INVESTIGATING FLEXIBILITY, REVERSIBILITY, AND MULTIPLE REPRESENTATIONS IN A CALCULUS ENVIRONMENT

by

Steven Todd Flanders

Bachelor of Science, Grove City College, 2004

Master of Education, Gannon University, 2006

Submitted to the Graduate Faculty of

the School of Education in partial fulfillment

of the requirements for the degree of

Doctor of Education

University of Pittsburgh

2014

UNIVERSITY OF PITTSBURGH

SCHOOL OF EDUCATION

This dissertation was presented

by

Steven T. Flanders

It was defended on

October 21, 2014

and approved by

James Greeno, PhD, Visiting Professor

Beverly Michael, PhD, Senior Lecturer

Charles Munter, PhD, Assistant Professor

Margaret Smith, EdD, Professor

Dissertation Advisor: Ellen Ansell, PhD, Associate Professor

INVESTIGATING FLEXIBILITY, REVERSIBILITY, AND MULTIPLE REPRESENTATIONS IN A CALCULUS ENVIRONMENT

Steven Todd Flanders, EdD

University of Pittsburgh, 2014

This study investigates the development of flexibility and reversibility in a calculus environment that attends to linking multiple representations. Reversibility was studied through Krutetskii's framework of reversibility of two-way processes and reversibility of the mental process in reasoning. The study was conducted over approximately four months in a high school calculus classroom in an urban school district in a mid-Atlantic state. Instruction attended to linking multiple representations whenever possible. Four types of data were collected: 1) a pre-test, 2) a post-test, 3) daily assessments, and 4) clinical interviews. Twenty-one students completed a pre-test and post-test that together assessed development of flexibility over the course of the study. They also completed daily assessments that were collected to provide evidence of the development of reversibility during the course of the study. Six students participated in four clinical interviews each, spread throughout the study. Inferential statistics were used to compare the results of the pre-test and post-test for significant differences and to determine significant differences in the presence of reversibility on the daily assessments over the course of the study. The clinical interviews were analyzed for evidence of students' thought processes while solving reversible questions. Analysis revealed that over the course of the study, students demonstrated significant increases in both flexibility and reversibility. Two-way reversibility seemed to develop with relative ease for most students and often developed simultaneously with learning a forward process. Developing reversibility of the mental process in reasoning was difficult and tended to develop simultaneously with learning in a forward direction for students with high levels of flexibility. For students who did not develop reversibility simultaneously with forward learning, both two-way reversibility and reversibility of the mental process in reasoning were able to develop through multiple opportunities to solve reversible tasks of similar content. Analysis of the clinical interviews indicated that students typically followed a 4-step thought process when using reversibility to solve problems. Implications and limitations of the study and areas of further research were discussed.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGMENTS

"I can do all things through Christ who strengthens me" (Philippians 4:13, New King James Version).

I would first like to thank God for the opportunity to pursue a lifelong goal while working full-time, learning to be a husband, and learning to be a father. The blessings on my life over the past seven years are too numerable to count and after marrying my wife and the birth of our daughter, I consider the opportunity to study mathematics education at the University of Pittsburgh to be tops among them.

There are so many people whom I must thank for their help, support, words of encouragement, and general interest in my program during my time as a doctoral student. First and foremost, I must thank my wife of seven years and four months for her limitless love, support, and patience. To Rebekah: thank you for your initial support of me going back to school, saying "this is something that you have always wanted to do; go for it", for your undying support during the subsequent years where weekends were often sacrificed and late nights became the norm, and for your boundless love and patience when completion of my dissertation became the biggest question in both of our lives. I would also like to thank my daughter Liana for her patience and unconditional love when I chose to work on my dissertation instead of spending time with her. As much as this is for your mother and me, it is also for you.

I would also like to thank my parents, Dr. Thomas and Katharine Flanders. You taught me persistence, diligence, and that we do not give up when things become difficult. That I have completed this degree is a testament to the work ethic that you instilled in me from the time that I was a child. Thank you also for your unending words of encouragement and prayers of support. To my mother-in-law, Jennifer Entwistle, thank you for being there for Rebekah and Liana when I was not.

I would like to express gratitude to the members of my dissertation committee: To Dr. Peg Smith for her support, knowledge, and ability to teach during the three different courses in which you were my instructor. In as much as my abilities as a mathematics instructor have improved during my time in the program, much of it is due to watching you in class and then implementing the behaviors that you modeled. Thank you also for your feedback on my overview. I would like to thank Dr. Beverly Michael for agreeing to sit on my committee and for extending your knowledge of the Rule of Four and best practices of calculus instruction. To Dr. Jim Greeno: thank you for agreeing to sit on my committee and for challenging me to consider issues of cognitive development that I otherwise would not have considered, both in class and when writing my dissertation. To Dr. Chuck Munter: thank you for the thoughtful, deep questions that you posed during my comprehensive exam and my overview; they caused me to consider issues of calculus instruction and learning from a different perspective and directly resulted in a better product than what I otherwise would have produced. Finally, to my adviser of seven years and dissertation director, Dr. Ellen Ansell, thank you for your considerable patience with me as I have grown during my program. Thank you for your extensive time investment in me and my professional growth as a teacher and researcher. Your feedback and suggestions for revision at every step of the dissertation process have been invaluable and I will forever be indebted to you. In my experience,

your ability to engage in others' work, internalize what is and is not being said, and then restate the work in a single, concise statement is unequaled. Thank you for all of the questions that you have posed and the recommendations that you have made. To whatever degree this work furthers the body of knowledge and can be applied to future research, I owe it to you for your help in designing the study and analyzing the results. It is my professional goal and hope to be as much of an asset to future graduate students as you have been to me.

I would also like to thank the school district, superintendent, and principal who allowed me to conduct research in their building with their students. I also offer a special thank you to Kelsay, Michael, Fred, Jill, Kirsten, and Marcus. Thank you for giving up your free time to do more calculus and to help me complete my degree.

Finally, I would like to thank Dr. Mercedes McGowen for her advice and consultation on previous drafts of this document and Rick Campbell, Bill Coles, Scott Layden, Dr. Michelle Switala, and Dr. Adam Vrabel for their help at various times during the completion of this study. Each of you has played an integral part in the completion of my dissertation and my degree and I thank you for that.

1.0 INTRODUCTION

What does it mean to understand the first-year of calculus? While textbook authors have doubtless wrestled with this question for decades, Sofronas et al. (2011) recently conducted empirical research attempting to answer this very question. Four over-arching end-goals of calculus presented: "(a) mastery of the fundamental concepts and-or skills of the first-year calculus, (b) construction of connections and relationships between and among concepts and skills, (c) the ability to use the ideas of the first-year calculus, and (d) a deep sense of the context and purpose of the calculus" (Sofronas et al., 2011, p. 134). Without diminishing the other three end-goals, this study takes great interest in the second end-goal of calculus, constructing connections and relationships between calculus concepts and skills. Constructing connections and relationships between concepts and skills is what Hiebert and Carpenter (1992) defined as building conceptual understanding. Using Hiebert and Carpenter's definition of understanding, this study examined the extent to which Advanced Placement (AP) Calculus students develop understanding of calculus concepts.

This study agreed with the stance taken by the National Council of Teachers of Mathematics (NCTM) that building understanding is a constructive process (NCTM, 1991, 2000). Constructivism assumes that students construct their own knowledge of mathematics (Richards & von Glasersfeld, 1980; von Glasersfeld, 1981). Within a constructivist framework, Hiebert and Carpenter (1992) conceptualized understanding as building or making connections between existing knowledge structures. Thus, understanding develops as networks of mental representations become larger and more connected, and as existing connections are strengthened (Hiebert & Carpenter, 1992). These connections can exist between facts, skills, concepts and procedures (Gray & Tall, 1994; Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986).

This framework for building mathematical understanding is entirely dependent on a student's ability to construct mental connections between learned concepts and procedures or skills, where a mathematical concept can be described as "a complex web of ideas developed from mathematical definitions and mental constructs" (Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013, p. 70) and a procedure or skill is a "specific algorithm for implementing a process" (Gray & Tall, 1994, p. 117). Thus, a worthy goal of mathematics education research is to investigate the kinds of processes that help students to acquire mathematical concepts and procedures and to develop connections between them.

The Russian psychologist V.A. Krutetskii (1976) identified three problem solving processes that may help students to construct connections between learned concepts and procedures (Confrey, 1981; Confrey & Lanier, 1980; Norman & Prichard, 1994) and that are particularly salient to constructing understanding of calculus concepts (Norman & Prichard, 1994). The Krutetskiian problem solving processes of generalizability, flexibility, and reversibility "can be seen as the basic processes in the acquisition of concepts in mathematics" (Confrey, 1981, p. 10) and should be a focus of development in the mathematics classroom. Acknowledging the importance of generalizability, this study took an interest in the development of flexibility and reversibility as problem solving processes and how students make use of reversibility to solve problems in a calculus environment.

In the subsequent sections, I briefly describe the Krutetskiian problem solving processes of generalizability, flexibility, and reversibility, and the potential that developing reversibility holds as a mechanism for developing the kinds of connections between learned concepts and procedures that are the hallmark of mathematical understanding (Hiebert & Carpenter, 1992). I then discuss the use of multiple representations in studying functions and the potential for constructing mathematical understanding by linking multiple representations. A possible overlapping between the concepts of generalizability, flexibility, reversibility, and multiple representations concludes the discussion. A conceptual framework for the present study is presented and the motivation and research questions are then described. The chapter concludes with a discussion of the possible significance of this study and an outline of the remaining chapters.

1.1 KRUTETSKIIAN CONSTRUCTS

Over the span of two decades, Krutetskii (1976) observed the problem-solving behaviors of students described as “capable”, “average”, and “incapable”. One outcome of his research was the identification of three problem-solving processes that capable mathematics students demonstrate: 1) generalizability, 2) flexibility, and 3) reversibility.

1.1.1 Generalizability

Generalizability consists of two aspects: “(1) a person’s ability to see something general and known to him in what is particular and concrete … and (2) the ability to see something general and still unknown to him in what is isolated and particular (to deduce the general from particular cases, to form a concept)" (Krutetskii, 1976, p. 287). Generalizability as described by Krutetskii is similar to what Lesh, Post, and Behr (1987) later described as evidence of understanding. They note that a student's ability to "recognize the idea within given representational systems" (p. 8) is a key evidence of understanding. A student who exhibits generalizability would be able to recognize a particular mathematical concept or idea within an unfamiliar system. Generalizability often surfaces in spaces where students generalize a rule from a specific example or from a series of examples (Norman & Prichard, 1994). For example, calculus students will often derive the formula for the derivative of several polynomial functions before abstracting a general equation (the simple power rule):

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(x^4) = 4x^3$$

following the pattern, students will typically propose that $\frac{d}{dx}(x^n) = nx^{n-1}$.

1.1.2 Flexibility

Flexibility is defined as "an ability to switch rapidly from one operation to another, from one train of thought to another" (Krutetskii, 1976, p. 222). Teachey (2003) viewed Krutetskiian flexibility as "the ability to change from one perspective to another perspective (e.g., change from an algebraic perspective to a graphical perspective)" (p. 6). Lesh et al. (1987) described an ability similar to flexibility as part of understanding when they observe that “part of what we mean when we say that a student ‘understands’ an idea like ‘1/3’ is that … he or she can accurately translate the idea from one system to another” (p. 8). An inability to demonstrate flexibility in thinking is often related to difficulties in working with multiple representations (Lesh et al., 1987) of functions (Norman & Prichard, 1994).

Flexibility is a key construct within the Common Core State Standards for Mathematics (CCSS-M). A review of the CCSS-M by grade level indicates that the use of different representations to make sense of problems and as a modeling technique should take place from kindergarten through twelfth grade and that the representations available to students ought to increase in complexity and number as students progress through the grade levels (NGA, 2010). In this study, I will restrict my discussion to the representations and connections between representations that the CCSS-M recommends at the high school level, namely the algebraic, graphical, numerical, and verbal representations (collectively referred to as multiple representations) of functions (NGA, 2010).

Flexibility with multiple representations of functions is a primary focus of this study because of their importance as a prerequisite for successfully completing Advanced Placement (AP) Calculus AB in high school (Collegeboard, 2010a) and because of the centrality of the function concept to understanding the functional properties of rate of change (often called the process of differentiation) and accumulation of area (often called the process of integration) (Sofronas et al., 2011).

1.1.3 Reversibility

Reversibility is one kind of network connection where a learner develops an ability to move back and forth between an input and an output or between a process and an outcome. “The reversibility of a mental process here means a reconstruction of its direction in the sense of switching from a direct to a reverse train of thought” (Krutetskii, 1976, p. 287). Lesh et al. (1987) describe a similar process as evidence of what it means to understand a mathematical concept. They identified two features of understanding a mathematical idea as 1) using a model about an original situation to make predictions and 2) translating predictions through a model back to the original situation. Krutetskii (1976) identified two processes that comprise reversibility: 1) the establishment of two-way processes; and, 2) the reversibility of the mental process in reasoning.

I suggest that we can conceptualize reversibility as a kind of network connection within the framework of conceptual learning and building understanding proposed by Hiebert and Carpenter (1992). In this viewpoint, reversibility serves as a two-way bridge between nodes within the network of knowledge and possibly as a two-way bridge between two networks of knowledge. As a learner develops reversibility, he/she develops a stronger, closer knit network. Thus, developing a reversible thought process, thinking about a mathematical process from beginning to end and end to beginning, may correlate with improved mathematical understanding.

In his research, Krutetskii (1976) interviewed 62 classroom teachers in order to classify the teachers’ students as capable, average, and incapable. Capable students are similar to what might be described as gifted students today (Teachey, 2003) and typically performed well in previous math classes. Average students demonstrated some mathematical ability, but did not stand out to their teachers as exceeding or failing to meet expectations. Incapable students were described as poor math students with low mathematical abilities. For capable students, the network connections (what Krutetskii called "bonds") "and their systems established in a straightforward direction took on a reversible character immediately ('on the spot'). Establishing or forming direct associations meant a simultaneous (or almost so) formation or establishment of reverse associations" (p. 288). However, for the average mathematics pupil to develop reversibility, special exercises accompanied by instruction were necessary. Incapable students were not able to develop reversible bonds even after receiving special exercises designed to help the students develop reversible bonds. Special exercises were typically a reverse problem following a direct problem (Krutetskii, 1976). A simple example of a Krutetskiian special exercise would be having a student solve $4 + 3 = \_\_\_$ and then having the student solve $4 + \_\_\_ = 7$.

Gray and Tall (1994) suggested that reversing a process is a key step in building "proceptual encapsulation" (p. 135). Proceptual encapsulation occurs when the reversing of a process is no longer seen as creating a new process at a different level of a mathematical hierarchy of relationships, but instead is seen as an equivalent process on the same level of the hierarchy. For example, consider addition and subtraction, two reversible processes. The student who has proceptually encapsulated addition would not see subtraction as a new process to be learned but rather addition in reverse. A student who lacks proceptual encapsulation would view subtraction as a new process, separate from addition (Gray & Tall, 1994).

Norman and Prichard (1994) proposed using generalizability, flexibility, and reversibility as a framework for analyzing conceptual understanding of calculus students. They suggested investigating each construct by designing problem-solving items that require generalizability, flexibility, and reversibility in order to solve. Examples of Krutetskiian constructs in calculus include deriving the simple power rule from a series of examples (generalization), making connections between the algebraic, graphical, and numerical representations of functions in order to solve problems involving limits, continuity, derivatives, and integrals (flexibility), and understanding the inverse relationship between differentiation and integration (reversibility) (Norman & Prichard, 1994). The reversible connection between the derivative as the rate of change of a function and the integral as the total change (also called the accumulating change or accumulating area) has been identified as key sub-goal of constructing connections and relationships between calculus concepts and skills (Sofronas et al., 2011). Figure 1 shows the reversible relationship between differentiation and integration.

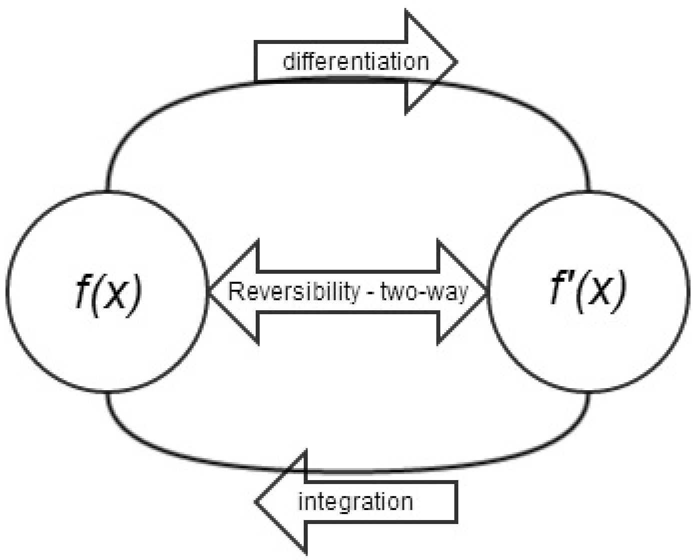


Figure 1. The reversible relationship of differentiation and integration

The Krutetskiian problem-solving processes of generalizability, flexibility, and reversibility are three kinds of mental processes that help to build the connections that support mathematical understanding. Researchers have suggested that these processes be developed in the mathematics classroom in order to promote conceptual learning and understanding (Norman & Prichard, 1994; Rachlin, 1981; Teachey, 2003). The potential that generalizability, flexibility, and reversibility hold for promoting mathematical understanding suggests a research agenda investigating how students develop these processes and how they access the processes in problem solving.

1.2 MULTIPLE REPRESENTATIONS

The CCSS-M identifies “analyze functions using different representations” (NGA, 2010, p. 68) as a standard of learning for high school students within the Functions domain. Moschkovich, Schoenfeld, and Arcavi (1993) and later Knuth (2000) described the “Cartesian Connection” as the need for students to make connections and understand relationships between functions presented algebraically, graphically, and as a tabular representation (the tabular representation is often referred to as the numerical representation).

One instructional approach fully immersed in the use of multiple representations is what the Calculus Consortium at Harvard (CCH) named “The Rule of Three” (Gleason & Hughes-Hallett, 1992, p. 1). The Rule of Three is a premise that all calculus instruction should attend to the graphical, numerical, and analytical representations of calculus concepts whenever appropriate. Beginning in 1995, the CCH produced a calculus textbook, *Calculus: single variable* in which the graphical, numerical, and analytical representations of calculus concepts are “emphasized throughout” (Gleason & Hughes-Hallett, 1992, p. 1). Beginning with the second edition, the CCH included a fourth representation, the verbal representation, which is the use of verbal descriptions to explain mathematical actions and as the traditional “word problem”. Thus, “The Rule of Three” became “The Rule of Four”, which has been extended beyond calculus to functions.

The Rule of Three/Four is not limited in applicability to calculus. At approximately the same time (late 1980’s and early 1990’s) as the CCH was coining the phrase “Rule of Three”, other researchers were reaching similar conclusions regarding the study of functions in algebra classes. Williams (1993) concluded that an emergent theme of mathematical practices necessary for understanding functions and graphs is "the importance of being able to move comfortably between and among the three different representations of function: algebraic, graphical, and tabular" (p. 329). As part of the interpreting functions standard, the CCSS-M recognizes the rule of four, recommending that high school students be able to "compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions)" (NGA, 2010, p. 70). The AP Calculus course description now states that an end goal of AP Calculus is that "students should be able to work with functions represented in a variety of ways: graphical, numerical, analytical, or verbal. They should understand the connections among these representations" (Collegeboard, 2010b, p. 6). This suggests that a key part of learning calculus includes developing flexibility with the multiple representations of functions, the end goal of the Rule of Four.

The ability to move between multiple representations whenever appropriate is a defining feature of mathematical understanding (Tall, 1992). Janvier (1987a) referred to the act of switching from one representation into another as *translation*. Lesh et al. (1987) described three actions, all of which require proficiently translating between multiple representations, that are evidence of student understanding, saying that

> part of what we mean when we say that a student 'understands' … is that: (1) he or she can recognize the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within given representational systems, and (3) he or she can accurately translate the idea from one system to another. (Lesh et al., 1987, p. 36)

I use the phrase “linking multiple representations” synonymously with the phrase “translating between representations”. The word linking helps to connote the image presented in figure 2 and is meant to indicate that the learner understands a relationship between the representations involved in the translation.

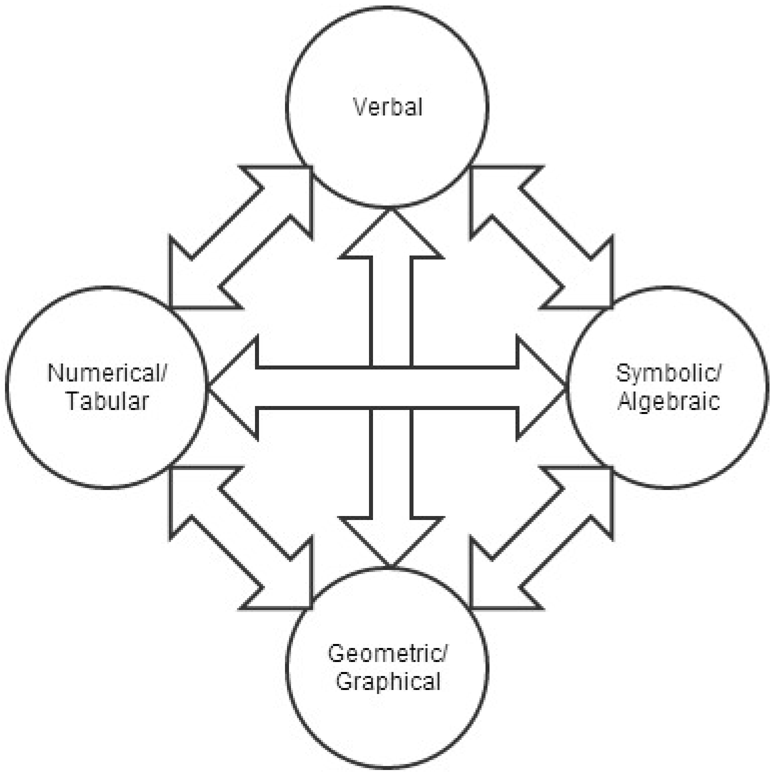


Figure 2. A model of the translations between the four representations of “The Rule of Four”

The NCTM (2000) recommended that students be able to “select, apply, and translate among mathematical representations to solve problems” (p. 63). CCSS-M lists the ability to analyze functions in multiple representations as a high school standard of learning. There is evidence that shows that the use of multiple representations of mathematical objects is one way to build networks of mental constructions (Dick & Edwards, 2008). As an evidence of student understanding and as an end goal of mathematics instruction, fluency with multiple representations is a highly desired outcome of mathematics education.

The Krutetskiian constructs of generalizability, flexibility, and reversibility are an integral part of an analysis of functions through multiple representations. Generalizability typically presents through a symbolic notation, such as when conjecturing a formula from a list of examples, or proving a theorem by using induction. Flexibility is the ability to move between the multiple representations as necessary. A student who has flexibility can switch between thinking in a symbolic representation to a graphical representation without difficulty. Reversibility, within a multiple representations context, is the ability to translate a symbolic representation into a graphical representation and the ability to translate the graphical representation back into a symbolic representation. I define the term *representational reversibility* to refer to the ability to make reversible translations between representations. Representational reversibility falls under Krutetskii's second kind of reversibility process, reversibility of the mental process in reasoning. Thus, I suggest sub-diving "reversibility of the mental process in reasoning" into two distinct categories: 1) reversibility of the mental process in reasoning without reversible translations, and 2) reversibility of the mental process in reasoning with reversible translations (representational reversibility). Figure 3 shows the framework through which I view reversibility.

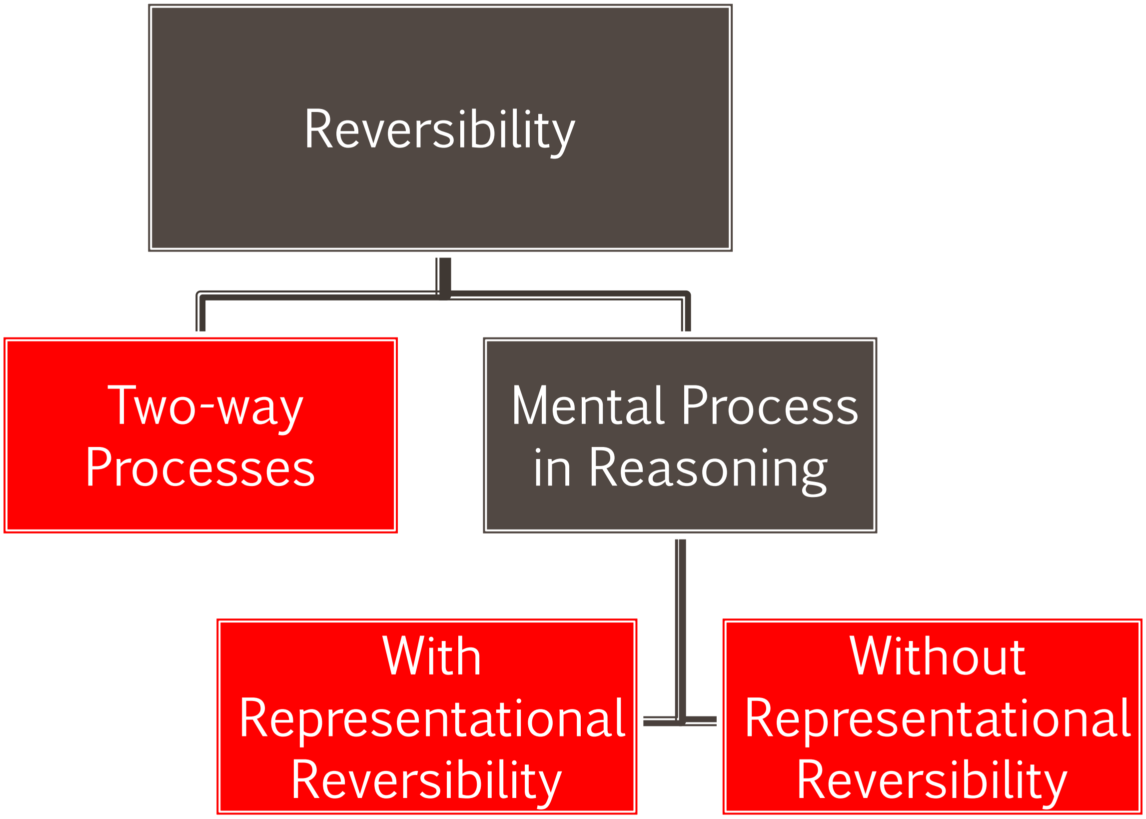


Figure 3. The three kinds of reversibility studied in this investigation

While closely related, there is a distinction between flexibility and reversibility within multiple representations: flexibility is the ability to switch between representations as necessary, reversibility is the ability to move back and forth between two specific representations. Flexibility does not require a bidirectional translation, reversibility does. Figure 2 shows an example of reversibility within multiple representations. The bidirectional arrows represent reversible links between representations. Figure 4 shows a visualization of flexibility within multiple representations. Flexibility can be seen when considering a one-way translation from tables to equations or from equations to table to graphs. Reversibility is represented by the bi-directional arrows that exist between each set of two representations.

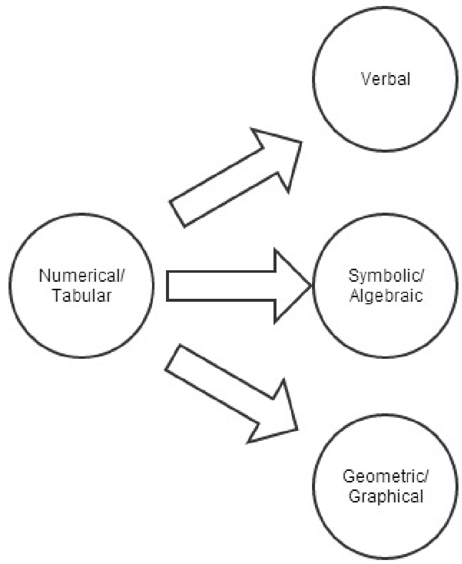


Figure 4. An example of flexibility within multiple representations

The distinction described between flexibility and reversibility within multiple representations describes the distinction I draw between the two categories of reversibility of the mental process in reasoning. The phrase "reversibility of the mental process in reasoning without reversible translations" does not preclude one-directional translation (i.e. flexibility), for example translating a table of values into a graph. This distinction precludes reversing the translation (i.e. reversibility of representation). Thus, direct and reverse set of problems that assesses reversibility of the mental process in reasoning without reversible translations may require the same directional translation in both the forward and reverse problems. A direct and reverse set of problems that assesses representational reversibility must require a one-directional translation in the forward task and the reverse translation in the reverse task.

1.3 CONCEPTUAL FRAMEWORK

Mathematical understanding consists of connections between mathematical concepts, processes, and procedures (Hiebert & Carpenter, 1992). Thus, in order to develop mathematical understanding, learners need to build a network of connections between concepts, processes, and procedures (also called a network of knowledge). In this section, I describe a current learning that describes how networks of knowledge are constructed. I then overlay reversibility onto the learning theory to show its possible role in building networks of connections.

A current theory about building a network of knowledge is the Action, Process, Object, Schema (APOS) framework (Asiala et al., 1996). APOS proposes that

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (Asiala et al., 1996, p. 5)

Within this framework, understanding a mathematical concept is first dependent on a student's ability to access and adjust (if necessary) previously constructed mental or physical objects to create an *action*. An action can be thought of as a procedure that transforms a mathematical object. When a student learns an action well enough that the student can reproduce the procedure without needing cues or a step-by-step recipe to follow, the student is said to have *interiorized* the action into a process. A key evidence of interiorizing an action conception into a process conception is that the student can reverse the procedure from output to input (Asiala et al., 1996).

Students move from a process conception of a mathematical concept to an object conception through the process of *encapsulation*. When a student encapsulates a process into an object, the student can now perform actions on the object. The object is transformable (Asiala et al., 1996). The researchers note the importance of de-encapsulation – the act of deconstructing an object into its constituent processes. Thus, reversibility is a key evidence of forming a process conception and reversibility is the process used to deconstruct an object into its constituent processes.

The fourth level of the APOS framework is schema construction. Schemas are organized structures of processes, objects, and other schemas (Asiala et al., 1996). It is reasonable to think of a schema as a network of knowledge. There is a noticeable similarity between schemas as defined by Asiala et al. (1996) and conceptual knowledge as defined by Hiebert and Carpenter (1992) and this similarity helps to explain how APOS theory can be viewed as a framework for the development of conceptual understanding. Building a schema requires linking processes, objects, and existing schemas (Asiala et al., 1996), much like the building of conceptual knowledge consists of the constructing of links between networks of existing mental representations and new mental constructions (Hiebert & Carpenter, 1992).

Schemas can themselves be treated as objects. When a student is able to perform actions and processes on a schema, the student has thematized a schema. An object created by thematizing a schema and then acted upon by an action or process can be de-thematized back to its original schema (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997). The Krutetskiian construct of reversibility is necessary to de-thematize an object into its schema.

Figure 5 shows the relationships within the APOS framework and where the Krutetskiian construct of reversibility fits within APOS.

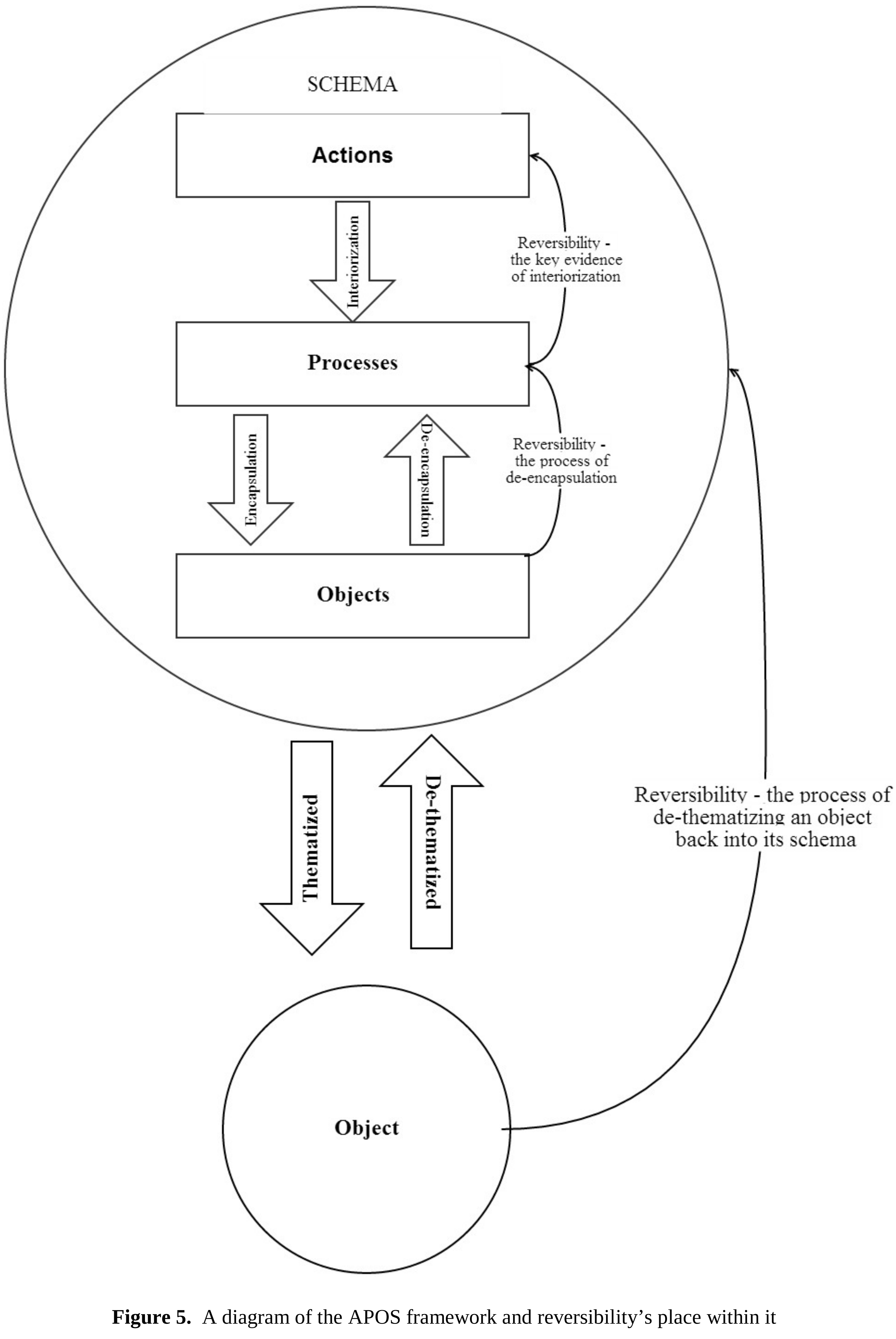


Figure 5. A diagram of the APOS framework and reversibility's place within it

Thus, reversibility is necessary to develop a process conception from an action conception, an object conception from a process conception, and is necessary to de-thematize a transformed object into its constituent schema(s). Since reversibility is necessary for constructing schema within the APOS framework, and since schema construction is an example of building conceptual understanding, we can conclude that developing the reversibility process is necessary for developing conceptual understanding. In order to help demonstrate the relationships within the APOS framework and reversibility's place within it, I present a mathematical example put forth by Breidenbach, Dubinsky, Hawks, and Nichols (1992) as descriptive evidence of the differences between the action and process conceptions of functions. I extend the example by describing possible features of an object conception of function and a function schema thematized into a new object.

An action conception of a function would likely be limited to substituting numbers into symbolic equations and evaluating the resulting expression. Students limited to an action conception of function can only solve one-step problems involving functions, such as evaluating the function at one input value. Considering evaluating the function on an infinite domain would be out of reach for a student limited to an action conception. The student limited to an action conception sees a function as a discrete input/output machine, not as describing a continuous relationship. Also, an action conception would not permit the composition of functions except in such a case where the student could evaluate a composition such as $f(x) + g(x)$ by substituting a value for x into each equation one step at a time (Breidenbach et al., 1992).

When an action conception of function has been interiorized into a process conception, the student can now evaluate functions across domains instead of at isolated points, can compose functions, and can reverse the effects of a function. This means that rather than only producing an output from a given input, the student could consider an output and identify a possible input. In this sense, the concepts of a one-to-one function and an invertible function become accessible (Breidenbach et al., 1992). Here, reversibility becomes a key evidence of interiorizing an action into a process. Without reversibility, the idea of an inverse function or the ability to move from output back to input is inconceivable.

When a student encapsulates a process conception of function into an object conception, the student can now perform actions on the function. Thus, the function is no longer a machine that modifies inputs into outputs, but is itself an object to be transformed. For example, a student with an object conception of a function would be able to perform an operation on the function, such as differentiation, and understand that the differentiation process inputs a function and outputs a new function. Similarly, with an object conception, a student can perceive of a rational function as a polynomial function divided by another polynomial function. Reversibility presents as de-encapsulation – the process of deconstructing an object into its constituent processes. One example of de-encapsulation of functions (and thus reversibility) would be deconstructing the composition $H(x) = \sin^2(3x)$ into three separate functions, $f(x) = \sin x$, $g(x) = x^2$, and $h(x) = 3x$. Thus, $H(x) = g(f(h(x)))$.

The function schema itself could then be thematized into an object and transformed. When thematizing a schema, the student is able to consider the effects of a transformation on the whole of the schema and consider what elements would be affected by the transformation and what elements would be left unaffected. For example, a student who has thematized a schema of polynomial functions would be able to consider what would happen to a polynomial function if we restricted its domains and created a piecewise function consisting of three polynomials on separate domains. Recent research has shown that schema can be thematized, however, incidence of thematization is rare (Cooley, Trigueros, & Baker, 2007).

1.4 MOTIVATION, PURPOSE AND RESEARCH QUESTION

1.4.1 Motivation

Since Krutetskii's (1976) research, there has been little research examining how students develop reversibility. Lamon (2007) observed that "researchers know very little about reversibility" and that reversibility could be the subject "for a valuable microanalysis research agenda" (p. 661). Recently, a research agenda began attempting to explicate the development of reversibility (Haciomeroglu, Aspinwall, & Presmeg, 2009; Ramful & Olive, 2008). I hypothesized that a key learning activity in developing reversibility lies in Krutetskii's construct of flexibility. Although Krutetskii (1976) drew a clear distinction between reversibility and flexibility, he noted that there exists an overlap of flexibility of thinking and reversibility. This overlap exists in the necessity for a student to make a "sharp turn" (Krutetskii, 1976, p. 287) in his/her mental construction from moving in the forward direction to the reverse direction. According to Krutetskii (1976), this kind of sharp turn would necessarily fit his definition of flexible thinking – switching from one train of thought to another. Building on Norman and Pritchard's (1994) assertion that issues with flexibility often relate to a student's inability to move flexibly between multiple representations of functions, I hypothesized that flexibility and reversibility may develop as a result of instruction that attends to linking multiple representations. Haciomeroglu et al. (2009) found that students with a strong preference for graphs or a strong preference for algebraic expressions tend to develop one-sided thinking processes and do not show evidence of developing reversible conceptions, a conclusion predicted by Krutetskii's (1976) research. I suggest that considering mathematical concepts from visual, analytic, and numerical perspectives (i.e. learning mathematics from a Rule of Three perspective) will support the development of all three processes within reversibility.

1.4.2 Purpose and research questions

This study investigated reversibility and linking multiple representations in a calculus environment. Specifically, this study attempted to answer the following research questions:

1) To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?
2) To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:
    i. To what extent does reversibility of two-way reversible processes develop?
    ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?
    iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?
3) What are the thought processes that students utilize when using reversibility to solve problems?

1.5 SIGNIFICANCE OF THE STUDY

This study purports to inform mathematics education researchers and mathematics teachers about the development of flexibility and reversibility when students are engaged in a course that attends to linking multiple representations. I expect that both flexibility and reversibility will develop as the students consistently engage in activities that require linking multiple representations. This research study has the potential to identify instructional activities that help to develop flexibility and reversibility.

This study may help to address some current calls for research investigating learning mathematics in a multiple representations framework. Haciomeroglu, Aspinwall, and Presmeg (2010) called for research on "classroom instruction that identifies and then challenges students' preferred modes of thinking" (p. 174). Lesser and Tchoshanov (2005) asserted that "studying an effect that representations have on students' understanding is critical for effectiveness of teaching mathematics" (p. 1). The current study has the potential to offer insights into these research questions. Cheng (1999) said that a long term goal of mathematics education research should be to show that translating between representations improves students' conceptual learning. This study may help the research agenda to move forward in accomplishing this goal.

This study also has the potential to increase our understanding of reversibility, how it develops, and how students reason with reversibility. Researchers have identified reversibility as a key problem solving process (Confrey, 1981; Confrey & Lanier, 1980; Norman & Prichard, 1994), however we know little in the way of how students develop reversible conceptions, how students use reversibility to solve problems, or how students think about problems when using reversibility. This research study may offer insights into how students think about reversible conceptions, how they translate those thoughts into problem solving processes, and help inform the research community in how students develop reversibility, thus helping to answer the calls for a research agenda investigating the development of reversibility.

Finally, since rate of change and accumulation of area are both properties of the same function, and given the importance of reversibility in making connections between the concept of rate of change and the concept of accumulating area, and thus understanding the first-year of calculus (Sofronas et al., 2011), this study may reveal new insights into what it is about the function concept that students learn when developing reversibility and possibly when developing flexibility. For example, can a student consider the effects that a vertical transformation on a function would have on the function's rate of change or accumulation of area? Does flexibility of representations aide in considering these effects? Is reversibility of representational translations involved in this kind of thought process? If so, how? This study has potential to offer insight into these questions.

1.6 LIMITATIONS OF THE STUDY

There are several factors that limit the generalizability of this study: 1) the participants in the study, 2) the content lens of the study, and 3) several of the assessment items only test a learning goal one time.

The participants in this study consist of AP Calculus AB students. Thus, this population consists of students who have been sufficiently successful in mathematics as to take a college level math course while in 10^{th}, 11^{th}, or 12^{th} grade. It is possible that the results of this study can only be extended to similar populations. However, if the results are limited in generalizability, the results would still strengthen calls for reversibility research with other populations.

This study is completely situated within calculus. It is possible that reversibility is context specific; as such, it could be the case that if students develop reversibility, it may be due to the nature of calculus as an ideal content lens for investigating the relationship between reversibility and linking multiple representations. Results indicating that a relationship exists would suggest that future research that examines reversibility and linking multiple representations should use other mathematical content to assess if the results of this study are replicable in other mathematical contexts.

Two of the assessment items, the flexibility pre-test and the differentiation competency test, each test 18 instances of flexibility and 6 instances of representational reversibility. It is possible that a student may have some amount of flexibility and reversibility in another mathematical content domain and that the assessment items used here do not offer the student the opportunity to demonstrate flexibility and/or reversibility. Given this limitation, there is a need for further research in reversibility and flexibility in other mathematical content domains.

1.7 OUTLINE OF THE STUDY

My primary purposes in Chapter 1 were to describe the theoretical assumptions that underlie this research study and to present reversibility and the linking of multiple representations as abilities that help students to construct mathematical understanding. The literature reviewed should serve to accomplish two main goals: 1) to present the research started by Krutetskii as worthy of continuing and that the development of reversibility in particular, should be examined, and 2) to show that there is reason to believe that engaging in a mathematics course that attends to linking multiple representations has the potential to foster the development of flexibility and reversibility.

In Chapter 2, I provided a thorough review of research conducted on developing reversibility and present a review of the literature published on the use of multiple representations. I attempted to make the argument, informed by the existing bodies of literature on reversibility and multiple representations, that engaging in a mathematics course that attends to linking multiple representations has the potential to foster the development of flexibility and reversibility.

Chapter 3 described the methodology by which I examined the extent to which flexibility and reversibility developed when students engaged in a mathematics course that attended to linking multiple representations. I described the students that participated in the study, the mathematical content presented through multiple representations, the data sources, the procedures for collecting the data, and the methods of data analysis.

Chapter 4 reported the results of the study and data analysis. Chapter 5 discussed the findings of the study and implications for future research.

2.0 LITERATURE REVIEW

The literature review of this study consists of four sections: (1) a review of the literature on reversibility, (2) a review of the literature on multiple representations, (3) an argument for why linking multiple representations will help students to develop reversibility, and (4) a brief summary of the chapter. This chapter positioned my study within the research on the development of reversibility and will presented evidence for why I suggest that instruction linking multiple representations will help students develop reversibility. This chapter will also help to situate my data set within the relevant literature conducted within a calculus classroom.

2.1 REVIEW OF REVERSIBILITY LITERATURE

The concept of reversibility has recently received renewed attention within the mathematics education literature. However, reversibility was originally proposed and studied by cognitive psychologists, most famously (the Russian psychologist) V.A. Krutetskii (1976) and Jean Piaget (Inhelder & Piaget, 1958; Sparks, Brown, & Bassler, 1970). It is upon the foundation laid by Krutetskii and to a lesser extent Piaget (Inhelder & Piaget, 1958) that research on how reversibility develops stands. In this section, I reviewed the body of research on reversibility and discussed the possible existence of an overlap between reversibility and flexibility. I began by discussing definitions of the key terms and then discussed the major conclusions about reversibility and how reversibility develops. I concluded the section with a discussion of current calls for a research agenda investigating various elements of reversibility.

2.1.1 Definition of reversibility

Krutetskii (1976) defined reversibility as a mental process; "the reversibility of a mental process here means a reconstruction of its direction in the sense of switching from a direct to a reverse train of thought" (p. 287). To distinguish between direct and reverse trains of thought, Krutetskii referred to a sequence of thoughts from A to E as a direct bond and sequence of thoughts from E to A as a reverse bond. Krutetskii suggests that reversibility is composed of two different, yet interrelated processes: 1) the establishment of bidirectional bonds (two-way processes), such as a biconditional theorem, as opposed to only constructing a one-way bond such as in a conditional theorem; 2) the reversibility of the mental process in reasoning.

Krutetskii (1976) described the second process as "thinking in a reverse direction from the result of the product of the initial data" (p. 287). One example of thinking in a reverse direction would be the construction of a converse to a theorem. He specifically observed that a reverse train of thought need not follow the same thought processes as the original thought. As an example, Krutetskii said that if a student learns a six-step process, call the steps A, B, C, D, E, F, then the student has learned the process from A to F. Reversibility would then require learning the process from F to A. However, reversing the process from F to A would not need to follow the steps F, E, D, C, B, A. Reversibility only requires that the process begin at F and conclude at A. Thus, Krutetskii proposed that reversibility can present in two separate ways. Reversibility can exist as a direct and reverse association, $A \leftrightarrow D$. In other cases, the process cannot be reversed across the same path and leads only to a reversal of general thought, $A \rightleftharpoons D$.

Krutetskii (1976) cautioned that often reversibility cannot be reduced to simple reverse associations. For example, consider factoring the difference of squares, $a^2 - b^2$. An algebra student who has learned how to factor the difference of squares answers $a^2 - b^2 = (a - b)(a + b)$. However, when a student is asked to multiply $(a - b)(a + b)$ a student is likely to follow the FOIL (First, Outer, Inner, Last) procedure and conclude that $(a - b)(a + b) = a^2 - ab + ab - b^2 = a^2 - b^2$. This procedure produces a correct answer and is an example of reversibility. However, it does not simply reverse the process of factoring the difference of squares.

Consider a second example from calculus. Differentiation and integration are completely reversible operations (Norman & Prichard, 1994), meaning that they are inverse operations. However, the relationship can only be reduced to simple reverse associations in trivial, procedural examples, such as $\frac{d}{dx}[x^2] = 2x$ and $\int 2x\, dx = x^2 + C$. In this example, the simple power rule for differentiation, $\frac{d}{dx}[x^n] = n\, x^{n-1}$, and the simple power rule for integration, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, are simple reverse associations. Reversibility should be used to generate and justify integration formulas from known derivatives; however, research indicates that calculus students tend not to use reversibility to develop connections between differentiation and integration (Norman & Prichard, 1994).

A non-trivial, procedural example of differentiating a function that requires the product rule, such as $f(x) = e^x x^2$, serves as evidence of how reversible properties do not only consist of reversing the sequence of a known procedure. In this case, $f'(x) = xe^x(2 + x)$. Integrating $f'(x)$ would by definition produce $f(x) = e^x x^2 + C$. However, the mathematical procedures necessary to integrate $f'(x) = xe^x(2 + x)$ to prove that $f(x) = e^x x^2$ are far more complicated than the product rule for differentiation and require an integration schema that includes integration by parts.

A common conceptual example of the use of reversibility in calculus involves describing the characteristics of functions. Many calculus students are adept at looking at a graph of a function and describing the characteristics of the function and the function's derivative; however, constructing the graph of a function from a discrete list of characteristics of the function and the function's derivative is extremely difficult for students (Baker, Cooley, & Trigueros, 2000; Norman & Prichard, 1994). Thus, it may be the case that reversibility requires a complex network of mental constructions and connections.

The two processes that comprise Krutetskiian reversibility have a commonality. Krutetskii (1976) noted that both processes require a "sharp turn" (p. 287) in the student's mental construction, from moving in the forward direction to the reverse direction. This sharp turn is particularly difficult for students; but, Krutetskii concluded from an extensive review of literature that "numerous psychological studies in our country and elsewhere have shown that reverse bonds can be formed at the same time that direct ones are established" (p. 288).

Krutetskii's (1976) definition of reversibility has been adopted without modification by other researchers in subsequent decades and applied to various age levels and levels of mathematics. Rachlin (1981), in his dissertation, studied reversibility as defined by Krutetskii in college algebra students. Later, Rachlin extended the definition of reversibility saying, "rather than refer merely to working backward through the steps of an algorithm, reversibility in thinking refers to finding an alternative path for meeting the conditions of the problem" (p. 471). Norman and Prichard (1994) identified Krutetskiian reversibility as a key process in the learning of calculus. Steffe (1994) positioned Krutetskii's (1976)definition of reversibility as a key process in children's counting. Fuson (1992) implicitly agreed with Krutetskii's definition of reversibility by saying that "reversibility knowledge enables one to think about a situation in reverse" (p. 257).

Krutetskii (1976) concluded his definition of reversibility by noting its distinction from Piaget's conception; "it is clear … that we are not attaching to the concept of reversibility the importance that Piaget does" (p. 287). For Piaget, reversibility locates within inversion (also called negation) and equilibration (also called conservation) (Flavell, 1963; Inhelder & Piaget, 1958). As a matter of inversion, reversibility presents as the ability to return a product to its starting point by canceling an operation that has already been performed. In this instance, the outcome of the direct operation and its inverse, thus the outcome of reversible processes, is the identity operation (Inhelder & Piaget, 1958). Rachlin (1981) described reversibility as an outcome of inversion as Piagetian "operational reversibility" (p. 17).

Flavell (1963) reported that Piaget used the terms *reversibility* and *equilibrium* nearly interchangeably. Piaget viewed reversibility as a necessary outcome of a complete understanding of reciprocal operations and the process of negation. The Piagetian perspective identifies that students should become aware that an operation can be undone (reversed) through an appropriate action or sequence of actions. When the forward and reverse actions are taken together, the student perceives equilibration (or reversibility).

Sparks et al. (1970) further developed Piaget's idea of reversibility. They characterized reversibility as the realization that prior to an action changing an object, there exists an inverse operation that will restore the object to its prior state. Even when actions taken result in objects that cannot be physically restored (such as in what takes place in a chemical reaction) to its original state, the reversible thought process is still present and is an entirely theoretical exercise. Thus, Sparks et al. (1970) concluded that in the Piagetian conservation experiment, reversibility presents when the child can mentally restore the object to its original state and note the invariance. This description by Sparks et al. (1970) seems to identify an overlap between the definitions given by

Krutetskii and Piaget. The linking similarity lies in Krutetskii's (1976) second process, "thinking in a reverse direction from the result of the product of the initial data" (p. 287). In both descriptions of reversibility, a key process is the mental reconstruction of taking an output and returning it to its constituent input(s) through negation, reciprocal operations, etc.

In this document, I attempted to continue in the footsteps of other reversibility researchers (Fuson, 1992; Norman & Prichard, 1994; Rachlin, 1981; Steffe, 1994) and build on Krutetskii's (1976) definition of reversibility. Krutetskii's definition is particularly salient for my research as this dissertation examined reversibility through a calculus lens and Norman and Prichard (1994) have noted the usefulness of Krutetskiian reversibility as a framework for observing calculus learning.

In this study, the development of reversibility will account for the development of two-way directional processes and for the development of a reversible thought process. When it is necessary to draw a distinction between the two aspects of Krutetskiian reversibility, I refer specifically to the act of constructing two-way directional processes or to developing a reversible thought process.

2.1.2 Definition of flexibility

A process that some researchers (Gray & Tall, 1994; Kendal & Stacey, 2003; McGowen, 2006) believe to be closely related to reversibility is flexibility. Krutetskii (1976) first identified flexibility as a key problem-solving process of capable mathematics students. Krutetskii defined flexibility as "an ability to switch from one mental operation to another" (p. 88). Kilpatrick (1978), who edited the English translation of Krutetskii's (1976) research, observed that, within Krutetskii's work, flexibility presents in two ways: 1) the degree to which a correct solution method on a prior problem does not limit the student's approach to a subsequent problem; and, 2) a student's ease in switching between multiple successful solution methods to the same problem.

Researchers have used Krutetskii's (1976) definition of flexibility as a framework for studying various elements of how students learn (Hashimoto & Becker, 1999; Kendal & Stacey, 2003; Norman & Prichard, 1994; Rachlin, 1981). Gray and Tall (1994) offered a nuanced definition of flexibility, relating it to the mathematics learning of processes and concepts. They defined flexibility as the ability to move freely between a symbolic notation's representation as either a process or concept. This definition suggests that flexible thinking situates in the cognitive shift from seeing a symbolic notation as a procedural directive to seeing the same symbolic notation as a mental object to be manipulated. Gray and Tall noted specifically that flexible thinking is a necessity in mathematics due to an inherent ambiguity in mathematical notation. Many mathematical notations can represent both a procedure and an object. For example, consider the calculus student who sees the symbolic notation $\frac{d}{dx}[x^2]$. A procedural thought process sees this notation as a symbol for the simple power rule and thus the output $2x$. Flexible thinking, as described by Gray and Tall (1994), requires that a student sees $\frac{d}{dx}[x^2]$ as a mental object upon which further transformations can be performed. Gray and Tall defined the difference between students who can use flexible thinking to "produce new known facts from old, giving a built-in feedback loop that acts as an autonomous knowledge generator" (p. 132) and students who lack flexible thinking and only have procedures as the "proceptual divide" (p. 132). They suggested that the proceptual divide "is one of the most significant factors in the difference between success and failure" (p. 132).

2.1.3 Relationship between reversibility and flexibility

Krutetskii (1976) positioned reversibility and flexibility as separate problem-solving processes but then also concluded that they and other processes are "closely interrelated, influencing one another and forming in their aggregate a single integral system, … the mathematical cast of mind" (p. 351). A look at the types of questions Krutetskii used demonstrates the proposed differences between reversibility and flexibility. He only used one kind of problem to assess reversibility – direct/reverse question pairs. Krutetskii asked a question in a forward direction and then asked the same question in the reverse direction. For example, a direct question would be "a saw in a sawmill saws off a 1 m piece of log every minute. How many minutes will it take to saw 16 m of log" (Krutetskii, 1976, p. 144). The reverse question is "in 3 minutes a log is sawed up into half-meter pieces, with each cutting taking 1 minute. Find the length of the log" (Krutetskii, 1976, p. 144). To test flexibility, Krutetskii used three different kinds of problems: 1) problems with multiple solution methods, 2) problems with changing content, and 3) problems requiring the reconstruction of an operation. Krutetskii argued that all three types of questions assess the same mental construction, the act of interrupting a "just-established solution pattern and replacing it with a new one" (Krutetskii, 1976, p. 277). An example of a question Krutetskii used to assess flexibility is "in how many ways can 78 rubles be paid if the money is in 3-ruble and 5-ruble notes" (Krutetskii, 1976, p. 136). The student solved the problem and then was tasked with finding the maximum number of solution methods to the aforementioned problem. These examples serve to demonstrate the difference between reversibility and flexibility as portrayed by Krutetskii. Reversibility requires a turn in the direction of thought, a change from thinking in the direction of input to output to thinking output back to input. Flexibility allows for the same direction of thought but requires a different kind of input to get to the same output.

Teachey (2003) followed Krutetskii's design dichotomy by testing students' reversibility and flexibility using separate kinds of problems. Teachey assessed gifted high school students' conceptual and procedural knowledge of polynomial functions. She noted that both reversibility and flexibility require connections as described by Hiebert and Lefevre (1986), but that each requires a different kind of connection. Reversibility requires connecting knowledge in appropriate sequences while flexibility requires connecting "pieces of knowledge so that a student can move easily from one representation to another or can connect a new representation to an existing concept" (Teachey, 2003, p. 12). She assessed reversibility by asking questions that required students to think in a reverse direction; she assessed flexibility by requiring students to make transitions between different functional representations. Thus, Teachey's research design distinguishes between reversibility and flexibility in the same way that Krutetskii did.

Other researchers have suggested that a relationship exists between reversibility and flexibility. Piaget contended that reversibility is essential to cognitive flexibility (Inhelder & Piaget, 1958). Usiskin (1999) and McGowen (2006) both portrayed reversibility as a specific kind of flexibility. Although not drawing this conclusion himself, it appears that Rachlin (1981) also viewed reversibility as a specific kind of flexibility, as he concluded that his "subjects' patterns of reversibility were also reflected in their ability to switch from one approach to another" (p. 244).

Gray and Tall (1994) present reversibility as a key element in flexible thinking. The relationship exists within the process of "proceptual encapsulation" (p. 135). Proceptual encapsulation requires a student to view a forward and reverse process as equivalent processes within a mathematical hierarchy of relationships. For example, in calculus, differentiation and integration are two reversible processes. The student who has proceptually encapsulated differentiation would not see integration as a new process to be learned but rather differentiation in reverse. A student who lacks proceptual encapsulation would view integration as a new process, separate from differentiation (Gray & Tall, 1994). From this viewpoint, reversibility is necessary for proceptual encapsulation.

Based on the research of Usiskin (1999), McGowen (2006), Gray and Tall (1994), and Rachlin (1981), it is reasonable to argue that reversibility is a kind of flexibility. The possibility that reversibility is a specific instance of flexibility offers a mechanism that may improve reversibility. If reversibility is a kind of flexibility, then it may be the case that activities designed to improve flexibility (i.e. linking multiple representations) would necessarily improve reversibility.

2.1.4 Kinds of reversibility

Since Krutetskii (1976) and Piaget (Flavell, 1963; Inhelder & Piaget, 1958) introduced the concept of reversibility, the research community has attempted to parse reversibility into multiple forms.

As previously noted, Krutetskii (1976) proposed that reversibility presents in two separate processes: the establishment of bidirectional bonds (two-way processes) and the reversibility of the mental process in reasoning. Piaget also characterized reversibility in two different forms: negation and reciprocity. Negation, also referred to as inversion, situates in the study of classes and encompasses the concept that every forward operation has a reverse operation, typically called an inverse, which cancels the forward operation. Typical mathematical examples of negation would be addition-subtraction and multiplication-division, where addition is negated by subtraction and multiplication is negated by division. Reciprocity refers to structural relationships such as inequalities; if $a < b$, then the reciprocal relationship is $b > a$ (Flavell, 1963; Inhelder & Piaget, 1958). A significant distinction between negation (inversion) and reciprocity is that in negation the outcome of two reversible operations is the null set, whereas in reciprocity, the outcome of two reversible operations is an equivalence (Inhelder & Piaget, 1958).

Pinard (1981) noted that Piaget also associated reversibility with physical actions that undo previous actions. When reversibility refers to a physical action instead of a mental construction, it is referred to as revertibility (Pinard, 1981). "The essential difference, if one really does want to make a difference between revertibility and reversibility, is that between an action carried out *physically* and the same action carried out *mentally*" (Pinard, 1981, p. 30). The distinction Pinard (1981) draws lies in the fact that physical actions are "never reversible" (p. 30). For example, a physical subtraction, (Pinard (1981) gives the examples of decapitation, suicide, and murder) cannot be reversed by an additive action. However, even these extreme examples are all reversible as mental operations. This example helps to illuminate Pinard's (1981) description that physical actions are not reversible. He means that physical actions do not imply an immediate, reversible action. However, mental operations always simultaneously imply an inverse operation that reverses the mental action. Another way to state the difference between revertibility and reversibility is that reversibility always leads to the conservation of the original state; revertibility does not.

Schnall, Alter, Swanlund, and Schweitzer (1972) drew a distinction between what they term empirical analogues of reversibility (also referred to as empirical return) and logical reversibility. Empirical analogues of reversibility refers to a child's ability to solve a problem through some sensory-motor action, as can occur in experiments involving children and concrete objects. For example, consider Piaget's balance beam experiment. A weight is placed on one end of the beam, thus unbalancing the beam. The child is then expected to restore equilibrium, either by adding an equivalent weight to the other side of the beam (compensation) or by removing the weight (negation). It would be possible that a child could solve this problem without actually thinking about reversibility. Schnall et al. (1972) would call this an empirical analogue of reversibility; they also note that an empirical analogue of reversibility can apply to thought, but would necessarily require a “bidirectional tension” (p. 1013). In this description, an empirical analogue of reversibility of thought is congruent with Krutetskii’s (1976) first kind of reversibility, the construction of two-way bonds. Logical reversibility refers to the reversing of a thought process within the system’s properties. Thus, logical reversibility is synonymous with Krutetskii’s reversibility of the mental process in reasoning.

2.1.5 Reversibility as a key understanding in APOS theory

Researchers have identified reversibility as a key to understanding multiple mathematical principles and formulas (Flavell, 1963; Fuson, 1992; Kang, 2012; Krutetskii, 1976; Lamon, 1993; Norman & Prichard, 1994). One particular framework in which reversibility is a key factor is the APOS framework of schema development. APOS theory was developed in the early 1990’s by a group of researchers with the purpose of explaining how learning takes place in mathematics[1] (Asiala et al., 1996). APOS theory is a model of cognition that suggests a possible pathway of mental constructions that a learner may use to construct understanding of a particular mathematical concept. The pathway of mental constructions is referred to as a genetic decomposition. The genetic decomposition is proposed by the researchers and then informed and adjusted as necessary by collected data.

---

[1] For a complete discussion of the design, purpose, and implementation of APOS Theory, the reader is directed to Asiala et al. (1996)

APOS theory proposes a four-stage, hierarchy of human tendencies to handle mathematical problems. APOS theory assumes that learners construct mental actions, processes, objects, and schemas that allow a learner to reason through a mathematical problem (Dubinsky & McDonald, 2002).

Actions are transformations of an object that require an external stimulus of the learner; actions typically require step-by-step instructions for the learner to carry out successfully (Asiala et al., 1996; Dubinsky & McDonald, 2002). An action conception of a transformation requires that a student receive a direct stimulus and a detailed procedure to successfully carry out the transformation. For example, a student limited to an action conception of function can only see a function as a defined input-output machine. The student cannot make use of any properties of a function other than evaluating specific points and possibly manipulating the algebraic formula of the function.

When an action has been performed successfully enough times such that the learner can perform the action without the need of an external stimulus, the researchers say that the action has been *interiorized* into a *process* (Asiala et al., 1996). A key feature of a learner having a process conception of a transformation is when the process can be completed mentally, without the need for writing out all of the steps. For example, a student who has interiorized the concept of function from action conception to a process conception can now evaluate a function over intervals of input values instead of at specific points and can compose functions together to create new functions. Asiala et al. (1996) cite the example of evaluating $f(x) = \sin x$ as an example of interiorizing the function concept from action to process. $f(x) = \sin x$ does not indicate in any way how to evaluate $f(x)$ at specific x-values. It is incumbent upon the student to imagine $\sin x$ mapping onto the real number line, thus mapping x-values with the appropriate $\sin x$ value.

In order for a student to demonstrate a process conception of a transformation, the student must demonstrate reversibility of the transformation, meaning that when a learner interiorizes an action into a process, s/he is also able to perform the transformation in reverse, thus desconstrucing the process (Asiala et al., 1996). Here, reversibility becomes a key understanding in the APOS framework. For example, "with a process conception of function, an individual can …reverse the process to obtain inverse functions" (Asiala et al., 1996, p. 11). Extending the example of $f(x) = \sin x$ from above, one evidence of a process conception of function would be the ability to solve problems such as "find an angle θ such that $\sin\theta = \frac{\sqrt{3}}{2}$". In this case, the student would likely solve the problem using one of two methods: 1) s/he may use the inverse function $f(x) = \arcsin x$ to solve $f\left(\frac{\sqrt{3}}{2}\right) = \arcsin\frac{\sqrt{3}}{2}$, or 2) s/he may use reversibility to answer the question "the sine of what angle would give me the answer $\frac{\sqrt{3}}{2}$?" The first method would serve as an example of the first kind of Krutetskiian reversibility – the establishment of a two-way process. The learner would understand the inverse relationship between $\sin x$ and $\arcsin x$. The second method is an example of the second kind of reversibility – the reversing of a mental process in reasoning. Since the student does not use an inverse process to solve the problem, the student reasons from a known output (in this case, $\frac{\sqrt{3}}{2}$) to find an input.

A learner progresses from a process conception to an *object* conception when the process is no longer viewed as a group of isolated steps, but is now one coherent body that can then itself be transformed (Asiala et al., 1996). Asiala et al. (1996) refer to the act of understanding that a process is one coherent object upon which transformations may be performed as *encapsulating*. Encapsulation has also been referred to as reification (Sfard, 1991). The researchers make a special note to emphasize that encapsulation is a particulary difficult process for students and that in general, we have very few life models in which we perform actions upon other processes.

De-encapsulation is the ability to deconstruct a mathematical object into its constituent procresses. "In many mathematical operations, it is necessary to de-encapsulate an object and work with the process from which it came" (Asiala et al., 1997, p. 400). What Asiala et al. (1997) described as "de-encapsulation" is the Krutetskiian construct of reversibility. Thus, I suggest that similar to being the key understanding in developing a process conception, reversibility is the key understanding in developing an object conception of mathematical processes as well. Cuoco (1994) first proposed de-encapsulation of "functions-as-objects into the underlying processes" (p. 123) as a cornerstone of mathematical mastery. The definition of de-encapsulation given by Asiala et al. (1997) is nearly identical to Cuoco's (1994) and both are restatements of Krutetskiian reversibility.

Finally, a *schema* is an organized structure of objects, processes, and actions. Schemas consist of all of the connections (Hiebert & Lefevre, 1986) between objects, processes, and actions. Schemas themselves become objects which can be composed with other objects (Asiala et al., 1996). Asiala et al. (1996) further describe a schema as encompassing all of a learner's knowledge that is connected to a mathematical topic. "As with encapsulated processes, an object is created when a schema is *thematized* to become another kind of object which can also be *de-thematized* to obtain the original contents of the schema" (Asiala et al., 1997, p. 400). The process of de-thematizing a schema into its supporting contents is a further example of reversibility embedded within the APOS framework.

The description of the constructs within the APOS framework reveal that Krutetskiian reversibility is a key understanding of moving from an action to a process conception and from a process to an object conception. Reversibility is also the evidence of a thematized schema.

It should be noted that not all mathematics researchers have adopted the APOS approach for knowledge building and schema construction. Tall (2010) suggested that APOS is too limited a framework for capturing how people construct networks of mathematical knowledge. Specifically, Tall proposes that objects need not only proceed from encapsulation of actions; rather, an object should be created from other objects. For example, Tall offers that in calculus, a derivative graph (a visible object) can be created by operating on the graph of a function (another visible object). Tall intimated that for students to build understanding of the derivative as a function, it makes more sense to create a derivative function (an object) from an existing function (also an object) than to try to convince students to encapsulate the limiting process applied to the slope equation into an object called the derivative.

Tall (2011) attempted to offer a unifying theory of how people construct knowledge in mathematics because “mathematical operations are not arbitrary, but have inevitable relationships that work in a particular context” (p. 2). Tall (2011) referred to this theory as a theory of crystalline concepts. A *crystalline concept* is “a concept that has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context” (p. 3). He suggested that a primary difference between crystalline concepts and objects as described by APOS is that objects must be encapsulated from an action or thematized from a schema, whereas crystalline concepts can be abstracted from any mathematical situation dependent on the contextual relationships of the structures.

Acknowledging that these kinds of competing theories for knowledge construction are of great interest to mathematics educators, the significant differences between the realization of crystalline concepts (Tall, 2011), encapsulation of actions into objects (Asiala et al., 1996), the process of reification (Sfard, 1992), and the process of proceptual development (Gray & Tall, 1994) do not diminish the importance of developing reversibility. The importance of reversibility can be seen within each competing theory. As discussed above, reversibility presents at multiple levels within the APOS framework. In the creation of objects from other objects or the recognition of an object as a crystalline concept (Tall, 2010, 2011), reversibility seems to be a highly desired learning outcome. For instance, in the example Tall (2011) cites of creating a derivative graph from the graph of a function, it would seem that the ability to create a function graph from a derivative graph is a highly desired learning goal. The contextual relationships and structures involved with creating a functional graph from a derivative graph could be construed as a crystalline concept. Creating a derivative graph from a function graph and creating a function graph from a derivative graph is a clear instance of the reversibility process. As previously noted, reversibility is a key understanding in learning to proceptually encapsulate processes (Gray & Tall, 1994). Thus, several current frameworks for knowledge construction value the reversibility process.

2.1.6 How reversibility has been studied and conclusions about students' reversibility

Reversibility and its role in flexibility have long been considered a key requirement in a variety of mathematics problems (Inhelder & Piaget, 1958). In this section, I describe the mathematical lenses used to study reversibility and reversibility as an element of flexibility. I also review the significant conclusions reached by researchers after conducting research on students' reversibility.

Many of the lenses researchers use to investigate reversibility seem unrelated. Krutetskii (1976) noted that many mathematical tasks fall into completely different categories and kinds of problems (for example, reciprocal relationships, direct and reverse operations, and direct and converse theorems); however, “their internal psychological basis in each case is a reconstruction of the direction of the mental process, a change from a direct to a reverse train of thought, and the establishment of two-way (reversible) associations” (p. 143). Furthermore, Krutetskii and Piaget (Flavell, 1963; Inhelder & Piaget, 1958) both specifically observe that reversibility presents at all stages of development. Thus, we should expect that a review of the kinds of questions used in reversibility studies show a wide variety of mathematical levels and content.

Krutetskii (1976) studied reversibility with students in grades 2-10 in a variety of mathematical disciplines. He used what he termed “paired problems” (p. 143) in arithmetic, algebra, and geometry to ascertain a student’s understanding of reversibility. A paired problem consisted of a direct and a reverse problem. Reverse problems were characterized by “problems in which the subject matter of the original problem is retained, with the original unknown becoming part of the terms, and one or several elements of the original terms becoming unknowns” (p. 143). Krutetskii asserted that “reverse problems and questions present a certain difficulty for pupils, but that they have a great value for developing active, independent, and creative thinking, since they develop the ability to switch from direct to reverse operations” (p. 187).

Students were given reverse problems immediately after solving a direct problem and independently of the direct problem. The reverse problems given independently were administered in an altered form about one month before the experiment was conducted. In the experiment, the reverse problem immediately followed the direct problem. In geometry, Krutetskii (1976) supplemented the direct/reverse problems with direct/converse problems and theorem/converse theorem problems. Theorem/converse theorem problems required students to prove each theorem. In the table below, I provide one example of Krutetskii's (1976) reversibility items from each mathematical content area[2].

Table 1. Krutetskii's (1976) reversibility items

Content Area	Direct Question	Reverse Question
Arithmetic	Sixteen liters of water were poured into a tank, filling the tank to $2/5$ of its volume. What is the volume of the tank?	Water was poured into a tank of 80-liter capacity to $2/5$ of its volume. How many liters of water were poured into the tank? (p. 144)
Algebra (contextual problems)	How many days should a worker spend working to earn b rubles if he gets c rubles per day?	How much does a worker earn in d days if he earns a rubles in one day?
Algebra (symbolic problems)	$(a-b)^2 =$	$a^2 - 2ab + b^2 =$ (p. 145)
Geometry (direct/converse)	How many degrees does $\frac{1}{24}$ of a straight angle contain?	What part of a circle is an arc of $22°30'$? (p. 145)
Geometry (direct/reverse)	Calculate the sum of the interior angles of a convex heptagon.	The sum of the interior angles of a convex polygon is $1{,}800°$. How many angles does this polygon have? (p. 145)
Geometry (theorem/converse)	If two oblique lines drawn to a straight line from a single point are equal, then their projections are also equal.	If two oblique lines are drawn to a straight line from a single point have equal projections, then they are equal to each other. (p. 146)

Krutetskii (1976) found that students varied widely in their ability to solve reversibility problems, a conclusion supported by subsequent research (Rachlin, 1981). Krutetskii concluded that the differences in students' reversibility is due to each student's innate mathematical ability and cites reversibility as a marker of the student's mathematical ability. Concluding from his own literature review that reverse bonds and direct bonds can be created at the same time, Krutetskii (1976) concluded that capable pupils solve reverse problems without any difficulties or need for

[2] For a full review of Krutetskii's (1976) items that test reversibility, the reader is directed to pages 144-146.

instruction. The capable students were able to identify reverse problems as such and in the cases where the reverse problem immediately followed the direct problem, the reverse problem was often solved more quickly and easily than when the reverse problem was given independent of the direct problem. These results led Krutetskii to conclude that capable students were able to create direct associations and reverse associations simultaneously.

Average students were also able to solve reverse problems without needing special exercises. However, average students struggled with a reverse problem that immediately followed a direct problem. Krutetskii (1976) determined that the direct problem influenced the students' thoughts and processes on the reverse problem. He found that when an average student solved a reverse problem immediately after solving the direct problem, s/he struggled to abandon the solution method of the direct problem. The solution method used to solve the direct problem just prior to solving the reverse problem inhibited the use of a correct solution method on the reverse problem. However, average students solved the same reverse problem independent of the direct problem with great ease. After learning how to solve the reverse problems, the average students excelled at solving the reverse problems. Thus, Krutetskii concluded that average students require instruction and special exercises to develop reverse bonds. Furthermore, average students develop reverse bonds separately from direct bonds: "first a direct bond is formed and then, as a result of appropriate exercises, a reverse bond" (Krutetskii, 1976, p. 289). Current learning theory suggests updating this conclusion. Current learning theory now suggests that for average students to develop connections within a schema that allow the traversing from the result to the constituent inputs, they would need instruction and special exercises.

The students who Krutetskii (1976) considered to be incapable students lacked any significant ability to determine that the second question in the direct/reverse pair was actually a reverse problem. The only situation in which incapable students could recognize the reverse problem as the reverse of the direct problem were in very elementary cases, "and they judged this by purely external signs ('there this was asked, but now it is given')" (Krutetskii, 1976, p. 289). Incapable students always solved reverse problems better when administered independently of the direct problem.

Krutetskii (1976) also observed that incapable students can establish direct and reverse bonds with great difficulty but only after receiving repeated exercises in both directions. The development of reverse bonds is a completely independent task, isolated from the development of direct bonds. Although Krutetskii does not make this observation, his conclusions suggest that incapable students simply develop two separate direct bonds with no understanding of the reversible relationship and thus may not develop reversibility at all.

In his dissertation research, Rachlin (1981) used clinical interviews with 4 participants to study the participants' reversibility in a basic algebra course. He interviewed 4 volunteers two times per week throughout the entire semester. Each interview lasted 45 minutes. Rachlin's (1981) conclusions agreed with Krutetskii (1976). Rachlin (1981) found that all four subjects showed unique patterns of reversibility with respect to the Piagetian constructs of negation and compensation. One student demonstrated complete reversibility of algebraic operations; one student was proficient with negation but not compensation. One student was proficient with compensation but very unsure of using negation and the fourth student could not use either negation or compensation. It is also important to note that Rachlin (1981) considered the ability to use either negation or compensation at will to solve a problem as an example of flexibility in using reversibility. He concluded that algebra students may show flexibility in how they apply the reversible actions of negation and compensation.

Subsequent researchers studying reversibility have largely built on the types of questions that Krutetskii (1976) asked. Tzur (2004), for example, studied how teachers and two fourth-grade students work together to develop a reversible fraction conception through the reversible operations of partitioning and iteration. The work Tzur (2004) described fits within Krutetskii's (1976) direct/reverse model of reversibility questions. Tzur (2004) began by asking students to solve a problem that required partitioning of a whole first and iteration of a unit part last and then had students solve a problem that began with iteration and ended with partitioning. For example, using computer generated manipulatives, students were given a length that represented $5/8$ths of a whole and were asked to construct a length representing 1 whole. The students were later asked to partition a non-unit fraction, $\frac{m}{n}$, into equal and non-equal parts. As an example, students were asked to identify different ways that 10 pieces of pizza could be reassembled in an oven. Solutions included three sections of 3 pieces each and one section of 1 piece or two sections of five pieces each.

Tzur (2004) concluded that students were able to develop a reversible fraction conception but that the development was entirely dependent on "the three principal activities of teaching: analyzing students' current conceptions, deciphering the mathematical conceptions to be taught, and selecting/using tasks that intentionally foster transitions from the former to the latter" (p. 108). His conclusions suggest a way to interpret Krutetskii's (1976) "special exercises". Tzur's conclusions indicate that appropriate special exercises require the teacher to first diagnose the student's current conception(s) of reversibility within the mathematical domain, determine the desired learning outcome(s) of the domain (in our discussion, the outcome is complete reversibility within the domain), and then intentionally design mathematical tasks to facilitate the development of the desired outcome.

Kang (2012) observed that reversibility of thought is required for advanced mathematical thinking and proposed a number of questions designed to promote it. Each question posed by Kang reflected Krutetskii's (1976) direct/reverse model of reversibility questions.

Ramful and Olive (2008) investigated reversibility in multiplicative tasks (proportional reasoning) by using Inhelder and Piaget's (1958) balance beam problem. Ranful and Olive tested reversibility using direct/reverse problems. The direct problem required students to develop an inverse relationship between two inputs to create an output. The second problem began with an output and one input and required students to find the missing input. They concluded that a student's ability to solve reversible problems within a mathematical domain depends on the quality of the student's "reversibility template" (Ramful & Olive, 2008, p. 149). The reversibility template is an instance of a "cognitive template" (Ramful & Olive, 2008, p. 149) which is a student's operational structure in symbolic form that related to a particular theorem or set of theorems. For example, the formula $c^2 = a^2 + b^2$ could be construed as a student's cognitive template for the Pythagorean Theorem. Ramful and Olive suggested that students construct cognitive templates for reversible actions and processes and that a student's ability to apply reversibility to solve problems is dependent upon the robustness and flexibility of the reversibility template. One of the participants, Ron, developed a reversibility template based on the equation $w_1 d_1 = w_2 d_2$ and could apply this template in a wide variety of examples. The researchers concluded that Ron had an advanced reversibility template and that his template had a high degree of flexibility.

Davis and McGowen (2002) studied reversibility as an element of flexible thinking. While they purported to study flexible thinking, the questions they chose follow Krutetskii's (1976) direct/reverse model of reversibility questions. The researchers reported using paired questions such as "(a) what is $f(5)$? … (b) for what value or values of x is $f(x) = 0$? Essentially this is the difference between evaluating a function and solving for a value" (p. 2).

In their study, Davis and McGowen (2002) studied developmental algebra students at a two-year college. They tested the claim that presenting functions as machines that receive an input and produce an output helps students to gain a process conception of functions. Their results indicated that teaching the function concept through a "function machine" approach did correlate with improvement in students' algebraic reversibility. The researchers hypothesized that reversibility with algebraic expressions is jointly and highly correlated with three processes: 1) the ability to visualize functions as machines, 2) using a graphing calculator as a concrete example of a function machine, and 3) the use of data to help students generate formulas through finite differences and ratios.

Norman and Prichard (1994) proposed using the Krutetskiian constructs of reversibility and flexibility to analyze conceptual understanding of calculus. They designed tasks purposed to elicit student use of reversibility and flexibility while solving calculus tasks.

Reversibility completely describes the inverse relationship between the two calculus processes of differentiation and integration (Norman & Prichard, 1994). However, "the role of reversibility … in the understanding of the relationship between differentiation and integration … has not been adequately described" (Haciomeroglu et al., 2009, p. 81). There is broad agreement among calculus researchers that students do not use reversibility to reason through the process of integration or use reversibility to help them understand the motivation behind the integration algorithms (Berry & Nyman, 2003; Haciomeroglu et al., 2009; Norman & Prichard, 1994).

Since integration is typically introduced by asking students to find a function F whose derivative is f, students often use reversibility to solve the first integration problems to which they are exposed (Norman & Prichard, 1994). However, shortly after being made aware of formulas that help to calculate integrals, students stop using reversibility. Norman and Prichard (1994) asserted that the use of integration formulas (algorithms) is typically a one-way understanding. For example, many calculus students correctly evaluate $\int x^2\,dx = \frac{x^3}{3} + C$. However, "the latter expression is never converted into, nor is it thought of in terms of the former. Students who are unable to reverse their thinking about the relationship of these expressions are unlikely to recognize their essential equivalence" (Norman & Prichard, 1994). The description given by Norman and Prichard is an example of how students often lack proceptual encapsulation (Gray & Tall, 1994). Without reversibility, calculus students will likely not view differentiation and integration as equivalent processes. Students exhibit similar weakness when working with the graphical representations of functions and derivatives.

The use of reversibility in calculus is not limited to the inverse relationship between differentiation and integration. Clark et al. (1997) used the APOS framework to analyze how students learn the chain rule, $\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) * g'(x)$. Based on the genetic decomposition determined by Clark et al. (1997) and the previous analysis of reversibility in APOS, the research suggests that a well-functioning chain rule schema requires that students develop reversibility of a function schema, reversibility of a function composition schema, and reversibility of a differentiation schema.

Flexibility presents in the calculus classroom through the processes of limit, derivative, and integral. Since the function concept is central to each process, part of calculus learning depends on the understanding of the function concept, which is typically a precalculus concept. Students who enjoy significant flexibility with multiple representations of functions are better prepared and have flexible tools available for problem solving in calculus (Norman & Prichard, 1994).

Flexibility is also required for processes that are uniquely calculus related. Kendal and Stacey (2003) asserted that the end goals of a differential calculus class should require students to "know how to formulate, calculate, and interpret the three types of derivatives … 'at a point' or as a function … Students also need to know how to translate between the different representations of derivative and recognize equivalent derivatives" (p. 28). The different representations of derivatives and translation between them will be discussed extensively in section 2.2.

Norman and Prichard (1994) noted that calculus textbooks and calculus instructors pay very little attention to the development of reversibility and flexibility but that students with greater facility with reversibility and flexibility tend to succeed in calculus more than students who lack proficiency with reversibility and flexibility. However, Norman and Pritchard also noted that as of the early 1990's, there was considerable attention directed towards revising calculus curricula to include the Krutetskiian problem-solving processes and multiple representations of functions. The research of Selden, Selden, and Mason (1994) supported the call for using a calculus curriculum that embeds reversibility, flexibility, and multiple representations. Selden et al. (1994) concluded that calculus itself cannot be reduced to just procedures and routine problems, thus a more effective instructional approach augments students' abilities to solve non-routine problems. Thus, research suggests that one possible method of helping calculus students to develop problem-solving skills is to promote reversibility, flexibility, and fluency with multiple representations in the calculus curriculum.

2.1.7 Development of reversibility and teacher actions that may support development

Krutetskii (1976) and Rachlin (1981) largely made conclusions regarding students' abilities to develop reversibility. There has also been research that attempts to identify *how* reversibility develops. In this section, I review this research and its implications for how reversibility develops and what teachers might do to help students develop reversibility.

Krutetskii (1976) concluded that reversibility either develops innately with no special attention paid by the teacher, as in the case of capable students, or develops through special exercises. These special exercises largely represent instruction in how to move from E back to A. Often the special exercises included a direct/reverse paired problem that the instructor and pupil talked through and worked together. It is reasonable to question if this method of instruction is aiding the student in developing reversible connections or if the student is merely developing a new direct connection. If the latter is the case, it would then be incumbent upon the student to eventually realize the reversible properties, presumably through repeated examples moving in both directions. An example of a Krutetskiian special exercise given to an incapable student follows (Krutetskii, 1976, p. 290):

> *Experimenter*: "Solve the problem: $5 * 5 =$ [pupil gives the right answer]. Now do this one: What numbers must we multiply to get 25? [Pupil gives the right answer.] Now watch: $5 * 5 = 25$, and $25 = 5 * 5$. The second problem is the reverse of the first. Do the problem $(2x + y)(2x - y) =$ [pupil gives the right answer]. Correct. But if $(2x + y)(2x - y) = 4x^2 - y^2$, then can we say that $4x^2 - y^2 = (2x + y)(2x - y)$? [Pupil gives an affirmative answer.] Well, to what is $9x^2 - 4y^2$ equal?"
>
> *Pupil*: "I don't know. These are odd problems. We haven't done these."

Experimenter: "No, you haven't done them, but we are learning to do them. Now you think: What is the product of the sum of two numbers by their difference equal to? You know this."

Pupil: "The product of the sum of two numbers by their difference is equal to the square of the first minus the square of the second."

Experimenter: "Right. Can it be said in reverse? To what is the difference of squares equal? What is $a^2 - b^2$ equal to?"

Pupil: "$a^2 - b^2 = (a + b)(a - b)$."

Experimenter: "And what is $9x^2 - 4y^2$ equal to?"

Pupil: "$(9x + 4y)(9x - 4y)$ …"

… Only after repeated explanations and exercises did the pupil learn to do problems of this type, and then only elementary ones.

This example serves to show the role of the teacher within Krutetskii's model. The teacher chooses and/or designs appropriate exercises that help students to develop reversible connections.

Confrey (1981) rejected Krutetskii's (1976) conclusion that reversibility (and flexibility and generalizability) develop spontaneously and suggested that it is a mathematical process to be developed. Recently, researchers (Kang, 2012; Ramful & Olive, 2008; Tzur, 2004) have extended the role of the teacher to include not just selecting appropriate exercises, but also facilitating discussion that makes reversible connections explicit.

Other researchers have agreed with Krutetskii's (1976) special exercises approach to developing reversibility. Although Inhelder and Piaget (1958) identified that children of the age of ten think of solving problems through reversible thought processes and do not need explicit instruction to form mentally reversible constructions, in Flavell's (1963) major study of Piaget, Flavell concluded that Piaget strongly supported specific instruction in the properties of reversibility.

> In the case of reversibility, for instance, this would imply that multiplication and division, as well as addition and subtraction, should be taught together in alternation. Thus, one should follow a problem like $(10 * 5 = ?)$ with its inverse $(50 \div 5 = ?)$. Similarly, if one traces a causal series in the usual cause-to-effect direction, one should not fail to trace it in the inverse, effect-to-cause direction. (Flavell, 1963, p. 368)

The Piagetian examples cited by Flavell (1963) are similar in format and thought process to the special exercises described by Krutetskii (1976).

Rachlin (1998) suggested that classroom teachers develop "non-routine *routine*" (p. 471) tasks that challenge students to develop reversibility. As an example, he offered the addition problem $\boxdot + \vartriangle = \odot$ and required students to find a different unknown in three different examples. For instance, in the first example, values could be given for $\boxdot$ and $\vartriangle$ and the student must find the value for $\odot$. In the second example, values could be given for $\vartriangle$ and $\odot$ and the student would find the value for $\boxdot$. In the third example, values are given for $\boxdot$ and $\odot$ and the student must find the value for $\vartriangle$. Kang (2012) and Vilkomir and O'Donoghue (2009) have similarly proposed a variety of problems that teachers can use to help students develop reversibility. These groups of problems follow the direct/reverse format originally proposed by Krutetskii (1976).

As noted earlier, Piaget (Inhelder & Piaget, 1958) and Krutetskii (1976) both assert that reversibility presents at all stages of mathematical development. Rachlin (1998) was more specific saying, "much of precollege mathematics refers to actions that can be reversed, including, but not limited to, the actions of graphing, making a table, solving an equation or problem, and simplifying an expression by performing indicated operations" (p. 471). Research should commence investigating the kinds of activities and exercises that will help students develop the kinds of reversible connections needed to relate graphs, tables, and algebraic representations of functions.

There is recent agreement that beyond just offering special exercises designed to foster reversibility, the approach of the teacher is of great importance to developing reversibility (Kang, 2012; Rider, 2007; Vilkomir & O'Donoghue, 2009). Often, the teacher's instructional methods do not encourage the development of reversibility as teachers typically only present solutions in one direction and assume that students will reverse the process without receiving instruction in the reverse direction (Kang, 2012). Rider (2007) observed this phenomenon in her own classroom, noting that "I made the implicit assumption that when I gave students a graph, they would be able to reverse the process to find the slope and the y-intercept from that graph and write the equation in slope-intercept form" (p. 495). She made similar assumptions regarding students' abilities to analyze the features of a linear graph by looking at a table of values. She noted that her students did not begin to develop reversibility until she began to introduce concepts from the result and work back to the beginning and from the beginning and work towards the result. For example, she began to introduce a graphing topic by starting with the graphical representation and developing a table of values as well as starting with a table of values to create a graph.

Rider's (2007) recommendations, informed by work in her own classroom, suggest that one possible mechanism by which reversibility develops is through flexibility in representations. The description Rider gave is one of helping students to develop flexible ways of thinking about mathematical concepts (in this case, graphing). Other researchers have also suggested that developing flexibility aids in developing reversibility (McGowen, 2006; Ramful & Olive, 2008). However, there is a need for empirical research beyond Rider's (2007) anecdotal evidence to specifically investigate this claim.

2.1.8 Calls for studying reversibility

Since Krutetskii's (1976) work, with the exception of Rachlin's (1981) dissertation research, there has been very little research investigating reversibility. However, in the past ten years there has been a frequent call for a renewed research agenda in reversibility. Vilkomir and O'Donoghue (2009) and Teachey (2003) both researched reversibility as a means of identifying mathematically gifted (what Krutetskii called "capable") students. Teachey noted that prior to her dissertation research, "no follow-up studies to apply Krutetskii's findings to the gifted population have been attempted" (p. 15).

Lamon (2007) specifically called for research on how reversibility develops, saying "researchers know very little about reversibility or about multiplicative operations and inverses and these could be subjects for a valuable microanalysis research agenda" (p. 661). While Lamon's observation was in relation to her work with rational numbers and proportional reasoning, other researchers have extended this call to wider mathematical arenas. Ramful and Olive (2008) noted that while researchers have examined reversibility peripherally to other research agendas, reversibility has not been the focus of research and that there is a specific need for research on reversibility at the secondary school level. Teachey (2003) cited the need for more research on how capable (gifted) students develop the Krutetskiian problem-solving processes (generalization, flexibility, and reversibility).

Finally, some researchers have called for studying reversibility as a need for improving pre-service mathematics teacher education. According to McGowen and Davis (2001), pre-service elementary teachers often lack reversibility and flexible thinking, which severely limits their ability to teach these problem-solving processes in a mathematics class. Since problem solving may be considered the essence of mathematics (Rachlin, 1998), the need for pre-service elementary teachers to develop reversibility and flexibility becomes paramount. Usiskin (1999) observed that many mathematics teachers not only lack awareness of reversibility, flexibility, and generalization but indeed teach against these traits. Thus, Usiskin's (1999) research strengthens the call for helping pre-service teachers to develop reversibility, flexibility, and generalization and to develop methods of instruction that facilitate reversibility, flexibility, and generalization.

2.2 REVIEW OF LITERATURE ON MULTIPLE REPRESENTATIONS

Krutetskiian flexibility has been repeatedly linked to the use of multiple representations (McGowen, 2006; Norman & Prichard, 1994; Rider, 2004) and flexibility with representations is a key aspect of the definition of understanding given by Lesh et al. (1987).

> Part of what we mean when we say that a student 'understands' … is that: (1) he or she can recognize the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within given representational systems, and (3) he or she can accurately translate the idea from one system to another. (Lesh et al., 1987, p. 36)

This definition of student understanding incorporates the Krutetskiian construct of flexibility and the need for students to understand mathematical content in multiple representations. In this section, I review the body of research surrounding multiple representations. I begin by discussing definitions of the key terms and concepts associated with multiple representations and then discuss the importance of multiple representations in mathematics education. Research on how students build connections between representations is considered. Finally, I conclude this section with a discussion of research supported teacher actions that promote the linking of multiple representations to develop conceptual understanding.

2.2.1 Definitions

In this section, I define several of the key terms prevalent in the research on multiple representations. Typically, when we refer to the term *representation*, we are referring to the three or four external representations of a mathematical concept, often named algebraic (also referred to as symbolic), graphical (also referred to as visual or geometric), numerical (also referred to as tabular), and verbal (Kendal & Stacey, 2003; Lesser & Tchoshanov, 2005). A common example of multiple representations involves functions. The algebraic representation uses functional notation and algebraic rules to specify a function. The graphical representation is the visualization of a function on the Cartesian plane, commonly referred to as a graph. The numerical representation is a function represented only by numerical values, typically a table of values (Kendal & Stacey, 2003). The verbal representation of a function refers to describing functional relationships in words (Brenner et al., 1997).

Each representation offers different information about a function and has a unique purpose in telling the story of the function (Greeno & Hall, 1997; Steketee & Scher, 2012). The tabular representation presents the function as input/output machine and is often the most concrete representation for students. The algebraic representation allows the student to make generalizations about the behavior of the function and is often the most useful for evaluating functional inputs. The strength of the graphical representation is in its appeal to the visual sense, allowing students to see relationships between the independent and dependent variables (Steketee

& Scher, 2012). The verbal representation has been lauded for its relationship to contextual situations and the physical embodiment of real world mathematics (Dick & Edwards, 2008).

Moschkovich et al. (1993) described the ability to make connections across the algebraic, numerical, and graphical representations the "Cartesian Connection" (p. 73). Janvier (1987a) defined *translation* as the act of switching from one representation into another. Knuth (2000) recommended that whenever possible, each mathematical concept should be presented in each representation and that learning to translate effectively between representations is a key to developing understanding of concepts. Knuth also adopted the terms "Rule of Three" and "Rule of Four" that the CCH debuted in 1992.

The desired learning outcome of emphasizing the Cartesian Connection and the Rule of Four is what Zbiek, Heid, Blume, and Dick (2007) define as *representational fluency*. Representational fluency goes beyond being aware of different representations and incorporates the "meaningful and fluent interaction" (p. 1196) with each of the representations as appropriate. Representational fluency requires an interaction between the student and the representation wherein the student draws inferences about the function from each representation and is able to generalize across multiple representations. Zbiek et al. (2007) sum up their discussion of representational fluency noting that the key understanding in multiple representations is "not so much that the representations exist but that the student interacts in meaningful ways with those representations" (p. 1196). Amit and Fried (2005) observed that teachers have a tendency to point students toward using, adapting, selecting, applying, and translating among the different representations. When students are using, adapting, selecting, applying, and translating at will between the different representations, they are exhibiting representational fluency. These actions require that students learn the meaning of each representation and the value of each representation independent of the others.

2.2.2 The importance of multiple representations

After nearly twenty years of research supporting the use of multiple representations in mathematics education, the importance of multiple representations in mathematics education has become almost axiomatic (Amit & Fried, 2005; Lesser & Tchoshanov, 2005). The NCTM (2000) recommended that students be able to "select, apply, and translate among mathematical representations to solve problems" (p. 63). The Common Core State Standards lists the ability to analyze functions in multiple representations as a high school standard of learning. The CCSS-M mathematical practice standard, CCSS.Math.Practice.MP4, Model with mathematics, states that students should be "able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas" (NGA, 2010, p. 7). Beyond the importance that the NCTM and CCSSM place on multiple representations, research reveals strong motivations for teaching mathematics in a multiple representations environment. In this section, I briefly review some of the research that has led researchers to conclude that "the general case for multiple representations in mathematics education hardly needs defending anymore" (Amit & Fried, 2005). I first consider a major learning deficiency identified by research as a result of either not using multiple representations or not using them well and then I review some of the purported benefits of using multiple representations.

The inability to translate between representations hinders the development of conceptual understanding and can be linked to the complexity involved in changing representations (Duval, 2006; Lesh et al., 1987). There is considerable agreement that the majority of students have a strong preference for using the algebraic representation; students often force the use of the algebraic representation when an algebraic representation is either impossible or impractical (Brenner et al., 1997; Dreyfus & Eisenberg, 1990; Hiebert & Carpenter, 1992; Knuth, 2000). One cause linked to students' failures to make connections and develop understanding of functions is the preference for the symbolic representation (Brenner et al., 1997; Knuth, 2000). Researchers suggest that the reason students prefer the symbolic representation lies in an instructional preference for explicitly or implicitly promoting the symbolic representation at the expense of the other representations (Brenner et al., 1997; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Hiebert & Carpenter, 1992; Knuth, 2000; Rider, 2004, 2007). Explicitly promoting the symbolic representation lies within teachers' common preference to introduce mathematical concepts first in an algebraic means (Brenner et al., 1997; Elia et al., 2007; Hiebert & Carpenter, 1992), then use the algebraic representation to generate a table of values, which is in turn used to create a graph. Finally, a verbal description, often given as a word problem is solved (or given as a homework problem) to show the students how the function has a real-life application. One can surmise that teachers prefer the symbolic notation for its mathematical power through generalization; research indicates that students in calculus courses prefer the symbolic notation "because they [the algebraic expressions] have been very powerful for them [the students] in the past" (Berry & Nyman, 2003, p. 495). The fact that teachers rarely introduce concepts using graphs or verbal descriptions and almost never introduce a concept with the tabular representation results in students devaluing the graphical, verbal and numerical representations (Rider, 2004). An implicit promotion of the symbolic representation is shown in Rider's (2007) description of activities in her own classroom, where she found that students did not value the graphical and numerical representations despite a focused instructional approach that weighted each representation evenly. Rider (2007) concluded that her students began to value the graphical and numerical representations as equal to the symbolic when she began to assess each representation equally on exams. Keller and Hirsch (1998) proposed that exposure to multiple representations may lessen the preference for the algebraic representation; however, Rider's (2007) work corroborates others who have concluded that "multiple representations do not by themselves help students develop mathematical understanding" (Gagatsis, Christous, & Elia, 2004, p. 157). These findings suggest that merely presenting concepts through multiple representations is insufficient to help students develop understanding.

Significant benefits for learning mathematics through the use of multiple representations have been identified (Brenner et al., 1997; Hashimoto & Becker, 1999; Keller & Hirsch, 1998; Lesh et al., 1987; Rider, 2004, 2007; Steketee & Scher, 2012; Zbiek et al., 2007). As noted earlier, one key benefit is the building of conceptual understanding as described by Hiebert and Carpenter (1992) and Lesh et al. (1987). Zbiek et al. (2007) suggested that teaching through multiple representations with the goal of representational fluency creates an ideal environment for the building of rich meaning.

Another benefit of studying (functions) through multiple representations is the subsequent improvement in problem solving (Brenner et al., 1997; Rider, 2004, 2007; Zbiek et al., 2007). Problem solving tends to improve as a student's repertoire of problem-solving techniques increases. For example, as students are able to recognize the features of a function that are invariant across representational forms, students begin to realize that each representational form offers information that may be useful in solving problems (Rider, 2004, 2007). The availability of multiple representations provides students with multiple avenues by which they can navigate novel problems. Lesh et al. (1987) suggested that fluency in multiple representations allows students to work around difficulties with one particular representation. For example, a student who has difficulty translating from an algebraic to a graphical representation could consider translating from an algebraic to a numerical representation and then from a numerical representation to a graphical representation. Zbiek et al. (2007) concluded that representational fluency with multiple representations allows students to work around unexpected results. If a student cannot explain a result in one particular representation, then the student can generate an equivalent representation and attempt to explain the result through the lens of the new representation. For example, suppose a student cannot explain why $x = 2$ is not a part of the domain of the function $y = \frac{1}{x-2}$. A student who can construct a graph of this function would then realize that a vertical asymptote exists at $x = 2$. This may allow the student to explain why $x = 2$ is not in the domain.

Research has shown that studying functions through multiple representations widens students' perceptions of what kind of answer is acceptable (Keller & Hirsch, 1998; Rider, 2004). When students come to understand that the four representations of a function are all equivalent, they begin to see that solutions are typically not limited to just a number or function. Rider (2004) suggested that due to each representation requiring a different kind of input and producing a different kind of output, "when viewed together, the representations make input and outputs of functions more salient than any one representation alone" (p. 3). These conclusions suggest that studying functions through multiple representations increases students' flexibility with functional representations.

2.2.3 Links between multiple representations

The key understanding in the use of multiple representations is the creation of links between the representations. Kaput (1992) identified two reasons for linking representations of a concept: "(i) to expose different aspects of a complex idea, and (ii) to illuminate the meanings of actions in one notation by exhibiting their consequences in another notation" (p. 542). How and in what manner these connections are made is an unsettled question. There is agreement among some researchers that the links between representations develop with a directionality (Dick & Edwards, 2008; Janvier, 1987b; Kaput, 1992). This directionality results in representational preference among students. Janvier (1987a) contended that translation processes are better learned as pairs. For example, students learn translation processes better if they learn to translate table to graph and graph to table simultaneously as opposed to learning each translation independently. What Janvier describes parallels Krutetskiian reversibility. Janvier (1987a) is suggesting that reversibility is a key in developing a strong network of connections between representations.

Kaput (1992) suggested that a two-way translational link between representations may not be possible in all cases, noting that two linked representations do not necessarily have an equal degree of correspondence. For example, it would always be possible to translate an algebraic representation into a unique numerical representation; however, the reverse is not possible. As such, Kaput (1992) emphasized the development of unidirectional links, called "hot links" (p. 530). The defining feature of a hot link goes beyond just making a linkage between two representations; a hot link requires that when a transformation acts on an object in representation A, the effect of the transformation is automatically understood in representation B. An example of this would be expecting a student to understand the graphical result of dividing polynomial A by polynomial B. In their discussion of hot links, Zbiek et al. (2007) postulate that hot links can be bidirectional, but that a bidirectional hot link would mean that any action taken in either representation would reflect the same consequence in both representations.

Amit and Fried (2005) proposed a different approach to multiple representations altogether. They concluded that instruction linking multiple representations may be too difficult for students who are likely not fluent in one particular representation. Rather than making the linking of multiple representations the instructional framework, they suggested the linking of multiple representations to be an end goal of a mathematical learning trajectory. It may be the case that the Krutetskiian constructs of flexibility and reversibility may aid in this proposed approach of viewing links between representations as an instructional end goal instead of a framework for instruction. However, I am not aware of any research that has tested this hypothesis nor am I aware of any ongoing research investigating Amit and Fried's (2005) initial claim. More research is necessary to substantiate or refute the effectiveness of delaying linkage between representations until after students have had considerable experience with each representation in isolation.

2.2.4 How to teach multiple representations

Researchers have proposed instructional techniques that may help students to generate the kinds of connections across representations associated with conceptual understanding (Greeno & Hall, 1997; Hiebert & Carpenter, 1992; Keller & Hirsch, 1998). Greeno and Hall (1997) recommended emphasizing interpretation of each representation rather than teaching each separate representation as an end goal. Hiebert and Carpenter (1992) stressed the use of mathematical activities highlighting the similarities and differences of the representation in order to "stimulate the construction of useful connections at all levels of expertise" (p. 68). As these connections increase and grow stronger, the students are likely to begin to transition from viewing each representation as separate entities to seeing each representation as a different form of the same entity.

Hiebert and Carpenter (1992) recommended that teachers make the connections between the representations explicit and should encourage students to reflect on the connections. Thompson's (1994) research supported this recommendation. He found that if the teacher fails to make the connections explicit for students, then the mathematics curriculum becomes in essence a traditional curriculum reducing each representation to another set of learned procedures.

As noted earlier, students often have a preferred representation. Keller and Hirsch (1998) asserted that this preference is an essential component for researchers to understand when investigating students' connection-building between representations. The researchers identified several factors that influence students' representation preferences. These factors include the students' prior experiences with each representation, the perceived acceptance of the representation, the difficulty of the task, the context of the representation, and the language of the task. For example, formal language in a task is associated with a student preference for an algebraic representation while a more intuitive, less formal wording engenders greater use of the graphical and numerical representations. This research has possible ramifications for teachers. For instance, mathematics teachers may be able to harness the potential of students' innate situational preferences for representations. Keller and Hirsch suggested the likeliness that the strongest links between representations will include the preferred representation. As such, "attempts to situate the development of mathematics in context and attempts to enable students to approach mathematics as a sense-making activity may do well to build from students' preferences for representations" (p. 14).

In her dissertation research, Rider (2004) investigated the benefits of presenting each representation in equal amounts without introducing a bias towards any particular representation. She concluded that a curriculum fully immersed in multiple representations increases students' flexibility. One of the benefits of this approach is that it "allowed students the freedom to decide which representation is most effective in certain situations and with which representation they were most comfortable" (Rider, 2004, p. 132). Unlike Keller and Hirsch (1998), Rider (2004) recommended requiring students to battle with each representation and develop a sufficient comfort level with each. However, she also noted the benefits of being able to translate representations when necessary, saying "if students were not comfortable with a representation, then they needed to know how to translate into an alternate representation that they were comfortable with, thus making connections among the representations and showing a higher level of understanding" (Rider, 2004, p. 132).

Rider (2004) also recommended strategic assessment writing as an instructional method supporting connection building between representations. By designing assessments that ask mathematical concepts in any of the representations, the assessments require students to develop a proficiency with all representations and in the event that they cannot solve a problem in a preferred representation, the students have to be able to translate the concept into an appropriate representation.

Rider (2004) cautioned teachers to be aware of the order in which representations are introduced. She emphasizes the importance of starting each subsequent problem in a class using a different representation, especially when first introducing a mathematical concept. She found that students inherently value the representation used to introduce a concept. Research has termed the explicit or implicit promoting of one representation over another as teacher *privileging* (Kendal

& Stacey, 1999, 2001). Research indicates a link between the representation(s) privileged by the teacher and the subsequent use of each representation by the students, suggesting that students are more likely to develop proficiency with a representation that a teacher privileges (Kendal & Stacey, 2001).

Rider (2004) agreed with Hiebert and Carpenter (1992) and recommended that teachers should specifically emphasize the connections between representations. Rider also cautioned that students typically do not develop reversibility between representations without receiving instruction in both directions. For example, after Rider's students received instruction translating from an algebraic equation to a graph, they were not able to translate from a graph back to an algebraic expression. It may be the case that particularly advanced students can develop reversibility between representations without receiving "special exercises" as Krutestkii observed, however Rider did not address this possibility. The average and incapable student will almost certainly need instruction and practice in translating in both directions between two representations.

2.2.4.1 The rule of four

One instructional approach that has gained significant traction in the past 20 years, especially in calculus, is instruction using the Rule of Three (or Four). The Rule of Three is the "belief that … three aspects of calculus – graphical, numerical, and analytical – should all be emphasized throughout [the calculus course] … students will repeatedly by confronted with the graphical and numerical meaning of what they are doing" (Gleason & Hughes-Hallett, 1992, p. 1). The purpose of the Rule of Three "is to produce a course where the three points of view are balanced and students see each major idea from several angles" (Hughes-Hallett et al., 1995, p. 121). Hughes-Hallett and Gleason, part of the Calculus Consortium at Harvard University (CCH), first proposed the Rule of Three in the late 1980's and published the first edition of a calculus textbook, *Calculus*, based on the Rule of Three in 1994-1995. Subsequent editions of the textbook expanded the Rule of Three to the Rule of Four. The Rule of Four incorporates a fourth representation, the verbal representation, which incorporates multiple facets: 1) the use of words to describe functions, such as "the function is increasing and concave up" or "a linear function", 2) the use of words to describe what a student did to solve a problem, and 3) the traditional "word problem", which are now considered "contextual problems" or "application problems" (Hughes-Hallett et al., 1998). Wilson (1997) reported that by 1997 the "Harvard Calc" book ("Harvard Calc" is the colloquial term given to the CCH textbook) had been adopted by over 500 institutions and was the most popular of the calculus textbooks that emphasized the use of multiple representations. By the fall of 1995, the Rule of Three based *Calculus* textbook had become the best-selling calculus textbook in the United States (Rublein, 1995).

One example of how teachers can incorporate the Rule of Three in calculus is in finding extreme values. Graphically, a graphing calculator can determine the maximum or minimum value of a function over an interval by using the TRACE feature of or using the maximum/minimum command. In this case, the student would see that the maximum value refers to the greatest $f(x)$ value over an interval. The numerical representation would allow for finding an extreme value by comparing the values of a function through the table menu of a graphing calculator. The analytical representation would be used to find extremes through the traditional method of differentiating, setting the resultant function equal to zero, solving for x and then conducting a first derivative test (Gleason & Hughes-Hallett, 1992).

2.2.4.2 Multiple representations within calculus

Using the Rule of Three/Four to inform calculus instruction has become one of the defining features of the calculus reform movement. “The emphasis on multiple representations of concepts is now the fundamental difference between traditional and reform calculus curricula” (Goerdt, 2007, p. 5). Thus, the use of multiple representations has gained significant traction with calculus reformers (Dick & Edwards, 2008; Haciomeroglu, 2007; Haciomeroglu et al., 2009; Rider, 2004). The use of the graphical, numerical, and verbal descriptions have increased dramatically in the calculus classroom and with them the opportunities for students to build stronger and more frequent network connections between representations. As noted earlier, the AP Calculus course description identifies developing flexibility with the four representations of functions as a goal of AP Calculus (Collegeboard, 2010b). However, within calculus, using multiple representations is not limited to only an understanding of functions. Linking multiple representations is also necessary for a complete understanding of the derivative and integral (Sofronas et al., 2011). Since this study focused on linking representations with functions and derivatives, I restricted my discussion of linking multiple representations in calculus to derivatives. Ross (1996) suggested that students who demonstrate a conceptual understanding of the derivative will be able to perform differential calculus and interpret the results on functions presented through symbolic, numerical, graphical, and verbal representations. Kendal and Stacey (2003) proposed a concept map of differentiation, linking the multiple representations of the derivative (see figure 6). The researchers list the physical (synonymous with the verbal) representation within their concept map; however, their empirical research focuses on the numerical, symbolic, and graphical representations. The concept map of differentiation shows how the derivative concept can be represented in multiple ways and how the multiple representations can be linked together.

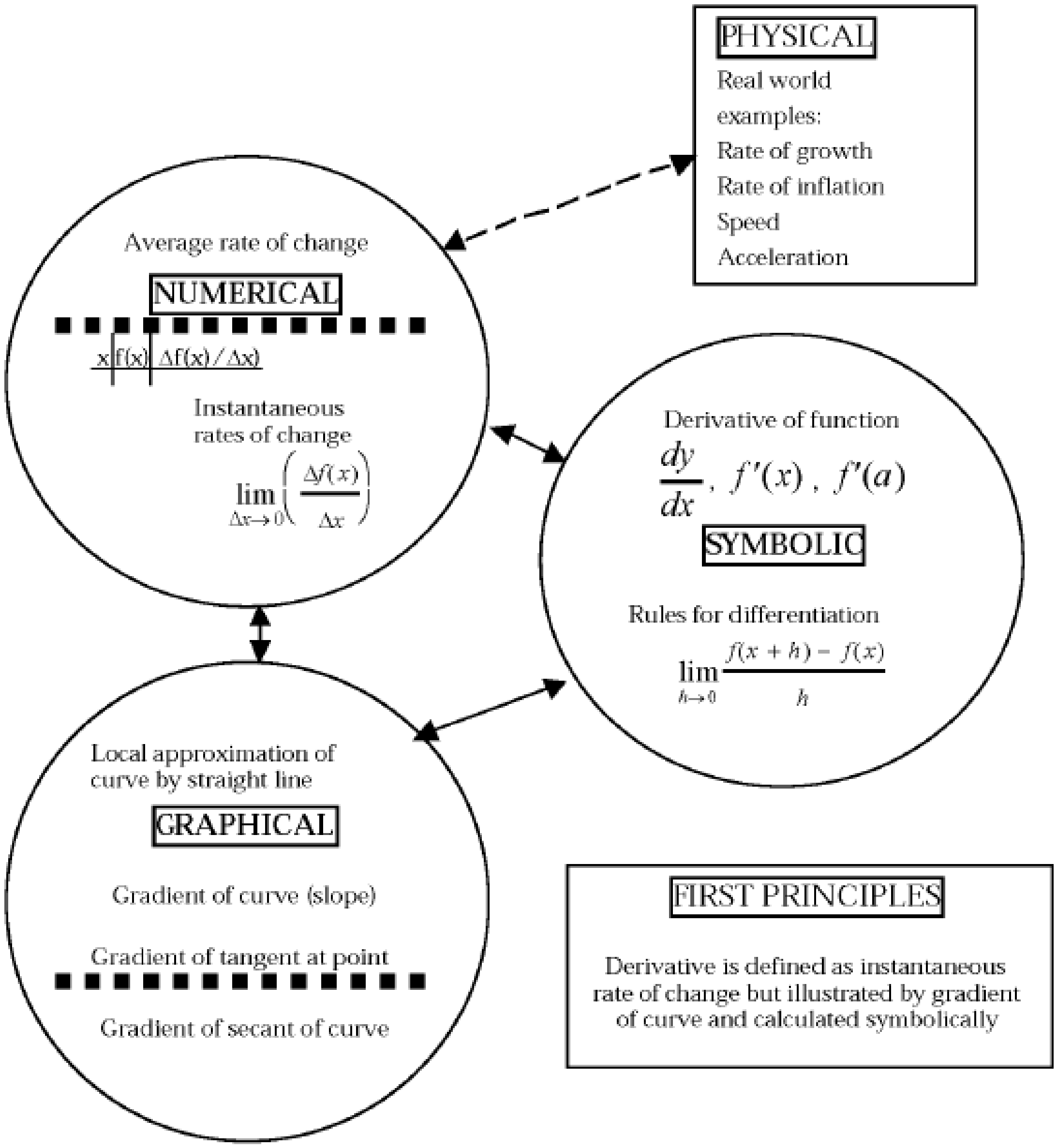


Figure 6. Kendal and Stacey's concept map of representations of differentiation.

Reprinted from "Tracing learning of three representations with the differentiation competency framework", by M. Kendal & K. Stacey, 2003, Mathematics Education Research Journal, 15, p. 26. Copyright 2003 by Mathematics Education Research Group of Australasia. Reprinted with permission.

The concept map, figure 6, proposes that reversible links exist between each representation. Each representation is discussed below.

In 2011, Sofronas, et al., authored a first-of-its-kind empirical piece identifying the overarching end goals of the first-year of calculus at the college/university level. Within these goals, the understandings needed to understand the derivative concept are identified. In the article, the authors surveyed 24 experts in the field of calculus, all of whom were Ph.D. mathematicians who had received national and/or international recognition for work in the field of mathematics or in the field of teaching mathematics. Sixteen of the experts were also textbook authors. To determine what it means for a student to understand the first-year of calculus, the researchers surveyed the 24 experts through an 11-question questionnaire and then applied a statistical analysis to determine 4 overarching end goals and 19 sub-goals that students should understand by the end of their first year of calculus.

One-hundred percent of the experts cited student understanding of the derivative as a central concept of first-year calculus. Student mastery and demonstrated comprehension of the derivative is a linchpin to deep comprehension of the first-year of calculus (Sofronas et al., 2011). To identify sub-goals of understanding of the derivative, the researchers required that at least 25% of the experts identify the same topic as essential to understanding the derivative. Sofronas et al. (2011) reported that three sub-goals of derivatives emerged: 1) understanding the derivative as rate of change was identified by 50% of those surveyed, 2) graphical understanding of the derivative was identified by 29% of those surveyed, and 3) facility with derivative computations was identified by 67% of those surveyed. These sub-goals present a framework by which the literature on instruction and learning of the derivative can be analyzed. These sub-goals are synonymous with the numerical, graphical, and symbolic representations as presented in Kendal and Stacey's (2003) concept map (figure 6), respectively. Each of these sub-goals is described in further depth below.

Before discussing in detail the components of understanding the derivative identified by Sofronas, et. al (2011), I should review the AP Calculus curriculum put forth by the College Board (2010a) and compare the treatments of the derivative as defined by Sofronas, et. al. and the AP program. This comparison is necessary as the recommendations made by Sofronas et al. (2011) were made largely by collegiate mathematics professors and could be interpreted as targeted towards collegiate, undergraduate instruction. The AP Calculus program, however, is targeted entirely to high school students and high school calculus teachers. Currently, at least half of all calculus instruction in the United States now takes place in the high school classroom (Bressoud, 2009a, 2009b), thus to inform our definition of the derivative, we should also consider how the AP calculus program defines the derivative.

The AP Calculus syllabus says that "students should understand the meaning of the derivative in terms of a rate of change and local linear approximation and should be able to use derivatives to solve a variety of problems" (Collegeboard, 2010a, p. 6). Furthermore, the syllabus identifies four sub-goals of understanding the meaning of the derivative: 1) concept of the derivative, 2) derivative at a point, 3) derivative as a function, and 4) computation of derivatives. These four goals will now be mapped onto the framework presented by Sofronas et al. (2011) and the concept map of Kendal and Stacey (2003).

Understanding the derivative as a rate of change – the numerical representation of a derivative

Fifty-percent of the experts (twice the required threshold of 25%) surveyed identified the derivative as a rate of change as an important concept for students to master during the first-year of calculus. To understand the derivative as rate of change, students must demonstrate knowledge of three separate but related elements (Sofronas et al., 2011). First, students must understand the foundational concepts of function and limit as they relate to the central idea of the derivative as the rate of change. It should be noted that the experts surveyed by the researchers intimate that students should enter a calculus course with a thorough understanding of functions. The research exploring calculus students' understanding of functions strongly indicates that an instructor should not assume such an understanding (Gur & Barak, 2007; Habre & Abboud, 2006; Horvath, 2008; Tall, 1992; Thompson, 1994). Secondly, students should be able to make a qualitative distinction between the instantaneous rate of change (i.e. the derivative) and the average rate of change (an estimate of the derivative). Also, students need to be able to make these distinctions when the independent variable is a quantity other than just time. Finally, students should be able to recognize that the derivative is not just change, but that it is change in a describable way (Sofronas et al., 2011).

Within the AP syllabus, several sub-goals seem to agree with the descriptions given by Sofronas et al. (2011). Under the sub-goal of "Concept of the derivative" the AP syllabus describes student understanding as recognizing the derivative as the instantaneous rate of change and as the limit of the difference quotient. Falling under the category of "Derivative at a point," students should understand the derivative as the instantaneous rate of change as the limit of the average rate of change (Collegeboard, 2010a).

Figure 6, the concept map of differentiation, shows that Kendal and Stacey (2003) referred to the numerical representation of the derivative as finding a rate of change arbitrarily close to a value, typically by taking the limit of a difference quotient. Within the numerical representation, students must understand the distinction between the average rate of change (the difference quotient without a limiting process) and the instantaneous rate of change (the difference quotient with a limiting process).

When I refer to understanding the derivative as the rate of change, I refer to students' understanding of the derivative as the end-result of the limiting process applied to the difference quotient that represents the average rate of change. It should be noted that the definition I use when referring to understanding the derivative as the rate of change is similar to the definition of students' graphical understanding of the derivative as offered by Asiala et al. (1997). The researchers describe students' graphical understanding of the derivative as characterized by understanding the connections between the process of taking the limit of the slopes of a set of secant lines to produce the slope of a tangent line at a point and the process of taking the limit of the values of the average rate of change over successively smaller time intervals to produce the instantaneous rate of change. In the following section, I offer an alternative definition of graphical understanding of the derivative. For the purposes of this document, and to maintain fidelity with the viewpoints supported by Sofronas et al. (2011), the learning goals stated by the Collegeboard (2010a), and the concept map of Kendal and Stacey (2003), when referring to understanding the derivative as the rate of change, I will be referring to students' understanding of the derivative as the end result of the limiting process applied to the difference quotient that represents the average rate of change – the numerical representation of the derivative.

Graphical understanding of the derivative – the graphical representation of a derivative

Twenty-nine percent of the experts surveyed by Sofronas et al. (2011) reported placing an emphasis on students' ability to interpret the derivative graphically as a key to understanding the derivative concept. Graphical understanding of the derivative includes the concepts of slope and several common applications of derivatives such as identifying relative maxima and minima, identifying points of inflection, and finding intervals of concavity.

The AP syllabus identifies key concepts of graphical understanding of the derivative as including an understanding of the derivative as the slope of a curve at an identified point, as the slope of the line tangent to a curve at a given point, as relating the corresponding characteristics between the graph's function and the function's derivative, and the understanding of the relationship between the increasing and decreasing behavior of a function and the positive or negative values of the derivative of a function (Collegeboard, 2010a).

Asiala et al. (1997) offered a particularly concise description of what is necessary to understand the graphical representation of the derivative.

> A key issue in a graphical understanding of the derivative [is] the relationship between the derivative of a function at a point and the slope of the line tangent to the graph of the function at that point. This forms a foundation for understanding the derivative as a function which, among other things, gives for each point in the domain of the derivative the corresponding value of the slope. (Asiala et al., 1997, p. 414)

This view is congruent with the views of other calculus researchers. Dreyfus (1990) offered that students should be able to demonstrate that they know that differentiation produces a new function from an old one, and that this new function returns the value of the old function's instantaneous rate of change at each input.

Kendal and Stacey (2003) described the graphical representation of differentiation as "finding the gradient of the curve" (p. 24). The gradient of a curve is typically referred to as the slope of the line tangent to a curve at a given point.

When I refer to a student's "graphical understanding of the derivative" I refer to a multi-faceted definition. In this document, a complete "graphical understanding of the derivative" incorporates three fundamental understandings:

1) a student's ability to make inferences regarding the behavior of a function based on information provided by the derivative, such as intervals of increase and decrease as well as intervals of concavity,

2) understanding that the act of differentiation of a function produces a new function and that this new function returns an output value that uniquely describes the slope of the line tangent to the original curve at the input value;

3) the ability to construct both the graph of a derivative given a function and to be able to graph a function, given a derivative.

The third element of a complete graphical understanding of the derivative is an example of how reversibility presents within a particular representation. In this case, the student must be able to construct the graph of f' given the graph of f and construct a possible graph of f given the graph of f'.

Facility with derivative computations – the symbolic representation of differentiation

Sofronas et al. (2011) reported that two-thirds of the experts surveyed listed computation of derivatives of elementary functions as vital to understanding the derivative concept. The experts placed the emphasis on student facility with the differentiation rules of elementary functions and the product rule, quotient rule, and chain rule.

The AP syllabus identifies "Computation of derivatives" (Collegeboard, 2010a) as a list of procedures that support understanding of the derivative. Specifically, AP calculus students are expected to be able to differentiate elementary functions, to be able to take the derivatives of sums, products, quotients, composite functions, and functions defined implicitly.

It should be noted that the experts surveyed by Sofronas et al. (2011) and the AP Calculus syllabus appear to place different weights on the importance of student facility with computational derivatives. A high percentage (67%) of mathematics experts surveyed by Sofronas et al. (2011) listed the importance of students being able to take derivatives of elementary functions and sums, products, quotients, and compositions of elementary functions. One expert wrote, “the basic rules of differentiation, you’ve got to have that cold, absolutely cold … If you can’t calculate … you’re going to get hung up there and not be able to see the conceptual picture that we are trying to paint. Those things fill in the details in the big picture” (Sofronas et al., 2011, p. 138). Sixty-seven percent is a higher percentage of agreement among the experts surveyed than for either of the topics “derivative as a rate of change” (50%) or “graphical representation of the derivative” (29%). The AP syllabus, however, seems to place the least weight of all the derivative sub-goals on facility with derivative computations. Kennedy (2003) noted that “the most significant thing about this topic is that it is listed last, consistent with the philosophy that the emphasis of the course is not on manipulation” (p. 19).

Kendal and Stacey (2003) defined “symbolic representation” as referring to finding the derivative of a function given in symbolic form by using known differentiation formulas. In Kendal and Stacey’s concept map of differentiation, figure 6, computation of derivatives falls in the symbolic representation.

When I refer to computational fluency with derivatives, I refer to a student’s ability to find a function f', that represents the derivative of a function f, where f is an elementary function or a sum, product, quotient, or composition of elementary functions. Much like what has been reported in algebra classes, calculus students exhibit a strong preference for the algebraic representation and have profound difficulties when an algebraic expression is not available (Asiala et al., 1997; Berry & Nyman, 2003; García, Llinares, & Sánchez-Matamoros, 2011; Knuth, 2000).

Using words to interpret differential relationships – the verbal representation of differentiation

Although not mentioned in Kendal and Stacey's (2003) differentiation competency framework nor discussed by Sofronas et al. (2011), the verbal representation of the derivative is also included in a discussion of multiple representations of the derivative concept by Ross (1996) and Goerdt (2007) and is an integral part of the CCH curriculum. The verbal representation "refers to the use of English language to interpret mathematical concepts" (Goerdt, 2007, pp. 46-47). Applying this definition to the derivative concept, the verbal representation refers to describing the relationship between a function and its derivative in words and interpreting the meaning of the derivative of a function within the context of the problem. An example of the kind of question that requires a verbal representation of the derivative follows.

Ex. 1: Suppose $T(x)$ is the function that models the air temperature in degrees Fahrenheit in Rome, Italy during one day in the month of July. Let x represent the number of hours since midnight (i.e. $x = 0$ means 12:00 am, $x = 1$ means 1:00 am, $x = 2$ means 2:00 am, etc.). Interpret the result that $T'(11) = 2.6$ using correct units.

Research indicates that calculus students' conceptions of the derivative and integral improve as students' facility with multiple representations, especially the graphical representation, improves (Berry & Nyman, 2003; García et al., 2011; Goerdt, 2007). One reason for this is that curve sketching, such as sketching a function f from a numerical representation of f', requires students to make connections between the algebraic and graphical representations or between the numerical and algebraic representations. Habre and Abboud (2006) taught a differential calculus course with a reformed curriculum focusing on the algebraic and graphical representations of functions and derivatives. They found that most students developed a "very good understanding of the idea of the derivative" (p. 67) and an appreciation of the information that each representation affords the reader.

Kendal and Stacey (2003) authored a report of empirical research in which they evaluated a proposed framework called the Differentiation Competency Framework (DCF) that organizes concept knowledge about differentiation into 18 competencies spread across the algebraic (symbolic), numerical, and graphical representations of the derivative. As noted earlier, the researchers place great value on the flexible use of multiple representations in differential calculus.

In the DCF, graphical differentiation refers to finding the slope of the graph using differentiation or approximating the slope of a graph using local linearity or by finding the slope of an appropriate secant line. Symbolic differentiation refers to finding the derivative of an algebraic expression, most likely by following well-known differentiation rules. Numerical differentiation refers to the difference quotient and the instantaneous rate of change (Kendal & Stacey, 2003). The DCF presupposes that reversible translations exist between the numerical, symbolic, and graphical representations.

Kendal and Stacey (2003) used the DCF to create the eighteen item Differentiation Competency Test (DCT), which contains one validated question for each competency identified in the DCF. The purpose of the DCT is to monitor students' conceptual understanding of the derivative across the symbolic, numerical, and graphical representations. It also measures the students' ability to move flexibly across multiple representations and tracks representational preference. The results of the DCT indicate that only the very best students could successfully translate between the three representations of derivatives and only half of the students demonstrated proficiency with two of the representations. These results suggest that developing flexibility with the three representations of derivatives is difficult for students and serves as a caution for calculus teachers that students do not easily develop flexibility in a calculus classroom. The researchers suggested that one possible instructional approach that may help calculus students is to focus on the symbolic and graphical representations first and incorporate the numerical representation after students are fluent with the symbolic and graphical representations. This recommendation seems to have been reached by the relative performance of students on the DCT. The students were far weaker in solving problems involving the numerical representation than the graphical or symbolic. One factor that may mitigate this recommendation in regards to the present study is that Kendal and Stacy only allotted 22 days for instruction. The instructional course in this study will be approximately 48 days.

In her dissertation research, Goerdt (2007) examined the differences in calculus students' understanding of the derivative concept when learned in a traditional (emphasizing the symbolic notation) classroom versus a reform (emphasizing connections between and within symbolic, graphical, numerical, and verbal representations of functions and derivatives) classroom. She found that "the mean understanding of derivative of the reform students is greater than that of the traditional students, when understanding is considered the ability to translate between and within representations of the concept" (Goerdt, 2007, p. 117). Furthermore, Goerdt (2007) concluded "that reform calculus curricula promote a more flexible and transferable understanding of the concept of derivative" (p. 120). This research suggests that calculus instruction linking multiple representations benefits students' understanding of the derivative and students' flexibility with multiple representations of the derivative.

2.3 WHY THE USE OF MULTIPLE REPRESENTATIONS HELPS STUDENTS TO DEVELOP REVERSIBILITY

In the previous sections, I reviewed the research on reversibility and multiple representations. In this section, I attempt to present an argument using the aforementioned research to support the hypothesis that there exists a relationship between linking multiple representations and developing reversibility. It is important to note that I am not limiting reversibility to just moving forward and backward between representations, I am proposing that as students learn to move flexibly between representations, they will develop an ability to think reversibly in such a way that they develop reversible connections when learning new mathematical processes in the forward direction.

The NCTM (1989) states that “students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept will have … a powerful, flexible set of tools for solving problems” (p. 146). Lesh et al. (1987) noted that a characteristic of successful problem solvers is an instinctive ability to switch to the most convenient representation at any time in the solution process. This ability necessarily requires a well-developed flexibility. The NCTM implied that proficiency in translating between multiple representations (flexibility) helps to develop problem-solving processes. Since reversibility is a problem-solving process (Krutetskii, 1976), it may be that proficiency in translating between multiple representations helps to develop reversibility.

Greeno and Hall (1997) agreed with the stance taken by the NCTM (1989) that multiple representations are tools useful for constructing understanding and are adaptable for the mathematical task at hand. Since I adopt the view that understanding requires the building of connections between mathematical concepts (Hiebert & Carpenter, 1992) and reversibility is a kind of connection, linking multiple representations may be useful for constructing reversible connections.

If students must consider each concept through multiple representations, specifically reversing thought processes across representations, then students may develop a reversible thought process that can be adapted to other mathematical processes. Krutetskii's (1976) research supports this claim. Since, as Krutetskii (1976) stated, "reverse problems … have a great value for developing … the ability to switch from direct to reverse operations" (p. 187), then continually rehearsing how to translate back and forth (or direct and reverse) between multiple representations should help students to develop "the ability to switch from direct to reverse operations".

According to Knuth (2000), flexibility with multiple representations requires understanding how to move in both directions between two representations, in other words, reversibility is necessary for flexibility. Rider's (2004, 2007) research supported this assertion. Janvier (1987a) proposed that students will learn to translate between representations best if they learn the translations in symmetric pairs, such as learning to move from table to graph and graph to table at the same time. What Janvier proposed is Krutetskiian reversibility between representations. Janvier (1987b) created a table of translations to show the actions necessary to translate from one representation into another (for example, translating from a symbolic representation into a graphical representation). Figure 7 is a reproduction of Janvier's table. As an example, consider how to translate from a table to a verbal description. Janvier's table shows that reading is the action required to translate from a table to a verbal description. According to Janvier's table, if a function is presented in a symbolic representation (formulae), then sketching is the necessary action to translate to a graphical representation.

From \ To	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Figure 7. Janvier's table of translation processes.

Dick and Edwards (2008) make a particularly salient observation of Janvier's (1987b) table of translations between representations:

> Note that reversing the direction of a translation between functions representations is not simply a matter of 'inverting' the steps in an algorithm. Each of the activities Janvier identifies in his table has its own unique set of cognitive skills and understandings associated with it. (p. 263)

This observation intimates that reversibility between representations requires far more than reversing the steps taken in the first translation. Rather, reversibility between representations requires the development of understandings specific to each representation and then applying these understandings. In this sense, reversibility can be viewed as a discipline to be developed. In this discipline, the students learn to identify the processes that must be reversed in order to move from the end product back to the beginning and then learn how to carry out these processes. From this perspective, multiple representations provide a particularly fertile learning ground for practicing with reversibility. Translating back and forth between multiple representations may train students to think reversibly, thus developing reversibility. Furthermore, if translations between representations are best learned in reversible pairs (Janvier, 1987b) but are not simple reverse associations (Krutetskii, 1976), then developing flexibility between two representations requires Krutetskii's second kind of reversibility, wherein a change in the direction of thought is required, but a simple reverse tracing of a direct procedure is insufficient. For clarity, I refer to reversible translations between representations as representational reversibility.

Flexibility holds promise for developing reversibility by appealing to different students' representational preferences. Krutetskii (1976) associated a student's representational preference with his/her learning style. For example, he considers students who prefer algebraic representations to be analytic learners; students who prefer graphical and/or geometric approaches to solving problems are called visual learners. Krutetskii termed students who use both analytical and graphical problem-solving methods as harmonic learners. He noted that both analytic and visual learners are limited by their learning preferences and benefit from an ability to switch between analytic and visual problem-solving approaches. Krutetskii's (1976) descriptions foretell the body of research now promoting the benefits of translating between multiple representations. Since analytical, visual, and harmonic thinkers' problem-solving skills seem to benefit from exposure to multiple representations and that reversibility is a key process in problem solving, then it is reasonable to surmise that exposure to multiple representations may help to develop reversibility.

Rider (2004) noted that within APOS theory, when a student develops a process conception of functions, she/he comes to see each representation as a unique form of the same entity (flexibility). As discussed earlier, reversibility is the key evidence of moving from action to process (interiorization) and reversibility is evidence of moving from a process conception to an object conception (encapsulation and de-encapsulation). This suggests that if we can identify activities that aid students in moving from action to process or from process to object, we can likely find evidence of activities that help students to develop reversibility. Since Rider observed the development of flexibility as a mechanism by which students gain a process conception of functions, we can again deduce that developing flexibility helps students to develop reversibility.

"Examining and re-examining the consequences of a mathematical action when the object is represented in multiple ways potentially increases the possible conceptual connections exponentially" (Dick & Edwards, 2008, p. 258). The researchers' observation holds potential for why instruction in multiple representations may help to develop reversibility. When students consider how a transformation acting on one representation effects a different representation and then continues to consider these possibilities from all combinations of representations, the students are in actuality being trained to consider the effects of an action in the forward and reverse directions. In other words, the students are training their thought processes to develop reversible connections. It may be the case that after receiving this training, the students are better equipped and more likely to develop reversible connections in other areas of mathematics.

There are noticeably few research studies that have considered the effects of reversibility and multiple representations on calculus learning. At this time, I am only aware of one such study. Haciomeroglu et al. (2009) investigated how students' representational preferences interact with reversibility and flexibility in the sketching of antiderivative graphs. The researchers noticed that each student's representational preference greatly influenced her/his thought processes and "resulted in a one-sidedness in their understandings" (Haciomeroglu et al., 2009, p. 86).

The participants all demonstrated a noticeable lack of flexibility and none of the participants attempted to reverse any thought processes. The researchers concluded that the representational interpretations, either visual or analytic, of the participants served as an example of a one-way relationship. There was no evidence that any of the participants considered whether or not her/his antiderivative graph would produce the correct derivative graph (Haciomeroglu et al., 2009). Haciomeroglu et al. (2009) suggested that teachers can help students to gain a "wider and more robust perspective" (p. 87) of mathematics by encouraging students to develop reversibility and flexibility.

Representational fluency (Zbiek et al., 2007) suggests a possible mechanism by which learning to link multiple representations may help to develop reversibility. Representational fluency refers to "meaningful and fluent interaction" (p. 1196) with representations when necessary. Representational fluency incorporates translation between representations, drawing inferences from each representation of a mathematical concept, and "the ability to generalize across different representations" (p. 1192). Students with representational fluency are able to use multiple representations interactively, by that Zbiek et al. (2007) mean that a student working in one representation can translate into a different representation when encountering a difficulty, work around the difficulty in the new translation, and then translate back to the original representation to finish solving the problem. Zbiek et al. suggested that representational fluency has great potential as a research construct because "a central set of research questions occurs at the interface of representational fluency and other constructs" (p. 1197). This research study exists in that space, the interface of representational fluency and reversibility.

Linking multiple representations helps students to develop connections between representations and flexible thinking. As connections increase in number and grow stronger, conceptual understanding develops (Hiebert & Carpenter, 1992; Zbiek et al., 2007). As conceptual understanding develops, reversibility may develop as well.

2.4 SUMMARY

Conceptual understanding consists of the building of rich network connections between mathematical concepts (Hiebert & Carpenter, 1992). Krutetskii (1976) proposed two problem-solving processes that serve as kinds of network connections – reversibility and flexibility. As students develop reversibility, they develop stronger network connections and a deeper understanding of the mathematical processes in question. As students develop flexibility, they create new links within their network of mathematical knowledge. Translating functions between multiple representations helps students to develop flexibility. Thus, as students develop reversibility and flexibility, conceptual understanding should increase. I suggest that the flexible use of multiple representations will help students to develop reversibility. The literature suggests that this may be the case because flexibility across multiple representations is linked to the development of reversibility. I suggest that as students develop flexibility between representations, their ability to make reversible connections within other mathematical domains will increase as well.

3.0 METHODS

This study investigated reversibility and linking multiple representations in a calculus environment. In the two decades since Moschkovich et al. (1993) called for research investigating the benefits of the Cartesian Connection, there have been some studies involving the use of multiple representations and attempting to identify the educational benefits of developing representational flexibility (Amit & Fried, 2005; Lesser & Tchoshanov, 2005). There have also been a limited number of studies that have examined whether or not students exhibit reversibility (Davis & McGowen, 2002; Haciomeroglu et al., 2009; Rachlin, 1981; Ramful & Olive, 2008; Tzur, 2004) since Krutetskii (1976) first identified reversibility as a problem solving process. What makes this study unique is its investigation of reversibility on a developmental trajectory, its investigation of reversibility as three separate but related entities (reversibility of two-way reversible processes, reversibility of the mental process in reasoning without reversible translation, and representational reversibility), and its investigation into the thought processes that students use when using reversibility to solve problems. I am unaware of any existing research that has examined the thought processes that students use when using reversibility to solve problems.

As an element of mathematical understanding, the reversibility process is likely content specific (Krutetskii, 1976; Rachlin, 1981; Ramful & Olive, 2008; Teachey, 2003). This study used calculus as its content lens. Calculus has served as a research lens for several research studies regarding multiple representations (Dick & Edwards, 2008; Haciomeroglu, 2007; Haciomeroglu et al., 2009; Rider, 2004). Despite agreement that calculus is a fertile research field for investigating reversibility (Berry & Nyman, 2003; Haciomeroglu et al., 2009; Norman & Prichard, 1994), I am only aware of one empirical study (Haciomeroglu et al., 2009) of reversibility situated within a calculus classroom. Calculus is a particularly appropriate content lens through which to investigate linking multiple representations and developing reversibility because a study of calculus requires translation between multiple representations of functions, between multiple representations of derivatives, and from representations of functions to representations of derivatives and vice versa; also, as Norman and Prichard (1994) have noted, the relationship between differentiation and integration is a reversible relationship, that is that they are inverse operations. Specifically, differentiation and integration represent two properties of functions, the rate of change and the accumulation of area, that are reversible properties. Thus, a foundational understanding of calculus, the relationship between differentiation and integration (Sofronas et al., 2011), offers a lens through which one can investigate how reversibility develops.

This study used a calculus course as a lens through which the relationship between linking multiple representations and developing flexibility and reversibility was investigated. Specifically, this study attempted to answer the following research questions:

1) To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?
2) To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:
    i. To what extent does reversibility of two-way reversible processes develop?

ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?

iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?

3) What are the thought processes that students utilize when using reversibility to solve problems?

To answer these questions, this study used two kinds of data, 1) class-level data, and 2) individual interview data. The class level data was collected from all students enrolled in the course during the length of the study (n = 21). The individual interview data was collected from six (6) students through think-aloud interviews.

To document the development of flexibility with multiple representations (research question 1), this study collected and analyzed class-level data from two sources of evidence: 1) a class-wide flexibility pre-test and 2) a class-wide flexibility post-test. The students' ability to link representations was assessed and documented at the class level through a pre-test and post-test model. A flexibility pre-test (FPT) measuring students' flexibility with functions was administered after the pre-requisite chapter, in which functions are reviewed and at the start of the study. The post-test is the Differentiation Competency Test (DCT), which was created and validated by Kendal and Stacey (2003). The DCT was administered at the end of the study and was intended to quantify the extent to which students have developed an ability to demonstrate flexibility of multiple representations of derivatives.

To document the development of reversibility (research question 2), this study used both class-level data and individual interview data. I collected the class-level data through the use of exit slips and opening activities. One exit slip and one opening activity were collected approximately daily throughout the course of the study on instructional days (i.e. exit slips and opening activities were not administered on testing days). Each set of one exit slip and one opening activity formed a Krutetskiian paired-problem set. The exit slip measured the current day's direct learning and the subsequent class period's opening activity measured the previous day's learning in a reverse direction. One-third of the exit slip and opening activity pairs required the use of reversibility as a two-way process, one-third of the exit slip and opening activity pairs required the use of reversibility of the mental process in reasoning without reversible translation, and one-third of the exit slip and opening activity pairs required the use of representational reversibility.

The individual interview data was collected through think-aloud interviews. *Think-aloud* interviews refers to a specific kind of interview, in which the participant is instructed to verbalize her/his thoughts while attempting to solve the interview questions. The interviewer interjects very little other than to encourage the participant to "describe what you are doing" in the event that the participant stops talking while solving the interview questions (Willis, DeMaio, & Harris-Kojetin, 1999). Six (6) students were selected to participate in four think-aloud interviews each. The think-aloud interviews used calculus problems designed to elicit evidence of reversible thinking. The four interviews took place at different intervals during the study and were positioned to allow for the development of reversibility between interviews. The think-aloud interviews were used to investigate the thought processes that students use when solving reversible problems (research question 3).

In order to increase the potential power and generalizability of the conclusions drawn from the interview data, I chose a multiple-case study research design consisting of two cases of three different groups. The three groups are differentiated by demonstrated achievement. Based on the results of the flexibility pre-test, the entire class of students was divided into thirds. Two students from the highest scoring third, two students from the middle third, and two students from the lowest scoring third were selected to participate in the think-aloud interviews.

Three groups were chosen to increase variability in the results of the interviews. Two cases within each group were chosen for the perceived benefits of the opportunity for direct replication between cases and the expected increase in power that accompanies reaching analytic conclusions arising independently from multiple cases. Two students were selected as representative of each group and participated in the interviews allowing for a multiple-case study within each group. "A case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (Yin, 2009, p. 18). In this study, the contemporary phenomenon is the development of reversibility and the context is learning calculus in a calculus classroom. I selected a multiple-case study design because it is well-suited to offer insights into phenomena (in this instance, the development of reversibility) consistent across multiple cases (two students in each groups) and because multiple-case study allows for examination of a phenomenon in its naturally occurring context (learning calculus in a calculus classroom) (Stake, 1995; Yin, 2009). A particular strength of case study research is that it allows for the collection and analysis of multiple sources of data with the intended outcome of a convergence of the data through triangulation (Stake, 1995; Yin, 2009). In this study, the multiple sources of data in the case study consist of the data from the four separate interviews that elicit evidence of the use of reversibility in four separate content areas within calculus.

The class level data and the individual interview data complement and augment one another. The class level data offers a broad picture of development of reversible conceptions and flexibility. The individual interview data served to confirm the patterns observed at the class level and offer insights and further details into the students' development of reversibility of two-way processes, reversibility of the mental process in reasoning without reversible translation, and representational reversibility. The individual interview data also offered a window into the kinds of thought processes that students utilized when using reversibility to solve problems.

3.1 PARTICIPANTS

The participants in this study were twenty-one students enrolled in AP Calculus AB at an urban school district in the mid-Atlantic region of the United States. The school district is economically diverse, with approximately 70% of students qualifying for the federal free and reduced lunch program. The school district is also ethnically diverse with approximately 70% African-American students and 30% White students. The students enrolled in the course ranged in grade level from 10^{th} grade to 12^{th} grade. Table 2 summarizes the demographic information of the 21 participants; the column "prior achievement" reports the students' year average in the prior mathematics class, Trigonometry & Advanced Math.

Table 2. 21 Participants' Demographic Information

Gender		Race			Grade		
Male	Female	Black	White	Other	10	11	12
13	8	4	15	2	3	11	7

I used data from all 21 students to attempt to answer research questions 1 and 2. All 21 students attempted the FPT and the DCT. Also, all 21 students were expected to complete an exit slip and opening activity approximately 33 times during the study. However, due to absence, on average 18-19 students completed an exit slip and opening activity on each day that one was administered.

A subset of the 21 students were selected for task-based, think-aloud interviews. In an effort to increase the likelihood of gaining multiple perspectives, I used a maximum variation sampling (Patton, 2002) of relatively high-achieving, average-achieving, and low-achieving students to select students for case study. This method of selection is consistent with how other researchers conduct case study research on reversibility (Krutetskii, 1976; Teachey, 2003). The selection of relatively high-achieving, average-achieving, and low-achieving students should offer variation consistent with the variation observed in Krutetskii's study.

This study recruited two students from each group of students, relatively high-achieving, average-achieving, and low-achieving to participate in the task-based, think-aloud interviews. I used the students' flexibility pre-test scores as the selection factor. Flexibility was chosen as the selection factor because, as described earlier, the extant body of research suggests that flexibility may be related to the development of reversibility. After administering and grading the FPT, I separated the scores into 3 groups based on score. The high group had 8 scores. The middle group had 7 scores and the low group had 6 scores. Kelsay and Michael both scored the median score in the high group. Fred and Jill both scored the median score in the middle group. Kirsten scored the median score in the low group. Marcus was in the low group, but did not score the median score. He was the highest score in the low group. He was selected because, of the students who were willing to participate in the interviews, his grade was closest to the median score of the low group. Table 2 below displays the six interview participants and each participant's relevant scores from the course.

Table 3. Interview participants' testing data

Name	Group	FPT %	DCT %	AP Score (out of 5)	Practice AP Score – Unscaled (%)	Practice AP Score – Scaled (%)	Class Test Average – Unscaled (%)	Class Test Average – Scaled (%)
Kelsay	High	65	89.5	5	66	90	78	94
Michael	High	65	84.2	5	55	87	71	92
Fred	Middle	40	60.5	3	39	82	51	86
Jill	Middle	40	71.1	3	51	86	52	87
Kirsten	Low	20	42.1	None	26	75	27	76
Marcus	Low	30	29.0	None	36	81	33	81

It should be noted that although the practice AP score column and the class test average column seem to indicate that all 6 participants have low overall scores, this course uses only released AP Calculus items on all exams and thus uses the AP grade scale to score the exams. Using the most recently released data, Table 3 shows how the students' grades are scaled to a $0 - 100\%$ grading scale, where x represents the student's grade on a test.

Table 4. Scaled scores based on AP grade scale

% Score – Unscaled	Corresponding AP Exam Grade	% Score Scaled for use in class
$x \geq 64\%$	5	$x \geq 90\%$
$47\% \leq x < 64\%$	4	$85\% \leq x < 90\%$
$32\% \leq x < 47\%$	3	$80\% \leq x < 85\%$
$21\% \leq x < 32\%$	2	$70\% \leq x < 80\%$
$x < 21\%$	1	$x < 70\%$

Six students participated in four interviews each, for a total of 24 task-based, think-aloud interviews. The number of participants and the number of interviews used in this study are consistent with other dissertation research on reversibility. Teachey (2003) interviewed ten participants one time each. Rider (2004) interviewed eight participants twice each, for a total of 16 interviews. Haciomeroglu (2007) interviewed three participants one time each, and Rachlin (1981) interviewed four participants on average 10 times each, resulting in approximately 40 interviews.

3.2 INSTRUCTIONAL SETTING

In order to investigate the existence of a relationship between linking multiple representations and developing reversibility, I (the researcher and calculus instructor) designed a 48-day instructional plan that emphasizes linking multiple representations of functions and linking multiple representations of derivatives in a differential calculus course.

Differential calculus is a natural setting for viewing the development of reversibility as differentiability and integration are reversible operations (Norman & Prichard, 1994). Thus, one would expect that if reversibility is present, students would develop some conceptions of integration while learning differentiation. Differential calculus is also well suited for instruction that links multiple representations (Goerdt, 2007; Habre & Abboud, 2006; Heid, 1988; Kendal & Stacey, 2003).

The course met five times per week, for 45-minutes per class. The calculus course used a reform calculus textbook, *Calculus: Early Transcendentals Single Variable*, 9th ed. (Anton, Bivens, & Davis, 2009). The authors noted that the rule of four, presenting concepts from the numerical, symbolic, verbal, and graphical perspectives, is incorporated "whenever appropriate" (Anton et al., 2009, p. viii). The textbook is described as a reform textbook because "the emphasis on multiple representations of concepts is now the fundamental difference between traditional and reform calculus curricula" (Goerdt, 2007, p. 5).

The portion of the course covered during this study consisted of Chapter 2: The Derivative, Chapter 3: Topics in Differentiation, and Chapter 4: The Derivative in Graphing and Application. A course calendar and list of topics is included as an appendix (Appendix A). During the 48-day study, 8 days consisted of administering tests, 4 days consisted of returning and discussing tests, and 36 days consisted of instruction of differential calculus. Chapter 2: The Derivative lasted 15 class periods. Chapter 3: Topics in Differentiation lasted 13 class periods. Chapter 4: The Derivative in Graphing and Application lasted 20 class periods.

During the 48-day focus of this study, 47 days (35 instructional days and 12 testing days) required translation between representations. The only day during the study that did not require translation is a one-day study of a differentiation technique known as logarithmic differentiation. To ensure that linking representations was emphasized during the course of study, the instructor selected tasks from the textbook and/or supplemental materials such as AP preparation workbooks that required students to link multiple representations and to move flexibly between them. As an example of how tasks were used to require students to attempt to link multiple representations and to move flexibly between them, I present three examples taken from section 2.6: The Chain Rule. In the following examples, students must translate from the verbal representation to the numerical representation (Ex. 1), from the symbolic and numerical representation to the numerical representation (Ex. 2), and from the graphical and symbolic representation to the numerical representation (Ex. 3).

Ex. 1: Suppose your motorcyle is known to get 60 miles per gallon when you drive around town. If gas currently costs $3 per gallon, how much does it cost per mile to drive to school?

In example 1, students are trying to find the rate of change of money with respect to miles traveled, $\frac{d\$}{d(miles)}$. However, money is not presented as a function of miles traveled. Money is a function of gallons of gas consumed, and gallons of gas consumed can be expressed as a function of miles driven. Thus, the rate of change of money with respect to miles traveled can be found by using the chain rule to determine that the rate of change of money with respect to the number of miles traveled, $\frac{d\$}{d(miles)}$, is equal to the rate of change of money with respect to gallons of gas consumed, $\frac{d\$}{d(gal)}$, multiplied by the rate of change of gallons of gas consumed with respect to the number of miles traveled, $\frac{d(gal)}{d(miles)}$. Symbolically, $\frac{d\$}{d(miles)} = \frac{d\$}{d(gal)} * \frac{d(gal)}{d(miles)}$. Numerically, $\frac{d\$}{d(miles)}|_{\frac{m}{gal}=60, \frac{\$}{gal}=3} = \frac{1}{60} * \frac{3}{1} = \frac{1}{20} \frac{\$}{mile}$. In this example, some students may translate directly from the verbal expression to the numerical answer by comparing the units of the problem. Other students may translate from the verbal to the symbolic and then to the numerical representation as I described.

Ex. 2: Suppose that it is known that $f(2) = 4, f'(2) = -1, f(3) = 1$, and $f'(3) = -1$. Also, it is known that $g(2) = 3, g'(2) = 8, g(3) = 21$, and $g'(3) = 7$. Find $F'(2)$, where $F(x) = f(g(x))$.

In example 2, students are expected to use the numerical values in the given table and the symbolic expression for $F(x)$ to evaluate $F'(2)$. The chain rule must be used to determine an expression for $F'(x)$, $F'(x) = f'\big(g(x)\big) * g'(x)$. Then, the numerical representation must be used to evaluate the expression $F'(x)$ when $x = 2$, $F'(2) = f'\big(g(2)\big) * g'(2) = f'(3) * 8 = -1 * 8 = -8$.

Ex. 3: Use the graph of the function f in the accompanying figure to evaluate $\frac{d}{dx}[\sqrt{x - f(x)}]|_{x=1}$

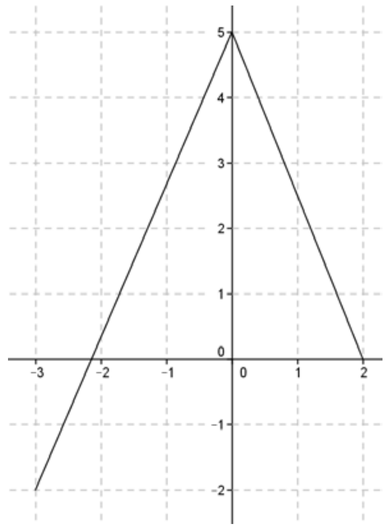


Figure 8. Example 3 graph of $\boldsymbol{f(x)}$

In this example, the students must interpret the derivative of a symbolic function and the derivative of a function presented graphically to evaluate the derivative of $\sqrt{x - f(x)}$ when $x = 1$. In this case, $\frac{d}{dx}\left[\sqrt{x - f(x)}\right]\Big|_{x=1} = \frac{1}{2\sqrt{x-f(x)}}\left(1 - f'(x)\right)\Big|_{x=1} = \frac{1}{2\sqrt{1-f(1)}}\left(1 - f'(1)\right)$. At this point, the problem has been entirely algebraic; however, to move forward, the student must read $f(1)$ from the graph, $f(1) = \frac{5}{2}$, and find $f'(1)$. There are two possible ways that a student could find $f'(1)$. First, a student could make use of the fact that $f'(1)$ equals the slope of the line tangent to the curve $f(x)$ at $x = 1$. Since $f(x)$ is a piecewise linear function, $f'(1)$ necessary equals the slope of the line segment passing through $x = 1$. Reading the slope off of the graph, $f'(1) = -\frac{5}{2}$. Alternatively, a student with a strong preference for algebraic expressions may choose to write an algebraic definition for $f(x) = \begin{cases} \frac{7}{3}x + 5, -3 \le x \le 0 \\ -\frac{5}{2}x + 5, 0 < x \le 2 \end{cases}$ and then find an algebraic definition for

$f'(x)$ by differentiating the piecewise function, $f'(x) = \begin{cases} \frac{7}{3}, -3 < x < 0 \\ -\frac{5}{2}, 0 < x < 2 \end{cases}$ and conclude that $f'(1) = -\frac{5}{2}$. Thus, a student would conclude that $\frac{d}{dx}\left[\sqrt{x - f(x)}\right]|_{x=1} = \frac{1}{2\sqrt{1-\frac{5}{2}}}\left(1 + \frac{5}{2}\right)$.

During the course of the study, the students were exposed to 183 mathematical problems embedded within the class notes during instruction and 470 homework problems. Of the 183 in-class problems, 121 (66%) problems required translation from the input representation to the output representation. Of the 470 homework problems, 332 items (70.6%) required translation from the input representation to the output representation. Table 5 summarizes the distribution of items by input and output representations. Table 6 provides examples of items that require various translations. In table 5, the input column reports if an item contains the representation within the question. For example, a question that utilizes a functional description that includes a symbolic expression, graphical representation, and verbal description would be classified GSV and will be included in all three categories G, S, and V. Similarly, an output that requires both a verbal description and a graphical representation would be classified VG and be counted in both the verbal and graphical output categories.

Table 5. Distribution of the items used in the study, classified by functional representation

Input	#	% of Total	Output	#	% of Total
G	150	23.0	G	62	9.5
N	124	19.0	N	347	53.1
S	395	60.5	S	171	26.2
V	250	38.3	V	152	23.3

Table 6. Examples of problems that require translations between functional representations

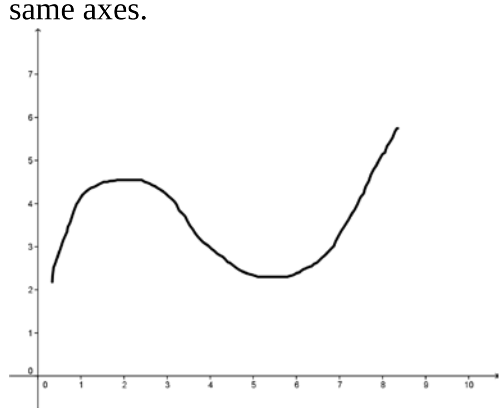
Representational Translation	Example Problem
S-S	Let $f(x) = 3x^2 + 2x$. Find $f'(x)$.
S-N	Let $f(x) = 3x^2 + 2x$. Find $f'(2)$.
S-G	Let $f(x) = 3x^2 + 2x$. Sketch a graph of $f(x)$ and $f'(x)$ on the same axes.
S-V	Let $f(x) = 3x^2 + 2x$. Explain the meaning of $f'(2)$.
N-N	Suppose two functions f and g are known to have the following values: $f(1) = 3$, $f'(1) = 4$, $g(1) = 2, g'(1) = -2$. Find $h'(x)$ if $h(x) = f(x)g(x)$.
N-S	Suppose $f(2) = 5$ and $f'(2) = 1$. Write the equation of the line tangent to f at $x = 2$.
N-G	Sketch a curve that satisfies the following requirements: $\lim_{x\to-\infty} f(x) = 1$, $\lim_{x\to\infty} f(x) = 3, f(0) = 1. f'(0) = f'(2) = f'(4) = 0$.
N-V	True or False: If $\lim_{x\to2}\frac{f(x)-f(2)}{x-2} = 5$, then $\lim_{h\to0}\frac{f(2+h)-f(2)}{h} = 5$. Justify your response.
G-G	Given the graph of f sketched below, sketch a possible graph of $f'(x)$ on the same axes. 

Table 6 (continued)

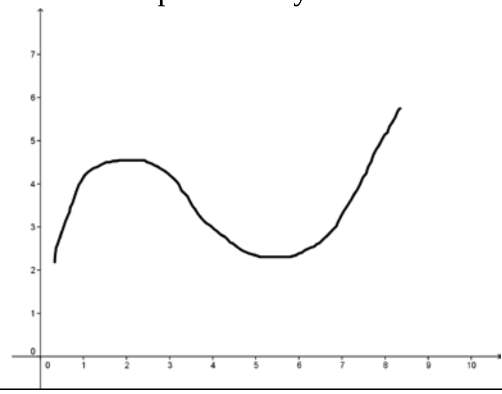
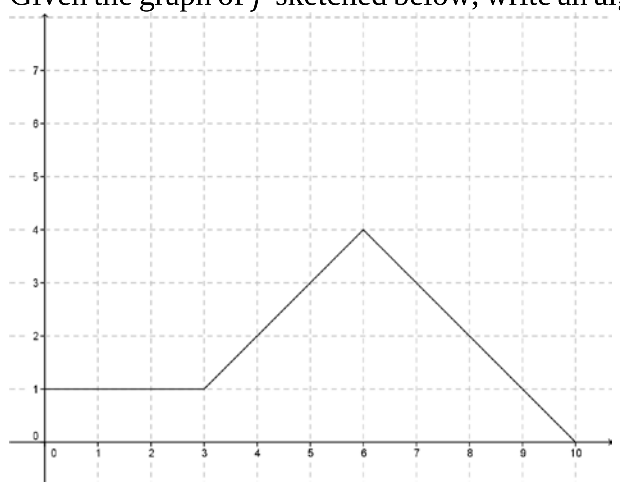
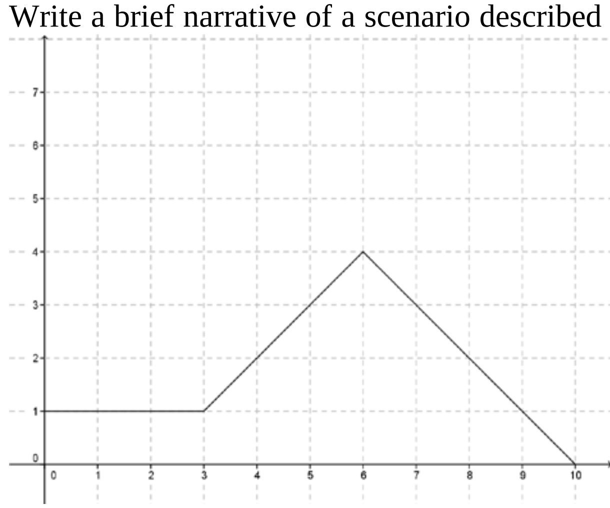
G-N	Given the graph of f sketched below, estimate $f'(2), f'(4)$, and $f'(5.5)$. Show or explain how you determined each estimation. 
G-S	Given the graph of f sketched below, write an algebraic expression for $f'(x)$. 
G-V	Write a brief narrative of a scenario described by the graph of $f'(x)$ below. 
V-V	True or false: One particular kind of secant line is a tangent line to a curve. Justify your response.

Table 6 (continued)

V-G	Suppose a particle's position function is known to be the Heaviside function. Sketch a graph of the particle's velocity function.
V-N	Suppose a particle's position function is known to be the Heaviside function. Evaluate the following: $s(1), v(1),$ and $a(1)$.
V-S	Suppose a particle's position function is known to be the Heaviside function. Write the equation of the particle's velocity function.

It should not be surprising that a high percentage of items involved the symbolic representation. The relatively high number of items that involve the symbolic representation was largely due to the fact that Chapter 2: The Derivative and Chapter 3: Topics in Differentiation each contained sections dedicated almost entirely to learning to use differentiation rules and procedures applied to algebraic expressions. Also, the graphical representation is used in relatively few outputs, however, this possible inequity is largely due to the fact that questions that require a graphical output are typically longer questions than those that require a symbolic or numerical output.

It is important to note that although the title Chapter 4: The Derivative in Graphing and Application may indicate that graphing is only included in chapter 4, this is not the case. The chapter is titled Chapter 4: The Derivative in Graphing and Application because the chapter emphasizes how differential calculus can be used to analyze functions presented symbolically and numerically to create accurate graphs of the functions. Also, the application part of the chapter focuses on the coupling of the symbolic and verbal, graphical and verbal, and numerical and verbal representations to describe real-world phenomena that can be analyzed with differential calculus. There are 293 mathematical tasks in chapter 4. Of those 293 items, 268 items required translation from the input to the output representation. Table 7 reports the distribution of items by representation in chapter 4. In table 7, the input column reports if an item contains the representation within the question. For example, a question that utilizes a functional description that includes a symbolic expression, graphical representation, and verbal description would be classified GSV and will be included in all three categories G, S, and V. Similarly, an output that requires both a verbal description and a graphical representation would be classified VG and be counted in both the verbal and graphical output categories.

Table 7. Distribution of the items used in chapter 4, classified by functional representation

Input	#	% of Total	Output	#	% of Total
G	102	34.8	G	39	13.3
N	57	19.5	N	191	65.2
S	117	39.9	S	11	3.8
V	175	59.7	V	125	42.7

Taken together, tables 5 and 7 indicate that while there is a relatively higher percentage of graphical representations in chapter 4 than overall (34.8% versus 23.0%), the graphical representation is present throughout chapters 2 and 3 as well. There are 48 graphical input tasks in chapters 2 and 3, indicating that there are on average four graphical input tasks per section in chapters 2 and 3. Table 7 also shows that the numeric, symbolic, and verbal representations are abundantly present throughout chapter 4 and are in no way neglected during Chapter 4: The Derivative in Graphing and Application.

The course also attended to studying the derivative through the four representations: algebraic, numeric, graphical, and verbal as previously described. Table 8 reports the number of items that require each output representation of the derivative.

Table 8. Distribution of the items used in the study, classified by derivative representation

Derivative Representation	#	%
G	275	42.1
N	88	13.5
S	235	36.0
V	157	24.0

There is a significant distinction between the categories of representations of function and the categories of representations of derivatives. While the multiple representations of both functions and derivatives have the same names: symbolic, graphical, numerical, and verbal, the names apply differently to functions and derivatives. For example, the solution to a problem that requires writing the equation of a line tangent to a curve $f(x)$ at $x = c$ is an example of the symbolic representation of a function; however, writing the equation of the tangent line is an example of the graphical representation of the derivative if the problem requires finding the slope of the tangent line at $x = c$. Table 9 presents examples of the kinds of questions that elicit responses in each derivative representation.

Table 9. Examples of the multiple representations of the derivative

Solution Representation	Problem	Rationale
Graphical	Find the equation of the line tangent to $f(x) = 3x^2 - 2x$ at $x = 2$.	A correct solution requires finding the slope of a line tangent to a curve at $x = c$.
Numerical	The position of a falling body t seconds after being dropped from 100 ft. is given by $s(t) = -16t^2 + 100$. Find the rate of change of the position of the body at $t = 2$.	A correct solution requires finding a rate of change of a function at $x = c$.
Symbolic	Find $\frac{dy}{dx}$ if $y = 3x^2 - 2x$.	A correct solution requires finding a symbolic expression for the derivative of a function.
Verbal	Suppose that $M(x)$ is the function that describes the miles per gallon of gas that a truck gets when traveling at x miles per hour. Using correct units, explain the meaning of $M'(50) = 1.3$.	The correct solution requires describing the contextual interpretation of the derivative of a function.

3.2.1 Description of a typical instructional period

The course instructor primarily used a task-based lecture approach to instructing the class. A sample lesson plan used by the instructor is included as an appendix (Appendix B). A typical class began with a 5-10 minute opening activity, which was collected. Students then volunteered to describe how they attempted to solve the opening activity. Various solutions and solution methods were described by students. If an acceptable solution had not been presented after 3-4 minutes, the instructor provided limited guidance to help students consider alternative solution methods. Discussion continued until a correct solution method is found. Described below is a vignette that serves to show how the opening activity is discussed.

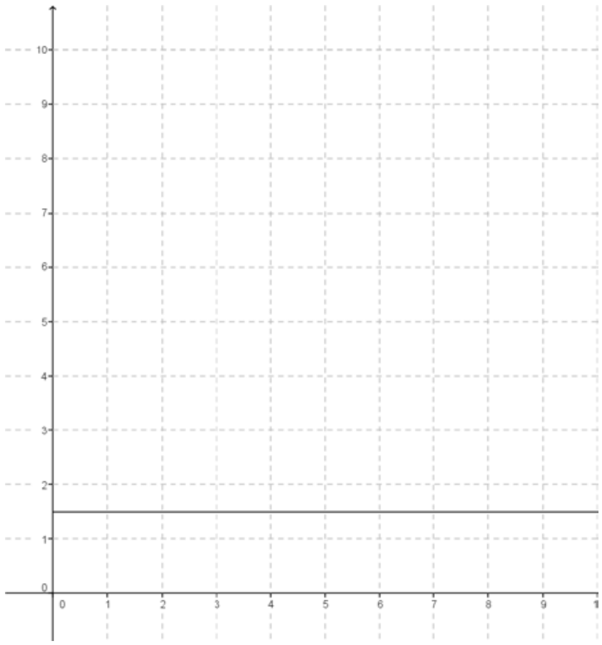
3.2.1.1 Opening activity (5-10 minutes)

The activity begins by students attempting to solve the following problem.

Opening activity: The following velocity versus time graph consists of a horizontal line segment from $t = 0\ s$ to $t = 10\ s$. Construct a position versus time graph on the same axes. Show or explain how you determined the position graph.

Are there any other possible answers? Explain why or why not.

Velocity (ft/s)



$v(t)$

time (s)

Figure 9. Opening activity example

The students worked on the opening activity for about four or five minutes. During this time, the teacher took attendance and walked around the room observing the students' attempted solution methods. The teacher wrote down the names of two or three students to share their solutions. After four or five minutes, the activities were collected. Ben, Jennifer, and Sasha were selected to share their responses.

Ben shared first. He drew the following line segment on the Promethean Board.

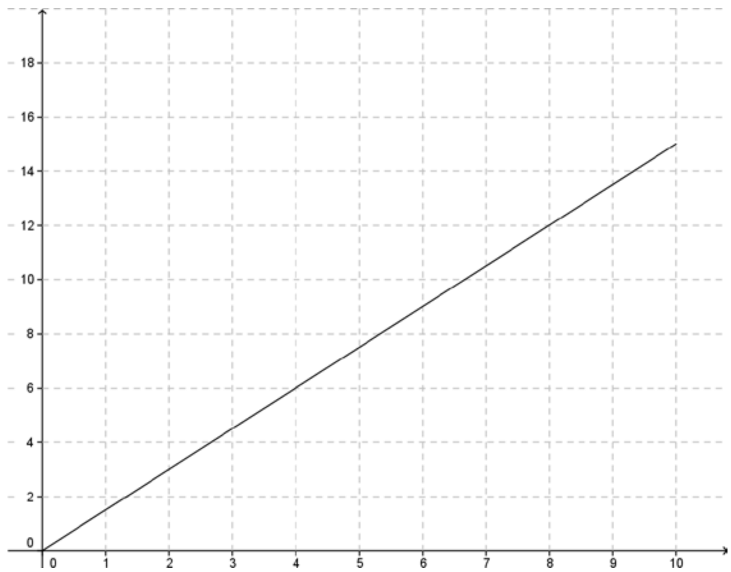


Figure 10. Ben's opening activity

Ben said, "I know that distance equals rate times time, so I just multiplied 1.5 by 10 and realized that the ending position would be at $(10,15)$, so I drew a line from the starting point to the point $(10,15)$."

The instructor asked, "are there any other possibilities for the position graph?"

Ben answered, "the fact that you asked the question, makes me think that there is, but I don't know how there would be one."

Jennifer shared next. She drew the following graph on the Promethean Board:

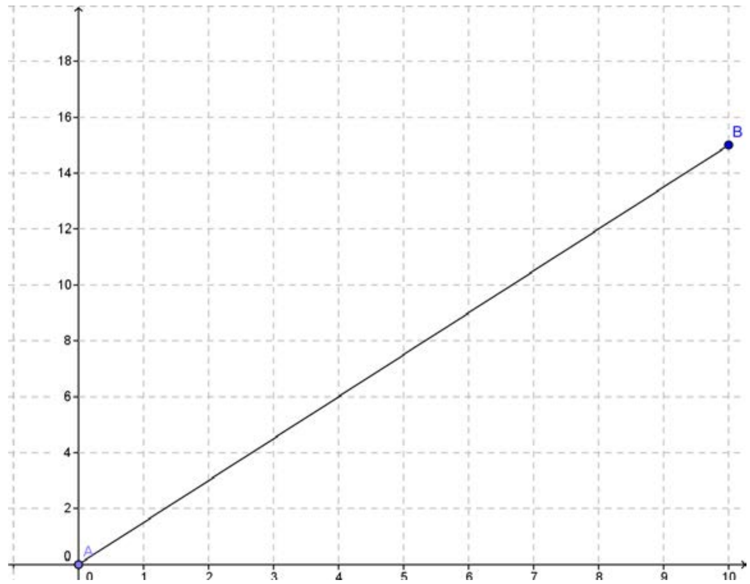


Figure 11. Jennifer's opening activity

Jennifer said "I started by making a table of positions. I first thought about when I drive a car, if I drive for one hour at 50 mph, then I have driven 50 miles, so if something is moving at 1.5 feet per second, then every second its position increases by 1.5 feet. If we call the starting position (0,0), then my table looked like this:

Table 10. Jennifer's table of values

t	0	1	2	3	4
$s(t)$	0	1.5	3	4.5	5.0

At this point, I could see that the pattern was linear. So, when $t = 10$, the position is 15. So I drew a linear graph with a slope of 1.5 and a y-intercept of 0.

The instructor asked Jennifer, "Do you think that there are any other possible position graphs?"

Jennifer replied, "I don't think so, the position has to be linear and run from 0 to 15."

Sasha shared his response last. He began by saying that "since we learned yesterday that velocity is the rate of change of position, I tried to think of what would have a rate of change of

1.5. Then I realized that the answer would be a line with a slope of 1.5. And, I would say that there would be infinitely many position functions because all of the parallel lines with a slope of 1.5 would produce a velocity function of $v(t) = 1.5$. So, here is my graph."

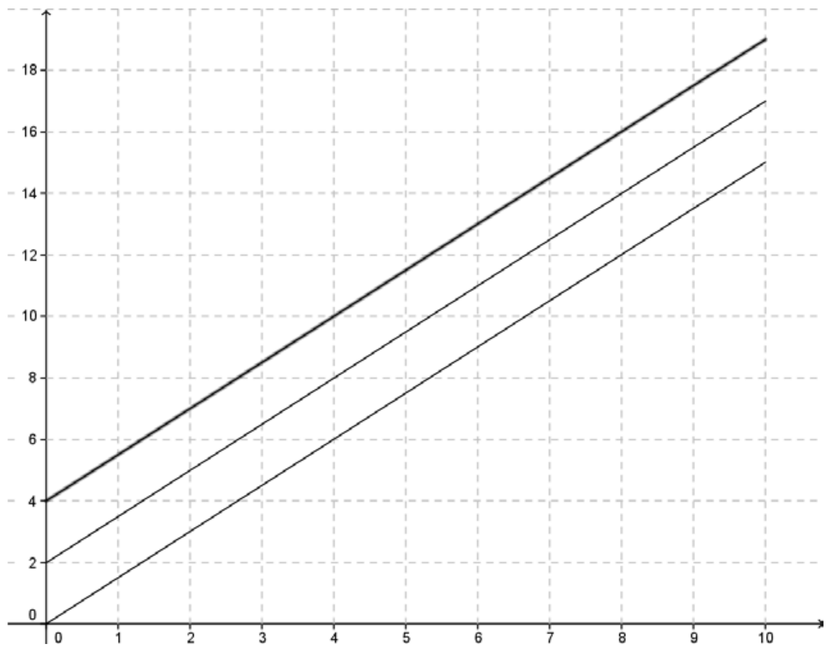


Figure 12. Sasha's opening activity

The instructor concluded the discussion by making the following observation, "so we see three different approaches to reaching the same conclusion: Ben solved the problem by using a physics formula; Jennifer solved the problem by making a table of values and then noticing a pattern in the table, and Sasha solved the problem by creating an algebraic expression by thinking in reverse. Are there any questions?"

Finally, the instructor asked the class "so, do we think that there is one or more possible position functions?"

The class answered, "More because any line with a slope of 1.5 works."

This vignette showed how many mathematical tasks were presented and discussed in the classroom. The vignette also showed how representations were intentionally linked in the mathematical activities. In this case, the function was originally presented in a G-V, graphical and verbal, representation. Since the function was presented as a graph, it was coded as a graphical representation and because the descriptive functional word "velocity" is included in the description of the function and is integral to understanding the problem, the functional representation of the input is also coded as verbal. The final output representation is a graph. However, the students' solution methods indicate that there are multiple translation methods to move from a graph of velocity to a graph of position. Ben translated the graph of velocity into a verbal equation that he knew from his physics class. He then used the equation to create numerical values for the position function and then translated these numerical values into a graphical representation. Jennifer used personal experiences to translate the graphical representation of the velocity graph into a tabular representation of the position function. From there, she translated the table into a graphical representation. Sasha appears to translate straight from the graph of velocity to the graph of position by using reversibility.

3.2.1.2 Discussion of homework (5-10 minutes)

The following 5-10 minutes were spent discussing questions that students asked about the previous night's homework. The length of time spent discussing homework varied and was dependent on the number of questions that the students ask. 5-10 minutes was an average. Occasionally, the remaining 35-40 minutes of class were spent discussing homework. After each question in the homework, students were asked to explain how they answered the question and to defend why they believed that they were correct. The following scenario demonstrates a typical discussion of a homework problem. The problem discussed comes from section 1.5 and follows instruction on continuity.

HW Example: Find values of the constants a and b, if possible, that will make the function f continuous everywhere.

$$f(x) = \begin{cases} x^2 + 1, & x > 1 \\ a(x-1) - b, & -2 \leq x \leq 1 \\ x^2 - x - 2, & x < -2 \end{cases}$$

The instructor began by asking "what does it mean to be continuous everywhere?" Several students responded. The responses included "it means you can draw the graph without lifting your pencil … the graph has no holes or breaks in it … there are no asymptotes." The instructor responded, "all of those answers describe the graph of a continuous function, what does it mean from an algebraic perspective to be continuous everywhere?"

Damian answered "that would mean that the domain of the function is $(-\infty, \infty)$."

The instructor asked, "is that all that is necessary?"

The students mumbled to one another and Morgan said "we have that theorem that says that polynomials and rational functions are continuous on their domains, so if the domain is $(-\infty, \infty)$, then the function must be continuous everywhere."

The instructor replied "I agree that the domain would have to be $(-\infty, \infty)$ in order to be continuous everywhere, but I am not convinced that a function whose domain is the entire real line is necessarily continuous. Can anyone name a function whose domain is $(-\infty, \infty)$ but is not everywhere continuous?"

Robert said, "What about the greatest integer function? Its domain is the entire real line, but it has jump discontinuities at every integer."

The instructor said to the class "Okay, so the greatest integer function is a counter-example to the claim that the domain $(-\infty, \infty)$ is a sufficient requirement to be continuous everywhere. So what else do we need?"

Jamie said "we need the limit to exist everywhere and the value of the function to equal the limit – I got that from looking at our definition of continuity at a point."

The instructor replied, "Alright, Jamie is correct. Are we then going to check the limit and functional value of every point from negative infinity until positive infinity?"

The class correctly decided that checking every point is impossible, at which point Morgan said, "but since we know that polynomials are continuous on their domains and since each part of the piecewise function is a polynomial, don't we know that the function is continuous everywhere except where the piecewise function changes definitions?" The class nodded in agreement.

The instructor asked, "So how can we write that correctly? Take a minute and write a mathematical sentence describing what Morgan just told us." After one minute, Jamele volunteered the following answer, "Since $f(x)$ consists of polynomials on $(-\infty,-2) \cup (-2,1) \cup (1,\infty)$, $f(x)$ is continuous on $(-\infty,-2) \cup (-2,1) \cup (1,\infty)$."

The class and instructor agreed that Jamele's description was appropriate. The instructor then posed the question, "Okay, so what about $x = -2$ and $x = 1$?" Daniel answered saying, "well now we can just use the definition of continuity at a point to setup a system of two equations." The instructor then directed the students to take a few minutes to attempt to use the definition of continuity at a point to setup and then solve a system of two equations for the variables a and b. After four minutes, a student (Daryle) demonstrated the correct solution on the Promethean board and explained his work. The instructor answered any lingering questions before moving on to the next phase of the class.

3.2.1.3 Discussion of new material (20-30 minutes) and the exit slip (5-7 minutes)

The next 20-30 minutes were spent in the new learning phase of the class. During this time, the students and teacher filled in the guided notes together. The students solved each example either individually or in groups of 2 or 3. The teacher strategically selected students to share their solutions and then various solution methods were discussed. The discussion of the in-class examples followed the patterns described in the previous discussions of the opening activity and the homework problem. After working individually or in small groups of their own choosing, the students proposed possible solution methods. If there was broad agreement on a solution method, then the final answers were compared for agreement or disagreement. If multiple solution methods were proposed, then the merits of each method were compared by the students with the instructor interjecting as appropriate. After discussing the multiple solution methods, final answers were compared for agreement or disagreement.

In 100% of the in-class examples, the students solved the problems to whatever extent they could before the instructor offered guidance. In the event that a particular student could not begin a problem, the instructor offered limited prompts to encourage the student to try to solve the problem. Often, the instructor prompted students by asking students to tell him “what the question is asking” or by asking students “does the problem look like anything we have done previously?” The final 5 minutes of class were spent distributing, administering, and collecting an exit slip activity. During the administration of the exit slip, the instructor walked around the room ensuring that each student worked individually on the exit slip

3.3 DATA SOURCES AND INSTRUMENTS

To document the development of flexibility and reversibility during the instructional period, this study used two kinds of data: 1) class level data, and 2) individual data. The class level data was collected through three data sources: 1) a flexibility pre-test, 2) exit slips and opening activities, and 3) a flexibility post-test. The individual data was collected through think-aloud interviews. Below, I give an overview of the data sources that were used in the study and how they address the research questions. The collection of each type of data and the instruments used to collect the data are described individually in the sections that follow.

3.3.1 Data sources

The class level data was collected from all students enrolled in the course during the length of the study ($n = 21$). Two kinds of class level data were collected: 1) pre-test and post-test data designed to elicit evidence of the development of flexibility, and 2) daily class activities consisting of exit slips and opening activities designed to elicit evidence of reversibility. The flexibility pre-test (FPT) is a teacher/researcher created exam that attempted to quantify the extent to which the AP Calculus AB students demonstrated flexibility with functions after reviewing pre-calculus content at the start of the course. Additionally, the FPT offered a base-line data of the extent to which the students exhibit representational reversibility. The post-test is the Differentiation Competency Test (DCT), which was created and validated by Kendal and Stacey (2003). The DCT was administered at the end of the course and was intended to quantify the extent to which students have developed an ability to demonstrate flexibility of multiple representations of derivatives and to offer a data point indicating the extent to which the students exhibit representational reversibility at the end of the study. The results of the FPT and the DCT were analyzed and used to address research question 1 and research sub-question 2.iii.

The exit slip and opening activity data were collected 33 times during the study. The 33 exit slips and 33 opening activities form respective forward-reverse pairs (consistent with the terminology used by Krutetskii (1976), I will use the words "direct" and "forward" interchangeably in reference to reversible assessment items), in which the exit slip assessed the new material learned during the class period in a forward direction and the opening activity, administered at the start of the next class, assessed the same material in a reverse direction. Fourteen of the exit slip and opening activity pairs required reversibility of a two-way process. Ten of the exit slip and opening activity pairs required reversibility of the mental process in reasoning without reversible translation. Twenty-one of the exit slip and opening activity pairs required representational reversibility. It should be noted that the total number of exit slip and opening activity pairs (33) does not equal the sum of the number of each pair in each of the three categories $(14 + 10 + 21 = 45)$. This is because many of the exit slip and opening activity pairs contained elements that assessed more than one type of reversibility. Thus, the exit slip and opening activity pair is counted in two or three categories. The data from the exit slip and opening activity pairs was used to gauge the development of reversibility throughout the course of study and thus inform research question 2.

The individual data was collected through think-aloud interviews from six students, purposefully chosen to provide variability in demonstrated levels of flexibility. The purpose of the interviews was to provide insight into the thought processes that students use to solve problems that require reversibility, that is, to address research question 3. To increase the expected power and generalizability of the conclusions drawn from the interview data, I chose a multiple-case study design consisting of two cases of three different groups. The three groups were differentiated by demonstrated levels of flexibility with functions. Based on the results of the flexibility pre-test, the entire class of students was divided into thirds. Two students from the highest scoring third, two students from the middle third, and two students from the lowest scoring third were selected to participate in the think-aloud interviews. Three groups were chosen to increase variability in the results of the interviews.

Table 11 indicates the data sources that were used to answer each research question.

Table 11. Data sources used to answer research questions

	Flexibility pre-test – whole class data	DCT – whole class data	Exit slips and opening activities – whole class data	Question-based think-aloud interviews – 2 cases of 3 different groups
To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?	Yes	Yes	No	No
To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations?	Yes (representational reversibility only)	Yes (representational reversibility only)	Yes	Yes
What are the thought processes that students utilize when using reversibility to solve problems?	No	No	Yes	Yes

3.3.2 Flexibility pre-test

A flexibility pre-test was administered to all of the students at the conclusion of the pre-calculus chapter, which largely focused on reviewing many topics related to functions. During the pre-calculus review, the following pre-calculus topics were discussed in class: 1) families of functions including linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions, 2) properties of functions, 3) transformations of functions, 4) building new functions from old functions through algebraic operations and through composition, 4) the graphs of all of the families of functions, 5) the effects of transformations on the graphs of parent functions, 6) the language of functions including terms such as domain and range, roots, intercepts, etc., and 7) four different representations of functions, the verbal, numerical, symbolic, and graphical representations.

3.3.2.1 Purpose and description

To assess the students' flexibility with functional representations, I designed a flexibility pre-test (FPT) that is included as an appendix (Appendix D). The FPT is designed to assess the extent to which students can translate between the symbolic, graphical, and numerical representations of functions within two mathematical domains, composition of functions and inverses of functions. The pre-test does not include the verbal representation because the pre-test is also designed to fully align with the translations assessed by the DCT. The DCT does not include the verbal representation so the FPT was designed without the use of the verbal representation as well.

The flexibility pre-test has eighteen items (appendix D). Each item is designed to test one of eighteen different competencies of flexibility with functions. Each competency is represented by a three character code consisting of two upper-case letters and one lower case letter, such as CNg or ISs. Appendix D contains a thorough discussion of the coding of the flexibility pre-test and each item on the flexibility pre-test is coded in Appendix D. Here, I present a brief description of how to read and interpret the coding of the flexibility pre-test.

Each item in the pre-test assessed one competency of flexibility with functions. The first nine competencies relate to flexibility with compositions of functions and the second nine competencies relate to flexibility with inverse functions. The first character in the three character code indicates the content domain of the item: the letter *C* refers to a composition item and the letter *I* refers to an inverse item. The second letter in the three character code indicates the input representation, *N* for numerical, *G* for graphical, and *S* for symbolic. The third letter in the three character code indicates the output representation, *n* for numerical, *g* for graphical, and *s* for symbolic. Thus, for the FPT, an item coded *ISg* would indicate that the item is an inverse item in which the representational input of the function is symbolic and the representational output is graphical. An item coded *CNn* would be a composition item whose input representation is a numerical representation and whose output representation is a numerical representation. Table 12 reports the coding of the flexibility pre-test items.

Table 12. Coding of the flexibility pre-test items

Process	Input Representation	Competency
Composition: without-translation	Numerical	CNn
	Graphical	CGg
	Symbolic	CSs
Composition: with-translation between two representations	Numerical	CNg
		CNs
	Graphical	CGn
		CGs
	Symbolic	CSn
		CSg
Inverse: without-translation	Numerical	INn
	Graphical	IGg
	Symbolic	ISs
Inverse: with-translation between two representations	Numerical	INg
		INs
	Graphical	IGn
		IGs
	Symbolic	ISn
		ISg

The FPT provided base-line data of the students' existing levels of flexibility before the instructional portion of the study began. The results of the FPT were compared with the results of the flexibility post-test (the DCT) to quantify the extent that flexibility developed during the study. By designing the translations on the FPT to fully align with the translations on the DCT, I was able to compare the development of each kind of translation between numerical, graphical, and symbolic representations. I was also able to compare the extent to which representational preference existed at the beginning of the study and the extent to which representational preference existed at the end of the study.

The FPT also provided base-line data of the extent to which the students exhibited representational reversibility at the start of the study. The pre-test offered insight into existing levels of representational reversibility by providing scores indicating what percent of the students could successfully translate from a given input representation to an output representation and then reverse the order of translation. For example, the pre-test assessed the students' proficiency with translating from symbolic to numerical and from numerical to symbolic. If the class shows high levels of success in both directions, then we have evidence that the students may have representational reversibility between the symbolic and numerical representations.

The two mathematical domains of composition of functions and inverses of functions were chosen because of the importance of both domains in calculus. Understanding composition of functions is foundational to understanding and correctly using the chain rule for differentiation (Clark et al., 1997) and for understanding the chain rule's reverse construction, integration by substitution. Both the chain rule and integration by substitution are specifically noted as topics to be learned in AP Calculus (Collegeboard, 2010b) and Sofronas et al. (2011) identified the chain rule and techniques of integration (including substitution) as end-goals of understanding the first-year of calculus. Inverses of functions are particularly important in calculus for their use in deriving and analyzing inverse trigonometric functions and for investigating the nature of the relationship between exponential and logarithmic functions.

Each of the content domains contained nine questions, six questions required translations (between representation problems) and three questions did not require translations (within representation problems): symbolic to symbolic, symbolic to numerical, symbolic to graphical, graphical to graphical, graphical to symbolic, graphical to numerical, numerical to numerical, numerical to graphical, and numerical to symbolic. Figure 13 represents how the representations and translations were assessed by the flexibility pre-test.

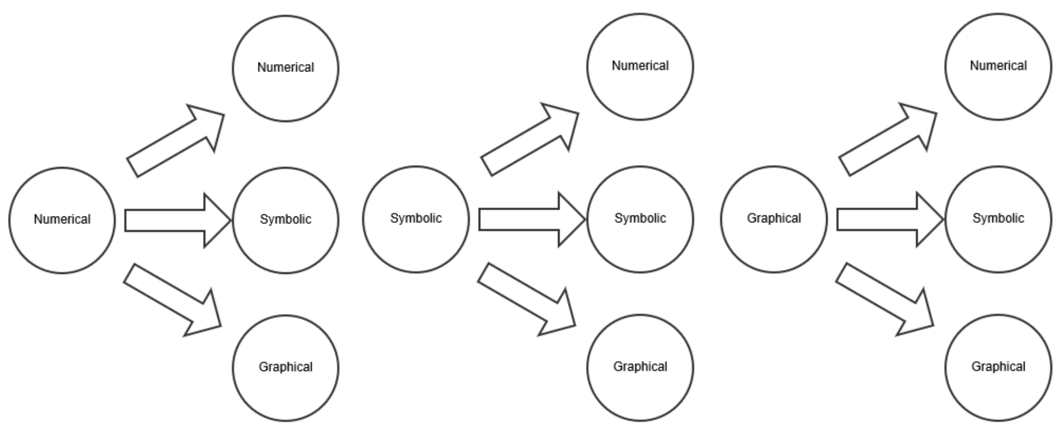


Figure 13. Model of the 9 questions for each mathematical content on the flexibility pre-test

To further explicate the design of the flexibility pre-test, I have included below, as figure 14, the first question on the flexibility pre-test.

1. Let $f(x) = |x|$ and $g(x) = \sqrt[3]{x} + 1$

   a. Find the algebraic expression for $f(g(x))$.

   b. Evaluate $g(f(-1))$.

   c. Sketch the graph of $f(g(x))$ on the axis below.

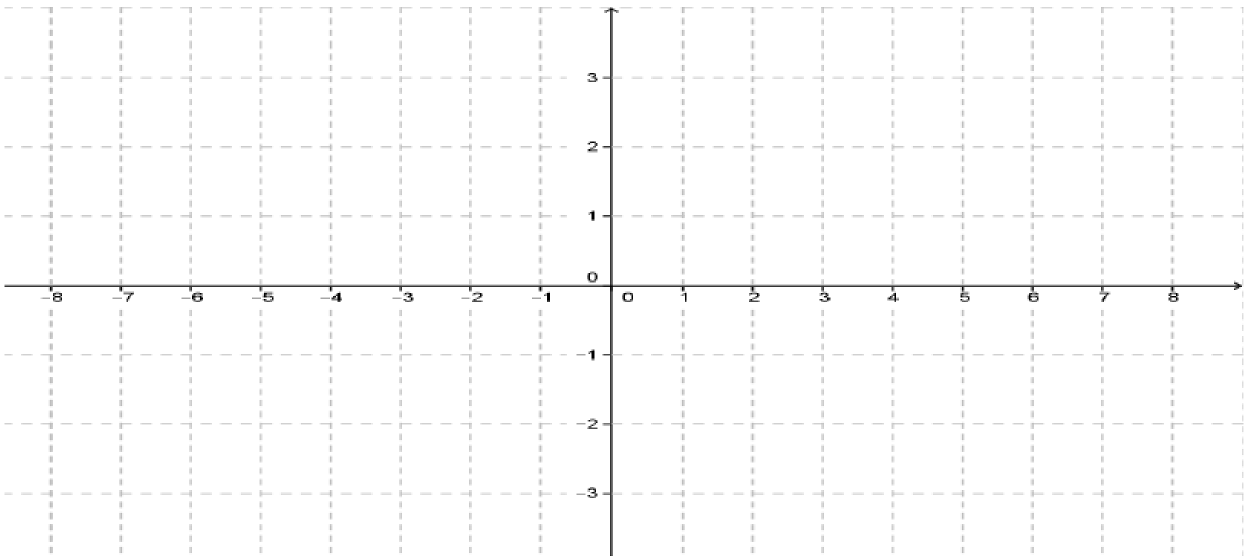


Figure 14. Question 1 on the flexibility pre-test

Part (a) required the students to compose two algebraic expressions to create a new function in its symbolic form. Thus, part (a) assessed composition of functions whose input representation is the symbolic representation and whose output representation is also the symbolic representation. Thus, this item did not require translation.

Part (b) required the students to evaluate a composition of two functions at a specific x-value. Here, the input is the symbolic representation and the output is the numerical representation. Translation from the symbolic to the numerical representation was required.

Part (c) required students to sketch the graph of a composite function. Thus, students must translate a symbolic representation into a graphical representation. Questions 2 and 3 followed a similar pattern of assessing flexibility with compositions of functions. Question 1 presented the functions symbolically, question 2 presented the functions numerically, and question 3 presented the functions graphically. The format was repeated for the nine questions about inverses.

3.3.2.2 Design and validation of the flexibility pre-test

To design the items in the flexibility pre-test, I consulted multiple rule of four based calculus (Anton et al., 2009; Finney, Demana, Waits, & Kennedy, 2011; Hughes-Hallett et al., 1995, 1998; Hughes-Hallett et al., 2005) and pre-calculus (Demana, Waits, Foley, & Kennedy, 2004; Larson & Hostetler, 1997) textbooks to familiarize myself with the kinds of items that textbook authors use to present composition of functions and inverses of functions from a rule of four perspective. I then designed items intended to assess students' flexibility with multiple representations of composite functions and inverses. The item design phase was an iterative process of 1) designing the item, 2) solving the item, 3) evaluating the extent to which the item tested only one construct (either composition or inversion) and required only one kind of translation (for example, graphical to numerical), 4) modifying the item to address any deficiencies identified in step 3, and then 5) repeating the process.

To assess the validity of the pre-test items, a second mathematics educator who has taught pre-calculus for four years and high-school mathematics for nine years reviewed the items for face and content validity. His review of the items concluded that the items did in fact assess a student's flexibility with multiple representations of composite functions and inverses of functions.

To assess the internal reliability of the pre-test, I calculated Cronbach's alpha. Cronbach's alpha is noted as the most common measure of the internal reliability of multiple items that assess the same construct (Trochim, 2002). George and Mallery (2003) recommend the following guidelines when interpreting Cronbach's alpha:

Table 13. Levels of Cronbach's alpha

Alpha level	Interpretation of the assessment measure's internal reliability (or consistency)
$1 \geq \alpha \geq 0.9$	Excellent
$0.9 > \alpha \geq 0.8$	Good
$0.8 > \alpha \geq 0.7$	Acceptable
$0.7 > \alpha \geq 0.6$	Questionable
$0.6 > \alpha \geq 0.5$	Poor
$0.5 > \alpha$	Unacceptable

It is also recommended that when an assessment measure contains more than one construct, Cronbach's alpha should be calculated for each construct separately (Tavakol & Dennick, 2011). I calculated Cronbach's alpha on the entire pre-test to assess the internal reliability of the pre-test as an assessment of flexibility. Table 14 reports the alpha level of the flexibility pre-test. The alpha level of 0.929 indicates excellent internal reliability of the pre-test as a measure of flexibility.

Table 14. Flexibility – Cronbach's alpha

Cronbach's Alpha	Number of Items
.929	18

Table 15 reports Cronbach's alpha for flexibility of compositions of functions. The alpha level of 0.847 indicates a good level of internal reliability of the pre-test as a measure of flexibility of compositions of functions.

Table 15. Flexibility of compositions of functions – Cronbach's alpha

Cronbach's Alpha	Number of Items
.847	9

Table 16 reports Cronbach's alpha for flexibility of inverses of functions. The alpha level of 0.892 indicates a good level of internal reliability of the pre-test as a measure of flexibility of inverses of functions.

Table 16. Flexibility of inverses of functions – Cronbach's alpha

Cronbach's Alpha	Number of Items
.892	9

As the minimum acceptable alpha level of 0.70 is necessary to retain an assessment instrument, I considered the flexibility pre-test to be a valid and reliable instrument by which to measure the extent of students' flexibility with functions after one chapter of instruction in the classroom.

3.3.2.3 Limitations of the instrument

A significant limitation of the flexibility pre-test is that it only tested each translational competency within each content domain once. It may be the case that a student may exhibit flexibility with multiple representations of functions in some other content domain such as symmetry, algebraic combinations of functions, transformations, etc. The use of two content domains, composition of functions and inverses of functions, allows for some amount of replication evidence. For example, a student who can translate from the symbolic to the graphical representation in both compositions of functions and inverses of functions may well be able to move flexibly from the symbolic to the graphical representation. Similarly, a student who is not able to translate from the symbolic to the graphical representation in either domain may lack fluency in translating from a symbolic to graphical representation.

3.3.3 Differentiation competency test (DCT)

The DCT is a measurement instrument created by Kendal and Stacey (2003) to gauge calculus students' ability to formulate or interpret derivatives when the given input is in one representation and the output requires translation to a different representation. Kendal and Stacey (2003) used the DCT to monitor students' ability to translate representations of the derivative across the symbolic, numerical, and graphical representations. It also measures the students' ability to move flexibly across multiple representations and tracks representational preference. The required differentiation skills necessary to solve the problems are considered elementary as the purpose of the exam is to test understanding of the different representations of differentiation and the ability to move flexibly between them (Kendal & Stacey, 2003). The DCT is attached as an appendix (Appendix C).

3.3.3.1 Purpose and description

The DCT was used in this study to help inform the first and second research questions. The DCT provided evidence of the extent to which students developed flexibility in a class that attended to linking multiple representations. The results of the FPT (pre-test) and DCT (post-test) provided evidence of the extent to which flexibility developed during the course of the study. The DCT also provided evidence that informed the development of representational reversibility.

The DCT has eighteen items (appendix C). Each item is designed to test one of eighteen different differentiation competencies. Each competency is represented by a three character code consisting of two upper-case letters and one lower case letter, such as INg or FSs. Appendix C contains a thorough discussion of the coding of the DCT and each item on the DCT is coded in

Appendix C. Here, I present a brief description of how to read and interpret the coding of the DCT.

Each item in the DCT is coded with a three character code. Each character aligns with one characteristic of the item. The first characteristic (formulation or interpretation) refers to the cognitive process required by the item to reach the intended output from the given input. Formulation refers to recognizing the need for a particular differentiation procedure and correctly executing the procedure. Interpretation "is the ability to reason about the input derivative supplied or to explain it in natural language, or to give it meaning including its equivalence to a derivative in a different representation." (Kendal & Stacey, 2003, p. 28). The second characteristic (an upper-case letter) indicates the input representation and the third characteristic (a lower-case letter) indicates the output representation. Thus, an item coded as ISg would be an interpretation item whose input (question stem) is given in a symbolic representation and whose output (expected answer) is in a graphical representation. Table 17 presents the coding of the DCT items.

Table 17. Coding of the DCT items

Process	Input Representation	Competency
Formulation: without-translation	Numerical	FNn
	Graphical	FGg
	Symbolic	FSs
Formulation: with-translation between two representations	Numerical	FNg
		FNs
	Graphical	FGn
		FGs
	Symbolic	FSn
		FSg
Interpretation: without-translation	Numerical	INn
	Graphical	IGg
	Symbolic	ISs

Table 17 (continued)

Interpretation: with-translation between two representations	Numerical	INg
		INs
	Graphical	IGn
		IGs
	Symbolic	ISn
		ISg

The DCT provided data at the class level and at the individual level. At the class level, the DCT scores were compared with the FPT scores using a paired-samples t-test to quantify the extent to which the students' flexibility with representations improved. Also, because the DCT assessed reversible pairs of representational translations (such as graphical to numerical and numerical to graphical), the DCT provided quantifiable evidence of the extent to which the students developed representational reversibility of derivatives.

At the individual level, the FPT and the DCT provided valuable information into whether students' difficulties answering the interview questions were due to translational difficulties or due to reversibility of the differentiation process. For example, a student who could not solve any of the reversibility questions in the task-based interviews, but scored highly on the FPT and the DCT was likely struggling with developing reversibility. Alternatively, a student who struggled to solve any reversibility questions correctly in the task-based interviews and who performed poorly on the FPT and DCT likely had very little understanding of representational translation and likely did not understand the forward direction of differentiation, rendering the possibility of reversibility moot. Thus, the FPT and the DCT offered explanatory potential when interpreting the results of the think-aloud interviews.

I administered the DCT to all students at the end of the 48-day instructional unit. It was expected that students exposed to an instructional curriculum emphasizing links between multiple representations of functions and derivatives would develop flexibility in moving between representations (Goerdt, 2007; Haciomeroglu, 2007). Researchers note that the amount of flexibility and conceptual understanding of differentiation developed is often less than the teacher and researchers desire (Goerdt, 2007; Habre & Abboud, 2006; Kendal & Stacey, 2003). Comparing the results of the FPT with the DCT offered insight into the extent that students developed flexibility of representations during the instructional unit. The results of the DCT indicated if students developed an ability to translate between multiple representations of derivatives.

Kendal and Stacey (2003) designed the DCT for administration in Australian high schools. As such, some of the language used in the original DCT is inconsistent with the language common to typical calculus classes in the United States. I made surface changes to the language of some of the DCT questions in order to equate the language used in the DCT to language used in calculus classes in the United States. For example, I translated the word "gradient" to "slope of the line tangent to the curve at a point" and the phrase "gradient function" was translated to "derivative function". Also, the phrase "Bush Walk" is used in the DCT to refer to an Australian family taking a recreational walk. The phrase "Bush Walk" was translated to "holiday walk".

3.3.3.2 Limitations of the instrument

Kendal and Stacey (2003) note that conclusions drawn from using the DCT should be done so with caution due to the fact that the DCT only tests each translational competency one time. However, the competencies were also tested on in-class assessments during the course of the semester and the researchers observe that the results on the class tests "confirmed the class achievements demonstrated on the DCT" (Kendal & Stacey, 2003).

3.3.4 Exit slips and opening activities

Research suggests that reversibility may develop on a day-by-day, task-by-task continuum and not necessarily as a one-time development (Krutetskii, 1976; Rachlin, 1981). For example, Krutetskii (1976) observed that the most capable mathematical students can develop reversibility immediately upon learning a concept; however, average and weak students need repeated examples and practice to develop reversible conceptions. Krutetskii's conclusion suggests that reversibility can develop along a continuum as students work with problems requiring reversibility. I used exit slips and opening activities to capture this type of development.

3.3.4.1 Purpose and description

I collected and analyzed exit slip and opening activity data in an effort to determine if reversibility was developing during the course of the study, that is, to address the second research question. Also, the exit slips and opening activities were purposefully designed to elicit evidence of the thought processes that students utilize when using reversibility to solve problems. Thus, the exit slip and opening activity data also contributed to informing research question 3.

Across the 48 days, I collected 33 exit slip and opening activity pairs. The 33 pairs assessed 45 specific instances of reversibility. Fourteen exit slip and opening activity pairs were administered that required the use of two-way reversible processes to solve. The data from these fourteen exit slips and opening activities informed research sub-question 2.1: does reversibility of two-way reversible processes develop? Ten direct and reverse exit slip and opening activity pairs were administered that required the use of reversibility of a mental process in reasoning without reversible translation. The data from this set of twelve exit slips and opening activities was used to address research sub-question 2.ii: does reversibility of the mental process in reasoning without reversible translation develop? Twenty-one direct and reverse exit slip and opening activity pairs were administered that require the use of reversibility of a mental process in reasoning with translation (representational reversibility). The data from this set of twenty-one exit slips and opening activities was used to address research sub-question 2.iii: does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?

Daily, except on test days, during the last five to seven minutes of class, the students attempted to solve problems on exit slips to measure direct learning of the content taught during the class. The following day, the students solved a reverse problem of the content taught during the previous class session as an opening activity. Upon collection of the opening activity, a correct solution was briefly discussed with the class.

The exit slips and opening activities are numbered to align with the class section in which they were administered. ES 2.1.1 indicates the exit slip administered after the first day of teaching section 2.1. OA 2.1.2 indicates the opening activity administered at the start of the class after the second day of teaching section 2.1.

The results of the exit slips and opening activities are numbered 1-45. As previously mentioned, there were 33 administered exit slips and opening activities. However, some exit slips and opening activities assessed multiple instances of reversibility within one exit slip and opening activity. Table 18 below reports the exit slips and opening activities by section of instruction, the numbered results of each pair, and the type of reversibility assessed. Exit slips are abbreviated ES; opening activities are abbreviated OA. Thus the exit slip administered after section 2.1.1 is labeled ES 2.1.1. When I refer to an exit slip and opening activity set, I call it a set of paired problems or abbreviate it ESOA.

Table 18. List of exit slips and opening activities

Exit Slip and Opening Activity Number	Results Number(s)	Type of Reversibility: 1 = 2-way, 2 = Reasoning, 3 = Representational
2.1.1	1	2
2.1.2	2	3
2.1.2	3	3
2.2.1	4	2
2.2.2	5	2
2.3.1	6	1
2.3.2	7	1
2.4.1	8	1
2.5.1	9	3
2.6.1	10	1
2.6.1	11	2
2.6.2	12	1
2.6.3	13	3
3.1.1	14	1
3.2.1	15	3
3.2.2	16	1
3.3.1	17	1
3.3.1	18	2
3.3.2	19	1
3.3.2	20	2
3.4.1	21	3
3.4.2	22	3
3.5.1	23	2
3.6.1	24	3
4.1.1	25	3
4.1.1	26	3
4.1.2	27	3
4.1.2	28	3
4.2.1	29	3
4.2.1	30	3
4.2.1	31	3
4.2.2	32	2
4.3.1	33	3
4.3.2	34	3
4.3.2	35	3
4.4.1	36	3
4.4.1	37	3
4.5.1	38	1
4.5.2	39	1
4.6.1	40	1

Table 18 (continued)

4.6.2	41	1
4.6.2	42	2
4.7.1	43	1
4.7.1	44	2
4.8.1	45	3

All of the exit slips and opening activities are included in Appendix E. The following are two examples of direct and reverse paired problems that appeared as exit slips and opening activities. The direct question was the exit slip and the reverse question was the subsequent day's opening activity:

Ex. 1: Direct exit slip (ES 2.1.1). Administered at end of Day 1 – Section 2.1. This paired problem set assesses reversibility of a mental process in reasoning.

The following position versus time graph consists of a linear position function, $s = f(t)$, construct a velocity versus time graph on the same set of axes. Show or explain how you determined the velocity graph.

Are there any other possible answers? Explain why or why not.

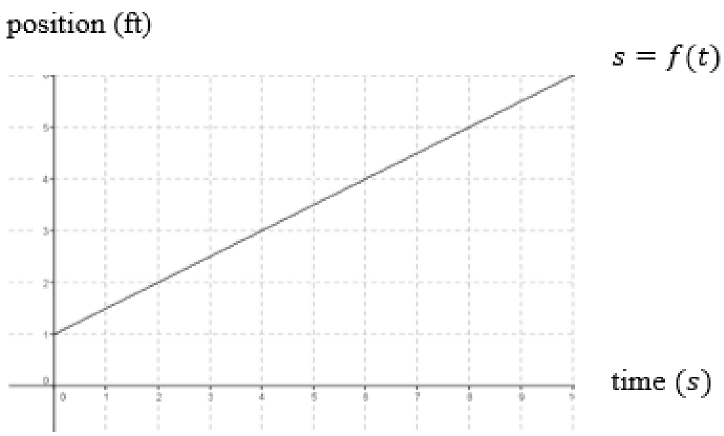


Figure 15. Exit slip 2.1.1

Ex. 1: Reverse opening activity (OA 2.1.1). Administered at the start of Day 2 – section 2.1.

The following velocity versus time graph consists of a horizontal line segment from $t = 0\ s$ to $t = 10\ s$. Construct a position versus time graph on the same axes. Show or explain how you determined the position graph.

Are there any other possible answers? Explain why or why not.

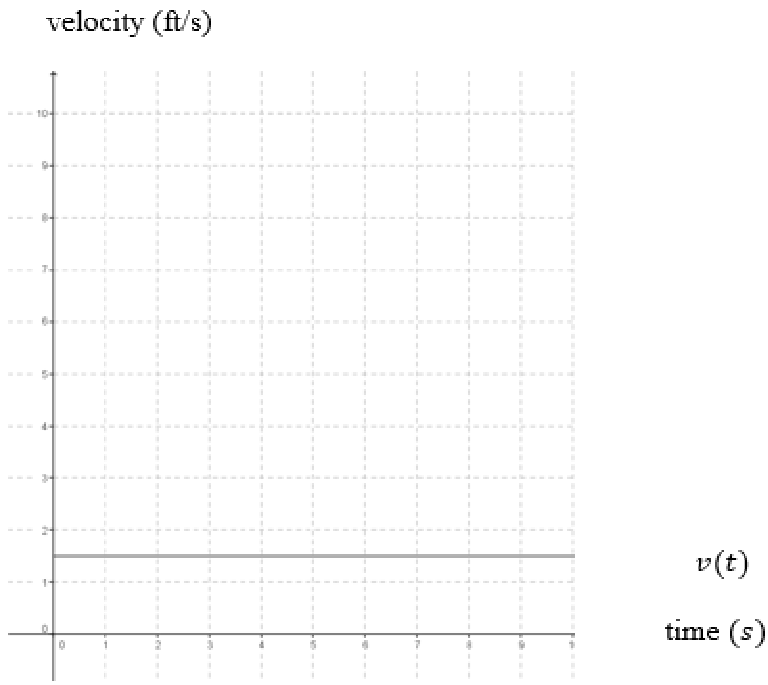


Figure 16. Opening activity 2.1.1

These two problems formed a direct reverse pair similar to what Krutetskii (1976) called "paired problems". The exit slip question is a direct application of learning the graphical relationship between position and velocity. Students should have learned in class that velocity is the rate of change of position; as such, to find velocity from a linear position graph, the students need to find the slope of the position graph and then understand that the slope is the velocity because the position graph is linear. The students were asked to show or explain how they determined the velocity graph in order to gather more information attending to the students' thought processes while solving the problem. During the administration of the exit slips and opening activities, I consistently encouraged the students to give a description of how they determined their answers.

At the start of the next class, the students attempted to solve the opening activity question, which is a reverse of the previous day's exit slip. In the reverse question, the students were presented a constant velocity function and asked to construct the graph of the position function. Students who correctly answered the direct question should be able to reverse their thinking and consider that since the velocity graph represents the slope of the position, and since the velocity is constant, then the slope is constant, which means that the position function is linear. Thus, the student would sketch a linear function with a slope of $\frac{3}{2}$. An alternative solution method would be to translate the graph of $v(t)$ into an algebraic representation: $v(t) = \frac{3}{2}, 0 \leq t \leq 10$. The student could then consider that $v(t)$ is the slope of $s(t)$, so $s(t)$ must be a line with slope $\frac{3}{2}$. Thus, $s(t) = \frac{3}{2}x + b$. The student would then translate the algebraic expression back into a graphical expression. The follow-up question, "are there any other possible answers" is designed to reveal if the students understand that b, in the algebraic expression, can be any value.

This is an example of reversibility as a reversing of the mental process in reasoning. Depending on the solution method, this problem may or may not require representational reversibility. There is no two-way process to be learned and reversed; thus, this cannot be an instance of reversibility of a two-way reversible process. The student must problem solve around the fact that s/he does not know of an "anti-slope" formula. Here, "anti-slope" refers to the reverse of finding the slope in the direct problem. Since no process is available to the students, they will necessarily have to reverse a mental process while reasoning. If the student solves the entire problem graphically by imagining a function whose slope is constantly $\frac{3}{2}$, then this would be an example of reversing of the mental process in reasoning without representational reversibility. On the other hand, if the student translates the graph of the velocity function into an algebraic expression, creates an algebraic expression for the position function, and then translates the algebraic expression of the position function back into a graphical expression, then solving this problem would be an example of using representational reversibility within reversing a mental process while reasoning.

Ex. 2: Direct exit slip (ES 2.3.2). Administered on Day 6 – Section 2.3.

This paired problem set assessed students' reversibility of a two-way reversible process.

Direct problem: Given at the end of class on Day 6: Section 2.3.

Find $y''(x)$ if $y = a\,x^2 + bx + c$. Show or explain your work.

This is a direct use of the simple-power rule for derivatives. This example provides evidence of which students learned the key concept of the day's lesson.

Reverse opening activity (OA 2.3.2): Given at the start of the next class on Day 7: Section 2.4.

Suppose $y''(x) = 3x - 4$. What could be y? Show your work or explain how you know that you are correct.

In order to answer this question correctly, the students had to apply the simple-power rule in reverse. Instead of subtracting one from the exponent, they had to add one to the existing exponent. Instead of multiplying the coefficient by the previous exponent, they had to divide the coefficient by the new exponent. This direct reverse pair is an example of a pair that required reversibility of a two-way process.

3.3.4.2 Design and validation of the exit slips and opening activities

Krutetskii's (1976) model for analyzing the presence of reversibility was to use paired problems that consisted of one question given in a forward direction and one question given in a reverse direction. The exit slips and opening activities were designed to be consistent with the procedures that Krutetskii used to test for the existence of reversibility. These activities were designed to help illumine when reversibility develops during the instructional unit.

I designed each of the exit slips and opening activities by first reviewing the calculus content and objective of each lesson. I took into account the examples used in class and the homework problems assigned when designing the exit slips and opening activities so that the exit slips would not be replicas of the examples used in class but could closely resemble the learning activities used in class. I designed the exit slips to assess the learning of the day's objective in a forward direction.

I designed each reverse problem by examining the question and answer portion of the associated exit slip and then essentially switching their respective roles. By this I mean that I used the solution of the exit slip to design a question prompt with different surface features but the same conceptual underpinnings and the question prompt of the exit slip became part of the solution to the opening activity. As discussed above, the exit slip on Day 6 prompts students to begin with y and find y'' while the opening activity on Day 7 prompts students to begin with y'' and find y.

After designing each direct reverse pair, I solved each paired problem to gauge the effectiveness of the pair. Pairs of problems that did not evidence face validity of requiring a reversible conception of the exit slip to solve the opening activity were revised until acceptable. A second mathematics educator also reviewed each pair of items and made recommendations which were then incorporated into revising each pair of items.

3.3.4.3 Limitations of the exit slips and opening activities

A possible confounding variable in the exit slips and opening activities is the time lapse between the direct question and the reverse question. It is possible that a student who could correctly solve the direct problem at the end of class would also have been able to solve the reverse problem correctly at the end of class but not be able to solve the reverse problem at the start of the following day's class, presumably because the student has forgotten some portion of the previous day's learning. While this hypothetical event is possible, it is doubtful that this event would greatly influence the interpretation of the data in this study. In a worst-case scenario, all 21 students would have developed reversibility of the day's content at the end of each class, correctly solved the direct question, and then forgot a sufficient amount of learning so as to not be able to solve the reverse problem at the start of the next day's class. This would result in the exit slips and opening activities indicating that reversibility does not exist, even though it actually does. In this event, the students should still be able to solve reversible questions in the task-based interviews. Thus, I would have interview evidence indicating that the student(s) have developed reversibility, which conflicts with the exit slip and opening activity data. I would then consider the time lapse between the exit slip and opening activity as a possible factor explaining why reversibility was not present in the exit slips and opening activities but was present in the think-aloud interviews. The results, discussed in chapter 4, indicate that this concern did not materialize and that students who developed reversibility were able to solve the reverse question.

A second limitation of the exit slip and opening activity data is that each pair only assesses reversibility of the day's learning using one item, a limitation imposed by time constraints. It is hoped that by assessing reversibility 33 times during the course of the study, the inherent variability of results on a single item assessment of the presence or lack thereof of reversibility was mitigated over the course of the study.

3.3.5 Think-aloud interviews

Researchers have consistently used similar methods to investigate reversibility in multiple contexts. Krutetskii (1976), Rachlin (1981), Rider (2004), and Teachey (2003) all incorporated a qualitative aspect into her/his research. In each case, the qualitative data collection involved qualitative task-based interviews. In this study, I continued in that tradition.

Six participants each participated in four think-aloud interviews. The purpose of the think-aloud interview is "to gain insight into the child's thinking and learning potential" (Ginsburg, 1997, p. 70). When conducting a think-aloud interview, the interviewer presents the problem and then essentially removes him/herself except to encourage the participant to describe what s/he is thinking while solving the problem (Willis et al., 1999). Each interview was designed to last 20-30 minutes, was video recorded, and consisted of at least two interview questions, one direct and one reverse, designed to elicit reversibility within a calculus concept. The think-aloud interviews provided data that, when analyzed, indicated the kinds of thought processes that the students utilized when using reversibility to solve problems, thus addressing the third research question. The interviewing procedures are described below.

3.3.5.1 Interviewing procedure

The four think-aloud interviews took place after learning content that specifically lends itself well to assessing reversibility. Table 19 reports the dates of all four interviews and the content covered during each interview.

Table 19. Interview content and schedule

Interview #	Date range of interview	Content assessed by interview questions	Kind of reversibility required to solve the question(s)
1	12/4/13 – 12/11/13	1) Differentiation using the simple power rule 2) Reversibility of the simple power rule	1) Two-way process 2) Representational reversibility
2	12/16/13 – 12/20/13	1) Differentiation using the chain rule 2) Reversibility of the chain rule 3) Finding numerical derivatives from the graph of a function 4) Sketching a function given a table of its derivative values	1) Reversibility of the mental process in reasoning without reversible translation 2) Representational reversibility
3	2/19/14 – 3/10/14	1) Graphical interpretation of the derivative and curve sketching	1) Reversibility of the mental process in reasoning without reversible translation 2) Representational reversibility
4	3/12/14 – 3/19/14	1) Reversibility of position and velocity presented in a numerical representation without translation 2) Reversibility of position and velocity presented in a symbolic representation requiring translation to a numerical representation 3) Reversibility of f and f' requiring translation from the graphical representation to a symbolic representation	1) Reversibility of the mental process in reasoning without reversible translation 2) Representational reversibility

Interviewing protocol

Each interview followed the think-aloud interviewing protocol described by Willis et al. (1999). The interviewer practiced a non-trivial sample problem with the participant at the start of the first interview in order to allow the participant an opportunity to practice thinking-aloud. After completion of the practice problem, the interviewer gave the participant the interview question(s) and then essentially removed himself from the equation except to encourage the participant to describe her/his thoughts while solving the problem(s). I did not answer any questions pertaining to solving the problems during the interview. After each interview, I discussed the interview questions with the students, at each student's behest.

Description of interview questions

In this section, I describe the interview questions that will be used during the think-aloud interviews. The interview items are all researcher created items designed to elicit reversible thought processes. The items that were used in the third interview were previously used in a pilot study that I conducted in 2012 exploring AP Calculus BC students' reversible conceptions of calculus graphing. The results of the pilot study indicated that the paired questions in interview 3 do effectively elicit reversible thought processes (in this instance, representational reversibility and reversibility of the mental process in reasoning without reversible translation) from calculus students. The remaining items have been independently analyzed by other mathematics educators to verify the face validity of each item as an assessment of a calculus construct and each pair of direct reverse interview questions has been analyzed to verify the face validity that the pair of problems are indeed reversible pairs. The recommendations and critiques of the mathematics educators who evaluated the interview questions were used to refine the interview questions. A table of all of the interview questions is included in Appendix F.

Interview #1

The first interview took place after teaching section 2.3: Techniques of Differentiation, which took place on November 25, 2013. The six interview participants were interviewed between December

4, 2013 and December 11, 2013. The interview consisted of three questions. The first problem is a direct problem designed to elicit evidence of an action conception of the simple power rule ($\frac{d}{dx}[x^n] = nx^{n-1}$).

Interview 1, Question 1 (IQ 1.1): Let $f(x) = 6x^3$. What is $f'(x)$? Are there any other possibilities for $f'(x)$?

This question provided evidence of the participant's existing understanding of the simple power rule for differentiation. It was expected that most, if not all, students would be able to successfully complete this question. The remaining questions required reversibility (and thus a process conception) of the simple power rule. The second question has two parts (IQ 1.2.a and IQ 1.2.b) and is an algebraic reverse problem of the first question.

Interview 1, Question 2 (IQ 1.2): Suppose a function has a known derivative of $f'(x) = x^5$.

a. What could be the function $f(x)$?

b. Can you think of any other possible functions for $f(x)$?

The participant who solves IQ 1.2.a correctly will most likely use reversibility of the simple power rule. This would be an instance of reversibility of a two-way process as the simple power rule is a procedure that can be learned and then reversed by using the inverse operations in reverse order. The follow-up question (IQ 1.2.b) offered insight into the participant's ability to consider the effects of differentiating a constant function in reverse, without having first differentiated a constant function.

The third interview question is similar to question 2 with respect to reversibility.

Interview 1, Question 3: The derivative of a polynomial function, $f'(x)$, is graphed below.

a. Sketch a possible graph of a polynomial function $f(x)$ whose derivative, $f'(x)$, is shown.

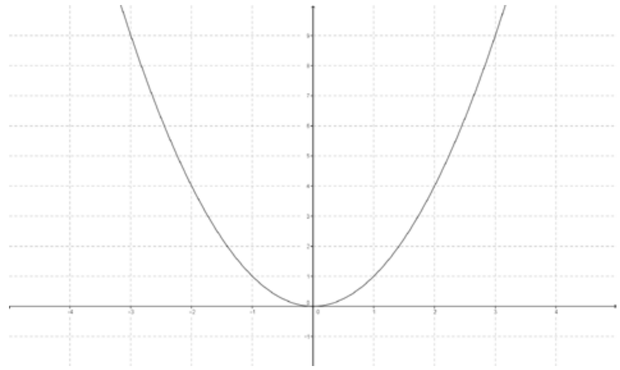


Figure 17. IQ 1.3 graph

b. Are there any other possibilities for $f(x)$?

The difference between questions IQ 1.2 and IQ 1.3 is representational. While IQ 1.2 is entirely symbolic, IQ 1.3 will require translation between graphical and symbolic representations. The student who correctly solves problem IQ 1.3.a will have most likely first translated the graphical representation into an algebraic expression and then used reversibility to find an expression for $f(x)$. The student could then translate the algebraic expression back into a graphical expression. This would also serve to demonstrate an instance of representational reversibility. The follow-up question (IQ 1.3.b), similar to question IQ 1.2.b, offered insight into a student's ability to consider the effects of differentiating a constant function in reverse, without having first differentiated a constant function.

Interview #2

The second interview took place after teaching section 2.6: The Chain Rule, which took place on December 5, 6, and 9, 2013. The six interviews were conducted between December 16, 2013 and December 20, 2013. The interview consisted of four questions, two sets of direct and reverse problems. The first two problems dealt specifically with reversibility of the chain rule and only use the symbolic representation. The students were required to solve the reverse problem first.

Interview 2, Question 1: Suppose a function $f(x)$ has the known derivative $f'(x)$ shown below.

$$f'(x) = x\sin(x^2)$$

What could be the function $f(x)$? Are there any other possibilities for $f(x)$?

The students must reverse the thought processes necessary to carry out the chain rule. Since the students had not learned integration by parts, the students were not able to use reversibility of a two-way process to solve this problem. Also, it is difficult to imagine a solution method by which the students could use representations other than the symbolic to solve this problem. Thus, this interview question required the use of reversibility of a mental process in reasoning without reversible translation to solve.

The second problem is a direct use of the chain rule.

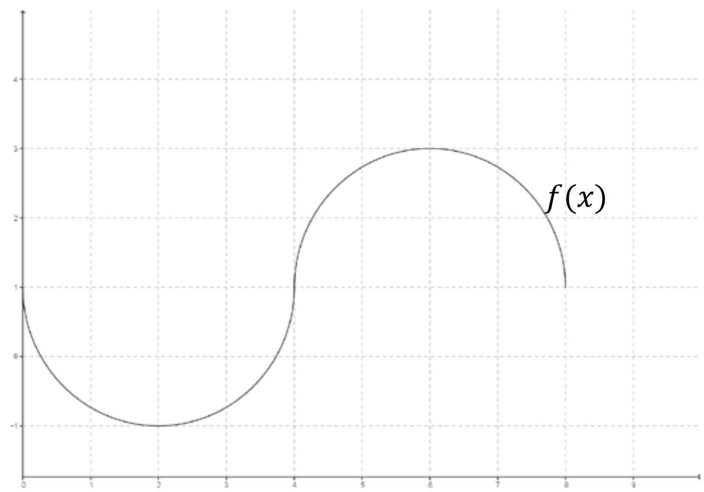
Interview 2, Question 2: Let $f(x) = \cos(x^2)$. Find $f'(x)$.

This question required students to demonstrate knowledge of the chain rule on a problem that is very common to calculus textbooks, but was not an example used in the present classwork or in homework.

In this interview, the direct reverse paired problems were presented reverse and then direct. This order was chosen to eliminate any influence from the direct problem on solving the reverse problem. Krutetskii (1976) interviewed students using direct reverse paired problems by presenting the pairs in multiple ways. He asked the direct question first and immediately followed with the reverse question. He also asked the reverse question first and then asked the direct question. He used this interviewing approach to make inferences regarding the effects of solving a direct problem before solving a reverse problem. Typically, the direct problem influenced the middle and weak students' solutions to the reverse problem.

Interview questions 3 and 4 are a direct reverse pair that required students to coordinate reversibility between $f(x)$ and $f'(x)$ with translating between graphical and numerical representations. IQ 2.3 is a direct question eliciting understanding of the relationship between the graph of f and the graphical representation of f'. IQ 2.4 required students to interpret f' values presented in a table to create a possible graph of f.

Interview 2, Question 3: The graph of $f(x)$ below consists of two complete semi-circles that intersect at [4,1].



x	$f'(x)$
2	
4	
5	
6	
7	

Figure 18. IQ 2.3: Graph and table of values

Estimate or give an exact value, if one exists, of $f'(x)$ at the x-values indicated in the table. In this interview question, the students were expected to demonstrate understanding of the graphical representation of the derivative by estimating $f'(x)$ by estimating the slope of the line tangent to the curve at each x −value. The question also required translating from a graphical representation of a function, $f(x)$, to a numerical representation of a new function, $f'(x)$.

Interview 2, Question 4: The table below gives selected values of $f'(x)$, the derivative of $f(x)$.

a. Sketch a possible curve for $f(x)$ on the axis below.

x	$f'(x)$
0	1
1	2
2	-1
3	0
4	-1

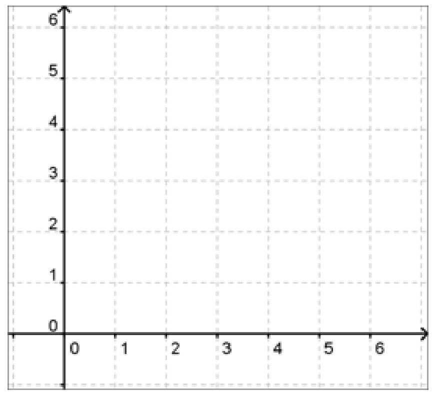


Figure 19. IQ 2.4: Table of values and graph

b. Could you sketch another function that would satisfy the table of values?

IQ 2.4 is the reverse problem to IQ 2.3 and required students to sketch a curve based on known derivative values presented through a numerical representation. IQ 2.4 required translation from a numerical representation to a graphical representation and required reversibility of the graphical interpretation of the derivative.

Taken as a direct reverse pair, IQ 2.3 and IQ 2.4 required reversibility of the mental process in reasoning without reversible translation and representational reversibility. Students who can solve IQ 2.3 will be able to read derivative values from a graph of a function. IQ 2.4 required these students to sketch a graph given derivative values. Focusing on the calculus element, given a graph of f, find $f'(2), f'(4)$, etc. and then given $f'(2), f'(4)$, etc. sketch a function f, this kind of reversibility is an example reversibility of the mental process in reasoning without reversible translation. The representational reversibility presents in the ability to translate from a graph to a table and then from a table to a graph.

Interview #3

The third interview took place after teaching section 4.4 between February 19, 2014 and March 10, 2014. Sections 4.1-4.4 focus on the graphical interpretation of the derivative and curve sketching. The interview consisted of two reversible graphing questions. The IQ 3.1 presents students with a graph of a function f and the students must draw inferences regarding f' and f'', thus indicating student understanding of calculus graphing. IQ 3.2 assessed the same kinds of calculus knowledge as IQ 3.1; however, IQ 3.2 required students to coordinate both a graphing schema and interval schema to create a graph (Baker et al., 2000). In the fall of 2012, I ran a pilot study using IQ 3.1 and IQ 3.2. The pilot study revealed that high school students who have completed a course in differential calculus maintained a strong understanding of how to analyze graphs from a calculus perspective; however, the students exhibited very little reversibility of their understanding of calculus graphing.

IQ 3.1: Consider the graph of $f(x)$ on the interval $[-4,4]$. $f(x)$ consists of two semi-circles and two line segments, as shown below.

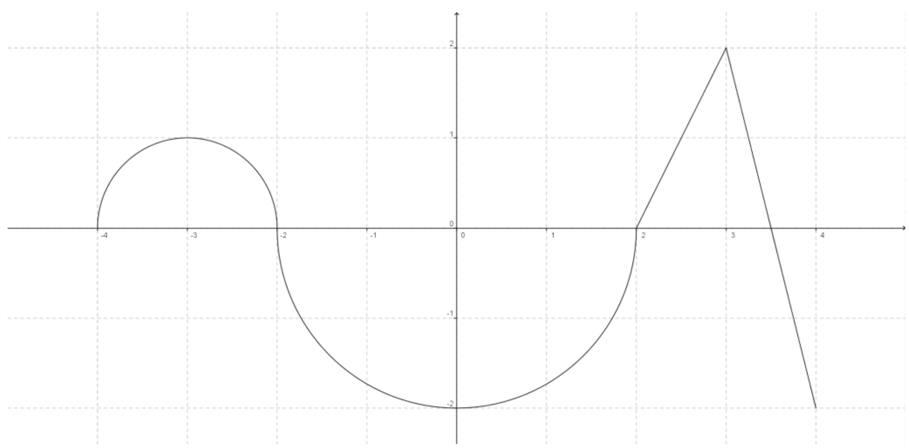


Figure 20. IQ 3.1: Graph

a. On what intervals, if any, is $f'(x) > 0$ and $f''(x) > 0$?

b. On what intervals, if any, is $f'(x) < 0$ and $f''(x) > 0$?

c. On what intervals, if any, is $f'(x) > 0$ and $f''(x) < 0$?

d. On what intervals, if any, is $f'(x) < 0$ and $f''(x) < 0$?

e. At what x −value(s), if any, does $f'(x) = 0$?

f. At what x −value(s), if any, does $f''(x) = 0$?

g. At what x −value(s), if any, does $f'(x)$ not exist?

h. Justify your response to question (g).

i. At what x −value(s), if any, does $f''(x)$ not exist?

j. Justify your response to question (i).

IQ 3.1 required students to determine intervals given calculus properties and a graph of a function. IQ 3.2 required students to sketch the graph of a function given the calculus properties and the intervals.

IQ 3.2: Sketch a possible graph of a function f that satisfies the following conditions:

f is continuous;

$f(0) = 1, f'(-3) = f'(2) = 0$, and $\lim_{x \to 0} f'(x) = \infty$;

$f'(x) > 0$ when $-5 < x < -3$ and when $-3 < x < 2$;

$f'(x) < 0$ when $x < -5$ and when $x > 2$;

$f''(x) < 0$ when $x < -5$, when $-5 < x < -3$, and when $0 < x < 5$;

$f''(x) > 0$ when $-3 < x < 0$ and when $x > 5$;

$\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -1$.

Taken as a direct reverse pair, IQ 3.1 and IQ 3.2 required two different kinds of reversibility. Representational reversibility is present in that the direct problem, IQ 3.1, required students to translate from a graphical representation to a numerical representation. IQ 3.2, the reverse question required students to translate from a numerical representation to a graphical representation. The calculus elements required reversibility of the mental process in reasoning without reversible translation as students had to determine f' and f'' values from f in IQ 3.1, but then had to determine f from a list of f' and f'' values in IQ 3.2.

Interview #4

The fourth interview took place at the conclusion of chapter 4, between March 12, 2014 and March 19, 2014. Sections 4.5-4.8 focus on the relationship between position, velocity, and acceleration. After section 4.8, students should have been able to change position into velocity and velocity into acceleration by differentiation. They had not discussed moving in the reverse direction.

The fourth interview consisted of three sets of reversible questions. IQ 4.1 and IQ 4.2 are a direct reverse pair designed to assess understanding of the numerical representation of functions and differentiation and to investigate student understanding of reversibility between position and velocity.

Interview 4, Question 1.a is shown below in figure 21:

The table below gives the distance a car has traveled, measured in miles, at selected time measurements in hours.

t (hours)	Distance (miles)
0	0
0.2	8
0.4	17
0.5	21
0.8	35
1.0	44
2.0	100

t (hours)	Average Velocity (mph)
0.1	
0.6	
0.9	
1.5	

If the car only moves in a positive direction, fill in the accompanying table by estimating the velocity of the car in miles per hour at the times indicated. Show how you determined the average velocity.

Figure 21. IQ 4.1.a

Interview 4, question 2.a is shown below in figure 22.

Suppose a car's velocity in $\frac{m}{s}$ is measured at intervals and recorded in the following chart. Estimate the distance traveled from $t = 0\ s$ to $t = 60\ s$. Show how you calculate the distance traveled.

t	$v(t)$
0	0
20	5
30	8
50	4
60	10

Figure 22. IQ 4.2.a

Taken as a direct reverse pair of problems, IQ 4.1.a and IQ 4.2.a required reversibility. Depending on how a student solves the problem, the student's answer may have indicated reversibility of a two-way process or the answer may have indicated reversibility of the mental process in reasoning without reversible translation. Most students were expected to solve IQ 4.1.a by using the slope formula, $v_{ave} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$. To solve IQ 4.2.a, students may have used several solution methods. Students could conceivably use the v_{ave} formula above in reverse by substituting $8\frac{m}{s}$ in for v_{ave}, $0\ s$ in for t_1, $60\ s$ in for t_2, and $0\ m$ in for $s(t_1)$, and then solve for $s(60)$. This would be an example of reversibility of a two-way process. Alternatively, a student could conceptualize distance as the product of velocity and time. Following this line of reasoning, a student may determine an approximate distance traveled over each time interval and then sum up the distances. This solution process would be an example of reversibility of the mental process in reasoning without reversible translation.

IQ 4.1.b and IQ 4.2.b also investigated reversibility between position and velocity; however, these questions were presented in symbolic representations instead of numerical representations. IQ 4.1.b is shown below in figure 23.

Suppose we know a function $s(t)$ that gives the position of the car in miles after t hours, $0 \leq t \leq 2$.

$$s(t) = t^3 + 3t^2 + 40t$$

Fill in the accompanying table of instantaneous velocities at the times indicated.

t (hours)	Instantaneous Velocity (mph)
0.1	
0.6	
0.9	
1.5	

Figure 23. IQ 4.1.b

IQ 4.1.b required students to find an equation for $s'(t)$ and then evaluate $s'(t)$ at the indicated values of t.

IQ 4.2.b: Suppose we know a velocity function, $v(t)$, for a vehicle in motion in meters per second.

$$v(t) = 4t^3 - 3t^2 + t$$

Assuming that the vehicle started at a position of zero meters, find the position of the vehicle at $t = 3$.

IQ 4.2.b required students to use the velocity function to find a position. IQ 4.1.b and IQ 4.2.b are a direct reverse pair. IQ 4.1.b required using differentiation, the simple power rule, to move from position to velocity and IQ 4.2.b required reversing the simple power rule to move from velocity to position. This direct reverse pair required reversibility of a two-way process in regards to the simple power rule and reversibility of a mental process in reasoning by recognizing the reversible relationship between position and velocity.

IQ 4.3.a and IQ 4.3.b required translation from a graphical representation to an algebraic representation and are a direct reverse pair. IQ 4.3.a required analysis of a graph of $f(x)$ to write an algebraic expression for $f'(x)$. IQ 4.3.b required analysis of a graph of $f'(x)$ to write an algebraic expression for $f(x)$.

IQ 4.3.a is shown below in figure 24:

The function $f(x)$ is graphed below on [0,50], write an algebraic expression for $f'(x)$ on [0,50].

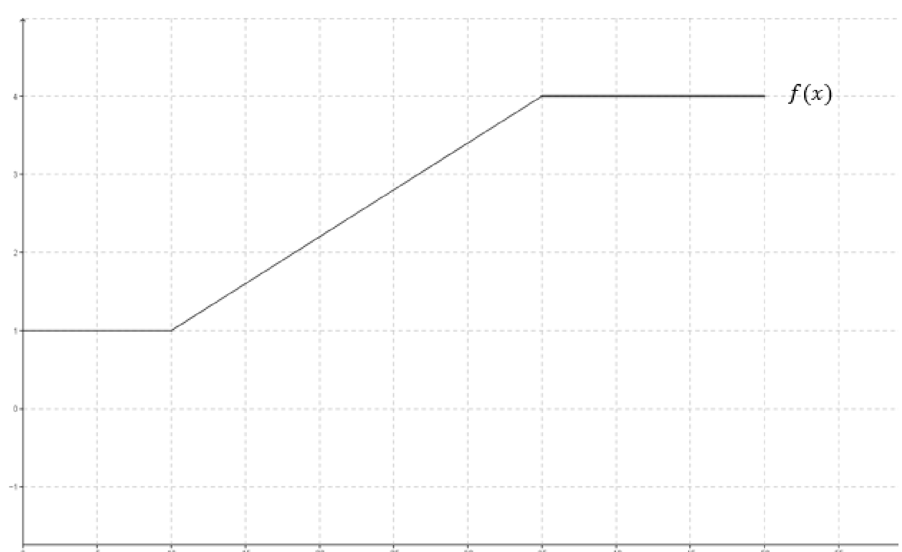


Figure 24. IQ 4.3.a

IQ 4.3.a was expected to be solved in one of two ways. One solution method is to note that $f(x)$ consists of three line segments, thus, $f'(x)$ consists of three constant functions, each defined by the slope of the respective line segment. Thus a student could find a piecewise-defined function for $f'(x)$ by using a graphical interpretation of the derivative and finding the slope of each line segment. Alternatively, a student who prefers an algebraic representation of $f(x)$ over the graphical representation of $f(x)$ may translate the graphical representation into the piecewise-defined function $f(x) = \begin{cases} 1, & 0 \leq x \leq 10 \\ \frac{3}{25}x - \frac{1}{5}, & 10 < x < 35 \\ 4, & 35 \leq x \leq 50 \end{cases}$. Then, the student may differentiate $f(x)$ using the simple power rule to determine that $f'(x) = \begin{cases} 0, & 0 < x < 10 \\ \frac{3}{25}, & 10 < x < 35 \\ 0, & 35 < x < 50 \end{cases}$.

IQ 4.3.b is shown below in figure 25.

The function $f'(x)$ is graphed below on [0,50], write an algebraic expression for $f(x)$ on [0,50].

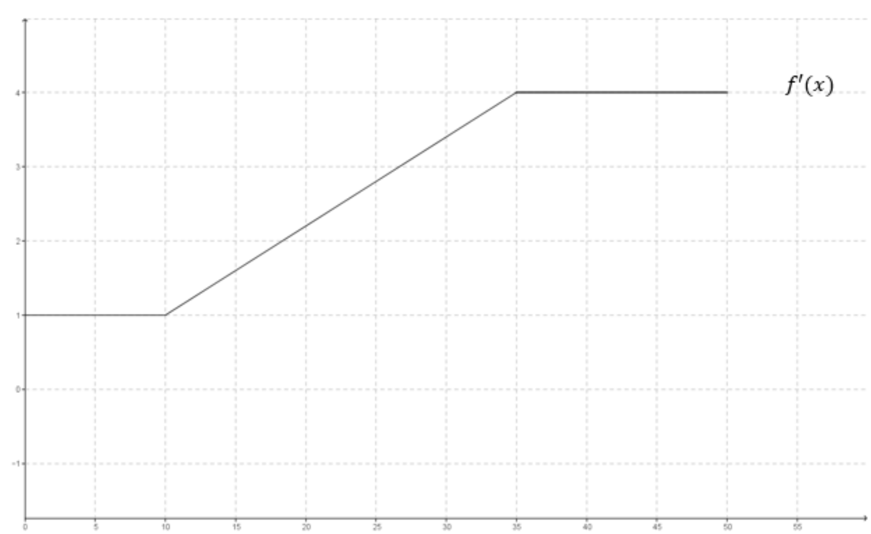


Figure 25. IQ 4.3.b

IQ 4.3.b, multiple solutions were available to the students. The most likely solution method is to attempt to translate the graphical representation of $f'(x)$ into an algebraic expression, $f'(x) = \begin{cases} 1, & 0 \leq x \leq 10 \\ \frac{3}{25}x - \frac{1}{5}, & 10 < x < 35 \\ 4, & 35 \leq x \leq 50 \end{cases}$, and then use reversibility of a two-way process to determine an algebraic expression for $f(x)$. Alternatively, students could use reversibility of the mental process in reasoning without reversible translation and try to think of a function that would have a constant slope on $(0,10)$, a linear slope function on $(10,35)$, and a constant slope on $(35,50)$. Since both the forward and reverse problems required translation from a graphical representation to an algebraic representation, representational reversibility was not required to solve this problem.

3.3.6 Timeline of data collection

The following timeline (figure 26) is a visual representation of the key data collection dates within the study. The date axis refers to days 1-48 in the course calendar. Day 1 fell on November 11, 2013. Day 48, the administration of the DCT, fell on March 10, 2014. All markers above the date axis referred to events that effect the entire class. There are three categories of events above the date axis: 1) instructional date markers ( ), which indicate the expected dates that chapters will begin and end, 2) exit slips and opening activity collection markers ( ), which indicate the dates that exit slips and opening activities will be administered and collected, and lastly, 3) the DCT marker ( ), which indicates the date that the DCT will be administered. The events that lie below the date axis are the days of the think-aloud interviews. They are marked with a . The flexibility pre-test was administered approximately three weeks before the first day of instruction.

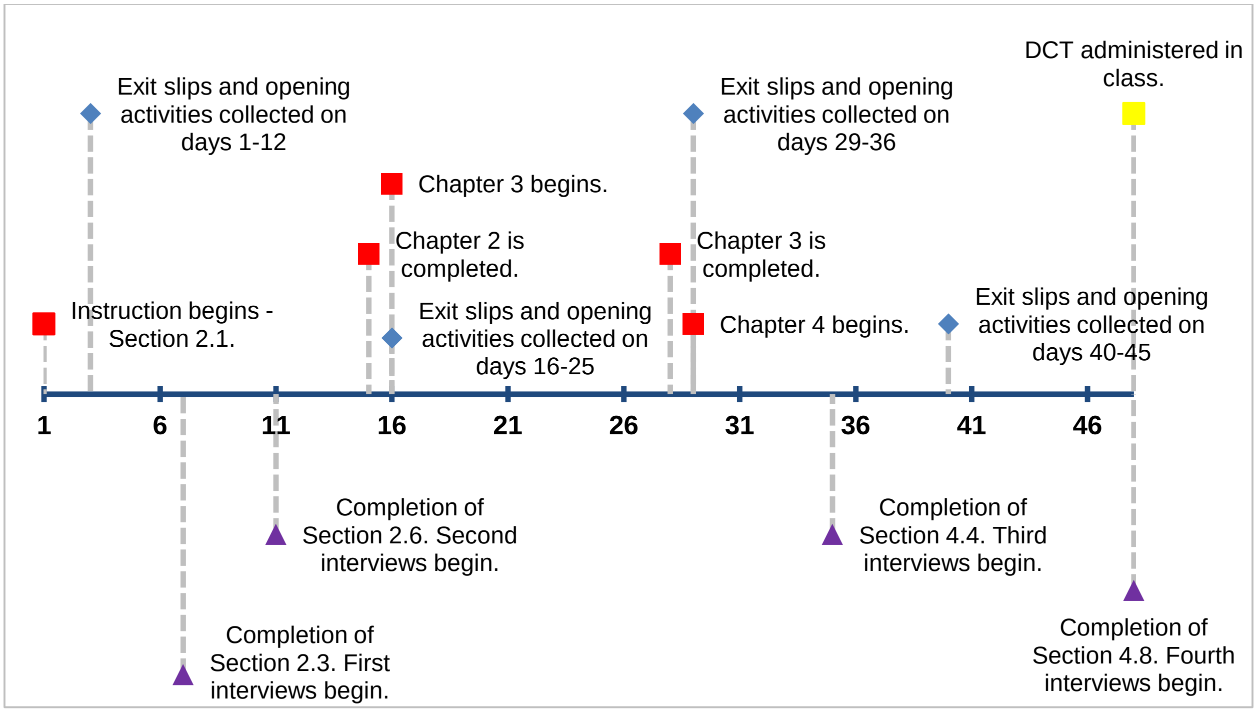


Figure 26. Timeline of data collection and instructional markers

3.4 DATA CODING AND MANAGEMENT

In this section, I describe how the data collected from the four sources, the flexibility pre-test, the DCT, the exit slips and opening activities, and the think-aloud interviews, was coded and managed to prepare for analysis. I describe the coding and management of the DCT before I describe the coding and management of the FPT because the pre-test was designed to align with the DCT and was coded and managed using a similar procedure.

3.4.1 Differentiation competency test (DCT)

The DCT was scored for accuracy and coded according to the description given by Kendal and Stacey (2003). Each item was worth one point, thus a perfect score on the DCT was $18/18$. There was no partial credit assigned, consistent with the grading rubric created and validated by Kendal and Stacey (2003). The results of the DCT are reported in Appendix G.

3.4.2 Flexibility pre-test (FPT)

The FPT was scored for accuracy and coded in much the same way as the DCT. Each item was worth one point, thus a perfect score on the FPT was $18/18$. There was no partial credit assigned, consistent with the grading rubric used on the DCT. The results of the FPT are reported in Appendix H.

3.4.3 Exit slips and opening activities

At the conclusion of each instructional class, the students solved an exit slip problem that assesses the day's learning in a forward direction. At the start of the next class, the students solved an opening activity problem that assessed the previous day's learning in a reverse direction. There were a total of 33 exit slip and opening activity pairs that assessed 45 specific instances of reversibility. Each daily item was designed to take approximately 5 minutes.

Each set of paired problems was analyzed twice on an individual level and at a class level. The first level of analysis attempted to determine the relative presence of reversibility at the class level at 33 separate data points. Each set of paired problems had four possible outcomes: Correct-

Correct, Correct-Incorrect, Incorrect-Correct, Incorrect-Incorrect. These possible outcomes and their implications for the presence of reversibility are summarized in Table 20.

Table 20. Possible outcomes and implications for reversibility

Direct	Reverse	Implication
Correct	Correct	Student demonstrates reversibility.
Correct	Incorrect	Student developed learning in forward direction but does not exhibit reversible conception of mathematical concept
Incorrect	Correct	Result offers no insight into reversibility. Student may have developed a forward and reverse conception of content while working outside of classroom. Result is unexpected.
Incorrect	Incorrect	Student does not demonstrate understanding of concept. Result offers no insight into reversibility.

The following table represents how the exit slip and opening activity data were coded and managed. Analysis of the data will be discussed in the analysis section. Table 21 represents how each set of paired problems will be coded at the individual level.

Table 21. Individual student data on single set of exit slip and opening activity

Student Identifier:	
Date:	
Description of Content:	What kind of reversibility is required? 1) Two-Way Process 2) Mental reasoning without representational reversibility 3) Mental reasoning with representational reversibility
Direct Outcome	Reverse Outcome
Correct or Incorrect	Correct or Incorrect

Each student's individual exit slip and opening activity data was aggregated for the entire instructional period. Table 22 is a sample table showing how the aggregated data was managed.

Table 22. Record of individual student data on all exit slips and opening activities

Student Id:					
Day	Content	Direct Outcome	Reverse Outcome	Does the student demonstrate reversibility?	What kind of reversibility is present?
1	Average rate of change	Correct or Incorrect	Correct or Incorrect	Yes or No	1) Two-Way Process 2) Mental reasoning without representational reversibility 3) Mental reasoning with representational reversibility
2	Average velocity	Correct or Incorrect	Correct or Incorrect	Yes or No	1) Two-Way Process 2) Mental reasoning without representational reversibility 3) Mental reasoning with representational reversibility
3	Limit definition of derivative	Correct or Incorrect	Correct or Incorrect	Yes or No	1) Two-Way Process 2) Mental reasoning without representational reversibility 3) Mental reasoning with representational reversibility
4	Limit definition of derivative	Correct or Incorrect	Correct or Incorrect	Yes or No	1) Two-Way Process 2) Mental reasoning without representational reversibility 3) Mental reasoning with representational reversibility

Table 23 represents how each set of paired problems were coded at the class level.

Table 23. Class data on single set of exit slip and opening activity

Date:	Group:
Description of Content:	What kind of reversibility is present?
Outcome on Paired Problems (Direct-Reverse)	Number of Students
Correct-Correct	
Correct-Incorrect	
Incorrect-Correct	
Incorrect-Correct	

The second level of data analysis on the exit slips and opening activities was used in tandem with the think-aloud interview data to help inform research question 3: what are the thought processes that students utilize when using reversibility to solve problems. In an effort to attend to students' thinking while solving the exit slips and opening activities, I noted the specific instances within the students' work that address the following questions:

1) What specific instances of reversibility exist within the exit slip and opening activity data?
   i. What specific instances of reversibility of a two-way reversible process are present?
   ii. What specific instances of reversibility as a reversing of the mental process in reasoning without reversible translation are present?
   iii. What specific instances of representational reversibility are present?
2) What specific instances of flexibility exist within the exit slip and opening activity?

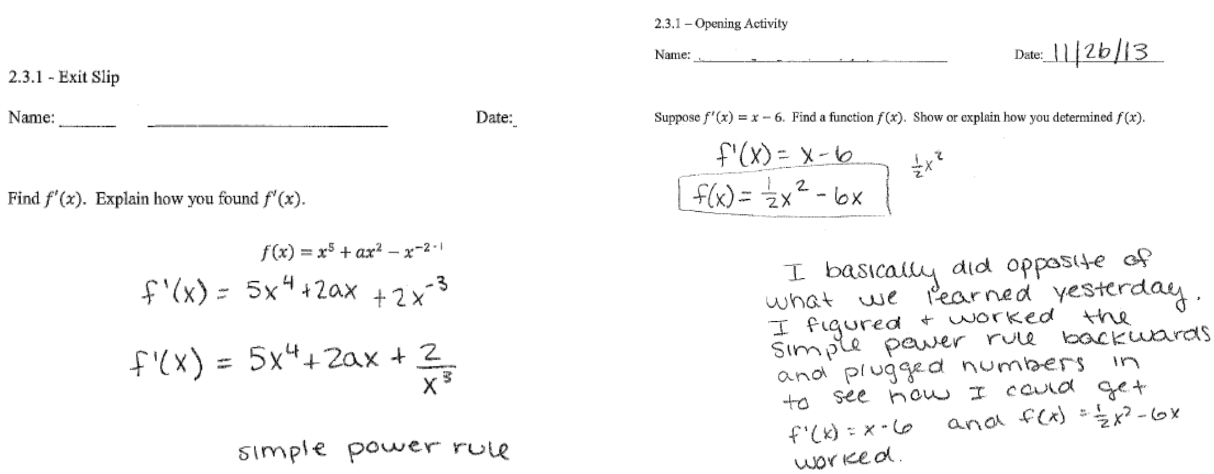
i. What specific instances of translations between representations of functions are present?

ii. What specific instances of translations between representations of derivatives are present?

3) What specific instances of translations between representations of functions and representations of derivatives are present?

The answers to these questions were categorized at the class level to attempt to identify over-arching thematic elements of how the class thinks about solving reversible problems. The interview participants' solutions to the exit slips and opening activities were used to strengthen or question the conclusions drawn from analyzing the think-aloud interview data.

Figure 27 shows an example of one student's work on exit slip and opening activity pair 2.3.1.


2.3.1 - Exit Slip

Name: ______ Date:

Find $f'(x)$. Explain how you found $f'(x)$.

$f(x) = x^5 + ax^2 - x^{-2}$

$f'(x) = 5x^4 + 2ax + 2x^{-3}$

$f'(x) = 5x^4 + 2ax + \frac{2}{x^3}$

simple power rule

2.3.1 – Opening Activity

Name: ______ Date: 11/26/13

Suppose $f'(x) = x - 6$. Find a function $f(x)$. Show or explain how you determined $f(x)$.

$f'(x) = x - 6$ $\frac{1}{2}x^2$

$\boxed{f(x) = \frac{1}{2}x^2 - 6x}$

I basically did opposite of what we learned yesterday. I figured + worked the simple power rule backwards and plugged numbers in to see how I could get $f'(x) = x - 6$ and $f(x) = \frac{1}{2}x^2 - 6x$ worked.

Figure 27. Sample student work on ESOA 2.3.1

On exit slip 2.3.1, the student received a grade of "correct" because she correctly found $f'(x)$ by using the simple power rule for differentiation. On opening activity 2.3.1, she correctly found $f(x)$ by reversing the simple power rule for differentiation. Thus, at the first level of analysis, the student's work would be scored "Correct-Correct" indiciating the presence of reversibility. At the second level of analysis, the student's description on the opening activity indicates a specific instance of when a student is thinking about using two-way reversibility when she says "I basically did opposite of what we learned yesterday".

3.4.4 Think-aloud interviews

The think-aloud interviewing data consists of two parts: 1) the written responses to the interview questions and, 2) the interview transcripts. In this section, I describe how the written work was coded and then how the interview transcript data was coded. Analysis of the written responses and transcript data is discussed in the analysis section.

3.4.4.1 Coding and management of written responses to the interview questions

The participants' written responses were coded along multiple dimensions. The written responses were first coded for evidence of the existence of reversibility between differentiation and integration and then coded for evidence of the existence of representational reversibility. Finally, each written response was coded for evidence of flexibility. The written solutions were coded using the following tables. Each table presented here is an excerpt from the full table. Each full table is presented in Appendix H.

I used the following tables to code and manage the written responses to interview questions for evidence of the existence of reversibility of between differentiation and integration. Interview questions 1.1 (Interview 1, Question 1) & 1.2, 1.1 & 1.3, 2.1 & 2.2, 2.3 & 2.4, 4.1.a & 4.2.a, 4.1.b & 4.2.b, and 4.3.a & 4.3.b form direct (differentiation) and reverse (integration) pairs. Table 24 reports the results of the interview items that assess reversibility of differentiation and integration, arranged by participant. The data reported in this table was used to answer whether any reversibility that develops is limited to just two-way reversible processes, just reversibility of the mental process in reasoning without reversible translation, or if it extends to both aspects of reversibility.

Table 24. Reversibility of differentiation and integration, arranged by participant

Participant	Interview Question (Interview #, Question #)	Direct Function → Derivative	Reverse Derivative → Function	Is reversibility of a two-way process present?	Is reversibility of the mental process in reasoning without reversible translation present?
FRED	Interview 1, Questions 1 and 2	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	Interview 1, Questions 1 and 3	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	Interview 2, Questions 1 and 2	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	Interview 2, Questions 3 and 4	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	Interview 4, Questions 1.a and 2.a	Correct/Incorrect Distance → Velocity	Correct/Incorrect Velocity → Distance	Yes or No	Yes or No
	Interview 4, Questions 1.b and 2.b	Correct/Incorrect Position → Velocity	Correct/Incorrect Velocity → Position	Yes or No	Yes or No
	Interview 4, Questions 3.a and 3.b	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No

Table 25 reports the results of the interview items that assess reversibility of differentiation and integration, arranged by interview question. The data reported in this table was used to answer whether any reversibility that develops is limited to just two-way reversible processes, just reversibility of the mental process in reasoning without reversible translation, or if it extends to both aspects of reversibility.

Table 25. Reversibility of differentiation and integration, arranged by interview question

Interview Question (Interview #, Question #)	Participant	Direct Function → Derivative	Reverse Derivative → Function	Is reversibility of a two-way process present?	Is reversibility of the mental process in reasoning without reversible translation present?
Interview 1, Questions 1 and 2	P. 1	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	P. 2	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	P. 3	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	P. 4	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	P. 5	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No
	P. 6	Correct/Incorrect $f \to f'$	Correct/Incorrect $f' \to f$	Yes or No	Yes or No

The participants' responses were then coded for evidence of representational reversibility. The written work was using the following two tables.

Table 26 reports the results of the interview items that assess representational reversibility, arranged by participant. The following interview questions require representational reversibility: 1.3, 2.3 & 2.4, and 3.1 & 3.2. The data reported in this table helped to inform whether or not students were developing representational reversibility.

Table 26. Representational reversibility, arranged by participant

Participant Name	Interview Question (Interview #, Question #)	Direct Translation	Reverse Translation	Is representational reversibility present?
FRED	Interview 1, Question 3	In order to solve problem, the participant will likely translate function from graphical to symbolic representation, differentiate the symbolic representation, then translate the resultant symbolic representation into a graphical representation. Thus, the reversible translation is G→S and then S→G. Any participant effort towards translation will be noted for Interview 1, Question 3.		Yes or No
	Interview 2, Questions 3 and 4	Correct/Incorrect G→N	Correct/Incorrect N→G	Yes or No
	Interview 3, Questions 1 and 2	Y/N Function Graph & Calculus Properties → Intervals	Y/N Intervals and Calculus Properties → Function Graph	Yes or No

Table 27 reports the results of the interview items that assess representational reversibility, arranged by interview question. The data reported in this table helped to inform whether or not students were developing representational reversibility.

Table 27. Representational reversibility, arranged by interview question

Interview Question (Interview #, Question #)	Participant Name	Direct Translation	Reverse Translation	Is representational reversibility present?
Interview 1, Question 3	FRED	In order to solve problem, participant will likely translate function from graphical to symbolic representation, differentiate the symbolic representation, then translate the resultant symbolic representation into a graphical representation. Thus, the reversible translation is G→S and then S→G. Any participant effort towards translation will be noted for Interview 1, Question 3.		Yes or No
	JILL	In order to solve problem, participant will likely translate function from graphical to symbolic representation, differentiate the symbolic representation, then translate the resultant symbolic representation into a graphical representation. Thus, the reversible translation is G→S and then S→G. Any participant effort towards translation will be noted for Interview 1, Question 3.		Yes or No

Due to the complex nature of interview questions 3.1 and 3.2, I developed a separate table to record analysis of the written work in conjunction with the interview transcript of interview 3. Table 28 reports the coding plan for interview 3.

Table 28. Analysis plan for interview 3

3.1: Forward: $G \to N$, $f \to f'$, $f \to f''$	3.2: Reverse: $N \to G$, $f' \to f$, $f'' \to f$	Is reversibility of the mental process in reasoning present in graph?	Are descriptions consistent with graph?	Is representational reversibility evident?
$f'(x) > 0, f''(x) > 0$	$(-3,0)$			
$f'(x) < 0, f''(x) > 0$	$(5, \infty)$			
$f'(x) > 0, f''(x) < 0$	$(-5,-3) \cup (0,2)$			
$f'(x) < 0, f''(x) < 0$	$(-\infty,-5) \cup (2,5)$			
$f'(x) = 0$	$x = -3, x = 2$			
$f''(x) = 0$	$x = -3, x = 5$			
$f'(x)\ DNE$	$x = -5, x = 0$			

Each line in the table was scored as $0, \frac{1}{2}$, or 1 point. A completely correct answer received one point, an answer that contained a partially correct answer received $\frac{1}{2}$ point, and an incorrect answer received zero points. Thus, the interview participant's response to IQ 3.1 was scored out of 7 points and the interview participant's response to IQ 3.2 was scored out of 7 points. Below, in figure 28, I have included a sample of an interview participant's work on IQ 3.1 and IQ 3.2 and shown how it was scored.

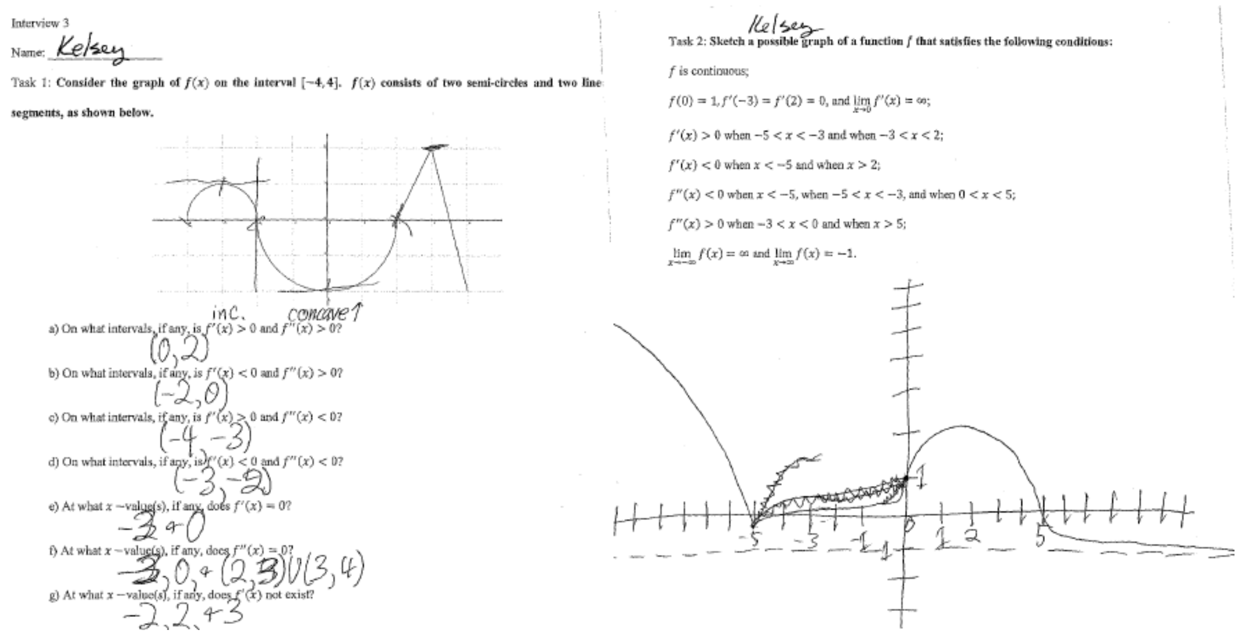


Figure 28. Kelsay's solutions to interview questions 3.1 and 3.2

Kelsay received a score of $6.5/7$ on IQ 3.1. All of her answers were completely correct except for a partially correct answer to part (f). She wrote that $f''(x) = 0$ at $x = -3, 0, (2,3) \cup (3,4)$. Kelsay is correct that $f''(x) = 0$ on the intervals $(2,3) \cup (3,4)$ but is incorrect when she writes that $f''(x) = 0$ at $x = -3$ and $x = 0$. In fact, $f''(-3) < 0$ and $f''(0) > 0$. On IQ 3.2, Kelsay drew a perfect sketch, thus she earned a score of $7/7$ because her graph is correct on each of the intervals and at each discrete x-value listed in table 28.

Finally, the participants' written responses were coded for evidence of flexibility. Evidence of translations between representations of functions, representations of derivatives, and/or representations of functions and representations of derivatives were recorded in Table 29, by participant.

Table 29. Evidence of flexibility, arranged by participant

Participant Name	Interview Question	Translations Present
FRED	1.1	
	1.2	
	1.3	
	2.1	
	2.2	
	2.3	
	2.4	
	3.1	
	3.2	
	4.1.a	
	4.1.b	
	4.2.a	
	4.2.b	
	4.3.a	
	4.3.b	

Table 30 reports evidence of flexibility, arranged by interview question.

Table 30. Evidence of flexibility, arranged by interview question.

Interview Question	Participant Name	Translations Present
1.1	FRED	
	JILL	
	KELSAY	
	KIRSTEN	
	MARCUS	
	MICHAEL	
1.2	FRED	
	JILL	
	KELSAY	
	KIRSTEN	
	MARCUS	
	MICHAEL	

3.4.4.2 Coding and management of interview transcript data

I transcribed the interviews and then analyzed the transcripts. The transcripts were analyzed to help explicate students' thought processes while solving the reversible questions. Interview transcripts were analyzed by searching the transcripts for terms that indicated consideration of reversibility. I examined the text of the transcripts to answer the following questions:

1) What specific instances of reversibility exist within the interview data?
   i. What specific instances of reversibility of a two-way reversible process are present?
   ii. What specific instances of reversibility as a reversing of the mental process in reasoning without reversible translation are present?
   iii. What specific instances of representational reversibility are present?
2) What specific instances of flexibility exist within the interview data?
   i. What specific instances of translations between representations of functions are present?
   ii. What specific instances of translations between representations of derivatives are present?
   iii. What specific instances of translations between representations of functions and representations of derivatives are present?

To answer these questions, I searched the transcripts for the phrases reported in the codebook in Appendix I. A codebook systematically sorts the coded text into categories, types, and relationships of meaning (Guest, MacQueen, & Namey, 2012). I developed a codebook that identifies key phrases indicating evidence of reversibility and/or flexibility. I used the results from the pilot study, reversibility literature, and flexibility literature to develop the codebook.

Creation of the codebook followed the guidelines prescribed by Guest et al. (2012). The first step in creation of the codebook was to label the identified themes of two-way reversibility, reversibility of the mental process in reasoning, representational reversibility, and flexibility, and then to define what they are and what they are not. Words and/or actions that are coded as evidence of two-way reversibility consist of all of the words, phrases, or actions taken when solving an interview question that indicate the reversing of a process by working the steps backwards through inverse operations. Words and/or actions that are coded as evidence of reversibility of the mental process in reasoning consist of words, phrases, and/or actions that indicate the student's attempt at solving a problem by reversing a thought process without using the direct process in reverse. Words and/or actions that are coded as evidence of representational reversibility are all of the words, phrases, and/or actions that indicate that a student is proficiently translating back and forth between two different representations. Words and/or actions coded to flexibility consist of all the words, phrases, and/or actions that indicate that a student is translating from one representation to another. Note the distinction between representational reversibility and flexibility: flexibility only requires a unidirectional translation from one representation to another while representational reversibility requires a bidirectional translation between two representations.

Each code definition includes the following: 1) a code label that distinguishes one code from another, 2) a short definition, 3) a full definition, 4) clear indicators of when to use the code, and 5) clear indicators of when not to use the code. An iterative process of reading and coding the data, comparing the coded data, adjusting the codebook definitions as necessary and then repeating the process was continued until all of the interview data could be effectively coded.

To aid in the creation and refinement of the codebook, the piloting of the interview questions, the reversibility literature, and the flexibility literature were used to design a preliminary codebook. A second coder double coded 33% of the interview data. I met with the second coder and discussed our coding. We had 100% agreement on where reversibility presented in the transcripts.

The complete codebook is presented in Appendix I. Below, I present examples of words, phrases, and actions that indicate two-way reversibility, reversibility of the mental process in reasoning, representational reversibility, and flexibility.

Examples of phrases that indicate two-way reversibility

In this study, two-way reversibility was most likely to present when trying to find a function $f(x)$ from a given function $f'(x)$. Two-way reversibility was evident at any point where a participant noted that the function would be the reverse of the given derivative and then reversed a step-by-step process to find the function $f(x)$.

For example, in interview 1, question 1 (1.1), the participant is asked to find $f'(x)$ if $f(x) = 6x^3$. It was expected that all participants would be able to use the simple power rule for differentiation to correctly solve the problem. Specifically, $f'(x) = 6 * 3x^2 = 18x^2$. Question 1.2 (interview 1, question 2) asks the participant to find a function $f(x)$ whose derivative is known to be $f'(x) = x^5$. It was expected that any participant who correctly solved the problem will use two-way reversibility to reverse the simple power rule. Thus, an expected correct solution of $f(x) = \frac{1}{6}x^6$ likely included at least of the following phrases:

- "Ok this is going in reverse, so instead of subtracting one from the exponent, I am going to add one to the exponent."
- "Ok, differentiating is multiplying so I need to divide something in order to end with a one as the coefficient if I differentiate my answer."

Both of these phrases indicated that the participant used two-way reversibility because s/he was trying to undo the simple power rule by using a sequence of inverse operations. In the first phrase, the participant indicated that s/he understands that since differentiating reduces the power of the polynomial by one, then reversing differentiation should increase the power of the polynomial by one. Thus, we see the student using the inverse operation, in this case addition, to reverse the learned process of subtraction as a part of differentiation.

Examples of phrases that indicate reversibility of the mental process in reasoning

Reversibility of the mental process in reasoning is most often indicated by a student referring to doing something "backwards" or in "reverse" without using inverse operations to reverse a learned process.

For example, in question 2.1, the students are asked to find $f(x)$ given that $f'(x) = x sin(x^2)$. The students do not know a process for reversing the chain rule, thus any correct solution would require reversibility of the mental process in reasoning. Thus, a correct solution would likely include a phrase similar to:

- Since I'm given f' … let me pick some f's and see if when I differentiate them I can produce f'

This phrase indicates that the student is aware that s/he needs to reverse differentiation but does not have access to a process that is reversible through inverse operations.

Reversibility of the mental process in reasoning can also present in graphing questions by making inferences about $f'$and f'' based on the graph of f and vice versa. As such, a student using reversibility of the mental process in reasoning while solving a graphing question may say something like:

- If f is increasing, then f' is positive, so if f' is positive, then f should be increasing
- If f is concave up, then f'' is positive, so if f'' is positive, then f should be concave up

Both of these statements indicate that the student is using reversibility to consider the graphical effects of information provided by f, f', and/or f''.

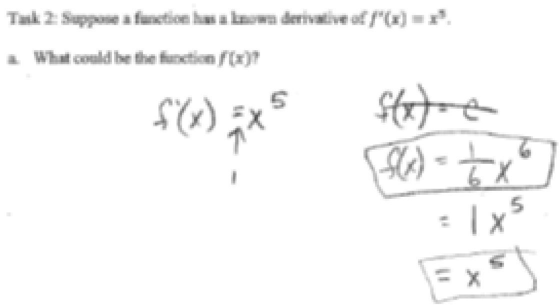
Examples of phrases that indicate representational reversibility

Representational reversibility is present any time that a student translates from one representation to another in the direct question and then reverses the direction of translation in the reverse problem. Examples of phrases that indicate representational reversibility include:

- Forward: "Is there a formula that describes this curve?"
- Reverse: "I am going to sketch of a graph of this function."

The forward statement indicates that the student has been provided a graph and the student is considering how to translate the graph into an algebraic expression. The reverse statement indicates that the student has been given an algebraic expression and recognizes the function as a known graph. Thus, these two statements taken in concert indicate representational reversibility between the graphical and symbolic representations. Further examples of phrases that indicate representational reversibility between other representations are included in Appendix I.

Figure 29, shown below, presents an example of student work on IQ 1.2.



Interview transcript
The function has a derivative of $f'(x) = x^5$ so this is going backwards.
So I just usually start testing things. It's kind of a guess and check for me. I try and think uh 'what would be'... so something would have to make what's right here 1 [Michael points at the coefficient in front of x^5]. So there would be nothing there and then so it would be $\frac{1}{5}x^6$ oh no, yes, no $\frac{1}{6}x^6$
because if you brought down this exponent 6 and multiplied it by 1/6 that would be 1 and then you subtract 1 from this exponent and that would be 5 and that would be x^5.
So then, that's the actual answer [Michael circles $f(x) = \frac{1}{6}x^6$]

Figure 29. Sample student work on IQ 1.2

On IQ 1.2, the student received a grade of “correct” because he correctly found $f(x)$ by reversing the simple power rule. His transcript indicates reversible thought when he says “this is going backwards”. Further, he indicates how he problem solves with reversibility by saying “I try and think uh ‘what would be”. A more detailed analysis of IQ 1.2 and the interview transcript are included in section 4.3.1.

To aid with interview data collection, management, and analysis, I used the computer software Nvivo10. Nvivo10 is a data management tool, noted for improving the efficiency of analyzing data in qualitative studies. All interview data was entered into NVivo10 and analyzed.

3.5 DATA ANALYSIS

In this section, I describe how the data was analyzed in order to answer the research questions. Accordingly this section is organized by research question. Each data source that was used to inform the appropriate research question will be discussed. This study attempted to answer the following research questions:

1) To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?
2) To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:
    i. To what extent does reversibility of two-way reversible processes develop?
    ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?
    iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?
3) What are the thought processes that students utilize when using reversibility to solve problems?

Table 31 indicates which data sources were analyzed to answer each research question.

Table 31. Data sources used to answer research questions

	Flexibility pre-test – whole class data	DCT – whole class data	Exit slips and opening activities – whole class data	Question-based think-aloud interview – 2 cases of 3 different groups
To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?	Yes	Yes	No	No
To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations?	No	No	Yes	Yes
What are the thought processes that students utilize when using reversibility to solve problems?	No	No	Yes	Yes

3.5.1 Research question 1

To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?

I used two data sources to inform the extent to which students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations. The results of the flexibility pre-test and the results of the DCT (a post-test) were compared to determine the extent to which flexibility develops.

The results of the pre-test established base-line data for students' flexibility with multiple representations of functions. The DCT (post-test), administered at the end of the instructional unit, indicated the extent to which students had developed flexibility with multiple representations of derivatives. I used a paired samples t-test to compare for a difference of mean scores on the pre-test and post-test. The paired samples t-test was chosen because the same participants are being tested twice on the same variable (flexibility) with a time lapse between tests. An a priori power analysis was conducted. Using the most conservative estimate of correlation, the minimum sample size necessary to reach the minimum acceptable power level of 0.80 (Huck, 2008) is $n = 19$. With $n = 21$, this research design was well-suited to identify a medium effect size. Using less conservative estimates of correlation (as suggested by previous years' correlation data between flexibility with functions and flexibility with derivatives), the power level increases to 0.9142.

The FPT and DCT post-test were compared to determine if any representational preference existed at the beginning of the study and if representational preference changed during the course of the study. To determine the extent to which students had a representational preference at the start of the study, I averaged the scores across competencies as shown in table 32, below.

Table 32. Class achievement (%) on specific groups of competencies (N = 21) on the flexibility pre-test

Grouped Competencies	Class Achievement (%) N = 21
All Competencies (18 items)	
Input Representation	
Symbolic (_S_) (6 items)	
Graphical (_G_) (6 items)	
Numerical (_N_) (6 items)	
Output Representation	
Symbolic (_S_) (6 items)	
Graphical (_G_) (6 items)	
Numerical (_N_) (6 items)	

Table 32 (continued)

Competencies without-translation (6 items)	
Competencies with-translation (12 items)	
Composition competencies (9 items)	
Inverse competencies (9 items)	

If one input (or output) representation was significantly higher or significantly lower than the other representations, we would be able to conclude that the students likely had a representational preference for or against that particular representation as an input representation (or output representation).

To determine the extent to which representational preference existed at the end of the study, I analyzed the results of the DCT using a table similar to that which was used to analyze the results of the pre-test. A table similar to Table 33 was used to report the relative percentages of correct answers within each representation.

Table 33. Class achievement (%) on specific groups of competencies (N = 21) on the DCT

Grouped Competencies	Class Achievement (%) N = 21
All Competencies (18 items)	
Input Representation	
Symbolic (_S_) (6 items)	
Graphical (_G_) (6 items)	
Numerical (_N_) (6 items)	
Output Representation	
Symbolic (_S_) (6 items)	
Graphical (_G_) (6 items)	
Numerical (_N_) (6 items)	
Competencies without-translation (6 items)	
Competencies with-translation (12 items)	
Formulation competencies (9 items)	
Interpretation competencies (9 items)	

Finally, because the FPT was designed to fully align with the DCT, I was able to run comparisons on a very fine-grained level, allowing me to compare relative improvement of flexibility within each translation. I conducted a paired samples t-test comparing the percentages of each translation (for example, I conducted a paired samples t-test on the percentage of students who successfully translated from symbolic to graphical on the pre-test and the percentage of students who successfully translated from symbolic to graphical on the post-test). If students developed flexibility when engaged in a course that attends to linking multiple representations, one would expect that the overall percentage of successful translation between representations will have improved and that likely most or perhaps all of the percentages of successful individual translations between representations will have improved over the course of the study.

3.5.2 Research question 2

To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:

i. To what extent does reversibility of two-way reversible processes develop?
ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?
iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?

I triangulated four data sources to help inform the extent to which students developed reversibility in a course that attends to linking multiple representations: 1) the FPT data, 2) the DCT data, 3) the exit slips and opening activities data, and 4) the think-aloud interview data.

3.5.2.1 FPT and DCT data

The results of the FPT and DCT were used to help inform the extent to which representational reversibility develops by comparing the relative amounts of representational reversibility present at the start of the study and at the end of the study. After calculating a mean score for representational reversibility for the FPT and the DCT, I used a paired samples t-test to test for a significant improvement in representational reversibility from the pre-test (FPT) to the post-test (DCT).

3.5.2.2 Exit slip and opening activity data

The exit slips and opening activities were analyzed across the entire study and within each kind of reversibility. To analyze the exit slips and opening activities for the presence of reversibility, I created a table that presented each student's results on each exit slip and opening activity pair attempted chronologically, which allowed for the observation of general trends in the presence of reversibility throughout the study.

In order for reversibility to be present, a student must first solve the direct exit slip correctly, and then correctly solve the opening activity at the start of the next class. Thus, the evidence of the existence or lack thereof of reversibility on a daily basis as measured by the exit slips and opening activities is limited by two factors: successfully solving the direct question and attendance. By definition, in order to demonstrate reversibility, a student must first correctly solve a problem assessing the day's learning in a forward direction in order to be eligible to demonstrate reversibility on the reversible task. Thus, for the purposes of analyzing the exit slips and opening activities, I define the term *eligibility* as follows: An exit slip and opening activity pair is defined to be *eligible* to show reversibility (and thus be included in analysis) if the direct exit slip was solved correctly and the opening activity was attempted. An eligible opening activity that was solved correctly received a score of 1. Any eligible opening activity that was solved incorrectly received a score of 0. Any direct exit slip and reverse opening activity pair that did not meet both of these requirements was deemed ineligible and subsequently removed from the data set.

During the study, the class attempted a total of 33 sets of paired problems, which assessed 45 instances of reversibility. The 45 instances of reversibility were each counted as separate paired problems. Thus, the 45 paired problems were administered to 21 students totaling 945 paired problems. On 245 (26%) of the direct exit slips, a student did not solve the forward problem correctly, and was thus ineligible to demonstrate reversibility. The average number of correct and incorrect responses per student on the direct question is reported below in table 34 and grouped by flexibility group.

Table 34. Average number of correct and incorrect responses per student on direct exit slip

Flexibility Group	Average # of Correct Responses	Average # of Incorrect Responses
High	28.375	8.125
Middle	26.571	12
Low	24	16.167

It is important to note that this study examines the presence and development of reversibility and that this study acknowledges that mathematics educators are interested in wondering why there were 245 instances when a student could not solve a direct problem after receiving instruction in a particular calculus topic. While that is no less important a question than investigating the development of reversibility and well worth exploring, this study limits itself to examining the development of reversibility.

If a student was absent on either the day of the exit slip or the day of the opening activity, then s/he was not eligible to demonstrate reversibility on the reversible set of paired problems. A student was absent on either the forward or reverse day 194 (20.5%) times during the study and thus her/his results on that direct reverse pair were removed from the data set. Table 35 reports the average number of times per student that a student's exit slip and opening activity data had to be removed from the data set due to absence, either on the day of the direct task or the day of the reverse task.

Table 35. Average number of paired problems that were deemed ineligible due to absence per student

Flexibility Group	Average # of paired problems per student removed from data set due to absence
High	11
Middle	8.57
Low	7.67

In sum, 439/945 = 46.6% of the paired problems were not eligible to be analyzed for the existence of reversibility. There were 506 paired problems that were analyzed for the existence of reversibility as exit slips and opening activities. The number of paired problems eligible to show the presence or lack of reversibility are displayed by flexibility group in table 36.

Table 36. Total number of eligible paired problems and average number of eligible paired problems per student

Flexibility Group	Total number of reversible paired problems eligible for analysis	Average number of reversible paired problems eligible for analysis per student
High	207	25.875
Middle	172	24.571
Low	127	21.167

Thus, at the student level, there is little or no difference in the opportunity to demonstrate reversibility between the high and middle flexibility group and just slightly less opportunity to demonstrate reversibility for the students in the low group. It is noteworthy that the low group had the lowest incidence of absence. The lower amount of paired problems eligible for analysis in the low group is almost entirely explained by the increased likelihood that a student in the low flexibility group would not be able to solve the direct problem.

To quantify the relative amount of reversibility present on each exit slip and opening activity pair, I used a measure I termed the *reversibility score*. The *reversibility score* is the ratio of eligible opening activities that were correctly solved to the number of eligible opening activities. For example, if 15 exit slips are solved correctly and all 15 of the students were present to attempt the opening activity, then there were 15 eligible opening activities. If 6 of the opening activities were solved correctly, then the reversibility score is $\frac{6}{15} = 40\%$.

To statistically measure the possible improvement in reversibility throughout the study, I divided the course into a first half and second half. The first half consists of exit slip and opening activity pairs 1-23 and the second half consists of exit slip and opening activity pairs 24-45. Exit slip and opening activity pair 12 (ESOA 2.6.2) was removed from analysis because no results were eligible for analysis due to twenty students incorrectly solving the exit slip and one student absent on the day of the exit slip. Thus, 22 exit slip and opening activity pairs are included in the first half of the study and 22 exit slip and opening activity pairs are included in the second half of the study.

In order to statistically analyze the reversibility scores of the exit slips and opening activities with a paired samples t-test, I had to account for the missing data due to ineligibility. I used multiple imputation, a process that estimates missing data, originally proposed by Rubin (1987). Rubin recommended that when 50% of the data was missing, five imputations were sufficient for an accurate estimation (Schafer, 1999); however, Graham, Olchowski, and Gilreath (2007) recommended 40 imputations when 50% of the data is missing. I used SPSS to run 50 multiple imputations. I then ran a paired samples t-test on the pooled data of the 50 imputations to test for a significant improvement in reversibility from the first half to the second half of the study.

I sorted the exit slips and opening activities into three groups, according to the kind of reversibility elicited by the opening activity. There were 14 paired problems that assessed two-way reversibility, 10 paired problems that assessed reversibility of the mental process in reasoning, and 21 paired problems that assessed representational reversibility. The results of the group of 14 two-way reversible process pairs were used to help answer question 2.i. The results of the group of 10 reversibility of the mental process in reasoning without reversible translation pairs were used to offer insight into research question 2.ii. The remaining set of 21 exit slips and opening activities were used to inform research question 2.iii by providing evidence of the class's usage of representational reversibility to solve problems. In each case I examined the table of results for general trends of development and then used a paired samples t-test to compare the class mean reversibility score during the first half of the study and the second half of the study. As described earlier, all statistical analysis of the exit slips and opening activities was conducted on the data set after using multiple imputation to fill in the missing data. The data set from the first half of the course consisted of the results of paired problems 1-23, with 12 removed. The data set from the second half of the course consisted of the results of paired problems 24-45.

3.5.2.3 Interview data

The written responses to the question-based interviews were analyzed for evidence of reversibility using tables 24, 25, 26, and 27. The written responses to the interview questions were parsed by content to identify if the reversibility present was limited to just two-way reversible processes, just reversibility of the mental process in reasoning without reversible translation, just representational reversibility or if it extended to all three aspects of reversibility or a combination of two kinds of reversibility. The data reported in tables 26 and 27 was used to help determine if the students demonstrate representational reversibility.

I then used data triangulation with the results of the independent analyses of the DCT, exit slips and opening activities, and interview data to develop a holistic answer to the research question. Data triangulation is the process of collecting multiple sources of data that corroborate the same phenomenon (Yin, 2009). Data triangulation is noted as a "major strength of case study data collection" (Yin, 2009, p. 114) and greatly increases the construct validity of the conclusions of the study.

To determine if the students developed reversibility of two-way processes, I analyzed the results of the 14 exit slip and opening activity pairs that were designed to assess reversibility of a two-way process. The results of the exit slips and opening activities were analyzed twice, first on a yes/no scale indicating if the result of the paired-problem set demonstrated reversibility and secondly for evidence of students' thought processes while using reversibility. The results of the yes/no analysis were aggregated as class data for evidence of reversibility. In this instance, the exit slip opening activity data provided evidence of the presence and development of reversibility of two-way processes. If reversibility of two-way processes developed throughout the course, we would expect to see an increase throughout the course of study of the percentage of students who demonstrate reversibility of two-way processes.

The think-aloud interview data was also used to draw conclusions regarding the development of reversibility of two-way processes. I analyzed the written solutions and the transcript data from the pairs of interview questions that assessed reversibility of a two way process: 1.1 (interview 1, question 1) and 1.2, 4.1.a and 4.2.a, 4.1.b and 4.2.b, and 4.3.a and 4.3.b.

Each pair of questions was analyzed for evidence of reversible conceptions. As described in section 3.4, the data reported in tables 24 and 25 provided evidence of the existence or non-existence of reversibility as a two-way reversible process and reversibility of the mental process in reasoning. We would expect that if reversibility of two-way processes develops throughout the study, then the number of students using reversibility of two-way processes to solve the interview questions would increase from interview 1 through interview 4. In this way, we would expect that the interview data would be a smaller scale ($n = 6$) representation of the continuum of development of reversibility of two-way processes observed through the exit slips and opening activities ($n = 21$).

To determine if reversibility of the mental process in reasoning without reversible translation developed, I consulted the exit slip opening activity data and the think-aloud interview data. I first analyzed the existence of reversibility on the 10 exit slip opening activity pairs that were designed to elicit reversibility of the mental process in reasoning without reversible translation. The results of the yes/no analysis were aggregated as class data for evidence of reversibility of the mental process in reasoning without reversible translation. If reversibility of the mental process in reasoning without reversible translation developed throughout the course, we would expect to see an increase throughout the course of study of the percentage of students who demonstrate reversibility of the mental process in reasoning without reversible translation.

The think-aloud interview data was also be used to draw conclusions regarding the development of reversibility of the mental process in reasoning without reversible translation. I analyzed the written solutions and the transcript data from the pairs of interview questions that assess reversibility of the mental process in reasoning without reversible translation: 2.1 (interview 2, question 1) and 2.2, 2.3 and 2.4, 3.1 and 3.2, 4.1.a and 4.2.a, 4.1.b and 4.2.b, and 4.3.a and 4.3.b.

Each pair of questions was analyzed for evidence of using reversibility of the mental process in reasoning without reversible translation to solve the problems. It is expected that the patterns of development of reversibility of the mental process in reasoning without reversible translation observed at the class level through the exit slips and opening activities will be similarly observed in the think-aloud interviews. I expect that if reversibility of the mental process in reasoning without reversible translation does indeed develop during the course of instruction, then the data should indicate that the percentage of students using reversibility of the mental process in reasoning without reversible translation will increase throughout the course of study.

To attempt to answer research sub-question 2.iii: does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop, I used data from the flexibility pre-test, the differentiation competency test, exit slip and opening activity data, and the think-aloud interview data. The FPT and the DCT provided additional data points that informed the extent to which representational reversibility develops in a course that attends to linking multiple representations. The FPT and the DCT show evidence of representational reversibility when the class performs well on two reversible translations (such as graphical to symbolic and symbolic to graphical). For example, if the class has a DCT group score of 88% on the FSg competency and 90% on the FGs competency, then we can likely suggest that the students have representational reversibility between the symbolic and graphical representations of derivatives. When we compare the results of the representational reversibility items on the flexibility pre-test and the DCT, we expect to see higher scores on the DCT items than on the flexibility pre-test, if representational reversibility develops when students are engaged in a course that attends to linking multiple representations.

The exit slip opening activity data and the think-aloud interview data was also used to draw conclusions regarding the extent to which representational reversibility develops during the course of study. I first analyzed the existence of reversibility on the 21 exit slip opening activity pairs that are designed to elicit representational reversibility. The results of the yes/no analysis was aggregated as class data for evidence of representational reversibility. If representational reversibility developed throughout the course, we would expect to see an increase throughout the course of study of the percentage of students who demonstrate representational reversibility on the exit slips and opening activities.

The think-aloud interview data was used to draw conclusions regarding the development of representational reversibility. I analyzed the written solutions and the transcript data from the pairs of interview questions that assessed representational reversibility: 1.3 (interview 1, question 3), 2.3 and 2.4, and 3.1 and 3.2. Each pair of questions was analyzed for evidence of using representational reversibility to solve the problems.

It is expected that the patterns of development of representational reversibility observed at the class level through the exit slips and opening activities will be similarly observed in the think-aloud interviews. I expect that if representational reversibility did indeed develop during the course of instruction, then the data should indicate that the percentage of students using representational reversibility will increase throughout the course of study.

Finally, I attempted to answer research question 2: to what extent do students develop reversibility when engaged in a course that attends to linking multiple representations, by triangulating the conclusions drawn when answering the three sub-questions above described. Table 37 reports possible outcomes to answering research sub-questions 2.i, 2.ii, and 2.iii.

Table 37. Possible combinations of outcomes answering research question 2

	R.Q. 2.i	*R.Q. 2.ii*	*R.Q. 2.iii*	*R.Q. 2*
Did reversibility develop?	Y	Y	Y	Strong evidence that reversibility does develop when students are engaged in a course that attends to linking multiple representations
	Y	Y	N	Evidence suggests that reversibility likely developed
	Y	N	Y	Evidence suggests that reversibility likely developed
	N	Y	Y	Evidence suggests that reversibility likely developed
	Y	N	N	Evidence suggests that reversibility may have developed in a limited amount and in a limited domain
	N	Y	N	Evidence suggests that reversibility may have developed in a limited amount and in a limited domain
	N	N	Y	Evidence suggests that reversibility may have developed in a limited amount and in a limited domain
	N	N	N	Evidence suggests that reversibility may not develop when students are engaged in a course that attends to linking multiple representations

3.5.3 Research question 3

What are the thought processes that students utilize when using reversibility to solve problems? The exit slip and opening activity data and the think-aloud interview transcript data were used to answer research question 3.

The exit slips and opening activities were specifically designed to elicit evidence of students' thinking while solving the questions. Each exit slip and opening activity contains a directive asking students to either explain how s/he solved the problem or to show work indicating how the student solved the problem. As described earlier, each exit slip and opening activity were analyzed twice. They were analyzed once for accuracy and then a second time for evidence of students' thought processes while using reversibility. The exit slips were analyzed for evidence that students used reversibility of two-way processes, reversibility of the mental process in reasoning without reversible translation, and representational reversibility. To find this evidence I looked for written evidence that a student was thinking reversibly, such as describing her/his work by saying, "I tried to think of a function that would have the given derivative" and for evidence of translations of representations.

The think-aloud interviews were video recorded and then transcribed. First, each interview was independently analyzed for thematic elements indicating consideration of reversible relationships. Possible verbal cues that may indicate a reversible thought process were listed in section 3.4.4. Each participant's four interview story was analyzed in sequence for thematic evidence of how the participant thinks about solving problems that require reversibility. Examples of possible themes that may present when analyzing the interview transcript data could include:

- How students use representational translations to support reversible thinking
- Do students consider changing representations to work around a stopping point, such as lacking a known procedure
- Evidence of representational preference when translating
- Evidence indicating different thought processes for different kinds of reversibility
    - For example, do participants follow similar thought processes when solving two-way reversible questions as they do when solving

reversible questions that require a reversing of the mental process in reasoning

    - Do reversible thought processes differ when translation is involved

- What features of functions do participants perceive as constant versus mutable when translating representations
    - If the translation is reversible, do participants' think about the translation differently than if the translation is a one-way translation

Commonalities and dissimilarities between questions were noted and categorized for possible patterns at the individual level. This included analyzing the text for evidence of how participants' thoughts about reversibility and translating between representations changed from the first interview through the last interview.

Same session interview analyses at the individual level were then compared for similarities and differences within groups (high flexibility, average flexibility, and low flexibility) to identify possible themes common across groups and more likely, significant differences across groups (Yin, 2009). It was expected that this portion of the analysis would reveal differences in how students with high flexibility, average flexibility, and low flexibility conceive of reversibility. I expected to find evidence of how participants think about reversibility to be unique at each group level, consistent with Krutetskii's (1976) conclusion that capable, average, and incapable students develop reversibility differently.

The DCT data was also used to help explain any difficulties that the interview participants had when trying to solve the interview questions. If the participants could not solve problems that required representational reversibility, there were two possible causes: 1) the participants were not able to translate representations to solve the problems, or 2) the participants were able to translate representations but were not able to apply calculus correctly. In order to isolate why students were not able to solve problems that required representational reversibility, I consulted the interview participants' DCT scores. If the DCT results indicated that the participant had a strong understanding of multiple representations of derivatives, then it is likely that the calculus content was the source of the difficulty. If the DCT results indicated that the participant had a low ability to translate between multiple representations of derivatives, then the lack of representational reversibility was likely responsible in part for the inability to solve questions that required representational reversibility.

The interview transcripts also helped to explain possible causes for why the participants were not able to solve questions that required representational reversibility. The transcripts were analyzed for verbal evidence of the participants having considered the need to translate representations. If there was evidence that the participant considered translating the representation, then the transcript was further analyzed to determine how the participant thought about translating the representation. I attempted to tease out from the interview data what it is about the representational reversibility that was difficult for the participant. To do so, I looked for verbal cues indicating difficulty with a representation such as:

- "I hate graphs"
- "Do we have a formula"
- "If there's no formula, then it's not a function"
- "What does the table mean"

If any verbal cues were found, I looked for patterns and/or dissimilarities across interview groups and within interview groups to attempt to identify stumbling blocks in regards to representational reversibility.

3.6 VALIDITY AND RELIABILITY

As described earlier, the assessment items used in this study have either been previously validated by other researchers (the DCT by Kendal and Stacey (2003)) or have been reviewed by multiple mathematics educators to ensure face and construct validity. The flexibility pre-test was analyzed for internal reliability and was found to have excellent internal reliability as a measure of flexibility with a Cronbach's alpha level of 0.929.

To protect the construct validity of the conclusion of the study, Yin (2009) recommended using multiple sources of evidence. In this study, data triangulation looked for convergence among four independent data sources, the FPT, the DCT, exit slips and opening activities, and think-aloud interviews to reach conclusions.

External validity requires "defining the domain to which a study's findings can be generalized" (Yin, 2009, p. 40). To address external validity in a multiple-case study, Yin recommended using replication logic. This study addresses external validity by selecting two participants in each interviewing group. By interviewing two students in each group, replication of findings allows for stronger claims and protects the external validity of the study. The total number of participants in this study, six, is consistent with the number of participants used in reversibility research. Teachey (2003) interviewed ten participants. Rider (2004) interviewed eight participants. Haciomeroglu (2007) interviewed three participants and Rachlin (1981) interviewed four participants.

Reliability is "demonstrating that the operations of a study – such as the data collection procedures – can be repeated, with the same results" (Yin, 2009, p. 40). To ensure reliability, Yin suggested using a case study protocol and developing a case study database. The purpose of the case study protocol and database is to document every step of the data collection and analysis so that another researcher could repeat the same investigation. I maintained a log of all data collection and analysis. A second rater double coded 10% $(n = 67)$ of the instructional items for representation. We had a 96% rate of agreement and the items on which we disagreed were discussed and resolved. A second rater double coded 33.3% $(n = 2)$ of the interview data. We had 100% agreement on coding the interview data. A second rater also double coded 28.6% $(n = 198)$ of the exit slips and opening activities. We had 100% agreement. A minimum inter-rater reliability threshold of 85% will be required throughout the analysis phase of the study.

3.7 SUMMARY

This investigation used a multiple-case study method to investigate the development of reversibility in a calculus class that attends to linking multiple representations. Four data sources were used to inform the conclusions of this study: 1) a flexibility pre-test, 2) a differentiation competency test, 3) 33 day-to-day activities consisting of exit slips and opening activities, and 4) think-aloud interviews.

The flexibility pre-test was administered before the start of the instructional part of the study to establish students' existing levels of flexibility with multiple representations and to provide initial evidence of students' existing levels of representational reversibility. The exit slips and opening activities were administered 33 times during the study and were analyzed for evidence of development of reversibility of two way processes, reversibility of the mental process in reasoning without reversible translation, and representational reversibility. Four think-aloud interviews with 6 participants were conducted and the results of the interviews were analyzed for evidence of development of reversibility of two way processes, reversibility of the mental process in reasoning without reversible translation, and representational reversibility. The research portion of the study concluded by administering Kendal and Stacey's (2003) differentiation competency test. The DCT provided evidence of the extent to which flexibility and representational reversibility have developed since the students attempted the flexibility pre-test.

The data management software program NVivo10 was used as a tool to manage and analyze the interview data. The interview transcripts were searched for phrases that indicated reversible conceptions. Themes common within groups and common across groups were used to attempt to build an explanation for the development of reversibility. Themes extremely divergent across groups were also considered.

Data triangulation from the flexibility pre-test, the DCT, exit slips and opening activities, and think-aloud interviews was used to address the following research questions:

1) To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?
2) To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:
    i. To what extent does reversibility of two-way reversible processes develop?
    ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?
    iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?
3) What are the thought processes that students utilize when using reversibility to solve problems?

4.0 RESULTS

The purpose of this study is to investigate reversibility and linking multiple representations in a calculus environment. Specifically, this study attempts to answer the following research questions:

1) To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations?
2) To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:
    i. To what extent does reversibility of two-way reversible processes develop?
    ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?
    iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?
3) What are the thought processes that students utilize when using reversibility to solve problems?

In chapter 4, I examine the results of the research study in relation to these questions. I discuss each question in turn and use the results of the appropriate instruments to reach conclusions regarding each question. I begin by discussing the first research question.

4.1 RESEARCH QUESTION 1

To what extent do students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations? Flexibility with multiple representations is "the ability to change from one perspective to another perspective (e.g., change from an algebraic perspective to a graphical perspective)" (Teachey, 2003, p. 6).

I used two data sources to inform the extent to which students develop flexibility with multiple representations when engaged in a course that attends to linking multiple representations. The results of the flexibility pre-test (FPT) and the results of the Differentiation Competency Test (DCT) were compared to determine the extent to which flexibility develops. Throughout the remainder of the document, I refer to the FPT as "pre-test" and the DCT as "post-test".

The results of the pre-test and the post-test, when taken together, suggest that flexibility with multiple representations has developed in a significant amount over the course of the study. In the subsequent sections, I examine the development of flexibility and representational preference.

4.1.1 Development of flexibility

In this section, I first examine the development of overall flexibility by reporting the class results on the pre-test and the post-test. I then focus my analysis on the individual translations and report the results of the pre-test and post-test at a more fine-grained level. The implications of the results in reference to the research question are discussed.

It is important to restate an earlier caveat, that flexibility is likely content specific (Krutetskii, 1976; Rachlin, 1981; Teachey, 2003). The pre-test assesses flexibility within the broad domain of functions and within the specific domains of composition of functions and inverses with 9 items testing each specific domain. The post-test assesses flexibility within the broad domain of derivatives and within the specific domains of formulating derivatives and interpreting derivatives with 9 items testing each specific domain.

I conducted a paired samples t-test to test for significant differences between the specific contents within the pre-test and the post-test. On both assessments, the class was significantly better at one content than the other. Table 38 reports the results of the paired samples t-test, where C represents composition, *Inv* represents inverse, F represents formulation, and *Int* represents interpretation.

Table 38. Paired samples t-test of content area differences on pre-test and post-test

Assessment	Content Area Mean (9 items)	Content Area Mean (9 Items)	Paired Difference t-value	Paired Difference p-value
Pre-test	$C = 5.4762$	$Inv = 3.4286$	4.580	$< 0.001^{*}$
Post-test	$F = 4.5714$	$Int = 5.5119$	-3.185	0.005^{*}

Thus, at the start of the study, the students were significantly more likely to demonstrate flexibility when presented a question involving compositions of functions than inverses of functions. At the end of the study, the students were significantly more likely to demonstrate flexibility when asked to interpret a derivative than when asked to formulate a derivative. The influence of the content on the ability to gauge the development of flexibility serves as a warning against drawing strong conclusions at the item level. It is not feasible to tease out the effects of flexibility versus content knowledge at the individual item level, thus making a conclusion about either flexibility or content knowledge invalid.

Previous flexibility researchers (Krutetskii, 1976; Rachlin, 1981; Rider, 2004) have shown that interpreting the results of an assessment that assesses flexibility across a mathematical domain allows for the drawing of valid conclusions regarding the existence of flexibility. Thus, while flexibility and content knowledge cannot be teased apart at the item level, they can be reliably teased apart at the assessment level and translational level, provided that flexibility is assessed across more than one content. Thus, at the test level (18 items) and at the translational level (2 items of different content), the pre-test and post-test both serve as reliable measures of flexibility (as discussed in chapter 3) and allow for the drawing of valid conclusions regarding the development of flexibility.

4.1.1.1 Development of flexibility – a broad view

Table 39 reports the group mean and standard deviation of the pre-test, post-test, and the difference between them.

Table 39. Group mean and standard deviation of the pre-test and post-test

Assessment	Mean	N	Standard Deviation
Pre-test	8.9048	21	3.30764
Post-test	10.0833	21	3.06628
Post-test – pre-test	1.17857	21	3.00431

A paired samples t-test was conducted to test for significance of the difference in class mean scores on the pre-test and post-test. The paired samples t-test indicated that the class mean score improvement of 1.1785 is a significant improvement with $t = 1.798, p = 0.0435$. I found a strong positive correlation between pre-test score and post-test score with $r = 0.558$. This result suggests that a student's pre-existing flexibility exhibited an influence on the amount of flexibility that a student showed at the end of the course and that 31% of the variability in a student's post-test score can be explained by the student's pre-test score. Furthermore, the strong correlation provides supporting evidence that the two assessments assessed the same concept, in this case flexibility.

I also examined the development of flexibility within each flexibility group. Table 40 reports the group mean results on the pre-test and post-test for the high, middle, and low flexibility groups and reports the results of a paired-samples t-test for significance of the difference in group mean scores on the pre-test and post-test.

Table 40. Mean flexibility score on pre-test and post-test reported by flexibility group

Flexibility Group	Pre-test Score	Post-test Score	Mean Difference (post-test – pre-test)	t-value, p-value
High	12.44	12.5	0.06	$t = 0.061, p = 0.477$
Middle	8.286	9.071	0.786	$t = 0.693, p = 0.257$
Low	4.917	8.042	3.125	$t = 2.915, p = 0.017^{*}$

Only the students in the low flexibility group showed a significant improvement in flexibility over the course of the study.

4.1.1.2 Development of flexibility at the translational level

Flexibility at the translational level refers to a student's ability to translate from a particular input representation to a different output representation. For example, one possible translation is graphical-numerical, meaning that a student was given a graph of a function and produced a numerical solution. I conducted a paired samples t-test comparing the mean score of each translation to test for significant differences in class performance on the pre-test and the post-test. Table 41 reports the class means for each translation on the pre-test and post-test. The translations are presented in alphabetical order.

Table 41. Class achievement (%) on specific translations on the pre-test and post-test

Translation (2 items each)	Pre-test Score (%)	Post-test Score (%)	Difference (Post-test – Pre-test)	t-value	p-value
Graphical-Numerical	55.95	35.72	-20.23	1.507	0.148
Graphical-Symbolic	26.19	61.91	35.72	3.807	0.001^{*}
Numerical-Graphical	73.81	38.10	-35.71	3.423	0.003^{*}
Numerical-Symbolic	23.81	45.24	21.43	2.905	0.009^{*}
Symbolic-Graphical	28.57	76.19	47.62	5.423	$< 0.001^{*}$
Symbolic-Numerical	59.53	78.57	19.04	2.609	0.017^{*}

The paired samples t-test revealed significant differences in class mean scores between pre-test score and post-test score in 5 out of 6 translations. There was no significant difference in pre-test class mean score and post-test class mean score when translating from the graphical to the numerical representation. There were significant improvements from the class mean score on the pre-test to the class mean score on the post-test on problems that required translating from graphical to symbolic, numerical to symbolic, symbolic to graphical, and symbolic to numerical. There was a significant decrease in the class mean score on the pre-test to the class mean score on the post-test on problems that required translation from numerical to graphical.

The calculus class showed improvement on four out of six translations, indicating a trend towards the development of flexibility. However, the two translations that did not show significant improvement, namely the graphical to numerical translation and the numerical to graphical translation, should be examined further to consider why the students' results on these questions differ from the overall picture painted by the data.

The calculus class' mean score on the pre-test on items that required translation from the graphical representation to the numerical representation was $1.119/2 = 55.95\%$. The class mean score on the post-test on items that required translation from the graphical representation to the numerical representation was $0.7144/2 = 35.72\%$, which while lower was not significantly different ($p = 0.148$) than the class mean score on the pre-test on items that required translation from the graphical representation to the numerical representation. In this instance, the decrease in class mean score may entirely be due to chance.

The decline in class mean score on items assessing translation from the numerical representation to the graphical representation cannot be attributed to chance as there is only a 3 out of 1,000 chance that the observed decline is a random occurrence. The role of the content differences between functions and derivatives may explain the significant decline. The pre-test assessed functions and the post-test assessed derivatives. I provide examples of the different contents below.

When translating functions from the numerical to the graphical representation, students typically use some form of plotting points (Janvier, 1987b). Question 2.c from the pre-test requires the plotting of points to sketch a graph of a composition function given a table of values. Question 2.c is displayed below in figure 30.

2) The table below reports selected values of two functions, f and g. Let $h(x) = f(g(x))$. The domain for all functions is all real numbers.

x	f	g	h
-3	9	-2	
-2	4	-1	
-1	1	0	
0	0	1	
1	1	2	
2	4	3	
3	9	4	

c) Make a possible graph of $h(x)$. Label the tick marks on the y –axis to indicate the scale that you use.

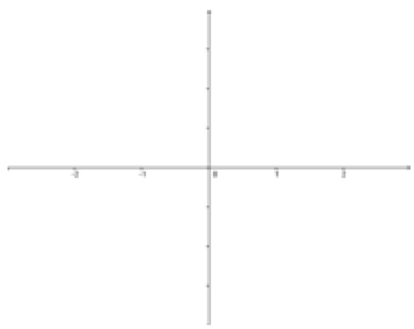


Figure 30. Pre-test question 2.c

In the most likely solution method, a student would fill-in the h column by using composition of functions and then plot the points and draw a curve through the plotted points.

Question 9 on the pre-test requires a similar translational skill set within the specific content domain of inversion. Question 9 is displayed below in figure 31.

9. If $f(x)$ is invertible on the entire real line and

x	0	1	2	3	4
$f(x)$	1	2	4	8	16

Sketch a possible graph of $f^{-1}(x)$.

Figure 31. Pre-test question 9

In question 9, the students have to first know that $f^{-1}(x)$ requires switching the domain and range of $f(x)$ and then they should plot the points and sketch a curve through the points; alternatively, a student could first sketch $f(x)$ and then reflect the graph of $f(x)$ through the line $y = x$ and produce a correct graph of $f^{-1}(x)$ as well. Both solutions require the act of plotting points displayed in a table as the translational aspect of the problem. As $31/42 = 73.81\%$ of the answers given to these two problems were correct, the results of the pre-test suggest that the calculus class had a strong grasp of plotting points (translating a function from a numerical representation to a graphical representation) and had a strong understanding of composing or inverting functions defined numerically.

The post-test assessed a student's ability to translate between multiple representations of derivatives. Translating a derivative from a numerical representation to a graphical representation requires a different set of skills and cognitive understandings than translating a function. As described earlier, the numerical representation of the derivative refers to understanding the derivative as the rate of change, as the end result of the limiting process applied to the difference quotient that represents the average rate of change. The graphical representation of the derivative incorporates the three understandings of 1) making inferences about functional behavior based on information provided by the derivative, 2) understanding that differentiation produces a new function whose functional values are determined by the slope of the line tangent to the curve $f(x)$ at each x-value, and 3) the ability to sketch both a function curve given a derivative curve and a derivative curve given a function curve. Thus, when translating from the numerical representation of a derivative to the graphical representation of a derivative, a student may be given an average rate of change and asked to determine a functional behavior, or the slope of the line tangent to the curve at an appropriate x-value, or to make a sketch of an appropriate function.

Post-test question 11 requires students to use a list of functional values approaching $x = 3$ (the numerical representation of the derivative) to determine the slope of the line tangent to the curve (the graphical representation of the derivative). Question 11 is displayed below in figure 32.

11) The CAS calculator was used to find values of the function $y = f(x)$ near $x = 3$.

$$(3.000, 0.000); (3.103, -0.701); (3.051, -0.353);$$
$$(3.011, -0.079); (2.990, 0.071); (2.999, 0.007)$$

Find the best estimate of the slope of the line tangent to the graph of $y = f(x)$ at $x = 3$.

Figure 32. Post-test question 11.

Reprinted from "Tracing learning of three representations with the differentiation competency framework", by M. Kendal & K. Stacey, 2003, Mathematics Education Research Journal, 15, p. 33. Copyright 2003 by Mathematics Education Research Group of Australasia. Reprinted with permission.

While it is the case that any two of the given points can be used to find an estimate of the slope of the line tangent to the graph of $y = f(x)$ at $x = 3$, *the best estimate* requires finding the slope of the secant line through the points $(3.000,0.000)$ and $(2.999,0.007)$, which results in the correct answer of $f'(3) \approx -7$. Finding the slope of the line through any of the other points would demonstrate knowledge of the translation from the numerical representation to the graphical representation but would not demonstrate an understanding of the limiting process of differentiation and thus not correctly answer the question by finding the *best estimate*. As such, I reviewed each student's work on post-test question 11 and found that $2/21$ students answered correctly, $5/21$ students translated correctly but used the incorrect points, and $14/21$ did not make a credible attempt at solving the problem, indicating that fully two-thirds of the class did not have a conception of how to translate from the numerical representation of the derivative to the graphical representation of the derivative. Thus, $7/21$ students demonstrated translational proficiency from the numerical representation to the graphical representation of the derivative.

Post-test question 17 also requires translating from the numerical representation of the derivative to the graphical representation of the derivative but has none of the ambiguities discussed regarding post-test question 11. Question 17 is presented below in figure 33.

17) A curve has the function rule $y = f(x)$. If the rate of change of y with respect to x is given by the function rule $5x + 7$, what is the derivative function of the curve?

Figure 33. Post-test question 17.

Reprinted from "Tracing learning of three representations with the differentiation competency framework", by M. Kendal & K. Stacey, 2003, Mathematics Education Research Journal, 15, p. 33. Copyright 2003 by Mathematics Education Research Group of Australasia. Reprinted with permission.

Question 17 should be solved by interpreting the phrase "rate of change of y" as the numerical representation of the derivative and making the connection that since the rate of change of y is defined by the function $5x + 7$ then the derivative function is also $5x + 7$. 14 out of 21 student correctly answered this question. When combining the results of the two questions while looking only at translation, the class total score was $19/42 = 45.24\%$, which still represented a significant decrease from the class performance on Ng questions on the pre-test, $t = 2.351, p = 0.029$.

4.1.2 Representational preference

Representational preference refers to a student's preference for one mathematical representation over another, such as a student's preference to analyze all functions in a symbolic form instead of the graphical and/or numerical representations. The pre-test and post-test were compared to determine if any representational preference existed at the beginning of the study and if representational preference changed during the course of study. To determine the extent to which students had a representational preference at the start of the study and at the end of the study, I averaged the scores across competencies as shown in table 42, below, consistent with the approach taken by Kendal and Stacey (2003).

Table 42. Class mean score on specific representations on the pre-test and post-test

Representation	Class Mean: Pre-test	Class Mean: Post-test	Difference (Post – Pre)	t-value	p-value
Input Representation					
Symbolic (_S_) (6 items)	2.905	4.095	1.190	4.365	$< 0.001^{*}$
Graphical (_G_) (6 items)	2.571	3.0833	0.512	1.432	0.168
Numerical (_N_) (6 items)	3.429	2.905	-0.524	-1.221	0.236
Output Representation					
Symbolic (_S_) (6 items)	2.143	3.143	1.000	3.981	0.001^{*}
Graphical (_G_) (6 items)	2.976	3.417	0.440	1.375	0.184
Numerical (_N_) (6 items)	3.786	3.524	-0.262	0.610	0.584

4.1.2.1 Input preference

A paired samples t-test indicated that at the start of the study (on the pre-test) there were no significant differences between the class mean scores of the symbolic and the graphical input representations, the symbolic and the numerical representations, and the graphical and numerical representations. However, the class mean score on the numerical input representation is significantly higher than the class mean score on the symbolic representation with $t = 2.257, p = 0.035$, indicating that the class as a whole did not have an input preference between the symbolic or numerical function representations but did prefer the numerical representation to the graphical representation.

A paired samples t-test was used to compare the class mean scores of each input representation on the post-test. The paired samples t-tests indicated that on the post-test, the students had a considerable preference for the symbolic representation as an input. The class mean score on post-test items whose input representation was symbolic was significantly higher than both the class mean score on post-test items whose input representation was graphical, $t = 4.104, p = 0.001$, and on post-test items whose input representation was numerical, $t = 6.367, p < 0.001$. There was no significant difference between class mean scores on post-test items whose input representations were numerical and graphical.

It is noteworthy that at the start of the study, the students exhibited a preference for the numerical representation over the graphical representation but at the end of the study, the students showed no preference between the graphical representation and the numerical representation. The students' preference for items given in the symbolic representation changed from being equal to the students' preference for either the numerical or graphical representation to being significantly greater than the students' preference for either the numerical or the graphical representation.

One possible explanation for the distinct changes in representational input preference is the interplay between content specificity and curricular design. The body of research suggests that flexibility is content specific (Krutetskii, 1976; Rachlin, 1981; Teachey, 2003) and it may be the case that these students preferred to be given a numerical representation when the problem required analysis of a function while they preferred to be given the symbolic representation when a problem required differentiation. Thus, the representational preference may be due in part to the content of the problem and not entirely due to a preference regarding the representation. The students' input preference for the symbolic representation of the derivative may be explained by the relative dominance of the symbolic input representation during the study. As noted earlier, 395 items during the course of the study used a symbolic representation for the input of a problem that required differentiation. This accounted for 60.5% of the total number of problems attempted.[3] It is reasonable to think that the students may have developed a certain comfort level with the symbolic representation when using derivatives before being fully exposed to solving derivative problems given in the numerical and graphical representations.

4.1.2.2 Output preference

As shown in table 42 above, the results of the pre-test indicate a strong representational output preference for the numerical representation. The class mean score for the numerical output representation was significantly larger than both the graphical output representation ($t = 2.615, p = 0.017$) and the symbolic output representation ($t = 5.101, p < 0.001$). The class mean score for the graphical output representation was also significantly larger than the class mean score for the symbolic output representation ($t = 2.764, p = 0.012$). These results indicate that at the beginning of the study, the calculus class had a clear order of representational output preference of numerical, then graphical, and then symbolic. This contrasts the post-test data which showed no significant differences between mean class scores on post-test items that required translation into symbolic, graphical, or numerical output representations.

The interplay between content specificity and curricular design may also explain the change in representational output preference. It appears to be the case that when students are analyzing functions, producing a numeric value is preferable to sketching a graph which is preferable to developing an algebraic expression. However, when solving a problem that uses differentiation, the students were equally likely to produce correct solutions in the symbolic,

[3] As explained earlier, the large number of symbolic input problems is due to the nature of learning the rules of differentiation in chapter 2 of the course curriculum.

graphical, and numerical representations of derivatives. The students' equality of representational output preference on problems involving differentiation is likely due in part to participating in a course that attends to linking multiple representations.

Thus, at the start of the study, there is strong evidence to suggest that the class of calculus students had a representational preference for working in the numerical representation, both in input and output. The students did not indicate a significant preference between the symbolic and graphical input representations but clearly preferred to translate into the graphical representation over the symbolic representation. At the conclusion of the study, the students' input preference had changed from the numerical representation to the symbolic representation and the students had no representational output preference.

Finally, it is also noteworthy that the students' performance on problems involving the symbolic representation increased significantly over the course of the study, both as an input representation and as an output representation. This suggests that over the course of the study, the students developed significant flexibility with the symbolic representation. As discussed earlier, the prevalence of items involving the symbolic representation (60.5%) as an input during the study likely explains a significant part of the increase in scores on problems with a symbolic input. The increase in scores on problems with the symbolic representation as an output occurred without a strong prevalence for the symbolic output representation (36%), indicating that flexibility with the symbolic representation may have improved while flexibility with the remaining representations remained the same.

4.1.3 Summary – the extent to which flexibility developed

Overall flexibility significantly improved over the course of the study. In particular, the students' proficiency level with individual translations significantly improved in four out of the six possible translations, symbolic to graphical, symbolic to numerical, graphical to symbolic, and numerical to symbolic, indicating a general trend towards improvement. There was a significant decrease in the numerical to graphical translation, indicating that the students had a more fully developed understanding of the numerical and graphical representations of functions and how to translate between them than they did the numerical and graphical representations of derivatives. The students did not develop flexibility between the numerical and graphical representations of the derivative in an amount similar to the other translations, which may have been due in part to the curriculum's relative lack of emphasis on the numerical representation of the derivative.

4.2 RESEARCH QUESTION 2

To what extent do students develop reversibility when engaged in a course that attends to linking multiple representations? In particular:

i. To what extent does reversibility of two-way reversible processes develop?

ii. To what extent does reversibility of the mental process in reasoning without reversible translation develop?

iii. To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop?

I triangulated four data sources to help inform the extent to which students develop reversibility in a course that attends to linking multiple representations: 1) the pre-test data, 2) the post-test data, 3) the exit slips and opening activities data, and 4) the think-aloud interview data. In this section, I first present the exit slip and opening activity data from the entire study. I then attempt to answer each sub-question individually and then use the answers to all three sub-questions to inform research question 2. In each case, I attempt to answer the question and then provide the evidence that supports the answer.

4.2.1 Results of exit slips and opening activities – overall reversibility

Over the course of the study, 33 exit slip and opening activity pairs were collected that assessed 45 separate instances of reversibility. Figure 34, shown below, presents the results of all of the exit slip and opening activity pairs, administered chronologically throughout the study.

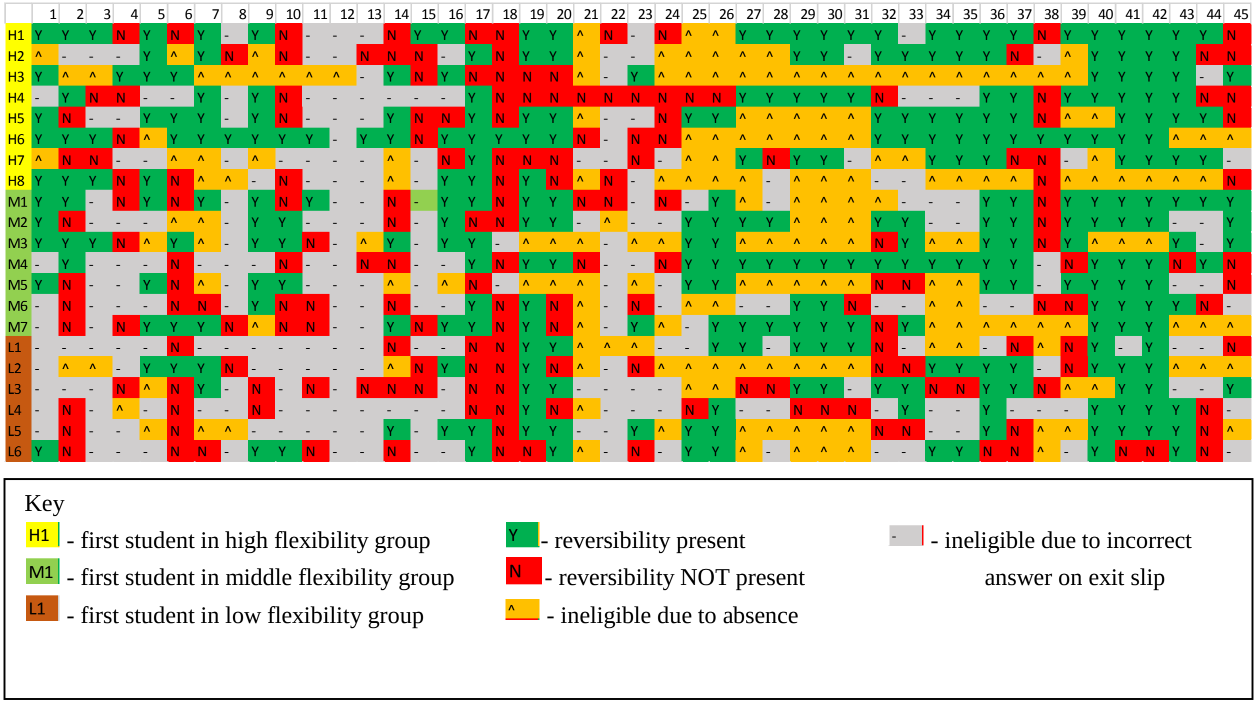


Figure 34. Results of the reversible pairs of exit slips and opening activities

Figure 34 shows a trend towards an increase in demonstrated reversibility from the first half (paired problems 1-23) of the study to the second half of the study (paired problems 24-45). The trend towards improvement can be seen in the increase in density of the green blocks and decrease in density of the red blocks over the course of the study. Table 43, shown below, reports the results of the presence or lack of reversibility on the paired exit slips and opening activities during the first half of the study and the second half of the study.

Table 43. Results of exit slips and opening activities

Outcome	First half: Paired problems 1-23	Second half: Paired problems 24-45	Totals
Reversibility Present	125	192	317
Reversibility Not Present	124	65	189
Ineligible: Absence	59	135	194
Ineligible: Incorrect Direct Exit Slip	175	70	245
Totals	483	462	945

Table 43 shows that during the first half of the course 51.5% of the exit slip and opening activity pairs were eligible to show reversibility. On these eligible pairs, reversibility was present 50.2% of the time. Thus, during the first half of the study, the class reversibility score on the exit slips and opening activities was 50.2%. During the second half of the course, 55.6% of the exit slip and opening activity pairs were eligible to show reversibility. On these eligible pairs, the class reversibility score was 74.2%. Multiple imputation was used to fill in the missing data due to ineligibility. Table 44, shown below, reports the class mean reversibility score during the first and second halves of the study and the results of a paired samples t-test for a significant difference.

Table 44. Class mean reversibility score on all exit slips and opening activities from the first half and second half of the study

	First half: Paired problems 1-23	Second half: Paired problems 24-45	Paired difference t-value	Paired difference p-value
Reversibility Score	52.78%	66.59%	$t = 3.548$	$p < 0.001$

The general trend towards development previously shown in figure 34 and the significant improvement in mean reversibility score at the class level suggest that reversibility as a problem solving process may have developed over the course of the study.

4.2.2 Development of two-way reversibility

To what extent does reversibility of two-way reversible processes develop? The data collected in this research study suggests that reversibility of two-way reversible processes has developed over the course of the study. To determine if the students developed reversibility of two-way processes, I analyzed the results of a sub-set of the exit slips and opening activities and a sub-set of the interview questions. I discuss the results of the exit slip and opening activity data and the interview data in turn.

4.2.2.1 Results of exit slips and opening activities – two-way processes

I analyzed the results of the 14 exit slip and opening activity pairs that were designed to assess the development of reversibility of a two-way process. Five of the exit slips and opening activities were administered during chapter 2, four were administered during chapter 3, and five were administered during chapter 4. In total, nine[4] of the two-way reversibility paired problems were administered during the first half of the study and five of the two-way reversibility paired problems were administered during the second half of the study.

Figure 35, shown below, presents the results of all of the two-way reversibility exit slip and opening activity pairs, administered chronologically throughout the study.

---

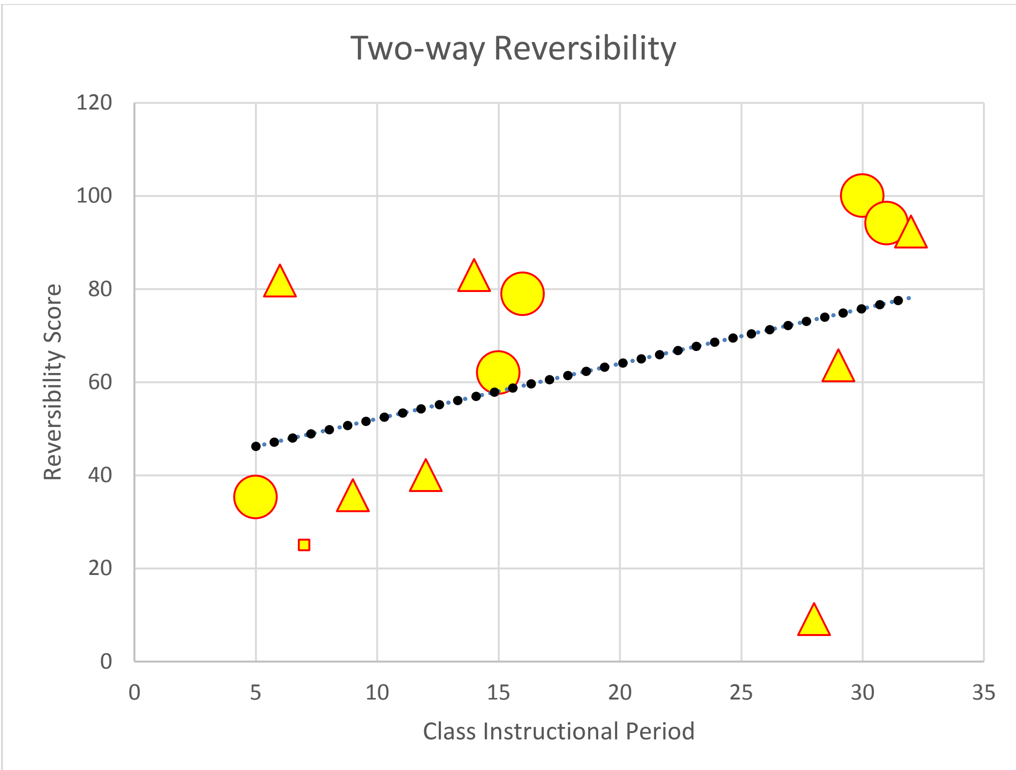
[4] Nine two-way reversibility exit slips and opening activities were administered during the 1st half of the study, however, ESOA 2.6.2 (12) was removed from analysis due to no eligible opening activities.

Day	5	6	7	9	10	12	14	15	16	28	29	30	31	32
Exit Slip #	6	7	8	10	12	14	16	17	19	38	39	40	41	43
Section	2.3.1	2.3.2	2.4.1	2.6.1	2.6.2	3.1.1	3.2.2	3.3.1	3.3.2	4.5.1	4.5.2	4.6.1	4.6.2	4.7.1
H1	N	Y	-	N	-	N	Y	N	Y	N	Y	Y	Y	Y
H2	^	Y	N	N	-	N	-	Y	Y	-	^	Y	Y	Y
H3	Y	^	^	^	^	Y	Y	N	N	^	^	Y	Y	Y
H4	-	Y	-	N	-	-	-	Y	N	N	Y	Y	Y	Y
H5	Y	Y	-	N	-	Y	N	Y	Y	N	^	^	Y	Y
H6	Y	Y	Y	Y	-	Y	Y	Y	Y	Y	Y	Y	Y	^
H7	^	^	-	-	-	^	N	Y	N	N	-	^	Y	Y
H8	N	^	^	N	-	^	Y	Y	Y	N	^	^	^	^
M1	N	Y	-	N	-	N	Y	Y	Y	N	Y	Y	Y	Y
M2	^	^	-	Y	-	N	Y	N	Y	N	Y	Y	Y	-
M3	Y	^	-	Y	-	Y	Y	Y	^	N	Y	^	^	Y
M4	N	-	-	N	-	N	-	Y	Y	-	N	Y	Y	N
M5	N	^	-	Y	-	^	^	N	^	-	Y	Y	Y	-
M6	N	N	-	N	-	N	-	Y	Y	N	N	Y	Y	Y
M7	Y	Y	N	N	-	Y	Y	Y	Y	^	^	Y	Y	^
L1	N	-	-	-	-	N	-	N	Y	^	N	Y	-	-
L2	Y	Y	N	-	-	^	Y	N	Y	-	N	Y	Y	^
L3	N	Y	-	-	-	N	-	N	Y	N	^	^	Y	-
L4	N	-	-	-	-	-	-	N	Y	-	-	Y	Y	Y
L5	N	^	^	-	-	Y	Y	Y	Y	^	^	Y	Y	Y
L6	N	N	-	Y	-	N	-	Y	N	^	-	Y	N	Y
R.S	35.3	81.8	25	35.7	N/A	40	83.3	61.9	78.9	9.09	63.6	100	94.4	92.3

Key		
H1 - first student in high flexibility group	Y - reversibility present	- - ineligible due to incorrect answer on exit slip
M1 - first student in middle flexibility group	N - reversibility NOT present	% - Ratio of Y to (Y + N)
L1 - first student in low flexibility group	^ - ineligible due to absence	R.S. – reversibility score

Figure 35. Results of the two-way reversibility exit slips and opening activities

Figure 35 shows a general trend towards increasing reversibility of two-way processes throughout the study. The trend towards improvement can be seen in the increase in density of the green blocks and decrease in density of the red blocks over the course of the study. Figure 36 displays the reversibility score of each exit slip and opening activity pair that assessed two-way reversibility as a scatter plot over time with a trend line imposed on the data. The data entries are plotted using various sized shapes to represent the amount of eligible exit slips and opening activities that comprise each reversibility score. The key, shown below the graph, describes the value of each shape.



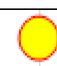
Key	
# of Eligible Exit Slips and Opening Activities	Symbol
0-5	
6-10	
11-15	
16-21	

Figure 36. Scatter plot of the reversibility score on the exit slip and opening activity pairs assessing two-way reversibility

The positive slope of the trend line indicates that there was a general increase in the amount of reversibility of two-way processes over the course of the study and this is corroborated by a strong positive correlation between the class instructional period and the percent of students who demonstrated reversibility with $r = 0.418$.

To further gauge the extent to which reversibility of two-way reversible processes develops when students are engaged in a course that attends to linking multiple representations, I compared the class mean reversibility scores during the first and second half of the course. Table 45, shown below, reports the class mean two-way reversibility score during the first and second halves of the study and the results of a paired samples t-test for a significant difference. The first half consisted of paired problems 6, 7, 8, 10, 14, 16, 17, and 19. The second half consisted of paired problems 38, 39, 40, 41, and 43.

Table 45. Class mean reversibility score on two-way exit slips and opening activities from the first half and second half of the study

	First half:	Second half:	Paired difference t-value	Paired difference p-value
Reversibility Score	53.17%	70.34%	$t = 2.854$	$p = 0.003$

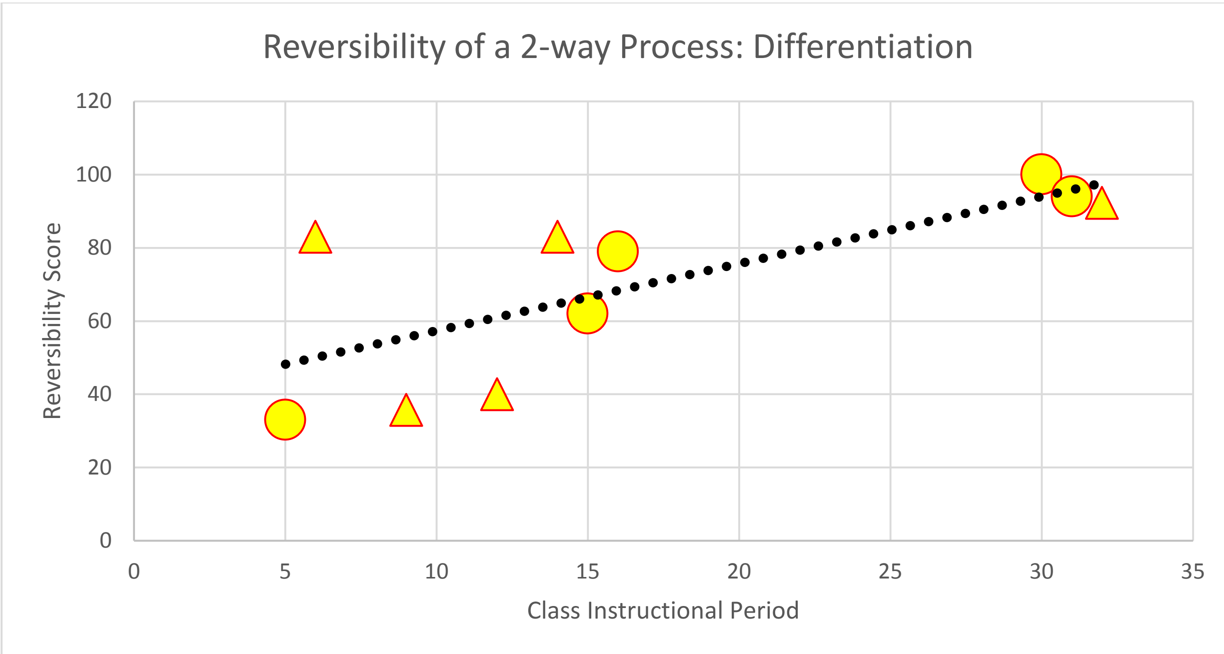
The general trend towards development previously shown in figures 25 and 26 and the significant improvement in mean two-way reversibility score at the class level suggest that two-way reversibility as a problem solving process may have developed over the course of the study.

I further analyzed the two-way reversibility exit slips and opening activities by content area. Ten of the thirteen paired problems assessed a differentiability process in the forward direction and the anti-differentiation process in reverse. When those ten paired problems are analyzed together, a clear pattern of development appears. Figure 37, shown below, presents the results of the two-way reversibility exit slip and opening activity pairs that assessed differentiation and anti-differentiation, administered chronologically throughout the study.

Day	5	6	9	12	14	15	16	30	31	32
Exit Slip #	6	7	10	14	16	17	19	40	41	43
Section	2.3.Y	2.3.2	2.6.Y	3.Y.Y	3.2.2	3.3.Y	3.3.2	4.6.Y	4.6.2	4.7.Y
HY	N	Y	N	N	Y	N	Y	Y	Y	Y
H2	^	Y	N	N	-	Y	Y	Y	Y	Y
H3	Y	^	^	Y	Y	N	N	Y	Y	Y
H4	-	Y	N	-	-	Y	N	Y	Y	Y
H5	Y	Y	N	Y	N	Y	Y	^	Y	Y
H6	Y	Y	Y	Y	Y	Y	Y	Y	Y	^
H7	^	^	-	^	N	Y	N	^	Y	Y
H8	N	^	N	^	Y	Y	Y	^	^	^
MY	N	Y	N	N	Y	Y	Y	Y	Y	Y
M2	^	^	Y	N	Y	N	Y	Y	Y	-
M3	Y	^	Y	Y	Y	Y	^	^	^	Y
M4	N	-	N	N	-	Y	Y	Y	Y	N
M5	N	^	Y	^	^	N	^	Y	Y	-
M6	N	N	N	N	-	Y	Y	Y	Y	Y
M7	Y	Y	N	Y	Y	Y	Y	Y	Y	^
LY	N	-	-	N	-	N	Y	Y	-	-
L2	Y	Y	-	^	Y	N	Y	Y	Y	^
L3	N	Y	-	N	-	N	Y	^	Y	-
L4	N	-	-	-	-	N	Y	Y	Y	Y
L5	N	^	-	Y	Y	Y	Y	Y	Y	Y
L6	N	N	Y	N	-	Y	N	Y	N	Y
R.S.	35.29	81.82	35.71	40	83.33	61.9	78.95	100	94.44	92.31

Figure 37. Results of the two-way reversibility exit slips and opening activities that assessed differentiation and anti-differentiation

Figure 37 shows a general trend towards increasing reversibility of two-way processes throughout the study. The trend towards improvement can be seen in the increase in density of the green blocks and decrease in density of the red blocks over the course of the study. Figure 38 displays the reversibility score of each exit slip and opening activity pair that assessed differentiation in the forward direction and anti-differentiation in the reverse direction as a scatter plot over time with a trend line imposed on the data.



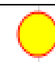
Key	
# of Eligible Exit Slips and Opening Activities	Symbol
0-5	
6-10	
11-15	
16-21	

Figure 38. Scatter plot of the reversibility score on the exit slip and opening activity pairs assessing two-way reversibility of differentiation and anti-differentiation

There is a very strong correlation between the instructional class period and the percent of students who demonstrate reversibility between differentiation and anti-differentiation, with $r = 0.738$. Table 46, shown below, reports the class mean reversibility score on paired problems that assessed reversibility of differentiation and anti-differentiation during the first and second halves of the study and the results of a paired samples t-test for a significant difference. The first half consisted of paired problems 6, 7, 10, 14, 16, 17, 19. The second half consisted of paired problems 40, 41, and 43.

Table 46. Class mean reversibility score on anti-differentiation exit slips and opening activities from the first half and second half of the study

	First half:	Second half:	Paired difference t-value	Paired difference p-value
Reversibility Score	55.9%	88.0%	$t = 4.723$	$p < 0.001$

The general trend towards development previously shown in figure 37, the strong positive correlation between date of instruction and reversibility score as seen in figure 38, and the significant improvement in mean reversibility score at the class level suggest that reversibility of the differentiation and anti-differentiation processes may have developed over the course of the study.

4.2.2.2 Results of interviews – two-way processes

I analyzed the written solutions and the transcript data from the pairs of interview questions that assess reversibility of a two way process: 1.1 (interview 1, question 1) and 1.2, 1.1 and 1.3, 2.1 and 2.2, 4.1.b and 4.2.b, and 4.3.a and 4.3.b. Each pair of questions was analyzed for evidence of two-way reversible conceptions. Table 47 reports the existence or absence of two-way reversibility on the interview questions. “Yes” indicates that the participant correctly solved both the forward and reverse questions; “No” indicates that the participant correctly solved the forward question but could not solve the reverse question. “N/A” indicates that the participant could not solve the forward question, thus rendering the question about the existence of reversibility moot.

Table 47. Is two-way reversibility present in the interview questions?

Flexibility Group	Participant	1.1 & 1.2	1.1 & 1.3	2.1 & 2.2	4.1.b & 4.2.b	4.3.a & 4.3.b
High	Kelsay	Yes	Yes	Yes	Yes	Yes
	Michael	Yes	Yes	Yes	Yes	Yes
Middle	Fred	Yes	Yes	Yes	Yes	Yes
	Jill	Yes	Yes	Yes	Yes	Yes
Low	Kirsten	No	No	N/A	No	N/A
	Marcus	No	No	Yes	Yes	N/A

The interview questions 1.2, 1.3, 4.2.b, and 4.3.b all require reversibility of the simple power rule in various instantiations. Interview questions 2.1 and 2.2 are a set of paired problems that require reversibility of the chain rule. Embedded within the chain rule in these questions is two-way reversibility of the derivative of $f(x) = \cos x$. Thus, all of the two-way reversibility interview items assessed reversibility of differentiation and anti-differentiation.

The interview questions 1.2, 1.3, 4.2.b, and 4.3.b are conceptually similar to the opening activities administered on days 5, 6, 30, 31, and 32. The interview participants' reversibility scores on these opening activities are presented in table 48 below (Y means that the student used reversibility to correctly solve the question, N means that the student did not use reversibility to correctly solve the question, A means that the student was absent and did not attempt the question).

Table 48. Results of two-way reversibility of the simple power rule in opening activities

Participant	2.3.1 – Day 5	2.3.2 – Day 6	4.6.1 – Day 30	4.6.2 – Day 31	4.7.1 – Day 32
Kelsay	Y	Y	Y	Y	A
Michael	Y	A	Y	Y	Y
Fred	N	Y	Y	Y	Y
Jill	Y	A	A	A	Y
Kirsten	N	N	Y	N	Y
Marcus	N	A	Y	Y	Y

Across the three flexibility groups, we see two stories. The students in the high and middle groups were able to correctly use two-way reversibility throughout all of the interviews. Similarly, the students in the high and middle group were largely able to use reversibility to find $f(x)$ from a given $f'(x)$ from the beginning of the study as shown by the exit slip and opening activity data. The first time that the interview participants were exposed to reversibility of the simple power rule was in exit slip and opening activity 2.3.1, which were administered on 11/25/2013 and 11/26/2013, respectively. The 2.3.1 opening activity and interview question 1.2 are conceptually identical, and are shown in table 49 below.

Table 49. Interview question 1.2 and opening activity 2.3.1

Interview 1, Question 2	2.3.1 Opening Activity
Suppose a function has a known derivative of $f'(x) = x^5$. What could be the function $f(x)$? Can you think of any other possible functions for $f(x)$?	Suppose $f'(x) = x - 6$. Find a function $f(x)$. Show or explain how you determined $f(x)$.

Kelsay, Michael, and Jill all correctly solved the opening activity 2.3.1 using reversibility of the simple power rule, indicating that they had developed reversibility "on-the-spot" (Krutetskii, 1976). Fred did not recognize that he needed to use reversibility to find $f(x)$ from $f'(x)$. However, one day later, on opening activity 2.3.2, Fred was able to use reversibility to correctly find $f(x)$ given $f''(x)$. Thus, beginning with opening activity 2.3.2, administered on day 6 on the study, all of the students in the high and middle groups were able to correctly use two-way reversibility to solve anti-differentiation problems throughout the remainder of the course. Thus, based on the evidence provided by the interview questions and the exit slips and opening activities, the students in the high and middle group largely developed two-way reversibility of the simple power rule immediately, or nearly immediately in the case of Fred, and were able to use reversibility to find $f(x)$ from a given $f'(x)$ by reversing the simple power rule throughout the course of the study.

The two students in the low group, Kirsten and Marcus, struggled to develop two-way reversibility. Neither student developed two-way reversibility on the spot, as evidenced by their results on the 2.3.1 and 2.3.2 opening activities. Furthermore, neither student was able to use two-way reversibility of the simple power rule to solve the interview questions 1.2 and 1.3.

On interview question 1.2, Kirsten indicated that she was aware that reversibility was necessary, saying "okay so this is going backwards" and attempted to reverse the simple power rule to determine that $f(x)$ would necessarily require an x^6 term, saying, "I know it's x^6 because … it has to be minus 1", referring to the fact that the degree of $f'(x)$ should be one less than the degree of $f(x)$. However, Kirsten could not resolve how to determine the correct coefficient. Kirsten was aware that the exponent of $f(x)$ would move out to the front of the expression, however she could not determine how to make the exponent of 6 change into the existing coefficient of 1. This confusion led Kirsten to change her exponent from the correct answer of 6 to $1/6$ and said "the exponent out front ... you would bring the 1 out front and then subtract 6 minus 1 and you would get the 5". In one sense, Kirsten seems to recognize that $1/6$ is necessary to the problem, however she does not indicate any knowledge that she needs to reverse the simple power rule in the exponent and then in the coefficient. As such, she does not reverse the process of the simple power rule. She was able to recognize that she needed to use reversibility but could not reverse the process.

Marcus had a similar experience to Kirsten on interview question 1.2. Marcus immediately noticed the need for reversibility saying "so the derivative is x^5 that means that you have to work backwards so that means that'll [referring to the exponent] have to be [a six]." However, shortly after writing the correct exponent of six, Marcus erased the exponent because he tried to differentiate x^6 and realized that the coefficient of $f'(x)$ would be a six. Marcus had no way to reconcile this issue. He described this conflict saying "normally for the exponent you do the exponent minus 1 for the derivative so if you are going backwards you would add one which would be 6, but there's a one in front of the x … that won't work." Marcus explored two other possible pathways to determine the coefficient. He first considered if using negative exponents would help him to find the coefficient but quickly dismissed the possibility after determining that $1/x^{-5}$ was an equivalent way of writing x^5. Secondly, he considered using a constant in front of the x^6 term but chose not to pursue this approach saying "well the derivative of a constant would be just zero, but there's no zero out front." In this case, Marcus has confused the derivative of a constant term, which is zero, and the derivative of a constant times a function, in which case the constant would carry through the derivative operation. Thus, Marcus, like Kirsten, was aware that he needed to use reversibility and was able to determine the correct exponent of $f(x)$ but could not resolve how to create the correct coefficient.

At the end of the study, there is conflicting evidence indicating whether or not Kirsten developed two-way reversibility. In the last interview, after the study had concluded, Kirsten was still not able to determine the appropriate coefficient when reversing the simple power rule. interview question 4.2.b says:

Suppose we know a velocity function, $v(t)$, for a vehicle in motion in meters per second.

$$v(t) = 4t^3 - 3t^2 + t$$

Find the position of the vehicle at $t = 3$.

Kirsten's work is shown below in figure 39.

$$S(t) = \frac{1}{3}t^4 - \frac{1}{2}t^3 + \frac{t^2}{2}$$

$$S(t) = 1 - 3.38 + 4.5$$

$$S(t) = 2.12$$

Figure 39. Kirsten's solution to interview question 4.2.b

Kirsten correctly found each exponent noting that "for this part you would have to take the backwards derivative". However, her work indicates that she only had a limited understanding of two-way reversibility. She did not correctly determine the coefficients in front of the t^3 and t^2 terms, but was able to correctly find the anti-derivative of t, which had a coefficient of 1.

Kirsten's inconsistency with correctly using two-way reversibility in interview 4 was also present in her opening activities at the end of the course. On day 30, Kirsten attempted opening activity 4.6.1, shown below in figure 40.

Let $v(t)$ be the velocity function of a particle where t is in seconds and s is measured in feet.

a. Find a position function, $s(t)$ of the particle if

$$v(t) = t^3 - t^2 + 5t + 6$$

$$\frac{t^4}{4} - \frac{t^3}{2} + 2.5t^2 + 6t$$

Figure 40. Kirsten's solution to opening activity 4.6.1

Kirsten used reversibility to correctly find three of the terms, but incorrectly concluded that the anti-derivative of t^2 was $\frac{t^3}{2}$. On the following day, day 31, Kirsten attempted opening activity 4.6.2, which required a repeated use of reversibility of the simple power rule. Her work is shown below in figure 41.

Let $a(t)$ be the acceleration function of a particle where t is in seconds and s is measured in feet.

a. Find a position function, $s(t)$ of the particle if

$$a(t) = t^2 + 3t - 6$$

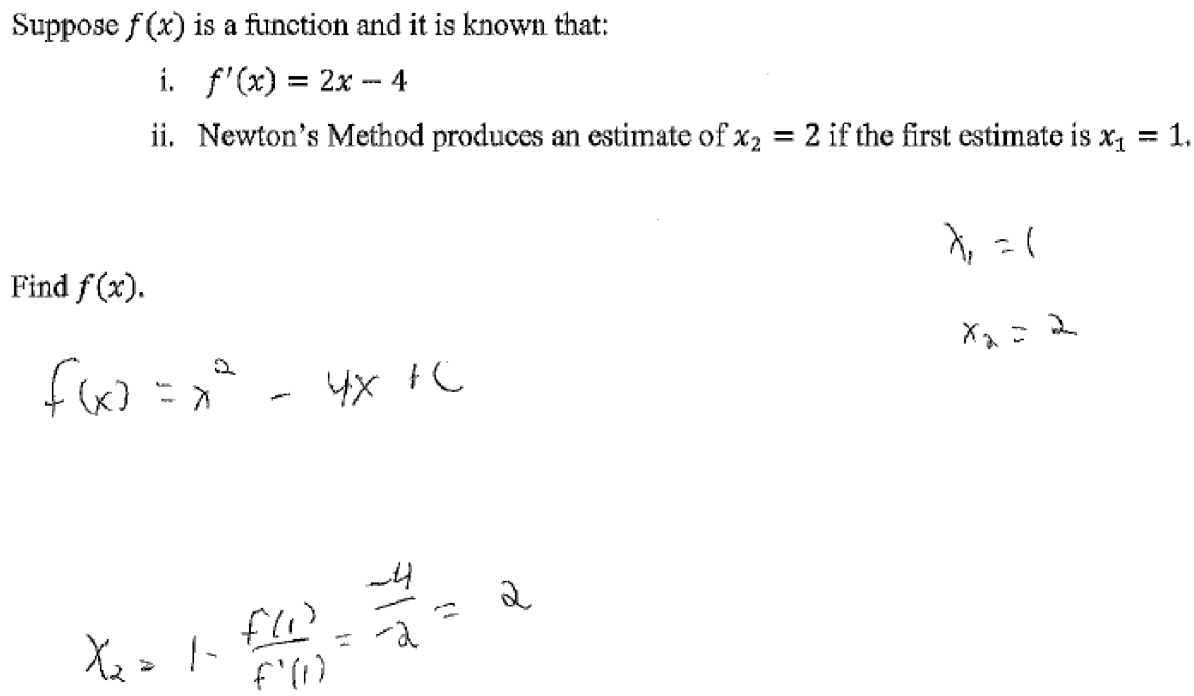
$$v(t) = \frac{1}{3}t^3 + \frac{3}{2}t^2 - 6t$$

$$s(t) =$$

Figure 41. Kirsten's solution to opening activity 4.6.2

Here, Kirsten again showed proficiency with two-way reversibility of the simple power rule by correctly finding a possible $v(t)$ from the given $a(t)$. However, she makes no effort to find $s(t)$ from her function for $v(t)$. Since she had shown that she understood the conceptual relationship between $s(t)$, the position function, and $v(t)$, the velocity function, in opening activity 4.6.1, it appears that Kirsten was not able to use reversibility to find $s(t)$ from $v(t)$ by reversing the simple power rule.

Finally, on opening activity 4.7.1, Kirsten correctly used reversibility of the simple power rule to find $f(x)$ from a given $f'(x)$ as shown below in figure 42.



Suppose $f(x)$ is a function and it is known that:

i. $f'(x) = 2x - 4$

ii. Newton's Method produces an estimate of $x_2 = 2$ if the first estimate is $x_1 = 1$.

$x_1 = 1$

$x_2 = 2$

Find $f(x)$.

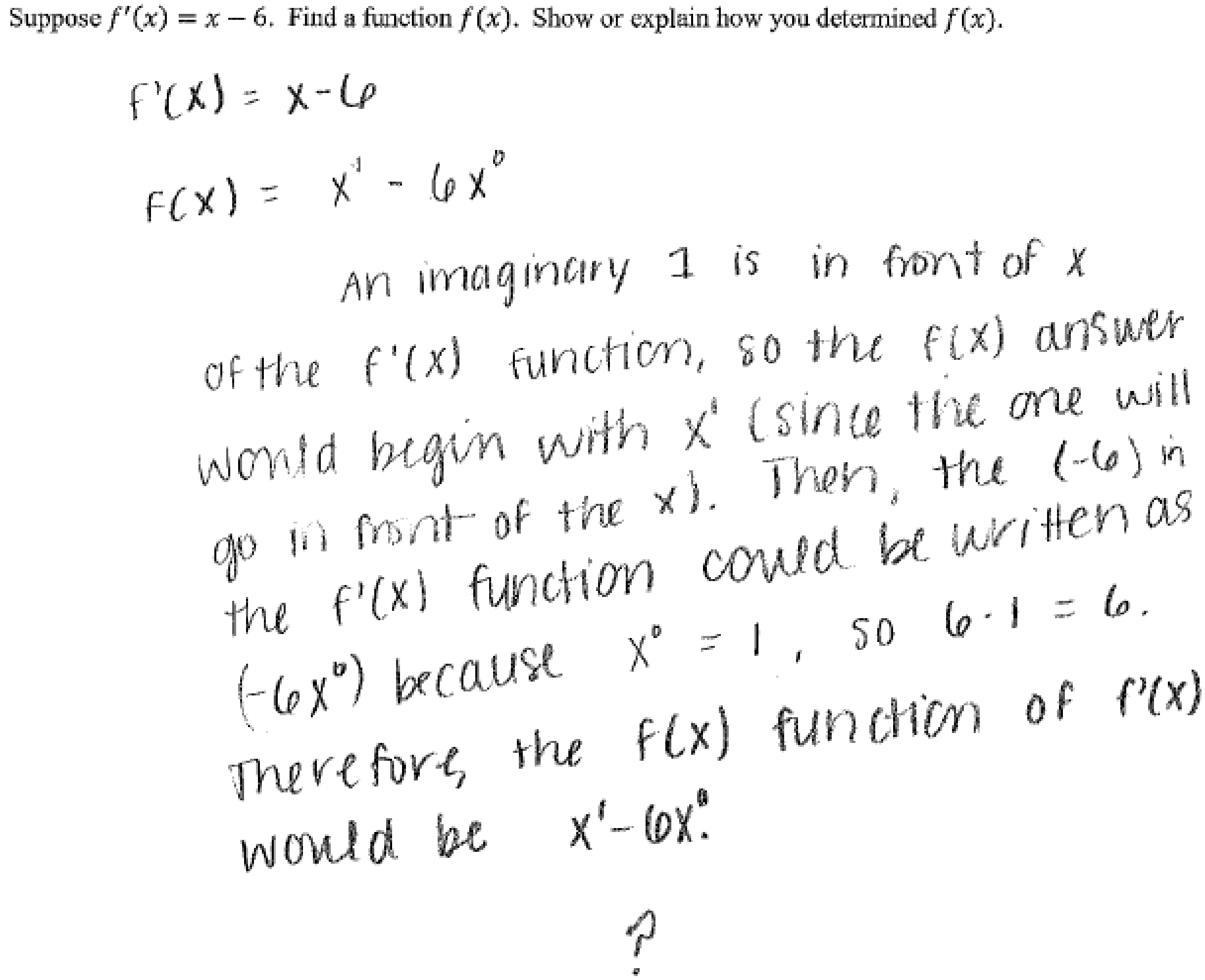
$f(x) = x^2 - 4x + C$

$X_2 = 1 - \frac{f(1)}{f'(1)} = \frac{-4}{-2} = 2$

Figure 42. Kirsten's solution to opening activity 4.7.1

Thus, Kirsten's work indicates that her ability to use reversibility of a two-way process improved over the course of the study; however, due to her inconsistency in correctly applying two-way reversibility, it appears that she has an under-developed sense of two-way reversibility.

Marcus demonstrated a much more consistent developmental trajectory. As discussed earlier, during his first interview, Marcus was not able to reverse the simple power rule to find $f(x)$ from a given $f'(x)$. Marcus's first attempt at two-way reversibility of the simple power rule was on the 2.3.1 opening activity. His work is shown below in figure 43.



Suppose $f'(x) = x - 6$. Find a function $f(x)$. Show or explain how you determined $f(x)$.

$f'(x) = x - 6$

$f(x) = x^1 - 6x^0$

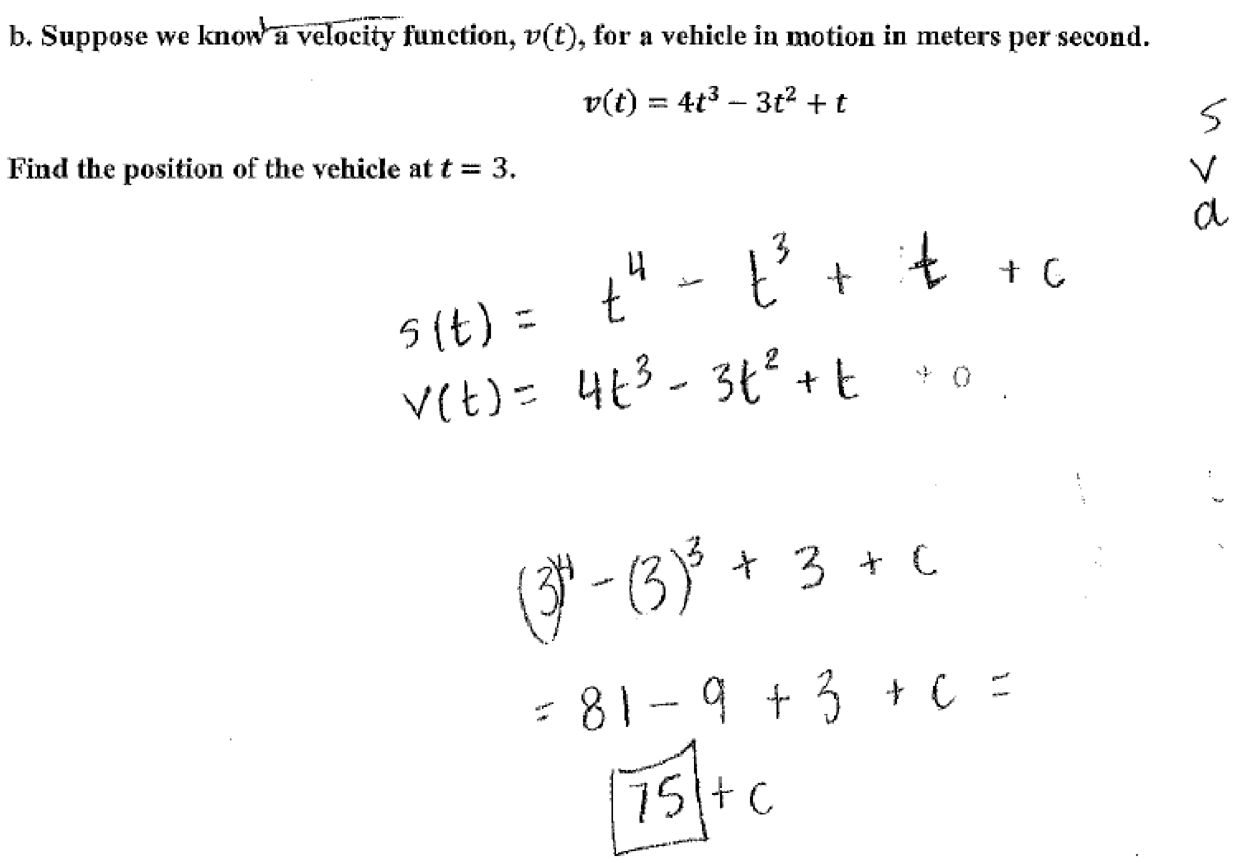
An imaginary 1 is in front of x of the $f'(x)$ function, so the $f(x)$ answer would begin with x^1 (since the one will go in front of the x). Then, the (-6) in the $f'(x)$ function could be written as $(-6x^0)$ because $x^0 = 1$, so $6 \cdot 1 = 6$. Therefore, the $f(x)$ function of $f'(x)$ would be $x^1 - 6x^0$.

?

Figure 43. Marcus's solution to opening activity 2.3.1

Marcus's work indicates that at the beginning of the course, he had no conception of how to reverse the simple power rule. He did not consider that the exponent(s) of the $f'(x)$ expression would need to increase by one, which is the reverse of differentiation causing the degree of the polynomial to decrease by one. When this evidence is considered in concert with Marcus's work on interview question 1.2, where he showed that he was now considering increasing the exponent by one, we see a slight improvement in two-way reversibility of the simple power rule from 11/26/2013 to 12/4/2013.

At the end of the study, Marcus had developed a strong conception of two-way reversibility of the simple power rule as he correctly reversed the simple power rule on opening activities 4.6.1, 4.6.2, and 4.7.1. In his final interview, Marcus showed that while he was confident in using reversibility to find a polynomial $f(x)$ from a polynomial $f'(x)$, his reversible conception did not generalize to all possible terms of a polynomial. His work on interview question 4.2.b is shown below in figure 44.


b. Suppose we know a velocity function, $v(t)$, for a vehicle in motion in meters per second.

$$v(t) = 4t^3 - 3t^2 + t$$

Find the position of the vehicle at $t = 3$.

s
v
a

$s(t) = t^4 - t^3 + t + c$
$v(t) = 4t^3 - 3t^2 + t + 0$

$(3)^4 - (3)^3 + 3 + c$
$= 81 - 9 + 3 + c =$
$\boxed{75} + c$

Figure 44. Marcus's solution to interview question 4.2.b

Marcus correctly determined the first two terms in $s(t)$; however, he was unable to determine a term whose derivative would be t. Marcus tried to think backwards by asking himself what he would differentiate to produce t, saying:

I'm still imagining a one in front so that one would go there but how the heck do you just get a t …so if you [had] $2t$, the derivative would just be 2, if you had t^2, the derivative would just be $2t$, so if you had a t, the derivative would be...? If there was a $1t$ then the 1 would go there. It couldn't be like t ... t to the no ... I don't know I'm just going to leave it as t but that's not right but I don't know what to do.

Marcus's solution suggests that he does have a working conception of two-way reversibility of the simple power rule at the end of the course; however, his conception does not generalize to all polynomials and thus is not fully developed. In this case, Marcus's results are similar to what Krutetskii (1976) found with students that he categorized as middle ability. He found that students who did not develop reversibility on the spot, could develop reversibility by working through similar examples and special exercises. As Marcus's reversibility with the simple power rule has improved markedly over the course of the study, he seems to have developed two-way reversibility as he worked with similar examples in the opening activities and interview questions.

The results of the interview questions that assessed two-way reversibility indicate that for students in the high and medium flexibility groups, reversibility of a two-way process likely developed simultaneously with learning the process in the forward direction. This conclusion is supported by the result that all of the students in the high and middle flexibility groups were able to reverse the simple power rule to find $f(x)$ from a given $f'(x)$ immediately after learning how to differentiate $f(x)$ to find $f'(x)$ by using the simple power rule.

Students in the low flexibility group did not develop two-way reversibility on the spot, but showed some development throughout the course. Neither of the students in the low group were able to apply two-way reversibility consistently across all items at the end of the study. However, both students in the low group showed improvement from the beginning of the study until the end, indicating that two-way reversibility had developed in a limited amount.

4.2.3 Development of reversibility of the mental process in reasoning without reversible translation

To what extent does reversibility of the mental process in reasoning without reversible translation develop? The data collected in this research study suggests that reversibility of the mental process in reasoning without reversible translation has developed in a limited and uneven amount over the course of the study. It is important to restate here that the phrase "reasoning without translation" does not preclude flexibility. It is expected that one directional translations would be utilized by students in both parts of the paired problems. "Reasoning without translation" indicates that the paired problems will not require representational reversibility. To determine if the students developed reversibility of the mental process in reasoning without reversible translation, I analyzed the results of a sub-set of the exit slips and opening activities and a sub-set of the interview questions. I discuss the results of the exit slip and opening activity data and the interview data in turn.

4.2.3.1 Results of exit slips and opening activities – mental process in reasoning without reversible translation

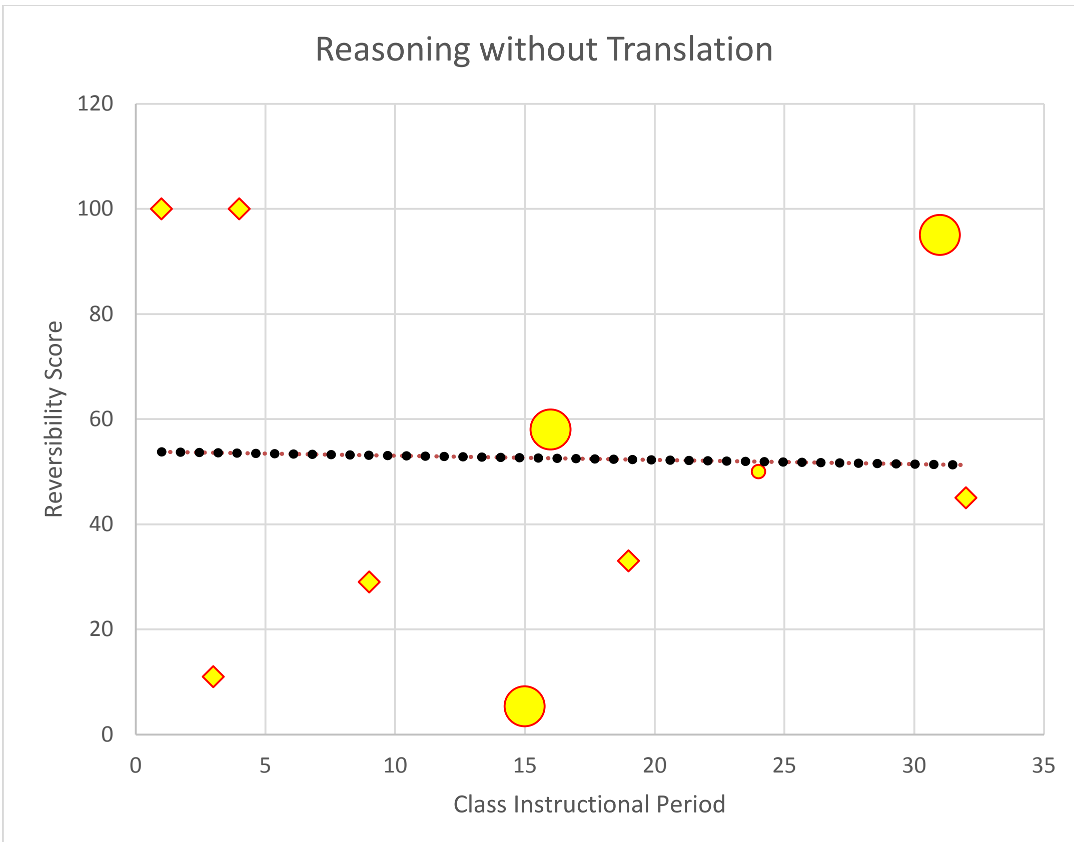
I analyzed the results of the 10 exit slip and opening activity pairs that were designed to assess the development of reversibility of the mental process in reasoning without reversible translation. Four of the exit slips and opening activities were administered during chapter 2, three were administered during chapter 3, and three were administered during chapter 4. Thus, the 10 exit slip and opening activity pairs were distributed approximately equally throughout the study. Figure 45, shown below, presents the results of all of the reversibility of the mental process in reasoning exit slip and opening activity pairs, administered chronologically throughout the study.

Day	1	3	4	9	15	16	19	24	31	32
Exit Slip	1	4	5	11	18	20	23	32	42	44
Section	2.3.1	2.2.1	2.2.2	2.6.1	3.3.1	3.3.2	3.5.1	4.2.2	4.6.2	4.7.1
H1	Y	N	Y	-	N	Y	-	Y	Y	Y
H2	^	-	Y	-	N	Y	-	Y	Y	N
H3	Y	Y	Y	^	N	N	Y	^	Y	-
H4	-	N	-	-	N	N	N	N	Y	N
H5	Y	-	Y	-	N	Y	-	Y	Y	Y
H6	Y	N	^	Y	Y	Y	N	Y	Y	^
H7	^	-	-	-	N	N	N	^	Y	Y
H8	Y	N	Y	-	N	N	-	-	^	^
M1	Y	N	Y	Y	N	Y	-	^	Y	Y
M2	Y	-	-	-	N	Y	-	Y	Y	-
M3	Y	N	^	N	-	^	^	N	^	-
M4	-	-	-	-	N	Y	-	Y	Y	Y
M5	Y	-	Y	-	-	^	^	N	Y	-
M6	-	-	-	N	N	N	N	-	Y	N
M7	-	N	Y	N	N	N	Y	N	Y	^
L1	-	-	-	-	N	Y	^	N	Y	-
L2	-	-	Y	-	N	N	N	N	Y	^
L3	-	N	^	N	N	Y	-	Y	Y	-
L4	-	^	-	-	N	N	-	-	Y	N
L5	-	-	^	-	N	Y	Y	N	Y	N
L6	Y	-	-	N	N	Y	N	-	N	N
R.S.	100.0	11.1	100.0	33.3	5.6	55.6	37.5	50.0	100.0	50.0

Key		
H1 - first student in high flexibility group	Y - reversibility present	- - ineligible due to incorrect answer on exit slip
M1 - first student in middle flexibility group	N - reversibility NOT present	% - Ratio of Y to (Y + N)
L1 - first student in low flexibility group	^ - ineligible due to absence	R.S. – reversibility score

Figure 45. Results of the exit slip and opening activity pairs assessing reversibility of the mental process in reasoning

Figure 45 shows an uneven presence of reversibility of the mental process in reasoning throughout the study as there is no clear trend towards an increasing density of green blocks and decreasing density of red blocks. Figure 36 displays the reversibility score of each exit slip and opening activity pair that assessed two-way reversibility as a scatter plot over time with a trend line imposed on the data.



Key	
# of Eligible Exit Slips and Opening Activities	Symbol
0-5	▫
6-10	◆
11-15	▲
16-21	●

Figure 46. Scatter plot of the reversibility score on the exit slip and opening activity pairs assessing reversibility of the mental process in reasoning

The slightly negative slope ($m = -0.08$) of the trend line coupled with the weak correlation between the class instructional period and the percent of students who demonstrated reversibility of $r = -0.03$ indicates that there was no observable development of reversibility of the mental process in reasoning without reversible translation over the course of the study at the whole class level.

To further gauge the extent to which reversibility of the mental process in reasoning develops when students are engaged in a course that attends to linking multiple representations, I compared the class mean reversibility scores during the first and second half of the course. Table 50, shown below, reports the class mean reversibility of the mental process in reasoning score during the first and second halves of the study and the results of a paired samples t-test for a significant difference. The first half consisted of paired problems 1, 4, 5, 11, 18, 20, and 23. The second half consisted of paired problems 32, 42, and 44.

Table 50. Class mean reversibility score on reversibility of the mental process in reasoning exit slips and opening activities from the first half and second half of the study

	First half:	Second half:	Paired difference t-value	Paired difference p-value
Reversibility Score	64.33%	62.10%	$t = 0.244$	$p = 0.808$

The paired samples t-test indicates that there was no improvement in reversibility of the mental process in reasoning over the course of the study. When the results in figures 45 and 46 and in table 50 are considered together, the data suggests that reversibility of the mental process in reasoning did not improve over the course of the study.

As with flexibility, the content in which reversibility was situated may help to explain in part the results of the exit slips and opening activities. To further investigate this possibility, I considered the results of the exit slips and opening activities that assessed reversibility of the mental process in reasoning that dealt with the same or very similar content. The exit slips and opening activities that assessed the chain rule (3 sets of paired problems), local linearization (2 sets of paired problems), and graphical tasks (3 sets of paired problems) are discussed below.

Chain rule tasks

The three opening activities addressing reversibility of the chain rule were administered on 12/9/2013, 1/14/2014, and 1/15/2014. Table 51 shows the progression of reversibility on opening activities that assess reversibility of the chain rule.

Table 51. Reversibility of the mental process in reasoning without reversible translation exit slips and opening activities assessing the chain rule

Section	Exit Slip ($n =$ # of correct answers)	Opening Activity ($n =$ # of correct answers)	Reversibility Score
2.6.1	Suppose $f(x) = 5\sin x$ and $g(x) = \sin 5x$. Which of the following is true? I. $f'(\pi) < g'(\pi)$ II. $f'(\pi) = g'(\pi)$ III. $f'(\pi) > g'(\pi)$ IV. Cannot be determined. Show your work or explain how you determined your answer. $n = 7$	If a function has a known derivative of $h'(x) = 5\cos x - 3\sin 3x$, what could be $h(x)$? Show or explain how you determined $h(x)$. $n = 2$	28.6
3.3.1	Find the derivative of the function. $f(x) = 3e^{4x}$ Show your work or explain how you determined $f'(x)$. Can you think of any other possible functions for $f'(x)$? $n = 19$	Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $f'(x) = e^{-5x}$ $n = 1$	5.3

Table 51 (continued)

3.3.2	Find the derivative of the function. $$f(x) = \sin^{-1}(e^x)$$ Show your work or explain how you determined $f'(x)$. Can you think of any other possible functions for $f'(x)$? $n = 19$	Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $$f'(x) = \frac{1}{\sqrt{1-(3x)^2}}$$ $n = 11$	57.9

Although it appears that the students' ability to use the chain rule improved from exit slip 2.6.1 to exit slip 3.3.1, there was no improvement in reversibility of the chain rule. However, there was a large increase in reversibility from opening activity 3.3.1 to opening activity 3.3.2, indicating that reversibility of the chain rule improved with repeated opportunities to use reversibility.

Two students were able to use reversibility immediately after learning the chain rule in section 2.6.1. Both of these students were interview participants. One student who developed reversibility on-the-spot was Kelsay, a member of the high flexibility group. The other student was Fred, a member of the middle flexibility group. On opening activity 3.3.1, the only student in the class who demonstrated reversibility was Kelsay. Of the seven students in the high flexibility group, only one student developed reversibility on the spot. This result suggests that there may be a high degree of difficulty in developing reversibility of the mental process in reasoning without reversible translation.

Local linearization tasks

Exit slips and opening activities 2.2.1 and 3.5.1 assessed local linearization. Both sets of activities are presented below in table 52.

Table 52. Reversibility of the mental process in reasoning without reversible translation exit slips and opening activities assessing local linearization

Section	Exit Slip (n = # of correct answers)	Opening Activity (n = # of correct answers)	Reversibility %
2.2.1	Write the equation of the line tangent to the curve $y = x^2 - x$ at $x = 3$. Make sure to show your work. $n = 9$	The line tangent to a curve $f(x)$ at $x = 2$ has the equation $y = 4 + 8(x - 2)$. Find a possible equation for $f(x)$. Show or explain how you determined $f(x)$. $n = 1$	11.1
3.5.1	What is the linearization of $f(x) = e^{2x}$ at $x = 1$? $n = 9$	What function has a linearization at $x = e^2$ of $y = 2 + \frac{1}{e^2}(x - 2)$? $n = 3$	33.3

On the 2.2.1 opening activity, the only student to exhibit reversibility was Michael, a member of the high flexibility group. On opening activity 3.5.1, administered approximately two months later, Michael again used reversibility to solve the problem. Of the other two students, one was from the middle flexibility group, and one was from the low flexibility group. Thus, the results of the paired problems on local linearization suggest that developing reversibility of the mental process in reasoning without reversible translation may be difficult for students.

Graphical tasks

The sets of paired problems from sections 2.1.1, 2.2.2, and 4.2.2 all assess the graphical relationship between $f(x)$ and $f'(x)$. Each exit slip assessed the direction $f \to f'$ and the opening activity assessed $f' \to f$. While the reversibility percentage appears to have taken a precipitous decline from 100% and 100% to 50%, the reason for the observed decline is not so much a decline in the number of students who correctly solved the reverse problem, which was 10-9-7, respectively, but an increase in the number of students who correctly solved the direct problem. On the 2.1.1 and 2.2.2 exit slips, 10 and 9 students, respectively, solved the direct question correctly. On the 4.2.2 exit slip, 14 students solved the problem correctly. What these results suggest is that understanding of the graphical relationship from f to f' seemed to increase throughout the study, however, reversibility did not. More specifically, the distribution of students who demonstrated reversibility on the three questions remained essentially constant. Table 53 below reports the number of students (n) and the percent of eligible students in the high, middle, and low groups who demonstrated reversibility on the paired problems from 2.1.1, 2.2.2, and 4.2.2.

Table 53. Number of students in each flexibility group who demonstrated reversibility

Section Number	High Group	Middle Group	Low Group
2.1.1	$n = 5, 100\%$	$n = 4, 100\%$	$n = 1, 100\%$
2.2.2	$n = 5, 100\%$	$n = 3, 100\%$	$n = 1{,}100\%$
4.2.2	$n = 4, 80\%$	$n = 2, 40\%$	$n = 1, 25\%$

What is most striking about the results of the paired problems that assessed reversibility of the graphical relationship between f and f' is that on the first two problems, which only dealt with constant and linear functions, every student who learned the relationship from f to f' was able to reverse the relationship from f' to f on the spot. However, when the degree of the function exceeded one on paired problem set 4.2.2, the percent of students demonstrating reversibility decreased in all flexibility groups. These results suggest that even though reversibility may exist within a particular content area, as the difficulty of the content increases, reversibility does not necessarily generalize from the less complicated instantiation to the more complicated instantiation of the same concept.

The results of the exit slip and opening activity pairs of problems assessing reversibility of the mental process in reasoning without reversible translation suggest that development of reversibility of mental processes is likely a difficult task for students and is likely related to the particular content in which the reversibility manifests.

4.2.3.2 Results of interviews – mental process in reasoning without reversible translation

I analyzed the written solutions and the transcript data from the pairs of interview questions that assess reversibility of the mental process in reasoning without reversible translation: 2.1 (interview 2, question 1) & 2.2, 2.3 & 2.4, 4.1.a & 4.2.a, 4.1.b & 4.2.b, and 4.3.a & 4.3.b. Each pair of questions was analyzed for evidence of reversibility of the mental process in reasoning without reversible translation. Table 54 reports the existence or absence of reversibility of the mental process in reasoning without reversible translation on the interview questions. "Yes" indicates that the participant correctly solved both the forward and reverse questions; "No" indicates that the participant correctly solved the forward question but could not solve the reverse question. "N/A" indicates that the participant could not solve the forward question, thus rendering the question about the existence of reversibility moot.

Table 54. Is reversibility of the mental process in reasoning without reversible translation present in the interview questions?

Flexibility Group	Participant	2.1 & 2.2	2.3 & 2.4	4.1.a & 4.2.a	4.1.b & 4.2.b	4.3.a & 4.3.b
High	Kelsay	Yes	Yes	Yes	Yes	Yes
	Michael	Yes	Yes	No	Yes	Yes
Middle	Fred	Yes	Yes	Yes	Yes	Yes
	Jill	No	Yes	Yes	Yes	Yes
Low	Kirsten	N/A	Yes	N/A	Yes	N/A
	Marcus	No	No	No	Yes	N/A

Consistent with the results that flexibility and reversibility of two-way processes are likely content specific, I grouped the interview questions by related content for the purposes of discussion of the results. Paired problems 2.1 & 2.2 assessed reversibility of the chain rule. Paired problems

2.3 & 2.4, 3.1 & 3.2, and 4.3.a & 4.3.b assessed reversibility of graphical analysis. Paired problems 4.1.a & 4.2.a and 4.1.b & 4.2.b assess reversibility of position and velocity. I discuss each content area within reversibility of the mental process in reasoning without reversible translation in the following sections. In each section, I consider the results of the interview questions by student and flexibility group. I begin by discussing the results of the chain rule questions in interview 2.

Chain rule interview questions

Interview questions 2.1 and 2.2 are a set of paired problems that require reversibility of the chain rule and are conceptually similar to the paired exit slips and opening activities from sections 2.6.1, 3.3.1, and 3.3.2. It should be noted that the interview questions were presented reverse problem first and then direct problem, unlike the exit slips and opening activities.

The results of the interview participants on all four chain rule questions in relation to the presence or absence of reversibility of the mental process in reasoning without reversible translation are reported in table 55. Y means that the student used reversibility to correctly solve the question. N means that the student did not use reversibility to correctly solve the question. A means that the student was absent and did not attempt the question. N/A means that the student could not solve the direct problem, rendering the question of reversibility moot. The results of high, middle, and low flexibility groups are discussed in detail below.

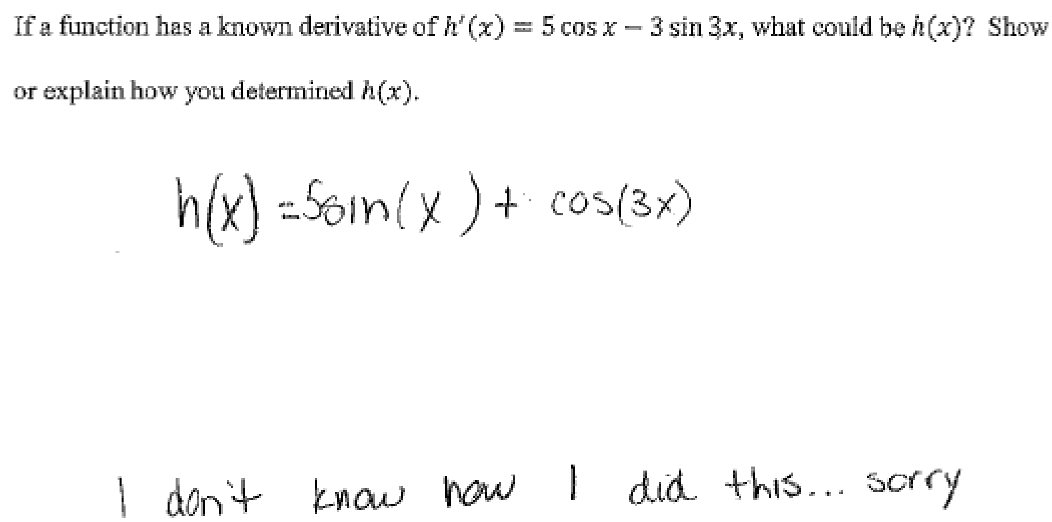
Table 55. Reversibility of the mental process in reasoning without reversible translation with the chain rule questions

Flexibility Group	Participant	2.6.1 – 12/13/14	Interview Questions 2.1 & 2.2 – 12/17/13	3.3.1 – 1/14/14	3.3.2 – 1/16/14
High	Kelsay	Y	Y	Y	Y
	Michael	A	Y	N	N
Middle	Fred	Y	Y	N	Y
	Jill	N	N	N/A	A
Low	Kirsten	N	N/A	N	Y
	Marcus	N/A	N	N	Y

High flexibility group – Kelsay and Michael

Kelsay, a member of the high group, developed reversibility of the mental process in reasoning without reversible translation with the chain rule immediately upon learning the chain rule and she maintained that reversibility throughout the study. In her 2nd interview, Kelsay made clear that she recognized the need for reversibility of the chain rule, saying, "so since $f'(x) \ = x * \sin(x^2)$, reversing the logic of this problem uh so this could possibly be $-\frac{1}{2}\cos(x^2) \ = \ f(x)$ because … taking the derivative … will equal $x\sin(x^2)$". Also, it is worth noting that Kelsay was able to reverse the chain rule through mental mathematics and then checked her answer through differentiation. Her work on the three exit slip and opening activity pairs was completely consistent with her approach to the interview question.

Michael, the second member of the high group, has a story that is not well captured by table 55 above. On exit slip and opening activity pair 2.6.1, Michael was absent on the day of the direct learning and thus did not attempt the exit slip. Therefore, his reversibility score was "A". However, Michael attended on the day of the 2.6.1 opening activity. His solution is shown below in figure 47.



If a function has a known derivative of $h'(x) = 5\cos x - 3\sin 3x$, what could be $h(x)$? Show or explain how you determined $h(x)$.

$h(x) = 5\sin(x) + \cos(3x)$

I don't know how I did this... sorry

Figure 47. Michael's solution to opening activity 2.6.1

Michael and I had a brief discussion about his solution when he turned in his opening activity. That discussion follows:

Michael: I don't know how I did this.

Teacher: Did you just puzzle that out?

Michael: Yeah

There are two possibilities that may explain how Michael could have solved the reverse problem without having been present to learn the direct application of the chain rule. One possibility is that Michael contacted another student in the class and inquired about what he had missed. Throughout the school year he proved to be a conscientious student who frequently made sure to copy any missed class notes from another student. A second possibility is that Michael used two-way reversibility to identify that since $f'(x)$ contained a negative $\sin 3x$ term, then it would be reasonable to conclude that $f(x)$ would have to contain a positive $\cos 3x$ term. In either event, what is significant is that even though his exit slip and opening activity cannot be scored as evidence of reversibility because there is no way to know if Michael had a functional conception of the chain rule at the time of solving the opening activity, Michael was able to find $f(x)$ from a given $f'(x)$ that required reversing the chain rule.

Michael's solutions to the interview questions indicate that within the week between opening activity 2.6.1 and interview 2, he had developed a fully reversible conception of the chain rule. His solution to interview question 2.1 follows in figure 48.

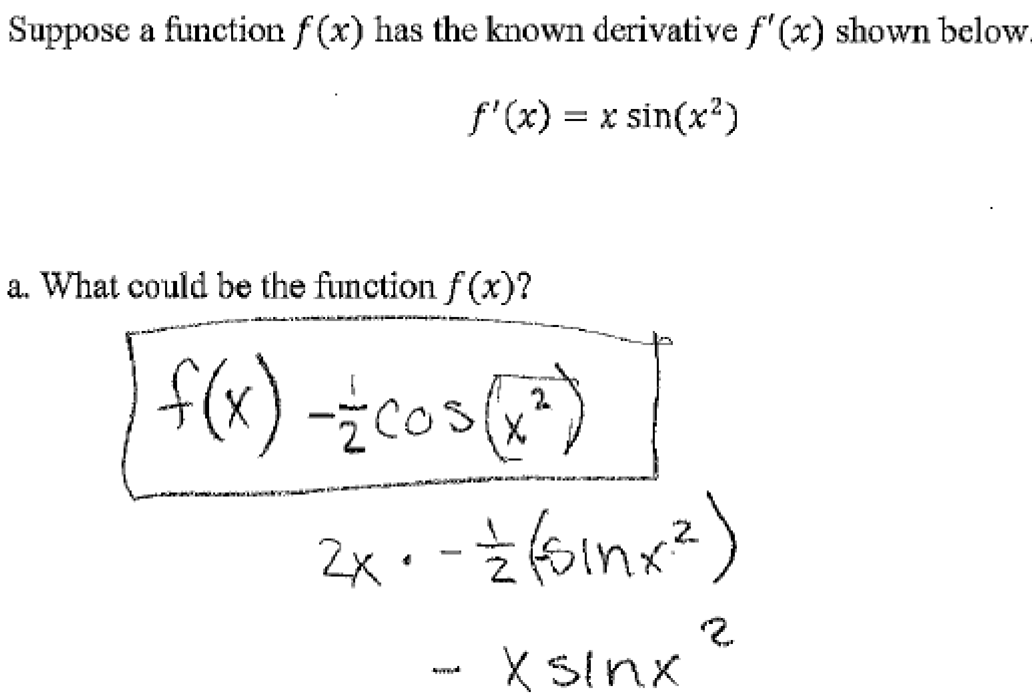


Figure 48. Michael's solution to interview question 2.1

When solving this problem, Michael explicitly used reversibility of the mental process in reasoning without reversible translation, saying:

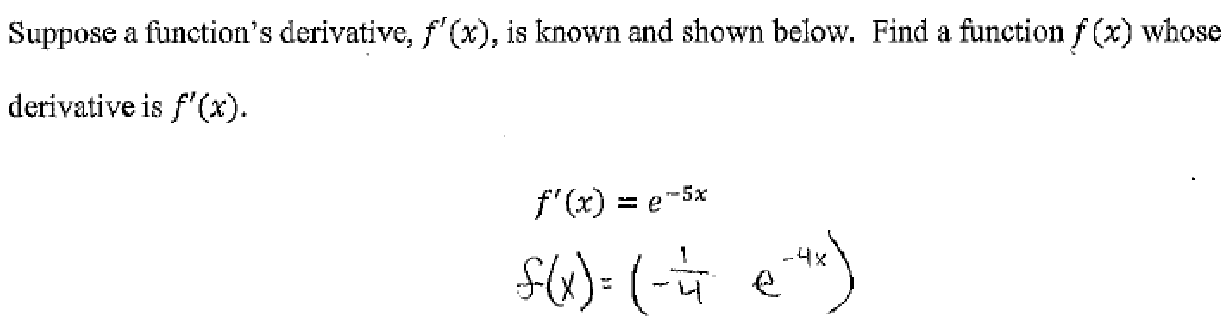
> so $f'(x) = x\sin(x^2)$ so to find $f(x)$ you'd kind of have to undo that, go backwards ... so I need to find something so the opposite of sine the derivative cosine is negative sine so it'd have to be negative cosine to be regular sine, so negative cosine of something ... um ... hmm ... I'm just going to kind of plug and chug and see if this works right now ok so if I had just this [referring to $f(x) = -1/2\cos(x^2)$], I would take there's the box

[Michael draws a box around x^2] the derivative of the box is $2x$ times $-\frac{1}{2}\sin(x^2)$ so 2 times $-1/2$ would be … yeah it would so it would be $x\sin(x^2)$ … I just kind of have to guess and check … I usually get a pretty good idea and then I check it to make sure.

This is an example of reversibility of the mental process in reasoning without reversible translation as Michael does not have a step-by-step procedure to reverse, rather he looked at the end of the procedure (in this case the chain rule) and proposed a possible starting point, informed by his knowledge of differential calculus and then evaluated the correctness of his answer through known procedures.

Michael's remaining paired problems involving the chain rule provide evidence that on its surface runs counter to the evidence provided in interview 2. However, two explanations are readily available to explain why Michael was not able to demonstrate reversibility on paired problems 3.3.1 and 3.3.2.

Michael correctly solved both direct problems. Michael's solution to opening activity 3.3.1 is shown below in figure 49.


Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$.

$$f'(x) = e^{-5x}$$

$$f(x) = \left(-\frac{1}{4}\ e^{-4x}\right)$$

Figure 49. Michael's solution to opening activity 3.3.1

This solution is marked as incorrect because the correct solution is $f(x) = -\frac{1}{5}e^{-5x}$. However, it is entirely possible that Michael misread the question. If $f'(x) = e^{-4x}$, his answer is correct and would be consistent with his solution to interview question 2.1. However, it is also possible that Michael incorrectly applied the simple power rule and reversed differentiation by adding one to the exponent and then dividing by the new exponent. This possibility does not seem as likely because the coefficient is $-\frac{1}{4}$, not $-\frac{1}{4x}$. Thus, it may be the case that Michael evidenced full reversibility of the chain rule in this problem or it is possible that Michael did not recognize the problem as involving the chain rule.

On opening activity 3.3.2, Michael was asked to find a function $f(x)$ whose derivative is known to be $f'(x) = \frac{1}{\sqrt{1-(3x)^2}}$. Michael noted that $f'(3x) = \sin^{-1}(3x) = \frac{1}{\sqrt{1-(3x)^2}}$. Michael left $f(x) =$ blank. This attempt at a solution suggests that Michael was aware that the derivative of $f(x) = \sin^{-1} x$ was involved but Michael did not recognize that the chain rule was involved in the problem. Taken together with the interview question and the other paired problems, it may be the case that Michael does have reversibility of the chain rule, provided that he is aware that the chain rule is involved. Thus, Michael's reversibility of the mental process in reasoning without reversible translation about the chain rule is limited by his flexibility to notice the result of the chain rule in its various instantiations.

Thus, when Kelsay and Michael's results are considered together, we see some variation within the high group. Kelsay developed reversibility on the spot and maintained complete reversibility of the chain rule throughout the study. Michael evidenced reversibility of the chain rule in his interview. However, there is insufficient evidence to conclude that the reversibility present in the interview generalized to other instantiations of the chain rule, including to the derivatives of the families of functions $f(x) = e^x$ and $f(x) = \sin^{-1} x$.

Middle flexibility group – Fred and Jill

The middle flexibility group, consisting of Fred and Jill as interview participants, exhibited inconsistency with reversibility of the mental process in reasoning without reversible translation about the chain rule. Fred and Jill's results on the reversibility of the chain rule interview questions are reported in table 56 below.

Table 56. Reversibility of the mental process in reasoning without reversible translation with the chain rule questions – middle flexibility group

Participant	2.6.1 – 12/13/14	Interview Questions 2.1 & 2.2 – 12/17/13	3.3.1 – 1/14/14	3.3.2 – 1/16/14
Fred	Y	Y	N	Y
Jill	N	N	N/A	A

Fred's work indicates that he largely developed reversibility of the chain rule on the spot. His solutions to opening activities 2.6.1 and 3.3.2 were perfect solutions and his 2nd interview also supports the claim that he has a well-developed reversible conception of the chain rule. Fred's solution to interview question 2.1 is presented below in figure 50.

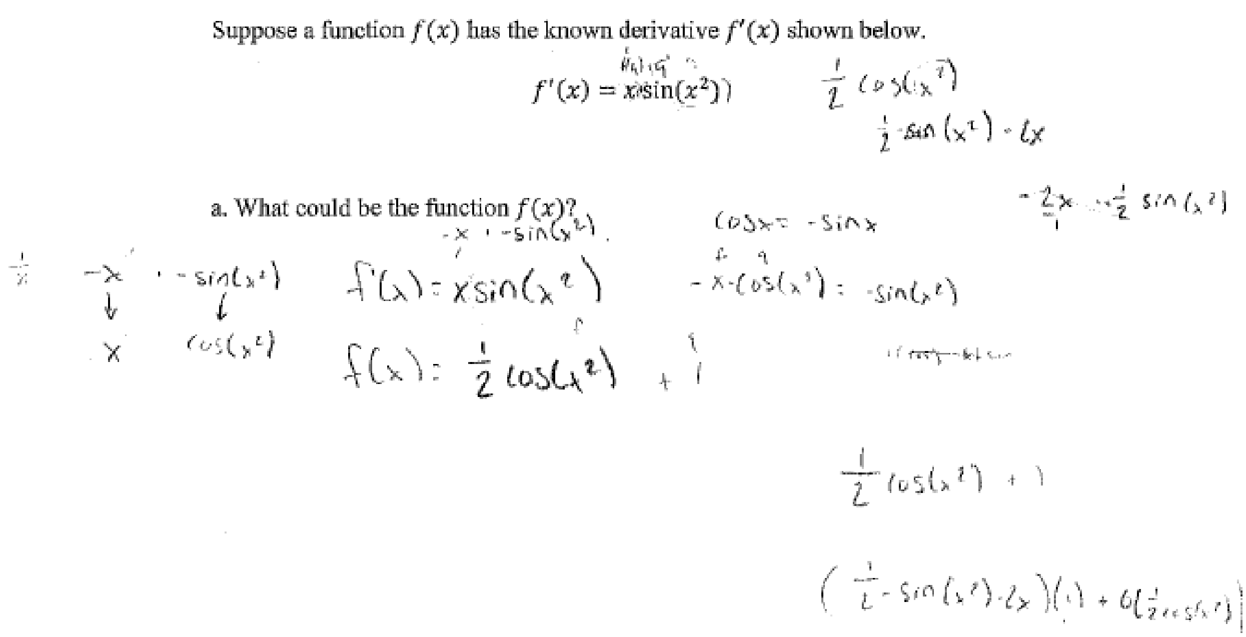


Figure 50. Fred’s solution to interview question 2.1

Fred began the problem by noting that “I know that the derivative of cosine equals negative $\sin x$, so since it's been given that the derivative … has a sine in it, then there should be a cosine in the … function.” Then Fred tested if the problem required reversibility of the product rule by placing an x in front of the $\cos(x^2)$ and differentiating the result. He rejected this approach saying “I just tried to put it [the x] in front to see if maybe the product rule gave the derivative but it didn't”. Having eliminated the product rule, Fred then conjectured that this problem may be the reverse of the chain rule and tested the conjecture, saying:

> “so I'm guessing it could be a chain rule. I'm going to try to see if that's what it is … I think $f(x)$ is $[1/2\cos(x^2)]$ because if you use chain rule, the outside is $\frac{1}{2}$ cosine and once you … apply the chain rule you have $-\frac{1}{2}\sin(x^2)\,(-2x)$ … the twos cancel out so you would be left with a positive $x\sin(x^2)$ ... so the normal function should be $\frac{1}{2}\cos(x^2)$.”

Fred's work showed a complete reversible understanding of the chain rule, although I should acknowledge that he incorrectly inserted a negative sign with the $(2x)$ term in his differentiation, thus explaining why his final answer lacks the necessary negative sign. Fred's approach to the problem of first recognizing that reversibility was necessary and then proposing a solution method (first by reversing the product rule and then by reversing the chain rule) and then testing the outcome of the method shows an understanding of reversibility of the mental process in reasoning without reversible translation. Fred could not reverse the chain rule step-by-step, so instead, he considered what starting point would be necessary to get him to the desired outcome and continued making informed adjustments until he reached the correct outcome.

Fred's incorrect solution to opening activity 3.3.1 should not be counted as evidence against his reversible conception of the derivative. His work, shown below in figure 51, suggests that Fred misread the question, thinking that he was trying to find the derivative instead of the function.

Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$.

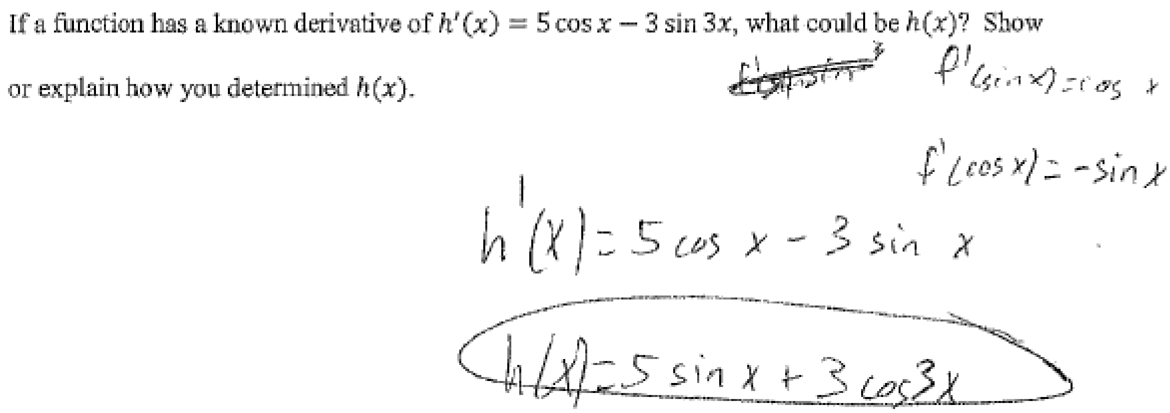
$$f'(x) = e^{-5x}$$

$$f(x) : e^{-5x} \cdot -5$$

Figure 51. Fred's solution to opening activity 3.3.1

Fred differentiated the given derivative and then labeled the result as $f(x)$. When this result is considered in the context of Fred's other work, especially his perfect solution on the same question with different surface characteristics in opening activity 3.3.2, the most reasonable conclusion is that Fred misunderstood the question.

Unlike Fred, Jill did not develop reversibility of the chain rule on the spot. Her initial response to opening activity 2.6.1, shown below in figure 52, indicates that Jill did not consider the effects of the chain rule when trying to find $f(x)$.



If a function has a known derivative of $h'(x) = 5\cos x - 3\sin 3x$, what could be $h(x)$? Show or explain how you determined $h(x)$.

$f'(\sin x) = \cos x$

$f'(\cos x) = -\sin x$

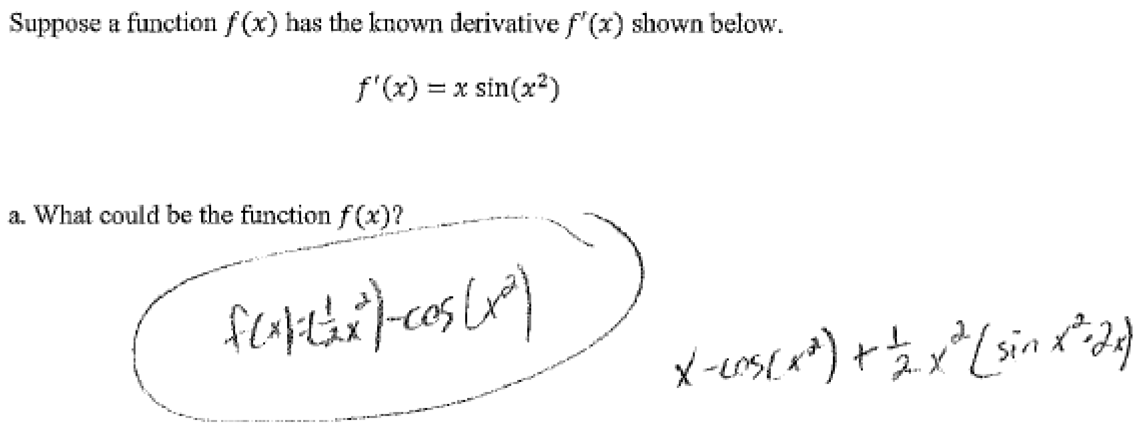
$h'(x) = 5\cos x - 3\sin x$

$h(x) = 5\sin x + 3\cos 3x$

Figure 52. Jill's solution to opening activity 2.6.1

The coefficient of 3 in front of the cosine term indicates that Jill did not use the chain rule to evaluate the quality of her answer and likely did not consider that the coefficient of 3 in $h'(x)$ was the result of the chain rule and not a pre-existing coefficient. It is possible that Jill did not recognize that the chain rule was involved in this problem.

In interview 2, Jill was able to recognize that reversibility of the chain rule was necessary to solve question 2; however, she evidenced no ability to reverse the chain rule. Her solution is shown below in figure 53.



Suppose a function $f(x)$ has the known derivative $f'(x)$ shown below.

$$f'(x) = x \sin(x^2)$$

a. What could be the function $f(x)$?

$f(x) = \left(\frac{1}{2}x^2\right) - \cos\left(x^2\right)$

$x - \cos(x^2) + \frac{1}{2}x^2\left(\sin x^2 \cdot 2x\right)$

Figure 53. Jill's solution to interview question 2.1

Jill began the problem by immediately considering that the chain rule is involved saying "so okay, that could be chain ... how could I make that chain?" However she was not able to propose a function that would differentiate to the given $f'(x)$. Specifically, Jill proposed the $f(x)$ shown above and then differentiated it to produce the expression on the right side of her solution. She knew this was incorrect saying "why is this stumping me so much … this should not be happening ... ok and that doesn't fit but this is the best that I've got." Jill's solution indicates that although she initially thought that the chain rule may be involved, she made no effort to consider the effects of the chain rule or how to reverse those effects. Furthermore, her solution indicates that she applied two-way reversibility to each factor in the derivative. Thus, the x led to the $\frac{1}{2}x^2$ in Jill's answer and the $\sin(x^2)$ in the derivative led to the $-\cos(x^2)$ in Jill's function. This result suggests that at the time of the 2nd interview, Jill had not developed any reversibility of the mental process in reasoning without reversible translation in relation to the chain rule.

Jill's score of "N/A" on the 3.3.1 paired problems are due to her incorrect use of the chain rule on the 3.3.1 exit slip; however, in this case, Jill's work on the exit slip and opening activity provide further evidence of her lack of reversibility of the mental process in reasoning without reversible translation of the chain rule. Jill's solutions to the 3.3.1 exit slip and opening activity are presented below in figures 54 and 55, respectively.

Find the derivative of the function.

$$f(x) = 3e^{4x}$$

Show your work or explain how you determined $f'(x)$. Can you think of any other possible functions for $f'(x)$?

$$f(x) = 3e^{4x}$$

$$f'(x) = 3e^{4x} \cdot (12)$$

Figure 54. Jill's solution to exit slip 3.3.1

Jill's work on exit slip 3.3.1 shows that Jill is aware of the derivative of the family of exponential functions; however, she is unable to correctly apply the chain rule.

Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$.

$$f'(x) = e^{-5x}$$

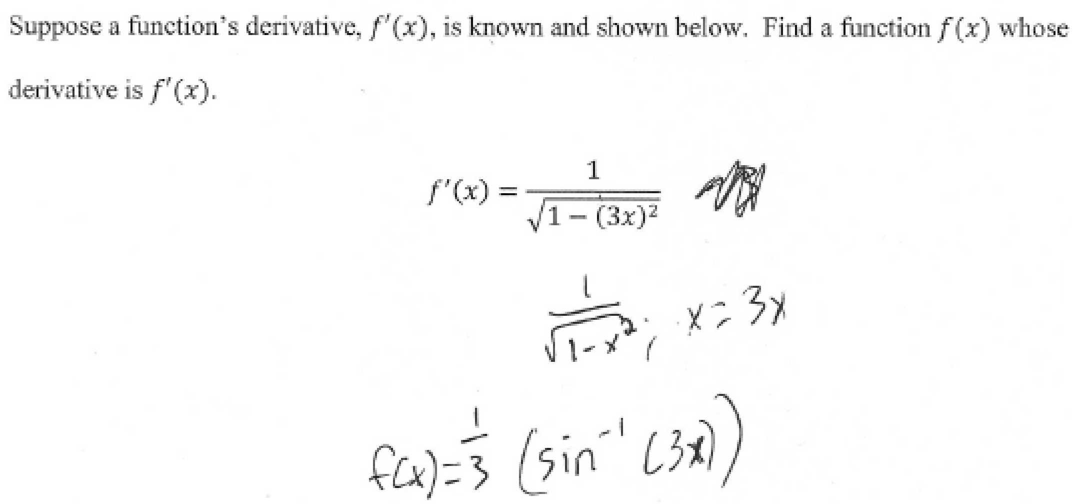
$$f'(x) = e^{-5x}$$

$$f(x) = e^{-5x}$$

Figure 55. Jill's solution to opening activity 3.3.1

On opening activity 3.3.1, we see that Jill seems to be aware of how to reverse the derivative of the exponential function, but does not account for reversing the derivative of the function within the exponent (i.e. reversing the chain rule). Thus, Jill does not provide evidence that she has a reversible conception of the chain rule.

After completing opening activity 3.3.1, the class received instruction regarding differentiation of the inverse trigonometric family of functions, including examples that required use of the chain rule. Upon completion of the instruction, the students attempted the 3.3.2 exit slip. Jill left class early that day, at the conclusion of receiving the notes. Thus, she did not attempt the exit slip even though she had been present for the day's learning. She did, however, attempt the 3.3.2 opening activity at the start of the subsequent class period. Her work is shown below in figure 56.


Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$.

$$f'(x) = \frac{1}{\sqrt{1-(3x)^2}}$$

$\frac{1}{\sqrt{1-x^2}}$; $x = 3x$

$f(x) = \frac{1}{3}\left(\sin^{-1}(3x)\right)$

Figure 56. Jill's solution to opening activity 3.3.2

Here, Jill shows that she has now developed a reversible conception of the chain rule. First, she recognized the form of the derivative of inverse sine and then by writing $x = 3x$, Jill shows that she recognizes that a function is composed inside of the inverse sine function. By writing $\frac{1}{3}$ in front of the expression, Jill shows that she understands that division is necessary to reverse the multiplicative factor created by using the chain rule.

Jill's work on the problems assessing reversibility of the mental process in reasoning without reversible translation in relation to the chain rule suggest that she did not develop reversibility on the spot and that development of reversibility of the chain rule took many examples which were spread over classroom activities, exit slips and opening activities, homework, tests, and an interview.

When Jill and Fred's work are viewed together, we see that for students in the middle flexibility group, there can be significant variability and inconsistency in the development of reversibility of the mental process in reasoning without reversible translation. Middle flexibility students can develop reversibility on the spot; they can also require a large amount of exercises and problem solving activities in order to develop reversibility of a particular mental process in reasoning without reversible translation.

Low flexibility group – Kirsten and Marcus

Kirsten and Marcus both struggled with developing a reversible conception of the chain rule. Neither student developed reversibility on the spot. However, there is some evidence to suggest that both Kirsten and Marcus developed some reversibility of the mental process in reasoning without reversible translation in relation to the chain rule over the course of the study. Kirsten and Marcus's results on the reversibility of the chain rule interview questions are reported in table 57 below.

Table 57. Reversibility of the mental process in reasoning without reversible translation with the chain rule tasks – low flexibility group

Participant	2.6.1 – 12/13/14	Interview Questions 2.1 & 2.2 – 12/17/13	3.3.1 – 1/14/14	3.3.2 – 1/16/14
Kirsten	N	N/A	N	Y
Marcus	N/A	N	N	Y

Both Kirsten and Marcus failed to demonstrate reversibility of the mental process in reasoning without reversible translation on the chain rule interview questions until the final question, at which point both students correctly used reversibility to solve opening activity 3.3.2. A review of each student's work, item by item, reveals a developmental trajectory for students in the low flexibility group.

Kirsten's four solutions to the reversible chain rule problems are shown below in table 58.

Table 58. Reversibility of the mental process in reasoning without reversible translation with the chain rule tasks – Kirsten's solutions

Question	Kirsten's Solution
2.6.1	If a function has a known derivative of $h'(x) = 5\cos x - 3\sin 3x$, what could be $h(x)$? Show or explain how you determined $h(x)$. $h(x) = 5\sin x + \cos(x^3)$
2.1	Suppose a function $f(x)$ has the known derivative $f'(x)$ shown below. $f'(x) = x\sin(x^2)$ a. What could be the function $f(x)$? $2x\sin(x^2)$
3.3.1	Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $f'(x) = e^{-5x}$ $f(x) = \frac{1}{e^{5x}}$

Table 58 (continued)

3.3.2	Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $$f'(x) = \frac{1}{\sqrt{1-(3x)^2}}$$ $f(x) = \frac{\sin 3x}{3}$

In opening activity 2.6.1, we see that Kirsten is not aware that the chain rule is involved. It is unclear why Kirsten wrote x^3 as the angle name for the cosine term. She is aware that reversibility is necessary and there is evidence that she uses two-way reversibility as she correctly reverses the trigonometric functions, but she exhibits no evidence of considering how to reverse the chain rule aspect of the derivative.

In interview question 2.1, Kirsten mixed using reversibility to find $f(x)$ and using the chain rule to find $f'(x)$. The results indicate that Kirsten struggled to determine which direction, forward or reverse, was called for by the problem. Kirsten began by correctly noting that "since the … derivative has the sine in it, you know the function has cosine in it", which is an example of two-way reversibility. However, she used the sine function in her solution instead of cosine. This may have just been a typographical error. Kirsten then says "it would be $2x$ out here … this should be sine and then x^2." Kirsten seems to indicate that she is now taking the derivative of x^2 and saying that the result of the chain rule should be included in the function. This result suggests that she no longer considers that she is trying to reverse differentiation and instead is just differentiating. Kirsten does not account for the effects of differentiating the angle name, x^2. Her solution suggests that she does not realize that the x in $f'(x) = x\sin(x^2)$ results from the chain rule instead of being part of the function. Furthermore, Kirsten makes no effort to differentiate her solution to test if it is correct.

In opening activity 3.3.1, Kirsten's work indicates that she may be aware of reversibility of the derivative of the exponential family of functions; however, there is no evidence of reversibility of the chain rule. The 3.3.1 exit slip clearly indicates that Kirsten is able to use the chain rule to take the derivative of a function of the form $f(x) = e^{g(x)}$; however, there is no evidence to suggest that Kirsten considers the need to reverse the chain rule.

In opening activity 3.3.2, Kirsten shows that she has realized that since the chain rule produces a multiplicative factor, then reversing the chain rule will necessarily require division. It is noted that she wrote $\sin(3x)$ instead of $\sin^{-1}(3x)$, this may have been a typographical error or she may have written the incorrect antiderivative. In either event, neither of those incorrect solution methods are evidence against reversibility of the mental process in reasoning without reversible translation. If Kirsten was not aware that $f(x)$ required an inverse sine term, then that would be evidence against two-way reversibility.

Marcus's four solutions to the reversible chain rule problems are shown below in table 59.

Table 59. Reversibility of the mental process in reasoning without reversible translation with the chain rule tasks – Marcus's solutions

Question	Marcus's Solution
2.6.1	If a function has a known derivative of $h'(x) = 5\cos x - 3\sin 3x$, what could be $h(x)$? Show or explain how you determined $h(x)$. $f(\square)\cdot\square'$ $h'(x) = 5\cos x - 3\sin 3x$ $h(x) = 5(\cos x) + 3\cos 3x$
2.1	Suppose a function $f(x)$ has the known derivative $f'(x)$ shown below. $f'(x) = x\sin(x^2)$ $\cos = -\sin$ $\sin = \cos$ a. What could be the function $f(x)$? $f'(x) = x\sin(x^2)$ $f\square\cdot\square'$ $f(x) = \frac{1}{2}x^2\cos 2x^{1/2}$ $f(x) = x^2$ $f'(x) = 2x^{1/2}$ $\frac{2}{1}\cdot\frac{1}{2} = \frac{2}{2} = 1$
3.3.1	Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $f'(x) = e^{-5x}$ $f(x) = e^{-5x} + 5$

Table 59 (continued)

3.3.2	Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $$f'(x) = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3$$ $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \cdot x'$ $\quad f(x) = \frac{1}{3}(\sin^{-1}(3x))$ $\hookrightarrow x = \frac{1}{3}$

On opening activity 2.6.1, Marcus shows that he expects the chain rule to be involved by writing the chain rule at the top of his paper. However, this is more likely due to the fact that the chain rule was the point of emphasis of the previous class rather than evidence of Marcus recognizing the chain rule within the problem. Marcus's solution to opening activity 2.6.1 shows no evidence of considering that the chain rule was used to create $h'(x)$ and thus there was no evidence that Marcus realized that the chain rule needed to be reversed.

On interview question 2.1, Marcus was able to identify that reversibility was necessary and that two-way reversibility was insufficient. He could not determine that the chain rule was involved. When describing how he was trying to solve the problem Marcus confused using the product rule, quotient rule, and chain rule, saying, "oh wait … pretty sure this is the product rule. Now I'm getting messed up between chapters ... well if this is the quotient rule, the quotient rule is f of the box times the box".[5]

[5] "The box" refers to the inside function of a composite function.

Once Marcus began considering the derivative as a result of the chain rule, he then confused how to reverse the chain rule, saying "if the derivative of the box is x^2 that means the box would have to be something to the 3rd". Marcus wrote x^3 and then erased that answer and changed it to x^2. Marcus continued to try to determine how taking the derivative effected the angle name, never realizing differentiation does not change the angle name. However, on the direct chain rule question, interview question 2.2, Marcus had no problem using the chain rule to differentiate $f(x) = \cos(x^2)$, which indicated that he knew that the chain rule did not change the angle name. Thus, reversing the chain rule caused significant confusion for Marcus.

Although his answer is incorrect, on opening activity 3.3.1, Marcus shows that he is now considering that the chain rule influences the solution to the problem. However, he does not reverse the chain rule correctly. By including the (-5) in his solution, Marcus shows that he is aware that the derivative of $-5x$ should be involved. However, he does not show that he considered reversing the multiplicative result of the chain rule. Thus, at this point, Marcus has not shown that he has considered how to reverse the chain rule in any of the reversible chain rule problems.

On opening activity 3.3.2, Marcus's work suggests that he has now developed some reversible conception of the chain rule. Here, Marcus works the two parts of the chain rule separately. He notes that since $3x$ is in the position of the angle name, there will be a 3 in the derivative. He then notes that since there is no three in the given derivative, there must be a $\frac{1}{3}$ in the function.

When Kirsten's work and Marcus's work on reversible chain rule problems are considered together, we see evidence of how difficult it can be for some students to develop reversibility of the mental process in reasoning without reversible translation. The students' initial exposure to the chain rule was on December 6, 2013 and the last opening activity assessing the chain rule was administered on January 15, 2014. In each of the 17 class meetings from the initial lesson on the chain rule until section 3.3.2, the chain rule was used to find derivatives of functions. It was not until the last day that the students in the low group began to think about the chain rule in a reversible mind set and it is reasonable to question whether or not Kirsten and Marcus would have correctly answered opening activity 3.3.2 if they had not tried opening activity 3.3.1. These results suggest that for low flexibility students, reversibility of the mental process in reasoning without reversible translation is not a natural or intuitive thought process, but instead may only develop as a result of the instructor requiring the students to engage with reversible problems in both directions.

Graphical analysis questions

Paired interview questions 2.3 & 2.4 and 4.3.a & 4.3.b examine reversible aspects of calculus graphing. Questions 2.3 & 2.4 assess reversibility of the graphical representation of the derivative at a point. Questions 4.3.a & 4.3.b assess reversibility of a graphical representation of the derivative on a continuous, finite domain.

Each pair of questions was analyzed for evidence of reversibility of the mental process in reasoning without reversible translation. Table 60 reports the existence or absence of reversibility of the mental process in reasoning without reversible translation on the interview questions. "Yes" indicates that the participant correctly solved both the forward and reverse questions; "No" indicates that the participant correctly solved the forward question but could not solve the reverse question. "N/A" indicates that the participant could not solve the forward question, thus rendering the question about the existence of reversibility moot.

Table 60. Is reversibility of the mental process in reasoning without reversible translation present in the interview questions?

Flexibility Group	Participant	2.3 & 2.4	4.3.a & 4.3.b
High	Kelsay	Yes	Yes
	Michael	Yes	No
Middle	Fred	Yes	Yes
	Jill	No	Yes
Low	Kirsten	Yes	N/A
	Marcus	No	N/A

The results of each flexibility group are discussed in detail below.

High flexibility group – Kelsay and Michael

Kelsay's solutions to questions 2.3 and 2.4 are presented below in figure 57.

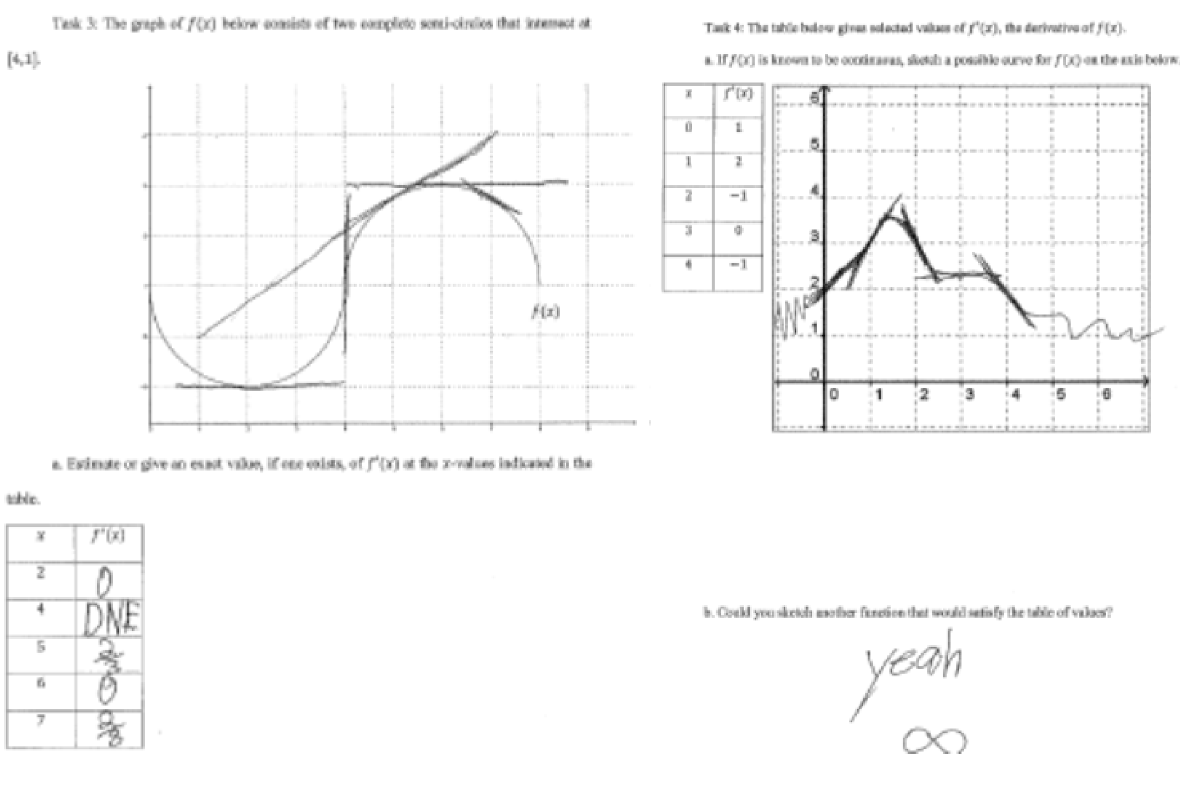


Figure 57. Kelsay's solutions to interview questions 2.3 and 2.4

Kelsay showed a complete reversibility of the mental process in reasoning without reversible translation about calculus graphing as she correctly solved all 4 questions. On the first set of paired problems, she showed her reversible understanding of the relationship between the graph of a function and the graphical representation of the derivative at a point.

On the forward question 2.3, Kelsay immediately invoked the graphical definition of the derivative as the slope of the line tangent to the curve at a specific x-value, saying, “if f' existed on this graph it would be of course equal to the slope” and then correctly found each f' value except for $f'(7)$ which she said was $\frac{2}{3}$ instead of $-\frac{2}{3}$. Kelsay’s mistake here was due to her correctly noting that $f'(7)$ would have the same magnitude as $f'(5)$ but not paying attention to the direction of the slope. On the reverse question 2.4, Kelsay noted that she could “sketch a million different functions” so long as the slope of the line tangent to the function is equal to the value in the table.

Kelsay’s solutions show her reversible understanding of the graphical representation of the derivative at a point. In the forward question, 2.3, Kelsay estimated each derivative value by sketching a tangent line and then finding the slope of the tangent line. On the reverse question, 2.4, Kelsay began by sketching line segments with the provided slopes at the appropriate x-value and then sketched a curve that was tangent to each line segment.

On questions 4.3.a and 4.3.b, Kelsay demonstrated reversibility of f and f' when provided a graph. On the forward question, 4.3.a, Kelsay was given the graph of f and asked to find an algebraic expression for f'. On the reverse question, 4.3.b, Kelsay was provided an identical graph that was now labeled f' and was asked to find an algebraic expression for f. Kelsay’s solutions to both questions are shown below in figure 58.

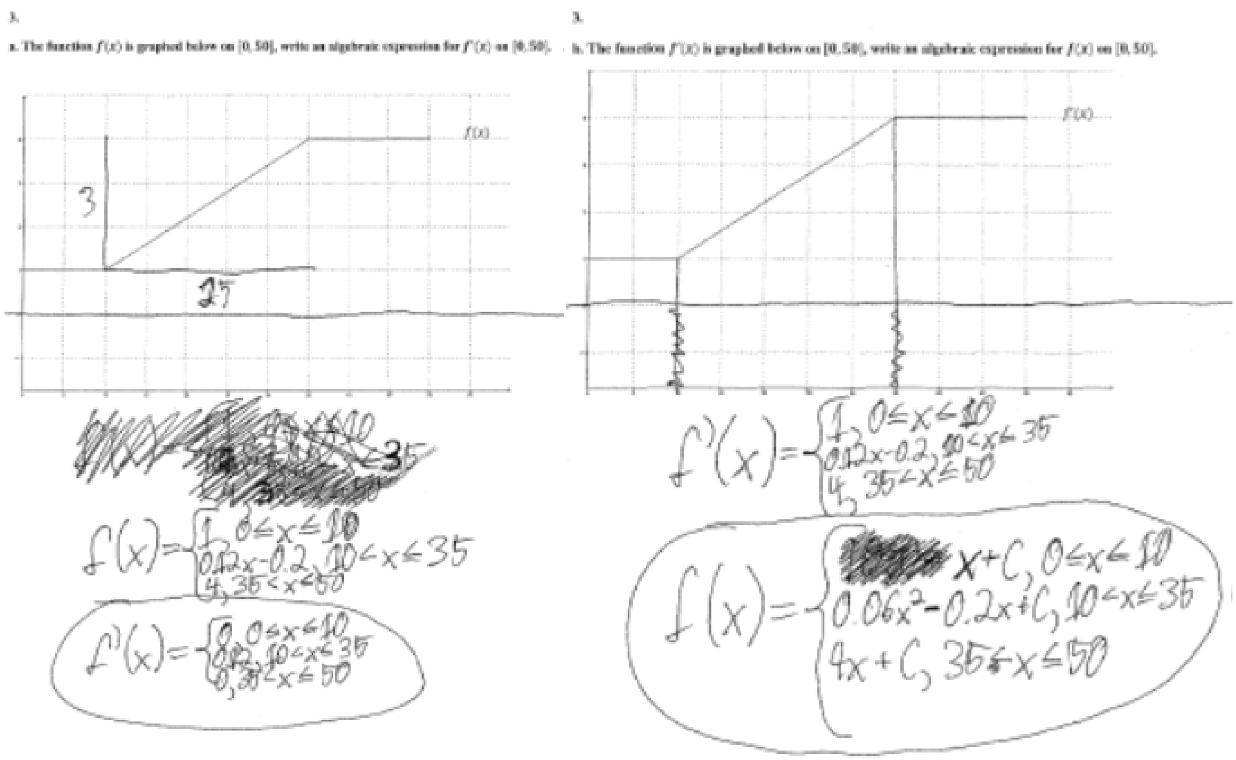


Figure 58. Kelsay's solutions to interview questions 2.3 and 2.4

Kelsay solved both problems by first translating the graphical representation into a piecewise algebraic expression. In the forward question, she took the derivative of the algebraic expression using simple differentiation rules. On the reverse question, Kelsay noted that she had two solution methods available. First, she said "I could just look at this [area under the curve] … and just make an equation out of that because these are definite lines, not curves." She then decided to write the algebraic expression for $f'(x)$ and then use two-way reversibility to find $f(x)$. Kelsay chose to reverse the method she had used on the previous question. She first used the slope and y-intercept of each line segment to produce a piecewise algebraic expression to represent f'. She then used two-way reversibility to find an acceptable equation for $f(x)$. It should be noted that Kelsay's awareness of how to find the equation of $f(x)$ by using the graphical interpretation of the reverse of the graphical derivative provides sufficient evidence to conclude that she has reversibility of the mental process in reasoning without reversible translation about graphing questions.

On questions 2.3 and 2.4, Michael was able to correctly solve the forward question, 2.3. He estimated the derivative at a point by sketching a tangent line and calculating the slope of the tangent line. On the reverse question, 2.4, Michael used the given derivative values to sketch line segments at the appropriate x-values and then sketched a continuous curve tangent to each of the line segments. Thus, as shown below in figure 59, Michael's solutions to questions 2.3 and 2.4 are nearly the same as Kelsay's.

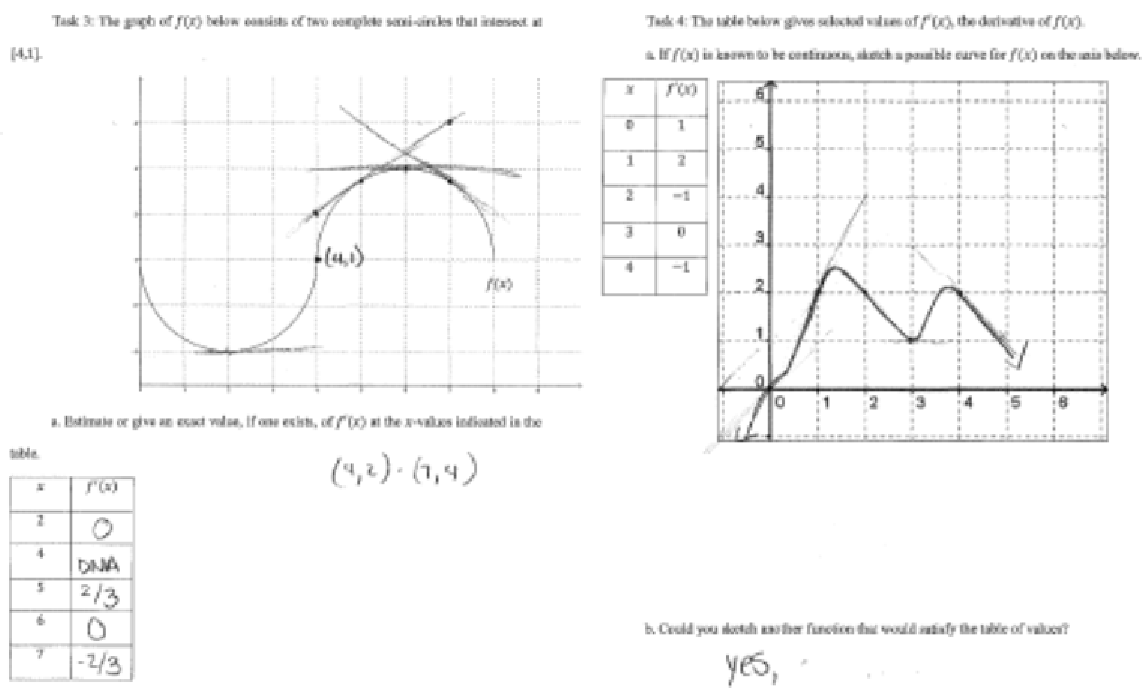


Figure 59. Michael's solutions to interview questions 2.3 and 2.4

Thus, Michael shows complete reversibility of the relationship between the graph of a function and the slope of the line tangent to the curve at a point. Even though he had only learned to read the derivative of a function from its graph by finding the slope of the tangent line, he was also able to use the slope of the tangent line to sketch the curve.

On the paired questions 4.3.a and 4.3.b Michael demonstrated a partial reversible understanding of the graphical representation of a derivative on an interval. Michael's solutions are shown below in figure 60 and then discussed.

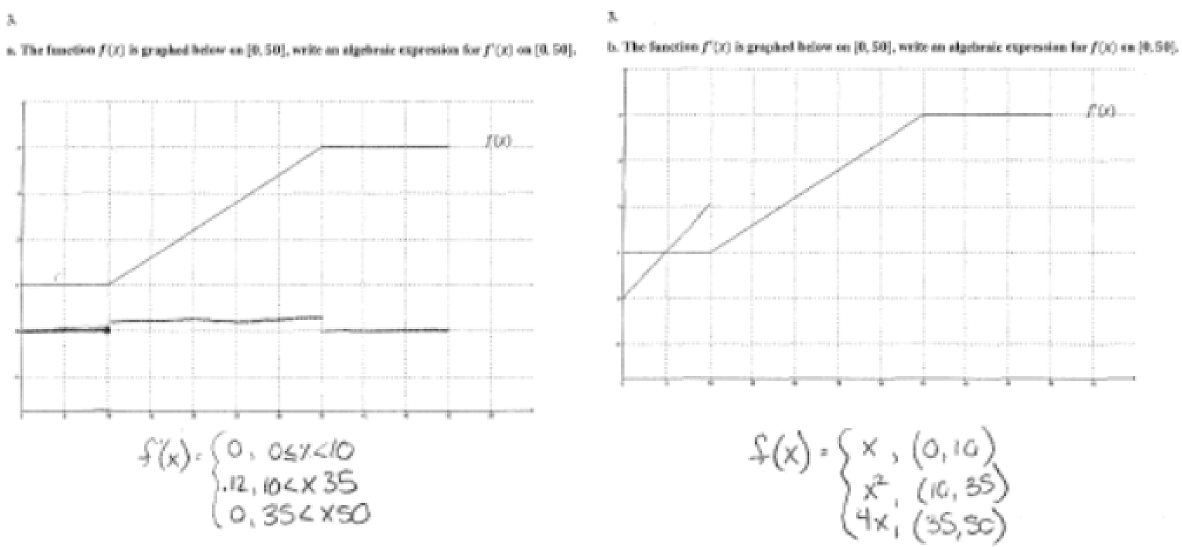


Figure 60. Michael's solutions to interview questions 4.3.a and 4.3.b

On forward question 4.3.a, Michael took a different approach to solving the question than Kelsay. Michael first graphed the derivative by noting that $f(x)$ is a linear piecewise function with three parts, thus the derivative consists of three horizontal line segments. Michael determined the height of each horizontal line segment in the derivative graph by finding the slope of each line segment in the function graph. He then translated the graph of $f'(x)$ into an algebraic expression to produce the correct answer.

On the reverse question, 4.3.b, Michael ran into difficulty finding an equation of a function whose derivative is a diagonal line. He again chose to try to sketch the desired graph first. He sketched the first part of the piecewise function correctly nothing that "so the slope on 0 to 10 is 1" and then correctly applied his line of thinking to the third line segment noting that the function would have to have $4x$ in it because the height of $f'(x)$ on $[35,50]$ is 4. However, Michael struggled to determine a function whose derivative would be the diagonal line shown in the question. Michael's thoughts on finding the algebraic expression for $f(x)$ on $[10,35]$ are well captured by the following excerpt from his interview.

> From 10 to 35 the slope it's not a definite slope, it would be a curve of some sort because it's increasing … this one is a little bit harder … you gotta assume it's going to be something squared … because if you take the derivative of that, it's a straight line … I guess I will just say x^2 on 10 to 35.

By observing that a second degree polynomial would be necessary because the derivative of a 2nd degree polynomial is a linear function, Michael has evidenced use of reversibility between the graphical representations of f and f'. However, he does not account for two additional pieces of information that are both necessary to determine an equation for $f(x)$ and are provided by the graph of $f'(x)$. First, Michael does not consider that the derivative of $f(x) = x^2$ would be $f'(x) = 2x$ and the graph shown on $[10,35]$ is not the graph of $2x$. The line segment has neither the slope of 2 nor a y-intercept of 0. Michael is correct that $f(x)$ should have a degree of 2, but $f(x) = x^2$ is insufficient. Secondly, Michael does not consider that a constant should be added to the function. This is noteworthy as Michael had noted on opening activities (2.3.1, 3.1.1, 4.6.1, 4.6.2) and interview questions, such as question 4.2.b, that reversing differentiation is an indefinite process and produces infinitely many solutions absent a known $f(x)$ value. Since $f'(x)$ exists for all x-values in $[0,50]$, $f(x)$ would have to be continuous, which will require a constant to be added to x^2 on $[10,35]$ in order for Michael's $f(x)$ to be continuous at $x = 10$. Similarly, although Michael correctly determined that the third segment of the piecewise function contained a $4x$, he did not find the missing constant that would make his solution correct. The lack of consideration of a constant suggests that Michael's use of reversibility of the mental process in reasoning without reversible translation with graphical questions exists to the extent that he realizes that the graph of the derivative will be one degree less than the graph of the function. If a graphical analysis requires further specificity beyond identifying the degree of the function given its derivative, Michael is unlikely to be able to use reversibility to solve the problem.

Kelsay's and Michael's interview questions assessing the reversible nature of the graphical representation of the derivative suggest that for students in the high group, reversibility of the graphical representation of the derivative, understanding that the derivative at a point is the slope of the line tangent to the curve at the given point, may develop on the spot. Both students' correct, well-described solutions to questions 2.3 and 2.4 indicate that both Kelsay and Michael fully understood how to determine a derivative at a point when given a graph of $f(x)$ and how to use the derivative at a point to sketch a possible graph of $f(x)$. Reversibility of an understanding of the graphical representation of the derivative on an interval may be more difficult for students to develop than a reversible understanding of the graphical representation of the derivative at a point. Kelsay was able to correctly solve questions 4.3.a and 4.3.b, although she avoided determining an equation for $f(x)$ directly from the graph of $f'(x)$. Rather, she first translated the graph of $f'(x)$ into an algebraic form and then used two-way reversibility to find an algebraic expression for $f(x)$. Michael attempted to find an algebraic expression for $f(x)$ directly from the graph of $f'(x)$. He was able to correctly identify the degree of each piece of the function. However, he could not produce an accurate answer for 2 out of the 3 parts of the piecewise function. It may have been the case that Michael would have been more successful solving the problem if he had pursued the same approach that Kelsay took.

Middle flexibility group – Fred and Jill

Fred and Jill showed similar degrees of reversibility of the mental process in reasoning without reversible translation situated in graphing questions during their interviews. Each correctly solved questions 2.3 and 2.4, using reversibility of the mental process on both of the reverse questions. Each student was also able to correctly solve the direct question 4.3.a. However, neither Fred nor Jill could completely solve question 4.3.b correctly. Each student partially solved the question correctly, indicating a present but not fully developed reversible conception of calculus graphing. I discuss each student's work in turn below.

Fred's solutions to interview questions 2.3 and 2.4 are shown in figure 61 and discussed below.

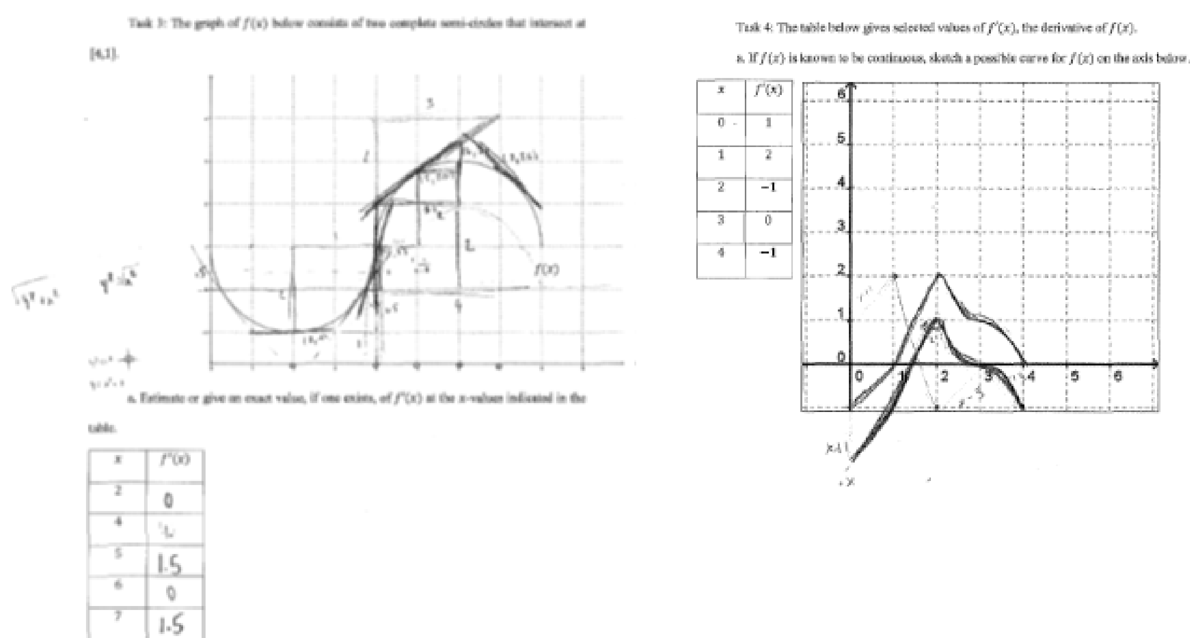


Figure 61. Fred's solutions to interview questions 2.3 and 2.4

On question 2.3, Fred began by trying to translate the graphical representation into an algebraic expression, saying, "If I remember a whole circle would be like $9^2 + x^2$ so I'm just trying to see what the semi-circle one was" but quickly determined this approach to be a dead-end saying "I have no idea what the equation is and I feel like if I did know it, it would probably help, but since I don't, I'm trying to figure it out some other way”. Fred was correct in thinking that an algebraic expression would make the problem simple; however, he was not able to create an algebraic expression for a semi-circle. Fred then said “since I have no idea what the equation is I'm going to try using the tangent line [Fred draws a line tangent to the curve] to figure out the derivative.” Fred then sketched in tangent lines at each of the appropriate x-values and found the slope of each tangent line. Thus, Fred showed a strong understanding of the graphical representation of the derivative in the forward direction.

On question 2.4, Fred did not use the given derivative values to sketch tangent lines and then sketch a curve suggested by the tangent lines. Instead, Fred began by noting that “I looked at … the derivative of f, the slope so I knew that if at 3 … the derivative was 0, it would have to somehow flatten out.” Fred used this connection to sketch out a piece of the curve that was horizontal at $x = 3$. Fred then used the signs of the derivative values to make inferences about the behavior of f, saying “the negative ones would mean that the $f(x)$ would be coming down and the $(0, 1)$ and the $(1, 2)$ would mean [the curve is] going up at around those points.” Fred then sketched a possible solution that is nearly correct. It should be noted that at $x = 2$, Fred’s graph appears to have a turning point when the turning point should be somewhere between $x = 1$ and $x = 2$.

Fred’s interview provides an example of an alternate method for how a student can reverse a learned process without traversing the learned steps in reverse. Kelsay and Michael solved the forward problem by starting with a given curve, sketching tangent lines, and then finding the slope of each line to fill in the table of derivative values. To solve the reverse problem, each student started with a table of derivative values, sketched tangent lines whose slopes were the derivative values, and then sketched a curve described by the tangent lines. Fred used the same approach for the direct problem; however, his approach to the reverse problem was completely different. On the reverse question, Fred started with the derivative (f') values, interpreted the f' value in relation to the behavior of f and then sketched the graph of f without the use of tangent lines. When comparing Kelsay's and Michael's solution to Fred's solution, we see one of the aspects of reversibility of mental processes in reasoning that differs from two-way reversibility. Two-way reversibility only has one possible solution method – the exact reverse of the forward solution method; whereas, reversibility of the mental process in reasoning without reversible translation often has multiple pathways by which a process can be reversed.

Fred's solutions to questions 4.3.a and 4.3.b are shown below in figure 62.

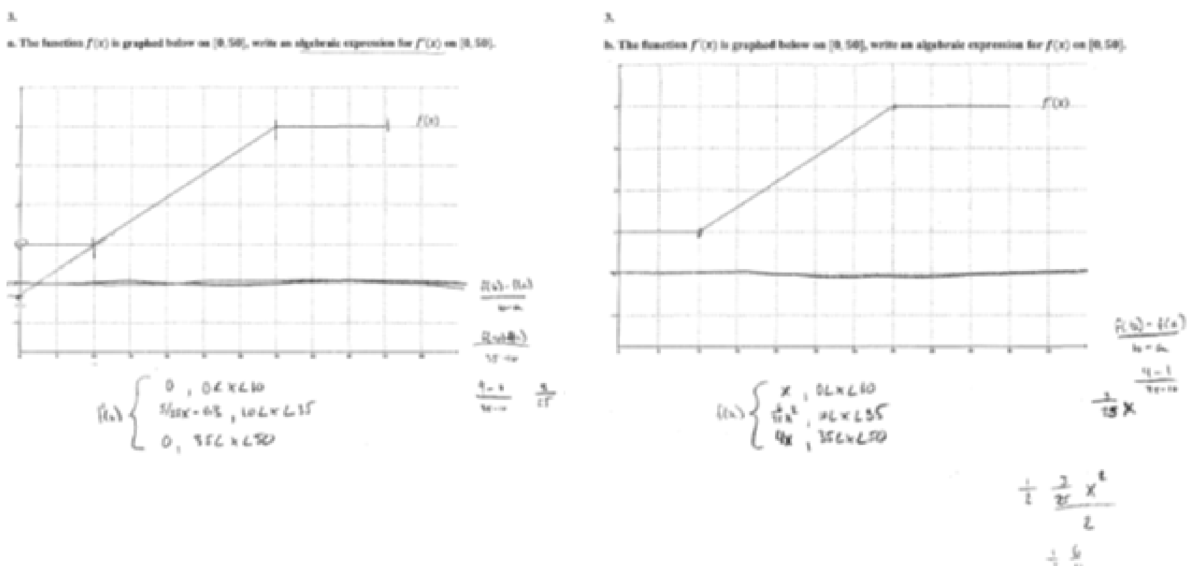


Figure 62. Fred's solutions to interview questions 4.3.a and 4.3.b

Fred solved the direct question 4.3.a by translating the graph of f to an algebraic expression of f' without needing to explicitly translate the graph of f into an algebraic expression, although his description of the question indicates that Fred mentally translated from a graphical representation of f into an algebraic expression of f. Fred first noted that he thought that the solution would be a piecewise function and that he needed to find the slope of each tangent line. He described the horizontal line segments on [0,10] and [35,50] as "constants" and then said "the derivative of a constant is zero" and thus concluded that $f'(x) = 0$ on [0,10] and [35,50].

Fred's solution to $f'(x)$on [10,35] was incorrect. He first found the slope of the line correctly to be $\frac{3}{25}$ using the slope formula. However, Fred seemed to lose sight of the fact that he was looking for $f'(x)$, not $f(x)$ as he attempted to estimate the y-intercept of the line segment to be -0.3, as shown by Fred's extending of the diagonal line until it intercepted the y-axis. Thus, he determined $f'(x)$ to be $\frac{3}{25}x - 0.3$ on [10,35].

On the reverse question, 4.3.b, Fred immediately noted, "this is basically the same exact graph. Now, the only difference here is that this is labeled now as the derivative, and it wants me to find an algebraic expression for just $f(x)$." Thus, Fred immediately noted that this question was the reverse of question 4.3.a. He began by writing the solution as a piecewise function and showed reversibility of his graphical understanding of the derivative on an interval saying that "since these two are constants, … [$f(x)$] would … be that number followed by x. So instead of it just being 1, it would be x. If … you reversed the derivative, … it would be x from $0 < x < 10$ … $4x, 35 < x < 50$." Fred then noted that he would need to reverse the simple power rule to find the function $f(x)$ on [35,50]. Fred posited that the function would have to have x^2 in it because the line has the equation $\frac{3}{25}x$. An incorrect use of two-way reversibility allowed Fred to conclude that on [10,35], $f(x) = \frac{6}{25}x^2$. Fred multiplied the numerator by 2 instead of multiplying the denominator by 2. Much like Michael, Fred did not account for the constants that would be necessary to correctly determine $f(x)$ from the graph of $f'(x)$.

Fred's solutions to questions 4.3.a and 4.3.b reveal a present but limited degree of reversibility. He was able to demonstrate reversibility of the graphical representation of the derivative by first using the slope of the linear function to find $f'(x)$ on the direct question, 4.3.a and then he used the y-value of the $f'(x)$ graph to determine the slope of the linear segments of the $f(x)$ graph on the reverse question 4.3.b. However, on the portion of the reverse question that was more complicated than the relationship between a constant and linear function, specifically, moving from a linear function to a quadratic function, Fred did not take into account the possible existence of a linear term or of a constant term.

Jill needed less that 90 seconds to solve question 2.3 and she did so without showing any work. When I asked Jill how she decided that $f'(2) = 0$, Jill replied, "it looks like a horizontal tangent right there". Jill later said that $f'(6) = 0$ because at "6 again that should be 0 because it looks like another horizontal tangent. Thus, Jill's description of how she solved question 2.3 make clear that she used the tangent line to the curve at a point to determine $f'(x)$ at the given x-values.

Jill also correctly solved question 2.4. Her solution is shown in figure 63 and discussed below.

Task 4: The table below gives selected values of $f'(x)$, the derivative of $f(x)$.

a. If $f(x)$ is known to be continuous, sketch a possible curve for $f(x)$ on the axis below.

x	$f'(x)$
0	1
1	2
2	-1
3	0
4	-1

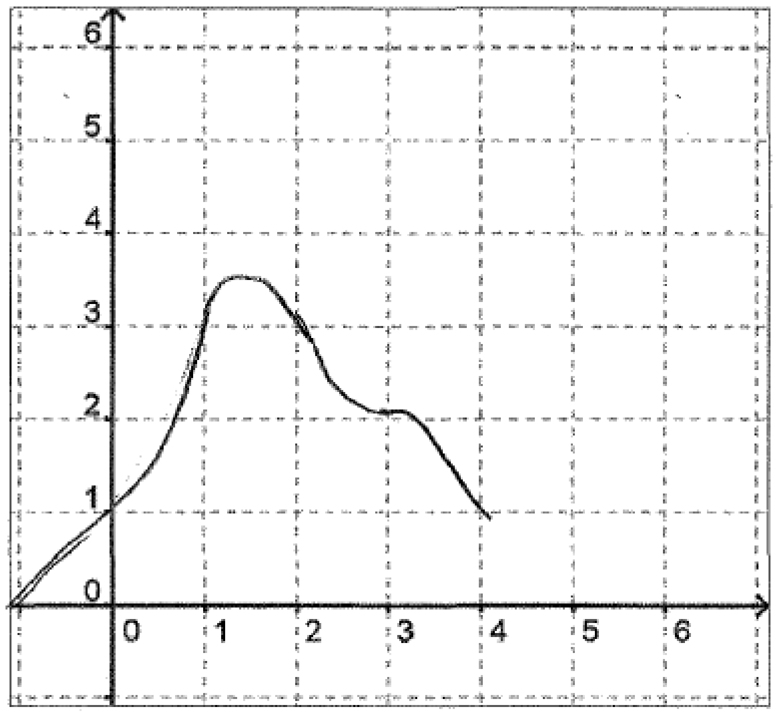


Figure 63. Jill's solution to interview question 2.4

Jill used the same approach used by Kelsay and Michael; she first sketched tangent lines at the given x-values with a slope of $f'(x)$. She then sketched a curve that was tangent to each line at the given x-value. Thus, Jill reversed her approach to question 2.3 in order to correctly solve question 2.4.

On paired questions 4.3.a and 4.3.b, Jill's work followed the same pattern shown by Michael and Fred. Jill correctly solved question 4.3.a by translating the graphical representation of $f(x)$ directly to the algebraic representation of $f'(x)$. She did so by noting that $f'(x) = 0$ on [0,10] and [35,50]. On [10,35], Jill said that "$f'(x)$ is average velocity [which] is the same as instantaneous for this part" and thus Jill determined that the slope of $f(x)$ was $\frac{3}{25}$ on [10,35] by using the difference quotient. Jill's solutions to questions 4.3.a and 4.3.b are shown below in figure 64.

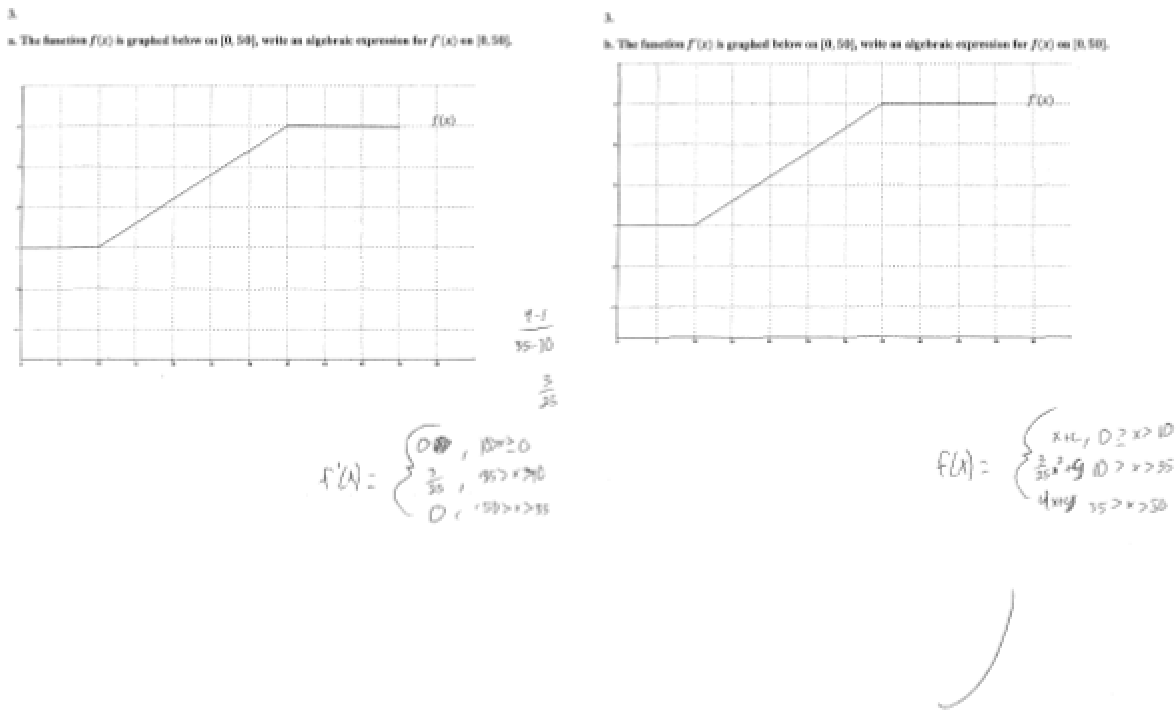


Figure 64. Jill's solutions to interview questions 4.3.a and 4.3.b

On the reverse question, 4.3.b, Jill attempted to translate directly from the graphical representation of $f'(x)$ to the algebraic representation of $f(x)$. She correctly noted that the horizontal line segment $f'(x) = 1, 0 \leq x \leq 10$ would become $f(x) = x + C, 0 \leq x \leq 10$ and that $f'(x) = 4, 35 \leq x \leq 50$ would become $f(x) = 4x + C, 35 \leq x \leq 50$. Jill then decided that since $f'(x) = \frac{3}{25}x$ on $10 \leq x \leq 35$, $f(x)$ should be $f(x) = \frac{3}{25}x^2 + C$ on $10 \leq x \leq 35$. This incorrect solution indicates that Jill was aware that the degree of $f(x)$ should increase, but she did not use two-way reversibility correctly to determine the coefficient in front of the x^2 term nor did she account for the fact that a linear term should exist as well. She made no effort to determine C and may not have realized that C was determined by her choice for C on the domain $0 \leq x \leq 10$.

Fred and Jill exhibited nearly identical amounts of reversibility on the paired graphing questions. Both students showed complete reversibility of the graphical representation of the derivative at a point; however, both students were only able to exhibit limited reversibility of the mental process in reasoning without reversible translation when using the graphical representation of the derivative on a continuous, finite domain. In the case of Fred and Jill, reversibility of the mental process in reasoning without reversible translation was limited to the case of a constant function and was not extended to a linear function. There is no evidence to suggest that either Fred or Jill could have used reversibility to find $f(x)$ from a graph of $f'(x)$ if $f'(x)$ consisted of any function more complicated than a constant function.

Low flexibility group – Kirsten and Marcus

Both Kirsten and Marcus had considerable difficulty with all or significant portions of the reversible graphing questions. Kirsten was able to successfully solve questions 2.3 and 2.4 but could not solve either question 4.3.a or 4.3.b. Marcus was not able to solve any of the four questions.

Kirsten's solutions to questions 2.3 and 2.4 are shown below in figure 65.

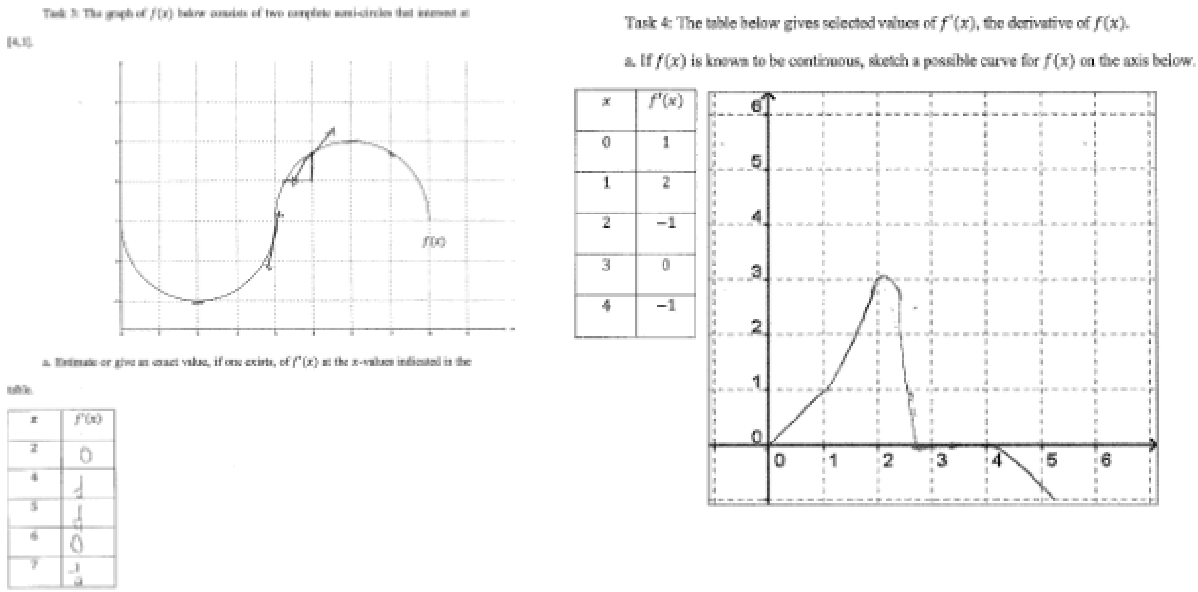


Figure 65. Kirsten's solutions to interview questions 2.3 and 2.4

She quickly solved question 2.3 saying "the derivative is just the slope" and then proceeding to estimate the slope of the line tangent to the curve at each given x-value. As can be seen in Kirsten's solution to question 2.3, her construction of the tangent line is poor at $x = 4$ and $x = 5$; however, her understanding of the graphical representation of the derivative appears to be sound.

Kirsten solved the reverse question 2.4 by using the same approach that we saw from Kelsay, Michael, and Jill. Kirsten first noted that the derivative tells her the slope, so she sketched in tangent lines with the given slopes at the appropriate x-values and then sketched a continuous curve that fit within the tangent lines. Thus, Kirsten showed reversibility of the mental process in reasoning without reversible translation with the graphical representation of the derivative.

During the 4th interview, Kirsten was not able to solve either the direct or reverse questions, 4.3.a and 4.3.b, respectively. Kirsten's answers to questions 4.3.a and 4.3.b are shown below in figure 66.

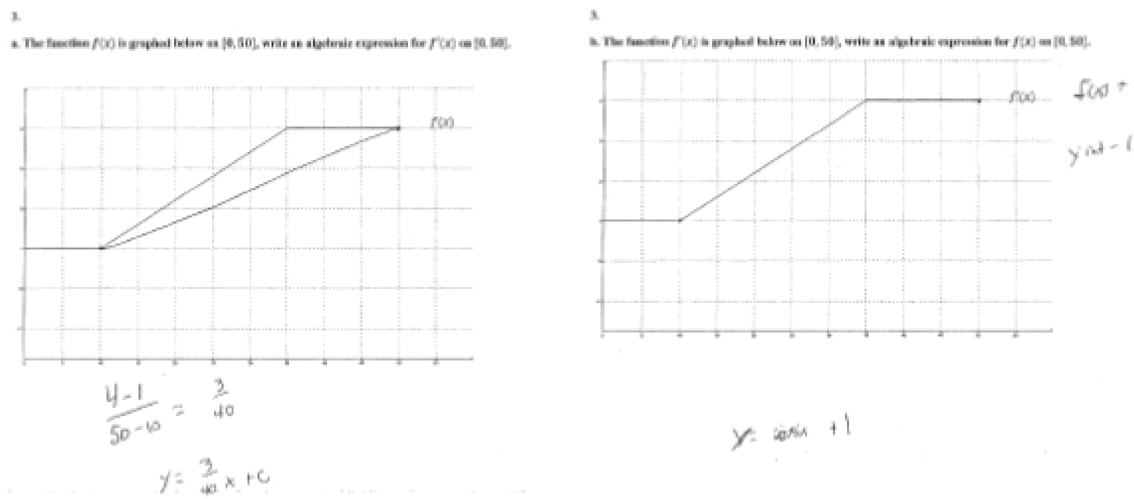
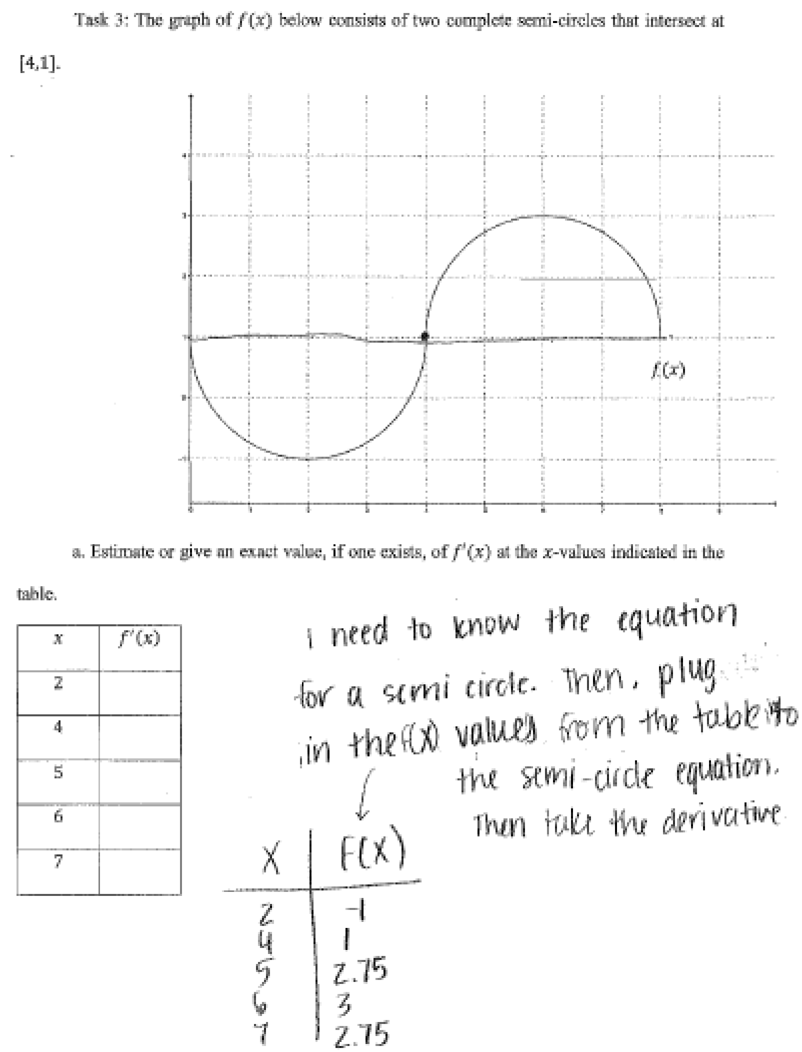


Figure 66. Kirsten's solutions to interview questions 4.3.a and 4.3.b

On the direct question 4.3.a, Kirsten correctly noted that $f'(x)$ would be defined by the slope of $f(x)$; however, Kirsten did not find the slope of each line segment. Instead, she constructed a secant line connecting the points at $x = 10$ and at $x = 50$ and found the slope of the secant line. She then used the slope of the secant line to write a linear equation for $f'(x)$. Her final answer of $y = \frac{3}{40}x + C$ is evidence against Kirsten's observation that "$f'(x)$ would just be the slope". Her final answer suggests that Kirsten thinks that the derivative of a linear function is itself a linear function.

On the reverse question 4.3.b, Kirsten showed no evidence of understanding the relationship between the graph of $f'(x)$ and $f(x)$. Kirsten began by saying "well whatever f is, it would be positive because this [$f'(x)$] is above the zero". This is inaccurate as $f'(x)$ does not influence the sign of $f(x)$. Kirsten completes the problem by using the same slope, $\frac{3}{40}$, as she did in the forward problem. When I asked her what the $\frac{3}{40}$ represented, Kirsten replied "the slope of this line right here". Kirsten was referring to the secant line that she constructed on question 4.3.a. This suggests that Kirsten did not recognize that the problems were a reverse pair.

Marcus showed no understanding of the graphical representation of the derivative. His attempt at the forward question 2.3 is shown below in figure 67.



Task 3: The graph of $f(x)$ below consists of two complete semi-circles that intersect at [4,1].

a. Estimate or give an exact value, if one exists, of $f'(x)$ at the x-values indicated in the table.

x	$f'(x)$
2	
4	
5	
6	
7	

i need to know the equation for a semi circle. Then, plug in the f(x) values from the table to the semi-circle equation. Then take the derivative

X	F(X)
2	-1
4	1
5	2.75
6	3
7	2.75

Figure 67. Marcus's solution to interview question 2.3

Marcus made no attempt to use the graph of $f(x)$ to determine the values of $f'(x)$. Consistently throughout the course, Marcus expressed discomfort with using graphs. He did not consider using the line tangent to the graph in order find the derivative; rather, he felt strongly that he would need the algebraic expression in order to solve the problem. Since he could not determine the equation of the curve, Marcus could not solve the problem.

On the reverse question 2.4, Marcus noted that the question was “the opposite” but had no means of using the $f'(x)$ values to sketch $f(x)$. Marcus attempted to graph $f'(x)$ but made no connections between the sketched graph of $f'(x)$ and a possible graph of $f(x)$. Marcus then tried to create an algebraic expression for $f'(x)$ and seemed to try to use the chain rule but was unsuccessful. His work is shown in figure 68 below.

Task 4: The table below gives selected values of $f'(x)$, the derivative of $f(x)$.

a. If $f(x)$ is known to be continuous, sketch a possible curve for $f(x)$ on the axis below.

x	$f'(x)$
0	1
1	2
2	−1
3	0
4	−1

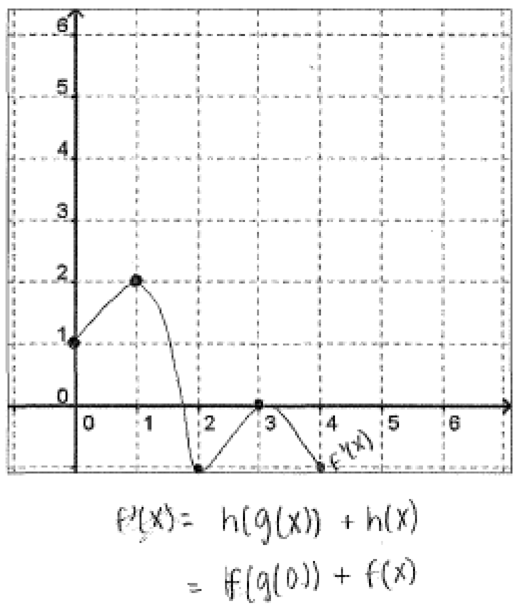


Figure 68. Marcus’s solution to interview question 2.4

At the time of the 2nd interview, Marcus showed no understanding of the graphical representation of the derivative. During the 4th interview, Marcus was to solve the forward question 4.3.a, but could not make sense of what the reverse question 4.3.b required. His solutions to questions 4.3.a and 4.3.b are shown below in figure 69.

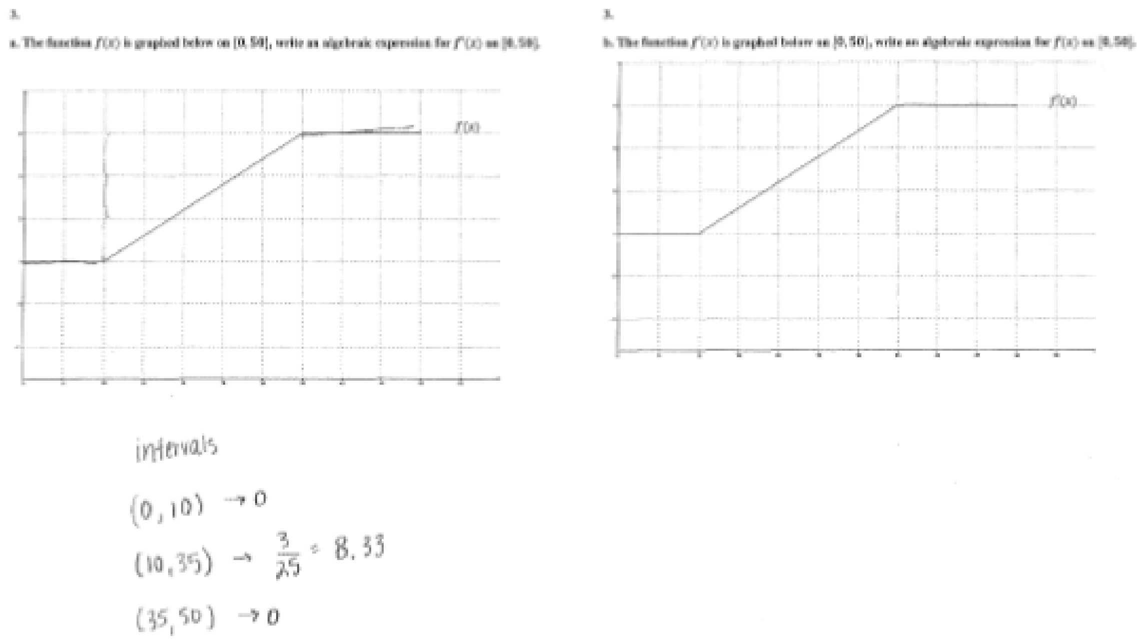


Figure 69. Marcus's solutions to interview questions 4.3.a and 4.3.b

On question 4.3.a, Marcus began by stating his disdain for graphs saying, "I just really hate graphs" but then noted that $f(x)$ was "constant, increasing, constant" and observed that "the derivative is the tangent line, which would just be zero (on $[0,10]$)". While Marcus's description of the derivative as "the tangent line" is incorrect, his conclusion suggests that he was thinking about the derivative as the slope of the tangent line, which is an improvement from the 2nd interview. He correctly found each of the slopes, although he divided $\frac{3}{25}$ incorrectly in his calculator. He was not confident that just finding the slope of each line segment was sufficient as the entirety of $f'(x)$ and it was not clear that Marcus was aware that the answer should be written as a piecewise function.

Marcus was not able to make any attempt at the reverse question 4.3.b, saying that "I don't know [what] information I would need to find to write an expression. I don't even understand what that means to do."

Marcus's results on questions 2.3 & 2.4 and 4.3.a & 4.3.b support the observation that reversibility does not develop on the spot for students in the low group. Marcus was not able to solve the direct question assessing the graphical representation of the derivative at a point during the 2nd interview but had developed enough understanding of the graphical representation of the derivative to be able to solve the direct question in the 4th interview. However, Marcus had developed no reversibility, which is consistent with his other interview and exit slip/opening activity data. Marcus needed multiple opportunities to engage with reversibility in a particular content area for reversibility to develop.

For the students in the low flexibility group, there was very limited evidence of reversibility of the mental process in reasoning without reversible translation. Kirsten showed reversibility of the mental process in reasoning without reversible translation about the graphical representation of the derivative at a point. She could not solve the direct question assessing understanding of the graphical representation of the derivative over a continuous, finite domain. Thus, there was no possibility that Kirsten could have reversibility about the graphical representation of the derivative over a continuous, finite domain. Marcus was not able to solve the direct question assessing reversibility of the mental process in reasoning without reversible translation about the graphical representation of the derivative at a point, thus he could not demonstrate reversibility of the concept. He was able to solve the direct question assessing understanding of the graphical representation of the derivative over a continuous, finite domain but could not start the reverse question.

Position and velocity questions

All of the interview participants demonstrated reversibility of the mental process in reasoning without reversible translation on questions 4.1.b and 4.2.b. As discussed earlier, these questions assessed reversibility of position and velocity. In section 4.2.1, I discussed the elements of the problem that required two-way reversibility. The element of this set of paired problems that requires reversibility of the mental process in reasoning without reversible translation is the reversibility of the relationship between position and velocity. The results here agree with the results of the 4.6.2 exit slip and opening activity pair. Specifically, the interview participants' understanding of the reversible relationship between position and velocity is well-developed and supports the observation that the class as a whole has a well-developed sense of the reversible relationship between positon and velocity.

Paired interview problems 4.1.a and 4.2.a also assessed reversibility of position and velocity; however, the results are not as conclusive as the result for questions 4.1.b and 4.2.b. Whereas for questions 4.1.b and 4.2.b in which all 6 participants demonstrated reversibility of the mental process in reasoning without reversible translation between position and velocity, only 3 participants demonstrated the same reversibility on questions 4.1.b and 4.2.b. Conceptually, both sets of paired questions assessed reversibility of position and velocity. The difference between the nature of the questions is representational. Questions 4.1.b and 4.2.b were only symbolic. The problem provided a symbolic expression and required an output that used a symbolic representation. Questions 4.1.a and 4.2.a used numerical representations. For the students, it was much more difficult to consider the reversible nature of position and velocity when the function was presented in a numerical representation. Table 61 records the six interview participants' results on interview questions 4.1.a and 4.2.a. "C" means correct, "I" means incorrect, "Y" means that reversibility is present, "N" means that reversibility is not present, and "N/A" means that the results of the forward and reverse questions do not allow me to draw a conclusion regarding the presence of reversibility.

Table 61. Results of questions 4.1.a and 4.2.a

Flexibility Group	Participant	4.1.a (forward)	4.2.a (reverse)	Is reversibility present?
High	Kelsay	C	C	Y
	Michael	C	I	N
Middle	Fred	C	C	Y
	Jill	I	C	N/A
Low	Kirsten	I	I	N/A
	Marcus	I	I	N/A

It was anticipated that students who correctly solved question 4.1.a would use the difference quotient, often written as $v_{ave} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$. Then on question 4.2.a, since the students are given velocities and times, they would need to use reversibility to find a distance. Typically, in a calculus class, one way to estimate distance traveled is to use a Riemann sum consisting of the product of a velocity and a time interval. That approach would be the reverse of the average velocity difference quotient.

Despite emphasis throughout the course on symbolic, graphical, and numerical representations of the average rate of change, only three of the students (both students in the high group and one student in the middle group) could correctly solve the direct question. Of those three students, only two students could correctly solve the reverse problem. Since the students' work on questions 4.1.b and 4.2.b clearly indicate that the students understand that position and velocity are reversible relationships through the calculus operations of differentiation and anti-differentiation, the relative lack of success on questions 4.1.a and 4.2.a suggest that the students are much less adept, or possibly not aware at all, about how to reverse the process of estimating position and velocity.

High flexibility group – Kelsay and Michael

As she did throughout the study, Kelsay correctly solved both questions. On the forward question, she immediately recognized that the question was calling for use of the average rate of change formula saying, “so this is just average velocity between these points and these points and these points and finally these points. Okay, that's easy enough.” She took a less straight-forward approach to the reverse problem. She first used a kinematics formula to correctly solve the problem. I then asked if she could have done the problem any other way and she identified that she could construct a graph of the given velocity points and then use the points to estimate the area under a possible velocity curve. When I asked why she would calculate the area under the curve, Kelsay replied, “because, taking a derivative of a position in physics would give you meters/second … and the [area] would take off a per second or multiply by seconds.” If Kelsay had limited her answer to using a physics formula, then we would not have evidence of reversibility of the mental process in reasoning without reversible translation because she would have used a direct physics equation to solve the problem. However, with Kelsay’s explanation of using a dimensional analysis as a reason for calculating the area under a curve, we see clear evidence of a reversible conception of moving between position and velocity that is not dependent on a learned two-way process.

Michael, also of the high flexibility group, had no difficulties with 4.1.a saying, “so it says estimate so I'm just going to use the average velocity …$\frac{d(t_2)-d(t_1)}{t_2-t_1}$. That's the general formula I would use to find the average velocity at each of these points.” Michael found question 4.2.a to be quite challenging. He attempted to find the average velocity of the graph, but then concluded that finding the average velocity was not relevant to the problem. He then observed that his struggle with this problem is “I can't remember what distance is, is the problem. I can't remember

I feel like it's the absolute value of position. I don't remember ... I can't figure it out." This statement suggests that Michael's difficulty may have been due to the vocabulary of the problem. However, he did not show any ability to move from velocity back to position or distance. It is more likely that his difficulty was the numerical representation and lack of a velocity equation.

Middle flexibility group – Fred and Jill

Fred's solution to question 4.2.a is shown in figure 70.

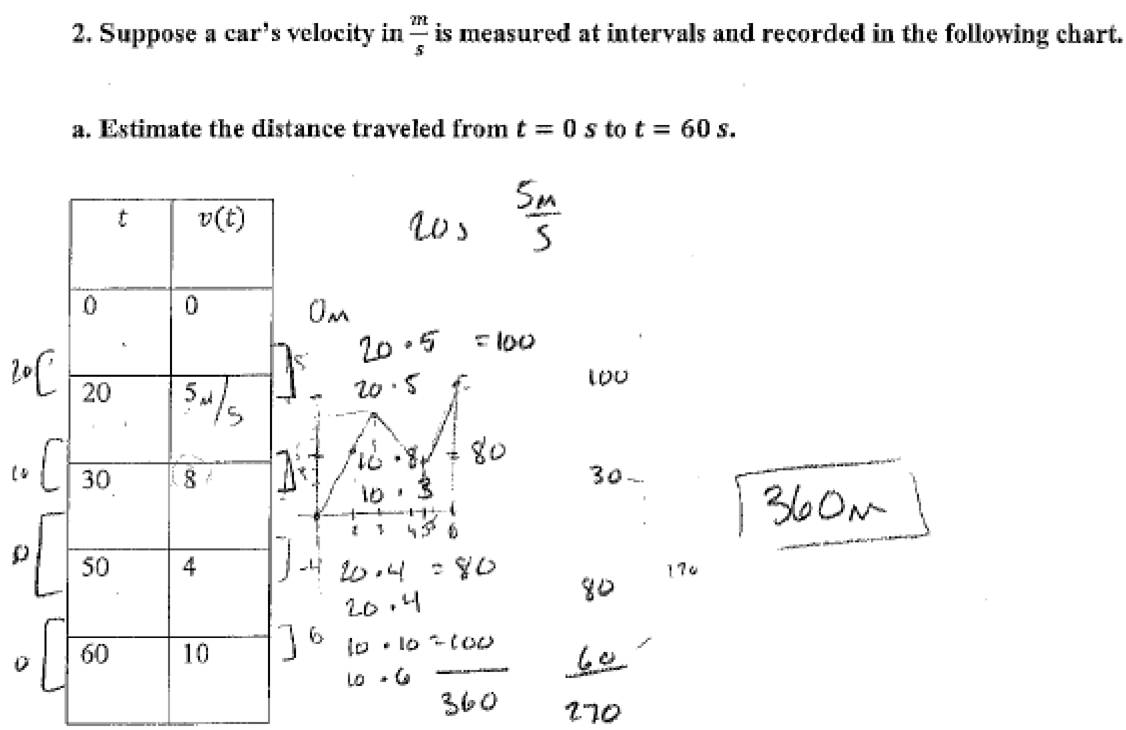


Figure 70. Fred's solution to interview question 4.2.a

He showed complete reversibility of the mental process in reasoning without reversible translation when estimating position and velocity. Fred immediately solved the direct question, 4.1.a, by using the average velocity formula. The reverse question caused Fred to hesitate as he spent nearly four minutes trying to make sense of the question. He then decided to translate the numerical table into a graphical representation, saying, "I'm just going to plot it real quick, to get an idea of what's happening." He then considered multiplying the width of the time interval by 1) the velocity at the end of the interval, resulting in a sum of 360 m, and 2) by the difference of the starting and ending velocity of each interval, resulting in a sum of 270 m. Fred chose to pick the answer 360 m. He expressed no particular reason for why he picked 360 m over 270 m. In effect, Fred chose to use the right hand Riemann estimation to find the distance traveled instead of using the equation $d = \Delta v * t$.

Jill's solution to question 4.1.a is shown below in figure 71.

1. The table below gives the distance a car has traveled, measured in miles, at selected time measurements in hours.

a. If the car only moves in a positive direction, fill in the accompanying table by estimating the velocity of the car in miles per hour at the times indicated.

t (hours)	Distance (miles)
0	0
0.2	8
0.4	17
0.5	21
0.8	35
1.0	44
2.0	100

Averaged y values at x=1 & x=.8 to find y value at x=.9

t (hours)	Average Velocity (mph)
0.1	20
0.6	25.6
0.9	43.9
1.5	72

$V = \frac{\Delta x}{t}$

$\frac{8-0}{.2-0}$

40

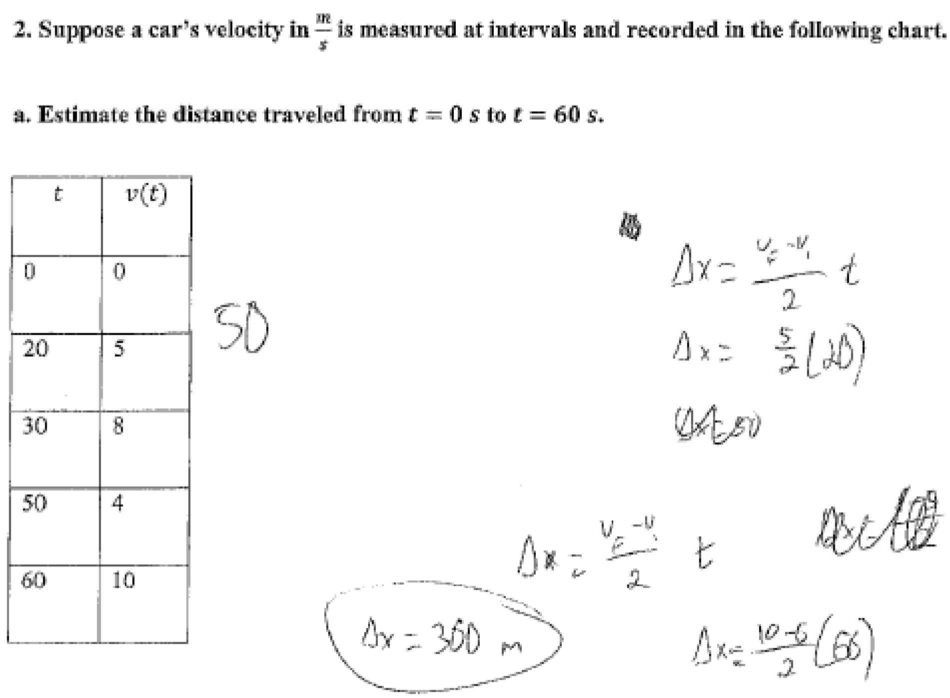
$\frac{35-21}{.8-.5}$

$46.\bar{6}$

Figure 71. Jill's solution to interview question 4.1.a

She originally took the correct approach to the forward question, 4.1.a, by using the equation $v = \frac{\Delta x}{t}$, however, the word "average" caused her difficulty. She decided to divide each of her results by 2 to account for finding the average, resulting in incorrect answers. In this case, Jill's incorrect answers are due to a limitation in vocabulary. From a conceptual perspective, Jill was fully aware of how to estimate a velocity from a table of observed positions.

Jill's solution to the reverse problem, question 4.2.a, is shown below in figure 72.



2. Suppose a car's velocity in $\frac{m}{s}$ is measured at intervals and recorded in the following chart.

a. Estimate the distance traveled from $t = 0\ s$ to $t = 60\ s$.

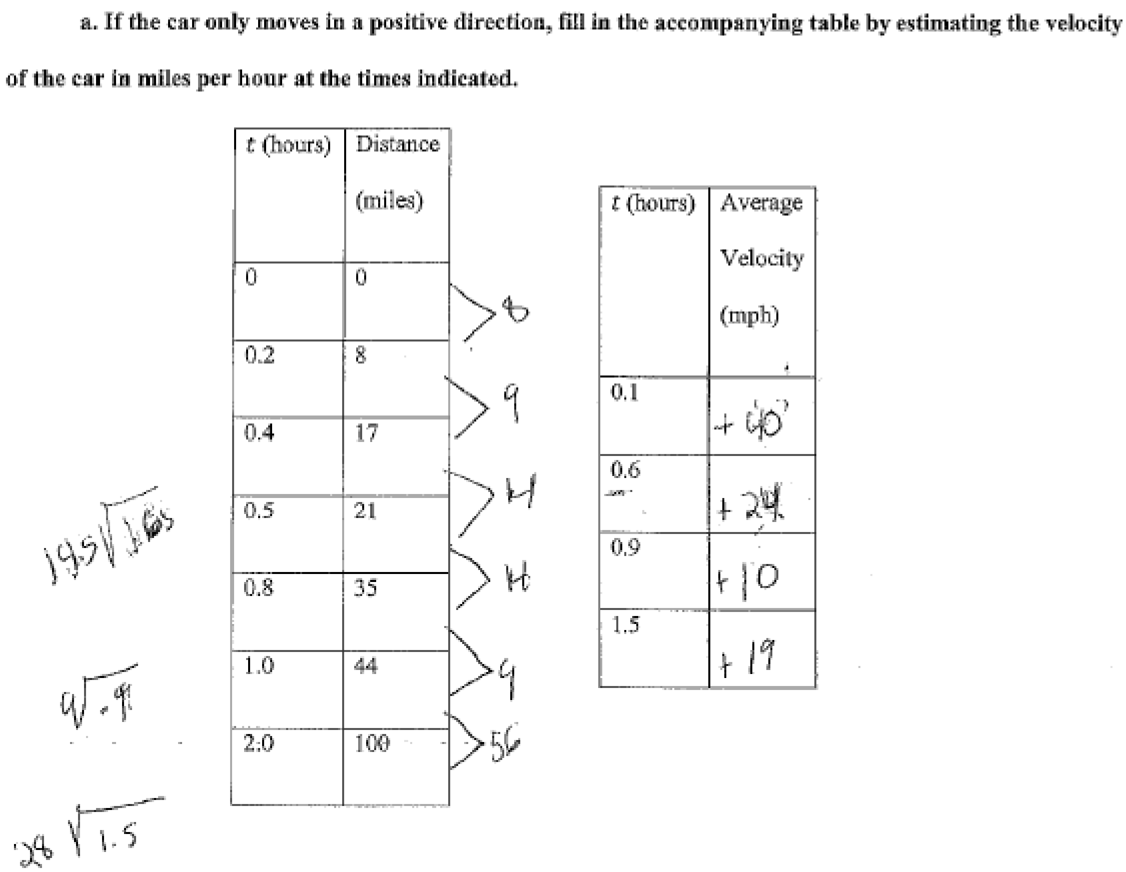
t	$v(t)$
0	0
20	5
30	8
50	4
60	10

Figure 72. Jill's solution to interview question 4.2.a

Jill took the same approach as Kelsay and used a physics formula. She began by writing the formula $\Delta x = \frac{v_f - v_i}{2} t$ and then making the appropriate substitutions. At this point, it is difficult to conclude that Jill used reversibility to solve this problem. Rather, she attempted to use a forward learning of a physics equation to solve the problem. Thus, we do not have evidence of reversibility of the mental process in reasoning without reversible translation. It should be noted that Jill's equation is incorrect. The correct kinematics equation is $\Delta x = \frac{v_f + v_i}{2} t$. However, in this particular instance, Jill's use of subtraction instead of addition did not matter because her initial velocity was zero mps.

Low flexibility group – Kirsten and Marcus

Neither Kirsten nor Marcus were able to solve the direct question, 4.1.a. Kirsten's solution to question 4.1.a is shown below in figure 73.



a. If the car only moves in a positive direction, fill in the accompanying table by estimating the velocity of the car in miles per hour at the times indicated.

t (hours)	Distance (miles)
0	0
0.2	8
0.4	17
0.5	21
0.8	35
1.0	44
2.0	100

t (hours)	Average Velocity (mph)
0.1	+40
0.6	+24
0.9	+10
1.5	+19

Figure 73. Kirsten's solution to interview question 4.1.a

She immediately expressed a desire for an algebraic expression saying, "well, it would help if I had an equation to plug back into … to find the velocity". Kirsten correctly found the change in distance over each time interval. However, she struggled with using the change in distance to find an average velocity. She then remembered that $d = \frac{v}{t}$. However, she could not make sense of what to use for time. She eventually settled on using the t-value listed in the average velocity chart and thus divided the change in distance by a specific time value instead of the change in time.

As such, she never accounted for the width of the time interval. She was able to recognize that the change in position was relevant to finding the average velocity.

Kirsten's solution to the reverse interview question 4.2.a is shown below in figure 74.

2. Suppose a car's velocity in $\frac{m}{s}$ is measured at intervals and recorded in the following chart.

a. Estimate the distance traveled from $t = 0\ s$ to $t = 60\ s$.

t	$v(t)$
0	0
20	5
30	8
50	4
60	10

$V = \frac{d}{t}$

$60 \cdot 10 = \frac{d}{60} \cdot 60$

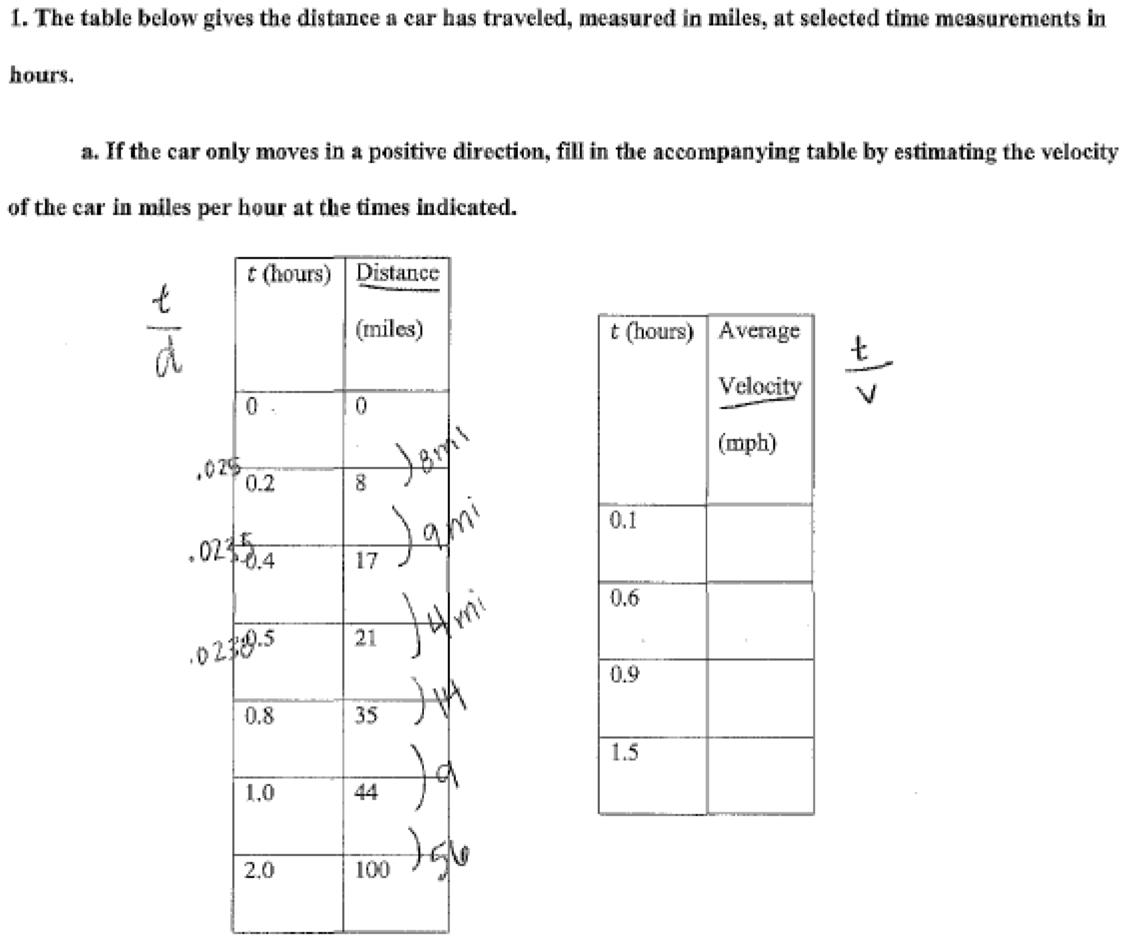
$600 = d$

Figure 74. Kirsten's solution to interview question 4.2.a

She showed some reversibility by writing the equation $v = \frac{d}{t}$ and saying "I'm going to plug back into that equation."

In effect, Kirsten used a right hand Riemann sum with only one rectangle, although she did not realize that at the time. She did not consider how velocity changing at each measured time effected distance traveled. Kirsten's use of reversibility was limited in this instance as she accounted for the change in distance on the direct problem but did not account for the change in velocity on the reverse problem. A fully developed conception of reversibility of the mental process in reasoning without reversible translation would have considered that if the change in a quantity effected the result in the direct question, then a change in a quantity would likely effect the reverse question. There is no evidence that Kirsten considered such a thought process.

Marcus was not able to make any sense of the direct question 4.1.a. His work is shown below in figure 75.



1. The table below gives the distance a car has traveled, measured in miles, at selected time measurements in hours.

a. If the car only moves in a positive direction, fill in the accompanying table by estimating the velocity of the car in miles per hour at the times indicated.

$\frac{t}{d}$

t (hours)	Distance (miles)
0	0
0.2	8
0.4	17
0.5	21
0.8	35
1.0	44
2.0	100

.025 .0235 .0238

8mi 9mi 4mi 14 9 56

t (hours)	Average Velocity (mph)
0.1	
0.6	
0.9	
1.5	

$\frac{t}{v}$

Figure 75. Marcus's solution to interview question 4.1.a

Marcus considered that the change in distance traveled over each time period may be relevant to finding the average velocity and thus calculated each difference. However, he then said, "I don't know if that helped at all" and subsequently ignored the differences. He then pursued the equation $v = \frac{t}{d}$, which is incorrect. After calculating several quotients (for example, $\frac{0.2}{8} = 0.025$), Marcus determined that those answers did not make sense within the context of the problem and left the problem unanswered.

On the reverse question, 4.2.a, Marcus began by sketching a graph of the given velocities as a sense-making activity, saying, "… not really sure so I feel like I should ... do a graph even though I hate graphs". Marcus's solution on question 4.2.a is shown below in figure 76.

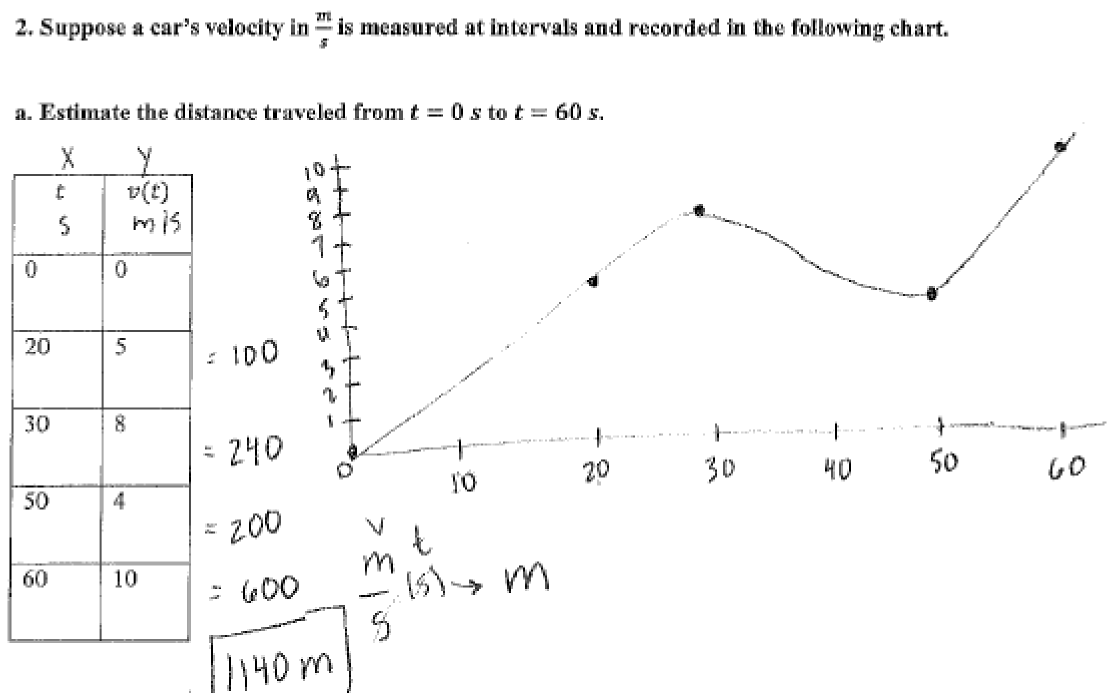


Figure 76. Marcus's solution to interview question 4.2.a

After completing the graph, Marcus was not able to use the graph to answer the question, saying "I don't really think that helped me at all". He then proceeded to a dimensional analysis saying, "well if that's meters per second and I want it to be in meters, then I am getting rid of the seconds, so that's like multiplying time and your velocity, so maybe I just multiply them." Marcus then added the four products to reach a total distance traveled of $1{,}140\ m$. Marcus did not account for the fact that the vehicle did not travel at $10\ m/s$ for the entire 60 s. He did not account for the different time intervals.

There is no indication that Marcus used reversibility in solving the reverse question. He used a dimensional analysis on the reverse question but had not used a dimensional analysis on the forward question. Furthermore, his approach to solving the forward question seems to have in no way influenced his approach to solving the reverse question.

4.2.4 Development of representational reversibility

To what extent does reversibility of the mental process in reasoning with reversible translations (representational reversibility) develop? The data collected in this research study suggests that representational reversibility has developed in a significant amount over the course of the study. To determine if the students developed representational reversibility, I compared the results of the pre-test and the post-test for evidence of improvement in representational reversibility. I also analyzed the results of a sub-set of the exit slips and opening activities and a sub-set of the interview questions. I discuss the results of the data collected below.

4.2.4.1 Pre-test and Post-test data – representational reversibility

The pre-test and the post-test provided data points that informed the extent to which representational reversibility develops in a course that attends to linking multiple representations. The pre-test and the post-test showed evidence of representational reversibility when the class performed well on two reversible translations (such as graphical to symbolic and symbolic to graphical). Table 62 reports the class mean score on representational reversibility across all three representations and by translation pair on the pre-test and the post-test. A paired samples t-test was conducted to test for a significant difference between the pre-test mean score and post-test mean score. Table 62 also reports the results of the paired samples t-test.

Table 62. Mean scores of representational reversibility on pre-test and post-test

Representations	Pre-test Score (%)	Post-test Score (%)	Mean Difference (post-test – pre-test)	t-value, p-value
Total	44.6	56.0	11.31^{*}	$t = 2.657, p = 0.015$
$N \leftrightarrow G$	64.88	36.9	-27.98^{*}	$t = -2.950, p = 0.008$
$N \leftrightarrow S$	41.67	61.9	20.24^{*}	$t = 3.600, p = 0.002$
$S \leftrightarrow G$	27.38	69.1	41.67^{*}	$t = 6.948, p < 0.001$

Overall representational reversibility significantly improved from the pre-test to the post-test. At the individual translation level, there was significant improvement between the numerical and symbolic representations and between the symbolic and graphical representations over the course of the study. There was a significant decrease in the amount of representational reversibility between the numerical and graphical representations. As discussed in section 4.1.1.2, the post-test mean scores on questions that involved translation from numerical to graphical and graphical to numerical were much lower than the other possible translations. The significant decrease in representational reversibility between the numerical and graphical representations is likely due to the difficulty of the content on the post-test. For a full discussion of this content, the reader is referred to the earlier discussion in section 4.1.1.2. Thus, the pre-test and post-test data reveal that representational reversibility significantly improved for two out of three possible representational combinations.

I further examined the extent to which representational reversibility improved from the pre-test to the post-test. Table 63 reports the pre-test and post-test representational reversibility mean scores of each flexibility group.

Table 63. Mean scores of representational reversibility on pre-test and post-test

Flexibility Group	Pre-test Score (%)	Post-test Score (%)	Mean Difference (post-test – pre-test)	t-value, p-value
High	65.1	74.0	8.85	$t = 1.295, p = 0.119$
Middle	41.7	47.6	5.95	$t = 0.813, p = 0.224$
Low	20.8	41.7	20.83^{*}	$t = 3.600, p = 0.025$

The students in the low flexibility group showed a significant improvement in representational reversibility over the course of the study.

4.2.4.2 Results of exit slips and opening activities – representational reversibility

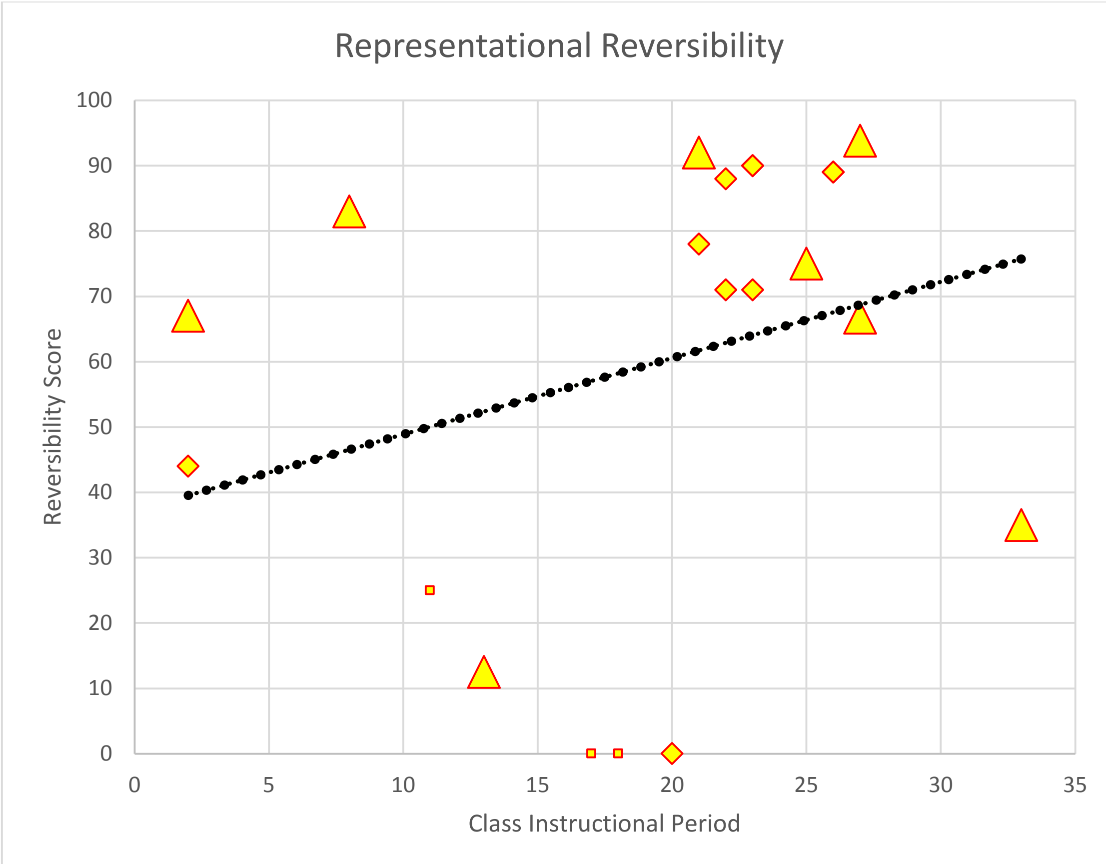
I analyzed the results of the 21 exit slip and opening activity pairs that were designed to assess the development of representational reversibility. Four of the exit slips and opening activities were administered during chapter 2, four were administered during chapter 3, and thirteen were administered during chapter 4. Due to the nature of the content of the course, representational reversibility was more prevalent during chapter 4 than during the rest of the study. Figure 77, shown below, presents the results of all of the representational reversibility exit slip and opening activity pairs, administered chronologically throughout the study.

Day	2	2	8	11	13	17	18	20	21	21	22	22	23	23	23	25	26	26	27	27	33
Exit Slip	2	3	9	13	15	21	22	24	25	26	27	28	29	30	31	33	34	35	36	37	45
Section	2.1.2	2.1.2	2.5.1	2.6.3	3.2.1	3.4.1	3.4.2	3.6.1	4.1.1	4.1.1	4.1.2	4.1.2	4.2.1	4.2.1	4.2.1	4.3.1	4.3.2	4.3.2	4.4.1	4.4.1	4.8.1
H1	Y	Y	Y	-	Y	^	N	N	^	^	Y	Y	Y	Y	Y	-	Y	Y	Y	Y	N
H2	-	-	^	N	N	^	-	^	^	^	^	^	Y	Y	-	Y	Y	Y	Y	N	N
H3	^	^	^	-	N	^	-	^	^	^	^	^	^	^	^	^	^	^	^	^	Y
H4	Y	N	Y	-	-	N	N	N	N	N	Y	Y	Y	Y	Y	-	-	-	Y	Y	N
H5	N	-	Y	-	N	^	-	N	Y	Y	^	^	^	^	^	Y	Y	Y	Y	Y	N
H6	Y	Y	Y	Y	N	N	-	N	^	^	^	^	^	^	^	Y	Y	Y	Y	Y	^
H7	N	N	^	-	-	-	-	-	^	^	Y	N	Y	Y	-	^	Y	Y	Y	N	-
H8	Y	Y	-	-	-	^	N	^	^	^	^	-	^	^	^	-	^	^	^	^	N
M1	Y	-	Y	-	^	N	N	N	-	Y	^	-	^	^	^	-	-	-	Y	Y	Y
M2	N	-	Y	-	-	-	^	-	Y	Y	Y	Y	^	^	^	Y	-	-	Y	Y	Y
M3	Y	Y	Y	^	-	^	-	^	Y	Y	^	^	^	^	^	Y	^	^	Y	Y	Y
M4	Y	-	-	N	-	N	-	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N
M5	N	-	Y	-	-	^	-	-	Y	Y	^	^	^	^	^	N	^	^	Y	Y	N
M6	N	-	Y	-	-	^	-	-	^	^	-	-	Y	Y	N	-	^	^	-	-	-
M7	N	-	^	-	N	^	-	^	-	Y	Y	Y	Y	Y	Y	Y	^	^	^	^	^
L1	-	-	-	-	-	^	^	-	-	Y	Y	-	Y	Y	Y	-	^	^	-	N	N
L2	^	^	-	-	N	^	-	^	^	^	^	^	^	^	^	N	Y	Y	Y	Y	^
L3	-	-	N	N	N	-	-	-	^	^	N	N	Y	Y	-	Y	N	N	Y	Y	Y
L4	N	-	N	-	-	^	-	-	N	Y	-	-	N	N	N	Y	-	-	Y	-	-
L5	N	-	-	-	-	-	-	^	Y	Y	^	^	^	^	^	N	-	-	Y	N	^
L6	N	-	Y	-	-	^	-	-	Y	Y	^	-	^	^	^	-	Y	Y	N	N	-
R.S.	46.7	66.7	81.8	25	12.5	0	0	0	75	90.9	87.5	71.4	90	90	71.4	75	87.5	87.5	100	73.3	38.5

Key		
H1 - first student in high flexibility group	Y - reversibility present	- - ineligible due to incorrect answer on exit slip
M1 - first student in middle flexibility group	N - reversibility NOT present	% - Ratio of Y to (Y + N)
L1 - first student in low flexibility group	^ - ineligible due to absence	R.S. – reversibility score

Figure 77. Scatter plot of the reversibility score on the exit slip and opening activity pairs assessing representational reversibility

Figure 77 shows a general trend towards increasing representational reversibility throughout the study. The trend towards improvement can be seen in the increase in density of the green blocks and decrease in density of the red blocks over the course of the study. Figure 78 displays the reversibility score of each exit slip and opening activity pair that assessed representational reversibility as a scatter plot over time with a trend line imposed on the data.



Key	
# of Eligible Exit Slips and Opening Activities	Symbol
0-5	▫
6-10	◇
11-15	△
16-21	○

Figure 78. Scatter plot of the reversibility score on the exit slip and opening activity pairs assessing representational reversibility

The positive slope ($m = 1.17$) of the trend line coupled with a weak, positive correlation ($r = 0.281$) between the class instructional period and the percent of students who demonstrated representational reversibility indicates that there was a slight observable development of representational reversibility over the course of the study at the whole class level.

To further gauge the extent to which representational reversibility develops when students are engaged in a course that attends to linking multiple representations, I compared the class mean reversibility scores during the first and second half of the course. Table 64, shown below, reports the class mean representational reversibility score during the first and second halves of the study and the results of a paired samples t-test for a significant difference. The first half consisted of paired problems 2, 3, 9, 13, 15, 21, and 22. The second half consisted of paired problems 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, and 45.

Table 64. Class mean reversibility score on representational reversibility exit slips and opening activities from the first half and second half of the study

	First half	Second half	Paired difference t-value	Paired difference p-value
Reversibility Score	33.24%	61.46%	$t = 4.891$	$p < 0.001$

The general trend towards development previously shown in figures 77 and 78 and the significant improvement in mean representational reversibility score at the class level suggest that representational reversibility as a problem solving process may have developed over the course of the study.

I further analyzed the representational reversibility exit slips and opening activities by content area. Three of the twenty-one paired problems assessed average rate of change and instantaneous rate of change. Two of the paired problems assessed related rates. Twelve of the paired problems assessed graphical analysis. Three of the paired problems assessed differentiation and one set of paired problems assessed L'Hospital's rule. Table 65 reports the reversibility score displayed on the paired problems that assessed a similar content two or three times.

Table 65. Reversibility score on representational reversibility exit slips and opening activities that assessed the same content two or three times

Section(s)	Content, # eligible opening activities, reversibility score on 1st set of paired problems, assessed representations	Content, # eligible opening activities, reversibility score on 2nd set of paired problems, assessed representations	Content, # eligible opening activities, reversibility score on 3rd set of paired problems, assessed representations
2.1.2, 4.8.1	Average Rate of Change $n = 16$, Reversibility Score: 44%, Representations: $N \leftrightarrow G$	Instantaneous Rate of Change $n = 6$, Reversibility Score: 66.7% Representations: $N \leftrightarrow G$	Mean-Value Theorem $n = 13$, Reversibility Score: 38% Representations: $N \leftrightarrow G$
3.4.1, 3.4.2	Related Rates $n = 4$, Reversibility Score: 0% Representations: $S \leftrightarrow V$	Related Rates $n = 4$, Reversibility Score: 0% Representations: $S \leftrightarrow V$	N/A
2.5.1, 2.6.3, 3.2.1	Differentiation $n = 12$, Reversibility Score: 83.3% Representations: $S \leftrightarrow G$	Differentiation $n = 4$, Reversibility Score: 25% Representations: $S \leftrightarrow G$	Differentiation $n = 8$, Reversibility Score: 12.5% Representations: $S \leftrightarrow G$

What is most significant about table 65 above is that representational reversibility did not improve in any of the three content areas as the content was repeated, which is a marked difference from the development of two-way reversibility and reversibility of the mental process in reasoning without reversible translation. In both two-way reversibility and reversibility of the mental process in reasoning, reversibility increased with repeated content.

The paired related rates problems from sections 3.4.1 and 3.4.2 seemed to cause great difficulty for all of the students. These problems required representational reversibility between the symbolic and verbal representations. Very few students were able to solve the direct problem either day. On the first day, four students correctly solved the direct exit slip. Two of the four students who were able to solve the direct problem were from the high group and two of the four students were from the middle group. None of these students were able to solve the reverse task. On the second day, four students correctly solved the direct exit slip, three of which were from the high group and one student was from the middle group. Again, none of these students were able to solve the reverse task. This is likely evidence of the difficulty that students may have when forced to use the verbal representation.

Table 66 reports the number of eligible opening activities and the reversibility score on the twelve graphical analysis paired problems that assessed representational reversibility.

Table 66. Reversibility score on exit slips and opening activities that assessed representational reversibility through graphical analysis

Class Period	21	21	22	22	23	23	23	25	26	26	27	27
Eligible	8	11	8	7	10	10	7	12	8	8	15	15
R.S.	78	92	88	71	90	90	71	75	89	89	94	67

All of the graphical analysis paired problems that assessed representational reversibility required translation between the graphical and numerical representations of functions. As can be seen in table 66, the class was consistently strong on graphical analysis paired problems that required representational reversibility. What is particularly noteworthy is that all of the paired problems presented in table 66 assessed representational reversibility between the numerical and graphical representations of functions. The overall class average score on graphical analysis paired problems assessing representational reversibility between the graphical and numerical representations was 83.1%, indicating a consistently high degree of representational reversibility between the graphical and numerical representations of functions.

4.2.4.3 Results of interviews – representational reversibility

The think-aloud interview data was used to draw conclusions regarding the development of representational reversibility. I analyzed the written solutions and the transcript data from the pairs of interview questions that assessed representational reversibility: 1.3 (interview 1, question 3), 2.3 & 2.4, and 3.1 & 3.2. Each pair of questions was analyzed for evidence of using representational reversibility to solve the problems. Table 67 reports the existence or absence of representational reversibility on the interview questions. "Yes" indicates that representational reversibility was present; "No" indicates that the participant did not use representational reversibility.

Table 67. Is representational reversibility present in the interview questions?

Flexibility Group	Participant	1.3	2.3 & 2.4	3.1 & 3.2
High	Kelsay	Yes	Yes	Yes
	Michael	Yes	Yes	Yes
Middle	Fred	Yes	Yes	Yes
	Jill	Yes	Yes	Yes
Low	Kirsten	Yes	Yes	No
	Marcus	No	No	No

The group of interview participants, as a whole, demonstrated a high degree of representational reversibility. All of the students in the high and middle flexibility groups successfully solved all of the questions that required representational reversibility. In the low flexibility group, Kirsten was able to solve the simpler questions 1.3 and 2.3 & 2.4 but was not able to demonstrate representational reversibility on 3.1 & 3.2. Marcus could not demonstrate representational reversibility on any of the interview questions.

Table 68 reports the specific translations used on questions 1.3, 2.3, and 2.4. S refers to the symbolic representation, N means numerical representation, and G means graphical representation. f means function and f' means derivative. Thus, the classification S_f represents the algebraic expression of the function; $S_{f'}$ indicates the algebraic expression of the derivative. G_f represents the graph of the function; $G_{f'}$ represents the graph of the derivative. N_f indicates a table or list of functional values; $N_{f'}$ represents a table or list of derivative values.

Table 68. Translations used on questions 1.3, 2.3, and 2.4

Flexibility Group	Participant	1.3	2.3	2.4
High	Kelsay	$G_{f'} \to S_{f'} \to S_f \to N_f \to G_f$	$G_f \to G_{f'} \to N_f{}'$	$N_{f'} \to G_f$
	Michael	$G_{f'} \to S_{f'} \to S_f \to N_f \to G_f$	$G_f \to G_{f'} \to N_f{}'$	$N_{f'} \to G_{f'} \to G_f$
Middle	Fred	$G_{f'} \to N_{f'} \to S_{f'} \to S_f \to N_f \to G_f$	$G_f \to G_{f'} \to N_{f'}$	$N_{f'} \to G_{f'} \to G_f$
	Jill	$G_{f'} \to S_{f'} \to S_f \to N_f \to G_f$	$G_f \to N_{f'}$	$N_{f'} \to G_f$
Low	Kirsten	$G_{f'} \to S_{f'} \to S_f \to G_f$	$G_f \to G_{f'} \to N_f{}'$	$N_{f'} \to G_f$
	Marcus	$G_{f'} \to S_{f'}$	$G_f \to N_f$	$N_{f'} \to G_{f'}$

The participants' results on question 1.3 indicate that for all of the participants except Marcus, representational reversibility between the graphical and symbolic representations was present during the first interview. Marcus was not able to find S_f from $S_{f'}$, thus he stopped before trying to convert a symbolic equation back into a graphical representation. Kelsay and Kirsten both translated from the symbolic representation directly to the graphical representation while Michael, Fred, and Jill all translated the symbolic representation through the numerical representation (by plotting points) to the graphical representation.

Paired problems 3.1 and 3.2 were scored on a 7 point rubric, shown below in table 69. Scoring of question 3.2 required marking each interval for correct behavior (increase/decrease and concavity) or marking specific x-values as extrema or inflection points.

Table 69. Scoring rubric on questions 3.1 and 3.2

3.1: Forward: $G \to N$, $f \to f'$, $f \to f''$	3.2: Reverse: $N \to G$, $f' \to f$, $f'' \to f$
1 pt. $f'(x) > 0, f''(x) > 0$	1 pt. $(-3,0)$
1 pt. $f'(x) < 0, f''(x) > 0$	1 pt. $(5,\infty)$
1 pt. $f'(x) > 0, f''(x) < 0$	1 pt. $(-5,-3) \cup (0,2)$
1 pt. $f'(x) < 0, f''(x) < 0$	1 pt. $(-\infty,-5) \cup (2,5)$
1 pt. $f'(x) = 0$	1 pt. $x = -3, x = 2$
1 pt. $f''(x) = 0$	1 pt. $x = -3, x = 5$
1 pt. $f'(x)\ DNE$	1 pt. $x = -5, x = 0$

The entire question was marked as correct if the participant scored 5 or higher total points on the question. Thus, a student who demonstrated representational reversibility scored $5/7$ or higher on both questions 3.1 & 3.2. Table 70 below reports each interview participant's scores on questions 3.1 and 3.2.

Table 70. Results of questions 3.1 and 3.2

Flexibility Group	Participant	3.1 Score (out of 7)	3.2 Score (out of 7)	Representational Reversibility
High	Kelsay	6.5	7	Yes
	Michael	7	7	Yes
Middle	Fred	5	5.5	Yes
	Jill	5.667	6.5	Yes
Low	Kirsten	4.333	2	No
	Marcus	1.667	1	No

The results of the six participants on interview questions 3.1 and 3.2 are discussed below by flexibility group.

High flexibility group – Kelsay and Michael

Both Kelsay and Michael showed representational reversibility on questions 3.1 and 3.2. Kelsay's solutions to questions 3.1 and 3.2 are shown below in figure 79.

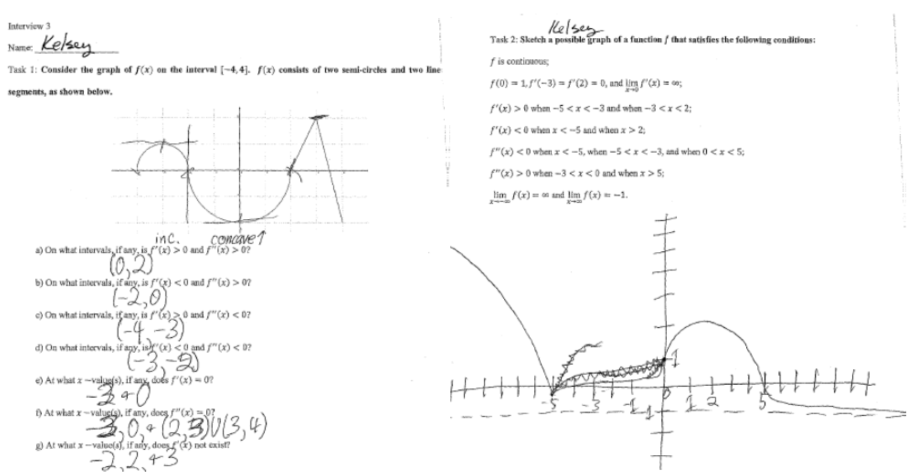


Figure 79. Kelsay's solutions to interview questions 3.1 and 3.2

Kelsay received a grade of $6.5/7$ because she offers a partially correct answer to part (f), saying that $f''(x) = 0$ at $x = -3, 0, (2,3) \cup (3,4)$. Kelsay is correct that $f''(x) = 0$ on the intervals $(2,3) \cup (3,4)$ but is incorrect when she writes that $f''(x) = 0$ at $x = -3$ and $x = 0$. In fact, $f''(-3) < 0$ and $f''(0) > 0$. Kelsay answered each section by determining the behavior of the function as described by the question prompt and then finding the interval of the curve with said behavior. For example, on part (a), Kelsay described the following thought process: "Because $f'(x)$ is greater than zero is whenever f is increasing and because f'' is greater than [zero], [$f(x)$] would be concave up so here [Kelsay points to the correct interval $(0,2)$]". Kelsay repeated this process of interpret whether f is increasing or decreasing by reading the sign of f' and then interpret if f is concave up or down by reading the sign of f'' and then choosing the interval containing both behaviors throughout question 3.1. Thus, Kelsay had no difficulties with translating functional behaviors from the graphical representation to the numerical representation.

On question 3.2, Kelsay drew a perfect sketch of a curve $f(x)$ that met each of the criteria presented in the numerical representation. She demonstrated a high ability to translate a numerical representation of a function into a graphical representation of the function. Thus, Kelsay showed complete representational reversibility between the graphical and numerical representations. It is noteworthy that Kelsay translated the functional information presented in a list of discrete characteristics and values directly to a graph. She was able to interpret and mentally organize the information presented in the list of functional characteristics and intervals and translate that information into a graphical representation of the function without writing down any organizing information such as using a table or regrouping the data to align the overlapping intervals.

Michael also demonstrated complete representational reversibility between the graphical and numerical representations on questions 3.1 and 3.2. His solutions to the questions are shown below in figure 80.

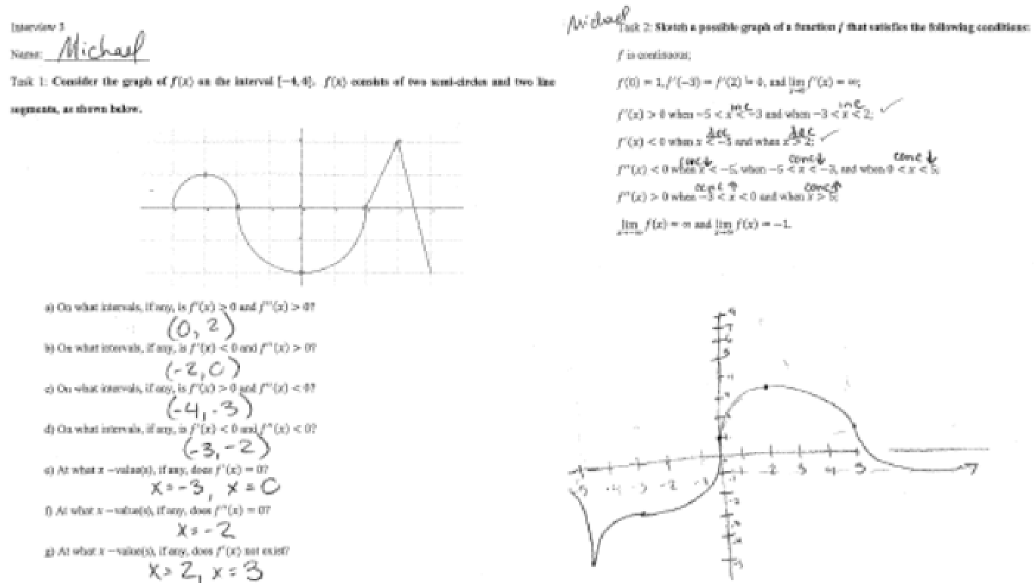


Figure 80. Michael's solutions to interview questions 3.1 and 3.2

On question 3.1, Michael solved each portion of the question by first interpreting the information about $f'(x)$ and $f''(x)$ in relation to the behavior of $f(x)$ and then finding the interval of the graph of $f(x)$ with the required behavior. For question d), Michael's response is indicative of how he solved each part of the question: "And then on what intervals, if any, is $f'(x) < 0$ and $f''(x) < 0$? That's when it is decreasing, concave down which would be $(-3, -2)$." Using this approach, Michael correctly solved each of the parts of the question, demonstrating a high ability to translate from the graphical representation to the numerical representation.

On question 2, Michael used several different approaches before settling on a method that produced a correct graph. First, Michael tried to sketch every detail individually. He correctly plotted the point $(0,1)$ given that $f(0) = 1$ and then he concluded that since "$f'(-3)$ is equal to $f'(2)$ which is equal to 0 … I know there are turning points at … -3, and 2". Here, Michael ran into his first issue. He assumed that since $f'(x) = 0$ at $x = -3$ and $x = 2$, $f(x)$ must have a turning point, which is incorrect. He then tried to sketch a section based only on the information provided by $f'(x)$, saying "$f'(x)$ is greater than 0 … between -5 and -3 so the slope is positive. It is increasing on -5 to -3 and we know that there is a turning point at -3." When Michael tried to sketch the section between $x = -5$ and $x = -3$ he realized that $f'(x)$ did not provide enough information and then said, "I just want to make sure I do it correctly, concave up or concave down so I'm going to skip around a little bit and look at f''." At this point, Michael began labeling his interpretations of the behavior of f as informed by f' and f'', saying "I should have just started labeling these from the beginning … I'm going to go back and do that real quick. $f'(x)$ is greater than 0 ... it's increasing here and here, decreasing here and here. It's concave up on these because $f''(x)$ is positive on these intervals." At this point, Michael was able to work through the rest of the graph and, once he had considered all of the information presented in the question, was able to sketch a correct graph. Michael correctly translated from the numerical representation to the graphical representation on question 3.2 and thus demonstrated complete representational reversibility between the numerical and graphical representations.

Middle flexibility group – Fred and Jill

Neither Fred nor Jill solved either question 3.1 or 3.2 completely correctly; however, each student answered at least 5 out of 7 parts correct, thus allowing me to conclude that representational reversibility on questions 3.1 and 3.2 was present for both Fred and Jill. Fred's solutions to questions 3.1 and 3.2 are shown below in figure 81.

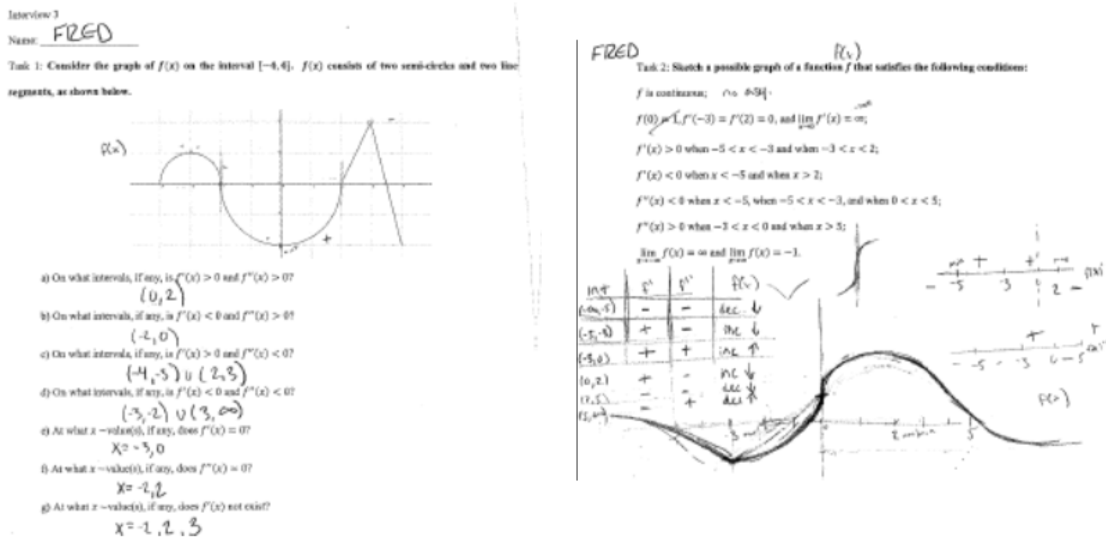


Figure 81. Fred's solutions to interview questions 3.1 and 3.2

On question 3.1, Fred began the question by first using + and – signs to mark on the graph the intervals on which $f(x)$ was concave up $(+)$ and concave down $(-)$. Fred had no difficulties with the intervals comprising $[-4,2]$. However, Fred was not sure how to handle the concavity of the linear portions of the graph. He marked both linear segments with a $(-)$ to indicate that he thought that the linear segment was concave down. Fred had no difficulties with parts (a) and (b) which required interpreting where $f(x)$ was increasing or decreasing and concave up. Due to his confusion about whether or not a linear function has concavity, Fred answered parts (c) and (d) incorrectly. He included the intervals $(2,3)$ and $(3,4)$ because he thought that a linear function was concave down. Thus, on parts (c) and (d), Fred's answers were marked as half correct to account for identifying the correct interval and for identifying the incorrect interval.

Fred struggled with part (f). Part (f) asks the student to identify where $f''(x) = 0$. Fred answered at $x = -2$ and $x = 2$. Both of these answers are incorrect because $f''(x)$ does not exist at $x = -2$ and at $x = 2$. However, when specifically asked later if there were any x-values for which $f''(x)$ does not exist, Fred said that $f''(x)$ would not exist at $x = -2,2,3$ because $f'(x)$

does not exist at those values. Thus, Fred's answer to part (f) contradicts his answer to where $f''(x)$ may not exist. Fred did not seem to be aware of this conflict.

Accounting for Fred's mistakes on parts (c), (d), and (f), Fred's score on question 3.1 was $5/7$, indicating that he had a reasonable grasp of calculus graphing questions and was able to successfully translate his interpretations from the graphical representation to the numerical representation.

On question 3.2, Fred scored $5.5/7$, indicating that he had a reasonably well-developed understanding of calculus graphing and translating from the numerical to the graphical representation. He began the question in much the same way Michael did, by first plotting the point $(0,1)$ and noting that there should be a maximum or minimum point at $x = -3$ and $x = 2$ because $f'(x) = 0$; again, this is a misconception. Fred's first sense-making activity was to construct a first-derivative number line. Fred plotted the intervals on the number line and wrote in $(+)$ or $(-)$ in each interval except for the interval $(0,2)$. It is not clear how Fred thought that he would handle that interval. Fred then constructed a number line for $f''(x)$ and labeled it as positive or negative to indicate whether $f(x)$ would be concave up or concave down. After completing his number lines, Fred read through the remaining characteristics and decided "to make a little table to put the information that I've ciphered so far and … find $f(x)$ out of it". Fred constructed a table consisting of the intervals described by the problem along with the sign of f' and the sign of f''. Fred then interpreted these signs together to determine the behavior of $f(x)$.

Having completed a chart of the behaviors of $f(x)$, Fred constructed his graph. What is noteworthy here is that all of the intervals described in Fred's table are correct. However, he incorrectly graphed $f(x)$ on $(-5, -3)$. Fred correctly identified that $f(x)$ should increase, concave down. However, he drew a graph that decreases concave down. Furthermore, he correctly identified that $f'(x)$ changes from negative to positive at $x = 5$, resulting in a local minimum at $x = 5$. However, his graph does not contain a local minimum at $x = 5$. Fred was aware that he was struggling with how to graph f on the interval $(-5, -3)$ saying "I'm trying to graph it but the concavity … it's like changing so I'm trying to figure how to how to graph it without it seeming to change." Fred did not realize that although the function was continuous it was not everywhere differentiable. Fred skipped over the interval $(-5, -3)$ and correctly sketched the remainder of the graph. When he revisited the interval $(-5, -3)$, he said that "I believe that -3 is a point that looks to be a minimum. There wasn't an inflection point around there." This observation, which agrees with the graph that Fred drew, directly conflicts with both of Fred's number lines. On his $f'(x)$ number line, Fred wrote that $f'(x)$ is positive on both sides of $x = -3$. Since $f(x)$ can only have a minimum if $f'(x)$ changes from negative to positive, Fred's number line indicates that $f(x)$ does not have a maximum or minimum at $x = -3$. Secondly, $f(x)$ has an inflection point at $x = -3$ if $f''(x)$ changes sign at $x = -3$. Since Fred's $f''(x)$ number line indicates that $f''(x)$ changes from negative to positive at $x = -3$, Fred should be aware that $f(x)$ has an inflection point at $x = -3$. His statement that "there wasn't an inflection point around there" suggests that he originally thought that $f(x)$ would have an inflection point at $x = -3$ but he could not reconcile the existence of an inflection point with the curve that he drew.

It is noteworthy that in two separate numerical representations, the number lines and the interpretative chart, Fred indicated that he understood that an inflection point existed at $x = -3$. However, when he translated his understanding of the function represented numerically to a graphical representation, his apparent discomfort with drawing a curve with a kink overruled what he knew to be true in the numerical representation. Thus, Fred was able to successfully translate most of the functional information from the numerical representation to the graphical representation; however, he did not correctly translate all aspects of the information that he had correctly interpreted from the list of characteristics and intervals.

Jill showed an amount of representational reversibility on questions 3.1 and 3.2 similar to the amount shown by Fred. She scored a $5.667/7$ on question 3.1 and a $6.5/7$ on question 3.2. Jill's solutions to questions 3.1 and 3.2 are shown below in figure 82.

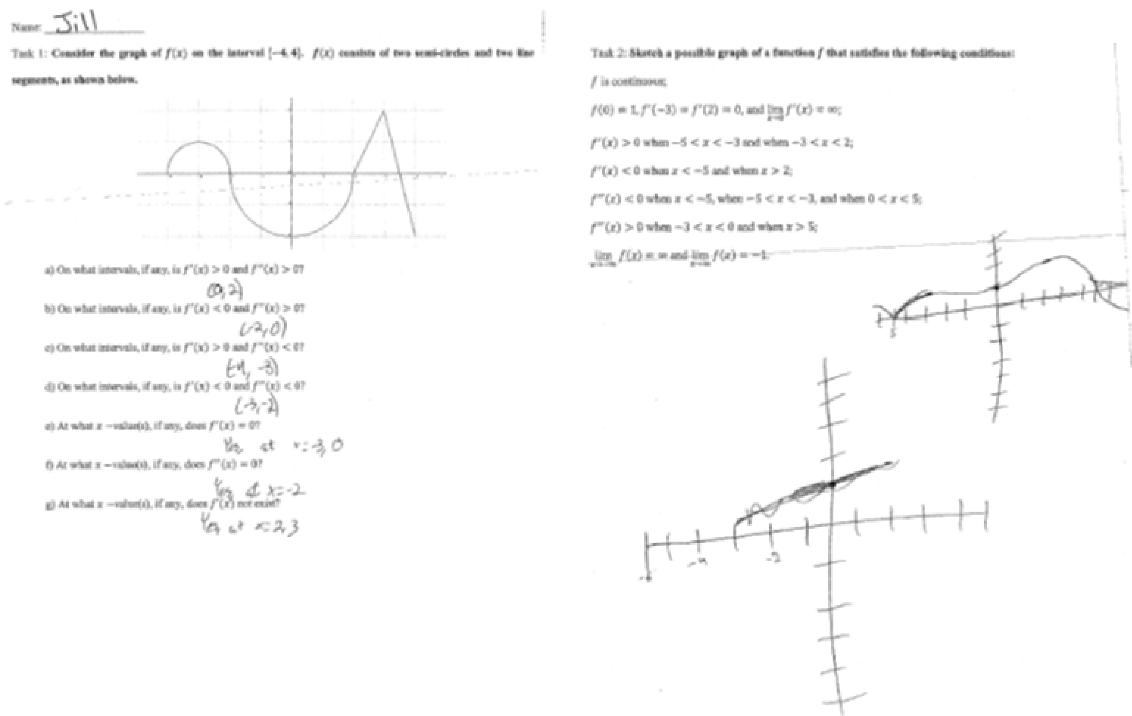


Figure 82. Jill's solutions to interview questions 3.1 and 3.2

Jill answered all of question 3.1 quite quickly (two minutes and seven seconds). She had no difficulties identifying the intervals on which $f(x)$ exhibited the behavior indicated by $f'(x)$ and $f''(x)$ in parts a-d. She also had no difficulty determining where $f'(x) = 0$ by looking at the graph. Jill made a subtle mistake when answering part f), where does $f''(x) = 0$? Jill selected $x = -2$. This selection is incorrect as $f''(x)$ does not exist at $x = -2$. However, it should be noted that $x = -2$ is an inflection point and it is likely the case that Jill identified the change in concavity as the place where $f''(x) = 0$. Jill made another mistake with $x = -2$ on part (g) when she failed to identify $x = -2$ as an x-value for which $f'(x)$ does not exist. Either Jill was not aware that $f(x)$ is vertical at $x = -2$ or she was not aware that $f'(x)$ does not exist when the line tangent to the curve is vertical.

On question 3.2, Jill correctly sketched each of the intervals pertaining to when $f(x)$ was increasing concave up, decreasing concave up, increasing concave down, and decreasing concave down. She also correctly sketched a graph $f(x)$ whose derivative $f'(x)$ was equal to zero at $x = -3$ and $x = 2$ and whose second derivative $f''(x)$ was equal to zero at $x = -3$ and $x = 5$. The only error in Jill's graph is at $x = 0$, where according to Jill's graph, $f'(0)$ clearly exists and is not vertical, as it should be. Also, it should be noted that Jill's sketch makes it appear that $f(x)$ is decreasing concave up on $(-3,0)$ instead of increasing concave up. However, Jill's transcript suggests that she meant for the graph to increase on $(-3,0)$, saying that the curve was "still increasing" from $x = -3$ to $x = 2$. When she drew the segment of the curve from $x = -3$ to $x = 0$, she said, "ehh, that's close enough … this is realism". She did note that she had placed the point at $x = -3$ too high on the graph, presumably because it was higher than $f(0) = 1$.

Jill solved question 3.2 in a method similar to the approach that Kelsay used. Jill worked on each interval individually and did so without re-organizing the information provided in the question prompt. Jill interpreted the behavior of f on each interval by coordinating the information provided by $f'(x)$ and $f''(x)$ on the various intervals mentally, unlike Fred and Michael who both wrote down organizational aids.

Jill exhibited a nearly complete understanding of representational reversibility between the numerical and graphical representations. She successfully identified intervals containing calculus properties by looking at the graph of f and then correctly sketched $f(x)$ by interpreting intervals of calculus properties. Her mistakes on both problems were limited to a misconception about when $f'(x)$ exists and were not related to a translational issue.

Low flexibility group – Kirsten and Marcus

Both Kirsten and Marcus had significant difficulties with interview questions 3.1 and 3.2. Both students scored worse on question 3.2 than they did on 3.1, with Kirsten decreasing from $4.333/7$ to $2/7$ and Marcus decreasing from $1.667/7$ to $1/7$.

Kirsten's solutions to questions 3.1 and 3.2 are shown below in figure 83.

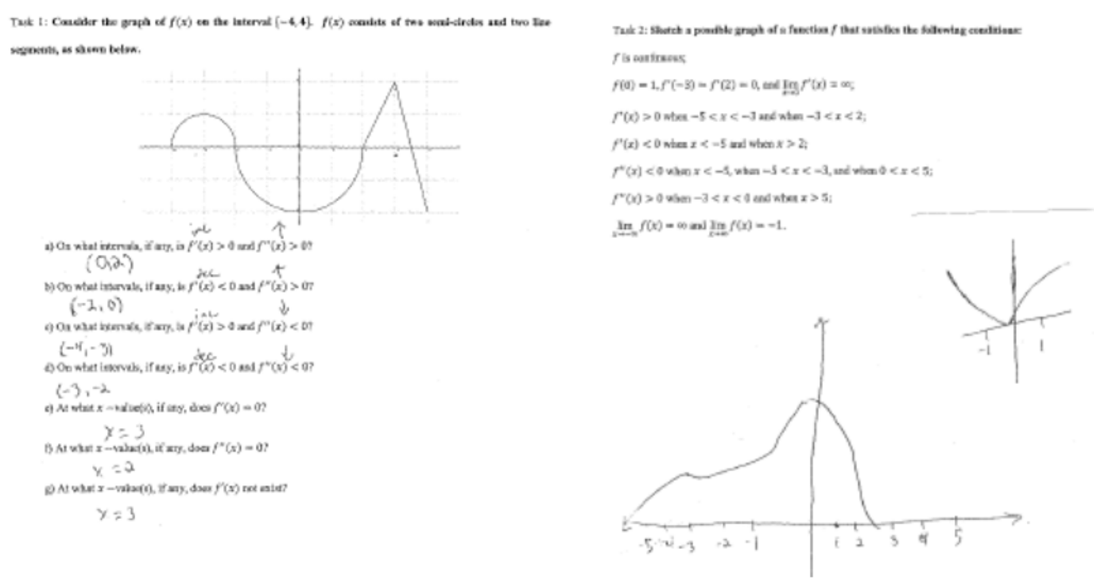


Figure 83. Kirsten's solutions to interview questions 3.1 and 3.2

On question 3.1, Kirsten successfully identified the intervals of functional behavior described by $f'(x)$ and $f''(x)$ in parts a-d. However, she was not able to identify where $f'(x) = 0$ or where $f''(x) = 0$ and was only able to identify one x-value at which $f'(x)$ does not exist. For each portion of the question that Kirsten correctly interpreted the calculus characteristic of the problem, she correctly translated her solution from the graphical representation to the numerical representation, indicating a well-developed flexibility to translate from the graphical representation to the numerical representation.

On question 3.2, Kirsten had significant difficulties translating the numerical representation of calculus properties into a graphical representation. Kirsten was able to use the information provided by $f'(x)$ to correctly determine where $f(x)$ was increasing or decreasing. She also correctly determined the concavity of $f(x)$ from the information provided by $f''(x)$. Kirsten had significant difficulties with translating the information she had gleaned about $f(x)$ from the numerical representation into a graphical representation. She correctly sketched the curve on the intervals $(-5,-3)$, $(-3,0)$, and $(0,2)$. However, she incorrectly sketched the curve on the intervals $(-\infty,-5), (2,5)$, and $(5,\infty)$. Consistent with her work on question 3.1, Kirsten did not sketch a curve with $f'(x) = 0$ or $f''(x) = 0$ at the appropriate x-values. Finally, neither her graph nor interview transcript indicated any consideration of the fact that $f'(0)$ does not exist.

Kirsten's inability to account for where $f'(x) = 0$, $f''(x) = 0$, or where $f'(x)$ does not exist in question 3.2 is not evidence of a lack of representational reversibility as she was not able to correctly solve the equivalent parts of question 3.1. However, her inability to correctly sketch three of the segments on which she had correctly described the behavior indicates that Kirsten had difficulty translating from the numerical representation to the graphical representation. Coupled with her work on question 3.1, I can conclude that Kirsten had very limited representational reversibility between the numerical and graphical representations. She was much stronger at translating from the graphical to the numerical representation than from the numerical to the graphical representation.

Marcus had significant difficulties with both questions, 3.1 and 3.2. On question 3.1, he scored $1.667/7$. On question 3.2, he scored $1/7$. His solutions to the questions are shown below in figure 84.

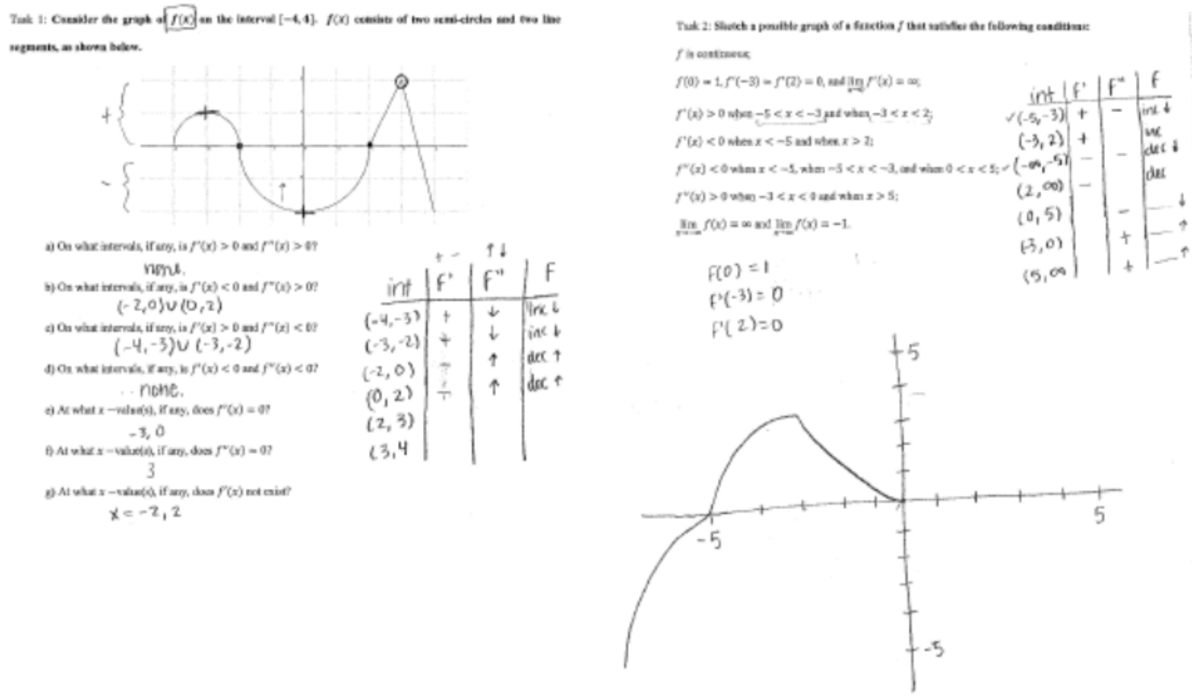


Figure 84. Marcus's solutions to interview questions 3.1 and 3.2

Marcus began the question by creating the organizational chart shown at the right side of his solution. He correctly identified that $f'(x)$ informs increasing and decreasing behavior and that f'' informs the concavity of the curve on the respective interval. However, Marcus's table shows that he is unaware of how to interpret increasing and decreasing behavior. Marcus was able to draw the correct calculus conclusion: For example, if Marcus identified an interval as "increasing and concave down", then he correctly concluded that $f'(x) > 0$ and that $f''(x) < 0$. Marcus's difficulty was in determining if $f(x)$ was increasing or decreasing by looking at the graph of $f(x)$. There are two possibilities for explaining Marcus's difficulties with increasing and decreasing behavior. One possibility is that he believes that increasing behavior means that the function lies above the x-axis and that decreasing behavior occurs when the function lies below the x-axis. Alternatively, Marcus may have confused himself regarding whether he was looking at the graph of f or f'. If Marcus thought that he was looking at the graph of f', then his answer would be accurate as anywhere that $f'(x) > 0$, $f(x)$ is increasing. His inability to correctly determine increasing and decreasing behavior resulted in incorrect responses to parts a-d. Since Marcus understood the calculus portion of the question, relating behavior to the sign of $f'(x)$ and $f''(x)$, this is evidence that his difficulty was in analyzing the graphical representation. This difficulty necessarily limited his ability to translate from the graphical representation to the numerical representation.

The only parts of question 3.1 that Marcus answered correctly were parts (e) and (f). In part (e), Marcus noted that "f' is slope … Here is where … the slope is zero. [Marcus draws in horizontal line segments tangent to the graph at $x = -3$ and $x = 0$] So I'm going to say -3 and 0." Here Marcus showed that he has some ability to translate from the graphical to the numerical representation. In part (f), he correctly identified that $f'(x)$ does not exist at $x = -2$ and $x = 2$ but did not identify $x = 3$. However, his reasoning for why $f'(x)$ does not exist at $x = -2$ and $x = 2$ suggests that he did not know what causes $f'(x)$ to not exist. Marcus said "I have absolutely no idea so I'm just going to guess these points: $x = -2, 2$" and then said that the reason that $f'(-2)$ and $f'(2)$ do not exist is because $f(x)$ changed from positive to negative or vice versa, which in fact, do not in any way determine the existence of $f'(x)$.

Marcus's answers to question 3.1 indicate that his understanding of calculus graphing is limited to superficial memorization: If f is increasing, then f' is positive. When f is decreasing, then f' is negative. If f is concave up, then f'' is positive. If f is concave down, then f'' is negative. Marcus was not able to correctly identify where f was increasing or decreasing by looking at the graph of f, indicating Marcus's difficulties with the graphical representation. Marcus's limited ability to analyze the graphical representation necessarily inhibited any ability he may have had to translate from the graphical representation to the numerical representation.

On question 3.2, Marcus had significant difficulties with nearly all parts of the question. He began the question by setting up a chart to organize the information. He correctly interpreted the effects of $f'(x)$ and $f''(x)$ on the behavior of $f(x)$. However, Marcus was not able to coordinate the various intervals described in the question prompt. He showed very little ability to coordinate the interval portion of the numerical representation. For Marcus, the fact that the intervals describing $f'(x)$ and $f''(x)$ did not form a partition but instead overlapped was an unresolvable point of confusion. The only interval on which Marcus sketched a correct curve was $(-5, -3)$, which was an interval on which both $f'(x)$ and $f''(x)$ were explicitly described: $f'(x) > 0$ on $-5 < x < -3$ and $f''(x) < 0$ when $-5 < x < -3$. It is interesting to note that on the interval $x < -5$, both $f'(x)$ and $f''(x)$ are explicitly described as well with both being negative. Marcus correctly labeled this in his table and correctly interpreted the behavior of $f(x)$ to be decreasing and concave down. However, Marcus sketched a curve that was increasing and concave down on $(-\infty, -5)$. Marcus also attempted to sketch the curve on $(-3,0)$. According to his table, the curve should have been increasing and concave up. He drew a curve that was decreasing and concave up. These errors are consistent with Marcus's difficulty in determining increasing and decreasing behavior in question 3.1 and are evidence of a difficulty in translating from the numerical to the graphical representation. Also, he noted that $f(0) = 1$, but his graph reflects that $f(0) = 0$. Marcus found translating from the numerical representation to the graphical representation to be a difficult question.

When viewed through a translational lens, Marcus's work on questions 3.1 and 3.2 shows significant difficulties with translating from the graphical representation to the numerical representation and from the numerical representation to the graphical representation. Given his inability to translate in either direction, Marcus's representational reversibility between the numerical and graphical representations is nearly non-existent.

Taken together, Kirsten's and Marcus's work on questions 3.1 and 3.2 suggest that for low flexibility students, representational reversibility is particularly difficult. Although based on the evidence present, I cannot conclude that it is the reversible nature that was difficult. For both students, there were significant difficulties with one-way flexibility. It is clear that if one directional flexibility is not present, than bi-directional flexibility (i.e. representational reversibility) will not be present either.

4.2.5 Summary – the extent to which reversibility developed

The results of the exit slips and opening activities suggest that two-way reversibility developed significantly at the class level over the course of the study. The interview data indicates that the high flexibility group and the middle flexibility group were largely able to learn a process in the forward direction and then, without any instruction, reverse the process immediately, thus developing two-way reversibility on the spot. For students in the low flexibility group, two-way reversibility may not have developed on the spot. However, for students who did not develop reversibility of a particular procedure on the spot, repeated use of a direct process, such as differentiation of the simple power rule, over the course of the study may have led to the development of reversibility as nearly the entire class was able to reverse differentiation of the simple power rule at the end of the course.

Developing reversibility of the mental process in reasoning without reversible translation was considerably more difficult than developing two-way reversibility for the students observed in this study. Reversibility of the mental process in reasoning without reversible translation was unlikely to develop on the spot and when it did, it was limited to a few students in either the high or middle flexibility groups. Students in the high flexibility group were often able to use reversibility of the mental process in reasoning without reversible translation to solve a reverse problem immediately after learning the forward direction of a process. Students in the middle flexibility group were occasionally able to use reversibility of the mental process in reasoning without reversible translation to solve a reverse problem immediately after learning the forward direction of a process. Students in the low flexibility group were not able to use reversibility of the mental process in reasoning without reversible translation to solve a reverse problem immediately after learning the forward direction of a process. After multiple opportunities to engage with the same learning, students in the low flexibility group were unlikely to use reversibility of the mental process in reasoning without reversible translation to solve a reverse task. The presence and development of reversibility of the mental process in reasoning without reversible translation was content specific, influenced by the functional representation in which the reversible task was presented, and improved as the students were given multiple opportunities to engage with reversible tasks assessing the same content.

Representational reversibility developed in a significant amount over the course of the study. The extent to which a student successfully used representational reversibility was likely influenced by the amount of flexibility that the student demonstrated at the start of the study. To that end, the students in the high flexibility group were consistently able to use representational reversibility whenever necessary. The middle flexibility group was also able to reliably use representational reversibility but not to the same extent as the high flexibility group. The low flexibility group consistently struggled to use representational reversibility to solve problems. As a distinct difference from the development of two-way reversibility and reversibility of the mental process in reasoning without reversible translation, the amount of representational reversibility demonstrated by the class did not improve over repeated opportunities to engage with the same content. The results of this study indicated that a student who demonstrated representational reversibility on the first day of learning a new content was also able to use representational reversibility when the content was revisited later in the course. Students who were not able to use representational reversibility on the first day of learning a new content were not able to use representational reversibility on later tasks involving the same content.

Finally, in order to answer research question 2: to what extent do students develop reversibility when engaged in a course that attends to linking multiple representations, I triangulated the conclusions drawn when answering the three sub-questions. In section 3.5.2, I proposed an evaluative table that would answer research question 2 based on the results of sub-questions 2.i, 2.ii, and 2.iii. For ease of access, that table is reprinted below as Table 71.

Table 71. Possible combinations of outcomes answering research question 2

	R.Q. 2.i	*R.Q. 2.ii*	*R.Q. 2.iii*	*R.Q. 2*
Did reversibility develop?	Y	Y	Y	Strong evidence that reversibility does develop when students are engaged in a course that attends to linking multiple representations
	Y	Y	N	Evidence suggests that reversibility likely developed
	Y	N	Y	Evidence suggests that reversibility likely developed
	N	Y	Y	Evidence suggests that reversibility likely developed
	Y	N	N	Evidence suggests that reversibility may have developed in a limited amount and in a limited domain
	N	Y	N	Evidence suggests that reversibility may have developed in a limited amount and in a limited domain
	N	N	Y	Evidence suggests that reversibility may have developed in a limited amount and in a limited domain
	N	N	N	Evidence suggests that reversibility may not develop when students are engaged in a course that attends to linking multiple representations

Since the results of the study indicate that two-way reversibility developed to a large degree, representational reversibility developed to a lesser degree, and reversibility of the mental process in reasoning without reversible translation may not have developed, I conclude that there is evidence to suggest that reversibility overall may have developed when students were engaged in a course that attends to linking multiple representations.

4.3 RESEARCH QUESTION 3

What are the thought processes that students utilize when using reversibility to solve problems?

The exit slip and opening activity data and the think-aloud interview transcript data were used to answer research question 3.

I analyzed the data collected in the study in two stages in order to inform research question 3. First, I analyzed class-wide data and individual interview data to look for recurrent themes indicating how students may think about reversibility. Secondly, I analyzed the interview transcripts for any possible evidence of a change in how a student thinks about reversible problems throughout the study. The results of the analysis suggest a pattern of thought processes present when students solve problems using two-way reversibility and reversibility of the mental process in reasoning without reversible translation. This pattern was consistent across flexibility groups. The data collected in this study also indicates that student do not use reversible thought processes when using representational reversibility. Finally, there was no evidence of a change in thought processes when using reversibility to solve problems during the course of the study. Each of these findings is discussed in turn.

4.3.1 Thought processes used when solving problems that require reversibility of a two-way process

Analysis of the exit slips and opening activities and interview questions that assessed reversibility of two-way processes indicates that when students use reversibility of two-way processes to solve problems, the students consistently used the following thought pattern: 1) recognize that the question requires reversibility of a known procedure, 2) reverse the steps of the known procedure by asking her/himself "what do I need to do to get back to …" to find a possible solution, 3) check the validity of the solution by using the known forward process, and 4) adjust the solution if necessary

Table 72 below reports the steps that each interview participant demonstrated on the interview questions that assessed two-way reversibility.

Table 72. Presence of 4-step thought process when using two-way reversibility

Participant	1.2	1.3	4.2.b	4.3.b
Kelsay	1) 2) 3)	1) 2)	1) 2)	1) 2) 3) 4)
Michael	1) 2) 3) 4)	1) 2) 3) 4)	1) 2)	1) 2) 3) 4)[a]
Fred	1) 2) 3)	1) 2)	1) 2) 3)	1) 2) 3)
Jill	1) 2)	1) 2)	1) 2) 3) 4)	1) 2) 3)
Kirsten	1) 2) 3) 4)	1) 2) 3) 4)	1) 2)	None
Marcus	1) 2) 3) 4)	1)	1) 2) 3) 4)	None

Michael's 4.3.b block is marked 1) 2) 3) 4)[a] to note that he solved the problem correctly by using reversibility; however, he did not use two-way reversibility. He used reversibility of the mental process in reasoning without reversible translation to solve the question. On question 4.3.b, Michael followed the 4-step thought process that I describe in the next section.

On question 1.2, Kelsay and Fred did not need to adjust their answers; thus, they only exhibited steps 1-3. Jill did not check her answer. Michael, Kirsten, and Marcus used all four steps to solve the problem. On question 1.3, Marcus only exhibited step 1. Jill, Fred, and Kelsay did not check their solutions. Michael and Kirsten showed all four steps of the proposed thought process. On question 4.2.b, Kelsay, Michael, and Kirsten did not feel the need to check their solutions; thus, they only showed steps 1 and 2. Fred checked his answer and determined that no adjustments were necessary, demonstrating steps 1-3. Jill and Marcus used all four steps of the two-way reversibility thought process. On question 4.3.b, Kirsten and Marcus did not demonstrate any thought processes that related to reversibility. Fred and Jill both used steps 1-3 and did not adjust their answers. Kelsay and Michael demonstrated all four steps.

Below, I present four interview vignettes of similar content. The vignettes serve to show how the 4-step process described above captures the thought processes of students when using two-way reversibility to solve a problem. Two vignettes are from the first interview and two vignettes are from the last interview. The two vignettes from interview 1 are from Michael and Marcus. Michael was in the high flexibility group and Marcus was in the low flexibility group. Interview 1, question 2 asks students to find a function $f(x)$ whose derivative is known to be $f'(x) = x^5$. Presented in table 73 below is Michael's interview transcript exemplifying the 4-step thought pattern of two-way reversibility.

Table 73. Michael's transcript from interview 1, question 2.

Michael's transcript	Step of two-way reversibility thought pattern
The function has a derivative of $f'(x) = x^5$ so this is going backwards.	Step 1: Michael recognizes that he has to work backwards.
So I just usually start testing things. It's kind of a guess and check for me. I try and think uh 'what would be'... so something would have to make what's right here 1 [Michael points at the coefficient in front of x^5]. So there would be nothing there and then so it would be $\frac{1}{5}x^6$ oh no, yes, no $\frac{1}{6}x^6$	Step 2: Michael tries to reverse the simple power rule by asking "what would be here". He indicates that he is reversing the simple power rule step-by-step by noting that the exponent increases by one and the coefficient must be divided by the new exponent.
because if you brought down this exponent 6 and multiplied it by $1/6$ that would be 1 and then you subtract 1 from this exponent and that would be 5 and that would be x^5.	Step 3: Michael checks his solution by using the simple power rule for differentiation
So then, that's the actual answer [Michael circles $f(x) = \frac{1}{6}x^6$]	Step 4: Michael determines that he does not need to adjust his solution

Presented below in table 74 is Marcus's interview transcript from interview 1, question 2 as an example of a student's thought process when using two-way reversibility to solve a problem.

Table 74. Marcus's transcript from interview 1, question 2.

Marcus's transcript	Step of two-way reversibility thought pattern
So the derivative is x^5 that means that you have to work backwards	Step 1: Marcus recognizes that he has to work backwards.
Well if you are working backwards normally for the exponent you do the exponent minus one for the derivative, so if you are going backwards you would add one which would be six.	Step 2: Marcus tries to reverse the simple power rule. Here, we see a difference between the low and high flexibility groups. Marcus was able to find the correct exponent but could not account for the coefficient.
hmm ... that won't work because then that would be $6x$ for the derivative [Marcus erases the exponent of 6]	Step 3: Marcus attempts to check his answer of $f(x) = x^6$. He notes that his answer would result in a 6 as a coefficient of x.

Table 74 (continued)

well if it was $\frac{1}{x^{-5}}$ that's the same I think that's the same thing as that [referring to x^5] so [Marcus erases x^5] ...oh but then the derivative ahhh ... [Marcus erases all of his work] ... umm well the derivative of a constant would just be zero, but there's no zero out front so ... I'm not sure	Step 4: Marcus attempted to adjust his solution because the derivative of x^6 was not equal to x^5. Marcus was not able find a viable solution.

The interview vignettes from the fourth interview are taken from question 2.b. Interview 4, question 2.b presents students with a velocity function of a vehicle in motion of $v(t) = 4t^3 - 3t^2 + t$ and asks students to find the position of the vehicle at $t = 3$. Two-way reversibility of the simple power rule is necessary to find the position function. Below, I present Fred's and Marcus's interview transcripts as examples of how students' thought processes when using two-way reversibility follow the proposed 4-step process. Fred is in the middle flexibility group and Marcus is in the low flexibility group. Table 75 below presents Fred's interview transcript from interview 4, question 2.b.

Table 75. Fred's transcript from interview 4, question 2.b.

Fred's transcript	Step of two-way reversibility thought pattern
Now I could … take this [$v(t)$] since … velocity is the derivative of s. So I could go backwards to find s.	Step 1: Fred recognizes that he has to work backwards.
Ok, so … this [$s(t)$] should be … power to the 4th, power to the 3rd, power to the 2nd ...so for the over here where it's t, as the s function would be $\frac{t^2}{2}$ cause when you would derive the 2 would come out front and it would cancel with the denominator.	Step 2: Fred increases the exponent of each term by one and then shows that he understands that he needs to divide by the new exponent as well. Here, he reverses the steps of the simple power rule.

Table 75 (continued)

The $-3t^2$ would be $\frac{9t^3}{3}$ and you would derive ... it would be 3, yeah when you would derive it would be $\frac{9t^2}{3}$ and the 9 and the 3 would reduce to be just 3. Lastly, the $4t^3$ would be $\frac{4t^4}{4}$ when you would derive it would be $\frac{16t^3}{4}$. The 16 and 4 would reduce to $4t^3$.	Step 3: Fred uses the simple power rule to differentiate each term of the position function in order to check his answer.

Fred did not demonstrate step 4 in this interview because his answer checked correctly and thus there was no need to adjust his solution. Marcus's transcript from interview 4, question 2.b is presented below in table 76. To aid in understanding the transcript, I have included the relevant portion of Marcus's solution to interview 4, question 2.b as figure 85.

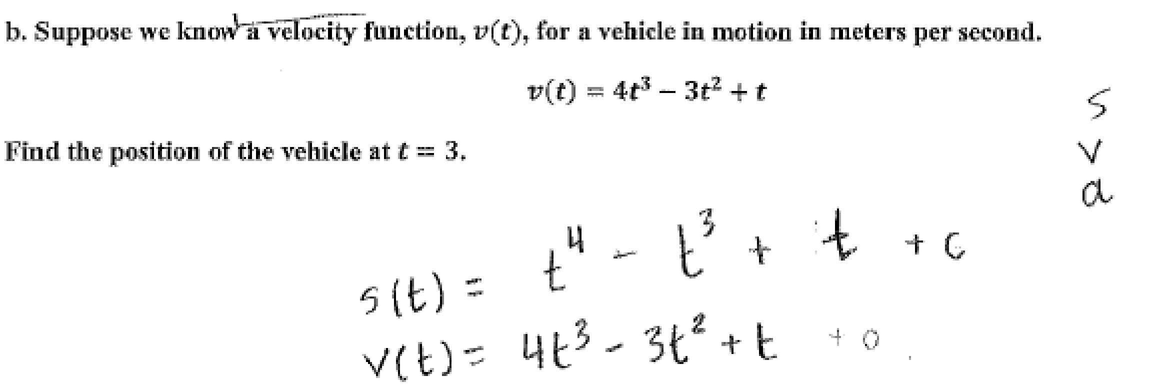


Figure 85. Marcus's solution to interview question 4.2.b

Table 76. Marcus's transcript from interview question 4.2.b.

Marcus's transcript	Step of two-way reversibility thought pattern
Suppose you know the velocity function $v(t)$ for a vehicle in motion in meters/second. Position, ahh, so this is going backwards	Step 1: Marcus recognizes that he has to work backwards.
Um 3 that means that that [the exponent of the first term] has to be 4 [inaudible calculations of the cubic term in $s(t)$]. I guess it would be $1t$ so it's just no no, just having the t that means $1t$ means that 1 would go there so it's just hmm I know there's a constant at the end so f' would be 0. Oh what is that one [Marcus references the linear term in $v(t)$]? It's 2t then the 2 would go there, when it's just t, then it's ... hmmm ... I'm blanking ... this is just t, I'm still imagining a 1 in front so that 1 would go there but how the heck do you just get a t?	Step 2: Marcus correctly reverses the simple power rule to find the first two terms of $s(t)$. However, he had significant difficulties with finding a term whose derivative would be t.
So if you [had] $2t$, the derivative would just be 2, if you had t^2, the derivative would just be $2t$, so if you had a t, the derivative would be...?	Step 3: Marcus uses differentiation to check multiple possibilities for the third term in $s(t)$ and consistently determines that his answer is incorrect.
My mind is totally blanking ... It couldn't be like t ... t to the no ... I don't know I'm just going to leave it as t but that's not right but I don't know what to do.	Step 4: Marcus attempts to adjust his answer but is unable to find a term whose derivative is $1t$.

These four vignettes demonstrate the 4-step thought process that I propose that students use when using two-way reversibility to solve problems. As shown above, the 4-step process was present during the first interview, administered in early December, near the beginning of the study, and during the 4th interview which was administered in March, after the instructional period of the study had concluded. Furthermore, the process was present in the interview transcripts from students in the high, middle, and low flexibility groups. These results suggest that the students' thought processes when using reversibility were largely consistent across flexibility groups and did not change over the course of the study.

The 4-step process shown above can also be seen in the students' responses to the exit slips and opening activities. Although, the exit slips and opening activities did not produce the rich descriptions of the interviews, evidence of the reversible thought process seen in the interview transcripts is present in the exit slips and opening activities. Opening activity 2.3.1 serves as an exemplar for how the students who used reversibility to solve the opening activity thought about using reversibility. Opening activity 2.3.1 asks students to find a function $f(x)$ whose derivative $f'(x) = x - 6$. Students were asked to explain or show how they determined $f(x)$. Table 77, shown below, presents the solutions of the six students who demonstrated reversibility along with their explanations. The students' explanations are reproduced verbatim. The steps of the 4-step two-way reversibility thought processes present in each explanation are noted in the last column.

Table 77. Evidence of reversible thoughts from opening activity 2.3.1.

Flexibility group	Solution	Student's explanation	Steps present
Low	$f'(x) = x - 6$ $f(x) = \frac{1}{2}x^2 - 6x$	I did the simple power root backwards by looking a $f'(x)$ and determining what powers were to make $f'(x) = x - 6$.	1) 2)
High	$f'(x) = x-6$ $f(x)\frac{1}{2}x^2 - 6x$	The derivative (if I did it correctly) of $\frac{1}{2}x^2$ is x and the derivative of $-6x$ is negative 6.	3)
Middle	$f(x) = \frac{1}{2}x^2 - 6x$	I found the reverse of the differentiated function, using the BPT [basic power theorem].	1) 2)
High	$f'(x) = x - 6$ $f(x) = \frac{x^2 + 12x}{2}$	I worked backwards.	1)

Table 77 (continued)

High	$f(x)=\frac{1}{2}x^2-6x$	I used the power rule, the constant multiple rule, & the sum & difference rules to logically discern how this derivative was found & differentiated & reversing the rules, how to find its' equation	1) 2)
Middle	$f'(x) = x-6$ $\boxed{f(x) = \frac{1}{2}x^2 - 6x}$ $\frac{1}{2}x^2$	I basically did opposite of what we learned yesterday. I figured & worked the simple power rule backward and plugged numbers in to see how I could get $f'(x) = x - 6$ and $f(x) = \frac{1}{2}x^2 - 6x$ worked.	1) 2) 3)

In five of the six explanations, the students reference working backwards/reverse/opposite, which is clear evidence of Step 1 in the two-way reversible thought process. Step 2, reversing the steps of the known procedure, in this case, the simple power rule, was referenced in four of the six opening activities. This should not be counted as evidence against students' consideration of step 2. Rather, it is likely an expected result of a less rich description of how a student solved a problem than what a clinical interview provides. Two students explicitly demonstrated step 3 by checking the solution by using the known procedure, the simple power rule. In this case, no students demonstrated step 4, which is to be expected as the opening activities only provide a window into the final product and not the development of the solution from beginning to end. If a student had checked her/his solution and then subsequently changed the answer, the opening activity would likely not reflect the change.

4.3.2 Thought processes used when solving problems that require reversibility of the mental process in reasoning without reversible translation

Analysis of the exit slips and opening activities and interview questions that assessed reversibility of the mental process in reasoning without reversible translation indicates that when students use reversibility of the mental process in reasoning without reversible translation to solve problems, the students consistently used the following thought pattern: 1) recognize that the question requires the use of reversibility, 2) propose a possible solution that is informed by knowledge of the forward process, 3) check the validity of the solution by using the known forward process, and 4) adjust the solution if necessary.

Table 78 below reports the steps that each interview participant demonstrated on the interview questions that assessed reversibility of the mental process in reasoning without reversible translation.

Table 78. Presence of 4-step thought process when using reversibility of the mental process in reasoning without reversible translation

Participant	2.1	2.4	3.2	4.2.a
Kelsay	1) 2) 3)	1) 2) 3)	1) 2) 3) 4)	None
Michael	1) 2) 3) 4)	1) 2)	1) 2) 3) 4)	None
Fred	1) 2) 3) 4)	1) 2) 3) 4)	1) 2) 3) 4)	None
Jill	1) 2) 3) 4)	1) 2)	1) 2) 3) 4)	None
Kirsten	1) 2) 3) 4)	1) 2) 3) 4)	None	None
Marcus	1) 2) 3) 4)	1)	None	None

On questions 2.1, 2.4, and 3.2, there were 18 opportunities to demonstrate the complete 4-step process. Eleven times, all four steps were present in the interview participants' responses. On question 4.2.a, which was designed to elicit reversible conceptions of position and velocity, none of the students used the 4-step thought process to attempt to solve the problem. In this particular instance, only 3 students (Kelsay, Fred, and Jill) were able to make a credible attempt at solving the problem and Kelsay and Jill both treated the problem as a forward application of physics knowledge. Fred solved the problem mathematically but he did so without providing any evidence that he conceived of the problem as a reverse of question 4.1.a. Fred's solution was discussed in section 4.2.2.2.

Unlike two-way reversibility, which is entirely limited to procedural understanding, reversibility of the mental process in reasoning without reversible translation can be used to solve procedural and conceptual problems. Below, I present four interview vignettes, the first two vignettes present students' thought processes while using reversibility of the mental process in reasoning without reversible translation to solve a procedural task and the second two vignettes exemplify students' thought processes while using reversibility of the mental process in reasoning without reversible translation while solving a conceptual task.

The first set of vignettes come from the 2nd interview, question 1. The question asks the interview participants to find a function $f(x)$ whose derivative $f'(x) = x\sin(x^2)$. In this case, the students are attempting to reverse the chain rule differentiation procedure. Both vignettes serve to exemplify the 4-step thinking process that students use when solving problems that require reversibility of the mental process in reasoning without reversible translation. Presented in table 79 below is Kelsay's interview transcript and a corresponding discussion of the related step.

Table 79. Kelsay's transcript from interview 2, question 1.

Kelsay's transcript	Step of reversibility of the mental process in reasoning without reversible translation thought pattern
Um so since $f'(x) = x \sin x^2$ reversing the logic of this problem	Step 1: Kelsay notes that she needs to use reversibility
this could possibly be $-\frac{1}{2}\cos x^2 = f(x)$	Step 2: Kelsay proposes a possible solution.
[using the] chain rule, the derivative of the first is $2x$ times the derivative of the function which would be $-\sin x^2$ … and then of course the constant $-\frac{1}{2}$ carries through and then it would [be] $-\frac{2}{2}\, x(-\sin x^2)$ … canceling out … creating 1, these negatives cancel out so leaving the derivative being $x \sin x^2$	Step 3: Kelsay checks her solution by differentiating her proposed $f(x)$ using the chain rule.
so $f(x)$ could be $-\frac{1}{2}\cos x^2$	Step 4: Kelsay does not need to adjust her answer because differentiation showed her answer to be correct.

Presented in table 80 below is Michael's interview transcript exemplifying the 4-step thought pattern of reversibility of the mental process in reasoning without reversible translation.

Table 80. Michael's transcript from interview 2, question 1.

Michael's transcript	Step of reversibility of the mental process in reasoning without reversible translation thought pattern
so to find $f(x)$ you'd kind of have to undo that, go backwards	Step 1: Michael recognizes that he has to work backwards.
So I need to find something … so it'd have to be negative cosine to be regular sine, so negative cosine of something ... um ... hmm ... I'm just going to kind of plug and chug and see if this works right now ok so if I had just this [referring to $f(x) = -\frac{1}{2}\cos x^2$].	Step 2: Michael proposes a solution, $f(x) = -\frac{1}{2}\cos x^2$

Table 80 (continued)

I would take there's the box. [Michael draws a box around x^2.] The derivative of the box is $2x$ times $-\frac{1}{2}\cos x^2$ err, $\sin x^2$ so 2 times $-\frac{1}{2}$ would be so that wouldn't work ... hmmm or wait, would it because it's negative sine? Yeah it would so it would be $x \sin x^2$ so yep.	Step 3: Michael uses the chain rule to check his answer
I just kind of have to guess and check … I usually get a pretty good idea and then I check it to make sure … so the function $f(x)$ would be $-\frac{1}{2}\cos x^2$.	Step 4: Michael determines that he does not need to adjust his solution

In the following two vignettes, I show how the same 4-step thought process was present in the students' transcripts on a question that required reversibility of the mental process in reasoning without reversible translation to solve a conceptual question. These vignettes come from interview 2, question 4, which is the reverse of interview 2, question 3. Interview 2, question 3, assessed the students' forward knowledge of the graphical representation of the derivative by asking the students to find the value of the derivative at a point when given the graph of a function. Interview 2, question 4 asks students to sketch a possible graph of $f(x)$ given selected $f'(x)$ values, thus reversing the interpretation of the graphical representation of the derivative. Fred's and Kelsay's interview transcripts are presented below. They were chosen because both answered interview 2, questions 3 and 4 correctly, thus demonstrating reversibility. Fred is in the middle flexibility group; Kelsay is in the high flexibility group. I attempt to show how the thoughts displayed in the transcripts are consistent with the proposed 4-step thought process. I first present Fred's interview transcript followed by Kelsay's. In each case, I include the student's written response for ease of reference.

Task 4: The table below gives selected values of $f'(x)$, the derivative of $f(x)$.

a. If $f(x)$ is known to be continuous, sketch a possible curve for $f(x)$ on the axis below.

x	$f'(x)$
0	1
1	2
2	−1
3	0
4	−1

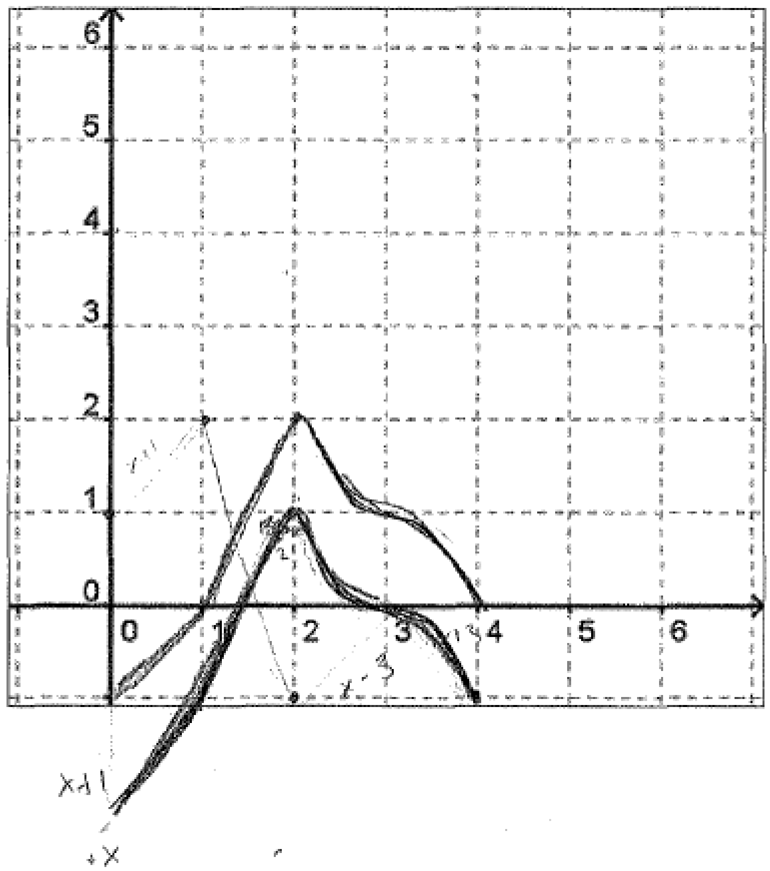


Figure 86. Fred's written response to interview question 2.4

Figure 86, above, shows Fred's written solution to interview question 2.4. Fred's transcript is presented below in table 81.

Table 81. Fred's transcript from interview question 2.4.

Fred's transcript	Step of reversibility of the mental process in reasoning without reversible translation thought pattern
this … problem's given me a table that has x-values and the derivative of f at the specific value so it's asking what the function … would be.	Step 1: Fred notes that he is starting with $f'(x)$ and needs to find $f(x)$
So I'm just gonna plot the points that the derivative of $f(x)$ would have. I'm just putting down what the equations of the derivative lines would be and probably use the secant idea for what $f(x)$ would be	Step 2: Fred proposes that $f(x)$ can be sketched by constructing secant lines.
[LONG PAUSE] …I looked at … the derivative of f, the slope. So I knew that if at [x equals] three, … the derivative was zero, it [$f(x)$] would have to somehow flatten out and the negative ones would mean that … $f(x)$ would be coming down and the $(0, 1)$ and the $(1, 2)$ would mean it's going up at around those points. So this is what I believe the $f(x)$ would look like.	Steps 3 and 4: After approximately six minutes of attempting to relate the line segments connecting the points plotted from the derivative table, Fred rejected this approach and instead sketched portions of $f(x)$ whose behavior matched that described by the derivative value. Fred then connected the portions of his sketch to make $f(x)$ continuous.

Kelsay's written solution to interview 2, question 4 is included below in figure 87. Kelsay's interview transcript immediately follows in table 82.

Task 4: The table below gives selected values of $f'(x)$, the derivative of $f(x)$.

a. If $f(x)$ is known to be continuous, sketch a possible curve for $f(x)$ on the axis below.

x	$f'(x)$
0	1
1	2
2	−1
3	0
4	−1

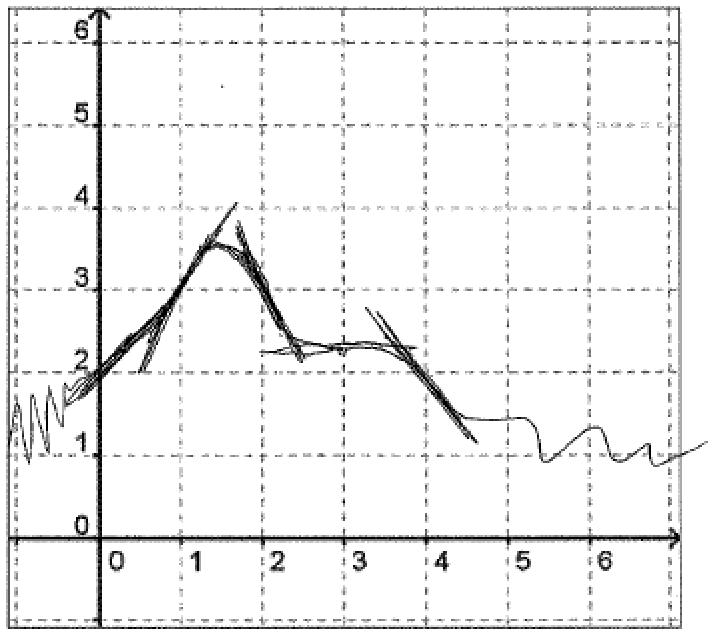


Figure 87. Kelsay’s written response to interview question 2.4

Table 82. Kelsay’s transcript from interview question 2.4.

Kelsay’s transcript	Step of reversibility of the mental process in reasoning without reversible translation thought pattern
so for this one … this [referring to the table of $f'(x)$ values] has to be the slope of the curve at these points	Step 1: Kelsay observes that she has to draw a curve whose slopes at the given x-values match the $f'(x)$ values in the table. Thus, she indicates that she is aware that she has to create $f(x)$ from information about $f'(x)$.
so at [x equals] zero, it [the slope of the tangent line] has to be one and then at two it has to be negative one … So let's see uh wiggly line [Kelsay draws a line that oscillates from $x = -1$ to $x = -0.3$ and then a line segment with a slope of about 1] … At [x equals] one, it has to be two, so that should work [She draws a line segment with a slope of about two], [x equals] two it has to be negative one … at [x equals] three, it has to be zero, that should work. Uh [x equals] four it has to be negative one.	Step 2: Kelsay proposes a sketch of $f(x)$ that has local behavior defined by $f'(x)$ values for $x = 0,1,2,3,4$.

Table 82 (continued)

That should work because it meets … the requirements for the tangent lines [Kelsay draws in the tangent lines at the appropriate x-values]	Step 3: Kelsay checks her answer by sketching tangent lines at $x = 0,1,2,3,4$. Kelsay did not see any need to adjust her answer; thus, she did not need to attempt step 4.

The 4-step thought process that students may use to solve problems that require reversibility of the mental process in reasoning without reversible translation can be observed in the students' responses to the exit slips and opening activities. As described earlier, the exit slips and opening activities did not produce the rich descriptions of the interviews. Opening activity 2.1.1 provided an example of how students were thinking about reversibility of the mental process in reasoning without reversible translation on a conceptual task at the start of the study. The day before opening activity 2.1.1 was administered, the class discussed velocity as the slope of the line tangent to the position graph. All instruction began with the position curve (in algebraic and graphical representations) and asked students to find velocities. Thus, opening activity 2.1.1 tests reversibility of the mental process in reasoning without reversible translation by presenting students with a graph of velocity and asking student to sketch a graph of position. Table 83, shown below, presents the solutions of six students who demonstrated reversibility and offered an explanation that provided insight into their thought processes. The students' explanations are reproduced verbatim. The steps of the 4-step two-way reversibility thought processes present in each explanation are noted in the last column.

Table 83. Evidence of reversible thoughts from opening activity 2.1.1.

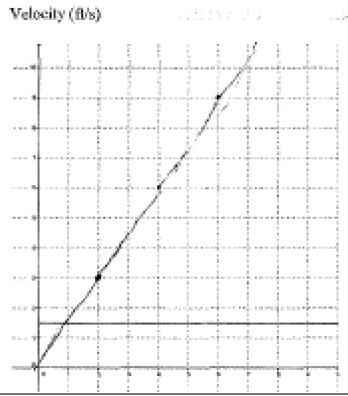
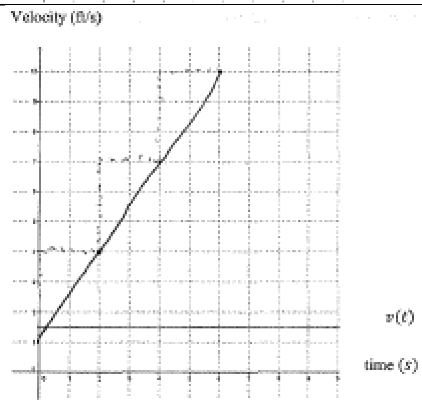
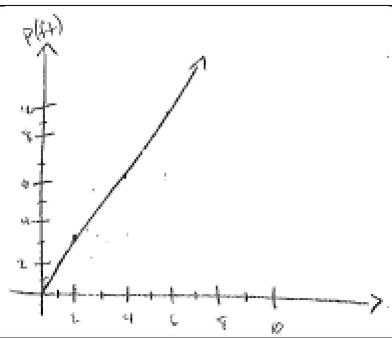
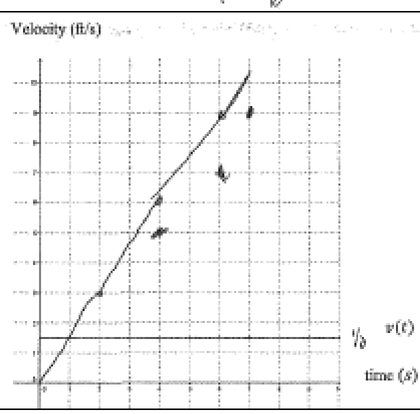
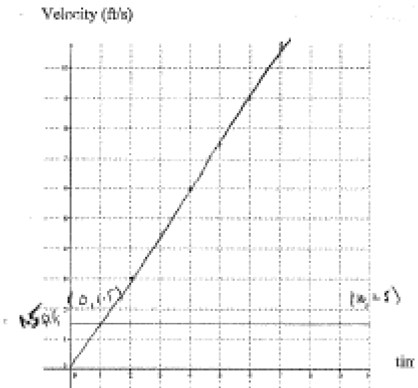
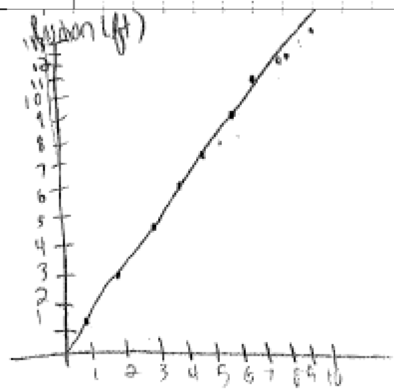
Flexibility group	Solution	Student's explanation	Steps present
High		The velocity graph is based on the slope of the position graph, so the slope of the position graph is based on the velocity graph. $y = 2x$ would be the position graph?	1) 2)
Middle		If the velocity is flat the position graph is constant and the $1.5\frac{ft}{s}$ is equal to the slope of the velocity graph.	1) 2)
High		The velocity is constant, therefore the slope of the position versus time graph is constant $(\frac{3}{2})$.	1) 2)
High		The $v(t)$ graph is constant at $1\frac{1}{2}$, so the corresponding $p(t)$ graph would have a slope of $1\frac{1}{2}$.	1) 2)

Table 83 (continued)

Middle		Velocity is constant so the position will change at a constant rate of 1.5 ft. per second.	1) 2)
High		The velocity was 1.5, so therefore the slope of the position graph was 1.5.	1) 2)

In this example, we see how step 1: recognition of the need for reversibility does not always present as a clear statement of "thinking backwards" or "working in reverse". In this case, the reference to the relationship of velocity and position is evidence that the students are aware that they are working backwards. Step 2, proposing a solution that is informed by knowledge of the forward process, is shown by noting that the velocity should represent the slope of the position graph. Therefore, the position graph should have a constant slope of 1.5. As described earlier in section 4.3.1, steps 3) and 4) are not readily observable in the opening activities as they only provide a window into the final answer and that any checking and adjusting of a solution would likely not be captured by an opening activity.

Consistent with the results of the two-way reversibility analysis, the thought processes that students used to solve questions requiring reversibility of the mental process in reasoning without reversible translation were the same at the start of the study and at the end of the study. Thus, this study did not provide evidence to suggest that students' thought processes regarding the use of reversibility of the mental process in reasoning without reversible translation changed over the course of the study.

4.3.3 Thought processes used when solving problems that require representational reversibility

Over the course of the study, I collected and analyzed 15 exit slips and opening activities that assessed representational reversibility. I also analyzed three sets of interview questions (1.3, 2.3 & 2.4, and 3.1, and 3.2) that assessed representational reversibility. In total, I analyzed 225 representational reversibility tasks for evidence of thought processes used when solving problems that require representational reversibility. There was not a single reference to the use of reversibility when translating back and forth between two representations. In each case where an interview participant demonstrated representational reversibility, there was no evidence to suggest that the students thought of the second translation as a reversing of the first translation. Rather, it appears that the students thought of each translation as an individual forward translation.

As an example of the interview participants' thought processes when solving problems that require representational reversibility, I provide excerpts from each participants' solution to interview 1, question 3. The question presents students with the graph of $f'(x)$ and asks students to sketch the graph of a function $f(x)$.

Kelsay solved the problem by translating the graphical representation of $f'(x)$ into an algebraic expression, saying "the … graph … x^2 so therefore $f(x)$ would have to be $\frac{1}{3}x^3$". Then she translated the algebraic expression of $f(x)$ into a numerical expression by "plotting points" and then drawing a cubic curve through the plotted points. Kelsay did not indicate any reversible thought processes in her translations. She completed a one-way translation from graphical to algebraic by recognizing the derivative as $f'(x) = x^2$. To translate $f(x)$ from algebraic back to graphical, Kelsay does not say anything to indicate that she is reversing her approach to translating from graphical to algebraic. Indeed, she took an entirely different approach by translating through the numerical representation.

Michael solved the problem by translating $f'(x)$ from a graphical expression to a numerical expression to an algebraic expression. This can be seen in Michael's description: "So $f'(x) = x^2$ … By looking at the graph you can see 1^2 is 1, 2^2 is 4, … etc." After determining that $f(x) = \frac{x^3}{3}$, Michael translated the algebraic expression into a numerical expression and then into a graph. He described this process saying, "so then I can't exactly remember the x^3 graph right now so I'm going to make a quick chart [Michael makes an $x - y$ chart] … ok now I remember how it goes". Michael followed the same translational pathway in the forward and reverse directions, graphical to numerical to symbolic and then symbolic to numerical to graphical. However, he did not indicate any sort of reversible thoughts. He treated each translational pathway as a distinct forward pathway.

Fred solved the problem using the same approach as Michael. Fred first translated the graphical representation of $f'(x)$ into a numerical representation by identifying the points on the curve, saying, "I'm just going to figure out the points that are given, and try to figure out what a possible $f(x)$ could be based on this". Then he translated the points into the algebraic expression $f'(x) = x^2$. After correctly determining that $f(x) = \frac{x^3}{3}$, Fred plotted points and then connected the points to create the graph of $f(x)$. Thus, he translated $f(x)$ from an algebraic representation to a numerical representation and finally to a graphical representation. Consistent with the other students, Fred never mentioned any words or phrases that indicated that he considered translation as a reversible act.

Jill solved the problem by translating $f'(x)$ from the graphical representation directly to the algebraic representation saying, “the derivative, that's your basic $f(x) = x^2$”. After correctly determining $f(x)$, Jill tried to translate $f(x)$ from the algebraic to the graphical representation but could not account for the effect of dividing x^3 by 3. Thus, she plotted points first and then sketched the curve. Jill did not make any reversible references while solving the question.

Kirsten, while not able to correctly solve the question, provided insight into her thoughts about translation. She began by observing, “well this is obviously … the x^2 graph so the derivative is $f'(x) = x^2$”. Thus, she began by translating the derivative from a graphical to an algebraic representation. She incorrectly determined that $f(x) = x^{\frac{1}{3}}$ and then demonstrated an inability to translate from the algebraic to the graphical representation by saying “the slope of $1/3$ you would start here [she points at the origin on the graph] go up 1 go over 3 go up 1 over 3. That's a really bad line [she erases line and redraws it]”. Kirsten thought that the graph of $f(x) = x^{\frac{1}{3}}$ was a linear function. She did not indicate that she considered translation to be a reversible action.

Marcus began by noting that the graph of $f'(x)$ “is a graph of x^2”, indicating that he could translate from the graphical representation to the algebraic representation. However, Marcus made no further progress on the problem and thus lacked the opportunity to demonstrate representational reversibility.

The students’ discussion of the representational aspects of paired problems 2.3 & 2.4 and 3.1 & 3.2 was commensurate with the discussions presented here regarding question 1.3. At no point in the study did the students say anything or show any work suggesting that they thought about two-way translations as an instance of reversibility. In each case, when a two-way translation was present, each translation was treated as an isolated problem, unrelated to the paired translation.

4.3.4 Summary – the thought processes that students utilize when using reversibility to solve problems

When students use two-way reversibility to solve a problem, they tend toward thinking about the problem in a 4-step pattern: 1) recognize that the question requires reversibility of a known procedure, 2) reverse the steps of the known procedure by asking her/himself "what do I need to do to get back to …" to find a possible solution, 3) check the validity of the solution by using the known forward process, and 4) adjust the solution if necessary. When students use reversibility of the mental process in reasoning without reversible translation, they use a similar but distinctly different 4-step pattern: 1) recognize that the question requires the use of reversibility, 2) propose a possible solution that is informed by knowledge of the forward process, 3) check the validity of the solution by using the known forward process, and 4) adjust the solution if necessary.

Students may not view representational reversibility as a reversible act. Rather, students seem to view two-way translation as two independent translations, both in the forward direction.

5.0 DISCUSSION

The purpose of this study was to investigate the development of the problem solving processes of flexibility and reversibility in a high school calculus class. Specifically, I examined the extent that flexibility and reversibility developed while students were engaged in linking multiple representations. Twenty-one high school calculus students from an urban school district in a mid-Atlantic state participated in the study.

Flexibility with representations in calculus significantly improved over the course of the study. In particular, the students' demonstrated flexibility with individual translations significantly improved in four out of the six possible translations, symbolic to graphical, symbolic to numerical, graphical to symbolic, and numerical to symbolic, indicating a general trend towards improvement.

Students developed reversibility of two-way processes in calculus at different rates depending on their flexibility level, consistent with Krutetskii's (1976) findings. Students in the high and medium flexibility groups seemed to develop reversibility of two-way processes in calculus simultaneously with learning the process in the forward direction and were able to consistently demonstrate reversibility of a two-way process in calculus throughout the course. Students in the low group were able to learn two-way calculus processes in the forward direction but did not develop reversibility simultaneously. However, through repeated interaction with solving problems that required two-way reversibility of the same calculus process over time, low flexibility students were able to develop two-way reversibility. Reversibility of a two-way calculus process, seemed to increase during the study and then maintain at a high level.

For most students, developing reversibility of the mental process in reasoning without reversible translation was difficult and typically did not happen simultaneously with learning the direct calculus pathway. For some high flexibility students, reversibility of the mental process in reasoning without reversible translation developed on the spot. If it did not, high flexibility students were likely to develop it quickly when given multiple opportunities to engage with reversible tasks in calculus, regardless of the calculus content area. For students in the middle flexibility group, if reversibility of the mental process in reasoning without reversible translation did not develop on the spot, the students required multiple opportunities to engage with reversible calculus tasks of the same or similar calculus content. If the content was presented using different representations, middle flexibility students tended to view the content as new and were not able to apply any reversibility demonstrated with the same content presented in a different representation. For students in the low flexibility group, reversibility of the mental process in reasoning without reversible translation did not develop on the spot and it may not have developed at all in a generalized capacity. Given multiple opportunities to engage with reversible calculus tasks, a low flexibility student may have developed reversibility of the mental process in reasoning without reversible translation. This reversibility likely could only be used to solve identical or nearly identical calculus problems.

Representational reversibility significantly improved between the numerical and symbolic representations and between the symbolic and graphical representations. Representational reversibility did not seem to improve over repeated learning opportunities with the same calculus content. Students who were able to use representational reversibility to solve a calculus problem after initial exposure to learning a new calculus concept were able to use representational reversibility to solve calculus problems with the same calculus content at a later date. Students who were not able to use representational reversibility to solve a calculus problem after initial exposure to learning a new calculus concept were not able to use representational reversibility to solve calculus problems with the same calculus content at a later date.

The results of this study suggest that representational reversibility developed over the course of the study. This result should be expected as representational reversibility is a specific example of linking multiple representations and the course attended to linking multiple representations. The students who had greater flexibility with multiple representations at the start of the study, the high and middle flexibility groups, were able to consistently solve problems that required representational reversibility and had little difficulty translating one representation into another and vice versa. Despite the fact that the students engaged in opportunities to link multiple representations on a nearly daily basis, use of representational reversibility proved very difficult for low flexibility students.

5.1 INTERPRETATION OF FINDINGS

The results of the study suggest that reversibility may develop as students have multiple opportunities to engage with problems within a particular content area that require reversibility to solve. For example, students' reversibility with the simple power rule and the chain rule increased throughout the course as students had multiple opportunities to reverse differentiation. This result serves to answer in part multiple researchers' calls for empirical research into how reversibility develops (Lamon, 2007; Ramful & Olive, 2008; Teachey, 2003) with specific attention paid to the call for research investigating the development of reversibility at the secondary level (Ramful & Olive). The results of this study show that reversibility may develop innately and immediately for some students; however, students who do not develop reversibility on the spot can develop reversibility of a procedure or concept through multiple opportunities to engage with reversible tasks of similar content. Furthermore, as shown by the results of the exit slips and opening activities assessing reversibility of the simple power rule and the chain rule, the opportunities to engage with reversible tasks need not be consecutive or necessarily near to one another chronologically.

This study proposes a model by which students think about using reversibility when solving problems. The distinct 4-step processes proposed here by which students think about using two-way reversibility and reversibility of the mental process in reasoning without reversible translation offer insights into what triggers a student to use reversibility and how students problem solve around the lack of an established cognitive pathway to solve a problem from output to input.

Underlying each of the thought processes for using reversibility is the ability to recognize a learned procedure or process present within an end result. The students had to first recognize that the problem presented an outcome of some forward action with which they had previous experience and an existing knowledge structure. This suggests that a pre-requisite for using reversibility is a reasonably well-developed conception of the forward process. One way to interpret this result is to conclude that reversibility of a process or procedure is a visible evidence of a well-functioning schema. Thus, reversibility requires learning beyond rote memorization of an algorithm, procedure, or process and requires recognition of the unseen procedures and processes that when enacted upon an input produced the present output.

5.1.1 Reversibility as an instance of transfer

The observed increase in reversibility through multiple opportunities to engage with reversibility of a particular process or concept may be an instance of transfer. Engle (2006) defined transfer as "the appropriate application of something that has been learned in one situation to a different but related situation" (p. 452). Thus, using reversibility to find a constituent input from a given output, when only the forward solution process has been learned would qualify as an instance of transfer.

During the study, the students used reversibility to reverse the simple power rule five different times, in various settings and contexts. In each case, the superficial characteristics were changed. The class improved significantly from their first experience with reversing the simple power rule to the fifth opportunity. This was evidence that the students had generalized reversibility of the simple power rule. The observed development of reversibility over repeated engagements with reversibility of the simple power rule can likely be explained by Wagner's (2006) theory that transfer does not happen in bulk, where all learning in a given context transfers, but rather piece by piece, through experiences with multiple problem solving opportunities requiring use of the same concept situated in various instantiations. In the present study, reversing the simple power rule was first presented as reversibility of differentiation and then as reversibility of multiple differentiations. Later in the course, reversibility of the simple power rule was situated within reversibility of rectilinear motion and finally within reversibility of Newton's Method. As the interviews helped to show, at the end of the study, students of all flexibility levels were able to recognize reversibility of the simple power rule within different contexts.

How the teacher positions the instruction on a day-by-day basis as part of an overall, connected body of knowledge in which the students are actively constructing knowledge has been identified as an integral piece in creating an environment in which transfer is able to occur (Engle, 2006). Within this study, the instructor/researcher endeavored to present differential calculus as a property of functions, namely the rate of change of a function at an x-value or on a finite or infinite domain within an environment that emphasizes linking multiple representations. With this view of instruction, students daily engaged with the derivative by comparing and contrasting three different representations of the instantaneous rate of change and three different representations of the average rate of change. These engagements included but were not limited to developing algebraic means for finding the instantaneous rate of change and comparing the result with the algebraic representation of the average rate of change, interpreting the slope of a graph of a function at an x-value as the instantaneous rate of change, contrasted with interpreting the slope of a secant line as the average rate of change, using the intersection of the graphical and algebraic representations of the derivative to analyze functional behavior and to solve physical application problems (i.e. related rates and optimization), and using graphical arguments to develop algebraic techniques of differentiation. As this list shows, the students had the opportunity to engage in a calculus class that research suggests was likely to foster transfer.

The reversibility that developed was not limited to just procedural knowledge such as the simple power rule and chain rule but also extended to conceptual knowledge that included curve sketching, functional analysis, and graphical analysis. Engle (2006) proposed that when learning takes place in an environment that links multiple topics during the same instructional time period, transfer is likely to happen. When reversibility is viewed as an instance of transfer, we see a possible explanatory factor describing why reversibility across multiple content domains within differential calculus has developed. By participating in a calculus course that attended to linking multiple representations at every opportunity, the students were positioned in a learning environment that was likely conducive to producing transfer, and thus reversibility.

5.1.2 Reversibility within the APOS framework

In section 1.3, I proposed a model of the APOS framework that includes reversibility. I proposed that reversibility served as the evidence of interiorization, the transition from an action conception to a process conception, and is the mechanism of de-encapsulation, digesting an object conception into constituent processes.

The results of this study suggest that reversibility of two-way processes and reversibility of the mental process in reasoning without translation developed over the course of the study. This result has particular implications when viewed through an APOS lens. Since actions are described as procedures that transform mathematical objects, reversibility of an action would require reversibility of a procedure. Reversing of a procedure can require either two-way reversibility or reversibility of a mental process in reasoning without reversible translation, depending on the nature of the procedure and the student's mathematical background knowledge. Since the successful interiorization of an action into a process is evidenced by the ability to reverse an action from end to beginning (Asiala et al., 1996), the development of two-way reversibility would likely improve interiorization.

The results of the study further suggest that interiorization may develop through multiple opportunities to engage with reversing actions situated in various contexts. Since the results of the study suggest that both two-way reversibility and reversibility of the mental process in reasoning without reversible translation can develop on the spot, applied to the APOS framework, it is reasonable to conclude that for some students, interiorization of an action into a process happens on the spot. For those students who do not interiorize on the spot, the results of this study suggest that interiorization may happen through multiple opportunities to engage with reversing the action in a variety of instantiations.

5.1.3 Reversibility as distinct from flexibility

Since Krutetskii (1976) first proposed flexibility and reversibility as separate problem solving processes related under the broad umbrella of "the mathematical cast of mind" (p. 351), researchers (Gray & Tall, 1994; McGowen, 2006; Rachlin, 1981; Usiskin, 1999) have positioned reversibility as a specific instance of flexibility. However, I am unaware of any research that has been previously conducted to indicate how reversibility fits within the cognitive processes underlying flexibility. This research study has made strides towards informing whether or not a dichotomy exists between reversibility and flexibility or if reversibility is contained within flexibility. The results of this study suggest that when students are engaged in a course that attends to developing flexibility (by linking multiple representations), reversibility develops. This result supports Rachlin's (1981) finding that his participants' patterns of reversibility were observable in their use of flexibility. However, this should not be counted as evidence that reversibility is a kind of flexibility. In fact, the results of this study suggest that although development of reversibility and flexibility may be related, reversibility may not be a kind of flexibility.

The results of investigating the kinds of thought processes used by students when solving reversible tasks produced evidence indicating similar trains of thought when solving problems that require two-way reversibility and reversibility of the mental process in reasoning without translation. In both cases, a reversible thought process began with a clear recognition of the need to work backwards. That observation was noticeably absent from all tasks that required representational reversibility. As discussed earlier, representational reversibility is two-way flexibility. Thus, the interview questions that assessed representational reversibility provided insight into the thought processes that students used when solving tasks that required flexibility. Since there was no evidence that any of the interview participants used thought processes that indicated a consideration of a reversible relationship, I suggest that the students were using flexibility to complete two separate one-way translations. This result suggests that reversibility and flexibility may be two distinct cognitive processes that do not overlap. Furthermore, if students are not using a reversible thought process to solve a problem that requires representational reversibility, then the proposed framework dividing reversibility of the mental process in reasoning into reversibility of the mental process in reasoning without reversible translation and representational reversibility is an unnecessary and perhaps incorrect dichotomy. The results of the interviews in this study suggest that representational reversibility may be an example of flexibility and not reversibility. To account for this result, I would modify the proposed framework originally presented as figure 3 in section 1.2, shown below as figure 88:

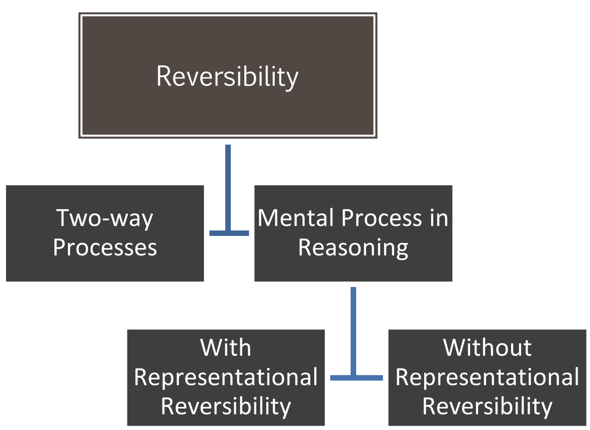


Figure 88. The three kinds of reversibility that will be studied in this investigation

and replace the reversibility framework with the original Krutetskiian framework shown below in figure 89.

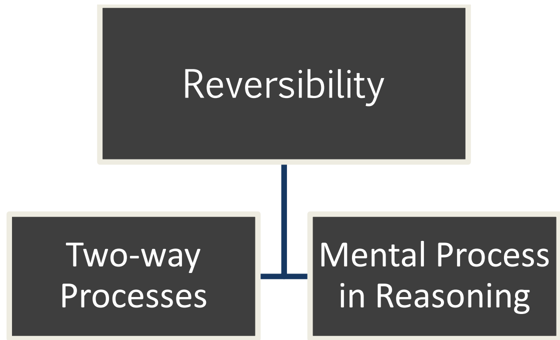


Figure 89. Visualization of the two component process within Krutetskiian reversibility

The possibility that reversibility and flexibility are distinct cognitive processes does not necessarily conflict with research suggesting that educational activities designed to improve flexibility will aid in the development of reversibility. Within Krutetskii's (1976) proposed mathematical cast of mind, reversibility and flexibility are positioned as distinct, but related problem solving processes, both of which fall under a single cognitive system. The results of the present study suggest that flexibility and reversibility may develop in tandem when students are engaged in a mathematics course that attends to linking multiple representations but are distinct cognitive processes.

It should again be noted that flexibility and reversibility are likely domain specific, and that the presence or lack thereof of an overlap between flexibility and reversibility when solving calculus problems may or may not be present when solving an algebra problem, or a geometry problem, or a trigonometry problem, etc. Furthermore, the conclusion that flexibility and reversibility are distinct cognitive processes is not supported by empirical data collected in this study. It is supported by the absence of data that supports the alternative hypothesis, in this case that reversibility is an instance of flexibility.

5.2 LIMITATIONS

There are several limitations to this study. First, the sample of students participating in this research study consisted of high school advanced placement calculus students. Advanced placement calculus students typically represent the strongest mathematics students in a high school and are likely not representative of the entire student body. It may be the case that the findings in this study do not generalize to the entire student population.

As previously noted, flexibility is content specific and reversibility is a problem solving process well suited for use in a calculus class. As such, calculus may have been an ideal content for investigating the development of flexibility and reversibility. It is possible that the significant development of flexibility and the significant development of reversibility may not happen to the same extent in a different mathematical content. Also, to the extent that students may not exhibit flexibility and/or reversibility in a calculus setting, they may exhibit flexibility and/or reversibility in a different mathematical setting. Thus, this research study does not suggest that flexibility and reversibility are processes that once developed in one content area will generalize to all other mathematical content areas.

The design of the exit slips and opening activities as well as the interview questions may have biased the students towards developing reversibility. The class attempted 33 exit slips and opening activities; each exit slip presented a task in the forward direction and each opening activity presented the same concept in the reverse direction. It is possible that the students noticed the pattern, which may have influenced their approach to the opening activities. In the event that this were true, the students would then view using reversibility to solve a problem as a classroom norm and would have an inherent bent toward using reversibility to solve opening activities as opposed to problem solving by first noticing that the problem prompt contains an end result of a learned process. This limitation could extend to the interviews. Since the interviews were designed to elicit reversible conceptions through the use of reversible pairs of questions, it is possible that more reversibility would be present in interview four than interview one by virtue of students' noticing that half of the questions in the interview are reverses of the other half of the questions. Varying the order of the questions in the interviews may have helped to mitigate this limitation.

Finally, there is the possibility of an inherent bias toward the development of reversibility and flexibility because the researcher was also the course instructor. The instructor/researcher took great strides towards not biasing instruction toward reversibility. Flexibility was the focus of all instruction related to the research questions in this study. However, it is reasonable to think that over the course of five months, the instructor's natural interest in using reversibility to solve problems would present during class discussions and other learning activities.

5.3 FUTURE RESEARCH

The results of this study have several implications for future research: 1) the study of development of reversibility and flexibility should commence in other mathematical content areas with other sample populations, 2) research should investigate the existence of a link between teaching that attends to linking multiple representations and the development of flexibility and/or reversibility, 3) reversibility as an integral part of the APOS framework should be explored, and 4) reversibility as an instance of transfer should be investigated. Each implication is discussed.

Since reversibility and flexibility are likely content specific, research should commence that investigates the presence and development of reversibility and flexibility in a variety of mathematical content domains including, but not limited to algebra 1, algebra 2, geometry, and trigonometry. If research similar to that conducted in this study were conducted in the disciplines listed, we would have a comprehensive body of knowledge informing the development of the problem solving processes of flexibility and reversibility in secondary mathematics. One could envision a microgenetic study within trigonometry due to the invertible properties of the families of functions studied in a trigonometry class. A researcher could follow the exit slip and opening activity design of this study to measure the development of reversibility over the course of an entire school year. A possible adjustment to the design could include administering the forward and reverse questions at the end of each class period, thus removing the overnight time lapse and issues with absence that were present in this study.

Students' thought processes when using reversibility within other mathematical content domains should be investigated and compared to the thought processes observed in this study. The 4-step process proposed here should be subject to review by other reversibility researchers in an effort to find either confirmation or a need for revision. The 4-step process could be used as a framework to evaluate the thought processes that students are using to solve problems likely to elicit reversible conceptions.

Researchers should investigate the existence of a link between teaching that attends to linking multiple representations and the development of flexibility and/or reversibility. Flexibility developed throughout the course and the students became equally proficient with translating into the numerical, graphical, and algebraic representations. This finding suggests that one instructional decision that may help students move beyond preference for the symbolic representation, which restricts understanding and fluency with the numerical and graphical representations (Brenner et al., 1997; Dreyfus & Eisenberg, 1990; Hiebert & Carpenter, 1992; Knuth, 2000), is to attend to linking multiple representations at every opportunity. Research should commence that explores the instructional decisions and learning opportunities that facilitate the development of flexibility and reversibility. For example, a detailed, descriptive study of a few students could allow for determining learning activities or events that foster the development of flexibility and/or reversibility. Researchers could also consider a comparative, experimental design to evaluate the effects of teaching with a curriculum that attends to linking multiple representations on flexibility and reversibility. As this study was an observational study and not a comparative experiment, causal links could not be explored. However, calculus researchers at the university level could offer two sections of differential calculus in which one section attended to linking multiple representations and the other section adopted a traditional, algebraic approach to instructing calculus. Then, by using the flexibility pre-test and the differentiation competency test, flexibility could be measured as a dependent variable. By using the interview questions and exit slips and opening activities, reversibility could be reliably measured in both sections as well.

Research should commence that examines reversibility within a fine-grained APOS framework. As previously discussed, reversibility situates within the APOS framework in at least three positions. The data set collected in this study could be parsed and analyzed entirely through an APOS lens. For example, Clark et al. (1997) proposed a chain rule schema wherein they proposed a genetic decomposition for understanding the chain rule. They did not address reversibility within their analysis of the chain rule (it is mentioned that recognizing a composite function requires reversing composition). Interview questions 2.1 & 2.2, as well as exit slips and opening activities 2.6.1, 3.3.1, and 3.3.2 could all be used as a data set to examine with the proposed chain rule schema. This research could confirm reversibility's place as the evidence of interiorization; it may also challenge the authors' contention that actions, processes, and objects were insufficient to describe how students construct knowledge of the chain rule. Clark et al.

(1997) also analyzed chain rule schema development through the Piagetian triad. By using the data set collected in this dissertation, one could attempt to determine where reversibility situates within the Piagetian triad, if at all.

Finally, research should investigate the educative benefits of reversible tasks on forward learning. This study did not attempt to evaluate whether or not solving reversible tasks nearly every day results in improvement in the students' forward learning. It is possible that by thinking reversibly, a student's knowledge of the forward direction improves. This research showed that over the course of the study, students' scores on exit slips of similar content improved; however, it is impossible to conclude that improvement is a result of, or even related to, solving reversible tasks. Research should investigate this matter.

APPENDIX A: COURSE CALENDAR

Key: N represents numerical, G represents graphical, V represents verbal, S represents symbolic, F represent function, D represents differentiation.

Table 84. Course Calendar

Day	Section Number	Topics Covered	Representations linked in Class			Representations linked in Homework		
			Input	Output	# of Examples	Input	Output	# of Examples
1-2	2.1	Tangent lines and rates of change; velocity, average velocity	S	S	2	G	G	1
			S	N	6	G	N	5
			VS	N	2	N	V	1
			VG	N	2	S	G	1
						S	N	3
						V	V	3
						VG	G	1
						VG	N	7
						VG	V	4
						VS	N	2

Table 84 (continued)

3-4	2.2	Derivatives of functions	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	7	G	G	12
			S	N	2	G	N	1
			S	VG	1	N	N	1
			S	VN	1	N	S	1
			VS	S	1	N	VN	1
						S	G	2
						S	S	4
						V	V	3
						VN	N	1
						VS	N	2
						VS	S	2
5-6	2.3	Techniques of differentiation, simple power rule	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	17	S	N	2
			S	N	1	S	S	8
			GS	N	1	VS	N	1
						VS	S	1
7	2.4	Product & quotient rule	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	4	NS	N	4
			S	G	1	S	S	7
8	2.5	Derivatives of trigonometric functions	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	2	S	S	6
			S	N	1			
			V	N	1			
			VS	VS	1			
9-11	2.6	Chain rule				Input	Output	# of Examples
			Input	Output	# of Examples	N	S	1
			S	S	5	NGS	N	1
			V	N	1	NS	N	3
			NS	N	1	S	N	5
			GS	N	1	S	S	16

Table 84 (continued)

12	Chapter 2 Review	Chapter 2 – Graded Homework Assignment	Input	Output	# of Examples			
			G	N	3			
			GS	N	1			
			N	G	1			
			NS	N	8			
			S	N	5			
			S	S	18			
			S	V	2			
			V	V	1			
			VS	N	4			
13	Chapter 2 Multiple Choice Test	Rules of differentiation, limits, continuity, position & velocity						
14	Chapter 2 Free-response Test	Rules of differentiation, limits, continuity, position & velocity						
15	Return Tests	Rules of differentiation, limits, continuity, position & velocity						
16	3.1	Implicit differentiation	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	6	S	N	2
			S	N	2	S	S	2
			S	G	1	VS	VS	1
17	3.2	Derivatives of logarithmic functions and inverse functions	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	3	S	S	7
			S	N	1	VS	N	1
			S	V	1			
18	3.2	Logarithmic differentiation	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	1	S	N	1
						S	S	2

Table 84 (continued)

19	3.3	Derivatives of inverse trigonometric functions	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	9	S	N	3
			N	S	1	S	S	9
			S	N	2	VS	N	1
			N	N	2			
20-21	3.4	Related rates	Input	Output	# of Examples	Input	Output	# of Examples
			VNS	N	1	VGN	V	1
			VN	N	2	VN	N	9
			VN	S	1			
			VGN	N	2			
			VGN	V	1			
22	3.5	Local linear approximation and differentials	Input	Output	# of Examples	Input	Output	# of Examples
			S	S	3	S	G	1
			S	N	2	S	N	2
			S	G	1	S	S	1
			NS	V	1	V	V	2
			NS	N	1	VN	N	2
			VN	N	2			
23-24	3.6	L'Hospital's rule	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	13	S	N	5
						VS	VN	1
25	Chapter 3 Review	Chapter 3 – Graded Homework Assignment	Input	Output	# of Examples			
			GNS	N	1			
			GNS	V	1			
			N	N	1			
			S	N	5			
			S	S	11			
			SN	N	1			
			VGN	N	2			
			VN	N	2			
			VNS	N	1			
			VS	N	3			

Table 84 (continued)

26	Chapter 3 Multiple Choice Test	Implicit differentiation, local linearization, derivatives of transcendental functions, related rates						
27	Chapter 3 Free-response Test	Implicit differentiation, local linearization, derivatives of transcendental functions, related rates						
28	Return Tests	Implicit differentiation, local linearization, derivatives of transcendental functions, related rates						
29-30	4.1	Functional analysis: Increasing, decreasing, and concavity	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	3	G	VN	18
			S	VN	7	N	V	1
			G	N	1	S	VN	4
			VS	VN	3	VN	G	3
						VN	N	2
						VN	VN	11
31-32	4.2	Relative extrema, graphing polynomials, 1^{st} and 2^{nd} derivative tests	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	2	G	G	2
			S	G	1	G	VN	4
			G	N	5	S	N	3
			S	VN	3	S	VN	3
						V	G	3
						V	V	2
						VN	G	5

Table 84 (continued)

33-34	4.3	Analyzing rational functions	Input	Output	# of Examples	Input	Output	# of Examples
			S	G	5	S	G	5
			S	VN	1	V	V	4
						VGN	VN	4
35	4.4	Absolute max/min	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	4	S	N	3
			S	VN	1	S	V	1
						V	V	4
36	Review of 4.1-4.4	4.1-4.4: Graded Homework Assignment	Input	Output	# of Examples			
			G	G	3			
			G	N	6			
			G	V	4			
			GS	N	1			
			S	N	4			
			S	S	1			
			S	V	5			
			V	G	1			
			V	N	1			
			VG	N	1			
			VG	S	1			
			VG	V	2			
			VG	VN	7			
			VN	G	1			
			VN	N	2			
			VN	V	1			
			VN	VN	1			
			VNS	N	1			
			VNS	S	2			
			VNS	V				
37	4.1-4.4 Multiple-Choice Test	Analyzing functions						
38	4.1-4.4 Free-Response Test	Analyzing functions						

Table 84 (continued)

39	Return Tests	Analyzing functions						
40	4.5	Optimization	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	1	V	N	4
			V	N	2	VS	N	2
			VG	N	3			
			VS	N	1			
41-42	4.6	Rectilinear motion	Input	Output	# of Examples	Input	Output	# of Examples
			GS	V	1	V	V	3
			VS	G	2	VG	G	6
			VS	S	2	VG	N	9
			VG	G	1	VG	V	4
						VN	V	6
						VS	N	12
						VS	S	1
43	4.7	Newton's method	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	2	S	N	2
						V	V	4
44	4.8	Mean-value theorem	Input	Output	# of Examples	Input	Output	# of Examples
			S	N	2	G	N	2
			VN	N	1	S	N	3
						V	V	1
						VN	N	4

Table 84 (continued)

45	4.5-4.8 Review	Ch. 4 Thought questions and graded homework	Input	Output	# of Examples	Input	Output	# of Examples
			V	V	1	G	V	1
			V	VN	1	S	N	3
			VN	V	1	V	N	1
			VN	VG	1	VG	N	12
			VN	VS	1	VG	S	1
			VNS	V	1	VGN	N	3
						VN	N	1
						VN	V	1
						VN	VN	1
						VS	N	11
						VS	S	2
						VS	V	2
						VS	VN	2
46	4.5-4.8 Multiple-Choice Test	Optimization, rectilinear motion, Newton's method, mean-value theorem						
47	4.5-4.8 Free-Response Test	Optimization, rectilinear motion, Newton's method, mean-value theorem						
48	Return Tests	Optimization, rectilinear motion, Newton's method, mean-value theorem						

APPENDIX B: SAMPLE LESSON PLAN – SECTION 2.1

Mathematical Goals/Objectives: The goal of this lesson is to show the significant relation of three seemingly unrelated ideas: 1) tangent lines to curves, 2) the velocity of an object moving along a line, and 3) the rate at which one variable changes relative to another.

Specific processes that students will develop: 1) limiting process of the slopes of a secant line as the secant line approaches a tangent line at $x = c$.

Specific connections to develop: 1) finding the slope of the line tangent to a curve at $x = c$, 2) finding the instantaneous velocity of an object in motion, and 3) finding the rate at which one variable changes relative to another are all instantiations of the same general process – the limiting process of the slope formula representing the average rate of change.

Method: Students and teacher will fill in guided notes – see below. Students will solve approximately 6 tasks (in the guided notes the tasks are called examples) on day 1 and 3 tasks on day 2. The tasks are designed to engage students in the limiting process of the average rate of change formula and then to make connections between the three separate representations described above. In each case, students work independently or in small groups (2-3) to attempt to solve each task. The teacher monitors the work by walking around the room and engaging with students. The teacher challenges students' thoughts, encourages students who are struggling, probes students who have completed the task, and makes suggestions as necessary. The teacher intentionally selects students to share solution methods, often multiple methods are shared. The process repeats for each task.

Homework: Homework problems have been intentionally selected to require students to solve tasks using the limiting process of the average rate of change to solve problems involving position and velocity, the slope of the line tangent to a curve at $x = c$, and finding the rate at which one variable changes relative to another. Day 1 homework problems are discussed at the beginning of Day 2. Day 2 homework problems are discussed at the beginning of the next day's class.

Guided Notes: The following notes are what the students and teacher will work through during section 2.1. All of the filled-in blanks are empty on the students' copies of the notes.

Chapter 2: The Derivative

One of the crowning achievements of calculus is its ability to capture
__capture___________________ motion mathematically, allowing that motion to be analyzed
__instant_______________ by **___instant_______________**.

2.1: Tangent Lines and Rates of Change – Day 1

In Section 1.1, we defined the slope of the secant line between a point $P(c, f(c))$ on $f(x)$ and a distinct point $Q(x, f(x))$ on $f(x)$ as $m_{PQ} =$__$\frac{f(x)-f(c)}{x-c}$______________. If we move Q closer and closer to P, we are in essence taking the limit as $x \to$ _c__. When $Q \to P$, the secant line PQ approaches its limiting position, the tangent line at P.

Thus, we define the slope of the line tangent to $f(x)$ at P,

$$m_{\tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

and we define the ***tangent line*** to the curve $y = f(x)$ at the point $(c, f(c))$ to be the line with the equation $\underline{L = f(c) + m_{tan}(x - c).}$

We refer to this line as the $\underline{\text{tangent line to } y = f(x) \text{ at } x = c.}$

Ex. 1: Find the equation of the line tangent to the parabola $y = x^2$ at the point $(1,1)$.

$y = f(1) + m_{tan}(x - 1) \qquad m_{tan} = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim\limits_{x \to 1} \frac{x^2-1}{x-1} = \lim\limits_{x \to 1}(x + 1) = 2$

$y = 1 + 2(x - 1)$

Begin: allow students approximately 4 minutes to solve problem.

1) Walk around classroom. If student has started problem in appropriate manner, then say “looks good” or nod and keep walking. If student has not started, ask student “what are you looking for” – Student should reply “the tangent line”. Teacher responds: “And what do you need to know in order to find the equation of a tangent line? If student lists 1) a point and 2) a slope, then encourage student to find the slope. If student does not know, ask student to draw a curve and a tangent line to the curve. Prompt student to explain what features define a tangent line. The student will likely now identify a point and a slope. Encourage student to find point and slope.
2) Identify students with partially correct and completely correct solution methods.
3) After 4 or 5 minutes, depending on progress of class, have a student with a partially correct solution present her/his solution on the promethean board. Require student to EXPLAIN all decisions and why s/he believes her/his work is correct. Then, have a student with fully correct solution present her/his solution and EXPLAIN all of her/his decisions and why s/he believes that her/his work is correct. Have presenting student answer any questions from other students regarding example.
4) Linking Question 1: After correct solution is presented and discussed, assign extra question: “NOW, SKETCH A GRAPH OF $F(X)$ AND THE LINE TANGENT TO $F(X)$ AT $X = 1$.”
    a. Have a student sketch the correct drawing on the Promethean Board. Ask students to explain the relationship between the algebraic representation and graphical representation of $f(x)$ and the tangent line.
5) Linking Question 2: Ask “what does a tangent line do for us? Why would we want one?” After taking a few answers, have students construct a table of values near $x = 1$. Ask question again. Give limited hints until large group discussion arrives at conclusion that the tangent line approximates the value of the curve at $x -$values near the $x -$coordinate of the given point.

The alternative form of the slope formula

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

There is a commonly used alternative form of this formula.

Let $h = x - c$. Now, substitute for all of the x’s in the formula: $m_{tan} = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Ex. 2: Compute the slope in example 1 again, this time using the alternative formula that we just derived.

$m_{tan} = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0} \frac{(1+h)^2-1}{h} =$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h}$$

$$= \lim_{h\to 0} \frac{h(2+h)}{h} = 2.$$

1) Allow approximately 4 minutes.
2) Walk around and look at students' work. Pay particular attention to how students handle $f(x+h)$. Students frequently have problems correctly evaluating/expanding compositions of functions.
3) Check to see if students are trying to evaluate limit before simplifying/canceling.
4) Check to see if anyone writes $\frac{0}{0}$ as final answer or attempts to draw conclusion from $\frac{0}{0}$. Redirect any student who does.
5) Select student who has correct solution and have student present solution to class. Make sure that students EXPLAINS/DESCRIBES his/her work and why s/he made the decisions that s/he made.
6) Follow-up question to class: what does this answer tell us?
   a. Have students discuss with each other and then in whole class discussion.
   b. Students will inevitably say "it means the slope is two."
      i. Follow-up by asking "well, what does that mean?"
   c. Student will likely say that "the tangent line has a slope of two"
      i. Follow-up by asking, "well, what does that have to do with the original function?"
   d. Class may become silent at this point.
   e. Continue asking and re-wording as appropriate, for example "What does the fact that the line tangent to the curve at $x = 1$ has a slope of 2 tell me about the curve?"
   f. Keep prompting until a student answers "it means that the y-value of the curve is increasing at a rate of 2 units for every 1-unit increase in x.
      i. Some possible prompts:
         1. "What does slope mean?"
         2. "What is rise over run?"
         3. What is "change in y over change in x?"

Ex. 3: Find an equation for the tangent line to the curve $y = \frac{2}{x}$ at the point $(2,1)$ on this curve.

$$y = f(2) + m_{tan}(x-2)$$
$$y = 1 + m_{tan}(x-2)$$
$$m_{tan} = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{x\to 2} \frac{\frac{2}{x}-1}{x-2} = \lim_{x\to 2} \frac{\frac{2}{x}-\frac{x}{x}}{x-2}$$
$$= \lim_{x\to 2} \frac{2-x}{x(x-2)} = \lim_{x\to 2} \frac{-1}{x}$$
$$= -\frac{1}{2}.$$

$$y = 1 - \frac{1}{2}(x-2)$$

1) Allow 5-7 minutes, students will likely struggle with correctly simplifying a complex fraction.
2) Walk around room and point out errors, specifically algebraic errors.
3) Select one student who uses the limit of the slope formula to find m and select one student who uses the limit of the difference quotient to find m. Have both students present solutions and describe their work.
4) Linking question: have students construct a graph of the function and the line tangent to the curve.

Ex. 4: Find the slopes of the tangent lines to the curve $y = \sqrt{x}$ at $c = 1, c = 4$, and $c = 9$.

$$m_{tan} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c}$$
$$= \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{m_{tan}}$
1	**1**	$\boldsymbol{\frac{1}{2}}$
4	**2**	$\boldsymbol{\frac{1}{4}}$
9	**3**	$\boldsymbol{\frac{1}{6}}$

1) Allow 4-5 minutes
2) Let students solve problem without finding slope generating formula
3) Walk around room, if any student solves problem by finding a slope generating formula, then have that student present answer.
   a. If not, then have three different students present answers for each x-value
   b. Then, I will construct the table of values and find the slope generating formula.
   c. After finding the equation $m_{tan} = \frac{1}{2\sqrt{c}}$ ask students what this algebraic expression represents. Continue prompting until a student or students identify that we have found a function that will return the slope of the line tangent to the curve at $x = c$.
4) Students will ask "if you had used the other formula, would you have come up with the same equation?"
   a. Have whole class find the slope generating function using the difference quotient.
   b. Have students discuss and identify if both approaches to finding the slope function have the same answer.

Velocity

What is the difference between speed and velocity?

Speed = |velocity|

1) Prompt until students reach conclusion
2) Students will likely suggest that "speed is a scalar and velocity is a vector"
   a. Reply by asking, "what does that mean mathematically?"

How do we measure velocity in rectilinear motion?

Meters/sec, ft/sec, mph, etc.

1) Allow large group discussion … students will soon identify that velocity is measured in units of length divided by units of time.

In this class, we will assume that motion only occurs along a straight line. That line is often referred to as the s-axis. Thus, we typically refer to the position function as $s = f(t)$, $f(t)$ returns the position of the object in motion at time t.

The Average Velocity

The average velocity of a particle in motion is defined as

$v_{ave} =$ $\frac{\text{change in position}}{\text{change in time}}$ $=$ $\frac{s(t_2)-s(t_1)}{t_2-t_1}$.

Does this formula look familiar? It looks like slope.

Have students discuss how to calculus average velocity. Usually through life experience (driving, etc.) or from a physics class, students give the average velocity formula pretty quickly.

How else could we write this formula? $\frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{\Delta s}{\Delta t}$.

Ex. 5: Suppose that $s = f(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the average velocities of the particle over the time intervals (a) [0,2] and (b) [2,3].

a)

$$v_{ave}=\frac{s(2)-s(0)}{2-0}$$

$$=\frac{3-0}{2-0}=\frac{3}{2}$$

Have students work through problem, about 2 minutes.
Have students check answers with one another.
Have a volunteer present her/his answer.
Show how this problem is scored <1> pt for difference quotient, <1> pt for final answer.

b)

$$v_{ave}=\frac{s(3)-s(2)}{3-2}$$

$$=\frac{-2-3}{3-2}=\frac{-5}{1}$$

Have students work through problem, about 2 minutes.
Have students check answers with one another.
Have a volunteer present her/his answer.

Average velocity describes a particle's motion over an interval of time, but we are interested in a particle's ___**instantaneous**_____________________ velocity, which describes the particle's behavior at an instant in time. We cannot use the average velocity formula because the length of the time interval is __**zero**___________. How could we work around this problem?

Allow students to volunteer answers. If prompting is needed, ask how we work around dividing by zero in a calculus class.

Ex. 6: Now find the instantaneous velocity at $t = 2\ s$ for the particle whose position is described by

$$s = f(t) = 1 + 5t - 2t^2.$$

$$v_{inst}=\lim_{t\to 2}\frac{s(t)-s(2)}{t-2}=\lim_{t\to 2}\frac{1+5t-2t^2-3}{t-2}$$

$$=\lim_{t\to 2}\frac{5t-2t^2-2}{t-2}=\lim_{t\to 2}\frac{(1-2t)(t-2)}{t-2}$$

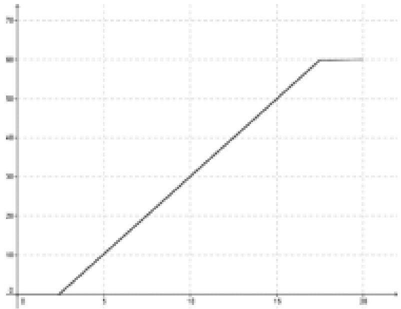
$$=\lim_{t\to 2}(1-2t)=-3.$$

1) Allow 4-5 minutes. Most issues that arise, if any, will be related to factoring the polynomial written out of traditional order.
2) Allow students to compare answers. At this point, nearly everyone will be getting the procedure correct.
3) KEY CONCEPT: Emphasize the difference between average velocity and instantaneous velocity. Have students verbalize the mathematical differences between average velocity and instantaneous velocity.

2.1: Tangent Lines and Rates of Change HW – Day 1

Limit homework time to no more than 10 minutes. Discuss #1, #4, #5, and if time permits #6.

1) The figure shows the position versus time curve for a ferry boat that moves passengers across a distance of 60 m across a river.



a. Estimate the instantaneous velocity of the ferry at $t = 10\ s$.
b. Sketch a possible curve that models the velocity of the ferry over the 20 minutes it takes to load the passengers, ferry across the river, and then unload the passengers.

Figure 90. 2.1 HW Day 1 - #1

a) At $t = 10$, the function is approximately linear and the slope appears to be $\frac{40}{10} = 4\frac{m}{s}$.
   a. Ask student to describe how s/he answered this problem.
b) Randomly call on student to present answer, in the likely event that the student replies “I didn’t know how to do this one”, give the entire class three minutes to think about the question and produce a graph. Then “randomly” call on exact same student to present answer.
c) After first student draws graph on Promethean board, allow two volunteers to draw graphs. Every person who draws a graph must explain how s/he determined her/his graph.
d) If no appropriate graph has been constructed, ask if anyone else has a different graph, if so, allow student to present, if

2)

The graph below shows the position curve of a car driving for 10 hours. Construct tangent lines at $t = 4\ h$ and $t = 8\ h$ and use the tangent line to estimate the instantaneous velocity of the car at $t = 4\ h$ and $t = 8\ h$.

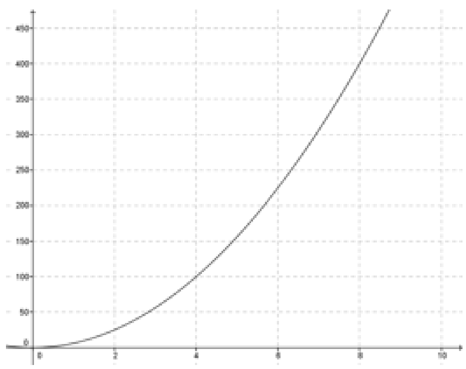
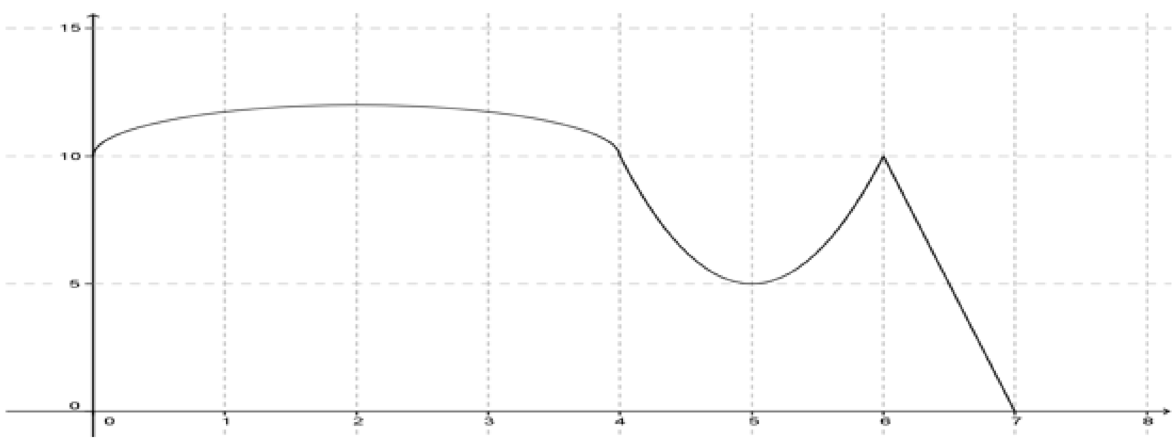


Figure 91. 2.1 HW Day 1 - #2

The position of a particle is modeled by the accompanying position versus time graph.

3)



Use the graph to estimate the following:

a) the average velocity on the interval $0 \leq t \leq 4$

b) when the particle is stopped

c) the times at which the particle's velocity is a maximum and minimum

d) the instantaneous velocity at $t = 6.5$

Figure 92. 2.1 HW Day 1 - #3

4) The four graphs below represent the position of a car driving on a straight track. In each graph, determine if the car's instantaneous velocity is increasing or decreasing as time increases.

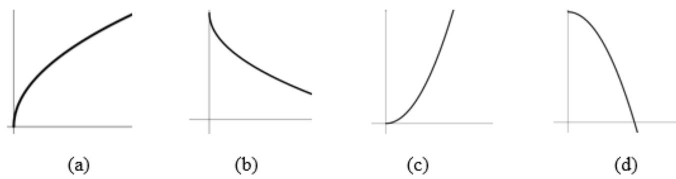


Figure 93. 2.1 HW Day 1 - #4

1) Ask class how one can read the instantaneous velocity from a position graph? Prompt for "slope of the tangent line"
2) Have students close eyes and vote on each graph individually to indicate if I.V. is increasing.
   a. Call on student after each vote to defend his/her vote

5) Suppose a particle's velocity is constant. What must be true of the particle's position graph?

1) Have all students write a sentence or two that answers the question and explains why the answer is correct.
2) Then, have all of the students construct a graph of a constant velocity function and the appropriate position curve.
3) Have students answer the question "are any other position curves possible?"

6) Suppose $f(x) = 2x^2$.

a) Find the average rate of change of $f(x)$ on $[0,1]$.

$$r_{ave} = \frac{2(1)^2 - 2(0)^2}{1 - 0} = 2$$

b) What is the instantaneous rate of change of $f(x)$ at $x = 0$?

$$r_{inst} = \lim_{x\to 0} \frac{2(x)^2 - 2(0)^2}{x - 0} = \lim_{x\to 0} \frac{2x^2}{x} = \lim_{x\to 0} 2x = 0.$$

1) Have students compare answers to parts a) and b). It is likely that the students who did the homework will have both parts correct.
2) Have a student volunteer present answer to part c. Volunteer should describe all work to the class while at the Promethean board.
3) Have all students answer part d). Re-emphasize the relationship of the average rate of change and the instantaneous rate of change. Use discussion of secant lines and tangent lines to introduce new learning.

c) Find the instantaneous rate of change of $f(x)$ at $x = c$.

$$r_{inst} = \lim_{x\to c} \frac{2(x)^2 - 2(c)^2}{x - c} = \lim_{x\to 0} \frac{2(x^2 - c^2)}{x - c} = \lim_{x\to c} \frac{2(x-c)(x+c)}{x - c} = \lim_{x\to c} 2(x + c) = 4c$$

d) Sketch the graph of $f(x)$, the line described in part a), and the line defined by part b).

2.1: Tangent Lines and Rates of Change – Day 2

Slopes and Rates of Change

Velocity can be viewed as a ___**rate of change**____ - it is the rate of change of __**position**__________________________ with respect to ____**time**_______________. Rates of change occur in all aspects of life:

For example:

A bridge engineer needs to know the rate at which the concrete and steel expand during the summer and contract during the winter.

A financial analyst is interested in the rate of change of the national deficit.

An epidemiologist might study the rate at which an infectious disease spreads among humans.

What do we mean when we say "rate of change of y with respect to x"?

Moderate discussion until a student answers "it is the amount of change in y per unit change in x"

In the case of a linear function, $y =$ ___$\boldsymbol{mx + b}$____________, __$\boldsymbol{m}$___ represents the rate of change of y with respect to x. This means that y changes ___$\boldsymbol{m}$______ units for each 1-unit increase in x.

Ex. 7: Find the rate of change of y with respect to x if:

a) $y = 3 + 2x$ b) $y = 14 - 5x$

Students will immediately answer 2 and -5. The question, "is that it" will inevitably follow.

Mention that slope is a rate of change.

Def: No matter what context we are using:

If $y = f(x)$, then we define the ***average rate of change of y with respect to x*** over the interval $[a, b]$ to be $\frac{f(b)-f(a)}{b-a}$.

The ***instantaneous rate of change of y with respect to x*** at $x = c$ to be $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.

Geometrically, the average rate of change of $f(x)$ on $[a, b]$ is the slope of the **_secant__________** line through the points $P(a, f(a))$ and $Q(b, f(b))$.

The instantaneous rate of change of $f(x)$ at point $P(a, f(a))$ is the slope of the **___tangent__________________** line at point P.

Ex. 8: Let $y = x^2 + 2$.

a) Find the average rate of change of y with respect to x over the interval [3,5].

$$r_{ave} = \frac{y(5) - y(3)}{5 - 3} = \frac{27 - 11}{2} = 8$$

b) Find the instantaneous rate of change of y with respect to x at $x = 4$.

$$r_{inst} = \lim_{x \to 4} \frac{y(x) - y(4)}{x - 4} = \lim_{x \to 4} \frac{x^2 + 2 - 18}{x - 4} = \lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

$$= \lim_{x \to 4} \frac{(x + 4)(x - 4)}{x - 4} = \lim_{x \to 4}(x + 4) = 8$$

1) Have students complete parts a) and b) and then compare the answers.
2) Have students write one or two sentences that interpret the results of parts a) and b) in the context of the problem.
   a. Call on two students at random to read their interpretations. If neither answer is correct, ask the class "does anyone have anything different?" Prompt until someone answers that a) describes the slope of the secant line to the curve drawn from $x = 3$ to $x = 5$ and that b) describes the slope of the line tangent to the curve at $x = 4$.
3) Have students sketch a graph of the function, average rate of change, and instantaneous rate of change on the same axes
4) Let a volunteer present his/her drawing on the Promethean board and describe the drawing.

Rates of Change in Applications

In applied problems, ___**average**_______________ and __**instantaneous**__________________ rates of change must be accompanied by appropriate ___**units**___________________. In general, the __**units**______________ for a rate of change of y with respect to x are obtained by ____**dividing**______________________ the units of y by the units of x.

Here are some common examples:

1) If we wanted to measure the instantaneous rate of change of the temperature at midnight, what would our units be?

1) Accept volunteer answers from the class, any answer that represents a measure of temperature divided by a measure of time is acceptable.

2) If we wanted to measure the average rate of change of our velocity $\left(\frac{m}{s}\right)$ over a $20s$ interval, what would the units of measurements be?

$\frac{m/s}{s}$

Students will solve this quickly. Give everyone 30 seconds and then compare answers.

2.1: Tangent Lines and Rates of Change HW – Day 2

True or False:

1) If $\lim_{x \to 2} \frac{f(x)-f(2)}{x-2} = 7$, then $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = 7$.

2) Tangent lines are specific kinds of secant lines.

3) Velocity is the change in position.

> Address all 3 T/F questions at the start of class.
> 1) Take class vote with eyes closed.
> 2) Call on one student who voted true and one student who voted false per questions and have each student tell the class why s/he voted true or false.
> 3) On question 3, call on students until someone recognizes that velocity represents a change in the objects velocity divided by the change in time.

4)

Suppose that the temperature in Duluth, Minnesota is measured on Veterans Day every hour and then graphed as a function of time, shown in the figure.

a) At about what time is the highest temperature recorded? Estimate the highest temperature.
b) The temperature graph from 8:00 am until noon is approximately linear. Estimate the rate of change of the temperature over this time interval.
c) At what time is the temperature changing most rapidly? Estimate the rate of change of temperature with respect to time at this time.

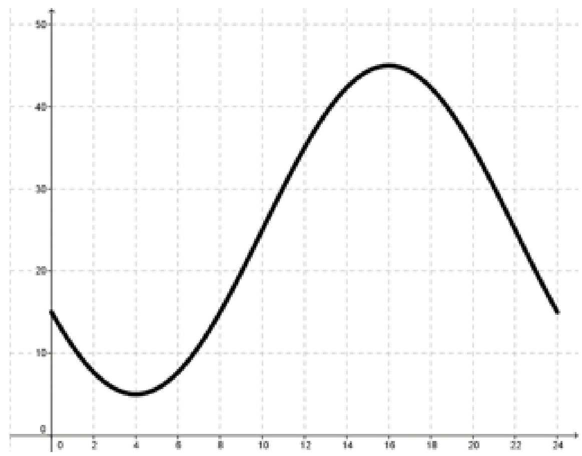


Figure 94. 2.1 HW Day 2 - #4

> Display answers to #4. Answer questions if students have any.

5)

The graph represents the height of a person in inches from birth to age 30.

a) Estimate the growth rate at age 10.
b) When does the growth rate appear to be the greatest?
c) What is the average growth rate between the ages of 0 and 20? Is there a time between 0 and 20 where the instantaneous growth rate is equal to the average growth rate? If so, where?
d) Draw a sketch of the growth rate graph as a function of age.

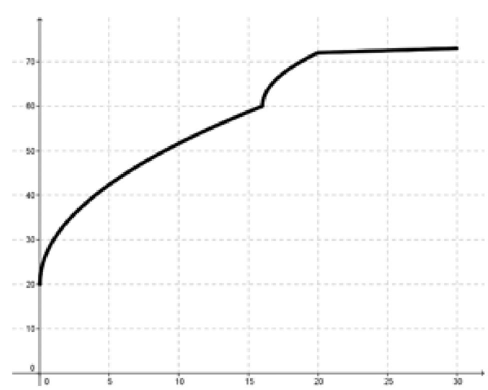


Figure 95. 2.1 HW Day 2 - #5

If time permits, have the class discuss parts a), b), and c) and then have two or three representatives of the class come to the Promethean board and present the class's answers.

1) Have ALL students attempt part d) in class.
2) Walk around the room and identify three different graphs that have been drawn, preferably the 3 most common.
3) The teacher then draws all 3 on the Promethean board in 3 different colors.
4) Have the class make an anonymous vote on which graph is correct.
5) Call on students to defend their answers.
6) Allow students to resolve disagreements and identify the correct curve.

6) A cyclist rides a bicycle down a straight highway. Her position in feet over a 20 second interval is given by $s(t) = 2t^2$.

a) Find the average velocity of the cyclist during the 20 second interval.

b) Find the instantaneous velocity of the cyclist at $t = 12\ s$.

As #6 is only superficially different from Example 8, most students will have no difficulty with #6.

1) Present answers to #6.
2) Answer any questions that student may have.

APPENDIX C: DIFFERENTIATION COMPETENCY TEST

Table 85. Differentiation Competency Test

Characteristic	Symbol	Meaning
1 – The cognitive *process* needed to achieve the *output* derivative indicates whether the goal of the question requires a derivative to be formulated or interpreted.	F	Formulation is the ability to recognize that a particular differentiation procedure is required using the data supplied and to know how to calculate it
1 – The cognitive *process* needed to achieve the *output* derivative indicates whether the goal of the question requires a derivative to be formulated or interpreted.	I	Interpretation is the ability to reason about the *input* derivative supplied or to explain it in natural language, or to give it meaning including its equivalence to a derivative in a different representation.
2 – The representation of the *input* derivative is determined by the data and is classified (using upper case letters) as numerical (N), graphical (G), or symbolic (S).	N	Numerical, if the data is a numerical derivative (instantaneous or average rate of change) or enables a difference quotient to be calculated (and possibly its limit) using ordered pair data or a table of values (by hand or with CAS)
2 – The representation of the *input* derivative is determined by the data and is classified (using upper case letters) as numerical (N), graphical (G), or symbolic (S).	G	Graphical, if the data is a graphical derivative, slope of the line tangent to the curve, or enables the slope of the line tangent to the curve "at a point" to be determined.

Table 85 (continued)

2 – The representation of the *input* derivative is determined by the data and is classified (using upper case letters) as numerical (N), graphical (G), or symbolic (S).	S	Symbolic, if the data is a symbolic derivative (algebraic function) or enables a symbolic derivative to be determined using the rules for symbolic differentiation.
3 – The representation of the *output* derivative is dependent on the goal of the question expressed in natural language and is classified (using lower case letters) as numerical (n), graphical (g), or symbolic (s).	n	Numerical, if the question requires finding or explaining a "rate of change".
3 – The representation of the *output* derivative is dependent on the goal of the question expressed in natural language and is classified (using lower case letters) as numerical (n), graphical (g), or symbolic (s).	g	Graphical, if the question requires finding or explaining a "slope of the line tangent to the curve or tangent".
3 – The representation of the *output* derivative is dependent on the goal of the question expressed in natural language and is classified (using lower case letters) as numerical (n), graphical (g), or symbolic (s).	s	Symbolic, if the question requires finding or explaining a "derivative".

Example: A question classified as IGs would be an interpretation (I) question whose input differentiation representation is graphical (G) and whose output representation is symbolic (s).

The DCT

Question #	Question	Classification
1	Find the derivative of $y = x^5 + 4x^2 - x + 10$.	FSs
2	Use a graph of $y = x^2 + x - 10$ to find the slope of the line tangent to the curve at $x = 3$.	FGg

Table 85 (continued)

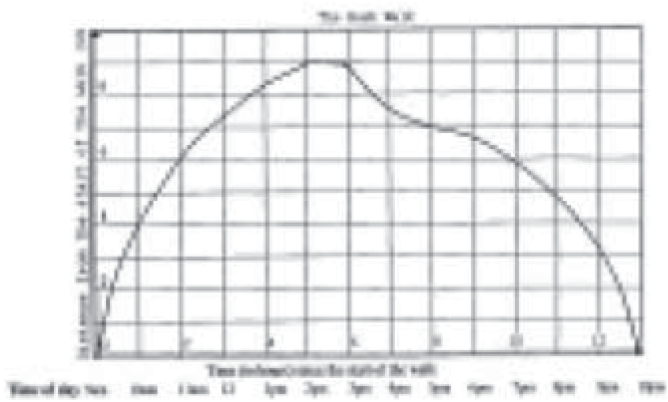
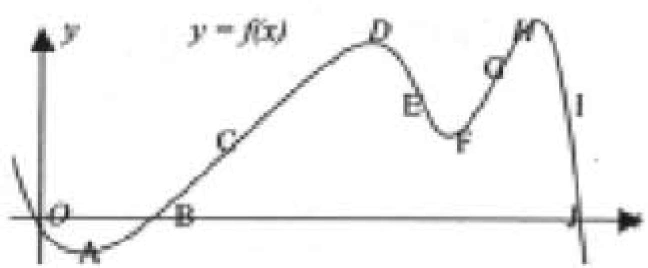
3	One day during the school holidays, a family went on a holiday walk. The graph below shows their distance from the start of the walk (in *kilometers*) as a function of the number of *hours* walked.  (a) What was the family's average rate of walking between 9am and 12 noon?	FNn
4	If y is a function of x, explain in words the meaning of the equation $\frac{dy}{dx} = 5$ when $x = 10$.	ISs
5	A graph of $y = f(x)$ is sketched below. A series of points A to j are marked along the curve. Consider the statements below and decide if they are true or false.  (a) The slope of the line tangent to the curve at F is greater than at B. (b) The slope of the line tangent to the curve at A is greater than at H. (c) The slope of the line tangent to the curve at I is less than at F. (d) The slopes of the lines tangent to the curve at O and J are approximately equal.	IGg
6	At 1:00 pm, the rate of change of the temperature in your house was +3 degrees Celsius (℃) per hour. Immediately after 1:00 pm, is the temperature most likely to: decrease, stay the same, or increase. Give a reason for your answer.	INn

Table 85 (continued)

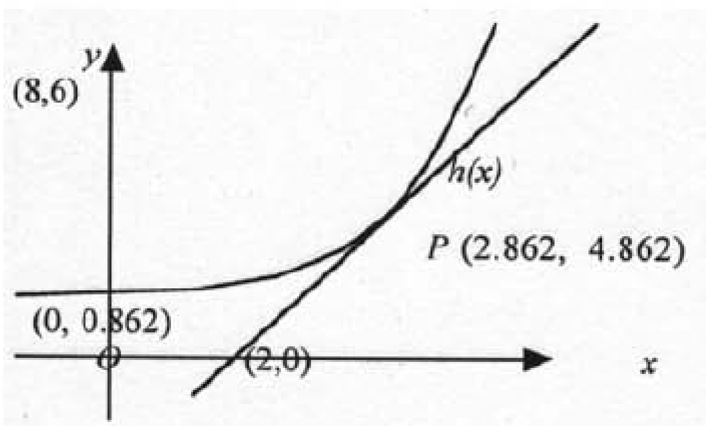
7	The height of a plant can be determined by the formula $H(t) = 7t^3 - 3t^2$ where H is the height of the tree in *meters*, and t is the number of *years* since the tree was first planted. Find the rate of increase of the plant's height 2 years after it was planted (i.e., $t = 2$).	FSn
8	A curve has the equation $g(x) = 5x^3 - 6x^2 + 3x - 6$. Find the slope of the line tangent to the curve at the point P, where $x = -1$.	FSg
9	Continuation of Question 3 above 'The Holiday Walk" 3(b). What was the family's speed (rate of walking) at 11:00 am?	FGn
10	The graph of the function $h(x)$ is sketched below. The tangent at point P, on the curve $y = h(x)$ has also been drawn. Find the value of the derivative of $h(x)$ at P.  	FGs
11	The CAS calculator was used to find values of the function $y = f(x)$ near $x = 3$. $(3.000, 0.000); (3.103, -0.701); (3.051, -0.353);$ $(3.011, -0.079); (2.990, 0.071); (2.999, 0.007)$ Find the best estimate of the slope of the line tangent to the graph of $y = f(x)$ at $x = 3$.	FNg
12	The values of a function close to $x = 5$ are shown in the table below. <table><tr><td>x</td><td>4.997</td><td>4.998</td><td>5.000</td><td>5.001</td><td>5.002</td></tr><tr><td>f(x)</td><td>15.470</td><td>15.482</td><td>15.500</td><td>15.508</td><td>15.515</td></tr></table> Find the best estimate of the derivative $f(x)$ at $x = 5$.	FNs

Table 85 (continued)

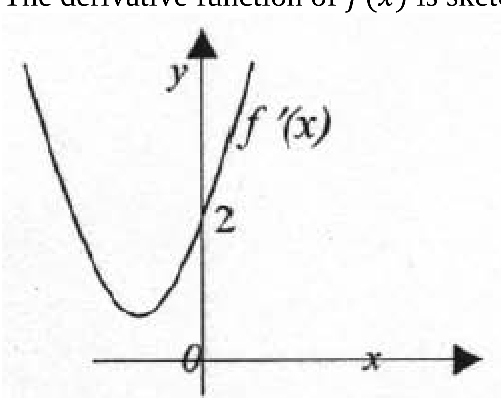
13	*Circle the letter corresponding to your answer.* The derivative of the function $g(t)$ is given by the rule $g'(t) = t^3 - 5t$. To find the rate of change of $g(t)$ at $t = 4$, you should: A. Differentiate $g'(t)$ and then substitute $t = 4$. B. Substitute $t = 4$ into $g'(t)$. C. Find where $g'(t) = 0$. D. Find the value of $g'(0)$. E. None of the above.	ISn
14	The derivative function of $f(x)$ is given by $f'(x) = x^3 - 5x + 3$. What is the slope of the line tangent to the curve $y = f(x)$ when $x = 1$?	ISg
15	$P(2,7)$ is a point on the curve $y = f(x)$, and at P, the slope of the line tangent to the curve is 3. $Q(2.001,7.351)$ is a second point on the curve P. What is the instantaneous rate of change of y with respect to x at the point P, when $x = 2$? (Note: an exact answer is required.)	IGn
16	*Circle the letter corresponding to your answer.* The derivative function of $f(x)$ is sketched below.  From the list of derivative function rules listed below, select the rule that best represents the derivative of $f(x)$. A. $f'(x) = x^3 + x^2 + 2$ B. $f'(x) = -x^3 + x^2 + 2$ C. $f'(x) = -3x^2 - 2x + 2$ D. $f'(x) = 3x^2 + 2x + 2$ E. $f'(x) = 3x + 2$	IGs
17	A curve has the function rule $y = f(x)$. If the rate of change of y with respect to x is given by the function rule $5x + 7$, what is the derivative function of the curve?	INg

Table 85 (continued)

18	An eagle follows a flight path where its height depends on the time since it flew out of its nest. The rule for finding the height of the bird (H in meters) above its nest is a function $H(t)$ of t, the flight time (in seconds). Find seconds after take-off, the 4kg eagle was observed to be 100m above its nest and climbing at the rate of 3 meters/second. What is the value of $H'(5)$?	INs

APPENDIX D: FLEXIBILITY PRE-TEST

Table 86. Flexibility Pre-test

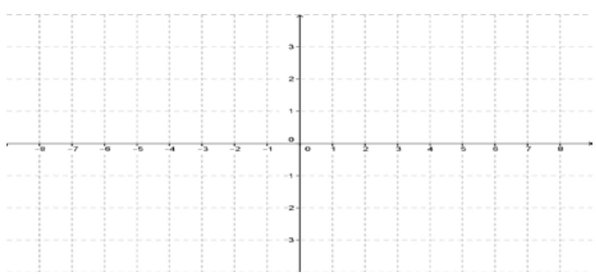
Characteristic	Symbol	Meaning
1 – The content domain of the function indicates if the problem requires composing functions together or if the problem requires analyzing the inverse of a function.	C	Composition is a non-algebraic means of combining multiple functions. The composition of f with g is the function $(f \circ g)(x) = f(g(x))$.
1 – The content domain of the function indicates if the problem requires composing functions together or if the problem requires analyzing the inverse of a function.	I	Inverse functions are functions that exchange the domain (input) and range (output) of a function. Algebraically, the function $f^{-1}(x)$ is the inverse of $f(x)$ if and only if $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.
2 – The representation of the *input* function is determined by the data and is classified (using upper case letters) as numerical (N), graphical (G), or symbolic (S).	N	Numerical, if the function is presented as a table of values or a list of discrete values.
2 – The representation of the *input* function is determined by the data and is classified (using upper case letters) as numerical (N), graphical (G), or symbolic (S).	G	Graphical, if the function is presented as a graph.
2 – The representation of the *input* function is determined by the data and is classified (using upper case letters) as numerical (N), graphical (G), or symbolic (S).	S	Symbolic, if the function is presented as an algebraic expression.

Table 86 (continued)

3 – The representation of the *output* function is dependent on the goal of the question expressed in natural language and is classified (using lower case letters) as numerical (n), graphical (g), or symbolic (s).	n	Numerical, if the question requires finding discrete values or filling in a table of values.
3 – The representation of the *output* function is dependent on the goal of the question expressed in natural language and is classified (using lower case letters) as numerical (n), graphical (g), or symbolic (s).	g	Graphical, if the question requires constructing a graph of a function.
3 – The representation of the *output* derivative is dependent on the goal of the question expressed in natural language and is classified (using lower case letters) as numerical (n), graphical (g), or symbolic (s).	s	Symbolic, if the question requires finding or creating an algebraic expression of a function.

Example: A question classified as IGs would be an inverse (I) question whose input functional representation is graphical (G) and whose output functional representation is symbolic (s).

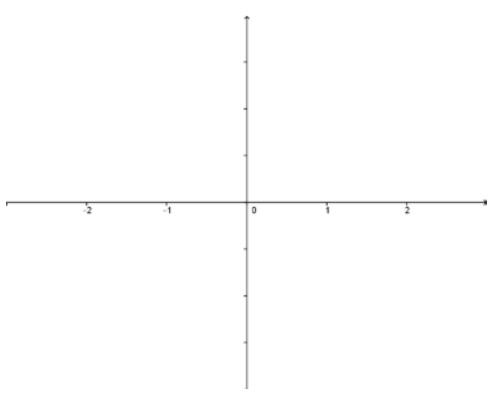
The flexibility pre-test

Question #	Question	Classification
1	Let $f(x) = \|x\|$ and $g(x) = \sqrt[3]{x} + 1$	
1.a	Find the algebraic expression for $f(g(x))$.	CSs
1.b	Evaluate $g\big(f(-1)\big)$.	CSn
1.c	Sketch the graph of $f(g(x))$ on the axis below. 	CSg

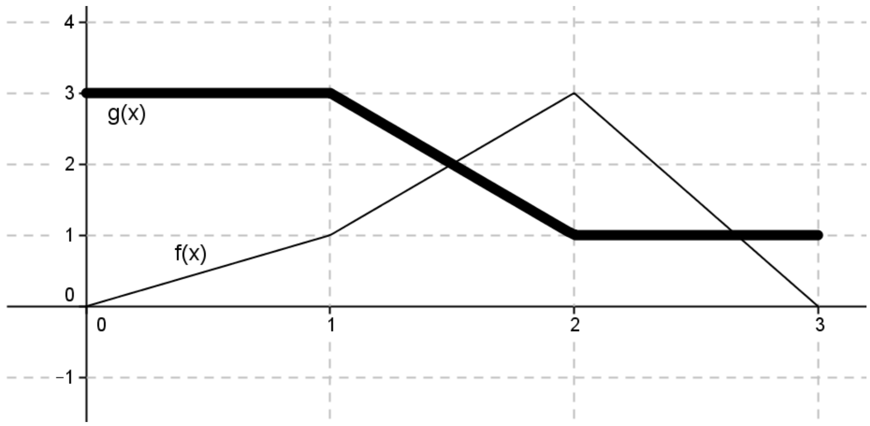
2 The table below reports selected values of two functions, f and g. Let $h(x) = f(g(x))$. The domain for all functions is all real numbers.

x	f	g	h
-3	9	-2	
-2	4	-1	
-1	1	0	
0	0	1	
1	1	2	
2	4	3	
3	9	4	

2.a	Fill in the column of values for $h(x)$. Explain how you determined $h(3)$.	CNn
2.b	Give an algebraic expression for $h(x)$.	CNs
2.c	Make a possible graph of $h(x)$. Label the tick marks on the y −axis to indicate the scale that you use.	CNg



3 Graphed below are the piecewise-defined functions, $f(x)$ and $g(x)$ on [0,3].



3.a On the axis below, sketch the graph of $h(x) = g(f(x))$ on [0,3]. CGg

3.b Fill in the following table of values for $i(x) = f(g(x))$. CGn

x	$i(x) = f(g(x))$
0	
1	
2	
3	

Table 86 (continued)

3.c	Give the algebraic expression for $i(x) = f(g(x))$.	CGs
4	Let $f(x) = 3\sqrt{x} + 2, x \geq 0$. Find an expression for $f^{-1}(x)$.	ISs
5	$f(x) = x + \sin x$ is an invertible function. Evaluate $f^{-1}(2\pi)$.	ISn
6	Let $f(x) = x^3 - 3$. Sketch a graph of $f^{-1}(x)$.	ISg
7	Find a function $f(x)$ such that:	INs

x	-2	-1	0	1	2
$f^{-1}(x)$	1	$3/2$	2	$5/2$	3

8	Let $g(x)$ be an invertible function such that:	Inn

x	0	1	2	3
$g(x)$	-2	4	0	3

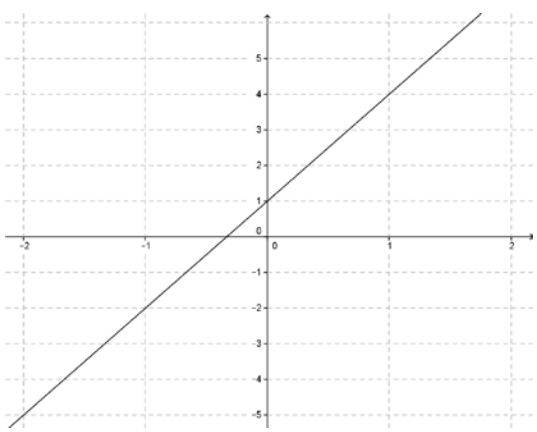
Find $g^{-1}(4)$.

9	If $f(x)$ is invertible on the entire real line and	INg

x	0	1	2	3	4
$f(x)$	1	2	4	8	16

Sketch a possible graph of $f^{-1}(x)$.

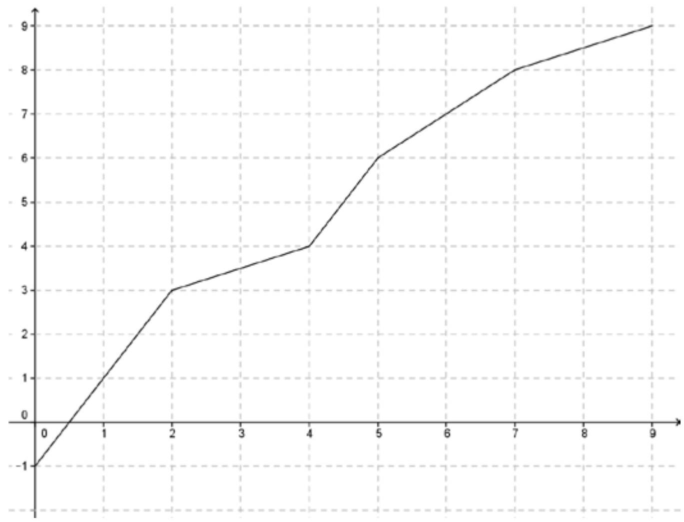
10	The graph of $f(x)$ is shown below.	IGs



Give an algebraic expression for $f^{-1}(x)$.

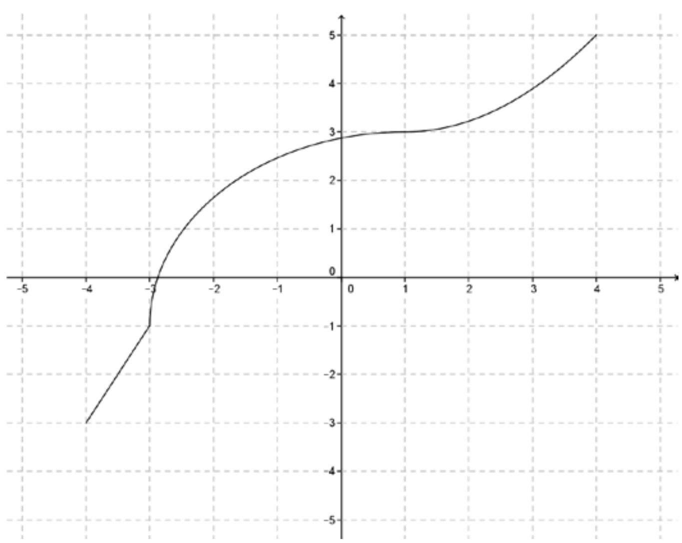
Table 86 (continued)

11 Use the graph of $f(x)$ below to fill in the table of values. IGn



x	0	3	5	7	8
$f^{-1}(x)$					

12 The graph of $g(x)$ is shown below. On the same set of axes, sketch the graph of $g^{-1}(x)$. IGg



APPENDIX E: EXIT SLIPS AND OPENING ACTIVITIES

Table 87. Exit slips and opening activities

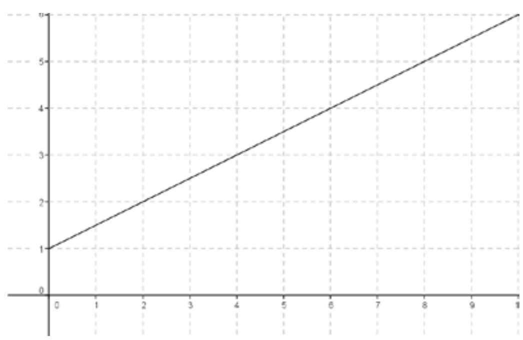
2.1.1 - Exit Slip

Name: ______________________________________ Date:__________________

The following position versus time graph consists of a linear position function, $s = f(t)$, construct a velocity versus time graph on the same set of axes. Show or explain how you determined the velocity graph.

Are there any other possible answers? Explain why or why not.

position (ft) $s = f(t)$



time (s)

Table 87 (continued)

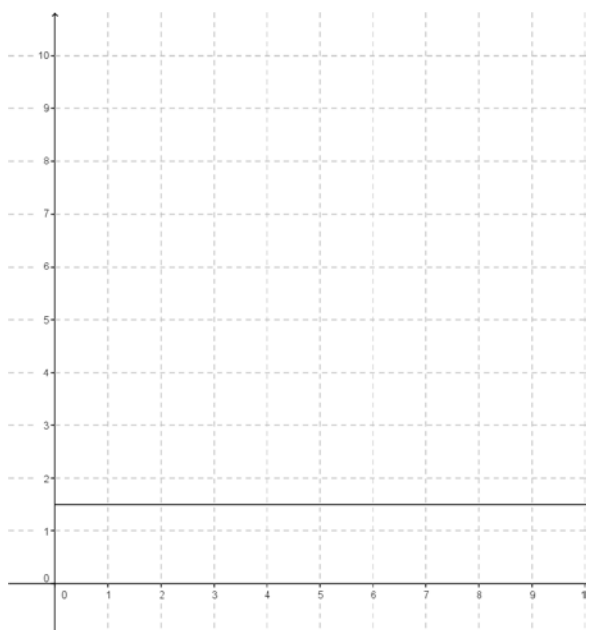
2.1.1 - Opening Activity

Name: ____ Date:___________________

The following velocity versus time graph consists of a horizontal line segment from $t = 0\ s$ to $t = 10\ s$. Construct a position versus time graph on the same axes. Show or explain how you determined the position graph.

Are there any other possible answers? Explain why or why not.

Velocity (ft/s)



$v(t)$

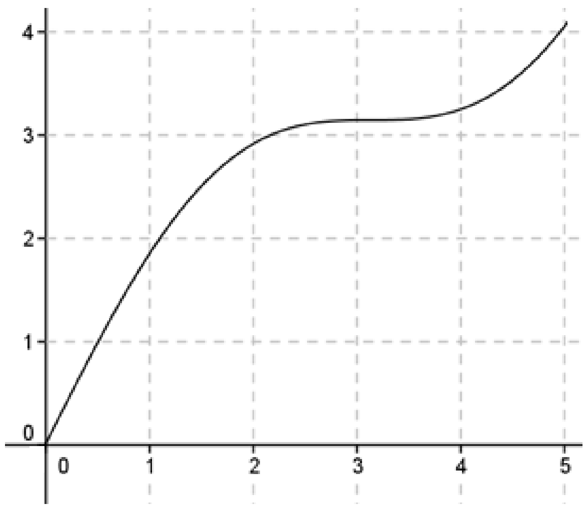
time (s)

2.1.2 - Exit Slip

Name: ______________________________________

Date:__________________

The following graph shows the function $f(x)$ on the interval [0,5].



Which is greater, the average rate of change of $f(x)$ on [0,5] or the instantaneous rate of change at $x = 3$? Show or explain how you determined your answer.

2.1.2 – Opening Activity

Name: ____

Date:____________________

Construct the graph of a function on the interval $[0,10]$ whose average rate of change on $[0,10]$ is 2 and whose instantaneous rate of change at $x = 5$ is approximately -2. Label the $y-$axis as appropriate. Explain why you drew the graph that you drew.

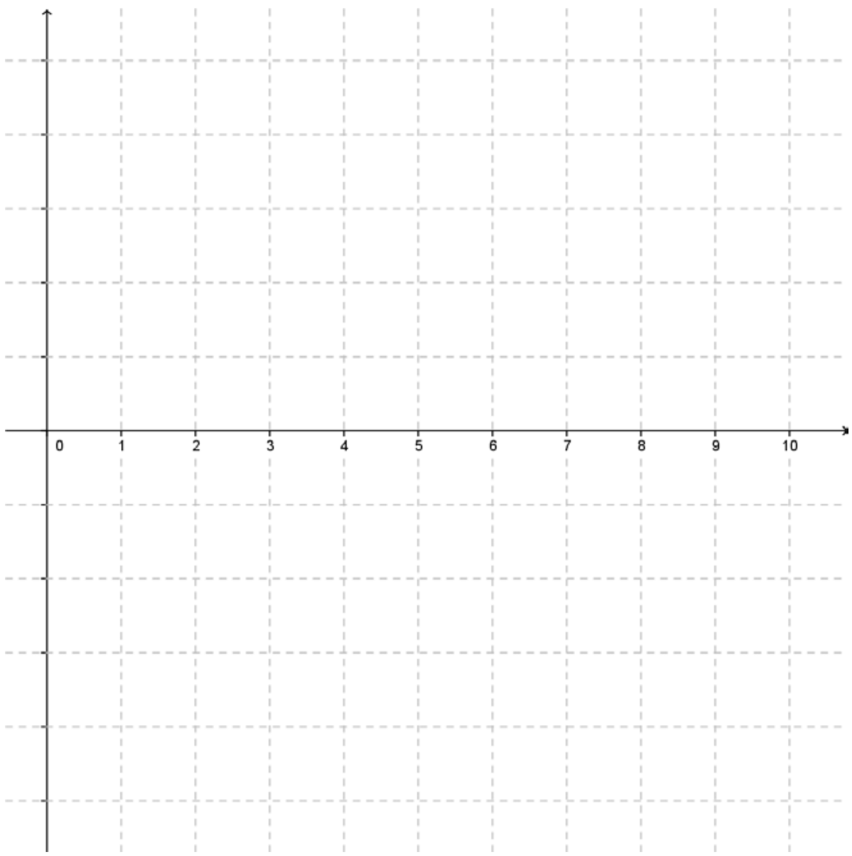


Table 87 (continued)

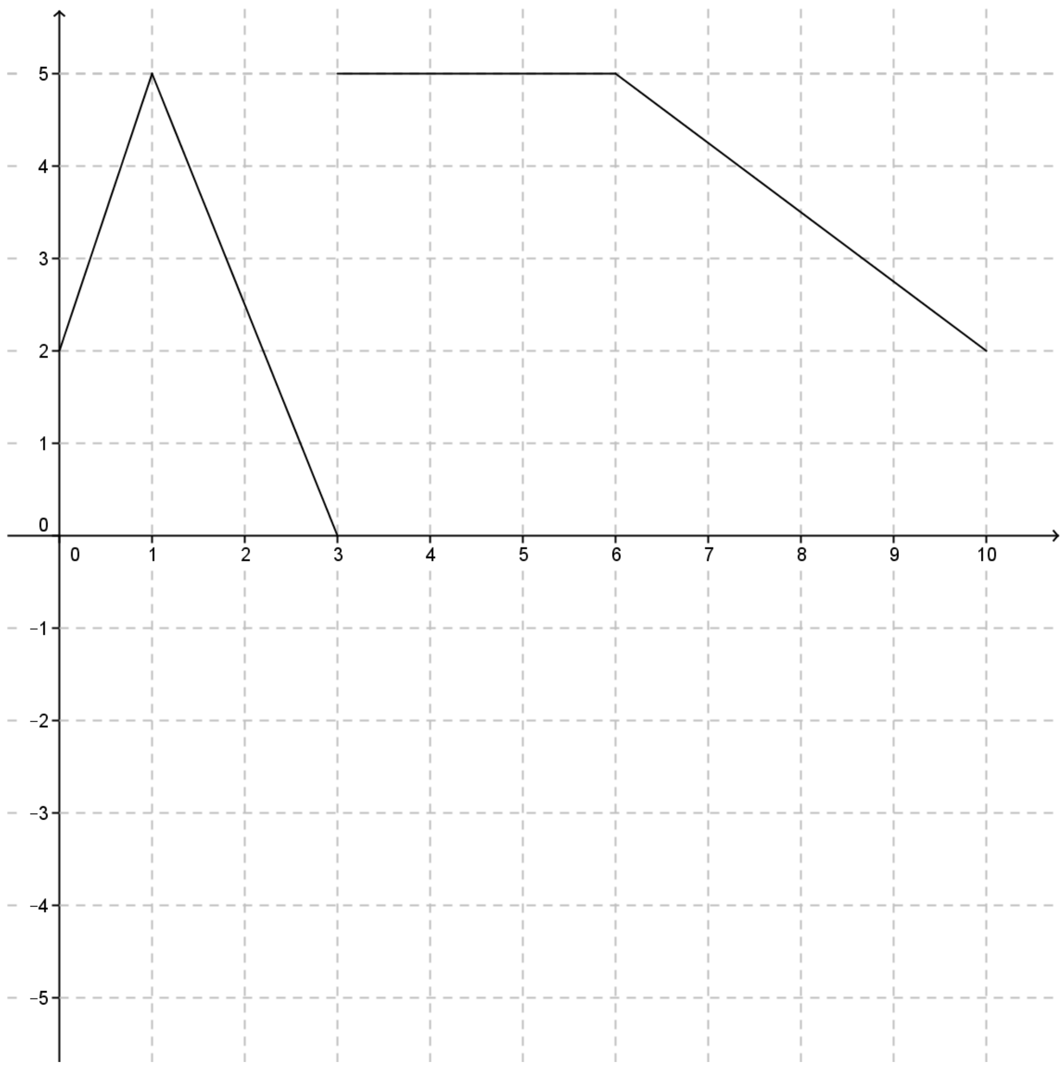
2.2.1 - Exit Slip Name: ____ Date:___________________ Write the equation of the line tangent to the curve $y = x^2 - x$ at $x = 3$. Make sure to show your work.
2.2.1 – Opening Activity Name: ____ Date:___________________ The line tangent to a curve $f(x)$ at $x = 2$ has the equation $y = 4 + 8(x - 2)$. Find a possible equation for $f(x)$. Show or explain how you determined $f(x)$.

2.2.2 - Exit Slip

Name: ____

Date:___________________

Sketch the derivative of the following function on the same axes. Describe how you determined the derivative.



2.2.2 – Opening Activity

Name: ____

Date:___________________

Sketch a continuous function $f(x)$ whose derivative $f'(x)$ is shown below on the same axes. Explain how you determined the continuous function $f(x)$.

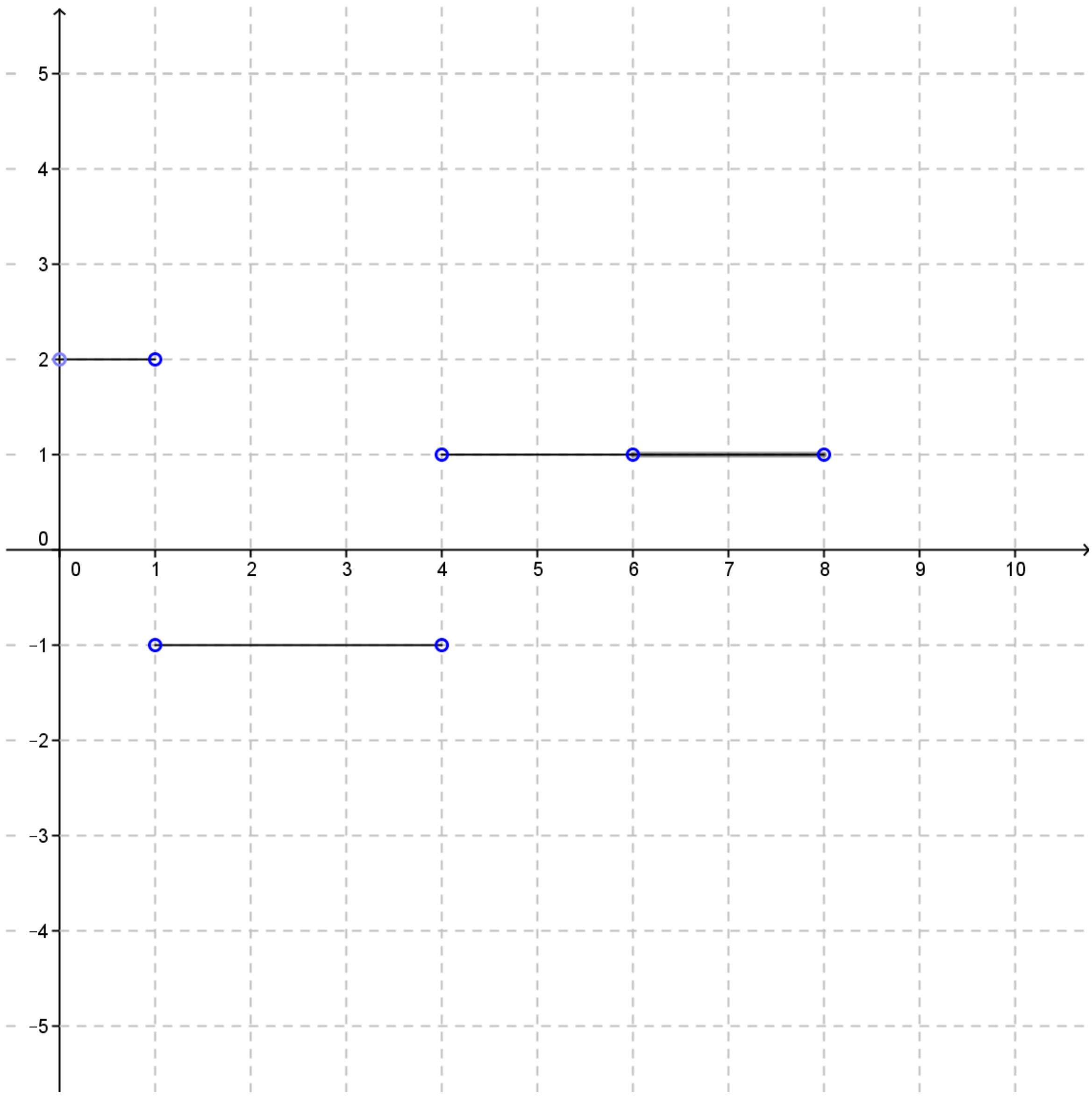


Table 87 (continued)

2.3.1 - Exit Slip Name: ______________________________________ Date:___________________ Find $f'(x)$. Explain how you found $f'(x)$. $f(x) = x^5 + ax^2 - x^{-2}$
2.3.1 – Opening Activity Name: ______________________________________ Date:___________________ Suppose $f'(x) = x - 6$. Find a function $f(x)$. Show or explain how you determined $f(x)$.
2.3.2 - Exit Slip Name: ______________________________________ Date:___________________ Find $y''(x)$ if $y = a\,x^2 + bx + c$. Show or explain your work.

Table 87 (continued)

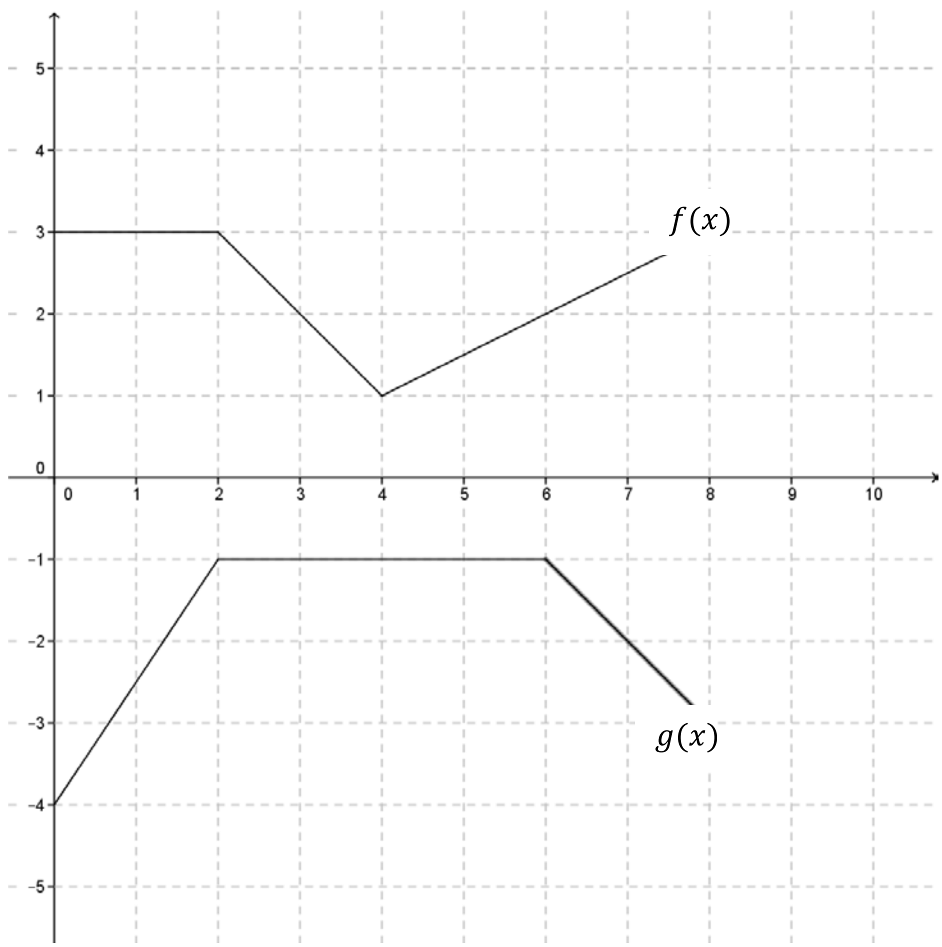
2.3.2 – Opening Activity Name: ____ Date:___________________   Suppose $y''(x) = 3x - 4$. What could be y? Show your work or explain how you know that you are correct.
2.3.2 – Opening Activity Name: ____ Date:___________________   Suppose $y''(x) = 3x - 4$. What could be y? Show your work or explain how you know that you are correct.

2.4.1 - Exit Slip

Name: ______________________________________

Date:__________________

The graph below shows functions $f(x)$ and $g(x)$.



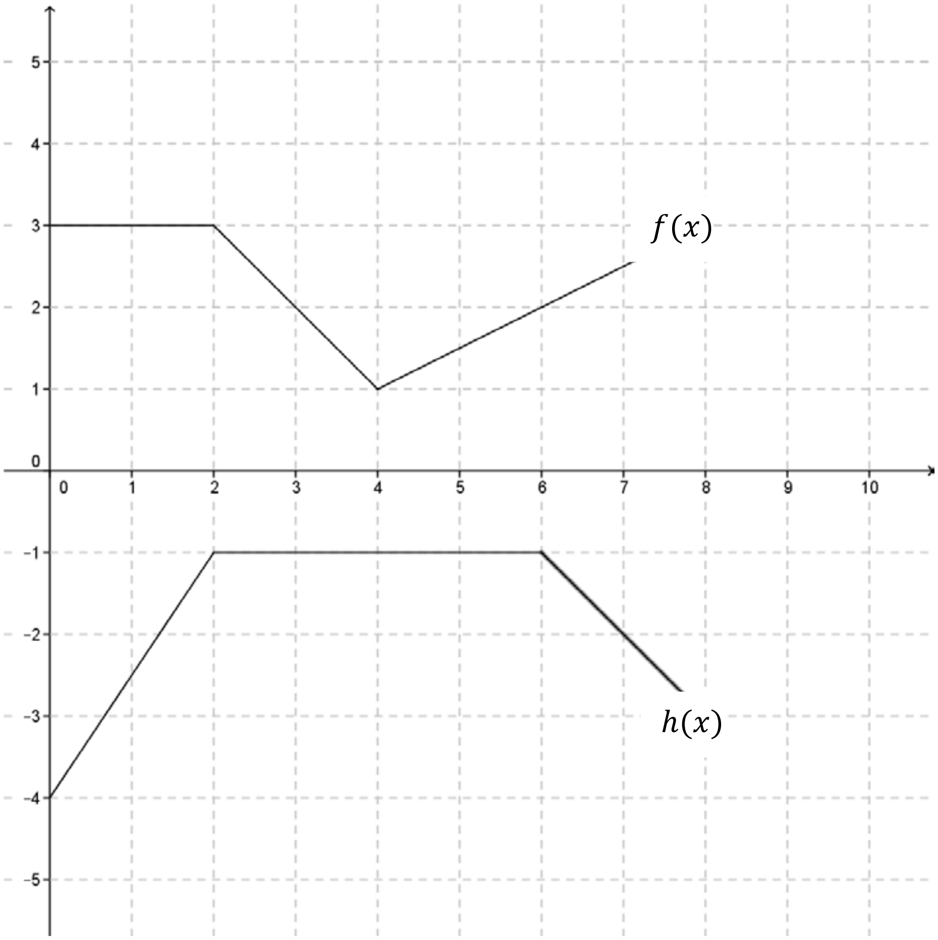
Let $h(x) = f(x) * g(x)$. Find $h'(7)$ if it exists. Show or explain your work.

2.4.1 – Opening Activity

Name: ____

Date:___________________

The graph below shows the functions $f(x)$ and $h(x)$, where $h(x) = f(x) * g(x)$.



Find $g'(1)$ if it exists. Show or explain your work.

2.5.1 - Exit Slip

Name: ____

Date:___________________

The graph of $f(x)$ is shown below. Give an algebraic expression for $f'(x)$. Show or explain how you determined $f'(x)$.

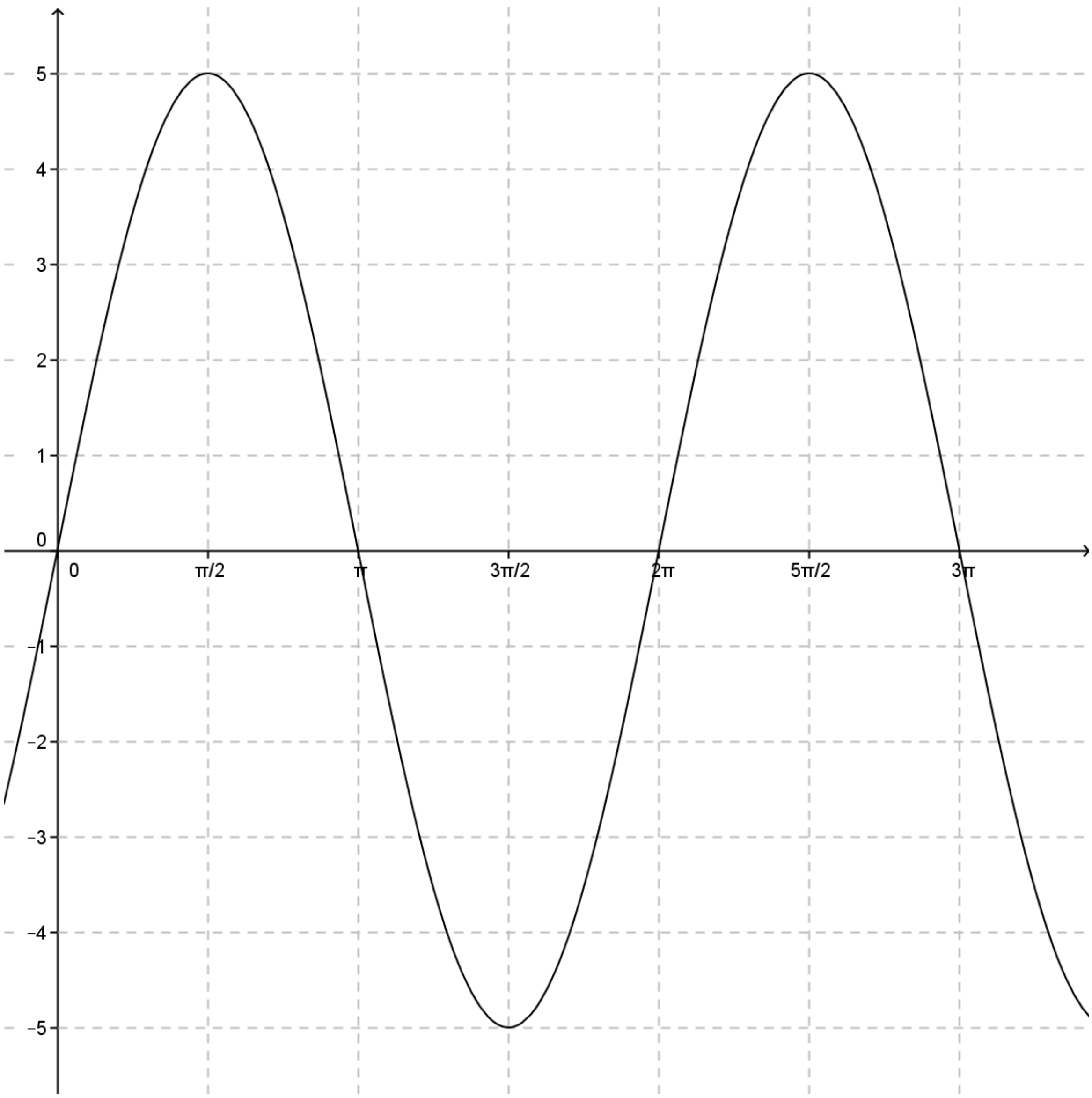


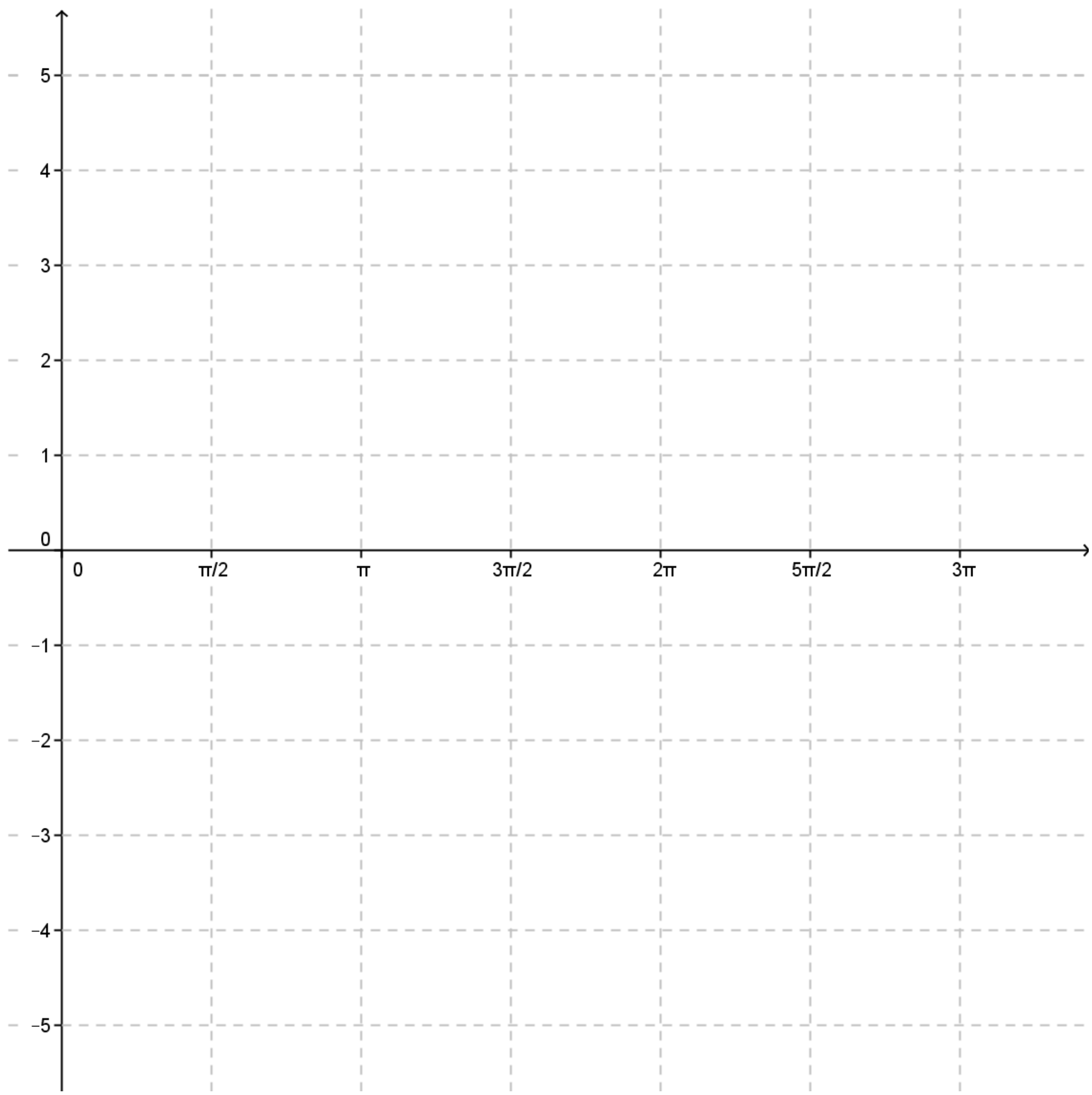
Table 87 (continued)

2.5.1 – Opening Activity

Name: ____

Date:___________________

Suppose $f(x) = -2\cos x$. Sketch a possible graph of $f'(x)$. Show or explain how you determined $f'(x)$.



2.6.1 - Exit Slip

Name: ______________________________________

Suppose $f(x) = 5\sin x$ and $g(x) = \sin 5x$. Which of the following is true?

I. $f'(\pi) < g'(\pi)$ II. $f'(\pi) = g'(\pi)$

III. $f'(\pi) > g'(\pi)$ IV. Cannot be determined.

Show your work or explain how you determined your answer.

2.6.1 – Opening Activity

Name: ______________________________________

If a function has a known derivative of $h'(x) = 5\cos x - 3\sin 3x$, what could be $h(x)$?

Show or explain how you determined $h(x)$.

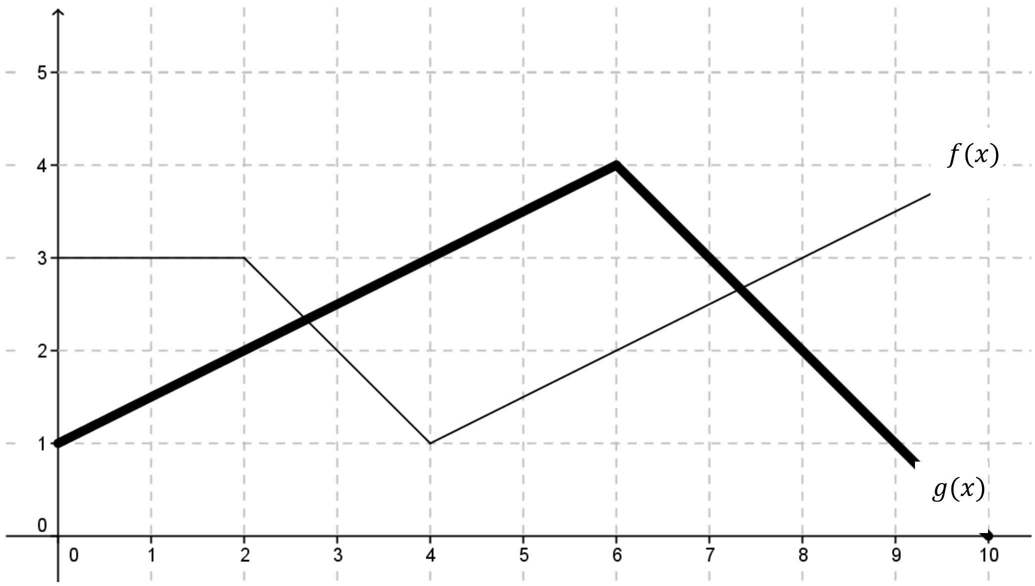
2.6.2 - Exit Slip

Name: ____

Date:___________________

Given the graphs of $f(x)$ and $g(x)$ below, fill in the table of values for $h'(x)$.

Let $h(x) = f\big(g(x)\big)$. Show or explain how you determined each value.

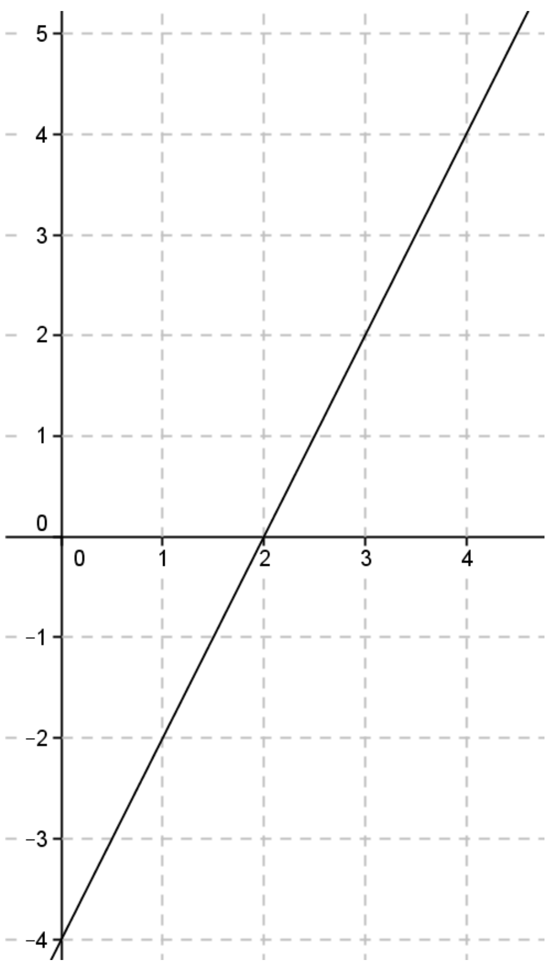


x	$h'(x)$
1	
4	
8	

2.6.2 – Opening Activity

Name: ______________________________

The graph of $g(x)$ is graphed below. Let $h(x) = f(g(x))$. Fill in the table of values for $f'(x)$. Show or explain how you determined each value.



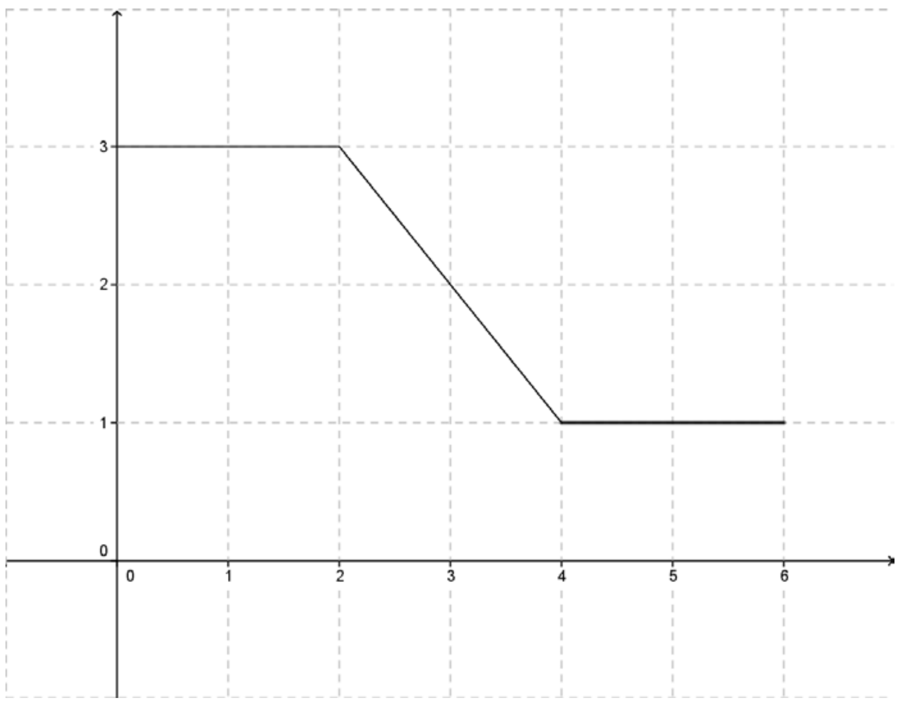
x	$h'(x)$	$f'(x)$
0	1	
2	1	
4	1	

2.6.3 - Exit Slip

Name: ____

Date:___________________

The graph of $f(x)$ is shown below. $f(x)$ is a piecewise function consisting of three linear segments.



Let $h(x) = [f(x)]^2$. Find $h'(3)$, if it exists. Show or explain your work.

2.6.3 – Opening Activity Name: ____ Date:___________________ Let $f(x) = \begin{cases} 3, & -2 \leq x < 0 \\ -x+5, & 0 \leq x \leq 2 \\ 1, & 2 < x \leq 4 \end{cases}$ Let $g(x) = f(x^2)$. Sketch the graph of $g'(x)$ on $[-2, 2]$. Label the $y-$ axis as you feel appropriate. Show or explain how you determined your solution.
Exit Slip – 3.1 Name: ____ Date:___________________ Find $\frac{dy}{dx}$ if $y^2 + y = x$. Show your work.
Opening activity – 3.1 Name: ____ Date:___________________ Find a relation involving x and y such that $$\frac{dy}{dx} = \frac{x-2}{y}$$ Be sure to show your work or explain how you found your solution.

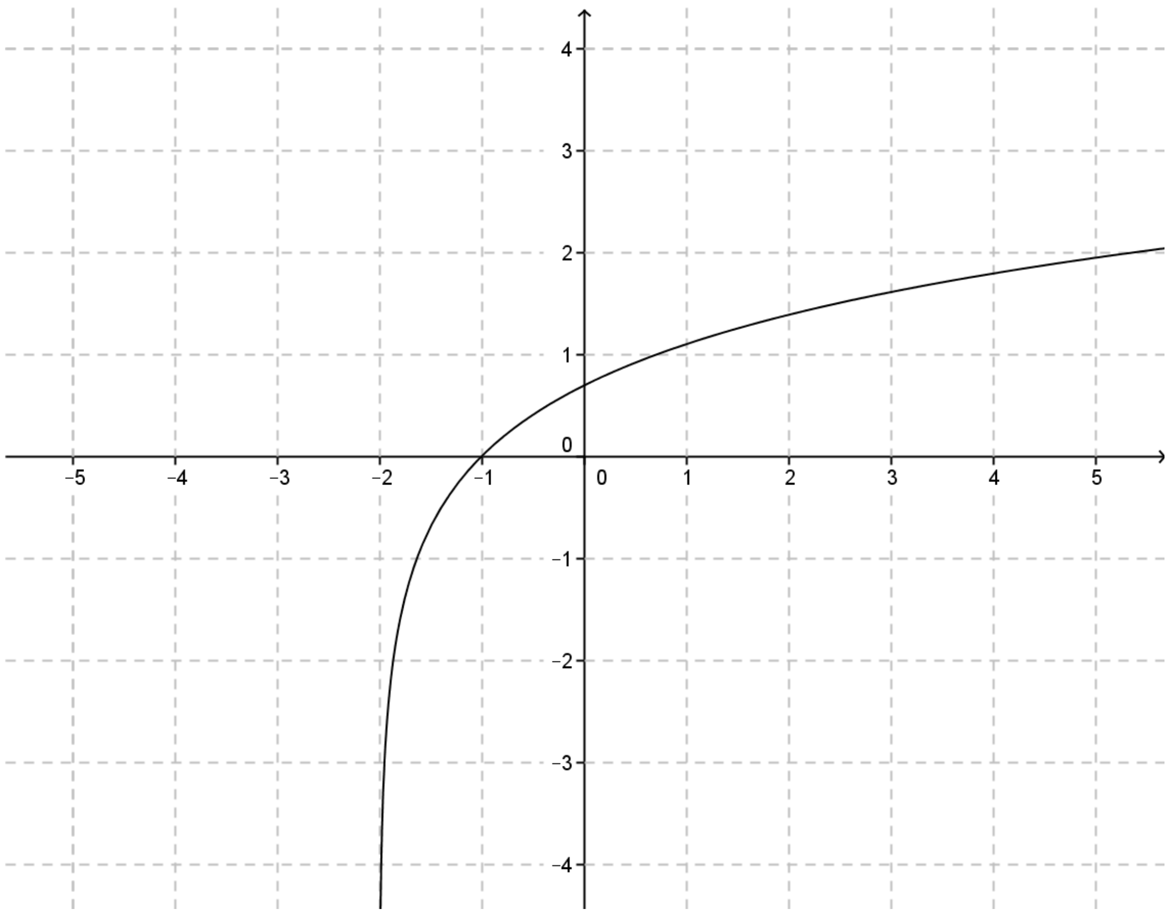
Exit Slip – 3.2.1 Name: ____ Date:___________________ Sketch the graph of the derivative of the function $y = \ln(x - 2)$.

Opening Activity – 3.2.1

Name: ____

Date:___________________

Find the derivative of the function graphed below. Show your work or explain how you determined your answer.



Exit Slip – 3.2.2 Name: ____ Date:___________________ Find the derivative of $f(x) = \ln(\frac{x-2}{x+3})$. Show your work or explain how you solved the problem.
Opening Activity – 3.2.2 Name: ____ Date:___________________ Suppose the derivative of a function is known to be $f'(x) = \frac{1}{x} - \frac{1}{x-2}$. Find a possible function for $f(x)$. Can you find a second function that could also be $f(x)$? Show or explain your work.
Exit Slip – 3.3.1 Name: ____ Date:___________________ Find the derivative of the function. $f(x) = 3e^{4x}$ Show your work or explain how you determined $f'(x)$. Can you think of any other possible functions for $f'(x)$?

Opening activity – 3.3.1

Name: ____

Date:___________________

Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$.

$$f^{'}(x) = e^{-5x}$$

Exit Slip – 3.3.2

Name: ____

Date:___________________

Find the derivative of the function.

$$f(x) = \sin^{-1}(e^{x})$$

Show your work or explain how you determined $f'(x)$. Can you think of any other possible functions for $f'(x)$?

Table 87 (continued)

Opening activity – 3.3.2 Name: ____ Date:___________________ Suppose a function's derivative, $f'(x)$, is known and shown below. Find a function $f(x)$ whose derivative is $f'(x)$. $$f'(x) = \frac{1}{\sqrt{1-(3x)^2}}$$
Exit Slip – 3.4.1 Name: ____ Date:___________________ What is the algebraic relationship between the rate of change of the area of a circle and the rate of change the circle's respective radius.
Opening Activity – 3.4.1 Name: ____ Date:___________________ Suppose that it is known that the temperature in degrees Fahrenheit is related to the temperature in degrees Celsius by the function: $$°\mathrm{F} = \frac{9}{5}°\mathrm{C} + 32$$ Interpret, using correct units, the meaning of the relation: $\frac{d(°\mathrm{F})}{dt} = \frac{9}{5}\frac{d(°\mathrm{C})}{dt}$.

Exit Slip – 3.4.2

Name: ____

Date:___________________

Suppose an isosceles right triangle is expanding at such a rate that the lengths of the legs of the triangle are always equal. Find an expression for the rate of change of the area of the triangle in terms of the length of the hypotenuse and the rate of change of the length of one of the legs of the triangle.

Opening Activity – 3.4.2

Name: ____

Date:___________________

The wind chill is the temperature, in degrees Fahrenheit (°F), a human feels based on air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). Suppose that the wind chill at wind velocity v is given by:

$$W(v) = 55 - 22v^{0.1}$$

Interpret, using correct units, the meaning of the relation: $W'(v) = -2.2v^{-0.9}$

when $v = 1$.

Exit Slip – 3.5.1

Name: ____

Date:___________________

What is the linearization of $f(x) = e^{2x}$ at $x = 1$?

Table 87 (continued)

Opening Activity – 3.5.1

Name: ____

Date:___________________

What function has a linearization at $x = e^2$ of $y = 2 + \frac{1}{e^2}(x - e^2)$?

Exit Slip – 3.6.1

Name: ____

Date:___________________

Evaluate the following limit:

$$\lim_{x \to 4} \frac{x^2 - 4}{4 \sin(x - 4)}$$

Opening Activity – 3.6.1

Name: ____

Date:___________________

Find two differentiable functions f and g with $\lim_{x \to 5} f(x) = \lim_{x \to 5} g(x) = 0$ that satisfies $\lim_{x \to 5} \frac{f(x)}{g(x)} = 12$.

Exit Slip – 4.1.1

Name: ____

Date:___________________

The table below reports the sign $(+, -)$ of $f'(x)$ and $f''(x)$ on selected intervals. Describe the behavior of $f(x)$ in regards to where $f(x)$ is increasing/decreasing and the concavity of $f(x)$.

Interval	Sign of $f'(x)$	Sign of $f''(x)$	Behavior of $f(x)$
$x < 1$	$+$	$-$	
$1 < x < 2$	$-$	$+$	
$2 < x < 3$	$-$	$-$	
$3 < x < 4$	$+$	$+$	

Opening Activity – 4.1.1

Name: ______________________________________ Date:__________________

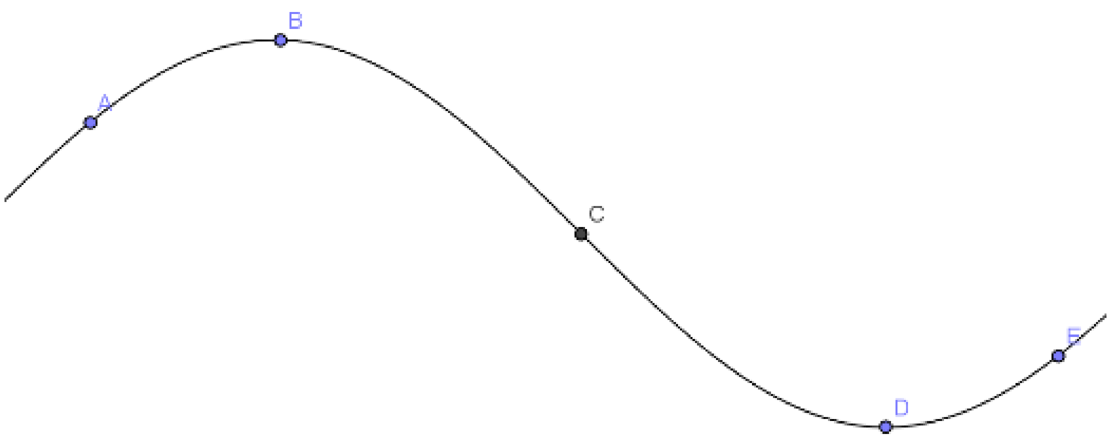
The table below describes the behavior of a function. Use the description to fill-in the sign $(+, -)$ of $f'(x)$ and $f''(x)$ on the selected intervals.

Interval	Sign of $f'(x)$	Sign of $f''(x)$	Behavior of $f(x)$
$x < 1$			f is increasing and is concave down
$1 < x < 2$			f is decreasing and is concave down
$2 < x < 3$			f is decreasing and is concave up
$3 < x < 4$			f is increasing and is concave up

Exit Slip – 4.1.2

Name: ____ Date:____________

Use the graph of $f(x)$ to fill in the table below at the selected points.



Point	Sign of $f'(x)$	Sign of $f''(x)$
A		
B		
C		
D		
E		
F		

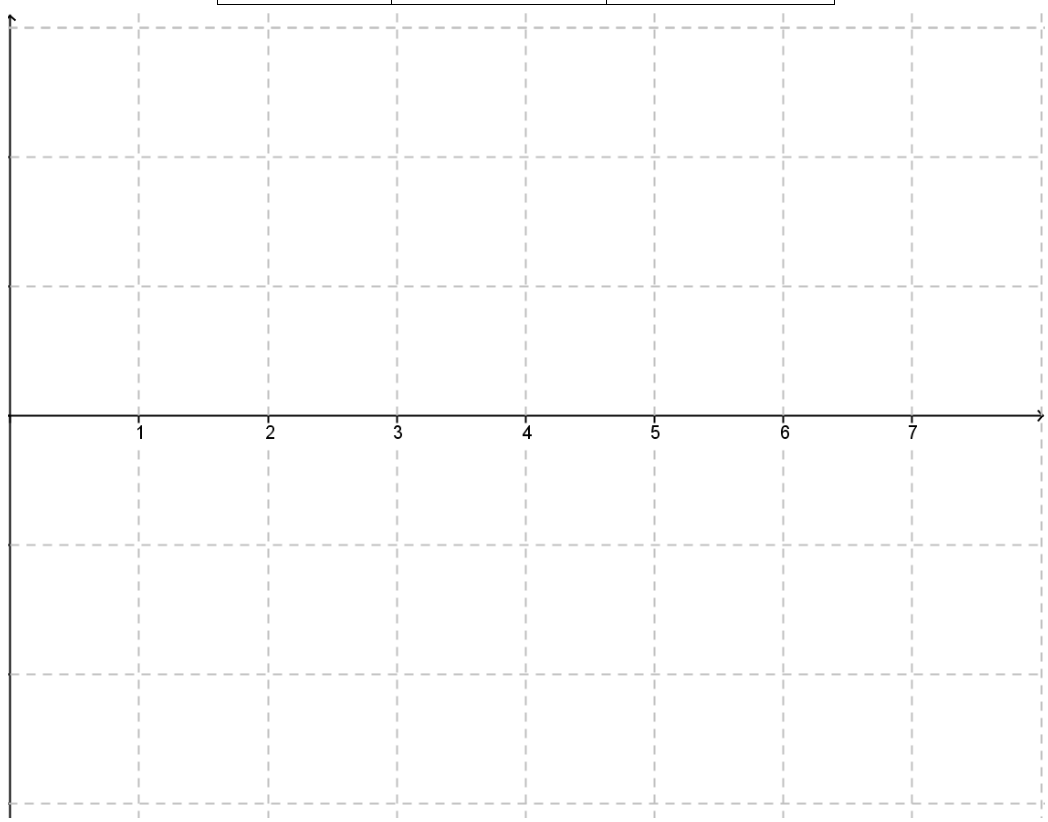
Opening Activity – 4.1.2

Name: ____

Date:___________________

Sketch a differentiable function on [0,8] that satisfies the table of f' and f'' values.

Interval	Sign of $f'(x)$	Sign of $f''(x)$
(0,2)	$-$	$+$
(2,4)	$+$	$+$
(4,6)	$+$	$-$
(6,8)	$-$	$-$

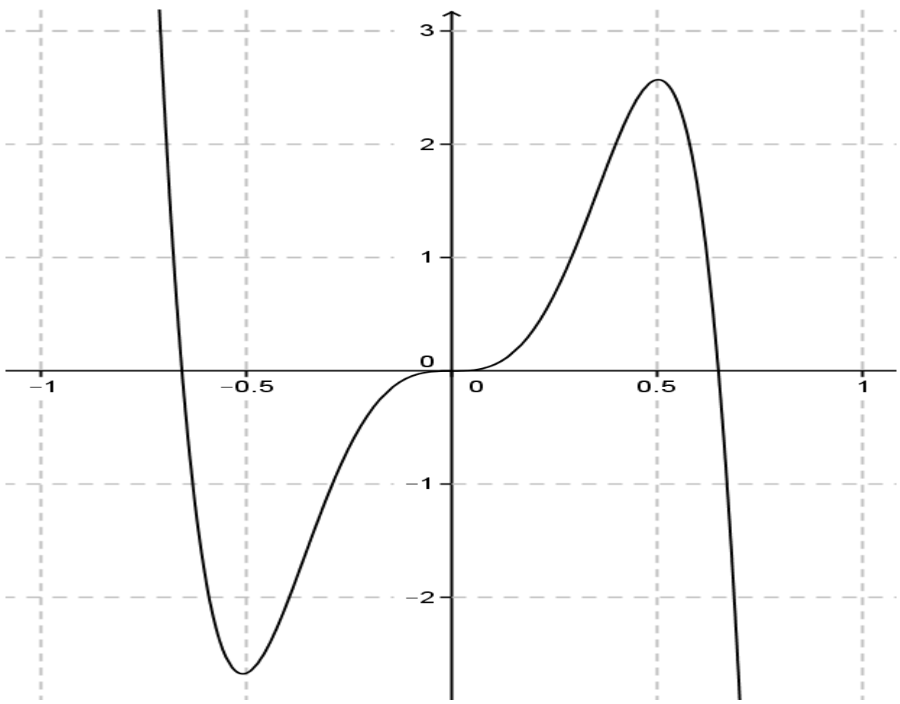


Exit Slip – 4.2.1

Name: ________________________________

Date:___________________

Given the graph of f, shown below, identify the critical numbers and classify each as a relative maximum, minimum, or neither.



Opening Activity – 4.2.1

Name: ____

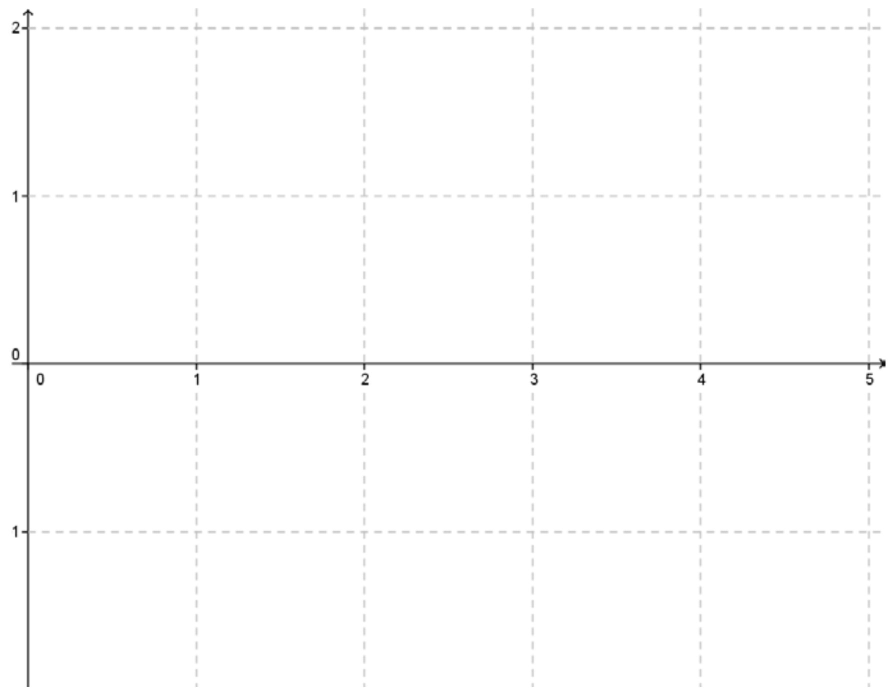
Date:___________________

Sketch a graph of a continuous function on the interval $0 \leq x \leq 5$ with the following properties:

1) f has a relative maximum at $x = 2$.

2) f has a relative minimum at $x = 4$.

3) f has a critical number at $x = 3$ that is neither a relative maximum nor relative minimum.

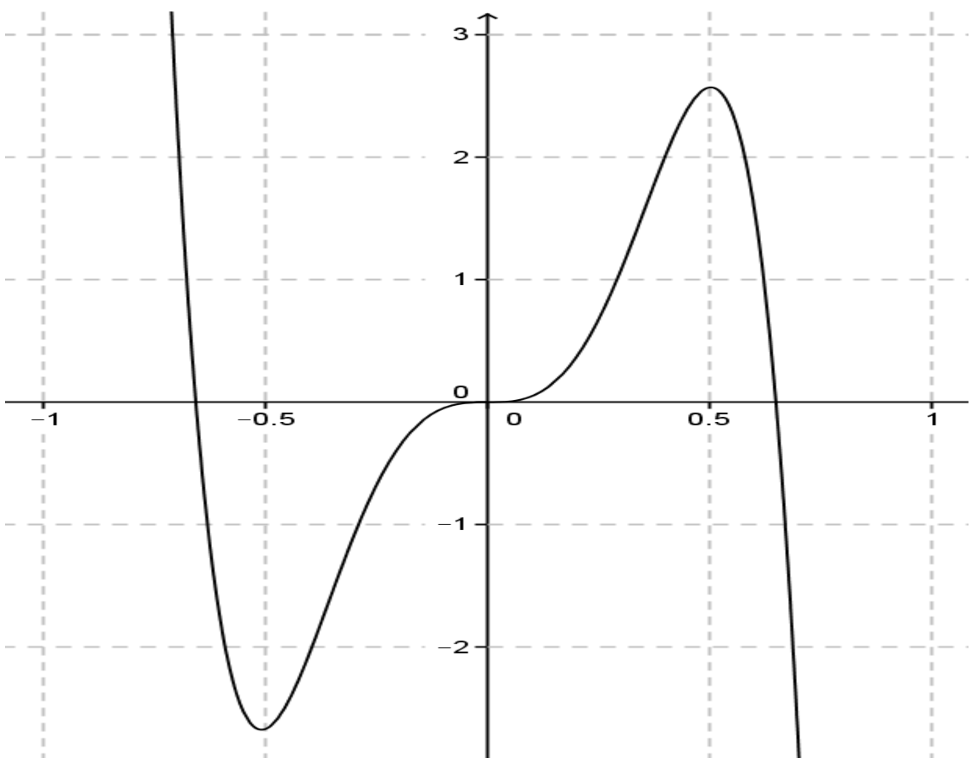


Exit Slip – 4.2.2

Name: ______________________________________

Date:__________________

Given the graph of f, shown below, sketch a possible graph of f' on the same axis.

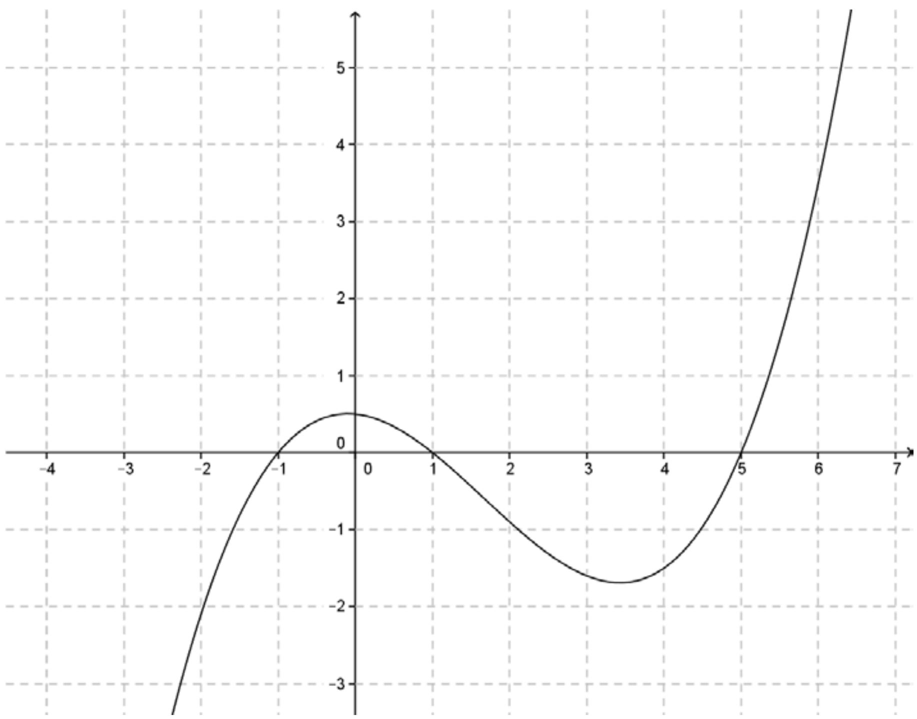


Opening Activity – 4.2.2

Name: ______________________________________

Date:__________________

Given the graph of f', shown below, sketch a possible graph of f on the same axis.



Exit Slip – 4.3.1

Name: ____

Date:___________________

$f(x)$ is a continuous function on $[0,4]$ with the following characteristics:

$f(0) = 0$

$f'(x) > 0$ on $[0,1) \cup (2,3)$

$f'(x) < 0$ on $(1,2) \cup (3,4]$

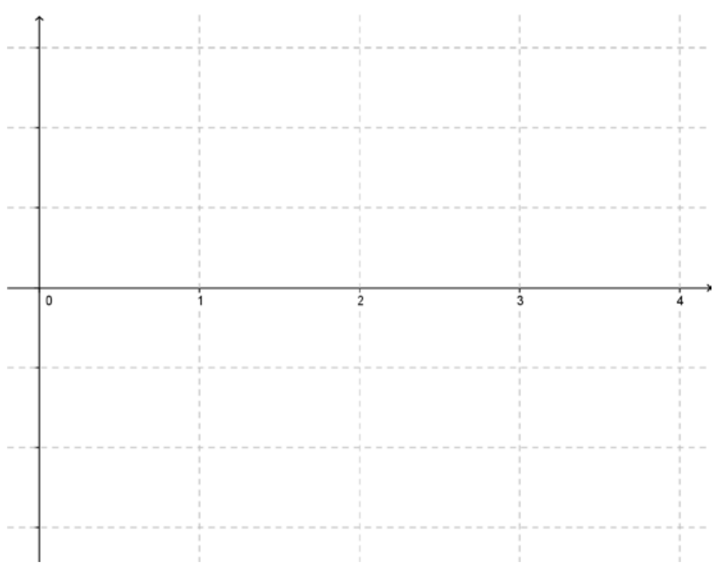
$f''(x) < 0$ on $[0,3)$

$f''(x) > 0$ on $(3,4]$

a) At what x-values, if any, does $f'(x)$ not exist?

b) At what x-values, if any, does $f''(x)$ not exist?

Sketch a possible graph of $f(x)$ on the axes below.

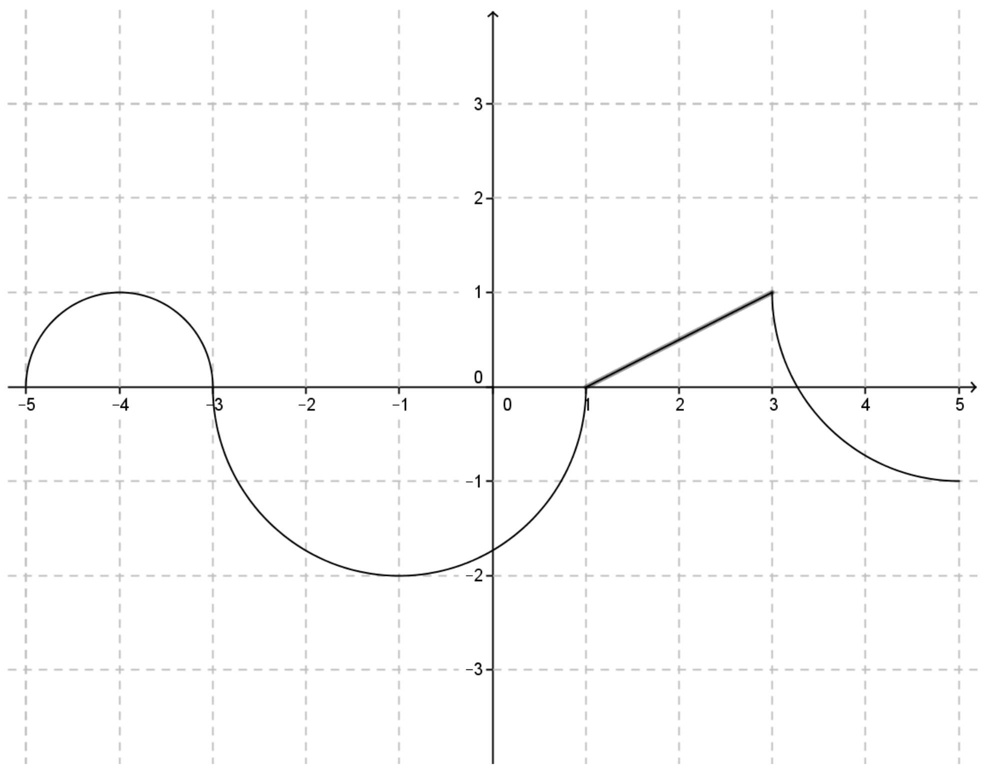


Opening Activity – 4.3.1

Name: ______________________________________

Date:__________________

The graph of $f(x)$ is shown below. $f(x)$ is a continuous function on$[-5,5]$ and consists of two semi-circles, one line segment and one quarter-circle.



a) On what interval(s), if any, is $f'(x) > 0$ and $f''(x) > 0$?

b) On what interval(s), if any, is $f'(x) > 0$ and $f''(x) < 0$?

c) On what interval(s), if any, is $f'(x) < 0$ and $f''(x) > 0$?

d) On what interval(s), if any, is $f'(x) < 0$ and $f''(x) < 0$?

e) At what x-value(s), if any, does $f'(x)$ not exist?

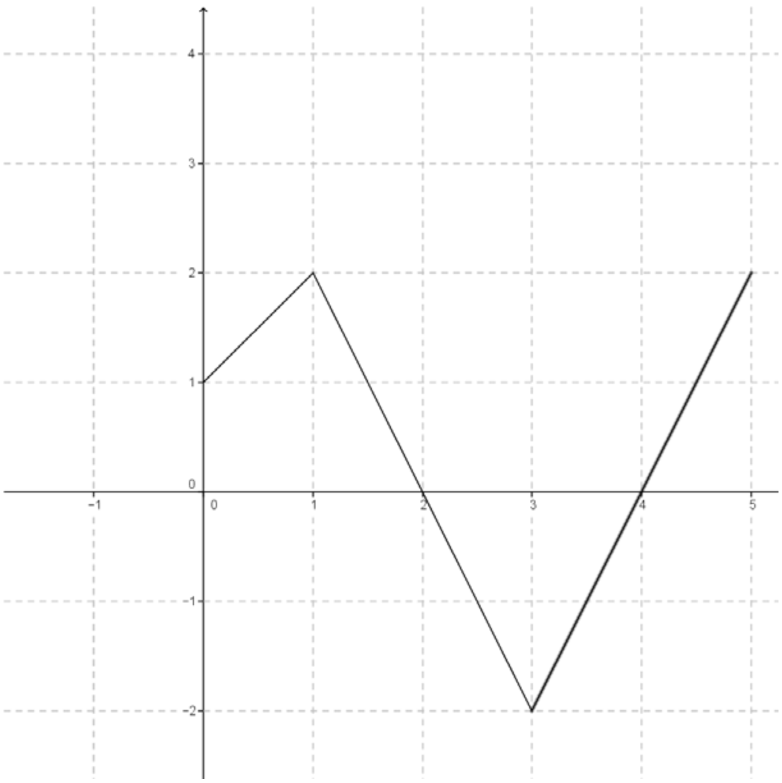
f) At what x-value(s), if any, does $f''(x)$ not exist?

Exit Slip – 4.3.2

Name: ______________________________________

Date:__________________

The graph of $f'(x)$, the derivative of $f(x)$ is shown below on $[0,5]$.



1) At what x-value(s), if any, does $f(x)$ have a relative max or min? Be sure to justify your response.
2) At what x-value(s), if any, does $f(x)$ have an inflection point? Be sure to justify your response.

Opening Activity – 4.3.2

Name: ______________________________________

Date:__________________

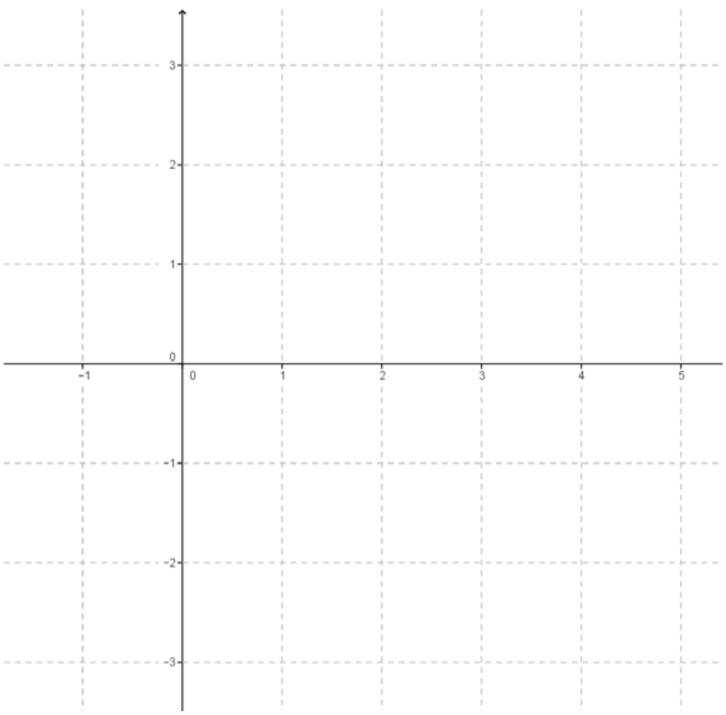
Sketch a possible graph of $f'(x)$, the derivative of $f(x)$, on $[0,5]$ on the axes below that meets the following requirements:

$f(x)$ has a relative minimum at $x = 2$

$f(x)$ has a relative maximum at $x = 4$

$f(x)$ has an inflection point at $x = 3$

Explain how you know that your graph of $f'(x)$ meets these requirements.



Exit Slip – 4.4.1

Name: ____

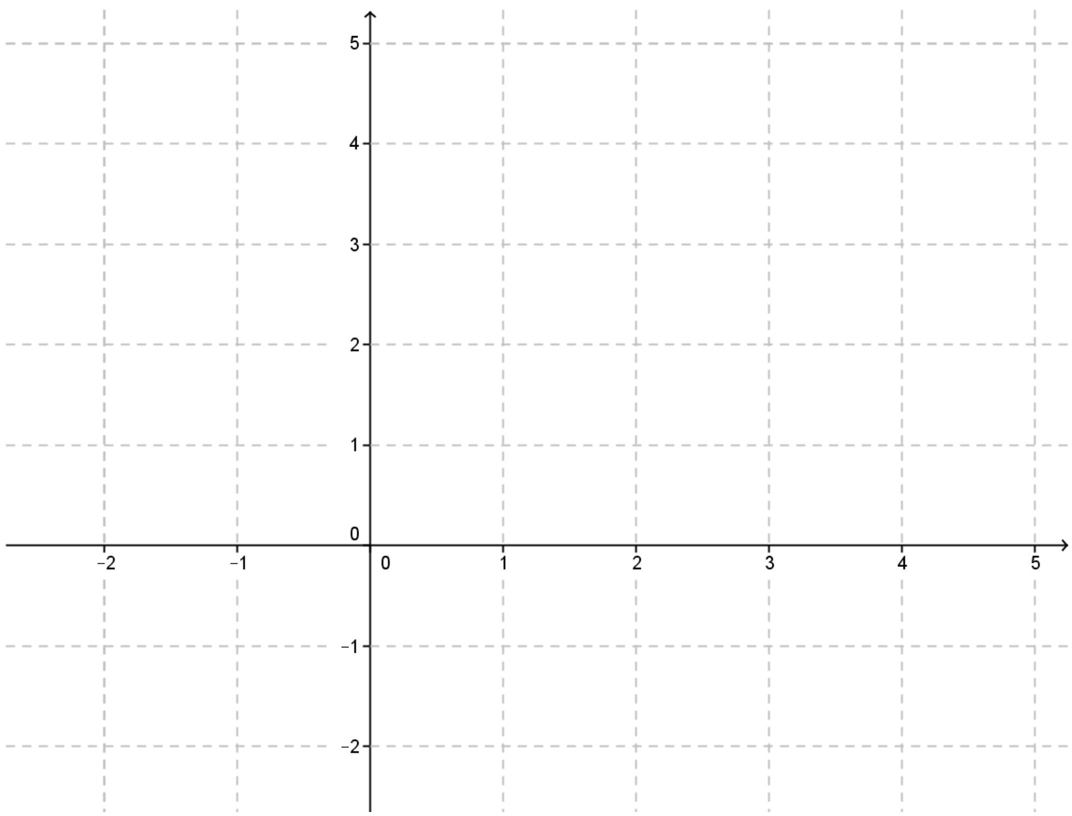
Date:___________________

A function f is continuous on its domain $[-2,4], f(-2) = 3, f(4) = -1$, and f' and f'' have the following properties:

x	$-2 < x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$3 < x < 4$
f'	$+$	DNE	$-$	0	$-$
f''	$+$	DNE	$+$	0	$-$

1) At what x-value(s), if any, does f have absolute extrema. Justify your answer.

2) Sketch a possible graph of $f(x)$ on the axes below.

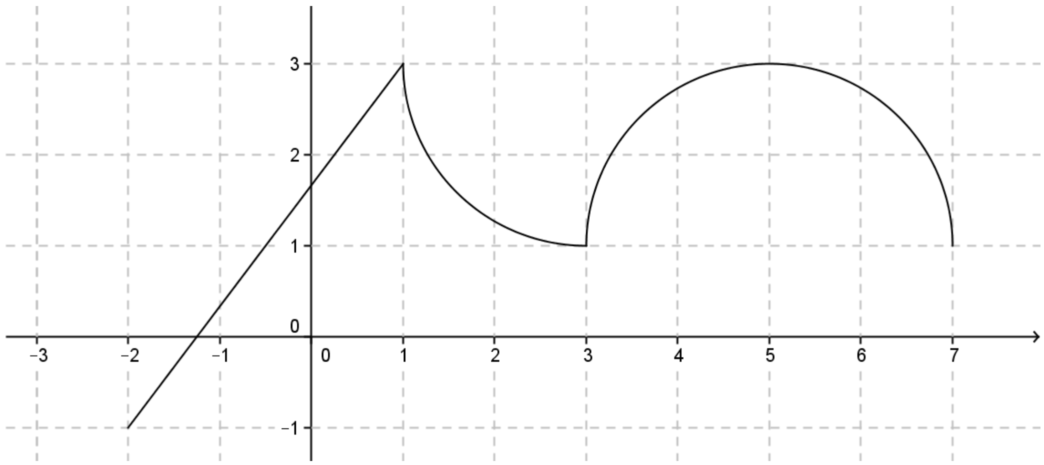


Opening Activity – 4.4.1

Name: ______________________________________

Date:__________________

The graph of $f(x)$ is shown below. $f(x)$ is a continuous function on$[-2,7]$ and consists of one line segment, one quarter-circle, and one semi-circle.



a) At what x-value(s), if any, does f have absolute extrema?

b) Fill in the table below with the symbols: $+,\ -,\ 0,\ DNE$ as appropriate.

x	$-2 < x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$3 < x < 5$	$x = 5$	$5 < x < 7$
f'							
f''							

Exit Slip – 4.5.1 Name: ____ Date:___________________ You have 200 feet of chain link fence available to fence off a rectangular garden. One side of the garden touches the side of a house and does not require fencing. What is the area of the largest garden that can be enclosed?
Opening Activity – 4.5.1 Name: ______________________________________ Date:__________________ A 216 ft^2 dog pen is going to be enclosed by a fence and divided into 2 equal areas by another fence parallel to one of the sides. What is the smallest amount of fence necessary to complete the job?
Exit Slip – 4.5.2 Name: ______________________________________ Date:__________________ Suppose that Jolly Green Giant has hired you to design a tin can in the shape of a right circular cylinder that holds exactly 300 cm^3 of brine. What is the minimum amount of tin needed to produce the can? (You may use a calculator for computational purposes.) (NOTE: $V_{cylinder} = \pi r^2 h, SA_{cylinder} = 2\pi r^2 + 2\pi rh$)

Opening Activity – 4.5.2

Name: ____

Date:___________________

Suppose we are designing a tin can for Jolly Green Giant in the shape of a right circular cylinder. If we have at most 100 cm^2 of tin to use, what is the maximum volume of the can? (You may use a calculator for computational purposes.)
(NOTE: $V_{cylinder} = \pi r^2 h, SA_{cylinder} = 2\pi r^2 + 2\pi rh$)

Exit Slip – 4.6.1

Name: ____

Date:___________________

Let $s(t)$ be the position function of a particle where t is in seconds and s is measured in feet.

a. Find the velocity function, $v(t)$ of the particle if

$$s(t) = 2t^3 - 10t^2 + 5t + 6$$

b. Are there any other possibilities for $v(t)$?

Opening Activity – 4.6.1

Name: ____

Date:___________________

Let $v(t)$ be the velocity function of a particle where t is in seconds and s is measured in feet.

a. Find a position function, $s(t)$ of the particle if

$$v(t) = t^3 - t^2 + 5t + 6$$

b. Are there any other possibilities for $s(t)$?

Exit Slip – 4.6.2

Name: ____

Date:___________________

Let $s(t)$ be the position function of a particle where t is in seconds and s is measured in feet.

a. Find an acceleration function, $a(t)$ of the particle if

$$s(t) = 4t^3 - 6t^2 + 7t - 10$$

b. Are there any other possibilities for $a(t)$?

Table 87 (continued)

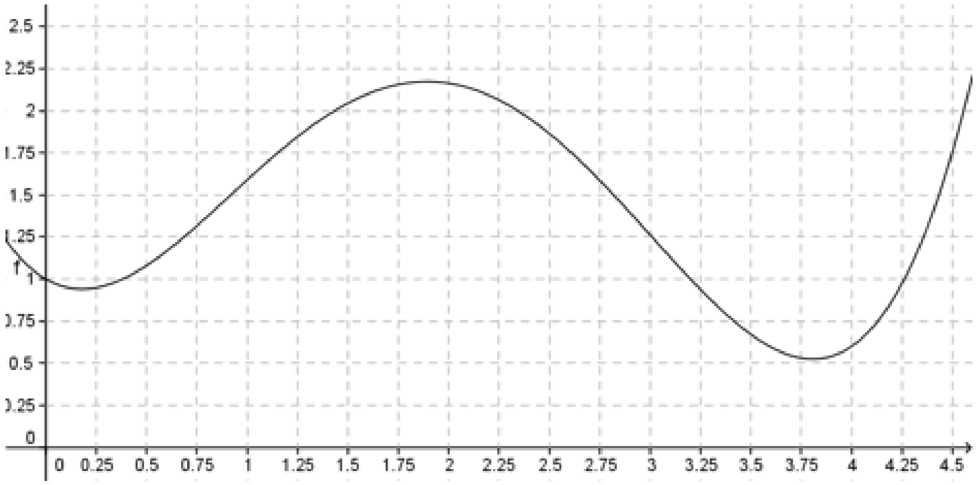
Opening Activity – 4.6.2 Name: ______________________________________ Date:__________________ Let $a(t)$ be the acceleration function of a particle where t is in seconds and s is measured in feet. a. Find a position function, $s(t)$ of the particle if $$a(t) = t^2 + 3t - 6$$ b. Are there any other possibilities for $s(t)$?
Exit Slip – 4.7.1 Name: ______________________________________ Date:__________________ If Newton's method is used to estimate a zero of $f(x) = -x^3 + x - 2$, what is the third estimate if the first estimate is $x_1 = 1$?
Opening Activity – 4.7.1 Name: ______________________________________ Date:__________________ Suppose $f(x)$ is a function and it is known that: i. $f'(x) = 2x - 4$ ii. Newton's Method produces an estimate of $x_2 = 2$ if the first estimate is $x_1 = 1$. Find $f(x)$.

Exit Slip – 4.8.1

Name: ______________________________

Date:________________

The function $f(x)$ is graphed below.

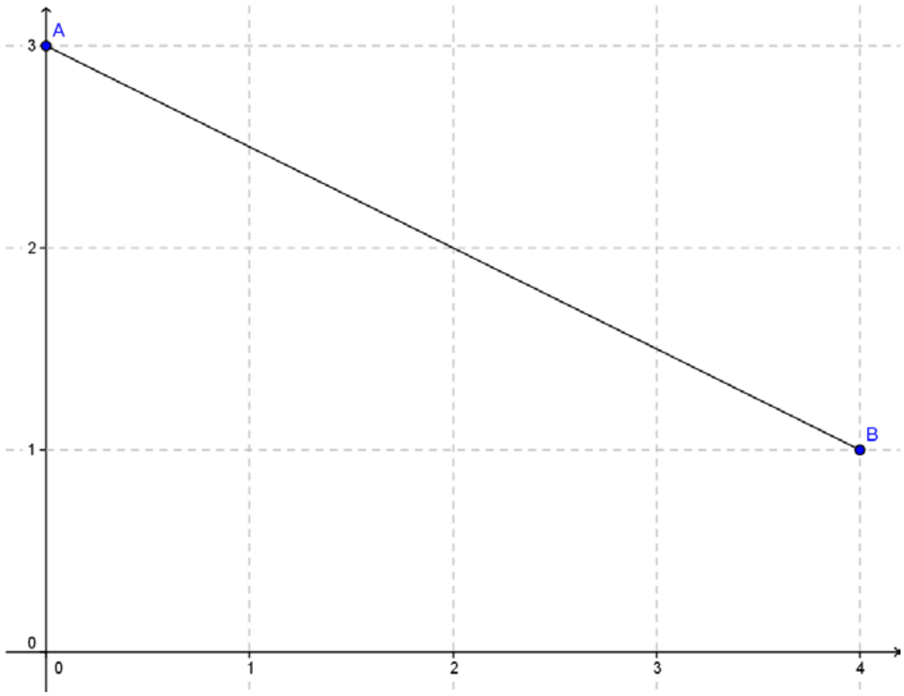


At what x-values, if any, is the Mean-Value Theorem satisfied on the interval $[0,4.5]$? Be sure to show or explain how you determined your answer(s).

Opening Activity – 4.8.1

Name: ________________________________ Date:________________

On the graph below, sketch a function $f(x)$ on [0,4] that satisfies the Mean-Value Theorem at $x = 1, x = 2$, and at $x = 3$, if line segment AB is known to be the secant line of $f(x)$ on [0,4].



APPENDIX F: INTERVIEW QUESTIONS

Table 88. Interview Questions

Interview #1

IQ 1.1. Let $f(x) = 6x^3$. What is $f'(x)$?

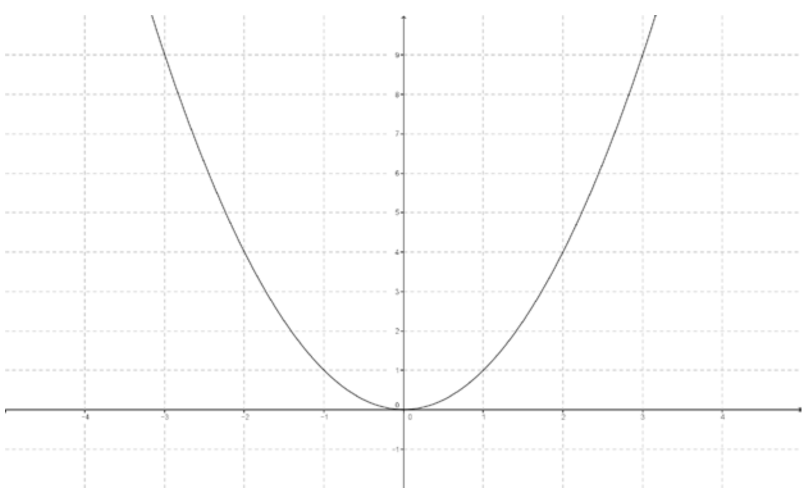
IQ 1.2. Suppose a function has a known derivative of $f'(x) = x^5$.

a. What could be the function $f(x)$?

b. Can you think of any other possible functions for $f(x)$?

IQ 1.3. The derivative of a polynomial function, $f'(x)$, is graphed below.

a. Sketch a possible graph of a polynomial function $f(x)$ whose derivative, $f'(x)$, is shown.



b. Are there any other possibilities for $f(x)$?

Interview #2

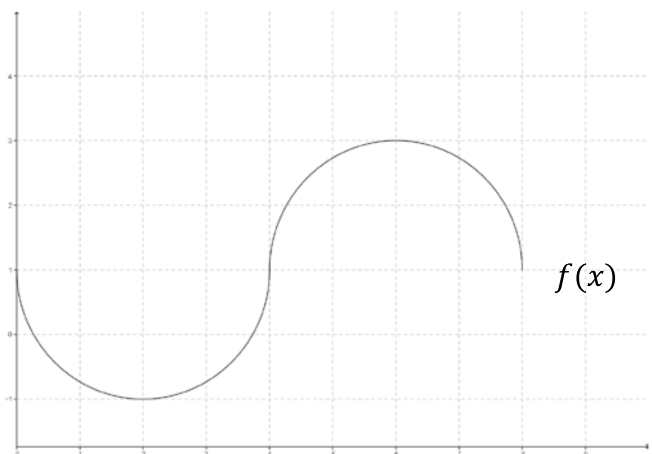
IQ 2.1. Suppose a function $f(x)$ has the known derivative $f'(x)$ shown below.

$$f'(x) = x\sin(x^2)$$

a. What could be the function $f(x)$?

IQ 2.2. Let $f(x) = \cos(x^2)$. Find $f'(x)$.

IQ 2.3. The graph of $f(x)$ below consists of two complete semi-circles that intersect at [4,1].



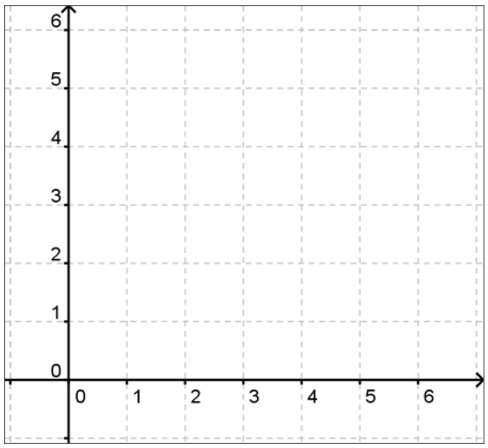
a. Estimate or give an exact value, if one exists, of $f'(x)$ at the x-values indicated in the table.

x	$f'(x)$
2	
4	
5	
6	
7	

IQ 2.4. The table below gives selected values of $f'(x)$, the derivative of $f(x)$.

a. If $f(x)$ is known to be continuous, sketch a possible curve for $f(x)$ on the axis below.

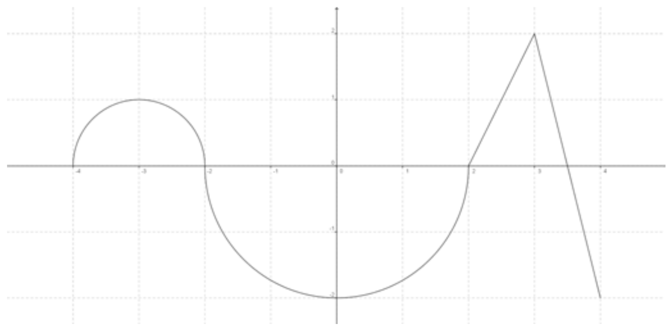
x	$f'(x)$
0	1
1	2
2	-1
3	0
4	-1



b. Could you sketch another function that would satisfy the table of values?

Interview #3

IQ 3.1. Consider the graph of $f(x)$ on the interval $[-4,4]$. $f(x)$ consists of two semi-circles and two line segments, as shown below.



a. On what intervals, if any, is $f'(x) > 0$ and $f''(x) > 0$?

b. On what intervals, if any, is $f'(x) < 0$ and $f''(x) > 0$?

c. On what intervals, if any, is $f'(x) > 0$ and $f''(x) < 0$?

d. On what intervals, if any, is $f'(x) < 0$ and $f''(x) < 0$?

e. At what x −value(s), if any, does $f'(x) = 0$?

f. At what x −value(s), if any, does $f''(x) = 0$?

g. At what x −value(s), if any, does $f'(x)$ not exist?

h. Justify your response to question (g).

i. At what x −value(s), if any, does $f''(x)$ not exist?

j. Justify your response to question (i).

IQ 3.2. Sketch a possible graph of a function f that satisfies the following conditions:

f is continuous;

$f(0) = 1, f'(-3) = f'(2) = 0$, and $\lim_{x\to 0} f'(x) = \infty$;

$f'(x) > 0$ when $-5 < x < -3$ and when $-3 < x < 2$;

$f'(x) < 0$ when $x < -5$ and when $x > 2$;

$f''(x) < 0$ when $x < -5$, when $-5 < x < -3$, and when $0 < x < 5$;

$f''(x) > 0$ when $-3 < x < 0$ and when $x > 5$;

$\lim_{x\to -\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -1$.

Interview #4

1.a

The table below gives the distance a car has traveled, measured in miles, at selected time measurements in hours.

t (hours)	Distance (miles)
0	0
0.2	8
0.4	17
0.5	21
0.8	35
1.0	44
2.0	100

t (hours)	Average Velocity (mph)
0.1	
0.6	
0.9	
1.5	

If the car only moves in a positive direction, fill in the accompanying table by estimating the velocity of the car in miles per hour at the times indicated. Show how you determined the average velocity.

1.b

Suppose we know a function $s(t)$ that gives the position of the car in miles after t hours, $0 \leq t \leq 2$.

$$s(t) = t^3 + 3t^2 + 40t$$

Fill in the accompanying table of instantaneous velocities at the times indicated.

t (hours)	Instantaneous Velocity (mph)
0.1	
0.6	
0.9	
1.5	

4.2.a

Suppose a car's velocity in $\frac{m}{s}$ is measured at intervals and recorded in the following chart.

Estimate the distance traveled from $t = 0\ s$ to $t = 60\ s$. Show how you calculate the distance traveled.

t	$v(t)$
0	0
20	5
30	8
50	4
60	10

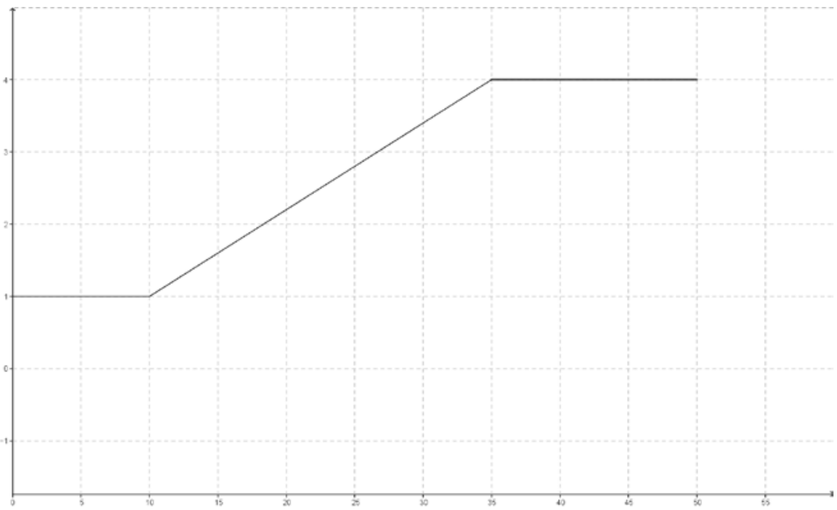
4.2.b. Suppose we know a velocity function, $v(t)$, for a vehicle in motion in meters per second.

$$v(t) = 4t^3 - 3t^2 + t$$

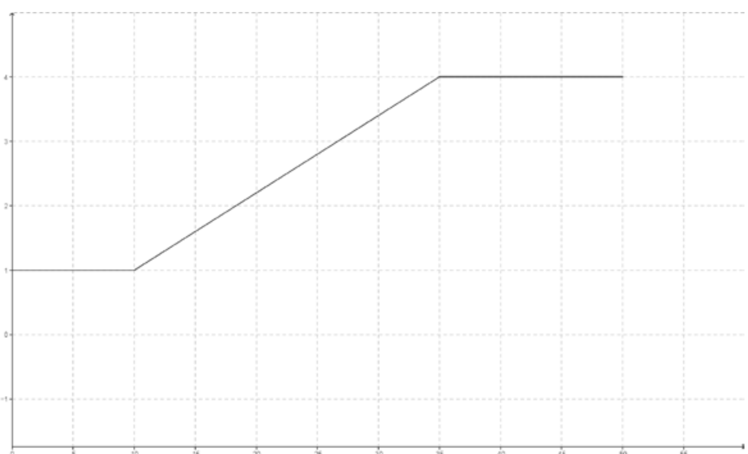
Find the position of the vehicle at $t = 3$.

IQ 4.3.

a. The function $f(x)$ is graphed below on [0,50], write an algebraic expression for $f'(x)$ on [0,50].



b. The function $f'(x)$ is graphed below on [0,50], write an algebraic expression for $f(x)$ on [0,50].



APPENDIX G: RESULTS OF THE DIFFERENTIATION COMPETENCY TEST

Table 89. Class achievement (%) on individual competencies on the DCT in each differentiation process and input representation by class

Process	Input Representation	Competency	Class Score (%) N = 21
Formulation			
without-translation	Numerical	FNn	57.14
	Graphical	FGg	28.57
	Symbolic	FSs	100
with-translation between two representations	Numerical	FNg	9.52
		FNs	14.29
	Graphical	FGn	38.10
		FGs	52.38
	Symbolic	FSn	71.43
		FSg	85.71
Interpretation			
without-translation	Numerical	INn	66.67
	Graphical	IGg	84.52
	Symbolic	ISs	0
with-translation between two representations	Numerical	INg	66.67
		INs	76.19
	Graphical	IGn	33.33
		IGs	71.43
	Symbolic	ISn	85.71
		ISg	66.67

Table 90. Class achievement (%) on specific groups of competencies (N = 21) on the DCT

Grouped Competencies	Class Achievement (%) N = 21
All Competencies (18 items)	56.02
Input Representation	
Symbolic (_S_) (6 items)	77.38
Graphical (_G_) (6 items)	48.81

Table 90 (continued)

Numerical (_N_) (6 items)	41.67
Output Representation	
Symbolic (_S_) (6 items)	53.57
Graphical (_G_) (6 items)	57.14
Numerical (_N_) (6 items)	57.14
Competencies without-translation (6 items)	56.15
Competencies with-translation (12 items)	49.21
Formulation competencies (9 items)	50.79
Interpretation competencies (9 items)	61.24

APPENDIX H: RESULTS OF THE FLEXIBILITY PRE-TEST

Table 91. Class achievement (%) on individual competencies of the flexibility pre-test

Process	Input Representation	Competency	Class Score (%) N = 21
Composition			
without-translation	Numerical	CNn	90.48
	Graphical	CGg	40.48
	Symbolic	CSs	95.24
with-translation between two representations	Numerical	CNg	80.95
		CNs	42.86
	Graphical	CGn	69.05
		CGs	14.29
	Symbolic	CSn	100
		CSg	14.29
Inverse			
without-translation	Numerical	INn	57.14
	Graphical	IGg	52.38
	Symbolic	ISs	19.05
with-translation between two representations	Numerical	INg	66.67
		INs	4.76
	Graphical	IGn	42.86
		IGs	38.10
	Symbolic	ISn	19.05
		ISg	42.86

Table 92. Class achievement (%) on specific groups of competencies (N = 21) on the flexibility pre-test

Grouped Competencies	Class Achievement (%) N = 21
All Competencies (18 items)	49.47
Input Representation	
Symbolic (_S_) (6 items)	44.05
Graphical (_G_) (6 items)	41.07
Numerical (_N_) (6 items)	48.81

Table 92 (continued)

Output Representation	
Symbolic (_S_) (6 items)	25.0
Graphical (_G_) (6 items)	51.19
Numerical (_N_) (6 items)	57.74
Competencies without-translation (6 items)	59.13
Competencies with-translation (12 items)	44.64
Composition competencies (9 items)	60.85
Inverse competencies (9 items)	38.10

APPENDIX I: CODEBOOK FOR INTERVIEW TRANSCRIPTS

Evidence of two-way reversibility consists of all of the words, phrases, or actions taken when solving an interview question that indicate the reversing of a process by working the steps backwards through inverse operations.

Table 93. Verbal indicators of two-way reversibility

Phrase	Rationale	Interview Questions (Interview #, Interview question) where phrase may appear. The word "and" indicates that the questions form a direct-reverse pair.
"if going from f to f' required … then to move backward, I should …" "Since I found f' from f by … then I should be able to find f from f' by doing the opposite"	When a question requires antidifferentiation, the students will likely describe the process of finding a function from its derivative as "backwards differentiation" or "by doing the opposite"	1.1 and 1.2, 1.1 and 1.3, 2.1 and 2.2, 2.3 and 2.4, 4.1.a and 4.2.a, 4.1.b and 4.2.b, 4.3.a and 4.3.b
"to find the derivative I subtracted one from the exponent, so to find the function, I should add one to the exponent"	When a student uses reversibility to find the function f when given the derivative f' and f' consists of a polynomial, then a correct reversing of the process would involve adding one to the existing exponent because differentiation requires subtracting one from the existing exponent.	1.1 and 1.2, 1.1 and 1.3, 4.1.b and 4.2.b, 4.3.a and 4.3.b

Table 93 (continued)

“to find the derivative I multiplied by the exponent, so to find the function, I should divide by the exponent”	When a student uses reversibility to find the function f when given the derivative f' and f' consists of a polynomial, then a correct reversing of the process would involve dividing term by the resultant exponent.	1.1 and 1.2, 1.1 and 1.3, 4.1.b and 4.2.b, 4.3.a and 4.3.b
Any phrase involving the word “reverse” or “direction”	Krutetskii (1976) said that reversibility requires a “sharp turn” (p. 287) in the student’s thought process. The words “reverse” or “direction” would seem to be indicators of making a sharp turn in the thought process.	1.1 and 1.2, 1.1 and 1.3, 2.1 and 2.2, 2.3 and 2.4, 4.1.a and 4.2.a, 4.1.b and 4.2.b, 4.3.a and 4.3.b

Evidence of reversibility of the mental process in reasoning consists of all of the words, phrases, and/or actions that indicate the student’s attempt at solving a problem by reversing a thought process without using the direct process in reverse.

Table 94. Verbal indicators of reversibility of the mental process in reasoning without reversible translation

Phrase	Rationale	Interview Questions (Interview #, Interview question) where phrase may appear. The word “and” indicates that the questions form a direct-reverse pair.
“Since I’m given f' … let me pick some f’s and see if when I differentiate them I can produce f'”	When a question requires antidifferentiation, but the student does not have access to a reversible process (i.e. when trying to antidifferentiate the result of the product rule without any knowledge of integration by parts)	2.1 and 2.2

Table 94 (continued)

Any phrase involving the word "reverse" or "direction"	Krutetskii (1976) said that reversibility requires a "sharp turn" (p. 287) in the student's thought process. The words "reverse" or "direction" would seem to be indicators of making a sharp turn in the thought process.	2.1 and 2.2, 2.3 and 2.4, 4.1.a and 4.2.a, 4.1.b and 4.2.b, 4.3.a and 4.3.b
"I found f' in the previous problem by estimating slopes … so, I should be able to sketch f by using the given f' values as my slopes"	The nature of this statement indicates that the student has determined that the two successive problems are a direct-reverse pair and thus the reverse problem can be solved be reversing what s/he did in the previous problem.	2.3 and 2.4
"When I was given distance, to estimate velocity I did … so to get distance from a velocity chart, I would need to …"	This statement indicates that the students is aware of the reversible relationship between distance and velocity.	4.1.a and 4.2.a
"Since velocity is the derivative of position, to get back to position from velocity, I need to do the reverse of the derivative"	This statement indicates that the students is aware of the reversible relationship between position and velocity.	4.1.b and 4.2.b
"What would cause a derivative that is a horizontal line" "What kind of function would have a linear derivative"	Both of these statements indicate that the student is considering using the graph of the derivative to inform the graph of the function.	4.3.a and 4.3.b
"What would the graph of f look like at $x = 2$ if $f'(2) = 0$"	Here, the student is visualizing the effect(s) of the derivative on the original function, which is the reverse of determining the value of the derivative by looking at the graph of the function.	2.3 and 2.4
"If f is increasing, then f' is positive, so if f' is positive, then f should be increasing"	This statement indicates that the student is reversing the information that f tells us about f' in order to make inferences about f from f'.	3.1 and 3.2

Table 94 (continued)

"If f is concave up, then f'' is positive, so if f'' is positive, then f should be concave up"	This statement indicates that the student is reversing the information that f tells us about f'' in order to make inferences about f from f''.	3.1 and 3.2

Evidence of representational reversibility consists of all of the words, phrases, and/or actions that indicate that a student is proficiently translating back and forth between two different representations.

Table 95. Verbal indicators of representational reversibility

Phrase	Rationale	Interview Questions (Interview #, Interview question) where phrase may appear. The word "and" indicates that the questions form a direct-reverse pair.
Forward: "I need to know an algebraic expression …" Reverse: "I'm going to make a table …" or "I'm going to sketch a graph"	The forward statement indicates that the student has been provided a list of functional values or a graph or both and the student is considering how to translate the given representation(s) into an algebraic expression. The reverse statement indicates that the student has been provided an algebraic expression and is thinking about how to translate the algebraic expression into a graphical or numerical representation of a function.	1.3 2.3 and 2.4 3.1 and 3.2 4.1.a and 4.2.a 4.3.a and 4.3.b

Table 95 (continued)

Forward: "Is there a formula that describes this curve?" Reverse: "I am going to sketch of a graph of this function."	The forward statement indicates that the student has been provided a graph and the student is considering how to translate the graph into an algebraic expression. The reverse statement indicates that the student has been given an algebraic expression and recognizes the function as a known graph.	1.3 2.3 3.1 and 3.2 4.3.a and 4.3.b
Forward: "I can use the table to plot some points …" Reverse: "I can fill in the table by reading the values off of the graph"	The forward statement indicates that the student has been provided a list of functional values and the student is considering how to translate the functional values into a graph. The reverse statement suggests that the student has been given a graph and is going to use the graph to determine the numerical representation of the function.	2.3 and 2.4 4.1.a and 4.2.a
Forward: "The value of the numerical derivative tells me the slope of the line tangent to the graph, so maybe I can sketch a graph that would have those slopes" Reverse: "To get the derivative at x-values, I need to look at the slope of the tangent line"	Here, the student is considering how to translate from a numerical representation of the derivative to a graphical representation of the function. This statement is the reverse of the forward statement indicating that the student is considering how to use the graph of the function to evaluate the derivative at specific x-values. Thus, the student is translating from graphical to numerical.	2.4

Table 95 (continued)

Forward: "I'm going to start by making a chart of the values …" Reverse: "Okay, I can use the table of values to …"	The forward statement indicates the student has been provided either a symbolic representation of the function or a graphical representation of the function and is going to translate it into a numerical representation. The reverse statement indicates that the student has been given a numerical representation and is attempting to translate the numerical representation into either a symbolic or graphical representation.	1.3 2.3 and 2.4 3.1 and 3.2 4.1.a and 4.2.a 4.1.b and 4.2.b

Table 96. Verbal indicators of flexibility: function to function or derivative to derivative

Phrase	Translation indicated by phrase	Interview Questions (Interview #, Interview question) where phrase may appear.
"I'm going to start by plugging in some values for x" "I need to substitute the given values in for t."	Symbolic → Numerical	2.3 4.1.a 4.1.b 4.2.a 4.2.b 4.3.a 4.3.b
"Ok, I'm going to sketch out the graph of $f(x)$"	Symbolic → Graphical	1.3 2.3 4.3.a 4.3.b
"Looking at the pattern (or relationship) between the x and y values the function should be …"	Numerical → Symbolic	1.3 4.1.a 4.2.a 4.3.a 4.3.b

Table 96 (continued)

"I'm going to plot these points and then see if I recognize a curve that passes through them"	Numerical → Graphical	2.4 3.2 4.1.a 4.2.a
"Reading the coordinates off of the graph …"	Graphical → Numerical	1.3 2.3 3.1 4.3.a 4.3.b
"That graph looks like the graph of …"	Graphical → Symbolic	1.3 2.3 3.1 4.3.a 4.3.b

Table 97. Verbal indicators of flexibility: function to derivative or derivative to function

Phrase	Translation indicated by phrase	Interview Questions (Interview #, Interview question) where phrase may appear.
"To find the instantaneous rate of change of $f(x)$, I need to find the derivative" "To find the average rate of change of $f(x)$, I need to use the slope formula" "To find the velocity function from the position function I need to take the derivative"	Symbolic Function → Numerical Derivative	4.1.b
"I need to find the slope of the line tangent to $f(x)$ at $x = c$" "So that's $f(x)$, to find where $f(x)$ is concave up, I need to find where $f''(x) > 0$"	Symbolic Function → Graphical Derivative	2.3.a

Table 97 (continued)

"Ok, so I'm just using the power rule here …"	Symbolic Function → Symbolic Derivative	1.1 2.2 4.3.a
"Looking at the pattern (or relationship) between the x and y values the function should be $y = f(x)$ and now I am going to take the derivative of $f(x)$ to find $f'(x)$."	Numerical Function → Symbolic Derivative	Unlikely to be found in interviews
"Given this list of values I need to estimate the slope of the line tangent to the graph at $x = c$"	Numerical Function → Graphical Derivative	Unlikely to be found in interviews
"Given this of values I need to find the average rate of change at $x = c$" "The position of the particle is given at several times, I need to estimate the velocity over this interval"	Numerical Function → Numerical Derivative	4.1.a
"I have the graph of f, I need to find the slope of the line tangent to $f(x)$ at $x = c$"	Graphical Function → Numerical Derivative	2.3 3.1 4.3.a
"So I have a graph of $f(x)$ and I want to find the formula for $f'(x)$"	Graphical Function → Symbolic Derivative	2.3 3.1 4.3.a
"I've been given the graph of $f(x)$ and need to sketch the graph of $f'(x)$" "I need to find where the slope of the tangent line to the graph is zero"	Graphical Function → Graphical Derivative	2.3 3.1 4.3.a
"Given $f'(x)$ I need to fill in the table of $f(x)$ values"	Symbolic Derivative → Numerical Function	4.2.b
"Given $f'(x)$ I need to sketch a graph of $f(x)$"	Symbolic Derivative → Graphical Function	1.3 4.3.b
"Given $f'(x)$, I need to find an expression for $f(x)$"	Symbolic Derivative → Symbolic Function	1.2 1.3 2.1 4.3.b

Table 97 (continued)

"Given the rate of change of $f(x)$, I need to find an algebraic expression for $f(x)$"	Numerical Derivative → Symbolic Function	4.2.b
"I know the rate of change of $f(x)$, so I need to sketch a possible graph of $f(x)$" "I know $f'(x)$ is at the listed x −values … what would be a graph of $f(x)$ look like"	Numerical Derivative → Graphical Function	2.4 3.2
"I know the rate of change of $f(x)$, so to evaluate $f(x)$ at $x = c$, I need to ..."	Numerical Derivative → Numerical Function	4.2.b
"The graph of $f'(x)$ tells me about the increase/decrease and concavity of $f(x)$, so in order to evaluate $f(x)$ at the given x −values …"	Graphical Derivative → Numerical Function	1.3
"I have the graph of $f'(x)$ and I want to find an expression for $f(x)$"	Graphical Derivative → Symbolic Function	1.3 4.3.b
"The graph of $f'(x)$ tells me about the increase/decrease and concavity of $f(x)$, so I can use that to sketch a possible graph of $f(x)$ by …"	Graphical Derivative → Graphical Function	1.3 2.4 4.3.b

BIBLIOGRAPHY

Amit, M., & Fried, M. N. (2005). Multiple representations in 8th grade algebra lessons: Are learners really getting it? In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 57-64). Melbourne: PME.

Anton, H., Bivens, I., & Davis, S. (2009). *Calculus: Early transcendentals single variable* (9th ed.). Jefferson City: John Wiley & Sons, Inc.

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. J. Kaput, A. H. Schoenfeld & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education* (Vol. 6, pp. 1-32). Providence: American Mathematical Society.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior, 16*(4), 399-431. doi: 10.1016/s0732-3123(97)90015-8

Baker, B., Cooley, L. A., & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education, 31*(5), 557-578.

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *Journal of Mathematical Behavior, 22*(2003), 481-497.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247-285.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Duran, R., Reed, B. S., & Webb, D. (1997). Learning by understanding: the role of multiple representations in learning algebra. *American Educational Research Journal, 34*(4), 663-689.

Bressoud, D. M. (2009a). AP Calculus: What We Know. Retrieved February 24, 2012, from http://www.maa.org/columns/launchings/launchings_06_09.html

Bressoud, D. M. (2009b). The Changing Face of Calculus: First-Semester Calculus as a High School Course. Retrieved March 1, 2009, from http://www.maa.org/features/faceofcalculus.html

Cangelosi, R., Madrid, S., Cooper, S., Olson, J., & Hartter, B. (2013). The negative sign and exponential expressions: Unveiling students' persistent errors and misconceptions. *The Journal of Mathematical Behavior, 32*(1), 69-82.

Cheng, P. C. H. (1999). Unlocking conceptual learning in mathematics and science with effective representational systems. *Computers & Education, 33*, 109-130.

Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St. John, D., . . . Vidakovic, D. (1997). Constructing a schema: The case of the chain rule? *The Journal of Mathematical Behavior, 16*(4), 345-364. doi: 10.1016/s0732-3123(97)90012-2

Collegeboard. (2010a). Calculus: calculus AB calculus BC course description. Retrieved February 25, 2012, from http://apcentral.collegeboard.com/apc/public/repository/ap-calculus-course-description.pdf

Collegeboard. (2010b). Calculus: calculus AB, calculus BC course description. Retrieved June 23, 2011, from http://apcentral.collegeboard.com/apc/public/repository/ap-calculus-course-description.pdf

Confrey, J. (1981). Concepts, processes, and mathematics instruction. *For the Learning of Mathematics, 2*(1), 8-12.

Confrey, J., & Lanier, P. (1980). Students' mathematical abilities: A focus for the improvement of teaching general mathematics. *School Science and Mathematics, 80*(7), 549-556.

Cooley, L., Trigueros, M., & Baker, B. (2007). Schema Thematization: A Framework and an Example. *Journal for Research in Mathematics Education, 38*(4), 370-392. doi: 10.2307/30034879

Cuoco, A. A. (1994). Multiple representations for functions. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning: Preliminary analyses and results* (Vol. 33, pp. 121-140). Washington, DC: Mathematical Association of America.

Davis, G. E., & McGowen, M. A. (2002). *Function machines & flexible algebraic thought*. Paper presented at the 26th International Group for the Psychology of Mathematics Education, University of East Anglia, Norwick, U.K.

Demana, F., Waits, B., Foley, G., & Kennedy, D. (2004). *Precalculus: Functions and graphs* (Sixth ed.). Boston, MA: Addison-Wesley.

Dick, T. P., & Edwards, B. S. (2008). Multiple representations and local linearity - research influences on the use of technology in calculus curriculum reform. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: cases and perspectives* (Vol. 2, pp. 255-275). Charlotte, NC: Information Age Publishing.

Dreyfus, T. (1990). Advanced mathematical thinking. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education* (pp. 113-134). Cambridge: Cambridge University Press.

Dreyfus, T., & Eisenberg, T. (1990). On diffficulties with diagrams: Theoretical issues. In *Proceedings of the 14th International Conference for the Psychology of Mathematics Education, 1* (pp. 27-34).

Dubinsky, E., & McDonald, M. A. (2002). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue, U. Kirchgräber, J. Hillel, M. Niss & A. Schoenfeld (Eds.), *The Teaching and Learning of Mathematics at University Level* (Vol. 7, pp. 275-282): Springer Netherlands.

Dunst, C. J., Hamby, D. W., & Trivette, C. M. (2004). Guidelines for calculating effect sizes for practice-based research syntheses. *Centerscope: Evidence-based approaches to early childhood development, 3*(1), 1-10.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1/2), 103-131.

Elia, I., Panaoura, A., Eracleous, A., & Gagatsis, A. (2007). Relations between secondary pupils' conceptions about functions and problem solving in different representations. *International Journal of Science and Mathematics Education, 5*(3), 533-556.

Engle, R. A. (2006). Framing interactions to foster generative learning: A situative explanation of transfer in a community of learners classroom. *The Journal of the Learning Sciences, 15*(4), 451-498.

Finney, R. L., Demana, F., Waits, B., & Kennedy, D. (2011). *Calculus: Graphical, Numerical, Algebraic, 4th ed.* Boston, MA: Prentice Hall.

Flavell, J. H. (1963). *The developmental psychology of Jean Piaget*. Princeton, NJ: D. Van Nostrand Company, Inc.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan Publishing Company.

Gagatsis, A., Christous, C., & Elia, I. (2004). The nature of multiple representations in developing mathematical relationships. *Quaderni di Ricerca in Didattica, 14*, 150-159.

García, M., Llinares, S., & Sánchez-Matamoros, G. (2011). Characterizing thematized derivative schema by the underlying emergent structures. *International Journal of Science and Mathematics Education, 9*(5), 1023-1045. doi: 10.1007/s10763-010-9227-2

George, D., & Mallery, P. (2003). *SPSS for Windows step by step: A simple guide and reference. 11.0 update* (4th ed.). Boston: Allyn & Bacon.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Gleason, A. M., & Hughes-Hallett, D. (1992). Introducing the calculus consortium based at Harvard University. *Focus On Calculus Newsletter*, (1), 1, 3. http://www.wiley.com/college/cch/Newsletters/issue1.pdf

Goerdt, S. (2007). *The effect of emphasizing multiple representations on calculus students' understanding of the derivative concept.* (Ph.D. dissertation), University of Minnesota, United States - Minnesota. Retrieved March 12, 2012, from Dissertations & Theses: Full Text. (Publication No. AAT 3277946)

Graham, J. W., Olchowski, A. E., & Gilreath, T. D. (2007). How many imputations are really needed? Some practical clarifications of multiple imputation theory. *Prevention science, 8*(3), 206-213.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: a "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140. doi: 10.2307/749505

Greeno, J. G., & Hall, R. P. (1997). Practicing representation: learning with and about representational forms. *The Phi Delta Kappan, 78*(5), 361-367. doi: 10.2307/20405797

Guest, G., MacQueen, K. M., & Namey, E. E. (2012). *Applied thematic analysis*. Los Angeles: SAGE Publications, Inc.

Gur, H., & Barak, B. (2007). The erroneous derivative examples of eleventh grade students. *Uygulamada Egitim Bilimleri (Educational Sciences: Theory & Practice), 7*(1), 473-480.

Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior, 25*(1), 57-72. doi: 10.1016/j.jmathb.2005.11.004

Haciomeroglu, E. (2007). *Calculus students' understanding of derivative graphs: Problems of representations in calculus.* (Ph.D. dissertation), The Florida State University, United States - Florida. Retrieved April 14, 2012, from Dissertations & Theses: Full Text. (Publication No. AAT 3308261)

Haciomeroglu, E., Aspinwall, L., & Presmeg, N. (2009). *The role of reversibility in the learning of the calculus derivative and antiderivative graphs.* Paper presented at the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Atlanta, GA.

Haciomeroglu, E., Aspinwall, L., & Presmeg, N. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning, 12*(2), 152-176.

Hashimoto, Y., & Becker, J. (1999). The open approach to teaching mathematics - creating a culture of mathematics in the classroom: Japan. In L. J. Sheffield (Ed.), *Developing mathematically promising students* (pp. 101-119). Reston, VA: National Council of Teachers of Mathematics.

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3-25.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan Publishing Company.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Horvath, A. (2008). *Looking at calculus students' understanding from the inside-out: The relationship between the chain rule and function composition*. Paper presented at the 11th Annual Conference on Research in Undergraduate Mathematics Education, San Diego, CA.

Huck, S. W. (2008). *Reading statistics and research* (Fifth ed.). Boston, MA: Pearson Education, Inc.

Hughes-Hallett, D., Gleason, A. M., Flath, D. E., Lock, P. F., Gordon, S. P., Lomen, D. O., . . . Tucker, T. W. (1995). *Calculus: Single and multivariable*. New York, NY: Wiley and Sons.

Hughes-Hallett, D., Gleason, A. M., Flath, D. E., Lock, P. F., Gordon, S. P., Lomen, D. O., . . . Tucker, T. W. (1998). *Calculus: Single and multivariable* (2nd ed.). New York, NY: Wiley and Sons.

Hughes-Hallett, D., McCallum, W. G., Gleason, A. M., Osgood, B. G., Flath, D. E., Quinney, D., . . . Tucker, T. W. (2005). *Calculus: Single variable* (Fourth ed.). Hoboken, NJ: John WIley & Sons.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking: from childhood to adolescence*. United States of America: Basic Books.

Janvier, C. (1987a). Representation and understanding: The notion of function as an example. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 67-71). Hillsdale, NJ: Lawrence Erlbaum Associates.

Janvier, C. (1987b). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27-32). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kang, W. (2012). *Implications from Polya and Krutetskii*. Paper presented at the 12th International Congree on Mathematical Education, Seoul, Korea.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbooks of research on mathematics teaching and learning* (Vol. 1, pp. 515-556). New York: Macmillan Publishing Company.

Keller, B. A., & Hirsch, C. R. (1998). Student preferences for representations of functions. *International Journal of Mathematical Education in Science and Technology, 29*(1), 1-17. doi: 10.1080/0020739980290101

Kendal, M., & Stacey, K. (1999). Varieties of teacher privileging for teaching calculus with computer algebra systems. *The International Journal of Computer Algebra in Mathematics Education, 6*(4), 233-247.

Kendal, M., & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning, 6*, 143-165.

Kendal, M., & Stacey, K. (2003). Tracing learning of three representations with the differentiation competency framework. *Mathematics Education Research Journal, 15*(1), 22-41.

Kennedy, D. (2003). Teacher's guide - AP Calculus. In CollegeBoard (Ed.), *Advanced Placement Program: Professional Development for Calculus AB and BC*. New York: The College Board.

Kilpatrick, J. (1978). *Research on mathematical abilities*. Paper presented at the Group in Science and Mathematics Education, the University of California, Berkeley.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: an exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500-507.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Lamon, S. J. (1993). Ratio and proportion: connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*(1), 41-61. doi: 10.2307/749385

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (Vol. 1, pp. 629-667). Charlotte, NC: Information Age Publishing.

Larson, R., & Hostetler, R. P. (1997). *Precalculus*. Boston, MA: Houghton Mifflin Company.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesser, L., & Tchoshanov, M. (2005). *The effect of representation and representational sequence on students' understanding*. Paper presented at the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Honolulu, HI.

McGowen, M. A. (2006). Developmental algebra. In N. Baxter-Hastings (Ed.), *MAA notes 69: A fresh start for collegiate mathematics* (pp. 369-275): University Publishing Online.

McGowen, M. A., & Davis, G. E. (2001). *Changing pre-service elementary teachers' attitudes to algebra*. Paper presented at the 12th ICMI Study Conference: the Future of the Teaching and Learning of Algebra, University of Melbourne, Australia.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating reseach on the graphical representation of functions* (pp. 69-100). Hillsdale, NJ: Lawrence Erlbaum Associates.

NCTM. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM.

NCTM. (1991). *Professional Standards for Teaching Mathematics*. Reston, VA: NCTM.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

NGA. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Norman, A., F., & Prichard, M. K. (1994). Cognitive obstacles to the learning of calculus: a Kruketskiian perspective. In J. J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning: Preliminary Analyses and Results* (pp. 65-77). Washington, DC: Mathematical Association of America.

Patton, M. Q. (2002). *Qualitative research & evaluation methods* (3rd ed.). Thousand Oaks: Sage.

Pinard, A. (1981). *The conservation of conservation: The child's acquisition of a fundamental concept*. Chicago, IL: University of Chicago Press.

Rachlin, S. (1981). *Processes used by college students in understanding basic algebra.* (Doctor of Education), University of Georgia. Retrieved from ProQuest Dissertations & Theses (8206303)

Rachlin, S. (1998). Learning to see the wind. *Mathematics Teaching in the Middle School, 3*(7), 470-473.

Ramful, A., & Olive, J. (2008). Reversibility of thought: An instance in multiplicative tasks. *Journal of Mathematical Behavior, 27*, 138-151.

Richards, J., & von Glasersfeld, E. (1980). Jean Piaget, psychologist of epistemology: A discussion of Rotman's Jean Piaget: Psychologist of the real. *Journal for Research in Mathematics Education, 11*, 19-36.

Rider, R. L. (2004). *The effect of multi-representational methods on students' knowledge of function concepts in developmental college mathematics.* (Doctor of Philosophy), North Carolina State, Retrieved from ProQuest Digital Dissertations. (3120252).

Rider, R. L. (2007). Shifting from traditional to nontraditional teaching practices using multiple representations. *The Mathematics Teacher, 100*(7), 494-500.

Ross, S. C. (1996). A modern course in calculus. In A. W. Roberts (Ed.), *Calculus: The Dynamics of Change*. Washington, DC: Mathematical Association of America.

Rubin, D. B. (1987). *Multiple imputation for nonresponse in surveys*. Hoboken, NJ: John Wiley & Sons.

Rublein, G. (1995). Conference report: Fourth annual conference on the teaching of mathematics. *Focus On Calculus Newsletter*(9), 1, 3.

Schafer, J. L. (1999). Multiple imputation: a primer. *Statistical Methods in Medical Research, 8*, 3-15.

Schnall, M., Alter, E., Swanlund, T., & Schweitzer, T. (1972). A sensory-motor context affecting performance in a conservation task: A closer analogue of reversibility than empirical return. *Child Development, 43*(3), 1012-1023.

Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve nonroutine problems. In J. J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning: Preliminary Analyses and Results* (pp. 19-26). Washington, DC: Mathematical Association of America.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflection on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification - the case of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of*

Epistemology and Pedagogy (pp. 59-84, MAA notes, no. 25). Washington, DC: Mathematical Association of America.

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, *30*(2), 131-148. doi: 10.1016/j.jmathb.2011.02.001

Sparks, B. E., Brown, J. A., & Bassler, O. C. (1970). The feasibility of inducing number conservation through training on reversibility. *Journal for Research in Mathematics Education, 1*(3), 134-143.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks: Sage.

Steffe, L. P. (1994). Children's multiplying scheme. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Steketee, S., & Scher, D. (2012). Using multiple representations to teach composition of functions. *The Mathematics Teacher, 106*(4), 260-268.

Tall, D. (1992). Students' difficulties in calculus. *Proceedings of working group 3 on students' difficulties in calculus*, ICME-7, Quebec, Canada, 13-28.

Tall, D. (2010). *A sensible approach to the calculus*. Paper presented at the Plenary session of the National and International Meeting on the Teaching of Calculus, Puebla, Mexico.

Tall, D. (2011). Crystalline concepts in long-term mathematical invention and discovery. *For the Learning of Mathematics, 31*(1), 3-8.

Tavakol, M., & Dennick, R. (2011). Making sense of Cronbach's alpha. *International Journal of Medical Education*(2), 53-55.

Teachey, A. L. (2003). *Investigations in conceptual understanding of polynomial functions and the impact of mathematical belief systems on achievement in an accelerated summer program for gifted students.* (Doctor of Philosophy), North Carolina State, Retrieved from ProQuest Digital Dissertations. (3085138).

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2/3), 229-274.

Trochim, W. M. (2002, August 2, 2000). Research methods knowledge base. Retrieved November 16, 2013, from http://csi.ufs.ac.za/resres/files/Trochim.pdf

Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *The Journal of Mathematical Behavior, 23*(1), 93-114.

Ubuz, B. (2001). First Year Engineering Students' Learning of Point of Tangency, Numerical Calculation of Gradients, and the Approximate Value of a Function at a Point through Computers. *Journal of Computers in Mathematics and Science Teaching, 20*(1), 113.

Usiskin, Z. (1999). The mathematically promising and the mathematically gifted. In L. J. Sheffield (Ed.), *Developing mathematically promising students* (pp. 57-69). Reston, VA: National Council of Teachers of Mathematics.

Vilkomir, T., & O'Donoghue, J. (2009). Using components of mathematical ability for initial development and identification of mathematically promising students. *International Journal of Mathematical Education in Science and Technology*, *40*(2), 183-199. doi: 10.1080/00207390802276200

von Glasersfeld, E. (1981). An attentional model for the conceptual construction of units and number. *Journal for Research in Mathematics Education, 12*, 83-94.

Wagner, J. F. (2006). Transfer in Pieces. *Cognition and Instruction*, *24*(1), 1-71. doi: 10.1207/s1532690xci2401_1

Williams, S. R. (1993). Some common themes and uncommon directions. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 313-337). Hillsdale, NJ: Lawrence Erlbaum Associates.

Willis, G., DeMaio, T., & Harris-Kojetin, B. (1999). Is the bandwagon headed to the methodological promised land? Evaluation of the validity of cognitive interviewing techniques. In M. Sirken, D. Herrmann, S. Schechter, N. Schwarz, J. Tanur & R. Tourangeau (Eds.), *Cognition and survey research* (pp. 133-153). New York: Wiley.

Wilson, R. (1997). A decade of teaching 'reform calculus' has been a disaster, critics charge. *The Chronicle of Higher Education*, *43*(22), A12.

Yin, R. (2009). *Case study research: Design and methods* (5th ed.). Los Angeles: Sage.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 1169-1207). Charlotte, NC: Information Age Publishing.