and if X is complete, then the Hausdorff metric is also complete [36]. So, if M is Polish then $\mathcal{K}(M)$ is Polish and the converse is also true since X embeds as a closed subspace in $\mathcal{K}(X)$.

Let $\mathcal{X}$ be the class of all homeomorphism classes of Tychonoff spaces. Then each $\mathcal{K}(X)$ is an element of $\mathcal{X}$, and we call a class map $\mathcal{D} : \mathcal{X} \to \mathcal{X}$ a $\mathcal{K}$-operator if for every X in $\mathcal{X}$ we have $X \subseteq \mathcal{D}(X) \subseteq \mathcal{K}(X)$. By the notation introduced in Chapter 0, $Sub([0,1]^\omega)$ is the set of all homeomorphism classes of subsets of $[0,1]^\omega$, or the set of all homeomorphism classes of separable metrizable spaces. By the above discussion, every $\mathcal{K}$-operator maps $Sub([0,1]^\omega)$ into $Sub([0,1]^\omega)$.

We upgrade the definition of $\mathcal{K}(Sub(X))$: for a space X and a given $\mathcal{K}$-operator $\mathcal{D}$, let $(\mathcal{D}(Sub(X)), \mathcal{K}(Sub(X)))$ be the poset of all relative Tukey equivalence classes of the form $[(\mathcal{D}(Y), \mathcal{K}(Y))]_T$ for Y from $Sub(X)$. We are particularly interested in cases when $X = \mathbb{R}$ or $X = \omega_1$ and $\mathcal{D}$ is defined by $X \mapsto X$, $X \mapsto \mathcal{F}(X)$ or $X \mapsto \mathcal{K}(X)$.

Note that $\mathcal{K}(X)$ is Dedekind complete: whenever $\mathcal{K} \subseteq \mathcal{K}(X)$ and $\bigcup \mathcal{K} \subseteq K$ for some $K \in \mathcal{K}(X)$, we have that $\overline{\bigcup \mathcal{K}}$ is compact and $\overline{\bigcup \mathcal{K}} \subseteq K$; so $\overline{\bigcup \mathcal{K}}$ is the least upper bound for a bounded $\mathcal{K} \subseteq \mathcal{K}(X)$. Therefore we may assume that relative Tukey quotients witnessing $(P', P) \geq_T (D, \mathcal{K}(X))$ are order-preserving.

We give two additional properties of $\mathcal{K}(X)$ that are also known [36]:

Lemma 25. *Let X be a space.*

(1) For any K in $\mathcal{K}(X)$, the set $\downarrow K = \{L \in \mathcal{K}(X) : L \subseteq K\}$ is a compact subset of $\mathcal{K}(X)$.

(2) For any compact subset $\mathcal{K}$ of $\mathcal{K}(X)$, its union, $\bigcup \mathcal{K}$, is a compact subset of X.

Proof. Note that the sets of the form $V(U) = \{K \in \mathcal{K}(X) : K \subseteq U\}$ and $W(U) = \{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$ form a subbase for the Vietoris topology.

(1) To show that $\downarrow K$ is compact, it suffices to show that every cover by sets from the subbase has a finite subcover. Suppose $\{V(U_i) : i \in I\} \cup \{W(U_j) : j \in J\}$ covers $\downarrow K$. The set $K \setminus \bigcup_{j \in J} U_j$ is a compact subset of K with the property that $K \setminus \bigcup_{j \in J} U_j \notin \bigcup_{j \in J} W(U_j)$.

Then there is $i \in I$ such that $K \backslash \bigcup_{j \in J} U_j \in V(U_i)$. Therefore $K \backslash \bigcup_{j \in J} U_j \subseteq U_i$. Then the compact set $K \backslash U_i \subseteq \bigcup_{j \in J} U_j$. So there exists $\{j_0, j_1, \cdots, j_n\} \subseteq J$ such that $K \backslash U_i \subseteq \bigcup_{k \leq n} U_{j_k}$. We claim that $\downarrow K \subseteq V(U_i) \cup \bigcup_{k \leq n} W(U_{j_k})$. To show this, let $L \subseteq K$. If $L \subseteq U_i$ then $L \in V(U_i)$. If $L \nsubseteq U_i$ then there is $k \leq n$ such that $L \cap U_{j_k} \neq \emptyset$ and therefore $L \in W(U_{j_k})$.

(2) Suppose $\mathcal{K} \subseteq \mathcal{K}(X)$ is compact. And let $\{U_i : i \in I\}$ be a an open cover of $\bigcup \mathcal{K}$. Then $\{V(\bigcup_{i \in F}(U_i)) : F$ is a finite subset of $I\}$ is a cover of $\mathcal{K}$. Pick a finite subcover $\{V(\bigcup_{i \in F_0}(U_i)), V(\bigcup_{i \in F_2}(U_i)), \cdots, V(\bigcup_{i \in F_n}(U_i)\}$ of $\{V(\bigcup_{i \in F}(U_i)) : F$ is a finite subset of $I\}$. Then the finite collection $\{U_i : i \in \bigcup_{k<n} F_k\}$ covers $\bigcup \mathcal{K}$. □

We start by giving variants and dual versions of a relative Tukey quotient of $(\mathcal{D}(X), \mathcal{K}(X))$ to $(\mathcal{D}(Y), \mathcal{K}(Y))$.

Lemma 26. *Fix two spaces X and Y and the $\mathcal{K}$-operator $\mathcal{D}$. The following are equivalent:*

(1) there is a relative Tukey quotient, ϕ, of $(\mathcal{D}(X), \mathcal{K}(X))$ to $(\mathcal{D}(Y), \mathcal{K}(Y))$,

(2) there is a map $\phi' : \mathcal{D}(X) \to \mathcal{K}(Y)$ such that $\phi'(X)$ is cofinal for $\mathcal{D}(Y)$, and if K is a compact subset of X then $\overline{\bigcup \phi'(\downarrow K \cap \mathcal{D}(X))}$ is compact,

(3) there is a relative Tukey map, ψ, of $(\mathcal{D}(Y), \mathcal{K}(Y))$ into $(\mathcal{D}(X), \mathcal{K}(X))$, and

(4) there is a map $\psi' : \mathcal{D}(Y) \to \mathcal{D}(X)$ such that if K is a compact subset of X then $\overline{\bigcup \psi'^{-1}(\downarrow K)}$ is compact.

Proof. Lemma 1 asserts that (1) and (3) are equivalent. Lemma 2 gives the equivalence of (1) and (2). Noting that a subset B of X is bounded in $\mathcal{K}(X)$ if and only if it has compact closure, we see that conditions (3) and (4) are the contra-positives of each other. □

Note that a collection $\mathcal{K}$ of compact subsets of a space X is 'cofinal for X in $\mathcal{K}(X)$' if and only if $\mathcal{K}$ is a compact cover of X.

Corollary 27. *Fix two spaces X and Y. The following are equivalent:*

(1) there is a relative Tukey quotient, ϕ, of $(X, \mathcal{K}(X))$ to $(Y, \mathcal{K}(Y))$,

(2) there is a map $\phi' : X \to \mathcal{K}(Y)$ such that $\phi'(X)$ is a cover of Y, and if K is a compact subset of X then $\overline{\bigcup\{\phi'(x) : x \in K\}}$ is compact,

(3) there is a relative Tukey map, ψ, of $(Y, \mathcal{K}(Y))$ into $(X, \mathcal{K}(X))$, and

(4) there is a map $\psi' : Y \to X$ such that if K is a compact subset of X then $\overline{\psi'^{-1}(K)}$ is compact.

The following two corollaries are versions of Lemma 26 for $\mathcal{K}(X) \geq_T (Y, \mathcal{K}(Y))$ and $(X, \mathcal{K}(X)) \geq_T \mathcal{K}(Y)$.

Corollary 28. *Fix two spaces X and Y. The following are equivalent:*

(i) there is a relative Tukey quotient, ϕ, of $\mathcal{K}(X)$ to $(Y, \mathcal{K}(Y))$,

(ii) there is a map $\phi' : \mathcal{K}(X) \to \mathcal{K}(Y)$ such that $\phi'(\mathcal{K}(X))$ is a cover of Y, and if K is a compact subset of X then $\overline{\bigcup \phi'(\downarrow K)}$ is compact,

(iii) there is a relative Tukey map, ψ, of $(Y, \mathcal{K}(Y))$ into $\mathcal{K}(X)$, and

(iv) there is a map $\psi' : Y \to \mathcal{K}(X)$ such that if K is a compact subset of X then $\overline{\psi'^{-1}(\downarrow K)}$ is compact.

Corollary 29. *Fix two spaces X and Y. The following are equivalent:*

(i) there is a relative Tukey quotient, ϕ, of $(X, \mathcal{K}(X))$ to $\mathcal{K}(Y)$,

(ii) there is a map $\phi' : X \to \mathcal{K}(Y)$ such that $\phi'(X)$ is cofinal in $\mathcal{K}(Y)$, and if K is a compact subset of X then $\overline{\bigcup\{\phi'(x) : x \in K\}}$ is compact,

(iii) there is a relative Tukey map, ψ, of $\mathcal{K}(Y)$ into $(X, \mathcal{K}(X))$, and

(iv) there is a map $\psi' : \mathcal{K}(Y) \to X$ such that if K is a compact subset of X then $\overline{\bigcup \psi'^{-1}(K)}$ is compact.

Note that both $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$ and $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$ are stronger than $\mathcal{K}(X) \geq_T (Y, \mathcal{K}(Y))$. But $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$ is independent from $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$. For example, if X is σ-compact and $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$ then Y is σ-compact. Therefore, by Theorem 76, if $(\mathbb{Q}, \mathcal{K}(\mathbb{Q})) \geq_T (Y, \mathcal{K}(Y))$ then Y is σ-compact but $\mathcal{K}(\mathbb{Q}) \geq_T \mathcal{K}(\mathcal{K}(\mathbb{Q}))$ and $\mathcal{K}(\mathbb{Q})$ is (coanalytic but) not σ-compact (or even Borel). Therefore $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$ does not imply $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$. On the other hand, by Theorem 76, $(\omega, \mathcal{K}(\omega)) =_T (\mathbb{Q}, \mathcal{K}(\mathbb{Q}))$ but, by Corollary 78, $\mathcal{K}(\omega) \not\geq_T \mathcal{K}(\mathbb{Q})$. So, $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$ does not imply $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$.

For any space Z, abbreviate $\mathcal{K}(\mathcal{K}(Z))$ to $\mathcal{K}^2(Z)$. The next lemma shows that, while moving from X to $\mathcal{K}(X)$ is likely to increase cofinal complexity, moving from $\mathcal{K}(X)$ to $\mathcal{K}^n(X)$ does not change it.

Lemma 30. *Let Z be a space, and $\mathcal{D}$ a $\mathcal{K}$-operator. Then $\mathcal{K}(Z) = (\mathcal{K}(Z), \mathcal{K}(Z))$ $=_T (\mathcal{D}(\mathcal{K}(Z)), \mathcal{K}^2(Z)) =_T (\mathcal{K}^2(Z), \mathcal{K}^2(Z)) = \mathcal{K}^2(Z)$.*

Hence for spaces X and Y, $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$ if and only if $(\mathcal{D}(\mathcal{K}(X)), \mathcal{K}^2(X)) \geq_T (\mathcal{D}(\mathcal{K}(Y)), \mathcal{K}^2(Y))$.

Proof. The second Tukey equivalence follows from the first by taking $\mathcal{D} = \mathcal{K}$, so we need to prove that for any $\mathcal{K}$-operator $\mathcal{D}$ we have $\mathcal{K}(Z) =_T (\mathcal{D}(\mathcal{K}(Z)), \mathcal{K}^2(Z))$. First define $\phi_1 : \mathcal{K}^2(Z) \to \mathcal{K}(Z)$ by $\phi_1(\mathcal{K}) = \bigcup \mathcal{K}$. Then ϕ_1 is order-preserving, and $\phi_1(\mathcal{D}(\mathcal{K}(Z))) \supseteq \phi_1(\mathcal{K}(Z)) = \mathcal{K}(Z)$. Thus $(\mathcal{D}(\mathcal{K}(Z)), \mathcal{K}^2(Z)) \geq_T (\mathcal{K}(Z), \mathcal{K}(Z))$.

For the reverse Tukey quotient define $\phi_2 : \mathcal{K}(Z) \to \mathcal{K}^2(Z)$ by $\phi_2(K) = \downarrow K$. Then ϕ_2 is order-preserving. It suffices to show that $\phi_2(\mathcal{K}(Z))$ is cofinal in $\mathcal{K}^2(Z)$. But take any $\mathcal{K}$ a compact subset of $\mathcal{K}(Z)$. Then $K = \bigcup \mathcal{K}$ is a compact subset of Z, and $\phi(K) = \downarrow \bigcup \mathcal{K} \supseteq \mathcal{K}$. □

The next few lemmas are existence and preservation results for Tukey quotients on $\mathcal{K}(X)$ and $(X, \mathcal{K}(X))$. Call a $\mathcal{K}$-operator, $\mathcal{D}$, *productive* if for any pair of spaces X and Y we have $(\mathcal{D}(X \times Y), \mathcal{K}(X \times Y)) =_T (\mathcal{D}(X) \times \mathcal{D}(Y), \mathcal{K}(X) \times \mathcal{K}(Y))$.

Lemma 31. *The operators $\mathcal{F}$, $\mathcal{K}$ and identity are productive.*

Proof. The desired relative Tukey quotients in all three cases are obtained by defining $\phi_1(K, L) = K \times L$ and $\phi_2(C) = (\pi_X(C), \pi_Y(C))$, where π_X and π_Y are projection maps onto X and Y, respectively. □

Lemma 32. *Let $\mathcal{D}$ be a productive $\mathcal{K}$-operator. Let X be any space and C a compact space. Then $(\mathcal{D}(X), \mathcal{K}(X)) =_T (\mathcal{D}(X \times C), \mathcal{K}(X \times C))$.*

Proof. By hypothesis $(\mathcal{D}(X \times C), \mathcal{K}(X \times C)) =_T (\mathcal{D}(X) \times \mathcal{D}(C), \mathcal{K}(X) \times \mathcal{K}(C))$. So it suffices to show $(\mathcal{D}(X) \times \mathcal{D}(C), \mathcal{K}(X) \times \mathcal{K}(C))$ is Tukey equivalent to $(\mathcal{D}(X), \mathcal{K}(X))$. Tukey quotients witnessing this are obtained by defining $\phi_1(K, L) = K$ and $\phi_2(K) = (K, C)$. □

Lemma 33. *Let A be a closed subspace of a space X. Let D be a subset of $\mathcal{K}(X)$. Then $(D, \mathcal{K}(X)) \geq_T (D \cap \mathcal{K}(A), \mathcal{K}(A))$.*

In particular, $(X, \mathcal{K}(X)) \geq_T (A, \mathcal{K}(A))$*,* $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(A), \mathcal{K}(A))$*,* $\mathcal{K}(X) \geq_T \mathcal{K}(A)$*, and* $\mathcal{K}(X) \geq_T (A, \mathcal{K}(A))$*.*

Proof. Define $\phi : \mathcal{K}(X) \to \mathcal{K}(A)$ by $\phi(K) = K \cap A$. Since A is closed, $K \cap A$ is in $\mathcal{K}(A)$. Clearly, ϕ is order-preserving. So to show that ϕ is the required relative Tukey quotient it suffices to show that $\phi(D)$ is cofinal for $D \cap \mathcal{K}(A)$. But this is clear since for any $K \in \mathcal{K}(A) \subseteq \mathcal{K}(X)$, $K = \phi(K)$. $\square$

Any continuous function $f : X \to Y$ induces a continuous function $\mathcal{K}f : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $\mathcal{K}f(K) = f(K)$. A map $f : X \to Y$ is said to be *compact-covering* if for every compact subset L of Y there is a compact subset K of X such that $f(K) \supseteq L$. Note that f is compact-covering if and only if $\mathcal{K}f$ is a surjection. A continuous surjective map f is said to be *perfect* if and only if f is closed and $f^{-1}(x)$ is compact for each $y \in Y$.

Lemma 34. *Let* $f : X \to Y$ *be a continuous map. Let* D *be a subset of* $\mathcal{K}(X)$*. Then* $(D, \mathcal{K}(X)) \geq_T (\mathcal{K}f(D), \mathcal{K}(Y))$*.*

If f *is surjective, then* $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$*,* $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$*, and* $\mathcal{K}(X) \geq_T (Y, \mathcal{K}(Y))$*. If* f *is compact-covering, then* $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$*. If* f *is perfect, then* $(X, \mathcal{K}(X)) =_T (Y, \mathcal{K}(Y))$*,* $(\mathcal{F}(X), \mathcal{K}(X)) =_T (\mathcal{F}(Y), \mathcal{K}(Y))$ *and* $\mathcal{K}(X) =_T \mathcal{K}(Y)$*.*

Proof. From the definition $\mathcal{K}f : \mathcal{K}(X) \to \mathcal{K}(Y)$ is a order-preserving relative Tukey quotient witnessing $(D, \mathcal{K}(X)) \geq_T (\mathcal{K}f(D), \mathcal{K}(Y))$.

Suppose $f : X \to Y$ is a continuous surjection. Then $\mathcal{K}f(X) = Y$ and $\mathcal{K}f(\mathcal{F}(X)) = \mathcal{F}(Y)$. So we get $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$, $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$. The third inequality, $\mathcal{K}(X) \geq_T (Y, \mathcal{K}(Y))$, follows from either of the two.

Now suppose f is also compact-covering. So, for each $L \in \mathcal{K}(Y)$, there is $K \in \mathcal{K}(X)$ such that $L \subseteq f(K) = \phi(K)$, which implies $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$.

If f is perfect, then $f^{-1}(L)$ is compact for each $L \in \mathcal{K}(Y)$. So, f is compact-covering (and surjective), which gives $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$, $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$ and $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$. For the other inequalities, the map $\phi : \mathcal{K}(Y) \to \mathcal{K}(X)$ define by $\phi(L) = f^{-1}(L)$ is order-preserving. As $K \subseteq f^{-1}(f(K))$, $\phi(\mathcal{K}(Y))$ is cofinal in $\mathcal{K}(X)$ and

$\mathcal{K}(Y) \geq_T \mathcal{K}(X)$. Clearly, $\phi(Y)$ is cofinal for X and $\phi(\mathcal{F}(Y))$ is cofinal for $\mathcal{F}(X)$. Therefore, $(X, \mathcal{K}(X)) \leq_T (Y, \mathcal{K}(Y))$ and $(\mathcal{F}(X), \mathcal{K}(X)) \leq_T (\mathcal{F}(Y), \mathcal{K}(Y))$. □

Lemma 35. *Suppose X is a subspace of compact K, Y is a subspace of compact L, and $f : K \to L$ is continuous such that $X = f^{-1}Y$. For any subset D of $\mathcal{K}(X)$ we have $(D, \mathcal{K}(X)) =_T (\mathcal{K}f(D), \mathcal{K}(Y))$.*

In particular, $(X, \mathcal{K}(X)) =_T (Y, \mathcal{K}(Y))$, $(\mathcal{F}(X), \mathcal{K}(X)) =_T (\mathcal{F}(Y), \mathcal{K}(Y))$ and $\mathcal{K}(X) =_T \mathcal{K}(Y)$.

Proof. Let f be as above. From Lemma 34 we know that $(D, \mathcal{K}(X)) \geq_T (\mathcal{K}f(D), \mathcal{K}(Y))$. Since $X = f^{-1}(Y)$, $f^{-1}(C) \subseteq X$ for each $C \in \mathcal{K}(Y)$. Since K is compact $f^{-1}(K)$ is compact and we can define a map $\phi : \mathcal{K}(Y) \to \mathcal{K}(X)$ by $\phi(C) = f^{-1}(C)$ for $C \in \mathcal{K}(Y)$. Since $L \subseteq f^{-1}(f(L)) = \phi(f(L))$ for each $L \in D$, we get that ϕ witnesses $(D, \mathcal{K}(X)) \leq_T (\mathcal{K}f(D), \mathcal{K}(Y))$.

The last statement follow from the fact that $\mathcal{K}f(X) = Y$, $\mathcal{K}f(\mathcal{F}(X)) = \mathcal{F}(Y)$ and $\mathcal{K}f(\mathcal{K}(X)) = \mathcal{K}(Y)$. □

Lemma 36. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces. Then $\mathcal{K}(\prod_{\lambda\in\Lambda} X_\lambda) =_T \prod_{\lambda\in\Lambda} \mathcal{K}(X_\lambda)$.*

Proof. The two maps $K \mapsto (\pi_\lambda(K))_{\lambda\in\Lambda}$ and $(K_\lambda)_{\lambda\in\Lambda} \mapsto \prod_{\lambda\in\Lambda} K_\lambda$ are the required Tukey quotients. □

The next lemma gives us freedom to work with arbitrary separable metrizable spaces while studying structure of $\mathcal{K}(Sub(\mathbb{R}))$.

Lemma 37. *If M is a separable metrizable space then there is M_0 a subset of the Cantor set $\{0,1\}^\omega$ (and therefore zero-dimensional) such that $\mathcal{K}(M) =_T \mathcal{K}(M_0)$, $(M, \mathcal{K}(M)) =_T (M_0, \mathcal{K}(M_0))$ and $(\mathcal{F}(M), \mathcal{K}(M)) =_T (\mathcal{F}(M_0), \mathcal{K}(M_0))$.*

In particular, $\mathcal{K}(Sub([0,1]^\omega)) = \mathcal{K}(Sub(\mathbb{R}))$, $(Sub([0,1]^\omega), \mathcal{K}(Sub([0,1]^\omega))) = (Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub([0,1]^\omega)), \mathcal{K}(Sub([0,1]^\omega))) = (\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$.

Proof. The space M is homeomorphic to a subspace of the Hilbert cube, $[0,1]^\omega$. So we assume that M is in fact a subspace of $[0,1]^\omega$. Fix a continuous surjection of the Cantor set, $\{0,1\}^\omega$ to $[0,1]^\omega$, and set $M_0 = f^{-1}M$. Then M_0 is zero-dimensional, and the preceding lemma immediately yields the desired conclusion. □

Theorem 38. *Let $\mathcal{D}$ be a $\mathcal{K}$-operator. Then there is an order-embedding, $\Phi = \Phi_{\mathcal{D}}$, of $\mathcal{K}(Sub(\mathbb{R}))$ into $(\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$ such that $\Phi(\mathcal{K}(Sub(\mathbb{R})))$ is cofinal in $(\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$. Hence $\mathcal{K}(Sub(\mathbb{R})) =_T (\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$.*

In particular, $\mathcal{K}(Sub(\mathbb{R}))$, $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$ are all Tukey equivalent.

Proof. Fix the $\mathcal{K}$-operator $\mathcal{D}$ and define $\Phi : \mathcal{K}(Sub(\mathbb{R})) \to (\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$ by $\Phi([\mathcal{K}(M)]_T) = [(\mathcal{D}(\mathcal{K}(M)), \mathcal{K}^2(M))]_T$. By Lemma 30, $\mathcal{K}(M)$ and $\mathcal{K}(M')$ are in the same Tukey class if and only if $(\mathcal{D}(\mathcal{K}(M)), \mathcal{K}^2(M))$ and $(\mathcal{D}(\mathcal{K}(M')), \mathcal{K}^2(M'))$ are in the same relative Tukey class. Combining this with Lemma 37 gives that Φ is well-defined.

By Lemma 30, $\mathcal{K}(M) \geq_T \mathcal{K}(M')$ if and only if $(\mathcal{D}(\mathcal{K}(M)), \mathcal{K}^2(M)) \geq_T (\mathcal{D}(\mathcal{K}(M')), \mathcal{K}^2(M'))$. Hence Φ is an order-embedding.

Take any member, $[(\mathcal{D}(M), \mathcal{K}(M))]_T$ of $(\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$. By Lemma 30, $(\mathcal{D}(\mathcal{K}(M)), \mathcal{K}^2(M)) \geq_T (\mathcal{K}(M), \mathcal{K}(M))$, and since $(\mathcal{K}(M), \mathcal{K}(M)) \geq_T (\mathcal{D}(M), \mathcal{K}(M))$, we have $(\mathcal{D}(\mathcal{K}(M)), \mathcal{K}^2(M)) \geq_T (\mathcal{D}(M), \mathcal{K}(M))$. Thus $[\mathcal{K}(M)]_T$ is in $\mathcal{K}(Sub(\mathbb{R}))$ and $\Phi([\mathcal{K}(M)]_T) \geq_T [(\mathcal{D}(M), \mathcal{K}(M))]_T$, and $\Phi(\mathcal{K}(Sub(\mathbb{R})))$ is cofinal in $(\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$. By Lemma 3, $\Phi(\mathcal{K}(Sub(\mathbb{R}))) =_T (\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$ and since Φ is order-embedding we have $\mathcal{K}(Sub(\mathbb{R})) =_T (\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$. □

To clear the way for applying results of section 1.3 we make some additivity calculations. A space X is *ω-bounded* if and only if whenever $\{x_n : n \in \omega\}$ is a sequence in X, then $\overline{\{x_n : n \in \omega\}}$ is compact. A space X is *strongly ω-bounded* if and only if whenever $\{K_n : n \in \omega\}$ is a countable family of compact subsets of X, then $\overline{\bigcup\{K_n : n \in \omega\}}$ is compact. Every metrizable ω-bounded space is compact. Note that ω_1 is strongly ω-bounded.

Lemma 39. *Let X be a space. Then:*

1. *The additivity of $\mathcal{K}(X)$ is $\aleph_0$ if and only if X is not strongly ω-bounded.*
2. *The additivity of $(X, \mathcal{K}(X))$ is $\aleph_0$ if and only if the additivity of $(\mathcal{F}(X), \mathcal{K}(X))$ is $\aleph_0$ if and only if X is not ω-bounded.*

In particular, if X is metrizable then the additivity of $\mathcal{K}(X)$ is $\aleph_0$ if and only if the additivity of $(X, \mathcal{K}(X))$ is $\aleph_0$ if and only if the additivity of $(\mathcal{F}(X), \mathcal{K}(X))$ is $\aleph_0$ if and only if X is not *compact.*

In the following chapters we will often encounter the space of irrational numbers, ω^ω, and will often consider Tukey order in relation to cardinal numbers. We finish this subsection with two lemmas that will help make arguments tidy later.

Lemma 40. *Suppose D is a subset of $\mathcal{K}(\omega^\omega)$ such that $\omega^\omega \subseteq D$. Then $(\omega^\omega, \leq) =_T (D, \mathcal{K}(\omega^\omega))$. In particular, $(\omega^\omega, \leq) =_T \mathcal{K}(\omega^\omega)$.*

Proof. For each $f \in \omega^\omega$, define $K(f) = \{g \in \omega^\omega : \ g \leq f\} \in \mathcal{K}(\omega^\omega)$. For any compact $K \subseteq \omega^\omega$ and $n \in \omega$, $p_n(K) \subseteq \omega$ is finite, where p_n is a projection on the n-th coordinate and we can define $f_K = (\max\{p_n(K)\})_{n\in\omega} \in \omega^\omega$.

Then, the maps $f \mapsto K(f)$ and $K \mapsto f_K$ are order-preserving and cofinal in their respective posets. $\square$

Lemma 41. *Let κ be a cardinal and let D be such that $\kappa \subseteq D \subseteq \mathcal{K}(\kappa)$. Then $\kappa =_T (D, \mathcal{K}(\kappa))$.*

In particular, $\kappa =_T \mathcal{K}(\kappa) =_T (\kappa, \mathcal{K}(\kappa))$.

Proof. Define $\phi : \kappa \to \mathcal{K}(\kappa)$ by $\phi(\alpha) = [0, \alpha]$. Then ϕ is order-preserving and $\phi(\kappa)$ is cofinal for $\mathcal{K}(\kappa)$ (therefore for D), since each compact subset of κ is contained in some initial segment $[0, \alpha]$. So, $\kappa \geq_T (D, \mathcal{K}(\kappa))$. Define $\phi' : \mathcal{K}(\kappa) \to \kappa$ by $\phi'(K) = \sup(K)$. The map ϕ' is also order-preserving and since $\kappa \subseteq D$, $\phi'(D)$ is cofinal for κ. Thus, $\kappa \leq_T (D, \mathcal{K}(\kappa))$. $\square$

1.6 THE KEY LEMMA

The following lemma is the key in studying structure of $\mathcal{K}(Sub(\mathbb{R}))$. It provides means for constructing antichains and for determining which sets are bounded. Recall that a space X is called *Fréchet-Urysohn* if for each $A \subseteq X$ and $x \in \overline{A}$ there is a sequence $\{x_n\}_{n\in\omega}$ in A that converges to x.

Lemma 42. *Let X be a space such that $\mathcal{K}(X)^2$ is Fréchet-Urysohn and let Y and Z be subspaces of X. Note that $\mathcal{K}(Y)$ and $\mathcal{K}(Z)$ are subspaces of $\mathcal{K}(X)$. Let D be a subset of $\mathcal{K}(Y)$ and E be a subset of $\mathcal{K}(Z)$.*

If $(D, \mathcal{K}(Y)) \geq_T (E, \mathcal{K}(Z))$ *then there is a closed subset* C *of* $\mathcal{K}(X)^2$ *such that* $C[\mathcal{K}(Y)] = \bigcup\{C([K]) : K \in \mathcal{K}(Y)\}$ *is contained in* $\mathcal{K}(Z)$ *and* $C[D] \supseteq E$.

In the case when X *is compact, a (strengthened) converse also holds: if there is a closed subset* C *of* $\mathcal{K}(X)^2$ *such that* $C[\mathcal{K}(Y)] \subseteq \mathcal{K}(Z)$ *and* $C[D]$ *is cofinal for* E *in* $\mathcal{K}(Z)$ *then* $(D, \mathcal{K}(Y)) \geq_T (E, \mathcal{K}(Z))$.

Proof. To start fix an order-preserving map ϕ of $\mathcal{K}(Y)$ to $\mathcal{K}(Z)$ witnessing the relative Tukey quotient $(D, \mathcal{K}(Y)) \geq_T (E, \mathcal{K}(N))$. Let $C_0 = \{(K, L) : K \in \mathcal{K}(Y) \text{ and } L \subseteq \phi(K)\}$. Let $C = \overline{C_0}$. Then C is closed in $\mathcal{K}(X)^2$.

To verify that $C[\mathcal{K}(Y)] \subseteq \mathcal{K}(Z)$ we need to show that if (K, L') is in C, where K is in $\mathcal{K}(Y)$, then L' is in $\mathcal{K}(Z)$. As $\mathcal{K}(X)^2$ is Fréchet-Urysohn, there is a sequence, $(K_n, L_n)_n$ in C_0 converging to (K, L'). Note that for each n we have that $L_n \subseteq \phi(K_n)$. Let $K_\infty = \{K\} \cup \bigcup\{K_n : n \in \omega\}$. Then K_∞ is compact and contains every K_n. So, for each n, we see that $L_n \subseteq \phi(K_\infty)$. Since $\downarrow\phi(K_\infty)$ is compact, the limit, L', of the L_n's is in $\downarrow\phi(K_\infty) \subseteq \mathcal{K}(Z)$.

Take any L in E, and pick K from D such that $L \subseteq \phi(K)$. Then (K, L) is in C_0, and clearly $L \in C[D]$. Thus $C[D] \supseteq E$.

Now suppose X is compact and C is a closed subset of $\mathcal{K}(X)^2$ such that $C[\mathcal{K}(Y)] \subseteq \mathcal{K}(Z)$ and $C[D]$ is cofinal for E in $\mathcal{K}(Z)$. Define $\phi : \mathcal{K}(Y) \to \mathcal{K}(Z)$ by $\phi(K) = \bigcup \pi_2(C \cap (\downarrow K \times \mathcal{K}(X)))$, where π_2 is the projection on the second coordinate. Since π_2 is continuous and C, $\downarrow K$ and $\mathcal{K}(X)$ are all compact, $\pi_2(C \cap (\downarrow K \times \mathcal{K}(X)))$ is a compact subset of $\mathcal{K}(X)$, and $\phi(K)$ is indeed an element of $\mathcal{K}(Z)$. We show that ϕ is the desired relative Tukey quotient. Clearly ϕ is order-preserving. Hence it remains to show that $\phi(D)$ is cofinal for E in $\mathcal{K}(Z)$.

Take any L in E. By hypothesis on C there is a K in D such that $L \in C([K])$. Then (K, L) is in $C \cap (\downarrow K \times \mathcal{K}(X))$, and by definition of ϕ we have, as desired, that $L \subseteq \phi(K)$. $\square$

We record the most useful instances of the above lemma.

Corollary 43. *Let* M *and* N *be subspaces of* $[0,1]^\omega$. *Then the following equivalences hold:*

(1) $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ *if and only if there is a closed subset* C *of* $\mathcal{K}([0,1]^\omega)^2$ *such that* $C[\mathcal{K}(M)] = \mathcal{K}(N)$,

(2) $(\mathcal{F}(M), \mathcal{K}(M)) \geq_T (\mathcal{F}(N), \mathcal{K}(N))$ *if and only if there is a closed subset* C *of* $\mathcal{K}([0,1]^\omega)^2$ *such that* $\mathcal{F}(N) \subseteq C[\mathcal{F}(M)]$*,*

(3) $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$ *if and only if there is a closed subset* C *of* $\mathcal{K}([0,1]^\omega)^2$ *such that* $\bigcup C[\mathcal{K}(M)] = N = \bigcup C[M]$*, and*

(4) $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$ *if and only if there is a closed subset* C *of* $\mathcal{K}([0,1]^\omega)^2$ *such that* $\bigcup C[\mathcal{K}(M)] = N$*.*

Proof. Since $[0,1]^\omega$ is metrizable, $\mathcal{K}([0,1]^\omega)^2$ is also metrizable and therefore $\mathcal{K}([0,1]^\omega)^2$ is Fréchet-Urysohn.

Then (1) and (2) follow immediately by setting $D = \mathcal{K}(M)$, $E = \mathcal{K}(N)$ and $D = \mathcal{F}(M)$, $E = \mathcal{F}(N)$, respectively. For (3), note that $N \subseteq C[M]$ implies $\bigcup C[\mathcal{K}(M)] = N = \bigcup C[M]$ and $N = \bigcup C[M]$ implies that $C[M]$ is cofinal for N. Similarly, for (4), note that $N \subseteq \mathcal{K}(M)$ implies $\bigcup C[\mathcal{K}(M)] = N$ and $\bigcup C[\mathcal{K}(M)] = N$ implies that $C[\mathcal{K}(M)]$ is cofinal for N. $\square$

2.0 ORDER PROPERTIES OF ELEMENTS OF $\mathcal{K}(Sub(\mathbb{R}))$ AND $\mathcal{K}(Sub(\omega_1))$

The main purpose of this chapter is to investigate spectra and calibres of elements of $Sub(\mathbb{R})$ and $Sub(\omega_1)$. The letters M and N will denote elements of $Sub(\mathbb{R})$ (and separable metrizable spaces in general). The letters S and T will denote elements of $Sub(\omega_1)$. We will begin with calculations of upper and lower bounds of various $\mathcal{K}(M)$ and $\mathcal{K}(S)$, as well as their additivities and cofinalities. Section 2.3 focuses on the spectrum of ω^ω, while other spectra results follow as corollaries to bound calculations. Section 2.4 is devoted to calibres, with the main cases being calibre ω_1 for $\mathcal{K}(M)$'s and calibre $(\omega_1, \omega_1, \omega)$ for $\mathcal{K}(S)$'s.

2.1 SIZE AND BOUNDS OF $\mathcal{K}(M)$ AND $\mathcal{K}(S)$

We start with a simple but useful observation about the elements of $\mathcal{K}(S)$.

Lemma 44. *Every compact subset K of ω_1 is contained in some initial segment, $[0, \alpha]$. Hence K is countable, scattered and Polish.*

Consequently, $\mathcal{K}(\omega_1) = \bigcup\{\mathcal{K}([0, \alpha]) : \alpha < \omega_1\}$, and for any $S \subseteq \omega_1$, $\mathcal{K}(S) = \bigcup\{\mathcal{K}([0, \alpha] \cap S) : \alpha < \omega_1\}$.

Proof. The family $\mathcal{U} = \{[0, \alpha] : \alpha < \omega_1\}$ is an increasing open cover of ω_1. So any compact subset K of ω_1, will be contained in a finite subcollection, and hence in the largest member of that subcollection, say $[0, \alpha]$. Scatteredness of K is immediate from the Baire category theorem. Since $[0, \alpha] \setminus K$ is countable, K is a G_δ subset of the compact, metrizable space $[0, \alpha]$, and so is Polish. The decompositions of $\mathcal{K}(\omega_1)$ and $\mathcal{K}(S)$ are now clear. □

We record this well known fact on the elements of $\mathcal{K}(M)$.

Lemma 45. *Every compact subset K of M is either countable, in which case K is scattered and Polish, or K has cardinality $\mathfrak{c}$, and contains a Cantor set.*

Hence if $|M| < \mathfrak{c}$ then every element of $\mathcal{K}(M)$ is countable.

Lemma 46. *For any space X, $\mathcal{K}(X)$ is finite if and only if X is finite.*

Suppose S and M are infinite. If S and M contain a sequence together with its limit then $|\mathcal{K}(M)| = |\mathcal{K}(S)| = \mathfrak{c}$. If M contains no convergent sequence then $|\mathcal{K}(M)| = \omega$. If S contains no convergent sequence then $|\mathcal{K}(S)| = \omega$ for bounded S and $|\mathcal{K}(S)| = \omega_1$ for unbounded S.

Proof. A separable metrizable space M can only have $\mathfrak{c}$-many closed subsets, so $|\mathcal{K}(M)| \leq \mathfrak{c}$. For any countable subset T of ω_1, let $\alpha_T = \sup T$. Then $\mathcal{K}(T) \subseteq \mathcal{K}([0, \alpha_T])$, and so $|\mathcal{K}(T)| \leq \mathfrak{c}$. For any $S \subseteq \omega_1$ we know $\mathcal{K}(S) = \bigcup_{\alpha \in \omega_1} \mathcal{K}(S \cap [0, \alpha])$, and hence $|\mathcal{K}(S)| \leq \mathfrak{c}.\omega_1 = \mathfrak{c}$.

It is clear that if a space X contains a sequence and its limit then $|\mathcal{K}(X)| \geq \mathfrak{c}$. So we get $|\mathcal{K}(M)| = |\mathcal{K}(S)| = \mathfrak{c}$ in this case. If a space does not contain a limit point then it is discrete and if it is separable, it must be countable. So, in this case, $\mathcal{K}(M) = [\omega]^{<\omega}$. Now suppose S is discrete. Then either S is bounded and $\mathcal{K}(S) = [\omega]^{<\omega}$, or S is uncountable and $\mathcal{K}(S) = [\omega_1]^{<\omega}$. The conclusions follow. □

Upper bounds: any poset of size $\leq \mathfrak{c}$ is a Tukey quotient of $[\mathfrak{c}]^{<\omega}$, by Lemma 6. So, $[\mathfrak{c}]^{<\omega}$ bounds each $\mathcal{K}(M)$ and $\mathcal{K}(S)$ from above. We can refine the upper bound for $\mathcal{K}(S)$. Since $\mathcal{K}(S) = \bigcup_{\alpha \in \omega_1} \mathcal{K}(S \cap [0, \alpha])$ and each $\mathcal{K}(S \cap [0, \alpha]) \leq_T \omega^\omega$, we know by Lemma 7 that $\mathcal{K}(S) \leq_T \omega^\omega \times [\omega_1]^{<\omega}$ for any $S \subseteq \omega_1$. In particular, if $\mathfrak{d} = \omega_1$, for each S, $\mathcal{K}(S) \leq_T \omega^\omega \times [\omega_1]^{<\omega} \leq_T [\omega_1]^{<\omega} \times [\omega_1]^{<\omega} = [\omega_1]^{<\omega}$.

Lower bounds: For any non-locally compact M, $\omega^\omega \leq_T \mathcal{K}(M)$ (Lemma 79). This also helps when we consider $\mathcal{K}(S)$'s. Notice that since ω_1 is locally compact, a subset S of ω_1 is locally compact if and only if S is open in its closure, which happens if and only if $\overline{S} \setminus S$ is closed in ω_1.

Lemma 47. *Let $S \subseteq \omega_1$, not necessarily unbounded. Then the following are equivalent:*

(1) S is locally compact;

(2) $\overline{S} \setminus S$ is closed;

(3) S does not contain a metric fan as a closed subspace;

(4) $\omega^\omega \not\leq_T \mathcal{K}(S)$.

Proof. Clearly, S is locally compact if and only if $S \cap [0, \alpha]$ is locally compact for each $\alpha \in \omega_1$. Also, since a metric fan is countable and each $S \cap [0, \alpha]$ is closed, S contains a metric fan as a closed subspace if and only if there is some $\alpha \in \omega_1$ such that $S \cap [0, \alpha]$ contains a metric fan as a closed subspace. For any $\alpha \in \omega_1$, $S \cap [0, \alpha]$ is separable and metrizable and therefore it is locally compact if and only if it contains a metric fan as a closed subspace. Therefore (1) and (3) are equivalent.

Lastly, if S contains a metric fan as a closed subspace then $\omega^\omega \leq \mathcal{K}(S)$. If not, then $\overline{S} \backslash S$ is closed, which means $\mathcal{K}(S)$ is Tukey equivalent to either $\mathbf{1}$, ω or $\omega \times \omega_1$, none of which are above ω^ω in the Tukey order (see Lemma 102). □

It turns out that when we study various groups of subsets of ω_1 (stationary, co-stationary, closed and unbounded, etc.) it is convenient to work with $\overline{S} \backslash S$ instead of with cases of whether or not S is locally compact. As a demonstration it is worthwhile to give a direct proof that conditions (2) and (3) from Lemma 47 are equivalent:

Proof. If $\overline{S} \backslash S$ is not closed, then there exists a sequence $\{\alpha_n : n \in \omega\}$ in $\overline{S} \backslash S$ that converges to some $\alpha \in S$. For each $n > 0$ there is an increasing sequence, $S_n \subseteq S \cap (\alpha_{n-1}, \alpha_n)$ that converges to α_n. Then $\{\alpha\} \cup \bigcup_{n>0} S_n$ is homeomorphic to a metric fan and is a closed subspace of S.

On the other hand, suppose S contains a metric fan as a closed subspace, say F. Let $F = \{\alpha\} \cup \bigcup_{n \in \omega} S_n$, with α as the only non-isolated point of F and with $S_n = \{\alpha_{n,i} : i \in \omega\}$'s as sequences converging to α. We may assume all S_n's lie below α. Let $\beta_i = \sup\{\alpha_{n,i} : n \in \omega\}$ for each i. Since F is a metric fan, for each i there is an open subset of F that contains α but does not intersect $\{\alpha_{n,i} : n \in \omega\}$. Then each β_i is different from α and each of them is a limit point of F. Therefore, for each i, $\beta_i \in \overline{S} \backslash S$. But $\{\beta_i : i \in \omega\}$ converges to $\alpha \in S$ and therefore $\overline{S} \backslash S$ is not closed. □

One more important lower bound for $\mathcal{K}(S)$ is ω_1.

Lemma 48. *If $S \subseteq \omega_1$ is unbounded then $\omega_1 \leq_T \mathcal{K}(S)$.*

Proof. Enumerate S as $\{\beta_\alpha : \alpha \in \omega_1\}$. Since all compact subsets of ω_1 are countable, a map $\psi : \omega_1 \to \mathcal{K}(S)$ given by $\psi(\alpha) = \beta_\alpha$ is a Tukey map. □

2.1.1 Bounds of $\mathcal{K}(B)$ when B is Totally Imperfect

Recall the definition of a *scattered* space. For a space X, let X' be the set of all isolated points of X. Let $X^{(0)} = X$ and define $X^{(\alpha)} = X \backslash \bigcup_{\beta<\alpha}(X^{(\beta)})'$ for each $\alpha > 0$. Then a space X is called scattered if $X^{(\alpha)} = \emptyset$ for some ordinal α. This α is called the *scattered height* of X and is denoted by $\mathrm{h}(X)$. Every countable separable metrizable compact space is scattered (with countable scattered height). We know that $\mathbb{Q}$ contains compact subsets of arbitrarily large countable scattered height (every countable ordinal embeds in $\mathbb{Q}$). And every uncountable separable metrizable space contains a copy of $\mathbb{Q}$. Recall that a separable metrizable space is totally imperfect if and only if it contains no Cantor set. Now we present a lower bound.

Lemma 49. *Let B be homeomorphic to $\mathbb{Q}$ or a totally imperfect uncountable separable metrizable space. Then $\omega_1 \leq_T \mathcal{K}(B)$.*

Proof. Fix $\alpha \in \omega_1$. Since B contains $\mathbb{Q}$, we can pick a compact subset K_α of B such that $\mathrm{h}(K_\alpha) > \alpha$. Define $\psi : \omega_1 \to \mathcal{K}(B)$ by $\psi(\alpha) = K_\alpha$. Consider an unbounded subset S of ω_1. Suppose there is $K \in \mathcal{K}(B)$ that bounds $\psi(S)$ from above. Since S is unbounded in ω_1, there exists $\alpha \in S$ with $\mathrm{h}(K) < \alpha < \mathrm{h}(K_\alpha)$. Then $K_\alpha \subseteq K$ contradicts the fact that if $X \subseteq Y$ then $\mathrm{h}(X) \leq \mathrm{h}(Y)$. Therefore, the map ψ is a Tukey map. $\square$

Let B be any totally imperfect separable metrizable space. Then there exists a countable base, $\mathcal{B} = \{B_n : n \in \omega\}$, consisting of sets that are closed and open. We may also assume that the base is closed under complements, finite intersections and finite unions. Note that for a compact scattered space K there is α such that $K^{(\alpha)}$ is finite. So, if $K \in \mathcal{K}(B)$, then there is $\alpha \in \omega_1$ such that $K^{(\alpha)}$ is finite. For a fixed α and (finite) subset F of B, let $\mathcal{K}_\alpha^F = \{K \in \mathcal{K}(B) : K^{(\alpha)} \subseteq F,\ F \subseteq K\}$, and $\mathcal{K}_\alpha = \bigcup\{K_\alpha^F : F \subseteq B\}$. Suppose we have described elements of $\mathcal{K}_\beta$ for each $\beta < \alpha$. Suppose $K \in \mathcal{K}_\alpha^{\{x\}}$ for some $x \in B$. Pick a decreasing local base at x, $\{B'_{x,n}\}_{n\in\omega}$. Let $B_{x,0} = B \backslash B'_{x,0}$ and $B_{x,n} = B'_{x,n-1} \backslash B'_{x,n}$ for each $n \in \omega$. Then each $B_{x,n}$ is in $\mathcal{B}$. If we let $K_n = K \cap B_{x,n}$, we get $K = \{x\} \cup \bigcup_{n\in\omega} K_n$. Note that each K_n is compact (since elements of $\mathcal{B}$ are closed) and is an element $\mathcal{K}_{\beta_n}$ for some $\beta_n < \alpha$.

Lemma 50. *Let B be totally imperfect separable metrizable space and $\kappa = \max\{\omega_1, |B|\}$. Then for each α in ω_1, (1) $([\kappa]^{<\omega})^\omega \geq_T \mathcal{K}_\alpha$. Hence (2) $([\kappa]^{<\omega})^\omega \geq_T \mathcal{K}(B)$.*

Proof. Since $\mathcal{K}(B) = \bigcup_{\alpha\in\omega_1} \mathcal{K}_\alpha$, from the first part and Lemma 7 we get $\mathcal{K}(B) \leq_T ([\kappa]^{<\omega})^\omega \times [\omega_1]^{<\omega} \leq_T ([\kappa]^{<\omega})^\omega \times [\kappa]^{<\omega} =_T ([\kappa]^{<\omega})^\omega$.

We prove $([\kappa]^{<\omega})^\omega \geq_T \mathcal{K}_\alpha$ by induction on α. We know $\mathcal{K}_0 = [B]^{<\omega}$, so $([\kappa]^{<\omega})^\omega \geq_T \mathcal{K}_0$. Define β_n's for $n \in \omega$, as follows: if α is a limit then pick an increasing sequence, $\{\beta_n\}$, converging to α, otherwise let $\beta_n = \alpha - 1$ for each n. Let $\mathcal{K}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{K}_\beta = \bigcup_{n\in\omega} \mathcal{K}_{\beta_n}$. By the inductive hypothesis, for each n, $([\kappa]^{<\omega})^\omega \geq_T \mathcal{K}_{\beta_n}$. Hence by Lemma 7, $([\kappa]^{<\omega})^\omega \geq_T ([\kappa]^{<\omega})^\omega \times [\omega]^{<\omega} \geq_T \mathcal{K}_{<\alpha}$.

Suppose that, for each x in B, we have $([\kappa]^{<\omega})^\omega \geq_T (\mathcal{K}_\alpha^{\{x\}}, \mathcal{K}_\alpha)$. Then for any $F \subseteq B$, we see that $([\kappa]^{<\omega})^\omega =_T (([\kappa]^{<\omega})^\omega)^{|F|} \geq_T \prod_{x\in F}(\mathcal{K}_\alpha^{\{x\}}, \mathcal{K}_\alpha) \geq_T (K_\alpha^F, \mathcal{K}_\alpha)$ (for the last relation take the union). Since $\mathcal{K}_\alpha = \bigcup\{K_\alpha^F : F \subseteq B\}$, and $([\kappa]^{<\omega})^\omega \geq_T \mathcal{K}_\alpha^F$, by Lemma 7, we have $([\kappa]^{<\omega})^\omega =_T ([\kappa]^{<\omega})^\omega \times [\kappa]^{<\omega} \geq_T ([\kappa]^{<\omega})^\omega \times [[B]^{<\omega}]^{<\omega} \geq_T \mathcal{K}_\alpha$.

Fix, then, x in B. Recall that associated with x we have a sequence $\{B_{x,n}\}$ of basic clopen sets. For each n, fix $\phi'_n : ([\kappa]^{<\omega})^\omega \to \mathcal{K}_{<\alpha}(B)$ and define $\phi_n : ([\kappa]^{<\omega})^\omega \to \mathcal{K}_{<\alpha}(B_n)$ by $\phi_n(\tau) = \phi'_n(\tau) \cap B_n$. Since each B_n is closed, each ϕ_n is a Tukey quotient. For $\sigma \in ([\kappa]^{<\omega})^{\omega\times\omega}$ and $n \in \omega$, define $\sigma_n \in ([\kappa]^{<\omega})^\omega$ by $\sigma_n(m) = \sigma(m,n)$. Now define $\phi : ([\kappa]^{<\omega})^{\omega\times\omega} \to \mathcal{K}_\alpha$ by $\phi(\sigma) = \{x\} \cup \bigcup_{n\in\omega} \phi_n(\sigma_n)$. Then ϕ is order-preserving, and from our description of elements of $\mathcal{K}_\alpha$, we see that its image is cofinal for $\mathcal{K}_\alpha^{\{x\}}$ in $\mathcal{K}_\alpha$. □

The following corollary follows immediately from Proposition 9.

Corollary 51. *Suppose B is totally imperfect or homeomorphic to $\mathbb{Q}$ and $|B| = \aleph_n$ for some $n \in \omega$. Then $\mathcal{K}(B) \leq_T \omega^\omega \times [\kappa]^{<\omega}$. In particular, $\mathcal{K}(\mathbb{Q}) \leq_T \omega^\omega \times [\omega_1]^{<\omega}$.*

2.2 ADDITIVITY AND COFINALITY OF $\mathcal{K}(M)$ AND $\mathcal{K}(S)$

If X is compact then add$(\mathcal{K}(X))$ is undefined. We compute add$(\mathcal{K}(M))$ and add$(\mathcal{K}(S))$ otherwise.

Lemma 52. *For any non-compact M,* $\text{add}(\mathcal{K}(M)) = \omega$*. If S is closed and unbounded then* $\text{add}(\mathcal{K}(S)) = \omega_1$*. If S is not closed then* $\text{add}(\mathcal{K}(S)) = \omega$*.*

Proof. If M is not compact, then M contains a countably infinite closed discrete subset $\{x_i : i \in \omega\}$. Then $\{\{x_i\} : i \in \omega\}$ is unbounded in $\mathcal{K}(M)$.

If S is closed and unbounded, every countable subset of $\mathcal{K}(S)$ is bounded, but collection of all singletons of S is not. So, $\text{add}(\mathcal{K}(S)) = \omega_1$. On the other hand if S is not closed, pick a sequence $\{x_n : n \in \omega\}$ in S that does not converge in S. Then $\{\{x_n\} : n \in \omega\}$ is unbounded in $\mathcal{K}(S)$ and $\text{add}(\mathcal{K}(S)) = \omega$. □

Corollary 53. *Let M be separable metrizable.*

(1) If M is compact, then $\text{cof}(\mathcal{K}(M)) = 1$*.*

(2) If M is locally compact, then $\text{cof}(\mathcal{K}(M)) = \omega$*.*

(3) If M is not locally compact, then $\text{cof}(\mathcal{K}(M)) \geq \mathfrak{d}$*.*

Proof. These statements follow directly from Corollary 78 and Lemma 79 and the fact that $\text{cof}(\mathcal{K}(\omega^\omega)) = \text{cof}(\omega^\omega) = \mathfrak{d}$. □

It is interesting what possible values $\text{cof}(\mathcal{K}(M))$ can take in $[\mathfrak{d}, \mathfrak{c}]$. We know that $\text{cof}(\mathcal{K}(\mathbb{Q})) = \text{cof}(\mathcal{K}(\omega^\omega)) = \mathfrak{d}$. The following lemma gives a partial answer.

Lemma 54. *Let $\kappa \in [\omega_1, \mathfrak{c}]$ and let B_κ be a κ-sized totally imperfect separable metrizable space. Then* $\text{cof}(\mathcal{K}(B_\kappa)) \geq \max\{\kappa, \mathfrak{d}\}$*. If $\kappa = \aleph_n$ for some $n \in \omega$, then* $\text{cof}(\mathcal{K}(B_\kappa)) = \max\{\kappa, \mathfrak{d}\}$*.*

Proof. Since each compact subset of B_κ is countable, we need at least κ-many of them to cover B_κ. So B_κ is not locally compact and $\text{cof}(\mathcal{K}(B_\kappa)) \geq \kappa$, which together implies $\text{cof}(\mathcal{K}(B_\kappa)) \geq \max\{\kappa, \mathfrak{d}\}$.

On there other hand, if $\kappa = \aleph_n$ for some $n \in \omega$, we know that $\omega^\omega \times [\kappa]^{<\omega} \geq_T \mathcal{K}(B_\kappa)$ and since $\text{cof}(\omega^\omega \times [\kappa]^{<\omega}) = \max\{\kappa, \mathfrak{d}\}$ we have $\text{cof}(\mathcal{K}(B_\kappa)) \leq \max\{\kappa, \mathfrak{d}\}$. □

Lemma 55. *There are four possibilities for* $\text{cof}(\mathcal{K}(S))$*.*

(1) If S is compact then $\text{cof}(\mathcal{K}(S)) = 1$*.*

(2) If $\overline{S} \setminus S$ is closed and S is bounded, then $\text{cof}(\mathcal{K}(S)) = \omega$*.*

(3) If $\overline{S}\setminus S$ is closed and S is unbounded, then $\mathrm{cof}(\mathcal{K}(S)) = \omega_1$.

(4) If $\overline{S}\setminus S$ is not closed, then $\mathrm{cof}(\mathcal{K}(S)) = \mathfrak{d}$.

Proof. (1) is clear. (2) follows from Lemma 47. For (3), note that $\mathcal{K}(S) = \bigcup_{\alpha\in\omega_1} \mathcal{K}(S\cap[0,\alpha])$. But by Lemma 47, $\mathrm{cof}(\mathcal{K}(S\cap[0,\alpha])) \leq \omega$ for each α and therefore $\mathrm{cof}(\mathcal{K}(S)) \leq \omega_1$. Since S is uncountable and all compact subsets are countable $\mathrm{cof}(\mathcal{K}(S)) \geq \omega_1$ and we are done.

For (4), again by Lemma 47 we have $\mathcal{K}(S) \geq \mathfrak{d}$. But $\mathcal{K}(S) = \bigcup_{\alpha\in\omega_1} \mathcal{K}(S\cap[0,\alpha])$ and $\mathrm{cof}(\mathcal{K}(S\cap[0,\alpha])) \leq \mathfrak{d}$ for each α, as each $S\cap[0,\alpha]$ is Polish. So $\mathrm{cof}(\mathcal{K}(S)) = \mathfrak{d}$. □

2.3 SPECTRA OF $\mathcal{K}(M)$ AND $\mathcal{K}(S)$

This section is dedicated to spectra calculations for $\mathcal{K}(M)$ and $\mathcal{K}(S)$. Notice that, by Corollary 23, $\mathrm{spec}(K(S)), \mathrm{spec}(\mathcal{K}(M)) \subseteq [\omega,\mathfrak{c}]^r$ since these posets are at most $\mathfrak{c}$-sized. We will start the calculations with the most important spectrum, $\mathrm{spec}(\omega^\omega)$. In light of Lemmas 79 and 47 we have the following corollary, and we see why ω^ω is so important. Note that $\mathrm{spec}(\omega^\omega) = \mathrm{spec}(\mathcal{K}(\omega^\omega))$.

Corollary 56. *If M is a non-locally compact separable metrizable space, then* $\mathrm{spec}(\omega^\omega) \subseteq \mathrm{spec}(\mathcal{K}(M))$. *If $S \subseteq \omega_1$ and $\overline{S}\setminus S$ is not closed, then* $\mathrm{spec}(\omega^\omega) \subseteq \mathrm{spec}(\mathcal{K}(S))$.

2.3.1 The Spectrum of ω^ω

We know that $\mathrm{add}(\omega^\omega) = \omega$ and $\mathrm{cof}(\omega^\omega) = \mathfrak{d}$. Then Lemma 22 implies that $\omega, \mathrm{cof}(\omega^\omega) \in \mathrm{spec}(\omega^\omega) \subseteq [\omega,\mathfrak{d}]^r$. Since the additivity of all non-compact M is ω, it will be included in all of their spectra. Therefore we would like to know what is the first uncountable element of the spectrum of ω^ω. Of course, if we look at $(\omega^\omega, \leq^*)$, ω^ω ordered by eventual domination ($f \leq^* g$ if and only if there is $n \in \omega$ such that for all $m > n$, $f(m) \leq g(m)$), the answer is $\mathfrak{b}$, the additivity of $(\omega^\omega, \leq^*)$. Next we establish the relation between the spectrum of ω^ω and the spectrum of $(\omega^\omega, \leq^*)$. Note that $\omega \in \mathrm{spec}(\omega^\omega)\setminus \mathrm{spec}((\omega^\omega, \leq^*))$.

Lemma 57. $\mathrm{spec}(\omega^\omega) = \mathrm{spec}((\omega^\omega, \leq^*)) \cup \{\omega\}$.

Proof. Notice that the function defined by $f \mapsto [f]$ (equivalence class of f in $(\omega^\omega, \leq^*)$) is order-preserving and cofinal. So $\omega^\omega \geq_T (\omega^\omega, \leq^*)$ and therefore $\mathrm{spec}((\omega^\omega, \leq^*)) \subseteq \mathrm{spec}(\omega^\omega)$.

On the other hand, suppose $\kappa \leq_T \omega^\omega$ and κ is uncountable. Let $\psi : \kappa \to \omega^\omega$ be a Tukey map. Define $\psi^* : \kappa \to (\omega^\omega, \leq^*)$ by $\psi^*(\kappa) = [\psi(\kappa)]$. Suppose there exists an unbounded subset of κ, say U, such that $\psi^*(U)$ is bounded in $(\omega^\omega, \leq^*)$ by some $[f]$. Since κ is regular, U has size κ. Also, for each $\alpha \in U$, there is $g \in [f]$ such that $\psi(\alpha) \leq g$. Since $[f]$ is countable and κ is regular, there exists $g \in [f]$ and κ-sized $U' \subseteq U$ such that for each $\alpha \in U'$, $\psi(\alpha) \leq g$. But then U' is an unbounded subset of κ with bounded image under ψ, which contradicts the Tukeyness of ψ. $\square$

Hence $\mathfrak{b} \in \mathrm{spec}(\omega^\omega)$ and $\mathfrak{b}$ is the least uncountable element of $\mathrm{spec}(\omega^\omega)$. We immediately have the following:

Corollary 58. *The cardinal ω_1 is in the spectrum of ω^ω if and only if $\omega_1 = \mathfrak{b}$.*

Now we look at what are the possibilities for the spectrum of $(\omega^\omega, \leq^*)$. Note that $\mathfrak{b}$ is regular, $\mathfrak{b} \leq \mathrm{cof}(\mathfrak{d}) \leq \mathfrak{d}$, and $\mathfrak{d}$ need not be regular. We look at what can happen in the interval $[\mathfrak{b}, \mathfrak{d}]^r$. First we need the following theorem by Hechler [30].

Theorem 59 (Hechler)**.** *Let A be a poset without the largest element and suppose $\omega <$ $\mathrm{add}(A)$. Then it is consistent that there exists an order-embedding of A into $(\omega^\omega, \leq^*)$ with cofinal image.*

Theorem 60. *For any set I of uncountable regular cardinals it is consistent that $I \subseteq$ $\mathrm{spec}((\omega^\omega, \leq^*))$.*

If I is a strictly increasing sequence $\{\kappa_n : n \in \omega\}$ then it is consistent that $I =$ $\mathrm{spec}((\omega^\omega, \leq^*)) \cap \sup I$.

If I is finite then it is consistent that $\mathrm{spec}((\omega^\omega, \leq^*)) = I$ (and therefore $\mathfrak{b} = \min I$ and $\mathfrak{d} = \mathrm{cof}(\mathfrak{d}) = \max I$).

Proof. We apply Hechler's theorem to $A = \prod\{\kappa : \kappa \in I\}$. As all elements of I are regular and uncountable, all countable subsets of A are bounded and $\omega < \mathrm{add}(A)$. We now work in the model given by Hechler. As A is isomorphic to a cofinal subset of $(\omega^\omega, \leq^*)$, they are Tukey equivalent and so have the same spectrum, $\mathrm{spec}((\omega^\omega, \leq^*)) = \mathrm{spec}(A)$.

For any κ in I, applying the relevant projection, it is clear that $A \geq_T \kappa$. Hence $\kappa \in \text{spec}(A)$. So $\text{spec}((\omega^\omega, \leq^*))$ contains I.

Now suppose I is a strictly increasing sequence $\{\kappa_n : n \in \omega\}$. Let $A = \prod\{\kappa_n : n \in \omega\}$ Take any regular $\kappa < \sup I$ such that $\kappa \notin I$. There are two cases: (1) $\kappa < \kappa_0$ and (2) there is $i \in \omega$ with $\kappa_i < \kappa < \kappa_{i+1}$. We prove the second case, the first case uses the same argument. We show that A has calibre κ. Take any κ-sized $H \subseteq A$. Since $\kappa_i < \kappa$ and all cardinals involved are regular, there is κ-sized $H' \subseteq H$ such that all elements of H' take the same value on the first $i+1$-many coordinates. Again by regularity and $\kappa < \kappa_{i+1}$, the set $\{h \restriction \omega \backslash (i+1) : h \in H'\}$ is bounded in $\prod\{\kappa_n : n > i\}$. Hence H' is bounded by some $x \in A$. So A has calibre κ and $\kappa \notin \text{spec}(A)$.

If I is finite, enumerate T in increasing order: $T = \{\kappa_0 < \kappa_1 < \cdots < \kappa_n\}$, so $A = \kappa_0 \times \kappa_1 \times \cdots \times \kappa_n$. Then $\text{add}(A) = \kappa_1$, $\text{cof}(A) = \kappa_n$ and we have $\text{spec}(A) \subseteq [\kappa_1, \kappa_n]$. Then whenever κ is not an element of T and $\kappa_1 < \kappa < \kappa_n$, there exists $i \leq n$ such that $\kappa_i < \kappa < \kappa_{i+1}$. By the same argument as in the countable case, we have $\kappa \notin \text{spec}(A)$. $\square$

2.3.1.1 Proof of Hechler's Theorem Hechler's proof of Theorem 59 uses an older forcing notation. Here we include a proof of cofinally embedding A into $(\omega^\omega, \leq^*)$ in standard notation. We will force over a model of ZFC+GCH with all conditions of Lemma IV.3.11 of [42] satisfied so that we have $\mathfrak{c} = \kappa_n$. The proof is technical, so, for simplicity, we will only do the case when $A = \omega_1 \times \omega_3$. The general case is similar. We will follow the notation of [42].

By $\omega^{<\omega}$ we mean all finite partial functions from ω to ω. Let $\mathbb{P}$ be a poset. For $p, q \in \mathbb{P}$, if $p \leq q$ we say that p extends q. If p and q have a common extension then they are said to be compatible. For the purposes of *the next proof only* we define A to be an antichain if and only if no two elements of A are compatible (in the rest of the text by an antichain we mean a set of pairwise incomparable elements). Recall that for a poset $\mathbb{P}$ and a $\mathbb{P}$-name τ, a *nice* $\mathbb{P}$*-name* for a subset of τ is a $\mathbb{P}$-name of the form $\{\{\sigma\} \times A_\sigma : \sigma \in \text{dom}(\tau)\}$, where each A_σ is an antichain in $\mathbb{P}$. We know that all sets can be named by nice names and we may assume that antichains in the definition of a nice name are maximal.

Theorem 61. *Let $A = \omega_1 \times \omega_3$. Then it is consistent that A embeds isomorphically and cofinally into $(\omega^\omega, \leq^*)$.*

Proof. For each $x = (\alpha, \beta) \leq (\omega_1, \omega_3)$ let $\downarrow\! x = \{y \in A : y < x\}$. To each $x \leq (\omega_1, \omega_3)$ we assign a forcing poset $\mathbb{P}_x$ recursively as follows:

- $\mathbb{P}_{(0,0)} = \{\emptyset\}$;

- For $(0,0) < x \leq (\omega_1, \omega_3)$:

- $p \in \mathbb{P}_x$ if and only if p is a function with properties that
  - dom$(p) \subseteq \downarrow\! x$ and dom(p) is finite;
  - for each $y \in$ dom(p), $p(y) = (\sigma_{p,y}, f^{p,y})$ where $\sigma_{p,y} \in \omega^{<\omega}$ and $f^{p,y}$ is a nice $\mathbb{P}_y$-name for a subset of $(\omega \times \omega)$ such that $p \restriction \downarrow\! y \Vdash_{\mathbb{P}_y} f^{p,y} \in \omega^\omega$.
- $p \leq q$ if and only if
  - dom$(q) \subseteq$ dom(p);
  - for each $y \in$ dom(q), $\sigma_{q,y} \subseteq \sigma_{p,y}$ and $p \restriction \downarrow\! y \Vdash_{\mathbb{P}_y} f^{p,y} \geq f^{q,y} \ \wedge \ \forall n \in \text{dom}(\sigma_{p,y}) \setminus \text{dom}(\sigma_{q,y}) \ (\sigma_{p,y}(n) \geq f^{q,y}(n))$.

To obtain the necessary embedding, we need to show the following:

(1) $\mathbb{P}_{(\omega_1,\omega_3)}$ is ccc;

(2) From a $\mathbb{P}_{(\omega_1,\omega_3)}$-generic set G we derive a function $\Psi : A \to \omega^\omega$ with the following properties:

   (a) Ψ is 1-to-1;

   (b) Ψ is incomparability-preserving;

   (c) $\Psi(A)$ is cofinal in ω^ω;

   (d) Ψ is order-preserving.

Notice that if $x \leq y$ then $\mathbb{P}_x \subseteq \mathbb{P}_y$; and if $p, q \in \mathbb{P}_x$ and $x \leq y$, then $p \leq_{\mathbb{P}_x} q$ if and only if $p \leq_{\mathbb{P}_y} q$, so we will drop the subscript in $\leq_{\mathbb{P}_x}$. Then following facts are immediate from the definitions.

Fact (1) We have $p \leq q$ if and only if $p \restriction \downarrow\! x \leq q \restriction \downarrow\! x$ for all x.

Fact (2) If $p \restriction \downarrow\! x \leq q \restriction \downarrow\! x$ and p and q coincide outside $\downarrow\! x$ then $p \leq q$.

Fact (3) If $q \subseteq p$ then $p \leq q$.

(1) $\mathbb{P}_{(\omega_1,\omega_3)}$ is ccc. Suppose P is an uncountable subset of $\mathbb{P}_{(\omega_1,\omega_3)}$. We may assume that $\{\mathrm{dom}(p) : p \in P\}$ forms a Δ-system with root r. If $r = \emptyset$ we are done since for each $p, q \in P$, $p \cup q \leq p, q$ by Fact (3). So assume r is non-empty. Since r is finite and $\omega^{<\omega}$ is countable, we may refine P so that P is still uncountable and for each $x \in r$ there is $\sigma_x \in \omega^{<\omega}$ with $\sigma_{p,x} = \sigma_x$ for each $p \in P$.

We want to show that in this new P all elements are compatible. Pick arbitrary $q, q' \in P$. Define p step by step. Let $\mathrm{dom}(p_0) = (\mathrm{dom}(q) \cup \mathrm{dom}(q'))\backslash r$. For $x \in \mathrm{dom}(q)\backslash r$, let $p_0(x) = q(x)$ and for $x \in \mathrm{dom}(q')\backslash r$, let $p_0(x) = q'(x)$. Note that p_0 does not have to be an element of $\mathbb{P}_{(\omega_1,\omega_3)}$, but if x is a minimal element of r, then $p_0 \restriction \downarrow\! x$ is in $\mathbb{P}_x$.

Pick a minimal $x \in r$. Since $q \restriction \downarrow\! x \Vdash_{\mathbb{P}_x} f^{q,x} \in \omega^\omega$ and $q' \restriction \downarrow\! x \Vdash_{\mathbb{P}_x} f^{q',x} \in \omega^\omega$ and $p_0 \restriction \downarrow\! x \leq q \restriction \downarrow\! x, q' \restriction \downarrow\! x$, we have $p_0 \restriction \downarrow\! x \Vdash_{\mathbb{P}_x} f^{q',x} + f^{q,x} \in \omega^\omega \ \wedge \ f^{q',x} + f^{q,x} \geq f^{q',x}, f^{q,x}$. Pick an arbitrary $\mathbb{P}_x$-generic filter H_x with $p_0 \restriction \downarrow\! x \in H_x$ and let f^x be a nice $\mathbb{P}_x$-name for $f^{q',x}_{H_x} + f^{q,x}_{H_x}$; then there is $p'_0 \in H_x$ such that $p'_0 \leq p_0 \restriction \downarrow\! x$ and $p'_0 \Vdash_{\mathbb{P}_x} f^x \in \omega^\omega \wedge f^x \geq f^{q',x}, f^{q,x}$. Now let $p_1 = p'_0 \cup \{(x, (\sigma_{q,x}, f^x))\}$ (notice that since x was minimal in r, $\mathrm{dom}(p_1)$ does not contain any other element of r). Then $p_1 \restriction \downarrow\! x \Vdash_{\mathbb{P}_x} f^{p_1,x} \geq f^{q,x} \ \wedge \ f^{p_1,x} \geq f^{q',x}$ and we are done with x.

Next, pick a minimal element $y \in r\backslash\{x\}$ and repeat the argument above to define p_2 with $\sigma_{p_2,y} = \sigma_y$ and $p_2 \restriction \downarrow\! y \Vdash_{\mathbb{P}_y} f^{p_2,y} \geq f^{q,y} \ \wedge \ f^{p_2,y} \geq f^{q',y}$. If we continue like this we will get $p_n \leq q, q'$, where $n = |r|$.

(2) The map Ψ and its properties. To prove claims (a)-(d), fix a $\mathbb{P}_{(\omega_1,\omega_3)}$-generic filter G. To define a map $\Psi : A \to \omega^\omega$, we first need to prove the next lemma.

Lemma 62. *For each $x \in A$ and $n \in \omega$, sets $D_x = \{p \in \mathbb{P}_{(\omega_1,\omega_3)} : x \in \mathrm{dom}(p)\}$ and $D_{x,n} = \{p \in \mathbb{P}_{(\omega_1,\omega_3)} : x \in \mathrm{dom}(p),\ n \in \mathrm{dom}(\sigma_{p,x})\}$ are dense in $\mathbb{P}_{(\omega_1,\omega_3)}$.*

Proof. Let $q \in \mathbb{P}_{(\omega_1,\omega_3)}$ and $x \notin \mathrm{dom}(q)$. Let $p = q \cup \{(x, (\emptyset, \check{0}))\}$. Then $p \leq q$. So D_x is dense.

Now suppose $q \in \mathbb{P}_{(\omega_1,\omega_3)}$ and $x \in \mathrm{dom}(q)$ but $n \notin \mathrm{dom}(\sigma_{q,x})$. Since $q \restriction \downarrow\! x \Vdash_{\mathbb{P}_x} f^{q,x} \in \omega^\omega$

there is $r \in \mathbb{P}_x$ with $r \leq q \restriction {\downarrow}x$ and $m \in \omega$ such that $r \Vdash f^{q,x}(n) = m$. Define p as follows: $\mathrm{dom}(p) = \mathrm{dom}(r) \cup \mathrm{dom}(q)$, $p(x) = (\sigma_{q,x} \cup \{(n,m)\}, f^{q,x})$, if $y \in \mathrm{dom}(r)$ then $p(y) = r(y)$, and if $y \in \mathrm{dom}(q)\setminus(\mathrm{dom}(r) \cup \{x\})$ then $p(y) = q(y)$. Then $p \leq q$ and $p \in D_{x,n}$. □

Now define $\Psi(x) = \bigcup\{\sigma_{p,x} : p \in G, x \in \mathrm{dom}(p)\}$.

(a) Ψ is 1-to-1: We only need to consider the case when $x < x'$, since (b) takes care of the rest. For $x, x' \in A$ with $x < x'$ and $k \in \omega$, let $E_{x,x',k} = \{p \in \mathbb{P}_{(\omega_1,\omega_3)} : x, x' \in \mathrm{dom}(p),\ \exists n > k,\ n \in \mathrm{dom}(\sigma_{p,x}) \cap \mathrm{dom}(\sigma_{p,x'}),\ \sigma_{p,x}(n) < \sigma_{p,x'}(n)\}$. We show that $E_{x,x',k}$ is dense.

Let $q \in \mathbb{P}_{(\omega_1,\omega_3)}$. Since D_x and $D_{x'}$ are dense, we may assume that $x, x' \in \mathrm{dom}(q)$. Pick any $n > k, \sup(\mathrm{dom}(\sigma_{q,x})), \sup(\mathrm{dom}(\sigma_{q,x'}))$. Extend q by some p' adding n to the $\mathrm{dom}(\sigma_{q,x})$ as in the proof of Lemma 62. Then $p'(x') = p(x')$. In particular, $n \notin \mathrm{dom}(\sigma_{p',x'})$. As before, there is $r \in \mathbb{P}_{x'}$ with $r \leq p' \restriction {\downarrow}x'$ and $m \in \omega$ so that $r \Vdash_{\mathbb{P}_{x'}} f^{p',x'}(n) = m$. Let $l = m + \sigma_{p',x}(n)$, then $r \Vdash_{\mathbb{P}_{x'}} f^{p',x'}(n) < l$. Define p as follows: $\mathrm{dom}(p) = \mathrm{dom}(r) \cup \mathrm{dom}(p')$, $p(x') = (\sigma_{p',x'} \cup \{(n,l)\}, f^{p',x'})$, if $y \in \mathrm{dom}(r)$ then $p(y) = r(y)$, and if $y \in \mathrm{dom}(p')\setminus(\mathrm{dom}(r) \cup \{x'\})$ then $p(y) = p'(y)$. Then $p \leq p' \leq q$ and $p \in E_{x,x',k}$.

(b) Ψ is incomparability-preserving: Let $x, x' \in A$ be incomparable and $k \in \omega$. Let $F_{x,x',k} = \{p \in \mathbb{P}_{(\omega_1,\omega_3)} : x, x' \in \mathrm{dom}(p),\ \exists n, m > k,\ n, m \in \mathrm{dom}(\sigma_{p,x}) \cap \mathrm{dom}(\sigma_{p,x'}),\ \sigma_{p,x}(n) < \sigma_{p,x'}(n),\ \sigma_{p,x}(m) > \sigma_{p,x'}(m)\}$.

To prove that $F_{x,x',k}$ is dense, we proceed as in (a). We will assume that $x, x' \in \mathrm{dom}(q)$. Extend q by some p', adding n to $\mathrm{dom}(\sigma_{q,x})$ as in the proof of Lemma 62, and then extend p' by some p'' adding m to $\mathrm{dom}(\sigma_{p',x'})$. Since $x \notin {\downarrow}x'$ and $x' \notin {\downarrow}x$, we still have $n \notin \mathrm{dom}(\sigma_{p'',x'})$ and $m \notin \mathrm{dom}(\sigma_{p'',x})$. Now repeat the last extension step from (a) twice to get the necessary inequalities.

(c) $\Psi(A)$ is cofinal in ω^ω: Let f be a nice $\mathbb{P}_{(\omega_1,\omega_3)}$-name so that $f_G \in (\omega^\omega)^{M[G]}$. Then we may assume that $f = \{\{\check{n}\} \times A_n : n \in \omega\}$ where each A_n is an antichain in $\mathbb{P}_{(\omega_1,\omega_3)}$. Since $\mathbb{P}_{(\omega_1,\omega_3)}$ is ccc, each A_n is countable. So, there is $x \in A$ such that f is actually a nice $\mathbb{P}_x$-name.

Pick $p_0 \in G$ with $p_0 \Vdash f \in \omega^\omega$. Then, $p_0 \restriction {\downarrow}x \Vdash_{\mathbb{P}_x} f \in \omega^\omega$, and since $p_0 \restriction {\downarrow}x \geq p_0$, $p_0 \restriction {\downarrow}x \in G$.

Let $K_f = \{p \in \mathbb{P}_{(\omega_1,\omega_3)} : p \leq p_0 \restriction \downarrow x\,, x \in \mathrm{dom}(p),\ p \restriction \downarrow x \Vdash_{\mathbb{P}_x} f \leq f^{p,x}\}$. K_f is dense below $p_0 \restriction \downarrow x$: suppose $q \leq p_0 \restriction \downarrow x$ and $x \in \mathrm{dom}(q)$. Then, $q \restriction \downarrow x \leq p_0 \restriction \downarrow x$ and, therefore, $q \restriction \downarrow x \Vdash_{\mathbb{P}_x} f, f^{q,x} \in \omega^\omega$. Then, $q \restriction \downarrow x \Vdash_{\mathbb{P}_x} f + f^{q,x} \in \omega^\omega,\ f \leq f + f^{q,x}$. Pick an arbitrary $\mathbb{P}_x$-generic H_x with $q \restriction \downarrow x \in H_x$ and let f' be a nice $\mathbb{P}_x$-name for $f_{H_x} + f^{q,x}_{H_x}$. Then there is $r \in H_x$ such that $r \leq q \restriction \downarrow x$ and $r \Vdash_{\mathbb{P}_x} f' \in \omega^\omega,\ f^{q,x}, f \leq f'$. Define p as follows: $\mathrm{dom}(p) = \mathrm{dom}(q) \cup \mathrm{dom}(r)$, $p(x) = (\sigma_{q,x}, f')$, if $y \in \mathrm{dom}(r)$ then $p(y) = r(y)$ and otherwise $p(y) = q(y)$. So $p \in K_f$ and $p \leq q$.

Since $p_0 \restriction \downarrow x \in G$ and K_f is dense below $p_0 \restriction \downarrow x$, there is $p_1 \in K_f \cap G$. So $x \in \mathrm{dom}(p_1)$, $p_1 \restriction \downarrow x \Vdash_{\mathbb{P}_x} f \leq f^{p_1,x}$. Then, $p_1 \restriction \downarrow x \Vdash_{\mathbb{P}_{(\omega_1,\omega_3)}} f \leq f^{p_1,x}$, and therefore $p_1 \Vdash_{\mathbb{P}_{(\omega_1,\omega_3)}} f \leq f^{p_1,x}$

For $n \in \omega$, let $L_n = \{p \in \mathbb{P}_{(\omega_1,\omega_3)} : p \leq p_1, n \in \mathrm{dom}(\sigma_{p,x})\}$. L_n is dense below $p_1 \in G$. So for each $n > \sup(\mathrm{dom}(\sigma_{p_1,x}))$, we have $p \in G \cap L_n$, which means $p \leq p_1$ and, since $n > \sup(\mathrm{dom}(\sigma_{p_1,x}))$, we have $p \restriction \downarrow x \Vdash_{\mathbb{P}_{(\omega_1,\omega_3)}} \sigma_{p,x}(n) \geq f^{p_1,x}(n)$. So in $M[G]$, $\Psi(x) \geq^* f^{p_1,x}_G \geq f_G$.

(d) Ψ is order-preserving:

Let G be a $\mathbb{P}_{(\omega_1,\omega_3)}$-generic filter. Then $G \restriction \downarrow x = \{p \restriction \downarrow x : p \in G\}$ is a $\mathbb{P}_x$-generic filter. To add n to the domain of $\sigma_{p,x}$, we only have to alter the part of p below x and at x. So whenever $y < x$, $\Psi(y) = \{p \in G \restriction \downarrow x :\ p \in G,\ y \in \mathrm{dom}(p)\}$. So there is a nice $\mathbb{P}_x$-name for $\Psi(y)$. Now we can repeat the argument from (c) for f to show that $\Psi(y) \leq^* \Psi(x)$.

$\square$

2.3.2 Spectrum for Totally Imperfect Spaces (and $\mathbb{Q}$)

Next we compute spectra of some totally imperfect separable metrizable spaces.

Proposition 63. *Suppose $\omega_1 \leq \kappa \leq \mathfrak{c}$ and B_κ is κ-sized totally imperfect separable metrizable. Then $[\omega_1, \kappa]^r \cup \mathrm{spec}(\omega^\omega) \subseteq \mathrm{spec}(\mathcal{K}(B_\kappa))$. If $\kappa = \aleph_n$ for some $n \in \omega$, then $[\omega_1, \kappa]^r \cup \mathrm{spec}(\omega^\omega) = \mathrm{spec}(\mathcal{K}(B_\kappa))$.*

Proof. We already know that $\mathrm{spec}(\omega^\omega) \subseteq \mathrm{spec}(\mathcal{K}(B_\kappa))$. Let $B_\kappa = \{x_\alpha : \alpha < \kappa\}$ and pick $\lambda \in [\omega_1, \kappa]^r$. Consider a map $\psi : \lambda \to \mathcal{K}(B_\kappa)$ given by $\alpha \mapsto \{x_\alpha\}$. Then, whenever $U \subseteq \lambda$ is uncountable, $\psi(U)$ cannot be contained in a compact subset of B_κ, since all compact subsets of B_κ are countable. So, $\psi(U)$ is unbounded and ψ is a Tukey map, which implies that $[\omega_1, \kappa]^r \subseteq \mathrm{spec}(\mathcal{K}(B_\kappa))$.

On the other hand, if $\kappa = \aleph_n$ for some $n \in \omega$, we know that $\mathcal{K}(B_\kappa) \leq \omega^\omega \times [\kappa]^{<\omega}$ and therefore $\mathrm{spec}(B_\kappa) \subseteq \mathrm{spec}(\omega^\omega \times [\kappa]^{<\omega}) = \mathrm{spec}(\omega^\omega) \cup \mathrm{spec}([\kappa]^{<\omega}) = [\omega_1, \kappa]^r \cup \mathrm{spec}(\omega^\omega)$ by corollaries 8 and 20. $\square$

Now we compute the spectrum of $\mathcal{K}(\mathbb{Q})$.

Lemma 64. $\mathrm{spec}(\mathcal{K}(\mathbb{Q})) = \{\omega_1\} \cup \mathrm{spec}(\omega^\omega)$.

Proof. We already know that $\omega^\omega \times \omega_1 \leq_T \mathcal{K}(\mathbb{Q}) \leq_T \omega^\omega \times [\omega_1]^{<\omega}$, hence, $\mathrm{spec}(\omega^\omega \times \omega_1) \subseteq \mathrm{spec}(\mathcal{K}(\mathbb{Q})) \subseteq \mathrm{spec}(\omega^\omega \times [\omega_1]^{<\omega})$. But from the corollary to Lemma 19, it is clear that $\mathrm{spec}(\omega^\omega \times \omega_1) = \{\omega_1\} \cup \mathrm{spec}(\omega^\omega) = \mathrm{spec}(\omega^\omega \times [\omega_1]^{<\omega})$. $\square$

Note that if $\omega_2 < \mathfrak{b}$, $\omega_2 \in \mathrm{spec}(\mathcal{K}(B_\mathfrak{c}))$ but $\omega_2 \notin \mathcal{K}(\mathbb{Q})$, which implies $\mathcal{K}(\mathbb{Q}) \not\geq_T \mathcal{K}(B_\mathfrak{c})$. Also if $\omega_1 \leq \kappa < \lambda \leq \mathfrak{c}$, $\mathrm{spec}(\mathcal{K}(B_\lambda)) \nsubseteq \mathrm{spec}(\mathcal{K}(B_\kappa))$ and therefore $\mathcal{K}(B_\lambda) \not\leq_T \mathcal{K}(B_\kappa)$.

2.3.3 Spectrum of $\mathcal{K}(S)$

If S is a bounded subset of ω_1 then $\mathcal{K}(S)$ is Tukey equivalent to either $\mathcal{K}(\mathbf{1})$ or $\mathcal{K}(\omega)$ or $\mathcal{K}(\omega^\omega)$. So the interesting case for the spectrum of $\mathcal{K}(S)$ is when S is unbounded.

Theorem 65. *Suppose $S \subseteq \omega_1$ is unbounded. If $\overline{S} \backslash S$ is closed then* $\mathrm{spec}(\mathcal{K}(S)) = \{\omega_1\}$ *and if $\overline{S} \backslash S$ is not closed then* $\mathrm{spec}(\mathcal{K}(S)) = \{\omega_1\} \cup \mathrm{spec}(\omega^\omega)$.

Proof. For unbounded S, $\omega_1 \leq_T \mathcal{K}(S)$. If $\overline{S} \backslash S$ is closed then $\mathrm{cof}(\mathcal{K}(S)) = \omega_1$ and $\kappa \leq_T \mathcal{K}(S)$ implies $\mathrm{cof}(\kappa) \leq \omega_1$. So κ must be equal to ω_1 and $\mathrm{spec}(\mathcal{K}(S)) = \{\omega_1\}$.

If $\overline{S} \backslash S$ is not closed then $\omega^\omega \times \omega_1 \leq_T \mathcal{K}(S) \leq_T \omega^\omega \times [\omega_1]^{<\omega}$. And $\mathrm{spec}(\omega^\omega \times \omega_1) \subseteq \mathrm{spec}(\mathcal{K}(S)) \subseteq \mathrm{spec}(\omega^\omega \times [\omega_1]^{<\omega})$. But from the corollary to Lemma 19, it is clear that $\mathrm{spec}(\omega^\omega \times \omega_1) = \{\omega_1\} \cup \mathrm{spec}(\omega^\omega) = \mathrm{spec}(\omega^\omega \times [\omega_1]^{<\omega})$. $\square$

2.4 CALIBRES OF $\mathcal{K}(M)$ AND $\mathcal{K}(S)$

Elements of $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$ have size $\leq \mathfrak{c}$ and therefore the most natural calibres to consider for these posets are the calibres ω_1, $(\omega_1, \omega_1, \omega)$ and (ω_1, ω). Clearly, calibre ω_1 implies calibre $(\omega_1, \omega_1, \omega)$, which, in turn, implies (ω_1, ω).

2.4.1 Calibres of $\mathcal{K}(M)$

Lemma 66. *For any separable metrizable M, $\mathcal{K}(M)$ has calibre (ω_1, ω).*

Proof. Take any uncountable collection $\mathcal{K}$ of compact subsets of M. Since $\mathcal{K}(M)$ is a separable metrizable (and so has countable extent), $\mathcal{K}$ has an accumulation point K in $\mathcal{K}(X)$. So, because $\mathcal{K}(M)$ is separable metrizable (and so is first countable), there is an infinite sequence $(K_n)_n$ in $\mathcal{K}$ converging to K. Then, $\mathcal{K}_\infty = \{K_n : n \in \omega\} \cup \{K\}$ is a compact subset of $\mathcal{K}(M)$. Now, $K_\infty = \bigcup \mathcal{K}_\infty$ is an upper bound in $\mathcal{K}(M)$ of all the K_n's. $\square$

Lemma 67. *Let X be hereditarily separable, and let $\theta \geq \lambda \geq \omega_1$. If $\mathcal{K}(X)$ has calibre $(\theta, \lambda, \omega)$ then $\mathcal{K}(X)$ has calibre (θ, λ).*

Proof. Suppose $\mathcal{K}(X)$ has calibre $(\theta, \lambda, \omega)$ and let $\mathcal{K} \subseteq \mathcal{K}(X)$ be θ-sized. Then there is λ-sized $\mathcal{K}' \subseteq \mathcal{K}$ such that every countable subcollection of $\mathcal{K}'$ has an upper bound in $\mathcal{K}(X)$. We show that $\mathcal{K}'$ has an upper bound in $\mathcal{K}(X)$. Let $A = \bigcup \mathcal{K}'$. Pick a countable set D contained in A which is dense in A, so $\overline{D} = \overline{A}$. For each d in D, pick K_d from $\mathcal{K}'$ such that $d \in K_d$. Then the countable family $\{K_d : d \in D\}$ has an upper bound, say K_∞. Since $\overline{\bigcup \mathcal{K}'} = \overline{A} = \overline{D} \subseteq \overline{\bigcup\{K_d : d \in D\}} \subseteq K_\infty$, we see that K_∞ contains every $K \in \mathcal{K}'$. $\square$

Corollary 68. *Let M be separable metrizable. Then $\mathcal{K}(M)$ has calibre $(\omega_1, \omega_1, \omega)$ if and only if $\mathcal{K}(M)$ has calibre ω_1.*

Lemma 66 and corollary 68 make our objective clear and we devote the rest of this subsection to investigating when $\mathcal{K}(M)$ has calibre ω_1. If M is locally compact, then $\mathcal{K}(M) =_T \omega$ and $\mathcal{K}(M)$ trivially has calibre ω. On the other hand, recall that a poset has calibre κ if and only if κ is not its Tukey quotient. So we immediately deduce from our spectrum results what happens when M is Polish.

Corollary 69. *The poset ω^ω has calibre ω_1 if and only if $\omega_1 < \mathfrak{b}$.*

Next we consider what happens above ω^ω, or when M is not Polish. If $\omega_1 = \mathfrak{b}$, then ω^ω fails to have calibre ω_1 and therefore nothing above ω^ω can have this calibre. Moreover, by lemma 49, in ZFC, $\mathcal{K}(\mathbb{Q})$ fails to have calibre ω_1 and so does $\mathcal{K}(B)$ for any uncountable totally imperfect separable metrizable B. The next natural question is if it is consistently possible to have a non-Polish M such that $\mathcal{K}(M)$ has calibre ω_1. It turns out that such an M exists if we assume that $\omega_1 < \mathfrak{p}$. To show this we need a result by Fremlin [20].

Theorem 70 (Fremlin)**.** *Let M be a separable metrizable space. Let $\mathcal{K}$ be a family of compact subsets of M and $\mathcal{F}$ a family of closed subsets of M. Suppose $|\mathcal{K} \cup \mathcal{F}| < \mathfrak{p}$ and $\bigcup \mathcal{K} \cap \bigcup \mathcal{F} = \emptyset$. Then there is a sequence, $\{C_n\}_{n\in\omega}$, of compact subsets of M such that for each $F \in \mathcal{F}$, $F \subseteq C_n$ for some n and $\bigcup \mathcal{K} \cap \bigcup \{C_n\}_{n\in\omega} = \emptyset$.*

Corollary 71. ($\omega_1 < \mathfrak{p}$) *Let $M = 2^\omega$. Let $\mathcal{K} \subseteq \mathcal{K}(2^\omega)$ have size $\leq \omega_1$ and define $X = 2^\omega \setminus \bigcup \mathcal{K}$. Then $\mathcal{K}(X)$ has calibre ω_1.*

Proof. If $\mathcal{K}$ is countable then X is a G_δ subset of 2^ω. Therefore X is Polish and since $\omega_1 < \mathfrak{p} \leq \mathfrak{b}$, $\mathcal{K}(X)$ has calibre ω_1.

If $\mathcal{K} = \{K_\alpha : \alpha \in \omega_1\}$ and the K_α's are distinct, let $\mathcal{F}$ be a ω_1-sized subset of $\mathcal{K}(X)$. We will show that $\mathcal{F}$ contains an uncountable bounded subset. Apply Theorem 70 to $\mathcal{K}$ and $\mathcal{F}$ to get necessary $\{C_n\}_{n\in\omega}$. Each C_n is compact and misses $\bigcup \mathcal{K}$, which implies that each $C_n \in \mathcal{K}(X)$. Since each $F \in \mathcal{F}$ is contained in some C_n, there is $n_0 \in \omega$ such that C_{n_0} contains uncountably many $F \in \mathcal{F}$. □

Corollary 72. ($\omega_1 < \mathfrak{p}$) *Let $M = 2^\omega$. Let $\mathcal{K}$ be a collection of ω_1-many distinct singletons of 2^ω. Define $X = 2^\omega \setminus \bigcup \mathcal{K}$. Then X is not Polish and $\mathcal{K}(X)$ has calibre ω_1.*

Proof. Suppose X is Polish. Then X must be G_δ in 2^ω, and therefore $Y = \bigcup \mathcal{K}$ would be F_σ subset of 2^ω. Since closed subsets of 2^ω are compact, Y is, in fact, the union of countably many compact subsets. Since $|Y| = \omega_1 < \mathfrak{p} \leq \mathfrak{c}$, all compact subsets of Y are countable, which contradicts Y being uncountable. □

2.4.2 Calibres of $\mathcal{K}(S)$

Let S be a subset of ω_1. If S is bounded then $\mathcal{K}(S)$ is Tukey equivalent to one of $\mathbf{1}$, ω and ω^ω. In the first two cases $\mathcal{K}(S)$ has calibres ω_1, $(\omega_1, \omega_1, \omega)$ and (ω_1, ω); in the second case $\mathcal{K}(S)$ always has calibre (ω_1, ω) but has the other two calibres if and only if $\omega_1 < \mathfrak{b}$.

Now let S be unbounded. We showed that $\omega_1 \leq_T \mathcal{K}(S)$ and therefore $\mathcal{K}(S)$ fails to have calibre ω_1. The case of calibre (ω_1, ω) has already been settled by Todorčević in [58]. Recall that a subset of an ordinal is called stationary if and only if it meets every closed and unbounded subset of ω_1. Using the fact that $\mathcal{K}(S) \geq [\omega_1]^{<\omega}$ if and only if $\mathcal{K}(S)$ does not have calibre (ω_1, ω), Todorčević's theorem becomes:

Lemma 73 (Todorčević)**.** *Let $S \subseteq \omega_1$ be unbounded. Then $\mathcal{K}(S)$ has calibre (ω_1, ω) if and only if S is stationary if and only if $\mathcal{K}(S) \not\geq [\omega_1]^{<\omega}$.*

In fact, Todorčević shows that if S is not stationary then S contains an uncountable closed discrete subset, which gives an uncountable collection of singletons such that any infinite subcollection is unbounded in $\mathcal{K}(S)$. We need this result in the next proof, so we include the argument here.

Lemma 74 (Todorčević)**.** *Suppose S is unbounded, C is closed, unbounded and $S \cap C = \emptyset$. Then there exist strictly increasing sequences $\{s_\alpha : \alpha < \omega_1\} \subseteq S$ and $\{c_\alpha : \alpha < \omega_1\} \subseteq C$ such that for each $\alpha < \omega_1$, $s_\alpha < c_\alpha < s_{\alpha+1}$. Hence, S contains an uncountable closed discrete subset.*

Proof. Construct $\{s_\alpha : \alpha < \omega_1\} \subseteq S$ and $\{c_\alpha : \alpha < \omega_1\} \subseteq C$ inductively with a property that for each $\alpha < \omega_1$, $s_\alpha < c_\alpha < s_{\alpha+1}$. Pick any $s_0 \in S$ and pick any $c_0 \in C$ with $s_0 < c_0$. Suppose we have constructed $\{s_\beta : \beta < \alpha\}$ and $\{c_\beta : \beta < \alpha\}$. Since S is unbounded, we can pick $s_\alpha \geq \sup\{s_\beta : \beta < \alpha\}$ in S. (Since $\sup\{s_\beta : \beta < \alpha\} = \sup\{c_\beta : \beta < \alpha\} \in C$, s_α is strictly larger than $\sup\{s_\beta : \beta < \alpha\}$.) Now pick any c_α with $c_\alpha > s_\alpha$.

To show that $\{s_\alpha : \alpha < \omega_1\}$ is closed and discrete pick any limit point s of $\{s_\alpha : \alpha < \omega_1\}$. Then there is an increasing sequence $\{\alpha_n : n \in \omega\}$ such that $\{s_{\alpha_n} : n \in \omega\}$ converges to s. Then $\{c_{\alpha_n}\} \subseteq C$ also converges to s and since C is closed, $s \in C$. Therefore $s \notin S$ and we are done. $\square$

Next we show exactly when $\mathcal{K}(S)$ has calibre $(\omega_1, \omega_1, \omega)$. Recall that a subset of ω_1 is called co-stationary if it has the stationary complement. S is co-stationary if and only if it contains a cub (closed and unbounded) set. Note that if S is unbounded and $\overline{S} \backslash S$ is bounded then S is co-stationary. In particular, $S \backslash (\sup(\overline{S} \backslash S) + 1)$ is a cub subset of S.

Lemma 75. *Let $S \subseteq \omega_1$ be unbounded. Then $\mathcal{K}(S)$ has calibre $(\omega_1, \omega_1, \omega)$ if and only if $\overline{S} \backslash S$ is bounded and either $\overline{S} \backslash S$ is closed or $\omega_1 < \mathfrak{b}$.*

Proof. Let S be an unbounded subset of ω_1 and suppose $\overline{S} \backslash S$ is bounded. Let $\alpha = (\sup(\overline{S} \backslash S) + 1)$. Then $S \backslash \alpha$ is closed and unbounded in ω_1 and $S = S \cap \alpha \oplus S \backslash \alpha$ and $\mathcal{K}(S) = \mathcal{K}(S \cap \alpha) \times \mathcal{K}(S \backslash \alpha) = \mathcal{K}(S \cap \alpha) \times \omega_1$. Clearly, ω_1 has calibre $(\omega_1, \omega_1, \omega)$, so by Lemma 13, $\mathcal{K}(S)$ has calibre $(\omega_1, \omega_1, \omega)$ if and only if $\mathcal{K}(S \cap \alpha)$ does. Since $S \cap \alpha$ is Polish, $\mathcal{K}(S \cap \alpha)$ has this calibre if and only if either $\overline{S \cap \alpha} \backslash (S \cap \alpha)$ is closed or $\omega_1 < \mathfrak{b}$. Then, by the fact that $\overline{S} \backslash S = \overline{S \cap \alpha} \backslash (S \cap \alpha)$, $\mathcal{K}(S)$ has calibre $(\omega_1, \omega_1, \omega)$ if and only if either $\overline{S} \backslash S$ is closed or $\omega_1 < \mathfrak{b}$.

What is left to show is that if $\mathcal{K}(S)$ has calibre $(\omega_1, \omega_1, \omega)$ then $\overline{S} \backslash S$ is bounded. Suppose S has calibre $(\omega_1, \omega_1, \omega)$. First we show that S contains a cub set. Let $\mathcal{K} = \{\{\alpha\} : \alpha \in S\}$. Then there is an uncountable $\mathcal{K}' \subseteq \mathcal{K}$ with every countable subset bounded in $\mathcal{K}(S)$. If we let $T = \bigcup \mathcal{K}'$, then every limit point of T lies in S: otherwise pick $\beta \in \omega_1 \backslash S$ and $\{\alpha_n\}_{n \in \omega} \subseteq T$ with $\beta = \sup\{\alpha_n\}_{n \in \omega}$. Then $\{\{\alpha_n\} : n \in \omega\}$ does not have an upper bound in $\mathcal{K}(S)$. So, $C = \overline{T}$ is closed and unbounded subset of S.

Therefore $\omega_1 \backslash S$ is non-stationary. If $\omega_1 \backslash S$ is also unbounded, apply Lemma 74 to $\omega_1 \backslash S$ and C to get strictly increasing sequences $\{s_\alpha : \alpha < \omega_1\} \subseteq \omega_1 \backslash S$ and $\{c_\alpha : \alpha < \omega_1\} \subseteq C$ such that for each $\alpha < \omega_1$, $s_\alpha < c_\alpha < s_{\alpha+1}$.

Since $\overline{S} \backslash S$ is non-stationary in $\overline{S}$, which is homeomorphic to ω_1, we may assume that $\overline{S} = \omega_1$. *Then all successor ordinals are in S.*

Let $\alpha \in \omega_1$. By construction, $s_\alpha > \sup(\{s_\beta : \beta < \alpha\} \cup \{c_\beta : \beta < \alpha\})$. Since α is a limit ordinal, we can pick an increasing sequence, $\{s_{\alpha,m}\}_{m \in \omega}$, of successor ordinals in the interval $(\sup(\{s_\beta : \beta < \alpha\} \cup \{c_\beta : \beta < \alpha\}), s_\alpha)$ that converges to s_α.

For each infinite $\alpha \in \omega_1$ let $f_\alpha : \alpha \to \omega$ be a bijection. Fix infinite $\alpha \in \omega_1$ and define $K_\alpha = \overline{\{s_{\sigma, f_\alpha(\sigma)} : \sigma \in \alpha\}}$. For each $\sigma, \sigma' \in \alpha$ with $\sigma < \sigma'$, $s_{\sigma, f_\alpha(\sigma)} < c_\sigma < s_{\sigma', f_\alpha(\sigma')}$. So every limit point of $\{s_{\sigma,f_\alpha(\sigma)} : \sigma \in \alpha\}$ is also a limit point of $\{c_\sigma : \sigma \in \alpha\}$ and therefore lies in $C \subseteq S$. Therefore, since $\{s_{\sigma,f_\alpha(\sigma)} : \sigma \in \alpha\} \subseteq S$, K_α is in $\mathcal{K}(S)$.

If $T \subseteq \omega_1$ is uncountable, we will show that $\{K_\alpha : \alpha \in T\}$ contains a countable subset that is unbounded in $\mathcal{K}(S)$. For this it will suffice to find $\sigma \in \omega_1$ such that $A_\sigma = \{f_\alpha(\sigma) : \alpha \in T,\ \alpha > \sigma\}$ is infinite; because then for each $n \in A_\sigma$, we can pick $\alpha_n \in T$ with $f_{\alpha_n}(\sigma) = n$, which will imply that $\bigcup_{n \in A_\sigma} K_{\alpha_n}$ contains an infinite subset of $\{s_{\sigma,m} : m \in \omega\}$ and therefore is unbounded in $\mathcal{K}(S)$.

Suppose, to get a contradiction, that for each $\sigma \in \omega_1$ there is $n_\sigma \in \omega$ that bounds $\{f_\alpha(\sigma) : \alpha \in T,\ \alpha > \sigma\}$. Then there is uncountable $T_1 \subseteq \omega_1$ and $n \in \omega$ such that $n_\sigma = n$ for all $\sigma \in T_1$. Since T and T_1 are uncountable, there is $\alpha \in T$ such that $\alpha \cap T_1$ is infinite. Then we have $f_\alpha(\sigma) \leq n$ for all $\sigma \in \alpha \cap T_1$, which contradicts the fact that f_α is a bijection.

$\square$

3.0 STRUCTURES OF $(\mathcal{K}(Sub(\mathbb{R})), \leq_T)$ AND $(\mathcal{K}(Sub(\omega_1)), \leq_T)$

In this chapter we present the main results of this work. For convenience we often drop '$\leq_T$' in '$(\mathcal{K}(Sub(X)), \leq_T)$'. We establish various order properties (size, cofinality, additivity, calibres) of $\mathcal{K}(Sub(\mathbb{R}))$ and construct various subposets of $\mathcal{K}(Sub(\mathbb{R}))$, the most important of which is an antichain of size $2^{\mathfrak{c}}$. On the other hand, the size of $\mathcal{K}(Sub(\omega_1))$ has already been established by Todorčević through construction of a 2^{ω_1}-sized antichain and, as we show that $\mathcal{K}(Sub(\omega_1))$ has the largest element, questions about additivity, cofinality and calibres are no longer relevant. We determine Tukey classes associated with different groups of subsets of ω_1. The concluding section of the chapter investigates the relation between elements of $\mathcal{K}(Sub(\mathbb{R}))$ and elements of $\mathcal{K}(Sub(\omega_1))$.

3.1 THE STRUCTURE OF $\mathcal{K}(Sub(\mathbb{R}))$

Most arguments in this section depend heavily on Lemma 42, which gives an equivalent condition for the existence of Tukey quotients. Note that through Theorem 38 some of these results transfer immediately to $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R}))$. However, we often give direct proofs for $\mathcal{K}(Sub(\mathbb{R}))$, $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$. Recall that by Lemma 37, we are allowed to work with *arbitrary* separable metrizable spaces when we study $\mathcal{K}(Sub(\mathbb{R}))$.

3.1.1 Initial Structure

We begin by upgrading Fremlin's results [21] on the initial segment of $\mathcal{K}(Sub(\mathbb{R}))$ to initial segments of $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$.

Theorem 76. *Let M and N be separable metrizable spaces.*

(1) The minimum Tukey equivalence class in $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ is $[(\mathbf{1}, \mathcal{K}(\mathbf{1}))]_T$, and $(M, \mathcal{K}(M))$ is in this class if and only if M is compact.

(2) It has a unique successor, $[(\omega, \mathcal{K}(\omega))]_T$, which consists of all $(M, \mathcal{K}(M))$ where M is σ-compact but not compact.

(3) This has $[(\omega^\omega, \mathcal{K}(\omega^\omega))]_T = \{(M, \mathcal{K}(M)) : M$ is analytic but not σ-compact$\}$ as a successor.

(4) However it is consistent that there is a co-analytic N which is not σ-compact such that $(N, \mathcal{K}(N)) \not\geq_T (\omega^\omega, \mathcal{K}(\omega^\omega))$.

Proof. Claim (1) is trivial. For (2) note that if M is not compact then it contains a closed copy of ω, and so there is a reduction $(M, \mathcal{K}(M)) \geq_T (\omega, \mathcal{K}(\omega))$. And if $M = \bigcup_{n\in\omega} K_n$, where each K_n is compact, define $\phi : \mathcal{K}(\omega) \to \mathcal{K}(M)$ by $\phi(F) = \bigcup_{n\in F} K_n$. The map ϕ is well-defined since all compact subsets of ω are finite, ϕ is order-preserving and the $\phi(\omega)$ covers M. Conversely, if ϕ witnesses $(\omega, \mathcal{K}(\omega)) \geq_T (M, \mathcal{K}(M))$ then $\phi(\omega)$ is a countable cover of M by compacta.

Claim (3) relies on a result of Hurewicz implying that every analytic set which is not σ-compact contains a closed copy of the irrationals [31]. Suppose that M is not σ-compact but $(\omega^\omega, \mathcal{K}(\omega^\omega)) \geq_T (M, \mathcal{K}(M))$. By Lemma 42 there is a closed subset of $C \in \mathcal{K}([0,1]^\omega)^2$ such that $M \subseteq C[\omega^\omega]$. Since C is Borel and $\mathcal{K}([0,1]^\omega)^2$ is Polish, $C[\omega^\omega]$ is also analytic and since M is a closed subset of $C[\omega^\omega]$, it is also analytic. Hence ω^ω embeds as a closed set in M, so $(\omega^\omega, \mathcal{K}(\omega^\omega))$ and $(M, \mathcal{K}(M))$ are Tukey equivalent, and thus there is nothing in the Tukey order strictly between $(\omega, \mathcal{K}(\omega))$ and $(\omega^\omega, \mathcal{K}(\omega^\omega))$.

Assume $\omega_1 < \mathfrak{d}$ and 'there is a co-analytic subset N of $\mathbb{R}$ of size ω_1'. Then in this model the claim in (4) holds. For if ϕ is a Tukey quotient of $(M, \mathcal{K}(M))$ to $(\omega^\omega, \mathcal{K}(\omega^\omega))$, then $\phi(M)$ is a compact cover of ω^ω of size $\leq \omega_1$. But $\mathfrak{d}$ is the minimal size of a compact cover of ω^ω. $\square$

An almost identical proof gives an almost identical result for the initial structure of $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$.

Theorem 77. *For separable metrizable M and N:*

(1) the minimum Tukey equivalence class in $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$ is $[(\mathcal{F}(\mathbf{1}), \mathcal{K}(\mathbf{1}))]_T$, and $(\mathcal{F}(M), \mathcal{K}(M))$ is in this class if and only if M is compact;

(2) it has a unique successor, $[(\mathcal{F}(\omega), \mathcal{K}(\omega))]_T$, which consists of all $(\mathcal{F}(M), \mathcal{K}(M))$ where M is σ-compact but not compact;

(3) this has $[(\mathcal{F}(\omega^\omega), \mathcal{K}(\omega^\omega))]_T = \{(\mathcal{F}(M), \mathcal{K}(M)) : M$ is analytic but not σ-compact$\}$ as a successor;

(4) however it is consistent that there is a co-analytic N which is not σ-compact such that $(\mathcal{F}(N), \mathcal{K}(N)) \not\geq_T (\mathcal{F}(\omega^\omega), \mathcal{K}(\omega^\omega))$.

Since the Tukey relation $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ is a special case of the relative relation $(M', \mathcal{K}(M')) \geq_T (N', \mathcal{K}(N'))$ we can also recover the initial structure of $\mathcal{K}(Sub(\mathbb{R}))$.

Corollary 78 (Christensen, Fremlin [21])**.** *Below M' and N' are separable metrizable spaces.*

(1) The minimum Tukey equivalence class in $\mathcal{K}(Sub(\mathbb{R}))$ is $[\mathbf{1}]_T$, and $\mathcal{K}(M')$ is in this class if and only if M' is compact.

(2) It has a unique successor, $[\omega]_T$, which consists of all $\mathcal{K}(M')$ where M' is locally compact but not compact.

(3) This has $[\omega^\omega]_T = \{\mathcal{K}(M') : M'$ is Polish$\}$ as a unique successor.

Proof. According to Lemma 30, $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ if and only if $(\mathcal{K}(M), \mathcal{K}(\mathcal{K}(M))) \geq_T (\mathcal{K}(N), \mathcal{K}(\mathcal{K}(N)))$. Now apply the preceding theorem to $M = \mathcal{K}(M')$ and recall that $\mathcal{K}(M')$ is compact if and only if M' is compact, $\mathcal{K}(M')$ is σ-compact if and only if M' is locally compact, and Christensen showed that $\mathcal{K}(M')$ is analytic if and only if M' is Polish.

The class $[\omega^\omega]_T$ is the *unique* successor of $[\omega]_T$ in $\mathcal{K}(Sub(\mathbb{R}))$ by the next lemma. $\square$

Lemma 79 (Fremlin)**.** *Let M be a non-locally compact separable metrizable space. Then $\mathcal{K}(\omega^\omega) \leq_T \mathcal{K}(M)$.*

Proof. Since M is (first countable and) not locally compact it contains a closed copy of F, the metrizable Fréchet fan. So $\mathcal{K}(M) \geq_T \mathcal{K}(F)$. As F is Polish and not locally compact, $\mathcal{K}(F)$ and $\mathcal{K}(\omega^\omega)$ are Tukey equivalent. $\square$

3.1.2 Cofinal Structure

3.1.2.1 Down Sets and Cardinality

Lemma 80. *Fix a separable metrizable space M. Let $\mathcal{E}$ and $\mathcal{D}$ be $\mathcal{K}$-operators. Then*

(1) $D_{\mathcal{E},\mathcal{D}} = \{N \in Sub(\mathbb{R}) : (\mathcal{E}(M), \mathcal{K}(M)) \geq_T (\mathcal{D}(N), \mathcal{K}(N))\}$ has size $\mathfrak{c}$.

If $\mathcal{D}$ is productive, then

(2) $T_{\mathcal{E},\mathcal{D}} = \{N \in Sub(\mathbb{R}) : (\mathcal{E}(M), \mathcal{K}(M)) =_T (\mathcal{D}(N), \mathcal{K}(N))\}$ has size either 0 or $\mathfrak{c}$, and

(3) $T_{\mathcal{D}}(M) = T_{\mathcal{D}} = \{N \in Sub(\mathbb{R}) : (\mathcal{D}(M), \mathcal{K}(M)) =_T (\mathcal{D}(N), \mathcal{K}(N))\}$ has size $\mathfrak{c}$.

Proof. Note that $D_{\mathcal{E},\mathcal{D}} \subseteq D_{\mathcal{I},\mathcal{D}} = \{N \in Sub(\mathbb{R}) : (M, \mathcal{K}(M)) \geq_T (\mathcal{K}(N), \mathcal{K}(N))\}$. So first we show $|D_{\mathcal{I},\mathcal{K}}| \leq \mathfrak{c}$. We can assume, without loss of generality (replacing M with a homeomorphic copy, if necessary), that M is a subspace of $[0,1]^\omega$. Take any separable metrizable N such that $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$. Again we can assume N is a subspace of $[0,1]^\omega$, and so by Lemma 42, we have $N = \bigcup C[\mathcal{K}(M)]$ for some closed $C \subseteq \mathcal{K}([0,1]^\omega)^2$. Since there are at most $\mathfrak{c}$-many closed subsets of the separable metrizable space $\mathcal{K}([0,1]^\omega)^2$ we have the claimed upper bound.

Since for any $\mathcal{E}$ and $\mathcal{D}$, we clearly have $(\mathcal{E}(M), \mathcal{K}(M)) \geq_T (\mathcal{D}(1), \mathcal{K}(1))$, and $(\mathcal{D}(1), \mathcal{K}(1)) =_T (\mathcal{K}(1), \mathcal{K}(1))$, the set $D_{\mathcal{E},\mathcal{D}}$ contains $T_{\mathcal{K}}(1)$. So the proof of (1) is complete once we prove claim (3).

Now assume $\mathcal{D}$ is productive, and prove claim (2). Suppose $T_{\mathcal{E},\mathcal{D}}$ is not empty, say it contains N. We show it has size $\mathfrak{c}$. According to Lemma 37 there is a zero-dimensional separable metrizable space N_0 such that $(\mathcal{D}(N), \mathcal{K}(N)) =_T (\mathcal{D}(N_0), \mathcal{K}(N_0))$. Without loss of generality, then, we assume N is zero-dimensional.

It is well known that there is a continuum sized family, $\mathcal{C}$, of pairwise non-homeomorphic continua (compact, connected, metrizable spaces). Then for any C from $\mathcal{C}$, Lemma 32 tells us that $(\mathcal{D}(N), \mathcal{K}(N)) =_T (\mathcal{D}(N \times C), \mathcal{K}(N \times C))$. Since N is zero-dimensional the connected components of $N \times C$ are the sets $\{x\} \times C$, for x in N, which are all homeomorphic to C.

For distinct C and C' from $\mathcal{C}$, any homeomorphism of $N \times C$ with $N \times C'$ must carry connected components of $N \times C$ to connected components to $N \times C'$, which is impossible since C and C' are not homeomorphic. Hence the $N \times C$'s, for C in $\mathcal{C}$, are distinct (pairwise non-homeomorphic) members of each of $T_{\mathcal{E},\mathcal{D}}$.

Since M is always in $T_{\mathcal{D}}$, this latter set is never empty and so must have size $\mathfrak{c}$. This gives claim (3). $\square$

The first option of part (2) of the preceding result, that $T_{\mathcal{E},\mathcal{D}}$ can have size 0, can not be eliminated (at least consistently). Let $\mathcal{E}$ be the identity operator and $\mathcal{D} = \mathcal{K}$. Assume $\omega_1 < \mathfrak{d}$. Let M be a subspace of $\mathbb{R}$ of size ω_1. Note that all compact subsets of M are countable, so it is not σ-compact. We show there is no separable metrizable space N such that $(M, \mathcal{K}(M)) =_T \mathcal{K}(N)$, in other words, $T_{\mathcal{E},\mathcal{D}}$ is empty. For if ϕ_1 is a Tukey quotient of $(M, \mathcal{K}(M))$ to $\mathcal{K}(N)$, then $\phi_1(M)$ is a cofinal collection in $\mathcal{K}(N)$ of size $\leq \omega_1$. If N were not locally compact then $\omega^\omega =_T \mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(N)$, and $\mathrm{cof}(\mathcal{K}(N)) \geq \mathrm{cof}(\omega^\omega) = \mathfrak{d}$. So under $\omega_1 < \mathfrak{d}$, the space N must be locally compact, and $\mathcal{K}(N) =_T \omega$. But now a Tukey quotient of ω to $(M, \mathcal{K}(M))$ forces M to be σ-compact, which it is not.

Since there are $2^\mathfrak{c}$ homeomorphism classes of separable metrizable spaces, but each (relative) Tukey equivalence classes, $T_{\mathcal{D}}$ of productive $\mathcal{K}$-operators contains just $\mathfrak{c}$-many elements, we immediately deduce:

Corollary 81. *Let $\mathcal{D}$ be a productive $\mathcal{K}$-operator. Then $|(\mathcal{D}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))| = 2^\mathfrak{c}$.*

In particular, $\mathcal{K}(Sub(\mathbb{R}))$, $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$, and $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ all contain exactly $2^\mathfrak{c}$ elements.

Recall that all $\mathcal{K}(M)$'s have size $\mathfrak{c}$ (unless M is discrete). So Corollary 81 implies that there are — in ZFC — $2^\mathfrak{c}$-many Tukey classes of posets of size $\mathfrak{c}$. Using some axioms in addition to ZFC various such families of posets have been constructed by Dobrinen and Todorčević. In ZFC alone the best result prior to this work is from [58].

Theorem 82 (Todorčević)**.** *For each regular κ, there are at least 2^κ-many Tukey classes of posets of size $\kappa^{\aleph_0}$.*

We observe that this theorem does not give a $2^\mathfrak{c}$-sized family in ZFC.

Lemma 83. *It is consistent that* $\sup\{2^\kappa : \kappa \leq \mathfrak{c},\ \kappa \textit{ is regular}\} < 2^{\mathfrak{c}}$.

Proof. We will use Easton's theorem (V.2.7.) from [42]. Let $\lim \omega_1$ be the set of all limit ordinals in ω_1.

To define Easton index function, let $\mathrm{dom}(E) = \{\kappa : \kappa < \aleph_{\omega_1},\ \kappa \text{ is regular}\} = \{\aleph_\alpha : \alpha \in \omega_1 \setminus \lim \omega_1\}$ and for each $\alpha \in \omega_1 \setminus \lim \omega_1$, let $E(\aleph_\alpha) = \aleph_{\omega_1+\alpha}$. Then $\mathrm{cof}(E(\aleph_\alpha)) = \mathrm{cof}(\aleph_{\omega_1+\alpha}) = \aleph_{\omega_1+\alpha} > \aleph_\alpha$ for each successor $\alpha < \omega_1$ and $\mathrm{cof}(E(\aleph_0)) = \mathrm{cof}(\aleph_{\omega_1}) = \aleph_1 > \aleph_0$.

Then it is consistent that for each $\alpha \in \omega_1 \setminus \lim \omega_1$, $2^{\aleph_\alpha} = \aleph_{\omega_1+\alpha}$. In particular, $\mathfrak{c} = \aleph_{\omega_1}$. But then $\sup\{2^\kappa : \kappa \leq \mathfrak{c},\ \kappa \text{ is regular}\} = \sup\{2^\kappa : \kappa < \aleph_{\omega_1},\ \kappa \text{ is regular}\} = \sup\{\aleph_{\omega_1+\alpha} : \alpha \in \omega_1 \setminus \lim \omega_1\} = \aleph_{\omega_1+\omega_1}$. Since $\mathrm{cof}(2^{\mathfrak{c}}) > \omega_1$, we have $2^{\mathfrak{c}} \neq \aleph_{\omega_1+\omega_1}$. But by monotonicity of exponentiation, $2^{\mathfrak{c}} \geq \sup\{2^\kappa : \kappa < \mathfrak{c},\ \kappa \text{ is regular}\}$. So, $2^{\mathfrak{c}} > \sup\{2^\kappa : \kappa \leq \mathfrak{c},\ \kappa \text{ is regular}\}$. □

3.1.2.2 Bounded Sets; Cofinality, Additivity and Calibres Let M and N be separable metrizable spaces, and $\mathcal{C}$ a family of subspaces of M such that $|\mathcal{C}| \leq |N|$. We define the *weak join* of $\mathcal{C}$ (along N) as follows. Index (with repeats, if necessary) $\mathcal{C} = \{C_y : y \in N\}$. Define, $J(\mathcal{C}) = J_N(\mathcal{C}) = \bigcup\{C_y \times \{y\} : y \in N\}$, considered as a subspace of $M \times N$. The weak join operation on $\mathcal{C}$ gives an upper bound for $\mathcal{C}$ (but it is unclear if it gives the *least* upper bound).

Lemma 84. *For each* $C = C_y$ *from* $\mathcal{C}$*, the subspace* $C_y \times \{y\}$ *is a closed subspace of* $J_N(\mathcal{C})$ *homeomorphic to* C_y*. Hence, by Lemma 33, for every* C *in* $\mathcal{C}$*:* $\mathcal{K}(J(\mathcal{C})) \geq_T \mathcal{K}(C)$*,* $(\mathcal{F}(J(\mathcal{C})), \mathcal{K}(J(\mathcal{C}))) \geq_T (\mathcal{F}(C), \mathcal{K}(C))$ *and* $(J(\mathcal{C}), \mathcal{K}(J(\mathcal{C}))) \geq_T (C, \mathcal{K}(C))$*.*

Lemma 85. *A subset of* $\mathcal{K}(Sub(\mathbb{R}))$ *(respectively, a subset of* $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$*, or* $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$*) is bounded if and only if it has size* $\leq \mathfrak{c}$*.*

Proof. Suppose first that $\mathcal{C}$ is a $\leq \mathfrak{c}$-sized subset of $\mathcal{K}(Sub(\mathbb{R}))$. Pick a representative M_c, a subspace of $[0,1]^\omega$, for each $c \in \mathcal{C}$. Let $M = [0,1]^\omega$ and $N = [0,1]$. Then the above observation immediately says $J(\mathcal{C})$ works as an upper bound of $\mathcal{C}$ in $\mathcal{K}(Sub(\mathbb{R}))$. *Mutatis mutandis* the same argument works for $\leq \mathfrak{c}$-sized subsets of $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$, or $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$.

For the converse suppose the subset $\mathcal{C}$ of $\mathcal{K}(Sub(\mathbb{R}))$ has an upper bound $\mathcal{K}(M)$. Then $\mathcal{C}$ is a subset of $D = \{[\mathcal{K}(N)]_T : \mathcal{K}(M) \geq_T \mathcal{K}(N)\}$. Since the set $D_{\mathcal{K},\mathcal{K}}$ of Lemma 80 has size $\leq \mathfrak{c}$, so does D. The same argument applies to bounded subsets of $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$, or $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$. $\square$

Note that we can use the fact that there is no Tukey-largest element to deduce that every bounded set has a strict upper bound. The following result is immediate from Lemma 85.

Corollary 86.

(1) $\mathrm{add}(\mathcal{K}(Sub(\mathbb{R}))) = \mathfrak{c}^+$. *Also,* $\mathrm{add}(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R}))) = \mathfrak{c}^+$ *and* $\mathrm{add}(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R}))) = \mathfrak{c}^+$.

(2) $\mathrm{cof}(\mathcal{K}(Sub(\mathbb{R}))) = 2^{\mathfrak{c}}$. *Also,* $\mathrm{cof}(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R}))) = 2^{\mathfrak{c}}$ *and* $\mathrm{cof}(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R}))) = \mathfrak{c}^+$.

(3) $\mathcal{K}(Sub(\mathbb{R}))$ *(respectively,* $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ *and* $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$*) has calibre* (κ, λ, μ) *if and only if* $\mu \leq \mathfrak{c}$.

3.1.2.3 Antichains

Theorem 87. *Let B be a $\mathfrak{c}$-sized totally imperfect subset of $[0,1]$. There is a $2^{\mathfrak{c}}$-sized family, $\mathcal{A}$, of subsets of B such that for distinct M and N from $\mathcal{A}$ we have $\mathcal{K}(M) \not\geq_T (N, \mathcal{K}(N))$ and $\mathcal{K}(N) \not\geq_T (M, \mathcal{K}(M))$.*

Hence $\{[\mathcal{K}(M)]_T : M \in \mathcal{A}\}$ is an antichain in $\mathcal{K}(Sub(\mathbb{R}))$ of size $2^{\mathfrak{c}}$.

Further $\{[(M, \mathcal{K}(M))]_T : M \in \mathcal{A}\}$ and $\{[(\mathcal{F}(M), \mathcal{K}(M))]_T : M \in \mathcal{A}\}$ are $2^{\mathfrak{c}}$-sized antichains in $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$, respectively.

Proof. Fix a $\mathfrak{c}$-sized totally imperfect subset of $[0,1]$, B (for example, a Bernstein set would work). We construct a $\mathfrak{c}$-sized M_s inside B for each $s \in \mathfrak{c}$. Then for each $I \subseteq \mathfrak{c}$ we let $M_I = \bigcup_{s \in I} M_s$ and show that $I_2 \nsubseteq I_1$ and $I_1 \nsubseteq I_2$ imply $(M_{I_1}, \mathcal{K}(M_{I_1})) \not\leq_T \mathcal{K}(M_{I_2})$. Pick $2^{\mathfrak{c}}$-sized set $\mathcal{P} \subseteq \mathcal{P}(\mathfrak{c})$ with the property that for every distinct $I_1, I_2 \in \mathcal{P}$ we have $I_2 \nsubseteq I_1$ and $I_1 \nsubseteq I_2$. Then $\mathcal{A} = \{M_I : I \in \mathcal{P}\}$ works.

Let $\mathcal{H} = \{(s, C) : s \in \mathfrak{c}, C \text{ is a closed subset of } \mathcal{K}([0,1])^2\}$. Enumerate $\mathcal{H} = \{p_\alpha : \alpha \in \mathfrak{c}\}$ so that each element is repeated $\mathfrak{c}$-many times. Let $p_\alpha = (s_\alpha, C_\alpha)$.

We will construct $M_{\alpha,s}$ for each $\alpha \in \mathfrak{c}$ and $s \in \mathfrak{c}$, and then let $M_s = \bigcup_{\alpha\in\mathfrak{c}} M_{\alpha,s}$. We will also construct Out_α for each $\alpha \in \mathfrak{c}$ and set $Out_{<\alpha} = \bigcup_{\beta<\alpha} Out_\beta$ and $Out_{\leq\alpha} = \bigcup_{\beta\leq\alpha} Out_\beta$. Define $M_{<\alpha,s}$ and $M_{\leq\alpha,s}$ similarly.

For each stage β the following will be true:

1. $Out_{\leq\beta}$ is disjoint from $M_{\leq\beta,s}$ for each $s \in \mathfrak{c}$;
2. $|M_{\leq\beta,s}| \leq |\beta|$ for each $s \in \mathfrak{c}$ and $|Out_{\leq\beta}| \leq |\beta|$;
3. for each $s \in \mathfrak{c}$, $s \notin \beta$ implies $M_{\leq\beta,s} = \emptyset$ and $s \in \beta$ implies $M_{\beta,s} \setminus \bigcup_{t\in\mathfrak{c}} M_{<\beta,t} \neq \emptyset$;
4. if $s_\beta \in \beta$ there are two cases:

   (a) either for each $K \in \mathcal{K}(B\setminus Out_{<\beta})$ such that $\bigcup C_\beta[K] \setminus \bigcup_{s\in\mathfrak{c}} M_{<\beta,s} \neq \emptyset$, we have $\bigcup C_\beta[K] \subseteq K$;

   (b) or there is $K_\beta \in \mathcal{K}(B\setminus Out_{<\beta})$ such that $\bigcup C_\beta[K_\beta] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\beta,s} \cup K_\beta) \neq \emptyset$, and in this case $Out_\beta \cap (\bigcup C_\beta[\mathcal{K}_\beta] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\beta,s} \cup K_\beta)) \neq \emptyset$ and $K_\beta \subseteq M_{\beta,s_\beta}$.

Now suppose the conditions are true for all $\beta < \alpha$.

Step 1: If $s_\alpha \notin \alpha$, set $Out_\alpha = \emptyset$ and proceed to Step 2.

If $s_\alpha \in \alpha$ consider two cases. Case 1: if for each $K \in \mathcal{K}(B\setminus Out_{<\alpha})$ such that $\bigcup C_\alpha[K] \setminus \bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \neq \emptyset$, we have $\bigcup C_\alpha[K] \subseteq K$, then let $Out_\alpha = \emptyset$. Case 2: there is $K_\alpha \in \mathcal{K}(B\setminus Out_{<\alpha})$ such that $\bigcup C_\alpha[K_\alpha] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup K_\alpha) \neq \emptyset$. Pick $a_\alpha \in \bigcup C_\alpha[K_\alpha] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup K_\alpha)$ and let $Out_\alpha = \{a_\alpha\}$.

Step 2: For each $s \notin \alpha$ set $M_{\alpha,s} = \emptyset$. Let $M'_{\alpha,s_\alpha} = \emptyset$ if no K_α was chosen and let $M'_{\alpha,s_\alpha} = K_\alpha$ if it was. Since only at most α-many $M_{<\alpha,s}$'s are non-empty and those that are non-empty have size at most $|\alpha|$, $B\setminus(Out_{\leq\alpha} \cup \bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup M'_{\alpha,s_\alpha})$ is $\mathfrak{c}$-sized. Pick $|\alpha|$-many distinct points of $B\setminus(Out_{\leq\alpha} \cup \bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup M'_{\alpha,s_\alpha})$ and list them $\{x_{\alpha,s} : s \in \alpha\}$. Now for each $s \in \alpha$, if $s \neq s_\alpha$ let $M_{\alpha,s} = \{x_{\alpha,s}\}$ and for $s = s_\alpha$, let $M_{\alpha,s_\alpha} = \{x_{\alpha,s_\alpha}\} \cup M'_{\alpha,s_\alpha}$.

Since K_α is countable all conditions are satisfied. Condition 3 implies that each M_s is $\mathfrak{c}$-sized. Moreover, note that each M_s contains a $\mathfrak{c}$-sized subset that is disjoint from all other M_t-s. So if $I_1 \nsubseteq I_2$, $M_{I_1} \setminus M_{I_2}$ is $\mathfrak{c}$-sized.

We need to show that $I_2 \nsubseteq I_1$ and $I_1 \nsubseteq I_2$ imply $M_{I_1} \not\leq_T \mathcal{K}(M_{I_2})$. Suppose $I_2 \nsubseteq I_1$, $I_1 \nsubseteq I_2$ and pick $s \in I_2 \setminus I_1$. Take any closed subset C of $\mathcal{K}([0,1])^2$. Then there is $\alpha \in \mathfrak{c}$ such that $(s, C) = p_\alpha$ and $s \in \alpha$ (this is why we need $\mathfrak{c}$-repetitions). First suppose that for each $K \in \mathcal{K}(B\backslash Out_{<\alpha})$ such that $\bigcup C_\alpha[K]\backslash\bigcup_{t\in\mathfrak{c}} M_{<\beta,t} \neq \emptyset$ we have $\bigcup C_\alpha[K] \subseteq K$. Then $\bigcup C_\alpha[\mathcal{K}(M_{I_2})] \subseteq M_{I_2} \cup \bigcup_{t\in\mathfrak{c}} M_{<\alpha,t}$. This implies that if $M_{I_1} = \bigcup C_\alpha[\mathcal{K}(M_{I_2})]$ then $M_{I_1}\backslash M_{I_2} \subseteq \bigcup_{t\in\mathfrak{c}} M_{<\alpha,t}$, which is $<\mathfrak{c}$-sized. So, $M_{I_1} = \bigcup C_\alpha[\mathcal{K}(M_{I_2})]$ contradicts $I_1 \nsubseteq I_2$.

Now suppose there is $K_\alpha \in \mathcal{K}(B\backslash Out_{<\alpha})$ such that $\bigcup C[K_\alpha]\backslash(\bigcup_{t\in\mathfrak{c}} M_{<\alpha,t} \cup K_\alpha) \neq \emptyset$. Then at stage α we made sure that $K_\alpha \in \mathcal{K}(M_s) \subseteq \mathcal{K}(M_{I_2})$ so $a_\alpha \in \bigcup C[K_\alpha] \subseteq \bigcup C[\mathcal{K}(M_{I_2})]$; but $a_\alpha \in Out_\alpha$ and therefore it misses all M-s, namely it misses M_{I_1}. So $\bigcup C[\mathcal{K}(M_{I_2})]\backslash M_{I_1} \neq \emptyset$. □

3.1.2.4 Embeddings It is interesting to see what other posets embed in $\mathcal{K}(Sub(\mathbb{R}))$. We were motivated by papers by Knight, McCluskey, McMaster and Watson [41, 46] that studied what posets embed into $\mathcal{P}(\mathbb{R})$ ordered by homeomorphic embeddability. Note that any poset that does embed in $\mathcal{K}(Sub(\mathbb{R}))$ must have the property that the set of predecessors of any element has size no more than $\mathfrak{c}$. For example, $\mathcal{P}(\mathbb{R})$ does not embed in any of $\mathcal{K}(Sub(\mathbb{R}))$, $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$, or $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$ (while, interestingly, $\mathcal{P}(\mathbb{R})$ embeds into $\mathcal{P}(\mathbb{R})$ ordered by homeomorphic embeddability). On the other hand, it is immediate from Lemmas 18 and 80 that:

Corollary 88. *$\mathfrak{c}^+$ is the largest ordinal that embeds in $\mathcal{K}(Sub(\mathbb{R}))$ (respectively, $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ or $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$).*

We develop some machinery demonstrating that two natural partial orders of size continuum, the real line and $\mathcal{P}(\omega)$, do order-embed in $\mathcal{K}(Sub(\mathbb{R}))$, $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ and $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$.

Theorem 89. *Let B be a $\mathfrak{c}$-sized totally imperfect subset of $[0,1]$. Let $B_y = B \times \{y\}$ for each $y \in [0,1]$.*

Suppose $\mathcal{I} \subseteq \mathcal{P}([0,1])$ has size at most $\mathfrak{c}$. Then for each y in $[0,1]$ there is a subspace M_y of B_y such that, whenever $y \notin I \in \mathcal{I}$, writing M_I for $\bigcup_{y\in I} M_y$, we have $\mathcal{K}(M_I) \ngeq_T (M_y, \mathcal{K}(M_y))$.

Proof. Fix B as above. Let $B_I = \bigcup_{y\in I} B_y$ for each $I \subseteq [0,1]$. The construction will ensure that every M_y has size $\mathfrak{c}$.

Let $\mathcal{H} = \{(I, C) :\ C$ is a closed subset of $\mathcal{K}([0,1]^2)^2,\ I \in \mathcal{I}\}$. Enumerate $\mathcal{H} = \{p_\alpha : \alpha \in \mathfrak{c}\}$. Let $p_\alpha = (I_\alpha, C_\alpha)$.

We will construct $M_{\alpha,y}$ for each $\alpha \in \mathfrak{c}$ and $y \in [0,1]$, and then let $M_y = \bigcup_{\alpha\in\mathfrak{c}} M_{\alpha,y}$. We will also construct Out_α for each $\alpha \in \mathfrak{c}$ and set $Out_{<\alpha} = \bigcup_{\beta<\alpha} Out_\beta$, and $Out_{\leq\alpha} = \bigcup_{\beta\leq\alpha} Out_\beta$. Define $M_{<\alpha,y}$ and $M_{\leq\alpha,y}$ similarly.

For each stage β the following will be true:

1. $Out_{\leq\beta}$ is disjoint from $M_{\leq\beta,y}$ for each $y \in [0,1]$;
2. $|M_{\leq\beta,y}| \leq |\beta|$ for each $y \in [0,1]$ and $|Out_{\leq\beta}| \leq |\beta|$;
3. each $M_{\beta,y} \backslash M_{<\beta,y}$ is non-empty;
4. there are two cases:

    (a) either $\bigcup C_\beta[\mathcal{K}(B_{I_\beta} \backslash Out_{<\beta})] \subseteq \bigcup_{y\in[0,1]} M_{<\beta,y}$;

    (b) or there is $K_\beta \in \mathcal{K}(B_{I_\beta} \backslash Out_{<\beta})$ such that $\bigcup C_\beta[K_\beta] \backslash \bigcup_{y\in[0,1]} M_{<\beta,y} \neq \emptyset$, and in this case $Out_\beta \cap (\bigcup C_\beta[\mathcal{K}_\beta] \backslash \bigcup_{y\in[0,1]} M_{<\beta,y}) \neq \emptyset$ and $K_\beta \cap B_y \subseteq M_{\beta,y}$ for each $y \in I_\beta$.

Suppose the conditions are true for all $\beta < \alpha$. If $\bigcup C_\alpha[\mathcal{K}(B_{I_\alpha} \backslash Out_{<\alpha})] \subseteq \bigcup_{y\in[0,1]} M_{<\alpha,y}$, then let $Out_\alpha = \emptyset$. Otherwise, there is $K_\alpha \in \mathcal{K}(B_{I_\alpha} \backslash Out_{<\alpha})$ such that $\bigcup C_\alpha[K_\alpha] \backslash \bigcup_{y\in[0,1]} M_{<\alpha,y} \neq \emptyset$. Pick $a_\alpha \in \bigcup C_\alpha[K_\alpha] \backslash \bigcup_{y\in[0,1]} M_{<\alpha,y}$ and let $Out_\alpha = \{a_\alpha\}$.

Now for each $y \in [0,1]$ pick $x_{\alpha,y} \in B_y \backslash (M_{<\alpha,y} \bigcup Out_{\leq\alpha})$. For $y \notin I_\alpha$ let $M_{\alpha,y} = \{x_{\alpha,y}\}$; for $y \in I_\alpha$, if no K_α was chosen let $M_{\alpha,y} = \{x_{\alpha,y}\}$ and if K_α was chosen let $M_{\alpha,y} = \{x_{\alpha,y}\} \bigcup (K_\alpha \cap B_y)$.

Then $K_\alpha \cap B_y$ is countable for each $y \in [0,1]$, since B_y is a closed subset of B_{I_α} and all compact subsets of B_y are countable. So all conditions are satisfied. Condition 3 implies that each M_y is $\mathfrak{c}$-sized.

We need to show that $y \notin I \in \mathcal{I}$ implies $(M_y, \mathcal{K}(M_y)) \not\leq_T \mathcal{K}(M_I)$. Take any closed subset C of $\mathcal{K}([0,1]^2)^2$. Then there is $\alpha \in \mathfrak{c}$ such that $(I, C) = p_\alpha$. Suppose $\bigcup C[\mathcal{K}(B_I \backslash Out_{<\alpha})] \subseteq \bigcup_{x\in[0,1]} M_{<\alpha,x}$ is the case. Then since $\bigcup C[\mathcal{K}(M_I)] \subseteq \bigcup C[\mathcal{K}(B_I \backslash Out_{<\alpha})] \subseteq \bigcup_{x\in[0,1]} M_{<\alpha,x}$ and $M_{\alpha,y} \backslash M_{<\alpha,y} \neq \emptyset$ (which implies that $M_{\alpha,y} \backslash \bigcup_{x\in[0,1]} M_{<\alpha,x} \neq \emptyset$) we have $M_y \backslash \bigcup C[\mathcal{K}(M_I)] \neq \emptyset$.

Now suppose there is $K_\alpha \in \mathcal{K}(B_I \backslash Out_{<\alpha})$ such that $\bigcup C_\alpha[K_\alpha] \backslash \bigcup_{x\in[0,1]} M_{<\alpha,x} \neq \emptyset$. Then at stage α we made sure that $K_\alpha \in \mathcal{K}(M_I)$ and $a_\alpha \in \bigcup C[K_\alpha]$ but $a_\alpha \in Out_\alpha$ so it misses all

M-s, namely it misses M_y. So $\bigcup C[\mathcal{K}(M_I)]\backslash M_y \neq \emptyset$. $\square$

Corollary 90. *There is a copy of* $([0,1]^\omega, \leq)$ *in* $\mathcal{K}(Sub(\mathbb{R}))$*,* $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ *and* $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$*. So* $(\mathbb{Q}, \leq)$*,* $(\mathbb{R}, \leq)$ *and* $([0,1], \leq)$ *are also embedded.*

Proof. Let $\mathcal{I} = \{I_x = \bigcup_{n\in\omega}[\frac{1}{2^{2n+1}}, x(n)] : x \in \Pi_{n\in\omega}[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}]\}$ in Theorem 89. Then, for $x, y \in \Pi_{n\in\omega}[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}]$, $x \not\leq y$ implies that there is $n \in \omega$ such that $x(n) > y(n)$ and therefore $x(n) \notin I_y$. So, by Theorem 89, we get $(M_{x_n}, \mathcal{K}(M_{x_n})) \not\leq_T \mathcal{K}(M_{I_y})$. But since $M_{x(n)}$ is a closed subset of M_{I_x} we get $(M_{I_x}, \mathcal{K}(M_{I_x})) \not\leq_T \mathcal{K}(M_{I_y})$, which implies $\mathcal{K}(M_{I_x}) \not\leq_T \mathcal{K}(M_{I_y})$, $(M_{I_x}, \mathcal{K}(M_{I_x})) \not\leq_T (M_{I_y}, \mathcal{K}(M_{I_y}))$ and $(\mathcal{F}(M_{I_x}), \mathcal{K}(M_{I_x})) \not\leq_T (\mathcal{F}(M_{I_y}), \mathcal{K}(M_{I_y}))$. However, if $x \leq y$, I_x is a closed subset of I_y and therefore $\mathcal{K}(M_{I_x}) \leq_T \mathcal{K}(M_{I_y})$, $(M_{I_x}, \mathcal{K}(M_{I_x})) \leq_T (M_{I_y}, \mathcal{K}(M_{I_y}))$ and $(\mathcal{F}(M_{I_x}), \mathcal{K}(M_{I_x})) \leq_T (\mathcal{F}(M_{I_y}), \mathcal{K}(M_{I_y}))$. $\square$

Corollary 91. *There is a copy of* $\mathcal{P}(\omega)$ *in* $\mathcal{K}(Sub(\mathbb{R}))$*,* $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ *and* $(\mathcal{F}(Sub(\mathbb{R})), \mathcal{K}(Sub(\mathbb{R})))$*. Hence every countable partial order-embeds.*

Proof. Let $N = \{\frac{1}{n+1} : n \in \omega\}$ and $\mathcal{I} = \mathcal{P}(N)$ in Theorem 89. As in Corollary 90 if $I_1 \nsubseteq I_2$ then $(M_{I_1}, \mathcal{K}(M_{I_1})) \not\leq_T \mathcal{K}(M_{I_2})$. Since N is discrete, $I_1 \subseteq I_2$ implies M_{I_1} is a closed subset of M_{I_2}, so we get $\mathcal{K}(M_{I_1}) \leq_T \mathcal{K}(M_{I_2})$, and the relative versions, as well. $\square$

3.1.3 $\mathcal{K}(Sub(M))$

In this section we let M be separable and metrizable and we investigate $\mathcal{K}(Sub(M))$. Firstly we have a corollary to Lemma 37.

Corollary 92. *Suppose* M *is a separable metrizable space that contains a Cantor set. Then* $\mathcal{K}(Sub(M)) = \mathcal{K}(Sub(\mathbb{R}))$*.*

The case left to investigate is what happens to $\mathcal{K}(Sub(M))$ when M does not contain a Cantor set.

Lemma 93. *Let* M *be separable metrizable and countable. Then there are four possibilities for* $\mathcal{K}(Sub(M))$*:*

(1) M *is finite, which occurs if and only if* $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1)\}$*,*

(2) M *is infinite and does not contain a metric fan if and only if* $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1), \mathcal{K}(\omega)\}$*,*

(3) M contains a metric fan and is scattered if and only if $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1), \mathcal{K}(\omega), \mathcal{K}(\omega^\omega)\}$, or

(4) M is not scattered if and only if $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1), \mathcal{K}(\omega), \mathcal{K}(\omega^\omega), \mathcal{K}(\mathbb{Q})\}$.

Proof. If M is finite then clearly $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1)\}$. If M is infinite, first suppose it does not contain a metric fan. Then M it is scattered of height at most 2; moreover, M is homeomorphic to one of the ordinals in $[\omega, \omega \times \omega]$ and thus $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1), \mathcal{K}(\omega)\}$. On the other hand if M contains a metric fan then clearly $\{\mathcal{K}(1), \mathcal{K}(\omega), \mathcal{K}(\omega^\omega)\} \subseteq \mathcal{K}(Sub(M))$. But if M is also scattered then it must be Polish and so must all of its subspaces. So, $\mathcal{K}(Y) \leq_T \mathcal{K}(\omega^\omega)$ for each $Y \subseteq M$ and $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1), \mathcal{K}(\omega), \mathcal{K}(\omega^\omega)\}$.

Now if M is not scattered then $\mathbb{Q}$ embeds in M as a closed subspace and $\mathcal{K}(\mathbb{Q}) \leq_T \mathcal{K}(M)$. Also, as any countable metrizable space embeds in $\mathbb{Q}$ as a closed subspace, we have $\mathcal{K}(\mathbb{Q}) \geq_T \mathcal{K}(M)$. As any subspace of M should also fall into the four categories mentioned so far we get $\mathcal{K}(Sub(M)) = \{\mathcal{K}(1), \mathcal{K}(\omega), \mathcal{K}(\omega^\omega), \mathcal{K}(\mathbb{Q})\}$. □

If M is uncountable totally imperfect, we observe that $\{\mathcal{K}(1), \mathcal{K}(\omega), \mathcal{K}(\omega^\omega), \mathcal{K}(\mathbb{Q})\} \subseteq \mathcal{K}(Sub(M))$, since M contains a copy of $\mathbb{Q}$. Also whenever M is $\mathfrak{c}$-sized the proof of Lemma 87 works just as well inside M. So, we still have $2^\mathfrak{c}$-sized antichain inside $\mathcal{K}(Sub(M))$ and the following lemma holds.

Lemma 94. *Let $\{M_\beta : \beta < \mathfrak{c}\}$ be a family of subspaces of separable metrizable $\mathfrak{c}$-sized M. Then there is a subspace N of M such that for all $\beta < \mathfrak{c}$, we have $\mathcal{K}(M_\beta) \not\geq_T N$.*

Therefore if M is $\mathfrak{c}$-sized totally imperfect separable metrizable space, $|\mathcal{K}(Sub(M))| = 2^\mathfrak{c}$ and $\mathcal{K}(Sub(M))$ has no largest element. Next we would like to see if $\mathcal{K}(Sub(M))$ can be directed.

Let $B \subseteq [0,1]$ be a $\mathfrak{c}$-sized totally imperfect set and let M be the direct sum of finite products of B such that B^n is repeated infinitely many times for each $n \in \omega$ (B_0 is a singleton). Then $M \times M = M$ and M does not contain a Cantor set. Then $\mathcal{K}(Sub(M))$ is directed since for each $N, N' \subseteq M$, $N \times N' \subseteq M$ and therefore $\mathcal{K}(N) \times \mathcal{K}(N') =_T \mathcal{K}(N \times N') \in \mathcal{K}(Sub(M))$. The fact that $M \times M = M$ allows application of the weak join operator from Subsection 3.1.2.2 and therefore we can construct upper bounds for sets of size $\mathfrak{c}$. Then

Lemma 18 implies that $\mathfrak{c}^+$ embeds in $\mathcal{K}(M)$. Here we present direct construction of the embedding of $\mathfrak{c}^+$.

Theorem 95. *Let M be a $\mathfrak{c}$-sized totally imperfect separable metrizable space such that $M \times M = M$. Then the ordinal $\mathfrak{c}^+$ embeds in $\mathcal{K}(Sub(M))$.*

More precisely, there is a family $\{M_\alpha : \alpha < \mathfrak{c}\}$ of subspaces of M such that if $\beta < \alpha$ then (i) $\mathcal{K}(M_\alpha) \geq_T \mathcal{K}(M_\beta)$ but (ii) $\mathcal{K}(M_\beta) \not\geq_T M_\alpha$.

Proof. For $\alpha \in \mathfrak{c}^+$ we will define a subspace M_α of M. We will arrange that for $\beta < \alpha$ the space M_β is homeomorphic to a closed subspace of M_α — and so $\mathcal{K}(M_\alpha) \geq_T \mathcal{K}(M_\beta)$ — but $\mathcal{K}(M_\beta) \not\geq_T M_\alpha$.

Let M_0 be any subspace of M. And suppose $\{M_\beta : \beta < \alpha\}$ have been constructed so that $\beta' < \beta < \alpha$ implies $\mathcal{K}(M_\beta) \geq_T \mathcal{K}(M'_\beta)$, but $\mathcal{K}(M'_\beta) \not\geq_T M_\beta$.

We construct M_α as follows. Let $\mathcal{C}_\alpha = \{M_\beta : \beta < \alpha\}$. Then $J_M(\mathcal{C}_\alpha) \subseteq M$ is an upper bound of $\{\mathcal{K}(M_\beta) : \beta < \alpha\}$. Pick distinct $x, y \in M$ and define $M_\alpha = (J_M(\mathcal{C}_\alpha) \times \{x\}) \cup (M_\alpha^+ \times \{y\}) \subseteq M$, where M_α^+ is a subspace of M obtained from Lemma 94 applied to M and the family $\{M_\beta : \beta < \alpha\}$. Observe that, for each $\beta < \alpha$, we see that $\mathcal{K}(M_\beta) \leq_T \mathcal{K}(J_M(\mathcal{C}_\alpha)) \leq_T \mathcal{K}(M_\alpha)$ since $J_M(\mathcal{C}_\alpha)$ is a closed subspace of M_α; also, by Lemma 94, $\mathcal{K}(M_\beta) \not\geq_T M_\alpha^+$ — and so $\mathcal{K}(M_\beta) \not\geq_T M_\alpha$, as required. □

3.2 THE STRUCTURE OF $\mathcal{K}(Sub(\omega_1))$

The key to the results on $\mathcal{K}(Sub(\mathbb{R}))$ was Lemma 42. Since $\mathcal{K}(\omega_1)$ is first countable, and therefore Fréchet-Urysohn, Lemma 42 still applies: for $S, T \subseteq \omega_1$, $\mathcal{K}(S) \geq_T \mathcal{K}(T)$ if and only if there is a closed subset C of $\mathcal{K}(\omega_1)^2$ such that $C[\mathcal{K}(S)] = \mathcal{K}(T)$. But there are 2^{ω_1}-many closed subsets of $\mathcal{K}(\omega_1)^2$, which is more than there are points in ω_1. Therefore the diagonalization constructions from Section 3.1 are unlikely to work and we resort to different approaches.

3.2.1 The Largest Element, Antichains and Size of $\mathcal{K}(Sub(\omega_1))$

We showed earlier that $\omega^\omega \times [\omega_1]^{<\omega}$ is an upper bound of $\mathcal{K}(Sub(\omega_1))$. But it is, in fact, the largest element of $\mathcal{K}(Sub(\omega_1))$. This implies that the additivity is not defined, cofinality is 1 and all calibre properties are present.

Proposition 96. *Let D be a subspace of ω_1 that consists of all isolated points of ω_1 and the point $\omega \cdot \omega$. Then $\mathcal{K}(D) = \omega^\omega \times [\omega_1]^{<\omega}$, and therefore $[\mathcal{K}(D)]$ is the largest element of $\mathcal{K}(Sub(\omega_1))$.*

Proof. Note that $D = F \oplus I$, where F is the metric fan and I has the discrete topology on a set of size ω_1. Hence $K(D)$ is Tukey equivalent to $K(F) \times K(I)$, which is $\omega^\omega \times [\omega_1]^{<\omega}$. $\square$

If $\omega_1 = \mathfrak{d}$ then Lemma 55 implies that $[\omega_1]^{<\omega}$ and $\omega^\omega \times [\omega_1]^{<\omega}$ are Tukey equivalent, so $\mathcal{K}(D)$ and $\mathcal{K}(I)$ are Tukey equivalent, where I is the set of isolated points of ω_1, and $\mathcal{K}(I)$ is also in the maximal class.

As for antichains, evidently, $\mathcal{K}(Sub(\omega_1))$ has size $\leq 2^{\omega_1}$. To construct a 2^{ω_1}-sized antichain in $\mathcal{K}(Sub(\omega_1))$, Todorčević proved in [58] the following theorem.

Theorem 97 (Todorčević)**.** *Let S and S' be unbounded subsets of ω_1. Then $\mathcal{K}(S) \geq \mathcal{K}(S')$ implies that $S \backslash S'$ is non-stationary.*

In the proof the author shows that if $S \backslash S'$ is stationary then for any function $g : \mathcal{K}(S') \to \mathcal{K}(S)$ there is a collection of *singletons* in $\mathcal{K}(S')$ such that their image under g is bounded. So, in fact, the author proves that if $S \backslash S'$ is stationary, then there is no *relative* Tukey map from S' to $\mathcal{K}(S)$. Now the fact that ω_1 splits into ω_1-many pairwise disjoint stationary subsets gives the following theorem.

Theorem 98 (Todorčević)**.** *There is a 2^{ω_1}-sized family, $\mathcal{A}$, of subsets of ω_1 such that for distinct S and T from $\mathcal{A}$ we have $\mathcal{K}(S) \not\geq_T (T, \mathcal{K}(T))$ and $\mathcal{K}(T) \not\geq_T (S, \mathcal{K}(S))$.*

Corollary 99. $|\mathcal{K}(Sub(\omega_1))| = 2^{\omega_1}$.

3.2.2 Special Classes in $\mathcal{K}(Sub(\omega_1))$

The subsets of ω_1 fall into various classes: locally compact (or not), bounded (or not), stationary (or not), co-stationary (or not). We would like to see if these correspond to classes in $\mathcal{K}(Sub(\omega_1))$. First we prove a proposition that singles out the most problematic class.

Proposition 100. *Let S be a subset of ω_1 that contains a cub set and $\overline{S}\backslash S$ is unbounded. Then $\mathcal{K}(S) =_T \Sigma(\omega^{\omega_1})$.*

Proof. Fix S as above and let $C \subseteq S$ be a cub set. We want to construct a cub set $C' = \{\beta_\alpha : \alpha \in \omega_1\} \subseteq C$ such that for each $\alpha \in \omega_1$, $\mathcal{K}([\beta_\alpha, \beta_{\alpha+1}] \cap S) =_T \omega^\omega$. Suppose we have constructed the desired β_γ for each $\gamma < \alpha$. First let α be a successor. Then since $(\overline{S}\backslash S) \cap (\alpha - 1, \omega_1)$ is not closed, there exists $\beta_\alpha \in C$ such that $[\beta_{\alpha-1}, \beta_\alpha] \cap S$ contains a metric fan as a closed subspace and therefore $\mathcal{K}([\beta_{\alpha-1}, \beta_\alpha] \cap S) =_T \omega^\omega$. If α is a limit, let $\beta_\alpha = \sup\{\beta_\gamma : \gamma < \alpha\}$. This sequence clearly works.

For each $K \in \mathcal{K}(S)$, there exists the smallest $\alpha_K \in \omega_1$ such that $K = \bigcup_{\gamma<\alpha_K} K \cap [\beta_\gamma, \beta_{\gamma+1}]$. Clearly, each $K \cap [\beta_\gamma, \beta_{\gamma+1}] \in \mathcal{K}([\beta_\gamma, \beta_{\gamma+1}] \cap S)$. And for any choice of $\alpha \in \omega_1$ and $K_\gamma \in \mathcal{K}([\beta_\gamma, \beta_{\gamma+1}] \cap S)$ for $\gamma < \alpha$, $\overline{\bigcup_{\gamma<\alpha} K_\gamma} \in \mathcal{K}(S)$.

We now show that $\mathcal{K}(S) =_T \Sigma((\omega^\omega)^{\omega_1})$. Since $\Sigma((\omega^\omega)^{\omega_1})$ is clearly order-isomorphic to $\Sigma(\omega^{\omega_1})$, that will complete the proof.

To show $\mathcal{K}(S) \geq_T \Sigma((\omega^\omega)^{\omega_1})$, fix Tukey quotients $\phi_\alpha : \mathcal{K}([\beta_\alpha, \beta_{\alpha+1}] \cap S) \to \omega^\omega$ for each $\alpha \in \omega_1$. Define $\phi : \mathcal{K}(S) \to \Sigma((\omega^\omega)^{\omega_1})$ by $\phi(K) = \Pi_{\gamma<\alpha_K} \phi_\gamma(K \cap [\beta_\gamma, \beta_{\gamma+1}])$. Clearly, ϕ is order-preserving. It is also cofinal since for any choice of functions $f_\gamma \in \omega^\omega$ for $\gamma < \alpha$, there is $K_\gamma \in \mathcal{K}([\beta_\gamma, \beta_{\gamma+1}] \cap S)$ such that $\phi_\gamma(K_\gamma) \geq f_\gamma$. Then $\phi(\overline{\bigcup_{\gamma<\alpha} K_\gamma}) \geq \Pi_{\gamma<\alpha} f_\gamma$.

For the other direction, fix Tukey quotients $\phi'_\alpha : \omega^\omega \to \mathcal{K}([\beta_\alpha, \beta_{\alpha+1}] \cap S)$ for each $\alpha \in \omega_1$. Define $\phi' : \Sigma((\omega^\omega)^{\omega_1}) \to \mathcal{K}(S)$ by $\phi'(\Pi_{\gamma<\alpha_K} f_\gamma) = \overline{\bigcup_{\gamma<\alpha} \phi'(f_\gamma)}$. Clearly, ϕ' is order-preserving and cofinal. $\square$

The next proposition was proven in [26].

Proposition 101. *For any separable metrizable M, $\mathcal{K}(M) \not\geq_T \Sigma(\omega^{\omega_1})$.*

Lemma 102 (Tukey classes in $\mathcal{K}(Sub(\omega_1))$). *For each $S \subseteq \omega_1$, $\mathcal{K}(S)$ falls into one of the following Tukey equivalence classes.*

(1) $[\mathbf{1}]_T = \{\mathcal{K}(S) : S \textit{ is compact}\}$;

(2) $[\omega]_T = \{\mathcal{K}(S) : S \textit{ is bounded}, \overline{S}\backslash S \textit{ is closed and non-empty}\}$;

(3) $[\omega_1]_T = \{\mathcal{K}(S) : S \textit{ is closed and unbounded}\}$;

(4) $[\omega_1 \times \omega]_T = \{\mathcal{K}(S) : S \textit{ is unbounded}, \overline{S}\backslash S \textit{ is closed, bounded and non-empty}\}$;

(5) $[\omega^\omega]_T \supseteq \{\mathcal{K}(S) : S \textit{ is bounded}, \overline{S}\backslash S \textit{ is not closed}\}$ *and with equality if* $\omega_1 < \mathfrak{b}$;

(6) $[\omega_1 \times \omega^\omega]_T \supseteq \{\mathcal{K}(S) : S \textit{ is unbounded}, \overline{S}\backslash S \textit{ is not closed but bounded}\}$ *with equality if* $\omega_1 < \mathfrak{b}$;

(7) $[[\omega_1]^{<\omega}]_T \supseteq \{\mathcal{K}(S) : S \textit{ is unbounded, not stationary}, \overline{S}\backslash S \textit{ is non-empty and closed}\}$ *with equality if* $\omega_1 < \mathfrak{d}$;

(8) $[[\omega_1]^{<\omega} \times \omega^\omega]_T \supseteq \{\mathcal{K}(S) : S \textit{ is unbounded, not stationary}, \overline{S}\backslash S \textit{ is not closed}\}$ *with equality if* $\omega_1 < \mathfrak{d}$;

(9) $[\Sigma(\omega^{\omega_1})]_T = \{\mathcal{K}(S) : S \textit{ is stationary, not co-stationary}, \overline{S}\backslash S \textit{ is unbounded}\}$;

(10) 2^{ω_1}*-many Tukey classes of the form* $\mathcal{K}(S)$*, where* S *is stationary and co-stationary.*

The classes are ordered as follows:

(a) $\mathbf{1} <_T \omega <_T \omega^\omega$;

(b) $\mathbf{1} <_T \omega_1 <_T \omega_1 \times \omega <_T \omega_1 \times \omega^\omega$*, also* ω *and* ω_1 *are incomparable;*

(c) $\omega <_T \omega_1 \times \omega$, $\omega <_T [\omega_1]^{<\omega}$ *and* $\omega_1 <_T [\omega_1]^{<\omega}$;

(d) $\omega_1 \not\geq_T \omega^\omega$ *and* $\omega_1 <_T \omega^\omega$ *if and only if* $\omega_1 = \mathfrak{b}$;

(e) $\omega_1 \times \omega \not\geq_T \omega^\omega$ *and* $\omega_1 \times \omega <_T \omega^\omega$ *if and only if* $\omega_1 = \mathfrak{b}$;

(f) $\omega^\omega \not\geq_T [\omega_1]^{<\omega}$ *and* $\omega^\omega <_T [\omega_1]^{<\omega}$ *if and only if* $\omega_1 = \mathfrak{d}$;

(g) $\omega^\omega \leq_T \omega_1 \times \omega^\omega$ *with equality if and only if* $\omega_1 = \mathfrak{b}$;

(h) $\omega_1 \times \omega^\omega \not\geq_T [\omega_1]^{<\omega}$ *and* $\omega_1 \times \omega^\omega <_T [\omega_1]^{<\omega}$ *if and only if* $\omega_1 = \mathfrak{d}$;

(i) $[\omega_1]^{<\omega} \leq_T [\omega_1]^{<\omega} \times \omega^\omega$ *with equality if and only if* $\omega_1 = \mathfrak{d}$;

(j) $\omega_1 \times \omega^\omega <_T \Sigma(\omega^{\omega_1}) <_T [\omega_1]^{<\omega} \times \omega^\omega$;

(k) If S *is stationary and co-stationary, then* $\omega_1 \times \omega^\omega <_T \mathcal{K}(S) <_T [\omega_1]^{<\omega} \times \omega^\omega$;

(l) If S *is stationary and co-stationary, then* $\mathcal{K}(S) \not\leq_T \Sigma(\omega^{\omega_1})$;

(m) $[\omega_1]^{<\omega} \not\leq_T \Sigma(\omega^{\omega_1})$ *and for stationary, co-stationary* S, $[\omega_1]^{<\omega} \not\leq_T \mathcal{K}(S)$.

Proof. (a) Clearly, $\mathbf{1} \le_T P$ for any poset P. The map $\psi : \omega \to \omega^\omega$ given by $\psi(n) = (n, 0, 0, 0, \ldots)$ is a Tukey map. But since $\mathrm{cof}(\mathbf{1}) = 1 < \mathrm{cof}(\omega) = \aleph_0 < \mathrm{cof}(\omega^\omega) = \mathfrak{d}$ we get that the inequalities are strict.

(b) We know $\omega_1 \le_T \omega_1 \times \omega$, but since ω_1 has no countable unbounded subsets, there cannot be a Tukey map from $\omega_1 \times \omega$ to ω_1. So, $\mathbf{1} <_T \omega_1 <_T \omega_1 \times \omega$. The map $\psi : \omega_1 \times \omega \to \omega_1 \times \omega^\omega$ defined by $\psi((\alpha, n)) = (\alpha, (n, 0, 0, 0, \ldots))$ witnesses $\omega_1 \times \omega \le_T \omega_1 \times \omega^\omega$. Since both ω and ω_1 have calibre $(\omega_1, \omega_1, \omega)$ then $\omega_1 \times \omega$ must also have this calibre. So if $\omega_1 \times \omega \ge_T \omega_1 \times \omega^\omega$, $\omega_1 \times \omega^\omega$ must have calibre $(\omega_1, \omega_1, \omega)$ as well. But $\omega_1 \times \omega^\omega$ has calibre $(\omega_1, \omega_1, \omega)$ if and only if ω^ω does, which happens if and only if $\omega_1 < \mathfrak{b}$. However, when $\omega_1 < \mathfrak{b}$, $\mathrm{cof}(\omega_1 \times \omega) = \aleph_1 < \mathfrak{d} = \mathrm{cof}(\omega_1 \times \omega^\omega)$ and we cannot have $\omega_1 \times \omega \ge_T \omega_1 \times \omega^\omega$.

Since ω_1 does not have countable unbounded subsets, $\omega \not\le_T \omega_1$. And since $\mathrm{cof}(\omega) < \mathrm{cof}(\omega_1)$, $\omega_1 \not\le_T \omega$.

(c) Clearly, $\omega <_T \omega_1 \times \omega$ and $\omega <_T [\omega_1]^{<\omega}$. The map $\psi : \omega_1 \to [\omega_1]^{<\omega}$ defined by $\psi(\alpha) = \{\alpha\}$ is a Tukey map and the strict inequality, $\omega_1 <_T [\omega_1]^{<\omega}$, follows from the fact that ω_1 does not have countable unbounded subsets.

(d) That $\omega_1 \not\ge_T \omega^\omega$, again, follows from the fact that ω_1 does not have countable unbounded subsets. While $\omega_1 <_T \omega^\omega$ if and only if $\omega_1 = \mathfrak{b}$ is immediate from the spectrum results for ω^ω.

(e) Since $\omega_1 \times \omega \ge_T \omega_1$ and ω_1, ω^ω are Dedekind complete, $\omega_1 \times \omega \ge_T \omega^\omega$ implies $\omega_1 \times \omega \ge_T \omega_1 \times \omega^\omega$, which is not true. So $\omega_1 \times \omega \not\ge_T \omega^\omega$. Also, since $\omega <_T \omega^\omega$, $\omega_1 \times \omega <_T \omega^\omega$ if and only if $\omega_1 <_T \omega^\omega$, which happens if and only if $\omega_1 = \mathfrak{b}$.

(f) ω^ω has calibre (ω_1, ω) but $[\omega_1]^{<\omega}$ does not, hence $\omega^\omega \not\ge_T [\omega_1]^{<\omega}$. We know $\omega^\omega <_T [\omega_1]^{<\omega}$ if and only if $\mathrm{cof}(\omega^\omega) \le \omega_1$, which happens if and only if $\omega_1 = \mathfrak{d}$.

(g) Clearly, $\omega^\omega \le_T \omega_1 \times \omega^\omega$. If $\omega_1 < \mathfrak{b}$ then $\omega_1 \not\le_T \omega^\omega$ but $\omega_1 \le_T \omega_1 \times \omega^\omega$ and $\omega^\omega \neq_T \omega_1 \times \omega^\omega$. If $\omega_1 = \mathfrak{b}$, then ω^ω contains an uncountable subset $U = \{f_\alpha : \alpha < \omega_1\}$ such that no uncountable subset of U is bounded. Define $\psi : \omega_1 \times \omega^\omega \to \omega^\omega$ by $\psi((\alpha, f)) = f_\alpha + f$, which is clearly a Tukey map and $\omega_1 \times \omega^\omega \le_T \omega^\omega$.

(h) Since every countable subset of ω_1 is bounded and ω^ω has calibre (ω_1, ω), $\omega_1 \times \omega^\omega$ also has calibre (ω_1, ω), while $[\omega_1]^{<\omega}$ does not have it, hence $\omega_1 \times \omega^\omega \not\ge_T [\omega_1]^{<\omega}$. We know $\omega_1 \times \omega^\omega <_T [\omega_1]^{<\omega}$ if and only if $\mathrm{cof}(\omega_1 \times \omega^\omega) \le \omega_1$, which happens if and only if $\omega_1 = \mathfrak{d}$.

(i) Clearly, $[\omega_1]^{<\omega} \leq_T [\omega_1]^{<\omega} \times \omega^\omega$. And the cofinalities of the two posets are equal if and only if $\omega_1 = \mathfrak{d}$.

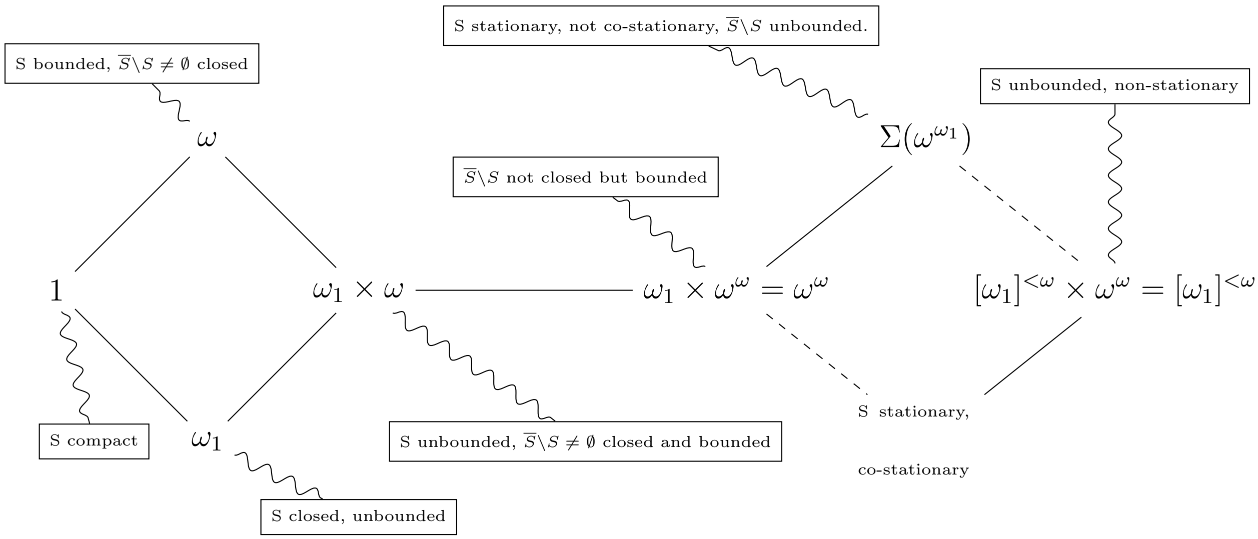

Figure 1: Classes of $\mathcal{K}(S)$ under $\omega_1 = \mathfrak{d}$

(j) We have already proved that $\Sigma(\omega^{\omega_1}) =_T \mathcal{K}(S)$, where S is stationary, not co-stationary and $\overline{S}\setminus S$ is unbounded. In this case $\overline{S}\setminus S$ is not closed and therefore S contains a metric fan as a closed subset. So $\omega^\omega \leq_T \Sigma(\omega^{\omega_1})$. On the other hand, for every unbounded S, $\omega_1 \leq_T \mathcal{K}(S)$. So, $\omega_1 \times \omega^\omega \leq_T \Sigma(\omega^{\omega_1})$. If $\Sigma(\omega^{\omega_1}) \leq_T \omega_1 \times \omega^\omega$ then $\Sigma(\omega^{\omega_1}) \leq_T \mathcal{K}(\mathbb{Q})$, which contradicts Proposition 101. For $\Sigma(\omega^{\omega_1}) <_T [\omega_1]^{<\omega} \times \omega^\omega$, recall that $[\omega_1]^{<\omega} \times \omega^\omega$ is the largest element of all $\mathcal{K}(S)$'s and the inequality is strict because $[\omega_1]^{<\omega} \leq_T \mathcal{K}(S)$ if and only if S is not stationary. If $\omega_1 < \mathfrak{b}$, then $\omega_1 \times \omega^\omega$ has calibre $(\omega_1, \omega_1, \omega)$ but $\Sigma(\omega^{\omega_1})$ does not. So, $\omega_1 \times \omega^\omega < \Sigma(\omega^{\omega_1})$.

(k) For a stationary, co-stationary S, $\overline{S}\setminus S$, is not closed and by the same argument as in (j), $\omega_1 \times \omega^\omega \leq_T \mathcal{K}(S) <_T [\omega_1]^{<\omega} \times \omega^\omega$. To show that the first inequality is also strict, recall that $\omega_1 \times \omega^\omega \leq_T \mathcal{K}(\mathbb{Q})$. Therefore $\mathcal{K}(S) \leq_T \omega_1 \times \omega^\omega$ implies $S \leq_T \mathcal{K}(\mathbb{Q})$ and S should not be co-stationary, but it is.

(l) Let S' be stationary and not co-stationary and suppose $\mathcal{K}(S) \leq_T \mathcal{K}(S')$. Then by Todorčević's theorem $S'\setminus S$ must be non-stationary. But $S'\setminus S = S' \cap \omega_1\setminus S$ and since S' contains a cub set and $\omega_1\setminus S$ is stationary, their intersection should also be stationary.

Therefore we get $\mathcal{K}(S) \not\leq_T \mathcal{K}(S') =_T \Sigma(\omega^{\omega_1})$.

(m) This follows from Todorčević's theorem that $[\omega_1]^{<\omega} \leq_T \mathcal{K}(S)$ if and only if S is not stationary.

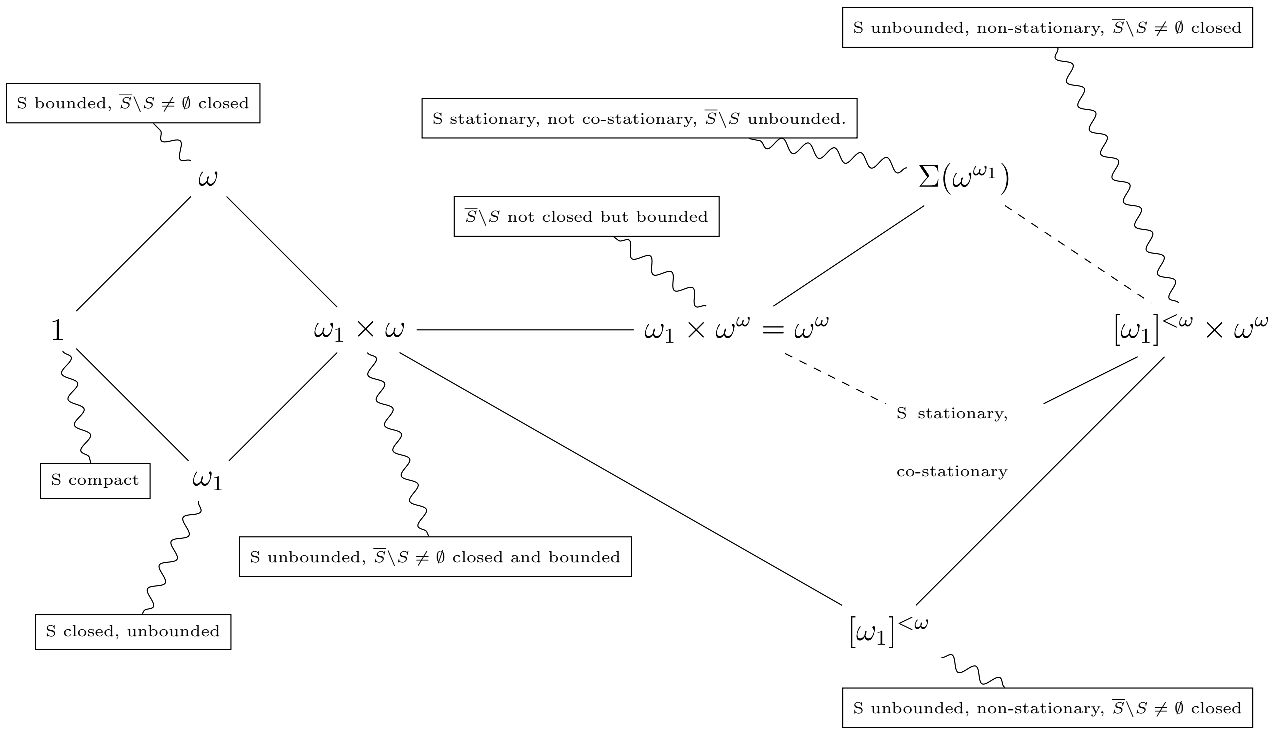

Figure 2: Classes of $\mathcal{K}(S)$ under $\omega_1 = \mathfrak{b} < \mathfrak{d}$

Claims (1), (2) and the 'inclusion' part of (5) are clear and these cases account for all bounded S's. So for the rest of the proof all S's are unbounded and therefore $\omega_1 \leq_T \mathcal{K}(S)$. For (3), notice that any closed unbounded set S is homeomorphic to ω_1 and therefore $\mathcal{K}(S) =_T \omega_1$. To show that nothing else is in this class, notice that if S is not cub, then $\overline{S}\setminus S$ is non-empty and therefore $\mathcal{K}(S)$ contains a countable unbounded subset.

For (4), if $\overline{S}\setminus S$ is closed bounded and non-empty, then $S = N \oplus C$ where N is bounded and locally compact and C is cub. So $\mathcal{K}(S) =_T \mathcal{K}(N) \times \mathcal{K}(C) =_T \omega \times \omega_1$. To show that nothing else is in this class, note that the case when $\overline{S}\setminus S = \emptyset$ was already accounted for (then S is compact or a cub set). If $\overline{S}\setminus S$ is not closed then $\omega^\omega \leq_T \mathcal{K}(S)$ and if $\overline{S}\setminus S$ is closed and unbounded then S is not stationary and $\mathcal{K}(S) \geq_T [\omega_1]^{<\omega}$.

For (6)-(8) we will show that the inclusions hold. The inclusion in (9) was already proven in the previous lemma. The class of sets in (10) is simply everything left over and it contains 2^{ω_1}-many classes by Todorčević's theorem. Parts (a)-(m) already specify positions of the classes and imply the 'equality' parts.

For (6), if $\overline{S}\backslash S$ is bounded but not closed, then $S = N \oplus C$ where N is bounded and non-locally compact and C is cub. So $\mathcal{K}(S) =_T \mathcal{K}(N) \times \mathcal{K}(C) =_T \omega^\omega \times \omega_1$.

For (7), if $\overline{S}\backslash S$ is closed we get that $\mathcal{K}(S \cap [0,\alpha]) =_T \omega$ for each $\alpha < \omega_1$. Since $\mathcal{K}(S) = \bigcup_{\alpha<\omega_1} \mathcal{K}(S \cap [0,\alpha])$, we get that $\mathcal{K}(S) \leq_T [\omega_1]^{<\omega}$. Since S is not stationary, $[\omega_1]^{<\omega} \leq_T \mathcal{K}(S)$.

For (8), if $\overline{S}\backslash S$ is not closed we get that $\omega^\omega \leq_T \mathcal{K}(S)$ and since S is not stationary, $[\omega_1]^{<\omega} \leq_T \mathcal{K}(S)$. Since $[\omega_1]^{<\omega} \times \omega^\omega$ is the largest element in $\mathcal{K}(Sub(\omega_1))$, $[\omega_1]^{<\omega} \times \omega^\omega =_T \mathcal{K}(S)$. $\square$

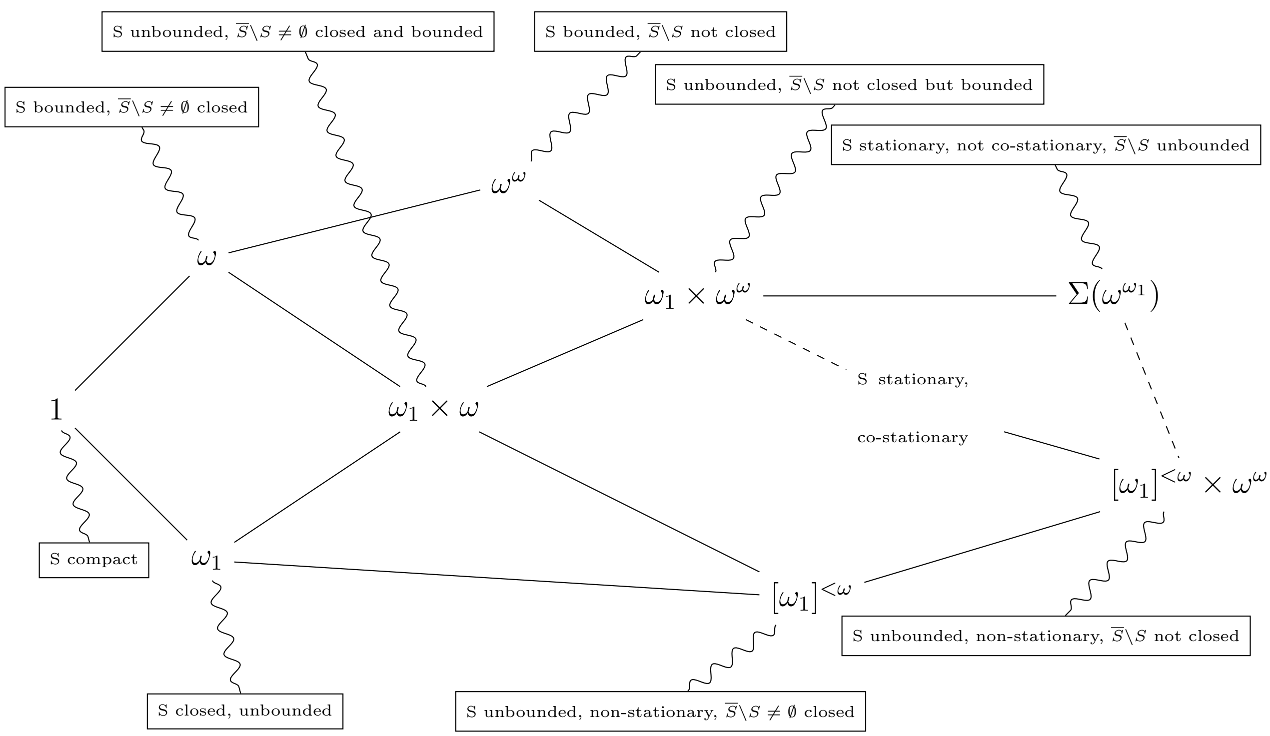

Figure 3: Classes of $\mathcal{K}(S)$ under $\omega_1 < \mathfrak{b}$

The Figures 3.2.2, 3.2.2 and 3.2.2 summarize Lemma 102 in the three cases when $\omega_1 = \mathfrak{d}$, $\omega_1 = \mathfrak{b} < \mathfrak{d}$ and $\omega_1 < \mathfrak{b}$, respectively. The lines indicate that node to the right is strictly above the one on the left. Solid lines indicate there is nothing strictly between the connected classes. Text in the boxes describe the corresponding equivalence classes. Note that the maximal antichain in Todorčević's theorem falls in the 'stationary, co-stationary' category.

3.3 COMPARING $\mathcal{K}(M)$ AND $\mathcal{K}(S)$

In this section we investigate under what circumstances we have $\mathcal{K}(S) \geq_T \mathcal{K}(M)$ and $\mathcal{K}(M) \geq_T \mathcal{K}(S)$. First we consider for which M and S we have $\mathcal{K}(S) \geq_T \mathcal{K}(M)$. The section above gives us some information about this situation but before we can make use of the special classes in $\mathcal{K}(Sub(\omega_1))$ we give a lemma that was proved in [21] for $\mathcal{K}(\mathbb{Q})$. The argument, however, works just as well for an arbitrary M with $\mathcal{K}(\omega^\omega) < \mathcal{K}(M)$.

Lemma 103. *If M is separable metrizable and $\mathcal{K}(\omega^\omega) < \mathcal{K}(M)$ then $\mathcal{K}(M) \not\leq_T \omega_1 \times \omega^\omega$.*

Proof. Let $\psi : \mathcal{K}(M) \to \omega_1 \times \omega^\omega$ be any function and let ϕ_1 and ϕ_2 be its components. Since $\omega^\omega =_T \mathcal{K}(\omega^\omega) < \mathcal{K}(M)$, ψ_2 cannot be a Tukey map, so there exists, $\mathcal{U}$, an unbounded subset of $\mathcal{K}(M)$ and $f \in \omega^\omega$ such that $\psi_2(K) \leq f$ for each $K \in \mathcal{U}$. Let D be a countable dense subset of $\bigcup\mathcal{U}$ and for each $x \in D$ pick $K_x \in \mathcal{U}$ with $x \in K_x$. Then $\overline{\bigcup\mathcal{U}} = \overline{\{K_x : x \in D\}}$ and therefore $\{K_x : x \in D\}$ is unbounded in $\mathcal{K}(M)$. Since $\{K_x : x \in D\}$ is countable, $\psi_1(\{K_x : x \in D\})$ is bounded in ω_1. Therefore $\psi(\{K_x : x \in D\})$ is also bounded. This implies ψ is not a Tukey map and we are done. $\square$

Now all relations between equivalence classes of $\mathcal{K}(Sub(\omega_1))$ that lie below $\omega_1 \times \omega^\omega$ and $\mathbf{1}$, ω and ω^ω are depicted on the diagrams for classes of $\mathcal{K}(Sub(\omega_1))$. We know that $\omega \leq [\omega_1]^{<\omega}$ and $\mathcal{K}(M) \leq_T [\omega_1]^{<\omega}$ if and only if $\mathrm{cof}(\mathcal{K}(M)) \leq \omega_1$. So for any non-locally compact M we need ω_1 to be at least $\mathfrak{d}$ for $\mathcal{K}(M) \leq_T [\omega_1]^{<\omega}$ to be possible. We also know that $\mathcal{K}(\mathbb{Q}) \leq_T [\omega_1]^{<\omega} \times \omega^\omega$ and for ω_1-sized totally imperfect B, $\mathcal{K}(B) \leq_T [\omega_1]^{<\omega} \times \omega^\omega$.

On the other hand, the $\mathcal{K}(M) \geq_T \mathcal{K}(S)$ situation does not happen very often. Since $\mathcal{K}(M)$ always has calibre (ω_1, ω), it is not possible to have $\mathcal{K}(M) \geq_T [\omega_1]^{<\omega} \times \omega^\omega$ or $\mathcal{K}(M) \geq_T [\omega_1]^{<\omega}$. Proposition 101 says that $\mathcal{K}(M) \geq_T \Sigma(\omega^{\omega_1})$ never happens. The next lemma further narrows down the possibilities.

Lemma 104. *Suppose $S \subseteq \omega_1$ is unbounded and there is separable metric M such that $\mathcal{K}(M) \geq_T (S, \mathcal{K}(S))$. Then S is not co-stationary.*

Proof. Suppose $\phi : \mathcal{K}(M) \to \mathcal{K}(S)$ is order-preserving and the image of ϕ covers S. Then as in Proposition 2.6 of [11] let $\mathcal{B}$ be a countable base of M that is closed under finite unions and finite intersections and for each $B \in \mathcal{B}$ define $G(B) = \bigcup\{\phi(K) : K \subseteq B,\ B \in \mathcal{K}(M)\}$.

There is $x \in M$ such that for each $x \in B \in \mathcal{B}$, $G(B)$ is unbounded in ω_1. (Otherwise $\mathcal{B}' = \{B \in \mathcal{B} :\ G(B) \text{ is bounded}\}$ is also a base of X that is closed under finite intersections and unions. Therefore the $G(B)$'s cover ω_1, but this is a contradiction since there are only countably many of them).

For a cardinal θ let $H(\theta)$ be the set of all sets with $< \theta$-sized transitive closure [42]. We know that if θ is regular and uncountable, all axioms of ZFC, with the exception of the Power Set Axiom, are true in $H(\theta)$.

Suppose S is co-stationary and θ is a regular cardinal large enough so that $H(\theta)$ contains all sets we need in this argument. As in the proof of Lemma 1 in [58], let E be a countable elementary submodel of $H(\theta)$ such that $\phi, S, M, \mathcal{B}, G \in E$ and $\omega_1 \cap E \in \omega_1 \backslash S$. (For the last part: construct a sequence of countable elementary submodels of $H(\theta)$, $\{E_\alpha : \alpha \in \omega_1\}$, so that for each successor $\alpha = \beta + 1$, $\phi, S, M, \mathcal{B}, G \in E_\alpha$, $E_\beta \subseteq E_\alpha$ and for each limit γ, $E_\gamma = \bigcup_{\alpha<\gamma} E_\alpha$. Then $\{\omega_1 \cap E_\alpha : \alpha \in \omega_1\}$ is a cub set and therefore meets $\omega_1 \backslash S$).

By elementarity there is $x \in M \cap E$ with decreasing local base $\{B_n : n \in \omega\} \subseteq \mathcal{B}$ at x such that each $G(B_n)$ is unbounded in ω_1. Then, by elementarity, for each n and $\alpha \in \omega_1 \cap E$, since $G(B_n)$ is unbounded there is $K_{n,\alpha} \in \mathcal{K}(M) \cap E$ such that $K_{n,\alpha} \subseteq B_n$ and $\sup(\phi(K_{n,\alpha})) \geq \alpha$. $K_{n,\alpha} \in E$ and $K_{n,\alpha}$ is countable, so $K_{n,\alpha} \subseteq \omega_1 \cap E$. Pick $\{\alpha_n : n \in \omega_1\}$ such that $\{\alpha_n\}_n$ converges to $\omega_1 \cap E$ and let $K = \{x\} \cup \bigcup_{n\in\omega} K_{n,\alpha_n}$. Then $K \in \mathcal{K}(M)$ and $\phi(K_{n,\alpha_n}) \subseteq \phi(K)$ for each n. But this contradicts $\phi(K) \in \mathcal{K}(S)$ and $\omega_1 \cap N \notin S$. □

Proposition 105. *Let S be a subset of ω_1 that contains a cub set. Then $\omega_1 \times \omega \geq_T (S, \mathcal{K}(S))$. Hence $\mathcal{K}(\mathbb{Q}) \geq_T (S, \mathcal{K}(S))$.*

Proof. Fix $S \subseteq \omega_1$ and a cub set $C \subseteq S$. Let $C = \{\beta_\alpha : \alpha \in \omega_1\}$ be the increasing enumeration of C. For each $\alpha \in \omega_1$ enumerate $[\beta_\alpha, \beta_{\alpha+1}] \cap S$ as $\{x_{\alpha,n} : n \in \omega\}$, with repetitions if necessary, and let $F_{\alpha,n} = \{x_{\alpha,0}, x_{\alpha,1}, \ldots, x_{\alpha,n}\}$.

Define $\phi : \omega_1 \times \omega \to \mathcal{K}(S)$ by $\phi((\alpha, n)) = \overline{\bigcup_{\gamma \leq \alpha} F_{\gamma,n}}$. Since C is a cub set, the only limit points of $\bigcup_{\gamma \leq \alpha} F_{\gamma,n}$ outside $\bigcup_{\gamma \leq \alpha} F_{\gamma,n}$ are in C, so $\phi((\alpha, n))$ is indeed in $\mathcal{K}(S)$. Clearly, ϕ is order-preserving and the image covers S. $\square$

Corollary 106. *For $S \subseteq \omega_1$, there exists a separable metrizable M with $\mathcal{K}(M) \geq_T S$ if and only if S is in the cub filter.*

Corollary 107. *For $S \subseteq \omega_1$, there exists a separable metrizable M with $\mathcal{K}(M) \geq_T \mathcal{K}(S)$ if and only if $\overline{S} \backslash S$ is bounded.*

Proof. If $\overline{S} \backslash S$ is bounded then $S = C \oplus N$, where C is a closed unbounded set or an empty set and N is countable (i.e. $\overline{S} \backslash S$ is bounded). Then, since cub sets are homeomorphic to ω_1, $\mathcal{K}(\mathbb{Q}) \geq_T \mathcal{K}(C)$. Since N is Polish, $\omega^\omega \geq_T \mathcal{K}(N)$. Now set $M = \mathbb{Q}$ and we have $\mathcal{K}(S) = \mathcal{K}(C \oplus N) = \mathcal{K}(C) \times \mathcal{K}(N) \leq_T \mathcal{K}(\mathbb{Q}) \times \mathcal{K}(\omega^\omega) \leq_T \mathcal{K}(\mathbb{Q}) \times \mathcal{K}(\mathbb{Q}) =_T \mathcal{K}(Q)$.

On the other hand, if $\mathcal{K}(M) \geq_T \mathcal{K}(S)$ for some M, then S contains a closed unbounded set. If in addition $\overline{S} \backslash S$ is unbounded, $\mathcal{K}(S) =_T \Sigma(\omega^{\omega_1})$, which contradicts Proposition 101.

$\square$

4.0 APPLICATIONS

The results of this section show why the relative Tukey order is the natural setting to study posets of the form $\mathcal{K}(X)$. First we establish a connection between the relative Tukey order and function spaces, and use the antichains in $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$ to construct large families of 'incomparable' function spaces. Next we explore the connection between the relative Tukey order and the Lindelöf Σ property established in [11] and present two applications.

4.1 FUNCTION SPACES C_p AND C_k

For any space X let $C(X)$ be the set of all real-valued continuous functions on X. Let $\mathbf{0}$ be the constant zero function on X. For any function f from $C(X)$, subset E of X and $\epsilon > 0$ let $B(f, E, \epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in E\}$. Write $C_p(X)$ for $C(X)$ with the pointwise topology (so basic neighborhoods of an f in $C_p(X)$ have the form $B(f, F, \epsilon)$ where F is finite and $\epsilon > 0$). Write $C_k(X)$ for $C(X)$ with the compact-open topology (so basic neighborhoods of an f in $C_p(X)$ have the form $B(f, K, \epsilon)$ where K is compact and $\epsilon > 0$).

The spaces $C_p(X)$ and $C_k(X)$ are connected to $\mathcal{K}(X)$. For $C_k(X)$ this is evident from the definition of the basic open sets, and the connection is very tight and topological.

Let Z be a space, and z a point in Z. Write $\mathcal{T}_z^Z$ for the family of all neighborhoods of z in Z ordered by reverse inclusion. The next lemma is a simple preservation result for $\mathcal{T}_z^Z$.

Lemma 108. *If f is a continuous open surjection of X to Y, then for any x from X, we have $\mathcal{T}_x^X \geq_T \mathcal{T}_{f(x)}^Y$.*

Similarly, if Y embeds in X then, for any y from Y, we have $\mathcal{T}_y^X \geq_T \mathcal{T}_y^Y$.

Proof. Given f define $\phi_1 : \mathcal{T}_x^X \to \mathcal{T}_{f(x)}^Y$ by $\phi_1(U) = f(U)$. Note that ϕ_1 is well-defined because f is open and onto, and then by continuity of f, ϕ_1 is obviously a Tukey quotient, $\mathcal{T}_x^X \geq_T \mathcal{T}_{f(x)}^Y$.

If Y is a subspace of X and y is in Y, then define $\phi_2 : \mathcal{T}_y^X \to \mathcal{T}_y^Y$ by $\phi_2(U) = U \cap Y$. Again it is immediate that ϕ_2 witnesses $\mathcal{T}_x^X \geq_T \mathcal{T}_{f(x)}^Y$. $\square$

Lemma 109. *For any space X we have that $\mathcal{K}(X) \times \omega$ is Tukey equivalent to $\mathcal{T}_{\mathbf{0}}^{C_k(X)}$, where $\mathbf{0}$ is the constant zero function.*

If X is not strongly ω-bounded, then $\mathcal{K}(X)$ is Tukey equivalent to $\mathcal{T}_{\mathbf{0}}^{C_k(X)}$.

Proof. Observe first that $\mathcal{B} = \{B(\mathbf{0}, K, 1/n) : K \in \mathcal{K}(X),\ n \in \omega \backslash \{0\}\}$ is cofinal in $\mathcal{T}_{\mathbf{0}}^{C_k(X)}$. It is easy to check that $B(\mathbf{0}, K', 1/n') \subseteq B(\mathbf{0}, K, 1/n)$ if and only if $K \subseteq K'$ and $n \leq n'$, and hence $\mathcal{B}$ is clearly Tukey equivalent to $\mathcal{K}(X) \times \omega$. Now recall (Lemma 3) that if C is a cofinal subset of a directed set P then P and C are Tukey equivalent.

When X is not strongly ω-bounded, $\mathcal{K}(X)$ has countable additivity (Lemma 39), and $\mathcal{K}(X) =_T \mathcal{K}(X) \times \omega$ (Lemma 16(2)). $\square$

Recalling that $C_k(Y)$ is homogeneous, so $\mathcal{T}_{\mathbf{0}}^{C_k(Y)} =_T \mathcal{T}_f^{C_k(Y)}$ for every f from $C_k(Y)$, we combine the previous two lemmas.

Proposition 110. *Suppose X and Y are spaces such that either there is a continuous open surjection of $C_k(X)$ onto $C_k(Y)$ or $C_k(Y)$ embeds in $C_k(X)$.*

Then $\mathcal{K}(X) \times \omega \geq_T \mathcal{K}(Y) \times \omega$, and if neither X nor Y are strongly ω-bounded spaces then $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$.

Proposition 110, along with the $2^{\mathfrak{c}}$-sized antichain of Theorem 87 directly implies the following.

Theorem 111. *There is a $2^{\mathfrak{c}}$-sized family $\mathcal{A}$ of separable metrizable spaces such that whenever M, N are distinct elements of $\mathcal{A}$, then $C_k(M)$ is not the continuous open image of $C_k(N)$ and does not embed in $C_k(N)$.*

As in the case of $\mathcal{K}(Sub(\mathbb{R}))$, we may use antichains to derive families of pairwise non-homeomorphic $C_k(S)$-s. As the spaces that give the antichain in Theorem 98 are not ω-bounded we immediately get the following corollary.

Corollary 112. *There is 2^{ω_1}-sized family $\mathcal{S}$ of subsets of ω_1 such that if S and T are distinct elements of $\mathcal{S}$, $C_k(S)$ is not a continuous open image of $C_k(T)$ and does not embed in $C_k(T)$.*

The connection between $\mathcal{K}(X)$ and $C_p(X)$ is more indirect, and associated with the linear topological structure. The weak dual of $C_p(X)$ is denoted $L_p(X)$. The space X embeds in $L_p(X)$ as a closed subspace which is a Hamel basis. Let $\hat{X} = \bigoplus_{n\in\omega}(X^n \times \mathbb{R}^n)$. There is a natural continuous map $p : \hat{X} \to L_p(X)$, namely $p((x_1, \ldots, x_n), (\lambda_1, \ldots, \lambda_n)) = \sum_{i=1}^{n} \lambda_i x_i$. As X is a Hamel basis, p is surjective. (See [4] for proofs of all these claims about $C_p(X)$ and $L_p(X)$.)

Proposition 113. *Let X and Y be spaces.*

(1) If X is not strongly ω-bounded and there is a linear embedding of $C_p(Y)$ into $C_p(X)$ then $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$.

(2) If X and Y are metrizable and there is a continuous linear surjection of $C_p(X)$ onto $C_p(Y)$ then (a) $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$ and (b) $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$.

Proof. For claim (1), suppose $\psi : C_p(Y) \to C_p(X)$ is a linear embedding. Then the dual map $\psi^* : L_p(X) \to L_p(Y)$ is a continuous linear surjection. Since Y is a closed subspace of $L_p(Y)$, combining the map p from $\hat{X}$ onto $L_p(X)$, ψ^* and tracing down onto Y, it follows that $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X})) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$. We verify that $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X}) =_T (\mathcal{F}(X), \mathcal{K}(X))$.

Since X embeds as a closed set in $\hat{X}$, evidently $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X})) \geq_T (\mathcal{F}(X), \mathcal{K}(X))$. The reverse Tukey quotient also holds. To see this first define $\phi_1 : \mathcal{K}(X) \times \omega \to \mathcal{K}(\hat{X})$ by $\phi_1(K, n) = \bigoplus_{m\leq n}(K^m \times [-n, +n]^m)$. Then it is straightforward to verify ϕ_1 is a relative Tukey quotient of $(\mathcal{F}(X) \times \omega, \mathcal{K}(X) \times \omega)$ to $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X}))$. As X is not ω-bounded, $\mathcal{F}(X)$ has countable additivity in $\mathcal{K}(X)$ (Lemma 39), so according to Lemma 16, we have $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(X) \times \omega, \mathcal{K}(X) \times \omega)$. Combining these two relative reductions gives the claim.

For claim (2) we use [6]. Let ψ be a continuous linear surjection of $C_p(X)$ onto $C_p(Y)$.

For any y in Y, let ψ_y be the element of $L_p(X)$ obtained by setting $\psi_y(f) = \psi(f)(y)$. As X is a Hamel basis for $L_p(X)$ there is a finite set, $\operatorname{supp} y = \{x_1, \ldots, x_n\}$, of elements of X such that ψ_y is a linear combination of the x_i's. Lemma 3.1 of [6] says that if K is a compact subset of X then the set $\phi(K) = \{y \in Y : \operatorname{supp} y \subseteq K\}$ is a compact subset of Y. Clearly ϕ is an order-preserving map from $\mathcal{K}(X)$ to $\mathcal{K}(Y)$. The map ϕ is cofinal, since by Proposition 2.2 of [6], the set $\operatorname{supp} L = \bigcup\{\operatorname{supp} y : y \in L\}$ has a compact closure in X, for any $L \in \mathcal{K}(Y)$ and, clearly, $L \subseteq \phi(\overline{\operatorname{supp} L})$. This argument was given on the page 881 of [6]. We have proven part (a) of claim (2).

To establish part (b) of claim (2), we show $\phi(\mathcal{F}(X))$ is cofinal for $\mathcal{F}(Y)$ in $\mathcal{K}(Y)$. Take any finite subset G of Y. Set $F = \bigcup_{y \in G} \operatorname{supp} y$. Then F is a finite subset of X, and clearly, by definition of ϕ, we have $G \subseteq \phi(F)$. $\square$

Proposition 113, along with the $2^{\mathfrak{c}}$-sized antichain of Theorem 87 directly imply the following.

Theorem 114. *There is a $2^{\mathfrak{c}}$-sized family $\mathcal{A}$ of separable metrizable spaces such that whenever M, N are distinct elements of $\mathcal{A}$, then $C_p(M)$ is not the continuous linear image of $C_p(N)$ and does not linearly embed in $C_p(N)$.*

Marciszewski in his article in [32] gave an example of a $\mathfrak{c}$-sized family of compact metrizable spaces such that if M, N are distinct elements of the family then $C_p(M)$ and $C_p(N)$ are not linearly homeomorphic.

To get a similar result for $C_p(S)$-s, we need to prove a variant of Lemma 113 that works for subspaces of ω_1.

Lemma 115. *Let S and T be subsets of ω_1.*

(1) If S is not closed and there is linear embedding of $C_p(T)$ into $C_p(S)$ then $(\mathcal{F}(S), \mathcal{K}(S)) \geq_T (\mathcal{F}(T), \mathcal{K}(T))$.

(2) If S and T are co-stationary and there is a continuous linear surjection of $C_p(S)$ onto $C_p(T)$ then (a) $\mathcal{K}(S) \geq_T \mathcal{K}(T)$ and (b) $(\mathcal{F}(S), \mathcal{K}(S)) \geq_T (\mathcal{F}(T), \mathcal{K}(T))$.

Proof. Note that if S is not closed then it is not ω-bounded and Lemma 113 implies the first claim.

For the second claim, we may not use results from [6] since S and T do not have to be metrizable. But there are similar results for arbitrary spaces in [3]. We use the same definitions as in the proof of Lemma 113. Let X and Y be any spaces such that there is a continuous linear surjection from $C_p(S)$ onto $C_p(T)$. Then for any compact $L \subseteq Y$, $\overline{\operatorname{supp} L}$ is a compact subset of X and for any closed and functionally bounded $K \subseteq X$, $L = \{y \in Y : \operatorname{supp} y \subseteq K\}$ is closed and functionally bounded. Here $A \subseteq X$ is called functionally bounded if and only if $f(A)$ is bounded for any $f \in C_p(X)$. For subsets of co-stationary $S \subseteq \omega_1$ being closed and functionally bounded is equivalent to being compact. To see this, take a closed subset of S, say C. If C is not closed in ω_1, then C contains an increasing sequence that converges to a point outside S and we can find $f \in C_p(S)$ such that $f(C)$ is unbounded. Therefore, C must be closed in ω_1 and since S is co-stationary it must be bounded. So C is compact. Now the map $\phi : \mathcal{K}(S) \to \mathcal{K}(T)$ defined by $K \mapsto \{y \in Y : \operatorname{supp} y \subseteq K\}$ is well-defined, order-preserving and since for each $L \in \mathcal{K}(T)$, $L \subseteq \phi(\overline{\operatorname{supp} L})$, it is also cofinal.

Just as in the proof of Lemma 113, $\phi(\mathcal{F}(S))$ is cofinal for $\mathcal{F}(T)$ in $\mathcal{K}(T)$, which establishes part (b) of claim (2). □

Corollary 116. *There is 2^{ω_1}-sized family $\mathcal{S}$ of subsets of ω_1 such that if S and T are distinct elements of $\mathcal{S}$, then there is no linear surjection of $C_p(S)$ onto $C_p(T)$ and no linear embedding of $C_p(S)$ in $C_p(T)$.*

4.2 ORDER PROPERTIES OF $\mathcal{K}(X)$

There is a strong connection between the Lindelöf Σ property and the relative Tukey order. Recall that a space is Lindelöf Σ if it has a countable network modulo some compact cover ($\mathcal{W}$ is a network modulo $\mathcal{C}$ if and only if for each $C \in \mathcal{C}$ and open U with $C \subseteq U$, there exists $W \in \mathcal{W}$ such that $C \subseteq W \subseteq U$). Also recall another characterization of Lindelöf Σ spaces: a space X is Lindelöf Σ if and only if there is a separable metrizable space M and some space Z such that M is a perfect image of Z and X is a continuous image of Z. This equivalent condition immediately implies that $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$ [10].

Lemma 117. *If X is Lindelöf Σ, then there is a separable metrizable M such that $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$.*

Proof. Pick a separable metrizable M and a space Z such that there is a continuous onto map $f : Z \to X$ and a perfect map $g : Z \to M$. Define $\phi : \mathcal{K}(M) \to \mathcal{K}(X)$ by $\phi(K) = f(g^{-1}(K))$. Since g is perfect, $g^{-1}(K)$ is compact and therefore $\phi(K)$ is indeed an element of $\mathcal{K}(X)$. Since f is onto, the image of ϕ covers X and, clearly, ϕ is order-preserving. $\square$

Observe that if $\mathcal{K}(\omega) \geq_T (X, \mathcal{K}(X))$ then X is σ-compact, and hence easily seen to be Lindelöf Σ. However, in general the converse to the preceding lemma is not true. Indeed when we move up to the next level of the Tukey hierarchy, $\mathcal{K}(\omega^\omega)$, we know that $\mathcal{K}(\omega^\omega) \geq_T \mathfrak{b}$, and the ordinal space $\mathfrak{b}$ is not Lindelöf, and so not Lindelöf Σ.

Nevertheless it was proven in [11] that a weak converse of this lemma does hold: if there is a separable metrizable M with $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$, then X has a countable network modulo some cover of X with countably compact sets (X is 'almost' Lindelöf Σ).

Further, it was shown in [11] that when $X = C_p(Y)$ we have the full converse: there exists separable metrizable M with $\mathcal{K}(M) \geq_T (C_p(Y), \mathcal{K}(C_p(Y)))$ if and only in $C_p(Y)$ is Lindelöf Σ.

Next we investigate the situation with the full Tukey relation, $\mathcal{K}(M) \geq_T \mathcal{K}(X)$, rather than the relative case, $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$. To do so we introduce a natural strengthening of the Lindelöf Σ property, 'Lindelöf cofinally Σ'. We show $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$ for some separable metrizable M if and only if $C_p(X)$ is Lindelöf cofinally Σ. Then we answer a question of Cascales, Orihuela and Tkachuk, [11], by showing that if X is compact and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$ then X is countable.

In the last section we move from examining how the order structure of the compact subsets of $C_p(X)$ affects X, to the inverse problem: how the order structure of X impacts $\mathcal{K}(C_p(X))$.

4.2.1 Lindelöf Σ Property for $C_p(Y)$

Let a space X be called Lindelöf cofinally Σ if and only if it has a countable network modulo some compact cover that is *cofinal* in $\mathcal{K}(X)$. First we show that, as in the case of the Lindelöf Σ property, there is a condition equivalent to the Lindelöf cofinally Σ property that is closely related to the Tukey ordering.

Lemma 118. *A space X is Lindelöf cofinally Σ if and only there is a space Z and a separable metrizable M such that M is a perfect image of Z and X is a compact-covering image of Z.*

Proof. Suppose there is Z, separable metrizable M, a perfect $f : Z \to M$ and a compact-covering $g : Z \to X$. Suppose $\mathcal{B}$ is a countable base of M that is closed under finite unions and intersections. Let $\mathcal{C} = \{g(f^{-1}(K)) : K \in \mathcal{K}(M)\}$ and $\mathcal{W} = \{g(f^{-1}(B)) : B \in \mathcal{B}\}$. Then $\mathcal{C}$ is a cofinal subcollection of $\mathcal{K}(X)$ and $\mathcal{W}$ is a network modulo $\mathcal{C}$.

Now suppose X is Lindelöf cofinally Σ and $\mathcal{W}$ is a countable network modulo $\mathcal{C}$, where $\mathcal{C}$ is some cofinal subset of $\mathcal{K}(X)$. Let $D(\mathcal{W})$ be $\mathcal{W}$ with the discrete topology. Define $M = \{m \in D(\mathcal{W})^\omega : \ \exists\, C \in \mathcal{C},\ C = \bigcap\{m(n) : \ n \in \omega\}\}$. Then M is separable and metrizable. For each $m \in M$ pick $C_m \in \mathcal{C}$ with $C_m = \bigcap\{m(n) : \ n \in \omega\}$. Let βX be the Stone-Čech compactification of X and consider a subset of $M \times \beta X$, $Z = \bigcup_{m \in M}\{m\} \times C_m$. The space Z is closed in $M \times \beta X$. Since βX is compact, π_M is a closed map and therefore $f = \pi_M|Z$ is a perfect map. On the other hand, $g = \pi_X|Z$ is compact-covering since $g(\{m\} \times C_m) = C_m$ and the C_m's are cofinal in $\mathcal{K}(X)$. $\square$

Lemma 119. *If X is Lindelöf cofinally Σ, then there is a separable metrizable M with $\mathcal{K}(M) \geq_T \mathcal{K}(X)$.*

Proof. As in the proof of Lemma 117 pick separable metrizable M and a space Z such that there is a compact-covering map $f : Z \to X$ and a perfect map $g : Z \to M$. Define $\phi : \mathcal{K}(M) \to \mathcal{K}(X)$ by $\phi(K) = f(g^{-1}(K))$. As before, ϕ indeed maps into $\mathcal{K}(X)$ and ϕ is order-preserving. To show cofinality, let K be a compact subset of X. Since f is compact-covering there is $L \in \mathcal{K}(Z)$ such that $f(L) = K$. Then $g(L)$ is a compact subset of M and $\phi(g(L)) = f(g^{-1}(g(L))) \supseteq f(L) = K$. $\square$

Corollary 120. *$C_p(X)$ is Lindelöf cofinally Σ if and only if there is a separable metrizable M with $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$.*

Proof. We have already proven one direction. For the other direction, suppose $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$. Then by the proof of Theorem 2.15 of [11], νX is Lindelöf Σ (here νX is the realcompactification of X). Therefore by Proposition IV.9.10 of [4], every countably compact subset of $C_p(X)$ is compact. By the proof of Proposition 2.6 of [11], $C_p(X)$ has a countable network modulo some $\mathcal{C}$, where each element of $\mathcal{C}$ is countably compact and for each $K \in \mathcal{K}(C_p(X))$ there is $C \in \mathcal{C}$ with $K \subseteq C$. But then each element of $\mathcal{C}$ is compact as well and we are done. $\square$

We conclude this section by answering a question posed in [11]. Using this result and Corollary 120 we discover a Lindelöf cofinally Σ counterpart of a well-known result about Gul'ko compacta. Recall that one of the many characterizations of Gul'ko compact spaces is given in terms of Lindelöf Σ property: a compact space K is Gul'ko if and only if $C_p(K)$ is Lindelöf Σ. We will show that a compact space K is countable if and only if $C_p(K)$ is Lindelöf cofinally Σ.

It was proven in [11] that, under CH, if X is compact and $\mathcal{K}(M) \geq \mathcal{K}(C_p(X))$, then X is countable. The authors asked if this was true in ZFC. We give a positive answer to this question. This question was answered independently and using a different approach in [29].

Theorem 121. *In ZFC, if X is compact, M is separable metrizable and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$, then X is countable.*

Proof. We extract the part of the proof of Theorem 3.10 of [11] that does not use CH. Here is the sketch of it: suppose $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$, then $C_p(X)$ is Lindelöf Σ and therefore X is Gul'ko compact. First, we show that X has to be scattered. Suppose X is not scattered, and pick a countable $A \subseteq X$ with no isolated points. Then by Theorem 7.21 and Theorem 4.1 from [32], $K = \overline{A}$ is compact, second countable, metrizable and $C_p(K)$ embeds as a closed subspace in $C_p(X)$. Therefore $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(K))$.

By Theorem 3.6 in [11] and Proposition 10.7 from [47] whenever $iw(C_p(X)) \leq \omega$ and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$, X has to be countable. Here $iw(Z)$ is defined to be the smallest cardinality of a coarser Tychonoff topology on Z, and $nw(Z)$ is defined to be the smallest cardinality of a network modulo the set $\{\{z\} : z \in Z\}$. We know that $nw(C_p(K)) = nw(K)$ and since K is metrizable $nw(K) = \omega$. Since $iw(C_p(K)) \leq nw(C_p(K)) = \omega$, K is countable. But since K is a compact set with no isolated points, it has to be uncountable. This shows that X must be scattered.

If D is the set of all isolated points in X, D is open and since X is scattered $\overline{D} = X$. If D is countable then $iw(C_p(X)) = d(X) = \omega$ and X is countable (here $d(X)$ is the cardinality of the smallest dense subset of X). So let D be uncountable and for what follows we may assume $|D| = \omega_1$.

Consider $F = X \backslash D$. Then F is closed in X. Let Y be a quotient space of X with F shrunk to a point. Then Y is a closed continuous image of X and therefore $C_p(Y)$ embeds as a closed subspace into $C_p(X)$. Thus $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(Y))$.

Since X is compact, Y is also compact and F is the only isolated point of Y. Therefore $Y = A(\omega_1)$. But $C_p(A(\omega_1))$ is homeomorphic to $\Sigma_*(\mathbb{R}^{\omega_1}) = \{(x_\alpha)_\alpha : \forall \epsilon > 0,\ \{\alpha : |x_\alpha| \geq \epsilon\}$ is finite$\}$. So $\mathcal{K}(M) \geq_T \mathcal{K}(\Sigma_*(\mathbb{R}^{\omega_1}))$.

So from [11], we get the following: in ZFC, if X is compact, uncountable and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$ for some separable metrizable M, then $\mathcal{K}(M) \geq_T \mathcal{K}(\Sigma_*(\mathbb{R}^{\omega_1}))$. To complete the proof, recall that $\mathcal{K}(M)$ has calibre (ω_1, ω) and that $\mathcal{K}(M) \geq_T \mathcal{K}(\Sigma_*(\mathbb{R}^{\omega_1}))$ implies that $\mathcal{K}(\Sigma_*(\mathbb{R}^{\omega_1}))$ must have calibre (ω_1, ω) as well. The next lemma gives the desired contradiction. $\square$

Lemma 122. *There exists an uncountable $\mathcal{K} \subseteq \mathcal{K}(\Sigma_*(\mathbb{R}^{\omega_1}))$ such that each countably infinite subset of $\mathcal{K}$ is unbounded in $\mathcal{K}(\mathbb{R}^{\omega_1})$. Hence $\mathcal{K}(\Sigma_*(\mathbb{R}^{\omega_1}))$ fails to have calibre (ω_1, ω).*

Proof. Let $\Sigma_* = \Sigma_*(\mathbb{R}^{\omega_1})$. We construct $\mathcal{K}$ as follows. For each infinite $\alpha \in \omega_1$, let $\mathcal{C}_\alpha = \{n\chi_{\{\omega^\alpha\}} : n \in \omega\} \subseteq \Sigma_*$ and let $f_\alpha : \mathcal{C}_\alpha \to \alpha$ be a bijection. For any infinite subset of $\mathcal{C}_\alpha$, its projection on the ω^α-th coordinate is infinite, so it cannot be contained in a compact set.

For each $\beta \in \omega_1$, let $K_\beta = \{\mathbf{0}\} \bigcup \{f_\alpha^{-1}(\beta) : \beta \in \alpha\}$. Clearly, all K_β's are distinct. For each K_β, elements of K_β have disjoint supports. Therefore for each K_β and each open set $U \subseteq \Sigma_*$ that contains $\mathbf{0}$, all but finitely many elements of K_β are in U. Therefore each K_β is a compact subset of Σ_*.

Suppose $(\beta_n)_{n\in\omega} \subseteq \omega_1$ is a strictly increasing sequence. There is $\alpha \in \omega_1$ such that $(\beta_n)_{n\in\omega} \subseteq \alpha$. Then for each $n \in \omega$ we have $f_\alpha^{-1}(\beta_n) \in K_{\beta_n}$. Also, for each $n \in \omega$, $f_\alpha^{-1}(\beta_n) \in \mathcal{C}_\alpha$. So $\bigcup_{n\in\omega} K_{\beta_n}$ contains an infinite subset of $\mathcal{C}_\alpha$ and, therefore, cannot be contained in a compact set. $\square$

Now by Corollary 120 we can re-phrase Theorem 121 as follows.

Theorem 123. *For a compact space X, $C_p(X)$ is Lindelöf cofinally Σ if and only if X is countable.*

4.2.2 Baturov's Theorem

Lindelöf Σ spaces have been widely studied and there are many interesting theorems about these spaces. It is natural to ask whether in any of the theorems the condition 'X is Lindelöf Σ' can be weakened to '$\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$'. Here we consider one particular well-known theorem, Baturov's theorem, and show that, at least consistently, such substitution is not possible. Recall that the extent of a space is the supremum of cardinalities of closed discrete subspaces. The lindelöf number of a space X is the least cardinal κ such that every open cover of X has a subcover of size $\leq \kappa$. The extent is less than or equal to the Lindelöf number for any space.

Theorem 124 (Baturov)**.** *Suppose X is Lindelöf Σ and $Y \subseteq C_p(X)$. Then the Lindelöf number and the extent of Y are equal.*

In [9] it was proven that for $W = \{\alpha \leq \omega_2 : \mathrm{cof}(\alpha) \neq \omega_1\}$, the extent of $C_p(W)$ is equal to ω while the Lindelöf number of $C_p(W)$ is ω_2. We show that consistently there exists a separable metrizable M such that $\mathcal{K}(M) \geq_T (W, \mathcal{K}(W))$.

Recall X is called strongly ω-bounded if and only if for any $\{K_n\}_{n\in\omega} \subseteq \mathcal{K}(X)$ there is $K \in \mathcal{K}(X)$ such that $\bigcup_{n\in\omega} K_n \subseteq K$. Note that if X is strongly ω-bounded then $\mathcal{K}(X)$ has calibre (ω_1, ω) and therefore $\mathcal{K}(X)$ is a good candidate to sit below some $\mathcal{K}(M)$ in the Tukey order.

Lemma 125. *Let $W = \{\alpha \in \omega_2 + 1 : \mathrm{cof}(\alpha) \neq \omega_1\}$. Then W and $W \backslash \{\omega_2\}$ are strongly ω-bounded.*

Proof. Suppose $\{K_n\}_{n\in\omega} \subseteq \mathcal{K}(W)$. Let $\beta \in (\omega_2+1)\backslash W$. Then for each n there is $\beta_n \in \beta$ such that $K_n \cap [\beta_n, \beta] = \emptyset$. Since $\mathrm{cof}(\beta) = \omega_1$, $\beta' = \sup\{\beta_n : n \in \omega\} \in \beta$. Let $U_\beta = [\beta', \beta]$. Then U_β is open in ω_2+1 and $U_\beta \cap \bigcup_{n\in\omega} K_n = \emptyset$. Let $U = \bigcup_{\beta\in\omega_2+1\backslash W} U_\beta$. Then U is open in ω_2+1, so $K = \omega_2+1\backslash U$ is a closed, hence compact, subset of ω_2+1 and $\bigcup_{n\in\omega} K_n \subseteq K \subseteq W$.

To show that $W\backslash\{\omega_2\}$ is strongly ω-bounded, again, take $\{K_n\}_{n\in\omega} \subseteq \mathcal{K}(W\backslash\{\omega_2\})$ and construct U the same way. Let $C = \omega_2\backslash U$. Again $\bigcup_{n\in\omega} K_n \subseteq C \subseteq W\backslash\{\omega_2\}$ and C is closed in ω_2 but perhaps not compact. Since each $K_n \in \mathcal{K}(\omega_2)$, it is bounded and thus $\bigcup_{n\in\omega} K_n$ is bounded by some $\alpha \in \omega_2$. So we can let $K = C \cap [0, \alpha] \in \mathcal{K}(W\backslash\{\omega_2\})$. (Actually, C has to be bounded because if it is not, then it is a cub set that does not intersect $\omega_2\backslash W$, which is stationary.) □

Next we establish a connection between strongly ω-bounded spaces and $\mathfrak{c}$-sized totally imperfect separable metrizable spaces.

Lemma 126. *Suppose X is strongly ω-bounded, $\mathrm{cof}(\mathcal{K}(X)) \leq \mathfrak{c}$ and B is a $\mathfrak{c}$-sized totally imperfect separable metrizable space. Then $\mathcal{K}(B) \geq \mathcal{K}(X)$.*

Proof. Let $\{K_x : x \in B\}$ be a cofinal subset of $\mathcal{K}(X)$. Define $\phi : \mathcal{K}(B) \to \mathcal{K}(X)$ by $C \mapsto \overline{\bigcup_{x\in C} K_x}$. This works because compact subsets of B are countable and X is strongly ω-bounded. The map ϕ is clearly order-preserving and since $\{K_x : x \in B\}$ is cofinal so is ϕ. □

The next lemma shows that under $\mathfrak{c} = 2^{\omega_1}$, we cannot weaken the hypothesis of the Baturov's theorem from 'X in Lindelöf Σ' to 'there is a separable metrizable M with $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$'.

Lemma 127. *Let $W = \{\alpha \in \omega_2+1 : \mathrm{cof}(\alpha) \neq \omega_1\}$. Then $\omega_2 \leq \mathrm{cof}(\mathcal{K}(W)) \leq 2^{\omega_1}$. Therefore, if $\mathfrak{c} = 2^{\omega_1}$ then $\mathcal{K}(B) \geq \mathcal{K}(W)$. But, under CH, there is no separable metrizable M with $\mathcal{K}(M) \geq \mathcal{K}(W)$.*

Proof. That $\mathrm{cof}(\mathcal{K}(W)) \geq \omega_2$ follows immediately from the fact that for every $K \in \mathcal{K}(W)$, $K\backslash\{\omega_2\}$ is bounded in ω_2. Indeed, suppose not, then since K is closed in ω_2+1, $K\backslash\{\omega_2\}$ is closed and unbounded in ω_2 and misses stationary $\omega_2\backslash W$.

On the other hand, if $K \in \mathcal{K}(W)$ then there is $\alpha \in \omega_2$ such that $K \subseteq [0,\alpha] \cup \{\omega_2\}$. Otherwise, K would contain a strictly increasing sequence $\{x_\alpha : \alpha < \omega_1\}$. Then, since the limit of $\{x_\alpha : \alpha < \omega_1\}$ has cofinality ω_1, it cannot be an element of W or K. Each $[0,\alpha]$ has 2^{ω_1}-many compact subsets and therefore $\mathrm{cof}(\mathcal{K}(W)) \leq \omega_2 \times 2^{\omega_1} = 2^{\omega_1}$.

The previous lemma and Lemma 5 imply the last two conclusions. $\square$

5.0 OPEN PROBLEMS

The results in this work have suggested new directions of research and have raised new questions. Among the most prominent are:

- Investigate the possible values of the spectrum of $(\omega^\omega, \leq^*)$;
- Investigate $\mathcal{K}(M)$ when M contains the Cantor set, and determine the position of $\mathcal{K}(\mathbb{Q})$ in $\mathcal{K}(Sub(\mathbb{R}))$;
- Determine what other posets embed in $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$;
- Investigate X when $P \geq_T (X, \mathcal{K}(X))$ and P is 'nice';
- Use antichains of $\mathcal{K}(Sub(\mathbb{R}))$ to construct large 'antichains' of Gul'ko compacta.

We discuss each of these problems in more detail.

Infinite sets realized as $\text{spec}((\omega^\omega, \leq^*))$**.** We saw that $\text{spec}((\omega^\omega, \leq^*))$ is fundamental for all spectrum calculations and we would like to know what it can possibly be. From Theorem 60, all finite sets of uncountable regular cardinals can be realized as $\text{spec}((\omega^\omega, \leq^*))$. Now let I be a countably infinite collection of uncountable regular cardinals. In the proof of Theorem 60 we defined $A = \prod\{\kappa : \kappa \in I\}$ and, using the fact that it is consistent to embed A cofinally into $(\omega^\omega, \leq^*)$, we were allowed to work with $\text{spec}(A)$ instead. The poset A was a natural choice since it is clear $I \subseteq \text{spec}(A)$. Another reason why the poset A is a natural choice is the following: Lemma 14 and the fact that $(\omega^\omega, \leq^*)$ is countably additive imply that if $I \subseteq \text{spec}((\omega^\omega, \leq^*))$ then $A \leq_T (\omega^\omega, \leq^*)$. So the main question is: what is $\text{spec}(A)$?

Properties of $\mathcal{K}(M)$ **when** M **contains the Cantor set, and the position of** $\mathcal{K}(\mathbb{Q})$**.** The main goal of this work, understanding the cofinal structure of $\mathcal{K}(Sub(\mathbb{R}))$, was largely achieved through totally imperfect spaces. Therefore, understandably, we paid special attention to these spaces, and studied the internal structure and Tukey relations of posets associated with these spaces.

However, spaces that contain the Cantor set open new and exciting avenues of research as well. One source of questions about these spaces is the problem of determining the position of $\mathcal{K}(\mathbb{Q})$ in $\mathcal{K}(Sub(\mathbb{R}))$. We know that $\mathcal{K}(1) <_T \mathcal{K}(\omega) <_T \mathcal{K}(\omega^\omega)$ and $\mathcal{K}(\mathbb{Q})$ lies strictly above $\mathcal{K}(\omega^\omega)$. It seems reasonable to conjecture that $\mathcal{K}(\mathbb{Q})$ might be the unique immediate successor of $\mathcal{K}(\omega^\omega)$. There are two types of potential counter-example to this conjecture: (1) there is a separable metrizable M such that $\mathcal{K}(\omega^\omega) <_T \mathcal{K}(M) <_T \mathcal{K}(\mathbb{Q})$, or (2) there is an M such that $\mathcal{K}(M)$ is Tukey incomparable with $\mathcal{K}(\mathbb{Q})$.

It was proven in [27] that for every uncountable totally imperfect B, $\mathcal{K}(B) \geq_T \mathcal{K}(\mathbb{Q})$, so examples of type (1) or (2) must contain (many) Cantor sets.

We know that there *is* a counter-example to the conjecture. Indeed the non-Polish space, X, from Corollary 72, with the property that $\mathcal{K}(X)$ has calibre ω_1 provided that $\omega_1 < \mathfrak{p}$, has $\mathcal{K}(X)$ strictly above $\mathcal{K}(\omega^\omega)$, but $\mathcal{K}(\mathbb{Q}) \not\leq_T \mathcal{K}(X)$. Oddly we do not know if $\mathcal{K}(X)$ is of type (1) or (2)!

Are there type (1) examples? Are there type (2) examples? Where does the space X sit and is it type (1) or (2)? The space X is a consistent example. Are there examples in ZFC? Does the existence of an example of type (1), or (2), or with the special properties of X imply some small cardinal inequality?

More on $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$. By embedding $\mathcal{P}(\omega)$ into $\mathcal{K}(Sub(\mathbb{R}))$, we showed that every countable partially ordered set also embeds into $\mathcal{K}(Sub(\mathbb{R}))$. Since all bounded subsets of $\mathcal{K}(Sub(\mathbb{R}))$ are $\mathfrak{c}$-sized, $\mathcal{P}(\mathbb{R})$ cannot embed in $\mathcal{K}(Sub(\mathbb{R}))$ and we wonder what posets of size $\mathfrak{c}$ embed in $\mathcal{K}(Sub(\mathbb{R}))$. Does every poset of size $\leq \mathfrak{c}$ embed in $\mathcal{K}(Sub(\mathbb{R}))$? Does ω_1 with the reverse order embed in $\mathcal{K}(Sub(\mathbb{R}))$? What about $\mathfrak{c}$ with the reverse order?

One more question suggested by the somewhat discrete structure of $\mathcal{K}(Sub(\omega_1))$ is the following. *Above* ω^ω are there, in ZFC, *gaps*: M_0, M_1 such that $\mathcal{K}(M_0) <_T \mathcal{K}(M_1)$ but for no N do we have $\mathcal{K}(M_0) <_T \mathcal{K}(N) <_T \mathcal{K}(M_1)$?

Subposets of $\mathcal{K}(Sub(\omega_1))$ are largely unknown. We know that there are 2^{ω_1}-many Tukey classes associated with stationary co-stationary subsets of ω_1 but we do not know how this part of $\mathcal{K}(Sub(\omega_1))$ is structured. What partial orders can or cannot embed in $\mathcal{K}(Sub(\omega_1))$?

As for comparing elements of $\mathcal{K}(Sub(\mathbb{R}))$ and $\mathcal{K}(Sub(\omega_1))$ there are still a few interesting questions left. We know that $\mathcal{K}(\mathbb{Q}) \leq_T [\omega_1]^{<\omega} \times \omega^\omega$ and $\mathcal{K}(B) \leq_T [\omega_1]^{<\omega} \times \omega^\omega$ for every totally imperfect space of size ω_1. For what other M do we have $\mathcal{K}(M) \leq_T [\omega_1]^{<\omega} \times \omega^\omega$? Is it possible to have $\mathcal{K}(M) \leq_T \Sigma(\omega^{\omega_1})$? Do we have $\mathcal{K}(\mathbb{Q}) \leq_T \Sigma(\omega^{\omega_1})$?

Investigating X when $P \geq_T (X, \mathcal{K}(X))$ and P is 'nice'. When $P = \mathcal{K}(M)$ for some separable metrizable M, we already know that $P \geq_T (X, \mathcal{K}(X))$ implies that X is very close to being Lindelöf Σ. We would like to weaken the condition '$P = \mathcal{K}(M)$' to 'P has calibre (ω_1, ω)'. One motivation for doing so is that the proof of Theorem 121 relies heavily on the fact that $\mathcal{K}(M)$ has calibre (ω_1, ω), which makes us wonder whether calibre (ω_1, ω) is all that is required. We would like to single out an internal property of X that is equivalent to 'there exists P with calibre (ω_1, ω) such that $P \geq_T (X, \mathcal{K}(X))$'.

Another reason to weaken the condition '$P = \mathcal{K}(M)$' to 'P has calibre (ω_1, ω)' is related to certain collections of compact subsets arising in Analysis. Recall the characterizations of some special compact subsets of Banach spaces: Eberlein, Talagrand and Gul'ko compacta. All three of these classes have been characterized as those that embed in $C_p(X)$ for some X with the property that $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$. In particular, Eberlein compacta are precisely the compact spaces that embed in $C_p(X)$ when X is compact, or $\mathcal{K}(\mathbf{1}) \geq_T (X, \mathcal{K}(X))$; Talagrand compacta are the ones that embed in $C_p(X)$ for some X with $\mathcal{K}(\omega^\omega) \geq_T (X, \mathcal{K}(X))$; Gul'ko compacta are the ones embedded in $C_p(X)$ with $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$ for some M. A natural next step in this sequence is to consider compact subsets of $C_p(X)$ where $P \geq_T (X, \mathcal{K}(X))$ for some poset P with calibre (ω_1, ω). Suppose K is a compact subset of such $C_p(X)$: is it Fréchet-Urysohn? If K is separable then is it metrizable? Is it Corson? We know that in case of Eberlein, Talagrand and Gul'ko compacta we can choose X so that it has only one non-isolated point. Is it true that if K is as above then K embeds in $C_p(X(p))$ where $P' \geq_T (X(p), \mathcal{K}(X(p)))$ for some P' with calibre (ω_1, ω) and $X(p)$ that has only one non-isolated point?

In [40] Eberlein, Gul'ko and other special compacta were studied using elementary submodel techniques. We would like to use the techniques developed in Appendix B to study the class of compact spaces associated with posets with calibre (ω_1, ω).

'Incomparable' Gul'ko compacta. We already saw that there is a close connection between Gul'ko compacta and the relative Tukey order. A further connection established in [25] motivates an attempt to use a $2^{\mathfrak{c}}$-sized antichain in $(Sub(\mathbb{R}), \mathcal{K}(Sub(\mathbb{R})))$ to construct large families of Gul'ko compacta that are in some sense 'incomparable'. In [5] powerful machinery was developed for constructing Gul'ko compacta of high complexity. It was shown in [25] that the complexity can be restated in relative Tukey order terms and, using the $\mathfrak{c}^+$-sized chain in $\mathcal{K}(Sub(\mathbb{R}))$, one can derive a $\mathfrak{c}^+$-sized 'chain' of Gul'ko compacta, where spaces that come later in the chain are strictly more complex than the spaces that come earlier. All Gul'ko compacta involved in these arguments have weight $\mathfrak{c}$, and therefore it is natural to attempt a construction of a $2^{\mathfrak{c}}$-sized 'antichain' of Gul'ko compacta.

APPENDIX A

STRENGTHENINGS OF ARC-CONNECTEDNESS

The work on strengthenings of arc-connectedness was done in collaboration with Benjamin Espinoza, Paul Gartside and Merve Kovan-Bakan. Two papers were written on this subject — one is published and the other is accepted [17, 18]. Results included here are excerpts from these papers.

A.1 n-ARC CONNECTEDNESS, $\aleph_0$-ARC CONNECTEDNESS

A topological space X is called *n–arc connected* (n–ac) if for any points $p_1, p_2, \ldots, p_n$ in X, there exists an arc α in X such that $p_1, p_2, \ldots p_n$ are all in α. Here an arc is a space homeomorphic to $[0,1]$. If a space is n–ac for all $n \in \mathbb{N}$, then we will say that it is *ω–ac*. Note that this is equivalent to saying that for any finite F contained in X there is an arc α in X containing F. Call a space $\aleph_0$–ac if for every countable subset, S, there is an arc containing S. Evidently a space is arc connected if and only if it is 2–ac, and '$\aleph_0$–ac' implies 'ω–ac' implies '$(n+1)$–ac' implies 'n–ac' (for any fixed n).

Thus we have a family of natural strengthenings of arc connectedness, and the main aim of this section is to characterize when 'nice' spaces have one of these strong arc connectedness properties. Secondary aims are to distinguish 'n–ac' (for each n), 'ω–ac' and '$\aleph_0$–ac', and to compare and contrast the familiar arc connectedness (i.e. 2–ac) with its strengthenings.

Observe that any Hausdorff image of an n–ac (respectively, ω–ac, $\aleph_0$–ac) space under a continuous injective map is also n–ac (respectively, ω–ac, $\aleph_0$–ac). Below, unless explicitly stated otherwise, all spaces are continua — compact connected metric spaces.

It turns out that 'sufficiently large' (in terms of dimension) arc connected spaces tend to be ω–ac. Indeed, it is not hard to see that manifolds (with or without boundary) of dimension at least 2 are ω–ac. Thus we focus on curves (1–dimensional continua) and especially on graphs (those connected spaces obtained by taking a finite family of arcs and then identifying some of the endpoints).

To motivate our main results consider the following examples.

(A) The arc is $\aleph_0$–ac.

(B) The open interval, $(0, 1)$; and ray, $[0, 1)$, are ω–ac.

(C) From (A) and (B), all continua which are the continuous injective images of the arc, open interval and ray are ω–ac. It is easy to verify that these include: (a) the arc, (b) the circle, (c) figure eight curve, (d) lollipop, (e) dumbbell and (f) theta curve.

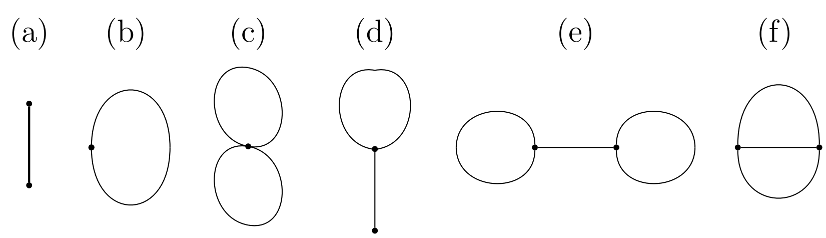

Figure 4: ω-ac graphs

(D) The Warsaw circle; double Warsaw circle; Menger cube; and Sierpinski triangle, are ω–ac.

(E) The simple triod is 2–ac but not 3–ac. It is minimal in the sense that no graph with strictly fewer edges is 2–ac not 3–ac.

The graphs (a), (b) and (c) below are: 3–ac but not 4–ac, 4–ac but not 5–ac, and 5–ac but not 6–ac, respectively. All are minimal.

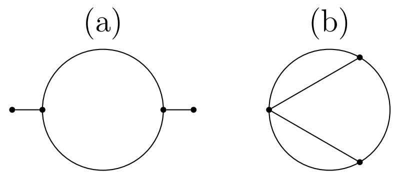

Figure 5: (a) 3-ac, not 4-ac; (b) 4-ac, not 5-ac; (c) 5-ac, not 6-ac

(F) The Kuratowski graph $K_{3,3}$ is 6–ac but not 7–ac. It is also minimal.

(G) The graphs below are all 6–ac and, by Theorem 128, none is 7–ac. Unlike $K_{3,3}$ all are planar. It is unknown if the first of these graphs (which has 12 edges) is minimal among planar graphs. A minimal example must have at least nine edges.

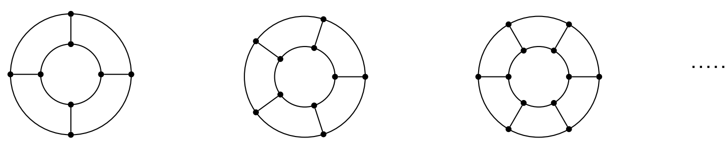

Figure 6: 6-ac, not 7-ac graphs

In this section we will characterize the ω–ac graphs and characterize the $\aleph_0$–ac continua by proving the following theorems.

Theorem 128. *For a graph G the following are equivalent:*

(1) G is 7–ac,

(2) G is ω–ac,

(3) G is the continuous injective image of a sub–interval of the real line,

(4) G is one of the following graphs: the arc, simple closed curve, figure eight curve, lollipop, dumbbell or theta curve.

Theorem 129. *For any continuum K (not necessarily metrizable) the following are equivalent:*

(1) K is $\aleph_0$–ac,

(2) K is a continuous injective image of a closed sub–interval of the long line,

(3) K is one of: the arc, the long circle, the long lollipop, the long dumbbell, the long figure eight or the long theta-curve.

A.1.1 Characterization of ω–ac Graphs

As noted in Example (C) the graphs listed in part (4) of Theorem 128 are all the continuous injective image of a closed sub–interval of the real line, giving (4) implies (3), and all such images are ω–ac, yielding (3) implies (2) of Theorem 128. Clearly ω–ac graphs are 7–ac, and so (2) implies (1) in Theorem 128.

It remains to show (1) implies (4) in Theorem 128, in other words that any 7–ac graph is one of the graphs listed in (4). This is established in Theorem 144 below. We proceed by establishing an ever tightening sequence of restrictions on the structure of 7–ac graphs.

Proposition 130. *Let G be a finite graph, and let $H \subseteq G$ be a subgraph of G such that $G - H$ is connected, $\overline{G - H} \cap H = \{r\}$ and r is a branch point of G. If G is n–ac, then $\overline{G - H}$ is n–ac.*

Proof. First note that $\overline{G - H} = (G - H) \cup \{r\}$. Hence every connected set intersecting $G - H$ and $H - \{r\}$, must contain r.

Let $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$ be a set of n points in $\overline{G - H}$. Then, since G is n–ac, there exists an arc α in G containing $\mathcal{P}$. If $\alpha \subseteq \overline{G - H}$, we are done. So assume α intersects $H - \{r\}$. Let $t_0, t_1 \in [0, 1]$ such that $\alpha(t_0) \in G - H$ and $\alpha(t_1) \in H - \{r\}$, assume without loss of generality that $t_0 < t_1$. Hence there exists $s \in [t_0, t_1]$ such that $\alpha(s) = r$. Then $\alpha([0, s])$ is an arc in $\overline{G - H}$ containing $\mathcal{P}$, otherwise $r \in \alpha((s, 1])$ which is impossible since α is an injective image of $[0, 1]$. This proves that $\overline{G - H}$ is n–ac. □

The reverse implication of Proposition 130 does not hold. To see this, let G be a simple triod and H be one of the edges of G. Clearly G is not 3–ac but $\overline{G - H}$ (an arc) is 3–ac.

Definition 131. *Let G be a finite graph. An edge e of G is called a terminal edge of G if one of the vertices of e is an end–point of G.*

Definition 132. *Let G be a finite graph, and let $I = \{e_1, e_2, \dots e_m\}$ be the set of terminal edges of G. Let G^* be the graph given by $\overline{G - I}$. Clearly this operation can be applied to G^* as well. We perform this operation as many times as necessary until we obtain a graph G' having no terminal edges. We called the graph G' the reduced graph of G.*

The following is a corollary of Proposition 130.

Corollary 133. *Let G be an n–ac finite graph. Then the reduced graph of G is an n–ac finite graph containing no terminal edges.*

Proof. Observe that the reduced graph of G can also be obtained by removing terminal edges one at a time.

Now, from Proposition 130, if G is an n–ac finite graph and e is a terminal edge of G, then $\overline{G - e}$ is n–ac. This implies that each time we remove a terminal edge we obtain an n–ac graph. This and the observation prove the corollary. □

Remark 1. *Note that if X is an n–ac space and $\{p_1, p_2, \dots, p_n\}$ are n different points of X, then there is an arc α such that $\{p_1, p_2, \dots, p_n\} \subseteq \alpha$ and such that the end–points of α belong to $\{p_1, p_2, \dots, p_n\}$. To see this, let β be the arc containing $\{p_1, p_2, \dots, p_n\}$, given by the fact that X is n–ac. Let $t_0 = min\{\beta^{-1}(p_i) \mid i = 1, \dots, n\}$ and $t_1 = max\{\beta^{-1}(p_i) \mid i = 1, \dots, n\}$. Then $\beta([t_0, t_1])$ satisfies the conditions of α.*

From now on, if X is an n–ac space, $\{p_1, \dots, p_n\}$ are n different points and α is an arc passing through $\{p_1, p_2, \dots, p_n\}$, then we will assume that the end–points of α belong to $\{p_1, p_2, \dots, p_n\}$.

Lemma 134. *Let G be a finite graph. Assume that G contains a simple triod $T = L_1 \cup L_2 \cup L_3$ (with $\{q\} = L_i \cap L_j$, for $i \neq j$) such that for each i, $L_i - \{q\}$ contains no branch points of G. For each $i = 1, 2, 3$, let $p_i \in int(L_i)$. If α is an arc containing $\{p_1, p_2, p_3\}$, then*

(1) $q \in int(\alpha)$, and

(2) at least one of the end points of α lies in $[q, p_1] \cup [q, p_2] \cup [q, p_3]$.

Proof. Let G, T and p_1, p_2, p_3 as in the hypothesis of the lemma. Let $\alpha \subseteq G$ be an arc containing $\{p_1, p_2, p_3\}$, and denote, for each $i = 1, 2, 3$, by $[q, l_i]$ the arc L_i.

(1) Assume, without loss of generality, that $\alpha(t_i) = p_i$ and that $t_1 < t_2 < t_3$. Then $p_2 \in \operatorname{int}(\alpha)$ and $\alpha = \alpha([0, t_2]) \cup \alpha([t_2, 1])$.

We consider two cases: $q \notin \alpha([0, t_2])$ and $q \in \alpha([0, t_2])$. Assume $q \notin \alpha([0, t_2])$, then, since $L_2 - \{q\}$ contains no branch points of G and $p_2 \in \operatorname{int}(L_2)$, we have that $l_2 = \alpha(s)$ for some s with $0 < s < t_2$. Hence $[l_2, p_2] \subseteq \alpha([0, t_2])$. Therefore, since $\{p_2, p_3\} \subseteq \alpha([t_2, 1])$, $p_2 \in \operatorname{int}(L_2)$, $L_2 - q$ has no branch points of G, and α is a $1-1$ function, we have that $[p_2, q] \subseteq \alpha([t_2, 1))$. This implies that $q \in \operatorname{int}(\alpha)$.

Now suppose that $q \in \alpha([0, t_2])$. If $q \in \alpha((0, t_2])$, then we are done. So assume that $q = \alpha(0)$, i.e. q is an end-point of α. Using the same argument as in the previous case, we can conclude that $[l_2, p_2] \subseteq \alpha([0, t_2])$. This implies, as before, that $[p_2, q] \subseteq \alpha([t_2, 1))$ wich contradicts the fact that α is a $1-1$ function. Hence $q \in \operatorname{int}(\alpha)$.

(2) First, assume that $\alpha(t_i) = p_i$ and that $t_1 < t_2 < t_3$. We will show that one end point of α lies on either $[q, p_1]$ or $[q, p_3]$. The other cases (rearrangements of the t_is) are done in the same way as this case, the only difference is the conclusion: the end point lies either on $[q, p_1]$ or $[q, p_2]$, or the end point lies either on $[q, p_2]$ or $[q, p_3]$.

By (1), $q \in \operatorname{int}(\alpha)$ and if $q = \alpha(s)$, then $s < t_3$; otherwise the arc $\alpha([0, t_3])$ would contain p_1, p_2, p_3 and $q \notin \operatorname{int}(\alpha([0, t_3]))$ which is contrary to (1). Similarly, $t_1 < s$. Hence $t_1 < s < t_3$.

If $s < t_2$, then $p_1, q \notin \alpha([t_2, 1]) = \alpha([t_2, t_3]) \cup \alpha([t_3, 1])$. Now, since $L_3 - \{q\}$ has no branch points of G, $q \in \alpha([0, t_2])$, and $p_3 \in \operatorname{int}(L_3)$, we have $l_3 \in \alpha([t_2, t_3])$. Thus, since α is a $1-1$ function, $\alpha([t_3, 1]) \subseteq (q, p_3]$. This shows that $\alpha(1)$ lies in $[q, p_3]$.

If $t_2 < s$, then a similar argument using $-\alpha$ (α traveled in the opposite direction) shows that one of the end points of α lies on $[q, p_1]$.

$\square$

We obtain the following corollaries.

Corollary 135. *With the same conditions as in Lemma 134. If α is an arc containing $\{p_1, p_2, p_3\}$, and $q = \alpha(s)$, $p_i = \alpha(t_i)$ for $i = 1, 2, 3$, then $t_j < s < t_k$ for some $j, k \in \{1, 2, 3\}$.*

Proof. To see this, note that if q does not lie between two of the p_is, then either $s < t_i$ for all i, or $t_i < s$ for all i. Then either $\alpha([s,1])$ or $\alpha([0,s])$ are arcs containing $\{p_1, p_2, p_3\}$ for which q is an end-point, this contradicts (1) of Lemma 134. $\square$

Corollary 136. *Let G be a finite graph, and let $\{p_1, p_2, \ldots p_n\} \subseteq G$ be n different points. In addition, let α be an arc containing $\{p_1, p_2, \ldots p_n\}$, with end–points belonging to $\{p_1, p_2, \ldots p_n\}$. If there are three different indexes i, j, k such that p_i, p_j and p_k belong to a triod T satisfying the conditions of (Lemma 134), and such that $([q, p_i] \cup [q, p_j] \cup [q, p_k]) \cap \{p_1, p_2, \ldots p_n\} = \{p_i, p_j, p_k\}$, then either p_i, p_j or p_k is an end–point of α.*

Proof. By (2) of Lemma 134, at least one of the end points of α lies in $[q, p_i] \cup [q, p_j] \cup [q, p_k]$. Hence, since the end-points of α belong to $\{p_1, p_2, \ldots p_n\}$ and $([q, p_i] \cup [q, p_j] \cup [q, p_k]) \cap \{p_1, p_2, \ldots p_n\} = \{p_i, p_j, p_k\}$, one of p_i, p_j or p_k is an end-point of α $\square$

Proposition 137. *Let G be a finite graph. If G is 5–ac, then G has no branch point of degree greater than or equal to five.*

Proof. Assume, by contradiction, that G contains at least one branch point, q, of degree at least 5. Then, since G is a finite graph, G contains a simple 5-od, $T = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$, such that $\{q\} = L_i \cap L_j$ for $i \neq j$, and such that $L_i - \{q\}$ contains no branch points of G.

For each $i = 1, \ldots, 5$, let $p_i \in \mathrm{int}(L_i)$. Then, since G is 5–ac, there exists an arc $\alpha \subseteq G$ such that $\{p_1, p_2, \ldots, p_5\} \subseteq \alpha$. Note that T contains a triod satisfying the conditions of Lemma 134, hence $q \in \mathrm{int}(\alpha)$. Let $t_0 \in (0,1)$ be the point such that $\alpha(t_0) = q$. Then $\alpha - \{q\} = \alpha([0, t_0)) \cup \alpha((t_0, 1])$, and either $\alpha([0, t_0])$ or $\alpha([t_0, 1])$ contains three points out of $\{p_1, p_2, p_3, p_4, p_5\}$, note that q is an end–point of $\alpha([0, t_0])$ and of $\alpha([t_0, 1])$. Without loss of generality, suppose that $p_1, p_2, p_3 \subseteq \alpha([0, t_0])$; then L_1, L_2, L_3 and the corresponding p_is satisfy the conditions of Lemma 134 implying that any arc containing those points contains q in its interior, a contradiction, since q is an end point of $\alpha([0, t_0])$. This shows that G does not contain a branch point of degree greater than or equal to five. $\square$

From Proposition 137 we obtain the following corollaries.

Corollary 138. *Let G be a finite graph. If G is n–ac, for $n \geq 5$, then G has no branch point of degree greater than or equal to five.*

The following proposition is easy to prove.

Proposition 139. *Let G be a finite connected graph. If G has at least three branch points, then there is an arc α such that the end-points of α are branch points of G and all the points of the interior of α, except for one, are not branch points of G. So α contains exactly three branch points of G.*

Theorem 140. *A finite graph with three or more branch points cannot be 7–ac.*

Proof. Let G be a finite graph with at least three branch points.

By Proposition 139, there is an arc α in G containing exactly three branch points of G such that two of them are the end–points of α. Denote by q_1, q_2, and q_3 these branch points, and assume without loss of generality that q_1 and q_3 are the end-points of α.

Let p_3 be a point between q_1 and q_2, and let p_5 be a point between q_2 and q_3. Since G is a finite graph, we can find, in a neighborhood of q_1, two points p_1 and p_2 such that p_1, p_2, p_3 belong to a triod T_1 satisfying the conditions of Lemma 134, and such that q_1 is the branch point of T_1. Similarly, we can find a point p_4, in a neighborhood of q_2, such that p_3, p_4 and p_5 belong to a triod T_2 satisfying the conditions of Lemma 134, and such that q_2 is the branch point of T_2. Finally, we can find two points p_6, and p_7, in a neighborhood of q_3, such that p_5, p_6 and p_7 belong to a triod T_3 satisfying the conditions of Lemma 134, and such that q_3 is the branch point of T_3.

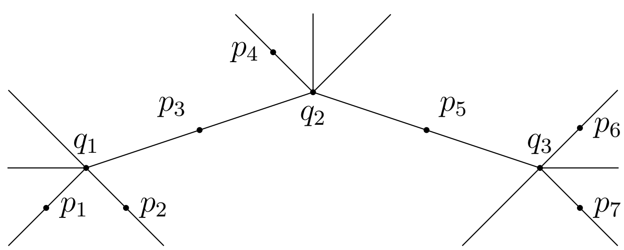

Figure 7: Three branch points

We show by contradiction that there is no arc containing $\{p_1, p_2, \ldots, p_7\}$. Suppose that there is an arc $\beta \subseteq G$ containing the points $\{p_1, p_2, \ldots, p_7\}$, using the same argument from Remark 1, we can assume that the end points of β belong to $\{p_1, p_2, \ldots, p_7\}$.

Now, by Corollary 136, one of $\{p_1, p_2, p_3\}$ is an end point of β. Similarly, one of $\{p_3, p_4, p_5\}$ is an end–point of β, and one of $\{p_5, p_6, p_7\}$ is an end–point of β. So, since β is an arc with end–points in $\{p_1, p_2, p_3, \ldots, p_7\}$, we have that either

(i) p_1 or p_2 and p_5 are the end–points of β, or

(ii) p_6 or p_7 and p_3 are the end–points of β or

(iii) p_3 and p_5 are the end–points of β,

are the only possible cases. We will prove that every case leads to a contradiction.

(i) Assume that p_1 and p_5 are the end–points of β. Since the arc between q_2 and q_3 contains no branch points of G, we have that either $[q_2, p_5] \subseteq \beta$ or $[p_5, q_3] \subseteq \beta$.

Assume first that $[q_2, p_5] \subseteq \beta$. Then, by the way p_4 was chosen and the fact that $p_4 \in \mathrm{int}(\beta)$, the arc $[p_4, q_2] \subseteq \beta$; similarly, since the arc $[q_1, q_2]$ contains no branch points of G and by the fact that $p_3 \in \mathrm{int}(\beta)$, the arc $[p_3, q_2] \subset \beta$. Then $([p_3, q_2] \cup [p_4, q_2] \cup [q_2, p_5]) \subseteq \beta$, which is a contradiction since $([p_3, q_2] \cup [p_4, q_2] \cup [q_2, p_5])$ is a nondegenerate simple triod.

Assume that $[p_5, q_3] \subseteq \beta$. Then, by the way p_6 was chosen and the fact that $p_6 \in \mathrm{int}(\beta)$, the arc $[q_3, p_6] \subseteq \beta$. Using the same argument we can conclude that the arc $[q_3, p_7] \subseteq \beta$. Hence $([p_5, q_3] \cup [p_6, q_3] \cup [q_3, p_7]) \subseteq \beta$, which is a contradiction.

The case when p_2 and p_5 are the end–points of β is similar the case we just proved. So (i) does not hold.

(ii) This case is equivalent to (i), therefore (ii) does not hold.

(iii) Assume that p_3 and p_5 are the end-points of β. Then, since the arc $[q_1, q_2]$ contains no branch points of G and p_3 is an end-point of β, either $[q_1, p_3] \subseteq \beta$ or $[p_3, q_2] \subseteq \beta$.

Suppose that $[q_1, p_3] \subseteq \beta$. As in (i), since $p_1, p_2 \in \mathrm{int}(\beta)$, we have that the arcs $[p_1, q_1]$ and $[q_1, p_2]$ are contained in β. This implies that the nondegenerate simple triod $([q_1, p_3] \cup [p_1, q_1] \cup [q_1, p_2]) \subseteq \beta$, which is a contradiction.

Now assume that $[p_3, q_2] \subseteq \beta$. Then the arc $[p_5, q_3] \subseteq \beta$. Again, the same argument as in (i) leads to a nondegenerate simple triod being contained in β since $p_6, p_7 \in \mathrm{int}(\beta)$. Hence (iii) does not hold.

This proves that there is no arc containing $\{p_1, p_2, \ldots, p_7\}$. Therefore G is not 7–ac. $\square$

Since every $(n+1)$–ac space is n–ac, we have the following corollary.

Corollary 141. *A finite graph with three or more branch points cannot be a n–ac, for $n \geq 7$.*

Lemma 142. *If G is a finite graph with only 2 branch points each of degree greater than or equal to 4, then G is not 7–ac.*

Proof. Let q_1 and q_2 be the two branch points of G. Then there exists at least one edge e having q_1 and q_2 as vertices. Let $p_1 \in \text{int}(e)$. Since G is a finite graph, and q_1 and q_2 have degree at least 4, we can chose three points p_2, p_3, p_4 in a neighborhood of q_1 such that $T_1 = [q_1, p_1] \cup [q_1, p_2] \cup [q_1, p_3] \cup [q_1, p_4]$ is a simple 4-od, and three points p_5, p_6, p_7 in a neighborhood of q_2 such that $T_2 = [q_2, p_1] \cup [q_2, p_5] \cup [q_2, p_6] \cup [q_2, p_7]$ is a simple 4–od, and they are such that $T_1 \cap T_2 = \{p_1\}$.

We show by contradiction, that there is no arc $\alpha \subseteq G$ containing $\{p_1, p_2, \ldots, p_7\}$. For this suppose that there exists such an arc α, assume further that the end–points of α belong to $\{p_1, p_2, \ldots, p_7\}$. Then, since $\{p_2, p_3, p_4\}$ satisfy the conditions of Corollary 136, we can assume without loss of generality that p_4 is an end–point of α. Similarly for the set $\{p_5, p_6, p_7\}$, so we can assume without loss of generality that p_5 is an end–point of α. On the other hand, the set $\{p_1, p_2, p_3\}$ also satisfies the conditions of Corollary 136, hence p_1, or p_2 or p_3 is an end-point of α which is impossible since α only has two end-points. This shows that there is no arc in G containing $\{p_1, p_2, \ldots, p_7\}$. This proves that G is not 7–ac. $\square$

Corollary 143. *If G is a finite graph with only 2 branch points each of degree greater than or equal to 4, then G is not n–ac, for $n \geq 7$.*

Theorem 144. *Let G be a finite graph. If G is 7–ac, then G is one of the following graphs: arc, simple closed curve, figure eight, lollipop, dumbbell or theta–curve.*

Proof. Let G be a finite graph. Suppose that G is n–ac, for $n \geq 7$. We will show that G is (homeomorphic to) one of the listed graphs.

Let K be the reduced graph of G. By Corollary 133 K is n–ac and contains no terminal edges. By Theorem 140 K has at most two branch points, and by Corollary 138 the degree of each branch point is at most 4. We consider the cases when K has no branch points, one branch point or two branch points.

In this case K is either homeomorphic to the arc, I, or to the simple closed curve, S^1.

Assume first that K is homeomorphic to I, then, by the way K is obtained, $K = G$. Otherwise, reattaching the last terminal edge that was removed gives a simple triod which is not 7–ac, contrary to the hypothesis. In this case G is on the list.

Next assume K is homeomorphic to S^1. If $K = G$, then G is on the list. So assume $G \neq K$, and let e denote the last terminal edge that was removed. Then $K \cup e$ is homeomorphic to the lollipop curve. Furthermore, $G = K \cup e$, otherwise reattaching the penultimate terminal edge will give a homeomorphic copy of the graph (a) of Example (E) which is not 7–ac, or a simple closed curve with two arcs attached to it at the same point at one of their end points which is not 7–ac either. Hence, again, G is on the list.

Note that the only possibility for K to have a single branch point of degree 3 is for K to be homeomorphic to a simple triod or to the lollipop curve, the former is not 7–ac and the latter is not a reduced graph. Hence the degree of the branch point of K is 4. In this case K is homeomorphic to either a simple 4–od, a simple closed curve with two arcs attached to it at the same point at one of their end points, or to the figure eight curve. The first two cases are not 7–ac. Therefore K must be homeomorphic to the figure eight curve. If $G = K$, then G is on the list. In fact, since attaching an arc to the figure eight curve yields a non 7–ac curve, we must have that $G = K$.

Since the sum of the degrees in a graph is always even and K has no terminal edges, then K can not have one branch point of degree 3 and another of degree 4. Hence the only options are that K has two branch points of either degree 3 or degree 4. However, by Corollary 143, K has only two branch points of degree 3.

If K has two branch points of degree 3, then it could be homeomorphic to one of the following graphs.

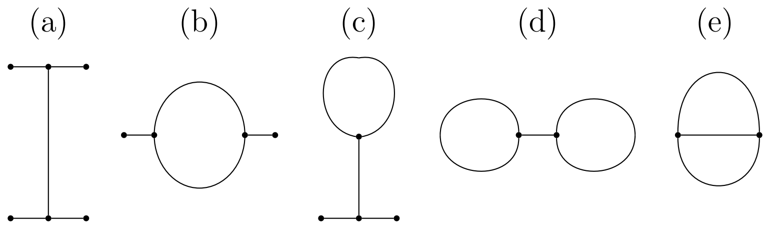

Figure 8: Two branch points of degree 3

However the graphs (a), (b), and (c) contain terminal edges. So K can only be homeomorphic to the dumbbell (d) or the θ–curve (e); in any case if $G = K$, then G is on the list. Note that neither curve, (d) nor (e), can be obtained from a n–ac graph ($n \geq 7$) by removing a terminal edge since by Theorem 140 the edge has to be attached to one of the existing branch points; it is easy to see that such a graph is not 4–ac, just take a point in the interior of each edge. Hence $G = K$. This ends the proof of the theorem. $\square$

A.1.2 Characterizing $\aleph_0$–ac Continua

Call a space κ–ac, where κ is a cardinal, if every subset of size no more than κ is contained in an arc. Note that for finite $\kappa = n$ and $\kappa = \aleph_0$ this coincides with the earlier definitions. For infinite κ we have a complete description of κ–ac continua (not necessarily metrizable), extending Theorem 129. To start let us observe that the arc is κ–ac for every cardinal κ. We will see shortly that the arc is the only separable κ–ac continuum when κ is infinite. In particular, the triod and circle are not $\aleph_0$–ac, and so any continuum containing a triod or a circle is also not $\aleph_0$–ac. This observation will be used below.

To state the theorem precisely we need to make a few definitions. Note that a subset of ω_1 is bounded if and only if the set is countable. The *long ray*, R, is the lexicographic product of ω_1 with $[0, 1)$ with the order topology. We can identify ω_1 (with its usual order topology) with $\omega_1 \times \{0\}$. Evidently ω_1 is cofinal in the long ray. Write R^- for R with each point x relabeled $-x$. The *long line*, L, is the space obtained by identifying 0 in the long ray, R, with -0 in R^-. The topology on the long ray and long line ensures that for any $x < y$ in R (or L) the subspace $[x, y] = \{z \in R : x \leq z \leq y\}$ is (homeomorphic to) an arc.

Note that any countable subset of the long ray, or the long line, is bounded, hence both the long ray and long line are $\aleph_0$–ac. To see this for the long ray take any countable subset S then since ω_1 is cofinal in R the set S has an upper bound, x say, and then S is contained in $[0, x]$, which is an arc.

Let αR be the one point compactification of R, and γL be the corresponding two point compactification of L. The *long circle* and *long lollipop* are the spaces obtained from αR by identifying the point at infinity to 0, or any other point, respectively. The *long dumbbell*, *long figure eight* and *long theta* curves come from γL by respectively identifying the negative ($-\infty$) and positive ($+\infty$) endpoints to -1 and $+1$, 0 and 0, or $+1$ and -1. As continuous injective images of the $\aleph_0$–ac spaces R and L, all the above spaces are also $\aleph_0$–ac.

Theorem 145. *Let K be a continuum.*

(1) If K is separable and $\aleph_0$–ac then K is an arc.

(2) If K is non–separable, then the following are equivalent:

(i) K is $\aleph_0$–ac, (ii) K is a continuous injective image of a closed sub–interval of the long line, and (iii) K is one of: the long circle, the long lollipop, the long dumbbell, the long figure eight, or the long theta–curve.

(3) If K is κ–ac for some $\kappa > \aleph_0$, then K is an arc.

For part (1) just take a dense countable set, then any arc containing the dense set is the whole space. Part (2) is proved in Proposition 146 ((i) $\implies$ (ii)), Proposition 149 ((ii) $\implies$ (iii)), while (iii) $\implies$ (i) was observed above with the definition of the curves in (2) (iii). For part (3) note that all non–separable $\aleph_0$–ac spaces (as listed in part (2) (iii)) have a dense set of size $\aleph_1$, and so are not $\aleph_1$–ac. Thus κ–ac continua for $\kappa \geq \aleph_1$ are separable, hence an arc, by part (1).

It is traditional to use Greek letters (α, β et cetera) for ordinals. Consequently we will use the letter 'A' and variants for arcs, and because in Proposition 146 we need to construct a map, in this subsection by an 'arc' we mean any homeomorphism between the closed unit interval and a subset of a given space. If K is a space, then by 'A is an arc in K' we mean the arc A maps into K. When A is an arc in a space K, then write $im(A)$ for the image of A (it is, of course, a subspace of K homeomorphic to the closed unit interval). For any function f, we write dom f, for the domain of f.

Proposition 146. *Let K be an $\aleph_0$–ac non–separable continuum. Then there is a continuous bijection $A_\infty : J_\infty \to K$ where J_∞ is a closed unbounded sub–interval of the long line, L.*

We prove this by an application of Zorn's Lemma. The following lemmas help to establish that Zorn's Lemma is applicable, and that the maximal object produced is as required.

Lemma 147. *Let K be an $\aleph_0$–ac non–separable continuum. If $\mathcal{K}$ is a countable collection of separable subspaces of K then there is an arc A in K such that $\bigcup \mathcal{K} \subseteq im(A)$.*

Proof. Let $\mathcal{K} = \{S_n : n \in \mathbb{N}\}$ be a countable family of subspaces of K, and, for each n, let D_n be a countable dense subset of S_n. Let $D = \bigcup_n D_n$ — it is countable. Since K is $\aleph_0$–ac there is an arc A in K such that $D \subseteq im(A)$. As D is dense in $\bigcup \mathcal{K}$ and $im(A)$ is closed, we see that $\bigcup \mathcal{K} \subseteq im(A)$. $\square$

Lemma 148. *Let K be an $\aleph_0$–ac non–separable continuum. Suppose $[a, b]$ is a proper closed subinterval of L (or R), $A : [a, b] \to K$ is an arc in K and $y \in K \setminus im(A)$. Then either (i) for every $c > b$ in L there is an arc $A' : [a, c] \to K$ such that $A' \restriction_{[a,b]} = A$ and $A'(c) = y$, or (ii) for every $c < a$ in L there is an arc $A' : [c, b] \to K$ such that $A' \restriction_{[a,b]} = A$ and $A'(c) = y$.*

Proof. Fix a, b, the arc A and y. Let $\mathcal{K} = \{im(A), \{y\}\}$, and apply Lemma 147 to get an arc $A_0 : [0, 1] \to K$ in K such that $im(A_0) \supseteq im(A) \cup \{y\}$. Let $J = A_0^{-1}(im(A))$, $a' = \min J$, $b' = \max J$ and $c' = A_0^{-1}(y)$. Without loss of generality (replacing A_0 with $A_0 \circ \rho$ where $\rho(t) = 1 - t$ if necessary) we can suppose that $A_0(a') = A(a)$ and $A_0(b') = b$.

Since $y \notin im(A)$, either $c' > b'$ or $c' < a'$. Let us suppose that $c' > b'$. This will lead to case (i) in the statement of the lemma. The other choice will give, by a very similar argument which we omit, case (ii). Take any c in L such that $c > b$. Let A_1 be a homeomorphism of the closed subinterval $[a, c]$ of L with the subinterval $[a', c']$ of $[0, 1]$ such that $A_1(a) = a'$, $A_1(b) = b'$ and $A_1(c) = c'$. Set $A_2 = A_0 \circ A_1 : [a, c] \to K$. So A_2 is an arc in K such that $A_2(a) = A(a)$, $A_2(b) = A(b)$, $A_2(c) = y$ and $A_2([a, b]) = im(A)$. The arc A_2 is almost what we require for A' but it may traverse the (set) arc $im(A)$ at a 'different speed' than A. Thus we define $A' : [a, c] \to K$ to be equal to A on $[a, b]$ and equal to A_2 on $[b, c]$. Then A' is the required arc. $\square$

Proof. **(Of Proposition 146)** Let $\mathcal{A}$ be the set of all continuous injective maps $A : J \to K$ where J is a closed subinterval of L, ordered by: $A \le A'$ if and only if $\operatorname{dom} A \subseteq \operatorname{dom} A'$ and $A' \restriction_{\operatorname{dom} A} = A$. Then $\mathcal{A}$ is the set of all candidates for the map we seek, A_∞. We will apply Zorn's Lemma to $(\mathcal{A}, \le)$ to extract A_∞. To do so we need to verify that $(\mathcal{A}, \le)$ is non–empty, and all non–empty chains have upper bounds.

As K is $\aleph_0$–arc connected we know there are many arcs in K, so the set $\mathcal{A}$ is not empty. Now take any non–empty chain $\mathcal{C}$ in $\mathcal{A}$. We show that $\mathcal{C}$ has an upper bound. Let $\mathcal{J} = \{\operatorname{dom} A' : A' \in \mathcal{C}\}$. Since $\mathcal{J}$ is a chain of subintervals in L, the set $J = \bigcup \mathcal{J}$ is also a subinterval of L. Define $A : J \to K$ by $A(x) = A'(x)$ for any A' in $\mathcal{C}$ with $x \in \operatorname{dom} A'$. Since $\mathcal{C}$ is a chain of injections, A is well–defined and injective. Since the domains of the functions in $\mathcal{C}$ form a chain of subintervals, any point x in J is in the J–interior of some $\operatorname{dom} A'$ (there is a set U, open in J such that $x \in U \subseteq \operatorname{dom} A'$), where $A' \in \mathcal{C}$, and so A coincides with A' on some J–neighborhood of x, thus, since A' is continuous at x, the map A is also continuous at x. If J is closed, then we are done: A is in $\mathcal{A}$ and $A \ge A'$ for all A' in $\mathcal{C}$.

If the interval J is not closed then it has at least one endpoint (in L) not in J. We will suppose $J = (a, \infty)$. The other cases, $J = (a, b)$ and $J = (-\infty, a)$, can be dealt with similarly. We show that we can continuously extend A to $[a, \infty)$. If so then A will be injective, hence in $\mathcal{A}$, and an upper bound for $\mathcal{C}$. Indeed, the only way the extended A could fail to be injective was if $A(a) = A(c)$ for some $c > a$, and then $A([a, c])$ is a circle in K, contradicting the fact that K is $\aleph_0$–ac.

Evidently it suffices to continuously extend $A' = A \restriction_{(a,b]}$ to $[a, b]$. Let $\mathcal{K} = \{A((a, b])\}$ and apply Lemma 147 to see that A' maps the half open interval, $(a, b]$, into $I_\mathcal{K}$, a homeomorphic copy of the unit interval. Let $h : [0, 1] \to I_\mathcal{K}$ be a homeomorphism. So we can apply some basic real analysis to get the extension. Indeed, the map $A' \circ h^{-1}$ is continuous and injective, and hence strictly monotone. By the inverse function theorem, A' has a continuous inverse, and so is a homeomorphism of $(a, b]$ with some half open interval, $(c, d]$ or $[d, c)$ in the closed unit interval. Defining $A(a) = h(c)$ gives the desired continuous extension.

Let A_∞ be a maximal element of $\mathcal{A}$. Then its domain is a closed subinterval of the long line, L. We first check that $\operatorname{dom} A_\infty$ is not bounded. Then we prove that A_∞ maps onto K.

If A_∞ has a bounded domain, then it is an arc. So it has separable image. As K is not separable we can pick a point y in $K \setminus im(A_\infty)$. Applying Lemma 148 we can properly extend A_∞ to an arc A'. But then A' is in $\mathcal{A}$, $A_\infty \leq A'$ and $A_\infty \neq A'$, contradicting maximality of A_∞.

We complete the proof by showing that A_∞ is surjective. We go for a contradiction and suppose that instead there is a point y in $K \setminus im(A_\infty)$. Two cases arise depending on the domain of A_∞.

Suppose first that $\operatorname{dom} A_\infty = L$. Pick a point x in $im(A_\infty)$. Pick an arc A from x to y. Taking a subarc, if necessary, we can suppose $A : [0,1] \to K$, $A(0) = x$ and $A(t) \notin im(A_\infty)$ for all $t > 0$. Let $x' = A_\infty^{-1}(x)$. Pick any a', b' from L such that $a' < x' < b'$. Then the subspace $A_\infty([a', b']) \cup A([0,1])$ is a triod in K, which contradicts K being $\aleph_0$–sac.

Now suppose that $\operatorname{dom} A_\infty$ is a proper subset of L. Let us assume that $\operatorname{dom} A_\infty = [a, \infty)$. (The other case, $\operatorname{dom} A_\infty = (-\infty, a]$, follows similarly.) Pick any $b > a$, and apply Lemma 148 to $A = A_\infty \restriction_{[a,b]}$ and y. If case (ii) holds then pick any $c < a$ and A can be extended 'to the left' to an arc A' with domain $[c, b]$. This gives a proper extension of A_∞ defined on $[c, \infty)$ (which is A' on $[c, a]$ and A_∞ on $[a, \infty)$), contradicting maximality of A_∞.

So case (i) must hold. Pick any $c > b$, and we get an arc $A' : [a, c] \to K$ in K extending A. Let $T = A_\infty([a, c]) \cup A'([a, c])$. Observe that T has at least three non cutpoints, namely $A'(a) = A_\infty(a)$, $A_\infty(c)$ and $A'(c)$. So T is not an arc, but it is a separable subcontinuum of the $\aleph_0$–ac continuum K, which is the desired contradiction. $\square$

To complete the proof of Theorem 145 it remains to identify the continuous injective images of closed sub–intervals of the long line. Recall that a countable intersection of closed and unbounded subsets of ω_1 is closed and unbounded (see [42], for example). The set Λ of all limit ordinals in ω_1 is a closed and unbounded set. The Pressing Down Lemma (also known as Fodor's lemma) states than if S is a stationary set and $f : S \to \omega_1$ is regressive (for every α in S we have $f(\alpha) < \alpha$) then there is a β in ω_1 such that $f^{-1}(\beta)$ is cofinal in ω_1.

Proposition 149. *If K is a non–separable continuum and is the continuous injective image of a closed sub–interval of the long line, then K is one of: the long circle, the long lollipop, the long dumbbell, long figure eight, or the long theta–curve.*

Proof. The closed non–separable sub–intervals of the long line are (up to homeomorphism) just the long ray and long line, itself.

Let us suppose for the moment that the K is the continuous injective image of the long ray, R. We may as well identify points of K with points in R. Note that on any closed subinterval, $[a, b]$ say, of R, (by compactness of $[a, b]$ in R, and Hausdorffness of K) the standard order topology and the K–topology coincide. It follows that at any point with a bounded K–open neighborhood the standard topology and K–topology agree. We will show that there is a point x in R such that every K–open U containing x contains a tail, (t, ∞), for some t. Assuming this, then by Hausdorffness of K, every point distinct from x has bounded neighborhoods, and so x is the only point where the K–topology differs from the usual topology. Then K is either the long circle or long lollipop depending on where x is in R (in particular, if it equals 0). The corresponding result for continuous injective images of the long line follows immediately.

Suppose, for a contradiction, that for every x in R, there is a K–open set U_x containing x such that U_x contains no tail. By compactness of K, some finite collection, $U_{x_1}, \ldots, U_{x_n}$, covers K. Let $S_i = U_{x_i} \cap \Lambda$, where Λ is the set of limits in ω_1. Then (since the finitely many S_i cover the closed unbounded set Λ) at least one of the S_i is stationary. Take any α in S_i, and consider it as a point of the closed subinterval $[0, \alpha]$ of R, where we know the standard topology and the K–topology agree. Since α is a limit point which is in $U_{x_i} \cap [0, \alpha]$, and this latter set is open, we know there is ordinal $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \subseteq U_{x_i}$. Thus we have a regressive map, f, defined on the stationary set S_i, so by the Pressing Down Lemma there is a β such that $f^{-1}(\beta)$ is stationary (and therefore cofinal) in ω_1. Hence U_{x_i} contains $\bigcup\{(f(\alpha), \alpha] : \alpha \in f^{-1}(\beta)\} = (\beta, \infty)$, and so U_{x_i} does indeed contain a tail. □

A.2 STRONG ARC CONNECTEDNESS

In this section we present a further strengthening of n–ac property. We strengthen the condition 'there is an arc containing the points' by requiring the arc to traverse the points in a given order. We call this property *n-strong arc connectedness* (abbreviated n-sac), and we call a space which is n-sac for all n an ω-strongly arc connected space (ω-sac).

Evidently a space is 2-sac if and only if it is arc connected. Many naturally occurring examples of arc connected spaces, especially those of dimension at least two, are ω-sac. For instance it is easy to see that manifolds of dimension at least 3, with or without boundary, and all 2-manifolds without boundary, are ω-sac. But note that the closed disk is 3-sac but not 4-sac (there is no arc connecting the four cardinal points in the order *North*, *South*, *East* and then *West*). We are led, then, to focus on one-dimensional spaces, and in particular on *curves:* one-dimensional continua (compact, connected metric spaces).

To further sharpen our focus, we observe that there is a natural obstruction to spaces being 3-sac. Suppose a space X contains a point x_1 so that $X \setminus \{x_1\}$ is not arc connected, and fix points x_2 and x_3 for which there is no arc in $X \setminus \{x\}$ from x_2 to x_3. Then no arc in X visits the points x_1, x_2, x_3 in the given order, and thus X is not 3-sac. More generally, see Lemma 150, if removing some $n-2$ points from a space renders it arc disconnected, then it is not n-sac. A continuum is said to be *regular* if it has a base all of whose elements have a finite boundary. It is well known that all regular continua are curves. From our observation it would seem that regular curves could only 'barely' be n-sac for $n \geq 3$, if, indeed, such spaces exist at all.

This section investigates the n-sac and ω-sac properties in graphs and regular curves. The section is divided into three subsections, in Subsection A.2.1 we formally introduce n-strong arc connectedness, give restrictions on spaces being 4-sac, or more generally n-sac. In Subsection A.2.2 we study n-strong arc connectedness in graphs noting that graphs are never 4-sac, and giving a simple (in a precisely defined sense) characterization of those graphs which are 3-sac. In Subsection A.2.3 we observe that regular curves are never ω-sac, but that there exist, for every n, a regular curve which is n-sac but not $(n+1)$-sac.

A.2.1 Preliminaries

In this section we introduce the basic definitions and notation use throughout the section. Most of the basic notions are taken from [51].

A topological space X is *n-strongly arc connected* (n-sac) if for every distinct $x_1, \ldots, x_n$

in X there is an arc $\alpha : [0,1] \to X$ and $t_1 < t_2 < \cdots < t_n$ from $[0,1]$ such that $\alpha(t_i) = x_i$ for $i = 1, \ldots, n$ — in other words, the arc α 'visits' the points in order. Note that we can assume that $t_1 = 0$ and $t_n = 1$, or even that $t_i = (i-1)/(n-1)$ for $i = 1, \ldots, n$. A topological space is called ω-sac if it is n-sac for every n.

In connection with n-arc connectedness, observe that n-strong arc connectedness implies n-arc connectedness. On the other hand, a simple closed curve is ω-arc connected but is not 4-strongly arc connected, thus the class of n-strongly arc connected spaces is a proper subclass of n-arc connected spaces.

Lemma 150. *Let X be a topological space. If there is a finite F such that $X \setminus F$ is disconnected, then X is not $(|F| + 2)$-sac.*

Proof. If F is empty then X is disconnected and hence not 2-sac. So suppose $F = \{x_1, \ldots, x_n\}$ for $n \geq 1$. Let U and V be an open partition of $X \setminus F$. Pick x_{n+1} in U and x_{n+2} in V. Consider an arc α in X visiting $x_1, \ldots, x_n$, and then x_{n+1}. Then α ends in U and can not enter V without passing through F. Thus no arc extending α can end at x_{n+2} — and X is not $n + 2$-sac, as claimed. $\square$

Suppose α is an arc in X with endpoints a and b. Recall that α is called a free arc if $\alpha \setminus \{a, b\}$ is open in X.

Corollary 151. *Let X be a topological space.*

(1) If there is an open set U with finite boundary, then X is not $(|\partial U| + 2)$-sac.

(2) A continuum containing a free arc is not 4-sac.

(3) No compact continuous injective image of an interval is 4-sac.

Proof. (1) is simply a restatement of Lemma 150. For (2), apply (1) to an open interval inside the free arc. While for (3) note that, by Baire Category, a compact continuous injective image of an interval contains a free arc, so apply (2). $\square$

Call an arc α in a space X a '*no exit arc*' if every arc β containing the endpoints of α, and meeting α's interior must contain all of α.

Lemma 152. *If a space contains a no exit arc then it is not 4-sac.*

Proof. Let x_1 and x_2 be the endpoints of α. Pick x_3 and x_4 so that x_1, x_3, x_4, x_2 are in order along α. Suppose, for a contradiction, β is an arc visiting the x_i in order. Since x_3 and x_4 are in the interior of α, by hypothesis, β contains α. Now we see that if β enters the interior of α from x_1 then it visits x_3 before x_2. While if β enters the interior of α from x_2 it visits x_4 before x_3. Either case leads to a contradiction. $\square$

A.2.2 Graphs

From Corollary 151 (2) it is immediate that no graph is 4-sac. Since only connected graphs will be considered, all graphs are 2-sac. In this section we give a characterization of 3-sac graphs. In fact, we show that for a general continuum X the property of being 3-sac is equivalent to the intensively studied property of being cyclicly connected (any two points in X lie on a circle).

We begin this section by noticing that the triod and the figure eight continuum are not 3-sac, while the circle and the theta curve continuum are 3-sac. In [8], Bellamy and Lum proved:

Theorem 153. *For a continuum X, the following are equivalent:*

(1) X is cyclicly connected;

(2) X is arc connected, has no arc-cut point, and has no arc end points.

Using the previous theorem we obtain the following characterization of 3-sac continua.

Proposition 154. *For a continuum X, the following are equivalent:*

(1) X is cyclicly connected;

(2) X is 3–sac;

(3) Any three points in X lie either on a circle or on a theta curve.

Proof. (3) $\Rightarrow$ (2): this follows from the fact that the circle and theta curve are both 3-sac.

(2) $\Rightarrow$ (1): if X is 3-sac, then for any $x \in X$, there is an arc that contains x in its interior. So X has no endpoints. If X has an arc–cut point then, by Lemma 1, X is not 3-sac. Now by Theorem 153 we have that 3-sac implies cyclically connected.

(1) $\Rightarrow$ (3): Let $x, y, z \in X$. By (1) there is a circle C in X containing x and y. If $z \in C$ we are done. If $z \notin C$, then by (1) there are two arcs, α and β, from z to x, that only meet at the endpoints. Let a and b be the points when α and β first intersect C and let α' and β' be parts of α and β from z to a and b respectively. If $a \neq b$, $C \cup \alpha' \cup \beta'$ is the desired theta curve. If $a = b$, it should not be an arc cut point by the above theorem, so there is an arc γ from z to some point on C other than a that misses a. Let γ' be part of γ that starts in $(\alpha' \cup \beta') - \{a\}$, ends in $C - \{a\}$ and does not meet $C \cup \alpha' \cup \beta'$ otherwise. Then $C \cup \alpha' \cup \beta' \cup \gamma'$ contains a theta curve that passes through x, y, z. $\square$

Notice, if X is a finite graph then 'X is arc-connected, has no arc–cut point, has no endpoints' is equivalent to 'X has no cut points' so we get:

Corollary 155. *For a finite graph X, the following are equivalent:*

(1) X has no cut points;

(2) X is 3-sac;

(3) Any three points in X lie either on a circle or on a theta curve.

A.2.3 n-Strongly Arc Connected Regular Curves

In this section we construct, for every $n \geq 3$, an n-sac regular continuum, then using the Finite Gluing Lemma (Lemma 157) we show that for any $n \geq 2$ there is a regular continuum that is n-sac but not $(n+1)$-sac.

We start the section by introducing the basic elements needed to construct an n-sac regular continuum.

Fix $N \geq 3$. Suppose $v_1, \ldots, v_k$ are affinely independent points in $\mathbb{R}^{N-1}$. Denote by $\langle v_1, \ldots, v_k \rangle$ the convex span of v_1 through v_k. Then $\langle v_1, \ldots, v_k \rangle$ is a k-simplex. We call the points $v_1, \ldots, v_k$ the vertices of $\langle v_1, \ldots, v_k \rangle$. For any $i \neq j$, we call $\langle v_i, v_j \rangle$ the edge from v_i to v_j, and we let $v_i \wedge v_j$ be the midpoint between v_i and v_j. Note that the space of all edges, $\bigcup_{i<j\leq k} \langle v_i, v_j \rangle$, of $\langle v_1, \ldots, v_k \rangle$, is a complete graph on the vertices $v_1, \ldots, v_k$.

Fix $v_1, \ldots, v_N$ affinely independent points in $\mathbb{R}^{N-1}$, for example let v_1 through v_{N-1} be the standard unit coordinate vectors, and $v_N = \mathbf{0}$. Define the operation $Trix$ taking a simplex

$\langle v_1, \ldots, v_N \rangle$ and returning a set of simplices, $\{\langle v_i, v_i \wedge v_j : i \neq j \rangle : i = 1, \ldots, N\}$. Inductively define sets of N-simplices as follows: $\mathcal{T}_0^N = \{\langle v_1, \ldots, v_N \rangle\}$, and $\mathcal{T}_{m+1}^N = \bigcup_{S \in \mathcal{T}_m^N} Trix(S)$. Let $T_m^N = \bigcup \mathcal{T}_m^N$, and $T^N = \bigcap_m T_m^N$. Then T^N is a regular continuum we call the N-trix. Observe that the 3–trix is the Sierpinski triangle, and the 4–trix is the tetrix (hence our name for these continua).

Some additional notation. Given a simplex $S = \langle v_1, \ldots, v_N \rangle$, let $\mathcal{T}_1 = Trix(S)$, and $T_1 = \bigcup \mathcal{T}_1$. Take any element of $\mathcal{T}_1$, say $S_i = \langle v_i, v_i \wedge v_j : j \neq i \rangle$. Call the point v_i the external vertex of S_i, and call the points $v_i \wedge v_j$, for $j \neq i$, the internal vertices of S_i. For any S in $\mathcal{T}_1$, denote the external vertex of S by $v(S)$. For any two elements, S and S' of $\mathcal{T}_1$, denote the (unique) internal vertex common to S and S' by $S \wedge S'$. Note that $\{S \wedge S'\} = S \cap S'$. Further for any x in T_1, fix an element, $S(x)$, of $\mathcal{T}_1$ containing x. It is easy to verify directly, or by applying Theorem 153 and Proposition 154 that all N-trixes are 3-sac.

Lemma 156. *Fix $n \geq 3$. Let $N = 6n^2 + 12n + 1$. The N-trix is n-sac.*

Proof. Let $T = T^N$, the N-trix, and — since N is fixed to be $6n^2 + 12n + 1$ — otherwise suppress the superscript N. Take any n points in T, say $x_1, \ldots, x_n$. Then there is a minimum $m \geq 1$ such that the x_i are in distinct simplices in $\mathcal{T}_m$. Further there is a maximum m' so that all the x_i are in the same simplex S of $\mathcal{T}_{m'}$. If there is an arc in $S \cap T$ visiting the points in order, then that same arc visits the points in order inside T. So without loss of generality, we can suppose that $m' = 0$, $S = T_0$, and the points $x_1, \ldots, x_n$ (obviously) each lie in an element of $\mathcal{T}_1$, but not all in the same element. Now call m the height of the points, $x_1, \ldots, x_n$.

There are $N = 6n^2 + 12n + 1$ elements of $\mathcal{T}_1$. Each of the n points, x_i, can only be in at most 2 members of $\mathcal{T}_1$. Hence we can find a subset $\mathcal{E}$ of $\mathcal{T}_1$ such that $\mathcal{E}$ has at least $3n + 1$ members, and no point x_i is in any element of $\mathcal{E}$. The lemma now follows from the next claim, which we prove by induction on m.

Claim: For each $m \geq 0$, points $x_1, \ldots, x_n$ of height m, and subset $\mathcal{E}$ of $\mathcal{T}_1$, such that $|\mathcal{E}| > 3n$ and $\mathcal{E} \cap \{S(x_i) : i \leq n\} = \emptyset$ (for any choice of $S(x_i)$), there is an arc α visiting the points $x_1, \ldots, x_n$ in order, and, for $i = 1, \ldots, n$, disjoint arcs (called '*spurs*'), β_i from x_i to the external vertex, $v(E)$, of some E in $\mathcal{E}$.

Base Step, $m = 1$**.** Since $m = 1$, we can assume that the sets $S(x_i)$, for $i = 1, \ldots, n$, are distinct.

Pick some C in $\mathcal{E}$. Let $\mathcal{E}' = \mathcal{E} \setminus \{C\}$. For each $i \leq n$, and $j = 1, 2, 3$, pick distinct $E_{i,j}$ from $\mathcal{E}'$. For each $i \leq n$, pick three disjoint arcs in $S(x_i)$: α_i^- from x_i to $S(x_i) \wedge E_{i,1}$, α_i^+ from $S(x_i) \wedge E_{i,2}$ to x_i, and β_i from x_i to $S(x_i) \wedge E_{i,3}$. Extend β_i by following the edge in $E_{i,3}$ to $v(E_{i,3})$. These β_i are the required 'spurs'. Denote by Ω the set of simplices, $E_{i,3}$, containing these spurs.

For $i < n$, let α_i be the arc formed by following the natural edges (of elements of $\mathcal{T}_1$) between these vertices in the prescribed order: $S(x_i) \wedge S(E_{i,1})$, $S(E_{i,1}) \wedge S(C)$, $S(C) \wedge S(E_{i+1,2})$ and $S(E_{i+1,2}) \wedge S(x_{i+1})$. Let α be the path obtained by following these arcs in the given order: $\alpha_1^-, \alpha_i, \alpha_2^+$, $\alpha_2^-, \alpha_2, \alpha_3^+$, $\ldots, \alpha_i^-, \alpha_i, \alpha_{i+1}^+, \ldots$, and finally $\alpha_{n-1}^-, \alpha_{n-1}, \alpha_n^+$. Since all the vertices appearing in the definition of the α_i's are distinct, α is a path which does not cross itself, and so is an arc, which, by construction, visits the points $x_1, \ldots, x_n$ in order.

Inductive Step. We assume the claim is true when the points come from a level $< m$. Prove for points on level m. First observe that for any S in $\mathcal{T}_1$, $S \cap T_m$ is homeomorphic to T_{m-1}.

We will use I_l to denote the set $\{1, 2, \ldots, l\}$. Let $x_1, x_2, \ldots, x_n$ be n points in T of height m and $\{S^1, S^2, \ldots, S^k\} = \{S(x_1), \ldots, S(x_n)\}$. For each $i \in I_k$ let $x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,k_i)}$ be a reenumeration of all the x_j's in S^i such that if $x_t = x_{(i,s)}$ and $x_l = x_{(i,r)}$ then $s < r$ if and only if $t < l$. For all S in $\mathcal{T}_1$, let $\mathcal{S}_{m-1}$ denote $S \cap \mathcal{T}_m$. In each S^i pick $6n + 1$-many simplices of $\mathcal{S}_1^i$ that do not contain any of $x_{(i,j)}$'s, none of them share external vertices, and none contain the external vertex of S^i; this can be done since $\mathcal{S}_1^i$ consists of $6n^2 + 13n + 1$ simplices and S^i contains at most $n - 1$ elements of $\{x_1, x_2, \ldots, x_n\}$. Let this set of simplices be $\mathcal{E}_i$.

In the next step we will choose the simplices that will allow us to construct an arc between two consecutive $x_i s$, whenever they lie on different elements of $\mathcal{T}_1$.

For each $i \in I_k$ let Υ_i be a set of simplices $Y_{(i,j)}$ in $\mathcal{T}_1$ given as follows:

1. For $j < k_i$,

   a. If $x_{(i,j)}$ and $x_{(i,j+1)}$ are not consecutive points in $\{x_1, x_2, \ldots, x_n\}$, then pick $Y_{(i,j)}$ such that

$$Y_{(i,j)} \notin \{S^1, S^2, \ldots, S^k\} \cup \{\bigcup_{t=1}^{i-1} \Upsilon_t\} \cup \{Y_{(i,1)}, Y_{(i,2)}, \ldots, Y_{(i,j-1)}\},$$

$Y_{(i,j)} \cap \{x_1, x_2, \ldots, x_n\} = \emptyset$, and such that $Y_{(i,j)} \wedge S^i$ lies in an element of $\mathcal{S}_{m-1}$ different from the elements containing the $x_{(i,j)}$'s.

b. If $x_{(i,j)}$ and $x_{(i,j+1)}$ are consecutive points in $\{x_1, x_2, \ldots, x_n\}$, then do nothing.

2. For $j = k_i$,

   a. If $x_{(i,k_i)} \neq x_n$, then pick $Y_{(i,k_i)}$ as above, satisfying the conditions on (a).

   b. If $x_{(i,k_i)} = x_n$, then do nothing.

3. For $j = 1$ (in some cases we are selecting twice for $j = 1$),

   a. If $x_{(i,1)} \neq x_1$, then pick $Y_{(1,0)}$ as in (1), satisfying the conditions on (a).

   b. If $x_{(i,1)} = x_1$, then do nothing.

Denote by $y_{(i,j)}$ the vertex $S^i \wedge Y_{(i,j)}$.

Now, in each sequence $\{x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,k_i)}\}$ insert the points $y_{(i,j)}$ (if they exist) as follows: $y_{(i,0)}$ before $x_{(i,1)}$, and $y_{(i,j)}$ immediately after corresponding $x_{(i,j)}$. So, for each $i \in I_k$, we have constructed a sequence l_i in S^i such that a point $y_{(i,j)}$ lies between $x_{(i,j)}$ and $x_{(i,j+1)}$ only if $x_{(i,j)}$ and $x_{(i,j+1)}$ are not consecutive points on $\{x_1, x_2, \ldots, x_n\}$. Observe that for each $i \in I_k$, the set of points $x_{(i,1)}, x_{(i,2)}, \ldots x_{(i,k_i)}$ have, in S^i, height at most $m-1$, hence, by the choice of y's, the sequence of points l_i also has height at most $m-1$.

By the Inductive Hypothesis, applied to S^i, l_i and $\mathcal{E}_i$, there is an arc α_i in S^i visiting the points of l_i in order, and disjoint spurs β_a for each $a \in l_i$ to external vertices of some elements in $\mathcal{E}_i$.

Construction of an arc through $x_1, x_2, \ldots, x_n$: Pick $C \in \mathcal{E}$ containing none of the y's or x's. For each $i \in I_{n-1}$, let γ_i be the arc connecting x_i to x_{i+1} given as follows: If x_i and x_{i+1} are in the same S^t then γ_i is the subarc of α_t connecting them. If not, then $x_i = x_{(p,j)}$, $x_{i+1} = x_{(r,l)}$ and $y_{(p,j)}, y_{(r,l-1)}$ exist. Let $\gamma_i = \gamma_i^1 \cup \gamma_i^2 \cup \gamma_i^3 \cup \gamma_i^4 \cup \gamma_i^5$, where $\gamma_i^1, \gamma_i^2, \gamma_i^3, \gamma_i^4, \gamma_i^5$ are as follows:

1. γ_i^1 is the subarc of α_p from $x_{(p,j)}$ to $y_{(p,j)}$ if possible or else a spur from $x_{(p,j)}$ to some vertex u of S^p, whichever is unused yet. In any case, there is a simplex U in $\mathcal{T}_1$ such that $u = S^p \wedge U$ or $y_{(p,j)} = S^p \wedge U$. Let γ_i^2 be the edge in U connecting $S^p \wedge U$ to $U \wedge C$.

2. similarly as in (1), γ_i^3 is the subarc α_r from $x_{(r,l)}$ to $y_{(r,l-1)}$ if possible or else a spur from $x_{(r,l)}$ to some vertex v of S^r, whichever is unused yet. In any case, there is a simplex V in $\mathcal{T}_1$ such that $v = S^r \wedge V$ or $y_{(r,l-1)} = S^r \wedge V$. Let γ_i^4 be the edge in V connecting $S^r \wedge V$ to $V \wedge C$.

3. let γ_i^5 be the edge in C connecting $U \wedge C$ and $V \wedge C$.

Let $\alpha = \bigcup_{i=1}^{n-1} \gamma_i$. Because of how y's and spur destinations were picked, α is an arc that visits the points $x_1, x_2, \ldots, x_n$ in order.

Construction of spurs to external vertices of elements of $\mathcal{E}$: suppose we have constructed spurs for all x_l, $l < i$ and $x_i \in S^j$. If spur β_{x_i} of x_i in S^j is not contained in α then it only intersects it at x_i, extend β_{x_i} as follows: let v_i be the other endpoint of β_{x_i}. Let V_i be the simplex in $\mathcal{T}_1 \setminus (\{\bigcup_{i=1}^{k} \Upsilon_i\} \cup (\bigcup_{i=1}^{k} S^i) \cup C)$ that intersects S^j at v_i. Pick any simplex E_i in $\mathcal{E} \setminus C$ that has not been picked for previous spur constructions. The spur β_i consist of β_{x_i}, followed by the edge in V_i connecting v_i and $E_i \wedge V_i$, and the edge in E_i connecting $E_i \wedge V_i$ with $v(E_i)$.

Suppose β_{x_i} is contained in α. Observe that in this case $x_i = x_{(j,r)}$ and $x_{(j,r-1)}$ are not consecutive points of $\{x_1, x_2, \ldots, x_n\}$, otherwise β_{x_i} would not be contained in α. Hence $y_{(j,r-1)}$ exists. Let δ be the subarc of α_j connecting $x_{(j,r)}$ to $y_{(j,r-1)}$, let γ be the subarc of α connecting $x_{(j,r-1)}$ to $y_{(j,r-1)}$, and let ω be the other end point of the spur $\beta_{y_{(j,r-1)}}$. By construction, $\alpha \cap \delta = \{x_{(j,r)}, y_{(j,r-1)}\}$ and $\alpha \cap \beta_{y_{(j,r-1)}} = \{y_{(j,r-1)}\}$.

Since the diameters of the simplices in $\mathcal{T}_t$ approach zero as t increases, there exists, for a sufficiently large t, a simplex Λ in $\mathcal{S}_t^j$ with the following properties:

1. $y_{(j,r-1)}$ is a vertex of Λ,

2. $x_{(j,r)},\ x_{(j,r-1)},\ \omega \notin \Lambda$,

3. $\Lambda \cap \alpha$ is connected, and

4. Λ does not intersect any spur, except for $\beta_{y_{(j,r-1)}}$.

By the choice of Λ, the arcs δ, γ and $\beta_{y_{(j,r-1)}}$ intersect Λ at different vertices of Λ, say $a,\ b,\ c$ respectively. Then revise α to go form b to $y_{(j,r-1)}$ through an edge of Λ and let β consist of the following parts: the subarc of δ from $x_i = x_{(j,r)}$ to a, followed by the edge in

Λ from a to c, and followed by the subarc of $\beta_{y_{(j,r-1)}}$ from c to ω. Now extend β as in the previous case to get the spur β_i for x_i. $\square$

Lemma 157 (Finite Gluing)**.** *If X and Y are $(2n-1)$-sac, and Z is obtained from X and Y by identifying pairwise $n-1$ different points of X and Y, then Z is n-sac but not $(n+1)$-sac.*

Proof. Let $z_1, z_2, \dots, z_n$ be any n points in Z. For each i, if $z_i \in X \setminus Y$ and $z_{i+1} \in Y \setminus X$ or $z_i \in Y \setminus X$ and $z_{i+1} \in X \setminus Y$, pick $z_{(i,i+1)} \in (X \cap Y) \setminus \{z_1, z_2, \dots, z_n, z_{(1,2)}, z_{(2,3)}, \dots, z_{(i-1,i)}\}$ (if $z_{(1,2)}, z_{(2,3)}, \dots, z_{(i-1,i)}$ were picked). This is possible since $|X \cap Y| = n-1$. Let $\mathcal{Z}$ be the sequence of z_j's with $z_{(i,i+1)}$'s inserted between z_i and z_{i+1} whenever they exist. And let $\mathcal{Z}_{\mathcal{X}}$ be the sequence derived from $\mathcal{Z}$ by deleting the terms that do not belong to X. Define $\mathcal{Z}_{\mathcal{Y}}$ similarly. Since elements of $\mathcal{Z}_{\mathcal{X}}$ come either from $\{z_1, z_2, \dots, z_n\}$ or from $X \cap Y$, $|\mathcal{Z}_{\mathcal{X}}| \leq 2n-1$. Similarly, $|\mathcal{Z}_{\mathcal{Y}}| \leq 2n-1$. Let β be an arc in X going through elements of $\mathcal{Z}_{\mathcal{X}}$ in order and γ be an arc in Y going through elements of $\mathcal{Z}_{\mathcal{Y}}$ in order. Let $a_1, a_2, \dots, a_k$ be $z_{(1,2)}, z_{(2,3)}, \dots, z_{(n-1,n)}$ whenever they exist, respectively. Without loss of generality, suppose $z_1 \in X$. Define α to be the union of the following arcs:

1. the subarc of β from z_1 to a_1;
2. the subarc of γ from a_1 to a_2;
3. the subarc of β from a_2 to a_3 …
4. the subarc of β or γ (depending on whether k is even or odd) from a_k to z_n.

Note that α is an arc visiting the points $z_1, z_2, \dots, z_n$ in order. Hence Z is n-sac. The fact that Z in not $(n+1)$-sac follows by Lemma 150. $\square$

Observe that for $n \geq 2$, Lemma 156 implies the existence of a $(2n-1)$-sac regular continuum G. Hence by Lemma 157 applied to two disjoint copies of G, there exists an n-sac regular continuum that is not $(n+1)$-sac. We summarize this in the following theorem.

Theorem 158. *For every $n \geq 2$ there exists a n-sac regular continuum that is not $(n+1)$-sac.*

APPENDIX B

OTHER PROJECTS

B.1 SEPARATORS AND GENERATORS

Fix a (Tychonoff) space X and let G be a subspace of of $C_p(X)$. Call G a *separator* if for every distinct x, x' in X there is g in G such that $g(x) \neq g(x')$. Call G a *generator* if for every point x not in a closed subset C of X there is g in G such that $g(x) \neq \overline{g(C)}$. Recall that a subset C of X is a zero set if there is $f \in C_p(X)$ such that $C = f^{-1}\{0\}$. Call G a *zero set generator* if for every zero subset C of X there is g in G such that $C = g^{-1}\{0\}$.

Clearly 'zero set generator' implies 'generator', and 'generator' implies 'separator'. Also, for every space X, $Z = C_p(X)$ is a zero set generator for X. Various variants of these ideas are also useful. Let G be a *$(0, \neq 0)$-separator* if for distinct x, x' there is g in G such that $g(x) = 0$ but $g(x') \neq 0$. Let G be a *$(0, \neq 0)$-generator* if for x not in closed C there is g in G such that $g(C) = 0$ but $g(x) \neq 0$. Let G be a *$(0, 1)$-separator* if for distinct x, x' there is g in G such that $g(x) = 0$ but $g(x') = 1$. And let G be a *$(0, 1)$-generator* if for x not in closed C there is g in G such that $g(C) = 0$ but $g(x) = 1$.

Recall from Section 0.5 of [4] that given any subspace F of $C_p(X)$, the map $\psi_F : X \to C_p(F)$ defined by $\psi_F(x)(f) = f(x)$ is continuous. Even when F is a generator it is not guaranteed that $\psi_F(X)$ is closed in $C_p(F)$. For example, let $X = \mathbb{R}$, $F = \{\arctan x\}$, then F is a generator since $\arctan x$ is a homeomorphism, $C_p(F) = \mathbb{R}$, but $\psi_F(X) = (-\pi/2, \pi/2)$ while $L_p^F(X) = \mathbb{R}$. The next lemma explores linear independence of $\psi_F(X)$.

Lemma 159. *Suppose $F \subseteq C_p(X)$. Then (1) if F is not a separator then $\psi_F(X)$ is linearly dependent, (2) if F is a zero set generator then $\psi_F(X)$ is linearly independent and (3) F being a generator (or a separator) does not guarantee linear independence of $\psi_F(X)$.*

Proof. For (1) pick $x, y \in X$ such that for each $f \in F$, $f(x) = f(y)$, or equivalently, for all $f \in F$, $\psi_F(x)(f) = \psi_F(y)(f)$. So $\psi_F(x) = \psi_F(y)$.

For (2) suppose $\psi_F(X)$ is not linearly independent and we have non-zero $a_1, \ldots, a_n \in \mathbb{R}$, $x, x_1, \ldots, x_n \in X$, with $a_1\psi_F(x_1) + \ldots + a_n\psi_F(x_n) = \psi_F(x)$, meaning $a_1 f(x_1) + \ldots + a_n f(x_n) = f(x)$, $\forall f \in F$. Now since X is Tychonov, $C = \{x_1, x_2, \ldots, x_n\}$ is closed and it is also an intersection of zero sets in X. So there is a zero set Z such that $C \subseteq Z$ and $x \notin Z$. Since F is a zero set generator, there is $f \in F$ sich that $f_{-1}\{0\} = Z$. Thus $f(x_i) = 0, \forall i$ and $f(x) \neq 0$. But $f(x) = a_1 f(x_1) + \ldots + a_n f(x_n) = 0$, a contradiction.

For (3) let $X = \mathbb{R}$ and let i be the identity function on X. Let $F = \{i\}$. Then F is a generator (for each C closed in X and $x \notin C$, we have $f(x) = x \notin C = cl(i(C))$). But $C_p(F) = \mathbb{R}$ and $\psi_F(X) = \mathbb{R}$ which is not linearly independent. $\square$

We know from [4] that if F is a separator then ψ_F is a continuous injection (and so X condenses onto a subspace of $C_p(F)$). But the converse is also true. If ψ_F is injective then for each pair of distinct $x, y \in X$ we have $\psi_F(x) \neq \psi_F(y)$ as functions on F. So there is $f \in F$ such that $\psi_F(x)(f) \neq \psi_F(y)(f)$, i.e., $f(x) \neq f(y)$. So F is a separator. Again from [4] we know that if F is a generator then ψ_F is an embedding (and so X is homeomorphic to a subspace of $C_p(F)$). There are many results describing a duality between properties of X and properties of $C_p(X)$. The above observations suggest that perhaps it is possible to improve these results to duality results between X and its separators or generators.

The following two theorems are well-known [4].

Theorem 160. *For a space X the following are equivalent:*

(1) $C_p(X)$ is second countable, (2) $C_p(X)$ is first countable and (3) X is countable.

Theorem 161. *For a space X the following are equivalent:*

(1) $C_p(X)$ has a coarser second countable topology, (2) points of $C_p(X)$ are G_δ and (3) X is separable.

From proofs of these theorems we immediately see that existence of a 'nice' generator is all that is necessary. Let $\mathbf{0}$ be the constantly zero function on X.

Corollary 162. *For a space X the following are equivalent:*

(1) X has a second countable $(0,1)$-generator containing $\mathbf{0}$, (2) X has a first countable $(0,1)$-generator containing $\mathbf{0}$ and (3) X is countable.

Corollary 163. *For a space X the following are equivalent:*

(1) X has a $(0,\neq 0)$-generator with coarser second countable topology that contains $\mathbf{0}$, (2) X has a $(0,\neq 0)$-generator containing $\mathbf{0}$ with all points G_δ and (3) X is separable.

The natural next step is to see what spaces have 'nice' generators. We present two examples. Recall that the Tangent Disc Space is the set $\mathbb{R}\times[0,\infty)$ with the topology generated by the following base: if $y>0$ then basic open sets around (x,y) are $\mathbb{R}^2$-open balls centered at (x,y); basic open sets containing $(x,0)$ are of the form $V_{n,x}\cup\{(x,0)\}$ where $V_{n,x}$ is the $\mathbb{R}^2$-open ball of radius $1/n$ and centered at $(x,1/n)$. By '$\mathbb{R}^2$-open' we mean 'open in the usual topology of $\mathbb{R}^2$'. Recall that a subset of $C_p(X)$, $B(f,F,\epsilon)$, is defined to be $\{g\in C_p(X):|f(x)-g(x)|<\epsilon\ \forall x\in F\}$.

Proposition 164. *Let X be the Tangent Disk Space. Then X has a first countable $(0,\neq 0)$ generator containing $\mathbf{0}$.*

Proof. Let $C_{n,x}$ be the boundary of $V_{n,x}$ in the usual topology of $\mathbb{R}^2$. We define $g_{n,x}: X\to\mathbb{R}$ by describing its graph: graphs of $g_{n,x}$ consists of the point $((x,0),1/n)$, planar region $V_{n,x}^c\times\{0\}$, line segments connecting the point $((x,0),1/n)$ with points $(A,0)$ where $A\in C_{n,x}\backslash\{(x,0)\}$. Clearly, each $g_{n,x}$ is continuous on X.

Let $G_n=\{g_{n,x}:x\in X\}$. The map ϕ_n taking x to $g_{n,x}$ is a homeomorphism from $\mathbb{R}$ with discrete topology to G_n: $\phi_n(\{x\})=\{g_{n,x}\}=B(g_{n,x},(x,0),1/(2n))$, which is an open set in G_n.

The open upper half plane has countable discrete $(0,\neq 0)$ generator. We can pick an injection $i:\mathbb{Q}^3\to\omega\backslash\{0\}$ such that for each n, $i(p,q,n)>n$ whenever $p,q\in\mathbb{Q}$. Let $U_{p,q,r}$ be $\mathbb{R}^2$-open ball of radius r that is centered at (p,q). For $p,q\in\mathbb{Q}$, $q>0$ assume that n is large enough so that $U_{p,q,1/n}$ does not intersect $\mathbb{R}\times\{0\}$. Define $g_{p,q,n}$ by describing the graph: let the graph of $g_{p,q,n}$ consist of the point $((p,q),1/i(p,q,n))$, planar region $U_{p,q,1/n}^c\times\{0\}$, line segments connecting the point $((p,q), 1/i(p,q,n))$ with points $(A,0)$ where A belongs to the $\mathbb{R}^2$-boundary of $U_{p,q,1/n}$. Then each $g_{p,q,n}$ is continuous. Let $G_0 = \{g_{p,q,n} : p,q \in \mathbb{Q},\ q > 0,\ n \in \omega\setminus\{0\}\}$. Then it is easy to see that G_0 is discrete.

Now define $G' = \bigcup_{n\in\omega} G_n$. It is easy to see that each G_n is open in G'. Therefore G' is still discrete and hence first countable. Let $G = G' \cup \{\mathbf{0}\}$.

To show that G is first countable we construct a local base at $\mathbf{0}$. Pick n and pick $p,q,r \in \mathbb{Q}$ with r positive and q non-negative. Draw the infinite grid in $\mathbb{R}^2$ with side length $1/n$ having a vertex at (p,q). Let $y_1,\dots,y_k$ be vertices of the grid that fall in $U_{p,q,r+1}$. Then let $B_{p,q,r,n} = B(\mathbf{0}; y_1,\dots,y_k; 1/n) \cap G$. The sets of the form $B_{p,q,r,n}$ give a countable base at $\mathbf{0}$. Let $B = B(\mathbf{0}, x_1,\dots,x_k,\epsilon) \cap G$ be any basic open set in G. Pick p,q,r so that all points $x_1,\dots,x_k$ fall in $U_{p,q,r}$. Pick n so that $1/n < \epsilon/4$ and rectangles of the grid containing points $x_1,\dots,x_k$ sit inside $U_{p,q,r}$. If $g \in G_0 \cap B_{p,q,r,n}$, then the graph of G is a cone of slope < 1 (because $i(p,q,m) > m$ for all m and $p,q \in \mathbb{Q}$), therefore it must also fall in B. On the other hand, suppose $g_{m,x} \in B_{p,q,r,n}$ for some m and some x. If m is large enough so that $V_{m,x}$ contains at most four y_j's, then $1/m < \epsilon$ and $g \in B$. But if $V_{m,x}$ contains at least five y_j's then the grid associated with $B_{p,q,r,n}$ is fine enough to ensure that f falls in B. So, $B_{p,q,r,n} \subseteq B$.

G is a $(0, \neq 0)$ generator since for each point $x \in X$ and each basic open set $x \in U \subseteq X$ there is a function $g \in G$ such that $f(U^c) = \{0\},\ f(x) \neq 0$. □

Recall that the Bow-tie Space is the set $\mathbb{R}^2$ with the topology generated by the following base: if $y \neq 0$ then basic open sets around (x,y) are $\mathbb{R}^2$-open balls centered at (x,y); basic open sets containing $(x,0)$ are of the form $W_{n,x} \cup \{(x,0)\}$ where $W_{n,x}$ is the $\mathbb{R}^2$-open set that lies between lines with slope of $\pm 1/n$ through $(x,0)$ and vertical lines going through $(x-1/n,0)$ and $(x+1/n,0)$.

Proposition 165. *Let X be the Bow-tie Space. Then X has a second countable $(0,\neq 0)$ generator containing $\mathbf{0}$.*

Proof. Pick $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and define $h_{n,x} : X \to \mathbb{R}$ as follows: the graph of $h_{n,x}$ consists of planar regions $W_{n+1,x} \times \{1/n\}$ and $W^c_{n,x} \times \{0\}$ and the rest of the graph consists of trapezoidal faces connecting the following pairs of sides:

- $\{(t, \frac{t}{n} - \frac{x}{n}) : x < t \le x + 1/n\} \times \{0\}$ and $\{(t, \frac{t}{n+1} - \frac{x}{n+1}) : x < t \le x + 1/(n+1)\} \times \{1\}$;
- $\{(t, -\frac{t}{n} + \frac{x}{n}) : x < t \le x + 1/n\} \times \{0\}$ and $\{(t, -\frac{t}{n+1} + \frac{x}{n+1}) : x < t \le x + 1/(n+1)\} \times \{1\}$;
- $\{(t, \frac{t}{n} - \frac{x}{n}) : x - 1/n \le t < x\} \times \{0\}$ and $\{(t, \frac{t}{n+1} - \frac{x}{n+1}) : x - 1/(n-1) \le t < x\} \times \{1\}$;
- $\{(t, -\frac{t}{n} + \frac{x}{n}) : x - 1/n \le t < x\} \times \{0\}$ and $\{(t, -\frac{t}{n+1} + \frac{x}{n+1}) : x - 1/(n-1) \le t < x\} \times \{1\}$;
- $\{(x + 1/n, t) : -1/n^2 \le t \le 1/n^2\} \times \{0\}$ and $\{(x + 1/(n+1), t) : -1/(n+1)^2 \le t \le 1/(n+1)^2\} \times \{1\}$;
- $\{(x - 1/n, t) : -1/n^2 \le t \le 1/n^2\} \times \{0\}$ and $\{(x - 1/(n+1), t) : -1/(n+1)^2 \le t \le 1/(n+1)^2\} \times \{1\}$.

Clearly, each $h_{n,x}$ is continuous on X. Let $G_n = \{h_{n,x} : x \in X\}$. The map ψ_n taking x to $h_{n,x}$ is a homeomorphism from $\mathbb{R}$ with usual topology to G_n and $G = \cup G_n$ is the free sum of countably many $\mathbb{R}$'s.

Note that $\mathbb{R}^2$ with the x-axis removed has a discrete countable $(0, \neq 0)$ generator. Let $U_{p,q,r}$ be $\mathbb{R}^2$-open ball of radius r that is centered at (p, q). For $p, q \in \mathbb{Q}$, $q \neq 0$ assume that n is large enough so that $U_{p,q,1/n}$ does not intersect $\mathbb{R} \times \{0\}$. Define $g_{p,q,n}$ the same way as in the proof of Proposition 164. Then, again, each $g_{p,q,n}$ is continuous and $G_0 = \{g_{p,q,n} : p, q \in \mathbb{Q},\ q \neq 0,\ n \in \omega \backslash \{0\}\}$ is discrete.

Let $G' = \bigcup_{n \in \omega} G_n$. It is easy to see that G' is the free sum of countably many copies of $\mathbb{R}$ and countably many points. So G' is second countable.

Let $G = G' \cup \{\mathbf{0}\}$. Then G is first countable (thus second countable since G' is second countable). The proof of first countability at $\mathbf{0}$ is similar to the proof for Proposition 164, except q is allowed to be negative. The resulting collection, G is a $(0, \neq 0)$ generator since for each point $x \in X$ and each basic open set $x \in U \subseteq X$ there is a function $g \in G$ such that $f(U^c) = \{0\},\ f(x) \neq 0$. □

B.2 ELEMENTARY SUBMODELS AND $C_p(X)$

This line of research was inspired by the way elementary submodels of set theory were used to study Corson, Valdivia and Eberlein compact spaces in [40] as well as by the study of upwards and downwards preservation of properties of spaces by elementary submodels from

[37, 38]. We are particularly interested in studying elementary submodels in connection with the spaces of the form $C_p(X)$.

Recall that for a cardinal θ, $H(\theta)$ is the set of all 'hereditarily $< \theta$-sized' sets, or, more precisely, the set of all sets with $< \theta$-sized transitive closure [42]. When θ is regular and uncountable all axioms of ZFC, with the exception of the Power Set Axiom, are true in $H(\theta)$. For the rest of this section assume that θ is large enough regular cardinal so that $H(\theta)$ contains all sets (and all power sets) required for arguments of this section to go through. From now on $(E, \in)$ is an elementary submodel of $(H(\theta), \in)$.

Recall that every countable element of E is also a subset of E and every finite subset of E is also an element of E. Any set defined using elements of E is also an element of E. Since almost all of ZFC is true in E, all natural numbers, ω, $\mathbb{Q}$, $\mathbb{R}$, ω_1 and etc. are elements of E. For any E, $E \cap \omega_1$ is an ordinal.

B.2.1 Y_E and $C_p(X)$

Let Y be a topological space and $\mathcal{B}$ be a base for the topology on Y. Let E be such that $Y, \mathcal{B} \in E$. Define $Y_E = Y \cap E$ with the topology generated by the base $\{B \cap E : B \in \mathcal{B} \cap E\}$. It is well-known (see [38]) that if Y is first countable then the topology of Y_E coincides with the subspace topology of $Y \cap E$. This is not true in general even if 'first countable' is replaced by 'Fréchet-Urysohn' [38]. The space Y_E does not have to be a subspace of Y even when $Y \cap E$ is second countable. For example, let $Y = \omega_1 + 1$ and E be countable. Then $Y \cap E = \alpha \cup \{\omega_1\}$ for some countable α but Y_E is homeomorphic to $\alpha + 1$.

We know that if Y is T_i then Y_E is also T_i, for $i = 0, 1, 2, 3, 3\frac{1}{2}$, while higher separation axioms are not preserved. For several cardinal functions f (for instance, cellularity, hereditary density, hereditary Lindelöf number, character, pseudo-character, weight), $f(Y_E) \leq f(Y)$. But covering properties like compactness and Lindelöfness are not preserved. All these results can be found in [38]. We add two more cases of preserving cardinal functions. Recall that the network weight of X, $nw(X)$, is the smallest cardinality of a network of X (modulo $\{\{x\} : x \in X\}$, by our earlier definition).

Lemma 166. *Let Y be a space and E be arbitrary. Then $nw(Y_E) \leq nw(Y)$ and $iw(Y_E) \leq$*

$iw(Y)$. The first inequality can be strict.

Proof. If $\mathcal{W}$ is a network for Y, then $\{W \cap E : W \in \mathcal{W}\}$ is a network for Y_E because the topology on Y_E is coarser than the subspace topology. The inequality can be strict since, by elementarity, $\{W \cap E : W \in \mathcal{W} \cap E\}$ is a network for Y_E as well, and any countable E and any Y with uncountable network weight will witness the strict inequality.

Let $\kappa = iw(Y)$. Then $\kappa \in E$ and by elementarity there is a base for a topology on X, $\mathcal{B}_\kappa$, of size κ such that the identity map $i_X : (X, \mathcal{B}) \to (X, \mathcal{B}_\kappa)$ is continuous. Let X_E^κ be $X \cap E$ with the topology generated by $\mathcal{B}_\kappa \cap E$ (this topology has weight $\leq \kappa$). By elementarity, $E \models \forall x \in X, U \in \mathcal{B}_\kappa (x \in U) \to \exists V \in \mathcal{B}(x \in V \subseteq U)$. Which translates to $\forall x \in X \cap E, U \in \mathcal{B}_\kappa \cap E (x \in U) \to \exists V \in \mathcal{B} \cap E (x \in V \cap E \subseteq U)$. So $i_X|_E : X_E \to X_E^\kappa$ is continuous. □

We want to investigate preservation of properties under the operation of taking Y_E when $Y = C_p(X)$. Let $\mathcal{B}$ be a base for the topology on X and $\mathcal{B}_p$ be the base for $C_p(X)$ that consists of all open sets of the form: $B(f, F, n) = \{g \in C_p(X) : |g(x) - f(x)| < \frac{1}{n}, \forall x \in F\}$ where $F \subseteq X$ is finite, $f \in C_p(X)$ and $n \in \mathbb{N}$. Note that, $X, \mathcal{B} \in E$ implies $C_p(X), \mathcal{B}_p \in E$.

We know that $C_p(X)$ is a topological ring and a topological vector space over $\mathbb{R}$. It is natural to ask if the same is true of $C_p(X)_E$. Let $+ \subseteq \mathbb{R}^3$ be the addition function and $\times \subseteq \mathbb{R}^3$ be the multiplication function on $\mathbb{R}$. Both of these functions are clearly absolute for E, so $+, \times \in E$.

Proposition 167. *$C_p(X)_E$ is a topological ring and topological vector space over $\mathbb{Q}$.*

Proof. Since the formula "$C_p(X)$ is a ring" can be stated using only elements of E, we immediately deduce that $C_p(X)_E$ is a ring. Similarly, $C_p(X)_E$ is a vector space over $\mathbb{Q}$.

We show that $+_p, \times_p, -_p$ are continuous from $C_p(X)_E^2$ to $C_p(X)_E$. By elementarity we have $E \models +_p$ is continuous. So $\forall f, g \in C_p(X)_E,\ f +_p g \in U \in \mathcal{B}_p \cap E,\ \exists V, W \in \mathcal{B}_p \cap E :$ $f \in V,\ g \in W,\ +_p(V \times W) \cap E = +_p((V \times W) \cap E) \subseteq U$. This precisely means that $+_p$ is continuous from $C_p(X)_E^2$ to $C_p(X)_E$. The rest is similar. □

It turns out that some relations between X and $C_p(X)$ still hold for X_E and $C_p(X)_E$. Let $w(Y)$ be the weight of Y and $\chi(Y)$ be the character of Y, or the smallest κ such that each point in Y has a local base of size κ. We know that $|X| = \chi(C_p(X)) = w(C_p(X))$ and $nw(X) = nw(C_p(X))$. Before proving these results, we prove a basic lemma.

Lemma 168. *(1) $\mathcal{B}_p \cap E = \{B(f, F, n) : f \in E,\ F \subseteq E\}$.*

(2) $C_p(X)_E$ densely embeds into $C_p(X_E)$ (hence in $\mathbb{R}^{X \cap E}$).

(3) The canonical embedding $j : X \to C_pC_p(X)$ is an element of E and $j|_E$ is an embedding of X_E into $(C_pC_p(X))_E$.

Proof. (1) If $F \subseteq E$ then $F \in E$ and together with $f \in E$ they give $B(f, F, n) \in E$. On the other hand, if B is a basic open set in E then there are f, F, n such that $B = B(f, F, n)$ and by elementarity, they can be picked in E.

(2) Define $\phi : C_p(X)_E \to C_p(X_E)$ to be the restriction map: $\phi(f) = f|_E$ for each $f \in C_p(X)_E$. By elementarity and the fact that standard base of $\mathbb{R}$ is a subset of E, ϕ indeed maps into $C_p(X_E)$. Further ϕ is injective: for $f, g \in C_p(X)_E$, $f \neq g$ elementarity requires that f and g be distinguished by a point in $E \cap X$, so $\phi(f) \neq \phi(g)$. Part (1) immediately implies that ϕ is an embedding. For denseness: take any finite $F \subseteq E \cap X$ and open interval I_x in $\mathbb{R}$ for each $x \in F$. Pick a rational number $q_x \in I_x$ for each $x \in F$. Now, there is $f \in C_p(X)$ such that $f(x) = q_x$ for each $x \in F$ and by elementarity, we can arrange for $f \in E$. So $\phi(C_p(X)_E)$ is dense in $C_p(X_E)$.

(3) The space $C_pC_p(X)$ and its standard base are in E and j is definable from $X, C_p(X)$, $C_pC_p(X)$ and thus is an element of E. Since $j : X \to C_pC_p(X)$ is an embedding and bases of X and $C_pC_p(X)$ are elements of E, elementarity implies that $j|_E : X_E \to (C_pC_p(X))_E$ is also an embedding. $\square$

Proposition 169. $|X_E| = \chi(C_p(X)_E) = w(C_p(X)_E)$.

Proof. We adjust the standard proof.

By Proposition 168 part (2), $\chi(C_p(X)_E) \leq w(C_p(X)_E) \leq w(\mathbb{R}^{X \cap E}) \leq |X \cap E| = |X_E|$.

Suppose $\chi(C_p(X)_E) < |X \cap E|$. Let 0_X be the zero function on X. Since $C_p(X)_E$ is a topological ring and $0_X \in E$, 0_X witnesses our assumption: there is a local base $\gamma \subseteq \mathcal{B}_p \cap E$ at 0_X consisting of sets of the form $B(0_X, F, n)$, such that $|\gamma| < |X \cap E|$. Let $W = \bigcup\{F : B(0_X, F, n) \in \gamma\}$. Then $|W| < |X \cap E|$. By Proposition 168 part (1), $W \subseteq X \cap E$. Pick

$x \in (X \cap E) \backslash W$ and let $B = B(0_X, \{x\}, 1)$. Then $B \in E \cap \mathcal{B}_p$. Fix any finite $F \subseteq W \subseteq E$. Then there is $g \in C_p(X)$ that maps all of F to zero and x to 1. Since all parameters are in E, we can assume $g \in E$. So $g \in C_p(X)_E \cap V$ for all $V \in \gamma$ but $g \notin B$. This contradicts γ being a local base. $\square$

Proposition 170. $nw(X_E) = nw(C_p(X)_E)$.

Proof. We know $nw(X_E) = nw(C_p(X_E))$. By Proposition 168 part (2), it follows that $nw(C_p(X)_E) \leq nw(C_p(X_E))$. So we have $nw(C_p(X)_E) \leq nw(X_E)$.

On the other hand, this inequality together with Proposition 168 part (3) gives $nw(X_E) \leq nw((C_pC_p(X))_E) \leq nw(C_p(X)_E)$. $\square$

A space Y is said to have Property K if and only if for any uncountable family, $\mathcal{U}$, of open sets, there is uncountable $\mathcal{V} \subseteq \mathcal{U}$ such that for all $V, W \in \mathcal{V}$ we have $V \cap W \neq \emptyset$. The space has the countable chain condition (ccc) if every pairwise disjoint collection of non-empty open subsets of Y is countable. Clearly, separable implies Property K, and Property K implies ccc. Any product of reals has Property K and, since Property K is preserved by dense subsets, $C_p(X)$ also has Property K and hence ccc.

Proposition 171. *$C_p(X)_E$ has Property K.*

Proof. We know that $C_p(X) \cap E$ is a $\mathbb{Q}$-vector subspace of $C_p(X)$. So the closure of $C_p(X) \cap E$ is a $\mathbb{R}$-vector space of $C_p(X)$. Hence $C_p(X) \cap E$ is dense in its closure which embeds densely in some $\mathbb{R}^\kappa$ (Lemma 5 in [24]). Hence $C_p(X) \cap E$ has Property K. Hence $C_p(X)_E$, as its continuous image, also has Property K. $\square$

By elementarity and the fact that a countable base for $\mathbb{R}$ is a subset of E, if $f \in C_p(X)_E$, then $f|X \cap E$ is continuous on X_E. Since $C_p(X)$ is a separator for X, by elementarity $\{f|X \cap E : f \in C_p(X)_E\}$ is a separator X_E.

Proposition 172. *$C_p(X)_E$ is a generator for X_E but not necessarily for $X \cap E$.*

Proof. Fix $x \in X \cap E$ and $U \in \mathcal{B} \cap E$ such that $x \in U$. Then $E \models \exists f \in C_p(X)(f(x) = 1,\ f(U^c) = \{0\})$ since all parameters are in E. So there is such $f \in C_p(X) \cap E$ and we are done.

On the other hand, consider $X = \omega_1 + 1$ and let E be countable. Then $E \cap \omega_1 = \alpha$ is a countable limit ordinal. $X \cap E = \alpha \cup \{\omega_1\}$ with subspace topology and thus $\{\omega_1\}$ is open. Let $x = \omega_1$ and $U = \{\omega_1\}$. Then the only functions that separate x and U^c are ones that are constantly zero on α and non-zero at ω_1. Pick any such $g \in C_p(X)$. We know g is eventually constantly $b = g(\omega_1)$. If $g \in E$ then, by elementarity, $E \models \exists \beta \in \omega_1 (\forall \gamma > \beta\ (g(\gamma) = b))$. This becomes $\exists \beta \in \alpha\ (\forall \gamma \in \alpha \backslash \beta\ (g(\gamma) = b))$. We have a contradiction. No such g can be in E and $C_p(X)_E$ is not a generator. $\square$

B.2.2 Y/E and $C_p(X)$

Let E be an elementary submodel of large enough $H(\theta)$. Let Y be a topological space with base $\mathcal{B}$ and $Y, \mathcal{B} \in E$. Recall that a subset U of Z is called cozero if and only if there is $f \in C_p(Z)$ such that $U = f^{-1}(0,1)$. Clearly, if $Z \in E$ then $U \in E$ if and only if $f \in E$. Define an equivalence relation on Y as follows: $x \sim_E y$ if and only if $x \in V \Leftrightarrow y \in V$ for all cozero sets $V \in E$. Then, by the above observation, $x \sim_E y$ if and only if $f(x) = f(y)$ for all $f \in C_p(Y) \cap E$ (article by Dow in [33]).

Let Y/E be the set of equivalence classes and let $\pi : Y \to Y/E$ be the quotient map. We give Y/E the topology generated by the set $\{\pi(V) : V \in E,\ V \text{ is cozero in } Y\}$. We call this topology the E-quotient topology. For every $f \in C_p(X) \cap E$, the map $f_E : Y/E \to \mathbb{R}$ defined by $f_E([x]) = f(x)$ is well-defined and $f = f_E \circ \pi$. Also, clearly, $\pi(f^{-1}((0,1))) = f_E^{-1}((0,1))$, which implies that E-quotient topology is coarser that the usual quotient topology and f_E is continuous on Y/E. Then π is also continuous. It was proven in [15] that if Y is Tychonoff then so is Y/E.

It is clear from the definition that there is a tight connection between Y/E and $C_p(Y)$. The case when Y is compact has been studied closely [40]. In this case the E-quotient topology and the standard quotient topology coincide on Y/E. Here are some other nice properties that hold when Y is compact (these can be found in [40]).

Theorem 173. *Let Y be compact. Then:*

(1) the set $\{\bar{f} \circ \pi : \bar{f} \in C(Y/E)\}$ is equal to the set $\overline{C(Y) \cap E}$ (the closure is uniform but the set also turns out to be closed in $C_p(Y)$);

(2) the map $j : (Y \times Z)/E \to Y/E \times Z/E$ defined by $[(y, z)] \mapsto ([y], [z])$ is a homeomorphism.

For non-compact Y things are not quite as nice. Let $A = \{\bar{f} \circ \pi_E : \bar{f} \in C_p(Y/E)\}$. Then A does not have to be closed in $C_p(Y)$; in fact it does not have to contain $\overline{C_p(Y) \cap E}$ (closure in $C_p(Y)$). However, as we showed above, if $f \in C_p(Y) \cap E$ then $f_E \in C_p(Y/E)$ and therefore we have $C_p(Y) \cap E \subseteq A$. Recall that the Michael Line is the space obtained from $\mathbb{R}$ by declaring irrational points isolated.

Proposition 174. *Let Y be the Michael Line and E be countable. Then $\overline{C(Y) \cap E} \nsubseteq A$.*

Proof. The elementary submodel E contains all intervals with rational endpoints and since all of them are cozero sets in Y, equivalence classes in Y/E are singletons (we will abuse notation and pretend $\{x\} = x$), then π_E is the identity map. However, the topology of Y/E is strictly weaker than the Michael line topology: since E is countable, there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ that is not in E so $\{x\}$ cannot be open in Y/E.

Note $\chi_{\{x\}} \in C_p(Y)$ and $\chi_{\{x\}} \circ \pi_E = \chi_{\{x\}}$ but $\chi_{\{x\}} \notin C_p(Y/E)$. So $\chi_{\{x\}} \notin A$. We will show that $\chi_{\{x\}} \in \overline{C_p(Y) \cap E}$. Pick four sequences of rationals converging to r, $\{p_n\}_n$, $\{p'_n\}_n$, $\{q_n\}_n$, $\{q'_n\}_n$ such that $\{p_n\}_n$, $\{p'_n\}_n$ are increasing, $\{q_n\}_n$, $\{q'_n\}_n$ are decreasing and $p_n < p'_n < q'_n < q_n$ for each n. Let the graph of $f_n : Y \to \mathbb{R}$ consist of $(-\infty, p_n) \cup (q_n, \infty) \times \{0\}$, $[p'_n, q'_n] \times \{1\}$, the line connecting $(p_n, 0)$ and $(p'_n, 1)$ and the line connecting $(q'_n, 1)$ and $(q_n, 0)$. Since we only used rationals to define the f_n-s, they are elements of E and continuous on Y (since they are continuous on $\mathbb{R}$ with usual topology). Then $\chi_{\{x\}}$ is pointwise limit of the f_n-s, so $\chi_{\{x\}} \in \overline{C_p(Y) \cap E}$. □

Proposition 175. *If $Y \times Z$ is Lindelöf then $j : (Y \times Z)/E \to Y/E \times Z/E$ defined by $[(y, z)] \mapsto ([y], [z])$ is injective.*

Proof. For any $g \in C_p(Y)$ let $\phi_f : Y \times Z \to \mathbb{R}$ be defined by $\phi_g(y, z) = g(y)$. If $[y_1] \neq [y_2]$, pick $f \in C_p(Y) \cap E$ such that $f(y_1) \neq f(y_2)$. Then the function $\phi_f \in E$ witnesses $[(y_1, z)] \neq [(y_2, z)]$. So, by symmetry, j is well-defined.

To show that j is injective we follow the proof in [40]. Suppose $[(y_1, z_1)] \neq [(y_2, z_2)]$ and suppose $F \in E$ witnesses it. Then we may assume that $F(y_1, z_1) < a < a' < b' < b <$

$F(y_2, z_2)$ and $a, a', b, b' \in E$. Let $C = F^{-1}(-\infty, a] \in E$ and $D = F^{-1}[b, \infty) \in E$. Since $Y \times Z$ is Lindelöf and C and D are closed subsets of $Y \times Z$, there are countable collections $\mathcal{U} \in E$ and $\mathcal{V} \in E$ of basic open subsets of $Y \times Z$ such that $C \subseteq \bigcup \mathcal{U} \subseteq F^{-1}(-\infty, a']$ and $D \subseteq \bigcup \mathcal{V} \subseteq F^{-1}[b', \infty)$. Then $\mathcal{U}$ and $\mathcal{V}$ are subsets of E and there are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $(y_1, z_1) \in U$ and $(y_2, z_2) \in V$. Then $U = U_Y \times U_Z$ and $V = V_Y \times V_Z$ and since $\overline{U} \cap \overline{V} = \emptyset$ we may assume, by symmetry, that $\overline{U_Y} \cap \overline{V_Y} = \emptyset$. Since Y is Lindelöf and Tychonoff it is normal and there exists $f \in C_p(Y) \cap E$ such that $f(U_Y) = \{0\}$ and $f(V_Y) = \{1\}$. Then f witnesses $[y_1] \neq [y_2]$. $\square$

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