CONTINUITY IN BANACH SPACES

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The main theme of this document and much of the author's research so far is to use porosity (as well as category) to describe how typical a variety of continuity conditions are within certain collections of functions. The intuition behind the work is that "most things should behave according to a pattern or principle of action". The pattern that the author has in mind is that things should: flow, be localized, and oscillate. Indeed, this theme occurs throughout the authors included work. In Chapter 1, we study what properties a typical bounded real valued derivative possesses, in terms of continuity. We first prove results for finite dimensional domains. Additionally, we obtain some results when the domain is a subset of a general Banach space with a Fréchet differentiable norm. In Chapter 2, we study porosity in relation to bounded variation. In particular, we show that when we suitably norm the space of functions with bounded variation, then the Cantor function becomes the typical example of a function in that space. In Chapter 3, we study how typical (in the sense of both category and porosity) it is for a function that is twice partial differentiable to have equal mixed partial derivatives. As it turns out, the ability to satisfy Clairaut's Theorem is infrequent. In Chapter 4, we study a general condition that implies we have an open, dense, co-porous set whenever we are looking at a set defined by a seminorm in a particular way. This allows us to prove a number of results. In Chapter 5, we introduce a few open questions that the author has recently been working on directly, or has formulated for further study. In the Appendix, we introduce the concept of porosity in an easy to follow format, with illustrative diagrams to guide the reader in their pursuit of intuition.

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PREFACE

"Imagination is the only weapon in the war against reality."
– Lewis Carroll, *Alice in Wonderland*

To me, mathematics is the most beautiful of subjects in that it is a combination of art and science, transforming imagination into methods of study... the study of the patterns of the universe. The joy that this study has provided me is an immeasurable gift given to me by many teachers, mentors and supporters over the years. As such, I would like to thank some of the people who made it possible.

I would first and foremost like to thank my PhD advisor Christopher Lennard for his teaching, support, encouragement, and friendship. I had always hoped to find an advisor brilliant enough that I could respect, laugh with, learn from, and look-up-to, and Dr. Lennard has exceeded any expectations I could have had in all those areas. I would not be the mathematician I am today without his help and influence, and I cannot thank him enough for always being there for me. Countless hours discussing mathematics, correcting my typos, helping with job interviews, endless advice and insight; Dr. Lennard always made himself available to talk about math, my work, life, and anything else. He is the epitome of the good natured professor that everyone wants to know, and I am honored to have him as my "academic father", but more importantly, I am proud to count him as a friend.

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Finally, this work is dedicated to my God, as my faith in Him has sustained me through many of life's ups and downs. *Caelitus mihi vires.*

1.0 BEHAVIOR OF TYPICAL DERIVATIVES ON CERTAIN SETS

1.1 INTRODUCTION

"Continuity is a rare property". This is a simple statement one often hears during introductory analysis classes. The precise statement of this notion is often repeated, and frequently treated from different perspectives. Often we show this by classifying how typical it is for a function to be continuous; for example, in the sense of category. Experience tells us that there is an interesting interplay between considering fixed sets in the domain of a space of functions and the rarity of continuity in said collections of functions.

A use of the Baire category ideas by Banach [3] (and also by S. Mazurkiewicz around the same time), showed that the set of nowhere differentiable functions is residual in $C[0,1]$. This category result was later strengthened using the notion of porosity.

The idea of porosity was first used by Dolzhenko [11] as a way of describing the boundary behavior of certain functions. Since then, this concept has shown itself to be useful in the study of quasiconformal mappings, functional analysis, harmonic analysis, and topology. As we will see later in our introduction and explanation of the topic, porosity is a way of combining the notions of "nowhere dense" and "measure zero" into one sharper notion. A great overview and treatment of the topic can be found in Zajíček's survey paper [28]. Porosity and σ-porosity, are often used to sharpen results that previously were proved only for sets of first category or sets of measure zero. For example, Banach's result has been strengthened by Gandini and Zucco [14] (also later by V. Anisiu) to show that the set of nowhere differentiable functions in $C[0,1]$ is the complement of a σ-porous set.

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Let $(D(X;Y), \|\cdot\|_D)$ be the Banach space of functions $f : B^o(X) \to Y$ with everywhere (on $B^o(X)$) existing derivatives such that for any $f \in D(X;Y)$, $\|f\|_D := \|f\|_\infty + \|f'\|_\infty = \sup_{x \in B(X)} \|f(t)\|_Y + \sup_{t \in B(X)} \|f'(t)\|_{L(X,Y)} < \infty$. Let $(D_N, \|\cdot\|_{D_N})$ be the Banach space of functions $f : [0,1]^N \to \mathbb{R}$ with everywhere (on $[0,1]^N$) existing derivatives such that for any $f \in D_N$, $\|f\|_{D_N} := \|f\|_\infty + \|f'\|_\infty = \sup_{x \in [0,1]^N)} \|f(t)\|_Y + \sup_{t \in [0,1]^N} \|f'(t)\|_{L(X,Y)} < \infty$. We can also consider $D[0,1] := \{f \in C[0,1] : f' \text{ exists and } f' \text{ is bounded on } [0,1]\}$ with norm $\|f\|_{D[0,1]} = \|f\|_\infty + \|f'\|_\infty$ as a space of differentiable functions that are "well behaved". Furthermore, let $\Lambda(f', a) := \lim_{\delta \to 0^+} \sup_{h,k \in B^o(a;\delta)} \|f'(h) - f'(k)\|_{L(X,Y)}$ be the tangential oscillation of the function f at the point a. Recall that f' is continuous at the point a if and only if $\Lambda(f', a) = 0$. Note that f' above may not be everywhere continuous, but that by a well known result of Baire we have that the set of these continuities is a dense G_δ set. In particular, A is the set of positive tangential oscillation of a real function if and only if A is a F_σ set (see [7]).

If we define $D^\sharp(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid \sup_{t \in \mathbb{R}} |f(t)| < \infty \text{ and } \exists F \text{ such that } F'(t) = f(t), \forall t \in \mathbb{R}\}$, then $(D^\sharp(\mathbb{R}, d(\cdot,\cdot))$ is a complete metric space with metric $d(f,g) := \sup_{t \in \mathbb{R}} |f(t) - g(t)|$. Then let $A := \{f \in D^\sharp(\mathbb{R}) : m(\{x : f \text{ is continuous at } x\}) > 0\}$. In 1978, Clifford Weil [26] proved A is a first category set in $D^\sharp(\mathbb{R})$. Weil's work is an extention of work done by Casper Goffman [16] in 1977, that gave existence of the generating function used in Weil's proof. However, much more can be said.

We proved the following result:

Theorem 1.1.1. *Let $\{x_n : n \in \mathbb{N}\}$ be a countable (possibly finite) set in Banach space $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ another Banach space. Then let $G := \{f \in D(X,Y) : \Lambda(f', x_n) > 0 \text{ for all } n \in N\}$ is a dense G_δ set in $D(X,Y)$ that is co-σ-porous, and if N is finite, then G is open and co-porous.*

Yet, we can in fact get much more when we restrict the domain of our functions to $\mathbb{R}^n$. In particular, we proved the following:

Theorem 1.1.2. *Let $E \subset [0,1]^N$ be a closed, nowhere dense set. Then $G := \{f \in D_N : \inf_{x \in E} \Lambda(f', x) > 0\}$ is a dense, open, co-porous set in D_N.*

One of our first results is that if E is an arbitrary closed nowhere dense subset of $[0,1]$, then $G_E := \{f \in D[0,1] : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$ is a dense open subset of $D[0,1]$; and furthermore, G_E is co-porous in $D[0,1]$. This result came about through asking the following question: "How typical is the Volterra-Cantor Function?". The Volterra-Cantor function, is often used as an example in analysis classes, but is it truly odd, or is it the type of function that we should expect? As our result shows, this example is actually what we should be thinking of when we think of an arbitrary function with bounded derivative.

Throughout the course of our research we had interesting conversations with many people. Specifically, the we would like to thank J. Holland for his initial conversations about the Volterra-Cantor type functions, as well as J. Sivek for useful discussions and ideas about differentiable norms as well as alerting us to Cartan's result [9]. We would also like to thank J.B. Turret for alerting us to the work of Weil, as well as pointing out some of the differences in the definitions and work.

1.2 PRELIMINARIES AND DEFINITIONS

We start with some definitions and preliminary theorems.

Definition We begin in the interval $[0, 1]$:

- $C[0, 1] := \{\text{The set of continuous functions on } [0, 1]\}$.
- $C^{(1)}[0, 1] := \{f \in C[0, 1] : f' \text{ exists, and } f' \text{ is continuous on } [0, 1]\}$
- $D[0, 1] := \{f \in C[0, 1] : f' \text{ exists and } f' \text{ is bounded on } [0, 1]\}$

  If $(X, \|\cdot\|)$ is any normed space, then we define the following subsets in the usual way for $x \in X$:

- $B(x; r) := B_r(x) = \{y \in X : \|x - y\| \leq r\}$
- $B^o(x; r) := B^o_r(x) = \{y \in X : \|x - y\| < r\}$

Further, define $\|f\|_D := \|f\|_\infty + \|f'\|_\infty$ for all $f \in D[0, 1]$.

Theorem 1.2.1. *Suppose that $f_n \to f$ uniformly on a set $E \subseteq M$ in a metric space (M, d), for f_n, f real functions (for all n). Let x be a limit point of E, and suppose that*

$$\lim_{\substack{t\to x\\ t\in E}} f_n(t) = a_n \text{ for all } n \in \mathbb{N}. \tag{1.1}$$

Then, a_n converges, and

$$\lim_{\substack{t\to x\\ t\in E}} f(t) = \lim_{\substack{t\to x\\ t\in E}} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{\substack{t\to x\\ t\in E}} f_n(t) = \lim_{n\to\infty} a_n \tag{1.2}$$

Proof. (See [22, p.152])

Fix $\epsilon > 0$. By the uniform convergence, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $\forall t \in E$,

$$|f_n(t) - f_m(t)| < \epsilon \tag{1.3}$$

$$|f_n(t) - f(t)| < \epsilon. \tag{1.4}$$

Letting $t \to x$ in 1.3, we get $|a_n - a_m| < \epsilon$. Thus $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$, converging to some $a \in \mathbb{R}$. Also, $[|a_n - a| \le \epsilon]$, $\forall n \ge N$. So, $\forall t \in E$,

$$|f(t) - a| \le |f(t) - f_N(t)| + |f_N(t) - a_N| + |a_N - a| \le \epsilon + |f_N(t) - a_N| + \epsilon.$$

Now by 3.1, there exists an open set $V \subset M$ such that $\forall t \in E \cap V$ with $t \neq x$, $|f_N(t) - a_N| < \epsilon$. So $\forall t \in E \cap V$ with $t \neq x$, $|f(t) - a| \le 3\epsilon$. Therefore, $\lim_{\substack{t \to x \\ t \in E}} f(t) = a$, and the conclusion holds. □

Theorem 1.2.2. *Fix $-\infty a < b < \infty$. Let $\{g_n\}$ be a sequence of functions such that for all $n \in \mathbb{N}$ we have $g_n : [a,b] \to \mathbb{R}$, g_n is differentiable on $[a,b]$, and such that $\alpha := \lim_{n\to\infty} g_n(x_0)$ exists in $\mathbb{R}$ for some $x_0 \in [a,b]$. Also, suppose that $\{g_n'\}$ converges uniformly on $[a,b]$ to some function $h : [a,b] \to \mathbb{R}$. Then $\{g_n\}$ converges uniformly on $[a,b]$ to some function $g : [a,b] \to \mathbb{R}$, and we have for all $x \in [a,b]$:*

$$g'(x) = \lim_{n\to\infty} g_n'(x) = h(x)$$

Proof. (See [22, p.153]) Fix $\epsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $\forall n, m \in N$, $|g_n(x_0) - g_m(x_0)| < \epsilon$; and also $\forall n, m \ge N$, $\forall x \in [a,b]$, $|g_n'(x) - g_m'(x)| < \epsilon$. So, fix $x \in [a,b]$ arbitrary. For all $n, m \ge N$, we have (using the Mean Value Theorem, for some η between x_0 and x):

$$|g_n(x) - g_m(x)| \le |g_n(x) - g_m(x) - (g_n(x_0) - g_m(x_0))| + |g_n(x_0) - g_m(x_0)|$$
$$\le \left|(g_n'(\eta) - g_m'(\eta))(x - x_0)\right| + \epsilon = \left|g_n'(\eta) - g_m'(\eta)\right| |x - x_0| + \epsilon \le \epsilon(b-a) + \epsilon.$$

So, for some $g(x) \in \mathbb{R}$, we have $g_n \underset{n}{\to} g$ uniformly on $[a,b]$. Fix $x \in [a,b]$ arbitrary, and let $E := [a,b] \setminus \{x\}$. Then for all $n \in \mathbb{N}$, for all $t \in E$, define:

$$\phi_n(t) := \frac{g_n(t) - g_n(x)}{t - x}$$
$$\phi(t) := \frac{g(t) - g(x)}{t - x}.$$

Then, for any $n \in \mathbb{N}$ we have $\phi_n : E \to \mathbb{R}$ and $\phi : E \to \mathbb{R}$. Observe that $\phi_n \underset{n}{\to} \phi$ uniformly on E. Indeed, $\forall t \in E$, $\forall n, m \ge N$ as above, using the Mean Value Theorem with ξ between x and x_0:

$$|\phi_n(t) - \phi_m(t)| = \frac{|(g_n(t) - g_m(x)) - (g_m(t) - g_m(x))|}{|t - x|}$$
$$= \frac{\left|(g_n'(\xi) - g_m'(\xi))(t - x)\right|}{|t - x|} = \left|g_n'(\xi) - g_m'(\xi)\right| \le \epsilon.$$

Thus, $a_n := \lim_{\substack{t\to x\\ t\in E}} \phi_n(t) = g_n'(x)$ exists in $\mathbb{R}$, for all $n \in \mathbb{N}$. Therefore, by 3.1,

$$g'(x) = \lim_{\substack{t\to x\\ t\in E}} \phi(t) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} g_n'(x) = h(x)$$

□

Proposition 1.2.1. *As defined above,* $(D[0,1], \|\cdot\|_D)$ *is a Banach space.*

Proof. That $\|\cdot\|_D$ is a norm is clear. In particular, we note that if $\|f\|_D := \|f\|_\infty + \|f'\|_\infty = 0$, then clearly $f(t) = 0, \forall t \in [0,1]$. Thus, we need only discuss completeness. Let $\{f_n\}$ be a $\|\cdot\|_D$-Cauchy sequence in $D[0,1]$. Then $\{f_n\}$ is a $\|\cdot\|_\infty$-Cauchy sequence in $(C[0,1], \|\cdot\|_\infty)$. So, as $(C[0,1], \|\cdot\|_\infty)$ is complete (see [22]), then there exists $f \in C[0,1]$ such that $\|f_n - f\|_\infty \underset{n}{\to} 0$. Also, $\{f_n'\}$ is a uniformly-Cauchy sequence, i.e. $\|f_k - f_m\|_\infty \xrightarrow[k,m\to\infty]{} 0$. Thus, there exists a function $h : [0,1] \to \mathbb{R}$ such that $f_n' \underset{n}{\to} h$ uniformly on $[0,1]$. Then by theorem 1.2.2, f is differentiable on $[0,1]$ and $f'(x) = h(x)$ for all $x \in [0,1]$. So, $f \in D[0,1]$. Also, $\|f_n - f\|_\infty = \|f_n - f\|_\infty + \|f_n' - f'\|_\infty \underset{n}{\to} 0 + 0 = 0$. Thus, $(D[0,1], \|\cdot\|_D)$ is complete. □

Remark We note here that the space we are using is a different space then the one that was used historically. Historically (as mentioned in Weil [26]), derivatives spaces don't include a norm of the primary function, only the derivative. Also, historically the spaces are just complete metric spaces, not Banach spaces. Our space uses both the primary function and its derivative in its norm.

Remark What if we had gone with defining our space as $D(0,1) := \{f \in C(0,1) : f'$ exists and f' is bounded on $(0,1)\}$? Let's investigate details of an interesting example. Let $f(0) := 0$ and $[f(x) := \sin(\ln(x)) + \cos(\ln(x))$, for all $x \in (0,1]]$. Note that f is differentiable, and therefore continuous at every $x \in (0,1]$. An interesting thing happens if we let $x_n := e^{-(\pi+2\pi n)} \in (0,1]$ and $y_n := e^{-2\pi n} \in (0,1]$. Note that $x_n, y_n \to 0$ as $n \to \infty$. However, for any $n \in \mathbb{N}$:

$$\begin{aligned} f(x_n) &= \sin\left(\ln(e^{-(\pi+2\pi n)})\right) + \cos\left(\ln(e^{-(\pi+2\pi n)})\right) \\ &= \sin(-(\pi + 2\pi n)) + \cos(-(\pi + 2\pi n)) \\ &= \sin(-\pi - 2\pi n) + \cos(-\pi - 2\pi n) = 0 + -1 = -1. \end{aligned}$$

and

$$f(y_n) = \sin\left(\ln(e^{-2\pi n})\right) + \cos\left(\ln(e^{-2\pi n})\right)$$
$$= \sin(-2\pi n) + \cos(-2\pi n) = 0 + 1 = 1.$$

Therefore, we can see that f is not continuous at zero, yet it is also a bounded function that is continuous at all but a finite set in $[0,1]$ and is therefore Riemann-integrable in $[0,1]$ (a fact we don't need, but is fun to note). Now define the function $F(0) := 0$ and $[F(x) := x\sin(\ln(x))$, for all $x \in (0,1]]$. Now, we have $F'(x) = f(x)$ for any $x \in (0,1]$. Therefore, $F \in D(0,1)$.

However, an interesting thing happens if we let $x_n := e^{-(\frac{\pi}{2}+2\pi n)} \in (0,1]$ and $y_n := e^{-(-\frac{\pi}{2}+2\pi n)} \in (0,1]$. Note that $x_n, y_n \to 0$ as $n \to \infty$. Let's investigate $\frac{F(x_n)-F(0)}{x_n-0}$ and $\frac{F(y_n)-F(0)}{y_n-0}$. However, for any $n \in \mathbb{N}$:

$$\frac{F(x_n) - F(0)}{x_n - 0} = \frac{e^{-(\frac{\pi}{2}+2\pi n)}\sin\left(\ln(e^{-(\pi+2\pi n)})\right) - 0}{e^{-(\frac{\pi}{2}+2\pi n)}}$$
$$= \sin\left(-(\frac{\pi}{2} + 2\pi n)\right) = -1.$$

and

$$\frac{F(y_n) - F(0)}{y_n - 0} = \frac{e^{-(-\frac{\pi}{2}+2\pi n)}\sin\left(\ln(e^{-(\pi+2\pi n)})\right) - 0}{e^{-(\frac{\pi}{2}+2\pi n)}}$$
$$= \sin\left(-(-\frac{\pi}{2} + 2\pi n)\right) = 1.$$

Therefore, we see that F' does not exist at zero. What does this give? Well, it gives that $F \in D(0,1)$, but $F \notin D[0,1]$. Hence, $D[0,1] \subsetneq D(0,1)$ as sets. Note that both $D[0,1]$ and $D(0,1)$ with the norm $\|\cdot\|_D$ are Banach spaces. However, it would seem to us as if the more intuitive notion of a D type space for $\mathbb{R}$ is to use $D[0,1]$, therefore, it is the one that we use.

Proposition 1.2.2. $C^{(1)}[0,1]$ *is a closed vector subspace of* $D[0,1]$.

Proof. Let $g \in D[0,1]$ be arbitrary, and suppose there exists $(f_n)_{n\in\mathbb{N}} \subseteq C^{(1)}[0,1]$ such that $\lim_{n\to\infty} \|f_n - g\|_D = 0$.

Therefore, $\lim_{n\to\infty} \|f_n - g\|_D = \lim_{n\to\infty} (\|f_n - g\|_\infty + \|f_n' - g'\|_\infty) = 0$. Now, each f_n' is continuous, and convergence in $\|\cdot\|_\infty$ is uniform convergence. So we have $(f_n')_{n\in\mathbb{N}} \subseteq C[0,1]$ converging uniformly to g'. Therefore, g' must be continuous. □

We now define the oscillation of a function on a normed space.

Definition Let $(X, \|\cdot\|)$ be a normed space. For any bounded functions $S : X \to \mathbb{R}$, define for $\delta > 0$ the function $\Lambda_\delta(S, x_0) := \sup_{h,k\in B^o_\delta(x_0)} \|S(h) - S(k)\|$, and $\Lambda(S, x_0) := \lim_{\delta\to 0^+} \Lambda_\delta(S, x_0)$, for all $x_0 \in X$.

Note the following fact.

Lemma 1.2.1. *Let $(X, \|\cdot\|)$ be a normed space with $A \subseteq X$, and let $S : X \to \mathbb{R}$ and $T : X \to \mathbb{R}$ be functions. Then the following holds true:*

- $\sup_{x\in A} (S(x) + T(x)) \leq \sup_{x\in A} S(x) + \sup_{x\in A} T(x)$
- $\inf_{x\in A} S(x) + \inf_{x\in A} T(x) \leq \inf_{x\in A} (S(x) + T(x))$

We now state a proposition that we use later.

Proposition 1.2.3. *Let $(X, \|\cdot\|)$ be a normed space with $A \subseteq X$, and let $Q : X \to \mathbb{R}$, $S : X \to \mathbb{R}$ and $T : X \to \mathbb{R}$ be functions. Further, let Λ be the oscillation function. Then the following holds true:*

1. $\Lambda(Q, t) \geq 0$.
2. $|\Lambda(S, t) - \Lambda(T, t)| \leq \Lambda(S + T, t)$ *for all* $t \in A$.

Proof. Let X, A, S, T, Q be as above. Then (1) is clear from the definition, so we will prove (2). Let $\delta > 0$, and $t \in A$. So we have that for $h, k \in B^o_\delta(t)$ then by the triangle inequality:

$$\begin{aligned} \|S(h) - S(k)\| &= \|(S(h) - S(k)) + (T(h) - T(k)) - (T(h) - T(k))\| \\ &\leq \|(S(h) - S(k)) + (T(h) - T(k))\| + \|(T(h) - T(k))\| \end{aligned}$$

Thus, by applying Lemma (1.2.1), we get $\Lambda_\delta(S, t) \leq \Lambda_\delta(S + T, t) + \Lambda_\delta(T, t)$. Therefore, we may now take a limit as $\delta \to 0^+$, and get $\Lambda(S, t) \leq \Lambda(S + T, t) + \Lambda(T, t)$ which easily implies the inequality in (2), and we are done. □

Consider now the case where $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ and $A = [0,1]$. Let $S \in D[0,1]$. We may refer to $\Lambda(S', a)$ as the ***tangential oscillation*** at $a \in [0,1]$.

We now summarize the facts we know by observing that $S \mapsto \Lambda(S', a)$ ***has seminorm properties. In particular:***

1. $\Lambda(cS', t) = |c|\Lambda(S', t)$ for any $t \in [0,1]$ and some $c \in \mathbb{R}$
2. $\Lambda(S', t) \geq 0$ for all $t \in [0,1]$.
3. $\Lambda(S' + T', t) \leq \Lambda(S', t) + \Lambda(T', t)$ for any $t \in [0,1]$
4. $|\Lambda(S', t) - \Lambda(T', t)| \leq \Lambda(S' + T', t)$ for all $t \in [0,1]$.
5. Also, $\Lambda(S', t) \leq 2\|S\|_D$

1.3 AN INTERESTING COLLECTION OF FUNCTIONS

We now define a certain auxiliary function, that we shall denote by $F(x, \alpha)$, which is often called the dampened topological sine function.

Definition Let $\alpha \in \mathbb{R}$ and let

$$F(x,\alpha) := \begin{cases} (x-\alpha)^2 \sin(\frac{1}{x-\alpha}) & : x \neq \alpha \\ 0 & : x = \alpha \end{cases}, \forall x \in \mathbb{R}$$

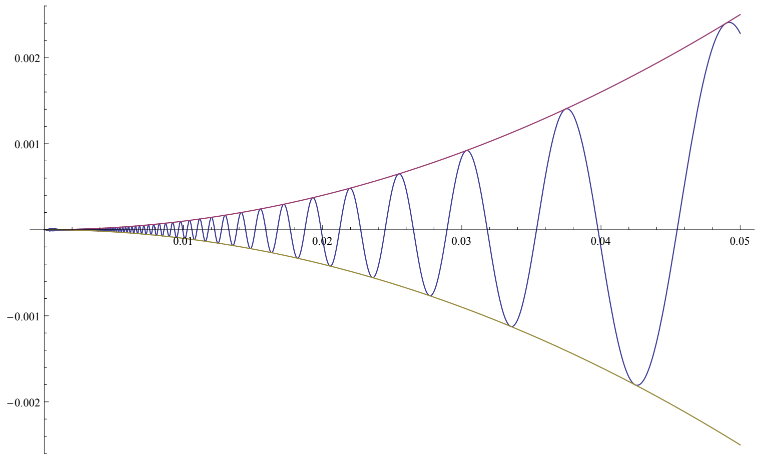

Figure 1: Dampened Topological Sine Between 2 Parabolas.

Proposition 1.3.1. *As defined above, $F(x,\alpha)$ is continuous and differentiable in x on $\mathbb{R}$. Moreover, we also have that $F \in D[0,1]$, i.e. F and it's derivative (which exists) are bounded on $[0,1]$.*

Proof. It is clear that $F(x,\alpha)$ is a continuous bounded function, and that it is differentiable away from $x = \alpha$. In fact, when $x \neq \alpha$ we see that $F'(x,\alpha) = 2(x-\alpha)\sin(\frac{1}{x-\alpha}) - \cos(\frac{1}{x-\alpha})$. So the only non-obvious thing to check is differentiability at $x = \alpha$. So we have

$$\lim_{|h|\to 0^+} \frac{F(\alpha+h,\alpha) - F(\alpha,\alpha)}{h} = \lim_{|h|\to 0^+} h \sin\left(\frac{1}{h}\right) = 0.$$

Thus, $F'(\alpha,\alpha) = 0$. □

1.3.1 The Yo-Yo

We can use this function to define a second auxiliary function $F_{(a,b)}(x)$ on the interval $(a,b) \subseteq [0,1]$ which we call the "Yo-Yo function", for reasons that shall become obvious. It is a smooth double-ending of the dampened topological sine. We develop this method from the classical example of Volterra, used in the construction of the Volterra function.

Definition We notice that $F'(x, a)$ has an infinite number of zeros in the interval $(a, \frac{a+b}{2}]$, so let $a + \gamma$ be the largest such value in $(a, \frac{a+b}{2}]$, where $\gamma \in \mathbb{R}, \gamma \geq 0$. We may now define

$$F_{(a,b)}(x) := \begin{cases} F(x, a) & : a < x \leq a + \gamma \\ F(a + \gamma, a) & : a + \gamma \leq x \leq b - \gamma \\ -F(x, b) & : b - \gamma \leq x < b \\ 0 & : x \in \mathbb{R} \setminus (a, b) \end{cases}$$

Now, by examining the definition of $F_{(a,b)}|_{[0,1]}$, we see that it is continuous, differentiable, and in $D[0, 1]$. To understand this definition, it is helpful to consider a typical Yo-Yo function, taken in the interval $(0, 1)$, i.e. to consider the graph of $F_{(0,1)}$.

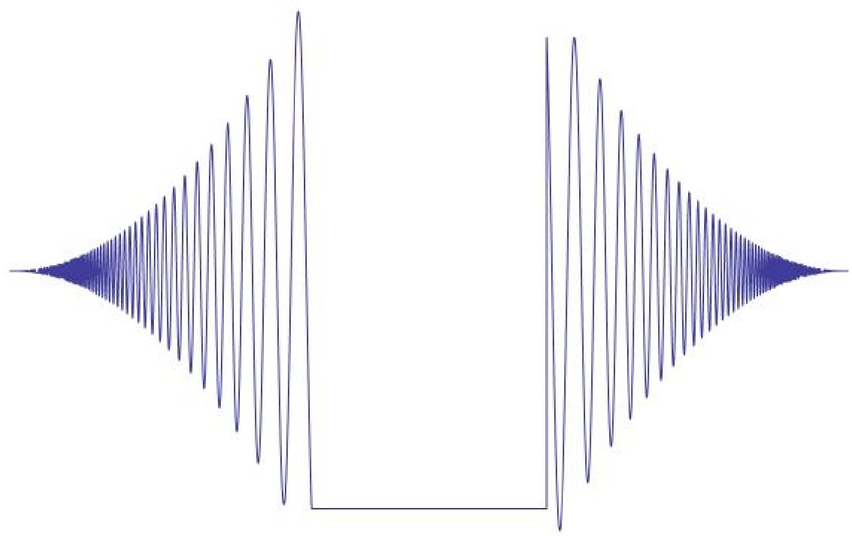

Figure 2: Typical Yo-Yo Function In An Interval.

1.3.2 Oscillation Theorem At A Point in $\mathbb{R}^1$

We are now ready to begin pursuing some interesting applications of these results. We start with an interesting result about what a "typical" function in $D[0, 1]$ really looks like.

Proposition 1.3.2. *Fix* $x_0 \in [0, 1]$. *Let* $G := \{f \in D[0, 1] : \Lambda(f', x_0) > 0\}$. *Then* G *is open and dense in* $D[0, 1]$.

Proof. First we prove open. To do so, we will show that $G^c := D[0,1]\backslash G = \{f \in D[0,1] : \Lambda(f', x_0) \leq 0\} = \{f \in D[0,1] : \Lambda(f', x_0) = 0\}$ is closed. Therefore, let $f \in D[0,1]$ and $(f_n)_{n\in\mathbb{N}} \subseteq G^c$ such that $\|f_n - f\|_D \underset{n}{\to} 0$. Thus, $(\|f_n - f\|_\infty + \|f_n' - f'\|_\infty) \underset{n}{\to} 0$, and $\Lambda(f_n', x_0) = 0$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$ be arbitrary. There exists $N_0 \in \mathbb{N}$ such that $(\|f_n - f\|_\infty + \|f_n' - f'\|_\infty) < \frac{\epsilon}{3}$, for all $n \geq N_0$. Then there exists $\delta_0 > 0$ such that $\Lambda_\delta(f_{N_0}, x_0) < \frac{\epsilon}{3}$ for all $\delta \leq \delta_0$. Now then, let $h, k \in (-\delta_0, \delta_0)$ for δ_0 above. Then,

$$\begin{aligned}
&|f'(x_0+h) - f'(x_0+k)| \\
&\quad \leq |f'(x_0+h) - f_{N_0}'(x_0+h)| + |f_{N_0}'(x_0+h) - f_{N_0}'(x_0+k)| + |f_{N_0}'(x_0+k) - f'(x_0+k)| \\
&\quad\quad < \frac{2\epsilon}{3} + |f_{N_0}'(x_0+h) - f_{N_0}'(x_0+k)| < \epsilon.
\end{aligned}$$

Now, apply the supremum within $(-\delta_0, \delta_0)$, and take the limit and we get $\lim\limits_{\delta\leq\delta_0} \sup\limits_{h,k\in(-\delta,\delta)} |f'(x_0+h) - f'(x_0+k)| \leq \epsilon$ for any $\epsilon > 0$. Therefore, $\Lambda(f', x_0) = 0$, and so G^c is closed... i.e. G is open.

Now we prove density of the set. Let $f \in D[0,1]$ be arbitrary. If $f \in G$, there is nothing to do. So without loss of generality, assume that $f \in G^c$. We wish to find $\widehat{g_\epsilon} \in G$ such that $\|f - \widehat{g_\epsilon}\|_D < \epsilon$ for any $\epsilon > 0$ with ϵ small. As such, we will define the following auxiliary function for $\nu \in \mathbb{R}, \nu > 0$ small:

$$f_{\beta,\nu}(x) := \beta F_{(0,\nu)}(x).$$

It is clear that $f_{\beta,\nu}(x)$ is just a β scaling of our Double Yo-Yo type function, so $f_{\beta,\nu}(x) \in D[0,1]$. Thus, we can say the following:

$$f_{\beta,\nu}'(x) = \begin{cases} 0 : x = 0 \\ \beta F_{(0,\nu)}'(x) : 0 < x < \nu \\ 0 : \nu \leq x \leq 1 \end{cases}$$

It will be useful to get a bound on $\Lambda(f_{\beta,\nu}', 0)$. Fix $\nu > 0$, arbitrary. While there is a best bound, we will be satisfied with a bound showing positivity of $\Lambda(f_{\beta,\nu}', 0)$. Let $x_n := \frac{1}{2\pi n}, y_n := \frac{1}{(2n+1)\pi}$ so that $\{x_n\}, \{y_n\}$ are real sequences. Eventually these sequences are entirely in $(0, \frac{\nu}{2}]$, so eventually

$f_{\beta,\nu}(x_n), f_{\beta,\nu}(y_n)$ behave as $F(x_n,0), F(y_n,0)$, where we recall that for $z > 0$, $F(z,0) = z^2\sin(\frac{1}{z})$. Then the following is true for any n large enough:

1. $0 < y_n < x_n$
2. $x_n, y_n \underset{n}{\to} 0$
3. $f'_{\beta,\nu}(x_n) = -\beta,\ f'_{\beta,\nu}(y_n) = \beta$

Therefore, for any $\delta > 0$ there exists $N_0 \in \mathbb{N}$ such that $\forall n > N_0$,

$$0 < y_n < x_n < \delta \text{ and } |f'_{\beta,\nu}(x_n) - f'_{\beta,\nu}(y_n)| = 2\beta. \tag{1.5}$$

Hence, we can conclude $2\beta \leq \Lambda(f'_{\beta,\nu}, 0)$, which is good enough for our purposes.

Now, we will use $f_{\beta,\nu}(x)$ to construct the required $h_\epsilon(x)$ to prove the density. First recall the Reverse Triangle Inequality:

$$||a| - |b|| \leq |a - b| \text{ for all } a, b \in \mathbb{R}.$$

Define (for $x_0 \in [0,1)$, where the case $x_0 = 1$ is similar) $g_{\beta,\nu}(x) := f(x) + f_{\beta,\nu}(x - x_0)$. Now we are legitimately prepared to go about showing density for the still fixed f. Let $\epsilon > 0$ be small and arbitrary. Then let $\beta := \frac{\epsilon}{4}$, $\nu = \frac{\epsilon}{6}$. We will show that $\widehat{g_\epsilon} = g_{\beta,\nu}$ "works". First, $\|f - g_{\beta,\nu}\|_D = \|f - g_{\beta,\nu}\|_\infty + \|f' - g'_{\beta,\nu}\|_\infty \leq \frac{\epsilon}{4}\left(\frac{\epsilon}{6}\right)^2 + \frac{\epsilon}{2} < \epsilon$. So $g_{\beta,\nu}$ is "close", and it is also clear that $g_{\beta,\nu} \in D[0,1]$.

Now we want to show that $g_{\beta,\nu} \in G$. Well, using the above,

$$\begin{aligned}
0 < \frac{\epsilon}{2} = 2\beta = 2\beta - 0 &\leq \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |f'_{\beta,\nu}(h) - f'_{\beta,\nu}(k)| - \sup_{h,k\in(-\delta,\delta)} |f'(x_0+h) - f'(x_0+k)| \\
&= \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |f'_{\beta,\nu}(h) - f'_{\beta,\nu}(k)| - \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |f'(x_0+h) - f'(x_0+k)| \\
&\leq \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} \Big||f'_{\beta,\nu}(h) - f'_{\beta,\nu}(k)| - |f'(x_0+h) - f'(x_0+k)|\Big| \\
&\leq \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |g'_{\beta,\nu}(x_0+h) - g'_{\beta,\nu}(x_0+k)| \\
&= \Lambda(g'_{\beta,\nu}, x_0).
\end{aligned} \tag{1.6}$$

Thus, $g_{\beta,\nu} \in G$ and so then G is open and dense in $D[0,1]$. □

By the Baire Category theorem, we get a theorem as an easy corollary.

Theorem 1.3.1. *Fix E a countable subset of* $[0,1]$. *Then* $G := \{f \in D[0,1] : \Lambda(f, x_0) > 0, \forall x_0 \in E\}$ *is a dense G_δ set in* $D[0,1]$.

1.4 WHAT IS A TYPICAL FUNCTION?

We would like to begin to answer in some sense, the following question: what is the behavior of a typical function in $D[0,1]$?

If we labor a little more, we will receive an interesting reward in the way of an answer to the above question. In particular, we would like to know about having our oscillation points as before be some arbitrary nowhere dense set instead of just a countable set.

Definition We recall that a set A a subset of a topological space (T, τ) is nowhere dense if $int_T(cl_T(A)) = \emptyset$.

1.5 VOLTERRA EXAMPLE

We start with a lemma that will lend itself to gaining intuition regarding the next problem. We will use some of our previously defined functions.

Lemma 1.5.1. *(A Generalization of the Volterra Function) Let E be an arbitrary nowhere dense subset of* $[0,1]$. *Let* $G_E := \{f \in D[0,1] : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$. *The set G_E is non-empty.*

Proof. (Lemma 1.5.1) We will use a generalization of an example of a function created by Volterra. Without loss of generality, we may take E to be closed. For if not, then we may let $A := cl(E)$, and show that $G_A \neq \emptyset$, for then as $G_A \subseteq G_E$ we know that $G_E \neq \emptyset$. So assume that E is closed.

Recall the fact that every closed, nowhere dense set is the boundary of an open set ($[0, 1] \setminus E$). Furthermore, recall that any open set in $\mathbb{R}$ can be written as the disjoint countable union of open intervals. So, let $J := [0, 1] \setminus E$ with $J = (\bigcup_{n\in\Gamma} I_n) \cap [0, 1]$ where each $I_n = (a_n, b_n)$ such that $a_n < b_n$ for any $n \in \Gamma$, and for any neighborhood N about a point $x \in E$ $\exists k \in \mathbb{N}$ such that $a_k \in N$ or $b_k \in N$. Here we have that $\Gamma \neq \emptyset$, where either $\Gamma = \{1, ..., t\}$ for some $t \in \mathbb{N}$, or else $\Gamma = \mathbb{N}$. Now, if Γ is finite, then it is easily seen to be similar to our earlier work. So without loss of generality, assume that $\Gamma = \mathbb{N}$.

Now, we will define our function $V(x)$ as follows:

$$V(x) := \begin{cases} 0 & : x \in [0, 1] \setminus J \\ F_{(a_n,b_n)}(x) & : x \in I_n, \forall n \in \mathbb{N} \end{cases}$$

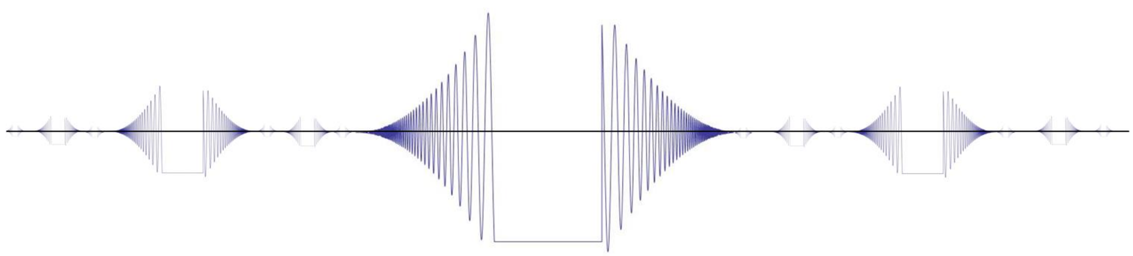

Figure 3: An Illustration Of A Typical V.

We shall now show that $V(x)$ has a derivative everywhere, as in [15, 107-108] and similar to [25, 165-166]. If x is any point of E and if y is any other point of $[0, 1]$, then either $V(y) = 0$ or y is a point of some removed interval $I_n = (a_n, b_n)$. In the former case,

$$\left|\frac{V(y) - V(x)}{y - x}\right| = \left|\frac{0}{y - x}\right| = 0 < |y - x| .$$

In the latter case, let d be the endpoint of I_n nearer to y, and in particular $|x - d| \leq |x - y|$. Then, we observe that $V(x) = 0$ and $V(y) = F_{(a_n,b_n)}(y)$ to get:

$$\left|\frac{V(y) - V(x)}{y - x}\right| = \left|\frac{V(y)}{y - x}\right| \leq \left|\frac{V(y)}{y - d}\right| = \left|\frac{F_{(a_n,b_n)}(y)}{y - d}\right| \tag{1.7}$$

Now, we recall the definition of $F_{(a_n,b_n)}$:

$$F_{(a_n,b_n)}(x) := \begin{cases} F(x, a_n) & : a_n < x \le a_n + \gamma_n \\ F(a_n + \gamma_n, a_n) & : a_n + \gamma_n \le x \le b_n - \gamma_n \\ -F(x, b_n) & : b_n - \gamma_n \le x < b_n \\ 0 & : x \in \mathbb{R} \setminus (a_n, b_n) \end{cases}$$

Therefore, as the derivative is zero in the interval $[a_n+\gamma_n, b_n-\gamma_n]$, we can assume our $y \in (a_n, a_n + \gamma_n] \cup [b_n - \gamma_n, b_n)$. Thus, $F_{(a_n,b_n)}(y) = \left|(y-d)^2 \sin(\frac{1}{y-d})\right| \le |y-d|^2$. Hence, we have:

$$\left|\frac{F_{(a_n,b_n)}(y)}{y-d}\right| \le \left|\frac{|y-d|^2}{y-d}\right| = |y-d| \le |y-x|. \tag{1.8}$$

Therefore, in either case, $\left|\frac{V(y)-V(x)}{y-x}\right| \le |y-x|$, and consequently $V'(x) = 0$ for any $x \in E$. Furthermore, as we have a uniform lower bound on the oscillation $\Lambda(V', a_n) \ge 2$ and $\Lambda(V', b_n) \ge 2$ at every endpoint, we can see that for every $x \in E$ (by taking neighborhoods about x and noting that in that neighborhood there is an endpoint whose tangential oscillation is at least 2), then we actually have $\Lambda(V', x) \ge 2$.

On the other hand, if x belongs to any removed interval $I_n = (a_n, b_n)$,

$$|V'(x)| \le |2z sin(1/z) - cos(1/z)| \le 3$$

for some z between 0 and 1, so that $V(\cdot)$ is everywhere differentiable on $[0, 1]$, and it's derivative is bounded everywhere. □

1.5.1 Category Results for Our Derivative Sets

Theorem 1.5.1. *Let E be an arbitrary nowhere dense subset of* $[0, 1]$. *Let* $G_E := \{f \in D[0, 1] : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$. *Then G_E is a dense open subset of* $D[0, 1]$.

Proof. (Theorem 1.5.1) Now, by the previous lemma, we know that the set G_E is non-empty. Therefore, the question of open and density "makes sense". We prove open first, and do so directly. Let $f \in G_E$ be arbitrary, and $\delta := \inf\limits_{x_0 \in E} \Lambda(f', x_0) > 0$. Now choose ϵ arbitrary such that $0 < \epsilon \leq \delta/4$. Then let $s \in D[0, 1]$ such that $\|s\|_D < \epsilon$, and $h := f + s$. We will show that $h \in G_E$. Well, for any $x \in E$,

$$\delta/2 < \delta - 2\epsilon < \Lambda(f', x) - 2\|s'\|_\infty < \Lambda(f', x) - \Lambda(s', x) < \Lambda(f' + s', x) =: \Lambda(h', x).$$

Therefore, $0 < \delta/2 \leq \inf_{x \in E} \Lambda(h', x)$, and so $h \in G_E$. Therefore, G_E is open.

Remark We comment that the above is equally easy to prove using sequences of functions.

Now we prove density of the set. Let $\epsilon > 0$, and $f \in D[0, 1]$. We will show that there exists a function g such that $g \in G_E$, and $\|f - g\|_D < \epsilon$, by creating a modified version of one of our previously defined auxiliary functions.

We can unevenly pair 2 Yo-Yo functions inside any interval, giving uneven tangential oscillation at the endpoints.

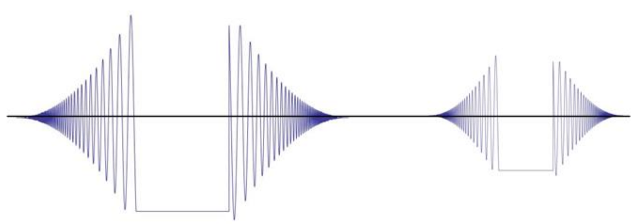

Figure 4: Unevenly Scaled Paired Yo-Yo's In A Interval.

Let $w := w_{(\bigcup_{n \in \mathbb{N}}(a_n,b_n),\nu,\vec{\eta},\vec{\beta})}$ where $0 < \nu$ and $\vec{\eta} = (\eta_n)_{n \in \mathbb{N}}, \vec{\beta} = (\beta_n)_{n \in \mathbb{N}}$ be defined in the following way: let $a_n + \gamma_n$ be the largest value in $(a_n, \min(\frac{a_n+b_n}{4}, \nu))$ such that $F'(a_n + \gamma_n, a_n) = 0$, and

$$w(x) := \begin{cases} 0 & : x \in [0,1] \setminus E \\ \eta_n F_{(a_n, a_n+\gamma_n)}(x) & : a_n < x \leq a_n + \gamma_n \\ 0 & : a_n + \gamma_n \leq x \leq b_n - \gamma_n \\ -\beta_n F_{(b_n-\gamma_n, b_n)}(x) & : b_n - \gamma_n \leq x < b_n. \end{cases}$$

Then this is just a scaled "Volterra", where instead of putting in oscillations at just the endpoints, we put in a Yo-Yo function at the endpoints. We will soon use this function $w(x)$ to create our function for density by choosing particular $\overrightarrow{\eta} = (\eta_n)_{n\in\mathbb{N}}, \overrightarrow{\beta} = (\beta_n)_{n\in\mathbb{N}}$.

Thus we let:

$$\eta_n := \begin{cases} 0 & : \frac{\epsilon}{8} \leq \Lambda(f', a_n) \\ \frac{\epsilon}{4} & : 0 \leq \Lambda(f', a_n) < \frac{\epsilon}{8} \end{cases} \qquad \beta_n := \begin{cases} 0 & : \frac{\epsilon}{8} \leq \Lambda(f', b_n) \\ \frac{\epsilon}{4} & : 0 \leq \Lambda(f', b_n) < \frac{\epsilon}{8} \end{cases}$$

Then η_n we use as the left scaling and β_n as the right scaling of a pairing of Yo-Yo's on $I_n = (a_n, b_n)$. Furthermore, let $\nu := \frac{\epsilon}{4}$. Then we let $g := f + w_{(E,\nu,\overrightarrow{\eta},\overrightarrow{\beta})} = f + w$, and we will show that $g \in G_E$.

Remark *What are we doing? We can think of w as a sum of functions with disjoint support on each I_n that looks like unevenly paired Yo-Yo's. So let's look at an example of our functions on an interval I_n:*

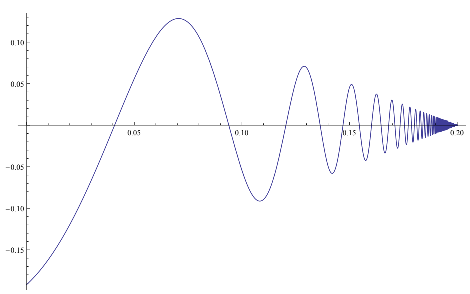

Figure 5: Example v on I_n

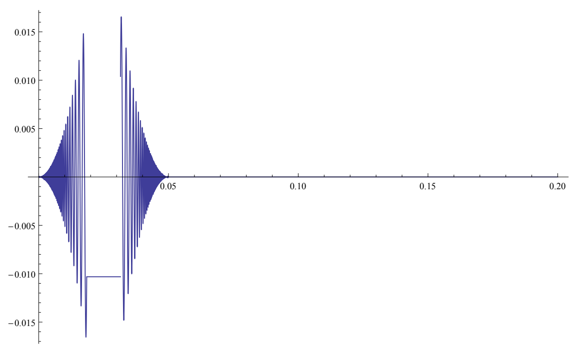

Figure 6: Example Uneven Scaled Yo-Yo's on I_n

So we want to perturb the tangential oscillation of v by adding a Yo-Yo on the left, as the tangential oscillation of v on the right is positive, so we don't want to cancel that out. Thus, we scale a Yo-Yo on the left and add it to our v.

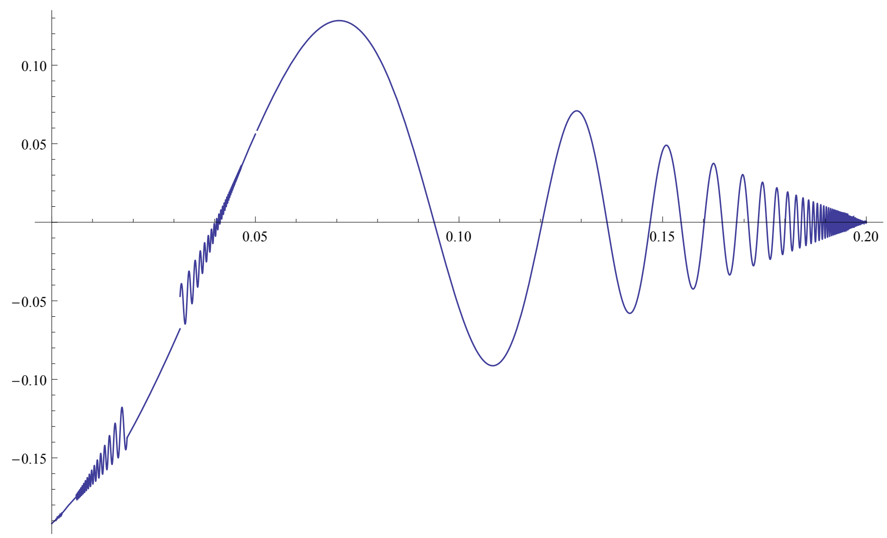

Figure 7: $z = w + v$ On I_n.

We see that we can maintain "closeness" to the original form of v with our perturbation, while still having tangential oscillation at both endpoints.

In fact we will show that $\Lambda(g', x) > \frac{\epsilon}{8}$ for all $x \in E$, but first, we will prove that $\Lambda(g', a_n) \geq \frac{\epsilon}{8}$, and $\Lambda(g', b_n) \geq \frac{\epsilon}{8}$, for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary. We prove for a_n first:

Case 1: Suppose $0 \leq \Lambda(f', a_n) < \frac{\epsilon}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{\epsilon}{8} \leq 2(\frac{\epsilon}{4}) - \frac{\epsilon}{8} < \Lambda(w', a_n) - \Lambda(f', a_n) \leq \Lambda(w' + f', a_n) = \Lambda(g, a_n) \tag{1.9}$$

Case 2: Suppose $\Lambda(f', a_n) \geq \frac{\epsilon}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{\epsilon}{8} \leq \Lambda(f', a_n) - 0 = \Lambda(f', a_n) - \Lambda(w', a_n) \leq \Lambda(w' + f', a_n) = \Lambda(g, a_n) \tag{1.10}$$

The proof for b_n is very similar:

Case 1: Suppose $0 \leq \Lambda(f', b_n) < \frac{\epsilon}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{\epsilon}{8} \leq 2(\frac{\epsilon}{4}) - \frac{\epsilon}{8} < \Lambda(w', b_n) - \Lambda(f', b_n) \leq \Lambda(w' + f', b_n) = \Lambda(g, b_n) \tag{1.11}$$

Case 2: Suppose $\Lambda(f', b_n) \geq \frac{\epsilon}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{\epsilon}{8} \leq \Lambda(f', b_n) - 0 = \Lambda(f', b_n) - \Lambda(w', b_n) \leq \Lambda(w' + f', b_n) = \Lambda(g, b_n) \tag{1.12}$$

Now, let $x \in E$, and $\delta > 0$ be arbitrary. Then within $B^o(x; \delta)$ there exists an endpoint of an interval, which without loss of generality, we say is a_n (as it would be similar if b_n instead), and there exists $\delta_0 > 0$ such that $B^o(a_n; \delta_0) \subseteq B^o(x; \delta)$. Then, as $\frac{\epsilon}{8} < \Lambda(g', a_n) := \lim_{\widetilde{\delta} \to 0^+} \sup_{h,k \in B^o(a_n; \widetilde{\delta})} |g'(h) - g'(k)| \leq \sup_{h,k \in B^o(a_n; \delta_0)} |g'(h) - g'(k)|$. Therefore, we have the following:

$$0 < \frac{\epsilon}{8} \leq \sup_{h,k \in B^o(a_n; \delta_0)} |g'(h) - g'(k)| \leq \sup_{h,k \in B^o(x; \delta)} |g'(h) - g'(k)|. \tag{1.13}$$

Now, as δ is arbitrary, and we may always find such an endpoint as a_n, then $0 < \frac{\epsilon}{8} \leq \lim_{\delta \to 0^+} \sup_{h,k \in B^o(x, \delta)} |g'(h) - g'(k)| =: \Lambda(g', x)$.

Now, we'll prove density. Well,

$$\begin{aligned} \|f - g\|_D &:= \|f - g\|_\infty + \|f' - g'\|_\infty = \|f - (f + w)\|_\infty + \|f' - (f' + w')\|_\infty = \|w\|_\infty + \|w'\|_\infty \\ &< \left(\frac{\epsilon}{4}\right)^3 + \left(\frac{\epsilon}{4}\right)\left[\sup_{x \in [0,1]} |2x \sin(1/x) + \cos(1/x)|\right] < \frac{\epsilon}{4} + 3\frac{\epsilon}{4} = \epsilon. \end{aligned} \tag{1.14}$$

Thus, we have that G_E is indeed dense in $D[0, 1]$. □

This leads to an additional result that further generalizes the ideas we are getting.

Theorem 1.5.2. *Let E be an arbitrary nowhere dense subset of* $[0, 1]$. *Let* $H_E := \{f \in D[0, 1] : \Lambda(f', x_0) > 0, \forall x_0 \in E\}$. *Then H_E is second category in* $D[0, 1]$, *and is indeed, residual.*

Proof. Let E be given. As $G_E \subseteq H_E$, and G_E is a dense open set, we then have that H_E is residual, and so second category. □

From this, we can extend further in order to get the result we have been after the whole time.

Theorem 1.5.3. *Let E be an arbitrary meagre (first category) set in* $[0, 1]$. *Let* $H_E := \{f \in D[0, 1] : \Lambda(f', x_0) > 0, \forall x_0 \in E\}$. *Then, H_E is residual (and so second category) in* $D[0, 1]$.

Proof. (Theorem 1.5.3) Well, we may write $E = \bigcup_{n \in \mathbb{N}} E_n$ where each E_n is a nowhere dense set. Let $A_n := cl(E_n)$ for all $n \in \mathbb{N}$. Then G_{A_n} is a dense open set, and so $J := \bigcap_{n \in \mathbb{N}} G_{A_n}$ is a dense G_δ set in $D[0, 1]$. So, as $J \subseteq H_E$, then H_E is residual, and so second category. □

1.6 POROSITY AND OSCILLATIONS OF DERIVATIVES

We now turn to a different way of looking at things... by introducing the concept of porosity. We first learned of porosity from a paper of Domínguez Benavides [12] regarding porosity of certain fixed point properties.

Remark Recall that in a metric space (M, d), a subset A is nowhere dense if its closure has empty interior. Note that A is nowhere dense if and only if $\overline{A} = cl(A)$ is nowhere dense. So a key case to consider is where A is closed. Assume that A is also closed. Then A is nowhere dense means that for every $r > 0$ and $x \in A$, there exists $z \in A^c$ such that $z \in B^o(x; r)$. Further, as A is closed, then for all w in A^c, there exists $t \in (0, \infty)$ such that $B^o(w; t) \subseteq A^c$. Therefore, for the particular $z \in A^c$, there exists $s \in (0, \infty)$ such that $B^o(z; s) \subseteq A^c$. So by shrinking s if necessary, then we can in fact guarantee that $B^o(z; s) \subseteq B^o(x; r)$, and so $B^o(z; s) \subseteq B^o(x; r) \cap A^c$.

Now notice that without loss of generality, as we really only care about small r to begin with, we could say that the condition on A is that there exists some $r_0 \in (0, \infty)$ such that for all $x \in A$ and for all $r \in (0, r_0]$, there exists $z \in A^c$ and there exists $s > 0$ such that $B^o(z; s) \subseteq B^o(x; r) \cap A^c$. Note that $s \leq r$. We can therefore rephrase this condition using $\beta := \frac{s}{r} \in (0, 1]$, to say: a closed set $A \subseteq (M, d)$ is nowhere dense if and only if

$$(\bigstar)\ [\exists r_0 \in (0, \infty) \text{ such that } \forall x \in A, \forall r \in (0, r_0], \exists \beta \in (0, 1]$$

$$\exists z \in A^c \text{ such that } B^o(z; \beta r) \subseteq B^o(x; r) \cap A^c]$$

Again, if S is an arbitrary set, then we see that it is nowhere dense precisely if $A := \overline{S}$ obeys the above condition ($\bigstar$). So we wonder about the following: what if there was a uniformity to the selection of β? In fact, let's look at the following definition (which turns out to be a strictly stronger statement then ($\bigstar$)).

Definition of Porosity Let (M, d) be a metric space, and $A \subseteq M$ such that A is closed. We say that A is porous if $\exists r_0 \in (0, \infty)$ and $\exists \beta \in (0, 1]$ such that $\forall x \in A$, $\forall r \in (0, r_0]$, $\exists z \in A^c$ such that

$B^o(z;\beta r) \subseteq B^o(x,r) \cap A^c$. Moreover, for arbitrary $S \subseteq M$, we say that S is porous if $\overline{S}$ is porous. Furthermore, we say that a set J is co-porous if J^c is porous.

Note that the uniformity in the definition of porosity compared to statement (★) comes in choosing the same $\beta \in (0,1]$, for all $x \in A$ and for all $r \in (0, r_0]$.

Definition of σ-Porosity A set S is σ-porous if and only if $S = \bigcup_{n\in\mathbb{N}} S_n$ such that each S_n is porous. A set A is co-σ-porous if and only if $A := S^c$ where $S = \bigcup_{n\in\mathbb{N}} S_n$ such that each S_n is porous.

Domínguez Benavides comments that this implies that a σ-porous set is first category, and that when $M = \mathbb{R}^n$, a σ-porous set is Lebesgue measure zero. This latter statement follows from the Lebesgue Density Theorem.

Theorem 1.6.1. *(See [27] for details and references) Each σ-porous subset of $\mathbb{R}^n$ is of the First category and of Lebesgue measure zero.*

Theorem 1.6.2. *(See [27] for details and references) There exists a closed nowhere dense set $F \subseteq \mathbb{R}^n$ which is not σ-porous*

Theorem 1.6.3. *(See [27] for details and references) There exists a non-σ-porous set $P \subseteq \mathbb{R}^n$ which is of the First category and is null for the Lebesgue measure μ.*

An Example: Let $Q := \mathbb{Q} \cap [0,1]$. Then Q is not porous in $[0,1]$, nor is $[0,1] \setminus Q$. However, as every singleton set (or in fact finite set) can be seen to be porous, then it is clear that $\mathbb{Q}$ is in fact σ-porous.

Please see the appendix for more information regarding porosity.

1.6.1 Porosity Applied to Volterra Example

Now, let's look at our set G from Theorem 1.5.1, and see what we can say about porosity of such a set.

Lemma 1.6.1. *For $G := \{f \in D[0,1] : \Lambda(f', x_0) > 0\}$ for some fixed $x_0 \in [0,1]$, G is co-porous.*

Proof. (Lemma 1.6.1) We will be using our previously defined auxiliary function $f_{\beta,v}(x)$. Let $\beta = \frac{1}{8}$ and $r_0 = 1/2$. Furthermore, take any $v \in D[0,1] \setminus G := \{u \in D[0,1] : \Lambda(u', x_0) = 0\}$ and let

$r \in (0, r_0]$. We take $z(x) := v(x) + f_{r/4,r/4}(x - x_0)$. Then just as in the proof of Proposition (1.3.2), we have that:

$$\begin{aligned} 0 < r/2 &= 2(r/4) - 0 \\ &\leq \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |f'_{r/4,r/4}(h) - f'_{r/4,r/4}(k)| - \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |v'(x_0 + h) - v'(x_0 + k)| \\ &\leq \lim_{\delta\to 0^+} \sup_{h,k\in(-\delta,\delta)} |z'(x_0 + h) - z'(x_0 + k)| = \Lambda(z', x_0). \end{aligned} \tag{1.15}$$

So $z \in G$. Now to show that $B^o(z; \beta r) \subseteq B^o(v, r) \cap G$. Let $g \in D[0, 1]$ such that $\|g\|_D \leq \beta r$. Then,

$$0 < r/4 = 2(r/4) - 2(r/8) < \Lambda(z', x) - 2\|g'\|_\infty < \Lambda(z', x) - \Lambda(g', x) \leq \Lambda(z' + g', x).$$

This proves that $z + g \in G$, as needed.

Now to show that $B(z; \beta r) \subseteq B(v, r)$. Well, let $s \in D[0, 1]$ such that $\|s\|_D \leq \frac{r}{4}$. Then,

$$\begin{aligned} \|(z + s) - v\|_D := \|(v + f_{r/4,r/4} + s) - v\|_D = \|f_{r/4,r/4} + s\|_D &\leq \|f_{r/4,r/4}\|_D + \|s\|_D \\ \leq \|f_{r/4,r/4}\|_D + \frac{r}{8} \leq \|f_{r/4,r/4}\|_\infty + \|f'_{r/4,r/4}\|_\infty &+ \frac{r}{4} \\ \leq \left(\frac{r}{4}\right)^2 + \frac{3r}{4} + \frac{r}{8} \leq \frac{r}{8} + \frac{3r}{4} + \frac{r}{8} &= r. \end{aligned} \tag{1.16}$$

Thus $B^o(z; \beta r) \subseteq B^o(v, r)$, and so then $B^o(z; \beta r) \subseteq B^o(v, r) \cap G$. Therefore, G is co-porous as claimed. □

Therefore, we get something for "free" from our proof above:

Corollary 1.6.1. *(To Lemma 1.6.1) Let E be any countable set in* $[0, 1]$. *Then* $G := \{f \in D[0, 1] : \Lambda(f', x_0) > 0, \forall x_0 \in E\}$ *is co-σ-porous.*

1.6.2 Porosity Theorems for Our Derivative Sets

It would be interesting to reach results like the ones before regarding porosity, and in fact we are about to see that this is possible.

Theorem 1.6.4. *Let $E \subseteq [0,1]$ be a closed nowhere dense set. Then $G_E := \{f \in D[0,1] : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$ is co-porous.*

Proof. (Theorem 1.6.4) Fix a nowhere dense set $E \subseteq [0,1]$ (which WLOG we assume to be closed), and similar to before, we take $E := [0,1] \setminus (\bigcup_{n\in\mathbb{N}} I_n) = [0,1] \setminus (\bigcup_{n\in\mathbb{N}} (a_n, b_n))$. Just as in Theorem (1.5.2), we will use one of our auxillary functions in the proof, specifically, the function $w := w_{(E, \frac{r}{4}, \vec{\eta}, \vec{\lambda})}$. Let $\beta = \frac{1}{32}$ and $r_0 = 1/2$. Furthermore, take any $v \in D[0,1] \setminus G_E := \{u \in D[0,1] : \inf_{x_0 \in E} \Lambda(u', x_0) = 0\}$ and let $r \in (0, r_0]$. We now define our sequences $\vec{\eta} = \{\eta_n\}, \vec{\beta} = \{\beta_n\}$

$$\eta_n := \begin{cases} 0 & : \frac{r}{8} \leq \Lambda(v', a_n) \\ \frac{r}{4} & : 0 \leq \Lambda(v', a_n) < \frac{r}{8} \end{cases} \qquad \beta_n := \begin{cases} 0 & : \frac{r}{8} \leq \Lambda(v', b_n) \\ \frac{r}{4} & : 0 \leq \Lambda(v', b_n) < \frac{r}{8}. \end{cases}$$

So let $w := w_{(E, v, \vec{\eta}, \vec{\beta})}$. We take $z(x) := v(x) + w(x)$. Then we proceed with the proof in a manner similar to the proof of Proposition (1.3.2). In fact we will show that $\Lambda(z', x) > \frac{r}{8}$ for all $x \in E$, but first, we will prove that $\Lambda(z', a_n) \geq \frac{r}{8}$, and $\Lambda(z', b_n) \geq \frac{r}{8}$, for all $n \in \mathbb{N}$. As such, let $n \in \mathbb{N}$ be arbitrary. We prove for a_n first:

<u>Case 1</u>: Suppose $0 \leq \Lambda(v', a_n) < \frac{r}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{r}{8} \leq 2(\frac{r}{4}) - \frac{r}{8} \leq \Lambda(w', a_n) - \Lambda(v', a_n) \leq \Lambda(w' + v', a_n) = \Lambda(z', a_n) \tag{1.17}$$

<u>Case 2</u>: Suppose $\Lambda(v', a_n) \geq \frac{r}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{r}{8} \leq \Lambda(v', a_n) - 0 = \Lambda(v', a_n) - \Lambda(w', a_n) \leq \Lambda(w' + v', a_n) = \Lambda(z', a_n) \tag{1.18}$$

The proof for b_n is very similar:

<u>Case 1</u>: Suppose $0 \leq \Lambda(v', b_n) < \frac{r}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{r}{8} \leq 2(\frac{r}{4}) - \frac{r}{8} \leq \Lambda(w', b_n) - \Lambda(v', b_n) \leq \Lambda(w' + v', b_n) = \Lambda(z, b_n) \tag{1.19}$$

<u>Case 2</u>: Suppose $\Lambda(v', b_n) \geq \frac{r}{8}$. Then using Proposition 1.2.3:

$$0 < \frac{r}{8} \leq \Lambda(v', b_n) - 0 = \Lambda(v', b_n) - \Lambda(w', b_n) \leq \Lambda(w' + v', b_n) = \Lambda(z, b_n) \tag{1.20}$$

Now, let $x \in E$, and $\delta > 0$ be arbitrary. Then within $B^o(x;\delta)$ there exists an endpoint of an interval, which without loss of generality, we say is a_n (as it would be similar if b_n instead), and there exists $\delta_0 > 0$ such that $B^o(a_n;\delta_0) \subseteq B^o(x;\delta)$. Then, as $\frac{\epsilon}{8} \leq \Lambda(z', a_n) := \lim_{\widetilde{\delta}\to 0^+} \sup_{h,k\in B^o(a_n;\widetilde{\delta})} |z'(h) - z'(k)| \leq \sup_{h,k\in B^o(a_n;\delta_0)} |z'(h) - z'(k)|$. Therefore, we have the following:

$$0 < \frac{r}{8} \leq \sup_{h,k\in B^o(a_n;\delta_0)} |z'(h) - z'(k)| \leq \sup_{h,k\in B^o(x;\delta)} |z'(h) - z'(k)|. \tag{1.21}$$

Now, as δ is arbitrary, and we may always find such an endpoint as a_n, then $0 < \frac{r}{8} \leq \lim_{\delta\to 0^+} \sup_{h,k\in B^o(x,\delta)} |z'(h) - z'(k)| =: \Lambda(z', x)$.

So $z \in G$. Now to show that $B^o(z;\beta r) \subseteq B^o(v, r) \cap G$. Let $g \in D(X)$ such that $\|g\|_D \leq \beta r = \frac{r}{32}$. Then,

$$0 < \frac{r}{16} = \frac{r}{8} - 2(\frac{r}{32}) < \Lambda(z', x) - 2\|g'\|_\infty < \Lambda(z', x) - \Lambda(g', x) \leq \Lambda(z' + g', x).$$

This proves that $z + g \in G$, as needed.

Now to show that $B^o(z;\beta r) \subseteq B^o(v, r)$. Well, let $s \in D[0, 1]$ such that $\|s\|_D \leq \beta r = \frac{r}{32}$. Then,

$$\begin{aligned} \|(z + s) - v\|_D := \|(v + w + s) - v\|_D = \|w + s\|_D \leq \|w\|_D + \|h\|_D \\ \leq \|w\|_D + \frac{r}{32} \leq \|w\|_\infty + \|w'\|_\infty + \frac{r}{32} \\ \leq \left(\frac{r}{4}\right)^2 + \frac{3r}{4} + \frac{r}{32} < \frac{r}{32} + \frac{3r}{4} + \frac{r}{16} < r. \end{aligned} \tag{1.22}$$

Thus $B^o(z;\beta r) \subseteq B^o(v, r)$, and so then $B^o(z;\beta r) \subseteq B^o(v, r) \cap G_E$. Therefore, G_E is co-porous as claimed. □

Now, utilizing the fact that porous sets are nowhere dense, we get the following corollary:

Corollary 1.6.2. *(To Theorem 1.6.4) Let E be a closed porous set. Then $G_E := \{f \in D[0, 1] : \inf_{x_0\in E} \Lambda(f', x_0) > 0\}$ is co-porous.*

1.7 ONWARD AND UPWARD - EXAMPLE SPACES AT A POINT

We shall now seek out some generalizations of the above to higher dimensional spaces. In particular, let's go to $[0,1]^N$, but we first need some definitions. We start with some definitions and preliminary theorems.

Definition If $(X, \|\cdot\|)$ is any normed space, then we define the following:

- $C(X;\mathbb{R}) :=$ {The set of continuous functions on X into $\mathbb{R}$}.
- $C^{(1)}(X) := \{f \in C(X) : f'$ exists, and f' is continuous on X into $\mathbb{R}\}$

Hence, if $(X, \|\cdot\|)$ is any normed space, then we define the following subsets in the usual way:

- $S(X) := \{x \in X : \|x\| = 1\}$
- $B(X) := \{x \in X : \|x\| \leq 1\}$
- $B^o(X) := \{x \in X : \|x\| < 1\}$

We will be using the idea of Fréchet derivative, so we will now define the topic.

Definition (See [18]) Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. For a function $f : X \to Y$, the Gâteaux derivative at a point $x_0 \in X$ is by definition a bounded linear operator $T : X \to Y$ such that for every $u \in X$,

$$\lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu.$$

The operator T is called the Fréchet derivative of f at x_0 if it is a Gâteaux derivative of f at x_0 and the limit above holds uniformly in u for all $u \in B(X)$ (or sometimes alternately defined for just $S(X)$, and extended to $B(X)$). We will identify T as either f' or df for notation.

1.7.1 D_N Space Definition

So in particular, we can define a D space on a unit cube:

Definition Let $N \in \mathbb{N}$ be fixed. Then we define D_N where:

$$D_N := \{f \in C_b([0,1]^N;\mathbb{R}) : f' \text{ exists, and } \|f\|_{D_N} < \infty\}$$

Here $\|f\|_{D_N} := \|f\|_\infty + \|f'\|_\infty$, where f' is the Fréchet or total derivative.

1.7.2 $D(X;Y)$ Space Definition

Now, we can see that similar to before, the space D_N is indeed a Banach space, and in fact we can see that $D(X)$ is a Banach space, where X is *any* Banach space.

Definition Let $(X, \|\cdot\|_X)$ be a non-trivial Banach space. Then we define the Banach space $D(X)$ in the following way:

$$D(X) := \{f \in C(B^o(X); \mathbb{R}) : f' \text{ (the Fréchet derivative) exists, and } \|f\|_D := (\|f\|_\infty + \|f'\|_\infty) < \infty\}$$

But we can in fact extend this definition to functions that map into another Banach space, which also follows by the above theorems.

Definition Let $(X, \|\cdot\|_X)$ be a non-trivial Banach space, and $(Y, \|\cdot\|_Y)$ be a non-trivial Banach space. Then we define $D(X;Y)$ in the following way:

$$D(X;Y) := \{f \in C(B^o(X), Y) : f' \text{ (the Fréchet derivative) exists, and } \|f\|_D := (\|f\|_\infty + \|f'\|_\infty) < \infty\}$$

When $Y = \mathbb{R}$, then we write $D(X;Y) = D(X)$.

Remark Please see the remark 1.2 where we discussed openness of the domain space of the functions we are using. The same idea seems to apply here in that we have a choice in whether we want to define the space on the open or closed ball. We used the open ball, as it seems more intuitive for general Fréchet differentiation in these cases. We will make no comparison on the possible definition of

$$D(X;Y) := \{f \in C(B(X), Y) : f' \text{ (the Fréchet derivative) exists, and } \|f\|_D := (\|f\|_\infty + \|f'\|_\infty) < \infty\}$$

Instead, we continue from here using the fact that openness works.

We now need to know about some definitions and theorems to work towards our primary generalization lemma.

Definition Let $(X, \|\cdot\|)$ be a Banach space. Then we say that X has Fréchet differentiable norm if for all $x \in B(X)$, $x \neq 0$ we have $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists uniformly for each $y \in B(X)$.

The differentiability of the norm away from zero is a prerequisite for how we would like to proceed to a generalization, and we are now ready to do so in the following part. However, we are going to start with some examples to build up to what we want.

1.7.3 Example Case: l_n^p, $1 \le p \le \infty$, $n \in \mathbb{N}$ Oscillation Theorem At A Point

Theorem 1.7.1. *Let $1 < p < \infty$ be a real number, and $n \in \mathbb{N}$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(X, \|\cdot\|) := (l_n^p, \|\cdot\|_p)$ which has a Fréchet differentiable norm, is a Banach space such that if $G := \{f \in D(X) : \Lambda(f', x_0) > 0\}$ for some fixed $x_0 \in B(X)$, then G is co-porous.*

Note that here $\Lambda(u, a) := \lim_{\delta \to 0^+} \sup_{h,k \in B_\delta^o(a)} \|(u(h) - u(k)\|$.

Proof. We check that $\|\cdot\| : B(X) \to \mathbb{R}$ is differentiable. Let $x = (x_1, ..., x_n) \in B(X)$, such that $x \neq 0$, then

$$d(\|x\|_p) = Grad(\|x\|_p) =$$
$$(|x_1|^p + ... + |x_n|^p)^{1/p-1}(|x_1|^{p-1} sgn(x_1), ..., |x_n|^{p-1} sgn(x_n)) =$$
$$(|x_1|^p + ... + |x_n|^p)^{1-p}(|x_1|^{p-1} sgn(x_1), ..., |x_n|^{p-1} sgn(x_n)) = \frac{1}{\|x\|_p^{p-1}}(|x_1|^{p-1} sgn(x_1), ..., |x_n|^{p-1} sgn(x_n)). \tag{1.23}$$

Now, let's also look at the norm (i.e. the X^*-norm) of $d(\|x\|) : B(X) \to \mathbb{R}$ for $x \neq 0$:

$$\begin{aligned} \|d(\|x\|_p)\|_{X^*} &= \|d(\|x\|)\|_{l_n^q} = \|Grad(\|x\|_p)\| \\ &= \|\frac{1}{\|x\|_p^{p-1}}(|x_1|^{p-1} sgn(x_1), ..., |x_n|^{p-1} sgn(x_n))\|_q \\ &= \frac{1}{\|x\|_p} \cdot \|(|x_1|^{p-1} sgn(x_1), ..., |x_n|^{p-1} sgn(x_n))\|_q \\ &= \frac{1}{\|x\|_p} \cdot \left(\sum_{j=1}^n (|x_j|^{p-1})^q\right)^{\frac{1}{q}} \\ &= \frac{1}{\|x\|_p} \cdot \|x\|_p^{p/q} = \|x\|_p^{1-p} \cdot \|x\|_p^{p-1} = 1. \end{aligned} \tag{1.24}$$

Now something we will need:

Definition Let $s \in (1, \infty)$. Let $N_s : B(X) \to \mathbb{R}$ be a function defined by $N_s(x) := \|x\|^s$.

Remark Let's check that N_p is differentiable. Well as a composition of Fréchet differentiable functions away from zero, N_p is differentiable away from zero. So we only need to check differentiability at zero. Well,

$$\lim_{\|z\|\to 0} \frac{N_p(0+z) - N_p(0)}{\|z\|} = \lim_{\|z\|\to 0} \frac{N_p(z)}{\|z\|} = \lim_{\|z\|\to 0} \frac{\|z\|^p}{\|z\|} = \lim_{\|z\|\to 0} \|z\|^{p-1} = 0.$$

Notice that $N_p(x)$ and $N_p'(x)$ are both bounded on $B(X)$. Indeed, $N_p(\cdot) \leq 1$ and N_p' is bounded by definition of Fréchet differentiable norm also by 1.

Thus, we are now ready to define a new auxiliary function.

$$F(x, \alpha) := \begin{cases} [N_3(x-\alpha)] \sin(\frac{1}{N_2(x-\alpha)}) & : x \neq \alpha, x \in B(X) \\ 0 & : x = \alpha, x \in B(X) \end{cases}$$

Now the Fréchet differentiability of $F(x, \alpha)$ is obvious, as it is the composition of Fréchet differentiable functions. Furthermore, we find that if $y(x)$ is the Fréchet derivative of $\|x\|$ away from zero,

$$F'(x, \alpha) := \begin{cases} 3N_2(x-\alpha)y(x-\alpha)\sin(\frac{1}{N_2(x-\alpha)}) - 2\cos(\frac{1}{N_2(x-\alpha)})y(x-\alpha) & : x \neq \alpha, x \in B(X) \\ 0 & : x = \alpha, x \in B(X) \end{cases}$$

Therefore, we see that $F'(x, \alpha) \leq 5$, and so we indeed have that $F(\cdot, \alpha) \in D(X)$. As we did before, we will use this type of function to define a second auxiliary function $F_r(x)$ on the ball $B(x; r)$ which we call the "Yo-Yo Function", similar to what we did before.

Definition Fix an $\tilde{r} > 0$. We notice that $F'(x, \alpha)$ has an infinite number of zeros in the ball $B(\alpha; \frac{\tilde{r}}{2})$ occurring on spheres $S(\alpha; \lambda)$, so let γ be the largest such value of λ in $(0, \frac{\tilde{r}}{2}]$. We may now define

$$F_{\tilde{r},\alpha}(x) := \begin{cases} F(x, \alpha) & : 0 \leq \|x-\alpha\| \leq \gamma, x \in B(X) \\ \gamma^3 \sin(\frac{1}{\gamma^2}) & : \gamma \leq \|x-\alpha\| \leq \tilde{r}-\gamma, x \in B(X) \\ (\tilde{r} - \|x-\alpha\|)^3 \sin(\frac{1}{(\|x-\alpha\|-\tilde{r})^2}) & : \tilde{r}-\gamma \leq \|x-\alpha\| \leq \tilde{r}, x \in B(X) \\ 0 & : x \in B(X) \setminus B(\alpha; \tilde{r}) \end{cases}$$

Let's look at $F'(\cdot,0)$ in more detail. In particular, let's examine the oscillation of $F'(\cdot,0)$ at zero, where it will be $\Lambda(F'(\cdot,0),0) := \lim_{\delta\to 0^+} \sup_{h,k\in B^o_\delta(a)} \|(F'(h,0) - F'(k,0)\|_{X^*} = \lim_{\delta\to 0^+} \sup_{h,k\in B^o_\delta(a)} \|(F'(h,0) - F'(k,0)\|_{l^q_n}$. Let $u_n := (\frac{1}{\sqrt{2\pi n}}, 0, ..., 0), z_n := (\frac{-1}{\sqrt{2\pi n}}, 0, ..., 0) \in B(X)$ for any $n \in \mathbb{N}$. Notice that $N_2(u_n) = N_2(z_n) = \frac{1}{2\pi n}$ for any $n \in \mathbb{N}$.

Now, for n large enough, we have the following: $F'(x_n, 0) = -2\cos(2\pi n)y(x_n) = -2y(x_n)$ and $F'(z_n, 0) = -2\cos(2\pi n)y(z_n) = -2y(z_n)$. Thus,

$$\begin{aligned}
&\|F'(u_n,0) - F'(z_n,0)\|_{X^*} = \|2y(z_n) - 2y(u_n)\|_{l^q_n} = \\
&= 2\left\|\frac{1}{\|u_n\|_p^{p-1}}\left(\left|\frac{1}{\sqrt{2\pi n}}\right|^{p-1} sgn\left(\frac{1}{\sqrt{2\pi n}}\right), 0, ..., 0\right) - \frac{1}{\|z_n\|_p^{p-1}}\left(\left|\frac{-1}{\sqrt{2\pi n}}\right|^{p-1} sgn\left(\frac{-1}{\sqrt{2\pi n}}\right), 0, ..., 0\right)\right\|_q \\
&= 2\left\|(\sqrt{2\pi n})^{p-1}\left(\left|\frac{1}{\sqrt{2\pi n}}\right|^{p-1}, 0, ..., 0\right) - (\sqrt{2\pi n})^{p-1}\left(-\left|\frac{-1}{\sqrt{2\pi n}}\right|^{p-1}, 0, ..., 0\right)\right\|_q \\
&= 2\left\|(\sqrt{2\pi n})^{p-1}\left(2\left|\frac{1}{\sqrt{2\pi n}}\right|^{p-1}, 0, ..., 0\right)\right\|_q \\
&= 2\|(2, 0, ..., 0)\|_q = 4.
\end{aligned} \tag{1.25}$$

Therefore,$\Lambda(F'(\cdot,0),0) := \lim_{\delta\to 0^+} \sup_{h,k\in B^o_\delta(a)} \|(F'(h,0) - F'(k,0)\|_{l^q_n} \geq 2$ as $\|u_n\|_X, \|z_n\|_X \to 0$ as $n \to \infty$. Hence, $F'_{\tilde{r},\alpha}$ has oscillation of at least 2 as well, as it is just a scaling and alteration.

Now, by examining the definition of $F_{\tilde{r},\alpha}$, we see that it is continuous, Fréchet differentiable, and in $D(X)$. We note that to understand this definition, it is helpful to consider the graph of a typical Yo-Yo Function for $D_2 := D(\mathbb{R}^2)$.

Now we are ready to talk about porosity. We want to show G^c is porous by showing $\exists\beta \in (0,1]$ and $\exists r_0 \in (0,\infty)$ such that $\forall x \in G^c$, $\forall r \in (0, r_0]$, $\exists y \in G$ such that $B^o(y;\beta r) \subseteq B^o(x,r) \cap G$.

Let $\beta = \frac{1}{16}$ and $r_0 = 1/2$, and let $\alpha := x_0$. Furthermore, take any $v \in D(X) \setminus G := \{u \in D(X) : \Lambda(u', x_0) = 0\}$ and let $r \in (0, r_0]$. Let $\frac{r}{8} > 0$. We take $z(x) := v(x) + \frac{r}{8}F_{\frac{r}{8},x_0}$. Then by Proposition (1.2.3), we have that:

$$\begin{aligned}
0 &< 2\frac{r}{8} = 2(\frac{r}{8}) - 0 \\
&\leq \Lambda(\frac{r}{8}F'_{\frac{r}{8},x_0}, x_0) - \Lambda(v', x_o) \leq \Lambda(v' + \epsilon F'_{\epsilon,x_0}, x_0) = \Lambda(z', x_0).
\end{aligned} \tag{1.26}$$

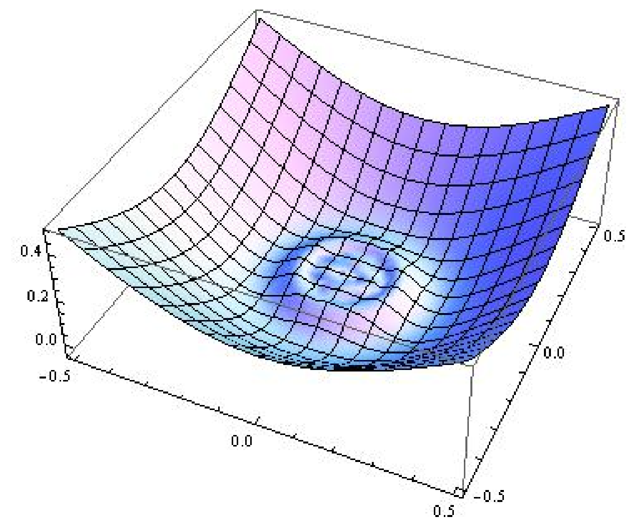

Figure 8: Typical Function

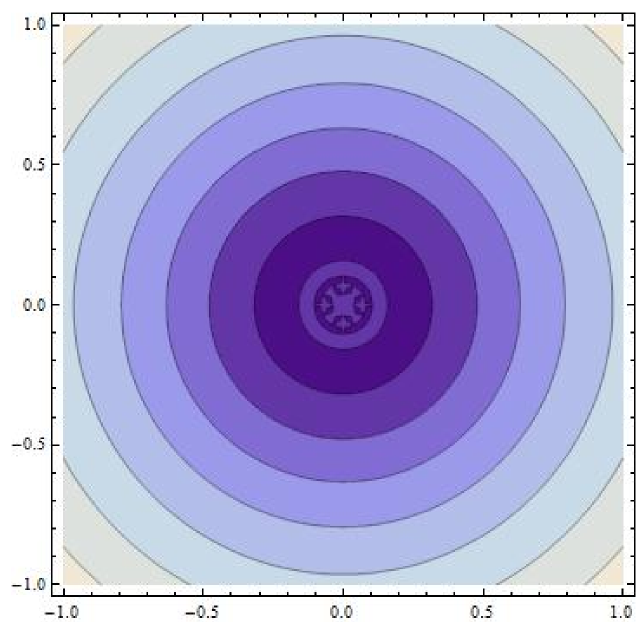

Figure 9: Contour Graph

So $z \in G$. Now to show that $B^o(z;\beta r) \subseteq B^o(v,r) \cap G$. Let $g \in D(X)$ such that $\|g\|_D \le \beta r = \frac{r}{16}$. Then,

$$0 < 2(\frac{r}{8}) - 2(\frac{r}{16}) < \Lambda(z', x) - 2\|g'\|_\infty < \Lambda(z', x) - \Lambda(g', x) \le \Lambda(z' + g', x).$$

This proves that $z + g \in G$, as needed.

Now to show that $B^o(z;\beta r) \subseteq B^o(v,r)$. Well, let $s \in D(X)$ such that $\|s\|_D \le \beta r = \frac{r}{16}$. Then, (we will use the inequality:

$$\begin{aligned} \|(z+s) - v\|_D &:= \|(v + \frac{r}{8}F_{\frac{r}{8},\alpha} + s) - v\|_D = \|\frac{r}{8}F_{\frac{r}{8},\alpha} + s\|_D \le \|\epsilon F_{\epsilon,\alpha}\|_D + \|s\|_D \\ &\le \|\frac{r}{8}F_{\frac{r}{8},\alpha}\|_D + \beta r \le \|\frac{r}{8}F_{\frac{r}{8},\alpha}\|_\infty + \|\frac{r}{8}F'_{\frac{r}{8},\alpha}\|_\infty + \beta r \\ &\le \left(\frac{r}{8}\right)^3 + 5\frac{r}{8} + \frac{r}{16} < r. \end{aligned} \tag{1.27}$$

Thus $B^o(z;\beta r) \subseteq B^o(v,r)$, and so then $B^o(z;\beta r) \subseteq B^o(v,r) \cap G$. Therefore, G is co-porous as claimed. □

1.7.4 An Auxiliary Function And Definitions

It will now be helpful to come up with some general results, as we will be using some ideas repeatedly. So for all of the following, we let $(X, \|\cdot\|)$ be a Banach space with Fréchet Differentiable Norm.

Definition Let $s \in (1,\infty)$. Let $N_s : B(X) \to \mathbb{R}$ be a function defined by $N_s(x) := \|x\|^s$.

Remark Let's check that N_p is differentiable. Well as a composition of Fréchet differentiable functions away from zero, N_p is differentiable away from zero. So we only need to check differentiability at zero. Well,

$$\lim_{\|z\|\to 0} \frac{N_p(0+z) - N_p(0)}{\|z\|} = \lim_{\|z\|\to 0} \frac{N_p(z)}{\|z\|} = \lim_{\|z\|\to 0} \frac{\|z\|^p}{\|z\|} = \lim_{\|z\|\to 0} \|z\|^{p-1} = 0.$$

Notice that $N_p(x)$ and $N'_p(x)$ are both bounded on $B(X)$. Indeed, $N_p(\cdot) \le 1$ and N'_p is bounded by definition of Fréchet differentiable norm.

Thus, we are now ready to define a auxiliary function.

$$F(x,\alpha) := \begin{cases} [N_3(x-\alpha)]\sin(\frac{1}{N_2(x-\alpha)}) & : x \neq \alpha, x \in B(X) \\ 0 & : x = \alpha, x \in B(X) \end{cases}$$

Now the Fréchet differentiability of $F(x,\alpha)$ is obvious, as it is the composition of Fréchet differentiable functions. Furthermore, we find that if $y(x)$ is the Fréchet derivative of $||x||$ away from zero,

$$F'(x,\alpha) := \begin{cases} 3N_2(x-\alpha)y(x-\alpha)\sin(\frac{1}{N_2(x-\alpha)}) - 2\cos(\frac{1}{N_2(x-\alpha)})y(x-\alpha) & : x \neq \alpha, x \in B(X) \\ 0 & : x = \alpha, x \in B(X) \end{cases}$$

Therefore, we see that $F'(x,\alpha) \leq 5$, and so we indeed have that $F(\cdot,\alpha) \in D(X)$.As we did before, we will use this type of function to define a second auxiliary function $F_r(x)$ on the ball $B(x;r)$ which we call the "Yo-Yo Function", similar to what we did before.

Definition Fix an $\tilde{r} > 0$. We notice that $F'(x,\alpha)$ has an infinite number of zeros in the ball $B(\alpha;\frac{\tilde{r}}{2})$ occurring on spheres $S(\alpha;\lambda)$, so let γ be the largest such value of λ in $(0,\frac{\tilde{r}}{2}]$. We may now define

$$F_{\tilde{r},\alpha}(x) := \begin{cases} F(x,\alpha) & : 0 \leq ||x-\alpha|| \leq \gamma, x \in B(X) \\ \gamma^3\sin(\frac{1}{\gamma^2}) & : \gamma \leq ||x-\alpha|| \leq \tilde{r}-\gamma, x \in B(X) \\ (\tilde{r}-||x-\alpha||)^3\sin(\frac{1}{(||x-\alpha||-\tilde{r})^2}) & : \tilde{r}-\gamma \leq ||x-\alpha|| \leq \tilde{r}, x \in B(X) \\ 0 & : x \in B(X) \setminus B(\alpha;\tilde{r}) \end{cases}$$

we will use these definitions in the soon to follow examples and general result.

1.7.5 Example Case: l^p, $1 \leq p \leq \infty$ Oscillation Theorem At A Point

Theorem 1.7.2. *Let $1 < p < \infty$ be a real number, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(X, \|\cdot\|) := (l^p, \|\cdot\|_p)$ which has a Fréchet differentiable norm, is a Banach space such that if $G := \{f \in D(X) : \Lambda(f', x_0) > 0\}$ for some fixed $x_0 \in B^o(X)$, then G is co-porous.*

Proof. We see that $\|\cdot\| : B(X) \to \mathbb{R}$ is differentiable, and for $x = \sum_{j=1}^{\infty} x_j e_j \in B(X)$, such that $x \neq 0$, then

$$\begin{aligned} d(\|x\|_p) &= (\sum_{j=1}^{\infty} |x_j|^p)^{1/p-1} \left(\sum_{j=1}^{\infty} |x_j|^{p-1} sgn(x_j) e_j \right) \\ &= \frac{1}{\|x\|_p^{p-1}} \left(\sum_{j=1}^{\infty} |x_j|^{p-1} sgn(x_j) e_j \right) \end{aligned} \tag{1.28}$$

Now, let's also look at the norm (i.e. the X^*-norm) of $d(\|x\|) : B(X) \to \mathbb{R}$:

$$\begin{aligned} \|d(\|x\|_p)\|_{X^*} &= \|d(\|x\|)\|_{l^q} \\ &= \left\| \frac{1}{\|x\|_p^{p-1}} \left(\sum_{j=1}^{\infty} |x_j|^{p-1} sgn(x_j) e_j \right) \right\|_q \\ &= \frac{1}{\|x\|_p^{p-1}} \left\| \sum_{j=1}^{\infty} |x_j|^{p-1} sgn(x_j) e_j \right\|_q \\ &= \frac{1}{\|x\|_p^{p-1}} \left(\sum_{j=1}^{\infty} \||x_j|^{p-1} sgn(x_j)|^q \right)^{1/q} \\ &= \frac{1}{\|x\|_p^{p-1}} \left(\sum_{j=1}^{\infty} (|x_j|^p \right)^{1/q} \\ &= \frac{1}{\|x\|_p^{p-1}} \|x\|^{p/q} = \|x\|_p^{1-p} \cdot \|x\|_p^{p-1} = 1. \end{aligned} \tag{1.29}$$

As before, we have N_s and F and $F_{r,\alpha}$. Furthermore, as the derivative of $\|\cdot\|$ is bounded 1 on $B(X) \setminus \{0\}$, then N_s and N_s' are bounded by the definition of the norm derivative, giving that $\|F_{r,\alpha}\|_\infty \leq r$ and $\|F_{r,\alpha}'\|_\infty \leq 4$. Let's look at the tangential oscillation of F at zero in more detail, in order to describe the tangential oscillation of $F_{r,\alpha}$ at zero. In particular, let's examine the oscillation of $F'(\cdot, 0)$ at zero, where it will be $\Lambda(F'(\cdot, 0), 0) := \lim_{\delta \to 0^+} \sup_{h,k \in B_\delta^o(a)} \|(F'(h, 0) - F'(k, 0)\|_{X^*} = \lim_{\delta \to 0^+} \sup_{h,k \in B_\delta^o(a)} \|(F'(h, 0) - F'(k, 0)\|_{l^q}$. Let

$u_n := \frac{1}{\sqrt{2\pi n}} e_1, z_n := \frac{-1}{\sqrt{2\pi n}} e_1 \in B(X)$ for any $n \in \mathbb{N}$. Notice that $N_2(u_n) = N_2(z_n) = \frac{1}{2\pi n}$ for any $n \in \mathbb{N}$.

Now, for n large enough, we have the following: $F'(x_n, 0) = -2\cos(2\pi n) y(x_n) = -2y(x_n)$ and $F'(z_n, 0) = -2\cos(2\pi n) y(z_n) = -2y(z_n)$. Thus,

$$\begin{aligned} \|F'(u_n,0) - F'(z_n,0)\|_{X^*} &= \|2y(z_n) - 2y(u_n)\|_{l^q} = \\ = 2\left\| \frac{1}{\|u_n\|_p^{p-1}} \left| \frac{1}{\sqrt{2\pi n}} \right|^{p-1} sgn\left(\frac{1}{\sqrt{2\pi n}} \right) e_1 - \frac{1}{\|z_n\|_p^{p-1}} \left| \frac{-1}{\sqrt{2\pi n}} \right|^{p-1} sgn\left(\frac{-1}{\sqrt{2\pi n}} \right) e_1 \right\|_q \\ &= 2\left\| (\sqrt{2\pi n})^{p-1} \left| \frac{1}{\sqrt{2\pi n}} \right|^{p-1} e_1 + (\sqrt{2\pi n})^{p-1} \left| \frac{1}{\sqrt{2\pi n}} \right|^{p-1} e_1 \right\|_q \\ &= 2\left\| (\sqrt{2\pi n})^{p-1} \left| \frac{2}{\sqrt{2\pi n}} \right|^{p-1} e_1 \right\|_q \\ &= 2\,\|2e_1\|_q = 4. \end{aligned} \tag{1.30}$$

Therefore,$\Lambda(F'(\cdot,0),0) := \lim_{\delta \to 0^+} \sup_{h,k \in B^o_\delta(a)} \|(F'(h,0) - F'(k,0)\|_{l^q_n} \geq 2$ as $\|u_n\|_X, \|z_n\|_X \to 0$ as $n \to \infty$. Therefore, the tangential oscillation of $F_{r,\alpha}$ is also at least 2 at zero, as it has the same behavior as F at the origin.

Now we are ready to talk about porosity. We want to show G^c is porous by showing $\exists \beta \in (0,1]$ and $\exists r_0 \in (0,\infty)$ such that $\forall x \in G^c$, $\forall r \in (0, r_0]$, $\exists y \in G$ such that $B^o(y;\beta r) \subseteq B^o(x,r) \cap G$.

Let $\beta = \frac{1}{16}$ and $r_0 = 1/2$, and let $\alpha := x_0$. Furthermore, take any $v \in D(X) \setminus G := \{u \in D(X) : \Lambda(u', x_0) = 0\}$ and let $r \in (0, r_0]$. Let $\frac{r}{8} > 0$. We take $z(x) := v(x) + \frac{r}{8} F_{\frac{r}{8},x_0}$. Then by Proposition (1.2.3), we have that:

$$\begin{aligned} 0 &< 2\frac{r}{8} = 2(\frac{r}{8}) - 0 \\ &\leq \Lambda(\frac{r}{8} F'_{\frac{r}{8},x_0}, x_0) - \Lambda(v', x_o) \leq \Lambda(v' + \epsilon F'_{\epsilon,x_0}, x_0) = \Lambda(z', x_0). \end{aligned} \tag{1.31}$$

So $z \in G$. Now to show that $B^o(z;\beta r) \subseteq B^o(v,r) \cap G$. Let $g \in D(X)$ such that $\|g\|_D \leq \beta r = \frac{r}{16}$. Then,

$$0 < 2(\frac{r}{8}) - 2(\frac{r}{16}) < \Lambda(z', x) - 2\|g'\|_\infty < \Lambda(z', x) - \Lambda(g', x) \leq \Lambda(z' + g', x).$$

This proves that $z + g \in G$, as needed.

Now to show that $B^o(z;\beta r) \subseteq B^o(v,r)$. Well, let $s \in D(X)$ such that $||s||_D \leq \beta r = \frac{r}{16}$. Then, (we will use the inequality:

$$\begin{aligned} ||(z+s)-v||_D &:= ||(v+\frac{r}{8}F_{\frac{r}{8},\alpha}+s)-v||_D = ||\frac{r}{8}F_{\frac{r}{8},\alpha}+s||_D \leq ||\epsilon F_{\epsilon,\alpha}||_D + ||s||_D \\ &\leq ||\frac{r}{8}F_{\frac{r}{8},\alpha}||_D + \beta r \leq ||\frac{r}{8}F_{\frac{r}{8},\alpha}||_\infty + ||\frac{r}{8}F'_{\frac{r}{8},\alpha}||_\infty + \beta r \\ &\leq \left(\frac{r}{8}\right)^3 + 5\frac{r}{8} + \frac{r}{16} < r. \end{aligned} \tag{1.32}$$

Thus $B^o(z;\beta r) \subseteq B^o(v,r)$, and so then $B^o(z;\beta r) \subseteq B^o(v,r) \cap G$. Therefore, G is co-porous as claimed. □

1.7.6 Example Case: L^p, $1 \leq p \leq \infty$ Oscillation Theorem At A Point

Theorem 1.7.3. *Let $1 < p < \infty$ be a real number, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(X, ||\cdot||) := (L^p[0,1], ||\cdot||_p)$ which has a Fréchet differentiable norm, is a Banach space such that if $G := \{f \in D(X) : \Lambda(f', x_0) > 0\}$ for some fixed $x_0 \in B^o(X)$, then G is co-porous.*

Proof. We see that $||\cdot|| : B(X) \to \mathbb{R}$ is differentiable, and for $x \in B(X)$, such that $x \neq 0$ a.e., then

$$d(||x(s)||_p) = \frac{1}{||x||_p^{p-1}}(|x(s)|^{p-1} sgn(x(s))) \tag{1.33}$$

Now, let's also look at the norm (i.e. the X^*-norm) of $d(||x||) : B(X) \to \mathbb{R}$, for $x \neq 0$:

$$\begin{aligned} ||d(||x||_p)||_{X^*} &= ||d(||x||)||_{L^q} \\ &= \left(\int_0^1 \left|\frac{1}{||x||_p^{p-1}}(|x(s)|^{p-1} sgn(x(s)))\right|^q ds\right)^{1/q} \\ &= \frac{1}{||x||_p^{p-1}}\left(\int_0^1 \left|(|x(s)|^{p(1-1/p)})\right|^q ds\right)^{1/q} \\ &= \frac{1}{||x||_p^{p-1}}\left(\int_0^1 (|x(s)|^{\frac{p}{q}})^q ds\right)^{1/q} = \frac{1}{||x||_p^{p-1}}\left(\int_0^1 |x(s)|^p ds\right)^{1/q} \\ &= \frac{1}{||x||_p^{p-1}}||x||_p^{p/q} = \frac{1}{||x||_p^{p-1}}||x||^{p/q} = ||x||_p^{1-p}\cdot||x||_p^{p-1} = 1. \end{aligned} \tag{1.34}$$

As before, we have N_s, F and $F_{r,\alpha}$. Therefore, as the derivative of $||\cdot||$ is bounded by 1 on $B(X)\backslash\{0\}$, then we have that N_s and N'_s are both bounded, and in fact $F_{r,\alpha} \leq r$ and $F'_{r,\alpha} \leq 5$. Let's look at the tangential oscillation of F at zero in more detail in order to get a bound on the tangential oscillation of $F_{r,\alpha}$. In particular, let's examine the oscillation of $F'(\cdot,0)$ at zero, where it will be $\Lambda(F'(\cdot,0),0) := \lim_{\delta\to 0^+} \sup_{h,k\in B^o_\delta(a)} \|(F'(a+h,0)-F'(a+k,0)\|_{X^*} = \lim_{\delta\to 0^+} \sup_{h,k\in B^o_\delta(a)} \|(F'(a+h,0)-F'(a+k,0)\|_{L^q[0,1]}$. Let $u_n := \frac{1}{\sqrt{2\pi n}}\mathbb{1}$, $z_n := \frac{-1}{\sqrt{2\pi n}}\mathbb{1} \in B(X)$ for any $n \in \mathbb{N}$. Notice that $N_2(u_n) = N_2(z_n) = \frac{1}{2\pi n}$ for any $n \in \mathbb{N}$.

Now, for n large enough, we have the following: $F'(x_n,0) = -2\cos(2\pi n)y(x_n) = -2y(x_n)$ and $F'(z_n,0) = -2\cos(2\pi n)y(z_n) = -2y(z_n)$. Thus,

$$\begin{aligned} \|F'(u_n,0)-F'(z_n,0)\|_{X^*} &= \|2y(z_n)-2y(u_n)\|_{L^q} = \\ = 2\left\| \frac{1}{\|u_n\|_p^{p-1}}\left|\frac{1}{\sqrt{2\pi n}}\right|^{p-1} sgn\left(\frac{1}{\sqrt{2\pi n}}\right)\mathbb{1} - \frac{1}{\|z_n\|_p^{p-1}}\left|\frac{-1}{\sqrt{2\pi n}}\right|^{p-1} sgn\left(\frac{-1}{\sqrt{2\pi n}}\right)\mathbb{1}\right\|_q \\ &= 2\left\|(\sqrt{2\pi n})^{p-1}\left|\frac{1}{\sqrt{2\pi n}}\right|^{p-1}\mathbb{1} + (\sqrt{2\pi n})^{p-1}\left|\frac{1}{\sqrt{2\pi n}}\right|^{p-1}\mathbb{1}\right\|_q \\ &= 2\left\|(\sqrt{2\pi n})^{p-1}\left|\frac{2}{\sqrt{2\pi n}}\right|^{p-1}\mathbb{1}\right\|_q \\ &= 2\|2\cdot\mathbb{1}\|_q = 4. \end{aligned} \tag{1.35}$$

Therefore,$\Lambda(F'(\cdot,0),0) := \lim_{\delta\to 0^+} \sup_{h,k\in B^o_\delta(a)} \|(F'(a+h,0)-F'(a+k,0)\|_{L^q[0,1]} \geq 2$ as $\|u_n\|_X, \|z_n\|_X \to 0$ as $n\to\infty$. Thus, the tangential oscillation of $F_{r,\alpha}$ is also at least 2, as $F_{r,\alpha}$ has the same behavior as F at the origin.

Now we are ready to talk about porosity. We want to show G^c is porous by showing $\exists\beta \in (0,1]$ and $\exists r_0 \in (0,\infty)$ such that $\forall x \in G^c$, $\forall r \in (0,r_0]$, $\exists y \in G$ such that $B^o(y;\beta r) \subseteq B^o(x,r)\cap G$.

Let $\beta = \frac{1}{16}$ and $r_0 = 1/2$, and let $\alpha := x_0$. Furthermore, take any $v \in D(X)\setminus G := \{u \in D(X) : \Lambda(u',x_0) = 0\}$ and let $r \in (0,r_0]$. Let $\frac{r}{8} > 0$. We take $z(x) := v(x) + \frac{r}{8}F_{\frac{r}{8},x_0}$. Then by Proposition (1.2.3), we have that:

$$\begin{aligned} 0 &< 2\frac{r}{8} = 2(\frac{r}{8}) - 0 \\ &\leq \Lambda(\frac{r}{8}F'_{\frac{r}{8},x_0},x_0) - \Lambda(v',x_o) \leq \Lambda(v'+\epsilon F'_{\epsilon,x_0},x_0) = \Lambda(z',x_0). \end{aligned} \tag{1.36}$$

So $z \in G$. Now to show that $B^o(z;\beta r) \subseteq B^o(v,r)\cap G$. Let $g \in D(X)$ such that $\|g\|_D \leq \beta r = \frac{r}{16}$. Then,

$$0 < 2(\frac{r}{8}) - 2(\frac{r}{16}) < \Lambda(z',x) - 2\|g'\|_\infty < \Lambda(z',x) - \Lambda(g',x) \leq \Lambda(z'+g',x).$$

This proves that $z + g \in G$, as needed.

Now to show that $B^o(z;\beta r) \subseteq B^o(v, r)$. Well, let $s \in D(X)$ such that $\|s\|_D \leq \beta r = \frac{r}{16}$. Then, (we will use the inequality:

$$\begin{aligned} \|(z+s)-v\|_D &:= \|(v+\frac{r}{8}F_{\frac{r}{8},\alpha}+s)-v\|_D = \|\frac{r}{8}F_{\frac{r}{8},\alpha}+s\|_D \leq \|\epsilon F_{\epsilon,\alpha}\|_D + \|s\|_D \\ &\leq \|\frac{r}{8}F_{\frac{r}{8},\alpha}\|_D + \beta r \leq \|\frac{r}{8}F_{\frac{r}{8},\alpha}\|_\infty + \|\frac{r}{8}F'_{\frac{r}{8},\alpha}\|_\infty + \beta r \\ &\leq \left(\frac{r}{8}\right)^3 + 5\frac{r}{8} + \frac{r}{16} < r. \end{aligned} \tag{1.37}$$

Thus $B^o(z;\beta r) \subseteq B^o(v, r)$, and so then $B^o(z;\beta r) \subseteq B^o(v, r) \cap G$. Therefore, G is co-porous as claimed. □

1.8 GENERAL FRÉCHET NORM OSCILLATION AT A POINT

Well, we will need to look at the Fréchet differentiable norm in a little more details. In particular, we are going to need a few of the properties that the derivative has, in order to use them for our proofs.

Norm Derivative Properties(NDP): As a reference for the following, please see [4, Part 3 Ch. 1] or [10, Ch. 2].

1. Let $H(x, h) := \lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$, where $h \in B(X)$. Then for every fixed $x \in B(X)$, $H(x, \cdot)$ is a norm-1 linear functional on $B(X)$.
2. For $\kappa \in \mathbb{R}$, $H(x, \kappa h) = \kappa H(x, h)$.
3. $H(x, \frac{x}{\|x\|}) = 1$ for every $x \neq 0$.
4. For all $x, h \in X$, $|H(x, h)| \leq \|h\|$.

Theorem 1.8.1. *Let $(X, \|\cdot\|)$ be a Banach space with a Fréchet differentiable norm. If $G := \{f \in D(X) : \Lambda(f', x_0) > 0\}$ for some fixed $x_0 \in B^o(X)$, then G is co-porous.*

Proof. As before, we have N_s, F and $F_{r,\alpha}$ as previously defined. Now, as $\|\cdot\|$ is bounded by 1 on $B(X)\setminus\{0\}$ then N_s, N_s' are both bounded, and $F_{r,\alpha} \leq r$ and $F_{r,\alpha}' \leq 4$. Let's look at the tangential oscillation of F at zero in more detail, in order to describe the tangential oscillation at zero of the similar function $F_{r,\alpha}$. In particular, let's examine the oscillation of $F'(\cdot, 0, h)$ at zero, where it will be:

$$\Lambda(F'(\cdot,0,\cdot),0) := \lim_{\delta\to 0^+} \sup_{u,k\in B_\delta^o(0)} \|(F'(u,0,h) - F'(k,0,h)\|_{X^*}$$

$$= \lim_{\delta\to 0^+} \sup_{u,k\in B_\delta^o(0)} \sup_{h\in B(X)} |(F'(u,0,h) - F'(k,0,h)|.$$

We will prove that $\Lambda(F'(\cdot,0,\cdot),0) > 0$ by using sequences to show a lower bound.

Let $c \in B(X)$ such that $c \neq 0$. Let $x_n := \frac{1}{\sqrt{2\pi n}}\frac{c}{\|c\|}, y_n := \frac{-1}{\sqrt{2\pi n}}\frac{c}{\|c\|}, h_n := \frac{x_n}{\|x_n\|} \in B(X)$ for any $n \in \mathbb{N}$. Notice that $\|x_n\| = \|y_n\| = \frac{1}{\sqrt{2\pi n}} \underset{n}{\to} 0$ and $\|h_n\| = 1$ for any $n \in \mathbb{N}$.

Now, for n large enough, we have the following: $F'(y_n, 0, h_n) = -2\cos(2\pi n)H(y_n, h_n) = -2H(y_n, h_n)$ and $F'(x_n, 0, h_n) = -2\cos(2\pi n)H(x_n, h_n) = -2H(x_n, h_n)$. Now, $-H(y_n, h_n) = H(y_n, -h_n) = H(y_n, \frac{-x_n}{\|x_n\|}) = H(y_n, \frac{y_n}{\|y_n\|})$ by NDP 2, and $H(y_n, \frac{y_n}{\|y_n\|}) = 1$ by NDP 3. Also, $H(x_n, h_n) = H(x_n, \frac{x_n}{\|x_n\|}) = 1$ by NDP 3.

Thus,

$$\begin{aligned} |F'(y_n,0,h_n) - F'(z_n,0,h_n)| &= |-2H(y_n,h_n) - (-2H(x_n,h_n))| \\ &= 2\,|H(x_n,h_n) - H(y_n,h_n)| = 2\,|H(x_n,h_n) - H(y_n,h_n)| \\ &= 2\,|H(x_n,h_n) + H(y_n,-h_n)| = 2|1+1| = 4. \end{aligned} \tag{1.38}$$

What does this mean? Well, it means that

$$\Lambda(F'(\cdot,0,\cdot),0) := \lim_{\delta\to 0^+} \sup_{u,k\in B_\delta^o(a)} \sup_{h\in B(X)} |(F'(u,0,h) - F'(k,0,h)| \geq 2.$$

Therefore, the tangential oscillation of $F_{r,\alpha}$ at zero is also at least 2, as $F_{r,\alpha}$ has the same behavior as F close to zero. Furthermore, we see that for any $a \in B(X)$, as we are now just translating our function, we have that

$$\Lambda(F_{\widetilde{r},a}'(\cdot,a,\cdot),a) := \lim_{\delta\to 0^+} \sup_{u,k\in B_\delta^o(a)} \sup_{h\in B(X)} |(F'(u,a,h) - F'(k,a,h)| \geq 2.$$

Now we are ready to talk about porosity. We want to show G^c is porous by showing $\exists\beta \in (0,1]$ and $\exists r_0 \in (0,\infty)$ such that $\forall x \in G^c$, $\forall r \in (0, r_0]$, $\exists y \in G$ such that $B^o(y;\beta r) \subseteq B^o(x,r) \cap G$.

Let $\beta = \frac{1}{16}$ and $r_0 = 1/2$, and let $\alpha := x_0$. Furthermore, take any $v \in D(X) \setminus G := \{u \in D(X) : \Lambda(u', x_0) = 0\}$ and let $r \in (0, r_0]$. Let $\frac{r}{8} > 0$. We take $z(x) := v(x) + \frac{r}{8}F_{\frac{r}{8},x_0}$. By Proposition (1.2.3), we have that:

$$\begin{aligned} 0 &< 2\frac{r}{8} = 2(\frac{r}{8}) - 0 \\ &\leq \Lambda(\frac{r}{8}F'_{\frac{r}{8},x_0}, x_0) - \Lambda(v', x_o) \leq \Lambda(v' + \epsilon F'_{\epsilon,x_0}, x_0) = \Lambda(z', x_0). \end{aligned} \tag{1.39}$$

So $z \in G$. Now to show that $B^o(z;\beta r) \subseteq B^o(v,r) \cap G$. Let $g \in D(X)$ such that $\|g\|_D \leq \beta r = \frac{r}{16}$. Then,

$$0 < 2(\frac{r}{8}) - 2(\frac{r}{16}) < \Lambda(z', x) - 2\|g'\|_\infty < \Lambda(z', x) - \Lambda(g', x) \leq \Lambda(z' + g', x).$$

This proves that $z + g \in G$, as needed.

Now to show that $B^o(z;\beta r) \subseteq B^o(v,r)$. Well, let $s \in D(X)$ such that $\|s\|_D \leq \beta r = \frac{r}{16}$. Then, (we will use the inequality:

$$\begin{aligned} \|(z+s) - v\|_D := \|(v + \frac{r}{8}F_{\frac{r}{8},\alpha} + s) - v\|_D = \|\frac{r}{8}F_{\frac{r}{8},\alpha} + s\|_D &\leq \|\epsilon F_{\epsilon,\alpha}\|_D + \|s\|_D \\ \leq \|\frac{r}{8}F_{\frac{r}{8},\alpha}\|_D + \beta r &\leq \|\frac{r}{8}F_{\frac{r}{8},\alpha}\|_\infty + \|\frac{r}{8}F'_{\frac{r}{8},\alpha}\|_\infty + \beta r \\ &\leq \left(\frac{r}{8}\right)^3 + 5\frac{r}{8} + \frac{r}{16} < \frac{r}{8} + \frac{3r}{4} + \frac{r}{8} = r. \end{aligned} \tag{1.40}$$

Thus $B^o(z;\beta r) \subseteq B^o(v,r)$, and so then $B^o(z;\beta r) \subseteq B^o(v,r) \cap G$. Therefore, G is co-porous as claimed. □

We can reach a extension for functions mapping into another Banach space, but let's first extend the general definition of Λ.

Definition Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed space. For any bounded function $S : X \to Y$, define for $\delta > 0$ the function $\Lambda_\delta(S, x_0) := \sup_{h,k \in B^o_X(x_0;\delta)} \|S(h) - S(k)\|_Y$, and $\Lambda(S, x_0) := \lim_{\delta \to 0^+} \Lambda_\delta(S, x_0)$, for all $x_0 \in X$.

Now we can state another theorem.

Corollary 1.8.1. *Let* $(X, \|\cdot\|)$ *be a Banach space with Fréchet differentiable norm, and let* $(Y, \|\cdot\|_Y)$ *be a nontrivial Banach space. Then fix some* $x_0 \in B^o(X)$*. Then for* $G := \{f \in D(X;Y) : \Lambda(f', x_0) > 0\}$*, G is co-porous.*

The proof is a slight modification of the previous one, where we do the following: let $y \in S(Y)$. Recall that for any $x \in B(X)$, we have that $F_{\widetilde{r},a} \in \mathbb{R}$. Therefore, $z := y \cdot F_{\widetilde{r},a} : B(X) \to Y$. Then the proof proceeds similar to the above, using z as the generating function.

We may now state the following result, which is now clear by the above:

Theorem 1.8.2. *Let* $\{x_n : n \in N\}$ *be a countable (possibly finite) set in Banach space* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *another Banach space. Then let* $G := \{f \in D(X,Y) : \Lambda(f', x_n) > 0$ *for all* $n \in N\}$ *is a dense* G_δ *set in* $D(X,Y)$ *that is co-*σ*-porous, and if N is finite, then G is open and co-porous.*

1.9 OSCILLATION ON THE BOUNDARY OF A CUBE

We would like to start getting some stronger results in higher dimensions. We have our "one-point" example in any Fréchet Differentiable norm Banach space, and we would like to get at least something more in higher dimensions.

Well, we first need a result:

Theorem 1.9.1. *[Dyadic Cube Theorem] Let* $d \in \mathbb{N}$*. Let U be an open set in* $\mathbb{R}^d$*. Then U is the countable union of disjoint half-open cubes of the form* $[x_1 - \lambda, x_1 + \lambda) \times ... \times [x_d - \lambda, x_d + \lambda)$ *for* $\lambda > 0$*. i.e.* $U = \bigcup_{j=1}^{\infty} [x_1^{(j)} - \lambda^{(j)}, x_1^{(j)} + \lambda^{(j)}) \times ... \times [x_d^{(j)} - \lambda^{(j)}, x_d^{(j)} + \lambda^{(j)})$*, so that* $(x_1, ..., x_d)$ *is the center of the cube and* 2λ *is the side length. Furthermore, we will guarantee that* λ *is at most* $\frac{1}{2}$*.*

So we will use this to try and get a stronger theorem for $\mathbb{R}^n$, but first we are going to build up some necessarily results and lemmas. This next lemma is not used in our work, but we discuss it because it is one of the intuitive ideas that might be tried, so we feel it is important to comment of the problem with this example.

Lemma 1.9.1. *Let* $C := [x_1^0 - \lambda, x_1^0 + \lambda] \times ... \times [x_n^0 - \lambda, x_n^0 + \lambda]$ *be a cube with center* $x_0 := (x_1^0, ..., x_n^0)$ *and radius* $\lambda > 0$ *in* $\mathbb{R}^n$*. Then there exists a ball* $A = B(x_0; \lambda)$ *inside C, and a function* s_A *such that:*

1. $s_A|_{\partial A} = s_A|_{\partial C} = 0$

2. $s_A \in D_n$

3. $\Lambda([s_A]', u) \geq 2\lambda > 0$ *for any* $u \in \partial A$

Proof. (1.9.1) We will be using $\|\cdot\| := \|\cdot\|_{l_2}$. We can now define a function $s_A : \mathbb{R}^n \to \mathbb{R}$:

Definition (Wiggle Function)

$$s_A(x) := \begin{cases} (\|x - x_0\|^2 - \lambda^2)^2 \sin\left[\frac{1}{(\|x-x_0\|^2-\lambda^2)}\right] & : \|x - x_0\| < \lambda \\ 0 & : \|x - x_0\| \geq \lambda \end{cases}$$

By our previous work, we know that $\|x - x_0\|^2 - \lambda^2$ is Fréchet differentable, thus as a composition, we see that s_A is as well. Thus, it is clear that $s_A \in D_n$, and so condition 1 and 2 in the lemma are easily seen to hold. Thus, we only need to show condition 3.

From here on out, we will suppress the input of λ. Furthermore, WLOG, $x_0 = \vec{0}$. Let's look at the derivative, where $y(x)$ is the derivative of the norm:

$$[s_A]'(x) := \begin{cases} \left(2(\|x\|^2 - \lambda^2) \sin\left[\frac{1}{(\|x\|^2-\lambda^2)}\right] - \cos\left[\frac{1}{(\|x\|^2-\lambda^2)}\right]\right)(2\|x\|)y(x) & : \|x\| < \lambda \\ 0 & : \|x\| \geq \lambda \end{cases}$$

Remark It can be shown that this function in fact has oscillation of 2λ, then we are done (minus certain details we skip).

□

Now this wasn't very optimal, and it won't lead us to what we want, because the oscillation lower bound of the family of functions may tend to zero. However, it is inspiring enough to get us started.

1.9.1 Saturn Ball Function

This next function is the real one to get us started.

Lemma 1.9.2. *Let* $0 < \lambda < 1$. *Let* $C := [-\lambda, \lambda] \times ... \times [-\lambda, \lambda]$ *be a n-cube with center* $\vec{0} := (0, ..., 0) \in \mathbb{R}^n$ *and radius* $\lambda > 0$. *Then there exists a ball* $A = B^o(\vec{0}; \lambda)$ *inside C, and a function* g_λ *such that:*

1. $supp(g_\lambda) \subseteq A$
2. $g_\lambda \in D_n$
3. $||g_\lambda||_\infty \leq \lambda^2 < 1$ *and* $||g'_\lambda||_\infty \leq 2\lambda + 1 < 3$
4. $\Lambda(g'_\lambda, u) \geq 1$ *for any* $u \in \partial A$

Proof. (1.9.2) We will be using $||\cdot|| := ||\cdot||_{l_2}$. We can now define a preliminary function $\widetilde{g} : \mathbb{R}^n \to \mathbb{R}$, that will be useful in getting our main function:

Definition (Pogo Function)

$$\widetilde{g}(x) := \begin{cases} (\lambda - ||x||)^2 \sin\left[\frac{1}{(\lambda - ||x||)}\right] & : ||x|| < \lambda \\ 0 & : ||x|| \geq \lambda \end{cases}$$

By our previous work, we know that $\lambda - ||x||$ is Fréchet differentable away from the origin, thus as a composition, we see that $\widetilde{g}$ is as well. So the only thing that is holding us back from differentiability everywhere is the center of our ball. Let's look at the derivative, where $y(x)$ is the derivative of the norm away from zero:

$$\left[\widetilde{g}\right]'(x) := \begin{cases} \left(2(\lambda - ||x||) \sin\left[\frac{1}{(\lambda - ||x||)}\right] - \cos\left[\frac{1}{(\lambda - ||x||)}\right]\right)(-y(x)) & : 0 < ||x|| < \lambda \\ 0 : ||x|| \geq \lambda \end{cases}$$

We are now ready for our primary function:

Definition (Saturn Ball Function) We notice that $\left[\widetilde{g}\right]'(x)$ has an infinite number of zeros in $[\frac{\lambda}{2}, \lambda)$, so let $\gamma > 0$ be the smallest such value. We may now define

$$g_\lambda(x) := \begin{cases} \widetilde{g}(x) & : ||x|| \geq \gamma \\ \widetilde{g}(\gamma) & : 0 \leq ||x|| < \gamma \end{cases}$$

Let's look at the derivative of g_λ, where $y(\cdot) : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R})$ is the Fréchet derivative of the norm:

$$g'_\lambda(x) := \begin{cases} 0 & : 0 \leq \|x\| \leq \gamma \\ \left(2(\lambda - \|x\|)\sin\left[\frac{1}{(\lambda - \|x\|)}\right] - \cos\left[\frac{1}{(\lambda - \|x\|)}\right]\right)(-y(x)) & : \gamma < \|x\| < \lambda \\ 0 & : \|x\| \geq \lambda \end{cases}$$

Let's check out $y(x)$ a little more carefully. We see that $\|\cdot\| : X \to \mathbb{R}$ is differentiable, and for $x = \sum_{j=1}^n x_j e_j \in X$, such that $x \neq 0$, then

$$\begin{aligned} d(\|x\|_2) &= (\sum_{j=1}^n |x_j|^2)^{1/2-1} \left(\sum_{j=1}^n |x_j|^{2-1} sgn(x_j) e_j \right) \\ &= \frac{1}{\|x\|_2^{2-1}} \left(\sum_{j=1}^n |x_j|^{2-1} sgn(x_j) e_j \right) \end{aligned} \tag{1.41}$$

Now, let's also look at the norm (i.e. the X^*-norm) of $d(\|x\|) : X \to \mathbb{R}$, for $x \neq 0$:

$$\begin{aligned} \|d(\|x\|_2)\|_{X^*} &= \|d(\|x\|)\|_{l^2} \\ &= \left\| \frac{1}{\|x\|_2^{2-1}} \left(\sum_{j=1}^n |x_j|^{2-1} sgn(x_j) e_j \right) \right\|_2 \\ &= \frac{1}{\|x\|_2^{2-1}} \left\| \sum_{j=1}^n |x_j|^{2-1} sgn(x_j) e_j \right\|_2 \\ &= \frac{1}{\|x\|_2^{2-1}} \left(\sum_{j=1}^n \||x_j|^{2-1} sgn(x_j)|^2 \right)^{1/2} \\ &= \frac{1}{\|x\|_2^{2-1}} \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \\ &= \frac{1}{\|x\|_2^{2-1}} \|x\|^{2/2} = \|x\|_2^{1-2} \cdot \|x\|_2^{2-1} = 1. \end{aligned} \tag{1.42}$$

So we now see that Conditions (1) and (2) are clear from our work... for $g_\lambda, [g_\lambda]'$ are clearly bounded. In fact, $\|g_\lambda\|_\infty \leq \lambda^2 < 1$, and $\|f'_R\|_\infty \leq 2\lambda + 1$. Now, we need to know about the tangential oscillation of g_λ. Let z be such that $\|z\| = \lambda$, and define the sequences $w^{(k)} := \left(\lambda - \frac{1}{2\pi k}\right)\frac{z}{\lambda}$, $v^{(k)} := (\lambda - a_k)\frac{z}{\lambda}$ where $\{a_k\}$ is a sequence of positive terms tending to zero that are critical points of the function $Q : \mathbb{R} \to \mathbb{R}$ defined by $x^2 sin(1/x)$ away from zero. Now, $\|w^{(k)}\| = \left(\lambda - \frac{1}{2\pi k}\right) \underset{n}{\to} \lambda$ and

$\|v^{(k)}\| = (\lambda - a_k) \underset{n}{\to} \lambda$. Therefore, $g'_\lambda(v^{(k)}) = 0$ for all $k \in \mathbb{N}$, and $g'_\lambda(w^{(k)}) = -y(w^{(k)})$. Therefore, $\|g'_\lambda(w^{(k)})\|_{l_2} = \|y(w^{(k)})\|_{l_2} = 1$. Thus, for any $z \in \partial B(0;\lambda)$ we have

$$\Lambda(g'_\lambda(\cdot), z) := \lim_{\delta\to 0^+} \sup_{u,q\in B^o_\delta(z)} \|g'_\lambda(u) - g'_\lambda(q)\|_{l_2} \geq \lim_{k\to\infty} \|g'_\lambda(w^{(k)}) - g'_\lambda(v^{(k)})\|_{l_2} = \lim_{k\to\infty} \|y\left(w^{(k)}\right)\|_{l_2} \geq 1.$$

We have shown condition (3) in the lemma as well, and hence g_λ works.

□

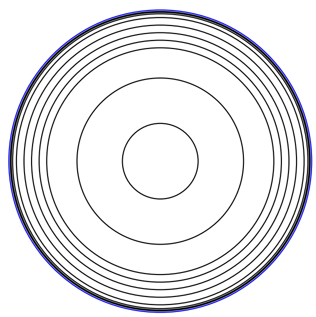

Figure 10: The Zero Set Of A Typical $\mathbb{R}^2 - D_2$ Saturn Ball

We can also shift the cube, and get a similar results.

Lemma 1.9.3. *Let* $0 < \lambda < 1$.*Let* $C := [x_1^0 - \lambda, x_1^0 + \lambda] \times ... \times [x_n^0 - \lambda, x_n^0 + \lambda]$ *be a n-cube with center* $x_0 := (x_1^0, ..., x_n^0)$ *and radius* $\lambda > 0$ *in* $\mathbb{R}^n$. *Then there exists a ball* $A = B^o(x_0; \lambda)$ *inside C, and a function* J_λ *such that:*

1. $supp(J_\lambda) \subseteq A$
2. $J_\lambda \in D_n$
3. $\|J_\lambda\|_\infty \leq \lambda^2 < 1$ *and* $\|J'_\lambda\|_\infty \leq 2\lambda + 1 < 3$
4. $\Lambda(J'_\lambda, u) \geq 1$ *for any* $u \in \partial A$

Proof. (1.9.3) Let g_λ be the function as defined above. We define $J_\lambda(x) := g_\lambda(x - x_0)$, and the rest follows. □

1.9.2 Subdivisions and Partitions

The Saturn Ball function is an interesting function because we may define it in exactly the same way as we did for *any* Banach space with a Fréchet differentiable norm, including infinite dimensional. We only used finite dimensions to describe the function inside an n-cube.

That means that we should be getting more out of the finite dimensionality for proving something, and in fact we do. Consider the following partitioned divisions of a cube, where we will describe the subdivision for the cube $C := [-1, 1] \times ... \times [-1, 1]$ with all others being a scaling and/or translation.

Let $K_0 := \{x \in C : \|x\|_\infty \leq 1/2\}$, $K_1 := \{x \in C : \|x\|_\infty \leq 1/2 + 1/4\}$, and in general $K_n := \{x \in C : \|x\|_\infty \leq \sum_{i=1}^{n+1} \frac{1}{2^i}\}$.

Let $C_{0,1} := K_0$, and $k_0 = 1$. Now, let $\widetilde{C_1} := K_1 \setminus K_0$, which can be written as the finite union of uniform cubes $C_{1,j}$ that are translations of the cube $[0, 1/4] \times ... \times [0, 1/4]$, so $\widetilde{C_1} = \cup_{j=1}^{k_1} C_{1,j}$ where $k_1 \in \mathbb{N}$. Likewise, $\widetilde{C_2} := K_2 \setminus K_1$, which can be written as the finite union of uniform cubes $C_{2,j}$ that are translations of the cube $[0, 1/8] \times ... \times [0, 1/8]$, so $\widetilde{C_2} = \cup_{j=1}^{k_2} C_{2,j}$ where $k_2 \in \mathbb{N}$. So in general, we define $\widetilde{C_n} := K_n \setminus K_{n-1}$, which can be written as the finite union of uniform cubes $C_{n,j}$ that are translations of the cube $[0, \frac{1}{2^{n+1}}] \times ... \times [0, \frac{1}{2^{n+1}}]$, so $\widetilde{C_n} = \cup_{j=1}^{k_n} C_{1,j}$ where $k_n \in \mathbb{N}$. Then the complete partition of C involves writing $C = \bigcup_{n=0}^{\infty} \cup_{j=1}^{k_n} C_{n,j}$, where we notice that the radius of the partition cubes goes to zero as we head towards the boundary of C.

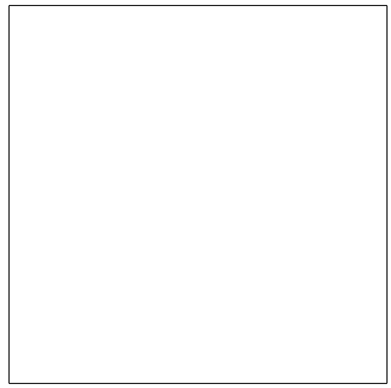

Figure 11: 0th Stage

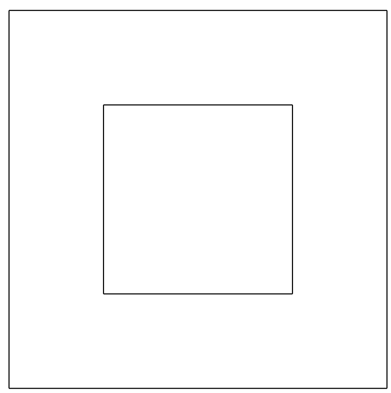

Figure 12: 1st Stage of the Subdividing

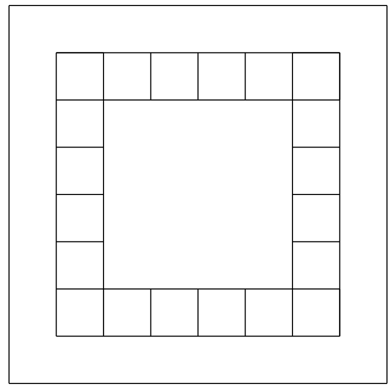

Figure 13: 2nd Stage

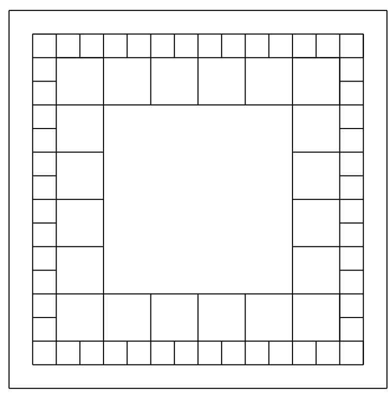

Figure 14: 3rd Stage of the Subdividing

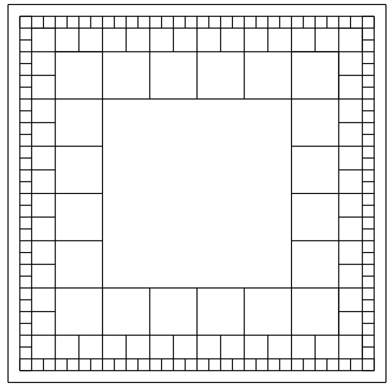

Figure 15: Another Stage of the Subdividing

We will call these partitions of the cube *Mayan* subdivisions, as in 2-dimensions they resemble a bird's eye view of the Mayan pyramids.

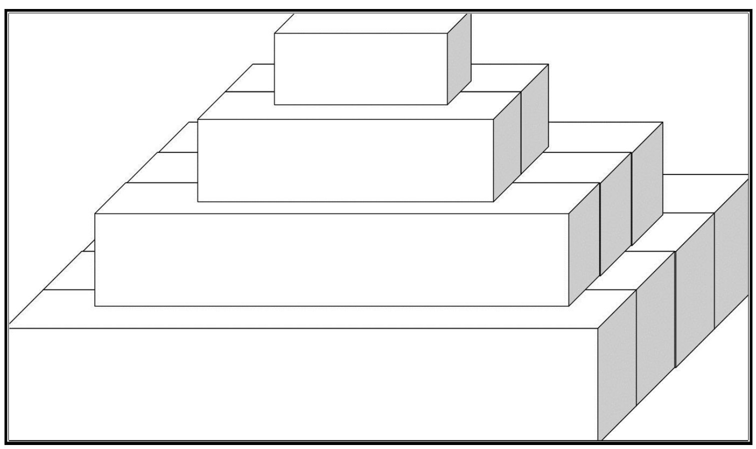

Figure 16: Illustration Of Mayan Step Pyramid.

1.9.3 Generating Function On The Boundary Of A Cube

It would be interesting to reach results like the ones before regarding porosity, and in fact we are about to see that this is possible.

Lemma 1.9.4. *Let* $0 < \lambda < 1$. *Let* $C \subseteq [0, 1]^n$ *be an n-cube with center* x_0 *and radius* λ. *Then there exists a function* T_C *such that:*

1. *$supp(T_C) \subseteq C$ and $T_C = 0$ on ∂C*
2. *$T_C \in D_n$*
3. *$\|T_C\|_\infty \leq \lambda^2$ and $\|T'_C\|_\infty \leq 2\lambda + 1$*
4. *$\Lambda(T'_C, x) \geq 1$ for all $x \in \partial C$*

Furthermore, we will call such a function a Mayan function.

Proof. We will prove this in a few steps using our previous work. Let g_λ be the Saturn ball function for λ. First, we observe that $C = \bigcup_{i=1}^{\infty} C_i$ where the $\{C_i\}_{i=1}^{\infty}$ are a Mayan subdivision of the cube, with each C_n having radius λ_n, and center x_n. Now, define $T_C : \mathbb{R}^n \to \mathbb{R}$ as $T_C(u) := \sum_{i=1}^{\infty} g_{\lambda_i}(u - x_i)$, where $g_{\lambda_i}(\cdot - x_i)$ is the Saturn Ball function for the ball $B^o(x_i; \lambda_i)$ within the n-cube C_i. Fix $x \in \partial C$. Let $\delta > 0$. Then within $B^o(x; \delta)$ there exists a cube, say C_k within the ball, as by construction, the size of the cubes goes to zero out to the boundary. Recall the tangential oscillation of g_{λ_i} is at least 1 on $\partial B^o(x_i; \lambda_i)$ for any $i \in \mathbb{N}$. Now, $T_C(x)$ agrees with $g_{\lambda_k}(x - x_k)$ on C_k, so within $B^o(x; \delta)$ we have that T_C attains a tangential oscillation of at least 1. Now, as this is a uniform lower bound for all such $\delta > 0$, we can see that the tangential oscillation of T_C is at least 1 at the point x. Thus (1),(2), (3) and (4) above clearly follow for T_C. □

Lemma 1.9.5. *Let E be a closed nowhere dense subset of $[0, 1]^n$. Then there exists a function f_E such that:*

1. *$f_E = 0$ on E*
2. *$f_E \in D_n$*
3. *$\|f_E\|_\infty \leq \frac{1}{2}$ and $\|f'_E\|_\infty \leq 3$*
4. *$\Lambda(f'_E, x) \geq 1$ for all $x \in E$*

Proof. We will prove this in a few steps using our previous work. Let g_λ be the Saturn ball function for λ. We mention that from here on, when we are defining our functions, we will not distinguish between half-open and closed n-cubes. The reason is that our functions are defined to be zero on the boundary of the cubes, therefore it doesn't matter. First, we observe that by the Dyadic Cubes Theorem, the open set $C := E^c = \bigcup_{i=1}^{\infty} K_i$ where the $\{K_i\}_{i=1}^{\infty}$ are half-open cubes at center z_i with radii d_i for each K_i. Furthermore, each $K_i = \bigcup_{j \in \mathbb{N}} A_{i,j}$ where the A_js are a Mayan subdivision. Therefore, we may re-enumerate all the $A_{i,j}$s as the union $C = \bigcup_{n \in \mathbb{N}} C_n$ where the C_n are cubes at center x_n with radii r_n for each C_n. We here make the important comment that because we have first done a dyadic subdivision, we have assured that the largest value of some r_n is $\frac{1}{2}$... therefore $\|g_{r_n}(\cdot - x_n)\|_\infty \leq \frac{1}{2}$.

Now, define $f_E(x) := \sum_{i=1}^{\infty} g_{r_i}(x - x_i)$, where g_{r_i} is the Saturn Ball function for the cube C_i. Fix $x \in \partial C$. Let $\delta > 0$. Then within $B^o(x; \delta/2)$ there is an edge of a dyadic cube. Furthermore, by the definition of the Mayan subdivision, there is a Mayan cube within $B^o(x; \delta)$, say C_k. Then f_E agrees with $g_{r_i}(x - x_i)$ on C_k. Now as the tangential oscillation of $g_{r_i}(\cdot - x_i)$ is at least 1 on the boundary of the ball $B^o(x_k; r_k)$, and as $B^o(x_k; r_k) \subseteq B^o(x; \delta)$, we have $1 \leq \sup_{h,k \in B^o(x;\delta)} \|f'_E(h) - f'_E(k)\|$. Now as $\delta > 0$ is arbitrary and we can uniformly bound below by 1, then taking a limit as $\delta \to 0^+$, we get that the tangential oscillation of f_E is at least 1 at x. Furthermore, (1),(2), and (3) above clearly follow for f_E.

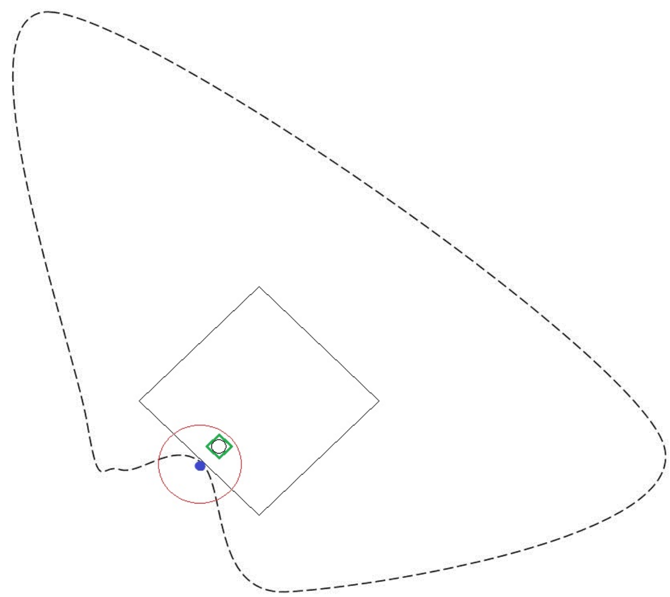

Figure 17: Illustration Of Density of Mayan Cube

□

1.10 CATEGORY RESULTS IN N-DIMENSIONS

Theorem 1.10.1. *Let* $N \in \mathbb{N}$. *Let* E *be an arbitrary closed nowhere dense subset of* $[0, 1]^N$. *Let* $G_E := \{f \in D_N : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$. *Then* G_E *is a dense open subset of* D_N.

Proof. (Theorem 1.10.1) Let $\|\cdot\| := \|\cdot\|_{l_2}$. We prove open first, and do so directly. First we note that G_E is non-empty, by the above lemma and scaling of the example function. Let $F \in G_E$ be arbitrary, and $\delta := \inf_{x_0 \in E} \Lambda(F', x_0) > 0$. Now choose ϵ arbitrary such that $0 < \epsilon \leq \delta/4$. Then let $s \in D_N$ such that $\|s\|_D < \epsilon$, and $h := F + s$. We will show that $h \in G_E$. Well, for any $x \in E$,

$$\delta/2 < \delta - 2\epsilon \leq \Lambda(F', x) - 2\|s'\|_\infty \leq \Lambda(F', x) - \Lambda(s', x) \leq \Lambda(F' + s', x) =: \Lambda(h', x).$$

Therefore, $0 < \delta/2 \leq \inf_{x \in E} \Lambda(h', x)$, and so $h \in G_E$. Therefore, G_E is open.

Now we prove density of the set. Let $\epsilon \in (0, 1)$, and $F \in D_N$. We will show that there exists a function g such that $g \in G_E$, and $\|F - g\|_{D_N} < \epsilon$, by creating a modified version of one of our previously defined auxiliary functions.

We will prove this in a few steps using our previous work. First, we observe that by the Dyadic Cubes Theorem, the open set $E^c = \bigcup_{i=1}^\infty K_i$ where the $\{K_i\}_{i=1}^\infty$ are half-open cubes at center z_i with radii d_i for each K_i. Furthermore, each $K_i = \bigcup_{j \in \mathbb{N}} A_{i,j}$ where the A_js are a Mayan subdivision. Therefore, we may re-enumerate all the $A_{i,j}$s as the union $E^c = \bigcup_{n \in \mathbb{N}} C_n$ where the C_n are cubes at center x_n with radii r_n for each C_n. Now again we will not distinguish between half-open and closed n-cubes because our functions are zero on the boundaries of the n-cubes.

Now, let g_λ be the Saturn Ball function for λ. Then we define the following function with $\alpha := \{\alpha_n\}$ a sequence of real scalars:

$$w(x) := \sum_{n \in \mathbb{N}} \alpha_n g_{r_n}(x - x_n)$$

We will soon use this function $w(x)$ to create our function for density by choosing particular $\overrightarrow{\alpha} = (\alpha_n)_{n \in \mathbb{N}}$.

Thus we let:

$$\alpha_n := \begin{cases} \frac{\epsilon}{4} & : 0 \leq \Lambda(g'_{r_n}(\cdot - x_n), x) < \frac{\epsilon}{8}, \forall x \in \partial B^o(x_n; r_n) \\ 0 & : \exists z_n \in \partial B^o(x_n; r_n) \text{ such that } \frac{\epsilon}{8} \leq \Lambda(g'_{r_n}(\cdot - x_n), z_n) \end{cases}$$

Then the sequences just defined are sequences for scaling the tangential oscillations.

Then we let $L := F + w$, and we will show that $L \in G_E$. We will first show that $\forall n \in \mathbb{N}$ there exists $y_n \in \partial B^o(x_n; r_n)$ such that $\Lambda(L', y_n) \geq \frac{\epsilon}{8}$. We will use this to show that $\Lambda(L', x) \geq \frac{\epsilon}{8}$ for all $x \in E$.

As such, let $k \in \mathbb{N}$ be fixed, we show existence of y_k.

<u>Case 1</u>: Suppose $0 \leq \Lambda(F', x) < \frac{\epsilon}{8}, \forall x \in \partial B^o(x_k; r_k)$. Then using Proposition 1.2.3, for any $y \in \partial B^o(x_k; r_k)$:

$$0 < \frac{\epsilon}{8} = \left(\frac{\epsilon}{4}\right) - \frac{\epsilon}{8} \leq \Lambda(\alpha_k g'_{r_n}(\cdot - x_n), y) - \Lambda(F', y) \leq \Lambda(w' + F', y) = \Lambda(L', y) \tag{1.43}$$

So choose any $y_k \in \partial B^o(x_k; r_k)$.

<u>Case 2</u>: Suppose $\exists z_k \in \partial B^o(x_k; r_k)$ such that $\frac{\epsilon}{8} \leq \Lambda(F', z)$. Then using Proposition 1.2.3:

$$0 < \frac{\epsilon}{8} \leq \Lambda(F', z_k) - 0 = \Lambda(F', z_k) - \Lambda(\alpha_k g'_{r_n}(\cdot - x_n), z_k) \leq \Lambda(\alpha_k g'_{r_n}(\cdot - x_n) + F', z_k) = \Lambda(L', z_k) \tag{1.44}$$

So we let $y_k := z_k$.

Now, let $x \in E$, and $\delta > 0$ be arbitrary. We will show that there is a point within $B^o(x; \delta)$ such that the tangential oscillation of L is at least $\frac{r}{8}$, for then $\sup_{h,k \in B(x;\delta)} \|L'(h) - L'(k)\| \geq \frac{\epsilon}{8}$.

Appealing again to our picture:

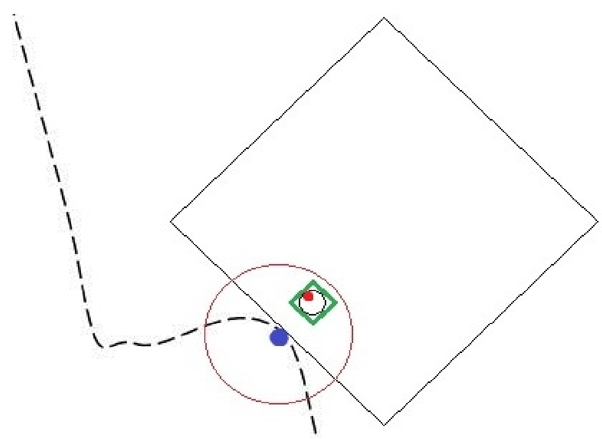

Figure 18: Selection Of Our Point

Then within $B^o(x; \delta/2)$ there is an edge of a dyadic cube. Furthermore, by the definition of the Mayan subdivision, there is a Mayan cube within $B^o(x; \delta)$, say C_m. So we have a point $y_m \in \partial B^o(x_m; r_m)$ with $B^o(x_m; r_m) \subseteq B^o(x; \delta)$, and $\frac{\epsilon}{8} \leq \Lambda(L', y_m)$. Therefore, $\frac{\epsilon}{8} \leq \sup_{h,k \in B^o(x;\delta)} \|L'(h) - L'(k)\|$.

Now, as δ is arbitrary, and we may always find such a point as y_m, then

$$0 < \frac{\epsilon}{8} \leq \lim_{\delta \to 0^+} \sup_{h,k \in B^o(x,\delta)} |L'(h) - L'(k)| =: \Lambda(L', x).$$

Now, we'll prove density. We here make the important comment that because we have first done a dyadic subdivision, we have assured that the largest value of some r_n is $\frac{1}{2}$... therefore $\|g_{r_n}(\cdot - x_n)\|_\infty \leq \frac{1}{2}$. Well,

$$\begin{aligned} \|F - L\|_D &:= \|F - L\|_\infty + \|F' - L'\|_\infty = \|F - (F + w)\|_\infty + \|F' - (F' + w')\|_\infty = \|w\|_\infty + \|w'\|_\infty \\ &< \left(\frac{\epsilon}{4}\right)^2 + \left(\frac{\epsilon}{4}\right)\left[\sup_{x \in [0,1]} |2x \sin(1/x) + \cos(1/x)|\right] < \frac{\epsilon}{4} \cdot \frac{1}{2} + 3\frac{\epsilon}{4} = \epsilon. \end{aligned} \tag{1.45}$$

Thus, we have that G_E is indeed dense in D_N. □

This leads to an additional result that further generalizes the ideas we are getting.

Theorem 1.10.2. *Let E be an arbitrary closed nowhere dense subset of* $[0, 1]^N$. *Let* $H_E := \{f \in D_N : \Lambda(f', x_0) > 0, \forall x_0 \in E\}$. *Then H_E is second category in D_N.*

Proof. Let E be given. As $G_E \subseteq H_E$, and G_E is a dense open set, we then have that H_E is second category. □

From this, we can extend further in order to get the result we have been after the whole time.

Theorem 1.10.3. *Let E be an arbitrary meagre set in* $[0, 1]^N$. *Let* $H_E := \{f \in D_N : \Lambda(f', x_0) > 0, \forall x_0 \in E\}$. *Then, H_E is residual in D_N.*

Proof. (Theorem 1.10.3) Well, we may write $E = \bigcup_{n \in \mathbb{N}} E_n$ where each E_n is a nowhere dense set. Let $A_n := cl(E_n)$ for all $n \in \mathbb{N}$. Then G_{A_n} is a dense open set, and so $J := \bigcap_{n \in \mathbb{N}} G_{A_n}$ is a dense G_δ set in D_N. So, as $J \subseteq H_E$, then H_E is second category. □

1.11 POROSITY RESULTS IN N-DIMENSIONS

Theorem 1.11.1. *Let $N \in \mathbb{N}$ be fixed. Let $E \subseteq [0,1]^N$ be a closed nowhere dense set. Then $G_E := \{f \in D_N : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$ is co-porous.*

Proof. (Theorem 1.11.1) Let $\|\cdot\| := \|\cdot\|_{l_2^N}$. Let $\beta = \frac{1}{32}$ and $r_0 = 1/2$, for the porosity constants. Furthermore, take any $F \in D_N \setminus G_E := \{u \in D_N : \inf_{x_0 \in E} \Lambda(u', x_0) = 0\}$ and let $r \in (0, r_0]$.

We will not be distinguishing be half-open and closed cubes, as the functions we will be using will be zero'd out on the boundary of all the cubes.

We will prove this in a few steps using our previous work. First, we observe that by the Dyadic Cubes Theorem, the open set $E^c = \bigcup_{i=1}^{\infty} K_i$ where the $\{K_i\}_{i=1}^{\infty}$ are cubes at center z_i with radii d_i for each K_i. Notice that the maximum n-cube radius is $\frac{1}{2}$ by the Dyadic cubes theorem. Furthermore, each $K_i = \bigcup_{j \in \mathbb{N}} A_{i,j}$ where the $A_{i,j}$'s are a Mayan subdivision by cubes with centers s_j^i and radii t_j^i.

So we may re-enumerate all the $A_{i,j}$'s as the union $E^c = \bigcup_{n \in \mathbb{N}} C_n$ where the C_n are cubes at center x_n with radii r_n.

Now, what is the idea? Well, we are going to use all these n-cubes, and place a Saturn ball function into each cube. These Saturn ball functions will then have positive tangential oscillation within the n-cube where they are nonzero defined, and if we can suitably control the amplitude, then we will have a "porously close generating function".

Now, let g_λ be the Saturn Ball function for $\lambda > 0$. Then within any neighborhood of $x \in E$ there is a translated Saturn Ball function whose derivative has oscillation of at least one along the boundary of a ball, according to the Mayan subdivision of the cube, and the definition of a Saturn ball. Then we define the following function with $\alpha := \{\alpha_n\}$ a sequence of real scalars:

$$w(x) := \sum_{n \in \mathbb{N}} \alpha_n g_{r_n}(x - x_n)$$

We will soon use this function $w(x)$ to create our function for porosity by choosing particular $\overrightarrow{\alpha} = (\alpha_n)_{n \in \mathbb{N}}$.

Thus we let:

$$\alpha_n := \begin{cases} 0 : r_n \geq \frac{r}{4} \\ \frac{r}{4} \qquad\qquad : r_n < \frac{r}{4} \text{ and } 0 \leq \Lambda(F', x) < \frac{r}{8}, \forall x \in \partial B^o(x_n; r_n) \\ 0 \qquad\qquad : r_n < \frac{r}{4} \text{ and } \exists z \in \partial B^o(x_n; r_n) \text{ such that } \frac{r}{8} \leq \Lambda(F', z) \end{cases}$$

Then the sequences just defined are sequences for the scaling of the tangential oscillations. Now let's explain why we zero'd out our translated Saturn ball is $r_n \geq \frac{r}{4}$. Well, if $r_n \geq \frac{r}{4}$, then the cube we are working in is not "close to the boundary of a dyadic cube", therefore it's tangential oscillation is unimportant for our concerns.

Then we let $J := F + w$, and we will show that $J \in G_E$. In fact we will show that $\Lambda(J, x) \geq \frac{r}{8}$ for all $x \in E$.

We will first show that $\forall n \in \mathbb{N}$ such that $\frac{r}{4} > r_n$ there exists $y_n \in \partial B^o(x_n; r_n)$ such that $\Lambda(J', y_n) \geq \frac{r}{8}$. We will use this to show that $\Lambda(J', x) \geq \frac{r}{8}$ for all $x \in E$.

As such, let $k \in \mathbb{N}$ be fixed, we show existence of y_k.

<u>Case 1</u>: Suppose $0 \leq \Lambda(F', x) < \frac{r}{8}, \forall x \in \partial B^o(x_k; r_k)$. Then using Proposition 1.2.3, for any $y \in \partial B^o(x_k; r_k)$:

$$0 < \frac{r}{8} = \left(\frac{r}{4}\right) - \frac{r}{8} \leq \Lambda(\alpha_k g'_{r_k}(\cdot - x_k), y) - \Lambda(F', y) \leq \Lambda(w' + F', y) = \Lambda(J', y) \tag{1.46}$$

So choose any $y_k \in \partial B^o(x_k; r_k)$.

<u>Case 2</u>: Suppose $\exists z_k \in \partial B^o(x_k; r_k)$ such that $\frac{r}{8} \leq \Lambda(F', z_k)$. Then using Proposition 1.2.3:

$$0 < \frac{r}{8} \leq \Lambda(F', z_k) - 0 = \Lambda(F', z_k) - \Lambda(\alpha_k g'_{r_k}(\cdot - x_k), z_k) \leq \Lambda(\alpha_k g'_{r_n}(\cdot - x_n) + F', z_k) \leq= \Lambda(w' + F', z_k) =: \Lambda(J', z_k) \tag{1.47}$$

So we let $y_k := z_k$.

Now, let $x \in E$, and $\delta > 0$ be arbitrary. We will show that there is a point within $B^o(x; \delta)$ such that the tangential oscillation of J is at least $\frac{r}{8}$, for then $\sup_{h,k \in B(x;\delta)} \|J'(h) - J'(k)\|_{\mathcal{L}(\mathbb{R}^N;\mathbb{R})} \geq \frac{r}{8}$.

We turn to an illustration to present clarity:

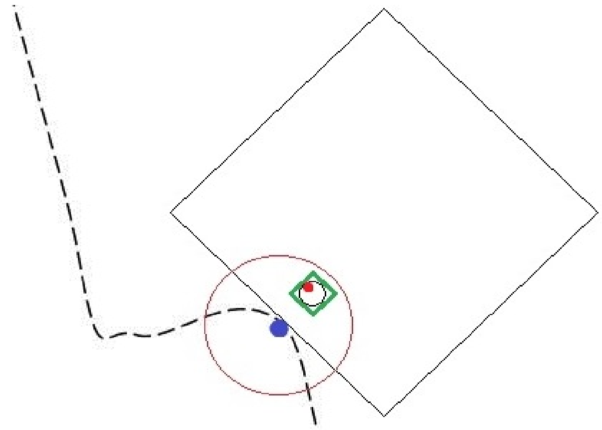

Figure 19: Selection Of Our Point

Then within $B^o(x;\delta/2)$ there is an edge of a dyadic cube. Furthermore, by the definition of the Mayan subdivision, there is a Mayan cube within $B^o(x;\delta)$, say C_m. Now, we may assume that $r_m < \frac{r}{4}$, for if not, there is a closer cube within $B^o(x;\delta)$. So we have a point $y_m \in \partial B^o(x_k;r_k)$ with $B^o(x_k;r_k) \subseteq B^o(x;\delta)$, and $\frac{r}{8} \le \Lambda(J', y_m)$. Therefore, $\frac{r}{8} \le \sup_{h,k\in B^o(x;\delta)} \|J'(h) - J'(k)\|_{\mathcal{L}(\mathbb{R}^N;\mathbb{R})}$.

Now, as δ is arbitrary, and we may always find such a point y_m, then

$$0 < \frac{r}{8} \le \lim_{\delta\to 0^+} \sup_{h,k\in B^o(x,\delta)} \|J'(h) - J'(k)\|_{\mathcal{L}(\mathbb{R}^N;\mathbb{R})} =: \Lambda(g', x).$$

So $J \in G_C$.

Now to show that $B^o(J;\beta r) \subseteq B^o(F,r) \cap G$. Let $v \in D_N$ such that $\|v\|_D \le \beta r = \frac{r}{32}$. Fix an arbitrary $x \in E$. Then,

$$0 < r/16 = (r/8) - 2(r/32) < \Lambda(J', x) - 2\|v'\|_\infty < \Lambda(J', x) - \Lambda(v', x) \le \Lambda(J' + v', x).$$

This proves that $J + v \in G_E$, as needed.

Now to show that $B^o(J;\beta r) \subseteq B^o(F,r)$. Well, let $s \in D_N$ such that $\|s\|_D \leq \beta r := \frac{r}{32}$. Then, recalling that $r \leq \frac{1}{2}$ so that $\frac{r^2}{16} \leq \frac{r}{32}$:

$$\begin{aligned}\|(J+s)-F\|_D &:= \|(F+w+s)-F\|_D = \|w+s\|_D \leq \|w\|_D + \|s\|_D \\ &\leq \|w\|_D + \frac{r}{32} = \|w\|_\infty + \|w'\|_\infty + \frac{r}{32} \\ &\leq \left(\frac{r}{4}\cdot\frac{r}{4}\right) + \frac{3r}{4} + \frac{r}{16} \leq \frac{r}{32} + \frac{3r}{4} + \frac{r}{16} < r.\end{aligned} \tag{1.48}$$

Thus $B^o(J;\beta r) \subseteq B^o(F,r)$, and so then $B^o(J;\beta r) \subseteq B^o(F,r) \cap G_E$. Therefore, G_E is co-porous as claimed. □

Now, utilizing the fact that porous sets are nowhere dense, we get the following corollary:

Corollary 1.11.1. *(To Theorem 1.11.1) Let E be a porous set in $[0,1]^N$. Then $G_E := \{f \in D_N : \inf_{x_0\in E} \Lambda(f',x_0) > 0\}$ is co-porous.*

We can also get another corollary of the above theorem for functions mapping into another Banach space.

Corollary 1.11.2. *Let $N \in \mathbb{N}$ be fixed. Let $(Y, \|\cdot\|_Y)$ be a non-trivial Banach space. Let $E \subsetneq [0,1]^N$ be a closed nowhere dense set. Then $G_E := \{f \in D([0,1]^N;Y) : \inf\limits_{x_0\in E} \Lambda(f',x_0) > 0\}$ is co-porous.*

Here, we have that if $f : [0,1]^N \to Y$, then $\|f'\|_\infty := \sup\limits_{x\in[0,1]^N} \|f'(x)\|_{\mathcal{L}(\mathbb{R}^N;Y)}$. Also, $\Lambda(f',x) := \lim\limits_{\delta\to 0^+} \sup\limits_{h,k\in B^o_{[0,1]^N}(x;\delta)} \|f'(h)-f'(k)\|_{\mathcal{L}(\mathbb{R}^N;Y)}$.

The proof uses the following: let $y \in S(Y)$. Then we take $s := wy : (0,1)^N \to Y$ as our generating function. Then, $s' = w'y : [0,1]^N \to \mathcal{L}(\mathbb{R}^N;Y)$, as $w' : [0,1]^N \to \mathcal{L}(\mathbb{R}^N;\mathbb{R})$, and the proof proceeds as before.

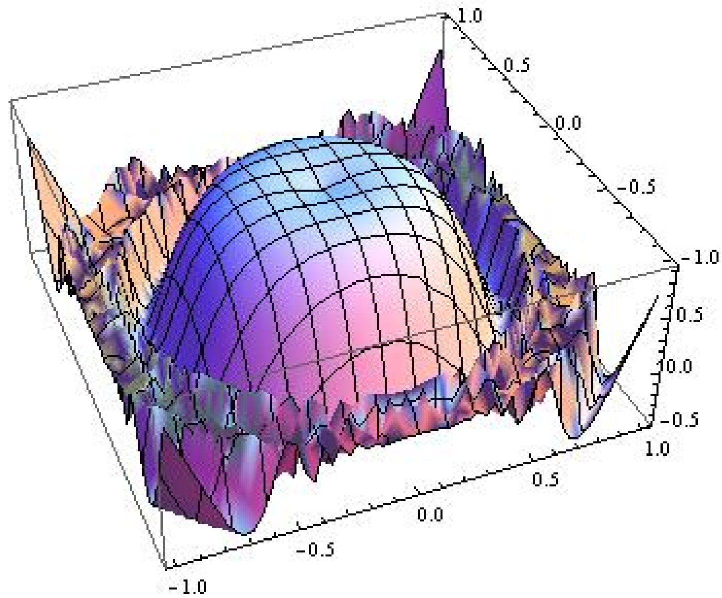

Figure 20: Saturn Ball Function

2.0 BOUNDED VARIATION AND POROSITY

2.1 INTRODUCTION

In the present article, we look at a way of describing the fundamental theorem in $BV[a,b]$, in terms of how typical the property of the fundamental theorem is within $BV[a,b]$ (sometimes referred to as the genericity of the property).

The history of the Fundamental Theorem of Calculus (FTC) is hard to precisely pinpoint. As noted in [5], contributions to the development of the FTC stretches at least as far as Eudoxus of Cnidus and has continued through Euler, Cauchy, and Weierstrass and includes modern day generalizations of the results. Many students are taught the FTC in their calculus class in the setting of Riemann Integrals, with a simple proof using partitions of the given interval. However, as they hopefully continue on in their studies, they eventually see the FTC again in a measure theoretic setting (see for instance [21]). There are also a number of generalizations of the FTC that the curious reader may investigate (See for example [13]).

We look within the space of bounded variation, and take this as a reasonable setting, as every function of bounded variation is differentiable almost everywhere (see [21]). Therefore, we may ask the question: How typical it is for a function u of bounded variation to satisfy $u(x) - u(a) = \int_{t=a}^{t=x} u'(t)dm(t)$?

2.2 THE SETTING

Definition of Porosity Let (M, d) be a metric space, and $A \subseteq M$ such that A is closed. We say that A is porous if $\exists r_0 \in (0, \infty)$ and $\exists \beta \in (0, 1]$ such that $\forall x \in A$, $\forall r \in (0, r_0]$, $\exists z \in A^c$ such that $B^o(z; \beta r) \subseteq B^o(x, r) \cap A^c$. Moreover, for arbitrary $S \subseteq M$, we say that S is porous if $\overline{S}$ is porous. Furthermore, we say that a set J is co-porous if J^c is porous.

A set S is σ-porous if and only if $S = \bigcup_{n \in \mathbb{N}} S_n$ such that each S_n is porous. A set A is co-σ-porous if and only if $A := S^c$ where $S = \bigcup_{n \in \mathbb{N}} S_n$ such that each S_n is porous.

We are now prepared to start towards our result, and now define the space we will be working in for the present article.

Definition Let $\mathfrak{F}(\mathbb{R}, \mathbb{R}) :=$ {functions $g : \mathbb{R} \to \mathbb{R}$}. Let $f \in \mathfrak{F}(\mathbb{R}, \mathbb{R})$. Let $a < b$ with $a, b \in \mathbb{R}$. Let $\mathcal{P}$ be the set of all Riemann partitions of $[a, b]$, such that if $P \in \mathcal{P}$, then $P = \{t_0 = a, t_2, t_3, \cdots, t_{m-1}, t_m = b\}$ is a partition of $[a, b]$. Then we define the variation of f for P:

$$V_a^b(f; P) := \sum_{n=1}^{m} |f(t_n) - f(t_{n-1})| .$$

Then, we define the total variation of f.

$$V_a^b(f) := \sup_{P \in \mathcal{P}} V_a^b(f, P).$$

Now, we can define the following linear vector space:

$$BV[a, b] := \{f \in \mathfrak{F}(\mathbb{R}, \mathbb{R}) : V_a^b(f) < \infty\}$$

Given the above, we now define a norm on $BV[a, b]$:

$$\|f\| := |f(a)| + V_a^b(f).$$

Theorem 2.2.1. *(See [1] for information) Under the above definitions,* $(BV[a, b], \|\cdot\|)$ *is a complete normed vector space over* $\mathbb{R}$*, therefore, it is a Banach space.*

2.3 CONTINUITY IN $BV[A, B]$

We start with an important example that we need in our work.

Cantor-Staircase/Cantor-Lebesgue Function, (See [13] among many others) Let C be the middle thirds Cantor set. Let x be a real number in $[0, 1]$ with the ternary expansion $0.a_1a_2a_3\ldots$, then let N be ∞ if no $a_n = 1$ and otherwise let N be the smallest value such that $a_n = 1$. Next let $b_n = \frac{1}{2}a_n$ for all $n < N$ and let $b_N = 1$. We define the Cantor function (or the Cantor ternary function) as the following:

$$f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}.$$

This function can be checked to be continuous and monotonic on $[0, 1]$, and $f'(x) = 0$ almost everywhere. Furthermore, $f(0) = 0$ and $f(1) = 1$.

As f is monotonic, we can see that $f \in BV[0, 1]$, as $V_0^1(f)$ is finite.

Proposition 2.3.1. *Define the function* $\Psi : BV[a, b] \to [0, \infty)$ *by*

$$\Psi(u) := \sup_{x\in[a,b]} \left| (u(x) - u(a)) - \int_{t=a}^{t=x} u'(t)dm(t) \right|.$$

Then for any $f, g \in X$ *and* $c \in \mathbb{R}$ *we have the following:*

(a) $\Psi(cf) = |c|\Psi(f)$

(b) $\Psi(f + g) \leq \Psi(f) + \Psi(g)$

(c) $|\Psi(f) - \Psi(g)| \leq \Psi(f + g)$

(d) $\Psi(f) \leq 2V_a^b(f) \leq 2\|f\|$

Proof. (a) Let $c \in \mathbb{R}$, and $f, g \in X$. Then $\Psi(cf) = \sup_{x\in[a,b]} \left| (cf(x) - cf(a)) - \int_{t=a}^{t=x} cf'(t)dm(t) \right|$

$$= |c| \sup_{x\in[a,b]} \left| (f(x) - f(a)) - \int_{t=a}^{t=x} f'(t)dm(t) \right| = |c|\Psi(f).$$

(b) Furthermore, we investigate $\Psi(f + g)$:

$$\Psi(f + g) = \sup_{x\in[a,b]} \left| ((f + g)(x) - (f + g)(a)) - \int_{t=a}^{t=x} (f + g)'(t)dm(t) \right|$$

$$\leq \sup_{x\in[a,b]} \left[\left| (f(x) - f(a)) - \int_{t=a}^{t=x} f'(t)dm(t) \right| + \left| g(x) - g(a) - \int_{t=a}^{t=x} g'(t)dm(t) \right| \right]$$

$$\leq \sup_{x\in[a,b]} \left| (f(x) - f(a)) - \int_{t=a}^{t=x} f'(t)dm(t) \right| + \sup_{x\in[a,b]} \left| g(x) - g(a) - \int_{t=a}^{t=x} g'(t)dm(t) \right| = \Psi(f) + \Psi(g)$$

(c) This is standard from the existence of the triangle inequality.

(d) For this, we will use some results from Royden (Ch. 5.2-3 [21]). First, we observe:

$$\sup_{x\in[a,b]} \left| (f(x) - f(a)) - \int_{t=a}^{t=x} f'(t)dm(t) \right|$$

$$\leq \sup_{x\in[a,b]} |f(x) - f(a)| + \sup_{x\in[a,b]} \left| \int_{t=a}^{t=x} f'(t)dm(t) \right| .$$

Now, let $P_x := \{a = x_0, x_1, ..., x_{n-1}, x_n = x\}$ be a partition of $[a, x]$. Then by the triangle inequality,

$$|f(x) - f(a)| = |f(x_n) - f(x_0)| \leq |f(x_n) - f(x_{n-1})| + |f(x_{n-1}) - f(x_0)| \leq ... \leq \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|.$$

Therefore, we can see that $|f(x) - f(a)| \leq V_a^x$, and so $\sup_{x\in[a,b]} |f(x) - f(a)| \leq \sup_{x\in[a,b]} V_a^x(f) \leq V_a^b(f)$.

Recall that $f(x) = P_a^x(f) - (N_a^x(f) - f(a))$ is the Jordan decomposition of f. Also, recall that $V_a^x(f) = P_a^x(f) + N_a^x(f)$. Additionally, $h_1(x) := P_a^x(f)$ and $h_2(x) = (N_a^x(f) - f(a))$ are both monotone increasing, and therefore differentiable. Furthermore, a version of the Fundamental Theorem of Calculus holds for h_1, h_2, i.e. for $i = 1, 2$, we have:

$$h_i(x) \geq \int_{t=a}^{t=x} h_i'(t)dm(t) + h_i(a).$$

Therefore,

$$\sup_{x\in[a,b]} \left| \int_{t=a}^{t=x} f'(t)dm(t) \right| \leq \sup_{x\in[a,b]} \int_{t=a}^{t=x} |f'(t)|\, dm(t)$$

$$= \sup_{x\in[a,b]} \int_{t=a}^{t=x} \left| V_a^{t\,\prime}(f) \right| dm(t) = \sup_{x\in[a,b]} \int_{t=a}^{t=x} \left| P_a^{t\,\prime}(f) - N_a^{t\,\prime}(f) \right| dm(t)$$

$$\leq \sup_{x\in[a,b]} \int_{t=a}^{t=x} P_a^{t}{}'(f) + N_a^{t}{}'(f) dm(t) = \sup_{x\in[a,b]} P_a^x(f) + N_a^x(f) \leq \sup_{x\in[a,b]} V_a^x(f) \leq V_a^b(f).$$

Therefore in sum we have that $\Psi(f) \leq 2V_a^b(f) \leq 2\,|f(a)| + 2V_a^b(f)$, and $\Psi(f) \leq 2V_a^b(f) \leq 2\|f\|_\infty + 2V_a^b(f)$. Therefore, $\Psi(f) \leq 2\|f\|$

□

Theorem 2.3.1. *Let $BV[a, b]$ be the Banach space of functions of bounded variation with norm $\|f\| := |f(a)| + V_a^b(f)$. Define $G := \{v \in BV[a, b] : \Psi(v) > 0\}$. Then G is a dense, open, Co-Porous set in $BV[a, b]$.*

Proof. We will prove the theorem for $BV[0, 1]$ for we may compose the usual Cantor function with a translation and scaling function to accomplish the same result for $BV[a, b]$. Let $f : [0, 1] \to [0, 1]$ be the Cantor function. So $0 < \|f\| \leq 1$. Additionally, we know that

$$\Psi(f) := \sup_{x\in[0,1]} \left| f(x) - f(0) - \int_{t=0}^{t=x} f'(t)dm(t) \right| = 1.$$

So $0 < \Psi(f) \leq 2\|f\| < \infty$ is true, and in fact we have $\Psi(f) = \|f\|$, but we will only use $\Psi(f) \leq 2\|f\|$. Let $\delta := \Psi(f) > 0, \gamma := \|f\| > 0$. We will use this later as our "generating function".

We will prove open first. Let $g \in G$ be arbitrary, and $\alpha := \Psi(g) > 0$. Now choose $\epsilon > 0$ such that $0 < \epsilon \leq \frac{\alpha}{4}$. Then let $s \in BV[0, 1]$ such that $\|s\| < \epsilon$, $h := g + s$, and we show that $h \in G$. Note that we know that $\Psi(s) \leq 2\|s\|$. Now we have the following:

$$0 < \frac{\alpha}{2} = \alpha - \frac{\alpha}{2} \leq \Psi(g) - 2\|s\| \leq \Psi(g) - \Psi(s) \leq \Psi(g + s) =: \Psi(h).$$

Now we need to show the porosity conditions. Let's see what we need to prove:

We need to show $\exists\beta \in (0, 1]$ and $\exists r_0 \in (0, \infty)$ such that $\forall v \in G^c$, $\forall r \in (0, r_0]$, $\exists S \in G$ such that $B^o(S; \beta r) \subseteq B^o(v, r) \cap G$.

Let $\beta = \frac{1}{600}$ and $r_0 = 1/2$. Furthermore, take any $v \in G^c := \{u \in BV[0,1] : \Psi(u) = 0\}$ and let $r \in (0, r_0]$. Let $c := \frac{r}{6} > 0$. We take $S := v + cf$, where f is the generating function. Then, we have that:

$$0 < \frac{r}{6} = c\Psi(f) - 0 = \Psi(cf) - \Psi(v) \leq \Psi(v + cf) = \Psi(S). \tag{2.1}$$

So $S \in G$. Now to show that $B^o(S;\beta r) \subseteq B^o(v,r) \cap G$. Let $s \in BV[0,1]$ such that $\|s\| \leq \beta r = \frac{r}{600}$.

Then, using $\frac{r}{6} \leq \Psi(S)$ and $\|s\| \leq \frac{r}{600}$:

$$\begin{aligned} 0 < \frac{r}{6} - \frac{r}{300} \leq \frac{r}{6} - 2\frac{r}{600} \\ < \frac{r}{6} - 2\|s\| \leq \Psi(S) - 2\|s\| < \Psi(S) - \Psi(s) \leq \Psi(S + s). \end{aligned} \tag{2.2}$$

This proves that $S + s \in G$, as needed.

Now to show that $B^o(S;\beta r) \subseteq B^o(v,r)$. Well, let $s \in BV[0,1]$ such that $\|s\| \leq \beta r = \frac{r}{600}$. Then, using $\|f\| \leq 1, c = \frac{r}{6}$:

$$\begin{aligned} \|(S + s) - v\| := \|(v + cf + s) - v\| = \|cf + s\| \leq \|cf\| + \|s\| \\ = c\|f\| + \|s\| \leq \left(\frac{r}{6}\right)\|f\| + \frac{r}{600} \leq \frac{r}{6} + \frac{r}{600} < r. \end{aligned} \tag{2.3}$$

Thus $B^o(S;\beta r) \subseteq B^o(v,r)$, and so then $B^o(S;\beta r) \subseteq B^o(v,r) \cap G$. Therefore, G is co-porous as claimed.

□

Corollary 2.3.1. *$AC[a,b]$ is a meagre subset of $BV[a,b]$.*

Proof. As every function f in $AC[a,b]$ satisfies $\Psi(f) = 0$ (see for example [21]), we are done. □

We can in fact relate our work to a different notion of porosity.

Definition Let $(X, \|\cdot\|)$ be a Banach space, and $A \subseteq X$ such that A is closed. We say that A is directionally porous if $\forall a \in A$ there exists $0 \neq v \in X$, $p > 0$ and a sequence $t_n \to 0$ of strictly positive numbers such that $B^o(a + t_n v; pt_n) \cap A = \emptyset$. Moreover, for arbitrary $S \subseteq M$, we say that S is directionally porous if $\overline{S}$ is directionally porous. Furthermore, we say that a set J is co-directionally porous if J^c is directionally porous.

A set S is σ-directionally porous if and only if $S = \bigcup_{n \in \mathbb{N}} S_n$ such that each S_n is directionally porous. A set A is co-σdirectionally porous if and only if $A := S^c$ where $S = \bigcup_{n \in \mathbb{N}} S_n$ such that each S_n is directionally porous.

Definition (From [18]) Let $(X, \|\cdot\|)$ be a separable Banach space. A Borel set $A \subseteq X$ is said to be *Haar null* if there is a Borel probability measure μ on X such that

$$\mu(A + x) = 0$$

for all $x \in X$. A possibly non-Borel set is called Haar null if it is contained in a Borel Haar null set.

Definition (From [18]) A Borel probability measure μ on a separable Banach space X is called *Gaussian* if for every $x^* \in X$ the measure $\nu = x^*\mu$ on $\mathbb{R}$ has a Gaussian distribution. The Gaussian measure μ is called nondegenerate if for every $x^* \neq 0$ the measure $\nu = x^*\mu$ has positive variance or equivalently, the measure μ is not supported on a proper closed hyperplane in X.

Definition (From [18]) A Borel set $A \subseteq X$ is said to be *Gauss null* if $\mu A = 0$ for every nondegenerate Gaussian measure μ on X.

Theorem 2.3.2. *([18], p.14) Let E be a Borel set in X (a separable Banach space) which is Lebesgue null on every line in the direction of a fixed vector $0 \neq u \in X$. Then E is Haar null. In particular, σ-directionally porous sets are Haar null.*

Theorem 2.3.3. *([18], p.14) σ-directionally porous sets are Gauss null in a separable Banach space.*

Theorem 2.3.4. *([18], p.33) In infinite dimensional Banach spaces, porous sets are not always null. In particular, by a result of Preiss and Tišer [20], porous sets are not Gauss small in any infinite dimensional separable Banach space, as any such space can be decomposed into two sets, one being σ-porous, and the other being Gauss null.*

Therefore, the two notions of porosity are in fact different. This provides motivation for the following result.

Theorem 2.3.5. *Let $BV[a,b]$ be the Banach space of functions of bounded variation with norm $\|f\| := |f(a)| + V_a^b(f)$. Define $G := \{v \in BV[a,b] : \Psi(v) > 0\}$. Then G is a dense, open, Co-Directionally-Porous set in $BV[a,b]$.*

The proof is similar to that of Theorem 2.3.1. Also, this result follows from Theorem 4.3.1, which we prove later.

3.0 BEHAVIOR OF TYPICAL MIXED PARTIALS

3.1 INTRODUCTION

In many different vector calculus classes the Schwartz-Clairaut Theorem is taught, giving a condition for when the mixed partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$ (for example) are equal. We could in fact refer to the equality of mixed partials as the Clairaut condition, which we know to hold when the second partial derivatives are continuous. Often we give out an example to students that illustrates that one must be careful to check the continuity condition, as the Clairaut condition must not always hold. A natural question which we are not aware of having been published before, is the following: "how typical is it for the Clairaut condition to hold for a function with bounded second derivatives"? In particular, if we address this question using the notion of porosity, does the typical function with bounded second partial derivatives obey the Clairaut condition?

The result that we reach is that the Clairaut condition is in fact rarely (from the sense of category and porosity) reached. Therefore, we are once again highlighting the intuition that continuity type properties are a rare trait among functions.

3.2 MIXED PARTIAL DERIVATIVES

We recall a previous result that we need again in order to move things further:

Theorem 3.2.1. *Suppose that $f_n \to f$ uniformly on a set $E \subseteq M$ in a metric space (M, d), for f_n, f*

real functions (for all n). Let x be a limit point of E, and suppose that

$$\lim_{t\to x} f_n(t) = a_n \textit{ for all } n \in \mathbb{N}. \tag{3.1}$$

Then, a_n converges, and

$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t) = \lim_{n\to\infty} a_n \tag{3.2}$$

Theorem 3.2.2. *Suppose that $\{g_n\}$ is a sequence of functions such that for any $n \in \mathbb{N}$, $g_n : [-1,1]^2 \to \mathbb{R}$, and $\{g_n\}$ converges uniformly to some function $g : [-1,1]^2 \to \mathbb{R}$. Also, suppose that $\{\frac{\partial g_n}{\partial x}\}$ converges uniformly on $[-1,1]^2$ to some function $h : [-1,1]^2 \to \mathbb{R}$. Then for all $(a,b) \in [-1,1]^2$,*

$$\frac{\partial g(a,b)}{\partial x} = \lim_{n\to\infty} \frac{\partial g_n(a,b)}{\partial x} = h(a,b)$$

Proof. Fix $(a,b) \in [-1,1]^2$, arbitrary. Let $E := [-1,1] \setminus \{a\} \subset [-1,1]$. Then we define for all $n \in \mathbb{N}$ and for all $t \in E$ the following:

$$\phi_n(t) := \frac{g_n(t,b) - g_n(a,b)}{t-a}$$
$$\phi(t) := \frac{g(t,b) - g(a,b)}{t-a}.$$

Then, $\forall n \in \mathbb{N}$, $\phi_n : E \to \mathbb{R}$ and $\phi : E \to \mathbb{R}$. Now, fix $n \in \mathbb{N}, m \in \mathbb{N}, t \in E$. By the Mean Value Theorem, for some ξ between t and a, we have:

$$\begin{aligned}
|\phi_n(t) - \phi_m(t)| &= \frac{|(g_n(t,b) - g_m(t,b)) - (g_n(a,b) - g_m(a,b))|}{|t-a|} \\
&= \frac{\left|\left(\frac{\partial g_n(\xi,b)}{\partial x} - \frac{\partial g_m(\xi,b)}{\partial x}\right)(t-x)\right|}{|t-x|} = \left|\frac{\partial g_n(\xi,b)}{\partial x} - \frac{\partial g_m(\xi,b)}{\partial x}\right| \\
&\le \sup_{(s,t)\in[-1,1]^2} \left|\frac{\partial g_n(s,t)}{\partial x} - \frac{\partial g_m(s,t)}{\partial x}\right| =: \gamma_{n,m}.
\end{aligned}$$

Now, as $\{\frac{\partial g_n}{\partial x}\}$ is uniformly convergent, then it is Cauchy, thus $\lim_{m,n\to\infty} \gamma_{n,m} = 0$. Hence, $|\phi_n(t) - \phi_m(t)| \xrightarrow[n,m\to\infty]{} 0$. Therefore, we get that $\phi_n \underset{n}{\to} \phi$ uniformly on E. Also, $a_n := \lim_{\substack{t\to a \\ t\in E}} \phi_n(t) = \frac{\partial g_n(a,b)}{\partial x}$ exists in $\mathbb{R}$, for all $n \in \mathbb{N}$. □

3.2.1 Completeness of P

We recall the following space, where f' denoted the Fréchet derivative:

$$D_2 := \{f \in C_b([-1,1]^2;\mathbb{R}) : f' \text{ exists, and } \|f\|_{D_2} := \|f\|_\infty + \|f'\|_\infty < \infty\}$$

We now recall a theorem. The space was defined a little differently before, but the proof is the same.

Theorem 3.2.3. *As defined above,* $(D_2, \|\cdot\|_{D_2})$ *is a Banach space.*

Theorem 3.2.4. *Let*

$$P := \{f \in D_2 : f_{xx}, f_{yy}, f_{xy}, f_{yx} \textit{ all exist and are bounded}\}.$$

Also, let $\|\cdot\|_P$ *be defined as* $\|f\|_P := \|f\|_D + \|f_{xx}\|_\infty + \|f_{yy}\|_\infty + \|f_{xy}\|_\infty + \|f_{yx}\|_\infty$. *Then* P *is a Banach space.*

Proof. Let $N \in \mathbb{N}$. Suppose that $f : [-1,1]^N \to \mathbb{R}$, then we note here that $\|f'\|_\infty := \sup\limits_{x\in[-1,1]^N} \|f'(x)\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})}$. We also note the following for any $x \in \mathbb{R}^N$ and for some $K \in \mathbb{R}$:

$$\|f'(x)\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})} := \left[(f_{x_1}(x))^2 + \ldots + (f_{x_N}(x))^2\right]^{\frac{1}{2}} \le K\left[\left|f_{x_1}(x)\right| + \ldots + \left|f_{x_N}(x)\right|\right]$$

Thus, $\|f'\|_\infty \le K\left[\|f_{x_1}\|_\infty + \ldots + \|f_{x_N}\|_\infty\right]$. But we also know that $\forall j \in \mathbb{N}$ and $\forall x \in [0,1]^N$ we have $\|f'\|_{\mathcal{L}(\mathbb{R},\mathbb{R})} \ge |f_{x_j}(x)|$. Therefore, $\|f'\|_\infty \ge \|f_{x_j}\|_\infty$ for any $j \in \mathbb{N}$. Hence, for some $\widetilde{K} \in \mathbb{R}$ we know:

$$\widetilde{K}\left(\|f_{x_1}\|_\infty + \ldots + \|f_{x_N}\|_\infty\right) \le \max_{1\le j\le N} \|f_{x_j}\|_\infty \le \|f'\|_\infty.$$

Thus, $\|f\|^*_{D_N} := \|f\|_\infty + \sum\limits_{j=1}^{N} \|f_{x_j}\|_\infty$ is an equivalent norm on D_N. We will use this fact shortly. Let $\|f\|^*_P := \|f\|_\infty + \|f_x\|_\infty + \|f_y\|_\infty + \|f_{xx}\|_\infty + \|f_{yy}\|_\infty + \|f_{xy}\|_\infty + \|f_{yx}\|_\infty$, an equivalent norm on P. Suppose that $\{f_n\}$ is a sequence in P such that $\|f_n - f_m\|_P \xrightarrow[m,n\to\infty]{} 0$. Well, for all $f \in P$, using our equivalent renorming of D_2 with the $\|\cdot\|_{D_2^*}$-norm

$$\|f_x\|_{D_2^*} = \|f_x\|_\infty + \|f_{xx}\|_\infty + \|f_{xy}\|_\infty$$

$$\|f_y\|_{D_2^*} = \|f_y\|_\infty + \|f_{yx}\|_\infty + \|f_{yy}\|_\infty$$

$$\|f\|_{D_2^*} = \|f\|_\infty + \|f_x\|_\infty + \|f_y\|_\infty$$

Clearly,

$$\exists h \in D_2 \text{ such that } \|(f_n)_x - h\|_{D_2^*}$$
$$\exists k \in D_2 \text{ such that } \|(f_n)_y - k\|_{D_2^*}$$
$$\exists g \in D_2 \text{ such that } \|f_n - g\|_{D_2^*}$$

Now, $\|(f_n)_x - g_x\|_\infty \underset{n}{\rightarrow} 0$ and $\|(f_n)_y - h\|_\infty \underset{n}{\rightarrow} 0$ we get $g_x = h$. Similarly, we get $g_y = k$. But, $h, k \in D_2$, so:

$$g_{xx} = h_x$$
$$g_{xy} = h_y$$
$$g_{yx} = k_x$$
$$g_{yy} = k_y$$

Putting this all together, we see that $g \in P$ and $\|f_n - g\|_P^* \underset{n}{\rightarrow} 0$, so P is complete. □

Now we need a definition for a new functional that we will use shortly.

Definition Fix $z_0 = (x_0, y_0) \in \mathbb{R}^2$. Let $\Gamma(\cdot, z_0) : P_2 \rightarrow \mathbb{R}^+$ be defined by $\Gamma(f, z_0) := \left|f_{xy}(z_0) - f_{yx}(z_0)\right|$

3.2.2 A Generating Function

Lemma 3.2.1. *Fix $z_0 \in [-1, 1]^2$. Then there is a function $f \in P$ such that $\Gamma(f, z_0) = 2$. In particular, $\|f\|_P \leq 96 < 100$. We shall call this function the unmixed function at z_0.*

Proof. Without loss of generality, we can assume that $z_0 = (0, 0)$, for if not, we can translate the example we are about to give. Let

$$f(x, y) := \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2} & : (x, y) \neq (0, 0) \\ 0 & : (x, y) = (0, 0) \end{cases}$$

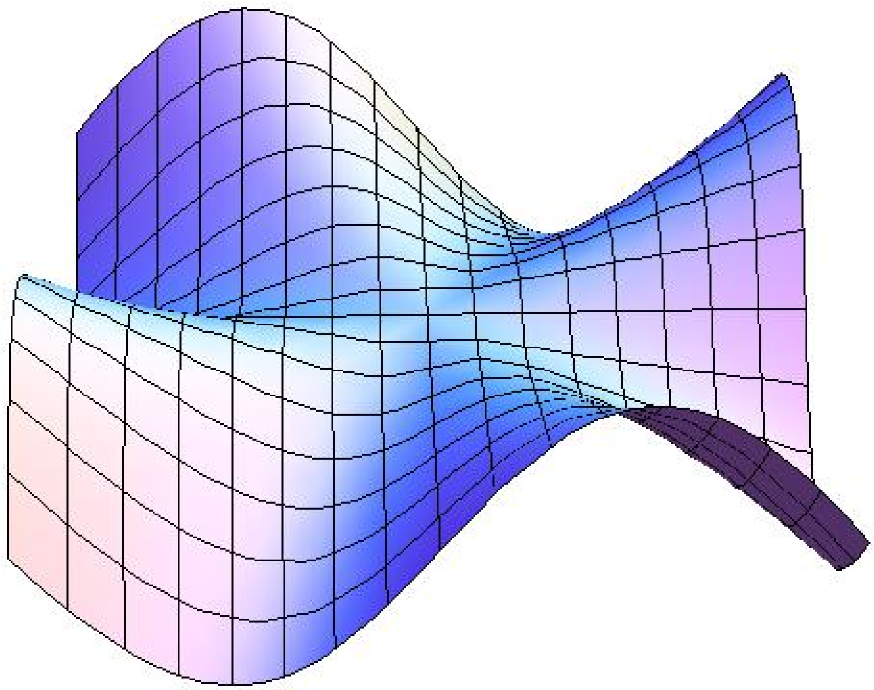

Figure 21: The Unmixed Function

Then,

$$f_y(x,y) := \begin{cases} \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ x & : x \neq 0, y = 0 \\ \lim_{k\to 0} \frac{f(0,k)}{k} = 0 & : (x,y) = (0,0) \end{cases}$$

and

$$f_x(x,y) := \begin{cases} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & : (x,y) \neq (0,0) \\ -y & : y \neq 0, x = 0 \\ \lim_{h\to 0} \frac{f(h,0)}{k} = 0 & : (x,y) = (0,0) \end{cases}$$

Hence, we have the following:

$$f_{xy}(x,y) = [f_x]_y(x,y) = \begin{cases} \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1 & : (x,y) = (0,0) \\ \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \end{cases}$$

.

$$f_{yx}(x,y) = [f_y]_x(x,y) = \begin{cases} \lim\limits_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim\limits_{h\to 0} \frac{h}{h} = 1 & : (x,y) = (0,0) \\ \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \end{cases}$$

Also,

$$f_{xx}(x,y) = [f_x]_x(x,y) = \begin{cases} 0 & : (x,y) = (0,0) \\ \frac{-4xy^3(x^2-3y^2)}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \end{cases}$$

$$f_{yy}(x,y) = [f_y]_y(x,y) = \begin{cases} 0 & : (x,y) = (0,0) \\ \frac{4x^3y(-3x^2+y^2)}{(x^2+y^2)^3} & : (x,y) \neq (0,0) \end{cases}$$

The function f is continuously differentiable, since both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ are continuous everywhere.

In particular, $\frac{\partial f}{\partial x}$ is continuous at the origin since, for $x^2 + y^2 \neq 0$, using polar coordinates of $x = r\cos(\theta), y = r\sin(\theta)$:

$$\begin{aligned} |f_x| &= \frac{\left|x^4y + 4x^2y^3 - y^5\right|}{(x^2+y^2)^2} \\ = \frac{r^5|\cos^4(\theta)\sin(\theta) + 4\cos^2(\theta)\sin^3(\theta) - \sin^5(\theta)|}{r^4} &\leq \frac{6(x^2+y^2)^{5/2}}{(x^2+y^2)^2} \\ &= 6\sqrt{x^2+y^2} \leq 6\sqrt{2} \end{aligned}$$

and

$$\begin{aligned} \left|f_y\right| &= \frac{\left|x^5 - 4x^3y^2 - xy^4\right|}{(x^2+y^2)^2} \\ \frac{r^5|\cos^5(\theta) - 4\cos^3(\theta)\sin^2(\theta) - \cos(\theta)\sin^4(\theta)|}{r^4} &\leq 6\sqrt{x^2+y^2} \\ &\leq 6\sqrt{2}. \end{aligned}$$

Now, we observe that $\|f\|_\infty \leq 1, \|f'\|_\infty = \left\|\left(f_x, f_y\right)\right\|_\infty \leq \left|\left(6\sqrt{2}, 6\sqrt{2}\right)\right| = \sqrt{2}\sqrt{36+36} < 23$. Using polar coordinates, $x = r\cos(\theta)$ and $y = \sin(\theta)$, we have:

$$
\begin{aligned}
|f_{xx}| &\leq \left|\frac{-4xy^3(x^2-3y^2)}{(x^2+y^2)^3}\right| \\
&= \frac{4r^6|\cos^3(\theta)\sin^3(\theta) - 3\cos(\theta)\sin^5(\theta)|}{r^6} \\
&= 4\left|\cos^3(\theta)\sin^3(\theta) - 3\cos(\theta)\sin^5(\theta)\right| \\
&\leq 4\cdot 4 = 16
\end{aligned}
$$

and

$$
\begin{aligned}
\left|f_{yy}\right| &\leq \left|\frac{4x^3y(-3x^2+y^2)}{(x^2+y^2)^3}\right| \\
&= \frac{4r^6|3\cos^5(\theta)\sin(\theta) - \cos^3(\theta)\sin^3(\theta)|}{r^6} \\
&= 4\left|3\cos^5(\theta)\sin(\theta) - \cos^3(\theta)\sin^3(\theta)\right| \\
&\leq 4\cdot 4 = 16.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left|f_{xy}\right| &\leq \left|\frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}\right| \\
&= \frac{r^6|\cos^6(\theta)+9\cos^4(\theta)\sin^2(\theta)-9\cos^2(\theta)\sin^4(\theta)-\sin^6(\theta)|}{r^6} \\
&= \left|\cos^6(\theta)+9\cos^4(\theta)\sin^2(\theta)-9\cos^2(\theta)\sin^4(\theta)-\sin^6(\theta)\right| \\
&\leq 20
\end{aligned}
$$

and

$$
\begin{aligned}
\left|f_{yx}\right| &\leq \left|\frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}\right| \\
&= \frac{r^6|\cos^6(\theta)+9\cos^4(\theta)\sin^2(\theta)-9\cos^2(\theta)\sin^4(\theta)-\sin^6(\theta)|}{r^6} \\
&= \left|\cos^6(\theta)+9\cos^4(\theta)\sin^2(\theta)-9\cos^2(\theta)\sin^4(\theta)-\sin^6(\theta)\right| \\
&\leq 20
\end{aligned}
$$

Observe that $||f||_P : ||f||_\infty + ||f'||_\infty + ||f_{xx}||_\infty + ||f_{yy}||_\infty + ||f_{xy}||_\infty + ||f_{yx}||_\infty \leq 1+23+16+16+20+20 = 96 \leq 100$

Thus, we see the last condition by examining that $\Gamma(f,(0,0)) := \left|f_{xy}(0,0) - f_{yx}(0,0)\right| = 2$. Hence the claim has been proved. □

Proposition 3.2.1. *Let $Q, T \in P$. Then the following holds true:*

1. $\Gamma(Q,t) \geq 0$ *for all* $t \in [-1,1]^2$, *and* $\Gamma(Q,t)$ *is a function of* t.
2. $\Gamma(aQ,t) = a\Gamma(Q,t)$ *for any* $a \in [0,\infty)$ *and any* $t \in [-1,1]^2$.
3. $\Gamma(Q+T,t) \leq \Gamma(Q,t) + \Gamma(T,t)$ *for all* $t \in [-1,1]^2$.
4. $|\Gamma(Q,t) - \Gamma(T,t)| \leq \Gamma(Q+T,t)$ *for all* $t \in [-1,1]^2$.

Proof. 3.2.1 Let T, Q be as above, and fix $t \in [-1,1]^2$. Then (1) and (2) are clear from the definition, so we will prove (3), and leave (4) to the reader. By the triangle inequality in $(\mathbb{R}, \|\cdot\|)$:

$$\begin{aligned}\Gamma(T+Q,t) = \left|(T+Q)_{xy}(t) - (T+Q)_{yx}(t)\right| \quad &= \left|T_{xy}(t) + Q_{xy}(t) - T_{yx}(t) - Q_{yx}(t)\right| \\ &\leq \left|T_{xy}(t) - T_{yx}(t)\right| + \left|Q_{xy}(t) - Q_{yx}(t)\right| \\ = \Gamma(T,t) + \Gamma(Q,t).\end{aligned}$$

Therefore, the desired inequality is now obvious. □

3.2.3 Unmixed At A Point

Theorem 3.2.5. *Fix $z_0 \in [-1,1]^2$. Let $G := \{u \in P : \Gamma(u,z_0) > 0\}$. Then G is open and dense in P, in fact it co-porous in P.*

Proof. Without loss of generality, we may assume that $z_0 = (0,0)$, for if not, then we use translations of the functions we will be discussing. We know by the previous theorem that G is non-empty, so it makes sense to discuss openness.

Let $g \in G$ be arbitrary, and $\delta := \Gamma(g,z_0) > 0$. Now choose $\epsilon > 0$ such that $0 < \epsilon \leq \delta/4$. Then let $s \in P$ such that $\|s\|_P < \epsilon$, $h := g + s$, and we show that $h \in G$. Note that we can see directly that $\Gamma(s,z_0) \leq 2\|s\|_P$. Now we have the following:

$$0 < \frac{\delta}{2} \leq \delta - 2\epsilon < \Gamma(g,z_0) - 2\|s\|_P \leq \Gamma(g,z_0) - \Gamma(s,z_0) \leq \Gamma(g+s,z_0) =: \Gamma(h,z_0).$$

Now we prove density of the set. Let $\epsilon \in (0,1)$ be arbitrary, and $g \in P$ be arbitrary. Without loss of generality, we assume that $g \in P \setminus G$. We will show that there is a function h such that $h \in G$ and $\|h - g\|_P < \epsilon$, by creating a modified version of our unmixed function. Let $J := \frac{\epsilon}{100} f$ where f is the unmixed function. Then, $\|J\|_P = \frac{\epsilon}{100}\|f\|_P \leq \epsilon$, and $\Gamma(J,z_0) = \frac{2\epsilon}{100} > 0$. Now recall that

$P \setminus G = \{v \in P : \Gamma(v, z_0) = 0\}$, so then we know that $\Gamma(g, z_0) = 0$. Then let $h := g + J$, which we claim is the dense function we need.

Well, $||h - g||_P = ||J||_P < \epsilon$. Also, $0 \leq \frac{2\epsilon}{100} - 0 = \Gamma(J, z_0) - \Gamma(g, z_0) \leq \Gamma(h, z_0)$. Therefore the claim is proved.

Now we need to show the porosity conditions. We want to prove that G^c is porous by showing that $\exists \beta \in (0, 1]$ and $\exists r_0 \in (0, \infty)$ such that $\forall x \in G^c$, $\forall r \in (0, r_0]$, $\exists y \in G$ such that $B^o(y; \beta r) \subseteq B^o(x, r) \cap G$.

Let $\beta = \frac{1}{4}$ and $r_0 = 1/2$. Furthermore, take any $v \in P \setminus G := \{u \in P : \Gamma(u, z_0) = 0\}$ and let $r \in (0, r_0]$. Let $c := \frac{r}{100} > 0$. We take $S := v + cf$, where f is the unmixed function. Then, we have that:

$$\begin{aligned} 0 &< \frac{r}{50} = 2c = 2(c) - 0 \\ &\leq \Gamma(cf, z_0) - \Gamma(v, z_o) \leq \Gamma(v + cf, z_0) = \Gamma(S, z_0). \end{aligned} \tag{3.3}$$

So $S \in G$. Now to show that $B^o(S; \beta r) \subseteq B^o(v, r) \cap G$. Let $g \in P$ such that $||g||_P \leq \beta r = \frac{r}{400}$. Then,

$$0 < \frac{3r}{200} = \frac{r}{50} - \frac{r}{200} = 2(\frac{r}{100}) - 2(\frac{r}{400}) < \Gamma(S, z_0) - 2||g||_P < \Gamma(S, z_0) - \Gamma(g, z_0) \leq \Gamma(S + g, z_0).$$

This proves that $S + g \in G$, as needed.

Now to show that $B^o(S; \beta r) \subseteq B^o(v, r)$. Well, let $s \in P$ such that $||s||_P \leq \beta r = \frac{r}{400}$. Then:

$$\begin{aligned} ||(S + s) - v||_P &:= ||(v + cf + s) - v||_P = ||cf + s||_P \leq ||cf||_P + ||s||_P \\ &= c||f||_P + ||s||_p \leq \left(\frac{r}{100}\right)||f||_P + \frac{r}{100} \leq \frac{96r}{100} + \frac{r}{400} < r. \end{aligned} \tag{3.4}$$

Thus $B^o(S; \beta r) \subseteq B^o(v, r)$, and so then $B^o(S; \beta r) \subseteq B^o(v, r) \cap G$. Therefore, G is co-porous as claimed.

□

Corollary 3.2.1. *Let E be a countable subset of $[-1, 1]^2$. Let $G_E := \{u \in P : \Gamma(u, z_0) > 0, \forall z_0 \in E\}$. Then G_E is a dense G_δ set in P, and if E is in fact finite, then G_E is co-porous in P.*

Proof. This is an easy modification of the above if E is finite, and if not, then we apply the Baire Category Theorem to get the first result using the above theorem. $\square$

4.0 SEMI-NORMS AND DENSE OPEN CO-POROUS SETS

4.1 INTRODUCTION

The present article is a result of observations made during research regarding derivatives and porosity of functions on certain sets (See for example [8]). In particular, we noticed that there was a general result that could be made on Banach spaces when we are looking at the set where a seminorm is positive when we have said seminorm is continuous with respect to the norm of the space.

Interestingly we did not need this result in their proof, however, they are unaware of the result being published, and find it interesting and useful for proving a number of related results. The result allows us to consider how typical a given condition is from the perspective of category and porosity, so long as we can phrase it within the perspective of a seminorm on a Banach space.

The idea of porosity was first used by Dolzhenko [11] as a way of describing the boundary behavior of certain functions. Since then, the definition and principle notions have shown themselves useful in the study of quasiconformal mappings, functional analysis, harmonic analysis, as well as topology. Porosity is a way of combining the ideas of category and measure into one notion. A great overview and treatment of the topic can be found in Zajíčeks survey paper [28]. Porosity (specifically σ-porosity, is often used as a way of sharpening results that previously only used sets of First Category or sets of measure zero.

4.2 SEMINORM DEFINED SET

We begin with some definitions to start things moving.

Definition Let V be a vector space, and $N(\cdot)$ be a real function on V over the field $\mathbb{K}$ such that:

1. $N(v) \geq 0$ for all $v \in V$.
2. $N(cv) = |c|N(v)$ for all $v \in V$ and for any $c \in \mathbb{K}$.
3. For any $w, v \in V$ we have that $N(w + v) \leq N(w) + N(v)$.

Then we say that $N(\cdot)$ is a seminorm on V.

We may also define the concept of Strong-Porosity. Throughout this paper for convenience we shall interchangeably use the term Porosity and Strong-Porosity, as Strong-Porosity is the only one we will be referencing.

Definition Let (U, d) be a metric space, and $A \subset U$. We say that A is porous if $\exists \beta \in (0, 1]$ and $\exists r_0 \in (0, \infty)$ such that $\forall x \in A$, $\forall r \in (0, r_0]$, $\exists y \in A^c := U \setminus A$ such that $B(y; \beta r) \subseteq B(x, r) \cap (U \setminus A)$. We say that a set J is Co-Porous, if J^c is Porous.

We are now ready for our result to be stated and proved.

Theorem 4.2.1. *Let $(X, \|\cdot\|)$ be any Banach space, and let $N(\cdot)$ be any seminorm on X that is continuous with respect to $\|\cdot\|$. i.e. There exists $C \in (0, \infty)$ such that $N(x) \leq C\|x\|$ for all $x \in X$.*

Define $G := \{x \in X : N(x) > 0\}$. If $G \neq \emptyset$, then G is a dense, open, Co-Porous set in X.

Proof. Assume the hypotheses of the theorem. Thus, we assume that $G \neq \emptyset$, so fix some $f \in G$. So by definition, $0 < N(f) \leq C\|f\| < \infty$. Let $\delta := N(f) > 0, \gamma := \|f\| > 0$. We will use this later as our "generating function".

We will prove open first. Let $g \in G$ be arbitrary, and $\alpha := \|g\| > 0$. Now choose $\epsilon > 0$ such that $0 < \epsilon \leq \frac{\alpha}{2C}$. Then let $s \in X$ such that $\|s\| < \epsilon$, $h := g + s$, and we show that $h \in G$. Note that we know that $N(s) \leq C\|s\|$. Now we have the following:

$$0 < \frac{\alpha}{2} = \alpha - \frac{\alpha}{2} \leq N(g) - C\|s\| \leq N(g) - N(s) \leq N(g + s) =: N(h).$$

Now we prove density of the set. Let $\epsilon \in (0,1)$ again be arbitrary, and $g \in G$ again be arbitrary. Without loss of generality, we assume that $g \in X \setminus G$. We will show that there is a function h such that $h \in G$ and $\|h - g\| < \epsilon$, by creating a modified version of our generating function.

By assumption we have $N(g) = 0$, and then we take $h := \frac{\epsilon}{2\gamma} f + g$. For $h \in G$ since we have that $0 < \frac{\epsilon}{2\gamma} N(f) = \frac{\epsilon}{2\gamma} N(f) - N(g) \leq N(\frac{\epsilon}{2\gamma} f + g) =: N(h)$. Furthermore, $\|h - g\| = \|\frac{\epsilon}{2\gamma} f\| = \frac{\epsilon}{2} < \epsilon$.

Now we need to show the porosity conditions. Let's see what we need to prove:

We need to show $\exists \beta \in (0,1]$ and $\exists r_0 \in (0,\infty)$ such that $\forall x \in G^c$, $\forall r \in (0, r_0]$, $\exists y \in G$ such that $B(y; \beta r) \subseteq B(x, r) \cap G$.

Let $\beta = \frac{\delta}{100(1+\delta)(1+C)}$ and $r_0 = 1/2$. Furthermore, take any $v \in G^c := \{u \in X : N(u) = 0\}$ and let $r \in (0, r_0]$. Let $c := \frac{r}{2(\delta+1)} > 0$. We take $S := v + cf$, where f is the generating function. Then, we have that:

$$0 < \frac{r}{2(\delta+1)}\delta = cN(f) - 0 = N(cf) - N(v) \leq N(v + cf) = N(S). \tag{4.1}$$

So $S \in G$. Now to show that $B(S; \beta r) \subseteq B(v, r) \cap G$. Let $g \in P$ such that $\|g\|_P < \beta r$. Then,

$$\begin{aligned} 0 < \frac{r\delta}{2(\delta+1)} - \frac{\delta r}{100(1+\delta)} &\leq \frac{r\delta}{2(\delta+1)} - \frac{C\delta r}{100(1+\delta)(1+C)} \\ < \frac{r\delta}{2(\delta+1)} - C\|g\| &< N(S) - C\|g\| < N(S) - N(g) \leq N(S+g). \end{aligned} \tag{4.2}$$

This proves that $S + g \in G$, as needed.

Now to show that $B(S; \beta r) \subseteq B(v, r)$. Well, let $s \in X$ such that $\|s\| < \beta r$. Then:

$$\begin{aligned} \|(S+s) - v\| &:= \|(v + cf + s) - v\| = \|cf + s\| \leq \|cf\| + \|s\| \\ = c\|f\| + \|s\| &\leq \left(\frac{r}{2(\delta+1)}\right)\|f\| + \frac{\delta r}{100(1+\delta)(1+C)} \leq \frac{r}{2} + \frac{r}{100} \leq r. \end{aligned} \tag{4.3}$$

Thus $B(S; \beta r) \subseteq B(v, r)$, and so then $B(S; \beta r) \subseteq B(v, r) \cap G$. Therefore, G is co-porous as claimed.

□

4.3 DIRECTIONALLY POROUS

We now define a different notion of porosity (see [28] for details, also [18, 29]).

Definition Let $(X, \|\cdot\|)$ be a Banach space, and $A \subseteq X$ such that A is closed. We say that A is directionally porous if $\forall a \in A$ there exists $0 \neq v \in X$, $p > 0$ and a sequence $t_n \to 0$ of strictly positive numbers such that $B^o(a + t_n v; pt_n) \cap A = \emptyset$. Moreover, for arbitrary $S \subseteq M$, we say that S is directionally porous if $\overline{S}$ is directionally porous. Furthermore, we say that a set J is co-directionally porous if J^c is directionally porous.

A set S is σ-directionally porous if and only if $S = \bigcup_{n\in\mathbb{N}} S_n$ such that each S_n is directionally porous. A set A is co-σ-directionally porous if and only if $A := S^c$ where $S = \bigcup_{n\in\mathbb{N}} S_n$ such that each S_n is directionally porous.

Now, one of the interesting things about directionally porous sets is that they are Aronszajn null, Haar null, and Γ-null in a separable Banach space (again see [28] and [18, 29] and others for details). We will however mention that a Borel subset A of a separable Banach space X is said to be Haar null if there is a Borel probability measure μ on X so that $\mu(A + x) = 0$ for every $x \in X$. We extend this definition to a general $A \subset X$ and say that it is Haar null if it is a subset of a Borel set with the same property. This gives the following theorem and interesting result.

Theorem 4.3.1. *Let $(X, \|\cdot\|)$ be any Banach space, and let $N(\cdot)$ be any seminorm on X that is continuous with respect to $\|\cdot\|$. i.e. There exists $C \in (0, \infty)$ such that $N(x) \leq C\|x\|$ for all $x \in X$.*

Define $G := \{x \in X : N(x) > 0\}$. If $G \neq \emptyset$, then G is a dense, open, Co-Directionally-Porous set in X.

Proof. As $G \neq \emptyset$, there exists $v \in X$, such that $N(v) > 0$ and so without loss of generality, we assume that $N(v) = 1$. Let $x \in G^c := \{y \in X : N(y) = 0\}$. Now, there exists $C > 0$ as above, so let $0 < p < \frac{1}{C}$, and $t_n := \frac{1}{2^n}$. Then we need to show that $B^o(x + t_n v; pt_n) \cap \{z \in X : N(z) = 0\} = \emptyset$.

Fix $n \in \mathbb{N}$. Then, let $s \in X$ such that $\|s\| < pt_n$. Then we will show that $N(x + t_n v + s) > 0$.

Well, we have the following:

$$0 < (1 - Cp)t_n = t_n - Cpt_n$$

$$\leq t_n - C\|s\| \leq t_n N(v) - N(s) = N(t_n v) - (N(x) + N(s))$$

$$\leq N(t_n v) - N(x + s) \leq N(x + t_n v + s).$$

Therefore, G is co-directionally porous as claimed. □

4.4 APPLICATIONS

We begin with some definitions to start things moving.

Definition Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $N(\cdot)$ be a real function on X over the field $\mathbb{K}$ such that:

(I) $N(v) \geq 0$ for all $v \in X$.

(II) $N(cv) = |c|N(v)$ for all $v \in X$ and for any $c \in \mathbb{K}$.

(III) For any $w, v \in X$ we have that $N(w + v) \leq N(w) + N(v)$.

Then we say that $N(\cdot)$ is a seminorm on X. Furthermore, if $\exists c \in [0, \infty)$ such that $\forall x \in X, N(x) \leq c\|x\|$, then we say that $N(\cdot)$ is continuous with respect to $\|\cdot\|$.

Proposition 4.4.1. *Let us consider the Banach space $(C[-1, 1], \|\cdot\|_\infty)$. Then the following are seminorms on $(C[-1, 1], \|\cdot\|_\infty)$ that are continuous with respect to $\|\cdot\|_\infty$:*

1. $N_1(f) := \sup_{t,s\in[-1,1]} \left| f\left(\frac{s+t}{2}\right) - \left(\frac{f(s)+f(t)}{2}\right)\right|$, *for all* $f \in C[-1, 1]$.
2. $N_2(f) := \sup_{t\in[-1,1]} |f(-t) - f(t)|$, *for all* $f \in C[-1, 1]$.
3. $N_3(f) := \sup_{t\in[-1,1]} |f(-t) + f(t)|$, *for all* $f \in C[-1, 1]$.

Note that a function $g \in C[-1, 1]$ is affine if and only if $N_1(g) = 0$ (i.e. $g((1 - \lambda)s + \lambda t) = (1 - \lambda)g(s) + \lambda g(t)$ for any $\lambda \in [0, 1], \forall s, t \in [-1, 1]$).

Also note that a function $g \in C[-1, 1]$ is even if and only if $N_2(g) = 0$ (i.e. $g(-s) = g(s)$ $\forall s \in [-1, 1]$).

Further note that a function $g \in C[-1,1]$ is odd if and only if $N_3(g) = 0$ (i.e. $g(-s) = -g(s)$ $\forall s \in [-1,1]$).

Proof of Proposition 4.4.1 Part 1. We will show the necessary properties systematically.

(I) Let $f \in C[-1,1]$. Then clearly $N_1(f) \geq 0$

(II) Let $f \in C[-1,1]$ and $c \in \mathbb{K}$. Then:

$$N_1(cf) := \sup_{t,s\in[-1,1]} \left| cf\left(\frac{s+t}{2}\right) - \left(\frac{cf(s)+cf(t)}{2}\right)\right|$$

$$= |c| \sup_{t,s\in[-1,1]} \left| f\left(\frac{s+t}{2}\right) - \left(\frac{f(s)+f(t)}{2}\right)\right| = |c| N_1(f)$$

(III) Let $f, g \in C[-1,1]$. Then fix $s, t \in [-1,1]$:

$$\left|\left(f\left(\frac{s+t}{2}\right) + g\left(\frac{s+t}{2}\right)\right) - \left[\left(\frac{f(s)+f(t)}{2}\right) + \left(\frac{g(s)+g(t)}{2}\right)\right]\right|$$

$$= \left|\left[f\left(\frac{s+t}{2}\right) - \left(\frac{f(s)+f(t)}{2}\right)\right] + \left[g\left(\frac{s+t}{2}\right) - \left(\frac{g(s)+g(t)}{2}\right)\right]\right|$$

$$\leq \left|f\left(\frac{s+t}{2}\right) - \left(\frac{f(s)+f(t)}{2}\right)\right| + \left|g\left(\frac{s+t}{2}\right) - \left(\frac{g(s)+g(t)}{2}\right)\right|$$

Now, taking the supremum of both sides of the above:

$$N_1(f+g) \leq N_1(f) + N_1(g)$$

(IV) Let $f \in C[-1,1]$. Then:

$$N_1(f) := \sup_{t,s\in[-1,1]} \left| f\left(\frac{s+t}{2}\right) - \left(\frac{f(s)+f(t)}{2}\right)\right| \leq 2\|f\|_\infty$$

□

Proof of Proposition 4.4.1 Part 2. We will show the necessary properties systematically.

(I) Let $f \in C[-1,1]$. Then clearly $N_2(f) \geq 0$

(II) Let $f \in C[-1,1]$ and $c \in \mathbb{K}$. Then:

$$N_2(cf) := \sup_{t\in[-1,1]} |cf(-t) - (cf(t))|$$

$$= |c| \sup_{t\in[-1,1]} |f(-t) - (f(t))| = |c| N_2(f)$$

(III) Let $f, g \in C[-1,1]$. Then fix $t \in [0,1]$:

$$\Big|(f(-t) + g(-t)) - [(f(t)) + (g(t))]\Big|$$

$$= \Big|[f(-t) - (f(t))] + [g(-t) - (g(t))]\Big|$$

$$\leq |f(-t) - (f(t))| + |g(-t) - (g(t))|$$

Now, taking the supremum of both sides of the above:

$$N_2(f+g) \leq N_2(f) + N_2(g)$$

(IV) Let $f \in C[-1,1]$. Then:

$$N_2(f) := \sup_{t\in[-1,1]} |f(-t) - (f(t))| \leq 2\|f\|_\infty$$

□

Proof of Proposition 4.4.1 Part 3. We will show the necessary properties systematically.

(I) Let $f \in C[-1,1]$. Then clearly $N_3(f) \geq 0$

(II) Let $f \in C[-1,1]$ and $c \in \mathbb{K}$. Then:

$$N_3(cf) := \sup_{t\in[-1,1]} |cf(-t) + (cf(t))|$$

$$= |c| \sup_{t\in[-1,1]} |f(-t) + (f(t))| = |c| N_3(f)$$

(III) Let $f, g \in C[-1,1]$. Then fix $t \in [0,1]$:

$$\left|(f(-t) + g(-t)) + [(f(t)) + (g(t))]\right|$$

$$= \left|[f(-t) + (f(t))] + [g(-t) + (g(t))]\right|$$

$$\leq |f(-t) + (f(t))| + |g(-t) + (g(t))|$$

Now, taking the supremum of both sides of the above:

$$N_3(f+g) \leq N_3(f) + N_3(g)$$

(IV) Let $f \in C[-1,1]$. Then:

$$N_3(f) := \sup_{t\in[-1,1]} |f(-t) + (f(t))| \leq 2\|f\|_\infty$$

□

Proposition 4.4.2. *Let us consider the Banach space* $(C[-1,1], \|\cdot\|_\infty)$. *Then,*

(I) There exists $f_1 \in C[-1,1]$ *such that* $N_1(f_1) > 0$.

(II) There exists $f_2 \in C[-1,1]$ *such that* $N_2(f_2) > 0$.

(III) There exists $f_3 \in C[-1,1]$ *such that* $N_3(f_3) > 0$.

Proof. We provide just the examples, as the details are easy to check.

(I) Let $f_1 \in C[-1,1]$ such that $N_1(f_1) > 0$ to be $f_1(t) := t^2$.

(II) Let $f_2 \in C[-1,1]$ such that $N_2(f_2) > 0$ to be $f_2(t) := t^3$.

(III) Let $f_3 \in C[-1,1]$ such that $N_3(f_3) > 0$ to be $f_3(t) := t^2$.

□

Corollary 4.4.1. *The following holds:*

(I) The continuous affine functions on $[-1,1]$ *are null in the sense of Haar, Gauss, and* Γ_n.

(II) The continuous even functions on $[-1,1]$ *are null in the sense of Haar, Gauss, and* Γ_n.

(III) The continuous odd functions on $[-1,1]$ *are null in the sense of Haar, Gauss, and* Γ_n.

We now have enough that we may state the following theorem:

Theorem 4.4.1. *Let $(C[-1,1], \|\cdot\|)$ be the Banach space of continuous functions on $[-1,1]$. There exists two sets, $E, O \subset C[-1,1]$ that are null in the sense of Haar and Gauss such that for all $f \in C[-1,1]$, we have that $f = f_e + f_o$ with $f_e \in E$ and $f_o \in O$.*

Proof. Let E be the even functions and O be the odd functions, then set

$$f_e(t) = \frac{1}{2}(f(t) + f(-t))$$

$$f_o(t) = \frac{1}{2}(f(t) - f(-t)).$$

The rest follows from straightforward checking. □

4.5 LOCAL CONDITIONS

We now extend some generalizations of ideas that we used directly in previous work. By way of working on related problems, we noticed a possible generalization, and what follows is the result of that work.

Definition Let $p \in \mathbb{N}$ be fixed. Let $(X, \|\cdot\|)$ be a Banach function space on $J := [-1,1]^p$, such that if $F \in X$, then $F : J \to \mathbb{R}$ and $\|F\| < \infty$ for the norm $\|\cdot\|$.

Furthermore, suppose that $N(\cdot,\cdot) : X \times J \to [0,\infty]$ is a function such that $N(\cdot, x)$ is a seminorm on X over $\mathbb{R}$ that is uniformly continuous with respect to $\|\cdot\|$. In particular, for some $A \in [1,\infty)$, we have for any $x \in J$:

1. $N(v,t) \geq 0$ for all $v \in X$.
2. $N(cv,t) = |c|N(v,t)$ for all $v \in X$ and for any $c \in \mathbb{R}$.
3. For any $w, v \in V$ we have that $N(w + v, t) \leq N(w,t) + N(v,t)$.
4. $N(v,t) \leq A\|v\|$ for all $v \in X$.

Define the following:

$$\Theta_\delta(f,x) := \sup_{t \in B_J(x;\delta)} N(f,t)$$

Observe that $\Theta_\delta(f,x) \leq A\|f\|$. Suppose that for some $\epsilon \in (0,1)$, and for all $x \in J$, $\lambda \in (0,1)$ there exists $f_{B(x;\lambda)} \in B_X$ such that $supp(f) \subset B_J(x;\lambda)$, and we have $\Theta_\lambda(f,x) > \epsilon$. We will call such a function the local seminormed (LS) function at the ball $B(x;\lambda)$.

Then let $\Theta(f,x) := \limsup_{\delta\to 0^+} \Theta_\delta(f,x) \leq A\|f\|$.

Under all the above assumptions, we will say that $(X,\|\cdot\|)$ is *Locally Continuously seminormed with respect to* $N(\cdot,\cdot)$. We may say that N is a Locally Continuous seminorm (LCS) for $(X,\|\cdot\|)$. We will call Θ the local gauge.

Theorem 4.5.1. *Let E be a closed dense set in J. Then $G_E := \{f \in X : \inf_{x\in E} \Theta(f,x) > 0\}$ is an open porous set in $(X,\|\cdot\|)$.*

Proof. (Theorem 4.5) Let $\epsilon \in (0,1)$ be given, and $\beta = \frac{1}{32A}$ and $r_0 = 1/2$, for the porosity constants. Furthermore, take any $f \in X \setminus G_E := \{u \in X : \inf_{x_0\in E} \Theta(u,x_0) = 0\}$ and let $r \in (0,r_0]$.

We will prove this in a few steps using our previous work. First, we observe that by the Dyadic Cubes Theorem, the open set $E^c = \bigcup_{i=1}^{\infty} C_i$ where the $\{C_i\}_{i=1}^{\infty}$ are cubes at center c_i with radii d_i for each C_i. Furthermore, each $C_i = \bigcup_{j\in\mathbb{N}} A_i, j$ where the A_js are a Mayan subdivision. Therefore, we may re-enumerate all the $A_{i,j}$s as the union $E = \bigcup_{n\in\mathbb{N}} S_n$ where the S_n are cubes at center z_n with radii r_n for each S_n.

Now, let $f_n := f_{B(z_n;,r_n)}$ be the LS function for the ball $B(z_n;r_n)$. Then within any neighborhood of $x \in E$ there is a LS function, according to the Mayan subdivision of the cube, and the definition of LS functions. Then we define the following function with $\alpha := \{\alpha_n\}$ a sequence of real scalars:

$$w(x) := \sum_{n\in\mathbb{N}} \alpha_n f_n(x)$$

We will soon use this function $w(x)$ to create our function for density by choosing particular $\overrightarrow{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$.

Thus we let:

$$\alpha_n := \begin{cases} \frac{r\epsilon}{4} & : 0 \leq \Theta(f,y) < \frac{r\epsilon}{8}, \forall y \in B^o(z_n;r_n) \\ 0 & : \exists z_0 \in B^o(z_n;r_n) \text{ such that } \frac{r\epsilon}{8} \leq \Theta(f,z_0) \end{cases}$$

Then the sequences just defined are sequences for the scaling of the tangential oscillations. Then we let $g := f + w$, and we will show that $g \in G_E$. In fact we will show that $\Theta(g, z) \geq \frac{r\epsilon}{8}$ for all $x \in E$.

We will first show that $\forall n \in \mathbb{N}$ there exists $s_n \in B^o(z_n; r_n)$ such that $\Theta(g, s_n) \geq \frac{r\epsilon}{8}$. We will use this to show that $\Theta(g, z) \geq \frac{r\epsilon}{8}$ for all $x \in E$.

As such, let $k \in \mathbb{N}$ be fixed, we show existence of s_k.

<u>Case 1</u>: Suppose $0 \leq \Lambda(f, z) < \frac{r\epsilon}{8}, \forall z \in B^o(z_k; r_k)$. Then using the seminorm properties of Θ, for any $s \in B^o(z_k; r_k)$:

$$0 < \frac{r\epsilon}{8} = \left(\frac{r\epsilon}{4}\right) - \frac{r\epsilon}{8} = \Theta(\alpha_k f_{R,S_k}, s) - \Theta(f, s) \leq \Theta(w + f, s) = \Theta(g, s) \tag{4.4}$$

So choose any $s_k \in B^o(z_k; r_k)$.

<u>Case 2</u>: Suppose $\exists t_k \in B^o(z_k; r_k)$ such that $\frac{r\epsilon}{8} \leq \Theta(f, t_k)$. Then using the seminorm properties of Θ:

$$0 < \frac{r\epsilon}{8} \leq \Theta(f, t_k) - 0 = \Theta(f, t_k) - \Theta(\alpha_k f'_{R,S_k}, t_k) \leq \Theta(\alpha_k v'_k + f', t_k) = \Theta(g, t_k) \tag{4.5}$$

So we let $s_k := t_k$.

Now, let $z \in E$, and $\delta > 0$ be arbitrary. We will show that there is a point within $B^o(z; \delta)$ such that $\Theta(g, \cdot)$ is at least $\frac{r\epsilon}{8}$.

We turn to an illustration to present clarity:

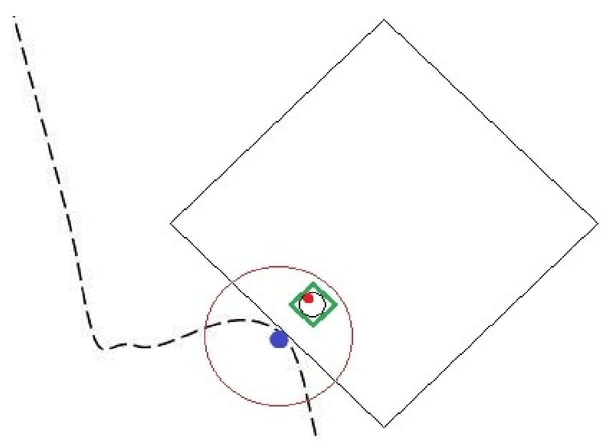

Figure 22: Selection Of Our Point

Within $B^o(z;\delta/2)$ there is an edge of a dyadic cube. Furthermore, by the definition of the Mayan subdivision, there is a Mayan square within $B^o(x;\delta)$, say S_m. So we have a point $s_m \in B^o(z_k;r_k)$ with $B^o(z_k;r_k) \subseteq B^o(z;\delta)$, and $\frac{r\epsilon}{8} \leq \Theta(g,s_m)$. Therefore, $\frac{r\epsilon}{8} \leq \sup_{u\in B^o(z;\delta)} N(g,u)$.

Now, as δ is arbitrary, and we may always find such a point s_m, then

$$0 < \frac{r}{8} \leq \lim_{\delta\to 0^+} \sup_{u\in B^o(z,\delta)} N(f,u) =: \Theta(g,z).$$

So $g \in G_E$.

Now to show that $B^o(g;\beta r) \subseteq B^o(f,r) \cap G$. Let $v \in X$ such that $||v|| \leq \beta r = \frac{r}{32}$. Fix an arbitrary $x \in E$. Then,

$$0 < \frac{r\epsilon}{16} = \frac{r\epsilon}{8} - \frac{r}{32} < \Theta(g,z) - A||v|| < \Theta(g,z) - ||v|| < \Theta(g,z) - \Theta(v,z) \leq \Theta(g+v,z).$$

This proves that $z + g \in G_C$, as needed.

Now to show that $B^o(z;\beta r) \subseteq B^o(v,r)$. Well, let $s \in P$ such that $||s|| \leq \frac{r\epsilon}{32A}$. Then,

$$||(z+s)-v|| := ||(v+w+s)-v|| = ||w+s|| \leq ||w|| + ||s|| \leq \frac{r\epsilon}{4} + \frac{r\epsilon}{32A} \leq \frac{r}{4} + \frac{r}{32} \leq r. \tag{4.6}$$

Thus $B^o(z;\beta r) \subseteq B^o(v,r)$, and so then $B^o(z;\beta r) \subseteq B^o(v,r) \cap G_C$. Therefore, G_C is co-porous as claimed. □

5.0 OPEN QUESTIONS

The following sections are questions that the author has been considering recently, some of which the author is actively investigating, some of which are designed by the author for future study. The author is not aware of these questions having been answered, and as such believes them to be open and related to the author's research and interests.

5.1 HP-SMALL AND POROUS EXTENSIONS

The first collection of questions comes from a paper that the author found recently ([17]). The paper utilizes some different definitions of porosity (as discussed in the main work, there are many notions of porosity), and as such stronger notions are also developed in this work. For our theorems on $D(X), D_N, P$, etc., we showed that certain sets are porous. There is a notion called HP-small, that would strengthen the results in the authors work, if they prove useful for the setting.

Definition Let A be a subset of $(X, \|\cdot\|)$ a Banach Space, and $c \in (0,1]$. We say that A is c-globally very porous if every $c' \in (0,c)$, $\forall x \in X$ and $r > 0$ there is a ball $B = B^o(y, c'r)$, $y \in \overline{B(x;r)}$, such that $B \cap A = \emptyset$. A set A is σ-c-globally very porous if it is a countable union of c-globally porous sets.

Definition Let A be a subset of a Banach Space $(X, \|\cdot\|)$ and $c \in (0,1]$. We say that A has property $HP_{(c)}$ if for every $c' \in (0,c)$ and $r > 0$ there exists $K > 0$ and a sequence of balls $\{B_i\} = \{B^o(y_i; c'r)\|$ with $\|y_i\| \leq r$, $i \in \mathbb{N}$ such that for every $x \in X$,

$$card\{i \in \mathbb{N} : (x + B_i) \cap A \neq \emptyset\} \leq K.$$

The set A is said to be HP-small if there is a porosity constant $c \in (0, 1]$ such that A is a countable union of sets with property $HP_{(c)}$.

We will say that a subset A of a Banach space is co-HP-small, or HP-large, if A^c is HP-small.

Definition A Borel subset A of s separable Banach Space X is said to be Haar null if there is a Borel probability measure μ on X so that $\mu(A + x) = 0$ for every $x \in X$. We extend this definition to a general $A \subset X$ and say that it is Haar null if it is a subset of a Borel set with the same property.

Theorem 5.1.1. *Every HP-small subset of a Banach space is σ-c-globally very porous, and hence meager. Furthermore,every HP-small subset of a separable Banach space is Haar Null.*

Now that the framework is laid out, we may now consider the following questions.

Question 5.1.1. *Can we find specific examples of sets that demonstrate the differences between the multiple definitions of porosity? i.e. can we find a set A that is porous, but not Globally-porous? Can we find a set that is porous, but not HP-null? Additionally can we do the above for the various notions of σ-porosity?*

Question 5.1.2. *Let $(X, \|\cdot\|)$ which is a Banach space and has Fréchet differentiable norm. Then for $G := \{f \in D(X) : \Lambda(f', x_0) > 0\}$ for some fixed $x_0 \in B(X)$, G is Globally-co-porous or HP-large? Do we need to restrict to particular Banach spaces (Separable or Hilbert)?*

Question 5.1.3. *Let $N \in \mathbb{N}$ be fixed. Let $E \subset (0, 1)^N$ be a closed nowhere dense set. Then is $G_E := \{f \in D_N : \inf_{x_0 \in E} \Lambda(f', x_0) > 0\}$ Globally-co-porous or HP-large? Is this even true in $\mathbb{R}$?*

Question 5.1.4. *Let $(X, \| \cdot \|)$ be a Banach space such that X^* is separable. Is there an extension of our finite dimension domain results to closed nowhere dense sets in X? In other words, can we get any results for our G_E set? The most pressing question is does there even exist a Volterra type function in this case?*

Question 5.1.5. *Can we define and use a space $D^n(X)$ to be the n-times bounded everywhere differentiable functions (where all n derivative are bounded)? So can we have a result like our results holding for the n^{th}-tangential oscillation? Also, if the previous questions holds, can we have $D^n(X; Y)$ work out as well?*

5.2 NEW AREAS FOR POROSITY

The following are a few of the questions that the author has been considering recently that seem to have interesting answers and appear to be questions that would make good use of the idea of porosity.

5.2.1 Bounded Variation

Question 5.2.1. *Let* $(BV[0,1], \|\cdot\|)$ *be the Banach space of functions of bounded variation with norm* $\|f\| := |f(0)| + V_0^1(f)$*. Let* $\mathfrak{C}$ *be the usual cantor* $1/3$ *set, and* $x \in \mathfrak{C}$*.*

Here is an idea related to self-similar fractal sets (such as $C :=$ *the usual middle-thirds Cantor set), and fractional derivatives.*

Let α *be the Hausdorff dimension of the Cantor set; i.e.,*

$$\alpha := \frac{\ln(2)}{\ln(3)} \approx 0.63093 .$$

Also, for all $u \in \mathbb{R}\backslash\{0\}$ *and for all* $\beta \in \mathbb{R}$*, we define*

$$u^{\beta} := |u|^{\beta}\,\mathrm{sgn}(u) .$$

Here, $\mathrm{sgn}(u) := u/|u|$*. Further, if* $\beta > 0$*, we set* $0^{\beta} := 0$*.*

For every function $g \in BV[0,1]$*, for every* $x \in C$*, we define*

$$Q(g;x) \quad := \quad \lim_{\delta \longrightarrow 0+} \sup_{y,z \in (x-\delta, x+\delta)\cap[0,1]} \left| \frac{g(z)-g(y)}{(z-y)^{\alpha}} - \frac{1}{(z-y)^{\alpha}} \int_y^z g'(t)\,dt \right| .$$

Note that the function $Q(g;\cdot)$ *maps* C *into the interval* $[0,\infty]$*. Also note that for all* $h \in AC[0,1]$*,* $Q(h;x) = 0$*, for all* $x \in C$*.*

Now, let $f : [0,1] \longrightarrow \mathbb{R}$ *be the usual Cantor function. We claim the following.*

Lemma 5.2.1. *The function* $Q(f;\cdot)$ *maps* C *into the interval* $(0,\infty)$*. Moreover,*

$$Q(f,x) \geq \frac{1}{2}\ , \textit{ for all } x \in C .$$

Now, let's define the subset W of $BV[0,1]$ by

$$W := \left\{ g \in BV[0,1] : \inf_{x \in C} Q(g;x) > 0 \right\} .$$

Is W a co-porous set in $BV[0,1]$?

Question 5.2.2. *What about if we define a function similar to Q for other closed nowhere dense sets?*

5.2.2 Other Topics

Question 5.2.3. *Are most (in the sense of category and porosity) series divergent? If yes, is there a stronger condition?*

Question 5.2.4. *Do most (in the sense of category and porosity) absolutely converging series fail the root test? i.e. Series a_n such that $\sum_{n=1}^{\infty} |a_n| < \infty$ and yet $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent, yet it fails the root test, is this common? If yes, is there a stronger condition?*

Question 5.2.5. *Can we use the notion of porosity to either define a new notion of integration, or to develop a strong notion of integration, or to reach some sort of strengthening of integration results for porous sets?*

Question 5.2.6. *Is the limit of most sequences of bounded functions (in the sense of category and porosity) discontinuous? If yes, is there a stronger condition?*

Question 5.2.7. *Let $I := [0,1]$, and $\#(\cdot)$ be the counting measure. Consider the Banach space $(X, \|\cdot\|_X) = (C(I;I), \|\cdot\|_\infty)$. In 1967, S. Sawyer proved [23] (among other things) that for a certain set A residual in X, we have that for any $f \in A$ there exists an interval (a,b) such that $\#\left(f^{-1}(y)\right) = \infty$ for any $y \in (a,b)$. Then we ask the following question: Do most functions $f \in X$, in the sense of directional porosity, possess a subinterval $(a,b) \subset [0,1]$ such that $J := f^{-1}((a,b))$ is uncountable? A positive answer would include measure theoretic notions, because X here is separable, therefore directionally porous sets are Haar, Gauss, etc. null.*

Question 5.2.8. *In the sense of Category and Porosity, is it typical that functions with proper iterated integrals (Riemann, or others) do not possess a double integral? i.e. Is Fubini's Theorem rare?*

Definition A function f on an open interval J is of the Pompeiu type if f has a bounded derivative and the sets on which f' is zero or does not vanish, respectively, are both dense in J. We let Δ be the collection of all bounded Pompeiu derivatives on $[0, 1]$.

Theorem 5.2.1. *(See [6])* $(\Delta, d(\cdot, \cdot))$ *is a complete metric space with metric* $d(g, f) := \sup_{t \in [0,1]} |g(t) - f(t)|$.

Question 5.2.9. *Is a p-typical (in the sense of porosity) Pompeiu derivative in* (Δ, d) *is such that there are dense sets* $A_1, A_2 \subset [0, 1]$ *with* $[f(x) \geq 0, \forall x \in A_1]$ *and* $[f(x) \leq 0, \forall x \in A_2]$.

Question 5.2.10. *Can we make the set of Pompeiu derivatives a Banach space as a subset of* $D[0, 1]$ *a Banach space, and label it* Δ^*. *If yes, can we then say that a p-typical (in the sense of porosity) Pompeiu derivative in* $(\Delta^*, \|\cdot\|_{D[0,1]}$ *such that there are dense sets* $A_1, A_2 \subset [0, 1]$ *with* $[f(x) \geq 0, \forall x \in A_1]$ *and* $[f(x) \leq 0, \forall x \in A_2]$.

Question 5.2.11. *Are most (in the sense of category and porosity) multiplicatively-integrable functions also integrable in the standard definition? What about if we use different notions of integrability (Riemann, Lebegue, etc.)?*

5.3 OTHER QUESTIONS OF IMMEDIATE INTEREST TO THE AUTHOR

There are additionally many questions that the author has been considering in other areas as well. The following are a few of the more recent ideas for research.

Question 5.3.1. *In recent work, Lennard and Dahma have proved that certain generalizations of* L^p *spaces for* $p \in (-\infty, 0)$ *are in fact F-spaces. Can some of the techniques used in the proof of this fact be used to define any additional integration methods analogous to product integration, such as what appears in the work of Dr. Pesi Masani?*

Question 5.3.2. *Let $\gamma(\cdot)$ be the measure of noncompactness. Let $(X, \|\cdot\|)$ be a Banach space, and $K \subset X$ be norm closed and bounded. Let $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ such that for all $\epsilon > 0$ the following are equivalent:*

1. $\gamma(K) < \epsilon$
2. *There exists a sequence $\{x_n\}_n$ in X such that $K \subset \overline{co}\{x_n : n \in \mathbb{N}\}$ and $\limsup_{n\to\infty} \|x_n\| < f(\epsilon)$.*

Does there exist a strictly increasing continuous function f such that the above holds? What is the best f that works? What are possible extensions if this holds?

Question 5.3.3. *Are there generalizations of the work of Grafakos/Lennard and Gröchenig/Heil on Frame Analysis to other functions spaces; in particular Hardy Spaces and Lorentz Spaces. Some work has been done is this area by others to extend in certain spaces and certain conditions, but I seek further generalizations and tightening of the bounds. In particular I seek a direct extension of the techniques used in the works above.*

APPENDIX

BACKGROUND: INVESTIGATING DEFINITIONS

We use this section to illustrate and compare porosity.

A.1 ILLUSTRATIVE DEFINITIONS

Porosity Let (M, d) be a metric space, and $A \subseteq M$ such that A is closed. We say that A is porous if and only if

$$(\clubsuit)\ [\exists r_0 \in (0, \infty) \text{ and } \exists \beta \in (0, 1] \text{ such that } \forall x \in A, \forall r \in (0, r_0]$$

$$\exists z \in A^c \text{ such that } B^o(z; \beta r) \subseteq B^o(x, r) \cap A^c]$$

Moreover, for arbitrary $S \subseteq M$, we say that S is porous if $\overline{S}$ is porous. Furthermore, we say that a set J is co-porous if J^c is porous.

Example of porous set:

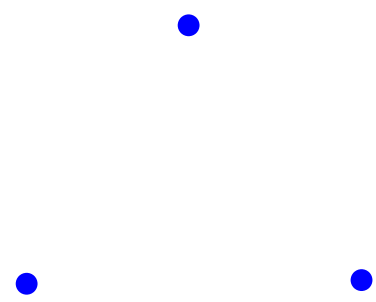

Figure 23: Porous Set “A” (the space is the whole slide, and “A” is the three points)

Example of porous set:

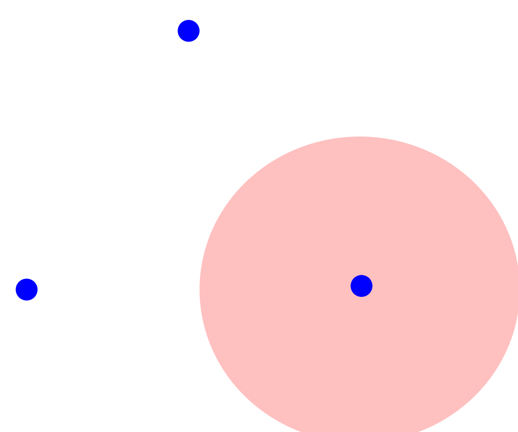

Figure 24: Ball around a point “x” of “A”

Example of porous set:

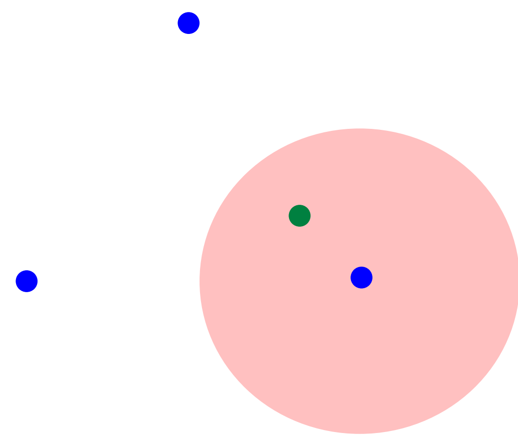

Figure 25: Value "z" inside the ball

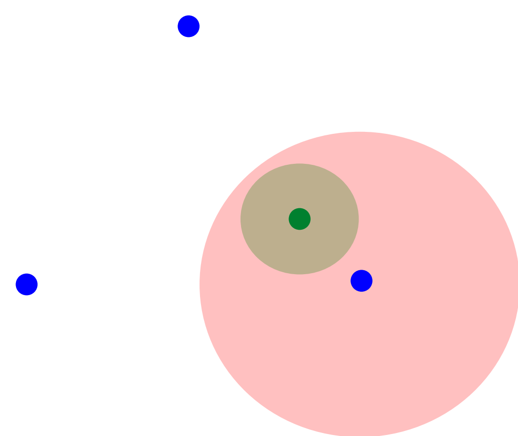

Figure 26: Sub-ball about "z" inside original ball

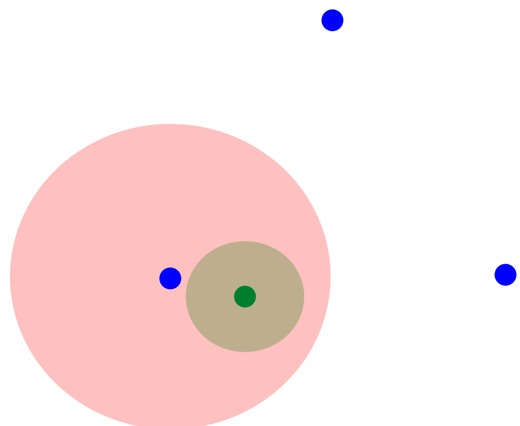

Figure 27: Moving the balls

Note the original ball can be moved to any other point of “A”, and the same sub-ball radius works uniformly.

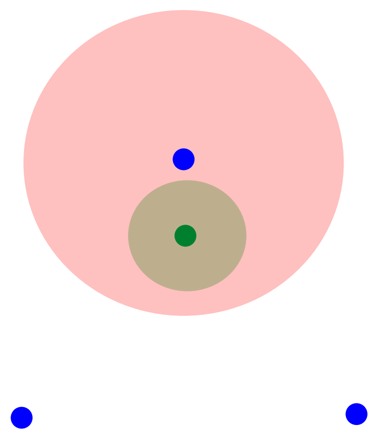

Figure 28: Moving the balls

Note the original ball can be moved to any other point of "A", and the same sub-ball radius works uniformly.

Definition of σ-Porosity A set S is σ-porous if and only if $S = \bigcup_{n \in \mathbb{N}} S_n$ such that each S_n is porous. A set A is co-σ-porous if and only if $A := S^c$ where $S = \bigcup_{n \in \mathbb{N}} S_n$ such that each S_n is porous.

Note: Porosity was first used by Dolzhenko in 1967 [11]. Also, see Zajíček [27, 28] for a thorough explanation of porosity.

Examples:

- Any finite set A is porous.
- If $x_n \to x$ then $A := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is σ-porous.
- The $1/3$ Cantor set $C_{\frac{1}{3}}$ is σ-porous.

We can think of porous sets as being "sponge-like", or "having no bulk to them".

Figure 29: A Seasponge

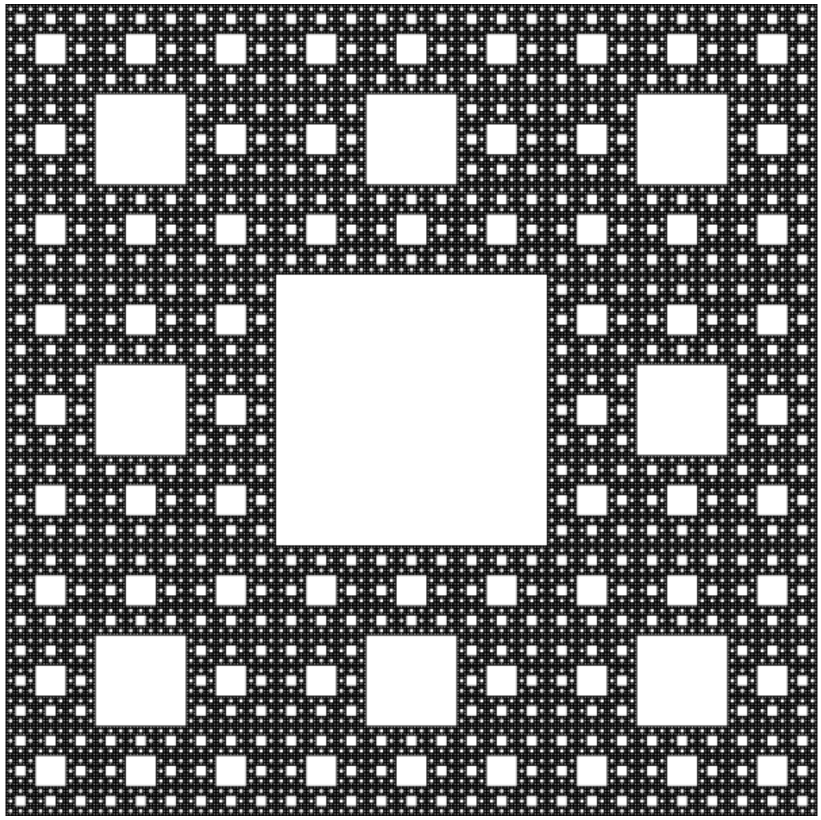

Figure 30: A Typical Porous set In $\mathbb{R}^2$

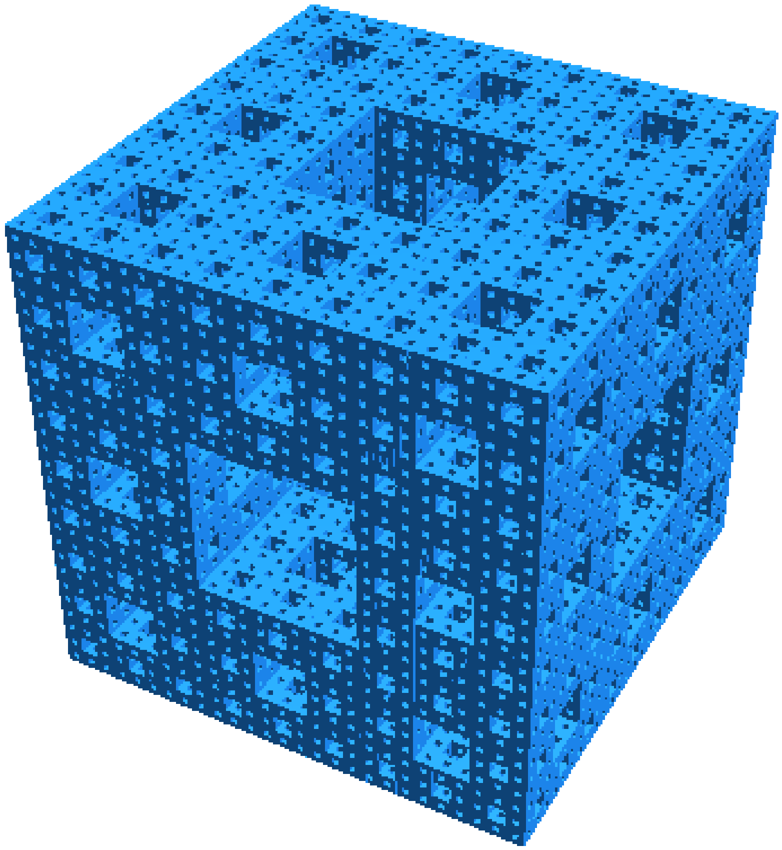

Figure 31: A Typical Porous set In $\mathbb{R}^3$

A.2 POROUS SETS ARE MEASURE ZERO

So if a set is closed and porous, then it is closed nowhere dense. However, more is true. As such we recall a consequence of the Lebesgue Density Theorem:

Lemma A.2.1. *Let $S \subseteq \mathbb{R}^n$ be a measure set. Then the following is true:*

1. $d_L(x) := \lim_{r \to 0^+} \frac{m_n(B^o(x;r) \cap S)}{m_n(B^o(x;r))} = \chi_S(x)$ *for a.a.* $x \in S$.
2. *If* $m_n(S) > 0$*, then for a.a.* $x \in S$, $d_L(x) = 1$.

Let's apply this to the next lemma:

Lemma A.2.2. *Let $A \subseteq \mathbb{R}^n$. Then if A is closed and porous, then it is measure zero for m_n the n-dimensional Lebesgue measure.*

Proof. Let $\beta \in (0,1], r_0 \in (0,\infty)$ such that $\forall r \in (0, r_0], \forall x \in A, \exists z \in A^c$ such that $B^o(z;\beta r) \subseteq B^o(x;r) \cap A^c$. Suppose to get a contradiction that $m_n(A) > 0$, then $\exists T, N$ such that $m_n(N) = 0, A = T \cup N$ (with $m_n(T) = m_n(A) > 0$) and for all $x \in T$, $d_L(x) = 1$.

Fix $x \in T$. Then $\exists z \in \mathbb{R}^n$ such that $B^o(z;\beta r) \subseteq B^o(x;r) \cap A^c$.
Notice that:

$$m_n(B^o(x;r)) = m_n(B^o(x;r) \cap A) + m_n(B^o(x;r) \cap A^c)$$

$$m_n(B^o(x;r)) - m_n(B^o(x;r) \cap A^c) = m_n(B^o(x;r) \cap A)$$

And $m_n(B^o(z;\beta r) \leq m_n(B^o(x;r) \cap A^c)$.

Therefore,

$$m_n(B^o(x;r) \cap A) \leq m_n(B^o(x;r)) - m_n(B^o(z;\beta r))$$

Hence,

$$d_L(x) := \lim_{r\to 0^+} \frac{m_n(B^o(x;r) \cap A)}{m_n(B^o(x;r))}$$

$$\leq \lim_{r\to 0^+} \left[1 - \frac{m_n(B^o(z;\beta r))}{m_n(B^o(x;r)}\right] = \lim_{r\to 0^+} [1 - \beta^n] < 1.$$

Thus we have a contradiction, and so $m_n(A) = 0$. □

A.3 DIFFERENT DEFINITIONS

The reader may be familiar with porosity through the work of a number of different researchers. Unfortunately, for as many researchers as there are in this area, there are that many definitions given of porosity. Not every notion of porosity is the same as all of the others. In this appendix, we comment on a few of the more commonly used definitions. We will state the definitions used by some of the more active researchers in the area, then we will compare which are known to be equivalent, and state a few results that point out some of the differences between a few of the definitions.

A.3.1 Lindenstrauss's Definitions and Results

We take the following definitions from Lindenstrauss ([18], p. 10):

Definition (Lindenstrauss) A set E in a Banach space $(X, \|\cdot\|)$ is called *L-porous* if there is $0 < c < 1$ such that for every $x \in E$ and every $\epsilon > 0$, there is a $y \in X$ with $0 < \|x - y\| < \epsilon$ and

$$B^o(y; c\,\|x - y\|) \cap E = \emptyset$$

In this situation, we say that E is *L-porous with constant* c.

Definition (From [18]) If Y is a subspace of X, then E is called *porous in the direction of* Y if there is $0 < c < 1$ such that for every $x \in E$ and $\epsilon > 0$ there is $y \in Y$ so that $0 < \|y\| < \epsilon$ and

$$B^o(x + y; c\|y\|) \cap E = \emptyset$$

We notice that the above definitions could be stated purely symbolically, and we would say that $E \subset X$ is *L-porous* if:

$$\exists c \in (0, 1),\ \forall x \in X,\ \forall \epsilon > 0,\ \exists y \in X \text{ such that } B^o(y; c\epsilon) \subseteq B^o(x; \epsilon) \setminus E$$

and E is *directionally L-porous in the Y direction* if:

$$\exists c \in (0, 1),\ \forall x \in X,\ \forall \epsilon > 0,\ \exists y \in X \text{ such that } B^o(x + y; c\epsilon) \subseteq B^o(0; \epsilon) \setminus E$$

We also say that $E \subseteq X$ is *directionally L-porous* if it is porous in some direction.

A set is σ-(L-porous), (L-directionally porous), or (L-porous in the direction of Y) if it is the union of sets that are either (L-porous), (L-directionally porous), or (L-porous in the direction of Y).

Definition (From [18]) Let $(X, \|\cdot\|)$ be a separable Banach space. A Borel set $A \subseteq X$ is said to be *Haar null* if there is a Borel probability measure μ on X such that

$$\mu(A + x) = 0$$

for all $x \in X$. A possibly non-Borel set is called Haar null if it is contained in a Borel Haar null set.

Definition (From [18]) A Borel probability measure μ on a separable Banach space X is called *Gaussian* if for every $x^* \in X$ the measure $\nu = x^*\mu$ on $\mathbb{R}$ has a Gaussian distribution. The Gaussian measure μ is called nondegenerate if for every $x^* \neq 0$ the measure $\nu = x^*\mu$ has positive variance or equivalently, the measure μ is not supported on a proper closed hyperplane in X.

Definition (From [18]) A Borel set $A \subseteq X$ is said to be *Gauss null* if $\mu A = 0$ for every nondegenerate Gaussian measure μ on X.

Theorem A.3.1. *([18], p.14) Let E be a Borel set in X (a separable Banach space) which is Lebesgue null on every line in the direction of a fixed vector $0 \neq u \in X$. Then E is Haar null. In particular, σ-L-directionally porous sets are Haar null.*

Theorem A.3.2. *([18],p.14) σ-L-directionally porous sets are Gauss null in a separable Banach space.*

Claim A.3.3. *([18],p.33)*

(I) The closure of a L-porous set is obviously nowhere dense, and thus σ-L-porous sets are of the first category.

(II) In a finite dimensional space, L-porous sets are, by the Lebesgue Density Theorem, sets of measure zero.

Claim A.3.4. *([18],p.33) In infinite dimensional Banach spaces, L-porous sets are not always null. In particular, by a result of Preiss and Tišer [20], L-porous sets are not Gauss small in any infinite dimensional separable Banach space, as any such space can be decomposed into two sets, one being σ-L-porous, and the other being Gauss null.*

A.3.2 Zajíček's Equivalent Definitions and Results

Now we use some definitions from Zajíček.

Definition Let $x \in M$. Then for $R > 0$, let

$$\gamma(x, R, M) := \sup\{r \geq 0 : \exists y \in X \text{ such that } B^o(y; r) \subseteq B^o(x; R) \setminus M\}$$

Definition Let $c > 0$. M is called *c-lower Z-porous* if for any $x \in M$, we have that

$$2 \liminf_{R \to 0^+} \frac{\gamma(x, R, M)}{R} \geq c$$

Definition We say that M is *lower Z-porous* if for any $x \in M$, we have that

$$\liminf_{R \to 0^+} \frac{\gamma(x, R, M)}{R} > 0$$

Definition Let $c > 0$. M is called *c-upper Z-porous* if for any $x \in M$, we have that

$$2 \limsup_{R \to 0^+} \frac{\gamma(x, R, M)}{R} \geq c$$

Definition We say that M is *upper Z-porous* if for any $x \in M$, we have that

$$\limsup_{R \to 0^+} \frac{\gamma(x, R, M)}{R} > 0$$

Theorem A.3.5. *[28] Let X be a metric space, and $A \subseteq X$.*

(I) A is σ-lower Z-porous [which is the same as ball-small].

(II) $A = \cup_{n \in \mathbb{N}} P_n$, where each $P := P_n$ has the following property:

$$\exists \alpha > 0,\ \exists R_0 > 0,\ \forall x \in X,\ \forall R \in (0, R_0),\ \exists y \in X \text{ such that } B^o(y; \alpha R) \subseteq B^o(x; R) \setminus P$$

(III) $A = \cup_{n \in \mathbb{N}} P_n$ where each $P := P_n$ has the following property:

$$\exists \alpha > 0,\ \forall x \in X,\ \forall R > 0,\ \exists y \in X \text{ such that } B^o(y; \alpha R) \subseteq B^o(x; R) \setminus P$$

Then $(I) \Leftrightarrow (II)$. Also, if X is a normed linear space, then $(I) \Leftrightarrow (II) \Leftrightarrow (III)$.

We now have enough that the following result can be checked:

Proposition A.3.1. *Let X be a metric space, and $A \subseteq X$ as above.*

(a) $\exists \alpha > 0,\ \exists R_0 > 0,\ \forall x \in X,\ \forall R \in (0, R_0),\ \exists y \in X$ such that $B^o(y; \alpha R) \subseteq B^o(x; R) \setminus P$.

(b) $\exists \beta > 0,\ \exists R_1 > 0,\ \forall x \in X,\ \forall R \in (0, R_1],\ \exists y \in P^c$ such that $B^o(y; \beta R) \subseteq B^o(x; R) \setminus P$.

(c) $\exists \beta \in (0, 1],\ \exists R_0 > 0,\ \forall x \in X,\ \forall R \in (0, R_0),\ \exists y \in P^c$ such that $B^o(y; \alpha R) \subseteq B^o(x; R) \setminus P$.

(d) For any $x \in P$, we have that

$$\liminf_{R \to 0^+} \frac{\gamma(x, R, P)}{R} > 0$$

Then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

A.3.3 Strobin's Comparisons

We now investigate some results from a paper of Filip Strobin [24], that compares two of the main notions of porosity.

Definition [24] Let $(X, \|\cdot\|)$ be a Banach space, and $M \subseteq X$. Then the following are equivalent:

(I) Let $R > 0$. We say that M is *R-ball porous* if

$$\forall x \in M,\ \forall \alpha \in (0,1),\ \exists y \in X\ \ (\|x-y\| = R \text{ and } B^o(y; \alpha\|x-y\|) \cap M = \emptyset)$$

(II) Let $R > 0$. We say that M is *R-ball porous* if

$$\forall x \in M,\ \forall \epsilon \in (0,R),\ \exists y \in X\ \ (\|x-y\| = R \text{ and } B^o(y; R-\epsilon) \cap M = \emptyset)$$

We say that a set A is *ball small* if it is the countable union of *ball porous* sets. i.e. $A := \cup_{n\in\mathbb{N}} S_n$ where each S_n is R_n *ball porous*.

Definition Let $(X, \|\cdot\|)$ be a Banach space, and $M \subseteq X$. We say that M is *O-porous* if

$$\forall \alpha \in (0,1),\ \exists R_0 > 0,\ \forall x \in M,\ \forall R \in (0,R_0),\ \exists y \in X\ \ (\|x-y\| = r \text{ and } B^o(y; \alpha R) \cap M = \emptyset)$$

Definition Let $x \in M$. Then for $R > 0$, let

$$\gamma(x,R,M) := \sup\{r \geq 0 : \exists y \in X \text{ such that } B^o(y;r) \subseteq B^o(x;R) \setminus M\}$$

Definition Let $c > 0$. M is called *c-lower porous* if for any $x \in M$, we have that

$$2 \liminf_{R\to 0^+} \frac{\gamma(x,R,M)}{R} \geq c$$

Definition We say that M is *lower porous* if for any $x \in M$, we have that

$$\liminf_{R\to 0^+} \frac{\gamma(x,R,M)}{R} > 0$$

Definition Let $c > 0$. M is called *c-upper porous* if for any $x \in M$, we have that

$$2 \limsup_{R\to 0^+} \frac{\gamma(x,R,M)}{R} \geq c$$

Definition We say that M is *upper porous* if for any $x \in M$, we have that

$$\limsup_{R \to 0^+} \frac{\gamma(x, R, M)}{R} > 0$$

Theorem A.3.6. *[24] The following implications hold:*

(I) c-lower porosity⇒lower porosity⇒upper porosity

(II) c-lower porosity⇒c-upper porosity⇒upper porosity

Theorem A.3.7. *Let $M \subseteq X$ and $c > 0$. The following conditions are equivalent:*

(I) *M is* c-lower porous*;*

(II) $\forall x \in M,\ \forall \beta \in (0, \frac{1}{2}c),\ \exists R_0 > 0,\ \forall R \in (0, R_0),\ \exists y \in X,\ (B^o(y; \beta R) \subseteq B(x; R) \setminus M)$

A set C is σ-☕-porous if $C := \cup S_n$ and each S_n is ☕-porous, where we use ☕to stand for a type of porosity.

Theorem A.3.8. *[24] Let $(X, \|\cdot\|)$ be a Banach space (or just a normed linear space). If $M \subseteq X$ is* R-ball porous*, then M is* r-ball porous *for all $r \in (0, R]$.*

Corollary A.3.1. *[24] Any* R-ball porous *subset of a Banach space (or just a normed linear space) is* O-porous.

Theorem A.3.9. *[24] Every* O-porous *subset M of a Banach space is 1-lower porous.*

Lemma A.3.1. *[24] There exists a 1-lower porous subset of* $\mathbb{R}$ *which is not* O-porous.

Proposition A.3.2. *[24] Any R-ball porous subset of* $\mathbb{R}$ *is countable.*

Proposition A.3.3. *[24] Let $M \subset \mathbb{R}$. Then M is ball small if and only if it is countable.*

Theorem A.3.10. *[24] In any nontrivial Banach space, there exists an* O-porous *subset which is not ball small.*

Lemma A.3.2. *[24] Let $(X, \|\cdot\|)$ be a normed linear space, and $R > 0$. If $M \subseteq X$ is* R-ball porous*, so is it's closure $\overline{M}$.*

A.3.4 Observations on Porosity

Now, we have moved through a number of author's definitions of porosity, let's bring it all together.

Proposition A.3.4. *We may now make the following 2 observations of the collected data:*

1. *Let A be a subset of Banach space X.*
   (a) *A is R-ball porous.*
   (b) *A is* O-porous.
   (c) *A is Z-porous.*
   (d) *A is L-porous*
   (e) *A is 1-lower porous.*
   (f) *A is lower porous.*
   (g) *A is upper porous.*

   Then $a \Rightarrow b \Rightarrow c \Leftrightarrow d \Rightarrow e \Rightarrow f \Rightarrow g$. *No claim is made of additional strictness of implications.*

2. *Let A be a subset of Banach space X.*
   (a) *A is* σ*-Z-porous.*
   (b) *A is* σ*-L-porous*
   (c) *A is* σ*-1-lower porous.*
   (d) *A is* σ*-lower porous.*

   Then $a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d$.

A.3.5 Interesting Porosity Results

There are two results that the author finds particularly interesting in terms of expressing the intuition behind the occurrence of porous sets. We quote these results here for the reader:

Theorem A.3.11. *[19] Every convex nowhere dense subset of a Banach space is O-porous.*

Theorem A.3.12. *[28] A convex nowhere dense subset of a Banach space is R-ball porous for every* $R > 0$

A.4 COMPLETENESS OF D_N

Let $N \in \mathbb{N}$. Suppose that $f : [0,1]^N \to \mathbb{R}$, then we note here that $\|f'\|_\infty := \sup_{x\in[0,1]^N} \|f'(x)\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})}$. We also note the following for any $x \in \mathbb{R}^N$ and for some $K \in \mathbb{R}$:

$$\|f'(x)\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})} := \left[(f_{x_1}(x))^2 + \ldots + (f_{x_N}(x))^2\right]^{\frac{1}{2}} \leq K\left[\left|f_{x_1}(x)\right| + \ldots + \left|f_{x_N}(x)\right|\right]$$

Thus, $\|f'\|_\infty \leq K\left[\|f_{x_1}\|_\infty + \ldots + \|f_{x_N}\|_\infty\right]$. But we also know that $\forall j \in \mathbb{N}$ and $\forall x \in [0,1]^N$ we have $\|f'\|_{\mathcal{L}(\mathbb{R},\mathbb{R})} \geq |f_{x_j}(x)$. Therefore, $\|f'\|_\infty \geq \|f_{x_j}\|_\infty$ for any $j \in \mathbb{N}$. Hence, for some $\widetilde{K} \in \mathbb{R}$ we know:

$$\widetilde{K}\left(\|f_{x_1}\|_\infty + \ldots + \|f_{x_N}\|_\infty\right) \leq \max_{1\leq j\leq N} \|f_{x_j}\|_\infty \leq \|f'\|_\infty.$$

Thus, $\|f\|^*_{D_N} := \|f\|_\infty + \sum_{j=1}^{N} \|f_{x_j}\|_\infty$ is an equivalent norm on D_N. We will use this fact shortly.

Theorem A.4.1. *Let $(X, \|\cdot\|_X)$ be a normed linear space, and $(Y, \|\cdot\|_Y)$ a Banach space. Then define:*

$$C(X;Y) := \{f : X \to Y \text{ such that } f \text{ is } \|\cdot\|_X \text{ to } \|\cdot\|_Y \text{ continuous function}\}.$$

Furthermore, define $\|f\|_\infty := \sup_{x\in X} \|f(x)\|_Y$. Let $C_b(X;Y) := \{f \in C(X;Y) : \|f\|_\infty < \infty\}$, then $C_b(X;Y)$ is a Banach space.

Theorem A.4.2. *Let $(X, \|\cdot\|_X)$ be a normed linear space, and $(Y, \|\cdot\|_Y)$ a Banach space. Then define:*

$$\mathcal{L}(X;Y) := \{A : X \to Y \text{ such that } A \text{ is } \|\cdot\|_X \text{ to } \|\cdot\|_Y \text{ bounded linear function}\}.$$

Furthermore, define $\|A\|_{\mathcal{L}(X,Y)} := \sup_{x\in B^o(X)} \|Ax\|_Y$. Then $(\mathcal{L}(X;Y), \|\cdot\|_{\mathcal{L}(X;Y)}$ is a Banach space.

Theorem A.4.3. *(See among others, [2] p.272) Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Let $f : X \to Y$ be Gâteaux differentiable on convex set $U \subset X$, i.e. $Df(x,h)$ the Gâteaux derivative of f at x in the direction of h exists for any $x \in U$. Then, for any $x, y \in U$:*

$$\|f(x) - f(y)\|_Y \leq \sup_{0\leq t\leq 1} \|Df(xt + (1-t)y, \cdot)\|_{\mathcal{L}(X,Y)} \cdot \|x - y\|_X$$

Corollary A.4.1. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Let $f : X \to Y$ be Fréchet differentiable on convex set $U \subset X$, i.e. $f'(x)$ the Fréchet derivative of f at x exists for any $x \in U$. Then, for any $x, y \in U$:*

$$\|f(x) - f(y)\|_Y \leq \sup_{u \in U} \|f'(u)\|_{\mathcal{L}(X;Y)} \cdot \|x - y\|_X$$

Theorem A.4.4. *(Cartan version 1)[9, p.44] Let U be an open convex set in a Banach space X, and let $\{f_n\}$ be a sequence of functions such that $f_n : U \to E$ where E is a Banach space, and f_n is Fréchet differentiable. We make the following assumptions:*

(a) There exists a point $a \in U$ such that the sequence $\{f_n(a)\} \subseteq E$ has a limit (call it f(a)).

(b) The sequence of mappings $\{f'_n : U \to \mathcal{L}(X; E)\}$ converges uniformly to $g : U \to \mathcal{L}(X; E)$

Then, for any $x \in U$ the sequence $\{f_n(x)\} \subseteq E$ converges to the limit function $f(x)$; further the convergence of f_n is uniform on each bounded subset of U; additionally, the function f is Fréchet differentiable, and $f'(x) = g(x)$.

Theorem A.4.5. *$(D_N, \|\cdot\|_{D_N})$ is a Banach space.*

Proof. Clearly D_N is a normed linear space over $\mathbb{R}$ with respect to $\|\cdot\|_{D_N}$. Let $\{f_n\}$ be a $\|\cdot\|_{D_N}$-Cauchy sequence in D_N, i.e. $\|f_n - f_m\|_\infty \xrightarrow[n,m\to\infty]{} 0$. So there exists $f \in C([0,1]^N; \mathbb{R})$ such that $\|f_n - f\|_\infty \xrightarrow[n]{} 0$. Now, $\{f'_n\}$ is uniformly Cauchy: $\forall \epsilon > 0$, there exists $M \in \mathbb{N}$ such that $\forall n, m \geq M$, $\|f'_n - f'_m\|_\infty := \sup_{x \in [0,1]^N} \|f'_n(x) - f'_m(x)\|_{\mathcal{L}(\mathbb{R}^N;\mathbb{R})} \leq \epsilon$. So, for all $x \in [0,1]^N$, there exists $L(x) \in \mathcal{L}(\mathbb{R}^N; \mathbb{R})$ such that $\|f'_n - L\|_\infty$. Now, we would like to say that $f'(x) = L(x)$ for any $x \in [0,1]^N$. We will show this through the following segment. Fix $\epsilon > 0$. Fix $x \in [0,1]^N$. Fix $h \in \mathbb{R}^N \setminus \{0\}$. Fix $m, n \in \mathbb{N}$. Let $\|\cdot\|_X := \|\cdot\|_{\mathbb{R}^N}$, and $\|\cdot\|_Y := \|\cdot\|_{\mathbb{R}}$.

$$\begin{aligned}
&\frac{\left\| f_m(x+h) - f_m(x) - (f_m'(x))\,h \right\|_Y}{\|h\|_X} \\
&\leq \frac{\left\| [f_m(x+h) - f_m(x) - (f_m'(x))\,h] - [f_n(x+h) - f_n(x) - (f_n'(x))\,h] \right\|_Y}{\|h\|_X} \\
&\quad + \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x))\,h \right\|_Y}{\|h\|_X} \\
&= \frac{\left\| [f_m(x+h) - f_n(x+h)] - [f_m(x) - f_n(x)] - [f_m'(x) - f_n'(x)]\,h \right\|_Y}{\|h\|_X} \\
&\quad + \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x) - f_m'(x))\,h \right\|_Y}{\|h\|_X} \\
&\leq \frac{\left\| [f_m(x+h) - f_n(x+h)] - [f_m(x) - f_n(x)] \right\|_Y}{\|h\|_X} + \frac{\left\| [f_m'(x) - f_n'(x)]\,h \right\|_Y}{\|h\|_X} \\
&\quad + \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x))\,h \right\|_Y}{\|h\|_X} \\
&\leq \text{ (using A.4.1) } \frac{\|f_m - f_n\|_\infty \|h\|_X}{\|h\|_X} + \frac{\left\| [f_m'(x) - f_n'(x)]\,h \right\|_Y}{\|h\|_X} \\
&\quad + \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x))\,h \right\|_Y}{\|h\|_X} \\
&\leq \frac{\|f_m - f_n\|_\infty \|h\|_X}{\|h\|_X} + \frac{\left\| f_m' - f_n' \right\|_\infty \|h\|_X}{\|h\|_X} \\
&\quad + \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x))\,h \right\|_Y}{\|h\|_X} \\
&\leq 2\,\|f_m - f_n\|_\infty + \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x))\,h \right\|_Y}{\|h\|_X}
\end{aligned}$$

Now, for any $j \in \mathbb{N}$, define $Q_j(\cdot,\cdot) : [0,1]^N \times [0,1]^N \to [0,\infty)$ as $Q_j(x,h) := \frac{\left\| f_n(x+h) - f_n(x) - (f_n'(x))h \right\|_Y}{\|h\|_X}$. Therefore, our above inequalities can be rephrased to say:

$$Q_m(x,h) \leq 2\|f_m' - f_n'\|_\infty + Q_n(x,h). \tag{A.4.1}$$

So, as $\{f_n'\}$ is a $\|\cdot\|_\infty$-Cauchy sequence, then for the fixed $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that $\forall n, m \geq K$, $\|f_n' - f_m'\|_\infty := \sup_{x \in [0,1]^N} \|f_n'(x) - f_m'(x)\|_{\mathcal{L}(\mathbb{R}^N;\mathbb{R})} \leq \epsilon$. So, in A.4.1, we take $n = K, m \geq K$ to get $Q_m(x,h) \leq 2\epsilon + Q_K(x,h)$. Now, we have the following as $m \to \infty$:

$$f_m(x+h) \xrightarrow[m\to\infty]{} f(x+h) \text{ in } Y$$

$$f_m(x) \xrightarrow[m\to\infty]{} f(x) \text{ in } Y$$

$$f_m'(x)h \xrightarrow[m\to\infty]{} L(x)h \text{ in } Y$$

Therefore, $E(x,h) := \frac{\|f(x+h)-f(x)-(L(x))h\|_Y}{\|h\|_X} = \lim_{m\to\infty} Q_m(x,h) \leq 2\epsilon + Q_K(x,h)$. Now, we consider the fact that f_K' exists, and thus $\lim_{\|h\|_X\to 0} Q_K(x,h) = 0$. Therefore, $\exists\delta > 0$ such that if $0 < \|h\|_X < \delta$, then $Q_K(x,h) < \epsilon$ and so:

$$E(x,h) \leq \lim_{\|h\|_X} 2\epsilon + Q_K(x,h) = 2\epsilon + \epsilon = 3\epsilon.$$

Hence, we have shown that $E(x,h) \xrightarrow[\|h\|_X\to 0]{h\neq 0} 0$, $\forall x \in [0,1]^N$. Thus, $f'(x)$ exists in $\mathcal{L}(\mathbb{R}^N;\mathbb{R})$, and $f'(x) = L(x) = \lim_{n\to\infty} f_n'(x)$. Therefore, $(D_N, \|\cdot\|_{D_N})$ is complete. □

Definition Let $(X, \|\cdot\|_X)$ be a non-trivial Banach space, and $(Y, \|\cdot\|_Y)$ be a non-trivial Banach. Then we define $D(X;Y)$ in the following way:

$$D(X;Y) := \{f \in C(B^o(X);Y) : f' \text{ exists, and } \|f\|_{D(X;Y)} := (\|f\|_\infty + \|f'\|_\infty) < \infty\}$$

When $Y = \mathbb{R}$, then we write $D(X;Y) = D(X)$.

Theorem A.4.6. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Then $D(X;Y)$ is a Banach space.*

Proof. (Method 1) Clearly $D(X;Y)$ is a normed linear space over $\mathbb{R}$ with respect to $\|\cdot\|_{D(X;Y)}$. Let $\{f_n\}$ be a $\|\cdot\|_{D(X;Y)}$-Cauchy sequence in $D(X;Y)$, i.e. $\|f_n - f_m\|_\infty \xrightarrow[n,m\to\infty]{} 0$. So there exists $f \in C_b(B^o(X);Y)$ such that $\|f_n - f\|_\infty \xrightarrow[n]{} 0$. Now, $\{f_n'\}$ is uniformly Cauchy: $\forall\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $\forall n,m \geq M$, $\|f_n' - f_m'\|_\infty := \sup_{x\in B^o(X)} \|f_n'(x) - f_m'(x)\|_{\mathcal{L}(X;Y)} \leq \epsilon$. So, for all $x \in B^o(X)$, there exists $L(x) \in \mathcal{L}(X;Y)$ such that $\|f_n' - L\|_\infty$. Now, we would like to say that $f'(x) = L(x)$ for any $x \in B^o(X)$. We will show this through the following segment. Fix $\epsilon > 0$. Fix $x \in [0,1]^N$. Fix $h \in X \setminus \{0\}$. Fix $m,n \in \mathbb{N}$.

$$\begin{aligned}
&\frac{\left\| f_m(x+h) - f_m(x) - (f'_m(x))\, h \right\|_Y}{\|h\|_X} \\
&\leq \frac{\left\| [f_m(x+h) - f_m(x) - (f'_m(x))\, h] - [f_n(x+h) - f_n(x) - (f'_n(x))\, h] \right\|_Y}{\|h\|_X} \\
&\qquad + \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x))\, h \right\|_Y}{\|h\|_X} \\
&= \frac{\left\| [f_m(x+h) - f_n(x+h)] - [f_m(x) - f_n(x)] - [f'_m(x) - f'_n(x)]\, h \right\|_Y}{\|h\|_X} \\
&\qquad + \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x) - f'_m(x))\, h \right\|_Y}{\|h\|_X} \\
&\leq \frac{\left\| [f_m(x+h) - f_n(x+h)] - [f_m(x) - f_n(x)] \right\|_Y}{\|h\|_X} + \frac{\left\| [f'_m(x) - f'_n(x)]\, h \right\|_Y}{\|h\|_X} \\
&\qquad + \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x))\, h \right\|_Y}{\|h\|_X} \\
&\leq \text{ (using A.4.1) } \frac{\|f_m - f_n\|_\infty \|h\|_X}{\|h\|_X} + \frac{\left\| [f'_m(x) - f'_n(x)]\, h \right\|_Y}{\|h\|_X} \\
&\qquad + \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x))\, h \right\|_Y}{\|h\|_X} \\
&\leq \frac{\|f_m - f_n\|_\infty \|h\|_X}{\|h\|_X} + \frac{\left\| f'_m - f'_n \right\|_\infty \|h\|_X}{\|h\|_X} \\
&\qquad + \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x))\, h \right\|_Y}{\|h\|_X} \\
&\leq 2\, \|f_m - f_n\|_\infty + \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x))\, h \right\|_Y}{\|h\|_X}
\end{aligned}$$

Now, for any $j \in \mathbb{N}$, define $Q_j(\cdot,\cdot) : B^0(X) \times X \to Y$ as $Q_j(x,h) := \frac{\left\| f_n(x+h) - f_n(x) - (f'_n(x))h \right\|_Y}{\|h\|_X}$. Therefore, our above inequalities can be rephrased to say:

$$Q_m(x,h) \leq 2\|f'_m - f'_n\|_\infty + Q_n(x,h). \tag{A.4.2}$$

So, as $\{f'_n\}$ is a $\|\cdot\|_\infty$-Cauchy sequence, then for the fixed $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that $\forall n, m \geq K$, $\|f'_n - f'_m\|_\infty := \sup_{x \in B^o(X)} \|f'_n(x) - f'_m(x)\|_{\mathcal{L}(X;Y)} \leq \epsilon$. So, in A.4.2, we take $n = K, m \geq K$ to get $Q_m(x,h) \leq 2\epsilon + Q_K(x,h)$. Now, we have the following as $m \to \infty$:

$$f_m(x+h) \xrightarrow[m\to\infty]{} f(x+h) \text{ in } Y$$
$$f_m(x) \xrightarrow[m\to\infty]{} f(x) \text{ in } Y$$
$$f'_m(x)h \xrightarrow[m\to\infty]{} L(x)h \text{ in } Y$$

Therefore, $E(x,h) := \frac{\|f(x+h)-f(x)-(L(x))h\|_Y}{\|h\|_X} = \lim_{m\to\infty} Q_m(x,h) \leq 2\epsilon + Q_K(x,h)$. Now, we consider the fact that f'_K exists, and thus $\lim_{\|h\|_X\to 0} Q_K(x,h) = 0$. Therefore, $\exists\delta > 0$ such that if $0 < \|h\|_X < \delta$, then $Q_K(x,h) < \epsilon$ and so:

$$E(x,h) \leq \lim_{\|h\|_X} 2\epsilon + Q_K(x,h) = 2\epsilon + \epsilon = 3\epsilon.$$

Hence, we have shown that $E(x,h) \xrightarrow[\|h\|_X\to 0]{h\neq 0} 0$, $\forall x \in B^o(X)$. Thus, $f'(x)$ exists in $\mathcal{L}(X;Y)$, and $f'(x) = L(x) = \lim_{n\to\infty} f'_n(x)$. Therefore, $(D(X;Y), \|\cdot\|_{D(X;Y)})$ is complete. □

We provide a second proof, as each proof gives a different type of intuition.

Proof. (Method 2) Clearly $D(X;Y)$ is a normed linear space over $\mathbb{R}$ with respect to $\|\cdot\|_{D(X;Y)}$. Thus, we need only discuss completeness. Let $\{f_n\}$ be a $\|\cdot\|_{D(X;Y)}$-Cauchy sequence in $D(X;Y)$. Then $\{f_n\}$ is a $\|\cdot\|_\infty$-Cauchy sequence in $(C_b(B^o(X);Y), \|\cdot\|_\infty)$. So, as $(C_b(B^o(X);Y), \|\cdot\|_\infty)$ is complete (see [22]), then there exists $f \in C_b([0,1]^N, \mathbb{R})$ such that $\|f_n - f\|_\infty \xrightarrow[n]{} 0$. We also have $\|f'_k - f'_m\|_\infty \xrightarrow[k,m\to\infty]{} 0$, where here $\|g'\|_\infty := \sup_{x\in B^o(X)} \|g'(x)\|_{\mathcal{L}(X;Y)}$.

Fix $x \in B^o(X)$. Then $\|f'_m(x) - f'_n(x)\|_{\mathcal{L}(X;Y)} \xrightarrow[m,n\to\infty]{} 0$. So $\{f'_n(x)\}$ is a Cauchy sequence in $\mathcal{L}(X;Y)$, which is complete, so there exists $\tilde{h}_x \in \mathcal{L}(X;Y)$ such that $\|f'_n(x) - \tilde{h}_x\|_{\mathcal{L}(X;Y,\mathbb{R})} \xrightarrow[n\to\infty]{} 0$. Now, define $h(x) := \tilde{h}_x$. Thus, $\forall x \in B^o(X)$ we have $\|f'_n(x) - h(x)\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})} \xrightarrow[n\to\infty]{} 0$, and so $\|f'_n - h\|_\infty \xrightarrow[n\to\infty]{} 0$. Then by theorem A.4.4, f is differentiable on $B^o(X)$ (which is a convex and open subset of a Banach space) and $f'(x) = h(x)$ for all $x \in B^o(X)$. So, $f \in D(X;Y)$. Also, $\|f_n - f\|_\infty = \|f_n - f\|_\infty + \|f'_n - f'\|_\infty \xrightarrow[n]{} 0 + 0 = 0$. Thus, $(D(X;Y), \|\cdot\|_{D(X;Y)})$ is complete. □

A.5 SEPARABILITY

Theorem A.5.1. $(BV[0,1], \|\cdot\|_{BV})$ *is not separable.*

Proof. Suppose not for a contradiction. Suppose that there is a countable dense subset of $A \subset BV[0,1]$. Then define a subset of $BV[0,1]$ in the following way: $C := \{f_t := \chi_{[0,t]} \in BV[0,1] : t \in [0,1]\}$. Furthermore, we claim that if $t \neq s$, then $\|f_t - f_s\|_{BV[0,1]} \geq 1$. Why? Well, let $s, t \in [0,1]$ with $s \neq t$ and we have two cases.

1. Suppose that one of s or t is zero. Without loss of generality, $s = 0$. Then $f_s = f_0 = 0$ and so $\|f_t - f_s\|_{BV[0,1]} = |f_s(0) - f_0(0)| + V_0^1(f_s - f_0) = |f_s(0)| + V_0^1(f_s) = \chi_{[0,t]}(0) + V_0^1(\chi_{[0,t]}) \geq 1 + 1 = 2.$
2. Suppose that neither of s, t are zero. Without loss of generality $t > s$, and so $f_t - f_s = \chi_{[s,t]}$. So $\|f_t - f_s\|_{BV[0,1]} = |\chi_{[s,t]}(0)| + V_0^1(\chi_{[0,t]} \geq 0 + 1.$

So, in any case, we have $\|f_t - f_s\|_{BV[0,1]} \geq 1$ whenever $t \neq s$. Also, note that C is uncountable. Now, fix $s, t \in [0,1]$ such that $s \neq t$. Then, by density of A there exists $a \in A$ such that $\|f_t - a\|_{BV[0,1]} < \frac{1}{2}$. Then, $2 \leq \|f_t - f_s\|_{BV[0,1]} \leq \|f_t - a\|_{BV[0,1]} + \|a - f_s\|_{BV[0,1]} < \frac{1}{2} + \|f_s - a\|_{BV[0,1]}$. Therefore, $\frac{3}{2} < \|f_s - a\|_{BV[0,1]}$ and as $s \in [0,1]$ is arbitrary, f_t is the unique element "close" to $a \in A$. But, for all $w \in [0,1]$ there must exist a unique $b \in A$ within $\frac{1}{2}$, and so we have shown that A is uncountable as it is in relation to uncountable set C. This is a contradiction, so $BV[0,1]$ is not separable. □

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