GEOMETRICAL PROBLEMS IN THE MATHEMATICAL STUDY OF PRESTRAINED MATERIALS

by

Pablo D. Ochoa

B. S. in Mathematics, Universidad de Cuyo, 2009

M. A., University of Pittsburgh, 2013

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This dissertation was presented

by

Pablo D. Ochoa

It was defended on

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and approved by

Professor Marta Lewicka, Department of Mathematics

Professor Amit Acharya, Department of Civil and Environmental Engineering, Carnegie Mellon University

Professor Juan Manfredi, Department of Mathematics

Professor Mohammadreza Pakzad, Department of Mathematics

Dissertation Director: Professor Marta Lewicka, Department of Mathematics

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Pablo D. Ochoa, PhD

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In my thesis, we derive a two dimensional energy model for deformations of unloaded elastic films as a result of the application of Γ-convergence to appropriate three dimensional models. The limiting model obtained in this way constitutes a Von Kármán type growth functional, attaining its minima at deformations $v \in W^{2,2}$ satisfying a Monge-Ampére constraint of the form $\det \nabla^2 v = f$, for some appropriate f. The main advantage of Γ-convergence is that it connects 2d theories with 3d nonlinear theory in the sense that minimizers of the 3d energy functionals converge to minimizers of 2d energy functionals.

Secondly, we study the variational behaviour of discrete lattice energies associated with a pre-strained elastic body, as a mathematical justification of the theoretical non-Euclidean energy model employed in this theses. Via Γ-convergence, we obtain asymptotic bounds on the Γ-limiting energy and, in the context of near and next-to-near interactions, we identify exactly the integral form of the limiting energy, comparing it to the theoretical model.

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1.0 GENERAL INTRODUCTION

In my thesis, we are concerned with the study of elastic bodies which assume rest configurations with residual stress, even in the absence of external forces or imposed boundary conditions. Examples of these structures can be found in nature: leaves, flowers, invertebrates, and in man-made structures: torn plastic sheets, engineered polymer gels, and many others (see Figure 1). Roughly speaking, the shape formation of these elastic bodies can be posted in the following terms: the elastic sheet seeks to reach a non-attainable equilibrium, and therefore necessarily adopts a final configuration different from the ideal rest state, but which minimizes its elastic energy. In geometrical terms, the ideal configuration can be interpreted as the realization of a Riemannian metric G (that is, an orientation preserving isometric immersion of G) in the Euclidean space. Thus, if the metric G is Euclidean, the sheet will adopt a stress free configuration. On the other hand, for G non-Euclidean, there will be no realization of G in the Euclidean space, so the ideal equilibrium is not attainable, and the elastic sheet hence adopts a configuration with a metric close to, but different from, the prescribed G. This lack of realizability introduces residual strain in the rest configuration. In order to formulate these phenomena in mathematical terms, it has been introduced a mathematical model ([17], [23], [24], and especially in [32]) called 'Incompatible elasticity', which measures, in some sense, how far the actual configuration of the sheet is from being a realization of the prescribed Riemannian metric G.

1.0.1 Organization of the Thesis

The first chapters provide the mathematical foundation and genesis of the ideas underlying in the mechanism of shape formation in unloaded elastic bodies:

Figure 1: Elastic structures assuming configurations with residual strain

The theory of non-linear three-dimensional elasticity used and developed in this work requires, to be well understood, some notions and results of Differential Geometry and Mathematical Analysis, such as existence of isometric immersions of Riemannian metrics, Theory of surfaces, Convex Analysis and Γ-convergence. Therefore, we shall dedicate Chapter 3 to introduce the necessary mathematical background.

In the next Chapter 4, the basic notions of the theory of non-linear elasticity are introduced. In between, the notions of elastic and hyperelastic bodies, stress tensors and strain energy function are rigorously defined. These are the starting points for the formulation of the mathematical energy model employed in this thesis.

In Chapter 5, the main motivation and application of non-Euclidean elasticity is studied: the unloaded growth of soft elastic tissues. We start by introducing the basic description of growth, and we provide explicit experiments which test the residual stress in rest configuration of grown tissues. In addition, throughout the chapter, the basic assumptions of the theory, i. e., the decomposition of the gradient of any deformation into a growth and an elastic part, and the dependence of the stored energy function only on the elastic response of the material, are carefully explained.

Following the findings in [23], we show in Chapter 6 that it is possible to prescribed metrics on unloaded plates, which guide the deformation process, and then to compare them to the actual metrics adopted by the deformed sheet. The numerical results show that, in general, the resulting deformed configuration induces a metric tensor which is close, in some sense, to the prescribed tensor G. Finally, these observations and results lead to the formulation of the mathematical model, called incompatible elasticity, which provides the non-Euclidean strain energy model measuring the discrepancy of the configuration from being an exact orientation-preserving isometric immersion of the ideal prescribed metric G.

In Chapter 7, we deal with the derivation of variational models for the morphogenesis of 2d thin elastic films, starting from three dimensional models. This dimension-reduction procedure is carried out through Γ-convergence. One of the main advantages of Γ-convergence is the convergence of the minimizers of the three dimensional models, to the minimizers of the appropriate lower dimensional limiting functional (see Theorem 22 in Chapter 3). We provide the explicit form of the limiting growth model. This limit consists of minimizing the bending content, relative to the ideal bending prescribed by the metric tensor, under the nonlinear Monge-Ampére constraint of the form $\det \nabla^2 v = f$, where f is a smooth function depending on the metric. In Section 7.3, we analyse the uniqueness of minimizers of the Monge-Ampére type functional.

Finally, in Chapter 8, we derive a variational study of the non-Euclidean energy model used in this work starting from an atomistic description. The main idea is to introduce lattice energies, taking into account the total interaction of the atoms in the discrete system, weighted by the prescribed metric G, and to analyse their asymptotic behaviour via Γ-convergence. In this work, we derive the lower and upper bounds on the Γ-limiting energy model $\mathcal{F}$ of the lattice energies. In the particular case of near or next-to-near interactions, these bounds coincide to each other, and we identify the exact integral form of the Γ-limiting energy $\mathcal{F}$. In Section 8.6, we provide a comparison between $\mathcal{F}$ and the energy model $\mathcal{E}$ through a series of examples.

The results of our authorship presented in this thesis, which are mainly contained in Chapter 7 and Chapter 8, were submitted as scientific articles in [30] and [31].

2.0 MAIN NOTATION

Throughout this section, we shall introduce the main notation conventions used in this thesis. Some extra notation may be introduced in the subsequent chapters.

2.0.2 General conventions

Unless otherwise indicated, all numbers, vectors, matrices, functions, functionals, etc., considered in this thesis are real.

The symbol C denotes a universal constant, independent of the relevant variables in the analysis under consideration.

2.0.3 Sets and mappings

The symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of natural, integers and real numbers, respectively. By $\overline{\mathbb{R}}$ we denote the extended set of real numbers, and by $\mathbb{R}_+$ we denote the set of non-negative real numbers.

The closure of a set Ω is denoted by $\overline{\Omega}$.

By id_2 we denote the map $id_2 : \mathbb{R}^2 \to \mathbb{R}^3$, given by:

$$id_2(x_1, x_2) = (x_1, x_2, 0).$$

Usually, points (x_1, x_2) in $\mathbb{R}^2$ are denoted by x'.

Let $f, g : X \to Y$, where X and Y are Banach spaces. Let $\alpha \geq 0$. Then we adopt the notation:

$$f(x) = \mathcal{O}(\|g(x)\|^\alpha),$$

if there exists a constant C and a neighborhood V of the origin in X such that:

$$\|f(x)\|_Y \leq C\|g(x)\|_X^\alpha, \quad \text{for all } x \in A \cap V.$$

Also,

$$f(x) = o(\|g(x)\|^\alpha),$$

if:

$$\lim_{x \to 0} \frac{\|f(x)\|_Y}{\|g(x)\|_X^\alpha} = 0.$$

2.0.4 Some function spaces

Let X, Y be normed spaces. We adopt the following conventions:

$$\mathcal{C}(X, Y)$$

denotes the set of all continuous mappings from X into Y.

$$\mathcal{C}^k(\Omega, Y)$$

is the space of all k times continuously differentiable mappings from the open set $\Omega \subset X$ into Y. If $Y = \mathbb{R}$, we denote these spaces by:

$$\mathcal{C}(X), \mathcal{C}^k(\Omega)$$

respectively. Moreover, $\mathcal{C}^k(\overline{\Omega})$, where Ω is a bounded and open subset of $\mathbb{R}^n$, denotes the space of real functions $f \in \mathcal{C}^k(\Omega)$ such that, for all multi-index α, with $|\alpha| \leq k$, there exists a continuous function $f^\alpha \in \mathcal{C}(\overline{\Omega})$ such that:

$$f^\alpha|_\Omega = \partial^\alpha f.$$

The Holder space $\mathcal{C}^{k,\beta}(\overline{\Omega})$, where $0 < \beta \leq 1$, is the space of all functions $f \in \mathcal{C}^k(\overline{\Omega})$ whose k-th partial derivatives satisfy on Ω a Holder condition with exponent β:

$$\|f\|_{\mathcal{C}^{k,\beta}(\overline{\Omega})} = \max_{|\alpha| \leq k} \sup_{x \in \Omega} |f^\alpha(x)| + \max_{|\alpha| = k} \sup_{x,y \in \Omega, x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\beta} < \infty.$$

The Sobolev spaces are:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}, \quad 1 \leq p \leq \infty.$$

2.0.5 Linear algebra

Let u, v be vectors in $\mathbb{R}^n$. Then:

$$u^T : \text{ transpose of the vector } u$$

$$u \cdot v = u^T v : \text{ Euclidean inner product}$$

$$u \otimes v = uv^T : \text{ tensor product.}$$

$$u \times v : \text{ exterior product in } \mathbb{R}^3.$$

For matrices, we use the following notation:

$$A^T : \text{ transpose of } A.$$

$$A^{-1} : \text{ inverse of } A.$$

$$A^{-1,T} = (A^{-1})^T = (A^T)^{-1}$$

$$\operatorname{tr} A : \text{ trace of } A.$$

$$\det A : \text{ determinant of } A.$$

$$\operatorname{sym} A = \frac{1}{2}(A + A^T) : \text{symmetric part of } A.$$

$$\operatorname{skew} A = \frac{1}{2}(A - A^T) : \text{ skew-symmetric part of } A.$$

$$A : B = \operatorname{tr} A^T B : \text{ matrix inner product.}$$

$$\operatorname{Cof} A = (\det A) A^{-1,T} : \text{ cofactor matrix.}$$

The following sets of matrices will be of particular interest in this work:

$\mathbb{R}^{n\times n}$: set of real square matrices of order n.

$\mathbb{R}^{n\times n}_+$: set of matrices of order n with positive determinant.

$SO(n)$: set of orthogonal matrices $AA^T = I$ of order n, with positive determinant.

3.0 MATHEMATICAL PRELIMINARIES

The theory of non-linear three-dimensional elasticity used and developed in this work requires, to be well understood, some necessary mathematical background. Therefore, we dedicate this chapter to the introduction of the main notions and results that we use in this thesis.

The chapter is divided into two main parts: the first section is dedicated to introduce all the notions from Differential Geometry that we use in our set up. The second section contains the necessary background in the field of Mathematical Analysis: convexity and quasiconvexity, a truncation result for Sobolev function, and the important notion of Γ-convergence.

3.1 PRELIMINARIES IN DIFFERENTIAL GEOMETRY

We start with the notion of orientation-preserving deformations, and then we discuss their main geometrical properties. We hence show how the metric notions, such as length, areas and volumes, of the image set of such mappings can be expressed in terms of quantities defined in the domain of these functions. We will see that these metric quantities can be obtained in terms of the gradient deformation tensor. This is the content of Theorem 1. We then turn to the important question of when a given metric tensor on a set can be induced by a deformation mapping, that is, when a given Riemannian metric has an isometric immersion in the Euclidean space. The answer to this question can be found in Theorem 3, and for weaker assumptions, in Theorem 4. The uniqueness issue is also treated, and it can be found in Theorem 5. The special interest is the role played by the theory of differential geometry of surfaces in the theory of three-dimensional elasticity. Hence, we also present a brief introduction to this theory. We begin with the introduction of the first and second fundamental form associated with a surface, and then we state the fundamental theorem of surface theory (see Theorem 7 and Theorem 10), which asserts that the Gauss-Codazzi-Mainardi equations (3.5) constitute a sufficient and necessary conditions for the existence of a surface with prescribed first and second fundamental forms.

3.1.1 Deformations and their geometrical properties

A central problem in non-linear, three-dimensional elasticity consists in finding the equilibrium position after a deformation process, of an elastic body that initially occupies a reference configuration $\overline{\Omega} \subset \mathbb{R}^3$. After deformation, the elastic body occupies a deformed configuration $u(\overline{\Omega})$, characterized by a mapping $u : \overline{\Omega} \to \mathbb{R}^3$ that must be orientation-preserving in $\overline{\Omega}$ and injective on the set Ω, to be physically acceptable.

Such mappings are called deformations, and the object of this section is to study their geometrical properties. It will be shown that the changes of volume, surface and length associated with a deformation u, are respectively governed by the scalar det ∇u, the matrix Cof ∇u, and the right Green-Cauchy strain tensor $C = (\nabla u)^T \nabla u$.

Let us start with an open, bounded, connected and Lipschitz domain $\Omega \subset \mathbb{R}^3$. We shall think of the closure $\overline{\Omega}$ as representing the volume occupied by the elastic body before it is deformed. For this reason, the set $\overline{\Omega}$ is called the reference configuration.

A deformation of the reference configuration $\overline{\Omega}$ is a map:

$$u : \overline{\Omega} \to \mathbb{R}^3,$$

which is regular enough, injective in Ω, and orientation-preserving, that is:

$$\det \nabla u(x) > 0, \quad \text{for } x \in \Omega.$$

Given a reference configuration $\overline{\Omega}$ and a deformation $u : \overline{\Omega} \to \mathbb{R}^3$, the set $u(\overline{\Omega})$ is called the deformed configuration. At every point $y = u(x)$, $x \in \Omega$, we have defined the vectors:

$$\partial_i u = \partial u / \partial x_i.$$

Each vector $\partial_i u$ measures the local deformation in the direction of the vector e_i. Thus, the knowledge of the deformation gradient ∇u completely defines the local deformation of the elastic body.

We now show how to compute lengths, areas and volumes in the deformed configuration in terms of the same quantities but defined over the reference configuration.

3.1.1.1 Volume, area and length in the deformed configuration. Let $u : \overline{\Omega} \to \mathbb{R}^3$ be a deformation which is differentiable at $x \in \Omega$. If $h = (h_1, h_2, h_3) \in \mathbb{R}^3$ has norm small enough so that the point $x + h$ belongs to Ω, then Taylor expansion gives:

$$u(x + h) = u(x) + \nabla u(x)h + o(h).$$

Let us define the vectors $g_i(x) \in \mathbb{R}^3$ as:

$$g_i(x) = \frac{\partial u}{\partial x_i}(x) = \begin{pmatrix} \partial_i u_1 \\ \partial_i u_2 \\ \partial_i u_3 \end{pmatrix} (x).$$

Then the expansion of u about x may be written as:

$$u(x + h) = u(x) + \sum_i h_i g_i(x) + o(h).$$

Therefore:

$$|u(x + h) - u(x)|^2 = h^T \nabla u(x)^T \nabla u(x)h + o(|h|^2).$$

In other words, the principal part with respect to h of the length between the points $u(x+h)$ and $u(x)$ is then given by:

$$\{\sum_{i,j} h_i g_i(x) \cdot g_j(x) h_j\}^{1/2}.$$

This observation suggests to define the matrix $G(x) = [g_{ij}(x)]$ by letting:

$$g_{ij}(x) = g_i(x) \cdot g_j(x) = \nabla u(x)^T \nabla u(x)_{ij}.$$

The elements $g_{ij}(x)$ of this symmetric matrix are called covariant components of the metric tensor G at the point $x^u = u(x)$.

We recall that a mapping $u : \Omega \to \mathbb{R}^3$ is an immersion at a point $x \in \Omega$ if it is differentiable at x and the matrix $\nabla u(x)$ is invertible. In this way, if u is an immersion at x, then the vectors $g_i(x) = \partial_i u(x)$ are linearly independent. Also, observe that, for an immersion u, the matrix field $G(x)$ is positive definite.

We now review fundamental formulas that show how volume, area, and length elements at a given point $x^u = u(x)$ in the set $u(\Omega)$ can be expressed either in terms of the deformation gradient $\nabla u(x)$, or in terms of the metric tensor $[g_{ij}(x)]$. These formulas will highlight the important role played by the matrix $G(x)$ for computing metric notions at the point $u(x)$.

Given a bounded, open, and connected subset $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary Γ, we let dx denote the volume element in Ω, $d\Gamma$ denote the area element along Γ, and $\mathbf{n}$ denote the unit outer normal vector along Γ. We have the following theorem:

Theorem 1. *Let Ω be an open subset of $\mathbb{R}^3$, let $u : \Omega \to \mathbb{R}^3$ be an injective and smooth enough immersion, and let $\Omega^u = u(\Omega)$. Then we have:*

1. *The volume element dx^u at $x^u = u(x)$ is given in terms of the volume element dx at $x \in \Omega$ by:*

$$dx^u = |det\ \nabla u(x)| dx = \sqrt{det\ G(x)}.$$

2. *Let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$. The area element $d\Gamma^u(x^u)$ at $x^u = u(x)$ is given in terms of the area element $d\Gamma(x)$ at $x \in \partial D$ by:*

$$d\Gamma^u(x^u) = |Cof\ \nabla u(x) \boldsymbol{n}(x)| d\Gamma(x) = \sqrt{det\ G(x)} \sqrt{\boldsymbol{n}(x)^T G(x)^{-1} \boldsymbol{n}(x)} d\Gamma(x),$$

   where $\boldsymbol{n}(x)$ denotes the unit outer normal vector at $x \in \partial D$.

3. *The length element dl^u at $x^u = u(x)$ is given by:*

$$dl^u = \{h^T \nabla u(x)^T \nabla u(x) h\}^{1/2} = \{h^T G(x) h\}^{1/2},$$

   where h is a general increment with small norm.

For a proof of this well-known result, see [9], and the references therein.

The relations found in Theorem 1 are used for computing metric notions such as: volumes, areas, and lengths inside the set $\Omega^u = u(\Omega)$ by means of integrals defined inside the reference set Ω.

For instance, let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$, and let $D^u = u(D)$. Let $f \in L(D^u)$ be a given function. Then, the change of variables formula gives:

$$\int_{D^u} f(x^u)dx^u = \int_D (f \circ u)(x)\sqrt{\det\,[g_{ij}]}dx.$$

Letting $f = \chi_{D^u}$ the characteristic function of D^u, we obtain, in particular, that the volume of D^u is given by:

$$|D^u| = \int_{D^u} dx^u = \int_D \sqrt{\det\, G}dx.$$

This can be also applied for subset of the boundary $d\Gamma^u$ of D^u: let ∂D the boundary of $D \subset \Omega$, let Σ be a ∂D-measurable subset of ∂D, let $\Sigma^u = u(\Sigma) \subset \partial D^u$, and let $h \in L(\Sigma^u)$ be given. Then:

$$\int_{\Sigma^u} h(x^u)d\Gamma^u(x^u) = \int_\Sigma (h \circ u)(x)\sqrt{\det\, G}\sqrt{\mathbf{n}^T G^{-1}\mathbf{n}}\, d\Gamma(x).$$

Which implies that the area of Σ^u is given by:

$$|\Sigma^u| = \int_{\Sigma^u} d\Gamma^u(x^u) = \int_\Sigma \sqrt{\det\, G}\sqrt{\mathbf{n}^T G^{-1}\mathbf{n}}\, d\Gamma(x).$$

We finally consider the case of a curve $C = \gamma(I)$, where I is a compact interval of $\mathbb{R}$ and $\gamma : I \to \Omega$ is a smooth enough injective mapping. Then, the length of the curve $C^u = u(C) \subset \Omega^u$ is given by:

$$|C^u| = \int_I \left|\frac{d}{dt}(u \circ \gamma)(t)\right|dt = \int_I \sqrt{\sum_{ij} g_{ij}(\gamma(t))\frac{d}{dt}\gamma_i(t)\frac{d}{dt}\gamma_j(t)}dt.$$

This relation shows that the lengths of curves inside the set $u(\Omega)$ are precisely those induced by the Euclidean metric space $\mathbb{R}^3$. That is, the length of any curve in the Riemannian manifold $(\Omega, [g_{ij}])$ is the same as the length of its image by the mapping u in the Euclidean space $\mathbb{R}^3$. In this way, the particular Riemannian manifold (Ω, G) possesses the remarkable property that its metric is the same as that of the surrounding space $\mathbb{R}^3$. In brief, the Riemannian metric (Ω, G) is isometrically immersed in the Euclidean space $\mathbb{R}^3$, in the sense of the following definition:

Definition 2. *Let* (Ω, G) *be a Riemannian manifold. An immersion* $u : \Omega \to \mathbb{R}^3$ *is an orientation-preserving isometric immersion of* (Ω, G) *if:*

$$G = \nabla u^T \nabla u, \quad \text{and} \quad \det \nabla u > 0.$$

It is remarkable that the components $g_{ij} = g_{ji} : \Omega \to \mathbb{R}$ of the metric tensor of an open set $u(\Omega)$, defined by a smooth enough immersion $u : \Omega \to \mathbb{R}^3$, cannot be arbitrary functions: they must satisfy relations (see Theorem 1.5-1 in [12]) that take the form:

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \sum_{p=1}^{3} \Gamma_{ij}^p \Gamma_{kqp} - \sum_{p=1}^{3} \Gamma_{ik}^p \Gamma_{jqp} = 0, \quad \text{in } \Omega, \tag{3.1}$$

for any $i, j, k, q \in \{1, 2, 3\}$. Here, the functions Γ_{ijq} and Γ_{ij}^p are the Christoffel symbols of the first and second kinds, respectively, and they are defined in terms of the functions g_{ij} and of some of their partial derivatives as follows:

$$\Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) = \partial \mathbf{g}_j \cdot \mathbf{g}_q = \Gamma_{jiq},$$

and:

$$\Gamma_{ij}^p = \sum_{q=1}^{3} g^{pq} \Gamma_{ijq} = \partial_i \mathbf{g}_j \cdot \mathbf{g}^p = \Gamma_{ji}^p, \quad [g^{ij}] = [g_{ij}]^{-1}.$$

The expressions in (3.1) are the covariant component of the Riemann curvature tensor of the set $u(\Omega)$ equipped with the metric tensor G, and they are denoted by:

$$R_{qijk} = \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \sum_{p=1}^{3} \Gamma_{ij}^p \Gamma_{kqp} - \sum_{p=1}^{3} \Gamma_{ik}^p \Gamma_{jqp}.$$

The condition $R_{qijk} = 0$ simply constitutes a re-writing of the relations $\partial_{ik} \mathbf{g}_j = \partial_{ij} \mathbf{g}_k$ in the form of the equivalent relations $\partial_{ik} \mathbf{g}_j \cdot \mathbf{g}_q = \partial_{ij} \mathbf{g}_k \cdot \mathbf{g}_q$.

Conversely, we can state the following question: given an open subset Ω of $\mathbb{R}^3$ and a positive-definite and symmetric matrix field $G = [g_{ij}] : \Omega \to \mathbb{R}^{3\times 3}$, when is the Riemannian manifold (Ω, G) flat, in the sense that it can be isometrically immersed in the Euclidean space $\mathbb{R}^3$? This will be the main topic of the next section.

3.1.2 Existence of an isometric immersion with a prescribed metric tensor

The answer to the question if a given Riemannian manifold (Ω, G) is flat, can be rephrased as follows: let Ω be a simply-connected open subset of $\mathbb{R}^3$. Then, a Riemannian manifold (Ω, G) with a Riemannian metric $G = [g_{ij}]$ of class $\mathcal{C}^2$ in Ω is flat if and only if its Riemannian curvature tensor vanishes in Ω. Moreover, if Ω is a connected open subset of $\mathbb{R}^3$, then isometric immersions of a flat Riemannian manifold are unique up to isometries of $\mathbb{R}^3$.

Recast as such, these two theorems together constitute a special case (that where the dimensions of the manifold and of the Euclidean space are both equal to three) of the **fundamental theorem of Riemannian Geometry**. This theorem addresses the same existence and uniqueness questions in the more general setting where Ω is replaced by a p-dimensional manifold and $\mathbb{R}^3$ is replaced by a $(p+q)$-dimensional Euclidean space. Another fascinating question (which will not be addressed here) is the following: given an open subset Ω of $\mathbb{R}^3$ equipped with a symmetric, positive-definite matrix field $[g_{ij}] : \Omega \to \mathbb{R}^{3\times 3}$, assume this time that the Riemannian manifold $(\Omega, [g_{ij}])$ is no longer flat, i.e., its Riemannian curvature tensor does not vanish in Ω. Can such a Riemannian manifold still be isometrically immersed, but this time in a higher-dimensional Euclidean space? That is, does there exist a Euclidean space $\mathbb{R}^d$ with $d > 3$ and does there exist an immersion $u : \Omega \to \mathbb{R}^3$ such that: $g_{ij} = \partial_i u \cdot \partial_j u$ in Ω? The answer is yes, according to the following Nash Theorem: Any p-dimensional Riemannian manifold equipped with a continuous metric can be isometrically immersed in a Euclidean space of dimension $2p$ with an immersion of class $\mathcal{C}^1$; it can also be isometrically immersed in a Euclidean space of dimension $(2p+1)$ with a globally injective immersion of class $\mathcal{C}^1$. See [43] for details.

We now return to the question of existence raised at the beginning, and we state the following theorem:

Theorem 3. *Let Ω be a connected and simply-connected open set in $\mathbb{R}^3$ and $G = [g_{ij}] \in \mathcal{C}^2(\Omega, \mathbb{R}^{3\times 3})$ be a symmetric, positive-definite matrix field that satisfies:*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \sum_{p=1}^{3} \Gamma_{ij}^p \Gamma_{kqp} - \sum_{p=1}^{3} \Gamma_{ik}^p \Gamma_{jqp} = 0, \quad \textit{in } \Omega, \tag{3.2}$$

where:

$$\Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) = \partial \boldsymbol{g}_j \cdot \boldsymbol{g}_q = \Gamma_{jiq},$$

and:

$$\Gamma_{ij}^p = \sum_{q=1}^{3} g^{pq}\Gamma_{ijq} = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^p = \Gamma_{ji}^p.$$

Then, there exists an immersion $u \in \mathcal{C}^3(\Omega, \mathbb{R}^3)$ such that:

$$G = \nabla u^T \nabla u \quad \text{in } \Omega.$$

Of course, this result is valid for any dimension $n \geq 2$, where the three-dimensional space $\mathbb{R}^3$ is replaced by any Euclidean space $\mathbf{E}^d$ of dimension d.

The proof of Theorem 3 relies on a simple, yet crucial, observation. When a smooth enough immersion $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ is a priori given, its components u_i satisfy the relations $\partial_{ij} u_l = \sum_p \Gamma_{ij}^p \partial_p u_l$, which are nothing but another way of writing the relations $\partial_i \mathbf{g}_j = \sum_p \partial_j \mathbf{g}_j \mathbf{g}_p$. This observation thus suggests to begin by solving the system of partial differential equations:

$$\partial_i F_{lj} = \sum_p \Gamma_{ij}^p F_{lp}, \quad \text{in } \Omega,$$

whose solutions $F_{lj} : \Omega \to \mathbb{R}$ then constitute natural candidates for the derivatives $\partial_j u_l$ of the unknown immersion u. See the reference [10] for a complete proof of Theorem 3.

The regularity assumption on the components of the metric tensor $G = [g_{ij}]$ made in Theorem 3 ($g_{ij} \in \mathcal{C}^2(\Omega)$ for all i, j) can be significantly weakened. More specifically, C. Mardare has shown in [37] that the existence theorem still holds if the components g_{ij} belong to the space $\mathcal{C}^1(\Omega)$ and, in this case, the resulting isometric immersion u is in the space $\mathcal{C}^2(\Omega, \mathbb{R}^n)$. After that, S. Mardare has shown that the existence theorem is also true if $g_{ij} \in W^{1,\infty}_{loc}(\Omega)$, and the resulting mapping u belongs to the space $W^{2,\infty}_{loc}(\Omega, \mathbb{R}^n)$. As expected, the sufficient condition $R_{qijk} = 0$ in the set Ω in Theorem 3 are then assumed to hold in the sense of distribution. More precisely, we quote the exact statement below:

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a connected and simply connected bounded open set. Let a metric be given in Ω by the means of a symmetric, positive definite matrix field $(g_{ij}) \in$*

$W^{1,\infty}_{loc}(\Omega, \mathbb{R}^{n\times n})$. *Assume that the corresponding Riemann curvature tensor vanishes in the sense of distributions, that is:*

$$\int_\Omega \{-\Gamma_{ikq}\partial_j\phi + \Gamma_{ijq}\partial_k\phi + \sum_p \Gamma^p_{ij}\Gamma_{kqp}\phi - \sum_p \Gamma^p_{ik}\Gamma_{jqp}\phi\}dx = 0,$$

for all $i, j, k, q \in \{1, ..., n\}$, *and for all real function* $\phi \in \mathcal{C}^\infty(\Omega)$, *with compact support included in* Ω. *Then, there exists a map* $u \in W^{2,\infty}_{loc}(\Omega, \mathbb{R}^n)$ *such that:*

$$\partial_i u \cdot \partial_j u = g_{ij}.$$

For a complete discussion of this result, see the reference [38].

3.1.3 Uniqueness up to isometries of immersion with the same metric tensor

In the previous section, we have established the existence of an immersion $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ giving rise to a set $u(\Omega)$ with a prescribed metric tensor, provided that the given metric tensor satisfies some compatibility relations. We now turn to the question of uniqueness of such immersions.

This uniqueness result is the object of the next theorem, aptly called a **rigidity theorem** in view of its geometrical interpretation: it asserts that if two immersions $u, \tilde{u} \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$ share the same metric tensor field, then the set $u(\Omega)$ is obtained by subjecting the set $\tilde{u}(\Omega)$ either to a rotation, or to a symmetry with respect to a plane followed by a rotation, then by subjecting the rotated set to a translation.

Theorem 5. *Let* Ω *be a connected open subset of* $\mathbb{R}^n$ *and let* $u, \tilde{u} \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$ *be two immersions such that their associated metric tensors satisfy:*

$$\nabla u^T \nabla u = \nabla \tilde{u}^T \nabla \tilde{u}, \quad in\ \Omega,$$

then there exist a vector $c \in \mathbb{R}^n$ *and an orthogonal matrix* Q *such that:*

$$u(x) = c + Q\tilde{u}(x), \quad for\ all\ x \in \Omega.$$

This theorem hence asserts that two immersions share the same metric tensor field over an open and connected set if and only if they are isometrically equivalent.

The uniqueness result is also true if we only assume that the components of the metric tensor g_{ij} just belong to the space $W^{1,\infty}_{loc}(\Omega)$. See Theorem 4.4 in [38] for a more complete discussion.

The special interest along this work will be the application of some notions from the differential geometry of surfaces, such as curvature and fundamental forms, to the study of unconstrained thin elastic bodies. Hence, in the next section, we introduce the fundamental notions that we shall use. Since the main application of the theory will be in three dimensions, we just present the theory in this framework.

3.1.4 Differential geometry of surfaces

3.1.4.1 The first fundamental form In the previous section, we saw that an open set $u(\Omega)$ in the three-dimensional space $\mathbb{R}^3$, where Ω is an open set in $\mathbb{R}^3$ and $u : \Omega \to \mathbb{R}^3$ is a smooth enough immersion, is unambiguously defined (up to isometries of $\mathbb{R}^3$) by a single tensor field, the metric tensor field, whose covariant components $g_{ij} : \Omega \to \mathbb{R}$ are given by:

$$g_{ij} = \partial_i u \cdot \partial_j u.$$

Instead of that situation, consider now a surface $\hat{\omega} = \theta(\omega)$ in $\mathbb{R}^3$, where ω is a two-dimensional open set in $\mathbb{R}^2$, and $\theta : \omega \to \mathbb{R}^3$ is a smooth injective immersion. Then, such a two-dimensional manifold requires two tensor fields for its definition: the first and second fundamental forms of $\hat{\omega}$. We start by introducing the first fundamental form.

Let ω be an open subset of $\mathbb{R}^2$ and let:

$$\theta : \omega \subset \mathbb{R}^2 \to \theta(\omega) = \hat{\omega} \subset \mathbb{R}^3,$$

be a mapping that is differentiable at a point $y \in \omega$. If h is such that $(y + h) \in \omega$, then:

$$\theta(y + h) = h + \nabla\theta(y)h + o(h),$$

where the 3×2 matrix $\nabla\theta(y)$ is defined by:

$$\nabla\theta(y) = \begin{pmatrix} \partial_1\theta_1 & \partial_2\theta_1 \\ \partial_1\theta_2 & \partial_2\theta_2 \\ \partial_1\theta_3 & \partial_2\theta_3 \end{pmatrix}(y).$$

Let the two vectors $\mathbf{a}_\alpha(y) \in \mathbb{R}^3$ be defined by:

$$\mathbf{a}_\alpha(y) = \partial_\alpha\theta(y) = \begin{pmatrix} \partial_\alpha\theta_1 \\ \partial_\alpha\theta_2 \\ \partial_\alpha\theta_3 \end{pmatrix}(y),$$

that is, $\mathbf{a}_\alpha(y)$ is the α-th column vector of the gradient $\nabla\theta(y)$. Then, the expansion of θ about y may be also written as:

$$\theta(y+h) = \theta(y) + \sum_\alpha h_\alpha \mathbf{a}_\alpha(y) + o(h).$$

From here, we can infer that:

$$|\theta(y+h) - \theta(y)|^2 = h^T\nabla\theta(y)^T\nabla\theta(y)h + o(|h|^2) = \sum_{\alpha,\beta=1}^{2} h_\alpha \mathbf{a}_\alpha(y) \cdot \mathbf{a}_\beta h_\beta + o(|h|^2).$$

In this way, the principal part with respect to h of the length between the point $\theta(y+h)$ and $\theta(y)$ is then given by:

$$\{\sum_{\alpha,\beta=1}^{2} h_\alpha \mathbf{a}_\alpha(y) \cdot \mathbf{a}_\beta h_\beta\}^{1/2}.$$

This observation suggests to define a symmetric matrix field $I = [a_{\alpha\beta}]$ of order two by letting:

$$a_{\alpha\beta}(y) = \mathbf{a}_\alpha(y) \cdot \mathbf{a}_\beta(y) = \nabla\theta(y)^T\nabla\theta(y)_{\alpha\beta}.$$

The symmetric matrix field I is called the first fundamental form of the surface $\hat{\omega}$ at the point $\hat{y} = \theta(y)$.

In the case where θ is an immersion, that is, in the case where the vectors $\mathbf{a}_\alpha$ are linearly independent, the matrix field I is also positive-definite.

The main application of the first fundamental form is that the metric notions on the surface $\hat{\omega}$, such as length and areas, can be written in terms of I. Hence, the first fundamental form well deserves metric tensor as its alias.

Theorem 6. *Let ω be an open subset of $\mathbb{R}^2$, let $\theta : \omega \to \mathbb{R}^3$ be an injective and smooth enough immersion, and let $\hat{\omega} = \theta(\omega)$. Then:*

1. *The area element $d\hat{a}(\hat{y})$ at the point $\hat{y} = \theta(y)$ on the surface $\hat{\omega}$ is given in terms of the area element dy at the point $y \in \omega$ by the formula:*

$$d\hat{a}(\hat{y}) = \sqrt{det\ [a_{\alpha\beta}(y)]}dy.$$

2. *The length element $d\hat{l}(\hat{y})$ at the point $\hat{y} \in \hat{\omega}$ is given by:*

$$d\hat{l}(\hat{y}) = \{\sum_{\alpha,\beta} h_\alpha a_{\alpha\beta}(y) h_\beta\}^{1/2}.$$

A proof of this well-known result can be found in the reference [12].

While the image $u(\Omega)$ of a three-dimensional open set Ω by a smooth enough immersion u is well defined by its metric tensor, uniquely up to isometries, provided some compatibility conditions that should be satisfied by the covariant components g_{ij} of the metric tensor, a surface given as the image $\theta(\omega)$ of a two-dimensional open set $\omega \subset \mathbb{R}^2$ cannot be defined by its metric along. To see this fact clearly, consider the following example: a flat surface $\hat{\omega}_0$ may be deformed into a portion $\hat{\omega}_1$ of a cylinder or a portion $\hat{\omega}_2$ of a cone without altering the length of any curve drawn on it. But it is clear that in general, the surfaces $\hat{\omega}_0$ and $\hat{\omega}_1$, or $\hat{\omega}_0$ and $\hat{\omega}_2$, or $\hat{\omega}_1$ and $\hat{\omega}_2$, are not identical surfaces modulo an isometry in $\mathbb{R}^3$. As this example suggests, the missing information is provided by the curvature of a surface. This question relies on the knowledge of the second fundamental form of a surface II. We now proceed, in the next subsection, to introduce this notion.

3.1.4.2 Curvature and the second fundamental form

We first recall the definition of curvature of a planar curve. Let γ be a smooth enough planar curve parametrized by its curvilinear abscissa $s \in \mathcal{O}$, where $\mathcal{O}$ is an open interval of $\mathbb{R}$. Consider two points on the curve $\gamma(s)$ and $\gamma(s+\Delta s)$ with curvilinear abscissae s and $s+\Delta s$, where the increment Δs is small enough so that the point $s+\Delta s$ still belongs to $\mathcal{O}$, and let $\Delta\phi(s)$ be the algebraic angle between the two normals $\mathbf{n}(s)$ and $\mathbf{n}(s+\Delta s)$ to γ at the points $\gamma(s)$ and $\gamma(s+\Delta s)$, respectively. When $\Delta s \to 0$, the ratio:

$$\frac{\Delta\phi(s)}{\Delta s}$$

has a limit, and it is called the curvature of the curve γ at the point $\gamma(s)$. If this limit is non-zero, its inverse R is called the algebraic radius of curvature of γ at $\gamma(s)$.

Consider as before a surface $\hat{\omega} = \theta(\omega)$ in $\mathbb{R}^3$, where ω in an open subset of $\mathbb{R}^2$ and $\theta : \omega \to \mathbb{R}^3$ is a smooth enough immersion. For each $y \in \omega$, the vector:

$$\mathbf{a}_3(y) = \frac{\mathbf{a}_1(y) \times \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \times \mathbf{a}_2(y)|}$$

is hence well defined, has Euclidean norm one, and is normal to the surface $\hat{\omega}$ at the point $\hat{y} = \theta(y)$. Fix now $y \in \omega$ and consider a plane P normal to $\hat{\omega}$ at the point $\hat{y} = \theta(y)$, i.e., a plane that contains the unit vector $\mathbf{a}_3$. Then, the intersection $\hat{C} = P \cap \hat{\omega} = \theta(C)$ is a planar curve on $\hat{w}$. It is remarkable that the curvature of the curve $\hat{C}$ can be computed in terms of the components $a_{\alpha\beta}$ of the first fundamental form I, together with the covariant components $b_{\alpha\beta}$ of the second fundamental form of $\hat{\omega}$: assume that, in a sufficiently small neighborhood of y, the restriction of the curve C to this neighborhood is the image $f(\mathcal{O})$ of an open interval $\mathcal{O} \subset \mathbb{R}$, where $f = \sum_\alpha f_\alpha e_\alpha$, and $\frac{df_\alpha}{dt}(t)e_\alpha \neq 0$, where $t \in \mathcal{O}$ is such that $y = f(t)$. Then, the curvature of the planar curve $\hat{C}$ at $\hat{y}$ (Theorem 2.4-1 in [9]) is given by the ratio:

$$\frac{1}{R} = \frac{\sum_{\alpha,\beta} b_{\alpha\beta}(f(t))\frac{df_\alpha}{dt}(t)\frac{df_\beta}{dt}(t)}{\sum_{\alpha,\beta} a_{\alpha\beta}(f(t))\frac{df_\alpha}{dt}(t)\frac{df_\beta}{dt}(t)},$$

where:

$$b_{\alpha\beta}(y) = \mathbf{a}_3(y) \cdot \partial_\alpha \mathbf{a}_\beta(y) = -\partial_\alpha \mathbf{a}_3(y) \cdot \mathbf{a}_\beta(y) = b_{\beta\alpha}.$$

The elements $b_{\alpha\beta}(y)$ of the symmetric matrix field $II = [b_{\alpha\beta}]$ defined in the previous theorem are called the covariant components of the second fundamental form of the surface $\hat{\omega}$ at the point $\hat{y} = \theta(y)$.

3.1.4.3 Gaussian curvature The analysis of the previous section suggests that precise information about the shape of a surface $\hat{w} = \theta(w)$ in a neighborhood of one of its points $\hat{y} = \theta(y)$ can be gathered by letting the plane P turn around the normal vector $\mathbf{a}_3(y)$ and by following in this process the variation of the curvatures at $\hat{y}$ of the corresponding planar curves $P \cap \hat{w}$.

It is well-known that these curvatures at a fix point $\hat{y} = \theta(y)$ span a compact interval in $\mathbb{R}$, denoted by:

$$\Big[\frac{1}{R_1(y)}, \frac{1}{R_2(y)}\Big].$$

Moreover, if:

$$\frac{1}{R_1(y)} \neq \frac{1}{R_2(y)},$$

there is a unique pair of orthogonal planes P_1 and P_2, normal to the surface $\hat{w} = \theta(\omega)$ at the point $\hat{y} = \theta(y)$, such that the curvatures of the associated planar curves:

$$P_1 \cap \hat{w} \quad \text{and} \quad P_2 \cap \hat{w},$$

are precisely the curvatures:

$$\frac{1}{R_1(y)} \quad \text{and} \quad \frac{1}{R_2(y)}.$$

In addition, we have the surprising fact that:

$$\frac{1}{R_1(y)R_2(y)} = \frac{\det\,[b_{\alpha\beta}(y)]}{\det\,[a_{\alpha\beta}(y)]}.$$

The real numbers $\frac{1}{R_1(y)}$ and $\frac{1}{R_2(y)}$ are called the principal curvatures of the surface $\hat{w}$ at the point $\hat{y}$. The **Gaussian curvature** κ of the surface $\hat{\omega}$ at $\hat{y}$ is defined as follows:

$$\kappa(\hat{y}) = \frac{1}{R_1(y)R_2(y)}.$$

A point on a surface is an elliptic, parabolic, or hyperbolic point according as its Gaussian curvature is positive, zero, or negative, respectively.

As we discussed above, a surface in $\mathbb{R}^3$ cannot be defined by its metric alone, i.e., through its first fundamental form alone since its curvature must be in addition specified through its second fundamental form. But it is quite surprising that the Gaussian curvature at a point can also be expressed solely in terms of the covariant components if the first fundamental form and their derivatives. This is the celebrated Gauss's Theorema Egregium, that we shall discuss in the next sections.

3.1.4.4 Necessary condition for the existence of a surface with prescribed first and second fundamental forms

It is remarkable that the components $a_{\alpha\beta} : \omega \to \mathbb{R}$ and $b_{\alpha\beta} : \omega \to \mathbb{R}$ of the first and second fundamental forms of a surface $\theta(\omega)$ cannot be arbitrary functions. As shown in the next theorem, they must satisfy relations that involve expressions in terms of the functions $a_{\alpha\beta}$ and of some of their partial derivatives. The regularity assumptions in the next theorem can be relaxed, as we shall see.

Theorem 7. *Let ω be an open subset of $\mathbb{R}^2$, let $\theta \in \mathcal{C}^3(\omega, \mathbb{R}^3)$ be an immersion, and let:*

$$a_{\alpha\beta} = \partial_\alpha \theta \cdot \partial_\beta \theta, \quad \textit{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta}\theta \cdot \frac{\partial_1 \theta \times \partial_2 \theta}{|\partial_1 \theta \times \partial_2 \theta|},$$

denote the covariant components of the first and second fundamental forms of the surface $\theta(\omega)$. Let the functions $\Gamma_{\alpha\beta\tau} \in \mathcal{C}^1(\omega)$ and $\Gamma^\sigma_{\alpha\beta} \in \mathcal{C}^1(\omega)$ be defined by:

$$\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),$$

and:

$$\Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau}, \quad \textit{where } [a^{\sigma\tau}] = [a_{\alpha\beta}]^{-1}.$$

Then, necessarily:

$$\partial_\beta \Gamma_{\alpha\sigma\tau} - \partial_\sigma \Gamma_{\alpha\beta\tau} + \sum_\mu \left[\Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu}\right] = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau}, \quad \textit{in } \omega, \tag{3.3}$$

and:

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \sum_\mu \left[\Gamma^\mu_{\alpha\sigma}b_{\beta\mu} - \Gamma^\mu_{\alpha\beta}b_{\sigma\mu}\right] = 0, \quad \textit{in } \omega. \tag{3.4}$$

See the reference [10] for a comprehensive and elementary proof of this result.

The equations (3.3) are called the Gauss equations, while the equations (3.4) are called the Codazzi-Mainardi equations.

The definitions of the functions:

$$\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_\tau = \Gamma_{\beta\alpha\tau},$$

and:

$$\Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta\tau} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}^\sigma = \Gamma^\sigma_{\beta\alpha}$$

imply that the sixteen Gauss equations are satisfied if and only if they are satisfied for $\alpha = 2$, $\beta = 1$, $\sigma = 2$, $\tau = 1$, and that the Codazzi-Mainardi equations are satisfied if and only if they are satisfied for $\alpha = 1$, $\beta = 2$, $\sigma = 1$ and $\alpha = 2$, $\beta = 2$, $\sigma = 1$. Of course, other choices of indices with the same properties are clearly possible. Therefore, the Gauss equations and the Codazzi-Mainardi equations in fact respectively reduce to one and two equations:

$$\begin{cases} \partial_2 b_{11} - \partial_1 b_{12} = b_{12}(\Gamma^2_{11} - \Gamma^1_{11}) + \Gamma^1_{12} b_{11} - \Gamma^2_{11} b_{22} \\ \partial_2 b_{21} - \partial_1 b_{22} = b_{12}(\Gamma^2_{22} - \Gamma^1_{21}) + \Gamma^1_{22} b_{11} - \Gamma^2_{21} b_{22} \\ S_{1212} = \partial_1 \Gamma_{221} - \partial_2 \Gamma_{211} + \Gamma^1_{21}\Gamma_{211} - \Gamma^1_{22}\Gamma_{111} + \Gamma^2_{21}\Gamma_{212} - \Gamma^2_{22}\Gamma_{122} = \det\,[b_{\alpha\beta}] \end{cases} \tag{3.5}$$

The system (3.5) will be referred as the Gauss-Codazzi-Mainardi system.

3.1.4.5 Theorema Egregium of Gauss

Letting $\alpha = 2$, $\beta = 1$, $\sigma = 2$ and $\tau = 1$ in the Gauss equations gives that:

$$S_{1212} = \det\,[b_{\alpha\beta}].$$

Consequently, the Gaussian curvature κ at each point $\theta(y)$ of the surface $\hat{\omega} = \theta(\omega)$ can be written as:

$$\kappa(y) = \frac{S_{1212}(y)}{\det\,[a_{\alpha\beta}(y)]}.$$

By inspection of the function S_{1212}, we thus reach the conclusion that, at each point of the surface, a notion involving the curvature of the surface, that is, the Gaussian curvature, is entirely determined by the knowledge of the metric of the surface at the same point, i. e., the components of the first fundamental form and their partial derivatives at the same point. This is the so-called Theorema Egregium of Gauss:

Theorem 8. *Let ω be an open subset of $\mathbb{R}^2$, let $\theta \in \mathcal{C}^3(\omega, \mathbb{R}^3)$ be an immersion, let $a_{\alpha\beta} = \partial_\alpha\theta \cdot \partial_\beta\theta$ denote the covariant components of the first fundamental form of the surface $\theta(\omega)$, and let the functions $\Gamma_{\alpha\beta\tau}$ and S_{1212} be given by:*

$$\begin{aligned}
&\Gamma_{\alpha\beta\tau} = \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}), \\
&S_{1212} = \frac{1}{2}(2\partial_{12}a_{12} - \partial_{11}a_{22} - \partial_{22}a_{11}) + \sum_{\alpha,\beta} a^{\alpha\beta}(\Gamma_{12\alpha}\Gamma_{12\beta} - \Gamma_{11\alpha}\Gamma_{22\beta}).
\end{aligned} \tag{3.6}$$

Then, at each point $\theta(y)$ of the surface $\theta(\omega)$, the Gaussian curvature $\kappa(y)$ satisfies:

$$\kappa(y) = \frac{S_{1212}(y)}{det\ [a_{\alpha\beta}(y)]}.$$

Therefore, the Gauss-Codazzi-Mainardi system (3.5) takes the form:

$$\begin{cases}
\partial_2 b_{11} - \partial_1 b_{12} = b_{12}(\Gamma^2_{11} - \Gamma^1_{11}) + \Gamma^1_{12}b_{11} - \Gamma^2_{11}b_{22} \\
\partial_2 b_{21} - \partial_1 b_{22} = b_{12}(\Gamma^2_{22} - \Gamma^1_{21}) + \Gamma^1_{22}b_{11} - \Gamma^2_{21}b_{22} \\
\kappa \ \det\ [a_{\alpha\beta}] = \det\ [b_{\alpha\beta}]
\end{cases} \tag{3.7}$$

We now provide a useful result which allows to compute the Gaussian curvature of a difference of appropriate metrics. The proof of this result can be found in [22].

Theorem 9. *Suppose that (M, g) is a smooth 2-dimensional Riemannian manifold and ϕ is a smooth function on M with:*

$$|\nabla\phi| < 1.$$

Then $g_1 = g - d\phi^2$ is a smooth Riemannian metric on M and the Gaussian curvature of g_1 is given by:

$$\kappa(g_1) = \frac{1}{(1 - |\nabla\phi|^2)}\Big[\kappa(g) - \frac{det\,\nabla^2 v}{(1 - |\nabla\phi|^2)det\,g}\Big]. \tag{3.8}$$

3.1.4.6 Existence of a surface with prescribed first and second fundamental forms

So far, we have considered that we are given an open set ω of $\mathbb{R}^2$ and a smooth immersion $\theta : \omega \to \mathbb{R}^3$, allowing us to define matrix field $I = [a_{\alpha\beta}]$ and $II = [b_{\alpha\beta}]$, which are the first and second fundamental forms. We now turn to the reciprocal questions: Given an open subset $\omega \subset \mathbb{R}^2$ and two smooth enough matrix fields $I = [a_{\alpha\beta}] : \omega \to \mathbb{R}^{2\times 2}$ and $II = [b_{\alpha\beta}] : \omega \to \mathbb{R}^{2\times 2}$, where I is symmetric and positive-definite, and II is symmetric, when are they the first and second fundamental forms of a surface $\theta(\omega) \subset \mathbb{R}^3$?. In other words, when does exist an immersion $\theta : \omega \to \mathbb{R}^3$ such that:

$$a_{\alpha\beta} = \partial_\alpha\theta \cdot \partial_\beta\theta, \quad \text{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta}\theta \cdot \frac{\partial_1\theta \times \partial_2\theta}{|\partial_1\theta \times \partial_2\theta|}.$$

Another interested question is: if such an immersion exists, to what extent is it unique?

The answers to these questions constitute the fundamental theorem of surface theory: if ω is simply-connected, the necessary conditions expressed via the Gauss and Codazzi-Mainardi equations, are also sufficient for the existence of such an immersion. If ω is connected, this immersion is unique up to isometries in $\mathbb{R}^3$. For future reference, we quote these results in the following theorem:

Theorem 10. *Let ω be a connected and simply connected open subset of $\mathbb{R}^2$ and let $[a_{\alpha\beta}] \in \mathcal{C}^2(\omega, \mathbb{R}^{2\times 2})$ be a symmetric and positive-definite matrix field, and let $[b_{\alpha\beta}] \in \mathcal{C}^2(\omega, \mathbb{R}^{2\times 2})$ be a symmetric matrix field. Suppose that both matrix fields satisfy the Gauss-Codazzi-Mainardi system. Then, there exists an immersion $\theta \in \mathcal{C}^3(\omega, \mathbb{R}^3)$ such that:*

$$a_{\alpha\beta} = \partial_\alpha\theta \cdot \partial_\beta\theta \quad \textit{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta}\theta \cdot \frac{\partial_1\theta \times \partial_2\theta}{|\partial_1\theta \times \partial_2\theta|}.$$

For a complete proof of this result, see [10].

The regularity assumptions made in the previous theorem on the matrix fields $[a_{\alpha\beta}]$ and $[b_{\alpha\beta}]$ can be significantly relaxed. S. Mardare (see [36]) was able to reduce these regularities, to those of the spaces $W^{1,p}_{loc}(\omega, \mathbb{R}^{2\times 2})$ and $L^p_{loc}(\omega, \mathbb{R}^{2\times 2})$ for any $p > 2$, with a resulting mapping θ in the space $W^{2,p}_{loc}(\omega, \mathbb{R}^3)$. For a proof of this result, see Theorem 9 in [36].

3.2 PRELIMINARIES IN MATHEMATICAL ANALYSIS

Throughout this section, we introduce the fundamental notions of Mathematical Analysis that we use in the thesis. We start with the notions of convexity and quasiconvexity, and their relations with integral functionals. We then introduce two useful results for Sobolev functions: a truncation theorem (Theorem 14) and the Brezis-Wainger inequality (Theorem 16). We finish this section with the definition of Γ-convergence. It is remarkable the role played by the notion and consequences of Γ-convergence in the derivation of the theory of elasticity presented here. Hence, we dedicate a main part of this section to the develop of this notion, and we state and prove their main consequences, such as compactness (Theorem 20) and convergence of minimizers (Theorem 22).

3.2.1 Convexity and quasiconvexity

In this section $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a function assumed to be Borel measurable, locally bounded and bounded from below. Recall that the convex and quasiconvex envelopes of f, i.e. $Cf, Qf : \mathbb{R}^{m\times n} \to \mathbb{R}$ are defined by:

$$Cf(M) = \sup\left\{g(M);\quad g:\mathbb{R}^{m\times n}\to\mathbb{R},\quad g \text{ convex}, \ g\le f\right\},$$
$$Qf(M) = \sup\left\{g(M);\quad g:\mathbb{R}^{m\times n}\to\mathbb{R},\quad g \text{ quasiconvex}, \ g\le f\right\}.$$

We say that f is quasiconvex, if:

$$f(M) \le \fint_D f(M+\nabla\phi(x))\ \mathrm{d}x \qquad \forall M\in\mathbb{R}^{m\times n}\quad \forall\phi\in W_0^{1,\infty}(D,\mathbb{R}^m),$$

on every open bounded set $D\subset\mathbb{R}^n$.

We now quote some results used in this work. for a proof, see the reference [13].

Theorem 11. *(i) When $m=1$ or $n=1$ then f is quasiconvex if and only if f is convex.*
(ii) For any open bounded $D\subset\mathbb{R}^n$ there holds:

$$Qf(M) = \inf\left\{\fint_D f(M+\nabla\phi(x))\ \mathrm{d}x;\ \phi\in W_0^{1,\infty}(D,\mathbb{R}^m)\right\}.$$

(iii) Assume that, for some $n_1 + n_2 = n$ *we have:*

$$f(M) = f_1(M_{n_1}) + f_2(M_{n_2}) \qquad \forall M \in \mathbb{R}^{m\times n},$$

where M_{n_1} *stands for the principal minor of* M *consisting of its first* n_1 *columns, while* M_{n_2} *is the minor of* M *consisting of its* n_2 *last columns. Assume that* f_1, f_2 *are Borel measurable and bounded from below. Then:*

$$Cf = Cf_1 + Cf_2, \qquad Qf = Qf_1 + Qf_2$$

The following classical results explain the role of convexity and quasiconvexity in the integrands of the typical integral functionals.

Theorem 12. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *and let* $f : \mathbb{R}^{m\times 1} \to \mathbb{R}$ *be lower semicontinuous (lsc). Then the functional:*

$$I(u) = \int_\Omega f(u(x))\ \mathrm{d}x \qquad \forall u \in L^2(\Omega, \mathbb{R}^m)$$

is sequentially lsc with respect to the weak convergence in $L^2(\Omega, \mathbb{R}^m)$ *if and only if* f *is convex.*

Theorem 13. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *and let* $f : \Omega\times\mathbb{R}^{m\times n} \to \mathbb{R}$ *be Caratheodory, and satisfying the uniform growth condition:*

$$\exists C_1, C_2 > 0 \quad \forall x \in \Omega \quad \forall M \in \mathbb{R}^{m\times n} \qquad C_1|M|^2 - C_2 \leq f(x, M) \leq C_2(1 + |M|^2). \tag{3.9}$$

Assume that the quasiconvexification Qf *of* f *with respect to the variable* M*, is also a Caratheodory function. Then for every* $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ *there exists a sequence* $\{u_\epsilon\} \in u + W_0^{1,2}(\Omega, \mathbb{R}^m)$ *such that, as* $\epsilon \to 0$*:*

$$u_\epsilon \rightharpoonup u \quad \textit{weakly in } W^{1,2} \qquad \textit{and} \qquad \int_\Omega f(x, \nabla u_\epsilon(x))\ \mathrm{d}x \to \int_\Omega Qf(x, \nabla u(x))\ \mathrm{d}x.$$

3.2.2 Some results on Sobolev functions

We now state a truncation result for Sobolev functions (see [35]):

Theorem 14. *Let Ω be a bounded, Lipschitz domain, $1 < p < \infty$, $k \in \mathbb{N}$ and $\lambda > 0$. Suppose that $u \in W^{k,p}(\Omega)$, and let:*

$$|u|_k(x) = \sum_{|\alpha| \leq k} |\nabla^\alpha u|(x).$$

Then there exists $u^\lambda \in W^{k,\infty}(\Omega)$ such that:

$$\begin{aligned} &\|u^\lambda\|_{W^{k,\infty}} \leq C(p,k,\Omega)\lambda; \\ &\big|\{x \in \Omega : u^\lambda(x) \neq u(x)\}\big| \leq \frac{C(p,k)}{\lambda^p} \int_{|u|_k \geq \lambda/2} |u|_k^p; \\ &\|u^\lambda\|_{W^{k,p}} \leq C(p,k,\Omega)\|u\|_{W^{k,p}}. \end{aligned} \tag{3.10}$$

In particular, it follows that:

$$\begin{aligned} &\lim_{\lambda \to \infty} \lambda^p \big|\{x \in \Omega : u^\lambda(x) \neq u(x)\}\big| = 0; \\ &\lim_{\lambda \to \infty} \|u - u^\lambda\|_{W^{k,p}} = 0. \end{aligned} \tag{3.11}$$

The next results are related to the Brezis-Wainger inequality. We refer the reader to the reference [49] for a detailed discussion. The first theorem (page 66 in [49]) shows that one can use Bessel Potentials to characterize Sobolev spaces. Recall that the space of Bessel potentials $L^{\alpha,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined as all functions u such that

$$F^{-1}(1 + |\xi|^2)^{\alpha/2} F u = f \in L^p(\mathbb{R}^n),$$

where F denotes the Fourier transform. The norm in this space is defined as $\|u\|_{\alpha,p} = \|F^{-1}(1 + |\xi|^2)^{\alpha/2} F u\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$. The next result also shows that in the case $\alpha > 1$ is a positive integer, then this norm is equivalent to the Sobolev norm of u.

Theorem 15. *If k is a positive integer and $1 < p < \infty$, then:*

$$L^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n).$$

Moreover, if $u \in L^{k,p}(\mathbb{R}^n)$ with $F^{-1}(1+|\xi|^2)^{\alpha/2} F u = f \in L^p(\mathbb{R}^n)$, then there exists a constant $C = C(\alpha, p, n) > 0$ such that:

$$C^{-1}\|f\|_p \leq \|u\|_{k,p} \leq C\|f\|_p.$$

Now we present the Brezis-Wainger inequality:

Theorem 16. *Let $u \in L^{l,q}(\mathbb{R}^n)$ with $lq > n$, $1 \leq q \leq \infty$ and let $\alpha p = n$, for $1 < p < \infty$. If $\|u\|_{\alpha,p} \leq 1$, then:*

$$\|u\|_\infty \leq C\big[1 + log^{1/p'}(1 + \|u\|_{l,q})\big].$$

3.2.3 Introduction to Γ-convergence

The notion of Γ-convergence was introduced in a paper by E. De Giorgi and T. Franzoni in 1975 [14], and was since then much developed especially in connection with applications to problems in the calculus of variation. In this section, we shall present a brief introduction to Γ-convergence, and we refer to the book [7] for a comprehensive introduction to the subject.

First, we start with an abstract definition of Γ-convergence on metric spaces.

Definition 17. *Let (X, d) be a metric space. For any natural number n, let $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ be a functional defined on X. We say that the sequence $\mathcal{F}_n$ Γ-converges to a functional $\mathcal{F} : X \to \overline{\mathbb{R}}$ if and only if the following statements are satisfied:*

1. *For any $x \in X$ and for any sequence x_n converging to x in (X, d), we have:*

$$\liminf_{n\to 0} \mathcal{F}_n(x_n) \geq \mathcal{F}(x). \tag{3.12}$$

2. *For any $x \in X$, there exists a sequence x_n converging to x in (X, d) such that:*

$$\limsup_{n\to\infty} \mathcal{F}_n(x_n) \leq \mathcal{F}(x) \tag{3.13}$$

Remark 18 (Some remarks on Γ-convergence)**.** *1) The inequality in (3.12) is called the liminf inequality or the lower bound inequality. On the other hand, the inequality (3.13) in Part 2, is called the limsup or upper bound inequality. Observe that (3.12) and (3.13) together imply the existence, for each point $x \in X$, of a sequence x_n converging to x in (X, d) and verifying:*

$$\lim_{n\to\infty} \mathcal{F}_n(x_n) = \mathcal{F}(x).$$

The sequence x_n with this property is referred as the recovery sequence for the Γ-limit at x.

2) One of the most remarkable issues in the above definition is that the Γ*-limit of a sequence of functionals is obtained via an optimization process. Indeed, if on one hand inequality (3.12) requires the search of an asymptotic local lower bound for the family of functionals* $\mathcal{F}_n$*, on the other hand in (3.13) such a bound is optimized.*

3) If $\mathcal{F}$ *is the* Γ*-limit of* $\mathcal{F}_n$ *in* (X,d)*, then* $\mathcal{F}$ *is lower semicontinuous with respect to the metric* d*, that is, for any* $x \in X$ *and for any sequence* x_n *converging to* x *in* (X,d)*, we have:*

$$\liminf_{n\to\infty} \mathcal{F}(x_n) \geq \mathcal{F}(x).$$

Indeed, let $x \in X$ *be given, and take any sequence* x_n *converging to* x *in* (X,d)*. For each* n*, there exists a recovery sequence* $x_m^n \in X$ *such that:*

$$x_m^n \to x_n, \quad \textit{and} \quad \lim_{m\to\infty} \mathcal{F}_m(x_m^n) = \mathcal{F}(x_n).$$

Let $y_n = x_{m_n}^n$*,* $m_n \in \mathbb{N}$*, be such that:*

$$d(x_n, y_n) < \frac{1}{n}, \quad \textit{and} \quad \mathcal{F}_{m_n}(x_{m_n}^n) < \frac{1}{n} + \mathcal{F}(x_n).$$

Therefore:

$$\mathcal{F}(x) \leq \liminf \mathcal{F}_{m_n}(y_n) \leq \liminf \Big(\mathcal{F}(x_n) + \frac{1}{n}\Big),$$

where the first inequality follows from the definition of Γ*-convergence. This proves the lower semicontinuity of the* Γ*-limit.*

In particular, for a constant sequence of functionals $\mathcal{F}_n = \mathcal{F} : X \to \overline{\mathbb{R}}$*, we then deduce that* $\mathcal{F}_n$ Γ*-convergence to* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *is lower semicontinuous. If* $\mathcal{F}$ *is not lower semicontinuous, then* $\mathcal{F}_n$ Γ*-converges to the lower semicontinuous envelope of* $\mathcal{F}$*, denoted by* $\Gamma_0(\mathcal{F})$ *and defined as follows:*

$$\Gamma_0(\mathcal{F})(x) = \inf\big\{\liminf_{n\to\infty} \mathcal{F}(x_n) : x_n \to x\big\}, \quad x \in X.$$

Thus, Γ_0 *is the greatest lower semicontinuous functions not greater that* $\mathcal{F}$*.*

The lower semicontinuity of the Γ-limit is one of the variational features of the theory.

We now introduce the notions of Γ-lower and Γ-upper limit of a sequence of functionals. In the next definition, the set of all open neighbourhoods of a given point $x \in X$ is denoted by $\mathcal{N}(x)$.

Definition 19. *The* Γ*-lower limit and the* Γ*-upper limit of the sequence of functionals* $\{F_n\}$ *are the functions from* X *into* $\overline{\mathcal{R}}$ *defined by:*

$$(\Gamma - \liminf_{n\to\infty} F_n)(x) = sup_{U\in\mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y\in U} F_n(y),$$
$$(\Gamma - \limsup_{n\to\infty} F_n)(x) = sup_{U\in\mathcal{N}(x)} \limsup_{n\to\infty} \inf_{y\in U} F_n(y).$$

Of course, the equality of the Γ-upper limit and the Γ-lower limit is equivalent to the existence of the Γ-limit for the sequence of functionals. Moreover, we can restrict the set of neighbourhoods $\mathcal{N}(x)$ of a point x in X by taking a base $\mathcal{B}(x)$ for the neighbourhood system of x. In that case, we have:

$$\begin{aligned}(\Gamma - \liminf_{n\to\infty} F_n)(x) &= \sup_{U\in\mathcal{B}(x)} \liminf_{n\to\infty} \inf_{y\in U} F_n(y),\\ (\Gamma - \limsup_{n\to\infty} F_n)(x) &= \sup_{U\in\mathcal{B}(x)} \limsup_{n\to\infty} \inf_{y\in U} F_n(y).\end{aligned} \tag{3.14}$$

We now provide an important topological property of Γ-convergence: Γ-convergence always occurs upon extracting subsequences, provided separability of X is assumed.

Theorem 20. *If* (X, d) *is a separable metric space, then any sequence of functionals* $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ *contains a* Γ*-convergent subsequence.*

Proof. Let $\{F_n\}$ be a sequence of functionals from X into $\overline{R}$ and let $\mathcal{B} = \{U_j\}$ be a countable basis for the topology of X. Since $\overline{R}$ is compact, for every $j \in \mathbb{N}$, there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that the limit:

$$\lim_{k\to\infty} \inf_{y\in U_j} F_{n_k}(y)$$

exists in $\overline{\mathbb{R}}$. By a diagonal argument, we can construct a subsequence $\{F_{n_k}\}$ such that:

$$\lim_{k\to\infty} \inf_{y\in U} F_{n_k}(y)$$

exists for every U in $\mathcal{B}$. For every $x \in X$, we define:

$$\mathcal{B}(x) = \{U \in \mathcal{B} : x \in U\}$$

and:

$$F(x) = \sup_{U \in \mathcal{B}(x)} \lim_{k\to\infty} \inf_{y \in U} F_{n_k}(y).$$

We now prove that the subsequence F_{n_k} Γ-converges to the functional F. This is straightforward, because by (3.14), we have the equalities:

$$F(x) = (\Gamma - \liminf_{k\to\infty} F_{n_k}) = (\Gamma - \limsup_{k\to\infty} F_{n_k}).$$

□

We now turn on the convergence of minimizers.

Definition 21. *A family of functionals $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ is equi-coercive on X, if for any $t \in \mathbb{R}$,*

$$\{\mathcal{F}_n \leq t\} \subset K_t,$$

where K_t is a sequencially compact set. Also, $\mathcal{F}_n$ is equi-mildly coercive on X if there exists a sequentially compact set $K \subset X$ such that:

$$\inf_X \mathcal{F}_n = \inf_K \mathcal{F}_n, \quad \textit{for all } n.$$

Theorem 22. *(Convergence of global minimizers) Suppose that the family of functionals $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ is equi-mildly coercive, and that it Γ-converges to a functional $\mathcal{F} : X \to \overline{\mathbb{R}}$. Then every sequence x_n of asymptotic minimizers, that is:*

$$\lim_{n\to\infty} \left(\mathcal{F}_n(x_n) - \inf_X \mathcal{F}_n \right) = 0, \tag{3.15}$$

is precompact and each cluster point x is a minimizer of $\mathcal{F}$, and moreover:

$$\lim_{n\to\infty} \left(\inf_X \mathcal{F}_n \right) = \mathcal{F}(x) = \inf_X \mathcal{F}.$$

Proof. Let $x_n \subset K$ be a sequence of asymptotic minimizers. Equi-mild coercivity implies that, up to a subsequence that we do not relabel, the sequence converges to a point $x_0 \in K$. The lower bound inequality (3.12) implies that:

$$\mathcal{F}(x_0) \leq \liminf_{n\to\infty} \mathcal{F}_n(x_n) = \liminf_{n\to\infty} \inf_X \mathcal{F}_n.$$

In addition, for any fix point x there exists a recovery sequence y_n so that:

$$\lim_{n\to\infty} \mathcal{F}_n(y_n) = \mathcal{F}(x).$$

Therefore:

$$\mathcal{F}(x_0) \leq \limsup_{n\to\infty} \inf_X \mathcal{F}_n \leq \limsup_{n\to\infty} \mathcal{F}_n(y_n) = \lim_{n\to\infty} \mathcal{F}_n(y_n) = \mathcal{F}(x).$$

In particular, if $x = x_0$, we get:

$$\mathcal{F}(x_0) = \lim_{n\to\infty} \inf_X \mathcal{F}_n.$$

Collecting the above inequalities, we can derive:

$$\mathcal{F}(x_0) = \inf_X \mathcal{F} = \lim_{n\to\infty} \inf_X \mathcal{F}_n.$$

□

Remark 23. *Observe that the role of equi-mild coercivity is to extract a converging subsequence from any minimizing sequence. One can replace this assumption with the following condition:*

Let x_n be a sequence in X and suppose that the sequence $\mathcal{F}_n(x_n)$ is uniformly bounded. Then, there exists a converging subsequence of x_n.

Observe that the compactness parts in our main theorems (Theorem 31) correspond to the condition that boundedness of $\mathcal{F}_n(x_n)$ implies the existence of a converging subsequence of x_n, while the estimates for lower bounds of scaled 3d energies (Theorem 31) and the construction of recovery sequences (Theorem 34 and Theorem 42) correspond to Γ-convergence.

4.0 DESCRIPTION OF THREE-DIMENSIONAL ELASTICITY

Throughout this chapter, we shall introduce the mathematical description of non-lineal three-dimensional elasticity. The notions and results presented here, together with the relevant observations and experiments described in the next chapter, will be the base for the formulation of the energy model for unconstrained elastic bodies studied in this work.

Firstly, we stablish the axioms of continuum mechanics for elastic bodies with applied external forces and imposed boundary conditions: the stress principle of Euler and Cauchy, the axiom of force balance and the axiom of moment balance. The main consequence of these axioms is the Cauchy's theorem (Theorem 24) which asserts the existence of the symmetric Cauchy stress tensor and the boundary value problem that it satisfies (equations 4.1). The equations in this boundary value problem constitute the equations of equilibrium of the elastic body. We finally provide the definitions of hyperelastic, homogeneous and frame-invariant elastic material, and we exhibit their main properties. In particular, the definition of hyperelastic materials allows to define the strain elastic energy in terms of an integral functional defined over the reference configuration.

4.1 THE STRESS PRINCIPLE OF EULER AND CAUCHY

We assume that in the deformed configuration $\overline{\Omega}^u = u(\overline{\Omega})$ associated with an arbitrary deformation u, the body is subjected to applied forces of two types:

1. Applies body forces, defined by a vector field:

$$f^u : \Omega^u \to \mathbb{R}^3.$$

2. Applied surface forces, defined by a vector field:

$$g^u : \Gamma_1^u \to \mathbb{R}^3,$$

on a measurable subset Γ_1^u of the boundary $\Gamma^u = \partial\Omega^u$.

Let $\rho^u : \Omega^u \to \mathbb{R}$ denote the mass density in the deformed configuration, so that the mass of every measurable subset A^u of $\overline{\Omega}^u$ is given by the integral:

$$\int_{A^u} \rho^u(x^u)dx^u.$$

We also assume that the mass density is positive in Ω^u.

The applied forces describe the action of the outside world on the body: an elementary force $f(x^u)dx^u$ is exerted on the elementary volume dx^u at each point x^u of the deformed configuration. This is exactly the case of the gravity field, for which $f^u(x^u) = -g\rho^u(x^u)e_3$ for all $x^u \in \Omega^u$, where g is the gravitational constant. Likewise, an elementary surface force $g^u(x^u)da^u$ is exerted on the elementary area da^u at each point of the subset of the boundary Γ_1^u. Such forces generally represents the action of another body along the portion of the boundary Γ_1^u.

Continuum mechanics for static problems is founded on the following axiom:

Axiom 1: stress principle of Euler and Cauchy. Consider a body occupying a deformed configuration $\overline{\Omega}^u$, and subjected to applied forces represented by densities f^u and g^u. Then, there exists a vector field:

$$t^u : \overline{\Omega}^u \times S^2 \to \mathbb{R}^3,$$

where: $S^2 = \{v \in \mathbb{R}^3 : |v| = 1\}$, such that:

1. For any subdomain A^u of $\overline{\Omega}^u$, and at any point $x^u \in \Gamma_1^u \cap \partial A^u$ where the unit outer normal vector n^u to $\Gamma_1^u \cap \partial A^u$ exists, the following holds:

$$t^u(x^u, n^u) = g^u(x^u).$$

2. **Axiom of force balance**: for any subdomain A^u of $\overline{\Omega}^u$, we have:

$$\int_{A^u} f^u(x^u)dx^u + \int_{\partial A^u} t^u(x^u, n^u)da^u = 0,$$

where n^u denotes the unit outer normal vector along ∂A^u.

3. **Axiom of moment balance:** for any subdomain A^u of Ω^u:

$$\int_{A^u} x^u \times f^u(x^u)dx^u + \int_{\partial A^u} x^u \times t^u(x^u, n^u)da^u = 0.$$

Basically, the stress principle of Euler and Cauchy asserts the existence of elementary surface forces $t^u(x^u, n^u)da^u$, $x^u \in \partial A^u$, along the boundary ∂A^u, with unit normal vector n^u, of any subdomain A^u of the deformed configuration $\overline{\Omega}^u$. Moreover, this principle asserts that at a point x^u of the boundary ∂A^u, the elementary surface force depends on the subdomain A^u only via the normal vector n^u to ∂A^u at x^u. Finally, the stress principle asserts that any subdomain of the deformed configuration is in static equilibrium, in the sense that the torsor formed by the elementary forces $t^u(x^u, n^u)da^u$, $x^u \in \partial A^u$, and the body forces $f^u(x^u)dx^u$, $x^u \in A^u$, is equivalent to zero. This means that the resultant vector vanishes (axiom of force balance) and that its resulting moment vanishes (axiom of moment balance).

Hence, the stress principle expresses the idea that the static equilibrium of any subdomain A^u of the deformed configuration, already subjected to given applied body forces $f^u(x^u)dx^u$, $x^u \in A^u$, and (possibly) to given applied surface forces $g^u(x^u)da^u$ at those points $x^u \in \Gamma_1^u \cap \partial A^u$ where the outer normal vector to $\Gamma_1^u \cap \partial A^u$ exists, is made possible by added effects of elementary surface forces of the form indicated, acting on the remaining part of the boundary ∂A^u.

4.2 THE CAUCHY STRESS TENSOR AND THE EQUATIONS OF EQUILIBRIUM

We now exhibit one of the most important results in continuum mechanics:

Theorem 24. *[Cauchy's Theorem] Assume that the applied body force density* $f^u : \overline{\Omega}^u \to \mathbb{R}^3$ *is continuous, and that the Cauchy stress vector field* t^u *is continuously differentiable with respect to the variable* x^u *for each* $n \in S^2$*, and continuous with respect to the variable* $n \in S^2$ *for each* $x^u \in \overline{\Omega}^u$*. Then the axiom of force and moment balance imply that there exists a continuously differentiable tensor field:*

$$T^u : \overline{\Omega}^u \to \mathbb{R}^{3\times 3},$$

such that the Cauchy stress vector satisfies:

$$t^u(x^u, n) = T^u(x^u)n, \quad \text{for all } x^u \in \overline{\Omega}^u \text{ and all } n \in S^2,$$

and such that the following relations hold:

$$\begin{cases} -div^u T^u(x^u) = f^u(x^u) & x^u \in \Omega^u \\ T^u(x^u) = T^u(x^u) & x^u \in \Omega^u \\ T^u(x^u)n^u = g^u(x^u) & x^u \in \Gamma_1^u \end{cases} \tag{4.1}$$

Essentially, the Cauchy's Theorem asserts that the dependence of the Cauchy stress vector $t^u(x^u, n)$ with respect to the second variable $n \in S^2$, is linear. Secondly, it asserts that at each point $x^u \in \overline{\Omega}^u$, the tensor $T^u(x^u)$ is symmetric. And finally, this result says that the tensor field $T^u : \Omega^u \to \mathbb{R}^{3\times 3}$, and the vector fields $f^u : \Omega^u \to \mathbb{R}^3$ and $g^u : \Gamma_1^u \to \mathbb{R}^3$ are related by a partial differential equation in Ω^u, and by a boundary condition on Γ_1^u, respectively. The proof of this result is omitted. However, the reader interested in the proof of the Cauchy's Theorem, can be find it in the reference [12].

The symmetric tensor $T^u(x^u)$ is called the Cauchy stress tensor at the point x^u in the deformed configuration. Observe that, for each point x^u the knowledge of the three vectors $t^u(x^u, e_i)$ completely determines the Cauchy stress vector $t^u(x^u, n)$ for any $n \in S^2$.

As shown in the previous section, the axioms of force and moment balance imply that the Cauchy stress tensor field T^u satisfies a boundary value problem expressed in terms of the variable x^u in the deformed configuration:

$$-\text{div}^u T^u = f^u,$$

in Ω^u, and the boundary data:

$$T^u n^u = g^u,$$

on Γ_1^u. One of the main properties of this boundary value problem is that, due to its divergence form, it can be written in a variational form as:

$$\int_{\Omega^u} T^u : \nabla^u \phi^u dx^u = \int_{\Omega^u} f^u \cdot \phi^u dx^u + \int_{\Gamma_1^u} g^u \cdot \phi^u da^u,$$

valid for all smooth vector mappings $\phi^u : \Omega^u \to \mathbb{R}^3$ which are zero on $\Gamma^u - \Gamma_1^u$.

The equations:

$$-\text{div}^u T^u = f^u,\ T^u = (T^u)^T \text{ in } \Omega^u;\ T^u n^u = g^u,\ \text{ on } \Gamma_1^u$$

constitute the equations of equilibrium in the deformed configuration.

4.3 THE PIOLA-KIRCHHOFF STRESS TENSORS AND THE EQUATIONS OF EQUILIBRIUM IN THE REFERENCE CONFIGURATION

One of the fundamental problems in elasticity is to determine the deformation field and the Cauchy stress tensor that arise in a body subjected to a given system of applied forces. In this respect, the equations of equilibrium in the deformed configuration are not much avail, since they are expressed in terms of the Euler variable $x^u = u(x)$, which is precisely one of the unknowns. To solve this difficulty, we shall rewrite these equations in terms of the Lagrange variable x which is attached to the initial or reference configuration Ω. The basic idea is to transform the quantities T^u, div^u, f^u and g^u in quantities defined in Ω.

We start defining the first Piola-Kirchhoff stress tensor $T : \overline{\Omega} \to \mathbb{R}^{3\times 3}$ by letting:

$$T(x) = (\det\, \nabla u(x))T^u(x^u)\nabla u(x)^{-1,T}, \quad x^u = u(x).$$

The main advantage of this transform, in that we have a simple relation between the divergences of both tensors:

$$\text{div}\; T(x) = (\det\, \nabla u(x))\text{div}^u T^u(x^u).$$

Observe that, while the Cauchy stress tensor $T^u(x^u)$ is symmetric, the first Piola-Kirchhoff stress tensor T is not symmetric in general. One has instead, the relation:

$$T(x)^T = \nabla u(x)^{-1} T(x) \nabla u(x)^{-1,T}.$$

To get a symmetric stress tensor (which is convenient to write the constitutive equation in the reference configuration, as we will see), we define the second Piola-Kirchhoff stress tensor $\Sigma(x)$ as follows:

$$\Sigma(x) = \nabla u(x)^{-1} T(x) = (\det\, \nabla u(x))\nabla u(x)^{-1} T^u(x^u)\nabla u(x)^{-1,T}, \quad x^u = u(x). \tag{4.2}$$

It now remains to transform the external forces f^u and g^u. To these vector fields, we associate densities $f : \Omega \to \mathbb{R}^3$ and $g : \Gamma_1^u \to \mathbb{R}^3$ such that:

$$f(x)dx = f^u(x^u)dx^u$$

for all $x^u = u(x)$, and:

$$g(x)da = g^u(x^u)da^u,$$

for all $x^u = u(x) \in \Gamma_1^u$. Therefore:

$$f(x) = (\det\, \nabla u(x))f^u(x^u), \quad g(x) = \det \nabla u(x)|\nabla u(x)^{-1,T} n| g^u(x^u).$$

We can now derive that the equations of equilibrium in the reference configuration take the form:

$$\begin{cases} -\text{div}\; T(x) = f(x) & x \in \Omega \\ \nabla u(x) T(x)^T = T(x) \nabla u(x)^T & x \in \Omega \\ T(x)n = g(x) & x \in \Gamma_1 \end{cases} \tag{4.3}$$

In terms of the second Piola-Kirchhoff stress tensor Σ (4.2), we easily derive the following system of equilibrium equations in the reference configuration:

$$\begin{cases} -\text{div } (\nabla u(x)\Sigma(x)) = f(x), & x \in \Omega \\ \Sigma(x) = \Sigma(x)^T, & x \in \Omega \\ \nabla u(x)\Sigma(x)n = g(x),\ x \in \Gamma_1. & x \in \Gamma_1 \end{cases} \tag{4.4}$$

4.4 ELASTIC AND HYPERLASTIC MATERIALS

In this section, we introduce the notions of: elastic materials, homogeneous elastic materials, frame-invariance elastic materials and hyperelastic materials. To provide a self-contained exposition, we shall quote all the important results that we will need in the sequel. However, we are not going to provide their proofs. A careful exposition, with all the details in the proofs, can be found in the reference [12].

We start with a general observation on the equations of equilibrium (4.3) or (4.1): while the equations of equilibrium are valid regardless of the particular material the body in consideration is made of, it is clear that the nature of the underlying material should be taken into account. For instance, in order to produce the same deformation in a body made of wood in one case or in a body made of iron in another case, it is clear that different systems of forces must be applied, and so, different stress tensors must arise. From a mathematical point of view, these system of equations is also incomplete. In fact, if we consider the equations of equilibrium in the reference configuration (4.1) as a boundary value problem, we have to deal with nine unknowns: the six components of the stress tensor, and the three components of the deformation. Therefore, it is clear that there is a discrepancy between the total number of unknown functions and the number of available equations.

To limit the class of materials involved, we are going to consider one category of materials. This class of materials require that the Cauchy stress tensor $T^u(x^u)$ at any point x^u in the deformed configuration is completely determined by the deformation gradient $\nabla u(x)$, at the corresponding point x in the reference configuration. More precisely, we have the following definition:

Definition 25. *We say that a material is elastic if there exists a mapping*

$$\hat{T}^D : (x, F) \in \overline{\Omega} \times \mathbb{R}_+^{3\times 3} \to \hat{T}(x, F),$$

called the response function for the Cauchy stress tensor, such that in any deformed configuration that a body made of this material occupies, the Cauchy stress tensor $T^u(x^u)$ at any point $x^u = u(x)$ is related to the deformation gradient $\nabla u(x)$ at the associated point x by the equation:

$$T^u(x^u) = \hat{T}^D(x, \nabla u(x)).$$

Notice that the response function $\hat{T}^D$ of an elastic material is by definition independent of the particular deformation considered. This is why the symbol u does not appear in the notation used for the response function. The superscript D reminds us that this function is used for computing a quantity in a deformed configuration.

Notice also that, by definition, the Cauchy stress tensor $T^u(x^u)$ at a point $x^u = u(x)$ of an elastic material depends on the deformation solely through its deformation gradient. In fact, the tensor $T^u(x^u)$ should not be a function of the values $u_i(x)$ themselves, for otherwise the Cauchy stress tensor would vary if the deformed configuration were rigidly translated.

Observe that, by the definition, the response function at each point of an elastic material must be defined for all matrices in $\mathbb{R}_+^{3\times 3}$. This implies that elastic materials have the property that given any point $x \in \Omega$ and any matrix $F \in \mathbb{R}_+^{3\times 3}$, there exists a deformation u of the body such that:

$$\nabla u(x) = F,$$

as the result of the application of appropriate applied forces and boundary conditions. Therefore, this definition leaves out materials that can only undergo a restricted class of deformations.

Taking the relations:

$$T = (\det \nabla u) T^u \nabla u^{-1,T},$$

and:

$$\Sigma = \nabla u^{-1} T,$$

we can guarantee the existence of mappings:

$$\hat{T} : \overline{\Omega} \times \mathbb{R}_+^{3\times3} \to \mathbb{R}_+^{3\times3} \text{ and } \hat{\Sigma} : \overline{\Omega} \times \mathbb{R}_+^{3\times3} \to \mathbb{R}^{3\times3},$$

given respectively by:

$$\begin{aligned} \hat{T}(x,F) &= (\det\, F)\hat{T}^D(x,F)F^{-1,T}, \\ \hat{\Sigma}(x,F) &= (\det\, F)F^{-1}\hat{T}^D(x,F)F^{-1,T}, \end{aligned} \tag{4.5}$$

for all points $x \in \overline{\Omega}$ and all matrix $F \in \mathbb{R}_+^{3\times3}$, which verify the relations:

$$T(x) = \hat{T}(x, \nabla u(x)) \quad \text{and } \Sigma(x) = \hat{\Sigma}(x, \nabla u(x)), \quad x \in \overline{\Omega}.$$

These relations can be taken as an equivalent definition of elastic materials. The mappings $\hat{T}$ and $\hat{\Sigma}$ are called response functions for the first and second Piola-Kirchhoff stress.

We now provide some additional definitions that we will use in this work.

Definition 26 (Homogeneous elastic material)**.** *A material in a reference configuration Ω is called homogeneous if its response function is independent of the particular point x. Hence, the constitutive equation takes the form:*

$$T^u(x^u) = \hat{T}^D(\nabla u(x)), \quad x \in \overline{\Omega}.$$

A general axiom in physics establishes that any observable quantity, such as mass density, velocity vector, acceleration, etc., must be independent of the particular orthogonal basis in which it is computed. This is the frame invariance principle. Expressed in terms of the response functions $\hat{T}$ and $\hat{\Sigma}$, we have the following related definition:

Definition 27 (Principle of material frame-indifference)**.** *Let the deformed configuration $\overline{\Omega}^u$ be rotated into another deformed configuration $\overline{\Omega}^v$, i.e., $v = Ru$, $R \in SO(3)$. Then:*

$$t^v(x^v, Rn) = Rt^u(x^u, n), \quad x \in \overline{\Omega},\, n \in S^2.$$

Here, $x^v = v(x)$, $x^u = u(x)$, and t^u, t^v denote the Cauchy stress tensors in the deformed configurations $\overline{\Omega}^u$ and $\overline{\Omega}^v$, respectively.

Expressed in terms of the response functions $\hat{T}$ and $\hat{\Sigma}$ for the first and second Piola-Kirchhoff stress tensors, a material is frame-invariance if and only if:

$$\begin{aligned} \hat{T}(x, RF) &= R\hat{T}(x, F), \quad \text{for all } F \in \mathbb{M}_3^+, R \in SO(3), \\ \hat{\Sigma}(x, RF) &= \hat{\Sigma}(x, F), \quad \text{for all } F \in \mathbb{M}_3^+, R \in SO(3). \end{aligned} \tag{4.6}$$

We now discuss the notion of hyperelastic materials. Combining the equations of equilibrium (4.3) in the reference configuration with the definition of elastic material, and imposing a boundary condition on the portion of the boundary $\Gamma_0 = \Gamma - \Gamma_1$, we find that the deformation u satisfies the following boundary value problem:

$$\begin{cases} -\text{div } \hat{T}(x, \nabla u(x)) = \hat{f}(x, u(x)), & x \in \Omega \\ \hat{T}(x, \nabla u(x))n = \hat{g}(x, \nabla u(x)) & x \in \Gamma_1 \\ u(x) = u_0(x), & x \in \Gamma_0 \end{cases} \tag{4.7}$$

where $\hat{T}$ is the response function for the first Piola-Kirchhoff stress tensor. It was mentioned that the first and second equations of equilibrium are together equivalent to the following variational form:

$$\int_\Omega \hat{T}(x, \nabla u(x)) : \nabla\phi(x)dx = \int_\Omega \hat{f}(x, u(x)) \cdot \phi(x)dx + \int_{\Gamma_1} \hat{g}(x, \nabla u(x)) \cdot \phi(x)da,$$

valid for smooth vector fields $\phi : \overline{\Omega} \to \mathbb{R}^3$ that vanishes on Γ_0. Also, it can be proved (see Sect. 2.7 in [12]), that the integrals appearing in the right-hand side can be written as Gateaux derivatives:

$$\begin{aligned} \int_\Omega \hat{f}(x, u(x)) \cdot \phi(x)dx &= F'(u)\phi, \\ \int_{\Gamma_1} \hat{g}(x, \nabla u(x)) \cdot \phi(x)da &= G'(u)\phi, \end{aligned}$$

of functionals F and G of the form:

$$F(v) = \int_\Omega \hat{F}(x, v(x))dx, \quad G(v) = \int_{\Gamma_1} \hat{G}(x, v(x), \nabla v(x))da.$$

A natural question is to ask whether the left-hand side in the variational formulation can be also written as the Gateaux derivative of an appropriate functional $\hat{W}$ as:

$$\int_\Omega \hat{T}(x, \nabla u(x)) : \nabla\phi(x)dx = W'(u)\phi.$$

If this is the case, the variational formulation of the equations of equilibrium is equivalent to expressing that the Gateaux derivatives of the functional $W - (F + G)$ is zero for all variations that vanish on the portion Γ_0. This motivates the following definition of hyperelastic materials:

Definition 28 (Hyperelastic materials)**.** *An elastic material with response function* $\hat{T}$: $\overline{\Omega} \times \mathbb{R}^{3\times3}_{+} \to \mathbb{R}^{3\times3}$ *is hyperelastic if there exists a function* $\hat{W} : \overline{\Omega} \times \mathbb{R}^{3\times3}_{+} \to \mathbb{R}$*, differentiable with respect to the variable* $F \in \mathbb{R}^{3\times3}_{+}$*, for each point* $x \in \overline{\Omega}$*, such that:*

$$\hat{T}(x, F) = \frac{\partial \hat{W}}{\partial F}(x, F), \quad \textit{for all } x \in \Omega,\ F \in \mathbb{R}^{3\times3}_{+}$$

.

The function $\hat{W}$ is called the stored energy function. In the case where the material is homogeneous, this function depends on F only.

It can be proven (see Theorem 4.1-2 in [12]) that for a hyperelastic material, any smooth enough mapping u that satisfies:

$$u \in \Phi = \{v : \overline{\Omega} \to \mathbb{R}^3,\ v = u_0 \text{ on } \Gamma_0\},$$

and:

$$I(u) = \inf_{v \in \Phi} I(v),$$

where:

$$I(v) = \int_{\Omega} \hat{W}(x, \nabla v(x))dx - F(v) - G(v),$$

solves the boundary value problem:

$$\begin{cases} -\text{div } \dfrac{\partial \hat{W}}{\delta F}(x, \nabla u(x)) = f(x, u(x)) & x \in \Omega \\ u(x) = u_0(x), & x \in \Gamma_0, \\ \dfrac{\partial \hat{W}}{\partial F}(x, \nabla u(x)) = g(x, \nabla u(x)), & x \in \Gamma_1 \end{cases} \tag{4.8}$$

In the language of the calculus of variations, this boundary value problem forms the Euler-Lagrange equations associated with the total energy I. Hence, any minimizer u of the total energy I over the set of admissible deformations or solutions, is a solution of the above boundary value problem.

4.4.1 The strain energy function

The functional W defined in (4.8) by:

$$W(v) = \int_{\Omega} \hat{W}(x, \nabla v(x))dx,$$

is called the strain energy. Another observation is that the stored energy function $\hat{W}$ of a hyperelastic material satisfies the principle of material frame-indifference (that is, the response function $\hat{T}^D$ for the Cauchy stress tensor is itself frame-indifferent) if and only if for all points $x \in \overline{\Omega}$:

$$\hat{W}(x, RF) = \hat{W}(x, F), \quad \text{for all } F \in \mathbb{M}_3^+,\ R \in SO(3).$$

Another interested property is that the stored energy function $\hat{W}$ is non-convex. In fact, it can be proven (Theorem 4.8-1 in [12]) that one of the consequences of a convex stored energy is incompatible with the property:

$$\hat{W}(x, F) \to +\infty, \text{ as } \det F \to 0^+,$$

that reflects the intuitive idea that infinite stress must accompany extreme strains, as suggested by the most immediate physical evidence. For more details, see for instance Sect. 4.7 and 4.8 in [12].

4.5 ISOTROPIC MATERIAL

We now close this chapter with another property of elastic material that, as the case of the axiom of frame-indifference, restricts the form of the response function. This property if called isotropy, and it corresponds to the intuitive idea that at a given point, the response function of the material is the same in all directions. To give a precise mathematical definition to this property, consider an arbitrary point $x^u = u(x)$ of a body occupying a deformed configuration $\Omega^u = u(\Omega)$. If the material is elastic, then by Definition 25, the Cauchy stress tensor at the point $x^u = u(x)$ is given by:

$$T^u(x^u) = \hat{T}^D(x, \nabla u(x)).$$

Let us rotate the reference configuration around the point x by a rotation R^T. Then introducing the mappings:

$$\theta(y) = x + R^T xy, \quad \text{for all } y \in \Omega.$$

and the mapping:

$$\tilde{u} = u \circ \theta^{-1} : \theta(\Omega) \to u(\Omega),$$

the same deformed configuration $u(\Omega)$ can be seen as the new reference configuration $\theta(\Omega)$. Now, the Cauchy stress tensor at the same point $x^u = x^{\tilde{u}}$ is given by:

$$T^{\tilde{u}}(x^{\tilde{u}}) = \hat{T}^D(x, \nabla\tilde{u}(x)) = \hat{T}^D(x, \nabla u(x)R).$$

After these observation, we thus led to the following definition:

Definition 29. *An elastic material is isotropic at a point x if its elastic response function for the Cauchy stress tensor satisfies:*

$$\hat{T}^D(x, FR) = \hat{T}^D(x, F),$$

for all $R \in SO(3)$ and $F \in \mathbb{R}^{3\times 3}_+$.

In this way, we have that a material is isotropic if the Cauchy stress tensor is left invariant when the reference configuration is subjected to an arbitrary rotation around a point.

Expressed in terms of the response functions $\hat{T}$ and $\hat{\Sigma}$ for the first and second Piola-Kirchhoff stress, isotropy at a point x is equivalent to either relation:

$$\hat{T}(x, FR) = \hat{T}(x, F)R,$$
$$\hat{\Sigma}(x, FR) = R^T\hat{\Sigma}(x, F)R,$$

for all $F \in \mathbb{R}^{3\times3}$, $\det F > 0$, and all $R \in SO(3)$.

Moreover, we say that a stored energy function is isotropic at a point x of the reference configuration if the corresponding response function is isotropic at x. In this case, the stored energy function $\hat{W}$ satisfies:

$$\hat{W}(x, F) = \hat{W}(x, FR),$$

for all matrix $F \in \mathbb{R}^{3\times3}_+$, $R \in SO(3)$.

5.0 APPLICATIONS OF NON LINEAR ELASTICITY

Throughout this chapter, we shall apply the theory of elasticity to growth of soft elastic tissues, which constitutes an example of deformations of unloaded elastic bodies. Hence, we shall dedicate the chapter to a detailed description of growth in tissues, and we shall also introduce the two main assumptions of the underlying theory: the decomposition of the gradient tensor into an elastic and a stretch part (see Section (5.1) below), and the dependence of the stored energy function only on the elastic response of the material (Section 5.2). These assumptions constitute the basic theoretical framework for the formulation of the mathematical model for the shape formation of unloaded sheets.

5.1 GROWTH IN SOFT ELASTIC TISSUES

Growth is usually the result of a continuous process on time scales varying from minutes to years depending on the specific system. After each incremental growth, relaxation and remodeling take place. Therefore, the process is represented by a series of small quasi-static incremental growth steps of an unloaded configuration followed by elastic relaxation at each step. Thus, generation of forms in biological tissues involves three distinct processes:

- Growth, which is defined as an increase of mass. It can occur through cell division (hyperplasia), cell enlargement (hypertrophy), secretion of extra-cellular matrix or accretion at external or internal surfaces. The removal of mass is referred to as atrophy and occurs through cell death, cell shrinkage or resorption.
- Remodeling, which involves changes in material properties, which lead to changes in

microstructure.

- Morphogenesis, which consists in a change in shape, involving both growth and remodeling, and usually refers to embryonic development, wound healing or organ regeneration.

The most general three-dimensional theory of stress-dependent volumetric growth for soft biological tissues was first presented by Rodríguez et al. [45]. That theory introduced a fundamental idea. Keeping in mind that the shape change of the unloaded tissue during growth can be described by a mapping u whose gradient is analogous to a deformation gradient tensor, the fundamental idea in [45] was that this gradient could be decomposed into the product of a volumetric growth stretch tensor A, which describes the addition of material at a point and the orientation of its deposition, and an elastic tensor F that represents the elastic accommodation of the body to the new material:

$$\nabla u = FA. \tag{5.1}$$

The theory presented in [45] was illustrated only by problems in which the initial configuration was stress-free. In addition, their interpretation of the theory required knowledge of a locally stress-free reference state for the grown material. That such a zero-stress state exists is suggested by the destructive experiments that are commonly used to determine the residual stress in biological tissues and other materials. When a piece of residually stressed material is cut into progressively smaller pieces, the residual stress is relieved as the cuts are made. For instance, we can quote the following easy experiments taking from [39], [33] and where residual stress in rest configurations can be tested: the first picture in Figure 2 shows a flat leaf and a leaf with a wavy pattern along the edges. We take both leaves and we cut carefully thin strips parallel to their edges, as it is shown in Figure 2. To see their geometry, flatten these strips between two glass plates. This reveals the difference in the leaves' intrinsic geometries. The strips cut from the flat leaf show the expected pattern of arcs, with the radius of the arcs increasing from the center. In contrast, strips from the edge of the wavy leaf exhibit a collection of decreasing radius as the edge of the leaf is approached. A geometry with a decreasing radius of curvature as the edge is approached cannot exist in a planar configuration, it requires a negative Gaussian curvature.

Figure 2: Residual stress in leaves. Images taken from [39], [33].

In the second picture of Figure 2, a similar experiment is carried out. We see now that the strips which are further from the midvein extend more. In other words, the non-uniform distribution of growth-induced strain is such that it generates compressive stresses along the leaf edge, which then lead to buckling instabilities.

These observations lead to the conclusion that in the absence of external forces or imposed boundary conditions, the shape formation during botanical growth in thin elastic bodies gives rise to structures which assume non-trivial rest configurations (for instance, wavy patterns along the edges) and which may exhibit residual strain. Essentially, as growth takes place locally, parts of the body need to be stretched or compressed to ensure integrity (no cavitation) and compatibility (no overlap) of the body. In turn, these strains are associated with stresses referred to as residual stresses. We introduce the term non-Euclidean elastic bodies for such sheets.

A volumetric growth deformation tensor is specified locally and is followed by an elastic response ensuring integrity and continuity of the body. However, the local specification of volumetric growth of a three-dimensional body may distort the constituent volume elements in such a way that the grown elements are unable to form a continuous body residing in Euclidean space. This situation is usually referred to as incompatible growth and corresponds to a growth deformation tensor which cannot be expressed as the gradient of a vector field. Geometrically speaking this means that the set of distances between points in the grown body does not agree with the set of Euclidean distances between points in any simply connected subset of three-dimensional Euclidean space. However, if an elastic response distorts the grown volume elements so that they again form a continuous Euclidean body this can give rise to so called residual stresses, that is stress that remain in the body in the absence of loading. To illustrate this idea, consider the following situation of circumferential growth, which serves as a first approximation for eccentric ventricular hypertrophy, a clinical term that describes ventricular enlargement in response to chronic volume overload (elevated filling pressure): consider a cylindrical tube model Ω composed of an isotropic elastic material (see Definition 29). The residual stress present in the cylinder after growth is calculated assuming that growth strains generate stress similar to those of loading (case where external forces are applied). Other than spatial position in $\mathbb{R}^3$, all vectorial and tensorial quantities discussed, such as forces and deformation gradients, will be constructed from tangent vectors and linear functionals on tangent vectors. Since each tangent space is a distinct vector space, each has a distinct local basis of vectors that span it. For the cylindrical coordinates (R, Θ, Z) that will be used in this example, we have the local orthonormal basis:

$$e_1(\Theta) = (\cos\,\Theta)e_1 + (\text{sen}\,\Theta)e_2, \quad e_2(\Theta) = (\text{-sen}\,\Theta)e_1 + (\cos\,\Theta)e_2, \quad \text{and} \quad e_3(\Theta) = e_3.$$

We use these vectors to define the position function for the reference configuration:

$$X(R, \Theta, Z) = Re_1(\Theta) + Ze_3.$$

When used to define a position function, this linear combination of vectors may be considered as a tangent vector anchored at the origin of $\mathbb{R}^3$ and pointing to the spatial position occupied by the material point with coordinates (R, Θ, Z). When used in expressions for deformation gradients and stress at a spatial point, however, they are to be understood as tangent vectors anchored at the spatial point in question. If we let Ω be the set of coordinates (R, Θ, Z) of all the material points of the body, then $X(\Omega)$ is the subset of $\mathbb{R}^3$ occupied by the body in its reference configuration, expressed in cylindrical coordinates. We consider a deformation of the body as a differentiable, invertible map $u : X(\Omega) \to \mathbb{R}^3$ whose value

$x^u = u(x)$ is the spatial position of the material point originally located at $x \in X(\Omega)$. A local, linear description of the deformation is provided by its gradient ∇u. This deformation gradient maps infinitesimal material lines in the reference configuration to their images in the deformed configuration. Since a material line is a tangent vector, this makes the deformation gradient a two-point tensor: it maps tangent vectors anchored at a point x in the reference configuration to tangent vectors anchored at $x^u = u(x)$, the image of x under the deformation. If the body is deformed so that it remains cylindrical and is neither stretched nor compressed along the vertical direction, the material point originally located at $X(R, \Theta, Z)$ is now located at:

$$x(R, \Theta, Z) = r(R)e_1(\Theta) + Ze_3.$$

A tangent space anchored at a point in the reference configuration has a local basis formed by the derivatives:

$$\frac{\partial X}{\partial R} = e_1(\Theta), \quad \frac{\partial X}{\partial \Theta} = Re_2(\Theta), \quad \frac{\partial X}{\partial Z} = e_3.$$

At the image under the deformation, the images of these derivatives are:

$$\frac{\partial x}{\partial R} = r'e_1(\Theta), \quad \frac{\partial x}{\partial \Theta} = re_2(\Theta), \quad \frac{\partial x}{\partial Z} = e_3.$$

The deformation gradient ∇u should have the following dot-products with the local basis vectors in the reference configuration:

$$\nabla u \cdot e_1(\Theta) = r'e_1(\Theta), \quad \nabla u \cdot (Re_2(\Theta)) = re_2(\Theta), \quad \nabla u \cdot e_3 = e_3.$$

Hence, the full deformation gradient is:

$$\nabla u = r'e_1(\Theta) \otimes e_1(\Theta) + \frac{r}{R}e_2(\Theta) \otimes e_2(\Theta) + e_3 \otimes e_3,$$

where the right vector in each tensor product is to be understood as a tangent vector anchored at a point in the reference configuration, while the left vector is a tangent vector anchored at a point in the final, deformed configuration.

The very special relationship between the coefficients of $e_1(\Theta)\otimes e_1(\Theta)$ and $e_2(\Theta)\otimes e_2(\Theta)$ in the deformation gradient reflects the fact that not all tensor fields of this form are equal to deformation gradients. Consider, for example, a growth tensor of the form:

$$A(R,\Theta,Z) = \gamma_1(R)e_1(\Theta)\otimes e_1(\Theta) + \gamma_2(R)e_2(\Theta)\otimes e_2(\Theta) + e_3\otimes e_3.$$

If A is the gradient of a cylindrical deformation of the form considered above, then $\gamma_1 = r$ and $\gamma_2 = r/R$, which implies:

$$\gamma_1 = \frac{d}{dR}(R\gamma_2).$$

If this is not satisfied, then G is not a deformation gradient of the type considered above. In fact, computing the Riemann curvature tensor that arises from $A^T A$ and observing that it does not vanish identically shows that A is not the gradient of any deformation (see Theorem 4). A two-point tensor field that is not equal to a deformation gradient will be called incompatible.

Each fixed tensor with positive determinant is equal to the local value of some deformation gradient; it is only in the failure of local values to patch together correctly that a non-gradient differs from a deformation gradient. Incompatible tensor fields have been used for decades in the theory of elastoplasticity where the gradient of any deformation ∇u is decomposed into the product of two incompatible tensor fields F and A as in (5.1) above. In this context, A describes incompatible volumetric growth. A local value of A describes the deformation that a growing element of matter would undergo if it were not constrained by the presence of neighboring matter. The grown elements are considered stress- free, but they no longer form a continuous solid. The tensor field F plays the role of the elastic deformation needed to re-establish a continuous solid. However, since A is not compatible and ∇u is, $F = \nabla u A^{-1}$ is not a deformation gradient. Thus the step from the grown, stress-free state to the final state, which may carry residual stress even in the absence of applied loads and body forces, cannot properly be called an elastic deformation. An interpretation of the multiplicative decomposition that avoids tearing the body into pieces is the following: the incompatible growth tensor A changes the metric tensor of the elastic body, and then the arclengths between points in the body preclude the body fitting into $\mathbb{R}^3$. The points of the body still form a continuous three-dimensional manifold with the growth metric defined by

$A^T A$ but, it is a manifold that cannot be isometrically immersed in $\mathbb{R}^3$. In two dimensions, we can consider the case of a flat disc that has its metric tensor changed by incompatible growth, resulting in a two-dimensional manifold that can no longer fit into the Euclidean plane. The subsequent elastic response restores the body's isometric immersibility into $\mathbb{R}^2$, but the resulting disc carries residual stress, even in the absence of applied loads and body forces.

As we will show in the next subsection, a hyperelastic constitutive relation (see Definition 28) should recognize only the elastic portion of the deformation gradient. In addition to the decomposition given in (5.1), the dependence of the constitutive relation (and, therefore, of the strain energy functional) only on the elastic response of the material, constitute the basic assumption of elastic growth, firstly formulated in [45].

5.2 HYPERELASTICITY AND INCOMPATIBLE GROWTH

As we mentioned before, in the work [45] it is conjectured that the stress which arises from the growth depends only on the elastic deformation that ensures the continuity of the body. We are now going to reproduce a rigorous argument of this conjecture, taken from the reference [15].

Consider a body composed of a compressible elastic material that occupies the domain $\Omega \subset \mathbb{R}^3$ at the current time. In a general setting, the body may undergo deformations due purely to external loads or deformations that include growth. We will characterize the material by a density function $\rho : \Omega \to \mathbb{R}_+$ and a response function $\hat{T}^D$. These functions are assumed to possess the smoothness needed in the analysis. Recall that the value $\hat{T}^D(x, \nabla u(x))$ gives the Cauchy stress tensor T^u at the point $x^u = u(x)$ (see Definition 25).

The value of the density function $\rho(x)$ gives the current density of the material at the point $x \in \Omega$. We will term a deformation that is produced purely by external loads a pure deformation to distinguish it from a deformation in which growth occurs. Note that the Cauchy stress $T(x)$ at x in the current configuration is given by

$$T(x) = \hat{T}^D(x, Id),$$

where Id is the identity tensor. Let Ω^w be the deformed configuration after the action of the mapping $w : \Omega \to \mathbb{R}^3$. Let $z : w(\Omega) \to \mathbb{R}^3$ be a further deformation. The Cauchy stress at the point $z(w(x))$ is then given by:

$$\hat{T}^D(x, \nabla z(w(x))\nabla w(x)).$$

Then the response function $\hat{T}^D_w$ in the configuration $\Omega^w = w(\Omega)$ at the point $w(x)$ is given by:

$$\hat{T}^D_w(w(x), F) = \hat{T}^D(x, F\nabla w(x)),$$

for all tensor $F \in \mathbb{R}^{3\times3}$, with det $F > 0$.

By the conservation of mass, we have that the density of the deformed body at the point $w(x)$ is then given by:

$$\rho^w(w(x)) = [\det\ \nabla w(x)]^{-1}\rho(x).$$

We now turn to the more general problem of describing the relation between the constitutive functions for both processes: pure deformations and growth (unloaded deformation).

To describe these relations, we first introduce the notion of points that have the same intrinsic material properties. By intrinsic properties of a material point, we mean the density and response functions at this point in a configuration with a prescribed stress state. Two material points will be said to be identical if their density functions and their response functions are the same. Roughly speaking, this says that it is not possible to distinguish between two identical material points with mechanical experiments.

From the assumptions made, during deformations (pure deformations and some types of growth), the intrinsic properties of a material point remain unchanged, although the forms of the density function and the response function at the material point will typically change. In order to identify materials which share intrinsic material properties, we now introduce the concept of equivalent material points:

Definition 30. *Two material points x_1 and x_2 will be said to be equivalent if there exists an orientation-preserving tensor $\tilde{F}$ so that:*

$$\rho_2(x_2) = [det\ \tilde{F}]^{-1}\rho_1(x_1)$$

and:

$$\hat{T}_2^D(x_2, F) = \hat{T}_1^D(x_1, F\tilde{F}), \quad \textit{for all tensor } F, \textit{ with det } F > 0,$$

where $(\rho_1, \hat{T}_1^D)$ *and* $(\rho_2, \hat{T}_2^D)$ *are the density and response function pairs associated with the material points* x_1 *and* x_2*.*

The tensor $\tilde{F}$ is called the equivalence transformation. Observe that the equivalence transformation tensor can be thought of as the gradient of the homogeneous deformation under which the material point x_1 becomes identical to the material point x_2.

It is clear that if two material points are identical, then they are equivalent with $\tilde{F} = I$. It is also clear that a material point x is equivalent to itself after any pure deformation w. In that case, $\tilde{F} = \nabla w(x)$.

We now use the concept of material equivalence developed above to study growth of soft biological tissues.

As it was mentioned in the beginning of this chapter, changes in size and shape of a biological tissue can involve both: growth, which is a change in mass, and re-modelling, which includes changes in internal structure, density and material properties. Often growth and remodelling are linked, since the process of mass alteration can change the mechanical properties of the tissue. For example, the newly deposited tissue may have different material properties than the original tissue; or only one component of the tissue may grow, thus changing the mechanical properties of the tissue as a whole. In this work, we will restrict our attention to growth processes that meet the following two requirements. First, the material points must be dense during growth. This implies that in any arbitrary neighborhood in the grown body, there will always be material points that existed before the growth took place. The second requirement is that the intrinsic mechanical properties of the material should not change during growth. In other words, the new material has the same properties as does the original material. This implies that not only is a material point equivalent (in the sense Definition 30) to itself after growth, but also that a new material point added during growth is equivalent to an original material point in an arbitrarily small neighborhood of the new material point. Growth processes that meet these restrictions are special forms of volumetric growth, which takes place in the volume of a tissue rather than on the surface.

Let $A : \overline{\Omega} \to \mathbb{R}^{3\times3}$ be the growth tensor, that describes the amount and the orientation of material deposition. We assume that the growth tensor is orientation-preserving: det $A(x) > 0$, for all $x \in \overline{\Omega}$. Let $\rho(x)$ and $\hat{T}^D(x, F)$ denote the density and response functions for the material in the current configuration. And let $\rho^u(u(x))$ and $\hat{T}^D(u(x), F)$ denote the density and response functions of the material at some later time after growth has occurred, as described by the deformation u. We now examine how $\rho^u(u(x))$ and $\hat{T}^D(u(x), F)$ are related to $\rho(x)$ and $\hat{T}^D(x, F)$.

By the definition of material equivalence and by the requirement that a material point be equivalent to itself at any time during a growth or deformation, there exists an equivalence transformation $\tilde{F}$ such that:

$$\rho^u(u(x)) = [\det\ \tilde{F}(x, A(x), \nabla u(x))]^{-1}\rho(x). \tag{5.2}$$

and:

$$\hat{T}^D(u(x), F) = \hat{T}^D(x, F\tilde{F}(x, A(x), \nabla u(x))),\ \text{for any tensor } F \text{ with det } F > 0. \tag{5.3}$$

Here, we have temporarily assumed that the equivalence transformation depends on the material point, the growth tensor, and the total deformation gradient. We now determine the form of the equivalence transformation $\tilde{F}(x, A, \nabla u)$, and show that the form of $\hat{F}$ does not depend directly on the material. By the properties of the growth tensor, we observe that if the values of the total deformation gradient and the growth tensor are the same at a point, then the stress and density at this point do not change. Consequently, such a material point is identical to itself after the growth. That is:

$$\tilde{F}(x, A(x), A(x)) = I,$$

at that point x. Observe that if the material, after the deformation u, undergoes a further pure deformation $w : u(\Omega) \to \mathbb{R}^3$, then the new equivalence tranformation can be obtained by:

$$\tilde{F}(x, A, \nabla w\nabla u) = \nabla w\tilde{F}(x, A, \nabla u). \tag{5.4}$$

Indeed, for a given growth tensor A, the value of the deformation gradient at a particular point $x \in \Omega$ can be made equal to any given tensor, with positive determinant, by applying appropriate loads to the body. Suppose that loads are chosen so that, at a point x:

$$\nabla u(x) = A(x).$$

Then, by (5.4), we get:

$$\tilde{F}(x, A(x), \nabla w(x)A(x)) = \nabla w(x)\tilde{F}(x, A(x), A(x)), \quad \text{for any deformation } w.$$

This can be written, with $F = \nabla w A$, as:

$$\tilde{F}(x, A(x), F(x)) = F(x)A(x)^{-1}, \tag{5.5}$$

for any tensor F with positive determinant. This equation gives the explicit form of the equivalence transformation $\tilde{F}$ in terms of the growth tensor. In particular, it indicates that it does not depend explicitly on either position or on the material. Rather, for a given growth tensor A and the deformation u on Ω, the value of $\tilde{F}$ depends on x implicitly through $A(x)$ and the gradient $\nabla u(x)$. Substituting (5.5) into the expressions (5.2) and (5.3), gives the formulas:

$$\rho^u(u(x)) = \det\, A(x)[\det\, \nabla u(x)]^{-1}\rho(x),$$

and:

$$\hat{T}^D(u(x), F) = \hat{T}^D(x, F\nabla u(x)A(x)^{-1}), \text{ for any } F \in \mathbb{R}^{3\times 3}, \det\, F > 0.$$

These equations describe the changes in the density function and the response function after growth. In particular, the Cauchy stress in the grown state is given by:

$$T^u(u(x)) = \hat{T}^D(u(x), Id) = \hat{T}^D(x, \nabla u(x)A(x)^{-1}). \tag{5.6}$$

The derivations leading to equations (5.5) and (5.6) certainly provide a rigorous foundation for two suggestions proposed in [45]: firstly, equation (5.5) can be rewritten as:

$$\tilde{F}(x, A, \nabla u) = \nabla u A^{-1},$$

so:

$$\nabla u = \tilde{F} A.$$

This is the decomposition (5.1) which was originally proposed in [45]. As we mentioned before, the decomposition was introduced in the context where A is defined as the growth from the original stress-free reference state to a new locally stress- free state, and $\tilde{F}$ is viewed as an elastic deformation that ensures the continuity of the body. In the current study, A corresponds to the growth of the body from the configuration Ω, which is not necessarily stress free. The introduction of the equivalence transformation $\tilde{F}$ is based on the requirement of material equivalence for growth. Since the equivalence of material points in local, $\tilde{F}$ is generally not the gradient of a deformation of the body. The second suggestion made in [45] is that stress which arises from the growth depends only on the elastic deformation that ensures the continuity of the body. The derivation of (5.6) provided here gives a rigorous proof of this conjecture.

6.0 THE NON-EUCLIDEAN ENERGY MODEL

As we explained in the previous chapters, non-Euclidean elastic bodies exhibit residual stress at their final configurations, even in the absence of external loads or imposed boundary conditions.

Mathematically speaking, the residual stress comes from an incompatible growth tensor A (in the sense specified in section 5), which induces a metric tensor $G = A^T A$, having Riamann curvature tensor not identically zero in the reference configuration. This implies, in particular, that there is no isometric immersion of the metric G in the Euclidean space (Theorem 4). In the first section of the chapter, we provide experiments describing deformations of unloaded sheets, where the growth tensor A can be given explicitly, and where it is possible to follow the influence of the metric tensor $G = A^T A$ in the mechanism of shape formation. The numerical results show that, in general, the resulting deformed configuration induces a metric tensor which is close to the prescribed tensor G. The observations and results developed here, together with those from the previous chapters, will lead to the formulation of the mathematical energy model for non-Euclidean elastic sheets.

6.1 BUILDING NON-EUCLIDEAN PLATES

We now provide an experimental model which allows to build non-Euclidean plates. We quote the reference [23] for a more complete and detailed exposition. The experimental system is presented in Figure 3. This experiment shows that the construction of elastic sheets with various target metrics G_{tar} is possible and results in spontaneous formation of non-trivial 3D structures. The free sheet will settle to a 3D configuration that minimizes its elastic

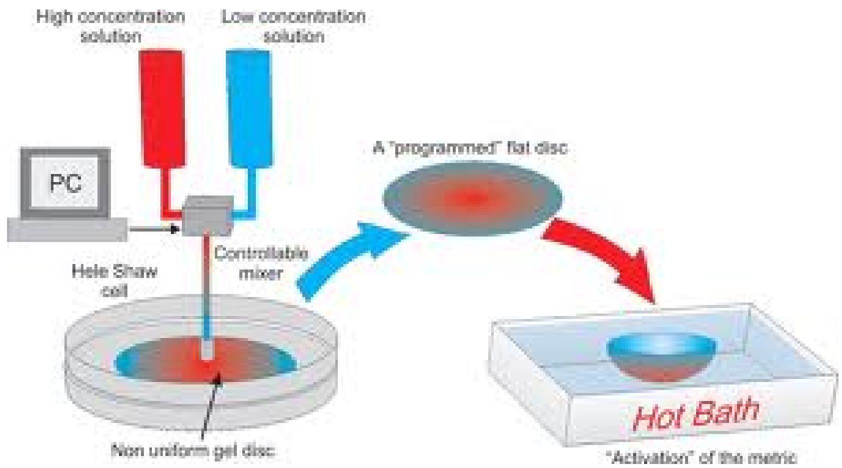

Figure 3: The experimental system to build non-Euclidean sheets. From [23]. Reprinted with permission of AAAS.

energy. In this mechanism the selected configuration is the result of a competition between the bending and the stretching energies, and, for non-Euclidean G_{tar}, its metric will be close to (but different from) G_{tar}. We now explain the experimental model: in the system, N-isopropylacrylamide (NIPA) gels are used. High and lower monomer concentrations of NIPA are mixed in a programmable mixer, and the resulting solution is injected into the gap between two flat glass plates throughout a central hole in one of them. Polymerization takes place in some minutes. The constructed gels are flat above a critical temperature $T_C = 33^0 C$, but are programmed to shrink differentially, with ratio $\eta(r)$ (where r is the radius) after activation at a temperature $T > T_C$. Calibration experiments described in [23] show that dilute gels shrink a lot, whereas gel with a high monomer concentration undergo moderate deformations. In Figure 4, we see that a radially decreasing monomer concentration (red line) prescribes a positive Gaussian curvature on the disc, which assumes an elliptic form. On the other hand, an increasing monomer concentration gives rise to a negative Gaussian curvature, and now the disc assumes a hyperbolic form, showing a wavy pattern. The induced deformation of a disc is an axially symmetric azimuthal deformation, which depends on the radius. Each element at radius r on the disc swells by a factor $\eta(r)$

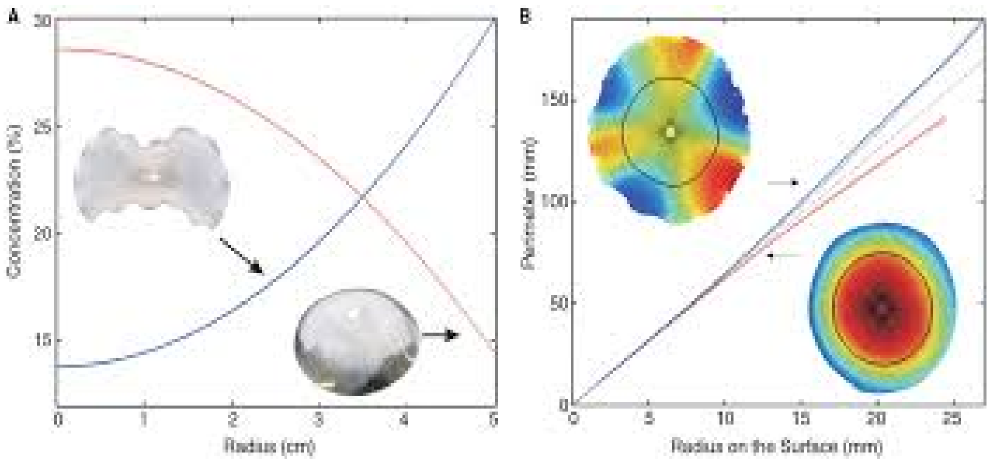

Figure 4: The effects of the monomer concentration and a comparison between the target metric (gray lines) and the actual metric (red and blue lines) on the deformed disc. From [23]. Reprinted with permission of AAAS.

only in the azimuthal direction. The equilibrium distance between points is now in polar coordinates (r, θ):

$$dl^2 = dr^2 + \eta(r)^2 r^2 d\theta^2.$$

And the target metric G_{tar} on the disc is given by:

$$G_{tar} = \begin{pmatrix} 1 & 0 \\ 0 & \eta^2(r)r^2 \end{pmatrix}.$$

It expresses the fact that in order to obey the new equilibrium distances, the perimeter of a circle of radius r on the disc is no longer $2\pi r$, but $2\pi\eta(r)r$. For non-constant η this cannot hold when the sheet is flat. It must take a non-flat configuration. In Figure 4, we present a comparison between the perimeter and radius prescribed by the target metric G_{tar} (gray lines), and these quantities measured on the actual disc. In general terms, we observe that the actual metric on the disc follows the prescribed metric G_{tar}. The deformed state adopts a non-trivial configuration as a result of the discrepancy between the ideal metric and the actual configuration of the sheet, exhibiting residual strain.

6.2 THE NON-EUCLIDEAN ENERGY MODEL

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected domain, viewed as the reference configuration of a three-dimensional elastic body $\mathcal{B}$. This elastic body is assumed to undergo a growth process (A, F) deforming the initial state Ω into the grown state Ω'. Let $u : \Omega \to \Omega'$ be the deformation mapping the original state Ω into the current or grown configuration Ω'. The increase in mass, described by A, changes the distances between points in the body $\mathcal{B}$. Therefore, the growth tensor A changes the intrinsic metric of Ω, and defines a new pull-back metric G in Ω given by:

$$G(x) = A(x)^T A(x), \qquad x \in \Omega.$$

In this way, we are given a new Riemannian structure to Ω. On the other hand, the current deformation $u : \Omega \to \mathbb{R}^3$ induces a pull-back metric $\tilde{G}_u$ on Ω given by:

$$\tilde{G}_u(x) = \nabla u(x)^T \nabla u(x), \qquad x \in \Omega,$$

which defines all metric notions (distances between points, areas and volumes, as Theorem 1 indicates) in the deformed state Ω'. As motivated by the experiments in the previous section, a natural problem is then to compare and measure, in some sense, the discrepancy between the metrics G and $\tilde{G}_u$. It is well-known that under the assumptions that Ω is connected and simply connected, a given metric G on Ω is flat, that is, there exists an orientation-preserving isometric immersion of G in Ω (in brief, a realisation of G in the given domain), if and only if the Riemann curvature tensor of G vanishes in Ω. When this condition fails, we need to determine how close G is from being flat. These observations lead to a mathematical model called 'Incompatible Elasticity'. In this model, we assume the following:

- The body $\mathcal{B}$ occupies a reference configuration Ω which is an open, connected and bounded domain in $\mathbb{R}^3$, with Lipschitz boundary.
- The body $\mathcal{B}$ is made of a hyperelastic material. Therefore, by Definition 25 and Definition 28, the Cauchy stress tensor depends only on the material variable x and on the gradient of the deformation, and, moreover, the response function is the gradient of a scalar function. From the analysis carried out in Section 5.2, a hyperelastic constitutive relation

should recognize only the elastic response F of the material in the face of the growth tensor A. Therefore, we assume that there exists a stored energy function $W : \mathbb{R}^{3\times 3} \to \overline{\mathbb{R}}_+$ such that the strain elastic energy $\mathcal{E}$ of a deformation u is written as follows:

$$\mathcal{E}(u) = \int_\Omega W(F)dx. \tag{6.1}$$

- We further assume that the stored energy density $W : \mathbb{R}^{3\times 3} \to \overline{\mathbb{R}}_+$ satisfies the following assumptions of frame invariance with respect to the group of proper rotations $SO(3)$, normalization, and non-degeneracy:

$$\forall F \in \mathbb{R}^{3\times 3}, R \in SO(3): \quad W(RF) = W(F), \quad W(R) = 0, \quad W(F) \geq c \,\mathrm{dist}^2(F, SO(3)), \tag{6.2}$$

  for some uniform constant $c > 0$. Further, we assume that W is $\mathcal{C}^2$-regular in a neighborhood of $SO(3)$.

- We also suppose that the decomposition:

$$\nabla u = FA$$

  holds for the gradient of any deformation. We refer to Section 5.2 for a discussion and analysis of conditions to be satisfy for the elastic material so that the above decomposition holds.

The resulting energy model $\mathcal{E}$ is hence given by:

$$\mathcal{E}(u) = \int_\Omega W(\nabla u A^{-1})dx. \tag{6.3}$$

Throughout this work, the expression (6.3) will be referred as the non-Euclidean energy model.

Observe that: $\mathcal{E}(u) = 0$ is equivalent to $\nabla u(x) \in SO(3)A(x)$ for almost every $x \in \Omega$. Further, in view of the polar decomposition theorem, the same condition is equivalent to: $(\nabla u)^T \nabla u = A^T A$ and $\det \nabla u > 0$ in Ω, i.e. $\mathcal{E}(u) = 0$ if and only if u is an isometric immersion of the imposed Riemannian metric $G = A^T A$. Hence, when G is not realizable (i.e. when its Riemann curvature tensor does not vanish identically in Ω), there is no u with $\mathcal{E}(u) = 0$. It has further been proven in [32] that in this case:

$$\inf\{\mathcal{E}(u);\ u \in W^{1,2}(\Omega, \mathbb{R}^n\} > 0.$$

This leads to the conclusion that, from a mathematical point of view, the lack of realizability of G introduces residual stress in the grown state. This justifies the name non-Euclidean bodies for those sheets exhibiting residual stress in their rest configurations.

7.0 MODELS FOR PRESTRAINED PLATES WITH MONGE-AMPERE CONSTRAINT

In this chapter, we derive a two-dimensional variational model for thin elastic structures starting from the three-dimensional theory of non-Euclidean elasticity. Such a dimension-reduction procedure is carried out through Γ-convergence, and it has been used in many works: we can quote the seminal works [21] and [20], where it was provided the analytical context and methodology for the derivation of the non-linear bending theory for plates, as the Γ-limit of classic theory of non-linear elasticity. Recently, in [32], [28], [29] [30] and [5] this methodology was applied to the context of non-Euclidean elasticity, obtaining Von-Kármán type growth functionals as energy models for thin elastic structures with residual stress at free equilibria. Among other features, this approach provides a rigorously justification of convergence of three-dimensional minimizers to minimizers of suitable two-dimensional limiting energies.

Our results show that under some assumptions on the growth tensors (see (7.1) and (7.6)), the limiting energy model for the two-dimensional elastic body Ω after growth is a pure bending functional, and that the asymptotic deformation of Ω is a pure bending deformation of the form $id_2 + h^{\gamma/2} v e_3$, where γ is the scaling exponent, h is the thickness parameter, and v is the out-of-plane displacement, subject to the Monge-Ampére constraint $\det \nabla^2 v = f$, for some smooth function f depending on the growth tensors.

Throughout this chapter, we will use the following notation. Given a matrix $F \in \mathbb{R}^{3\times 3}$, the $m\times n$ principal minor of a matrix F is denoted by $F_{m\times n}$. Also, for a given $F_{m\times n} \in \mathbb{R}^{m\times n}$, the matrix with $m \times n$ principal minor equal to $F_{m\times n}$ and all other entries equal to zero, is denoted by $(F)^*$. If $A = [A_{ij}]$ is invertible, we denote by $A^{-1} = [A^{ij}]$ its inverse. Finally, we denote by $\mathrm{curl}^T\mathrm{curl}$ the operator acting on matrices $F \in \mathbb{R}^{2\times 2}$ defined as:

$$\mathrm{curl}^T\mathrm{curl}\ F = \partial^2_{11}F_{22} - \partial^2_{12}(F_{12}+F_{21}) + \partial^2_{22}F_{11}.$$

7.1 THE MODEL INVOLVED AND THE LOWER BOUND IN THE Γ-LIMIT

Assume that $\Omega \subset \mathbb{R}^2$ is an open, connected, Lipschitz and bounded set. For any $h > 0$, define the three-dimensional plate Ω^h, with mid-surface Ω and thickness h, as follows:

$$\Omega^h = \Big\{ x = (x', x_3) : x' \in \Omega, x_3 \in \Big(-\frac{h}{2}, \frac{h}{2}\Big)\Big\}.$$

On each Ω^h a growth process is taking place. We assume that the growth tensors $A^h : \overline{\Omega^h} \to \mathbb{R}^3$ are given by:

$$A^h(x', x_3) = Id_3 + h^\gamma S_g(x') + x_3 h^{\gamma/2} B_g(x'), \quad (x', x_3) \in \Omega^h, \tag{7.1}$$

where the stretching and bending tensors S_g, $B_g : \overline{\Omega} \to \mathbb{R}^3$ are assumed to be smooth matrix fields in $\overline{\Omega}$, while γ is the scaling exponent.

Let F^h be the tensor representing the elastic response of the material after the action of A^h. Then, following the multiplicative decomposition (5.1) for the gradient of any deformation $u^h : \Omega^h \to \mathbb{R}^3$, we have:

$$\nabla u^h = F^h A^h.$$

Therefore, the total elastic energy per unit of thickness I^h_W associated with the deformation u^h is:

$$I^h_W(u^h) = \frac{1}{h}\int_{\Omega^h} W(F^h)dx = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h (A^h)^{-1})dx, \tag{7.2}$$

where the elastic energy density $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_+$, is assumed to satisfy the properties (6.2), and the $\mathcal{C}^2$-regularity in a neighborhood of $SO(3)$.

Given the smooth tensors A^h, a natural question is to analyse the behaviour of the minimizers of the energies I^h_W. Let us define:

$$e_h = \inf\{I^h_W(u) : u \in W^{1,2}(\Omega^h, \mathbb{R}^3)\}.$$

According to [32], $e_h > 0$ if and only if the Riemann curvature tensor of the metric $G^h = (A^h)^T A^h$ does not vanish identically on Ω^h. Thus, when G^h is not Euclidean, then $e_h > 0$ and this points to the existence of residual strain at free equilibria. In this chapter, we are concerned with the following related problems:

- Identify the scaling of the residual energy e_h in terms of the thickness parameter h and the exponent γ.
- Study the asymptotic behaviour of the minimizers as h converges to 0 and deduce the Γ-limit of their rescaled energies.

As a motivation for the scaling in Theorem 31, let us consider the following heuristic argument: assume that $W(F) = \text{dist}^2(F; SO(3))$, and let us consider the metric $G^h = (A^h)^T A^h$. Then:

$$G^h(x', x_3) = G^h(x', 0) + 2x_3 h^{\gamma/2} \text{sym} B_g(x') + \text{higher order terms},$$

where

$$G^h(x', 0) = Id_3 + 2h^\gamma \text{sym} S_g(x') + h^{2\gamma} S_g(x')^T S_g(x').$$

Suppose that we are in an optimal situation, that is, suppose that we have an isometric immersion $u_h : \Omega \to \mathbb{R}^3$ of the metric $G^h(x', 0)_{2\times 2}$ defined in the mid-surface Ω. We now consider the Kirchhoff-Love extension of u_h in the direction of the vector N^h to the surface $u_h(\Omega)$):

$$u^h(x', x_3) = u_h(x') + x_3 N^h(x'). \tag{7.3}$$

Expanding its elastic energy and ignoring higher order terms, we lead to:

$$\begin{aligned} I_W^h(u^h) &\approx \frac{1}{h} \int_{\Omega^h} |(\nabla u^h)^T \nabla u^h - G^h|^2 \\ &\approx \frac{1}{h} \int_{\Omega^h} |2h^{\gamma/2} x_3 \big((\text{sym } B_g(x') \big)_{2\times 2} - \Pi(x')|^2 \approx C h^{2+\gamma}, \end{aligned}$$

where $\Pi(x') = (\nabla u_h)^T \nabla N^h$ is, modulo higher order terms, the second fundamental form for the deformed surface $u_h(\Omega)$. This suggests the following scaling for the residual energies:

$$e_h \leq C h^{2+\gamma}, \qquad 0 < h \ll 1. \tag{7.4}$$

We now provide a short review of the current state of the theory of non-Euclidean elasticity. In [32], inspired by the experimental model discussed in Section 6.1, the authors considered the case of a metric G^h, given by a tangential Riemannian metric $[g_{\alpha\beta}]$ in Ω independent of both: the thickness parameter h and the normal direction x_3:

$$G^h(x', x_3) = \begin{pmatrix} [g_{\alpha\beta}(x')] & 0 \\ 0 & 1 \end{pmatrix}, \quad (x', x_3) \in \Omega^h.$$

It was proven that any sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ whose energies scale like

$$I_W^h(u^h) \leq Ch^2, \quad (\gamma = 0 \text{ in } (7.4)), \tag{7.5}$$

converges to a $W^{2,2}$ regular isometric immersion of the metric $[g_{\alpha\beta}]$. Moreover, the converse is also true: for any isometric immersion y of $[g_{\alpha\beta}]$, there exists a recovery sequence of deformations u^h converging to y, and whose energies scale like (7.5). The Γ-limit of the rescaled energies $h^{-2}I_W^h$ is then given by a curvature functional, defined on isometric immersions of the 2d metric $[g_{\alpha\beta}]$, which has the form:

$$\mathcal{I}_g(y) = \int_\Omega \mathcal{Q}_2(x')\Big(\sqrt{[g_{\alpha\beta}]}^{\,-1}(\nabla y)^T \nabla n\Big)dx',$$

where n denotes the unit normal vector to the surface $y(\Omega)$, while $\mathcal{Q}_2(x')$ are the quadratic forms, positive definite on the symmetric 2×2 tensors, which can be calculated from the Hessian of the stored energy function W. See [32] for the details. The case $\gamma = 0$ in the scaling (7.4) was generalized in the work [5]. In this case, the prescribed Riemannian metric G is now an arbitrary smooth field of 3×3 symmetric matrices, independent of h. The Γ-limit is a curvature functional depending, as above, on the second fundamental form of the surface $y(\Omega)$, where y is a $W^{2,2}(\Omega, \mathbb{R}^3)$ regular isometric immersion of the metric $G_{2\times 2}$. We finally mention the case $\gamma = 2$. This case was treated in [28], where the growth tensors have the form as in (7.1), with $\gamma = 2$, and the scaling is:

$$I_W^h(u^h) \leq Ch^4, \quad u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3).$$

The first main observation is that any sequence of deformations u^h scaling as above is, asymptotically, of the form:

$$u_h = id_2 + hve_3 + h^2 w,$$

where $v \in W^{2,2}(\Omega)$ is the out-of-plane displacement, and $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ is the in-plane displacement field. Moreover, the limiting energy functional $\mathcal{I}_g$ is given as a function of v and w as follows:

$$\mathcal{I}_g(v,w) = \frac{1}{24}\int_\Omega \mathcal{Q}_2\big(\nabla^2 v + \frac{1}{2}(B_g)_{2\times 2}\big) + \int_\Omega \mathcal{Q}_2\big(\text{sym } \nabla w + \frac{1}{2}\nabla v \otimes \nabla - \frac{1}{2}(S_g)_{2\times 2}\big),$$

while the quadratic form $\mathcal{Q}_2$ is given by (7.11) below. It was also proven that the growth functional $\mathcal{I}_g$ has a global minimizer at (w, v) if and only if the following holds:

$$\text{curl}(B_g)_{2\times 2} \equiv 0, \quad \text{and} \quad \text{curl}^T\text{curl}(S_g)_{2\times 2} = -\det \text{ sym}(B_g)_{2\times 2}.$$

In fact, this system constitutes the Gauss-Codazzi-Mainardi equations (see (3.5)) for the existence of a surface on the connected and simply connected set Ω, having first fundamental form $I = G_h(\cdot, 0)_{2\times 2} = Id_2 + 2h^2 \text{ sym } (S_g)_{2\times 2}$, and second fundamental form $II = h \text{ sym } (B_g)_{2\times 2}$. Therefore, the integral functional $\mathcal{I}_g$ measures the discrepancy between: the first order terms in h of the ideal second fundamental form II and of the second fundamental form of the surface $u_h(\Omega)$; and the discrepancy in the second order terms in h of the first fundamental form I and the first fundamental form under the displacement field u_h.

In this work we deal with the case where the scaling exponent γ belongs to the full range:

$$0 < \gamma < 2. \tag{7.6}$$

We now proceed to state our first result regarding the identification of the asymptotic behaviour of the minimizers. In fact, Theorem 31 deals with the more general case of any sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ whose energies $I_W^h(u^h)$ scale like $h^{2+\gamma}$. The lower bound for the Γ-limit of the rescaled energies is also provided. This bound will be optimized in Theorem 34 and Theorem 42.

Theorem 31 (Compactness and lower bound). *Assume that the growth tensors A^h are given by (7.1), with an arbitrary γ in the range:*

$$0 < \gamma < 2.$$

Then, for a given sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfying:

$$I_W^h(u^h) \leq Ch^{2+\gamma}, \tag{7.7}$$

for a uniform constant $C > 0$, there exist a sequence of proper rotations $\overline{R}^h \in SO(3)$ and a sequence of translations $c^h \in \mathbb{R}^3$ such that the normalized deformations:

$$y^h(x', x_3) = (\overline{R}^h)^T u^h(x', hx_3) - c^h : \Omega^1 \to \mathbb{R}^3, \tag{7.8}$$

satisfy the following properties:

1. *y^h converges to id_2 in $W^{1,2}(\Omega^1, \mathbb{R}^3)$.*
2. *The scaled displacements:*

$$V^h(x') = \frac{1}{h^{\gamma/2}} \int_{-1/2}^{1/2} [y^h(x', x_3) - x'] dx_3$$

converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to a vector field of the form $(0, 0, v)^T$, where $v \in W^{2,2}(\Omega, \mathbb{R})$ and verifies:

$$det\, \nabla^2 v = -curl^T curl\, (S_g)_{2\times 2}, \quad in\ \Omega. \tag{7.9}$$

In other words: $v \in \mathcal{A}_f$, where:

$$\mathcal{A}_f = \{v \in W^{2,2}(\Omega) : det\, \nabla^2 v = f\}, \quad f = -curl^T curl\, (S_g)_{2\times 2}.$$

3. Moreover:

$$\liminf_{h\to 0} \frac{1}{h^{2+\gamma}} I_W^h(u^h) \geq \mathcal{I}_f(v),$$

where $\mathcal{I}_f(v) : W^{2,2}(\Omega) \to \overline{R}_+$ *is defined as:*

$$\mathcal{I}_f(v) = \begin{cases} \dfrac{1}{24}\displaystyle\int_\Omega Q_2\big(\nabla^2 v + (sym\, B_g)_{2\times 2}\big)\ dx, & \text{if } v \in \mathcal{A}_f \\ +\infty & \text{if } v \notin \mathcal{A}_f. \end{cases} \tag{7.10}$$

The quadratic nondegenerate form $\mathcal{Q}_2 : \mathbb{R}^{2\times 2} \to \mathbb{R}_+$ *is:*

$$\begin{aligned} \mathcal{Q}_2(F) &= min\,\{\mathcal{Q}_3(\tilde{F}) : \tilde{F} \in \mathbb{R}^{3\times 3}, \tilde{F}_{2\times 2} = F\}, \\ \mathcal{Q}_3(\tilde{F}) &= \nabla^2 W(Id_3)(\tilde{F}, \tilde{F}). \end{aligned} \tag{7.11}$$

Remark 32. *Theorem 31 (1) states that, modulo rigid motions, any sequence of deformations satisfying the scaling (7.7) converges, up to a subsequence and in a suitable space, to the identity of the mid-surface. On the other hand, the convergence in part 2 asserts that the deformation* u_h *of the mid-surface is, asymptotically, of the form:*

$$u_h \approx id_2 + h^{\gamma/2} v e_3,$$

where $v \in W^{2,2}(\Omega)$ *is the out-of-plane displacement. Moreover, the Gaussian curvatures* κ *of the metric* $G^h(x', 0)_{2\times 2}$ *and of the surface* u_h *coincide at their highest order in the expansion in terms of h: this is exactly the meaning of the Monge-Ampére constraint* $v \in \mathcal{A}_f$ *in virtue of:*

$$\kappa\big(G^h(x', 0)_{2\times 2}\big) = -h^\gamma curl^T curl (S_g)_{2\times 2} + \mathcal{O}(h^{2\gamma}),$$

and:

$$\kappa\big(\nabla(id_2 + h^{\gamma/2} v)^T \nabla(id_2 + h^{\gamma/2} v)\big) = h^\gamma det\ \nabla^2 v + \mathcal{O}(h^{2\gamma}).$$

Hence, we do not expect a contribution of these curvatures to the limiting energy functional. In addition, Theorem 31 (3) provides the liminf inequality in the definition of Γ*-convergence. It shows that the functional* $\mathcal{I}_g$ *constitutes a lower bound for the variational behaviour of the rescaled energies* $h^{-(2+\gamma)} I_W^h$*. We will give an interpretation of this functional in Section 4.2.*

Before proving Theorem 31, we first state an approximation lemma from [32], which is just rephrasing Theorem 10 in [21] in the present context of non-Euclidean elasticity.

Lemma 33. *Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfies (7.7). Then there exist matrix fields $R^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$, such that $R^h(x') \in SO(3)$ a.e. $x' \in \Omega$, and:*

$$\frac{1}{h}\int_{\Omega^h} |\nabla u^h(x) - R^h(x')A^h(x)|^2 dx \leq C\Big(h^{2+2\gamma} + \frac{1}{h}\int_{\Omega^h} dist^2(\nabla u^h (A^h)^{-1}; SO(3))dx\Big),$$

$$\int_\Omega |\nabla R^h|^2 \leq C\Big(h^{2\gamma} + \frac{1}{h^3}\int_{\Omega^h} dist^2(\nabla u^h (A^h)^{-1}; SO(3))dx\Big).$$

The proof of Theorem 31 is almost the same as the proof of Theorem 1.2 in [28], however, we quote all the details in the present context, where the scaling exponent belongs now to a different range. Moreover, the constraint (7.9) is new in the current analysis.

Proof of Theorem 31:

1. From Lemma 33, there exists a sequence of matrix fields $R^h \in W^{1,2}(\Omega, SO(3))$, such that:

$$\begin{aligned}\frac{1}{h}\int_{\Omega^h} |\nabla u^h(x) - R^h(x')A^h(x)|^2 dx &\leq C\Big(h^{2+2\gamma} + \frac{1}{h}\int_{\Omega^h} \mathrm{dist}^2(\nabla u^h (A^h)^{-1}; SO(3))dx\Big) \\ &\leq C(h^{2+2\gamma} + h^{2+\gamma}) \leq Ch^{2+\gamma},\end{aligned} \tag{7.12}$$

and:

$$\begin{aligned}\int_\Omega |\nabla R^h|^2 dx' &\leq C\Big(h^{2\gamma} + \frac{1}{h^3}\int_{\Omega^h} \mathrm{dist}^2(\nabla u^h (A^h)^{-1}; SO(3))dx\Big) \\ &\leq C(h^{2\gamma} + h^{\gamma}) \leq Ch^{\gamma}.\end{aligned} \tag{7.13}$$

Define the averaged rotations:

$$\tilde{R}^h = \mathbb{P}_{SO(3)} \fint_\Omega R^h.$$

The rotations R^h are well-defined because, in view of (7.13):

$$\mathrm{dist}^2\Big(\fint_\Omega R^h, SO(3)\Big) \leq \fint_\Omega |R^h(x) - R^h(x_0)|^2 \leq C\int_\Omega |\nabla R^h|^2 \leq Ch^\gamma.$$

By (7.13) again and Poincaré inequality, we further obtain:

$$\int_\Omega |R^h - \fint_\Omega R^h|^2 \leq Ch^\gamma,$$

which yields that $\tilde{R}^h$ is well-defined. Moreover, the L^2-distance of R^h and $\tilde{R}^h$ is hence bounded by:

$$\int_\Omega |R^h - \tilde{R}^h|^2 \leq C\Big(\int_\Omega |R^h - \fint_\Omega R^h|^2 + \mathrm{dist}^2\Big(\fint_\Omega R^h, SO(3)\Big)\Big) \leq Ch^\gamma. \tag{7.14}$$

Let us consider now the projection:

$$\widehat{R}^h = \mathbb{P}_{SO(3)} \fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h dx, \tag{7.15}$$

which is well-defined in view of the estimates (7.12) and (7.14), and:

$$\begin{aligned} &|\fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h - Id|^2 \leq C \fint_{\Omega^h} |\nabla u^h - \tilde{R}^h|^2 \\ &\leq C\Big(\fint_{\Omega^h} |\nabla u^h - R^h A^h|^2 + \fint_{\Omega^h} |A^h - Id|^2 + \fint_{\Omega^h} |R^h - \tilde{R}^h|^2\Big) \leq Ch^\gamma. \end{aligned} \tag{7.16}$$

Moreover, by (7.16), we deduce the estimate:

$$|\widehat{R}^h - Id|^2 \leq C|\mathrm{skew} \fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h|^2 \leq Ch^\gamma. \tag{7.17}$$

The first inequality follows from the fact that for any matrix F close to Id it holds that:

$$\mathbb{P}_{SO(3)}(\mathrm{sym}\ F) = Id,$$

and hence:

$$|\mathbb{P}_{SO(3)}F - Id| \leq C|F - \mathrm{sym}\, F| \leq C|\mathrm{skew}\ F|.$$

Let now:

$$\overline{R}^h = \tilde{R}^h \widehat{R}^h. \tag{7.18}$$

Then by (7.14) and (7.17):

$$\int_\Omega |R^h - \overline{R}^h|^2 \leq C\Big(\int_\Omega |R^h - \tilde{R}^h|^2 + \int_\Omega |\tilde{R}^h - \tilde{R}^h\widehat{R}^h|^2\Big) \leq Ch^\gamma, \tag{7.19}$$

and so:

$$\lim_{h\to 0} (\overline{R}^h)^T R^h = Id, \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^{3\times 3}). \tag{7.20}$$

Let $c^h \in \mathbb{R}^3$ be vectors such that for the rescaled averaged displacements V^h:

$$V^h(x') = \frac{1}{h^{\gamma/2}} \fint_{-h/2}^{h/2} [(\overline{R}^h)^T u^h(x', x_3) - c^h - x'] dx_3,$$

the following holds:

$$\int_\Omega V^h = 0, \qquad \text{skew} \int_\Omega \nabla V^h = 0. \tag{7.21}$$

In order to obtain the last requirement, notice that:

$$\text{skew} \fint_{\Omega^h} (\overline{R}^h)^T \nabla u^h = 0.$$

In fact:

$$(\overline{R}^h)^T \fint_{\Omega^h} \nabla u^h = (\widehat{R}^h)^T \fint_{\Omega^h} (\tilde{R}^h)^T \nabla u^h \quad \text{is symmetric,}$$

because for a matrix F close enough to $SO(3)$, its projection $R = \mathbb{P}_{SO(3)} F$ coincides with the unique rotation appearing in the polar decomposition of F:

$$F = RU, \quad \text{skew } U = 0.$$

Now we deal with the convergence of the normalized deformations:

$$y^h(x', x_3) = (\overline{R}^h)^T u^h(x', hx_3) - c^h,$$

where $\overline{R}^h \in SO(3)$ is given by (7.18). In virtue of (7.16) and (7.17), we get:

$$\begin{aligned} \|(\nabla y^h - Id)_{3\times 2}\|^2_{L^2(\Omega^1)} &\leq \fint_{\Omega^h} |(\overline{R}^h)^T \nabla u^h - Id|^2 \\ &\leq C\Big(\fint_{\Omega^h} |(\tilde{R}^h)^T \nabla u^h - Id|^2 dx + |\widehat{R}^h - Id|^2 \Big) \leq Ch^\gamma. \end{aligned} \tag{7.22}$$

Further, from (7.12):

$$\|\partial_3 y^h\|^2_{L^2(\Omega^1)} \leq Ch \int_{\Omega^h} |\nabla u^h|^2 \leq Ch^2.$$

Using the above two inequalities, together with:

$$\int_{\Omega^1} y^h(x) - x' = h^{\gamma/2} \int_\Omega V^h = 0,$$

and Poincaré inequality, we therefore get the strong convergence in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ of the sequence y^h to x'. This finishes the proof of (i).

2 Let us consider the matrix fields $D^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$ defined as follows:

$$D^h(x') = \frac{1}{h^{\gamma/2}} \fint_{-h/2}^{h/2} [(\overline{R}^h)^T R^h(x') A^h(x', x_3) - Id] dx_3.$$

Then:

$$D^h(x') = h^{\gamma/2}(\overline{R}^h)^T R^h(x') S_g(x') + \frac{1}{h^{\gamma/2}}[(\overline{R}^h)^T R^h(x') - Id],$$

and we therefore derive:

$$\|D^h\|^2_{L^2(\Omega)} \leq C, \ \|\nabla D^h\|^2_{L^2(\Omega)} \leq C, \tag{7.23}$$

where to get the first bound we have used (7.19), and in the last bound we have applied (7.13). As a result, up a subsequence:

$$D^h \to D \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3}) \ \text{ weakly in } W^{1,2}(\Omega, \mathbb{R}^{3\times 3}),$$

and then:

$$\frac{1}{h^{\gamma/2}}[(\overline{R}^h)^T R^h(x') - Id] \to D \ \text{ strongly in } L^q(\Omega, \mathbb{R}^{3\times 3}) \quad \forall q \geq 1. \tag{7.24}$$

Next, using (7.19) and (7.13):

$$\begin{aligned} h^{-\gamma/2}\|\text{sym } (\overline{R}^h)^T R^h - Id\|_{L^2(\Omega)} &= \frac{1}{2} h^{-\gamma/2}\|((\overline{R}^h)^T R^h - Id)^T((\overline{R}^h)^T R^h - Id)\|_{L^2(\Omega)} \\ &\leq C h^{-\gamma/2}\|(\overline{R}^h)^T R^h - Id\|^2_{L^2(\Omega)} \leq C h^{-\gamma/2}\|(\overline{R}^h)^T R^h - Id\|^2_{W^{1,2}(\Omega)} \leq C h^{\gamma/2}. \end{aligned}$$

This implies that:

$$\text{sym } D = \lim_{h\to 0} \text{sym } D^h = 0.$$

Observe that:

$$\frac{1}{h^{\gamma/2}}\text{sym } D^h = \text{sym } [(\overline{R}^h)^T R^h S_g] - \frac{1}{2h^\gamma}[(\overline{R}^h)^T R^h(x') - Id]^T[(\overline{R}^h)^T R^h(x') - Id].$$

Hence from (7.24) we can deduce:

$$\lim_{h\to 0} \frac{1}{h^{\gamma/2}}\text{sym } D^h = \text{sym } S_g - \frac{1}{2}D^T D = \text{sym } S_g + \frac{1}{2}D^2, \quad \text{in } L^q(\Omega, \mathbb{R}^{3\times 3}), \ q \geq 1. \tag{7.25}$$

Let us study now the convergence of V^h. First, observe that the gradient of V^h can be written as:

$$\nabla V^h(x') = D^h_{3\times 2}(x') - \frac{1}{h^{\gamma/2}}(\overline{R}^h)^T \fint_{-h/2}^{h/2} R^h(x') A^h_{3\times 2}(x', x_3) - \nabla_{tan} u^h(x', x_3) dx_3. \tag{7.26}$$

Hence, by (7.12) and Jensen's inequality:

$$\begin{aligned}\|\nabla V^h - D^h_{3\times 2}\|^2_{L^2(\Omega)} &\leq \frac{C}{h^\gamma}\int_\Omega \Big| \fint_{-h/2}^{h/2} R^h(x')A^h(x',x_3) - \nabla_{tan} u^h(x',x_3)dx_3\Big|^2 dx' \\ &\leq \frac{C}{h^{\gamma+1}}\int_{\Omega^h} |\nabla u^h(x) - R^h(x')A^h(x)|^2 dx \leq Ch^2.\end{aligned}$$

Therefore, ∇V^h converges in $L^2(\Omega, \mathbb{R}^{3\times 2})$ to $D_{3\times 2}$, and so there exists a vector field $V \in W^{2,2}(\Omega, \mathbb{R}^3)$ so that:

$$\lim_{h\to 0} V^h = V \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^3),$$

and $\nabla V = D_{3\times 2}$. Since sym $D = 0$, we conclude sym $\nabla(V_{tan}) = 0$ and then by Korn's inequality, V_{tan} is a constant. But since:

$$\int_\Omega V^h = 0,$$

we finally get $V_{tan} = 0$.

Finally we prove (7.9). In a first step, we show that the scaled in-plane displacements $h^{-\gamma/2}V^h_{tan}$ converge weakly in $W^{1,2}(\Omega, \mathbb{R}^2)$ to an in-plane displacement field $w \in W^{1,2}(\Omega, \mathbb{R}^2)$. To do so, divide (7.26) by $h^{\gamma/2}$, take its symmetric part and pass to the limit, using the convergence:

$$\|h^{-\gamma/2}(\nabla V^h - D^h_{3\times 2})\|^2_{L^2(\Omega)} \to 0, \quad \text{as } h \to 0, \tag{7.27}$$

which is true in view of:

$$\begin{aligned}\|h^{-\gamma/2}(\nabla V^h - D^h_{3\times 2})\|^2_{L^2(\Omega)} &\leq \frac{C}{h^{2\gamma}}\int_\Omega \Big| \fint_{-h/2}^{h/2} R^h(x')A^h(x',x_3) - \nabla_{tan} u^h(x',x_3)dx_3\Big|^2 dx' \\ &\leq \frac{C}{h^{2\gamma+1}}\int_{\Omega^h} |\nabla u^h(x) - R^h(x')A^h(x)|^2 dx \leq Ch^{2-\gamma},\end{aligned}$$

and the assumption $0 < \gamma < 2$.

Using (7.25), (7.27), (7.21), and Poincaré inequality, we then obtain:

$$\|h^{-\gamma/2}V^h_{tan}\|_{W^{1,2}(\Omega)} \leq C\|\nabla(h^{-\gamma/2}V^h_{tan})\|_{L^2(\Omega)} = C\|\text{sym } (h^{-\gamma/2}\nabla V^h_{tan})\|_{L^2(\Omega)} \leq C.$$

Let $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ be the weak limit of the scaled displacements $h^{-\gamma/2}V^h_{tan}$. Consider now the tangential part of (7.26), divide it by $h^{\gamma/2}$, take its symmetric part and pass to the limit as h converges to 0. Using again (7.27), (7.25) and the weak convergence of $h^{-\gamma/2}V^h_{tan}$ to w in $W^{1,2}(\Omega, \mathbb{R}^2)$, we therefore get:

$$\text{sym } \nabla w = \text{sym } (S_g)_{2\times 2} - \frac{1}{2}(D^T D)_{2\times 2}.$$

Recalling that:

$$D = \begin{pmatrix} 0 & 0 & -\partial_1 v \\ 0 & 0 & -\partial_2 v \\ \partial_1 v & \partial_2 v & 0 \end{pmatrix}, \tag{7.28}$$

it follows that:

$$(D^T D)_{2\times 2} = \nabla v \otimes \nabla v.$$

Therefore, there exists a solution $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ to the equation:

$$\text{sym } \nabla w + \frac{1}{2}\nabla v \otimes \nabla v - \text{sym } (S_g)_{2\times 2} = 0.$$

Taking curt^{T}curl to both sides of the above equation, we hence deduce:

$$\text{curl}^T\text{curl } \Big(\frac{\nabla v \otimes \nabla v}{2} - \text{sym } (S_g)_{2\times 2}\Big) = 0.$$

Considering now the fact:

$$\text{curl}^T\text{curl } \Big(\frac{\nabla v \otimes \nabla v}{2}\Big) = -\text{det } \nabla^2 v,$$

we derive the constraint (7.9). This completes the proof of Theorem 31 (2).

3. We first define the rescaled strains $G^h : \Omega^1 \to \mathbb{R}^{3\times 3}$:

$$G^h(x', x_3) = \frac{1}{h^{1+\gamma/2}}\big[(R^h(x'))^T \nabla u^h(x', hx_3) A^h(x', hx_3)^{-1} - Id\big]. \tag{7.29}$$

Hence, by (7.12) it follows:

$$\begin{aligned} \int_{\Omega^1} |G^h|^2 &\leq \frac{C}{h^{2+\gamma}} \int_{\Omega^1} |\nabla u^h(x', hx_3) - R^h(x')A^h(x', hx_3)|^2 \\ &\leq \frac{C}{h^{3+\gamma}} \int_{\Omega^h} |\nabla u(x) - R^h(x')A^h(x)|^2 \leq C. \end{aligned} \tag{7.30}$$

Thus, up to extracting a subsequence:

$$\lim_{h\to 0} G^h = G, \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{7.31}$$

We now claim that:

$$\lim_{h\to 0} \frac{1}{h^{1+\gamma/2}}\big[\partial_3 y^h - he_3\big] = De_3, \quad \text{strongly in } L^2(\Omega^1, \mathbb{R}^3). \tag{7.32}$$

In fact:

$$\begin{aligned}
\frac{1}{h^{1+\gamma/2}}\big(\partial_3 y^h(x) - he_3\big) &= \frac{1}{h^{\gamma/2}}\big[(\overline{R}^h)^T \nabla u^h(x', hx_3) - Id\big]e_3 \\
&= \frac{1}{h^{\gamma/2}}(\overline{R}^h)^T\big[\nabla u^h(x', hx_3) - R^h(x')A^h(x', hx_3)\big]e_3 \\
&\quad + \frac{1}{h^{\gamma/2}}(\overline{R}^h)^T R^h(x')\big(A^h(x', hx_3) - Id\big)e_3 + \frac{1}{h^{\gamma/2}}\big[(\overline{R}^h)^T R^h(x') - Id\big]e_3.
\end{aligned} \tag{7.33}$$

In view of (7.12), the first term in the last equality converges to 0 in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$. From the definition of A^h (7.1), the second term converges to 0 in L^∞, and in virtue of (7.24), the last term converges to De_3.

We now derive a formula for the limiting strain G. Consider the difference quotients:

$$f^{s,h}(x) = \frac{1}{h^{1+\gamma/2}}\frac{1}{s}\big[y^h(x + se_3) - y^h(x) - hse_3\big]. \tag{7.34}$$

Since:

$$f^{s,h}(x) = \frac{1}{h^{1+\gamma/2}} \fint_0^s \frac{d}{dt} y^h(x + te_3) - he_3 \, dt, .$$

we obtain from (7.32):

$$\lim_{h\to 0} f^{s,h} = De_3, \quad \text{in } L^2(\Omega^1, \mathbb{R}^3).$$

Similarly, again by (7.32):

$$\lim_{h\to 0} \partial_3 f^{s,h} = 0, \quad \text{in } L^2(\Omega^1, \mathbb{R}^3),$$

while for $\alpha = 1, 2$, by (7.20) and (7.31):

$$\begin{aligned}
\partial_\alpha f^{s,h}(x) &= \frac{1}{h^{1+\gamma/2}}\frac{1}{s}(\overline{R}^h)^T\big[\nabla u^h(x', hx_3 + hs) - \nabla u^h(x', hx_3)\big]e_\alpha \\
&= \frac{1}{s}(\overline{R}^h)^T R^h(x')\big[\big(G^h(x', x_3 + s) - G^h(x', x_3)\big)A^h(x', hx_3)\big]e_\alpha \\
&\to \frac{1}{s}\big[G(x', x_3 + s) - G(x', x_3)\big]e_\alpha, \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3).
\end{aligned}$$

Hence:

$$f^{s,h} \to De_3, \quad \text{weakly in } W^{1,2}(\Omega^1, \mathbb{R}^3),$$

and further:

$$\partial_\alpha(De_3) = \frac{1}{s}\big[G(x', x_3+s) - G(x', x_3)\big]e_\alpha.$$

Concluding:

$$G(x)_{3\times 2} = G_0(x')_{3\times 2} + x_3(\nabla\big(D(x')e_3\big) - B_g(x')),$$

for some $G_0 \in L^2(\Omega, \mathbb{R}^{3\times 3})$.

We now compute the lower bound of the rescaled energies. Define the 'good' set as follows:

$$\Omega_h^1 = \{x \in \Omega^1 : h^{\gamma/2}|G^h| \le 1\}.$$

From (7.30), it follows the convergence of the characteristic functions:

$$\chi_{\Omega_h^1} \to 1, \quad \text{in } L^1(\Omega^1).$$

Hence, by (7.31):

$$\chi_{\Omega_h^1} G^h \to G, \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{7.35}$$

Considering the frame invariance of W, we can write:

$$\frac{1}{h^{2+\gamma}}W\big(\nabla u^h (A^h)^{-1}\big) = \frac{1}{h^{2+\gamma}}W\big((R^h)^T\nabla u^h (A^h)^{-1}\big) = \frac{1}{h^{2+\gamma}}W\big(Id + h^{1+\gamma/2}G^h\big). \tag{7.36}$$

Using Taylor expansion of W around the identity, we obtain for all $A \in \mathbb{R}^{3\times 3}$ with sufficiently small $\|A\|_{L^\infty}$:

$$W(I + A) = \frac{1}{2}\mathcal{Q}_3(A) + \eta(A), \tag{7.37}$$

where:

$$\eta(A) = \int_0^1 (1-s)\langle \nabla^2 W(Id + sA) - \nabla^2 W(Id) : A \otimes A\rangle ds,$$

and it satisfies:

$$\frac{\eta(A)}{|A|^2} \to 0, \qquad \text{as } |A| \to 0. \tag{7.38}$$

Therefore:

$$\frac{1}{h^{2+\gamma}}W\big(\nabla u^h (A^h)^{-1}\big) = \frac{1}{h^{2+\gamma}}W\big(Id + h^{1+\gamma/2}G^h\big) = \frac{1}{2}\mathcal{Q}_3(G^h) + \frac{1}{h^{2+\gamma}}\eta\big(h^{1+\gamma/2}G^h\big).$$

From (7.35) we derive:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{h^{2+\gamma}}I_W^h(u^h) &\geq \liminf_{h\to 0}\frac{1}{h^{2+\gamma}}\int_{\Omega^1}\chi_{\Omega_h^1}W\big(\nabla u^h(x)A^h(x)^{-1}\big)dx\\ &= \liminf_{h\to 0}\Big(\frac{1}{2}\mathcal{Q}_3\big(\chi_{\Omega_h^1}G^h\big)+o(1)\int_{\Omega^1}|G_h|^2\Big)\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3(G(x))dx,\end{aligned}\tag{7.39}$$

where we also use that the quadratic form $\mathcal{Q}_3$ is positive definite. Furthermore:

$$\begin{aligned}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3(G(x))dx &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2(G_{2\times 2}(x))dx\\ &= \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2(G_0(x')_{2\times 2}+x_3(\nabla(D(x')e_3)-B_g(x'))_{2\times 2})dx\\ &\geq \frac{1}{2}\int_{\Omega^1}x_3^2\mathcal{Q}_2((\nabla(D(x')e_3)-B_g(x'))_{2\times 2})dx\\ &= \frac{1}{2}\Big(\int_{-1/2}^{1/2}x_3^2dx_3\Big)\int_{\Omega}\mathcal{Q}_2(-\nabla^2 v(x')-B_g(x')_{2\times 2})dx'\\ &= \frac{1}{24}\int_{\Omega}\mathcal{Q}_2(\nabla^2 v(x')+\mathrm{sym}\ B_g(x')_{2\times 2})dx'=\mathcal{I}_g(v),\end{aligned}\tag{7.40}$$

where we have used (7.28). Therefore, from (7.39) and (7.40) we derive the lower bound:

$$\liminf_{h\to 0}\frac{1}{h^{2+\gamma}}I_W^h(u^h)\geq \mathcal{I}_g(v).$$

□

7.2 THE RECOVERY SEQUENCE: THE UPPER BOUND IN THE Γ-LIMIT

In this section, we show that the energy lower bound $\mathcal{I}_f$ in Theorem 31 is optimal, and that the scaling of the elastic energies (7.7) is sharp. We first provide a proof of these results for a partial range of the scaling exponent γ (Theorem 34), and then we present a proof for the full range $0 < \gamma < 2$ (Theorem 42). The reason to proceed in that way is that for the full case, some extra assumptions on the domain Ω and on the metric tensors A^h are needed. See Subsection 7.2.2 for the details.

7.2.1 Range $1 < \gamma < 2$.

Theorem 34. *Assume (7.1), with γ in the range:*

$$1 < \gamma < 2.$$

Further, assume that Ω is simply connected. Then for any $v \in \mathcal{A}_f$, where:

$$\mathcal{A}_f = \left\{v \in W^{2,2}(\Omega) : det\, \nabla^2 v = f\right\}, \quad f = -curl^T curl\, (S_g)_{2\times 2},$$

there exists a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that the following statements hold:

1. *The sequence $y^h(x', x_3) = u^h(x', hx_3)$ converges to x', strongly in $W^{1,2}(\Omega^1, \mathbb{R}^3)$.*
2. *The rescaled displacements:*

   $$V^h(x') = h^{-\gamma/2} \fint_{h/2}^{h/2} (u^h(x', x_3) - x_3) dx_3$$

   converge to $(0, 0, v)^T$ in $W^{1,2}(\Omega, \mathbb{R}^3)$.
3. *Recalling definition (7.10):*

   $$\lim_{h\to 0} \frac{1}{h^{2+\gamma}} \int_{\Omega^h} W(\nabla u^h (A^h)^{-1}) dx = \mathcal{I}_f(v).$$

We now can rewrite the results in Theorem 31 and in Theorem 34 using the language of Γ-convergence in the following manner:

Theorem 35 (Γ-convergence)**.** *Let Ω be simply connected and assume that $\gamma \in (1, 2)$. Define the sequence of functionals $\mathcal{F}^h : W^{1,2}(\Omega^1, \mathbb{R}^3) \times W^{1,2}(\Omega^1, \mathbb{R}^3) \to \overline{\mathbb{R}}$ as follows:*

$$\mathcal{F}^h(y, V) = \begin{cases} \dfrac{1}{h^{2+\gamma}} I_W^h(y(x', hx_3)), & \textit{if } V(x') = h^{-\gamma/2} \fint_{-1/2}^{1/2} (y(x', hx_3) - x') dx_3 \\ +\infty & \textit{otherwise.} \end{cases} \tag{7.41}$$

Then $\mathcal{F}^h$ Γ-convergences, as $h \to 0$, to the functional $\mathcal{F} : W^{1,2}(\Omega^1, \mathbb{R}^3) \times W^{1,2}(\Omega^1, \mathbb{R}^3) \to \overline{\mathbb{R}}$ defined by:

$$\mathcal{F}(y, V) = \begin{cases} \mathcal{I}_g(v), & \textit{if } y(x', x_3) = x',\ V = (0, 0, v)^T, v \in \mathcal{A}_f. \\ +\infty & \textit{otherwise.} \end{cases} \tag{7.42}$$

The main consequence of the Γ-convergence result is the following: if u^h is a minimizing sequence for I_W^h (see 7.43 below) and if $v \in \mathcal{A}_f$ is the respective limiting out-of-plane displacement corresponding to u^h, then v will be a minimizer of the Von Kármán growth functional $\mathcal{I}_g$. This is the content of the following result:

Theorem 36. *Assume that Ω is simply connected and let $1 < \gamma < 2$ in (7.1). Then the following statements are true:*

1. *$\mathcal{A}_f \neq \emptyset$ if and only if there exists a uniform constant $C > 0$ such that:*

$$e_h = \inf I_W^h \leq Ch^{2+\gamma}.$$

2. *Assume $\mathcal{A}_f \neq \emptyset$. Then for any minimizing sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ for I_W^h, that is, any sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that:*

$$\lim_{h\to 0} \frac{1}{h^{2+\gamma}} \Big(I_W^h(u^h) - \inf I_W^h \Big) = 0, \tag{7.43}$$

the convergences (1), (2) of Theorem 31 hold up to subsequences and the limit $v \in W^{2,2}(\Omega)$ is a minimizer of the functional $\mathcal{I}_g$.

Conversely, for any global minimizer v of $\mathcal{I}_g$, there exists a minimizing sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfying (7.43), and the convergent statements (1), (2) and (3) of Theorem (31).

The next result will provide a geometric interpretation of the growth functional $\mathcal{I}_g$ in the context of theory of surfaces.

Lemma 37. *Suppose that Ω is a simply connected domain. Then, the following statements are equivalent:*

1. *There exists $v \in \mathcal{A}_f$ and:*

$$\mathcal{I}_g(v) = 0.$$

2. *The following holds:*

$$curl(sym(B_g)_{2\times 2}) = 0 \quad and \quad curl^T curl(S_g)_{2\times 2} = -det(sym(B_g)_{2\times 2}). \tag{7.44}$$

Corollary 38. *If:*

$$curl(sym\ (B_g)_{2\times 2}) \neq 0, \qquad or \qquad curl^T curl(S_g)_{2\times 2} + det\big((sym\, B_g)_{2\times 2}\big) \neq 0, \tag{7.45}$$

then there exists a constant $c > 0$ such that:

$$e_h = \inf I_W^h \geq ch^{2+\gamma}.$$

Remark 39. *The system (7.44) in Lemma 37 is the Gauss-Codazzi-Mainardi system (see (3.5)) which constitutes a sufficient and necessary condition for the existence of a surface on Ω with first fundamental form $I = G^h(\cdot, 0)_{2\times 2}$ and second fundamental form $II = h^{\gamma/2} sym(B_g)_{2\times 2}$. In fact, denoting the metric $G^h(\cdot, 0)_{2\times 2}$ by M^h, using the relations:*

$$\begin{aligned}
&(\Gamma^h)_{ij}^k = \frac{1}{2}(M^h)^{kl}(\partial_i M_{il}^h + \partial_i M_{jl}^h - \partial_j M_{ij}^h) = 1 + \mathcal{O}(h^\gamma) \\
&det\ M^h = 1 + \mathcal{O}(h^\gamma) \\
&\kappa(M^h) = -h^\gamma curl^T curl(S_g)_{2\times 2} + \mathcal{O}(h^{2\gamma}),
\end{aligned}$$

and putting:

$$I = M^h \quad and \quad II = h^{\gamma/2} sym(B_g)_{2\times 2},$$

in the Gauss-Codazzi-Mainardi system (3.5), we derive from the first two equations in (3.5), the following:

$$curl(sym(B_g)_{2\times 2}) = \mathcal{O}(h^{\gamma/2}),$$

and from the last equation in (3.5), the equality:

$$h^\gamma det(sym(B_g)_{2\times 2}) = \kappa(M^h) det\ M^h = -h^\gamma curl^T curl(S_g)_{2\times 2} + \mathcal{O}(h^{2\gamma}).$$

These equations yield the system (7.44). With this interpretation in mind, we can now say that the growth functional $\mathcal{I}_g$ defined in (7.11) constitutes an L^2-distance between the ideal second fundamental form $II = h^{\gamma/2} sym(B_g)_{2\times 2}$ prescribed by the growth tensors A^h, and the actual second fundamental form $II = (\nabla u_h)^T \nabla N^h = -h^{\gamma/2}\nabla^2 v +$ higher order terms, given by the deformation $u_h = x' + h^{\gamma/2} v e_3$ of Ω.

Proof of Theorem 34. In order to deal with deformations on a fix domain, we do the following change of variables: for $h > 0$, and for $(z_1, z_2, z_3) \in \Omega \times (-h/2, h/2)$, let:

$$x_1 = z_1,\ x_2 = z_2,\ x_3 = \frac{1}{h} z_3.$$

so that:

$$(x_1, x_2, x_3) \in \Omega^1 = \Omega \times \Big(-\frac{1}{2}, \frac{1}{2} \Big).$$

Rescale deformations u of Ω^h according to the rule:

$$y(x_1, x_2, x_3) = u(x_1, x_2, hx_3).$$

Thus:

$$\frac{1}{h} \int_{\Omega^h} W\big(\nabla u(z) A^h(z)^{-1}\big) dz = \int_{\Omega^1} W\big(\nabla_h y(x) A^h(x)^{-1}\big) dx,$$

where $\nabla_h y = (\partial_1 y, \partial_2 y, h^{-1}\, \partial_3 y)$.

For any matrix $F \in \mathbb{R}^{3\times 3}$, we also introduce the notation $l(F)$ for the unique vector in $\mathbb{R}^3$ for which:

$$\text{sym}\ \big(F - (F_{2\times 2})^*\big) = \text{sym}\ \big(l(F) \otimes e_3\big).$$

Now we prove Theorem 34: Let $v \in W^{2,2}(\Omega, \mathbb{R})$ be the out-of-plane displacement satisfying the constaint (7.9). Then there exists $w \in W^{1,2}(\Omega, \mathbb{R}^2)$, such that:

$$\text{sym}\ \nabla w + \frac{\nabla v \otimes \nabla v}{2} - \text{sym}\ (S_g)_{2\times 2} = 0. \tag{7.46}$$

By Korn's inequality: $w \in W^{1,q}(\Omega, \mathbb{R}^2)$, for all $1 \leq q < \infty$.

Define the truncation scale:

$$\lambda = 1 + \frac{\gamma}{2},$$

and the exponent:

$$q = \frac{2+\gamma}{\gamma - 1} > 4,$$

so that $w \in W^{1,q}(\Omega, \mathbb{R}^2)$. For an appropriated small constant $\epsilon_0 > 0$, Theorem 14 allows us for having truncation sequences $v^h \in W^{2,\infty}(\Omega, \mathbb{R})$, $w^h \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ such that:

$$\begin{aligned} &\|v^h\|_{W^{2,\infty}} \leq \epsilon_0 h^{-\lambda}, \quad \|w^h\|_{W^{1,\infty}} \leq \epsilon_0 h^{-2\lambda/q}. \\ &|\{x \in \Omega : v^h(x) \neq v(x)\}| \leq C \frac{\omega_1(h)}{h^{2\lambda}}. \\ &|\{x \in \Omega : w^h(x) \neq w(x)\}| \leq C \frac{\omega_2(h)}{h^{2\lambda}}. \end{aligned} \tag{7.47}$$

Here $\omega_1(h), \omega_2(h) > 0$ and:

$$\omega_1(h),\, \omega_2(h) \to 0, \quad \text{as } h \to 0.$$

Define the recovery sequence $y^h \in W^{1,\infty}(\Omega^1, \mathbb{R}^3)$ by:

$$y^h(x', x_3) = \begin{bmatrix} x' \\ 0 \end{bmatrix} + \begin{bmatrix} h^\gamma w^h \\ h^{\gamma/2} v^h \end{bmatrix} + h x_3 \begin{bmatrix} -h^{\gamma/2} \nabla v^h \\ 1 \end{bmatrix} + \frac{1}{2} h^{2+\gamma/2} x_3^2 (d_1^h + l(B_g)) + h^{1+\gamma} x_3 d_0^h,$$

where $d_0^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ is defined as:

$$d_0^h = l(S_g) - \frac{1}{2} |\nabla v^h|^2 e_3,$$

and $d_1^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ is a warping sequence satisfying:

$$\|d_1^h - d_1\|_{L^2} \to 0, \quad h^{\gamma/2} \|d_1^h\|_{W^{1,\infty}} \to 0, \tag{7.48}$$

where $d_1 : \Omega \to \mathbb{R}^3$ is given by:

$$\mathcal{Q}_2\big(\nabla^2 v + (B_g)_{2\times 2}\big) = \mathcal{Q}_3\big(-(\nabla^2 v + (B_g)_{2\times 2})^* + d_1 \otimes e_3\big).$$

The convergence statements in parts **1** and **2** of Theorem 34 easily follow.

3. Now we estimate the contribution of the deformations y^h to the energy I_W^h. In a first step, we calculate the deformation gradient of y^h, obtaining:

$$\begin{aligned} \nabla_h y^h = Id + h^{\gamma/2} D^h + h^\gamma (\nabla w^h)^* - h^{1+\gamma/2} x_3 (\nabla^2 v^h)^* + h^\gamma d_0^h \otimes e_3 \\ + h^{1+\gamma/2} x_3 (d_1^h + l(B_g)) \otimes e_3 + \rho^h, \end{aligned} \tag{7.49}$$

where:

$$D^h = \begin{pmatrix} 0 & -\nabla v^h \\ (\nabla v^h)^T & 0 \end{pmatrix}, \quad \rho^h = \frac{1}{2} h^{2+\gamma/2} x_3^2 [\nabla(d_1^h + l(B_g)) : 0] + h^{1+\gamma} x_3 [\nabla d_0^h : 0].$$

Now, for all sufficiently small h:

$$\|h^\gamma S_g + x_3 h^{1+\gamma/2} B_g\|_{L^\infty} \leq C_1 < 1,$$

which implies that the series:

$$\sum_{n=0}^{\infty} (-1)^n (h^\gamma S_g + x_3 h^{1+\gamma/2} B_g)^n$$

converges uniformly in Ω^1 to $(A^h)^{-1}$. Moreover, we have the estimates:

$$\|(A^h)^{-1}\|_{L^\infty} \leq \|(A^h)^{-1} A^h\|_{L^\infty} \|A^h\|_{L^\infty} \leq C, \quad \|(A^h)^{-1} - Id\|_{L^\infty} \leq Ch^\gamma, \tag{7.50}$$

for all h sufficiently small. Before going further, we obtain a better estimate for the L^∞-norm of ∇v^h. From the definition of v^h:

$$\|v^h\|_{W^{2,2}} \leq C\|v\|_{W^{2,2}} \leq C, \quad \|\nabla v^h\|_{W^{1,\infty}} \leq \epsilon_0 h^{-\lambda}.$$

From the Brezis-Wainger inequality (see Theorem 16) applied to the sequence $\nabla v^h \in W^{1,4}$, we deduce:

$$\|\nabla v^h\|_{L^\infty} \leq C\left(1 + \log^{1/2}(1 + \|\nabla v^h\|_{W^{1,4}})\right) \leq C\big(1 + \log^{1/2}(1 + \epsilon h^{-\lambda})\big) \leq C\log\,(1/h) \tag{7.51}$$

for every sufficiently small $h > 0$. In particular: $\|\nabla v^h\|_{L^\infty} \leq Ch^{-\gamma/4}$. As a result, we derive the bounds:

$$\|D^h\|_{L^\infty} \leq Ch^{-\gamma/4}, \quad \|d_0^h\|_{L^\infty} \leq C(\|S_g\|_{L^\infty} + h^{-\gamma/2}), \quad \|\nabla d_0^h\|_{L^\infty} \leq C(\|\nabla S_g\|_{L^\infty} + h^{-\gamma/4-\lambda}), \tag{7.52}$$

which, together with the following estimates (see (7.47), (7.48) and (7.52)):

$$\begin{aligned}
&\|h^{1+\gamma/2}x_3(\nabla^2 v^h)\|_{L^\infty} \le C\epsilon_0,\\
&\|h^{1+\gamma/2}x_3 d_1^h \otimes e_3\|_{L^\infty} \le Ch,\\
&\|\rho^h\|_{L^\infty} \le Ch^{\gamma/4},\\
&\|h^\gamma \nabla w^h\|_{L^\infty} \le Ch,
\end{aligned} \tag{7.53}$$

imply that:

$$\|\nabla_h y^h - Id\|_{L^\infty} \le C\epsilon_0. \tag{7.54}$$

In virtue of (7.54) and (7.50), it follows that:

$$\begin{aligned}
\mathrm{dist}(\nabla_h y^h(x)(A^h(x))^{-1}, SO(3)) &\le \|\nabla_h y^h (A^h)^{-1} - Id\|_{L^\infty}\\
&\le \|\nabla_h y^h - Id\|_{L^\infty} + \|\nabla_h y^h[(A^h)^{-1} - Id]\|_{L^\infty} \le C(\epsilon_0 + h^\gamma) \le C\epsilon_0,\ x \in \Omega^1,
\end{aligned} \tag{7.55}$$

for all $h \le h_0$, h_0 sufficiently small. Therefore, choosing small enough ϵ_0 and h_0, we get that $\nabla_h y^h (A^h)^{-1}$ belongs to a compact neighborhood of $SO(3)$ where W is $\mathcal{C}^2$ regular, for all $h \le h_0$.

We now proceed to estimate the contribution of the rescaled deformations y^h to the elastic energies. Let us define:

$$\Omega_h = \{x \in \Omega : v^h(x) = v(x)\} \cap \{x \in \Omega : w^h(x) = w(x)\}.$$

Then, on the set of points $\Omega \setminus \Omega_h$, and considering the uniform boundedness of W near $SO(3)$, we have:

$$\begin{aligned}
\frac{1}{h^{2+\gamma}} \int_{\Omega\setminus\Omega_h \times(-1/2,1/2)} W(\nabla_h y^h (A^h)^{-1}) dx &\le C\frac{|\Omega \setminus \Omega_h|}{h^{2+\gamma}}\\
&= \frac{C}{\epsilon_0^2} h^{2\lambda} |\Omega \setminus \Omega_h| \to 0,\ \text{as } h \to 0,
\end{aligned} \tag{7.56}$$

in view of (7.47).

Let us study now the contribution of y^h to the elastic energy I_W^h in the set Ω_h. In order to measure the discrepancy between the prescribed metric tensors $G^h = (A^h)^T A^h$ and the actual metric $(\nabla_h y^h)^T \nabla_h y^h$, we introduce the strain:

$$F^h = (A^h)^{T,-1}(\nabla_h y^h)^T \nabla_h y^h (A^h)^{-1} - Id.$$

We now estimate the quantity $\|F^h\|_{L^\infty}$ in the set $\Omega_h \times (-1/2, 1/2)$. Firstly, observe that:

$$\begin{aligned}(\nabla_h y^h)^T \nabla_h y^h = Id + 2h^\gamma \mathrm{sym}\big[\nabla w^* - \frac{1}{2}D^2 + d_0 \otimes e_3\big] \\ + 2x_3 h^{1+\gamma/2}\mathrm{sym}\big[d_1^h \otimes e_3 + l(B_g) \otimes e_3 - (\nabla^2 v)^*\big] + \mathcal{R}^h,\end{aligned} \tag{7.57}$$

where the remainder $\mathcal{R}^h$ is given by:

$$\begin{aligned}\mathcal{R}^h = 2\ \mathrm{sym}\ \Big(&\rho^h + h^{\gamma/2}\rho^{h,T}D^h + h^\gamma \rho^{h,T}(d_0^h \otimes e_3 + \nabla w^{h,*}) - h^{3\gamma/2}D^h(\nabla w^{h,*} \\ &+ d_0^h \otimes e_3) + h^{1+\gamma/2}\big(x_3(e_3 \otimes d_1^h + e_3 \otimes l(B_g)) - \nabla^2 v^{h,*}\big)\rho^h + x_3 h^{1+\gamma} D^h(\nabla^2 v^{h,*} \\ &- d_1^h \otimes e_3) - l(B_g) \otimes e_3 + \frac{h^{2\gamma}}{2}\big((\nabla w^{h,T})^* \nabla w^{h,*} + (e_3 \otimes d_0^h)[d_0^h \otimes e_3 + 2\nabla w^{h,*}]\big) \\ &- h^{1+3\gamma/2}x_3\big(\nabla^2 v^{h,*}[\nabla w^{h,*} + d_0^h \otimes e_3] - (e_3 \otimes d_1^h + e_3 \otimes l(B_g))\nabla w^{h,*}(e_3 \otimes d_0^h) \\ &(d_1^h \otimes e_3 + l(B_g) \otimes e_3)\big) + \rho^{h,T}\rho^h + h^{2+\gamma}x_3^2\big(\nabla^2 v^{h,*}[\nabla^2 v^{h,*} - (d_1^h + l(B_g)) \otimes e_3] \\ &+ e_3 \otimes (d_1^h + l(B_g)(d_1^h + l(B_g)) \otimes e_3)\Big),\end{aligned} \tag{7.58}$$

In view of (7.53) and the assumption $\gamma > 1$, the remainder $\mathcal{R}^h$ satisfies:

$$h^{-(1+\gamma/2)}\chi_h|\mathcal{R}^h| \le f \in L^1(\Omega^1), \quad \text{for all } h \le h_0, \tag{7.59}$$

and:

$$\|\chi_h \mathcal{R}^h\|_{L^\infty(\Omega^1)} \le C\epsilon_0, \quad h^{-(1+\gamma/2)}\chi_h \mathcal{R}^h(x) \to 0, \text{ as } h \to 0\ a.e.\ x \in \Omega^1. \tag{7.60}$$

Here $\chi_h = \chi_{\Omega_h}$ is the characteristic function of the set $\Omega_h \times (-1/2, 1/2)$. Hence, with the help of equation (7.46) we deduce:

$$\begin{aligned}(\nabla_h y^h)^T \nabla_h y^h - (A^h)^T A^h &= 2x_3 h^{1+\gamma/2}\mathrm{sym}\big[d_1^h \otimes e_3 + l(B_g) \otimes e_3 - (\nabla^2 v)^* - B_g\big] + \mathcal{R}'^h \\ &= 2x_3 h^{1+\gamma/2}\mathrm{sym}\big[d_1^h \otimes e_3 - (\nabla^2 v)^* - (B_g)^*_{2\times 2}\big] + \mathcal{R}'^h,\end{aligned}$$

where:

$$\mathcal{R}'^h = \mathcal{R}^h - \big(h^\gamma S_g + x_3 h^{1+\gamma/2} B_g\big)^T \big(h^\gamma S_g + x_3 h^{1+\gamma/2} B_g\big).$$

Using the identity:

$$F^h = (A^h)^{T,-1}[(\nabla_h y^h)^T \nabla_h y^h - (A^h)^T A^h](A^h)^{-1}, \tag{7.61}$$

together with (7.60) and the estimates (7.50) and (7.53), we therefore obtain:

$$\|\chi_h F^h\|_{L^\infty(\Omega^1)} \leq C\epsilon_0, \tag{7.62}$$

and

$$\lim_{h\to 0} h^{-(1+\gamma/2)} \chi_h F^h = 2x_3 \mathrm{sym}\big[d_1 \otimes e_3 - (\nabla^2 v)^* - (B_g)^*_{2\times 2}\big], \ a.e. \text{ in } \ \Omega^1. \tag{7.63}$$

For sufficiently small ϵ_0 in (7.55), it now follows by polar decomposition theorem that $\nabla_h y^h (A^h)^{-1}$ is the product of a proper rotation and the well defined square root of:

$$(A^h)^{T,-1}(\nabla_h y^h)^T \nabla_h y^h (A^h)^{-1} = Id + F^h.$$

By frame indifference of W, we obtain:

$$W(\nabla_h y^h (A^h)^{-1}) = W\Big(\sqrt{(A^h)^{T,-1}(\nabla_h y^h)^T \nabla_h y^h (A^h)^{-1}}\Big) = W\Big(\frac{1}{2}F^h + O(|F^h|^2)\Big), \tag{7.64}$$

where the last equality follows by Taylor expansion. Recalling (7.62) and the Taylor formula (7.37), we can expand W around Id to get:

$$\frac{1}{h^{2+\gamma}} W(\nabla_h y^h (A^h)^{-1}) = \frac{1}{2}\mathcal{Q}_3\Big(\frac{1}{2h^{1+\gamma/2}}F^h + \frac{1}{h^{1+\gamma/2}}O(|F^h|^2)\Big) + \frac{1}{h^{2+\gamma}}\eta\Big(\frac{1}{2}F^h + O(|F^h|^2)\Big). \tag{7.65}$$

From (7.63), it follows that:

$$\frac{O\big(|F^h|^2\big)}{h^{1+\gamma/2}} = \frac{|F^h|^2}{h^{1+\gamma/2}} \frac{O\big(|F^h|^2\big)}{|F^h|^2} \to 0, \ \text{as} \ h \to 0.$$

Hence:

$$\chi_h \mathcal{Q}_3\Big(\frac{1}{2h^{1+\gamma/2}}F^h + \frac{1}{h^{1+\gamma/2}}O(|F^h|^2)\Big) \to x_3^2 \mathcal{Q}_2(\nabla^2 v + \mathrm{sym}\ (B_g)_{2\times 2}), \ h \to 0, \tag{7.66}$$

pointwise in Ω^1. On the other hand, applying (7.38) and (7.63), we therefore deduce:

$$\begin{aligned}\frac{1}{h^{2+\gamma}}\eta\Big(\frac{1}{2}F^h+O(|F^h|^2)\Big)\\ =\frac{\Big|\frac{1}{2}F^h+O(|F^h|^2)\Big|^2}{h^{2+\gamma}}\frac{\eta\Big(\frac{1}{2}F^h+O(|F^h|^2)\Big)}{\Big|\frac{1}{2}F^h+O(|F^h|^2)\Big|^2}\to 0,\ h\to 0.\end{aligned} \tag{7.67}$$

Therefore:

$$\lim_{h\to 0}\frac{1}{h^{2+\gamma}}\chi_h W\big(\nabla_h y^h(A^h)^{-1}\big)=x_3^2\mathcal{Q}_2\big(\nabla^2 v+\text{sym }(B_g)_{2\times 2}\big)\quad a.e.\text{ in }\ \Omega^1. \tag{7.68}$$

Furthermore, from (7.62) we obtain:

$$\frac{1}{2}|F^h|+O\big(|F^h|^2\big)\leq C(|F^h|+|F^h|^2)\leq C\epsilon_0.$$

Thus, for sufficiently small ϵ_0 and by (7.38), it follows now that:

$$\eta\Big(\frac{1}{2}F^h+O(|F^h|^2)\Big)\leq C\Big(\frac{1}{2}|F^h|+O\big(|F^h|^2\big)\Big)^2\leq C|F^h|^2,$$

for all $h\leq h_0$. Thus, by (7.59), the above estimates, and in view of the expansion (7.65), we can conclude that the function:

$$\frac{1}{h^{2+\gamma}}\chi_h W\big(\nabla_h y^h(A^h)^{-1}\big)$$

is bounded by an L^1-function in Ω^1, for all $h\leq h_0$. Therefore, it follows now by the Dominated Convergence Theorem that:

$$\lim_{h\to 0}\frac{1}{h^{2+\gamma}}\int_{\Omega^1}W\big(\nabla_h y_h(A^h)^{-1}\big)dx=\frac{1}{24}\int_{\Omega}\mathcal{Q}_2\big(\nabla^2 v+\text{sym }(B_g)_{2\times 2}\big)dx'.$$

□

Proof of Theorem 36.

1. Assume that the set $\mathcal{A}_f$ is not empty. Let $v \in \mathcal{A}_f$. By Theorem 34, there exists a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that:

$$\lim_{h \to 0} \frac{1}{h^{2+\gamma}} \int_{\Omega^h} W\big(\nabla u^h (A^h)^{-1}\big) dx = \mathcal{I}_f(v) < \infty.$$

Therefore, for all $h \leq h_0$, there is a uniform constant $C > 0$ such that:

$$e_h = \inf I_W^h \leq C h^{2+\gamma}.$$

To prove the converse, assume that there exists a constant $C > 0$ such that:

$$e_h \leq C h^{2+\gamma}.$$

For any h, there corresponds some $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that:

$$I_W^h(u^h) \leq 2C h^{2+\gamma}.$$

By Theorem 31, the rescaled displacements V^h converge to an out-of-plane displacement field of the form $(0, 0, v)^T$, with $v \in \mathcal{A}_f$. This proves the theorem.

2. Assume that u^h is a minimizing sequence. Then, for all $h \leq h_0$, we have:

$$I_W^h(u^h) \leq C h^{2+\gamma} + \inf I_W^h.$$

Since $\mathcal{A}_f \neq \emptyset$, by part **1**, there is a uniform constant $C > 0$ so that:

$$\inf I_W^h \leq C h^{2+\gamma}.$$

Hence, we deduce, for all small h, that:

$$I_W^h(u^h) \leq C h^{2+\gamma}.$$

This bound allows us to use Theorem 31 to get the following convergences:

$$y^h(x', x_3) = u^h(x', h x_3) \to x', \quad \text{in } W^{1,2}(\Omega^1, \mathbb{R}^3)$$

and:

$$V^h(x') = h^{-\gamma/2} \fint_{-1/2}^{1/2} (y^h(x', x_3) - x_3) dx_3 \to (0, 0, v)^T, \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^3).$$

Moreover, the limiting out-of-plane displacement v belongs to $\mathcal{A}_f$ and by Theorem 22, it is a global minimizer of the growth functional $\mathcal{I}_g$. Conversely, assume that $v \in \mathcal{A}_f$ is a global minimizer of the functional $\mathcal{I}_g$. Then, by Theorem 34, there exists a recovery sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that the convergences $y^h \to x'$ and $V^h \to (0,0,v)^T$ hold in the appropriate spaces and topologies, and moreover:

$$\lim_{h\to 0} \frac{1}{h^{2+\gamma}} I_W^h = \mathcal{I}_g(v).$$

By Theorem 22, we obtain:

$$\lim_{h\to 0} \left(\inf \frac{1}{h^{2+\gamma}} I_W^h \right) = \inf \mathcal{I}_g = \mathcal{I}_g(v).$$

Therefore, u^h is a minimizing sequence for the functional $\mathcal{I}_g$.

□

Proof of Lemma 37.

The proof of (1) implies (2) is straightforward in view of the definition (7.11) of the functional $\mathcal{I}_g$. Conversely, assume that:

$$\operatorname{curl}(\operatorname{sym}(B_g)_{2\times 2}) = 0 \quad \text{and} \quad \operatorname{curl}^T \operatorname{curl}(S_g)_{2\times 2} = -\det(\operatorname{sym}(B_g)_{2\times 2}). \tag{7.69}$$

The first assumption is equivalent, in a simply connected domain, to the existence of a function $v \in W^{2,2}(\Omega)$ such that:

$$\operatorname{sym}(B)_{2\times 2} = -\nabla^2 v. \tag{7.70}$$

This together with the second equality in (7.69) imply that $v \in \mathcal{A}_f$, with

$$f = \operatorname{curl}^T \operatorname{curl}(S_g)_{2\times 2}.$$

Moreover, (7.70) yields that $\mathcal{I}_g(v) = 0$. This ends the proof of the lemma. □

Proof of Corollary 38.

Observe that Lemma 37 implies that:

$$\inf \mathcal{I}_g > 0.$$

From the lower bound in Theorem 31, we obtain the desired lower bound for the inf I_W^h.

□

7.2.2 Full range: $0 < \gamma < 2$.

In view of the proof of Theorem 34, the Monge-Ampére constraint allows us to obtain a map $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ satisfying the equation:

$$\text{sym } \nabla w + \frac{1}{2}\nabla v \otimes \nabla v = \text{sym}(S_g)_{2\times 2}.$$

Writing the recovery sequence in the form:

$$u^h = u_h + \text{higher order term},$$

where: $u_h(x') = x' + h^{\gamma/2}v(x')e_3 + h^\gamma w(x')$, we end up with the matching (up to terms of order $\mathcal{O}(h^{3\gamma/2})$) of the ideal metric:

$$G^h(x',0)_{2\times 2} = Id_2 + 2h^\gamma \text{sym } (S_g)_{2\times 2} + h^{2\gamma}(S_g)^2_{2\times 2},$$

and the metric induced by the map u_h:

$$\nabla(x' + h^{\gamma/2}ve_3 + h^\gamma w)^T \nabla(x' + h^{\gamma/2}ve_3 + h^\gamma w) = Id_2 + 2h^\gamma \Big(\text{sym } \nabla w + \frac{1}{2}\nabla v \otimes \nabla v\Big) + \mathcal{O}(h^{3\gamma/2}).$$

As we discussed in Theorem 34, the resulting discrepancy does not exceed the residual energy bound:

$$I^h_W(u_h) \leq Ch^{2+\gamma},$$

for the range $1 < \gamma < 2$. To cover the full range for the exponent γ we need a better accuracy of the metrics. This is the content of the next theorem, where an extra term w^h depending on h in the expansion of u_h is needed to obtain an exact isometry of the metric $G^h(x',0)_{2\times 2}$.

Lemma 40. *Assume that Ω is simply connected and that $-curl^T curl(S_g)_{2\times 2} \geq c > 0$. Also, suppose that $v \in \mathcal{C}^{2,\beta}(\overline{\Omega}, \mathbb{R})$, $\beta \in (0,1)$, satisfies:*

$$det\nabla^2 v = -curl^T curl(S_g)_{2\times 2} \quad in \quad \Omega.$$

Then, for any $h > 0$, there corresponds $w^h \in \mathcal{C}^{2,\beta}(\overline{\Omega}, \mathbb{R})$ such that:

$$\nabla(x' + h^{\gamma/2}v + h^\gamma w^h)^T \nabla(x' + h^{\gamma/2}v + h^\gamma w^h) = G^h(x', 0_{2\times 2}). \tag{7.71}$$

and:

$$\sup_h \|w^h\|_{\mathcal{C}^{2,\beta}} < \infty. \tag{7.72}$$

The next result concerns the density of regular solutions to the elliptic two-dimensional Monge-Ampére equation in the space $\mathcal{A}_f$. For a proof of this result, see [27].

Theorem 41. *Let Ω be open, bounded, connected and star-shaped with respect to an interior ball. For a fixed constant $c_0 > 0$, define:*

$$\mathcal{A}_{c_0} = \{u \in W^{2,2}(\Omega) : det\, \nabla^2 u = c_0 \quad a.e.\ in\ \Omega\}.$$

Then the set $\mathcal{A}_{c_0} \cap \mathcal{C}^\infty(\overline{\Omega})$ is dense in $\mathcal{A}_{c_0}$ with respect to the strong topology of $W^{2,2}$.

We now state the main result of this section.

Theorem 42. *Assume that Ω is open, bounded, connected and star-shaped with respect to an interior ball. Assume that:*

$$f = -curt^T curl(S_g)_{2\times 2} = c_0 > 0 \qquad in \quad \Omega.$$

Let:

$$0 < \gamma < 2.$$

Then for any $v \in \mathcal{A}_f$, there exists a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that the following statements hold:

1. *The sequence $y^h(x', x_3) = u^h(x', hx_3)$ converges to x', strongly in $W^{1,2}(\Omega^1, \mathbb{R}^3)$.*
2. *The rescaled displacements:*

$$V^h(x') = h^{-\gamma/2} \fint_{h/2}^{h/2} (u^h(x', x_3) - x_3) dx_3$$

converge to $(0, 0, v)^T$ in $W^{1,2}(\Omega, \mathbb{R}^3)$.

3. *Recalling definition (7.10):*

$$\lim_{h\to 0} \frac{1}{h^{2+\gamma}} \int_{\Omega^h} W(\nabla u^h (A^h)^{-1}) dx = \mathcal{I}_f(v).$$

Moreover, all the assertions in Theorem 36, Lemma 37 and Corollary 38 hold as well.

We now prove Lemma 40. This proof is mostly the same as the proof of Theorem 4.1 in [27]. However, we present the proof in the current context.

Proof of Lemma 40.

For simplicity in the notation, we let $\epsilon = h^{\gamma/2}$ and $M^\epsilon = [M_{ij}]_{i,j=1,2} = G^\epsilon(\cdot, 0)_{2\times 2}$. Writing the unknown vector field $w^\epsilon \in \mathcal{C}^{2,\beta}(\overline{\Omega}, \mathcal{R}^3)$ into its tangential and normal parts:

$$w^\epsilon = w^\epsilon_{tan} + w^\epsilon_3 e_3,$$

we deduce that (7.71) is equivalent to:

$$\nabla(id_2 + \epsilon^2 w^\epsilon_{tan})^T \nabla(id_2 + \epsilon^2 w^\epsilon_{tan}) = M^\epsilon - \epsilon^2 \nabla(v + \epsilon w^\epsilon_3) \otimes \nabla(v + \epsilon w^\epsilon_3). \tag{7.73}$$

Denote by $v^\epsilon_0 = v + z_\epsilon$, where $z_\epsilon = \epsilon w^\epsilon_3$, and consider the metric:

$$g_\epsilon(z_\epsilon) = M^\epsilon - \epsilon^2 \nabla v^\epsilon_0 \otimes \nabla v^\epsilon_0. \tag{7.74}$$

The idea is to find z_ϵ so that the Gaussian curvature κ of the $2d$ metric $g_\epsilon(z_\epsilon)$ vanishes for any $\epsilon \in (-\epsilon_0, \epsilon_0)$. This is done through an application of the implicit function theorem. Further, invoking Theorem 4, the above will imply the existence of an isometric immersion ϕ_ϵ of the given metric g_ϵ. We shall finally prove that ϕ_ϵ is of the form:

$$\phi_\epsilon = id_2 + \epsilon^2 w^\epsilon_{tan},$$

and $\sup_\epsilon \|w^\epsilon\|_{\mathcal{C}^{2,\beta}} < \infty$.

We start computing the Gaussian curvature of $g_\epsilon(z_\epsilon)$. First, by a direct calculation, we deduce that the Christoffel symbols, the inverse of M^ϵ and the determinant of M^ϵ can be written as:

$$\begin{aligned} &(\Gamma^\epsilon)^k_{ij} = \frac{1}{2}(M^\epsilon)^{kl}(\partial_i M^\epsilon_{il} + \partial_i M^\epsilon_{jl} - \partial_j M^\epsilon_{ij}) = 1 + \mathcal{O}(\epsilon^2) \\ &(M^\epsilon)^{-1} = [(M^\epsilon)^{ij}] = \frac{1}{\det\, M^\epsilon}\mathrm{cof}\,[M^\epsilon_{ij}] = 1 + \mathcal{O}(\epsilon^2) \\ &\det\, M^\epsilon = 1 + \mathcal{O}(\epsilon^2). \end{aligned} \tag{7.75}$$

For the case where v^ϵ_0 is a smooth vector field, formula (3.8) allows us to write:

$$\begin{aligned} \kappa(g_\epsilon) &= \kappa\big(M^\epsilon - \epsilon^2 \nabla v^\epsilon_0 \otimes \nabla v^\epsilon_0\big) \\ &= \frac{1}{\big(1 - \epsilon^2 (M^\epsilon)^{ij}\partial_i v^\epsilon_0 \partial_j v^\epsilon_0\big)^2}\left(\kappa(M^\epsilon) \;-\; \frac{\epsilon^2 \det\,\big(\nabla^2 v^\epsilon_0 - [(\Gamma^\epsilon)^k_{ij}\partial_k v^\epsilon_0]_{ij}\big)}{\big(1 - h^\gamma (M^\epsilon)^{ij}\partial_i v^\epsilon_0 \partial_j v^\epsilon_0\big)^2 \det\, M^\epsilon}\right). \end{aligned} \tag{7.76}$$

This formula can be also obtain for $\mathcal{C}^{2,\beta}$ maps as follows: approximate $v_0^\epsilon \in \mathcal{C}^{2,\beta}$ by a smooth sequence $v_0^{\epsilon,n}$. Then, each Gaussian curvature:

$$\kappa_n = \kappa(M^\epsilon - \epsilon^2 \nabla v_0^{\epsilon,n} \otimes \nabla v_0^{\epsilon,n})$$

is given by the formula (7.76). Now, the sequence κ_n converges in $\mathcal{C}^{0,\beta}$ to the right hand side of (7.76). Moreover, from the definition of the Gaussian curvature (see 3.5):

$$\kappa = \frac{S_{1212}}{\det\, g},$$

we get that κ_n converges in the sense of distributions to the Gaussian curvature of the metric $M^\epsilon - \epsilon^2 \nabla v_0^\epsilon \otimes \nabla v_0^\epsilon$. This establishes the formula (7.76) for maps v_0^ϵ in $\mathcal{C}^{2,\beta}$.

Now, observe that:

$$\kappa(g_\epsilon) = 0$$

if and only if:

$$\kappa(M^\epsilon)\big(1 - \epsilon^2 (M^\epsilon)^{ij} \partial_i v_0^\epsilon \partial_j v_0^\epsilon\big)^2 \det\, M^\epsilon = \epsilon^2 \det\, \big(\nabla^2 v_0^\epsilon - [(\Gamma^\epsilon)_{ij}^k \partial_k v_0^\epsilon]_{ij}\big),$$

that is, if and only if the operator $\Phi : \mathcal{C}_0^{2,\beta}(\overline{\Omega}, \mathbb{R}) \times (-\epsilon, \epsilon) \to \mathcal{C}^{0,\beta}(\overline{\Omega}, \mathbb{R})$ defined as:

$$\begin{aligned} \Phi(z,\epsilon) = \frac{1}{\epsilon^2} \kappa(M^\epsilon)\big(1 - \epsilon^2 (M^\epsilon)^{ij} \partial_i (v+z) \partial_j (v+z)\big)^2 \det\, M^\epsilon \\ - \det\, \big(\nabla^2 (v+z) - [(\Gamma^\epsilon)_{ij}^k \partial_k (v+z)]_{ij}\big), \end{aligned} \tag{7.77}$$

satisfies:

$$\Phi(z_\epsilon, \epsilon) = 0. \tag{7.78}$$

We now look for solutions $z_\epsilon \in \mathcal{C}_0^{2,\beta}(\overline{\Omega}, \mathbb{R})$ of (7.78). In view of (7.75), and the expansion:

$$\kappa(M^\epsilon) = -\epsilon^2 \mathrm{curl}^T \mathrm{curl} (S_g)_{2\times 2} + \mathcal{O}(\epsilon^4),$$

we can write:

$$\begin{aligned} \Phi(z,\epsilon) = -\big(\mathrm{curl}^T \mathrm{curl}(S_g)_{2\times 2} + \mathcal{O}(\epsilon^2)\big)\big(1 + \mathcal{O}(\epsilon^2)|\nabla v + \nabla z|^2\big)^2 \big(1 + \mathcal{O}\big(\epsilon^2\big)\big) \\ - \det\, \big(\nabla^2 v + \nabla^2 z + \mathcal{O}(\epsilon^2)|\nabla v + \nabla z|\big). \end{aligned} \tag{7.79}$$

It follows now that:

$$\Phi(0,0) = -\text{curl}^T\text{curl}(S_g)_{2\times2} - \det\,\nabla^2 v = 0.$$

Moreover, the partial Fréchet derivative $\mathcal{L} = \partial\Phi/\partial z(0,0) : \mathcal{C}_0^{2,\beta}(\overline{\Omega},\mathbb{R}) \to \mathcal{C}_0^{0,\beta}(\overline{\Omega},\mathbb{R})$ is a linear continuous operator of the form:

$$\begin{aligned}\mathcal{L}(z) &= \lim_{\epsilon\to0}\frac{1}{\epsilon}\Phi(\epsilon z,0) = \lim_{\epsilon\to0}\frac{1}{\epsilon}\big(-\text{curl}^T\text{curl}(S_g)_{2\times2} - \det\,\big(\nabla^2 v + \epsilon\nabla^2 z\big)\big)\\ &= \lim_{\epsilon\to0}\frac{1}{\epsilon}\big(-\text{curl}^T\text{curl}(S_g)_{2\times2} - \det\,\nabla^2 v - \epsilon^2\det\,\nabla^2 z - \epsilon\,\text{cof}\,\nabla^2 v : \nabla^2 z\big)\\ &= -\text{cof}\,\nabla^2 v : \nabla^2 z,\end{aligned}\tag{7.80}$$

where we have used the formula:

$$\det\,(A+B) = \det\,A + \det\,B + \text{cof}\,A : B,$$

valid for 2×2 matrices A, B. Because of the uniform ellipticity of the matrix field $\nabla^2 v$ which follows from the assumption:

$$\det\,\nabla^2 v = -\text{curl}^T\text{curl}(S_g)_{2\times2} \equiv c_0 > 0,$$

we obtain that $\mathcal{L}$ is an invertible operator. Consequently, invoking the implicit function theorem we obtain the solution operator:

$$\mathcal{Z} : (-\epsilon_0,\epsilon_0) \to \mathcal{C}_0^{2,\beta}(\overline{\Omega},\mathbb{R})$$

such that the function $z_\epsilon = \mathcal{Z}(\epsilon)$ satisfies:

$$\Phi(z_\epsilon,\epsilon) = 0, \quad \forall\,\epsilon \in (-\epsilon_0,\epsilon_0).$$

We also have that the operator $\mathcal{Z}$ is differentiable at $\epsilon = 0$, and its derivatives at $\epsilon = 0$ is given by:

$$\mathcal{Z}'(0) = \mathcal{L}^{-1}\circ\Big(\frac{\partial\Phi}{\partial\epsilon}(0,0)\Big),$$

where:

$$\frac{\partial\Phi}{\partial h}(0,0) = 0.$$

Therefore,

$$\|w_h^3\|_{\mathcal{C}^{2,\beta}} = \frac{1}{\epsilon}\|z_\epsilon\|_{\mathcal{C}^{2,\beta}} \to 0, \quad \text{as } \epsilon \to 0.$$

In summary, we have so far obtained a uniformly bounded sequence of $\mathcal{C}_0^{2,\beta}$ out-of-plane displacements w_3^ϵ such that the Gaussian curvature of the metric $g_\epsilon(z_\epsilon)$ vanishes. According to Theorem 4, for each small ϵ, there corresponds exactly one, modulo rigid motions, orientation preserving isometric immersion $\phi_\epsilon \in \mathcal{C}^{2,\beta}(\overline{\Omega}, \mathbb{R}^2)$ of the metric $g_\epsilon(z_\epsilon)$:

$$\nabla\phi_\epsilon^T \nabla\phi_\epsilon = g_\epsilon(z_\epsilon), \qquad \text{and} \qquad \det \nabla\phi_\epsilon > 0. \tag{7.81}$$

From a direct calculation, (7.81) implies the equation:

$$\nabla^2\phi_\epsilon - [(\Gamma^\epsilon)_{ij}^k \partial_k \phi_\epsilon]_{ij} = 0. \tag{7.82}$$

From (7.81), we derive that:

$$\|\nabla\phi_\epsilon\|_{L^\infty} \le C,$$

which, together with equation (7.82) imply that:

$$\|\nabla^2\phi_\epsilon\|_{L^\infty} \le C.$$

Therefore:

$$\|\phi_\epsilon\|_{\mathcal{C}^{2,\beta}} \le C.$$

Note that by (7.75), the Christoffel symbols Γ^ϵ verifies:

$$\|(\Gamma^\epsilon)_{ij}^k\|_{\mathcal{C}^{0,\beta}} \le C\epsilon^2.$$

Hence, by equation (7.82) again, we obtain the bound:

$$\|\nabla^2\phi_\epsilon\|_{\mathcal{C}^{0,\beta}} \le C\epsilon^2. \tag{7.83}$$

By Poincaré inequality, there exists some matrix fields S_ϵ such that:

$$\|\nabla\phi_\epsilon - S_\epsilon\|_{\mathcal{C}^{1,\beta}} \le C\epsilon^2. \tag{7.84}$$

In fact, the matrix fields S_ϵ can be taken in the group $SO(2)$, or even as $S_\epsilon = Id$: for each $x \in \Omega$, we have:

$$\text{dist } (S_\epsilon, SO(2)) \leq |S_\epsilon - \nabla\phi_\epsilon(x)| + \text{dist } (\nabla\phi_\epsilon(x), SO(2)). \tag{7.85}$$

To estimate the second term, write:

$$\sqrt{\nabla\phi_\epsilon^T(x)\nabla\phi_\epsilon(x)} = Q_\epsilon(x)D_\epsilon(x)Q_\epsilon(x)^T,$$

where: $Q_\epsilon(x) \in SO(2)$, and $D_\epsilon(x) = \text{diag } (\lambda_1^\epsilon(x), \lambda_2^\epsilon(x))$, with λ_1^ϵ, $\lambda_2^\epsilon > 0$. Taking into account that ϕ_ϵ is orientation preserving, we hence get:

$$\begin{aligned}
\text{dist } (\nabla\phi_\epsilon(x), SO(2)) &= \left|\sqrt{\nabla\phi_\epsilon^T(x)\nabla\phi_\epsilon(x)} - Id\right| = \left|Q_\epsilon(x)D_\epsilon(x)Q_\epsilon(x)^T - Id\right| \\
&\leq C|D_\epsilon - Id| = C \ \max_i|\lambda_i^\epsilon - 1| \leq C \ \max_i|(\lambda_i^\epsilon)^2 - 1| \\
&\leq C|D_\epsilon^2(x) - Id| = C|Q_\epsilon(x)^T\nabla\phi_\epsilon^T\nabla\phi_\epsilon(x)Q_\epsilon(x) - Id| \\
&\leq C|\nabla\phi_\epsilon^T(x)\nabla\phi_\epsilon(x) - Id| = C|g_\epsilon(z_\epsilon) - Id| \leq C\epsilon^2.
\end{aligned}$$

By the above inequality and (7.85), (7.84), we certainly obtain:

$$\text{dist } (S_\epsilon, SO(2)) \leq C\epsilon^2.$$

In this way, we can assume without loss of generality, that $S_\epsilon = Id$, and so:

$$\|\nabla\phi_\epsilon - Id\|_{\mathcal{C}^{2,\beta}} \leq C\epsilon^2.$$

Therefore, we deduce that:

$$\phi_\epsilon = id + \epsilon^2 w_{tan}^\epsilon,$$

with:

$$\|w_{tan}^\epsilon\|_{\mathcal{C}^{2,\beta}} \leq C.$$

This ends the proof of Theorem 42.

□

Proof of Theorem 42.

Firstly, we show that the statements in Theorem 42 hold for $v \in \mathcal{A}_f$, $v \in \mathcal{C}^{2,\beta}(\overline{\Omega})$. The conclusion will hold for any $v \in \mathcal{A}_f$ by using Theorem 41 and a diagonal argument. By Lemma 40, for any $h > 0$, there exists $w_h \in \mathcal{C}^{2,\beta}(\overline{\Omega}, \mathbb{R}^3)$ such that:

$$(\nabla u_h)^T \nabla u_h = G^h(x', 0)_{2\times 2},$$

where:

$$u_h(x') = x' + h^{\gamma/2} v(x') e_3 + h^\gamma w_h(x'),$$

and $\sup_h \|w_h\|_{\mathcal{C}^{2,\beta}} < \infty$. Define the recovery sequence $u^h \in \mathcal{C}^{1,\beta}(\Omega^h, \mathbb{R}^3)$ by:

$$u^h(x', x_3) = u_h(x') + x_3 b^h(x') + \frac{1}{2} x_3^2 h^{\gamma/2} d^h,$$

where $b^h \in \mathcal{C}^{1,\beta}(\overline{\Omega}, \mathbb{R}^3)$ is defined so that:

$$[\partial_1 u_h \quad \partial_2 u_h \quad b^h]^T [\partial_1 u_h \quad \partial_2 u_h \quad b^h] = G^h(\cdot, 0).$$

More precisely, b^h is defined as follows:

$$b^h(x') = \alpha^h(x') N^h(x') + [\nabla u_h(x')] G^h(x', 0)^{-1}_{2\times 2} G^h(x', 0)_{13,23}.$$

Here N^h denotes the normal unit vector to the surface u_h:

$$N^h = \frac{\partial_1 u_h \times \partial_2 u_h}{|\partial_1 u_h \times \partial_2 u_h|} = \frac{-h^{\gamma/2}(\nabla v)^* + e_3 + \mathcal{O}(h^\gamma)}{\sqrt{1 + h^\gamma |\nabla v|^2 + \mathcal{O}(h^\gamma)}} = -h^{\gamma/2}(\nabla v)^* + e_3 + \mathcal{O}(h^\gamma), \tag{7.86}$$

and α^h is a smooth function satisfying:

$$(b^h)^T b^h = |G^h(\cdot, 0)_{13,23}|^2 + (\alpha^h)^2 = G^h(\cdot, 0)^2_{33}.$$

Moreover, the approximating warping sequence $d^h \in \mathcal{C}^{2,\beta}(\overline{\Omega}, \mathbb{R}^3)$ is defined by:

$$h^{\gamma/2} \|d_0^h\|_{\mathcal{C}^{1,\beta}} \leq C, \qquad d_0^h \to d \quad \text{in } L^\infty,$$

where $d \in \mathcal{C}^{0,\beta}(\overline{\Omega}, \mathbb{R}^3)$ verifies:

$$\mathcal{Q}_2(\nabla^2 v + \text{sym } B_g) = \mathcal{Q}_3\big[-(\nabla^2 v + \text{sym } B_g)^* + \text{sym } (d \otimes e_3)\big]. \tag{7.87}$$

Observe that the convergences in parts (1) and (2) hold for the sequence u^h. We now have that:

$$[\partial_1 u^h \quad \partial_2 u^h] = \nabla u_h + x_3 \nabla b^h + o(h^{1+\gamma/2}),$$

and:

$$\partial_3 u^h = b^h + x_3 h^{\gamma/2} d^h.$$

Therefore:

$$\begin{aligned}
[(\nabla u^h)^T \nabla u^h]_{tan} &= (\nabla u_h)^T \nabla u_h - 2x_3 h^{\gamma/2} (\nabla^2 v)^* + o(h^{1+\gamma/2}) \\
&= G^h(\cdot, 0)_{2\times 2} - 2x_3 h^{\gamma/2} (\nabla^2 v)^* + o(h^{1+\gamma/2}) \\
[(\nabla u^h)^T \nabla u^h]_{33} &= |b^h + x_3 h^{\gamma/2} d^h|^2 = (b^h)^T b^h + 2x_3 h^{\gamma/2} (d^h)_3 + o(h^{1+\gamma/2}) \\
&= G^h(\cdot, 0)_{33} + 2x_3 h^{\gamma/2} d_3^h + o(h^{1+\gamma/2}) \qquad (7.88) \\
[(\nabla u^h)^T \nabla u^h]_{13,23} &= (\nabla_{tan} u^h)^T (b^h + x_3 h^{\gamma/2} d^h) \\
&= [\nabla u_h + x_3 \nabla b^h]^T [b^h + x_3 h^{\gamma/2} d^h] + o(h^{1+\gamma/2}) \\
&= G^h(\cdot, 0)_{13,23} + h^{\gamma/2} x_3 d_{tan}^h + o(h^{1+\gamma/2}).
\end{aligned}$$

Consider now the strain:

$$F^h = (A^h)^{-1,T} (\nabla u^h)^T \nabla u^h (A^h)^{-1} = (A^h)^{-1,T} [(\nabla u^h)^T \nabla u^h - G^h] (A^h)^{-1}.$$

By the expansions in (7.88), we derive:

$$(\nabla u^h)^T \nabla u^h - G^h = G^h(\cdot, 0) - G^h + 2x_3 h^{\gamma/2} \text{sym } (-(\nabla^2 v)^* + d^h \otimes e_3) + o(h^{1+\gamma/2}). \quad (7.89)$$

Using that $G^h = G^h(\cdot, 0) + 2x_3 h^{\gamma/2} \text{sym } B_g + o(h^{1+\gamma/2})$, we obtain:

$$\begin{aligned}
(\nabla u^h)^T \nabla u^h - G^h &= 2x_3 h^{\gamma/2} \text{sym } (-(\nabla^2 v)^* - B_g + (d^h + l(B_g)) \otimes e_3) + o(h^{1+\gamma/2}) \\
&= 2x_3 h^{\gamma/2} \text{sym } (-(\nabla^2 v)^* - (B_g)_{2\times 2}^* + d^h \otimes e_3) + o(h^{1+\gamma/2}).
\end{aligned} \qquad (7.90)$$

Hence:

$$F^h = 2x_3 h^{\gamma/2} (A^h)^{-1,T} [\text{sym } (-(\nabla^2 v)^* - (B_g)_{2\times 2}^* + d^h \otimes e_3)] (A^h)^{-1} + o(h^{1+\gamma/2}).$$

Taylor expanding the energy W around Id gives:

$$\begin{aligned} W\big(\nabla u^h (A^h)^{-1}\big) &= W\big(\sqrt{Id + F^h}\big) = W\Big(\frac{1}{2}F^h + \mathcal{O}(|F^h|^2)\Big) \\ &= \frac{1}{2}\mathcal{Q}_3\Big(\frac{1}{2}F^h + \mathcal{O}(|F^h|^2)\Big) + \eta\Big(\frac{1}{2}F^h + \mathcal{O}(|F^h|^2)\Big). \end{aligned} \tag{7.91}$$

Observe that:

$$\mathcal{Q}_3\Big(\frac{1}{2}F^h + \mathcal{O}(|F^h|^2)\Big) = x_3^2 h^\gamma \mathcal{Q}_3\big((A^h)^{-1,T}[\text{sym } (-(\nabla^2 v)^* - (B_g)^*_{2\times 2} + d^h \otimes e_3)](A^h)^{-1} + o(1)\big),$$

and:

$$\eta\Big(\frac{1}{2}F^h + \mathcal{O}(|F^h|^2)\Big) \leq \omega(h)\Big|\frac{1}{2}F^h + \mathcal{O}(|F^h|^2\Big|^2,$$

where $w(h) \to 0$ as $|F^h| \to 0$. Hence, we conclude:

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h^{2+\gamma}} I_W^h(u^h) &= \int_{\Omega^h} W\big(\nabla u^h (A^h)^{-1}\big)dx \\ &= \frac{1}{2}\lim_{h\to 0}\frac{1}{h^{2+\gamma}}\Big(\fint_{-h/2}^{h/2} x_3^2 h^\gamma\Big)\int_\Omega \mathcal{Q}_3\big([\text{sym } ((\nabla^2 v)^* + (B_g)^*_{2\times 2} + d^h \otimes e_3)]\big) \\ &= \frac{1}{24}\int_\Omega \mathcal{Q}_2\big(\nabla^2 v + \text{sym } B_g\big)dx', \end{aligned}$$

where in the last equality, we have used (7.87). This proves that for every $v \in \mathcal{C}^{2,\beta}(\overline{\Omega})$, there exists a recovery sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ so that:

$$\lim_{h\to 0}\frac{1}{h^{2+\gamma}} I_W^h(u^h) = \mathcal{I}_g(v).$$

We now extend this conclusion to any $v \in \mathcal{A}_f$. Fix $v \in \mathcal{A}_f$ and let h_j be a sequence of positive real numbers converging to 0. From Theorem 41, there exists a sequence $v_j \in \mathcal{C}^{2,\beta}(\overline{\Omega}) \cap \mathcal{A}_f$ such that:

$$\|v_j - v\|_{W^{2,2}(\Omega)} \to 0, \quad \text{as } j \to \infty.$$

From the proof of the $\mathcal{C}^{2,\beta}$-case, for any j, there exists a deformation $u_j^{h_{n_j}} \in W^{1,2}(\Omega^{h_{n_j}}, \mathbb{R}^3)$ satisfying:

$$\|u_j^{h_{n_j}}(x', h_{n_j}x_3) - x'\|_{W^{1,2}(\Omega^1,\mathbb{R}^3)} < \frac{1}{j}, \quad \|V_j^{h_{n_j}} - (0,0,v_j)^T\|_{W^{1,2}(\Omega,\mathbb{R}^3)} < \frac{1}{j}, \tag{7.92}$$

where: $V_j(x') = h_{n_j}^{-\gamma/2} \fint_{-h_{n_j}/2}^{h_{n_j}/2} u_j^{h_{n_j}}(x', h_{n_j} x_3) - x' dx_3$, and:

$$\frac{1}{h_{n_j}^{2+\gamma}} I_W^{h_{n_j}}(u_j^{h_{n_j}}) - \mathcal{I}_g(v_j) < \frac{1}{j}. \tag{7.93}$$

By (7.92), we certainly get that the sequence $u_j^{h_{n_j}}$ satisfies the convergences in parts (1) and (2) of Theorem 42. In addition, by the L^2-convergence of $\nabla^2 v_j$ to $\nabla^2 v$, we obtain the convergence of $\mathcal{I}_g(v_j)$ to $\mathcal{I}_g(v)$, and so by (7.93), we can derive that:

$$\lim_{j\to\infty} \frac{1}{h_{n_j}^{2+\gamma}} I_W^{h_{n_j}}(u_j^{h_{n_j}}) = \mathcal{I}_g(v).$$

This ends the proof of Theorem 42.

□

7.3 ON THE UNIQUENESS OF THE MINIMIZERS TO THE MONGE-AMPÉRE TYPE ENERGY

In this section, we analyse the multiplicity of minimizers to the Monge-Ampére energy functional (7.10), in the following particular case: assume that the stored energy function $W : \mathcal{O} \subset \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}}_+$, where $\mathcal{O}$ is an open and bounded small neighborhood of $SO(3)$, is given by:

$$W(F) = \frac{1}{2}\mathrm{dist}^2(F, SO(3)), \quad F \in \mathcal{O}.$$

Now, for a matrix $F = Id + \epsilon A$, with small $\|A\|$, we get:

$$2W(F) = |\sqrt{(Id + \epsilon A)^T (Id + \epsilon A)} - Id|^2 = \epsilon^2 |\mathrm{sym}\ A|^2 + \mathcal{O}(\epsilon^3).$$

where in the last equality, we have used Taylor expansion. Therefore, the quadratic form $\mathcal{Q}_3$ is given by:

$$\mathcal{Q}_3(A) = |\mathrm{sym}\ A|^2.$$

This implies that the quadratic form $\mathcal{Q}_2$ (see (7.11)) is defined by:

$$\mathcal{Q}_2(F_{2\times 2}) = |\mathrm{sym}\ F_{2\times 2}|^2.$$

Given an open, bounded and simply connected domain Ω in $\mathbb{R}^2$, and a function $f \in L^1(\Omega)$, we consider the following Monge-Ampére type functional:

$$\mathcal{I}(v) = \int_\Omega |\nabla^2 v|^2 dx', \tag{7.94}$$

subject to the constraint:

$$\mathcal{A}_f = \{v \in W^{2,2}(\Omega) : \det \nabla^2 v = f\}. \tag{7.95}$$

Now the minimization problem for (7.94)-(7.95) may have multiple or unique solutions, depending on the choice of the function f.

Example 43. *Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Let $f = -1$. Let $v \in \mathcal{A}_f$. Then the quantity:*

$$|\nabla^2 v|^2 = (tr\ \nabla^2 v)^2 - 2det\ \nabla^2 v = (tr\ \nabla^2 v)^2 + 2,$$

is minimize when $tr\ \nabla^2 v = 0$. This is certainly satisfied with a Hessian of the form:

$$\nabla^2 v_\theta = \begin{pmatrix} cos\,\theta & sin\,\theta \\ sin\,\theta & -cos\,\theta \end{pmatrix}.$$

Therefore, the one-parameter family of functions:

$$v_\theta(x_1, x_2) = (cos\,\theta)\frac{x_1^2 - x_2^2}{2} + (sin\,\theta)(x_1 x_1),$$

constitutes a family of absolute minimizers for (7.94).

On the other hand, take the choice $f = 1$ in Ω. Then, for any $v \in \mathcal{A}_f$, we have:

$$|\nabla^2 v|^2 = (tr\ \nabla^2 v)^2 - 2det\ \nabla^2 v = (\lambda_1 + \lambda_2)^2 - 2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 - 2,$$

where λ_1 and λ_2 are the eigenvalues of the matrix $\nabla^2 v$. This quantity achieves its minimum, under the constraint $\lambda_1\lambda_2 = 1$, precisely when $\lambda_1 = \lambda_2 = 1$. Therefore, we have a unique minimizer:

$$v(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}.$$

We now want to derive conditions under which there is uniqueness for the minimizers to the problem (7.94)-(7.95). We start by defining the relaxed constraint:

$$\mathcal{A}_f^* = \{v \in W^{2,2}(\Omega) : \det \nabla^2 v \geq f\}.$$

We denote by $\mathcal{I}_f$ and $\mathcal{I}_f^*$ the restriction of the functional $\mathcal{I}$ to the constraints $\mathcal{A}_f$ and $\mathcal{A}_f^*$, respectively. It easily follows that:

$$\inf \mathcal{I}_f^* \leq \inf \mathcal{I}_f.$$

We first quote an existence result.

Lemma 44. *Assume that $\mathcal{A}_f \neq \emptyset$ ($\mathcal{A}_f^*$). Then the functional $\mathcal{I}_f$ ($\mathcal{I}_f^*$) admits a minimizer. Moreover, there must be $f \in L^1 log\, L^1(\Omega)$, that is:*

$$\int_{\Omega'} |f log\,(2+f)| < \infty.$$

Proof. Let $v_n \in \mathcal{A}_f$ be a minimizing sequence for the functional $\mathcal{I}_f$. It particular, it satisfies that:

$$\|v_n\|_{L^2(\Omega)} \leq C.$$

By modifying v_n by its average $\fint v_n$ and the affine function $(\fint \nabla v)x$, we obtain, by Poincaré inequality that:

$$\|v_n\|_{W^{2,2}(\Omega)} \leq C.$$

Therefore, v_n converges to $v \in W^{2,2}(\Omega)$ weakly. From the sequentially weak lower semicontinuity of $\mathcal{I}_f$, we hence obtain:

$$\mathcal{I}_f(v) \leq \liminf_{n\to\infty} \mathcal{I}_f(v_n). \tag{7.96}$$

We now prove that $v \in \mathcal{A}_f$. From the weak convergence of ∇v_n to ∇v in the space $W^{1,2}(\Omega)$, we obtain that:

$$\nabla v_n \to \nabla v, \quad \text{in } L^q(\Omega), \quad \text{for all } q \in [1,\infty).$$

This implies the convergence:

$$\nabla v_n \otimes \nabla v_n \to \nabla v \otimes \nabla v,$$

strongly in $L^2(\Omega)$. Applying $\mathrm{curl}^T\mathrm{curl}$, this yields the following convergence in the sense of distributions:

$$\det\, \nabla^2 v_n = -\mathrm{curl}^T\mathrm{curl}(\nabla v_n \otimes \nabla v_n) \to -\mathrm{curl}^T\mathrm{curl}(\nabla v \otimes \nabla v) = \det\, \nabla^2 v.$$

Therefore, $v \in \mathcal{A}_f$, which together with the inequality (7.96), show that v is a minimizer for the growth functional $\mathcal{I}_f$. A similar argument shows the existence of minimizers for $\mathcal{I}_f^*$.

The final assertion follows from the next result (see [42]): If $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ and it satisfies $\det\, \nabla u \geq 0$ in Ω, then $\det\, \nabla u \in L^1 log\, L^1(\Omega)$. □

We now prove the uniqueness of minimizers to the functional $\mathcal{I}_f^*$, up to affine functions. more precisely, we have the following result:

Theorem 45. *Assume that $f \geq c > 0$ in Ω. Let $v_1, v_2 \in \mathcal{A}_f^*$ be two minimizers of the functional $\mathcal{I}_f^*$. Then:*

$$\nabla^2 v_1 = \nabla^2 v_2.$$

In particular, the function:

$$\psi(f) = det\, \nabla^2(argmin\, \mathcal{I}_f^*) = det\, \nabla^2 v_1$$

is well defined and it satisfies:

$$\psi(f) \geq f, \quad and \quad \psi(f) \in L^1 log\, L^1(\Omega).$$

Proof. In the proof, we use the following result, which is Theorem 6.1 in [27]:

Theorem: Let $u \in W^{2,2}(\Omega)$ be such that:

$$\det\, \nabla^2 u = f\, in\, \Omega,$$

where $f : \Omega \to \mathbb{R}$ satisfies:

$$f(x) \geq c_0 > 0,\ a.e.\, x \in \Omega.$$

Then $u \in \mathcal{C}^1(\Omega)$, and modulo a global sign change, u is locally convex in Ω.

Therefore, without loss of generality, we may assume that $\nabla^2 v_1$ and $\nabla^2 v_2$ are strictly positive definite almost everywhere in the set Ω. For each $\lambda \in [0, 1]$, consider the convex combination:

$$v_\lambda = \lambda v_1 + (1 - \lambda) v_2.$$

By the Brunn-Minkowski inequality, we obtain:

$$(\det \nabla^2 v_\lambda)^{1/2} \geq \lambda (\det v_1)^{1/2} + (1 - \lambda)(\det \nabla^2 v_2)^{1/2} \geq \lambda \sqrt{f} + (1 - \lambda)\sqrt{f} = \sqrt{f}.$$

Therefore, $f \in \mathcal{I}_f^*$. We also have:

$$\mathcal{I}(v_\lambda) \leq \lambda \mathcal{I}(v_1) + (1 - \lambda)\mathcal{I}(v_2) = \inf \mathcal{I}_f^*.$$

Hence:

$$\mathcal{I}(v_\lambda) = \min \mathcal{I}_f^*.$$

Since the L^2-norm is a strictly convex function, we derive that $\nabla^2 v_1 = \nabla^2 v_2$. □

7.3.1 The radially symmetric case

In this section, we assume that $\Omega = B(0, 1) \subset \mathbb{R}^2$, and that:

$$f = f(r) \geq c > 0.$$

We also assume that $f \in L^1(\Omega)$:

$$\int_0^1 r f(r) dr < \infty.$$

We then have the following theorem regarding the uniqueness (up to affine maps) of minimizers in the radially symmetric case.

Theorem 46. *Assume that $\mathcal{A}_f^* \neq \emptyset$, and that f is a. e. nonincreasing: for a. e. $r \in [0, 1]$, for a. e. $x \in [0, r]$, we assume that:*

$$f(r) \leq f(x).$$

Then both problems $\mathcal{I}_f$ and $\mathcal{I}_f^*$ has a unique, modulo affine maps, minimizer. Moreover, the minimizer is common to both problems, necessarily radially symmetric, and it is given by:

$$v_f(r) = \int_0^r \Big(\int_0^s 2t f(t) dt \Big)^{1/2} ds.$$

We start by proving some preliminary results.

Lemma 47. *If a radial function $v = v(r) \in W^{2,2}(\Omega)$ satisfies:*

$$det\ \nabla^2 v = f,$$

then:

$$|v'(r)|^2 = \int_0^r 2sf(s)ds.$$

In particular, there exists at most one, modulo a constant, radial function $v = v_f$ satisfying the constraint $det\ \nabla^2 v = f$.

Proof. Writing the derivative: $\partial_r v = v'$, we have that the gradient of v in polar coordinates is given by:

$$\nabla v(r, \theta) = (v'(r) cos\, \theta, v'(r) sin\, \theta).$$

Then:

$$\det\ \nabla^2 v = \frac{1}{r} v' v'' = \frac{1}{2r}(|v'|^2)'. \tag{7.97}$$

Therefore:

$$|v'(r)|^2 = \int_0^r 2sf(s)ds + C,$$

where $C \geq 0$. We now prove that $C = 0$. First, observe that:

$$\Delta v = v'' + \frac{1}{r} v' \in L^2(\Omega), \tag{7.98}$$

in view of $v \in W^{2,2}(\Omega)$. By (7.98) and (7.97), we derive:

$$\int_0^1 \frac{2\pi C}{r} dr < 2\pi \int_0^1 \frac{1}{r} |v'(r)|^2 dr = \int_\Omega \frac{1}{r^2} |v'|^2 < \infty.$$

Hence, $C = 0$. □

We now derive a necessary and sufficient condition for the existence of a radial solution of $\det\ \nabla^2 v = f$.

Corollary 48. *A necessary and sufficient condition for the existence of a radial function $v = v(r) \in W^{2,2}(\Omega)$, solving the equation det $\nabla^2 v = f$, is:*

$$\int_0^1 r|log\, r|f(r)dr < \infty, \quad and \quad \int_0^1 \frac{r^3 f(r)^2}{\int_0^r sf(s)ds} dr < \infty. \tag{7.99}$$

The solution v_f is then given by:

$$v_f(r) = \int_0^r \Big(\int_0^s 2tf(t)dt \Big)^{1/2} ds. \tag{7.100}$$

In particular, (7.99) is satisfied for $f \in L^2(\Omega)$, and so $\mathcal{A}_f \neq \emptyset$.

Proof. By Lemma 47, the solution v is given by the expression (7.100). Also $\nabla v_f \in \mathcal{C}^1(\overline{\Omega})$. To check if effectively $\nabla^2 v_f \in L^2(\Omega)$, we now compute:

$$\begin{aligned} \int_\Omega |\nabla^2 v_f|^2 &= \int_\Omega |v_f''|^2 + \frac{1}{r^2}|v_f'|^2 = 2\pi \int_0^1 r|v_f''|^2 + \frac{|v_f'|^2}{r} dr \\ &= 2\pi \int_0^1 \frac{r^3 f(r)^2}{2\int_0^r sf(s)ds} dr + 2\pi \int_0^1 2r|log\, r|f(r)dr. \end{aligned} \tag{7.101}$$

This proves the first claim in the Corollary. In particular, when $f \in L^2(\Omega)$, the two terms in the last equality are finite, and then $\mathcal{A}_f \neq \emptyset$. □

Lemma 49. *1. Assume that $\mathcal{A}_f^* \neq \emptyset$. Then the unique, modulo affine maps, minimizer of $\mathcal{I}_f^*$ is radially symmetric, given by $v_{\psi(f)}$.*

2. Assume that $\mathcal{I}_f$ has a unique, up to affine maps, minimizer. Then, it is radially symmetric and hence given by the expression (7.100).

Proof. We prove (2). Let $v \in \mathcal{A}_f$ be the unique minimizer of $\mathcal{I}_f$. We modify it so that $v(0) = 0$, and:

$$⨍ v = 0.$$

For any parameter $\theta \in [0, 2\pi)$, let the rotation matrix:

$$R_\theta = \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}.$$

Then:

$$\nabla^2(v \circ R_\theta) = R_\theta^T((\nabla^2 v) \circ R_\theta)R_\theta,$$

hence:

$$\det\, \nabla^2(v \circ R_\theta) = \det\, (\nabla^2 v \circ R_\theta).$$

Since f is radially symmetric, we have that $v \circ R_\theta \in \mathcal{A}_f^*$, and, moreover, $\mathcal{I}(v \circ R_\theta) = \mathcal{I}(v)$. Therefore, by the assumption of uniqueness, we derive:

$$v = v \circ R_\theta.$$

So, v is radially symmetric, and then the result follows by Corollary 48.

The proof of part 1 is similar to the above proof. Observe that $\psi(f)$ satisfies the conditions (7.99) in view of Theorem 45. □

We now can prove Theorem 46:

Proof. Let $v_{\psi(f)}$ be the unique minimizer of $\mathcal{I}_f^*$. Let v_f given by (7.100). Since $\psi \geq f$, and:

$$\int_0^r 2sf(s)ds \geq r^2 f(r)$$

in view of that f is nonincreasing, we obtain:

$$\begin{aligned}
\mathcal{I}(v_\psi) - \mathcal{I}(v_f) &= 2\pi \int_0^1 \frac{r^3\psi(r)^2}{\int_0^r 2s\psi(s)ds} - \frac{r^3 f(r)^2}{\int_0^r 2sf(s)ds} dr + 2\pi \int_0^1 \frac{\int_0^1 2s(\psi - f)ds}{r} dr \\
&\geq -2\pi \int_0^1 \frac{r^3 f^2 \int_0^r 2s(f-\psi)ds}{(\int_0^r 2sf(s)ds)(\int_0^r 2s\psi(s)ds)} 2\pi \int_0^1 \frac{\int_0^1 2s(\psi - f)ds}{r} dr \\
&\geq -2\pi \int_0^1 \frac{r^3 f^2 \int_0^r 2s(f-\psi)ds}{(\int_0^r 2sf(s)ds)^2} + 2\pi \int_0^1 \frac{\int_0^1 2s(\psi - f)ds}{r} dr \\
&\geq -2\pi \int_0^1 \frac{r^3 f^2 \int_0^r 2s(f-\psi)ds}{(r^2 f(r))^2} + 2\pi \int_0^1 \frac{\int_0^1 2s(\psi - f)ds}{r} dr = 0.
\end{aligned} \tag{7.102}$$

Therefore:

$$\mathcal{I}(v_f) \leq \mathcal{I}(v_\psi).$$

By poincaré inequality, $v_f \in W^{2,2}(\Omega)$ and, by the uniqueness of minimizers, $v_f = v_\psi$.

□

8.0 FROM DISCRETE TO CONTINUUM MODELS

In this chapter, we analyse the problem of how to rigorously derive the continuum energy model $\mathcal{E}$ (6.3) from an appropriate atomistic description. The derivation of continuum models from discrete models by using Γ-convergence has been used in many articles such as [1], [2], [3], [18], [20], [21], [26], [41], [46], [48], among others, and in [31], [30], [32], [28], [29] for the particular context of non-Euclidean elasticity.

We start with an open, connected and bounded subset Ω of $\mathbb{R}^n$ considered as reference configuration, and equipped with a given smooth Riemannian metric $G \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^{n\times n})$. We then propose a discrete model E_ϵ (8.1) comparing the displacements of points in the discrete reference configuration $\epsilon\mathbb{Z}^n \cap \Omega$, due to the deformation field, and the ideal displacement lengths prescribed by the Riemannian metric G. In this way, the discrete models E_ϵ measure the discrepancy between the lengths of the displacements of the atoms of the initial state under the action of the deformation mapping u, that is:

$$|u(\alpha) - u(\beta)|, \quad \alpha,\, \beta \in \epsilon\mathbb{Z}^n \cap \Omega,$$

and the ideal displacement field given by the metric G:

$$G(\alpha)(\alpha-\beta)\cdot(\alpha-\beta) = A^T(\alpha)A(\alpha)(\alpha-\beta)\cdot(\alpha-\beta) = |A(\alpha)(\alpha-\beta)|^2.$$

where, we call $A = \sqrt{G}$, and without loss of generality we assume that A is symmetric and strictly positive definite in Ω.

The variational limit of the energy models E_ϵ is studied in the context of Γ-convergence, as the mesh parameter ϵ tends to 0.

The atomistic model proposed here follows the spirit of the models considered in [1], [41], [26], where the Euclidean case $G = Id$ was analysed. In these works, it has been proved that the proposed atomistic models Γ-converge to a continuum integral functional $\mathcal{F}$ of the form:

$$\mathcal{F}(u) = \int_{\Omega} f(\nabla u)dx, \quad u \in W^{1,2}(\Omega, \mathbb{R}^{n\times n}),$$

where the limiting density f is frame invariant with respect to the group of proper rotations and quasiconvex. Our main goal in this chapter is to analyse the variational behavior of the proposed atomistic functionals, deducing upper and lower bounds for their continuum limit $\mathcal{F}$, and to determine, whenever is possible, the exact integral expression of $\mathcal{F}$.

8.1 THE DISCRETE MODEL E_ϵ

We now describe the discrete model whose asymptotic behavior we intend to study. The total stored discrete energy of a given deformation acting on the atoms of the lattice in Ω, is defined to be the superposition of the energies weighting the pairwise interactions between the atoms, with respect to G. More precisely, given $\epsilon > 0$ and a discrete map $u_\epsilon : \epsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}^n$, let:

$$E_\epsilon(u_\epsilon) = \sum_{\xi\in\mathbb{Z}^n} \sum_{\alpha\in R^\xi_\epsilon(\Omega)} \epsilon^n \psi(|\xi|)\Big|\frac{|u_\epsilon(\alpha+\epsilon\xi) - u_\epsilon(\alpha)|}{\epsilon|A(\alpha)\xi|} - 1\Big|^2, \tag{8.1}$$

where

$$R^\xi_\epsilon(\Omega) = \{\alpha \in \epsilon\mathbb{Z}^n : [\alpha, \alpha+\epsilon\xi] \subset \Omega\}$$

denotes the set of lattice points in Ω interacting with the node α, and where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a smooth interaction potential with finite range:

$$\psi(0) = 0 \quad \text{and} \quad \exists M > 0 \quad \forall n \geq M \quad \psi(n) = 0.$$

The energy in (8.1) measures the discrepancy between lengths of the actual displacements and the ideal displacement length in the sense explained above. When $\epsilon \to 0$ and when sampling on sufficiently many interaction directions ξ, one might expect that (8.1) will

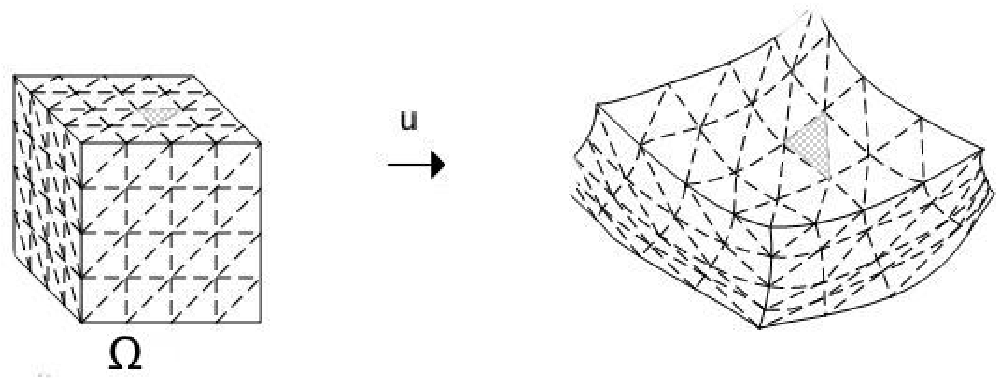

Figure 5: The standard triangulation of a three-dimensional set Ω and the action of a deformation u.

effectively measure the discrepancy between all lengths $|u(x) - u(y)|$ and the ideal lengths $|A(x)(x - y)|$ determined by the imposed metric.

Towards studying the energies (8.1), we first derive an integral representation for E_ϵ by introducing a family of lattices determined by each length of the admissible interactions (when $\psi \neq 0$); this is done in sections 8.2 and 8.3. Since the general formula for the integral representation uses quite involved notation, we first present its simpler versions, valid in cases of the near and next-to-near interactions. For each lattice, we define its n-dimensional triangulation and, as usual in the lattice analysis, we associate with it the piecewise affine maps matching with the original discrete deformations at each node. See Figure 5.

In section 8.4 we derive the lower and upper bounds I_Q and I of the Γ-limit $\mathcal{F}$ of E_ϵ, as $\epsilon \to 0$, in terms of the superposition of integral energies defined effectively on the $W^{1,2}$ deformations of Ω. The disparity between the upper and lower bounds reflects the fact that each lattice in the discrete description gives raise, in general, to a distinct recovery sequence of the associated Γ-limit.

On the other hand, each term in I_Q and I has the structure as in (6.3), but with G replaced by other effective metric induced by the distinct lattices. In case of only near or next-to-near interactions all the effective metrics coincide with one residual metric $\bar{G}$. This further allows to obtain the formula for $\mathcal{F}$, which is accomplished in section 8.5. In section 8.6 we compare $\mathcal{F}$ with $\mathcal{E}$ through a series of examples. The main observation here is that the realisability of G does not imply the realisability of $\bar{G}$, neither the converse of this statement is true.

8.1.1 Notation

Throughout this section, Ω is an open bounded subset of $\mathbb{R}^n$. For $s > 0$, we denote:

$$\Omega_s = \{x \in \Omega; \quad \mathrm{dist}(x, \partial\Omega) > s\}.$$

The standard triangulation of the n-dimensional cube $C_n = [0,1]^n$ is defined as follows (see Figure 6). For all permutations $\pi \in S_n$ of n elements, let T^π be the n-simplex obtained by:

$$T^\pi = \{(x_1, ..., x_n) \in C_n; \quad x_{\pi(1)} \geq \cdots \geq x_{\pi(n)}\}.$$

Note that T^π is the convexification of its vertices:

$$T^\pi = \mathrm{conv}\Big\{0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \ldots, e_{\pi(1)} + \cdots + e_{\pi(n)} = e_1 + \cdots + e_n\Big\},$$

and that all simplices T^π have 0 and $(1, \ldots, 1) = e_1 + \ldots + e_n$ as common vertices. The collection of $n!$ simplices $\{T^\pi\}_{\pi \in S_n}$ constitutes the standard triangulation of C_n, which can also be naturally extended to each cell $\alpha + \epsilon C_n$ where $\alpha \in \epsilon\mathbb{Z}^n$:

$$T^\pi_\alpha = \mathrm{conv}\Big\{\alpha, \big\{\alpha + \epsilon \sum_{i=1}^{j} e_{\pi(i)}\big\}_{j=1}^{n}\Big\}.$$

When $\pi = (i_1, \ldots, i_n)$ we shall also write $T_\alpha^{(i_1,\ldots,i_n)} = T_\alpha^\pi = \mathrm{conv}\Big\{\alpha, \big\{\alpha + \epsilon \sum_{k=1}^{j} e_{i_k}\big\}_{j=1}^{n}\Big\}$. Moreover, we call:

$$\mathcal{T}_{\epsilon,n} = \{T^\pi_\alpha; \quad \alpha \in \epsilon\mathbb{Z}^n, \ \pi \in S_n\}. \tag{8.2}$$

Finally, by C we denote any universal constant, depending on Ω and W, but independent of other involved quantities at hand.

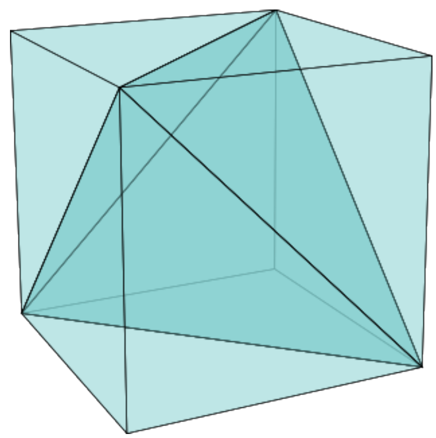

Figure 6: Standard triangulation of the unit cube

8.2 INTEGRAL REPRESENTATION OF DISCRETE ENERGIES (3.1) - SPECIAL CASES

Since the general formula for the integral representation of E_ϵ, given in section 8.3, uses a somewhat involved notation which may obscure the construction, we first present its simpler versions, valid in cases of the near and next-to-near interactions, which we further discuss in sections 8.5 and 8.6.

8.2.1 Case 1: near interactions in $\mathbb{R}^2$

Let $\Omega \subset \mathbb{R}^2$ and assume that $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\psi(1) = 1$ and $\psi(|\xi|) = 0$ for $|\xi| \geq \sqrt{2}$. The energy (8.1) of a deformation $u_\epsilon : \epsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}^2$, takes then the form:

$$E_\epsilon(u_\epsilon) = \sum_{i,j=1}^{2} \sum_{\alpha \in R_\epsilon^{(-1)^j e_i}(\Omega)} \epsilon^2 \left| \frac{|u_\epsilon(\alpha + (-1)^j \epsilon e_i) - u_\epsilon(\alpha)|}{\epsilon |A(\alpha) e_i|} - 1 \right|^2.$$

Let $U_\epsilon \subset \Omega$ be the union of those (open) cells in the lattice $\epsilon\mathbb{Z}^2$, which have non-empty intersection with the set $\Omega_{\sqrt{2}\epsilon}$. (See Figure 7). We consider the standard triangulation $\mathcal{T}_{\epsilon,2}$ of the lattice $\epsilon\mathbb{Z}^2$, as in (8.2), and we identify the discrete map u_ϵ with the unique continuous

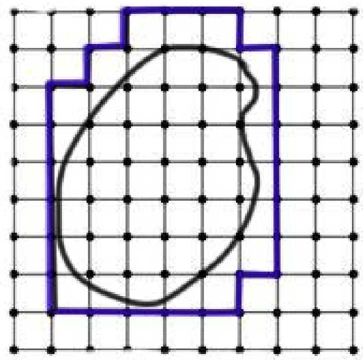

Figure 7: The sets U_ϵ (in blue) and the set $\Omega_{\sqrt{2}\epsilon}$ (in black).

function on U_ϵ, affine on all the triangles in $\mathcal{T}_{\epsilon,2} \cap U_\epsilon$, and matching with u_ϵ at each node. Define the function $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$:

$$W([M_{ij}]_{i,j=1..2}) = \sum_{j=1}^{2} \left(\Big(\sum_{i=1}^{2} |M_{ij}|^2 \Big)^{1/2} - 1 \right)^2 \qquad \forall M \in \mathbb{R}^{2\times 2}.$$

We easily see that for every $\alpha \in \epsilon\mathbb{Z}^2 \cap U_\epsilon$:

$$\begin{aligned} \epsilon^2 \bigg(\Big| \frac{|u_\epsilon(\alpha + \epsilon e_1) - u_\epsilon(\alpha)|}{\epsilon |A(\alpha) e_1|} - 1 \Big|^2 + \Big| \frac{|u_\epsilon(\alpha + \epsilon(e_1 + e_2) - u_\epsilon(\alpha + \epsilon e_1)|}{\epsilon |A(\alpha + \epsilon e_1) e_2|} - 1 \Big|^2 \bigg) \\ = 2 \int_{T_\alpha^{(1,2)}} W(\nabla u_\epsilon(x) \lambda_\epsilon(x)) \, \mathrm{d}x, \end{aligned}$$

where $\lambda_\epsilon : U_\epsilon \to \mathbb{R}^{2\times 2}$ is a piecewise constant matrix field, given by:

$$\begin{aligned} \forall x \in T_\alpha^{(1,2)} \cap U_\epsilon, \qquad & \lambda_\epsilon(x) = \operatorname{diag}\left\{ |A(\alpha) e_1|^{-1}, |A(\alpha + \epsilon e_1) e_2|^{-1} \right\} \\ \forall x \in T_\alpha^{(2,1)} \cap U_\epsilon, \qquad & \lambda_\epsilon(x) = \operatorname{diag}\left\{ |A(\alpha + \epsilon e_2) e_1|^{-1}, |A(\alpha) e_2|^{-1} \right\} \end{aligned}$$

while we recall that $T_\alpha^{(1,2)} = \operatorname{conv}\{\alpha, \alpha + \epsilon e_1, \alpha + \epsilon(e_1 + e_2)\}$ and $T_\alpha^{(2,1)} = \operatorname{conv}\{\alpha, \alpha + \epsilon e_2, \alpha + \epsilon(e_1 + e_2)\}$. Similarly, we get:

$$\begin{aligned} \epsilon^2 \bigg(\Big| \frac{|u_\epsilon(\alpha + \epsilon e_2) - u_\epsilon(\alpha)|}{\epsilon |A(\alpha) e_2|} - 1 \Big|^2 + \Big| \frac{|u_\epsilon(\alpha + \epsilon(e_1 + e_2)) - u_\epsilon(\alpha + \epsilon e_2)|}{\epsilon |A(\alpha + \epsilon e_2) e_1|} - 1 \Big|^2 \bigg) \\ = 2 \int_{T_\alpha^{(2,1)}} W(\nabla u_\epsilon(x) \lambda_\epsilon(x)) \, \mathrm{d}x. \end{aligned}$$

For the interactions in the opposite directions: $-e_1$ and $-e_2$, we obtain:

$$\epsilon^2 \left(\Big| \frac{|u_\epsilon(\alpha + \epsilon e_1) - u_\epsilon(\alpha + \epsilon(e_1 + e_2))|}{\epsilon |A(\alpha + \epsilon(e_1 + e_2))e_2|} - 1 \Big|^2 + \Big| \frac{|u_\epsilon(\alpha) - u_\epsilon(\alpha + \epsilon e_1)|}{\epsilon |A(\alpha + \epsilon e_1)e_1|} - 1 \Big|^2 \right)$$
$$= 2 \int_{T_\alpha^{(1,2)}} W(\nabla u_\epsilon(x) \bar{\lambda}_\epsilon(x)) \, \mathrm{d}x,$$

and:

$$\epsilon^2 \left(\Big| \frac{|u_\epsilon(\alpha + \epsilon e_2) - u_\epsilon(\alpha + \epsilon(e_1 + e_2))|}{\epsilon |A(\alpha + \epsilon(e_1 + e_2))e_1|} - 1 \Big|^2 + \Big| \frac{|u_\epsilon(\alpha) - u_\epsilon(\alpha + \epsilon e_2)|}{\epsilon |A(\alpha + \epsilon e_2)e_2|} - 1 \Big|^2 \right)$$
$$= 2 \int_{T_\alpha^{(2,1)}} W(\nabla u_\epsilon(x) \bar{\lambda}_\epsilon(x)) \, \mathrm{d}x,$$

where $\bar{\lambda}_\epsilon : U_\epsilon \to \mathbb{R}^{2\times 2}$ is given by:

$$\forall x \in T_\alpha^{(1,2)} \cap U_\epsilon, \qquad \bar{\lambda}_\epsilon(x) = \mathrm{diag}\left\{ |A(\alpha + \epsilon e_1)e_1|^{-1}, |A(\alpha + \epsilon(e_1 + e_2))e_2|^{-1} \right\}$$
$$\forall x \in T_\alpha^{(2,1)} \cap U_\epsilon, \qquad \bar{\lambda}_\epsilon(x) = \mathrm{diag}\left\{ |A(\alpha + \epsilon(e_1 + e_2))e_1|^{-1}, |A(\alpha + \epsilon e_2)e_2|^{-1} \right\}$$

Summing over all 2-simplices and noting that each interaction was counted twice, we obtain:

$$0 \leq E_\epsilon(u_\epsilon) - I_{\epsilon,1}(u_\epsilon) \leq \sum_{i,j=1}^{2} \sum_{\alpha \in R_\epsilon^{(-1)^j e_i}(\overline{\Omega \setminus U_\epsilon})} \epsilon^2 \Big| \frac{|u_\epsilon(\alpha + \epsilon(-1)^j \epsilon e_i) - u_\epsilon(\alpha)|}{\epsilon |A(\alpha)e_i|} - 1 \Big|^2, \tag{8.3}$$

where:

$$I_{\epsilon,1}(u_\epsilon) = \int_{U_\epsilon} \Big(W(\nabla u_\epsilon(x)\lambda_\epsilon(x)) + W(\nabla u_\epsilon(x)\overline{\lambda}_\epsilon(x)) \Big) \, \mathrm{d}x. \tag{8.4}$$

8.2.2 Case 2: near interactions in $\mathbb{R}^n$

Let now $\Omega \subset \mathbb{R}^n$, and assume that $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, satisfies $\psi(1) = 1$ and $\psi(|\xi|) = 0$ for $|\xi| \geq \sqrt{n}$. For small $\epsilon > 0$, define $U_\epsilon \subset \Omega$ as the union of all cells in $\epsilon\mathbb{Z}^n$, with the standard triangulation $\mathcal{T}_{\epsilon,n}$, that have nonempty intersection with $\Omega_{\epsilon\sqrt{n}}$. As in Case 1, we identify the given discrete deformation $u_\epsilon : \epsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}^n$ with its unique extension to the continuous function on U_ϵ, affine on all of the n-dimensional simplices in $\mathcal{T}_{\epsilon,n} \cap U_\epsilon$.

We also have $W : \mathbb{R}^{n\times n} \to \mathbb{R}$:

$$W([M_{ij}]_{i,j:1..n}) = \sum_{i=1}^n \left((\sum_{i=1}^n |M_{ij}|^2)^{1/2} - 1 \right)^2 \qquad \forall M \in \mathbb{R}^{n\times n}. \tag{8.5}$$

Note that for any permutation $\pi \in S_n$ one has:

$$\begin{aligned} &\epsilon^n \sum_{j=0}^{n-1} \Big| \frac{|u_\epsilon(\alpha + \epsilon\sum_{i=1}^{j+1} e_{\pi(i)}) - u_\epsilon(\alpha + \epsilon\sum_{i=1}^{j} e_{\pi(i)})|}{\epsilon|A(\alpha + \epsilon\sum_{i=1}^{j} e_{\pi(i)})e_{\pi(j+1)}|} - 1\Big|^2 \\ &\qquad = n! \int_{T_\alpha^\pi} W(\nabla u_\epsilon(x)\lambda_\epsilon(x))\,\mathrm{d}x, \end{aligned}$$

where the piecewise constant matrix field λ_ϵ is given by:

$$\forall x \in T_\alpha^\pi \cap U_\epsilon, \qquad \lambda_\epsilon(x) = \mathrm{diag}\left\{ |A(\alpha + \epsilon \sum_{i=1}^{\pi^{-1}(j)-1} e_{\pi(i)})e_j|^{-1} \right\}_{j=1}^n. \tag{8.6}$$

To include the interactions in $\{-e_i\}$ directions, as before, we write:

$$\begin{aligned} &\epsilon^n \sum_{j=0}^{n-1} \Big| \frac{|u_\epsilon(\alpha + \epsilon\sum_{i=1}^{j} e_{\pi(i)}) - u_\epsilon(\alpha + \epsilon\sum_{i=1}^{j+1} e_{\pi(i)})|}{\epsilon|A(\alpha + \epsilon\sum_{i=1}^{j+1} e_{\pi(i)})e_{\pi(j+1)}|} - 1\Big|^2 \\ &\qquad = n! \int_{T_\alpha^\pi} W(\nabla u_\epsilon(x)\bar\lambda_\epsilon(x))\,\mathrm{d}x, \end{aligned}$$

where:

$$\forall x \in T_\alpha^\pi \cap U_\epsilon, \qquad \bar\lambda_\epsilon(x) = \mathrm{diag}\left\{ |A(\alpha + \epsilon \sum_{i=1}^{\pi^{-1}(j)} e_{\pi(i)})e_j|^{-1} \right\}_{j=1}^n. \tag{8.7}$$

Summing over all of the n-simplices, and noting that each one-length interaction is counted $n!$ times, we obtain:

$$0 \leq E_\epsilon(u_\epsilon) - I_{\epsilon,1}(u_\epsilon) \leq \sum_{|\xi|=1} \sum_{\alpha \in R_\epsilon^\xi(\overline{\Omega\setminus U_\epsilon})} \epsilon^n \Big| \frac{|u_\epsilon(\alpha+\epsilon\xi) - u_\epsilon(\alpha)|}{\epsilon|A(\alpha)\xi|} - 1\Big|^2, \tag{8.8}$$

where $I_{\epsilon,1}$ is given by the same formula as in (8.4), with λ_ϵ and $\bar\lambda_\epsilon$ defined as in (8.6), (8.7).

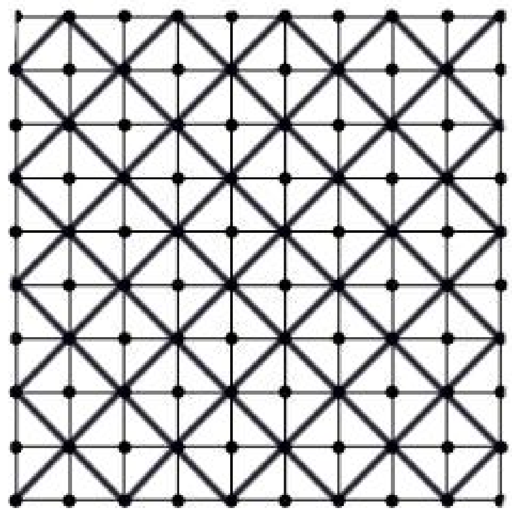

Figure 8: The lattices $\epsilon\mathbb{Z}^2$ and $\epsilon B\mathbb{Z}^2$.

8.2.3 Case 3: next-to-near interactions in $\mathbb{R}^2$

Let us assume now again that $\Omega \subset \mathbb{R}^2$, and that $\psi(\sqrt{2}) = 1$ and $\psi(|\xi|) = 0$ for $|\xi| \geq \sqrt{3}$ and $|\xi| \leq 1$. Our goal now is to obtain a similar representation and bound to (8.3) (8.4) for the discrete energy corresponding to the next-to-near interactions of length $\sqrt{2}$. The canonical lattice $\epsilon\mathbb{Z}^2$ is now mapped onto the lattice $\epsilon B\mathbb{Z}^2$, Figure 8, where:

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We will also need to work with the translated lattice $\epsilon(e_1 + B\mathbb{Z}^2)$. Let $U^0_{\epsilon,\sqrt{2}} \subset \Omega$ be the union of all open cells in the lattice $\epsilon B\mathbb{Z}^2$ which have nonempty intersection with $\Omega_{2\epsilon}$. Define $u^0_{\epsilon,\sqrt{2}}$ to be the unique continuous function on $U^0_{\epsilon,\sqrt{2}}$, affine on the triangles of the induced triangulation $B\mathcal{T}_{\epsilon,2} \cap U^0_{\epsilon,\sqrt{2}}$, matching with the original deformation u_ϵ at each node of the lattice $\epsilon B\mathbb{Z}^2 \cap U^0_{\epsilon,\sqrt{2}}$. Likewise, by $U^1_{\epsilon,\sqrt{2}} \subset \Omega$ we call the union of cells in the lattice $\epsilon(e_1 + B\mathbb{Z}^2)$ which have nonempty intersection with $\Omega_{2\epsilon}$, while $u^1_{\epsilon,\sqrt{2}}$ is the matching continuous piecewise affine (on triangles in $\epsilon e_1 + B\mathcal{T}_{\epsilon,2}$) extension of u_ϵ.

Denoting $\xi_1 = Be_1$ and $\xi_2 = Be_2$ we obtain, as before:

$$\epsilon^2\Big(\Big|\frac{|u_\epsilon(B(\alpha+\epsilon e_1)) - u_\epsilon(B\alpha)|}{\epsilon|A(B\alpha)\xi_1|} - 1\Big|^2 + \Big|\frac{|u_\epsilon(B(\alpha+\epsilon(e_1+e_2))) - u_\epsilon(B(\alpha+\epsilon e_1))|}{\epsilon|A(B(\alpha+\epsilon e_1))\xi_2|} - 1\Big|^2\Big)$$
$$= \frac{2}{|\det B|}\int_{BT_\alpha^{(1,2)}} W(\nabla u^0_{\epsilon,\sqrt{2}}(x)\lambda^0_{\epsilon,\sqrt{2}}(x))\ \mathrm{d}x,$$

where $\lambda^0_{\epsilon,\sqrt{2}} : U^0_{\epsilon,\sqrt{2}} \to \mathbb{R}^{2\times 2}$ is given by:

$$\forall x \in BT_\alpha^{(1,2)} \cap U^1_{\epsilon,\sqrt{2}} \qquad \lambda^0_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\mathrm{diag}\left\{|A(B\alpha)\xi_1|^{-1}, |A(B(\alpha+\epsilon e_1))\xi_2|^{-1}\right\}$$
$$\forall x \in BT_\alpha^{(2,1)} \cap U^1_{\epsilon,\sqrt{2}} \qquad \lambda^0_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\mathrm{diag}\left\{|A(B(\alpha+\epsilon e_2))\xi_1|^{-1}, |A(B\alpha)\xi_2|^{-1}\right\}.$$

Interactions in the opposite directions $-\xi_i$, yield the integrals:

$$\frac{2}{|\det B|}\int_{BT_\alpha^{1,2}} W(\nabla u^0_{\epsilon,\sqrt{2}}(x)\bar{\lambda}^0_{\epsilon,\sqrt{2}}(x))\ \mathrm{d}x,$$

where now $\bar{\lambda}^0_{\epsilon,\sqrt{2}} : U^1_{\epsilon,\sqrt{2}} \to \mathbb{R}^{2\times 2}$ satisfies:

$$\forall x \in BT_\alpha^{(1,2)} \cap U^1_{\epsilon,\sqrt{2}}$$
$$\bar{\lambda}^0_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\ \mathrm{diag}\left\{|A(B(\alpha+\epsilon e_1))\xi_1|^{-1}, |A(B(\alpha+\epsilon(e_1+e_2)))\xi_2|^{-1}\right\},$$
$$\forall x \in BT_\alpha^{(2,1)} \cap U^1_{\epsilon,\sqrt{2}}$$
$$\bar{\lambda}^0_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\ \mathrm{diag}\left\{|A(B(\alpha+\epsilon(e_1+e_2)))\xi_1|^{-1}, |A(B(\alpha+\epsilon e_2))\xi_2|^{-1}\right\}.$$

Similarly, we obtain the integral representations on the triangulation $\epsilon e_1 + B\mathcal{T}_{\epsilon,2}$ of the set $U^1_{\epsilon,\sqrt{2}}$:

$$\int W(\nabla u^1_{\epsilon,\sqrt{2}}(x)\lambda^1_{\epsilon,\sqrt{2}}(x))\ \mathrm{d}x \qquad \text{and} \qquad \int W(\nabla u^1_{\epsilon,\sqrt{2}}(x)\lambda^1_{\epsilon,\sqrt{2}}(x))\ \mathrm{d}x,$$

with the piecewise affine functions:

$$\forall x \in (\epsilon e_1 + BT_\alpha^{(1,2)}) \cap U^1_{\epsilon,\sqrt{2}}$$
$$\lambda^1_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\ \mathrm{diag}\left\{|A(\epsilon e_1 + B\alpha)\xi_1|^{-1}, |A(\epsilon e_1 + B(\alpha+\epsilon e_1))\xi_2|^{-1}\right\}$$
$$\forall x \in (\epsilon e_1 + BT_\alpha^{(2,1)}) \cap U^1_{\epsilon,\sqrt{2}}$$
$$\lambda^1_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\ \mathrm{diag}\left\{|A(\epsilon e_1 + B(\alpha+\epsilon e_2))\xi_1|^{-1}, |A(\epsilon e_1 + B\alpha)\xi_2|^{-1}\right\}$$
$$\forall x \in (\epsilon e_1 + BT_\alpha^{(1,2)}) \cap U^1_{\epsilon,\sqrt{2}}$$
$$\bar{\lambda}^1_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\ \mathrm{diag}\left\{|A(\epsilon e_1 + B(\alpha+\epsilon e_1))\xi_1|^{-1}, |A(\epsilon e_1 + B(\alpha+\epsilon(e_1+e_2)))\xi_2|^{-1}\right\}$$
$$\forall x \in (\epsilon e_1 + BT_\alpha^{(2,1)}) \cap U^2_{\epsilon,\sqrt{2}}$$
$$\bar{\lambda}^1_{\epsilon,\sqrt{2}}(x) = \sqrt{2}B\ \mathrm{diag}\left\{|A(\epsilon e_1 + B(\alpha+\epsilon(e_1+e_2)))\xi_1|^{-1}, |A(\epsilon e_1 + B(\alpha+\epsilon e_2))\xi_2|^{-1}\right\}$$

Consequently:

$$
\begin{aligned}
0 &\le E_\epsilon(u_\epsilon) - I_{\epsilon,\sqrt{2}}(u_\epsilon) \\
&\le \sum_{i,j=1}^{2} \sum_{\alpha \in R_\epsilon^{(-1)^j \xi_i}\overline{(\Omega\setminus\Omega_{2\epsilon})}} \epsilon^2 \Big| \frac{|u_\epsilon(\alpha + \epsilon(-1)^j \xi_i) - u_\epsilon(\alpha)|}{\epsilon |A(\alpha)\xi_i|} - 1 \Big|^2,
\end{aligned} \tag{8.9}
$$

where:

$$
\begin{aligned}
I_{\epsilon,\sqrt{2}}(u_\epsilon) = \ & \frac{1}{2} \int_{U^0_{\epsilon,\sqrt{2}}} \Big(W(\nabla u^0_{\epsilon,\sqrt{2}}(x) \lambda^0_{\epsilon,\sqrt{2}}(x)) + W(\nabla u^1_{\epsilon,\sqrt{2}}(x) \bar{\lambda}^1_{\epsilon,\sqrt{2}}(x)) \Big) \, \mathrm{d}x \\
& + \frac{1}{2} \int_{U^1_{\epsilon,\sqrt{2}}} \Big(W(\nabla u^1_{\epsilon,\sqrt{2}}(x) \lambda^1_{\epsilon,\sqrt{2}}(x)) + W(\nabla u^1_{\epsilon,\sqrt{2}}(x) \bar{\lambda}^1_{\epsilon,\sqrt{2}}(x)) \Big) \, \mathrm{d}x.
\end{aligned}
$$

8.3 INTEGRAL REPRESENTATION OF DISCRETE ENERGIES (3.1) - THE GENERAL CASE

Lemma 50. *Let $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{Z}^n \setminus \{0\}$. Let k denote the number of non-zero coordinates in ξ, and denote: $\xi^{i_1}, \ldots, \xi^{i_k} \neq 0$ with $i_1 < i_2 \ldots < i_k$, while $\xi^{j_1} = \ldots = \xi^{j_{n-k}} = 0$ with $j_1 < j_2 \ldots < j_{n-k}$. Fix $\bar{s} \in \{1 \ldots k\}$ and define n vectors $\xi_1, \ldots, \xi_n \in \mathbb{Z}^n$ by the following algorithm:*

$$
\begin{aligned}
&\xi_1 = \xi \\
&\forall p = 2, \ldots, k - \bar{s} + 1 \qquad \xi_p^{i_{\bar{s}-1+p}} = -\xi^{i_{\bar{s}-1+p}}, \quad \textit{and} \quad \xi_p^i = \xi^i \ \ \textit{for all other indices } i \\
&\forall p = k - \bar{s} + 2, \ldots, k \qquad \xi_p^{i_{\bar{s}-1+p-k}} = -\xi^{i_{\bar{s}-1+p-k}}, \quad \textit{and} \quad \xi_p^i = \xi^i \ \ \textit{for all other indices } i \\
&\forall p = k+1, \ldots, n \qquad \xi_p^{i_{\bar{s}}} = 0, \quad \xi_p^{j_{p-k}} = \xi^{i_{\bar{s}}}, \quad \textit{and} \quad \xi_p^i = \xi^i \ \ \textit{for all other indices } i.
\end{aligned}
$$

(In other words, given ξ and fixing one of its nonzero coordinates $i_{\bar{s}}$, we first change sign of all its nonzero coordinates but $\xi^{i_{\bar{s}}}$, in the cyclic order, starting from $\xi^{i_{\bar{s}}}$: this gives k vectors ξ_p. Then we permute the $\xi^{i_{\bar{s}}}$ coordinate with all the zero coordinates: this gives the remaining $n-k$ coordinates).

Then the n-tuple of vectors $\xi_1, \ldots, \xi_n$ is linearly independent.

Proof. Without loss of generality, we may assume that $i_p = p$ for all $p = 1, \dots, k$ and $\bar{s} = 1$.

Consider first the case when $k = n$, i.e. when all coordinates of the vector ξ are nonzero. Then the matrix $B = [\xi_1, \dots, \xi_n]$ is similar to the following matrix:

$$\tilde{B} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & -1 \end{bmatrix},$$

by the basic operations of dividing each row by $|\xi^i|$. The matrix $\tilde{B}$ above has nonzero determinant, which proves the claim.

Assume now that $k \neq n$, i.e. the last $n - k > 0$ coordinates of ξ are zero. Then, the $k \times k$ principal minor of the matrix $B = [\xi_1, \dots, \xi_n]$ is invertible, as in the first case above. The minor consisting of $n - k$ last rows and k first columns of B equals zero, hence B is invertibe if and only if its minor B_0 consisting of $n - k$ last rows and $n - k$ last columns is invertible. But $B_0 = \xi^{i_{\bar{s}}} \mathrm{Id}_{n-k}$ and hence the lemma is achieved. □

8.3.1 Case 4: interactions of a given length $|\xi_0| \neq 0$ in $\mathbb{R}^n$

Assume now that $\Omega \subset \mathbb{R}^n$ and let $\psi(|\xi_0|) = 1$ and $\psi(|\xi|) = 0$ for $||\xi| - |\xi_0|| > s$, and a small $s > 0$. Consider the following set of unordered n-tuples, which we assume to be nonempty:

$$S_{|\xi_0|} = \Big\{ \zeta = \{\zeta^1, ..., \zeta^n\} \subset \mathbb{Z}, \quad |\zeta|^2 = |\xi_0|^2 \Big\}. \tag{8.10}$$

Fix $\zeta \in S_{|\xi_0|}$ and let N_ζ be the set of all distinct signed permutations without repetitions of the coordinates of ζ, i.e.:

$$N_\zeta = \left\{ (\pm\zeta^{\pi(1)}, \pm\zeta^{\pi(2)}, \dots, \pm\zeta^{\pi(n)});\ \pi \in S_n \right\}. \tag{8.11}$$

Clearly: $|N_\zeta| = 2^k \frac{n!}{k_1! \dots k_n!}$, where $k_1, \dots, k_n$ denote the numbers of repetitions of distinct coordinates in ζ, and k is the number of non-zero coordinates in ζ.

For each $\xi \in N_\zeta$ and each of its k non-zero entries $\xi^{i_{\bar{s}}}$ we define the set of linearly independent vectors $\xi_1, \dots \xi_n$ using the algorithm described in Lemma 50. We call K_ζ the set of all matrices $B = [\xi_1, \dots \xi_n]$ obtained by this procedure; it corresponds to the set of lattices $\epsilon B\mathbb{Z}^n$ whose edges have lengths $\epsilon|\xi_0|$. Note that:

$$|K_\zeta| = k|N_\zeta| = 2^k k \frac{n!}{k_1! \dots k_n!}.$$

Lemma 51. *Let $\zeta \in S_{|\xi_0|}$ have k non-zero entries. Then every vector $\xi \in N_\zeta$ is included in exactly nk lattices B, as described above.*

Proof. Firstly, the number of lattices where ξ is one of the first k columns of B, equals k^2 (k possible columns and k choices of a non-zero entry $\xi^{i_{\bar{s}}}$). Secondly, the number of lattices where ξ is one of the last $n-k$ columns, equals $(n-k)k$ (given by $n-k$ possible columns and k choices of a non-zero entry which defines the first vector in B). We hence obtain nk total number of lattices, as claimed. □

Remark 52. *The total number of vectors (with repetitions) which are columns of lattices in the set K_ζ, equals $|K_\zeta|n = nk|N_\zeta|$. This is consistent with Lemma 51, as each vector in N_ζ is repeated nk times.*

We now construct the integral representation of the discrete energy in the presently studied Case 4. Fix $B \in K_\zeta$ as above, and define $U^{0,B}_{\epsilon,|\xi_0|} \subset \Omega$ to be the union of all open cells in $\epsilon B\mathbb{Z}^n$ that have nonempty intersection with $\Omega_{\epsilon\sqrt{n}|\xi_0|}$. We identify the discrete deformation u_ϵ with its unique continuous extension $u^{0,B}_{\epsilon,|\xi_0|}$ on $U^{0,B}_{\epsilon,|\xi_0|}$, affine on all the simplices of the induced triangulation $\epsilon B\mathcal{T}_{\epsilon,n}$. Following the same observations as in the particular cases before, we obtain, for any $\pi \in S(n)$:

$$\epsilon^n \sum_{j=0}^{n-1} \Big| \frac{|u_\epsilon(B(\alpha + \epsilon \sum_{i=1}^{j+1} e_{\pi(i)})) - u_\epsilon(B(\alpha + \epsilon \sum_{i=1}^{j} e_{\pi(i)}))|}{\epsilon |A(B(\alpha + \epsilon \sum_{i=1}^{j} e_{\pi(i)}))e_{\pi(j+1)}|} - 1 \Big|^2$$
$$= \frac{n!}{|\det B|} \int_{BT^\pi_\alpha} W(\nabla u^{0,B}_{\epsilon,|\xi_0|}(x) \lambda^{0,B}_{\epsilon,|\xi_0|}(x)) \, \mathrm{d}x,$$

where W is as in (8.5), and:

$$\forall x \in BT^\pi_\alpha \cap U^{0,B}_{\epsilon,|\xi_0|}$$
$$\lambda^{0,B}_{\epsilon,|\xi_0|}(x) = |\xi_0| B \ \mathrm{diag} \left\{ |A(B(\alpha + \epsilon \sum_{i=1}^{\pi^{-1}(j)-1} e_{\pi(i)}))Be_j|^{-1} \right\}_{j=1}^n .$$

In order to take into account all of the interactions of length $|\xi_0|$, we need to consider traslations of the lattice $\epsilon B\mathbb{Z}^n$. Define:

$$V_B = \epsilon\mathbb{Z}^n \cap \left(\left(\mathrm{Int}(\epsilon B C_n) \cup \bigcup_{i=1}^{n} \epsilon B\{(x_1 \dots x_n) \in C_n; \quad x_i = 1\} \right) \setminus \epsilon B V_n \right), \tag{8.12}$$

where V_n is the set of vertices of the unit cube C_n. For every $\tau \in V_B$, define $U^{\tau,B}_{\epsilon,|\xi_0|} \subset \Omega$ to be the union of all cells in $\tau + \epsilon B\mathbb{Z}^n$ that have nonempty intersection with $\Omega_{\epsilon\sqrt{n}|\xi_0|}$. We extend the discrete deformation u_ϵ to the continuous function $u^{\tau,B}_{\epsilon,|\xi_0|}$ on $U^{\tau,B}_{\epsilon,|\xi_0|}$, affine on all the simplices of the induced triangulation $\tau + B\mathcal{T}_{\epsilon,n}$. We then have:

$$\begin{aligned}\epsilon^n \sum_{j=0}^{n-1} \Big| \frac{|u_\epsilon(\tau + B(\alpha + \epsilon \sum_{i=1}^{j+1} e_{\pi(i)})) - u_\epsilon(\tau + B(\alpha + \epsilon \sum_{i=1}^{j} e_{\pi(i)}))|}{\epsilon |A(\tau + B(\alpha + \epsilon \sum_{i=1}^{j} e_{\pi(i)}))e_{\pi(j+1)}|} - 1 \Big|^2 \\ = \frac{n!}{|\det B|} \int_{\tau + BT^\pi_\alpha} W(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}(x) \lambda^{\tau,B}_{\epsilon,|\xi_0|}(x))\, \mathrm{d}x,\end{aligned}$$

where:

$$\begin{aligned}&\forall x \in (\tau + BT^\pi_\alpha) \cap U^{\tau,B}_{\epsilon,|\xi_0|} \\ &\qquad \lambda^{\tau,B}_{\epsilon,|\xi_0|}(x) = |\xi_0| B \ \mathrm{diag} \left\{ |A(\tau + B(\alpha + \epsilon \sum_{i=1}^{\pi^{-1}(j)-1} e_{\pi(i)}))Be_j|^{-1} \right\}_{j=1}^{n}.\end{aligned}$$

Summing now over all simplices in the triangulations, we obtain the functional:

$$I_{\epsilon,|\xi_0|}(u_\epsilon) = \sum_{\zeta \in S_{|\xi_0|}} \frac{1}{n!(nk)} \sum_{B \in K_\zeta} \frac{n!}{|\det B|} \sum_{\tau \in \{0\} \cup V_B} \int_{U^{\tau,B}_{\epsilon,|\xi_0|}} W(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}(x) \lambda^{\tau,B}_{\epsilon,|\xi_0|}(x))\, \mathrm{d}x, \tag{8.13}$$

and the bound:

$$\begin{aligned}0 &\leq E_\epsilon(u_\epsilon) - I_{\epsilon,|\xi_0|}(u_\epsilon) \\ &\leq \sum_{\xi \in \mathbb{Z}^n, |\xi| = |\xi_0|} \quad \sum_{\alpha \in R^\xi_\epsilon(\Omega \setminus \Omega_{\epsilon\sqrt{n}|\xi_0|})} \epsilon^n \Big| \frac{|u_\epsilon(\alpha + \epsilon\xi) - u_\epsilon(\alpha)|}{\epsilon |A(\alpha)\xi|} - 1 \Big|^2.\end{aligned} \tag{8.14}$$

In (8.13), k is the number of non-zero entries in the vector ζ, while the factor $n!$ in the first denominator is due to the fact that every edge in a given lattice is shared by $n!$ simplices in $\mathcal{T}_{\epsilon,n}$.

8.3.2 Case 5: the general case of finite range interactions in $\mathbb{R}^n$

Reasoning as in the previously considered specific cases, we get:

$$0 \leq E_\epsilon(u_\epsilon) - I_\epsilon(u_\epsilon) \leq \sum_{\xi\in\mathbb{Z}^n, 1\leq|\xi|\leq M} \quad \sum_{\alpha\in R_\epsilon^\xi(\Omega\setminus\Omega_{\epsilon\sqrt{n}M})} \epsilon^n \psi(|\xi|) \Big| \frac{|u_\epsilon(\alpha+\epsilon\xi) - u_\epsilon(\alpha)|}{\epsilon|A(\alpha)\xi|} - 1\Big|^2, \quad (8.15)$$

where:

$$I_\epsilon = \sum_{1\leq|\xi_0|\leq M} \psi(|\xi_0|) I_{\epsilon,|\xi_0|}. \tag{8.16}$$

8.4 BOUNDS ON THE VARIATIONAL LIMITS OF THE LATTICE ENERGIES

Consider the following family of energies:

$$F_\epsilon : L^2(\Omega, \mathbb{R}^n) \to \overline{\mathbb{R}}, \qquad F_\epsilon(u) = \begin{cases} E_\epsilon(u_{|\epsilon\mathbb{Z}^n\cap\Omega}) & \text{if } u \in \mathcal{C}(\Omega) \text{ is affine on } \mathcal{T}_{\epsilon,n}\cap\Omega \\ +\infty & \text{otherwise.} \end{cases}.$$

By Theorem 20, the sequence F_ϵ has a subsequence (which we do not relabel) Γ-converging to some lsc functional $\mathcal{F} : L^2(\Omega, \mathbb{R}^n) \to \overline{\mathbb{R}}$. Our goal is to identify the limiting energy $\mathcal{F}$ in its exact form, whenever possible, or find its lower and upper bounds. This will be accomplished in Theorem 56, and in the next section.

We first state some easy preliminary results regarding the quasiconvexification QW and the piecewise affine extensions $u^{\tau,B}_{\epsilon,|\xi_0|}$ of the discrete deformations u_ϵ.

Lemma 53. *The quasiconvexification $QW : \mathbb{R}^{n\times n} \to \mathbb{R}$ of W in (8.5), is a convex function, and:*

$$QW(M) = \sum_{i=1..n; |Me_i|>1} (|Me_i| - 1)^2 \qquad \forall M \in \mathbb{R}^{n\times n}. \tag{8.17}$$

Proof. By Theorem 11, we note that:

$$QW(M) = \sum_{i=1}^{n} Q\big(|Me_i| - 1\big)^2.$$

and that the convexification: and the quasiconvexification Qf of the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\xi) = (|\xi| - 1)^2$ coincide with each other. The claim follows by checking directly that:

$$Cf(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq 1 \\ (|\xi| - 1)^2 & \text{if } |\xi| > 1. \end{cases}$$

□

Lemma 54. *For every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, and every mesh-size sequence $\epsilon \to 0$, there exists a subsequence ϵ (which we do not relabel) and a sequence $u_\epsilon \in W_0^{1,2}(\mathbb{R}^n, \mathbb{R}^n)$ of continuous piecewise affine on the triangulation in $\mathcal{T}_{\epsilon,n}$ functions, such that:*

$$\forall 1 \leq |\xi_0| \leq M \quad \forall \zeta \in S_{|\xi_0|} \quad \forall B \in K_\zeta \quad \forall \tau \in \{0\} \cup V_B$$
$$u = \lim_{\epsilon \to 0} u_{\epsilon,|\xi_0|}^{\tau,B} \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^n).$$

Proof. Approximate u by $u_k \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, so that $u_k \to u$ in $W^{1,2}(\Omega, \mathbb{R}^n)$. Fix $|\xi_0| \leq M$, $\zeta \in S_{|\xi_0|}$, $B \in K_\zeta$ and $\tau \in V_B$. Then, by the fundamental estimate of finite elements [11], the $\mathbb{P}_1$-interpolation $u_{\epsilon,k}$ of u_k on $\mathcal{T}_{\epsilon,n}$, i.e. the continuous function affine on the simplices in $\mathcal{T}_{\epsilon,n}$ which coincides with u_k on $\epsilon\mathbb{Z}^n$, satisfies:

$$\|u_{\epsilon,k} - u_k\|_{W^{1,2}(\Omega)} \leq \frac{1}{k} \qquad \forall \epsilon \leq \epsilon_k.$$

Likewise, because the set of all involved quantities $|\xi_0|, \zeta, B, \tau$ is finite, it follows that:

$$\|(u_{\epsilon,k})_{\epsilon,|\xi_0|}^{\tau,B} - u_k\|_{W^{1,2}(\Omega)} \leq \frac{1}{k}$$

if only $\epsilon \leq \epsilon_k$ is sufficiently small. We set $u_\epsilon := u_{\epsilon_k,k}$ which satisfies the claim of the Lemma. □

We now observe a compactness property of E_ϵ, which together with the Γ-convergence of F_ϵ to $\mathcal{F}$, implies convergence of the minimizers of E_ϵ to the minimizers of $\mathcal{F}$ (see Theorem 22 and Remark 23).

Lemma 55. *Assume that $E_\epsilon(u_\epsilon) \leq C$ for some sequence of discrete deformations $u_\epsilon : \epsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}^n$, which we identify with $u_\epsilon \in \mathcal{C}(\Omega)$ that are piecewise affine on $\mathcal{T}_{\epsilon,n} \cap \Omega$ and agree with the discrete u_ϵ at each node of the lattice. Then there exist constants $c_\epsilon \in \mathbb{R}^n$ such that $u_\epsilon - c_\epsilon$ converges (up to a subsequence) in $L^2(\Omega, \mathbb{R}^n)$ to some $u \in W^{1,2}(\Omega, \mathbb{R}^n)$.*

Proof. Observe that for every $|\xi_0|, \tau, B$ as in (8.16), (8.13), and every $\epsilon \leq \epsilon_0$:

$$\int_{U^{\tau,B}_{\epsilon,|\xi_0|}} W(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}(x)\lambda^{\tau,B}_{\epsilon,|\xi_0|}(x))\,\mathrm{d}x \leq C. \tag{8.18}$$

Thus in particular, for some $\xi_0 \in \mathbb{Z}^n$ such that $\psi(|\xi_0|) \neq 0$, and for every $\eta > 0$:

$$\|\nabla u^{0,B}_{\epsilon,|\xi_0|}\|_{L^2(\Omega_\eta)} \leq C.$$

if only $\epsilon \leq \epsilon_0$ is small enough. Fix $\eta > 0$. The above bound implies that $\nabla u^{0,B}_{\epsilon,|\xi_0|}$ converges weakly (up to a subsequence) in $L^2(\Omega_\eta)$, which by means of the Poincaré inequality yields weak convergence of $u^{0,B}_{\epsilon,|\xi_0|} - c_\epsilon$ in $W^{1,2}(\Omega_\eta)$. We now observe that:

$$\|u^{0,B}_{\epsilon,|\xi_0|} - u_\epsilon\|_{L^2(\Omega_\eta)} \leq C\epsilon|\xi_0|\|u_\epsilon\|_{W^{1,2}(\Omega_\eta)}, \tag{8.19}$$

because $u^{\tau,B}_{\epsilon,|\xi_0|}$ is a $\mathbb{P}_1$ interpolation of u_ϵ on the lattice $\epsilon B\mathbb{Z}^n \cap \Omega_\eta$, allowing to use the classical finite element error estimate in [11, Theorem 3.1.6]. This ends the proof. □

We are now able to provide the main theorem of this chapter.

Theorem 56. *We have:*

$$\forall u \in W^{1,2}(\Omega, \mathbb{R}^n) \qquad I_Q(u) \leq \mathcal{F}(u) \leq I(u), \tag{8.20}$$

where:

$$\begin{aligned} I_Q(u) &= \sum_{1\leq|\xi_0|\leq M} \quad \sum_{\zeta\in S_{|\xi_0|}, B\in K_\zeta} \psi(|\xi_0|)\frac{(1+|V_B|)}{(nk)|\det B|}\int_\Omega QW(\nabla u(x)\lambda^B_{|\xi_0|}(x))\,\mathrm{d}x, \\ I(u) &= \sum_{1\leq|\xi_0|\leq M} \quad \sum_{\zeta\in S_{|\xi_0|}, B\in K_\zeta} \psi(|\xi_0|)\frac{(1+|V_B|)}{(nk)|\det B|}\int_\Omega W(\nabla u(x)\lambda^B_{|\xi_0|}(x))\,\mathrm{d}x, \end{aligned} \tag{8.21}$$

and where $\lambda^B_{|\xi_0|}(x)$ is given by:

$$\lambda^B_{|\xi_0|}(x) = |\xi_0| B \;\mathrm{diag}\left\{|A(x)Be_j|^{-1}\right\}_{j=1}^n. \tag{8.22}$$

Proof. **1.** Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ and consider the approximating sequence u_ϵ as in Lemma 54. Directly from the definition of Γ-convergence (see (17)), we obtain:

$$\mathcal{F}(u) \leq \liminf_{\epsilon \to 0} F_\epsilon(u_\epsilon) = \liminf_{\epsilon \to 0} E_\epsilon(u_\epsilon). \tag{8.23}$$

Further, in view of the boundedness of ψ, and of the sequence $\|\nabla u_\epsilon\|_{L^2(\Omega)}$, (8.15) implies:

$$\begin{aligned} 0 \leq E_\epsilon(u_\epsilon) - I_\epsilon(u_\epsilon) &\leq C \sum_{\xi \in \mathbb{Z}^n, 1 \leq |\xi| \leq M} \quad \sum_{\alpha \in R_\epsilon^\xi(\overline{\Omega \setminus \Omega_{\epsilon\sqrt{n}M}})} \epsilon^n \left(\left| \frac{u_\epsilon(\alpha + \epsilon\xi) - u_\epsilon(\alpha)}{\epsilon|\xi|} \right|^2 + 1 \right) \\ &\leq C \left(\|\nabla u_\epsilon\|^2_{L^2(\Omega \setminus \Omega_{\epsilon\sqrt{n}M})} + |\Omega \setminus \Omega_{\epsilon\sqrt{n}M}| \right) \to 0 \quad \text{ as } \epsilon \to 0. \end{aligned} \tag{8.24}$$

Indeed, the third inequality in (8.24) can be proven by the same argument as in the proof of Lemma 54. Alternatively, a direct proof can be obtained as follows. Since u_ϵ is piecewise affine, we have:

$$\left| \frac{u_\epsilon(\alpha + \epsilon\xi) - u_\epsilon(\alpha)}{\epsilon|\xi|} \right|^2 = \left| \int_0^1 \langle \nabla u_\epsilon(\alpha + t\epsilon\xi), \frac{\xi}{|\xi|} \rangle \ \mathrm{d}t \right|^2 \leq \int_0^1 q_\epsilon(\alpha + t\epsilon\xi)^2 \ \mathrm{d}t,$$

where $q_\epsilon(p) = \sup_i \langle \nabla u_\epsilon(p), v_i \rangle$ when p is an interior point of a face of the trangulation $\mathcal{T}_{\epsilon,n}$ spanned by unit vectors $v_1, \dots v_k$ (here $0 \leq k \leq n$). Note that:

$$q_\epsilon(p)^2 \leq \frac{n!}{\epsilon^n} \int_T |\nabla u_\epsilon|^2 \qquad \forall p \in T \in \mathcal{T}_{\epsilon,n}.$$

We hence obtain for all $1 \leq |\xi| \leq M$:

$$\begin{aligned} &\sum_{\alpha \in R_\epsilon^\xi(\overline{\Omega \setminus \Omega_{\epsilon\sqrt{n}M}})} \epsilon^n \left| \frac{u_\epsilon(\alpha + \epsilon\xi) - u_\epsilon(\alpha)}{\epsilon|\xi|} \right|^2 \leq \int_0^1 \sum_{\alpha \in R_\epsilon^\xi(\overline{\Omega \setminus \Omega_{\epsilon\sqrt{n}M}})} \epsilon^n q_\epsilon(\alpha + \epsilon\xi)^2 \ \mathrm{d}t \\ &\leq C \int_0^1 \left(\sum_\alpha \int_T |\nabla u_\epsilon|^2 \right) \ \mathrm{d}t \leq C \int_0^1 \|\nabla u_\epsilon\|^2_{L^2(\Omega \setminus \Omega_{\epsilon\sqrt{n}M})} \ \mathrm{d}t = \|\nabla u_\epsilon\|^2_{L^2(\Omega \setminus \Omega_{\epsilon\sqrt{n}M})}, \end{aligned}$$

which achieves (8.24).

Consequently, by (8.23), (8.24), we see that:

$$\mathcal{F}(u) \leq \liminf_{\epsilon \to 0} I_\epsilon(u_\epsilon),$$

From the strong convergence of the sequences $\{u^{\tau,B}_{\epsilon,|\xi_0|}\}$ to u in $W^{1,2}(\Omega,\mathbb{R}^n)$, we get, up to a subsequence, that $\nabla u^{\tau,B}_{\epsilon,|\xi_0|}$ converges to ∇u, a. e. in Ω. Moreover, there exists $g \in L^2(\Omega)$ such that:

$$\|\nabla u^{\tau,B}_{\epsilon,|\xi_0|}\|_{\mathbb{M}_{n\times n}} \leq g.$$

Using the continuity of W and the uniform convergence of $\lambda^{\tau,B}_{\epsilon,|\xi_0|}$ to $\lambda^B_{|\xi_0|}$ in Ω, it follows that:

$$W\big(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}\lambda^{\tau,B}_{\epsilon,|\xi_0|}\big) \to W\big(\nabla u \lambda^B_{|\xi_0|}\big), \quad a.e.\, \Omega.$$

In addition:

$$W\big(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}\lambda^{\tau,B}_{\epsilon,|\xi_0|}\big) \leq C\|\nabla u^{\tau,B}_{\epsilon,|\xi_0|}\|^2_{\mathbb{M}_{n\times n}} + 1 \leq Cg^2 + 1 \in L(\Omega).$$

Therefore, by the dominated convergence theorem:

$$\mathcal{F}(u) \leq \liminf_{\epsilon\to 0} \sum_{1\leq|\xi_0|\leq M} \sum_{\zeta\in S_{|\xi_0|},B\in K_\zeta} \psi(|\xi_0|)\frac{(1+|V_B|)}{(nk)|\det B|}\int_\Omega W(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}(x)\lambda^{\tau,B}_{\epsilon,|\xi_0|}(x))\, \mathrm{d}x = I(u),. \tag{8.25}$$

The proof of the upper bound for $\mathcal{F}$ in (8.20) is hence accomplished.

2. We now show the lower bound in (8.20). Let $u \in W^{1,2}(\Omega,\mathbb{R}^n)$; note that the upper bound proved above yields: $\mathcal{F}(u) < \infty$. Therefore, u has a recovery sequence $u_\epsilon \in \mathcal{C}(\Omega)$ affine on $\mathcal{T}_{\epsilon,n}\cap\Omega$, such that: $u_\epsilon \to u$ in $L^2(\Omega,\mathbb{R}^n)$ and $E_\epsilon(u_\epsilon) \to \mathcal{F}(u)$ as $\epsilon \to 0$.

As in the proof of Lemma 55, we see that (8.18) holds for every $|\xi_0|, \tau, B$ as in (8.16), (8.13). Thus, for every $\eta > 0$ we have:

$$\|\nabla u^{\tau,B}_{\epsilon,|\xi_0|}\|_{L^2(\Omega_\eta)} \leq C, \tag{8.26}$$

for every $\epsilon \leq \epsilon_0$ is small enough. Fix $\eta > 0$. The bound (8.26) implies that every $\nabla u^{\tau,B}_{\epsilon,|\xi_0|}$ converges weakly (up to a subsequence) in $L^2(\Omega_\eta)$. Next, we note that $u^{\tau,B}_{\epsilon,|\xi_0|}$ converges to u in $L^2(\Omega_\eta)$, which yields that the same convergence is also valid weakly in $W^{1,2}(\Omega_\eta)$.

Indeed, by [11, Theorem 3.1.6], we have:

$$\|u^{\tau,B}_{\epsilon,|\xi_0|} - u_\epsilon\|_{L^2(\Omega_\eta)} \leq C\epsilon|\xi_0|\|u_\epsilon\|_{W^{1,2}(\Omega_\eta)},$$

because $u^{\tau,B}_{\epsilon,|\xi_0|}$ is a $\mathbb{P}_1$ interpolation of u_ϵ on the lattice $\epsilon B\mathbb{Z}^n \cap \Omega_\eta$. Consequently, in view of (8.26):

$$\|u^{\tau,B}_{\epsilon,|\xi_0|} - u\|_{L^2(\Omega_\eta)} \leq \|u^{\tau,B}_{\epsilon,|\xi_0|} - u_\epsilon\|_{L^2(\Omega_\eta)} + \|u_\epsilon - u\|_{W^{1,2}(\Omega_\eta)} \leq C\epsilon + \|u_\epsilon - u\|_{W^{1,2}(\Omega_\eta)} \to 0, \quad \text{as } \epsilon \to 0.$$

Since $QW \geq W$, we further obtain:

$$\begin{aligned}
\mathcal{F}(u) &= \lim_{\epsilon\to 0} F_\epsilon(u_\epsilon) \geq \limsup_{\epsilon\to 0} I_\epsilon(u_{\epsilon|\Omega_\eta}) \\
&\geq \sum_{1\leq|\xi_0|\leq M} \sum_{\zeta\in S_{|\xi_0|}} \frac{\psi(|\xi_0|)}{(nk)} \sum_{B\in K_\zeta} \frac{1}{|\det B|} \sum_{\tau\in\{0\}\cup V_{l,B}} \liminf_{\epsilon\to 0} \int_{\Omega_\eta} QW(\nabla u^{\tau,B}_{\epsilon,|\xi_0|}(x)\lambda^{\tau,B}_{\epsilon,|\xi_0|}(x))\,\mathrm{d}x \\
&\geq \sum_{1\leq|\xi_0|\leq M} \sum_{\zeta\in S_{|\xi_0|}, B\in K_\zeta} \frac{\psi(|\xi_0|)}{(nk)} \frac{1+|V_{l,B}|}{|\det B|} \int_{\Omega_\eta} QW(\nabla u(x)\lambda^B_{|\xi_0|}(x))\,\mathrm{d}x = I_Q(u_{|\Omega_\eta}),
\end{aligned}$$

where the last inequality above follows by the lower semicontinuity of the functional

$$\int_\Omega QW(v(x))dx$$

with respect to the weak topology of $L^2(\Omega_\eta, \mathbb{R}^{n\times n})$ (see Theorem 12), and by the weak convergence of $\nabla u^{\tau,B}_{\epsilon,|\xi_0|}\lambda^{\tau,B}_{\epsilon,|\xi_0|}$ to $\nabla u\lambda^B_{|\xi_0|}$ in L^2. Since $\eta > 0$ was arbitrary, the proof is achieved.

□

Corollary 57. *We have: $\mathcal{F}(u) < +\infty$ if and only if $u \in W^{1,2}(\Omega, \mathbb{R}^n)$.*

Proof. By Theorem 56, $\mathcal{F}$ is finite on all $W^{1,2}$ deformations. Conversely, let $u \in L^2(\Omega, \mathbb{R}^n)$ and let $\mathcal{F}(u) < \infty$. Then there exists a recovery sequence $u_\epsilon \in \mathcal{C}(\Omega)$ affine on $\mathcal{T}_{\epsilon,n} \cap \Omega$, so that $u_\epsilon \to u$ in L^2 and $F_\epsilon(u_\epsilon)$ is uniformly bounded. This implies (8.18) so in particular $\|\nabla u_\epsilon\|^2_{L^2(\Omega)}$ is bounded and hence (up to a subsequence) u_ϵ converges weakly in $W^{1,2}(\Omega)$. Consequently, $u \in W^{1,2}(\Omega)$. □

Corollary 58. *Let $\Gamma_0(I)$ denote the sequentially weak lsc envelope of I in $W^{1,2}(\Omega, \mathbb{R}^n)$. Then:*

$$\mathcal{F}(u) \leq \Gamma_0(I)(u) \qquad \forall u \in W^{1,2}(\Omega, \mathbb{R}^n).$$

Proof. The proof is immediate since the Γ-limit F is sequentially weak lsc in $W^{1,2}(\Omega, \mathbb{R}^n)$. □

8.5 THE CASE OF NEAR INTERACTIONS

In this section we improve the result in (8.20) to the exact form of the limiting energy $\mathcal{F}$, in the special cases of near and next-to-near interactions.

Theorem 59. *(Case 1: near interactions in $\mathbb{R}^2$.) Let $\Omega \subset \mathbb{R}^2$ and let $\psi(1) = 1$ and $\psi(|\xi|) = 0$ for all $|\xi| \geq \sqrt{2}$. Denote: $\lambda(x) = \operatorname{diag}\{|A(x)e_1|^{-1}, |A(x)e_2|^{-1}\}$. Then:*

$$\mathcal{F}(u) = \begin{cases} 2\displaystyle\int_\Omega QW(\nabla u(x)\lambda(x))\mathrm{d}x & \text{for } u \in W^{1,2}(\Omega, \mathbb{R}^2) \\ +\infty & \text{for } u \in L^2 \setminus W^{1,2}. \end{cases} \tag{8.27}$$

Proof. From Theorem 56 and (8.4), we see that $I_Q(u) = 2\int_\Omega QW(\nabla u\lambda(x))\,\mathrm{d}x$ and $I(u) = 2\int_\Omega W(\nabla u\lambda(x))\,\mathrm{d}x$. By Corollary 58 it follows that:

$$\mathcal{F}(u) \leq G_0\Big(2\int_\Omega W(\nabla u(x)\lambda(x))\,\mathrm{d}x\Big) = 2\int_\Omega QW(\nabla u\lambda(x))\,\mathrm{d}x.$$

The last equality is a consequence of Theorem 13 because the function $f(x, M) = W(M\lambda(x))$ clearly satisfies the bounds (3.9) and also its quasiconvexification with respect to M equals:

$$Qf(x, M) = QW(M\lambda(x)).$$

The proof is now complete in view of Corollary 57. □

Theorem 60. *(Case 2: near interactions in $\mathbb{R}^n$.) Let $\Omega \subset \mathbb{R}^n$ and let $\psi(1) = 1$ and $\psi(|\xi|) = 0$ for $|\xi| \geq \sqrt{n}$. Denote: $\lambda(x) = \operatorname{diag}\{|A(x)e_j|^{-1}\}_{j=1}^n$. Then, the $\Gamma-$limit $\mathcal{F}$ has the form as in (8.27):*

$$\mathcal{F}(u) = \begin{cases} 2\displaystyle\int_\Omega QW(\nabla u(x)\lambda(x))\,\mathrm{d}x & \text{for } u \in W^{1,2}(\Omega, \mathbb{R}^n) \\ +\infty & \text{for } u \in L^2 \setminus W^{1,2}. \end{cases} \tag{8.28}$$

Proof. The proof follows exactly as in Theorem 59, using the representation developed in section 8.2.2. Alternatively, using the notation and setting of section 8.3, we see that $S_1 = \{e_i\}_{i=1}^n$ and:

$$\forall \zeta \in S_1 \quad N_\zeta = N = \{e_i, -e_i\}_{i=1}^n, \qquad \text{and } K = \bigcup_{\zeta \in S_1} K_\zeta = \{B = \pm[e_i, e_{i+1}, \ldots, e_{i-1}]\}_{i=1}^n,$$

so that $|K| = 2n$. Also, for every $B \in K$ as above: $V_B = \emptyset$, $|\det B| = 1$ and $\lambda_1^B(x) = B\mathrm{diag}\{|A(x)Be_j|^{-1}\}_{i=1}^n$, i.e. $\lambda_1^B(x)$ differs from $\lambda(x)$ only by the order and sign of its columns. Hence:

$$\forall B \in K \qquad QW(\nabla u(x)\lambda_1^B(x)) = QW(\nabla u(x)\lambda(x)), \quad W(\nabla u(x)\lambda_1^B(x)) = W(\nabla u(x)\lambda(x))$$

and so:

$$I_Q(u) = \sum_{\zeta \in S_1, B \in K_\zeta} \frac{1}{n} \int_\Omega QW(\nabla u(x)\lambda_1^B(x))\, \mathrm{d}x = 2\int_\Omega QW(\nabla u(x)\lambda(x))\, \mathrm{d}x.$$

Likewise: $I(u) = 2\int_\Omega W(\nabla u(x)\lambda(x))\, \mathrm{d}x$. The proof follows now by Corollary 58 and Theorem 13, as before. □

Using the integral representation of section 8.2.3, we also arrive at:

Theorem 61. *(Case 3: next-to-near interactions in $\mathbb{R}^2$.) Let $\Omega \subset \mathbb{R}^2$ and assume that $\psi(\sqrt{2}) = 1$ and $\psi(|\xi|) = 0$ for all $|\xi| \geq \sqrt{3}$ and $|\xi| \leq 1$. Denote:*

$$\lambda_{\sqrt{2}}(x) = \sqrt{2}B \ \mathrm{diag}\left\{|A(x)Be_1|^{-1}, |A(x)Be_2|^{-1}\right\}, \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then:

$$\mathcal{F}(u) = \begin{cases} 2\displaystyle\int_\Omega QW(\nabla u(x)\lambda_{\sqrt{2}}(x))\mathrm{d}x & \text{for } u \in W^{1,2}(\Omega, \mathbb{R}^2) \\ +\infty & \text{for } u \in L^2 \setminus W^{1,2}. \end{cases}$$

The functionals $\mathcal{F}$ obtained in Theorems 59, 60 and 61, measure the deficit of a deformation u from being an orientation preserving (modulo compressive maps, due to the quasiconvexification of the energy density W) realisation of the metric $\bar{G} = (\lambda^{-1})^T(\lambda^{-1})$. In the next section we compare these functionals with the non-Euclidean energy $\mathcal{E}$.

8.6 COMPARISON OF THE VARIATIONAL LIMITS AND THE MODEL $\mathcal{E}$

In this section we assume that Ω is an open bounded subset of $\mathbb{R}^2$. Our scope is to compare the following integral functionals:

$$\mathcal{F}_1(u) = \int_\Omega QW(\nabla u\lambda(x))dx,\ \mathcal{F}_{\sqrt{2}}(u) = \int_\Omega QW(\nabla u\lambda_{\sqrt{2}}(x))dx,\ \mathcal{E}(u) = \int_\Omega \overline{W}(\nabla u A(x)^{-1})dx,$$

where the stored energy density $\overline{W} : \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}}_+$ satisfies (6.2).

Lemma 62. *Assume that* $\min \mathcal{E}(u) = 0$*, so that the prestrain metric* G *is realisable by a smooth* $u : \Omega \to \mathbb{R}^2$ *with* $(\nabla u)^T\nabla u = G$*. Then:* $\mathcal{F}_1(u) = 0$*.*

Proof. Since $A = \sqrt{G} = \sqrt{(\nabla u)^T\nabla u}$, it follows that $A = R\nabla u$, for some rotation field $R : \Omega \to SO(2)$. Hence, $|A(x)e_i| = |\nabla u(x)e_i|$, and so both columns of the matrix:

$$\nabla u(x)\lambda(x) = \left[\frac{\nabla u(x)e_1}{|\nabla u(x)e_1|},\ \frac{\nabla u(x)e_2}{|\nabla u(x)e_2|}\right]$$

have length 1. The claim follows now by Lemma 53. □

The following example shows that G may be realisable, as in Lemma 62, but the metric $\bar{G} = \lambda^{-1,T}\lambda^{-1}$ is still not realisable. The vanishing of the infimum of the derived energy $\mathcal{F}_1$ is hence due to the quasiconvexification effect in the energy density.

Example 63. *Let* $g : \mathbb{R} \to (0, +\infty)$ *be a smooth function. Consider:*

$$G(x_1, x_2) = \begin{bmatrix} 1/2 & 1 \\ 1 & g(x_1) \end{bmatrix}, \quad \bar{G}(x_1, x_2) = diag\{|A(x_1)e_1|^2, |A(x_1)e_2|^2\} = \begin{bmatrix} 1/2 & 0 \\ 0 & g(x_1) \end{bmatrix},$$

where the formula for $\bar{G}$ *follows from the fact that* $|A(x)e_i|^2 = \langle e_i, A(x)^2e_i\rangle = \langle e_i, G(x)e_i\rangle$*. We now want to assign* g *so that the Gaussian cuvatures* κ *and* κ_1 *of* G *and* $\bar{G}$*, satisfy:*

$$\kappa = 0, \qquad \kappa_1 \neq 0. \tag{8.29}$$

By a direct calculation, we see that:

$$\kappa_1 = \frac{1}{\sqrt{g}}\left(\frac{g'}{\sqrt{g}}\right)' = \frac{-2gg'' + (g')^2}{2g^2}$$

$$(\frac{g}{2} - 1)^2\kappa = -\frac{1}{2}g''(\frac{g}{2} - 1) + \frac{1}{8}(g')^2 = \frac{1}{2}g'' + \frac{g^2}{4}\kappa_1.$$

Hence, (8.29) is equivalent to:

$$g > 2, \qquad g'' \neq 0, \qquad g'' = \frac{(g')^2}{2(g-2)}. \tag{8.30}$$

Clearly, the second order ODE above has a solution on a sufficiently small interval $(-\epsilon, \epsilon)$, for any assigned initial data $g(0) = g_0 > 2$ and $g'(0) = g_1 > 0$. Also, this local solution satisfies all three conditions in (8.30) by continuity, if $\epsilon > 0$ is small enough.

This completes the example. By rescaling $\tilde{g}(x_1) = g(\epsilon x_1)$, we may obtain the metric G on $\Omega = (0,1)^2$, with the desired properties.

The next example shows that the induced metric $\bar{G}$ can be realisable even when G is not. In this case, one trivially has: $\inf \mathcal{E}(u) > 0$ while $\min \mathcal{F}_1(u) = 0$.

Example 64. *Let $w : (0,1)^2 \to (0, \frac{\pi}{2})$ be a smooth function such that $w_{x_1,x_2} \neq 0$, and define:*

$$G(x) = \begin{bmatrix} 1 & \cos w(x) \\ \cos w(x) & 1 \end{bmatrix}, \qquad \bar{G}(x) = diag\{|A(x)e_1|^2, |A(x)e_2|^2\} = Id_2.$$

Clearly, $\kappa_1 \neq 0$. We now compute the Gaussian curvature of G:

$$\kappa = \frac{1}{\sin^4 w}\big((-(\cos w)w_{x_1}w_{x_2} - (\sin w)w_{x_1,x_2})\sin^2 w + (\sin^2 w)w_{x_2}(\cos w)w_{x_1}\big)$$
$$= -\frac{w_{x_1,x_2}}{\sin w} \neq 0.$$

The following simple observation establishes the relation between $\mathcal{F}_1$ and $\mathcal{F}_{\sqrt{2}}$.

Lemma 65. *Let $\Omega = B(0,1)$. Then, we have:*

$$\forall u \in W^{1,2}(\Omega, \mathbb{R}^2) \qquad \mathcal{F}_{\sqrt{2}}(u) = \overline{\mathcal{F}}_1(\sqrt{2}u \circ R),$$

where $\overline{\mathcal{F}}_1$ is defined with respect to the metric G_1 in:

$$G_1(x) = R^T G(Rx) R, \qquad R = \frac{1}{\sqrt{2}} B.$$

Proof. Note first that G_1 is the pull-back of the metric G under the rotation $x \mapsto Rx$. Thus:

$$\begin{aligned}
\mathcal{F}_{\sqrt{2}}(u) &= \int_\Omega QW(\nabla u(x)\lambda_{\sqrt{2}}(x))\ \mathrm{d}x \\
&= \int_\Omega QW\Big(\sqrt{2}\nabla u(Ry)\sqrt{2}R\ \mathrm{diag}\{|A(Ry)Be_1|^{-1}, |A(Ry)Be_2|^{-1}\}\Big)\ \mathrm{d}y \\
&= \int_\Omega QW\Big(\nabla(\sqrt{2}u \circ R)(y)\ \mathrm{diag}\{|A(Ry)Re_1|^{-1}, |A(Ry)Re_2|^{-1}\}\Big)\ \mathrm{d}y \\
&= \int_\Omega QW\Big(\nabla(\sqrt{2}u \circ R)(y)\bar{\lambda}(y)\Big)\ \mathrm{d}y = \overline{\mathcal{F}}_1(\sqrt{2}u \circ R),
\end{aligned}$$

because $|\sqrt{G_1(x)}e_i| = |A(Rx)Re_i|$, which implies:

$$\bar{\lambda}(x) = \mathrm{diag}\{|A(Rx)Re_1|^{-1}, |A(Rx)Re_2|^{-1}\}.$$

□

Finally, observe also that if $\mathcal{F}(u) = \mathcal{F}_1(u) = 0$, then the length of columns in the matrix $\nabla u(x)\lambda_{\sqrt{2}}(x)$ equals $\sqrt{2}$. Hence $\mathcal{F}_{\sqrt{2}}(u) \neq 0$.

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