FUNCTION CONCEPTIONS OF AP CALCULUS STUDENTS

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Functions are one of the most important topics in secondary school mathematics, especially for students who wish to take higher-level mathematics courses beginning with calculus. The prerequisites for Advanced Placement Calculus state that a thorough understanding of functions is needed for those who wish to succeed in the course and pass the AP Calculus Exam. However, research has shown that students' struggles with calculus concepts could be traced to inadequate prerequisite knowledge of functions. Since thousands of students take and pass the AP Calculus Exam every year, it is important to investigate which aspects of functions AP Calculus students generally do and do not understand, and to determine how well their understandings of functions relate to their performance on the exam.

In order to explore this, students from AP Calculus classes in three different schools were tested on their understandings of functions at the end of the course, after they had already taken the AP Calculus Exam. Additionally, some of the participants were also interviewed to further ascertain their understandings of functions. Finally, the AP Calculus Exam scores for all participants were collected and compared to their function understandings. It was found that 1) most participants' understandings of functions were less than sufficient for an AP Calculus course, and 2) there was a high positive correlation between function understandings and scores on the AP Calculus Exam. These results suggest that more work must be done in developing students' understandings of functions in the secondary mathematics curriculum, and greater

measures should be taken to ensure that students entering AP Calculus have a sufficient understanding of the prerequisite knowledge.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

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1.0 STATEMENT OF THE PROBLEM

1.1 INTRODUCTION

Functions are one of the most important topics in secondary school mathematics. They make up one of the primary high school standards listed in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010), and they are featured prominently in the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* (2000). There has also been much research that has highlighted their importance to the secondary mathematics curriculum (Baker, Hemenway, & Trigueros, 2001; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Confrey & Smith, 1991; Dubinsky & Harel, 1992; Eisenberg, 1992; Elia, Panaoura, Eracleous, & Gagatsis, 2005; Gerson, 2008; Knuth, 2000; Leinhardt, Zaslavsky, & Stein, 1990)**.**

While understanding functions is important for all students who wish to graduate high school, it is especially so for those who wish to further their education in mathematics by taking higher level courses, beginning with calculus (Carlson, 1998; Thompson, 1994). The prerequisites for Advanced Placement (AP) Calculus (College Board, 2010) highlight the fact that knowledge and skill with functions is needed for those who wish to succeed in the course:

> "Before studying calculus, all students should complete four years of secondary mathematics designed for college-bound students: courses in which they study algebra, geometry, trigonometry, analytic geometry and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric and piecewise-defined functions. In particular, before studying calculus, students must be familiar with the properties of functions, the

> algebra of functions and the graphs of functions. Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts and so on) and know the values of the trigonometric functions at the numbers, 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, and their multiples." (College Board, 2010, p. 6).

Additionally, researchers at Arizona State University have also discussed the fact that functions are important for students entering calculus. Oehrtman, Carlson, & Thompson (2008) indicated that they spend the beginning of college calculus 1 courses testing and strengthening their students' conceptions of functions, and have said that the time spent doing this "…is crucial for their understanding the major ideas of calculus" (p. 167). Carlson and Oehrtman were also part of the team that developed and validated the *Precalculus Concept Assessment Instrument* (Carlson, Oehrtman, & Engelke, 2010), a 25-item multiple choice exam used to measure students' knowledge of concepts that are central to precalculus and foundational to calculus 1. The majority of the exam's taxonomy is centered on the understanding of function concepts and functional reasoning.

Despite this importance, research has also shown that students at all levels generally have struggled with functions, that the development of a strong sense of functions can take a very long time, and students can maintain several common misconceptions about functions (Carlson, 1998; Eisenberg, 1992; Leinhardt, et al., 1990; Vinner & Dreyfus, 1989). There have also been several studies that have found that students' struggles with calculus concepts could be traced to inadequate prerequisite knowledge of functions (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Clark, et al., 1997; Ferrini-Mundy & Gaudard, 1992; Judson & Nishimori, 2005; Ubuz, 2007). Even students who take and do well in calculus courses have been shown to have a relatively weak or narrow conceptualization of functions (Carlson, 1998). In order to address this issue, there has been much research devoted to determining how students learn about functions and how to organize the teaching of functions such that students have a more sophisticated understanding of them by the time they are ready to take calculus. The following sections contain descriptions of two frameworks that arose from this body of research: NCTM's Big Ideas of Functions, and Action-Process-Object-Schema (APOS) Theory (Asiala, et al., 1996).

1.2 BACKGROUND

1.2.1 The Big Ideas of Functions

The first framework was published recently in NCTM's *Developing Essential Understanding of Functions* (Cooney, Beckmann, Lloyd, Wilson, & Zbiek, 2010). In this book, functions were organized into five "Big Ideas" that the authors argue teachers must focus on in their instruction: 1) The Function Concept; 2) Covariation and Rate of Change; 3) Families of Functions; 4) Combining and Transforming Functions; and 5) Multiple Representations of Functions. These Big Ideas serve as a framework that teachers can use as a way to shape and strengthen their students' learning and understanding of functions.

While NCTM developed and published these Big Ideas only recently, the framework itself is not entirely new. In 1991, Confrey and Smith presented a similar framework for organizing functions. They first provided two ways of defining and viewing functions: as a covariation between two quantities, and as a correspondence between values of two quantities. Second, they described three approaches to teaching functions. These approaches respectively emphasized function prototypes (similar to Big Idea #3), multiple representations (Big Idea #5), and transformations (Big Idea #4). While the framework proposed by Confrey and Smith (1991) is not quite as detailed as the Big Ideas, a comparison between the two reveals that they both address the same points (Table 1.1).

Table 1.1. Alignment of the Confrey and Smith (1991) framework and the CCSS with the Big Ideas.

NCTM Big Idea	***Confrey & Smith Framework***	***Sample Standard from CCSS***
Function Concept	Correspondence definition of functions	Write a function that describes the relationship between two quantities.
Covariation and Rate of Change	Covariation definition of functions	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.
Families of Functions	Function prototype approach	Distinguish between situations that can be modeled with linear functions and with exponential functions.
Combining and Transforming Functions	Function Transformations approach	Find inverse functions.
Multiple Representations	Multiple Representations approach	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Note: This table displays the alignment of aspects of the Confrey and Smith (1991) framework and certain standards of the CCSS with each Big Idea.

In addition to Confrey and Smith (1991), the Common Core State Standards Initiative (CCSSI) of 2010 also provided evidence of attention to the Big Ideas. The function strand of the Common Core State Standards (CCSS), while not organized in exactly the same way, consists of many standards that are very well aligned with the Big Ideas. For example, three of the categories that function standards are grouped under include the following:

1) Understand the concept of function and use function notation (p. 69)
2) Analyze functions using different representations (p. 69)
3) Build new functions from existing functions (p. 70)

It is not difficult to notice that these categories are related in some capacity to Big Ideas 1, 5, and 4 respectively. These and the other Big Ideas are also found sprinkled among the rest of the individual standards (Table 1.1). The strong alignment with such an authoritative document as the CCSS further supports the Big Ideas framework as a valid means for supporting the teaching and learning of functions, such that a student who understands the Big Ideas has a strong overall conception and understanding of functions in general.

The Big Ideas can also be seen in the fundamental concepts of calculus. For example, several aspects of the derivative (Sofronos, et al., 2011) are related or connected to one or more of the Big Ideas:

- The derivative as a rate of change (Big Idea 2)
- The various rules of derivatives (Big Ideas 3 and 4)
- The derivative and antiderivative are inverses of each other (Big Idea 4)
- Multiple representations of the derivative (Big Idea 5)
- The derivative as a limit vs. the derivative as slope. (Big Ideas 1, 2, and 5)

It could be inferred from these properties of derivative just how well connected it is to those of functions, and thus how important it is to have a strong conception of functions in order to better understand the derivative. This is also the case for other important aspects of calculus, such as limits and integrals. Essentially, students with a strong conceptual understanding of functions should succeed in calculus.

1.2.2 APOS Theory

There has also been much research that focused on the idea that how students view functions has a great influence on their learning of functions. Many researchers have discussed the notion that functions can be viewed as an action, a process, an object, or a schema. These function views or conceptions were defined by, among others, Breidenbach, et al. (1992), summarized by Dubinsky and Harel (1992), and eventually grew into what Asiala, et al. (1996) called APOS (Action-Process-Object-Schema) Theory. In short, students with an *action* view see functions as merely a means for performing a particular action, such as computation. Meanwhile, those with a *process* view see a function as a collection of actions all at once, and can comprehend the connections between those actions and what they can produce together. The *object* view of functions and the function *schema* are even more sophisticated. Several researchers (Breidenbach, et al, 1992; Carlson, et al., 2010; Dubinsky & Harel, 1992; Oehrtman, et al., 2008) have claimed that students need at least a process view in order to develop a strong understanding of functions, and have used APOS Theory to help explain students' impoverished function sense:

> "Unfortunately, most pre-calculus students do not develop beyond an action view, and even strong calculus students have a poorly developed process view that often leads only to computational proficiency." (Oehrtman, et al., 2008, p. 160)

Carlson, et al., (2010) asserted that a process view is necessary for students to develop covariational reasoning (Big Idea #2), which they cited as a critical component to a student's calculus readiness. To that end, many of the items on the *Precalculus Concept Assessment* were designed to measure whether or not the student holds a process view of functions. It is expected that students who hold at least a process view of functions should do well in a calculus course.

1.2.3 Functions, Calculus, and AP Calculus Students

While there is much research on students' understanding of functions, studies of functions and calculus learning have been primarily conducted at the college level. There has been very little research that focuses on AP Calculus students' understandings of functions. There is, however, one doctoral dissertation that does, by Kimani (2008), who found that most AP Calculus students have a low or superficial understanding of function transformations (Big Idea #4), which could suggest that they may have difficulty with functions in general. Despite these claims, thousands of students take and pass the AP Calculus exam every year. For instance, in 2012 over 159,000 students earned a 3 or higher on the AP Calculus AB exam (College Board, 2012). Research conducted by Eisenberg (1992) and Carlson (1998) suggested that it is possible that a significant number of AP Calculus students are passing the course and exam with a minimal understanding of functions. These claims were supported empirically by Judson and Nishimori (1995). In a study that compared 18 AP Calculus BC students with 26 Japanese students in an equivalent class, they found that "all students lacked a sophisticated understanding of functions," (p. 39) despite demonstrating a strong understanding of calculus concepts. This result conflicted with the notion that a high understanding of functions is a prerequisite to take AP Calculus, as the College Board stated (2010). It is important to note, however, that Judson and Nishimori did not focus on the entire scope of function understandings when they made the above assertion. Instead, they only focused on a few select aspects of functions, such as students' understandings of constant functions and function compositions. Therefore, there is still a need for an investigation of AP Calculus students' understandings of all aspects of functions, as well as a comparison of those understandings to their understandings of calculus concepts by way of their performance on the AP Calculus Exam.

1.3 PURPOSE AND RESEARCH QUESTIONS

The purpose of this dissertation is to ascertain AP Calculus students' understandings of functions (using the Big Ideas and APOS Theory as a framework), and to compare their understandings with their respective performances on the AP Calculus Exam. Specifically, the study is designed to answer the following research questions:

1) *What do students completing the AP Calculus AB course know and/or understand about functions?*
    a. *To what extent can AP Calculus students solve problems about functions?*
    b. *To what extent can AP Calculus students explain their thinking about functions?*
    c. *What view of functions do AP Calculus students hold?*
    d. *To what extent do AP Calculus students understand each of the Big Ideas of functions?*
2) *To what extent is there alignment between AP Calculus students' understanding of functions and their performance on the AP Calculus Exam?*
    a. *What is the relationship between students' overall understanding of functions and their performance on the AP Calculus Exam?*
    b. *What is the relationship between students' view of functions and their performance on the AP Calculus Exam?*
    c. *What is the relationship between students' understanding of each of the Big Ideas of functions and their performance on the AP Calculus Exam?*

1.4 WHY CALCULUS?

This study has been designed to understand the connection between function understanding and probable success in calculus. So why is the ability to understand calculus and succeed in it important? Calculus is often seen as the gateway to taking courses in higher-level mathematics, and is a necessity for those who wish to follow careers paths in mathematics, the hard sciences, engineering, and technology, among others. Most AP Calculus students are on the cusp of going to college, choosing a major, and following a path to a career. An understanding of calculus will leave students open to more choices and opportunities in both their education and careers than they likely will have otherwise. Therefore, the study of the different aspects of calculus education, such as the course itself, its prerequisites, its students and its teachers is an important pursuit of many mathematics education researchers.

1.5 SIGNIFICANCE OF THE STUDY

The results of this study will inform teachers, administrators, and researchers about the nature of AP Calculus and the students who take it. It is expected that there will be a high positive correlation between a student's understanding of functions and his or her score on the AP Calculus Exam. If this is the case, then it will be worth investigating what it takes for students to be admitted to AP Calculus in most schools. In 2012, approximately 40% of students earned less than a 3 on the AB exam (College Board, 2012). If most of these students are entering AP Calculus with a less than proficient understanding of functions, then it could mean that schools would need to strengthen the function content in preceding algebra and precalculus courses, and possibly begin administering a placement exam similar to the *Precalculus Concept Assessment* to would-be AP Calculus students.

However, if no correlation is found, then it raises questions about what the AP Calculus Exam is actually measuring, what the prerequisites for taking AP Calculus should actually be, and what the teachers of the course are doing in order to help their students achieve success in the course.

1.6 LIMITATIONS OF THE STUDY

This study has several limitations. First, it measures what students know about functions at the *end* of the AP Calculus course, as opposed to their understanding as they are entering the course. While this tells us much about students with a weak understanding of functions, it is more difficult to get an idea of the influence the course had on students with a strong understanding of functions at the end of the course.

Also, most of the participants will only be given the opportunity to take the Precalculus Concept Assessment (PCA), and will not be interviewed. Therefore, it will be more difficult to draw general conclusions about any understandings that are assessed by the interviews but not by the PCA. This includes such information as students' abilities to define functions, to identify functions and nonfunctions, and to transform functions. The assessment of students' ability to view functions as objects will also be difficult to generalize, as this is also only covered in the interviews. The fact that these differences in content between the PCA and interviews exist is also a limitation of the study, as not all concepts found on the PCA are also found in the interview. So while the interview offers information about the understanding of concepts that are not covered on the PCA, in some cases the ability to exhibit understanding of concepts in an interview environment, in comparison to the testing environment of the PCA, was not available for some of those concepts, such as the ability to use one representation to produce one or two others.

Finally, the selection of the participants for this study was based on convenience to the researcher, and may not necessarily be a completely representative sample of the national population of AP Calculus students. Similarly, the selection of interview participants was restricted only to students who were over the age 18 and agreed to be interviewed. Ideally, three students from each school would have been selected for the interviews, one for each level of understanding of functions based on the scores of the PCA. However, only two students from one of the schools signed up for the interview, and they had the same level of understanding on the PCA, so the preferred sample for the interview participants was unable to be met.

1.7 OVERVIEW

The upcoming chapters are organized as follows. Chapter Two contains a review of the literature on function understanding, organized around the five Big Ideas of functions. Chapter Three describes the methodology of the study, including the participants, data sources, and the coding and analysis of the data. Chapter Four contains the results of the data analysis, and Chapter 5 provides a discussion of the results, conclusions, and implications for further research.

2.0 REVIEW OF THE LITERATURE

The purpose of this study is to examine AP Calculus students' conceptual understanding of functions, and to compare that understanding to their success in calculus as measured by their score on the AP Calculus Exam. In order to follow through with this study, it is first important to review the literature to further understand what it really means for AP Calculus students to have a conceptual understanding of functions, and why it is important to further examine the relationship between calculus students' understanding of functions and their success in calculus.

This chapter is divided into three sections. The first section reviews the literature on the function concept (Big Idea #1), and provides a greater look at APOS Theory as well as other theories of function learning and understanding. The second section provides a look at literature on the other four Big Ideas of functions, including how APOS Theory relates to each Big Idea, common student difficulties and misconceptions, and what makes each important for calculus students. The third and final section reviews past literature that focused primarily on calculus students' understandings of functions.

2.1 THE FUNCTION CONCEPT

The *Function Concept* is the first of NCTM's Big Ideas (Cooney, et al., 2010), and it is the central idea that students must grasp in order to fully understand the others. Cooney, et al.'s discussion of the function concept focused primarily on a) how it is defined, and b) the fact that functions include many entities beyond those that can be expressed in a continuous mathematical formula or equation, such as matrices and sequences. However, these understandings of function, as essential as they are, are not the only ones, and some may argue that they are not even the most basic nor the most critical of understandings. Instead, how students *view* functions is often discussed as the key that can either promote or obstruct their learning and understanding of them (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Oehrtman, et al., 2008). This section begins with a brief explanation on how students are expected to develop their understandings of functions throughout their K-12 experiences. Then, literature on how students view functions is reviewed, followed by a discussion of function definitions and a look at what researchers have found to be common misconceptions and other difficulties students have in trying to understand the function concept.

2.1.1 Development of the Function Concept

The development of an understanding of the function concept is a slow process that spans the entirety of a student's K-12 schooling. The way in which this development takes place is outlined in two documents: NCTM's *Principles and Standards of School Mathematics* (2000), and the Common Core State Standards for Mathematics (CCSSI, 2010). According to these documents, the understanding of the function concept begins with the understanding of patterns. Up through second grade, students are expected to begin developing an ability to recognize, describe, and extend simple patterns and relationships. This includes identifying, sorting and ordering objects by their properties, such as size and number. At this age, they also begin to make sense of how basic patterns may be generated.

From grades 3 to 5, the reasoning about patterns and relationships moves from the concrete to the abstract, and students are expected to be able to make generalizations. They also learn how to analyze patterns in different representations such as tables and graphs, and also create representations of relationships when described to them. Finally, they begin to investigate how one variable may change in relationship to another, and to identify and describe rates of change as either constant or varying.

In middle school (grades 6 to 8), students are expected to learn about the concept of functional relationships, they learn about linear functions and their properties, and they can identify and compare linear functions with nonlinear functions. They can begin to relate and compare different representations of the same relationship, and to represent patterns algebraically as well as with tables, graphs and words.

Finally, when students reach high school they begin to learn more of the specific aspects of functions. They learn about different types of functions and their properties, and they learn how to combine, transform, and invert functions. They are taught to analyze functions by investigating their properties such as their rates of change, intercepts and roots, asymptotes, and behavior. They are also expected to be able to identify different functional relationships that occur within various contexts.

2.1.2 Views of Functions

As already stated in Chapter One, Action-Process-Object-Schema (APOS) Theory is a widely held model of the stages in the development of a student's understanding of functions (Breidenbach, et al., 1992; Dubinsky & Harel, 1992). Recall that students with an *action view* only see functions at the most superficial of levels. For instance, when given the functional formula $y = 3x^2 + 5$, these students can easily evaluate it at given *x* values (such as 1) and compute a new value (in this case, 8). Each action the function performs is done in a separate, static instance. Asiala, et al. (1996) claimed that students with this view think a function must have a single formula for computing values, and that they may have much difficulty with more advanced concepts such as function compositions, function inverses, and piecewise functions. A student with an action view may be able to execute a procedure for inverting the above function (such as switching the *x* and *y* and solving for *y*, or reflecting the graph about the line $y = x$), but this would not indicate that he or she has an understanding of why the procedure works (Oehrtman, et al., 2008).

The *process* view of function is more complicated. According to Dubinsky and Harel (1992):

> "A *process* conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done." (p. 85)

In other words, students with a process view are capable of internalizing the function. They see it more as a collection of actions at once, and can understand the relationship and connection between the inputs and outputs of the function. In the above example of $y = 3x^2 + 5$, a student with a process view sees the entire expression as an output. As Oehrtman, et al. (2008) put it, "the student can imagine a set of input values that are mapped to a set of output values by the defining expression" (p. 158). In the case of the inverse, a student with a process view understands it as a reversal of the initial process, and can make sense of the previously-discussed inverse procedure in a way that the student with the action view could not (Oehrtman, et al., 2008). Figure 2.1 depicts the differences between an action and process view of function inverses.

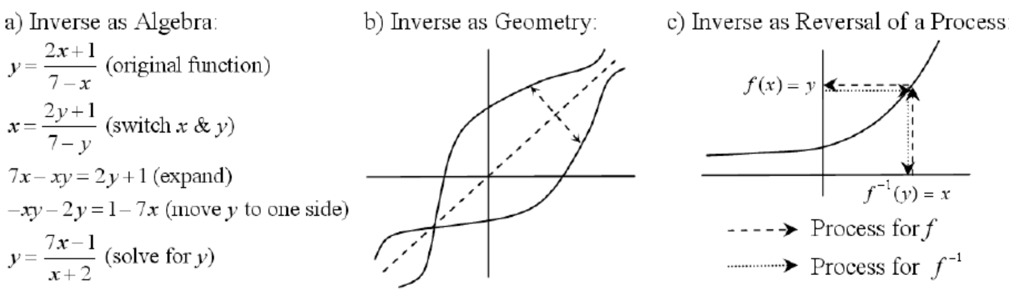


Figure 2.1. Action and process views of inverse.
In which a) and b) depict procedures that students with an action view may be able to execute, while (c) provides a sense of the understanding of inverse that a student with a process view has. (as depicted in Oehrtman, et al., 2008).

The next stage after the process view is the ability to see functions as objects. A student with an *object* view of functions is able to see that, in addition to the function performing actions, other entities can also perform actions on the function itself. With this understanding, students are capable of recognizing transformations of functions in both graphical and algebraic form. In the $y = 3x^2 + 5$ example, a student is capable of recognizing that it is a transformed version of the parent function $y = x^2$, and that the graph of that function would be wider and positioned higher than the standard quadratic graph. The student can also understand that the function's inverse is also a transformation, that the original function has been *inverted*. In general, APOS theory states that as students move from an action view up to an object view, their understanding of functions becomes stronger and more sophisticated (Asiala, et al., 1996; Breidenbach, et al., 1992; Dubinsky & Harel, 1992).

Finally, a student with a function *schema* has reached the most sophisticated level of understanding, which is what occurs when a student organizes all previous actions, processes, and objects, and the relationships between them. Dubinsky and McDonald (2002) explained schemas as such:

> "A *schema* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in

> the individual's mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not." (p. 3)

Students with a well-constructed function schema understand when a particular view of function is needed in ordered to solve a problem, whether it is action, process or object. They also understand the relationships between the different views and possess the flexibility to move between them fluidly. While there is still much to learn about schemas in general and the function schema in particular (Asiala, et al., 1996), it is easy to surmise that a student with a strong function schema has a well-formed conceptual understanding of functions, and is capable of succeeding in a calculus course.

However, while a well-constructed function schema may indicate the strongest of function conceptions, several researchers have speculated that all incoming calculus students really need is a process view (Oehrtman, et al., 2008). In fact, most researchers (e.g., Carlson, 1998; Dubinsky & Harel, 1992; Oehrtman, et al., 2008) who discussed the connections between function view and function understanding have focused on the differences in learning between students who maintain an action view and those who have reached a process view, and focus in particular on the benefits of the process view. In each case, it is argued that an action and process view are the difference between hindered and enhanced learning of functions. For example, Dubinsky and Harel (1992) identified several indicators that could help one discern whether a student has attained a process view. These included the ability of the student to recognize a) that inputs and outputs of a function do not have to be numbers, b) that functional graphs are not necessarily continuous, and c) the difference between the single-valued property of all functions (one y for every x) and the one-to-one property held by only some functions (no more than one x for every y). For each case, the process view carries a greater understanding of function than the action view. Similarly, an empirical study by Carlson (1998) compared high performing students in different college mathematics courses, and found that students in college algebra mostly held action views of functions, while those in second-semester calculus had developed a process view, albeit barely in some cases. Oehrtman, et al. (2008) discussed the importance of the process view for understanding the primary calculus concepts: the limit, derivative, and integral. As an example, they explained what a student must be able to do in order to understand limits:

> "In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process, and determine the corresponding region of input values. The action of a function on these values must be considered simultaneously since another process (one of reducing the size of the neighborhood in the range) must be applied while coordinating the results." (p. 160)

This type of reasoning can also be applied to the derivative and the integral, both of which are built on the limit concept. Essentially, in order to understand calculus, a student must be able to conceptualize functions as more than just a means to perform some sort of action.

While it appears that holding a process view of functions is key for students who wish to succeed in calculus, it is not the end of the road. The third view of functions students can hold is the aforementioned object view. Most researchers would agree that the object view can only be reached once a process view is in place (Breidenbach, et al., 1992; Dubinsky & Harel, 1992; Sfard, 1991; Thompson, 1994). Dubinsky and Harel (1992) stated that the object view is reached by "encapsulating a process," and that in many cases it is necessary to be able to move between the two conceptions. In any case, it is implied through the literature that if a process view is sufficient, then a student that holds an object view of functions (or a function schema, for that matter) should certainly have a strong enough conceptualization to thoroughly understand calculus concepts.

However, it is important to note that APOS Theory can be applied to many different ideas and aspects of function, and that there is not just one way of looking at all functions all at the same time. In fact, APOS Theory can be applied to each of NCTM's Big Ideas (Cooney, et al., 2010). For instance, when considering different function types (Big Idea #3), students can hold a process view of one or two types of functions, such as linear functions and quadratics, while still perceiving more complicated functions, such as rational functions or piecewise functions, with an action view (Baker, Hemenway, & Trigueros, 2001; Dubinsky & McDonald, 2002; Eisenberg & Dreyfus, 1994). Different representations of functions (Big Idea #5) such as formulas and graphs also seem to have their own respective APOS Theory constructions (Monk, 1992; Ronda, 2009), and it is possible for a student to concurrently possess different mental constructions for each (Asiala, et al., 1997). These applications of APOS Theory within the different Big Ideas, and what they mean for calculus students, will be discussed in further detail in the next section.

APOS Theory (Asiala, et al., 1996) may be the common and most referenced view of function discussed by researchers. Slavit (1997), however, discussed another way that students may think about functions. He theorized that students may also be able to think of functions by way of the properties they exhibit, such as intercepts, symmetries, extrema, and continuity. For example, a student who thinks this way would identify the function $y = 3x^2 + 5$ as being continuous, concave up, having a domain that spans the real numbers, a range with a lower bound at $y = 5$, etc. In his discussion of the property view, Slavit suggested that it is actually an extension of the object view, or at the very least, an object view is required to reach it. He said that students can "conceive of functions as abstract objects either possessing or not possessing these previously experienced properties," (p. 260). While Slavit did not discuss what this means for calculus students, it could be inferred that such a view would be beneficial for them. After all, there are important aspects of calculus that require students to recognize, understand, and work with such properties of a given function such as those mentioned above and many more (College Board, 2010). A student with a property-oriented view of function should be able to do this type of work.

2.1.3 Definitions of Functions

While students' views of function may have a very strong influence on their learning and understanding of functions, how they define functions can also say a lot about their conceptualizations. As Cooney, et al. (2010) indicated, there are many ways to define a function. In *Developing Essential Understanding of Functions*, Cooney, et al. listed eight different textbook definitions of a function, some of which came from the same publishers or even the same textbook. Cooney, et al. then separated those definitions into one of two specific categories, which they called the "mapping" definitions (one set is mapped to another set) and the "ordered pair" definitions (an element in one set is paired with exactly one element in another). Essentially, each of the definitions attempted to communicate the same basic idea, that a function is a rule that assigns exactly one element in a set for each element in another set. This rule is what Confrey and Smith (1991) called the *correspondence* definition of functions, which has long been the standard definition used at all levels of mathematics (Thompson, 1994). However, functions can also be defined as a *covariation* between quantities (Confrey & Smith, 1991; Thompson, 1994), which means changes to quantities in one set can be predicted by how quantities change in another set.

Understanding both the correspondence and covariation definitions for function is important to grasping the overall function concept. However, as functions are taught in schools, there is a clear imbalance in favor of the correspondence definition. Thompson (1994), while having recognized the benefits of the correspondence definition, called for the school mathematics curriculum to emphasize covariance first and most often:

> "The tension between thinking of function as covariation and of function as correspondence is natural. They are both part of our intellectual heritage, so they show up in our collective thinking…but we still face the question of how to reflect our heritage within a curriculum in a way that is coherent in regard to a conceptual development of the subject and at the same time respect current mathematical conventions. One way is to reflect the historical development within the curriculum – emphasize function as covariation in K-14, and then introduce function as correspondence as the need arises." (p. 11)

For students entering calculus, this suggested shift towards a curriculum that emphasizes the covariation definition (even just some of the time) can be beneficial to their learning. A majority of the key concepts in beginning calculus revolve around the idea of what is happening as a function is changing, whether it is as it approaches a particular point, moves through a given interval, or heads off to infinity (College Board, 2010).

In addition to how students define a function, another important aspect of their understanding is what they "see" when they think of a function. This is not the same as how they *view* a function as discussed previously, but rather the physical incarnation of a function that they picture when thinking about functions. For example, Vinner (1983) found that 14% of 146 students in grades 10 and 11 thought of a function as simply an equation, or as Thompson (1994) described it, "two written expressions separated by an equal sign" (p. 5). This is an example of what is referred to as a student's *concept image* of a function (Vinner, 1983; 1992; Vinner & Dreyfus, 1989). Ideally, a student's concept image of a function should align with how he or she defines a function, but this is often not the case. While 57% of the students in Vinner's (1983) study were able to properly define a function, only 34% of those students (19% of the entire sample) were "acting according to" (p. 304) the definition. Vinner and Dreyfus (1989) also found that 57 first-year college students defined functions with the standard correspondence definition, but 56% of those students did not actually use the definition when answering questions that required it. Instead they deferred to their concept images, which were different than the definition they originally gave. For example, they would indicate that a graph is not a function if it has a discontinuity somewhere (such as in $1/x$), or if the same rule does not hold for the entire domain of the function (such as in any piecewise function). This type of issue seems to be common in many students, which is troublesome for their overall understanding of function. For students about to take calculus, it is particularly important that their definition of function and concept image of function be the same and, of course, correct. Otherwise, it could lead to misconceptions, flaws and inconsistencies in their thinking, problem solving, and general understanding of functions themselves, which in turn could hinder their understandings of calculus concepts and solving of calculus problems (Tall, 1992).

2.1.4 Difficulties and Misconceptions

In order to further understand what it takes to develop a strong conception of functions, it is important to identify the kinds of difficulties students have as they attempt to understand them. Many of these difficulties were described in Leinhardt, Zaslavsky, and Stein (1990). At the top of their list was the fact that students often have trouble recognizing certain graphs or expressions as legitimate functions. Oehrtman, et al. (2008) also talked about this issue. One of the primary reasons for this is that students struggle with their concept image of a function, as discussed above. This struggle can lead to several problems. Not only do students often think of a function as two expressions between an equal sign (Vinner, 1983; Vinner and Dreyfus, 1989; Thompson, 1994), but this has also led to the common misconception that a constant expression, such as $y = 5$, is not a function because there is no x. Carlson (1998) encountered this problem first-hand when she found that only 7% of high-performing college-algebra students could correctly answer the question, “Does there exist a function all of whose values are equal to each other?” (p. 121). She also found that 25% of high-performing second-semester calculus students answered the same question by citing the function $y = x$ as an example. Oehrtman, et al. (2008) saw this problem as a byproduct of a curriculum that overemphasizes procedures and algorithms, so students tend to get the notion that dealing with functions requires that one plug in an x to get the y.

Piecewise functions are another example of something students often do not classify as a function (Leinhardt, et al., 1990; Vinner & Dreyfus, 1989). In fact, the reasons for this explain much about how students often think about functions: neither their graphs nor their algebraic forms look “nice”. Basically, students want their functions to a) be defined by a single formula (Carlson, 1998; Oehrtman, et al., 2008) and b) to have graphs that look smooth and follow a clear pattern (Leinhardt, et al., 1990; Vinner, 1983; Vinner & Dreyfus, 1989). This kind of thinking in students leads them to believe that other functions with similar properties, such as discontinuous and discrete functions, are also not functions. Once again, Leinhardt, et al. (1990) suggested that this problem could be a biproduct of the concept image problem:

> “In many cases, students may ‘know’ the accurate, formal definition of a function, but fail to apply it when deciding whether or not a graph represents a function…The majority of examples that students are exposed to are functions whose rules of correspondence are given by formulas that produce patterns that are obvious or easy to detect when graphed. Hence, students develop the idea that only patterned graphs represent functions; others look strange, artificial, or unnatural.” (p. 30-31)

While trouble with identifying functions seems to be one of the primary issues students encounter, it is by far not the only one. Other difficulties listed by researchers include the following:

- Confusion with correspondence, such as thinking a functional relationship only occurs if there is a one-to-one correspondence, or that the function must be symmetrical (Leinhardt, et al., 1990; Vinner, 1983).
- Students' tendencies to gravitate toward linearity when defining, describing, or graphing functions (Leinhardt, et al., 1990; Markovits, Eylon, & Bruckheimer, 1986).
- Difficulty constructing and expressing appropriate notation, both algebraically ($f(x)$ notation), and with the Cartesian coordinate axes and grid, where appropriate scaling can be especially difficult (Carlson, 1998; Leinhardt, et al., 1990; Oehrtman, et al., 2008).
- Difficulty interpreting graphical features of a function, such as focusing on a single point rather than an interval, confusing slope with height, and interpreting a graph as a literal picture, such as a hill (Clement, 1989; Leinhardt, et al., 1990; Monk, 1992; Oehrtman, et al., 2008; Schultz, Clement, & Mokros, 1986).
- Difficulty distinguishing between an equation and the algebraic form of a function (Carlson, 1998; Oehrtman, et al., 2008).

Overall, Leinhardt, et al. (1990) summarized students' difficulties with the function concept as falling into three distinct areas: "a desire for regularity, a pointwise focus, and difficulty with the abstractions of the graphical world" (p. 45). Difficulties with algebraic expressions and notation are a fourth area identified by Oehrtman, et al. (2008). While the first three seem to be mostly conceptual difficulties, the latter is more of a result of students relying too much on memory, rules, and procedures, rather than paying attention to the actual meaning of the function notation (Carlson, 1998).

Difficulty in any of these areas, let alone all of them, does not bode well for students entering calculus. They will frequently encounter unusual functions and graphs and must be able to work with them and understand their properties. They will be expected to interpret what is happening in various intervals and at and around discontinuities, extrema, and other interesting points. They will have to compare and analyze multiple graphs at once, such as those of a function and its derivative. They also must fully understand the standard notation, as it is ubiquitous in the study of calculus, and they will be expected to learn and work with more complex notations as they move forward in the calculus course (College Board, 2010; Sofronos, et al., 2011).

2.2 THE OTHER BIG IDEAS

This section examines the other Big Ideas (2010). It highlights the importance of these ideas and how students should be able to conceptualize them as they are entering calculus. Each subsection describes a) what the Big Idea is, b) how students tend to conceptualize the Big Idea, c) common student difficulties and misconceptions specific to the Big Idea, and d) what the Big Idea means for students entering calculus.

It should be noted that while Cooney, et al. (2010) has ordered the Big Ideas in a specific way, that is not the order in which they will be discussed here. Instead, *Families of Functions* (Big Idea 3) will come first, as it is the simplest of the Big Ideas, and the discussion on them is short. This is followed by discussions of *Covariation and Rate of Change* (Big Idea 2) and *Multiple Representations* (Big Idea 5), which are ordered the same as they were by Cooney, et al. Finally, *Combining and Transforming Functions* (Big Idea 4) was saved for last, as it is the most complex of the Big Ideas and requires a strong understanding of functions in general (Baker, Hemenway, & Trigueros, 2001; Lage & Gaisman, 2006) and a full comprehension of most of the other Big Ideas (Confrey & Smith, 1991).

2.2.1 Families of Functions

Families of Functions is the third of NCTM's Big Ideas (Cooney, et al., 2010). It is based on the idea that there are several different function types, referred to as families, each with their own characteristics and properties. These include linear functions, quadratics, exponentials, rational functions, and trigonometric functions, and they are defined by their unique rates of change (e.g., linear functions have a constant rate of change, while quadratics have a linear rate of change).

Understandings of each of these functions are specifically identified both in the prerequisites for AP Calculus (College Board, 2010) and in the Common Core State Standards (CCSSI, 2009), and it is expected that students entering calculus should know these functions well. Therefore, there is little need to delve too deeply into students' conceptualizations, views, and difficulties with specific function types. Covering them all would prove to be too extensive, time consuming, and would reveal little about students' *general* understanding of functions. It is much more important to instead discuss students' conceptualizations of functions through the other Big Ideas, as these conceptualizations apply to *all* functions.

However, it is important to note that evidence has turned up in the literature that discusses students' conceptualizations, views, and difficulties of specific function families. For instance, students need to have strong understandings of certain concepts in order to attain deeper conceptualizations of specific function families, such as rate for linear functions (Thompson & Thompson, 1992), recursion for exponential functions (Confrey & Smith, 1994), and periodicity for trigonometric functions (Dreyfus & Eisenberg, 1980). Also, it has been shown that students can conceptualize certain functions differently, as Eisenberg & Dreyfus (1994) and Baker, et al. (2001) found that students can view linear functions and quadratics as objects while simultaneously viewing other function types like higher-order polynomials as processes or actions. Finally, Baker et al. (2001) found that students tend to think that certain rational functions are linear because the x, though found in the denominator, has no exponent, when in actuality its exponent is -1. This finding is an example that students can have difficulties and misconceptions when dealing with specific types of functions.

For students entering calculus, understanding different function types is very important, as they will be working with essentially every standard type of function. An understanding of their respective properties will especially help them as they learn limit concepts, and every function type has both a differentiation and integration rule that goes with it.

2.2.2 Covariation and Rate of Change

The second of NCTM's Big Ideas (Cooney, et al., 2010) focuses on *Covariation and Rate of Change*. This idea has historically received little attention in the curriculum before calculus (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994; Thompson, 1994). Despite this issue, many researchers have emphasized covariational reasoning as extremely important, both as a prerequisite for calculus, and also as a means for understanding functions in general (Carlson, et al., 2002; Confrey & Smith, 1994; Herbert, 2008). For instance, Oehrtman, et al. (2008) identified covariational reasoning as one of two conceptualizations (a process view is the other) that students need in order to understand and use functions. Confrey and Smith (1994) advocated for covariation and rate of change to be taught as an entry point to learning about functions:

> "Arguing against the delaying of the introduction of rate of change issues until calculus, where it becomes studied as a 'property' of functions, we have shown that students exhibit strong intuitive understandings of change and can use this understanding to generate functional relationships." (p. 137)

They went on to suggest that students who learn a covariation approach (as opposed to the conventional correspondence approach) find it "easier and more intuitive" (p. 137), and that such an approach can lead to the construction of the correspondence definition of function. They also argued that such an approach illuminates the notion of rate of change, and makes it a more important part of their mathematical learning experience.

However, Herbert (2008) suggested that introducing rate for this purpose is no easy task. She noted that in order for students to learn the concept of rate, they must have strong proportional reasoning skills, an area many students struggle with (Carlson, et al., 2002). Thus, Herbert claimed that it is not surprising that many students also have difficulty with the rate concept. Essentially, she argued that, while using covariation and rate as a gateway to functions can be beneficial to students, actually doing so is not trivial, and difficulties should be expected.

One of the primary benefits students gain from a strongly developed covariational reasoning skillset is that it prepares them very well for calculus. Carlson, et al. (2002) identified it as one of the foundational skills needed for calculus, and others have discussed its importance for understanding specific calculus concepts such as the limit (Cottrill, et al., 1996), the derivative (Zandieh, 2000), and the Fundamental Theorem of Calculus (Thompson, 1994).

So what does it mean for students to understand covariation? Confrey and Smith (1994) identified three components of covariation that students should be able to learn. The first is that covariation is a "unit by unit comparison", in that as one unit changes, so does another. Second, students need to be able to intuitively understand variation in rates. They need to be able to recognize when a rate (such as the speed of a car) is increasing, decreasing, or constant, and to be able to compare one rate to another (such as varying gusts of wind) as "more", "less", or "the same." Finally, students need to be able to understand the above rate concepts graphically by their connection with slope. Confrey and Smith argued that combining these three conceptualizations together will give a student a rich understanding of rate and covariation which will serve them well as they enter calculus (Figure 2.2).

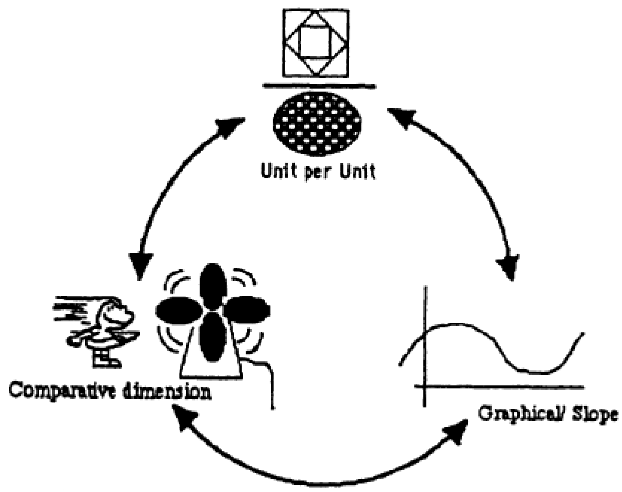


Figure 2.2. Conceptualization of rate as depicted in Confrey and Smith (1994).

Carlson, et al. (2002) also devised a framework that describes students' understanding of covariation. This framework identified five mental actions of increasing difficulty (Table 2.1), and five levels of covariational reasoning that students reach based on how many of those mental actions they can perform. The mental actions in question are as follows: 1) coordinating the dependence of one variable on another, 2) coordinating the direction of change of a variable as another changes, 3) coordinating the amount of change in a variable as another changes, 4) determining the average rate of change over a given interval, and 5) determining the instantaneous rate of change of a function and identifying its relationship to the rest of the function. The reasoning levels are aptly named *Coordination*, *Direction*, *Quantitative Coordination*, *Average Rate*, and *Instantaneous Rate*. It is expected from this framework that students entering calculus who exhibit an ability to apply Level 5 reasoning will do very well for themselves in the course.

Table 2.1. Mental Actions of Carlson, et al.'s Covariation Framework.

Mental Action	***Description of Mental Action***	***Behaviors***
Mental Action 1 (MA1)	Coordinating the dependence of one variable on another variable	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing a monotonic straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable	• Constructing secant lines for contiguous intervals in the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct)

Note: Table is as depicted in Carlson, et al. (2002).

2.2.3 Multiple Representations

That functions can be represented in multiple ways is the fifth of NCTM's Big Ideas (Cooney, et al., 2010). *Multiple Representations of Functions* gives students access to different aspects, properties, and understandings of functions. The ability to recognize, construct, interpret, and manipulate functions in multiple forms, such as formulas, graphs, and tables, helps students enhance their overall conceptual understanding of functions (Cooney, et al., 2010; Kaput, 1998; Keller & Hirsch, 1998). Other critical skills that contribute to this include moving fluidly from one representation to another (Cooney, et al., 2010; Moschkovich, Schoenfeld, & Arcavi, 1993), and identifying which representation is the most appropriate for a particular situation or problem (Cooney, et al., 2010).

Much of the research on multiple representations deals with identifying which types of representations students prefer. In a study of 178 high school students of varying levels (Algebra 1 to AP Calculus), Knuth (2000) found that over 75% of the students, regardless of level or year, tend to use algebraic or analytical representations. He also indicated that many of these students did not even consider graphical solutions as an option to solving problems. He identified several possible factors that may have contributed to his results, such as the following:

- The problems asked for an exact solution, which students felt more comfortable obtaining algebraically.
- A curricular and instructional emphasis on algebra has conditioned the students to rely on algebraic methods to solve problems.
- Students are used to performing formula-to-graph translations, but not the other way around.

Eisenberg (1992) also noted that students did not consider graphical solutions when solving problems, and suggested that this happens even when the problems were designed to force students to think this way:

> "Students have a strong tendency to think of functions algebraically rather than visually. Moreover, this is so, even if they are explicitly and forcefully pushed towards visual processing," (p. 165).

Eisenberg saw this as a big problem, especially since he insisted that the ability to visualize functions is an important component of a well-developed function sense. He suggested several reasons for students' reluctance to think visually, such as a belief that doing so is not a central feature of mathematics, or that thinking visually requires higher-level thought processes and skills than working in an algebraic representation does.

In a study of first-semester college calculus students, Keller and Hirsch (1998) also looked at students' preferences of three representations: tables, graphs, and algebraic formulas. However, unlike Knuth and Eisenberg, they found that these preferences depended on particular situations. When the problems were of a more pure mathematical form and/or used more formal mathematical language, 68% of students that had a significant preference for algebraic representations. In more contextualized problems with less formal language, 52% of such students preferred to use graphs. Also, the majority of students (56%) had no significant preference when the problem was situated within a context. In contrast, when the problems were purely mathematical, only 29% of the students had no significant preference of function representation. Keller and Hirsch summed up their findings as such:

> "It is possible that the availability of multiple representations within the mathematics classroom allows students to tie higher order thinking skills developed in contextualized settings to purely mathematical settings. It is also possible that the availability of multiple representations within the mathematics classroom changes the perceptions of students regarding the form of acceptable solutions." (p. 15)

However, it should be noted that the students in this study were asked to identify which of the three representations they would use to solve each problem they were given, rather than simply solving the problems. This differs from Knuth (2000), who had his participating students solve the problems in any way they chose without prior consideration of multiple representations.

Most studies that focused on multiple representations of functions, such as Knuth (2000) and Eisenberg (1992), have focused primarily on formulas and graphs. This is because it is these two representations, and the links between them, that Cooney, et al. (2010) identified as especially important for function understanding. So how do students conceptualize these two representations? What kinds of difficulties do students have with each? To answer these questions, each representation will be discussed separately.

Ronda (2009) addressed how students conceptualize functions in their algebraic form by identifying growth points in their developing understanding of them. In studying students from 8^{th} to 10^{th} grade, she found four distinct points of growth in students' understanding of formulas, using APOS Theory (Asiala, et al., 1996) as a framework. First, students see formulas as procedures that are used for generating values, consistent with the action view of function. At the second growth point, students begin to see a formula as a representation of a relationship, which is similar to moving from an action view to a process view. At the third growth point, students are able to use the formula to describe properties of the given functional relationship, such as being able to explain the meanings of the coefficient (slope) and the constant term (*y*-intercept) in a linear formula. At the final growth point, students are able to see formulas as objects that can be manipulated and transformed, and thus, they can see functions this way as well. Students who reached this point were capable of performing operations on an entire formula as if it were merely a number.

As for student difficulties, Oehrtman et al. (2008) described two such issues that students have had with the algebraic form of a function. First, students have often had trouble distinguishing between a function and an equation. This stems from the multiple uses of the equal sign, combined with the fact that it is not uncommon in the math community for the word "equation" to be used loosely, with even some of the other studies cited in this paper, such as Ronda (2009), using the word to denote functions in algebraic form. Oehrtman, et al. (2008) went on to describe why it is important to maintain this distinction:

> "For the student, this ambiguous use of the word equation appears to cause difficulty for them in distinguishing between the use of the equal sign as a means of defining a relationship between two varying quantities and a statement of equality of two expressions." (p. 152)

The other common difficulty students have with function formulas is with the notation that is used to denote them (Carlson, 1998; Oehrtman, et al., 2008). They have issues in both understanding the notation when it is in front of them, and using it when trying to express functional relationships. For example, Carlson (1998) found that 43% of 30 high-performing college algebra students, when asked to evaluate $f(x + a)$ for the function $f(x) = 3x^2 + 2x - 4$, simply added an a to the end of the function. Even those that gave a correct response explained in interviews that they were merely following some learned procedure, and showed no indication that they understood that $x + a$ was a new input value for the function f.

As for how students conceptualize graphs, the majority of the research in this area was focused on difficulties and misconceptions. Leinhardt, et al. (1990) described many of these difficulties. One of the most common problems students have with graphs was what Leinhardt, et al. referred to as "iconic interpretations," which means they see a graph as a literal picture.

This is an especially common problem when the graph is being used to model a real-world situation, such as how an object's position is changing over time. For example, Schultz, et al., (1986) found that exactly half of the students who were asked to graph the speed of a bicycle going over a hill (as shown in Figure 2.3) as a function of its position instead produced a graph that also looks like a hill. Similarly, Clement (1989) found that students believed that the position of the cars in the graph in Figure 2.4 would be the same at the point of intersection on the graph. In other words, they believed that car B would catch up to and either pass or collide with car A at the one hour mark. In actuality, the cars' positions would not necessarily be the same at that point, but rather their speeds would be the same. In these examples, students were confusing the points on the graph, which is modeling the velocity of the object, as the literal position of that object at the given time.

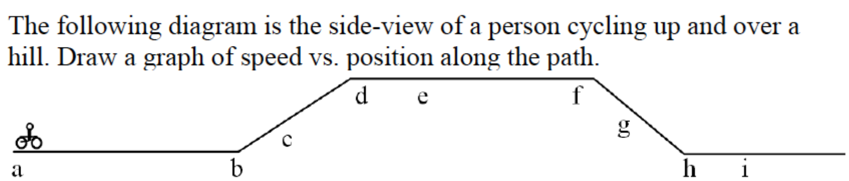


Figure 2.3. Bicycle graph problem as depicted in Oehrtman, et al. (2008).

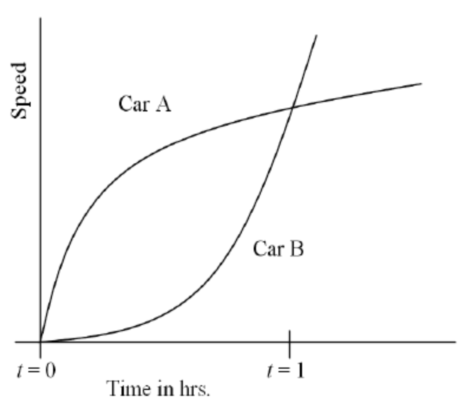


Figure 2.4. Speed vs. Time graph of two cars, as depicted in Oehrtman, et al. (2008).

The above difficulties are evidence of what Monk (1992) called a *Pointwise* view of functions, a conceptualization that sees a function as a series of isolated points or ordered pairs. Monk argued that students with such a conception are most likely to use a table to represent a function. A student with a conceptualization that is beyond a Pointwise approach is said to have an *Across-Time* view of functions. Much like Ronda's framework on algebraic conceptualizations (2009), Monk's framework fits in well with APOS Theory (Asiala, et al., 1996), with the Pointwise view equivalent to the action view, and Across-Time equivalent to process. While the framework applies to functions in general, it is especially strong at explaining students' conceptualizations of graphs, as points and intervals are primarily graphical features.

Having now looked at the various difficulties that students have as they attempt to understand formulas and graphs separately, there is another complication they encounter when linking the two together: moving from one representation to the other (Leinhardt, et al., 1990). More specifically, the problem is when they try to move from a graph to a formula. Essentially, students have little trouble performing translations from a formula to a graph because this can be done in a procedural fashion by producing the requisite ordered pairs by plugging in various values for the independent variable. On the other hand, performing translations from a graphical to an algebraic representation required higher level processing skills such as detecting patterns. This is in line with Eisenberg's (1992) suggestion that students are reluctant to visualize functions because the thought processes are more complex than when working with formulas. Another possible factor is that students are routinely asked to graph formulas by their teachers, but are rarely asked to produce a formula from its graph (Knuth, 2000). However, Markovits, et al. (1986) found this one-way difficulty was only true when the function was familiar to the student. When the given function was less familiar (such as a piecewise function), then the task was equally difficult in both directions.

So why does all this matter for beginning calculus students? For one, multiple representations are very prevalent throughout calculus, particularly in the graphical and algebraic forms, and even tables are commonly utilized to solve problems and construct concepts in calculus. Students are constantly moving between these representations in order to gain a well-formed understanding of concepts such as the derivative (Kendal, 2001; Sofronos, et al., 2011). In fact, Robert and Boschet (as cited in Tall, 1992, p. 6, 8) showed that calculus students who could use a multitude of representations for given problem situations were the most successful. Ideally, students entering calculus should be able to demonstrate an Across-Time view of graphs and view formulas as objects that can be manipulated. They should be able to move fluidly from one representation to another, fully understand function notation, and know how to recognize and choose the best representation for the situation they are working in at a given time.

2.2.4 Combining and Transforming Functions

Combining and Transforming Functions is the fourth of NCTM's Big Ideas (Cooney, et al., 2010), and the final one to be discussed. It is based on the idea that functions can be added, subtracted, multiplied, divided, inverted, and combined with other functions. When this happens, a function's graph can be shifted horizontally or vertically, stretched or contracted, flipped, rotated, or changed in some other way. This Big Idea was saved for last because it is the most complex, and understanding it requires both a strong conception of functions in general (Baker, et al., 2001; Lage & Gaisman, 2006) and a full comprehension of most of the other Big Ideas (Confrey & Smith, 1991). For example, talk of function transformations is usually in the context of the basic function families, such as linear functions, quadratics, trigonometric functions, and exponentials, and how the basic function prototypes of these families change when certain operations are performed on them (Confrey & Smith, 1991). Also, one needs to be able to move between multiple representations frequently in order to understand how a graph changes when a function's formula has been transformed or vice versa, and it has been shown that students have greater success with transformations when they are able to think about them visually (Eisenberg & Dreyfus, 1994). In their study, Eisenberg and Dreyfus gave seven high school seniors various transformation problems to solve. They found that 71% of the problems that were solved visually were correct, while only 22% of those that were solved using non-visual methods were correct.

In addition to having a strong understanding of the other Big Ideas, an object view of functions is required in order to best conceptualize transformations, since they are quite literally actions that are being taken on functions (Baker, et al., 2001; Eisenberg & Dreyfus, 1994). In a study of approximately 240 college precalculus students, Baker, et al. (2001) found that the few students who had an object view of transformations were much more successful with transformation problems than students who held action or process views. They stated that the students "…can identify transformations on any of the given functions and are able to describe the effects of the transformation on the graph and properties of the functions" (p. 45).

Essentially, a full conceptualization of transformations requires a high level of thinking. It is of no surprise, then, that most students have had difficulty understanding them. For example, while visual methods have helped students understand transformations, they preferred to solve such problems algebraically (Eisenberg & Dreyfus, 1994). A possible reason for this is that students are generally more fluent with algebraic methods, and naturally struggle with graphical methods (Lage & Gaisman, 2006).

It has also been found that most students had an easy time understanding transformations of linear and quadratic functions, but exhibited much more difficulty in doing the same thing with other kinds of functions (Baker, et al., 2001, Eisenberg & Dreyfus, 1994). This is because the students had achieved an object view of linear and quadratic functions, but were stuck on a process or action view of other function types, such as higher-order polynomials. This issue led to difficulties, such as seeing two completely different functions as opposed to a basic parent function and its transformation, and indicated that these students either had no conceptualization of transformation at all, or saw transformations as "a sequence of two static states rather than as a dynamic process" (Eisenberg & Dreyfus, 1994, p. 59).

In their study of 16 college precalculus and calculus students, Lage and Gaisman (2006) found that more students (7) held action views of transformations than process (5) or object (4) views, and that has inhibited those students' ability to recognize and explain transformations. The students were still capable of solving problems, particularly in algebraic form, but did so by making use of memorized rules and procedures. This was consistent with several other studies (Borba & Confrey, 1996; Eisenberg & Dreyfus, 1994; Zazkis, Liljedahl, & Gadowsky, 2003) that also found that students relied on rules and algorithms to solve transformation problems, as there were several instances found of students solving such problems correctly, but unable to demonstrate understanding of transformations later on. Eisenberg & Dreyfus (1994) also found the reverse effect of this, in that students who were able to at least recognize certain kinds of transformations still had difficulty solving problems that utilized them.

Another finding that came out of several studies was that certain kinds of transformations proved to be more difficult to comprehend than others. For example, Lage and Gaisman (2006) observed that students saw visual transformations as something that happens to the whole function, and thus failed to think about how the transformation would change each point on the graph. This led to students having little difficulty with "rigid" transformations, or those that moved the graph (e.g., left, right, up or down) but kept it the exact same shape. However, "dynamic" transformations, or those that changed the shape of the graph somehow (e.g., stretching, shrinking), were much more difficult for them, as not every point was transformed in exactly the same way. Also, several studies (Borba & Confrey, 1996; Eisenberg & Dreyfus, 1994; Zazkis, et al., 2003) found that students had an easy time with vertical translations, which move the graph in the direction that is expected (e.g., $f(x) + 5$ would move the graph up five units). However, they had a much more difficult time with horizontal translations, which move the graph in a counterintuitive manner (e.g., $f(x + 5)$ would move the graph five units to the *left*, not the right). Zazkis, et al. (2003) found that even when students get this concept right, they do so by relying on memorized rules such as "plus moves it to the left, minus to the right", and that they did not try to explore *why* this happened, even when prompted.

Finally, several common general issues with transformations found by Lage and Gaisman (2006) were that students had difficulties: a) identifying properties of a given transformation, b) seeing which kinds of transformations were applied to a basic function, and c) predicting how transformations would change certain properties of a given function.

It is easy to see how this collection of difficulties and misconceptions about function transformations highlights just how hard it is for students to reach a full comprehension of them. It is necessary to have an object conception of functions, and extremely helpful to be able to think about them visually. Both of these require a high level of thinking (Breidenbach, et al., 1992; Eisenberg, 1992). Also, it is necessary to have a healthy understanding of each of the other Big Ideas of functions, especially *Multiple Representations* and *Families of Functions.* This implies that a student with a strong conceptualization of function transformations has all of the above understandings and thinking processes, and thus has a strong sense for functions in general. Therefore, it can be insinuated that a student who thoroughly understands function transformations is definitely prepared to take calculus.

Function transformations are also important underlying concepts within the calculus curriculum, and thus it is important for students taking calculus to be able to understand them. Transformations can help students see how certain calculus concepts work, such as the Chain Rule, which is based on function compositions (Clark, et al., 1997), and the Fundamental Theorem of Calculus, which essentially illuminates the inverse relationship between the derivative and the integral (Sofronos, et al., 2011).

2.3 FUNCTIONS AND CALCULUS STUDENTS

Based on the analysis from the previous sections, it is clear that students entering calculus must have a very strong and deep understanding of the many function concepts covered in the Big Ideas. However, there is much research that has shown that calculus students still have many difficulties and misconceptions about functions both during the course and even after they have completed it. For example, in his discussion about visual processing, Eisenberg (1992) highlighted calculus students' reluctance to think of functions visually. Also, Monk (1992) found that the majority of calculus students displayed a Pointwise view of functions (as opposed to Across-Time), and that most of his participants had difficulty interpreting and representing such graphical features as concavity and inflection points. Carlson, et al. (2002) reported a similar finding in a study of 20 students who had completed a 2nd-semester college calculus course. They also showed that the majority of these students were unable to consistently apply covariational reasoning beyond that of Level 3 (Quantitative Coordination; refer to Table 2.1), and while they had occasional success at Level 4 (Average Rate of Change), they rarely if ever were able to apply Level 5 reasoning (Instantaneous Rate of Change). In other words, the students had difficulty "explaining why a curve is smooth and what is conveyed by an inflection point on a graph" (pg. 373).

Lage and Gaisman (2006) found that most of the eight calculus students in their study viewed function transformations as either actions or processes, and that almost every student who demonstrated an object view of transformations had already *completed* a calculus course. In a rare study of high school calculus students, Judson and Nishimori (2005) also found similar results. They compared 18 AP Calculus BC students and 26 Japanese *Suugaku* 3 (calculus) students, and noted that, "almost all students lacked a sophisticated understanding of functions" (p. 39). Finally, Williams (1998) conducted a study of 28 college calculus students that asked them to produce their respective concept maps of functions. Among Williams' findings were the following:

- Many students included trivial or irrelevant information related to functions, such as which letters are usually used to represent a variable;
- Few students considered including function families, properties, or transformation types in their concept maps;

- Most students tended to think of a function as an equation;
- Most students' definitions were based on the notion of "domain and range";
- There were no hierarchies or connections between concepts.

Basically, Williams found that calculus students' conceptions of functions based on their concept maps were superficial, disconnected, and extremely limited.

However, there is also evidence in the literature that suggests that students have been able to demonstrate success in calculus or with calculus concepts without necessarily fully understanding functions. For example, both Clark, et. al., (1997) and Cottrill (1999) conducted studies that showed that at least some students were capable of successfully applying the Chain Rule without understanding how its primary underlying principle, function compositions, actually work. Kimani (2008) also showed that many first year calculus students held minimal understandings of function transformations, compositions, and inverses, and the relationships between those concepts. Yet they were able to successfully solve problems while relying heavily on following memorized rules and procedures. Finally, Carlson (1998) found that even students who had just completed second-semester calculus with an A had achieved only limited process views of function, as they did not demonstrate many skills beyond a high proficiency with computations. They still had difficulties understanding several aspects of functions, such as covariation (particularly as it pertains to graphs), piecewise functions, and function notation, and they only tended to demonstrate a process view on less difficult problems:

> "Although second-semester calculus students had begun to demonstrate a more general view of functions, when confronted with more demanding problems, they had a tendency to regress to lower level skills and, at times, demonstrate weak understanding…[these] students appeared to be in transition from an action view to a process view of functions." (p. 139)

Carlson's findings were particularly significant, especially when combined with her comparisons to the function conceptions of students in a college algebra course (action views) and those in a first-year mathematics graduate course (strong process views):

> "Results indicate that as students progress through the undergraduate mathematics curriculum, function constructs develop slowly. Although students are eventually able to use concepts taught previously, even for our best students, complex concepts are slow to develop and new information is not immediately accessible." (pg. 137)

This result is in line with Eisenberg's (1992) assertion that developing a strong conception of functions can take a very long time, and seems to indicate that much actual learning of functions occurs during calculus and other higher level courses rather than in the courses that precede it, even though those are the courses where functions are explicitly taught.

Despite this evidence that students can succeed in calculus without a strong conception of functions, most studies of students' difficulties with calculus concepts have found strong correlations between skill with calculus concepts and conceptual understanding of functions. For example, Asiala, et al. (1997) analyzed 41 college calculus students' graphical understanding of function and of the derivative, using APOS Theory as a theoretical framework. Twenty-four of the students had taken a traditional calculus course, while the other 17 were in a reform calculus course that spent extra time developing students' understanding of functions. They found that all 17 reform students had developed a process conception of functions, and had demonstrated a more thorough graphical understanding of functions and their derivatives than those in the traditional course. The traditional students also could not solve certain problems without the use of a formula, which suggests they possessed an action view and a lack of flexibility with multiple representations. These results led them to make the following remarks:

> "Based on the results of this study and other recent work on the function concept, we believe that our instructional treatment of functions, or one similar to it, is necessary for most college freshmen entering calculus. This necessity cannot be over-emphasized: pre-college mathematics courses must address the well-documented problem of students' inadequate preparation for calculus (and other higher-mathematics courses) with respect to the function concept." (p. 427)

Ubuz (2007) also focused on students' graphical interpretations of the derivative, and similarly called for enhanced function understanding, particularly in making connections between graphical and symbolic representations. Clark, et al. (1997) found that students' difficulties with the Chain Rule can be attributed to their struggles with function compositions, and that the schema they build for the Chain Rule "must contain a function schema which includes at least a process conception of function, function composition and decomposition" (p. 13). Finally, Gur and Barak (2007) identified the inability to recognize the correct function type and difficulties with function compositions as two of the most common errors students tend to make when working with derivatives.

2.4 SUMMARY

The literature on students' conceptions and understandings of each Big Idea has helped paint a clearer picture of the kinds of understandings of functions students need in order to truly succeed in calculus. These understandings are summarized below:

- Students must be able to view functions as processes at the very least. They should understand both the covariance and the correspondence definitions of a function, which should be aligned with a well-defined concept image of a function. They also must demonstrate fluency in working with function notation.

- Students should have an understanding of the different function types and their respective properties. This includes linear functions, quadratics, higher-order polynomials, rational functions, trigonometric functions, exponentials, and logarithms.

- Students should be able to understand the notions of covariation and rate of change. They should be able to conceptualize rate in three different ways: they should recognize it is a unit-by-unit comparison, they should be capable of identifying variation in rate, and they should understand rate's graphical connection to slope (Confrey & Smith, 1994). They also should be able to consistently demonstrate Level 5 covariational reasoning skills (Carlson, et al., 2002).

- Students should be capable of visualizing functions and moving fluidly between different function representations in any direction. They should have an Across-Time view of graphs (Monk, 1992) and view formulas as objects that can be manipulated and acted upon. They should also be able to recognize and choose the best representation for the situation in which they are currently working.

- Students should be able to understand the different types of function transformations, and how algebraic transformations are connected to graphical ones. This includes predicting how a transformation has changed a function, and recognizing a transformation after it has occurred.

It should also be noted that a full understanding of function transformations implies that students perceive functions as objects, are capable of visualizing functions, can move fluidly between multiple representations, and know their function prototypes. Essentially, a student with these conceptualizations is primed to succeed in a calculus course.

However, the evidence from studies of actual calculus students imply that many students are getting through calculus with a less-than-ideal understanding of functions, even though it is also clear that certain function understandings are related to understandings of calculus concepts. However, the majority of these studies have taken place at the college level. There is still much to be learned about AP Calculus students and what they need to know about functions in order to succeed in the course. The proposed study has been designed to investigate this question, and to uncover more about what AP Calculus students know about functions and how that relates to their performance in the course and on the exam.

3.0 METHODOLOGY

This study was designed to determine AP Calculus students' understandings of functions, and to compare those understandings to their performance on the AP Calculus Exam. The specific research questions that this study addresses are:

1) *What do students completing the AP Calculus course know and/or understand about functions?*
    a. *To what extent can AP Calculus students solve problems about functions?*
    b. *To what extent can AP Calculus students explain their thinking about functions?*
    c. *What view of functions do AP Calculus students hold?*
    d. *To what extent do AP Calculus students understand each of the Big Ideas of functions?*
2) *To what extent is there alignment between AP Calculus students' understanding of functions and their performance on the AP Calculus Exam?*
    a. *What is the relationship between students' understanding of functions and their performance on the AP Calculus Exam?*
    b. *What is the relationship between students' view of functions and their performance on the AP Calculus Exam?*
    c. *What is the relationship between students' understanding of each of the Big Ideas of functions and their performance on the AP Calculus Exam?*

Details regarding the study are provided in the following sections. The first section provides information about the participants. The second section contains an explanation of the data collection instruments: the Precalculus Concept Assessment (PCA), the interview process, and the AP Calculus Exam scores. The third section contains explanations of how the data will be coded and analyzed in order to answer each of the research questions.

3.1 PARTICIPANTS

The participants for this study were 85 students from three different high schools enrolled in AP Calculus AB in the 2012-13 academic school year. Sixty-seven of the 85 participants took the 2013 AP Calculus Exam on May 8th, 2013. Sometime between May 8th and the end of the school year, they were administered the *Precalculus Concept Assessment*, (Carlson, Oehrtman, & Engelke, 2010), henceforth referred to as the PCA. In order to take the PCA and therefore take part in the study, either the participants (if at least 18 years old) or their parents (for those still under 18) had to agree to release to the researcher their AP Calculus Exam scores once they were made available. These scores were sent to the researcher by the teacher at the end of July, after all other data had been collected and the PCA had been scored.

Additionally, the participants who were at least 18 years old were also asked if they were willing to take part in the interview portion of the study. Of those who agreed to participate, the researcher attempted to select three from each school based on their scores on the PCA. However, at one of the three schools, only two participants volunteered to be interviewed. So in total, eight participants took part in the interviews. These participants were compensated $50 each for participating in the interview. The interviews took place between June 25th and July

22nd, after the administration and scoring of the PCA, but before the AP Exam scores were released to the researcher. Each interview was audio-recorded and transcribed for analysis.

Finally, background information was collected, most of which was at the outset of the study. The participants' demographic information (grade, gender, and race/ethnicity) was collected from the students themselves, as part of the PCA. Table 3.1 displays the collected demographic information. As can be seen in the table, each school site had at least 20 participants. The majority of the participants (73%) were seniors. School A consisted of all seniors, School B had mostly seniors and a few juniors, and School C had close to an even split between juniors and seniors. There were only slightly more females (52%) in the study than males, and all three sites had approximately the same number of each. Most of the study participants were Caucasian (72%). Among the minorities, all 6 African American participants, as well as the lone Native American, came from School C, while 6 of the 10 Asian/Pacific Islander participants came from School B. Six students identified themselves as "Other", and there were no Hispanic participants.

3.1.1 The Schools and Teachers

The three schools sites in this study will be denoted as Schools A, B, and C. The schools were selected to participate in the study due to convenience to the researcher. The AP Calculus teachers and principals at each school were contacted in order to gauge interest in the study and to receive permission to conduct it at the school site. All three schools are public high schools, two of which are suburban high schools in a greater metropolitan region, while the third is located in a more rural area. The AP Calculus classes are considered to be typical in that they are designed to prepare students to take the AP Calculus Exam.

Table 3.1. Demographic information of study participants.

		School A	School B	School C	Total
Total		20	27	38	85
Grade	Senior	20	22	20	62
	Junior	0	5	17	22
Gender	Male	10	11	19	40
	Female	10	16	18	44
Racial/Ethnic Background	Caucasian	17	17	27	61
	Asian/Pacific Islander	3	6	1	10
	African American	0	0	6	6
	Native American	0	0	1	1
	Hispanic	0	0	0	0
	Other	0	4	2	6

Note: One student from School C did not provide any demographic information.

Background information was collected from a survey distributed to the AP Calculus teacher at each school, and included the following:

- The size of the school (total number of students)
- Racial/ethnic breakdown, by percent, of each school's population
- The AP Calculus teacher's total years of experience, and his or her total years teaching AP Calculus
- Gender and racial/ethnic breakdowns of each AP Calculus class
- The frequency (times per week) and length of time (in minutes) of each meeting of the AP Calculus course.

Additional information was collected from the students who participated in the interviews about their perceptions of their AP Calculus class. Based on the information collected from the AP teacher and the interview participants, each school site is briefly described below.

School A: There were 20 students from School A that participated in the study. The school had a population of approximately 500 students, only about 1% of which were racial and ethnic minorities.

The AP Calculus teacher at School A had been teaching for 33 years, and she had been teaching AP Calculus for 20 years. Among the comments from the interview participants was that she was "vibrant and upbeat" and also "eclectic." They also said that she was very open to questions, even outside of class, and she commonly remained after school to assist students who wanted to get extra help.

The class itself met three times a week for 80 minutes each. The class had 23 total students, three of whom were absent on the day the PCA was given, and therefore did not participate in the study. All of the students were seniors, and there were slightly more male students than females. There were only three minorities in the class, all of whom were Asian or Pacific Islanders. Lessons were presented "by the book," according to one student. According to all three interview participants, the teacher would first go over previous homework, then present material along with 2 or 3 example problems, give the students opportunities to work through further problems, and then assign 20 to 30 problems for homework.

The interview participants also said that the teacher emphasized preparing for the AP Calculus Exam throughout the year, and that she only taught material that was going to be on the exam. Beginning in the second half of the year, she regularly handed out review packets in order to keep the material fresh for the students, and once a week she held sessions that specifically focused on preparing for and practicing the open-ended exam questions.

School B: There were 27 students that participated in the study from School B. The school had a population of 851 students in total, 12% of whom were minorities.

The AP Calculus teacher at School B was in his 6th year of teaching the course, and had been teaching for 20 years in total. Included in the comments from the interview participants at School B were that he was "engaging and fun", that he was very good at explaining concepts, and that he maintained a consistent classroom routine.

There were two sections of the AP Calculus course at School B, for a total of 34 students. Seven of these students were absent the day the PCA was administered and did not take part in the study as a result. Only six of the participating students were in their junior year, the rest were seniors. The female-to-male ratio was approximately 3-to-2. Almost two-thirds of the students were Caucasian, and ten of the twelve minority students were of Asian or Pacific Islander descent. Each class met 5 times a week for 40 minutes. According to the interview participants, the typical lesson was a "straightforward lecture" that involved the use of a smartboard, and it included going over material and trying examples with the students. Homework was assigned afterward and the students were given two days to complete it, with a chance to ask questions about it after the first day.

The interview participants also said that preparation for the AP Exam consisted of review and practice in the month leading up to the exam, along with brief explanations after each lesson throughout the year on how the new material could be applied to problems on the exam. One of the participants said that the exam problems were completely different than what she was used to seeing all year, and that she particularly struggled with the "complicated word problems" that appeared on the exam.

School C: There were 38 students that took part in the study from School C. The school was the largest of the three sites, with a population of 1243 students. The school was also the most diverse, with minorities making up between 55% and 60% of the student population.

The AP Calculus teacher at School C had been teaching the course for all 9 years of his teaching career. The interview participants from School C said they found him to be very organized, demanding, and "straight to the point." They also said that he moved at a quick pace, but was very good at explaining the materials and making sure his students got the help they needed.

There were two sections of AP Calculus at School C, with a total of 39 students. There were almost as many juniors in the course as there were seniors, and there was also an even split between males and females. There were only 9 minorities, most of whom were African Americans. Each section met 5 times a week for 45 minutes each. According to the interview participants, lessons often began with the presentation of a real-world context relevant to the material to be taught, followed by the presentation of content, an example or two, and then time given to the students to work on problems on their own, with help from the teacher as needed. Homework was assigned each night and was optional, although the teacher expected the students to do it.

Students in the course were not required to take the AP Calculus Exam, but the teacher offered as a final exam an official AP Calculus practice exam that is authored by the College Board and released only to AP certified teachers. The interview participants said that preparation for the AP Exam persisted throughout the year, and the two weeks preceding the exam consisted solely of exam preparation. This included reviewing all of the material, quizzes and practice exams. One student referred to the review period as “sort of intense” but “simple and straightforward.” Another student stated that he chose not to buy a test preparation booklet because he felt the course was preparing him well for the exam.

At Schools A and B, all students were required to take the AP Calculus Exam, but at School C, the exam was optional. Therefore, while 38 students from School C took the PCA and participated in the study, only 20 of them actually took the AP Exam.

3.1.2 Anonymity and Confidentiality of the Participants

With the exception of the interviewees, all AP Calculus students who participated in the study remain anonymous. This was ensured by the following procedures. First, the students’ AP Calculus teacher received copies of the PCA from the researcher. Each PCA was numbered, and the teacher recorded the numbers next to the corresponding names of the students. The numbered exams were returned to the researcher. Once the AP Exam scores were released, the teacher sent the scores alongside the corresponding exam number of each student. Using this process, the researcher was not able to attach student names to AP scores for the majority of students.

At the time the test was administered, students were given a separate sheet that also had the number corresponding to their test and provided information about the interview. Students checked “yes” or “no”, indicating their willingness to participate in the interview. They were told that they must be 18 and provide their name and contact information. This information was only shared with the researcher if the student was selected for an interview. The teacher let the researcher know which students agreed to be interviewed by placing a red dot or other mark on their PCA test. The teacher only provided the identity and contact information for the selected students.

The identities of the students who participated in the interview portion of the study remain confidential. They were given pseudonyms in all written reports of the research and the original tests and identifying information will be kept in a locked file cabinet. All files of the audio recordings will be stored on a data CD that will also be kept in the file cabinet.

3.2 DATA COLLECTION

Three primary data sources were used to answer the research questions: the PCA, interviews, and AP Calculus Exam scores. The PCA and interviews were used in order to determine the strength of each student's understanding of functions. Once these understandings were determined, they were then compared to the AP Exam scores as soon as those scores were released to the researcher. Table 3.2 shows how the data helped answer each research question.

3.2.1 The Precalculus Concept Assessment

The primary means for data collection was through the administration of the PCA. The PCA is a 25-item multiple-choice examination designed to assess students' knowledge that is said to be "foundational for students' learning and understanding of central ideas of beginning calculus" (Carlson, et al., 2010, p. 115).

Table 3.2. Collected data for analysis.

Research Questions	Data Sources	Data to be Analyzed
1a) To what extent can AP Calculus students solve problems about functions?	PCA	All participant responses to the PCA items
1b) To what extent can AP Caculus students explain their thinking about functions?	Interviews	Transcripts of participant responses to the interview questions along with any written work produced during the interview.
1c) What view of functions do AP Calculus students hold?	PCA, Interviews	All participant responses to the process view PCA items, transcripts of participant responses to interview question 1 along with any written work produced in the interview.
1d) To what extent do AP Calculus students understand each of the Big Ideas of functions?	PCA, Interviews	All participant responses to PCA Items for each Big Idea, transcripts of participant responses to each interview question for each Big Idea, along with any written work produced in the interview.
2a) What is the relationship between students' overall understanding of functions and their performance on the AP Calculus Exam?	PCA, Interviews, AP Exam scores	All participant responses to the PCA items, Transcripts of participant responses to the interview questions along with any written work produced in the interview, AP Exam scores
2b) What is the relationship between students' view of functions and their performance on the AP Calculus Exam?	PCA, interviews, AP Exam scores	All participant responses to the process view PCA items, transcripts of participant responses to interview question 1 along with any written work produced in the interview, AP Exam scores
2c) What is the relationship between students' understanding of each the Big Ideas of functions and their performance on the AP Calculus Exam?	PCA, interviews, AP Exam scores	All participant responses to PCA Items for each Big Idea, transcripts of participant responses to each interview question for each Big Idea, along with any written work produced in the interview., AP Exam scores

The PCA was selected because it is a well-developed instrument with a long history. It has undergone several iterations of development and validation over the past 15 years. This process included the conducting of several studies aimed at understanding the reasonings and understandings that are considered foundational to students' learning of precalculus and calculus. From those studies, the original draft of the PCA taxonomy was written, and a series of open-ended questions was constructed around that taxonomy. Carlson and her colleagues used the open-ended questions to gather test and interview data about how students commonly answered those questions, and used that data to help refine the taxonomy and revise the open-ended questions into multiple-choice items. Finally, the full 25-item multiple-choice assessment was validated and meaning was given to the scores. The current version of the PCA as it was used in this study is its eighth iteration.

The taxonomy of the PCA is primarily centered around functions, and is split into six categories: process view of function, covariational reasoning, computational abilities, function concept, growth rate of function types, and function representations. The first three categories were classified as *reasoning abilities* while the final three are considered to be *understandings.* Table 3.3 provides definitions for each category and highlights the items that help assess each particular reasoning ability or understanding. Every item on the exam assesses at least one reasoning ability and at least one understanding.

In Chapter 2, the argument was made that a student's understanding of functions in general is defined by his or her understanding of each of NCTM's Big Ideas of functions (Cooney, et al., 2010). Therefore, it was important to map the PCA items and taxonomy to the Big Ideas themselves, as well. *The Function Concept* (Big Idea #1) is covered by the *process view* items (R1 in the PCA taxonomy) and *function evaluation* (ME). *Covariation and Rate of Change* (BI #2) align with the *covariational reasoning* (R2) and *rate of change* (MR) items. *Families of Functions* (BI #3) are covered by each of the four understandings under *growth rate of function types* (GL, GE, GR, and GN). *Combining and Transforming Functions* (BI #4) is assessed by the *function composition* (MC) and *function inverse* (MI) understandings. Finally, *Multiple Representations* (BI #5) is aligned with the four understandings found in *understand functional representations* (RG, RA, RN, and RC). However, it is important to note that all twenty-five items are related to at least one of these understandings. Therefore, the items specifically identified for this Big Idea are those items that fall under at least two of the four understandings. For example, item #10 falls under both the graphical and algebraic understandings, so it would be included as an item that assesses *Multiple Representations.*

Table 3.3. PCA Taxonomy.

Reasoning Abilities	
R1	*Process view of function* (items 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 20, 22, 23) View a function as a generalized process that accepts input and produces output. Appropriate coordination of multiple function processes
R2	*Covariational reasoning* (items 15, 18, 19, 24, 25) Coordinate two varying quantities that change in tandem while attending to how the quantities change in relation to each other
R3	*Computational abilities* (items 1, 3, 4, 10, 11, 14, 16, 17, 21) Identify and apply appropriate algebraic manipulations and procedures to support creating and reasoning about function models
Understandings	
Understand meaning of function concepts	
ME	Function evaluation (items 1, 5, 6, 11, 12, 16, 20)
MR	Rate of change (items 8, 10, 11, 15, 19, 22)
MC	Function composition (items 4, 5, 12, 16, 17, 20, 23)
MI	Function inverse (items 2, 4, 9, 10, 13, 14, 23)
Understand growth rate of function types	
GL	Linear (items 3, 10, 22)
GE	Exponential (item 7)
GR	Rational (items 18, 25)
GN	General non-linear (items 15, 19, 24)
Understand function representations (interpret, use, construct, connect)	
RG	Graphical (items 2, 5, 6, 8, 9, 10, 15, 19, 24)
RA	Algebraic (items 1, 4, 7, 10, 11, 14, 16, 17, 18, 21, 22, 23, 25)
RN	Numerical (items 3, 12, 13)
RC	Contextual (items 3, 4, 7, 8, 10, 11, 15, 17, 18, 20, 22)

Note: Table is as depicted in Carlson, Oehrtman, & Engelke (2010), p. 120.

Note that the third reasoning ability, *computational abilities* (R3) is not aligned with any of the Big Ideas of Functions, as it merely describes the use of algebraic manipulation and procedures rather than any conceptual understanding of functions themselves. Table 3.4 contains the lists of the PCA items that are aligned with each of the Big Ideas based on the mapping described above.

Finally, the sixteen items under the R1 reasoning ability, *process view of functions*, in the PCA taxonomy are based on APOS Theory (Asiala, et al., 1996). These items were designed to assess whether a student maintains an action or process view of functions, and will be analyzed for the same purpose in this study.

Table 3.4. PCA Items as aligned with each Big Idea.

Big Idea	PCA Items
1. The Function Concept	1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 20, 22, 23
2. Covariation and Rate of Change	8, 10, 11, 15, 18, 19, 22, 24, 25
3. Families of Functions	3, 7, 10, 18, 22, 25
4. Combining and Transforming Functions	2, 4, 5, 9, 10, 12, 13, 14, 16, 17, 20, 23
5. Multiple Representations	3, 4, 7, 8, 10, 11, 15, 17, 18, 22

The PCA was administered to the participants by their AP Calculus teacher some time after they took the 2013 AP Calculus Exam, but before the end of the school year. Upon administration of the exam, the participants were given both written and oral instructions to not only indicate their answer choice for each question, but to explicitly show the work they did to reach that answer choice. Scratch paper was provided to the participants for this purpose. They also could show their work directly on the PCA.

3.2.2 Interviews

In addition to the responses to the PCA provided by all study participants, three participants from Schools A and B and two participants from School C were also selected for in-depth interviews in order to gain further insight into their understanding of functions. This was done for two reasons. First, the PCA does not cover the entire scope of function understanding. For example, the PCA has no questions that are designed to assess a student's understanding of function transformations, an important aspect of Big Idea 4. The interviews allowed the researcher to assess those aspects of functions that are missing from the PCA. Second, the interviews were designed so that the participants could explain their thinking and reasoning about functions in ways that they were not able to on the PCA. This included the opportunity to ask clarifying questions and to think out loud as they worked on tasks. In this way, the student had more ways to access their understandings of functions than on the PCA, and the researcher had more information that could be used to accurately assess a participant's understanding of functions.

The interview participants from each school were selected from those who agreed to be interviewed, and their selection was largely based on their performance on the PCA. That is, the selected participants were those with the median score for each of the three PCA functional understanding categories: *strong, moderate,* and *weak* (these categories will be described in greater detail in the coding and analysis section of this chapter). Therefore, at each school, one student who was coded as *strong, moderate,* and *weak,* respectively, based on their PCA scores were to be selected. If no student at a particular school was coded as *strong*, then the highest PCA score of those who agreed to interview was selected. Likewise, if no student was coded as *moderate*, then the overall median score was selected, and if no student was coded as *weak*, then the lowest score was selected. If three or fewer students at a particular school volunteered to be interviewed, then all of those students were selected regardless of their PCA scores. This happened at School C, where only two students volunteered to be interviewed, both of whom were classified as *weak* on the PCA. At Schools A and B, there were enough student volunteers that one from each category was selected to be interviewed at each school.

The interview began with some brief preliminary questions that related to their perceptions of their AP Calculus class, such as what a typical lesson is like and what they like and dislike about the class. This was done in order to gain further information about student perceptions of the course, the teacher, and the class environment, and the responses were already discussed earlier in this chapter. After that, the interview consisted of five parts: one corresponding to each of the Big Ideas. Each part had one or more questions and/or tasks that pertained to the respective Big Idea. Also, follow-up questions that focused on the students' thinking and decision-making, such as "Why did you do that?" and "How do you know?" were asked throughout each part of the interview.

In part 1, participants were asked to give their definition of a function (Breidenbach, et al., 1992), and then asked to identify which of several given relations are functions (Cooney, et al., 2010; Oehrtman, et al., 2008; Slavit, 1997; Vinner & Dreyfus, 1989). These relations were of various function types, including piecewise functions and sequences, and were shown in several different representations, such as graphical, algebraic, tabular, and contextual. The goal was to help determine whether students had an *action* or *process* view of functions (Breidenbach, et al., 1992) and to see how open-minded they were about what a function can be (Cooney, et al., 2010).

In part 2, students were asked to solve two covariation tasks (Cooney, et al., 2010; Oehrtman, et al., 2008). The first task involved interpreting how the graph of a function changes, and the second task involved filling in a table of values based on a description of the relationship between the input and output. Both of these tasks assess how the participant makes sense of a function's covariation and rate in these different contexts.

In part 3, the student was given six different representations of functions and asked to determine what type of functions they are (Cooney, et al., 2010). Three of the given functions were verbally described, while there was one each in tabular, graphical, and algebraic form. This task was chosen because it was designed to assess a student's ability to recognize and identify linear, quadratic, and exponential functions based on the covariation of the outputs with the inputs as well as what they look like in a given representation. While there are several items on the PCA that assess a student's understanding of a function's rate, they are never explicitly challenged to differentiate between function types. The ability to identify a function based on its rate type (e.g., a linear function's rate is always constant) is a key aspect to understanding Big Idea #3 (Cooney, et al., 2010).

In part 4, students were given four different function transformation tasks to solve and/or make sense of (Eisenberg & Dreyfus, 1994). These tasks were selected because they assess a student's ability to recognize and make sense of different types of function transformations, such as reflections, translations, and resizings. These types of transformations are not assessed on the PCA, as the only understandings related to Big Idea #4 that are found in the PCA taxonomy are function compositions and inverses. These tasks were also used to help determine if the student displayed any characteristics of an *object* view of functions, as the ability to easily recognize and perform transformations on functions is a defining characteristic of an *object* view in APOS Theory (Asiala, et al., 1996; Baker, et. al, 2001; Breidenbach, et al, 1992; Dubinsky & McDonald, 2002; Lage & Gaisman, 2006).

Finally, in part 5 the participants were given four different representations (graph, formula, table, and description) of the same function and were given two tasks to perform (Cooney, et al., 2010). In the first task, they were given five pieces of information about the function, and asked to identify which of the four representations best displays that information. In the second task, they were asked to identify where they can find the same pieces of information in each of the other representations. This task was selected because it assesses how well a student can interpret different representations of the same function, and to see if they could make sense of the connections between those representations. These aspects of Big Idea #5 are mostly absent on the PCA. While making sense of each type of representation is a common theme on the PCA, there are very few items that ask students to simultaneously make sense of different representations of the same function. Only one item on the PCA asks for students to make a connection between a graph and a formula, and the only table on the PCA is not connected to any of the other representations. All of the other items identified as being related to Big Idea #5 on the PCA (as shown in Table 3.4) highlight connections between a verbal description of the function and either a graph or a formula.

The interviews were scheduled in two-hour time slots. Each was audio-recorded and transcribed, and any written work by the participants was collected for analysis. The entire collection of interview tasks can be found in Appendix B.

3.2.3 The AP Calculus Exam

The final data source was the AP Calculus Exam. This is an exam given toward the end of the school year to all students in the United States that are enrolled in an AP Calculus AB course. It assesses students' understanding of material from a first-semester college course in differential and integral calculus. The test consists of two primary sections: multiple choice and free response. Both sections are also divided into two subsections: one where a calculator is required to solve the problems, and one where use of a calculator is forbidden. Students who take the AP Exam are scored on a scale of 1 to 5, with 1 representing the lowest proficiency and 5 representing the highest proficiency. The score reflects the student's understanding of first semester college calculus as compared to how college students perform in such a course, and therefore determines whether that student qualifies for college credit in calculus. For example, a 5 on the AP exam means that student's performance is similar to that of an A student in a college calculus class. Likewise, a 4 is similar to a B student, etc. In most cases, a student must earn at least a 3 on the exam to qualify for college credit in calculus, although it is possible to qualify for the credit with a score of 2. However, some colleges and universities have stopped accepting AP scores as credit (Ben-Achour, 2013).

3.3 DATA CODING AND ANALYSIS

In this section, the way in which the data were coded and analyzed will be explained. This will include how the PCA and interviews were scored and coded, and how each participant's overall understanding of functions, view of functions, and understanding of each Big Idea of functions were coded an analyzed. This section also contains the explanation of the reliability coding, and how each of the research questions were answered.

3.3.1 Scoring the PCA Items

Each item on the PCA has five possible answer choices (a through e), and exactly one correct answer. However, some of the incorrect answer choices may be considered better selections than others. Much of it depends on the student's work in solving the problem, as the reasoning for choosing a particular answer may provide insight into a participant's degree of function understanding. Therefore, a participant's performance on the PCA was scored in two different ways. First, each item was coded as *correct* or *incorrect*, based on the answer key for the PCA, which can be found on the PCA website (Arizona Board of Regents, 2007). Second, each item was scored on a 0-4 point scale, with the number of points awarded based on the answer selected and, in some cases, the work produced by the student on that item. Four points are awarded for a correct answer, while incorrect answers can receive anywhere from 0 to 3 points. Each item on the PCA has its own scoring rubric, which is primarily based on the official analysis done on each item answer by the PCA developers, which can also be found on the PCA website (Arizona Board of Regents, 2007). Table 3.5 shows the scoring rubric for PCA item 1, and includes the analysis from the PCA website for each incorrect answer.

Note that for Item 1, there is no answer worth 3 points. Instead, choices (b) and (d) are both worth 2 points, as they both are products of the same mistake made by the student and thus are essentially equivalent. Also, answer (a) could be worth either 1 or 2 points, depending on what the student may have done to produce it. The analysis suggests two different possibilities, the second of which suggests slightly greater understandings than the first. In the first case, the student is not certain of what to do with $x + a$, which suggests he or she earn only one point for knowing to somehow input x and a, whereas the second case is merely a computational error, in which case the student earned two points. A student who selects answer (e) has very little understanding of function inputs, evaluation, and/or notation, and thus will not earn any points for choosing that answer. The item rubrics are primarily based on the official analysis done by the PCA authors as displayed on the PCA website (Arizona Board of Regents, 2007). The rubrics themselves can be found in Appendix A.

Table 3.5. PCA Item 1 Scoring Rubric

1) Given the function *f* defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.		
Answer	**PCA Website Analysis**	**Points**
a) $3x^2 + 3a^2 + 2x + 2a - 4$	"Students see the notation $f(x{+}a)$ as instructing them to do the same thing to both 'x' and 'a'. Thus since they must square 'x' and multiply it by three, they must do the same thing to 'a'. Students who make this error do not have a process view of function and have difficulty interpreting what to do with the input '$x{+}a$'. Another common mistake on this problem is that the students are able to input '$x{+}a$' for all of the 'x' terms, but they decide to multiply all the terms out and forget the middle term that would come from squaring the '$x{+}a$' term."	1-2 points
b) $3x^2 + 6xa + 3a^2 + 2x - 4$	"Students who choose this answer get overwhelmed in calculating the first term of $f(x{+}a)$ and subsequently appear to forget to input '$x{+}a$' into the term '$2x$'. This mistake typically happens when students take a procedural approach to this problem on that they see '$x{+}a$' as something that needs to be "plugged in" and then simplified which is not necessary for this problem."	2 points
c) $3(x + a)^2 + 2(x + a) - 4$	Correct Answer	4 points
d) $3(x + a)^2 + 2x - 4$	"Here students have not input '$x{+}a$' into the second term. It is possible that they do not see '$x{+}a$' as an input that must be inserted everywhere there is an 'x' in the problem. This could be due to a weak understanding of inputs and function notation."	2 points
e) $3x^2 + 2\text{x} - 4 + a$	"Students interpret $f(x{+}a)$ as $f(x){+}a$. They sometimes note that 'a' has been added to $f(x)$, and when dealing with equations, you must do the same thing to both sides, so the subsequently "add a" to both sides."	0 points

Source: PCA Website, Item #1, http://mathed.asu.edu/instruments/pca/vH/pcaq1.shtml

On most of the PCA items, work does not need to be shown in order to earn partial credit for incorrect answers. On these items, inferences can be made about what the student likely was thinking when he or she chose a particular solution. For example, on Item 2 (Figure 3.1), the student is given the graph of a function $f(x)$, and asked to find x when $f(x) = -3$. Each incorrect answer was the result of an obvious mistake the student likely made. Two of the incorrect answers involved solving for the correct x-value, but either the student thought to use the entire point as the answer (3 points) or put down the output value of -3 instead (2 points). The other two incorrect answers both involved the student confusing -3 for the input, and answering either with the output or the entire point when $x = -3$, both of which were 1-point answers. While work shown on this problem would be more definitive, there is little guesswork needed in determining why a student chose a particular answer on this problem.

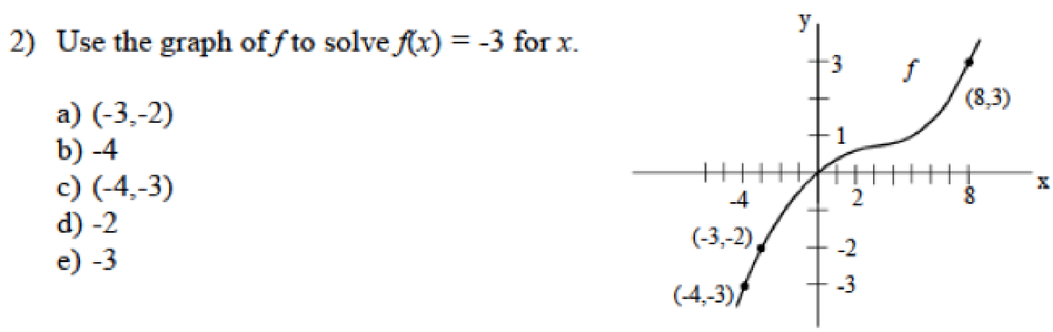


Figure 3.1. PCA Item #2.

However, there were some items, or even some answer choices for given items, that did require work to be shown in order for partial credit to be given. On these items, either fewer assumptions could be made about why a student chose a particular answer, or it was more difficult to tell whether or not the student was simply guessing. For example, on Item #11, the student is given the distance function $s(t) = t^2 + t$, and asked to find the average velocity from $t = 1$ to $t = 4$. The answer choices were 5, 6, 9, 10, and 11, all in feet per second. Some of these answer choices had likely reasons for why the student may have chosen them. For example, it was likely that the student who chose 11 simply took the average of the velocities at $t = 1$ and $t =$

4. Still, work was required on this item in order to see the calculations the student made when choosing an answer so that any mistakes could be identified and guesswork could be ruled out. The way in which it was determined whether or not an item required work to be shown for partial credit will be discussed in the section on reliability coding. The breakdown of which items required work for partial credit is shown in Table 3.6.

Table 3.6. PCA items that do and do not require work to be shown for partial credit.

	PCA Items
No work required.	2, 4, 7-10, 12, 13, 15, 16, 19-25
Work required.	3, 11, 14, 17, 18
Special conditions.	<u>Item 1:</u> Work required for answer choices (a) and (b) only. <u>Items 5 and 6:</u> Work required unless both are correct.

Finally, most correct answers were awarded 4 points regardless of whether there is supporting work or not. This is done in order to maintain consistency in the scoring of correct answers. However, for the items that required work for partial credit, the number of correct answers that were given without any work that supported the answer were also accounted for in the analysis, and had some influence on the participant's PCA score. This will be explained further in the analysis section.

3.3.2 Coding Interview Responses

Responses to each of the five parts of the interview were coded as *strong, moderate,* and *weak.* The coding was based on the responses given to each question or task, and were designed to reflect the participant's understanding of each of the Big Ideas of Functions. The general rubric for coding possible interview responses is shown in Table 3.7.

Table 3.7. General coding rubric for interview responses.

Part	Strong	Moderate		Weak
1	Q1: *Process view definition* given. That is, the definition highlights the function as a process that turns an input into an output, or as a correspondence or covariation between two quantities or sets. Q2: At least 14 functions correctly identified.	Q1: Process view definition given. Q2: No more than 13 functions correctly identified.	Q1: *Action view definition* given. That is, the function is defined as a procedure, algorithm used to perform a calculation, ties the function to a formula. Q2: At least 14 functions correctly identified.	Q1: Action view definition given Q2: No more than 13 functions correctly identified.
2	Both tasks solved and explained correctly.	Both tasks solved correctly, but unable to explain why.	One task solved and explained correctly.	At most one task solved correctly, neither task explained correctly.
3	5-6 of the function types correctly identified and explained.	All function types correctly identified, but unable to explain why.	3-4 of the function types correctly identified and explained.	0-2 of the function types correctly identified.
4	3-4 tasks correctly solved and explained.	3-4 correctly solved, unable to explain why.	2 tasks correctly solved and explained.	0-1 tasks correctly solved and explained.
5	T1: Strong support for all or most choices T2: All or most parts answered correctly.	T1: Most or all choices lack strong support. T2: All or most parts answered correctly.	T1: Strong support for most choices T2: Approx. half the parts answered correctly.	T1: Most or all choices lack strong support. T2: No more than half the parts answered correctly.

In order to further understand the meaning of Table 3.7, it is important to understand what constitutes a correct response and good explanation for each part of the interview. Most of the tasks in the interviews had definite correct answers, and those are shown in Table 3.8.

Elements of good explanations that support those correct answers are also found in Table 3.8. However, information about the correct answers for two of the interview tasks requires further explanation. One of these tasks is the first question in Part 1 of the interview. This question asks for the participant to give his or her definition of a function. As shown in Table 3.7, a “correct” answer to this question is given if the participant provides what Breidenbach, et al. (1992) called a *process view* definition. This type of definition focuses on the input-output nature of a function.

The following is a list of examples of process view definitions, as provided by Breidenbach, et al., (1992):

> “A function is a statement that when given values will operate with these values and return some result.
> A function is some sort of input being processed, a way to give some sort of output.
> A function is an algorithm that maps an input into a designated output.
> A function is an operation that accepts a given value and returns a corresponding value.” (p. 252)

Likewise, they defined an *action view* definition as a response that emphasized the performing of a calculation or operation, often tying the function to a formula or equation. Their examples of an action definition of functions were as follows:

> “A function is something that evaluates an expression in terms of *x*.
> A function is an equation in which a variable is manipulated so that an answer is calculated using numbers in place of that variable.
> A function is a combination of operations used to derive an answer.
> A function is an expression that will evaluate something when either variables or numbers are plugged into the function.” (p. 252)

A participant who gave this type of definition of a function was coded as *moderate* or *weak* for Part 1, depending on the responses given on the second question. Similarly, a participant who gave a process definition was coded as *strong* or *moderate* for Part 1, again depending on the outcome of the second question.

Table 3.8. Rubric for what makes a correct answer and good explanation for interview tasks.

<table>
<tr><th>Part</th><th colspan="3">Correct Answers and Explanations</th></tr>
<tr><td>1</td><td>1. Yes – each x has exactly one y
2. No – some inputs have more than one output
3. No – x is 3 for all y's.
4. Yes – each x has exactly one y.
5. Yes – Constant function
6. Yes – can be solved to get a rational function</td><td>7. Yes – equivalent to $y = x^2$
8. No – x's can have more than one y
9. No – two y's for x between 5 and 7
10. Yes – sum of functions is a function
11. Yes – each x has exactly one y
12. No – overlapping y values</td><td>13. Yes – each x has exactly one y
14. No – multiple y's in the loop
15. No – every x has two y's
16. Yes - maps increasing dot pattern to the natural numbers
17. No – sender as input can have more than one recipient as output
18. Yes – every person has exactly one mother</td></tr>
<tr><td>2</td><td colspan="3">Graph task - The slope of S increases as Q moves closer to P. Good explanations may include: example points and lines on the graph between P and Q that illustrate how the slope changes; that the rate at which x decreases as Q gets closer to P is higher than the rate at which y decreases; the curve between P and Q is concave down, indicating a rate decrease as x goes forward (and thus a rate increase going backward).

Table task - From top to bottom: 10, 2.5, 1.25. More hoses filling the vat means the vat will fill faster, and each hose fills the vat at the same rate. Therefore, doubling the number of hoses cuts the amount of time in half, and vice versa.</td></tr>
<tr><td>3</td><td colspan="3">a) Quadratic – The x term is squared, the graph makes a parabola

b) Exponential – Multiplying by the same growth rate for each year.

c) Linear – The rate of change is constant, for each ticket sold, $7.50 is made, so the rate is always 7.5. The graph is a line.

d) Linear – Each time D goes up by 7, M goes up by 0.14. The rate of change is constant. The graph is a line.

e) Exponential – The graph starts slowly than increases very quickly, increases exponentially.

f) Quadratic – The radius is squared in the area of a circle, graph is a parabola.</td></tr>
</table>

4

1) $y = f(2x)$ – The new graph is the same, but thinner. The x values are cut in half, so to get the same y value, the new x must be doubled. If the new graph is g, then $f(-2) = 0 = g(-1)$. Since $-2 = 2*-1$, $g(x)$ must be $f(2x)$. (This works for other test points as well).

2) The parameter d is the same in all three graphs. d represents a change in horizontal position of the graph, and all three graphs have the same horizontal position. They have different sizes and vertical positions, which are what parameters a and e, represent, respectively.

3) Much is made of *how* the graphs are produced. Graphs in parts b), c), and d) are correctly drawn and explained if they are based on the *transformations* of the original graph. These are not correct if the participant consistently redraws the graph without considering the transformations from the original graph.

   a) Correct graph for $y = |x^2 - 4|$.
   b) Shift the entire graph up by 1
   c) Shift the graph down by 4 and to the right by 1.
   d) Vertically resize by half. That is, each point on the new graph is twice as close to the x axis as the original, so the graph appears flatter than before.

4) *Graph features:* Vertical asymptotes at 2 and -2; in between them: vertex at (0, -1/2), concave down; outer pieces: concave up with positive y-values, x-axis as a horizontal asymptote.

Explanation - The asymptotes are where the zeroes are in the original function, since 0 has no reciprocal. As the original graph gets closer to zero, the reciprocal graph will get closer to closer to positive or negative infinity, depending on which side of the axis it is on. y-values of the original graph between -1 and 1 will be either less than -1 or greater than 1 in the reciprocal graph depending on its sign. Similarly, y values greater than 1 in the original graph will fall between 0 and 1 in the reciprocal graph, and those less than -1 in the original graph will fall between -1 and 0 in the reciprocal graph.

5

These are correct answers for each part of Task 2, with answers in bold identified as likely answers for Task 1.

Verbal: a) **Given in the first sentence.**
b) Calculate how many tickets must be sold to make $500 more than the operating cost.
c) Calculate how many tickets must be equal the operating cost, $1025.
d) Ticket cost represents the rate, since it's $7.50 *per* ticket.
e) Every ticket sold adds $7.5 more to the profit, a constant change.

Formula: a) -1025, when $T = 0$ b) Set $P = 500$, solve for T c) Set $P = 0$, solve for T
d) **7.5 is the linear coefficient, it represents the slope, which is the rate of change.**
e) **The power of *T* is 1, no exponent on *T*.**

Graph: a) When $T = 0$, where the graph crosses the y-axis b) **Point nearest *P* = 500 on the graph**
c) **Where the line crosses the *x*-axis** d) Slope of the line that connects the points
e) **Connecting the points makes a line, the graph is a line.**

Table: a) First line b) ***P* is 500 somewhere between *T* = 200 and 250, closer to 200.**
c) ***P* is 0 somewhere between *T* = 100 and 150, closer to 150.**
d) The rate is how P changes over how T changes from one row to the next
e) The way in which both P and T respectively change from row to row is constant

The other interview response that required further explanation is the first task of Part 5. In this task, the participant was asked to explain which of the four representations of the given function best highlights each of five pieces of information. A correct answer for each element in this task is more subjective than those of the other interview tasks, as it is mostly dependent on the explanation given for choosing a particular representation. For example, one participant may have decided that the graph best depicts the number of tickets sold to get a $500 profit, while another may have decided that the table does this best. Either way, as long as the participant was able to provide good support for his or her choice (examples of which can be found in Table 3.8), the response could still have been coded as *strong*. However, a choice with little support, such as "I just like it best," was seen as lacking strong support or evidence and coded as such.

3.3.3 Coding Function Understanding

Each participant was assessed based on their understanding of functions, which was determined by their scores on the PCA and their interview responses (if applicable). As with each interview response, students' general understandings of functions were also classified as *strong*, *moderate*, or *weak*.

For the PCA, first the categories of *strong*, *moderate*, and *weak* must be defined. A student with a *strong* understanding of functions earned the equivalent of an A on the PCA. A *moderate* understanding means the student earned at least a C. A *weak* understanding means the student earned less than a C.

Therefore, what does it mean to earn an A or a C on the PCA? This determination was based on one of two factors: the number of correct answers or the total score on the assessment.

The cutoff scores for total number of correct answers were determined by the correlations between performance on the PCA and success in calculus, as reported by Carlson, et al., (2010). In their study, 77% of the students who got at least 13 correct answers on the PCA went on to earn a C or higher in first-semester calculus, while 60% of those who scored less than 13 either earned a D or an F in calculus or they withdrew from the course. Therefore, 13 correct answers on the PCA are the lowest needed in order for a participant to have earned at least a C on the PCA, and therefore be classified as having a *moderate* understanding of functions. Since 13 correct answers are equivalent to 12 incorrect answers, it can be said that a *moderate* student misses no more than 12 items on the PCA. Also, since no more than 12 incorrect answers is at least a C, then using a scale of 4 incorrect answers per letter grade, at most 8 incorrect answers would be a B, and at most 4 incorrect answers is an A. Therefore, a student who had at least 21 correct answers (no more than 4 incorrect answers) will have earned an A on the PCA and be classified as having a *strong* understanding of functions.

The cutoff scores for the total PCA score was based on traditional cutoff scores in school mathematics. On a 100-point scale, a 90 is usually an A, an 80 is a B, and a 70 is a C. Since the PCA items total up to 100 points, a student who earned at least 90 points on the PCA based on the item rubrics was classified as having a *strong* understanding of functions, while a student who earned at least 70 points was classified as *moderate.*

The two determining factors for classification of function understanding was treated on an either/or basis. That is, a *strong* student either had 21 correct answers or 90 points on the PCA. In most cases, this happened simultaneously. However, there were a few situations where that was not the case. For example, it was possible to answer only 19 items correctly, earning 76 points, and earning 14 more points spread amongst the other 6 items (e.g., 2.33 points per item).

Also, one may have scored 21 correct answers for 84 points and averaged only 1 point per item among the remainder for a total score of 88, which falls short of 90 points. In either case, the participant still displayed enough proficiency on the assessment to be considered to have a *strong* overall functional understanding for the purposes of this study. Similarly, participants were considered to have a *moderate* understanding of functions if they scored at least 70 points on the PCA or answered at least 13 items correctly. If participants scored less than 70 points and fewer than 13 correct answers, their understanding of functions were coded as *weak*. Table 3.9 provides the rubric for coding participants' function understanding based on the PCA.

Table 3.9. Coding Function Understanding based on the PCA.

	Strong	**Moderate**	**Weak**
Total Score	≥ 90	≥ 70	Other
Correct Answers	≥ 21	≥ 13	

As stated previously, the number of correct answers that had no supporting work on items that required work were also accounted for in the analysis. This was done in order to control for the possibility that some correct answers with no supporting evidence may have been the result of a guess. Since each item has five answer choices and only one correct answer, the chance that an item was guessed correctly is 20%. Likewise, the chance of guessing two items correctly is only 4%, and guessing three items correctly is only a 0.8% chance. Since that is less than a 1% chance, it is considered unlikely that a participant will have guessed three items correctly. Therefore, for those items that required work, the first two correct answers with no supporting work were not counted toward that participant's total number of correct answers and received 0 points. Any subsequent correct answers with no supporting work continued to be counted toward the participant's total number of correct answers and received the full 4 points. Therefore, the coding of each participant's function understanding based on the PCA was reflected by this adjustment. There were 38 participants (45%) who lost credit for a correct problem in this manner, but it only effected the coding of understanding of six participants (7%).

For the interviews, function understanding depended on how each part of the interview was coded, and again was coded as *strong, moderate,* or *weak.* Since there are five parts to the interview, each participant had a 5-string code to reflect their performance on each part of the interview. For example, the code SMMWS is the code for a participant who was *strong* in parts 1 and 5, *moderate* in parts 2 and 3, and *weak* in part 4. The total number of each S, M, and W in the code determined the participant's overall code for the interview portion. A participant's function understanding was considered *strong* if they had either at least four S's or three S's and at least one M. Likewise, participants were coded as *weak* if they had at least four W's or three W's and at least one M. All other possible configurations were coded as *moderate.* All possible coding configurations are shown in Table 3.10.

Table 3.10. Coding Function Understanding based on interview response code configurations.

<table>
<tr><th></th><th>Total S's</th><th>Total M's</th><th>Total W's</th><th>Overall</th></tr>
<tr><td rowspan="7">Interview Code</td><td>5</td><td>0</td><td>0</td><td>Strong</td></tr>
<tr><td>4</td><td>1 or 0</td><td>0 or 1</td><td>Strong</td></tr>
<tr><td>3</td><td>2 or 1</td><td>0 or 1</td><td>Strong</td></tr>
<tr><td>1 or 0</td><td>1 or 2</td><td>3</td><td>Weak</td></tr>
<tr><td>1 or 0</td><td>0 or 1</td><td>4</td><td>Weak</td></tr>
<tr><td>0</td><td>0</td><td>5</td><td>Weak</td></tr>
<tr><td colspan="3">All Other Configurations</td><td>Moderate</td></tr>
</table>

3.3.4 Coding Function View and Understandings of the Big Ideas

In addition to determining each participant's overall understanding of functions, it was also pertinent to the study to identify whether each participant holds an *action* or *process* view of functions, and/or possibly an *object* view of functions, as defined by APOS Theory (Asiala, et al., 1996). It was also important to assess each participant's understandings of each of the five Big Ideas of functions (Cooney, et al., 2010). Coding of these views and understandings were done in a similar manner to those of each participant's general understanding of functions. However, the assessment was based on a subset of the PCA items and interview tasks, and the scoring and coding of these items were fit to the same scale as that of the overall scoring and coding of items and tasks. This was done in order to remain consistent with the overall scoring and coding of PCA items and interview tasks.

Function view: In order to determine a participant's function view based on the PCA, the focus was placed solely on the students' performance on the 16 PCA items that fall under category R1 in the taxonomy, *process view of functions*. Participants were identified as having a *process view* of functions if they answered at least 8 of the items correctly. Otherwise, they were considered to have an *action* view. Item scores were not counted for this determination, since the test authors stated that only correct answers on these items indicate a process view of functions (Carlson, et al., 2010). Eight was chosen as the cut-off because it is equivalent to the percentage needed for a *moderate* understanding of functions in the overall scoring. That is, 8 is the smallest number of correct answers that is equal to or greater than 50%. This was done because it is reasoned that a process view of functions requires at least a *moderate* understanding of functions. This reasoning is based on three ideas. The first idea is that a process view is necessary to succeed in calculus (Oehrtman, et al., 2008). The second is that Carlson et al., (2010) found that most students who answered over half of the PCA items correctly passed first semester calculus. Finally, the cut-off scores for a *moderate* understanding of functions are based on those that predicted calculus success as reported in Carlson, et al. (2010). Therefore, the scale for determining the cut-off score for a *process* view is based on the scale that is equivalent to that used for determining a *moderate* understanding of functions, which was at least half of the total number of items.

For the coding of the interviews, a participant's view of functions was identified as *action* or *process* based primarily on the part 1 responses. In these responses, the participant could give an action definition or process definition in question 1 (Breidenbach, et al., 1992), and the explanations of identifying certain relations as functions in question 2 also contributed to determining whether he or she has an action or process view. For example, stating a graph is a function because it passes the Vertical Line Test only suggests an action view, but the ability to explain why the Vertical Line Test works is evidence of a process view. Similarly, a student with an action view may not identify functions that do not look or behave "nicely", such as piecewise functions, sequences, those with discontinuities in the graph, or non-numerical relations. A participant with a process view will recognize these as functions and be able to explain why by identifying the correspondence between the inputs and outputs, and showing how there is no more than one output for each input.

The final coding of responses to part 1 of the interview was as follows. In question 1, the given definition was identified as either an *action* or *process* definition. In question 2, each relation that was a) correctly identified as a function or nonfunction, and b) explained using elements of the process view as described above was coded as *process*. Otherwise, that relation was coded as *action*. The combined total of *action* and *process* codes for all of the elements of questions 1 and 2 determined the participant's view of functions on the interview. That is, if the participant had more *process* codes, then he or she was coded as having a *process view* of functions. Since there were an odd number of elements in part 1 of the interview, it was guaranteed that the number of *action* and *process* codes for these elements was equal.

For students who did not participate in the interview, the final function view was only determined by their performance on the PCA. For those who did participate in the interview, their function view was determined by both the PCA and interview responses. If both had the same code, then the final view for that participant was that respective code. However, if one was *action* and the other was *process*, then a final determination needed to be made. This was done by adding the total number of correct answers on *process view* PCA items with the number of elements in the interview that were coded as *process*. Similarly, the total number of incorrect *process view* PCA items were added to the interview elements with *action* codes. The code with the greater total was the final code for the function view of that participant. The fact that the total number of *process view* PCA items and interview elements in part 1 was odd guaranteed that these two sums will never be equal. The rubric for coding between *action* and *process* view is shown in Table 3.11.

While each participant was categorized as having either an *action* or *process* view of functions, interview participants were also coded on whether they were showing characteristics of an *object* view, based on their responses to three items in part 1 and on all items in part 4. This was done anytime a student treated a function as an object, by performing some action on it or manipulating it in some way, or recognizing pieces of one function as other complete functions.

Table 3.11. Overall coding of action vs. process view.

	Process	**Action**
PCA – R1 Items	At least 8 correct items	Fewer than 8 correct items
Interview Part 1, Q1	Process definition given	Action definition given
Each element of Part 1, Q2	Correctly identified relations as function or nonfunction. Can identify the correspondence between input and output. Can demonstrate how the identified function (or nonfunction) has no more than one output per input (or has at least one input with two outputs). Can explain how the Vertical Line Test works.	Incorrectly identifies relations as functions or nonfunctions. Procedure-based explanations, such as it does or does not pass the Vertical Line test. The only relations that are functions are those that are "nice": it has an explicit numerical formula, smooth, continuous graph. Going by memory or guessing. Cannot explain why.
Entire Interview Part 1	Greater number of *process* elements.	Greater number of *action* elements.
Overall	Sum of correct PCA R1 items and *process* view elements from Interview Part 1 is **greater** than the sum of incorrect PCA R1 items and *action* view elements from Interview Part 1.	Sum of correct PCA R1 items and *process* view elements from Interview Part 1 is **less** than the sum of incorrect PCA R1 items and *action* view elements from Interview Part 1.

First, a participant's given definition of a function in part 1, question 1 was coded as *object* (along with its *action* or *process* code) if it included elements of an *object* view, such as the idea that actions can be performed on a function, or a function can be transformed. In part 1, question 2, item 10 could be seen as a sum of three separate known functions (a quadratic, an exponential, and a logarithm). In item 15, the given relation is not a function of y in terms of x, yet it is possible that a participant will recognize that the graph is actually a function of x in terms of y. These respective responses to items 10 and 15 could reflect an object view, since they required the ability to recognize familiar functions either as parts of a larger function (item 10), or one that was simply oriented differently (item 15). In each case, the response for that item will be coded as *object.*

Finally, since part 4 consisted of function transformation tasks, a strong performance on these tasks could suggest an object view. Responses to any of these tasks were coded as *object* if the participant was able to correctly solve the tasks by being able to explain the transformations taking place without resorting to checking specific points. Additionally, responses could also be given a *half object* code if the participant seemed to treat the function in the task as an object but still did not correctly solve or complete the task. For example, in task 4, the correct graph has three curves separated by two asymptotes. If the student was able to correctly produce at least one of the three curves, then he or she was given a *half object* code for that task. Also, it should be noted that the four items in task 3 were coded separately, since each required graphing a different transformation, and it is possible that a student could easily recognize and graph one transformation while having to work through steps in order to graph a different one. A student will be identified as displaying characteristics that suggest an *object* view if the total number of *object* codes (while treating *half object* codes as such) is greater than or equal to five. For example, a student with 3 *object* codes and 2 *half object* codes would not be classified as displaying characteristics of an object view, since the two *half object* codes only count as one whole code. The coding of the object view for these interview items is shown in Table 3.12.

Table 3.12. Coding of object view for interview items.

Interview Item	When to code for object view	Notes
Part 1, Question 1	Definition includes discussion of how functions are entities that can be manipulated, transformed, or acted upon in some way.	Coded in addition to *action* or *process* definition code.
Part 1, Question 2	Item 10: Is a function because it is a sum of three functions. Item 15: Is not a function of *y* in terms of *x*, but IS a function of *x* in terms of *y*.	Treated as *process* elements when determining overall function view (Table 3.11).
Part 4, Task 1	Correctly solved due to how the second graph is compressed horizontally, without having to test with points (though points can be used to *help explain it*). Count as half with an incorrect answer if there is evidence of understanding how changes in the formula affect the graph. This includes commentary about what certain formulas will do to the graph, or correctly identifying that a coefficient in the formula will change the width of the graph somehow.	
Part 4, Task 2	Correctly solved by explaining what each parameter does to the graph, again without having to test any points or examples. Count as half if incorrectly solved, but at least one parameter is correctly explained along the way.	
Part 4, Task 3	a): Easily graphs the function based on knowledge of the absolute value graph and how it was transformed, as opposed to plotting points. b) – d): Correctly identifies the transformation of the graph in (a), produces the correct graph based on that transformation, rather than point-plotting. Count as half for an incorrect graph that was produced in that manner (such as getting a transformation backwards).	Each individual part will be coded rather than the entire task.
Part 4, Task 4	Correct graph given with the correct explanation, or if the *inverse* was correctly produced instead. Count as half if at least one of the three primary parts of the graph (those divided by the asymptotes) is correctly produced and explained.	Refer to Table 3.8
Overall	Add the total number of codes and half codes together. If the total is at least 5 (at least half the items), then code the participant as *possibly having an object view*.	

Understandings of the Big Ideas: In order to code for understandings of each Big Idea on the PCA, only the items identified as related to each respective Big Idea were scored and coded. This was done in the same way that the whole PCA was coded. That is, the same scale was used to determine the cut-off scores for classifying the strength of understanding each participant had for each Big Idea. The collection of items aligned with each Big Idea was scored and the total number and the percent of correct answers in that set were counted. Participants were classified as being *strong* in that Big Idea if they scored at least 90% of the total points earned by those items, or answered approximately 84% of the items correctly. Similarly, they were coded as *moderate* for that Big Idea if they scored 70% of the points or answered at least half of the items correctly. Otherwise, they were considered to have a *weak* understanding of that Big Idea. The coding rubric for each Big Idea item set on the PCA is shown in Table 3.13.

Table 3.13. Rubric for coding item sets for each Big Idea on the PCA.

Big Idea	# of Items	Strong		Moderate		Weak
		Score	*Correct*	*Score*	*Correct*	
Function Concept	17	61	14	47	9	Other
Covariation	9	32	7	25	5	
Families	6	21	5	16	3	
Transformations	12	43	10	33	6	
Representations	10	36	8	28	5	

For those participants who were also interviewed, their understanding for each Big Idea on the PCA was paired with how they were coded for the same Big Idea in the interview in order to determine an overall understanding of that idea. Since the interview offers a greater opportunity to understand the participants' thought processes, the interview code took precedence. That is, a student who scored *strong* or *weak* on the PCA for a given Big Idea and *moderate* in the interview for the same Big Idea will be coded as having a *moderate* understanding of that Big Idea. Likewise, a *moderate* on the PCA and a *strong* (or *weak*) interview will be coded as *strong* (or *weak*). Finally, if the two outcomes are *strong* on the PCA and *weak* in the interview or vice versa, the participant will be coded as *moderate*. The coding of overall understanding of each Big Idea for interview participants is shown in Table 3.14.

Table 3.14. Overall understanding of each Big Idea for interview participants.

PCA	Interview	Overall
Strong	Strong	Strong
	Moderate	Moderate
	Weak	Moderate
Moderate	Strong	Strong
	Moderate	Moderate
	Weak	Weak
Weak	Strong	Moderate
	Moderate	Moderate
	Weak	Weak

3.3.5 Reliability Coding

In order to ensure reliability, a second coder was asked to code twenty PCA tests and one interview, which is approximately 20% of the data collected. This rater was trained to use the coding system as described in this section for the PCA and the interview tasks. This included scoring the PCA items using the item rubrics in Appendix A, and using the interview coding rubrics in Tables 3.7 and 3.8 for the interview questions and tasks. The rater also coded interview items as *action*, *process*, or *object*, as described in Tables 3.11 and 3.12. Pilot data was collected for use in this training. During this time, the researcher and the rater also went over the PCA items to determine which of them required work to be shown in order to earn partial credit for incorrect answers. They each identified items separately, then discussed any discrepancies until 100% agreement was reached.

The rater was given twenty PCA tests to score and code. Since all correct answers are automatically four points, only incorrect answers were compared for reliability, based on how many points were allotted to each incorrect answer, particularly for those items that required work to be shown. Among the twenty tests, there were 239 incorrect answers, and there was agreement on all but three of them (98.7%).

The rater also received copies of a transcript of one interview and the related work from each task from that interview to code. This transcript was divided into explicit sections that highlighted each interview task. This was done to help ease the coding process for the rater. For the interview, agreement was reached on 38 of 39 items (97%) that were coded as correct or incorrect. Also, there was 100% agreement on how each of the five parts of the interviews were coded for understanding (i.e., whether they were *strong*, *moderate*, or *weak*). For the items that were coded as *action* or *process* in part 1, agreement was reached on 18 of 19 items (95%), and for the items that were coded as *object* in parts 1 and 4, agreement was originally reached on 17 out of 20 items (85%). Further clarifications were made on the coding rubric for *object* view until 100% agreement was reached.

3.3.6 Analysis

This section contains the explanation for how the coded data will be analyzed in order to answer each of the research questions. The first research question asks about AP Calculus students' understanding of functions, and has four sub-questions. The second research question asks about the relationship between that understanding and performance on the AP Calculus exam, and has three subquestions. Both sets of subquestions ask about the same respective issues: students' ability to solve problems about functions, students' ability to explain their thinking about functions, students' view of functions, and students' understanding of each of the Big Ideas of functions. In this section, data analysis as it pertains to each of the seven research subquestions is explained. Additionally, an explanation for a comparison between the PCA results and interview results is featured at the end of this section.

RQ 1a – To what extent can AP Calculus students solve problems about functions? To answer this question, the total number and the percent of participants that were coded as *strong, moderate*, and *weak*, respectively on the PCA were recorded. The average PCA score and average number of correct answers, along with their respective standard deviations, were also be calculated and reported.

RQ 1b – To what extent can AP Calculus students explain their thinking about functions? Similarly to Research Question 1a, the total number and the percent of interview participants that were coded as *strong, moderate*, and *weak*, respectively, were reported. Their performances were compared to how they did on the PCA, and a breakdown of how each participant performed on each part of the interview was also presented.

RQ 1c – What view of functions do AP Calculus students hold? In order to answer this question, the total number and the percent of students with each view (*action* and *process*) were recorded, and also broken down by function understanding based on the PCA scores. Also, the average score for *process view* items on the PCA as well as the average number of correct answers, along with their respective standard deviations, were calculated and reported. Performance on the function view sections of the PCA and interviews were broken down and analyzed for each interview participant. Finally, the number and percent of interview participants who were coded as displaying characteristics of an *object* view were also reported and analyzed.

RQ 1d – To what extent do AP Calculus students understand each of the Big Ideas of functions? To answer this question, the total number and the percent of *strong, moderate,* and *weak* students for each Big Idea was calculated and reported. Also, the average score for each item set on the PCA, the average number of correct answers for each, and their respective standard deviations were also be calculated and reported. A Friedman's rank-order ANOVA test was run to determine if participants generally understood some Big Idea better than others, and a Spearman's rank correlation was run to determine if there were any significant correlations between the understandings of any two Big Ideas.

RQ 2a – What is the relationship between students' ability to solve problems about functions and their performance on the AP Calculus Exam? To answer this question, the average scores and standard deviations on the AP Calculus Exam for students with *strong, moderate,* and *weak* scores on the PCA, respectively, were calculated. Then, a 1-way Between-Subjects ANOVA was performed in order to determine if any significant differences existed between the AP Exam scores of students for each level of function understanding. Also, the number of students in each category were calculated and reported for all five AP exam scores, and a Chi-square test (df = 4) was run to determine if significant differences existed between the number of exam scores for each level of understanding.

Most of the same procedures outlined above were also followed for the eight interview participants. However, the ANOVA and the Chi-Square tests were not run for the interview participants due to the sample size being too small.

RQ 2b – What is the relationship between students' view of functions and their performance on the AP Calculus Exam? Data analysis for this question was handled similarly to those in RQ 2a, except it focused solely on students' function view. The average scores and standard deviations on the AP Calculus Exam were calculated and reported for students with *action* and *process views* of functions, respectively, as well as for those who were identified as showing *object* view characteristics. Then, a 1-way Between-Subjects ANOVA was performed in order to determine if a significant difference existed between the AP Exam scores of students with *action* and *process* views. Also, the number of students in each category (including *object*) was calculated for each of the five AP Exam scores (1-5). In other words, the total number and the percent of participants with an *action*, *process*, and *object* view who earned a 1 on the AP exam was determined. The same was done for those who earned a 2 on the exam, etc. Then a Chi-square test (df = 4) was run to determine if significant differences existed between the number of exam scores for each view of functions.

RQ 2c – What is the relationship between students' understanding of each of the Big Ideas and their performance on the AP Calculus Exam? To answer this question, the average scores and standard deviations on the AP Calculus Exam for students with *strong, moderate,* and *weak* understandings of each Big Idea, respectively, were calculated. Then, 1-way between subjects ANOVA tests were performed on all 5 Big Ideas in order to determine if any significant differences exist between the AP Exam scores of students who were *strong, moderate,* and *weak* in each Big Idea. Also, the total number and the percent of students in each category were calculated for all five AP exam scores, and a Chi-square test (df = 4) was run to determine if significant differences existed between the number of exam scores for each level of understanding for each Big Idea.

PCA vs. Interviews – In addition to answering the research questions, an important analysis for this study is a comparison between the results of the PCA and of the interview responses for the nine participants who interviewed for this study. For these participants, data from both the PCA and interviews were displayed and compared. These data include function understanding as determined by both the PCA and the interviews, function view, and function understanding of each Big Idea. The AP Calculus Exam scores for each respective participant were reported as well. This was done to determine if there were any noted discrepancies between the results of the PCA and of those from the interview responses.

The sections that follow will contain a review of the results of the analysis, followed by a discussion of the meaning of those results. The final section will contain the conclusions drawn from the results and analysis of the study, and ideas for further research.

4.0 RESULTS

The results of the data analysis described in Chapter 3 will be presented in this chapter and organized by each of the research questions. All statistical tests were performed using an α-level of .05, unless stated otherwise.

4.1 AP CALCULUS STUDENTS' UNDERSTANDINGS OF FUNCTIONS

The results in this section are in response to the first research question, which was divided into four subquestions. Those subquestions respectively focused on the overall PCA results, the interviews, the students' view of functions and their understandings of each of the Big Ideas of Functions.

4.1.1 Function Understandings on the PCA

RQ1a – To what extent can AP Calculus students solve problems about functions? This subquestion focuses on the results of the PCA. Overall, 85 AP Calculus students took the PCA. Recall from Chapter 3 that the PCA was scored by both the total number of correct answers out of 25, and on a 100-point scale to account for partial credit. The average number of correct answers was 12.66 ($\sigma = 4.584$), and the average partial credit score was 65.73 ($\sigma = 15.052$).

Note that these mean scores are less than 13 and 70, respectively, and thus lie in the range of *weak* scores. In total, 43 participants (approximately 51%) were classified as having a *weak* understanding of functions based on their PCA scores. Meanwhile, 35 participants (41%) were classified as having a *moderate* understanding of functions, and only 7 participants (8%) were classified as having a *strong* understanding of functions. In other words, over half of the AP Calculus students participating in this study demonstrated a *weak* understanding of functions as measured by performance on the PCA.

In some cases, the two PCA scoring methods produced different results. These discrepancies were most noticeable when discerning between codes of *moderate* and *weak*. Twelve of the participants who scored in the *moderate* range by answering at least 13 items correctly scored in the *weak* range (less than 70) when partial credit was factored in. In fact, every student who scored 13 correct answers had a *weak* partial credit score of less than 70, as did half of the students who had 14 correct answers. Also, one student who had only 12 correct answers had a partial credit score of 70 and was therefore classified as *moderate*. Every other student who had at most 12 correct answers scored less than 70 when partial credit was given and were classified as *weak*.

Based on the overall results, these numbers are not surprising. Recall that students received 4 points for each correct answer on the PCA, and 0-3 points for each incorrect answer, based on which answer was chosen and, if necessary, any supporting work that was shown. On average, students on the PCA earned about 1.22 points of partial credit for incorrect responses. Students who scored 14 correct items would need to earn 1.27 points per incorrect item to reach 70 points, which is more than the average of 1.22. For students with 13 correct responses, they would need 1.5 points per incorrect item to reach 70 points. So the fact that most of these students did not reach the *moderate* threshold of 70 points indicates that they were answering questions correctly more often than not, but on the questions they missed they earned few points in partial credit. For example, one such student with 14 correct answers earned 65 points, which means he earned 9 points in partial credit on 11 missed items, an average of 0.82 points per missed item. In fact, he earned no points on 6 of those items. This suggests that this student answered correctly more often than not, but when he was incorrect, he was choosing the responses that were not very close. This seemed to indicate that he, and others like him, had good understandings of some aspects of functions but struggled in other areas.

As for the student who only had 12 correct answers but earned 70 points, she was the exception, in that she got more questions wrong than she got right, but she was often close. She had earned 22 points in partial credit on the PCA, an average of 1.69 per item, which is well over the average. Only 11 participants had a higher average, and 8 of them had at least 18 correct answers to begin with. She earned either 2 or 3 points on 9 of the 13 items she missed. So, even though she was incorrect more often than she was correct, she still displayed enough understanding to still be classified as *moderate* on the PCA.

At the top end of the scale, all participants with at least 21 correct answers scored over 90 in partial credit as well, and thus were classified as *strong*. Likewise, those with scores from 15 to 18 correct answers all had partial credit scores between 70 and 89, and were classified as *moderate*. Meanwhile, six students scored either 19 or 20 correct answers, which fall just below the *strong* cutoff score of 21. Two of these participants scored a 90 on the partial credit scale and were classified as *strong*, while the rest scored below 90 and were classified as *moderate*. Again, these differences can be accounted for the fact that the students who "jump" to the *strong* level collected a lot of points in partial credit. The *strong* student with 19 correct items and 90 points earned 14 points in partial credit among 6 missed items, an average of 2.33 points per item. Meanwhile, the student with 20 correct answers but only 88 points overall only earned 8 points among 5 missed items, an average of 1.6 points per item. In order to get to 90 points, this participant needed to average at least 2 points per missed item. In general, the *strong* students were earning 2 or 3 points on each item they missed, while those who stayed in the *moderate* range had at least one or two items on which they earned 0 or 1 points in partial credit. This rarely if ever happened to *strong* students, as they were more likely to choose a "close" answer on those rare occasions when they did not choose the correct answer.

4.1.2 Function Understandings on the Interviews

RQ1b – To what extent can AP Calculus students explain their thinking about functions? The interviews are the focus of this subquestion. Only 8 of the participants were interviewed. Four of them were classified as having a *weak* understanding of functions based on the results of the PCA, while two each were classified as *moderate* and *strong* respectively. Based on the interview analysis described in Chapter 3, four of the participants (50%) were classified as *strong* and the other four were all classified as *moderate*. None of the interview subjects were classified as having a *weak* understanding of functions. In general, participants seemed to perform better in the interviews than on the PCA, which is illustrated in Table 4.1. All four of the students with a *weak* score on the PCA were classified as *moderate* in the interviews. Meanwhile, both the students with *moderate* and *strong* scores on the PCA were classified as *strong* in the interviews. . In other words, all interview participants were classified one category higher than they were on the PCA, unless they were *strong* to begin with. No participants were classified lower on the interview than they were on the PCA. These differences will be discussed further toward the end of this chapter.

Table 4.1. PCA and interview results for interview participants.

Participant	School	PCA # Correct/Points	PCA Partial Credit Score	PCA Code	Interview Code
Steve	A	23/92	97	Strong	Strong
Eric	A	15/60	77	Moderate	Strong
Amy	B	15/60	72	Moderate	Strong
Jessica	B	19/76	90	Strong	Strong
Natalie	B	10/40	49	Weak	Moderate
Heather	C	9/36	56	Weak	Moderate
Andrew	C	12/48	60	Weak	Moderate
Rachel	A	8/32	43	Weak	Moderate

Recall that in order to be classified as having a *strong* (or *weak*) understanding of functions based on the interviews, the participant needed to have a *strong* (or *weak*) score in at least three of the five parts of the interview (where each part corresponded with one of the Big Ideas), and a *moderate* score in at least one of the remaining parts. Of the four participants who were classified as *strong*, each of them received *strong* scores in parts 1, 3, and 5, and three of the four also had a *strong* score in part 2 and *moderate* in part 4. The remaining *strong* participant scored *moderate* in part 2 and *weak* in part 4.

The participants' codes for each part of the interview as well as their overall interview code can be seen in Table 4.2. Of the four participants who were classified as having a *moderate* understanding of functions, two of them had *moderate* scores on three of the parts of the interviews, and the other two had only two *moderate* scores. Each participant also had at least one *weak* score, and three of the four also had at least one *strong* score. The most notable difference between the *strong* and *moderate* participants was that all four *strong* participants scored *strong* in part 1, while none of the *moderate* participants did. Recall that Part 1 focuses on the first Big Idea, the *Function Concept.* In it, the participants were asked to first describe what a function is, and then identify the functions in a set of 18 relations. It appears that the students who did this well were also strong in most of the other aspects of functions. A more careful analysis of how the participants performed on each part of the interview will be described in the sections ahead.

Table 4.2. Interview function understanding codes for each participant.

Participant	Part 1	Part 2	Part 3	Part 4	Part 5	Overall
Steve	Strong	Strong	Strong	Moderate	Strong	Strong
Eric	Strong	Strong	Strong	Moderate	Strong	Strong
Amy	Strong	Strong	Strong	Moderate	Strong	Strong
Jessica	Strong	Moderate	Strong	Weak	Strong	Strong
Natalie	Moderate	Moderate	Strong	Weak	Strong	Moderate
Heather	Moderate	Moderate	Moderate	Weak	Strong	Moderate
Andrew	Weak	Moderate	Moderate	Weak	Strong	Moderate
Rachel	Weak	Moderate	Moderate	Weak	Moderate	Moderate

4.1.3 Function View

RQ1c – What view of functions do AP Calculus students hold? The PCA contained 16 items that were designed to measure whether a student had an *action* or *process* view of functions. Recall that in order to be classified as having a *process* view of functions, a participant needed to answer at least 8 of these 16 items correctly. The average number of correct answers on these items was 7.96 ($\sigma = 2.925$), so on average the participants were answering approximately half of the items correctly. There were 45 participants (53%) who answered at least 8 of the process items correctly, and thus were classified as having a *process view* of functions, while the remaining 40 participants (47%) were classified as having an *action view*. This is somewhat surprising at first glance, as the majority of participants had a *weak* function understanding based on their overall PCA scores. Recall that many researchers have claimed that significant growth in function understanding occurs when a student moves from an *action* view to a *process* view (e.g., Carlson, 1998; Dubinsky & Harel, 1992; Oehrtman, et al., 2008). It would be expected that students with *weak* understandings of functions have *action* views. Instead, the aforementioned result indicates that several of the students demonstrated a *process* view of functions, but a *weak* overall understanding of them. One possible explanation is that these students' views of functions are right on the boundary of *action* and *process*, and their understandings of functions as a whole have not quite reached a moderate level. A similar but opposite assertion could be made for the few students who had an *action* view but displayed a *moderate* understanding of functions.

The breakdown of function understanding by function view based on the PCA is shown in Table 4.3. As can be seen from the table, 7 students had a *process* view of functions but scored *weak* on the PCA. However, all but 4 of the students with an *action* view were classified as *weak* on the PCA, all 7 of the *strong* students had a *process* view, and the vast majority of students with a *process* view were classified as *moderate* on the PCA. Also of note is that the 11 participants with "mismatching" views and understandings all had *process* scores within 1 point of the cutoff score of 8 correct items, and their overall PCA scores were within 2 points of 13 correct items, the *moderate* cutoff score. Essentially, their understandings of functions seem to be at the boundary of *weak* and *moderate,* and their views at that of *action* and *process*. This, along with the fact that most of the students' views and understandings "match", support the idea that moving from an *action* view to a *process* view is akin to an upgrade in function understanding.

Table 4.3. Breakdown of function view by function understanding based on the PCA.

	Strong	**Moderate**	**Weak**
Process	7	31	7
Action	0	4	36

For the interviews, recall from Chapter 3 that function view was determined by how each participant responded to the items in part 1. That is, it was determined by how they defined functions, and how they explained why a certain relation is or is not a function. For example, Steve was classified as having a *process* view of functions for the interview based on his responses in part 1. He first gave the following definition of functions:

> **Steve:** "Well I would say a function's a relationship between a dependent variable and an independent variable, and the dependent variable changes based on the independent variable, and that for each independent variable there is… one independent variable only has one dependent outcome, whereas you can get multiple, like the same dependent outcome multiple times but one each, each *x*-value has one *y*-value, one distinct *y*-value."

This is a *process* definition because it focuses on the idea that the function is a relationship between two changing variables, and he mentions that no input may have more than one output. He then proceeded to use his definition to correctly classify 15 of the 18 given relations, giving him a total "interview process score" of 16 out of 19. For example, the following was his classification of the final relation, that of "every person to his or her mother:"

> **Steve:** "But the relationship between every person to his or her mother in #18, I'd say would be, that's definitely a function because each person has like one biological mother, so there's no way you can have, like that has to be a function because each person has one distinct mother, that, you can't have one person with 2, so that one's definitely a function."

In comparison, Rachel was classified as having an *action view* of functions. Her definition did not focus on the relationship between the input and output, but rather, the forms a function could take, and how it could be manipulated or changed:

> **Rachel: "**A function is, okay so it's like an equation you can manipulate doing different things, like you can take the integral of a function or a derivative of a function and it has a variable which affects the function whether, determining what you're doing with it, I would say. Because you're taking an integral and the variable would change based on the exponent and stuff, I think that's, and you can graph a function, and you can manipulate the graph of a function based on what you're doing with the function."

She proceeded to correctly identify 12 of the 18 relations as either a function or not a function. However, her interview process score was 0, as she did not apply a process view when discussing why any of the relations were or were not functions. Instead, she tended to focus primarily on whether the relation followed a rule or set of rules, or the form it could take. For example, for the function $xy = 1$, she correctly identified it as a function, "because you could solve, make it say y equals, and that would then be a function." In comparison, Steve said that it is a function because "you still have one y-value for each x-value." So where Steve noted that the relation fits the definition of a function, Rachel focused on the idea that it could have the “look” of a function, since it could be solved for *y*. Her response to the relation of “every person to his or her mother,” was also distinctly different from Steve’s and also incorrect:

> **Rachel: “**This one I want to say is not a function, because, just because the relationship to the mother, the children would be a like a direct relationship, so I don't know how unless you were saying like the amount of children, is like a function. I don't know, unless, I don't know. I wouldn't say it's a function, I would say it's a relationship.”

In this case, since she lacks an understanding that the function is a particular type of input-output relationship, she has trouble identifying the functional relationship inherent in the statement. So while she is capable of identifying equations and graphs of functions, she has difficulty extending the idea of function beyond basic mathematical contexts. Her fixation on the idea that functions follow certain rules and must take certain forms implies that she only has an *action* view of functions.

The full results of function view measures for the interview participants from both the PCA and the interviews are shown in Table 4.4. Overall, of the eight interview participants, five of them (62.5%) were classified as having a *process view*, while the other three were classified as having an *action view*. Seven of the eight participants were classified the same for both the PCA and in the interview. The remaining interview participant, Natalie, had different codes, *action* for the PCA and *process* for the interview. So just like with the function understandings, when a difference between the PCA and interview was recorded for a participant, it was the interview that yielded the stronger score. Natalie answered only 6 of the 16 *process* view items correctly on the PCA, but gave “process view” answers on 12 of the 19 relevant items in the interview. Having received a “process” score on just over half of the total items between the interview and the PCA (18 of 35), Natalie was classified as having an overall *process view*. Since there was only one noted change, it is still the case that the majority of the participants were classified as having a *process* view of functions when both the PCA and the interviews were taken into account. In Natalie's case, her function view seems to be right at the boundary of *action* and *process*. It is possible that Natalie simply demonstrates a greater understanding of functions when she is given the opportunity to vocalize her thinking and ask for clarifications.

Table 4.4. Function view results for interview participants.

Participant	PCA process score	Interview process score	Total process score	PCA View	Interview View	Overall View
Steve	14	16	30	Process	Process	Process
Eric	8	17	25	Process	Process	Process
Amy	8	16	24	Process	Process	Process
Jessica	12	13	25	Process	Process	Process
Natalie	6	12	18	Action	Process	Process
Heather	5	8	13	Action	Action	Action
Andrew	7	0	7	Action	Action	Action
Rachel	4	0	4	Action	Action	Action

Finally, exactly half of the interview participants (4 of 8) had responses that often displayed characteristics of an *object view*. That is, in situations where they could perceive and possibly treat functions as objects in some capacity, they did so at least half the time. For example, they easily predicted how a graph would change based on given changes to its corresponding formula in task 3 of part 4. In this case, they were able to see that the same graph was simply being manipulated by the formula, rather than seeing the formulas and graphs as completely different. They saw no need to recalculate and plot the new set of points.

Amy's work on this task is a good example of this type of thinking. In graphing the function $y = |x^2 - 4|$, she was able to use the operations present in the formula to produce an accurate graph, using point-plotting only as a supplementary strategy:

> **Amy:** "Ignoring the absolute value for just a second, I'm picturing just a parabola that goes down to -4, and it's just a regular parabola, and so then it would, this would be like -2 and 2 'cause it equals 0 there. So then, take the absolute value of that, and that flips up to the top, so it's going to be like -2, and 2, 4, a point here, a point here, and a point here, and it's gonna be like that."

She uses similar strategies on the subsequent graphs, each of which contain one distinct transformation. For example, on the next graph, $y = |x^2 - 4| + 1$, she recognizes that adding one to the original graph will simply shift her original graph up one unit:

> **Amy:** Okay, so that one, I would just move that up to 5, still 2 and -2, but up one, so a point there, a point there, and this is 5 instead of 4, it's 5 instead of 4, so I think all the points just shifted...yeah. That's my answer.

She did the same thing for the third and fourth graphs, although she graphed the transformation of the final graph horizontally rather than vertically. Even so, she was able to identify that the change in the formula transformed the graph in a particular way, and was able to apply that change to her original graph instead of trying to use point-plotting to produce a completely new graph. This implies that Amy is often displaying the characteristics of an *object view* of functions.

The *object* view classifications can be seen in Table 4.5, alongside their function understandings from the PCA and interviews and their function views as determined by both. Of the participants who were classified as showing characteristics of an *object* view, one each was classified as *strong* and *moderate* on the PCA, and the other two as *weak*. In the interviews, two were classified as *strong*, and the other two as *moderate.* Three of them were classified as having an overall *process* view of functions otherwise, while the fourth had an *action* view. The average number of correct answers on the PCA for these participants was 14 ($\sigma = 6.683$), and the average PCA score was 65.25 ($\sigma = 24.581$).

Table 4.5. Interview participants with object view results.

Participant	Object Score	Shows Object View Characteristics	Overall Function View	PCA Function Understanding	Interview Function Understanding
Steve	5.5	Yes	Process	Strong	Strong
Eric	3.5		Process	Moderate	Strong
Amy	6	Yes	Process	Moderate	Strong
Jessica	4		Process	Strong	Strong
Natalie	5	Yes	Process	Weak	Moderate
Heather	1.5		Action	Weak	Moderate
Andrew	0.5		Action	Weak	Moderate
Rachel	6	Yes	Action	Weak	Moderate

Note that Natalie and Rachel both displayed *object* view characteristics despite the fact that they had *weak* scores on the PCA, and Rachel, in particular, had an *action* view of functions. Given that the *object* view is considered an indication of a greater understanding of functions than a *process* view, it seems highly unlikely that students with a *weak* function understandings and an *action* view would actually have an *object* view. Natalie's and Heather's results were most likely made possible due to the fact that they were remembering particular rules they had learned about functions, such as what happens to a graph when its entire formula is multiplied by some constant. Rachel in particular focused on recalling rules and procedures throughout the entirety of the interview, so it is especially probable that she was simply remembering what certain changes to a formula do to its graph, rather than actually reasoning about function transformations. These results indicate that much more information is needed in order to definitively identify which students are showing characteristics of the *object* view and which students are applying rules they had previously learned and memorized.

4.1.4 Understandings of the Big Ideas

RQ1d – To what extent do AP Calculus students understand each of the Big Ideas of functions? This subquestion aims at uncovering any differences between the participants' understandings of the five Big Ideas: *Function Concept* (Big Idea 1), *Covariation and Rate of Change* (Big Idea 2), *Families of Functions* (Big Idea 3), *Combining and Transforming Functions* (Big Idea 4), *and Multiple Representations of Functions* (Big Idea 5). Recall that each Big Idea was covered by a defined set of problems on the PCA, and each part of the interview corresponded to a Big Idea. Tables 4.6 – 4.9 show student performance related to each of the Big Ideas. Table 4.6 shows the descriptive results for each Big Idea on the PCA, while Tables 4.7, 4.8, and 4.9 show the number of participants that were classified as *strong*, *moderate,* and *weak* for each Big Idea on the PCA, Interviews, and overall, respectively.

Table 4.6. PCA descriptive results for each Big Idea.

Big Idea	**# Items**	**Total Points**	**# Correct**			**Score**		
			Mean	**%**	**SD**	**Mean**	**%**	**SD**
1	17	68	8.93	52.5	2.975	46.81	68.8	9.225
2	9	36	3.85	42.8	1.918	22.86	63.5	5.978
3	6	24	2.19	36.5	1.547	13.26	55.3	5.15
4	12	48	6.05	50.4	2.104	31.94	66.5	6.753
5	10	40	3.52	35.2	2.292	22.24	55.6	7.359

Table 4.7. Frequency of PCA codes of understanding for each Big Idea.

Big Idea	Strong		Moderate		Weak	
	#	%	#	%	#	%
1	7	8.2	46	54.1	32	37.6
2	8	9.4	29	34.1	48	56.5
3	12	14.1	21	24.7	52	61.2
4	6	7.1	47	55.3	32	37.6
5	5	5.9	19	22.4	61	71.8

Table 4.8. Frequency of Interview codes of understanding for each Big Idea.

Big Idea	Strong		Moderate		Weak	
	#	%	#	%	#	%
1	4	50	2	25	2	25
2	3	37.5	5	62.5	0	0
3	5	62.5	3	37.5	0	0
4	0	0	3	37.5	5	62.5
5	7	87.5	1	12.5	0	0

Table 4.9. Frequency of Overall codes of understanding for each Big Idea.

Big Idea	Strong		Moderate		Weak	
	#	%	#	%	#	%
1	9	10.6	46	54.1	30	35.3
2	10	11.8	31	36.5	44	51.8
3	13	15.3	25	29.4	47	55.3
4	5	5.9	46	54.1	34	40.0
5	6	7.1	24	28.2	55	64.7

In Table 4.6, the mean, its percent of the total, and the standard deviation for the set of problems on the PCA that correspond to each Big Idea are shown, first for the number of correct answers, and then for the partial credit score. The percent columns display the mean as a percent of the total number of items (for # correct) or total points (for score) for that Big Idea. These percent columns are necessary since the scoring scales are otherwise different for each Big Idea. They allow for direct comparisons between the Big Ideas, as well as the overall performance on the PCA. So for Big Idea 1, the PCA had 17 items for 68 total points on the partial credit scale. The average number of correct answers among these 17 items was 8.93, which is 52.5% of the total number of items. The average partial credit score was 46.81, which is 68.8% of the 68 available points. Note that for Big Ideas 1 and 4, the average number of correct items were both over 50%, and their partial credit scores both fell between 65% and 70%. By comparison, the average number of correct items was less than 40% for Big Ideas 3 and 5, and the average partial credit score was approximately 55% for both. So on average, the participants fared the best on items related to Big Ideas 1 and 4, and the worst on those related to Big Ideas 3 and 5.

In Tables 4.7, 4.8, and 4.9, the number and percent of participants that were classified as *strong, moderate,* and *weak* in each Big Idea based on the PCA, interviews, and the overall codes, respectively, are displayed. Recall that the scoring scales for each Big Idea were adapted from the scales used for the PCA as a whole. So a participant was classified as *strong* for a Big Idea if he or she answered at least 84% of the items corresponding to that Big Idea correctly (equivalent to 21 out of 25 correct answers), or earned 90% or more of the total partial credit points for those items (equivalent to 90 points out of 100). Likewise, *moderate* scores for each Big Idea then required either answering over 50% of the items correctly or earning at least 70% of the partial credit points. Anything under those thresholds on both scales merits a *weak* classification for that Big Idea. The interview classifications were based on their performance on each part of the interview, and the overall coding was determined by a comparison between the two, with the interview code taking precedence when necessary.

In looking at Table 4.7, only 32 of the 85 participants (37.6%) were classified as *weak* for Big Ideas 1 and 4 on the PCA. Meanwhile, for each of Big Ideas 3 and 5, the number of *weak* participants on the PCA is over 60%. Again, it appears that the participants performed strongest on items related to Big Ideas 1 and 4 on the PCA, and the weakest on Big Ideas 3 and 5.

However, the results based on the interviews seem to show the reverse. That is, the interview participants had high success with Big Ideas 3 and 5, with at least five of the eight participants classified as *strong* for each. Meanwhile, Big Idea 4 had five participants classified as *weak*, and Big Ideas 1 and 4 were the only Big Ideas that had any interview participants classified as *weak*. Big Idea 4 was also the only one where none of the interview participants were classified as *strong*.

With eight interview participants and five Big Ideas, there were 40 total codes of understanding from the interviews. The majority of these interview codes (26) were different from the PCA codes of the same Big Idea for the same student. This led to several changes from the number of PCA codes of understanding for each Big Idea to the number of overall codes of understanding. These changes can be seen when comparing Tables 4.7 and 4.9. For example, for Big Idea 5, the number of students who were classified as *moderate* based on the PCA alone (Table 4.7) was 19 (22.4%). This number increased to 24 (28.2%) when the interviews were also taken into account (Table 4.9). The reasons for these various differences will be discussed further in the section that compares the interviews to the PCA.

In order to determine if there were any significant differences between the participants' performance on each Big Idea, a Friedman's ANOVA by ranks test was run. It was determined that there were significant differences between the five groups ($X^2 = 28.074$, df = 4, $p < .001$). To determine where these significant differences existed, a post-hoc analysis with Wilcoxon signed-rank tests was used on each pair of Big Ideas, with a Bonferroni correction applied, resulting in an α-level set at .005. With the exception of Big Idea 5, which differed significantly from each of the other Big Ideas, there were no significant differences found between any pair of Big Ideas. In other words, there was no indication that for any pair of Big Ideas among the first four, a student was likely to understand one significantly better or worse than the other, while his or her understanding of Big Idea 5 was likely to be lower when compared to each of the other Big Ideas. The results of the post-hoc analysis are displayed in Table 4.10.

In addition to the Friedman's test, Spearman's rank-order correlation test was run in order to determine the strength of the correlations between the understandings of different Big Ideas. All correlations were found to be significant (α-level = .01), though the strongest correlations were between Big Ideas 2 and 5 (.808), Big Ideas 3 and 5 (.750), and Big Ideas 2 and 3 (.746). The weakest correlations were between Big Ideas 3 and 4 (.378) and Big Ideas 4 and 5 (.416). In other words, participants were most likely to be classified as having the same levels of understandings for Big Ideas 2 and 5, and they were most likely to be classified as having different levels of understanding for Big Ideas 3 and 4. The full results of the Spearman's test can also be found in Table 4.10.

The results of the Friedman's test and the Spearman correlation test tell us a few things. First is that the participants' understanding of Big Idea #5 is significantly less than their understandings of the other Big Ideas. However, it also has very strong correlations with Big Ideas 2 and 3, respectively. In other words, it is very likely that a student's level of understanding of Big Idea #5 will be the same as his or her level of understanding of Big Ideas 2 and/or 3. However, when those understandings were different, Friedman's test shows that the participant was significantly more likely to have a greater understanding of Big Idea 2 and/or 3 than Big Idea 5, rather than the other way around. When comparing Big Ideas 2 and 3 to each other, Friedman's test indicates that the chances that one was understood more than the other were equal. That is, of the participants whose understandings of Big Ideas 2 and 3 differed, about half of them had a stronger understanding of Big Idea 2, and the other half had a stronger understanding of Big Idea 3. So while the Spearman correlations suggest that the level of understanding of Big Idea 5 was generally close to the levels of understandings of some of the other Big Ideas, Friedman's test shows that most participants had a weaker understanding of Big Idea 5 than of any of the other Big Ideas. Finally, Big Ideas 1 and 4 have a strong correlation (.644) and appear to yield the strongest levels of understanding based on the PCA results alone. However, according to Friedman's test, those understandings are not significantly greater than those of Big Ideas 2 and 3.

To summarize, students' understandings of each of the Big Ideas were generally about the same, with the exception of Big Idea 5, which was lower than any of the other Big Ideas.

Table 4.10. Results of Wilcoxon post-hoc tests and Spearman's correlation tests.

Big Idea	1	2	3	4
2	$Z = -2.335$ $p = .020$			
	$\rho = .579$			
3	$Z = -2.03$ $p = .042$	$Z = 0.00$ $p = 1.000$		
	$\rho = .496$	$\rho = .746$		
4	$Z = -1.706$ $p = .088$	$Z = -.845$ $p = .398$	$Z = -.729$ $p = .466$	
	$\rho = .644$	$\rho = .483$	$\rho = .378$	
5	$Z = -4.667$* $p < .001$	$Z = -3.638$* $p < .001$	$Z = -3.273$* $p = .001$	$Z = -3.162$* $p = .002$
	$\rho = .568$	$\rho = .808$	$\rho = .75$	$\rho = .416$

The top values in each cell show the results of the post-hoc Wilcoxon signed-rank tests with Bonferroni adjustment. *A *p*-value of less than .005 reflects a significant difference in participants' understandings of those two Big Ideas. The bottom value, ρ, is Spearman's correlation coefficient, which shows the strength of the correlation between understandings of the Big Ideas. All such correlations were significant.

4.2 COMPARISONS TO AP CALCULUS EXAM PERFORMANCE

This section contains the results of comparisons between AP Calculus students' understandings of functions and their performance on the AP Calculus exam. Before answering the research questions, it is first important to note that, while 85 AP Calculus students participated in this study, only 67 of them took the AP Calculus Exam, so the results reported here will reflect those numbers. Table 4.11 shows the function understandings on the PCA of the students who took the AP Exam, along with those of all participants and the difference between the two. So of the 18 students who did not take the AP exam, 12 of them (67%) were classified as *weak* on the PCA, and the other 6 as *moderate*. All 7 students classified as *strong* on the PCA took the AP exam.

Also, of those 67 participants who took the AP Exam, the number and percent of participants that received each score on the exam are shown in Table 4.12. The average AP score for the participants was 2.54 ($\sigma = 1.48$).

Table 4.11. Function understandings on the PCA of participants who took the AP Exam.

	Strong	**Moderate**	**Weak**	**Total**
All Participants	7	35	43	85
Took the AP Exam	7	29	31	67
Change	0	-6	-12	-18

Table 4.12. Number and percent of participants for each score on the AP Exam.

AP Score	**5**	**4**	**3**	**2**	**1**
Total	10	9	13	10	25
Percent	14.93	13.43	19.4	14.93	37.31

4.2.1 Function Understandings and the AP Calculus Exam

RQ2a – What is the relationship between students' ability to solve problems about functions and their performance on the AP Calculus exam? The results of both the PCA and interviews were compared to the AP Calculus exam scores in order to answer this question. First, Table 4.13 shows the means and standard deviations of the AP Calculus exam scores for students who were classified as *strong*, *moderate*, and *weak* on the PCA and then the interviews. The number of participants in each category is also shown in the table.

For the PCA, a 1-way Between-Subjects ANOVA was performed in order to determine if any significant differences exist between the AP Exam scores of students for each level of function understanding. It was determined that the there are significant differences between the AP scores for participants with different levels of understanding ($F = 13.286$, $p < .001$). In order to determine where those significant differences exist, a post-hoc analysis using Tukey's range test was also run, and it was found that the differences between AP Scores were significantly different for each pair of function understandings: *strong* vs. *moderate* ($p < .001$), *strong* vs. *weak* ($p < .001$), and *moderate* vs. *weak* ($p = .038$). Essentially, these results indicate that students who were classified as *strong* based on their performance on the PCA generally scored high on the AP Calculus exam, and those who were classified as *weak* generally scored low, with scores of *moderate*-classified students in the middle. This was shown to also be true for those students who were classified as *strong* and *moderate* based on their performance in the interviews. The differences between the AP Exam scores were significant between those two groups (F = 10.714, p = .017). While no students were classified as having a *weak* understanding of functions based on the interviews, it is assumed that any student who would have was also likely to score low on the AP Calculus exam.

Table 4.13. Descriptives for levels of understandings on the PCA and in the interviews.

	Strong			**Moderate**			**Weak**		
	N	μ	σ	N	μ	σ	**N**	μ	σ
PCA	7	4.57	.787	29	2.72	1.579	31	1.9	.978
Interview	4	4.5	1	4	2	1.155	NA		

It is also important to look at how the AP scores break down within each category of understanding. These distributions can be seen in Table 4.14. A Chi-Square test for independence (df = 4) was run in order to check for a relationship between the distributions. It was determined that these distributions are related (X^2 = 30.178, p < .001). That is, there is a significant relationship between exam score and function understanding. Higher scores correspond to stronger understandings, and lower scores with weaker understandings. Note how 5 of the 7 *strong* students scored a 5 on the AP exam, and none of them scored lower than a 3. Likewise, none of the 31 *weak* students scored a 5 on the exam, and only 2 of them scored a 4. Meanwhile, 22 of them scored either a 1 or a 2 on the exam.

Table 4.14. Breakdown of AP Exam scores for each level of understanding on the PCA.

AP Score	Strong	Moderate	Weak
5	5	5	0
4	1	6	2
3	1	5	7
2	0	2	8
1	0	11	14

The scores of the *moderate* students, however, were much more evenly distributed. Thirteen of the 29 students scored a 1 or a 2, while 11 of them scored a 4 or a 5. The reason for this phenomenon can be explained upon further inspection of the students' PCA scores. Of the *moderate* students who scored a 4 or a 5 on the AP exam, the average number of correct answers on the PCA was 17.18 ($\sigma = 2.483$), and their partial credit score mean was 80.27 ($\sigma = 6.695$). Meanwhile, for those who scored a 1 or a 2 on the AP Exam, the average number of correct answers was only 14 ($\sigma = .913$), and the average partial credit score was 69.69 ($\sigma = 4.404$), which just barely falls in the *weak* range. Also, the highest number of correct answers on the PCA for this group was 16, which only one student achieved. In contrast, eight students from the group with high AP Exam scores had at least 17 correct answers on the PCA. Essentially, the *moderate* students with high scores on the AP exam generally scored higher on the PCA than the *moderate* students with low scores on the AP exam.

For the interview participants, two of them scored a 1 on the AP exam, both of whom were classified as *moderate* in the interview. Three of them scored a 3 (two *moderate*, one *strong*), and three of them scored a 5 (all *strong*). The breakdown of how each interview participant fared on the PCA, in the interview, and on the AP exam is shown in Table 4.15. The participants who were *strong* on both the PCA and the interview both scored a 5 on the AP Exam. One of the two participants who were *moderate* on the PCA and *strong* on the interview also scored a 5, while the other scored a 3. The four students who were *moderate* on the interview were also *weak* on the PCA, and two of them scored a 3 on the AP Exam while the other two scored a 1. The only notable difference upon closer inspection was that both participants who scored a 1 had a partial credit score of less than 50 on the PCA, while the partial credit scores of the participants with a 3 were 56 and 60. So for the most part, the interview results were correlated with the AP Exam scores, and the differences between students who had similar interview results but different exam scores could be explained further when examining the PCA results.

Table 4.15. Performance on the PCA, interviews, and AP Exam.

Participant	School	PCA # Correct/Points	PCA Partial Credit Score	PCA	Interview	AP Score
Steve	A	23/92	97	Strong	Strong	5
Eric	A	15/60	77	Moderate	Strong	5
Amy	B	15/60	72	Moderate	Strong	3
Jessica	B	19/76	90	Strong	Strong	5
Natalie	B	10/40	49	Weak	Moderate	1
Heather	C	9/36	56	Weak	Moderate	3
Andrew	C	12/48	60	Weak	Moderate	3
Rachel	A	8/32	43	Weak	Moderate	1

4.2.2 Function View and the AP Calculus Exam

RQ2b – What is the relationship between students' view of functions and their performance on the AP Calculus Exam? For students who were classified as having an *action* view of functions, the average score on the AP Exam was 1.81, with a standard deviation of .962. For those classified as having a *process* view of functions, the mean AP score was 3.03, with a standard deviation of 1.577. Using a 1-Way Between-Subjects ANOVA, it was found that these scores were significantly different (F = 12.677, p = .001). In other words, students who were found to have a *process* view of functions scored much higher on the AP Calculus Exam than those with an *action* view. The breakdown of AP Exam scores by function view (including those that show *object* view characteristics based on the interviews) is shown in Table 4.16.

In order to check for a relationship between the distributions for the *action* and *process* views across AP score, a Chi-Square test for independence (df = 4) was run. It was determined that these distributions are related (X^2 = 13.882, p = .008). It is clear from the test and from the table that the majority of students with an *action* view had a low score on the AP Exam, as only 8 of the 27 students scored a 3 or higher, and none of them scored a 5. However, the distribution of students with a *process* view was not skewed toward the top. While 18 of the students scored a 4 or a 5 on the AP Exam, almost the same amount of students (16) scored only a 1 or a 2 on the exam, with the majority of that scoring a 1. Based on these results, it could be argued that while a *process* view of functions is likely necessary in order to succeed in calculus, it is not sufficient.

Table 4.16. Breakdown of AP Exam scores for each function view.

AP Score	Process	Action	Possibly Object
5	10	0	1
4	8	1	0
3	6	7	1
2	5	5	0
1	11	14	2
Did not take	6	12	

4.2.3 The Big Ideas and the AP Calculus Exam

RQ2c – What is the relationship between students' understanding of each of the Big Ideas and their performance on the AP Calculus Exam? For each Big Idea, the mean and standard deviations of AP scores were recorded for the participants who were classified as being *strong*, *moderate*, or *weak* in that respective Big Idea. These values are shown in Table 4.17. For each of the Big Ideas, the AP Score mean decreases along with the level of understanding.

Table 4.17. Descriptives of AP scores for levels of understanding in each Big Idea.

Big Idea	Strong			Moderate			Weak		
	N	μ	σ	N	μ	σ	N	μ	σ
1	9	4.44	.882	38	2.53	1.447	20	1.70	.865
2	9	4.67	.707	27	2.56	1.34	31	1.9	1.165
3	13	3.92	1.382	21	2.43	1.502	33	2.06	1.071
4	5	4.4	.894	39	2.69	1.472	23	1.87	1.18
5	6	4.83	.408	21	3.05	1.431	40	1.93	1.141

For each of the Big Ideas, a 1-Way Between-Subjects ANOVA test was conducted in order to determine significant differences between AP scores for each level of understanding of the respective Big Idea. All five tests yielded significant results, so for each Big Idea, a Tukey range post-hoc analysis was run to test for significant differences between the AP scores for each pair of understandings (*strong* vs. *moderate*, *strong* vs. *weak*, and *moderate* vs. *weak*). The results of the ANOVA and the Tukey post-hoc analyses are shown in Table 4.18. As can be seen from the table, all of the pairwise differences in AP Scores were significant, with the exception of scores between *moderate* and *weak* for Big Ideas 2, 3, and 4. Essentially, a student classified as *strong* in at least one of the Big Ideas had a significantly better score on the AP exam than a student who was not classified as *strong* in any Big Idea. Also, those classified as *moderate* for Big Ideas 1 and 5 also had significantly higher AP scores than those classified as *weak*, while for Big Ideas 2, 3, and 4, the *moderate* students scored higher than the *weak* students, but not significantly so. Finally, note that Big Idea 3 is the only Big Idea where the mean for the *strong* students is lower than 4, and it is also the only Big Idea where the mean for the *weak* students is higher than 2.

Table 4.18. ANOVA and post-hoc results for AP scores among understandings for each Big Idea.

Big Idea	**ANOVA**	***Strong* vs. *Moderate***	***Strong vs. Weak***	***Moderate vs. Weak***
1. Function Concept	$F = 15.285$ $p < .001$	Diff = 1.918 $p < .001$	Diff = 2.744 $p < .001$	Diff = .826 $p = .048$
2. Covariation	$F = 18.659$ $p < .001$	Diff = 2.111 $p < .001$	Diff = 2.763 $p < .001$	**Diff = .652** $\boldsymbol{p = .103}$
3. Families	$F = 9.351$ $p < .001$	Diff = 1.495 $p = .006$	Diff = 1.862 $p < .001$	**Diff = .368** $\boldsymbol{p = .582}$
4. Transformations	$F = 7.865$ $p = .001$	Diff = 1.708 $p = .026$	Diff = 2.53 $p = .001$	**Diff = .823** $\boldsymbol{p = .06}$
5. Multiple Representations	$F = 18.011$ $p < .001$	Diff = 1.786 $p = .006$	Diff = 2.908 $p < .001$	Diff = 1.123 $p = .003$

All differences are significant except those in **bold**.

Finally, for each Big Idea, the total number of students in each level of function understanding was calculated for all five AP exam scores, shown in Table 4.19. Chi-square tests for independence (df = 4) were also run for all Big Ideas, and all five tests were found to be significant. That is, understandings of each Big Idea were related to AP Exam scores. The results of these tests are also shown in Table 4.19. There are several noteworthy things in the table. First, there are two students with a *strong* understanding of Big Idea 3 that scored less than a 3 on the exam. No other Big Ideas had a *strong* student score of less than 3. Also, none of the students who scored a 4 or a 5 on the exam were *weak* in Big Idea 1, and only two of them were *weak* in Big Idea 4. At least four of them were *weak* in each of the other three Big Ideas. Finally, for Big Idea 5, all but one of the *strong* students scored a 5, with the lone exception scoring a 4. At the other end of the scale, all but six of the participants who scored a 1 or a 2 on the exam were *weak* in Big Idea 5. For the other four Big Ideas, at least ten participants with a 1 or a 2 had at least a *moderate* understanding of that Big Idea.

Table 4.19. Total participants to get each exam score for each level of understanding of each Big Idea.

AP Score	**Big Idea 1**			**Big Idea 2**			**Big Idea 3**			**Big Idea 4**			**Big Idea 5**		
	S	**M**	**W**	**S**	**M**	**W**	**S**	**M**	**W**	**S**	**M**	**W**	**S**	**M**	**W**
5	6	4	0	7	2	1	7	2	1	3	6	1	5	4	1
4	1	8	0	1	5	3	1	4	4	1	7	1	1	4	4
3	2	6	5	1	8	4	3	5	5	1	7	5	0	7	6
2	0	6	4	0	3	7	1	0	9	0	7	3	0	1	9
1	0	14	11	0	9	16	1	10	14	0	12	13	0	5	20
X^2 Test	$X^2 = 31.346$, $p < .001$			$X^2 = 39.012$, $p < .001$			$X^2 = 29.042$, $p < .001$			$X^2 = 16.437$, $p = .037$			$X^2 = 36.621$, $p < .001$		

Note: All X^2 tests were significant.

4.3 PCA VS. INTERVIEWS

One of the important aspects of these results is the noted difference between the PCA results and the interviews. While only eight of the participants actually took part in the interviews, there was much insight gained from those interviews, especially in comparison to the PCA. First, the interview performances were in general better than that of the PCA. One possible reason for this is due to the interview environment, which was significantly different than that of the PCA. The PCA was administered during their regular AP Calculus class, so the students had about 50 minutes to quietly work on it as their classmates did the same. In the interviews, each participant was only with the researcher, and they were allowed to take as much time as they needed on each task. The interviews were open and interactive, which granted them the opportunity to voice their thoughts and explain themselves out loud as they worked on the tasks, which they were not able to do as they took the PCA. They also could ask clarifying questions, and the interviewer could also ask them to further explain a given response if it was deemed unclear. These factors likely had at least some influence on the difference between their performance on the PCA and in the interviews as a whole.

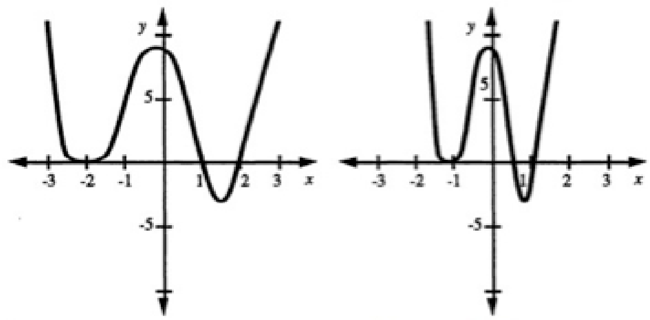
When focusing on each individual Big Idea, however, there are also some noticeable differences between the PCA and interview results. For example, as was stated previously, the participants' understandings of Big Ideas 1 and 4 seemed to be the strongest based on the results of the PCA. However, these were the only two Big Ideas in which some interview participants received *weak* codes. Likewise, participants had significantly lower levels of understanding with Big Idea 5 on the PCA than with any of the others. Yet on the interviews, all but one of the students were classified as *strong* for Big Idea 5.

These discrepancies can be traced to the differences between the related items on the PCA and those on the interviews. For example, Big Idea #4 is titled *Combining and Transforming Functions*, and is made up of function transformations, function compositions, and function inverses. Most of the items on the PCA that were related to Big Idea 4 focused on either function compositions (e.g., items 5, 12, and 16) or function inverses (e.g., items 13, 14, and 23). Since actual function transformations were not as well-represented on the PCA, the interview tasks that focused on Big Idea 4 were transformation tasks. As discussed in Chapter 2, transformations in particular tend to be much more difficult for students than other aspects of functions. The participants' performance on these transformation tasks reflected that. For example, only two of the participants correctly solved Task 1, which asked students to identify the corresponding equation of a transformed graph (see Figure 4.1). While most were able to narrow it down to the correct *type* of transformation, they usually resorted to guessing or trying to remember a rule, as Jessica demonstrated:

> **Jessica: "**So, the graph is just, it's shrinking in, but the like the *x* is moving in 1/2, the *y*'s aren't changing at all. So I immediately crossed out all the ones that are adding and subtracting because it's not shifting in anyway. It's not absolute value because it still has like negative values in it, so I crossed out all of those. It's not the opposite of it because it's still the same graph. And then I crossed out these two initially just because it seems like it switched the *y*-value, and these here, the *y*-value over here either decreased it by half or multiplied it by two. That just left me with either 2*x* or 1/2*x* in side the parentheses. And since it's decreasing by a half I said 1/2 *x*, it' just idunno. It took me a while because I know in the equations and stuff there's stuff like you need to do the opposite sometimes, and whenever I try, I always mess up with that stuff and do the wrong thing."

The other transformation tasks were met with the same amount of success. Only three students correctly solved Task 2, in which they were to identify a parameter in a stock equation that would be the same for three different given graphs. In Task 3, the students were asked to graph an absolute value equation, and then graph three different transformations of that equation. Only two students were able to correctly produce all four graphs, while four of them could not produce more than one correct graph. Finally, no student could produce the correct solution to Task 4, in which they were to graph the reciprocal of a given parabola.

Task 1 – In the figure on the left you are given the graph of the function $y = f(x)$



Which of the following formulas corresponds to the graph on the right?

$y = f(2x)$	$y = f(x+1)$	$y = f(\lvert x \rvert)$
$y = f\left(\frac{1}{2}x\right)$	$y = f(x-1)$	$y = f(-\lvert x \rvert)$
$y = 2f(x)$	$y = f(x)+1$	$y = f(-x)$
$y = \frac{1}{2}f(x)$	$y = f(x)-1$	$y = -f(x)$
	$y = \lvert f(x) \rvert$	

Figure 4.1. Task 1 in part 4 of the interview.

Overall, 5 of the students were classified as *weak* in Big Idea 4 while the other 3 were only *moderate*. While every other Big Idea had at least 3 participants classified as *strong* in the interview, Big Idea 4 did not have any. Also of note is the fact that five participants had the same level of understanding of Big Idea 4 on both the PCA and the interview, while the other four dropped a level. It was the only Big Idea in which some participants' understandings from the interview were lower than those of the PCA, and also the only one in which no participants had a higher level of understanding on the interview. This discrepancy likely can be attributed to the lack of transformation problems on the PCA.

On a similar note, the same argument can be made for Big Idea 5, but in the opposite direction. As noted, most students struggled with problems that related to Big Idea 5, *Multiple Representations*, on the PCA. For example, in item #7, a formula is given, and the students must identify a correct verbal interpretation of the formula. Only 33 of the 85 participants (39%) correctly answered this item. However, the students in the interviews thrived in this Big Idea, as seven of them were classified as *strong*, and the other as *moderate*. Again, this difference could be noted in the types of tasks the students are being asked to perform. In the interview, the task for Big Idea 5 centers on looking at different representations of the same function and identifying where in those representations lie important pieces of information about the function, and deciding which representation best displays each piece of information. This was something the participants were able to do with ease, especially since the function that was used for this problem was linear. In contrast, many of the items found on the PCA that relate to *Multiple Representations* used non-linear functions, and they involved the need to look at one representation and use that to create a different one in order to help solve the problem. For example, item #17 describes an effect in which circular ripples in water travel at a certain speed. No diagram is actually given, and a formula is asked for. It is up to the student to move from the contextual paragraph to the formula, with the possibility of producing a diagram or graph along the way. Also, the function itself is nonlinear. Only 12 of the 85 participants (14%) correctly answered this item. Essentially, producing representations and/or working with nonlinear functions seemed to give the students great difficulty, while moving between already-given representations and working primarily with linear functions was much more within their grasp.

So essentially, the differences between the understandings determined by the PCA and those determined by the interviews is most likely due to the different environments between the two settings, the opportunity for questions, explanations, and clarifications in the interviews, and the differences in the types of tasks the participants were asked to solve. The high correlation between function understanding on the PCA and performance on the AP Calculus Exam supports the idea that the PCA is a good tool for predicting AP Calculus Exam performance, despite the fact that the interviews may provide a more in-depth picture of what students understand about functions.

4.4 SUMMARY

The results of this study have shown that the majority of participants had a *weak* understanding of functions, and that a very small percentage of them had a *strong* understanding. However, participants who interviewed generally demonstrated a stronger understanding of functions than they did on the PCA, although some of that could be attributed to the differences between the types of tasks they were asked to solve in the interviews. Additionally, most of the participants were classified as having a *process view* of functions as opposed to an *action* view. The participants were most likely to have a weaker understanding of *Multiple Representations* when compared to any other Big Idea, but it was also unlikely that a participant understood two different Big Ideas at significantly different levels.

When comparing function understanding to AP Calculus exam scores, it was found that the level of understanding was directly correlated with the score. That is, a student with a high level of understanding likely scored a 4 or a 5 on the AP Exam, and a similar correlation existed between low levels of understanding and low scores. While students with an *action view* of functions primarily scored low on the exam, those with a *process view* produced mixed results.

Students with a *strong* understanding of at least one Big Idea generally scored higher on the AP Exam than those who did not. The level of understanding of *Multiple Representations* was positively related to exam score. Finally, students with a high exam score had at least a *moderate* understanding of the *Function Concept*.

The analysis and significance of these results, along with conclusions, will be discussed in Chapter 5.

5.0 DISCUSSION

The purpose of this research study was to examine AP Calculus students' understandings of functions, and to compare how their understandings related to their performance on the AP Calculus exam. It also aimed to determine if certain aspects of function conceptualization, such as function view or the understanding of any of the Big Ideas of functions, have significant correlations with the AP Calculus Exam scores.

The findings reported in Chapter 4 suggest that most AP Calculus students generally do not have a very strong understanding of functions. They also suggest that a student's level of proficiency with functions is highly correlated with his or her AP Calculus Exam score. In the following sections, the results of the study will be discussed in connection with existing research on function understandings, including their connection to students' success in calculus. This will be followed by a discussion of other aspects of the study, and end with general conclusions, implications, and ideas for follow-up studies. Specifically, Section 5.1 provides a detailed description of AP Calculus students' understandings and views of functions, and relates those understandings to previous research. Section 5.2 discusses the connections between the different views and understandings of functions and AP Calculus exam performance. Section 5.3 discusses other interesting findings that emerged from the study, such as noted differences in performance and understanding based on school site, grade, and gender. Finally, Section 5.4

provides information on the significance and implications of the study and ideas for further research.

5.1 AP CALCULUS STUDENTS' UNDERSTANDINGS OF FUNCTIONS

In this section, a description of AP Calculus students' understandings of functions is provided. It begins with a discussion of the general understandings of functions based on the results of the study. It is followed by a discussion of how most of the participants viewed functions, and concludes with a description of the specific difficulties and misconceptions of functions that most of the participants seemed to have.

5.1.1 General Function Understandings

Based on the results of the PCA in this study, just over half of the participants (51%) were classified as having a *weak* understanding of functions. This result was similar to those found by Carlson, et al. (2010) when they administered the PCA to 248 college students entering a beginning calculus course. In their study, 49% of the participants had no more than 12 correct answers on the PCA. Recall that a participant in the study reported herein had to score no higher than 12 in order to be classified as *weak*, and that this cut score was based on Carlson, et al.'s results. Carlson and her colleagues found that 77% of the students with a score higher than 12 on the PCA passed calculus with at least a C, while only 40% of the students who scored no more than 12 had done so.

Also, it is important to note that the partial credit scale developed in this study increased the likelihood that a student would be classified as *moderate* rather than *weak*. This is due to the fact that in order to be classified as *weak*, the participant had to be *weak* on both the correct-answer scale and the partial-credit scale. However, only one participant scored less than 12 on the PCA but earned enough partial credit to still be classified as *moderate*, so there was no difference between the results of the two studies. Basically, approximately half of the calculus students in both studies had a *weak* understanding of functions.

The results of both studies suggest that the majority of students completing calculus do not have the level of understanding of functions needed to be successful in calculus. Recall from Chapter 3 that the PCA was given shortly after the AP Calculus Exam had been administered, so it is difficult to know what the students' understandings of functions were at the beginning of the course. Similarly, the participants in Carlson, et al.'s (2010) study took the PCA before they began calculus, and thus it is difficult to know what their understandings of functions were at the end of calculus. In both cases, it seems likely that students' understandings of functions would either remain the same or slightly increase. Recall from Chapter 2 Carlson's (1998) study of students at various points in the undergraduate mathematics curriculum. The claim was made that understanding of the function concept develops very slowly over time, but does tend to get stronger as students move through the curriculum. So while it is possible that a calculus course had some effect on students' understanding of functions, the likelihood that the effect would be a *decline* in function understandings is low. One would expect this to be especially true for students who have reached at least a *process* view of functions, as their level of understanding is then based more on their ability to reason about functions and make sense of what is happening. Any student who seemed to show a decline in function understanding from the time the course began would likely be relying more on the memorization of rules and procedures, which are characteristics of an *action* view (Asiala, et al., 1996), in which case their understandings of functions were probably *weak* to begin with. Therefore, the high number of students with a *weak* understanding of functions at the end of AP Calculus is a strong indicator that there are too many students entering the course with insufficient understandings of the prerequisite knowledge needed to be successful.

5.1.2 The Participants' View of Functions

The results have also shown that just over half of the participants (54%) had a *process* view of functions at the end of the AP Calculus class. Recall from Chapter 2 that many researchers (e.g., Carlson, 1998; Dubinsky & Harel, 1992) have claimed that a *process* view is indicative of a well-formed understanding of functions. The results of this study supported these claims, as the majority of participants with a *process* view (83%) had at least a *moderate* understanding of functions, and 90% of students with an *action* view were classified as having a *weak* understanding of functions (refer to Table 4.3). In fact, based on the PCA results, 87% of the participants in the study had either a *process* view and at least a *moderate* understanding of functions, or an *action* view and a *weak* understanding of functions. Also, recall from Chapter 4 that all of the students who were either *action*-and-*moderate* or *process*-and-*weak* had PCA scores either just below or just above the cut score of 13 correct answers, which might suggest the development of their understandings and views of functions had not yet solidified and they were in a transitional stage. It supports the claim that a *process* view is generally one of the characteristics of a *moderate* understanding of functions.

5.1.3 Difficulties and Misconceptions

In order to obtain further insight into AP Calculus students' understandings of functions, special attention must be paid to particular aspects of their understandings. Much of the research on functions described in Chapter 2 referred to the common difficulties and misconceptions that students tend to have with functions. Such difficulties include, but are not limited to, struggles with interpreting function notation; working with functions whose equations and/or graphs do not look "regular" or "nice" (e.g., piecewise functions, functions with discontinuities, etc.); and tendencies to interpret graphs as a literal picture, referred to by Leinhardt, et al. (1990) as an *iconic representation* (Carlson, 1998; Leinhardt, et al, 1990; Oehrtman, et al., 2008). In examining the results of the PCA and interviews, the most common difficulties the AP Calculus students had with functions were as follows:

- Identifying functions and nonfunctions in algebraic form
- Transforming functions
- Misinterpreting function inverse notation
- Viewing graphs as iconic representations of phenomena
- Moving between real-world situations and the functional representations that model them

Each of these difficulties are described in detail below.

5.1.3.1 Identifying functions and nonfunctions in algebraic form

In Part 1 of the interview, each participant was given 18 relations, and was asked to determine whether each relation was a function or not. The relations came in several different representational forms, but the only ones that the participants as a group had difficulty with were those in the form of an algebraic expression. In particular, the two piecewise relations (#4, which was a function, and #9, which was not) were the most challenging for students. Even though some of the students recognized the relations as "piecewise", they still struggled in determining whether they each fit the definition of a function. There was a third algebraic relation (#8) that half the students incorrectly identified, and a fourth (#10) where one student misidentified it, and several others only guessed that it was a function, but were unable to explain why.

Most of these difficulties could likely be attributed to the fact that there is no simple rule to follow when identifying functions in algebraic notation. The students generally had little difficulty with the relations presented as graphs because all they needed to do was apply the Vertical Line Test, whether they understood it or not. For the tables, they needed to simply check for whether all of the input values were different. If so, then it was a function, otherwise it was not. However, the formulas required more sense-making. For example, item #10 could be identified as a function by recognizing that it was a sum of three different known functions: a quadratic, an exponential, and a logarithm. But for item #9, the students had to identify that the domains for the two pieces of the relation overlapped, a completely different approach from item #10. These formulas required an understanding of the notation and what it represents, and it seems as if the students generally preferred to look for a rule or a procedure that they could follow. Rachel even tried to explicitly do this, as she identified functions as those that could be solved for y. As a result, she incorrectly identified a formula for a circle (item #8) as a function, because she could still solve it for y. She was so fixated on using a rule that she did not notice that the existence of the y^2 term was significant. This supports the idea presented by Carlson

(1998) in that students' difficulties with algebraic expressions and notation is often a result of relying too much on the memorization of rules and procedures, and not enough on understanding the relationships they represent. Whereas other representations seemed to have an easy rule to apply, the lack of such a rule for relations in algebraic form seemed to make them more difficult for students to identify correctly, as they were required to actually make sense of what was going on in each relation instead.

5.1.3.2 Transforming functions

This difficulty can be seen in the interview participants' performance on the tasks of Part 4. Not a single student was classified as *strong* in this part of the interview. The most successful tasks in Part 4 were parts b) and c) of the third task, in which the participant was asked to graph a vertical and horizontal translation, respectively, of a previously defined graph. In both cases, five of the eight students produced a correct result. So on each of the other tasks in Part 4, no more than half the participants produced a correct result. This difficulty with transformations is punctuated by their performance on task 4, in which not one student was able to produce the correct graph of the reciprocal of a parabola. These struggles are further evidence that function transformations are extremely difficult and require a higher level of thinking than other aspects of functions (Baker, et al., 2001; Eisenberg & Dreyfus, 1994; Lage & Gaisman, 2006).

5.1.3.3 Misinterpreting function inverse notation

On the PCA, there were three items (#13, 14, and 23) that used function inverse notation (denoted $f^{-1}(x)$). On all three items, less than 25% of the participants got the correct answer, and in most cases they were taking the notation to refer to the reciprocal of the function, as they believed the raised -1 meant “to the negative one power”, rather than “inverse of the function.” This is a specific case of the problem with recognizing function notation as observed and discussed by Carlson (1998) and Oehrtman, et al. (2008).

5.1.3.4 Viewing graphs as iconic representations of phenomena

Items #8 and #15 on the PCA proved to be difficult for most students. In both cases, a graph was given and the students had to interpret what the graph represented. Only 21% and 36% of respondents had the correct answer, respectively, while an iconic representation was the most common answer choice (64% and 50% respectively). The students seemed to be selecting the answer choice that best represented the look or shape of the graph rather than interpreting what the graph actually represented. Both of these items also highlighted the struggles students seemed to have in making sense of rate and what it represents. For instance, item #8 presented a velocity vs. time graph, and most students may have been interpreting it as a position vs. time graph instead. They did not realize that one of the dimensions of the graph was actually a rate. In item #15, the key to getting the correct answer was identifying that the rate presented in the graph was steadily increasing. In choosing an iconic representation instead, either the students were not attending to the rate, or they misinterpreted it. Further possible reasons for students’ struggles with these problems are explained below.

5.1.3.5 Moving between real-world situations and functional representations

This was clearly the most common difficulty across participants. The problems that asked students to do this were well-represented on both the PCA and in the interviews. For example, items #8 and #15 discussed above can be included in this category, as they both require students to interpret the phenomenon represented in graphical form. There are four other PCA items that required students to move between a function and a situation in some capacity. Three of them (items #7, #20, and #22) required the students to interpret the meaning of a formula or part of a formula. For example, item #7 provided an exponential growth function that models the number of bacteria in a culture over time, $P(t) = 7(2)^t$. Students were asked to determine what happens when the base of the exponent in the function is changed from 2 to 3. Only 41% of respondents chose the correct answer, "the number of bacteria triples every day instead of doubling every day." The students' answers were somewhat evenly spread out among three of the other four choices, which indicates that, instead of a clear-cut misconception, the students as a group just were not sure of how to interpret what the function was actually doing. Overall, of these three problems, #20 was the only one on which over half of the respondents chose the correct answer.

In another item, #17, the students were required to create a function from a real-world phenomenon. They were asked to produce an expression for the area of a circular ripple in a lake traveling at a given rate, in terms of the amount of time that has passed since the ripple was created. Only 19% of respondents chose the correct answer. However, 50% of respondents chose the correct answer for #4, a similar problem in which they were also asked to produce an expression for area, this time for that of a square in terms of its perimeter. The reasons for the disparity between the results of the two items can be drawn from two major differences. First is that #17 describes a real-world situation, where #4 just asks about a square, a common shape.

Item #4 requires no mathematical interpretation of a real-world context. Second, #4 is static while #17 is dynamic. That is, a rate is given in #17, something the students do not have to deal with in #4. These differences are likely what led to the vast disparity between the number of correct answers between the two problems.

Overall, of the six items that required students to either move from a functional representation to a real-world situation or vice-versa, the maximum percentage of respondents to get the correct answer was 50%.

In addition to these PCA items, this difficulty was also highlighted by participants' performance on the interview tasks. For example, in Part 2 of the interview, the second task described a situation where hoses were filling up a tub of water, and the participant was asked to determine how long it would take a certain number of hoses to fill up the tank. Half of the participants got this wrong, citing that as the number of hoses grew, so did the time it took to fill up the tub, when in fact the opposite is true. Also, in Part 3, students were asked to determine what type of function a given representation portrayed. There were three contextual descriptions given along with one graph, one table, and one formula. Half of the students were either unable to identify the correct function type and/or simply guessed which one it was for the two contextual descriptions of nonlinear functions. For each of the other representations, including the description of a linear function, at least six of the eight participants were able to identify the correct function type.

Finally, the fact that the participants did not struggle when the function was linear is important to note. In Part 5 of the interview, the participants were asked to identify important pieces of information found within four different representations of a linear function, which includes a description of a real-world situation. All eight interview participants were coded as *strong* for this part of the interview, and they generally had no problem identifying the relevant information in the description as well as the other representations. This is different from similar items described above in the PCA (such as #7, #8, and #17) and in Parts 2 and 3 of the interview, where students were asked to essentially move either from a description of a phenomenon to a graph or formula or vice-versa. For one thing, each of the functions in those items is nonlinear, and students tend to struggle more with nonlinear functions (Baker, et al., 2001; Leinhardt, et al., 1990). Secondly, in most of the cases described above, the students are asked to *produce* or *interpret* the function, whereas in Part 5 of the interview, they only needed to *identify* the required information. It could be argued that the cognitive demand of each of the specified problems on the PCA is much higher than that found in the tasks of Part 5. This could also help explain why the participants mostly struggled with Big Idea 5 on the PCA, but did very well on it in the interviews. The disparity between the level of difficulty of the interview tasks and those found on the PCA seem to indicate the existence of a point where the understanding of Big Idea 5 concepts drop off significantly, as the skills assessed by the PCA items described above seem to be of greater difficulty to AP Calculus students than those assessed in Part 5 of the interview.

5.1.4 Summary

Based on the results of this study, it appears that about half of the students in an AP Calculus course have a *weak* understanding of functions, and slightly over half of the students have a *process* view of functions. One of the major difficulties most of them have is moving between a functional and contextual representation. This includes making sense of all or part of a functional formula or expression, and interpreting the meaning of graphs, in which mistaking them for an iconic representation is commonplace. Problems that include dynamic systems such as the hose problem from the interview and the ripple problem from the PCA also seem to cause issues for most AP Calculus students. Other difficulties included understanding function inverse notation, identifying functions from algebraic representations, and difficulties with transforming functions.

5.2 FUNCTION UNDERSTANDINGS NEEDED FOR SUCCESS IN AP CALCULUS

There was a very strong correlation between the participants' understandings of functions and their performance on the AP Calculus Exam. This correlation reinforces the idea that knowledge and understanding of functions is what is needed in order to succeed in an AP Calculus course (College Board, 2010). However, some researchers have given more specific parameters for just what aspects of functions are needed for calculus success. In particular, Oehrtman, et al. (2008) argued that a process view of functions (from Big Idea 1) and strong covariational reasoning skills (from Big Idea 2) are essential to understanding the primary concepts of calculus. First, of the 40 students with a *process* view who took the AP Exam, 24 of them (60%) scored a 3 or higher. However, of the 27 students with an *action* view who took the exam, just one student scored higher than a 3, and only 7 others (26%) scored a 3 (refer to Table 4.16). Therefore, it could be said that a process view is necessary for success in calculus, but it is not sufficient.

As for covariation, 24 of the 32 participants (75%) who scored at least a 3 on the exam had at least a *moderate* understanding of Big Idea 2. All but three of these students also had a *process* view of functions. However, there were also 9 students who scored less than 3 on the exam despite having both a *moderate* understanding of Big Idea 2 and a *process* view of functions. So perhaps a *moderate* understanding of covariation is not a strong enough predictor for success in calculus. Instead, it could be the case that having both a *process* view and a *strong* understanding of covariation is both necessary and sufficient for success in calculus. The results of this study support this theory, as 7 of the 9 students (78%) who had both a *process* view and a *strong* understanding of Big Idea 2 scored a 5 on the exam, with the other two scoring a 4 and a 3, respectively. It is likely that students who were classified as having a *strong* understanding of covariation are capable of adequately performing at least four of the five mental actions of covariational reasoning (refer to Table 2.1) as defined by Carlson, et al. (2002), and thus are likely to do well in a calculus course. Since every student with a *strong* understanding of Big Idea 2 also had a *process* view of functions, it is likely that the latter is necessary in order to develop the former. This is supported by the idea that a strong indicator of a *process* view is the ability to identify and explain how a function's input and output are related, and that part of that relationship is how the input and output covary. In general, the results of the study support the claims of Oehrtman, et al. (2008), in that both a process view of functions and high skill with covariational reasoning is essential to success in calculus. The full breakdown of exam scores for students based on their function view and understanding of Big Idea 2 can be seen in Table 5.1.

Table 5.1. Breakdown of exam scores by function view and Big Idea 2.

	Strong and Process	**Moderate**		**Weak**	
		Process	**Action**	**Process**	**Action**
5	7	2	0	1	0
4	1	5	0	2	1
3	1	5	3	0	4
2	0	2	1	3	4
1	0	7	2	4	12
Did not take	1	3	1	2	11

Note: There were no "strong" students in Big Idea 2 that had an action view, so that column was omitted.

It should also be mentioned that function view makes up a large part of understanding Big Idea 1. In fact, every student classified as having a *process* view of functions had at least a *moderate* understanding of Big Idea 1, and all students with a *strong* understanding of Big Idea 1 also had a *process* view. More information about the ties between function understanding and success in AP Calculus is revealed when the focus is switched from function view to the level of understanding of Big Idea 1, as now the difference between *strong* and *moderate* understandings of Big Idea 1 can be accounted for. For example, the average exam scores for participants with a *strong* and *moderate* understanding of Big Idea 1 (not including the small subset of *moderate* students with an *action* view) are 4.44 and 2.61 respectively, a difference of almost 2. When also accounting for covariation, there were 12 students who took the exam and were classified as *strong* in at least one of Big Ideas 1 and 2. For these students, the average AP Exam score was 4.5. For the 19 students with *moderate* understandings of both, the average score was only 2.47, a difference of over 2. This further supports the claims of Oehrtman et al. (2008), while shedding more light on the degree to which students should understand the function concept and covariation. It seems as if a good understanding of both is necessary, while a strong understanding of at least one is essential to success in calculus.

One of the other Big Ideas that is likely necessary for calculus success is Big Idea 5, *Multiple Representations.* All but one of the students classified as *strong* in Big Idea 5 scored a 5 on the AP Exam (with the remaining student earning a 4), and 29 of the 34 students who were *weak* in Big Idea 5 scored a 1 or 2 on the exam (refer to Table 4.19). Therefore, it seems as if students proficient in working in multiple representations of functions are more likely to have success in calculus. This is most likely due to the notion that fluency with different representations of functions is indicative of a strong understanding of functions in general

(Cooney, et al., 2010; Kaput, 1998; Keller & Hirsch, 1998; Moschkovich, et al., 1993), and includes the ability to easily move between representations and to choose an appropriate representation to work in.

Finally, while it is important to note which ideas of functions seem to be most related to success in AP Calculus, the overall results suggest that at least good understandings of all aspects of functions are correlated with strong performances in calculus. There were 21 participants who were not classified as *weak* in any of the five Big Ideas of functions. Of these participants, 17 of them (81%) scored a 3 or higher on the exam, 13 of whom (76%) scored at least a 4, and the average score for these students was 3.67. In contrast, of the 46 participants who were *weak* in at least one Big Idea, only 15 of them (33%) scored at least a 3 on the exam, only 6 of whom scored a 4 or a 5 (13%), and the average score was merely 2.02. This is an important result, as it reinforces the notion that functions are a key prerequisite to understanding and succeeding in calculus. This breakdown of AP Scores between students who had at least one *weak* Big Idea and those who did not can be seen in Table 5.2.

Table 5.2. AP Scores of students who did or did not have at least one *weak* Big Idea.

	N	**5**	**4**	**3**	**2**	**1**	**μ**	**σ**
No Weak BIs	21	8	5	4	1	3	3.67	1.39
≥ 1 Weak BIs	46	2	4	9	9	22	2.02	1.19

It should be noted that the specific relationships between understandings of different function concepts and performance on the AP Calculus Exam are all based on the students' understandings after having taken a full year of calculus, and it is not known whether these correlations would have been found if the participants' understandings of functions were measured at the beginning of the course instead. As stated previously, it is difficult to speculate on how much taking AP Calculus influenced the function understandings of the participants at the end of the course.

5.3 OTHER CONSIDERATIONS

This section addresses other aspects of the study that do not necessarily fit in with the primary findings. This includes a brief synopsis of the PCA and AP Score results when demographic information is accounted for, brief discussions of Big Idea 4 and the *object* view of functions, and an examination of the scoring adjustment on the PCA for items that required work to be shown to be counted. These are presented in order to provide further insight into the function understandings of the participants and how they are related to success on the AP Calculus Exam.

5.3.1 Demographic Splits

Recall that demographic information for each participant was collected on the front page of the PCA. This information includes their school, grade, gender, and race. This section includes a brief breakdown of PCA scores and AP Exam scores within each of these categories, with the exception of race. Recall from Chapter 3 that 62 of the 85 participants (73%) were Caucasian, and the rest of the participants were as follows: 10 Asians/Pacific Islanders (12%), 6 each of African Americans and students who circled "Other" (7% each) and 1 Native American (1%). Because there were too few of each minority, the data was too skewed to make any notable observations. Therefore, race was not included in this analysis.

School site: This study was conducted at three different school sites, A, B, and C. School B performed the best on the PCA, as it was the only school where less than half of its students were classified as *weak.* It also had the most students with a *strong* score, with 4, which was more than the other two schools combined (3). On the AP test, however, School B actually had the lowest average score, at 2.22. This was due to the fact that 14 of the 27 students (52%) from School B scored a 1 on the AP Exam, which again was more than the other two schools combined (11). It is possible that this happened because, according to the information supplied by the interview participants, the students spent less time specifically preparing for the AP Calculus Exam at School B than at the other two schools. At Schools A and C, the teacher employed strategies and reviews throughout the year in preparation for the exam, where at School B, preparation seemed to occur solely in the weeks preceding the exam. Another factor could be that the School B classes only met for 200 minutes per week, which is 25 minutes less than at School C, and 40 minutes less than at School A. Finally, it is also possible that differences in instruction may have also been a factor, although since no classes were observed at any of the schools, it is difficult to identify what these differences may have been.

School C, which had the most *weak* students on the PCA, had the highest average score on the exam, at 2.95. Along with preparation, this can be explained by the fact that School C did not require all of its AP Calculus students to take the exam, whereas Schools A and B did. Most of the students at School C who chose not to take the exam were categorized as *weak* on the PCA. In other words, the students who did choose to take the exam were mostly those with better function understandings. Thus, they were more likely to perform well on the exam. Finally, the number of 4s and 5s on the exam were about the same at each school, between 25% and 35%. The breakdown of PCA and AP Exam results by school site are shown in Table 5.3.

Table 5.3. PCA and AP Exam results by school site.

	PCA							**AP Exam**						
	# Correct		**Partial Credit**		**Category**									
	μ	**σ**	**μ**	**σ**	**S**	**M**	**W**	**5**	**4**	**3**	**2**	**1**	**μ**	**σ**
A	12.3	4.38	65.6	13.15	1	7	12	2	3	5	4	6	2.55	1.36
B	13.96	4.59	69.7	15.18	4	14	9	4	3	2	4	14	2.22	1.55
C	11.92	4.61	62.97	15.63	2	14	22	4	3	6	2	5	2.95	1.47

<u>Grade:</u> The participants of this study consisted of 22 juniors, 62 seniors, and one student that did not specify. On the PCA, juniors had approximately the same number of *moderate* and *weak* students, while seniors had many more *weak* students than *moderate*. However, 6 of the 7 students who were classified as *strong* on the PCA were seniors. On the AP Exam, the average exam scores and the distributions were approximately the same for both juniors and seniors. This is somewhat surprising, as the national average for juniors on the AP Calculus Exam in 2013 was 3.39, whereas for seniors it was only 2.77 (College Board, 2013). In general, it would be expected that most juniors would have a stronger understanding of functions and a better performance on the AP Calculus Exam than most seniors, as students who reach AP Calculus by their junior year have generally been those who are stronger at mathematics, and thus placed in accelerated tracks at some point in their schooling. The breakdown of PCA and AP Exam results by grade are shown in Table 5.4.

Table 5.4. PCA and AP Exam results by grade.

	PCA							AP Exam						
	# Correct		Partial Credit		Category									
	μ	σ	μ	σ	S	M	W	5	4	3	2	1	μ	σ
Jr.	12.82	4.19	67.32	12.99	1	11	10	2	3	3	2	6	2.56	1.5
Sr.	12.53	4.75	64.95	15.8	6	23	33	8	6	9	8	19	2.52	1.5

Gender: This study had 44 female participants and 40 male participants, with one student that did not specify. There were 28 females (64%) that were *weak* on the PCA, as opposed to only 15 males (38%). However, 5 of the 7 *strong* students on the PCA were also female. The males were mostly *moderate*, with 23 (58%). On the exam, males (2.94) performed better than females (2.26) on average, with the primary difference coming in the top scores. Only 6 females out of 35 (17%) that took the exam scored a 4 or a 5, compared to 13 males out of 31 (42%). This is consistent with the observation that significantly more males than females score in the highest percentiles, or the right tail of the distribution, on most major mathematics tests including the AP Calculus AB Exam (Hedges & Nowell, 1995; Niederle & Vesterlund, 2010). Females also seemed to perform significantly lower than the national average, as 38% of all females scored a 4 or a 5 on the AP Exam in 2013, and the average score was 2.82 (College Board, 2013). The males were much closer to their national averages, with 45% of all males scoring 4s and 5s, and their mean score was 3.06. In general, the results of the study are consistent with AP Score averages, in that males tend to yield higher average scores on the AP Exam than females. In the last five years, the average score for females has ranged from 2.6 to 2.81. For males, their average scores have ranged from 2.93 to 3.11 in the same time period (College Board, 2009; 2010; 2011; 2012; 2013). While there are likely several reasons for this, it may be worth examining whether there are any notable differences between the function conceptions of males and females. The breakdown of PCA and AP Exam results by gender are shown in Table 5.5.

Table 5.5. PCA and AP Exam results by gender.

	PCA							AP Exam						
	# Correct		Partial Credit		Category									
	μ	σ	μ	σ	S	M	W	5	4	3	2	1	μ	σ
M	13.45	4.87	67.08	16.38	2	23	15	7	6	5	1	12	2.84	1.66
F	11.84	4.23	64.2	13.82	5	11	28	3	3	7	9	13	2.26	1.29

5.3.2 On Big Idea 4 and Object View

Recall from Chapter 4 that Big Idea 4, *Combining and Transforming Functions*, had some of the better results on the PCA, but clearly had the worst results from the interviews. The reason for these differences, as stated previously, is because the PCA focused solely on function compositions and inverses, while the interview tasks centered on function transformations. While it was previously indicated that students also struggled with the function inverse notation, their performance with function compositions was generally superb. On three of the PCA items (#5, #12, and #16) that included a composition of functions, over 75% of respondents chose the correct answer. On two others (#4 and #20), at least half of the respondents answered correctly. This proficiency with function compositions likely drove up the overall understanding of Big Idea #4 on the PCA, although it was still somewhat balanced out by their struggles with inverse notation. Also, even though only eight participants had a chance to try their hand at transformations, their consistent struggles with them likely indicate that most of the rest of the students would also have some difficulties with transformations. The interview participants could solve no more than half of the tasks correctly, none of them were classified as *strong* and only three of them were *moderate*. Even Steve, who scored a 97 on the PCA and was classified as *strong* in every other part of the interview, was only classified as *moderate* for Big Idea 4. In displaying difficulties with two of the three essential aspects of Big Idea 4, it is likely that most of the participants, and by extension most AP Calculus students in general, do not have a very proficient understanding of it.

This is likely because most students have yet to develop an *object* view of functions. The results of the interview showed that half of the participants displayed tendencies toward an *object* view of functions. They were capable of recognizing how the graph of a function can be moved based on how its equation is manipulated. However, most of these students still had difficulties with other areas of functions, including Big Idea 4. Two of them were still coded as *weak* in Big Idea 4, and two of them scored a 1 on the AP Exam. So it is likely that these students had not fully developed an *object* view of functions, especially if such a view is in fact an indicator of a more sophisticated understanding of functions in general, and proficiency with transformations in particular, as claimed by several researchers (e.g., Asiala, et al., 1996; Dubinsky & Harel, 1992; Eisenberg & Dreyfus, 1994). Rather, their tendencies toward the object view are just that: tendencies.

5.3.3 PCA Score Adjustments for No Work Shown

Recall from Chapter 3 that for several PCA items, work was required in order for a participant to get credit for a correct answer. If a student correctly answered one or two of these items without showing any supporting work, then those items would not count in their PCA score. Any subsequent items would still count in their score, as the probability of simply guessing the correct answer for more than two items drops below 1%.

There were 38 students (45%) in the study who had at least one such item deducted from their PCA score. However, only 6 of these students (7%) actually had it affect how their understanding of functions was classified. Four of these six students were classified as *weak* rather than *moderate*, while the other two were classified as *moderate* when they otherwise would have been *strong*. The AP Scores of each of these students seems to reflect the classification after the adjustment. The four students who dropped to *weak* because of the no-work adjustment had AP scores of either 1 or 2 (two of each score), which reflects both a *weak* understanding (with the adjustment) and a low *moderate* understanding (without the adjustment). Also, both students who were *moderate* instead of *strong* earned a 4 on the AP Exam. In their case, both the exam scores and the fact the adjustment affected them show that their understanding of functions and calculus are both *near* the top, but not *at* the top. So in general, it appears as though the no-work adjustment had no discernable effect on the overall results of the study.

5.4 CONCLUSIONS, IMPLICATIONS, AND FURTHER RESEARCH

There were two major ideas that can be taken from the findings in this study. First, students with a good understanding of functions at the end of an AP Calculus course tend to score higher on the AP Calculus Exam than those without one. These are the students who can see functions as processes or even objects, identify and work with several different kinds of functions, work within and between different kinds of representations, connect functions to the situations they model, and reason about the changes that occur within and around functions. The second idea is that many students completing AP Calculus still need to significantly improve their understandings of functions, and they tend to have difficulty with many of the concepts described above.

Current research has explained the various ways in which students understand each of the different aspects of functions, and discussed common misconceptions and difficulties that students often have with functions (e.g., Leinhardt, et al., 1990; Oehrtman, et al., 2008; Thompson, 1994). The Big Ideas of functions were developed from much of this research as a way for teachers to organize their teaching of functions (Cooney, et al., 2010). There have also been studies that determined college calculus students' understandings of functions (e.g., Carlson, 1998; Carlson, et al., 2010). The current study is significant in that it is the first that produces similar insight at the high school level, and in AP Calculus in particular. It provides information about the importance of the understanding of functions for AP Calculus students, as well as which aspects of functions appear to be the most related to success in AP Calculus. Given the vast number of students taking AP Calculus every year, learning how to improve their understandings and readiness for the course is critical, especially since some colleges and universities have stopped accepting AP scores for credit, claiming that many former AP students enter college with a lack of the foundational knowledge and skills needed to succeed in their programs (Ben-Achour, 2013).

The findings of this study have several implications for teachers and developers of the secondary mathematics curriculum. First, they reinforce the idea that developing students' conceptualization of functions is very important in preparing them to take calculus. It is recommended that high school mathematics departments place an emphasis on the teaching of functions, with an aim toward especially developing a process view of functions, covariational reasoning skills, and skill with interpreting, producing, and moving between different representations of functions. Precalulus teachers may wish to consider using NCTM's Big Ideas of Functions as a means to organizing their teaching of functions, as this is what they were initially designed for (Cooney, et al., 2010). Recall from Chapter 1 that the Big Ideas are well-aligned with the Common Core State Standards (CCSSI, 2010), so teachers who use the Big Ideas to teach functions would not be straying from the standards and practices outlined in the Common Core. Additionally, teachers of AP Calculus may choose to implement a placement exam like the PCA for students who wish to enter the course. Recall that the PCA has shown to be a strong predictor of success in calculus, even though it was not specifically designed to be a placement exam (Carlson, et al., 2010). Implementing such an exam would help limit the number of students who take the course without a well-developed understanding of functions.

This study also leaves room for future research in order to address some of the limitations of the study as well as any new questions generated from the findings. For example, the participants in this study all took the PCA *after* they had taken the AP Calculus Exam. It is possible that their understandings of functions could have changed in some capacity from the time they entered the course. In learning calculus, some students may have strengthened their function understanding. It is also possible that some may have had their understandings and skills with certain aspects of functions regress. Therefore, a similar but more comprehensive study could be conducted in which AP Calculus students' function understandings are measured at the beginning of the course as well as at the end. Multiple interviews could also be conducted with a subset of students at given intervals during the year to gain further insight into their understandings. Again, their conceptualizations of functions would be compared to their scores on the AP Exam, and also with their performance in the class itself. The findings of such a study would help provide secondary mathematics teachers with further information about students' levels of proficiency with functions as they are entering AP Calculus, and it would give them an idea of how well the course itself influences their understandings of functions. That is, it would help answer the question, "How much learning of functions occurs *during* a calculus class?"

Another possible future study would be a similar investigation of AP Calculus BC students. It would be expected that those who take the BC course generally have a stronger understanding of functions than those in the AB course. The results would help highlight some of the similarities and differences between AP Calculus AB and BC students.

Also, as stated earlier in this chapter, the differences in function conceptions between males and females could be explored further in order to help explain and address the perennial gap between male and female averages on the AP Calculus Exam. Are there any notable disparities in how males and females understand functions? Does it take females longer to develop a process view of functions? Do males generally have a significantly stronger understanding of one or more of the Big Ideas of functions? Answers to these questions could have implications for secondary mathematics teachers, administrators, curriculum developers, and researchers as they work towards achieving gender equity in the secondary mathematics classroom. A similar study could also be conducted to identify differences between function understandings of different racial or ethnic groups in order to help explain any observed gaps between the average AP Calculus exam scores of those groups.

Finally, a larger study that focuses on the development of function understanding from the end of algebra 2 to the end of precalculus could also be very informative. How does a student's understanding of functions actually change over time? Does function understanding grow any faster with the use of one curriculum or textbook as opposed to another? The findings of such a study could help teachers and curriculum developers identify aspects of functions teachers need to focus on and how they might be organized and presented.

APPENDIX A

PCA ITEM SCORING RUBRICS

These are the scoring rubrics for each of the 25 items on the PCA. These rubrics contain information about possible reasoning a student might have for selecting each answer choice, and how many points such reasoning would earn. Most of the reasonings are based on the explanations given on the PCA website (Arizona Board of Regents, 2007). The PCA items can also be found on the website.

#1.

Answer	**Possible Reasoning**	**Points**
a)	Input $x + a$ correctly, multiply it out correctly, match to closest answer choice.	3
	Correctly input $x + a$, multiply it out but forget the middle term.	2
	Did the same thing to both x and a	1
b)	Input $x+a$ correctly in first term, overlooked the second term, multiplied out.	2
c)	Correct Answer	4
d)	Input $x+a$ correctly in first term, overlooked the second term.	2
e)	Simply added a to $f(x)$.	0

#2.

Answer	Possible Reasoning	Points
a)	Solved for *x* = -3, put down the whole point.	1
b)	Correct Answer	4
c)	Solved correctly, put down the whole point.	3
d)	Solved for *x* = -3 (confused input and output).	1
e)	Correctly identified the point, put down the wrong coordinate.	2
	Took -3 to be the answer since it was on the right side of the equation.	0

#3.

Answer	Possible Reasoning	Points
a)	Correctly identified that it's between 7 and 8, chose halfway between or guessed the wrong one.	3
b)	The wide cylinder will be 2 less than the thin one each time.	1
c)	Proportional approximation. 4 is 2/3 of 6, therefore 8 is almost 2/3 of 11.	2
d)	Correct Answer	4
e)	Guess, 11 appears in the problem.	0

#4

Answer	Possible Reasoning	Points
a)	Correct Answer	
b)	Technically correct, but not in terms of *p*. Plugged in values and saw that it works.	1
c)	Computation/algebra error. Forgot to square the denominator.	3
	Knew to square *p*, figured 4 needs to be in the problem since perimeter is 4 times the side length.	1
d)	Not in terms of *p*, squared the perimeter. Switched side length and perimeter.	0
e)	Technically correct, but this is *p(A)*, not *A(p)*. Think "in terms of *p*" means "*p* =".	2

#5

Answer	Possible Reasoning	Points
a)	Correctly evaluated *f*(2) and stopped. May not know what to do with *g*, or simply overlooked it.	2
b)	Correct Answer	4
c)	Correctly evaluated the functions in the wrong order.	3
d)	Took *g*(2) after tracing from the point (2, -2) on *f*.	1
e)	Evaluated *f* twice.	1

#6

Answer	Possible Reasoning	Points
a)	Correct Answer	4
b)	Evaluated *f* correctly, subtracted 0 instead of *g*(0).	1
c)	Evaluated both functions correctly, added instead of subtracted. Sign confusion.	2
d)	Just evaluated *g*(0), or picked 2 since it's in the problem. Got lost in the process.	0
e)	Switched *f* and *g*, otherwise evaluated correctly.	3

#7

Answer	Possible Reasoning	Points
a)	Can tell it's a multiplicative relationship, but not exponential.	3
b)	Misplaced where the initial value goes in the function.	1
c)	Correct Answer	4
d)	Understands that the function has a growth rate, but cannot make sense of how much the growth rate is.	2
e)	Depends on the solution given (if any) and the rationale behind it.	0-2

#8

Answer	Possible Reasoning	Points
a)	Interprets graph as a literal picture.	0
b)	Correct Answer	4
c)	Interprets the slope of the graph as the speed of the cars.	1
d)	Car B's speed is increasing, Car A is slowing down, B has a steeper slope after 1 hour.	2
e)	Sees the graph as position vs. time, and determines that the average speed of the cars is the same.	2

#9

Answer	Possible Reasoning	Points
a)	2 and 5 are the lowest and highest points of *f* while it's under *g*. Solved correctly for *y*, used *x*.	0
b)	Solved correctly for *x*, but used *y*.	3
c)	Ignores or forgets to include the lower bound.	2
d)	Solved correctly for *y*, lowest and highest points for *f* while it's under *g*.	1
e)	Correct Answer	4

#10

Answer	Possible Reasoning	Points
a)	Switched *t* and *v*, solved for *v* instead of *t*.	1
b)	Switched *t* and *v*. Put it in terms of *t* but found the slope of the line and put that as the coefficient for *v*.	2
c)	Correct Answer	4
d)	Correct formula, but solved for *v* instead of *t*.	3
e)	Depends on the solution given (if any) and the rationale behind it.	0-2

#11

Answer	Possible Reasoning	Points
a)	Took the total distance, divided by time. Took the initial position to be at $t = 1$ rather than $t = 0$.	2
b)	Correct Answer	4
c)	Found the correct change in distance, but divided by 2 (the number of *t* values) rather than the change in time.	3
d)	Took the average of the distances at times 1, 2, 3, and 4.	0
e)	Took the average of the distances at times 1 and 4.	1

#12

Answer	Possible Reasoning	Points
a)	Correct Answer	4
b)	Took the *x* value when $g(x) = 3$, or $g[f(2)] = g(3)$.	0
c)	Took the value of $g(3)$, did not compute *f*.	1
d)	Took the *x* value when $g(x) = 3$, computed *f*. Read the table backwards.	2
e)	Reversed the order of *f* and *g*, otherwise correctly calculated it.	3
	Found $g(3) = 0$, went to where $f(x) = 0$, took the number to the right of it.	1

#13

Answer	Possible Reasoning	Points
a)	Correctly evaluated $g^{-1}(-1)$, but then took the reciprocal.	3
b)	Took the reciprocal of $g(-1)$.	1
c)	Multiplied -1 and -1, calculated $g(1)$.	0
d)	Correct Answer	4
e)	Took $g(-1)$. Ignored the inverse.	2

#14

Answer	Possible Reasoning	Points
a)	Took the reciprocal instead of the inverse.	0
b)	Took the natural log of only the right side of the equation.	2
c)	Did not take the natural log at all.	1
d)	Correct Answer	4
e)	Took the natural log of only the left side of the equation.	2

#15

Answer	Possible Reasoning	Points
a)	Can tell the graph increases, but has difficulty interpreting the concavity.	2
b)	Correct Answer	4
c)	Thinks as the height gets bigger, so does the volume, chose something with larger volume toward the top.	2
d)	Most likely a guess.	0
e)	Thinks as the height gets bigger, so does the volume, has a similar shape to the graph itself.	1

#16

Answer	Possible Reasoning	Points
a)	Correctly evaluated $h(2)$, but multiplied that by 2 instead of squaring it.	1
b)	Reversed the order, evaluated g before h.	2
c)	Evaluated $h(2)$ and $g(2)$ and took the product.	0
d)	Correct Answer	4
e)	Computation error, got 6 instead of 5 for $h(2)$, then input 6 into g.	3

#17

Answer	Possible Reasoning	Points
a)	Did not square *t*. Either overlooked this or didn't realize it had to be squared.	2
b)	Provided the area formula for a circle with time as the radius, did not use the given rate at all.	1
c)	Correct Answer	4
d)	Did not square the rate, just time. Either overlooked this or didn't realize it had to be squared.	2
e)	Depends on the solution given (if any) and the rationale behind it.	0-2

#18

Answer	Possible Reasoning	Points
a)	Incorrect horizontal asymptote / upper bound.	1
b)	Correct Answer	4
c)	Incorrect horizontal asymptote, confused about the behavior.	0
d)	Incorrect horizontal asymptote / upper bound.	2
e)	Correct horizontal asymptote, but confused about the behavior.	2

#19

Answer	Possible Reasoning	Points
a)	Misinterprets or confuses increasing and decreasing rates.	2
b)	Correct Answer	4
c)	Only sees *y* as increasing, misinterprets the idea of rate.	1
d)	Focusing only on the fact that rate is decreasing, not what the graph itself is doing.	1
e)	Confuses the change in the *y*-values with the change in the rate.	2

#20

Answer	Possible Reasoning	Points
a)	Correct Answer	4
b)	Confused *m* with an initial salary, which changes after 12 months.	3
c)	Added 12 to the entire function.	1
d)	Difficulty identifying function output, needs something concrete.	1
e)	Depends on the solution given (if any) and the rationale behind it.	0-2

#21

Answer	Possible Reasoning	Points
a)	Can tell that there are only positive values for square roots, and no zeroes in the denominator. Does not realize that some negative values of *x* still work for this functions	2
b)	Recognizes that the function cannot have a 0 in the denominator, does not consider restrictions under the radical.	2
c)	Correct Answer	4
d)	Recognizes restrictions under the radical, but not in the denominator.	2
e)	Does not recognize any restrictions on the function.	0

#22

Answer	Possible Reasoning	Points
a)	Recognized I but not III. Easier time with integers.	2
b)	Does not understand multiplicative structure of "2.5 times greater".	0
c)	Only looks at one year at a time.	2
d)	Correct Answer	4
e)	Confuses multiplicative structure with additive structure.	2

#23

Answer	Possible Reasoning	Points
a)	Takes the reciprocal of the function, not the inverse.	1
b)	Took the reciprocal of the input.	0
c)	Took the reciprocal of the output.	1
d)	Correct Answer	4
e)	Recognizes the equivalence of (a) and (c), but is not a function inverse.	2

#24

Answer	Possible Reasoning	Points
a)	Correct, but does not discuss behavior as *x* increases.	2
b)	Correct, but does not discuss behavior as *x* goes to 0.	2
c)	Does not understand what it means for *x* and *f(x)* to approach 0. Considers any axis to mean 0.	0
d)	Correct Answer	4
e)	Confused about when *x* approaches 0 and when it does not.	1

#25

Answer	Possible Reasoning	Points
a)	Recognized the significance of 2 in the function.	0
b)	Correct Answer	4
c)	Recognized that the function approaches something as the denominator goes to 0, but misinterpreted what that means for the function as a whole.	1
d)	Can tell that something happens as something approaches 2.	1
e)	Can tell that the function is increasing and that something happens as the denominator goes to 0.	2

APPENDIX B

INTERVIEW QUESTIONS AND TASKS

Part 1 – The Function Concept

Question 1 – What is a function?

Question 2 – Which of these relations are functions?

1)

x	y
-10	100
-5	25
-2	4
0	0
2	4
5	25
10	100

2)

x	y
-10	-100
-5	-25
-2	-4
0	0
-2	4
-5	25
-10	100

3)

x	y
3	100
3	25
3	4
3	0
3	4
3	25
3	100

4) $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

5) $y = 5$

6) $xy = 1$

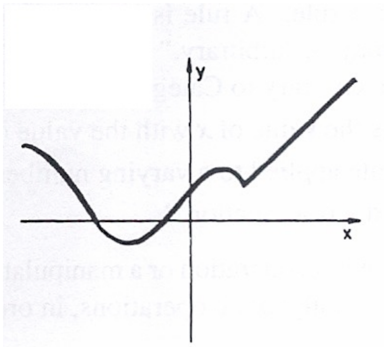
7) $f(x) = \sum_{k=1}^{n}(2k - 1)$

8) $x^2 + y^2 = 16$

9) $f(x) = \begin{cases} 3x^2 - 5, & x < 7 \\ 4x, & x \geq 5 \end{cases}$

10) $y = x^2 + 3e^x - 4\log(x + 2)$

11)



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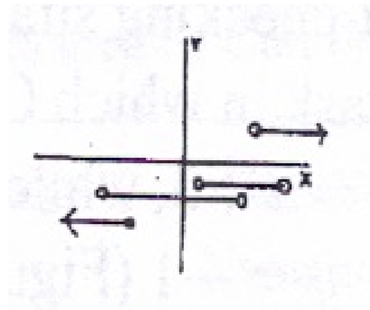

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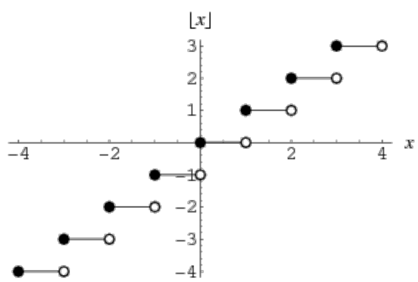

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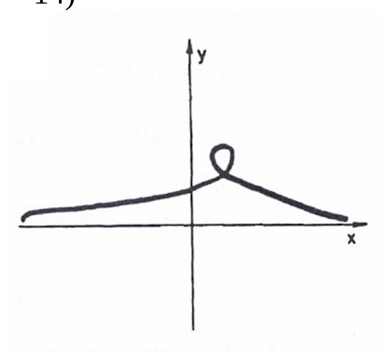

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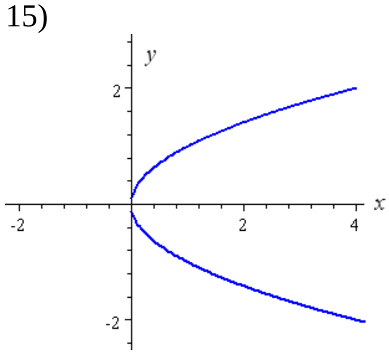

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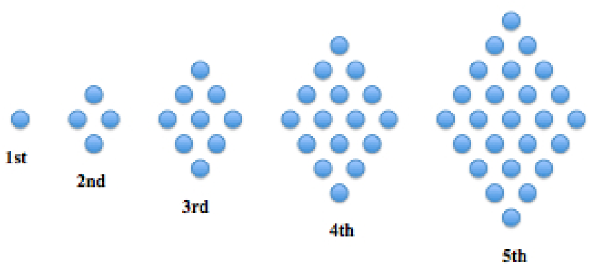
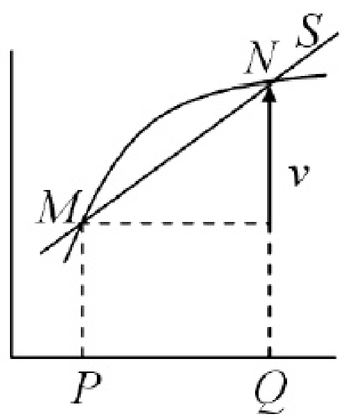


17) The relation of the sender of an email to its recipients.

18) The relation of every person to his or her biological mother.

Part 2 – Covariation and Rate of Change

Task 1 – When point *Q* moves toward point *P*, does the slope of line *S* increase or decrease?



Task 2 – Let *T*(*H*) model the amount of time it takes to fill a vat with water using *H* hoses, where each hose has water flowing through it at the same rate. Knowing that it takes 5 minutes for 6 hoses to fill the vat, use logical reasoning to fill in the rest of the entries in the table for the function *T*(*H*). Then explain why this function does not have a constant rate of change.

H	*T*
3	
6	5
12	
24	

Part 3 – Families of Functions

To what major family does each of these functions belong? How do you know?

a) $y = 5x^2 + 3$

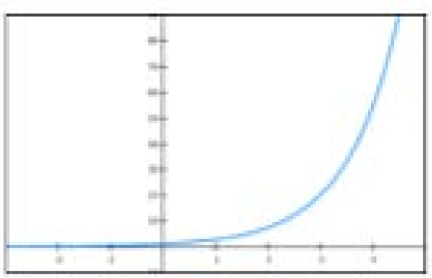
b) The 2010 Census shows that Smallville has a population of 40,000 people. Social scientists predict that Smallville will experience a growth rate of 5% per year over the next 20 years. Let P be the function such that $P(t)$ is the predicted population of Smallville t years from now (for t between 0 and 20).

c) A movie theater has operating costs of $1025 per day. Tickets cost $7.50 each. The theater's profit each day depends on the number of tickets sold. $P(T)$ is the profit that the movie theater will make on a day when it sells T tickets.

d)

D	M
7	15.29
14	15.43
21	15.57
28	15.71
35	15.85

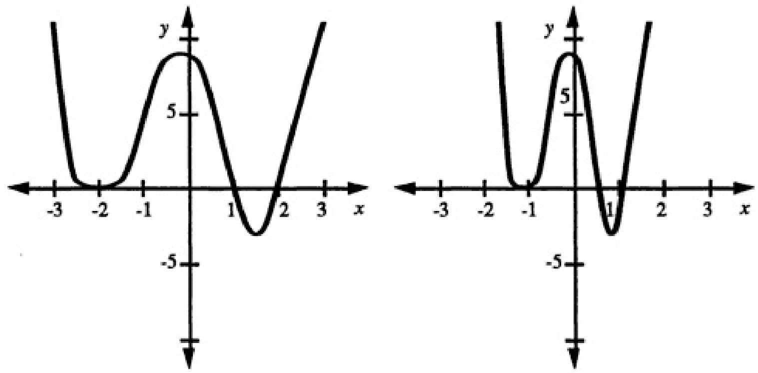
e)



f) Fred is deciding which size of pizza is the best buy. He wonders how the area of the pizza is related to its diameter. Let A be the function such that $A(r)$ is the area of a circular pizza of radius r.

Part 4 – Combining and Transforming Functions

Task 1 – In the figure on the left you are given the graph of the function $y = f(x)$.

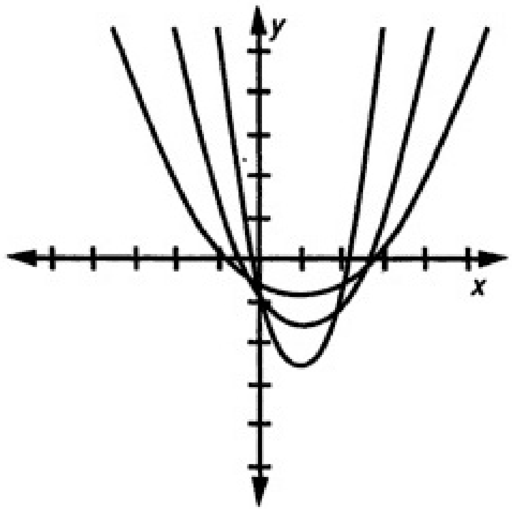


Which of the following formulas corresponds to the graph on the right?

$y = f(2x)$	$y = f(x+1)$	$y = f(\lvert x \rvert)$
$y = f\left(\frac{1}{2}x\right)$	$y = f(x-1)$	$y = f(-\lvert x \rvert)$
$y = 2f(x)$	$y = f(x) + 1$	$y = f(-x)$
$y = \frac{1}{2}f(x)$	$y = f(x) - 1$	$y = -f(x)$

$$y = \lvert f(x) \rvert$$

Task 2 - The three graphs correspond to a formula of the form $y = a(x-d)^2 + e$. Which of the parameters *a*, *d*, and *e* are identical for the three given graphs?



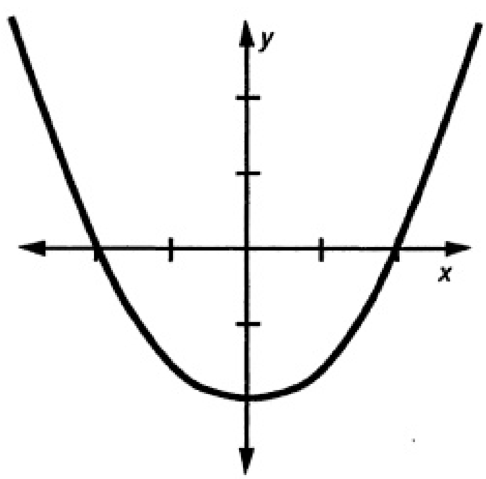
Task 3 – Given the function $y = |x^2 - 4|$

a) Graph the given function
b) Graph $y = |x^2 - 4|+1$

c) Graph $y = |(x-1)^2 - 4|$

d) Graph $y = \frac{1}{2}|x^2 - 4|$

Task 4 - Given the graph of the function $y = f(x)$, sketch the graph of the function $y = \frac{1}{f(x)}$.



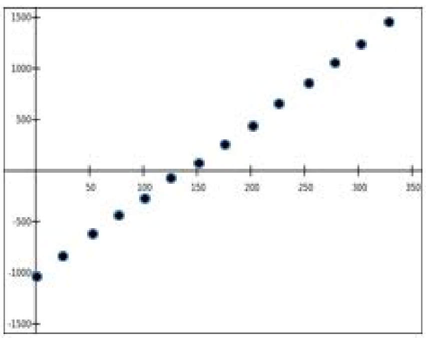
Part 5 – Multiple Representations of Functions

The function *P(T)* models the relationship between the *number of tickets sold* (*T*) by a movie theater and *profit* (*P*, in dollars). This relationship is represented in four different ways below.

(i) A movie theater has operating costs of $1025 per day. Tickets cost $7.50 each. The move theater's profit each day depends on the number of tickets sold.

(ii) $P = 7.5T = 1025$

(iii)



(iv)

T	*P*
0	-1025
50	-650
100	-275
150	100
200	475
250	850
300	1225

Task 1 – Which of the representations above would be most helpful for determining the following information:

a) The daily operating cost for the theater?
b) The number of tickets that must be sold for the theater to have a profit of $500?
c) The daily "break-even point" for the movie theater?
d) The rate of change in the relationship?
e) The major family of functions (e.g., linear, quadratic, exponential) to which this relationship belongs?

Task 2 – How does each of the pieces of information above (from (a) – (e)) appear in each of the four representations? For instance, what does the break-even point from part (c) look like on the graph? In the formula?

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