THE 3N+1 PROBLEM: SCOPE, HISTORY, AND RESULTS

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The $3n + 1$ problem can be stated in terms of a function on the positive integers: $C(n) = n/2$ if n is even, and $C(n) = 3n + 1$ if n is odd. The problem examines the behavior of the iterations of this function; specifically it asks if the long term behavior of the iterations depends on the starting point or if every starting point eventually reaches the number one.

We discuss the history of this problem and focus on how diverse it is. An intriguing aspect of this problem is the vast number of areas of mathematics that can translate this number theoretic problem into the language of their discipline and the result is still a meaningful question which requires proof.

In addition to its history and the scope of the problem we discuss a probability theoretic approach which gives a model to predict how many iterations it will take to reach 1 for any given starting value. We also present some major results on this problem, one which demonstrates that "most" numbers eventually reach 1 and another which shows that any cycle that exists must be extremely large.

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PREFACE

I would like to thank my advisor, Jeff, for his help throughout this year. You have been an amazing influence and it has been a joy to work with you. You have always known the best time to send along a reprimand, motivation, or praise. You have helped me turn this year, as well as this document, into something we can both be truly proud of.

My parents have also been a huge helping hand throughout my whole academic career. Mom, Dad, you have always been in my corner to help with tough situations and provide lap time when needed. I'm coming back with my shield.

1.0 INTRODUCTION

1.1 EXAMPLE

In the set of all mathematics problems there is a special subset which contains very easy to state problems that are still very difficult to solve. These problems are quite interesting because it seems the prerequisites for understanding the statement of the problem are much lower than the prerequisites for working on the problem. As an example consider problem 34 from Section 8.1 on sequences, in Stewart's *Essential Calculus* [12]:

Problem 1. *Find the first 40 terms of the sequence defined by*

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd} \end{cases}$$

and $a_1 = 11$*. Do the same if* $a_1 = 25$*. Make a conjecture about this type of sequence.*

Solution. The first 40 iterates with $a_1 = 11$ are:

$$\{11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots, 4\}$$

The first 40 iterates with $a_1 = 25$ are:

$$\{25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 \ldots, 4\}$$

Based on our work so far we may be tempted to draw the conclusion that as long as our initial choice is odd our sequence eventually repeats at $\{\ldots, 4, 2, 1, 4, 2, 1, \ldots\}$.

Often the iteration is stated through a function:

$$C(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Though not stated as such in Stewart's calculus book this problem, i.e., proving any conclusions about the sequence, is still an open problem. The sequence, as its given, is known as the Collatz sequence, and is part of a conjecture by Collatz (among others):

Conjecture 2 (Collatz Conjecture)**.** *Starting from any positive integer n, iterations of the Collatz function will eventually reach the number 1. Thereafter iterations will cycle, taking successive values* $1, 4, 2, 1, \ldots$.

1.2 HISTORY

The $3n + 1$ problem is an open problem dealing with a sequence of numbers, whose terms are based on the starting value of the sequence.

The problem has many names including the Collatz Conjecture (named after Lothar Collatz), the Hasse Algorithm (after Helmut Hasse), Ulam's Conjecture (after Stanisław Ulam), the Syracuse Problem, Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), etc. .

The problem is also occasionally referenced as the Hailstone Numbers, due to the sudden rising and falling of the numbers in a sequence, similar to how hailstones are formed via repeated risings and fallings in clouds.

It is generally agreed that the problem was distributed in the 1950's [7]. The common story is that Lothar Collatz circulated the problem (among others of his creation) at the International Congress of Mathematicians in Cambridge, Massachusetts in 1950. Quite a number of people who are credited with work on this problem were in attendance, including:

Harold Scott MacDonald Coxeter, Shizuo Kakutani, and Stanisław Ulam[7]. The true origins – much like the truth of the conjecture itself – are not clear.

The interest in this problem extends past the area of Number Theory; including Computer Science, via algorithms to help compute and find patterns in our iteration, into Logic as decision problems, and Dynamical Systems, by examining our iteration as a dynamical system on $\mathbb{Z}$. The problem can also be viewed from a Probability Theory and Stochastic Processes standpoint by creating and analyzing heuristic algorithms.

Other notables associated with this problem include John Conway and Jeffrey Lagarias, who has written numerous papers on the topic and edited the first book on the problem[7].

Ulam is considered as one of the major projector's of this problem, distributing the problem to any who might solve it. In fact a quote from Paul Stein, a collaborator of Ulam's, Paul said:

> ...it was [Ulam's] particular pleasure to pose difficult, though simply stated, questions in many branches of mathematics. Number theory is a field particularly vunerable to the "Ulam Treatment", and [Ulam] proposed more than his share of hard questions; not being a professional in the field, he was under no obligation to answer them.[11]

1.3 SUPPORTING EVIDENCE

The natural question, since we have no solution yet, would be "should we suspect this conjecture is true?". Why should one believe that this conjecture is true?

For starters there is an abundance of evidence that an n that does not eventually iterate to 1 does not exist. It has been computationally verified for all starting values $n < 5 \times 2^{60} \approx 5.7646 \times 10^{18}$ [9] the Collatz iteration eventually reaches 1. Of course this does not suffice as a proof, we need only look to the Pólya Conjecture (below) to see that having a large number of confirmed cases does not prove a conjecture true; the first proposed counter-example was 1.845×10^{361} (though a smaller counter example of 906,150,257 now exists).

Conjecture 3 (Pólya's Conjecture)**.** *For any $n > 1$, partition the positive natural numbers less than or equal to n into two sets A and B where A consists of those with an odd number of* *prime factors, and B consists of those with an even number of prime factors, then* $|A| \geq |B|$.

Further, fantastic mathematicians have worked on this problem including Conway and Tao. Even they have been unable to draw a definitive conclusion to this problem. Paul Erdős has been quoted numerous times as having said "Mathematics is not yet ready for such problems".

Following this, why should we not just abandon this problem? Because this problem is still a *good* one. Legarias uses the Hilbert criteria for a good problem and concludes:

- The $3n + 1$ problem is a clear, simply stated problem;
- The $3n + 1$ problem is a difficult problem;
- The $3n + 1$ problem initially seems accessible.

1.4 DIFFICULTY

If the difficulty of a problem were proportional to the sophistication of its statement then this should not be a difficult problem to solve; indeed a background in Calculus is a bit of overkill for the statement of this problem. Alas, this is not the case. Why then is this, as of yet, unsolved? In [7] Lagarias credits the difficulty of this problem to "pseudorandomness", in the sense that from a given randomly selected starting point predicting the parity of the nth iteration is a "coin flip random variable". Legarias also attributes the difficulty of this problem to non-computability, referencing a result of John Conway[1] which relates a generalized version of the Collatz function to unsolvability.

The difficulty of this problem is tangentially related to the difficulty of factoring integers. Given a prime factorization of the integer n, this factorization does not lend itself to the factorization of $n+1$, other than the parity. This relates the the $3n+1$ problem, due to the iteration; if we know n's factorization and it is even, $\frac{n}{2}$ changes the factorization very little. If n is odd $3n$ changes the factorization very little, but adding 1 to arrive at $3n + 1$ could change the factorization immensely. Thus after an iteration of our function C, we may have no clue as to the type of number we have, other than if n was odd $C(n)$ is even.

Since it is known that an even will follow an odd after an iteration of C we use a new function $T(n)$ that divides $C(n)$ by 2 when n is odd, essentially calling C twice (in this instance only).

$$T(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Compounded with not knowing the factorization of iterations of a starting number n, we have that our iteration grows by a factor of $\frac{3}{2}$ for each odd iterate and shrinks by a factor of $\frac{1}{2}$ for each even integer. Thus if the sequence, under the function T, continually has odd iterates the sequence would be growing rapidly. Likewise, if we have many even iterates the sequence would shrink quickly. Even less helpful, if the sequence goes between even and odd iterates then our sequence could grow and shrink many times, hence the name "Hailstone Numbers".

2.0 DEFINITIONS AND NECESSARY MATHEMATICS

This chapter is included to be a one-stop-shop for the main definitions and concepts necessary for later work in this paper. While not all terms are used later, they are terms used commonly in the literature and worth knowing. We also include some background information on some of the more complicated material, put here as a quick reference which can be skipped or focused on as necessary. Definitions which are specific to results are introduced in the respective sections.

2.1 BASIC DEFINITIONS

Definition 4 (Half or Triple plus one Process)**.** The process of dividing even integers by 2 or muliplying odd integers by 3 and adding 1 is called the **Half Or Triple Plus One** (HOTPO) process.

Definition 5 (Oneness)**.** The property that a number evenutally reaches 1 under the HOTPO process is called **oneness**. e.g., 4 has **oneness** since it eventually reaches 1 under the HOTPO process.

Definition 6 (Collatz function)**.** We refer to the function T as the **Collatz function**:

$$T(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Definition 7 (Collatz sequence)**.** We refer to any sequence $\{a_n\}$ whose terms are determined by the Collatz function as a **Collatz sequence**.

Definition 8 (trajectory)**.** The **trajectory** of a number, n, is the Colltaz sequence beginning at n.

Definition 9 (Stopping time)**.** The least positive k for which $T^{(k)}(n) < n$ is called the **stopping time**, $\sigma(n)$ of n. Or $\sigma(n) = \infty$ if no k occurs with $T^{(k)}(n) < n$.

Example 10. Examine the trajectory of 15:

$$(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, \ldots)$$

Thus $\sigma(15) = 7$, since $T^{(7)}(15) = 10$ is the first time our iteration is below 15

Definition 11 (Convergence)**.** We say that a Collatz sequence has **stopped** or **converged** if it reaches the number 1. Thus we stop considering values after the first instance of 1 in the sequence. That is we now write

$$(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, \ldots)$$

simply as

$$(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1)$$

Definition 12 (Total stopping time)**.** The least positive k for which $T^{(k)}(n) = 1$ is called the **total stopping time**, $\sigma_\infty(n)$ of n, or $\sigma_\infty(n) = \infty$ if no k occurs with $T^{(k)}(n) = 1$.

Example 13. With the same iteration:

$$(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1)$$

$\sigma_\infty(n) = 12$ since $T^{(12)}(15) = 1$ is the first time our iteration reaches 1.

2.2 FORMAL STATEMENT

We can now formally state our problem in terms of our new definitions:

Conjecture 14 ($3n + 1$ Conjecture)**.** *Every integer $n \geq 2$ has a finite total stopping time.*

In fact it is enough to show (by induction) that every integer has a finite stopping time.

2.3 P-ADIC NUMBERS

The use a p-adic numbers occurs occasionally when examining the Collatz conjecture, hence we define the system here.

Let p be a prime number, and define the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ as $|0|_p = 0$ and for every other rational write $r = p^k ab^{-1}$ where $a, b \in \mathbb{Z}$ with $(a, p) = (b, p) = 1$ then $|r|_p = p^{-k}$. This can be thought of as factoring out all powers of p from a fraction. This function is called the **p-adic valuation**, or p-adic absolute value, on $\mathbb{Q}$.

Example 15. Let us use $p = 2$. Then we can compute the 2-adic absolute value on rational numbers:

1. $|2|_p = \frac{1}{2}$.
2. $\left|\frac{1}{2}\right|_p = |2^{-1}|_p = 2$.
3. $\left|\frac{2}{3}\right|_p = \left|2 \cdot \frac{1}{3}\right|_p = \frac{1}{2}$.
4. $\left|\frac{1}{7}\right|_p = \left|2^0 \cdot \frac{1}{7}\right|_p = 1$.

The p-adic valuation measures how many powers of p are in the number. The p-adic valuation has the following properties:

1. $|x|_p \geq 0$ with equality if and only if $x = 0$,
2. $|x + y|_p \leq \max(|x|_p, |y|_p)$,
3. $|xy|_p = |x|_p |y|_p$,
4. $|-1|_p = |1|_p = 1$, and
5. $|-x|_p = |x|_p$.

Property 2 has equality if and only if $|x|_p \neq |y|_p$, when $|x|_p = |y|_p$ we have $x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and bd has no additional factors of p if b and d are relatively prime to p; thus the p-adic absolute value of the sum can be no more than the max of the p-adic absolute values. Property 3 is obvious, if a, b, c, and d share no common factors with p, then neither does ac or bd. The rest of the properties follow from the above. Property 2 is called the **ultrametric inequality** and implies $|x|_p + |y|_p \leq |x + y|_p$ (the triangle inequality).

The p-adic absolute value function gives a metric on $\mathbb{Q}$, for each prime p, as $d_p(x, y) = |x - y|_p$. It can be seen that $d_p(x, y)$ satisfies the conditions of a metric, for all prime p: Property 1 above gives $d_p(x,y) \geq 0$ and equal to 0 only when $x = y$, Property 5 gives $d_p(x,y) = d_p(y,x)$ and as mentioned above Property 2 gives the triangle inequality $d_p(x,z) \leq d_p(x,y)+d_p(y,z)$.

All of this gives that the fuction $d_p(x,y) = |x-y|_p$ forms a metric on $\mathbb{Q} \times \mathbb{Q}$, called the **$p$-adic metric** on $\mathbb{Q}$.

Equipped with the p-adic metric we may consider many of the notions we do with a distance function: specifically we can consider convergent and divergent sequences of rational numbers. In fact, under the p-adic metric the sequences which converge are drastically different than convergent sequences under the standard Euclidean Metric $d(x,y) = |x-y|$.

Example 16. Consider the sequence $\{2, 4, 8, \ldots, 2^n, \ldots\}$. Note that $d_2(2^n, 0) = |2^n|_2 = \frac{1}{2^n}$. Hence under the 2-adic metric we see that this sequence converges to 0.

Example 17. However the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \ldots \frac{1}{2^n}, \ldots\right\}$ does not converge under the 2-adic metric. This can be seen by recognizing that the sequence is not even Cauchy: Assume $n > m$

$$\begin{aligned}
\left|\frac{1}{2^n} - \frac{1}{2^m}\right|_2 &= \left|\frac{1}{2^n} - \frac{2^{n-m}}{2^n}\right|_2 \\
&= \left|2^{-n}(2^{n-m} - 1)\right|_2 \\
&= 2^n
\end{aligned}$$

Thus there is an $\epsilon > 0$ (say $\epsilon = 1$) such that no N exists with all $n, m \geq N$ has $\left|\frac{1}{2^n} - \frac{1}{2^m}\right|_2 < \epsilon$

It can be seen that the rationals under the p-adic metric are not complete, that is, there are Cauchy sequences which do not converge. In a similar sense to how the real numbers, $\mathbb{R}$, are the completion of $\mathbb{Q}$ under the standard metric, the completion of $\mathbb{Q}$ under a p-adic metric are called the p-adic numbers, $\mathbb{Q}_p$.

The p-adic numbers are briefly discussed in Section 5.1.

2.4 CONTINUED FRACTIONS

We introduce a method of approximating rational and real numbers with fractions. It is a standard result in analysis that the rational numbers are dense in the real numbers, and as such we can construct sequences of rational numbers to approximate real numbers. However for concreteness we can also explicitly construct rational numbers which can closely approximate real numbers. One method to do this is continued fractions.

To begin, we describe how to represent rational numbers as continued fractions. We do this with the Euclidean algorithm for, say, 56 and 17:

$$\begin{aligned}
56 &= 3 \cdot 17 + 5 \\
17 &= 3 \cdot 5 + 2 \\
5 &= 2 \cdot 2 + 1 \\
2 &= 2 \cdot 1 + 0
\end{aligned}$$

Now dividing each row by the next row gives the following expressions, which we write in a specific form:

$$\begin{aligned}
\frac{56}{17} &= 3 + \frac{5}{17} = 3 + \frac{1}{\frac{17}{5}} \\
\frac{17}{5} &= 3 + \frac{2}{5} = 3 + \frac{1}{\frac{5}{2}} \\
\frac{5}{2} &= 2 + \frac{1}{2}
\end{aligned}$$

Putting everything we have together as one *continued* fraction:

$$\begin{aligned}
\frac{56}{17} &= 3 + \cfrac{1}{\cfrac{17}{5}} \\
&= 3 + \cfrac{1}{3 + \cfrac{1}{\frac{5}{2}}} \\
&= 3 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}}
\end{aligned}$$

Using this we can define the following:

Definition 18 (Simple finite continued fraction)**.** A **simple finite continued fraction** is an expression of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

where $a_i \in \mathbb{Z}$. Since the above notation is very space consuming we adopt the standard notation of $[a_0; a_1; \ldots; a_n]$.

Simple finite continued fractions are useful, because via the method described above we have that every simple continued fraction corresponds uniquely to a rational number, as well as every rational number can be written as a simple finite continued fraction. See [10] for proofs of these claims.

However more interestingly, if we extend to infinite continued fractions, that is using an infinite sequences of integers $[a_0; a_1; a_2; \ldots]$ we can approximate any real number [10].

The simple continued fraction for a real number can be computed as follows. Let α be the number we try to approximate. For the sake of the algorithm set $\alpha_0 = \alpha$. Then we define the sequence $[a_0; a_1; \ldots]$ as $a_i = [\alpha_i]$ and $\alpha_{i+1} = 1/(\alpha_i - a_i)$ where $[n]$ represents the greatest integer less than n. We showcase this by writing the continued fraction of $\sqrt{18}$:

Example 19. Here $\alpha = \sqrt{18}$. So then we set $a_0 = \left[\sqrt{18}\right] = 4$ and then compute $\alpha_1 = \dfrac{1}{\sqrt{18}-4}$ and continue on. The results are summarized:

$$\begin{aligned}
a_0 &= \left[\sqrt{18}\right] = 4, & \alpha_1 &= \frac{1}{\sqrt{18}-4} = \frac{\sqrt{18}+4}{2}, \\
a_1 &= \left[\frac{\sqrt{18}+4}{2}\right] = 4, & \alpha_2 &= \frac{1}{\left(\frac{\sqrt{18}+4}{2}\right)-4} = \sqrt{18}+4, \\
a_2 &= \left[\sqrt{18}+4\right] = 8, & \alpha_3 &= \frac{1}{\left(\sqrt{18}+4\right)-8} = \frac{\sqrt{18}+4}{2} \ (=\alpha_1)
\end{aligned}$$

Since we see that $\alpha_3 = \alpha_1$ we get a repeating pattern and we can represent $\sqrt{18} = [4; 4; 8; 4; 8; 4; 8; \ldots]$

An interesting note, for those interested is that the continued fraction expansion for the golden ratio is: $\phi = [1; 1; 1; \ldots]$.

Definition 20 (Convergents)**.** When we are working with infinite continued fractions we adopt the notation that $C_k = [a_0; a_1; \ldots; a_k]$. As stated above this is a rational number, if we wish to know the exact numerator and denominator we identify them as: $\frac{p_k}{q_k} = [a_0; a_1; \ldots; a_k]$. The C_k are referred to as the **convergents**, or occasionally as **truncated continued fractions**.

A standard result in algebra on continued fractions is the following:

Lemma 21. *If C_k is a sequence of convergents to α then $C_{2m} < \alpha < C_{2m+1}$ for every m. That is the even indexed convergents will always under approximate α while the odd indexed convergents will over approximate α.*

As mentioned it is often desired to know the exact rational of our truncated continued fraction approximation. We can compute these using a recurrence relation, that is $\frac{p_k}{q_k} =$

$[a_0; a_1; a_2; \ldots; a_k]$ is given by the recurrence relations:

$$p_k = a_k p_{k-1} + p_{k-2} \tag{2.1}$$

$$q_k = a_k q_{k-1} + q_{k-2} \ \ (k \geq 0) \tag{2.2}$$

with the initial values $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$ [8]. These convergents have a useful property. First:

Definition 22 (Farey pair)**.** Two fractions (any fractions, not necessarily convergents) $\frac{p}{q}$ and $\frac{p'}{q'}$ with p, q, p', q' non-negative integers and in reduced form are a **Farey pair** if $pq' - p'q = \pm 1$.

Example 23. The following pairs form Farey pairs:

$$\frac{1}{4}, \frac{1}{3}$$
$$\frac{1}{3}, \frac{1}{2}$$
$$\frac{1}{2}, \frac{2}{3}$$
$$\frac{2}{3}, \frac{3}{4}$$

These pairs form the Farey Series (of order 4), proper fractions, namely $\frac{h}{k}$ with $\gcd(h, k) = 1$ with $k \leq 4$ [10].

With this definition we can demonstrate that each pair of consecutive convergents forms a Farey pair:

Theorem 24. *If $\{p_i\}$ and $\{q_i\}$ are defined as in (2.1) and (2.2) then*

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1} \quad (i \geq -1)$$

Proof. We prove this by induction on i. We can computationally show this result holds for $i = -1, 0$:

$$\begin{aligned} p_{-1}q_{-2} - p_{-2}q_{-1} &= 1(1) - 0(0) \\ &= 1 = (-1)^{-2} \\ p_0q_{-1} - p_{-1}q_0 &= a_0(0) - 1(1) \\ &= -1 = (-1)^{-1} \end{aligned}$$

Now we assume that the result holds for $i = k - 1$ and we show it holds for $i = k$, that is we show:

$$p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1}$$

what we know from their definition is:

$$p_k = a_kp_{k-1} + p_{k-2} \qquad q_k = a_kq_{k-1} + q_{k-2}$$

substituting into the left hand side of the desired relation:

$$\begin{aligned} p_kq_{k-1} - p_{k-1}q_k &= (a_kp_{k-1} + p_{k-2})q_{k-1} - p_{k-1}(a_kq_{k-1} + q_{k-2}) \\ &= a_kp_{k-1}q_{k-1} + p_{k-2}q_{k-1} - a_kp_{k-1}q_{k-1} - p_{k-1}q_{k-2} \\ &= p_{k-2}q_{k-1} - p_{k-1}q_{k-2} \\ &= (-1)(p_{k-1}q_{k-2} - p_{k-2}q_{k-1}) \end{aligned}$$

Since the Theorem holds for $i = k - 1$ we can replace the second term:

$$\begin{aligned} &= (-1)(-1)^{k-2} \\ &= (-1)^{k-1} \end{aligned}$$

Which proves the result. □

A helpful property of Farley pairs is if any fraction is in the middle of the pair it must have a larger denominator, this can be stated explicitly as:

Lemma 25. *Let $\frac{p}{q} < \frac{p'}{q'}$ form a Farley pair. Then any intermediate fraction with $\frac{p}{q} < \frac{x}{y} < \frac{p'}{q'}$ which has $y > 0$ is of the form:*

$$\frac{x}{y} = \frac{ap + bp'}{aq + bq'}$$

with a, b positive integers. In particular $x \geq p + p'$ and $y \geq q + q'$.

Proof. The proof is simple if our pair is $\frac{0}{1}$ and $\frac{1}{1}$ so we restrict to the case where neither numerator is zero. We now examine the matrix:

$$F = \begin{pmatrix} p' - p & p \\ q' - q & q \end{pmatrix}$$

notice that $\det F = p'q - pq' = \pm 1$ by assumption thus F is invertible, with inverse:

$$F^{-1} = \begin{pmatrix} q & -p \\ q - q' & p' - p \end{pmatrix}$$

We can consider F a linear map from $\mathbb{Z}^2 \to \mathbb{Z}^2$, by restricting the input we can think of this as a map from $\mathbb{Q} \to \mathbb{Q}$ and we can use this to introduce a bijection: $f : [0, 1] \cap \mathbb{Q} \to \left[\frac{p}{q}, \frac{p'}{q'}\right] \cap \mathbb{Q}$ defined as

$$f\left(\frac{u}{v}\right) = \frac{(p' - p)u + pv}{(q' - q)u + qv}$$

whose inverse is:

$$f^{-1}\left(\frac{u}{v}\right) = \frac{qu - pv}{(q - q')u - (p - p')v}$$

Back to our original problem if we have that $\frac{p}{q} < \frac{x}{y} < \frac{p'}{q'}$ we define $a = p'y - q'x$ and $b = qx - py$, then a and b are positive integers as seen by:

$$\frac{x}{y} < \frac{p'}{q'} \implies 0 < p'y - q'x$$
$$\frac{p}{q} < \frac{x}{y} \implies 0 < qx - py$$

We use a and b because it reduces the inverse of f to:

$$f^{-1}\left(\frac{x}{y}\right) = \frac{qx - py}{(q - q')x - (p - p')y} = \frac{b}{a + b}$$

Now by construction we have:

$$\begin{aligned}\frac{x}{y} &= ff^{-1}\left(\frac{x}{y}\right)\\ &= f\left(\frac{b}{a+b}\right)\\ &= \frac{ap+bp'}{aq+bq'}\end{aligned}$$

Which proves the claim, in particular since $a, b > 0$ we have that $x \geq p + p'$ and $y \geq q + q'$.

□

A summarization of our current standing with continued fractions is if $\frac{p_n}{q_n}$ is the nth truncated continued fraction $[a_0; a_q; \ldots; a_n]$ approximating the real number θ in (reduced) rational form then we have:

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1} \tag{2.3}$$

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} \cdots < \theta < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1} \tag{2.4}$$

We now focus in on the **upper convergents** namely the convergents with odd indices. While consecutive convergents form a Farey pair, in general two consecutive upper convergents do not. We now create a type of "intermediate" upper convergent such that this chain always forms a Farey pair. We define the intermediate terms as:

$$p_{n,i} = p_n + ip_{n+1} \qquad q_{n,i} = q_n + iq_{n+1}$$

where $n \geq -2$ and i is a non-negative integer. In particular $p_{n,0} = p_n$ and from our relation in equation (2.1) we have $p_{n+2} = p_n + a_{n+2} p_{n+1}$ so we restrict $i \leq a_{n+2}$. Under the above notation we will call a fraction

$$\frac{p_{n,i}}{q_{n,i}} = \frac{p_n + ip_{n+1}}{q_n + iq_{n+1}}$$

an **intermediate convergent** (to θ), and if n is odd we call it an upper intermediate convergent (to θ). Now we can state and prove some helpful results:

Lemma 26. *For $a, b, c, d \in \mathbb{Z}$ if $ad - bc > 0$ then we have that:*

$$\frac{a + (i+1)c}{b + (i+1)d} < \frac{a + ic}{b + id}$$

for any $i \in \mathbb{N}$.

Proof. We work algebraically:

$$0 < ad - bc \iff cb < ad$$

Then for any $i \in \mathbb{N}$:

$$\begin{aligned}
&\iff iad + (i+1)cb < (i+1)ad + icb \\
&\iff ab + iad + (i+1)cb + i(i+1)cd < ab + (i+1)ad + icb + i(i+1)cd \\
&\iff (a + (i+1)c)(b + id) < (a + ic)(b + (i+1)d) \\
&\iff \frac{a + (i+1)c}{b + (i+1)d} < \frac{a + ic}{b + id}
\end{aligned}$$

□

Theorem 27. *Under the above notation, if n is odd and we use a for a_{n+2} (to simplify notation) then:*

$$\frac{p_{n+2}}{q_{n+2}} = \frac{p_{n,a}}{q_{n,a}} < \frac{p_{n,a-1}}{q_{n,a-1}} < \cdots < \frac{p_{n,1}}{q_{n,1}} < \frac{p_n}{q_n}$$

and any two consecutive (upper) intermediate convergents form a Farey pair.

Proof. We can easily see that $p_{n+2}/q_{n+2} < p_n/q_n$ from Equation (2.4), we need to show that the (upper) intermediate convergents are decreasing. Notice that the inequalities we need to show fall exactly in to the case for Lemma 26 since we have that $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1} = 1$ by Equation (2.3) and n being odd. Which means in the context of Lemma 26 we can let $a = p_n$, $b = q_n$, $c = p_{n+1}$ and $d = q_{n+1}$ and we achieve the desired inequalities.

We now show that consecutive intermediate convergents form a Farey pair, that is $p_{n,i}q_{n,i+1} - p_{n,i+1}q_{n,i} = \pm 1$:

$$\begin{aligned}
p_{n,i}q_{n,i+1} - p_{n,i+1}q_{n,i} =& (p_n + ip_{n+1})(q_n + (i+1)q_{n+1}) - (p_n + (i+1)p_{n+1})(q_n + iq_{n+1}) \\
=& p_nq_n + (i+1)p_nq_{n+1} + ip_{n+1}q_n + i(i+1)p_{n+1}q_{n+1} - p_nq_n - ip_nq_{n+1} \\
& - (i+1)p_{n+1}q_n - i(i+1)p_{n+1}q_{n+1} \\
=& p_nq_{n+1} - p_{n+1}q_n \\
=& (-1)^{n+1}
\end{aligned}$$

□

Now we look at another (the main) useful component of the $p_{n,i}$, $q_{n,i}$. We will show that if we are approximating any irrational θ and we have a fraction $\theta < \frac{k}{l} < \lfloor\theta\rfloor + 1$ ($\lfloor\cdot\rfloor$ is the round down function, often called the floor function) where either k or l is minimal fitting in that region (meaning, choose the rational in the region so that every other fraction has a larger numerator (or denominator)) then $k = p_{n,i}$ and $l = q_{n,i}$.

Theorem 28. *Let $\theta > 0$ be an irrational number and let θ' be any number with $\theta < \theta' < \lfloor\theta\rfloor + 1$. If k, and l are positive integers so that*

$$\theta < \frac{k}{l} < \theta'$$

and if either k or l is minimal with this property then $\frac{k}{l}$ is an upper intermediate convergent to θ. That is $k = p_{n,i}$ and $l = q_{n,i}$ for some odd positive integer n and some integer $i = 0, 1, \ldots, a_{n+2} - 1$.

Proof. Since $\theta < \frac{k}{l}$ if k and l do not form an upper intermediate convergent to θ then it must lie between some two consecutive intermediate convergents to θ, say

$$\theta < \frac{p}{q} < \frac{k}{l} < \frac{p'}{q'}$$

But by Theorem 27, $\frac{p}{q}$ and $\frac{p'}{q'}$ form a Farey pair. And thus by Lemma 25 we have that $k \geq p + p'$ and $l \geq q + q'$ so in particular $k > p$ and $l > q$. This contradicts the minimality of k (or of l), thus $\frac{k}{l}$ is an upper intermediate convergent; that is $k = p_{n,i}$ and $l = q_{n,i}$ for n odd and $i = 0, 1, \ldots a_{n+2} - 1$. □

This concludes the results on continued fractions that will be used in this paper. While the proofs are necessary to justify the claims the results are what are important later in this paper. In particular we will make use of the notion of Farey pairs and and intermediate convergents. Lemma 25 and Theorem 28 are the results most important in later sections.

3.0 HEURISTIC PROOF

There are many reasons that lead one to suspect the Collatz problem should be true. One such justification is the existence of some heuristic proofs. As an example let us compute the expected growth between consecutive odd iterations in a sequence. We assume that the function T "mixes" evens and odds well enough that whether the output of T is even or odd happens with equal probability, i.e.,

Assumption 29. *For any $n \in \mathbb{N}$, selected at random, the probability that n is even is $\frac{1}{2}$ which is also the probability that n is odd.*

Assumption 30. *The function $T(n)$ "mixes" evens and odds equally, that is for any $n \in \mathbb{N}$, selected at random, the probability that $T(n)$ is even $= \frac{1}{2} =$ probability that $T(n)$ is odd.*

3.1 HEURISTIC ARGUMENT

Under these assumptions we begin a heuristic proof that every number should converge to 1 under the Collatz function. Choose an odd integer n_0, we iterate T until we arrive at the next odd integer n_1. Thus after one iteration $\frac{3n_0+1}{2}$ is even with probability $\frac{1}{2}$ and odd with a probability $\frac{1}{2}$ etc.

This gives that the next odd number in our iteration is $\frac{3n_0+1}{2}$ with probability $\frac{1}{2}$, $\frac{3n_0+1}{4}$ with probability $\frac{1}{4}$, $\frac{3n_0+1}{8}$ with probability $\frac{1}{8}$ etc. This is summarized in Figure 1. Then we can compute the expected growth factor between successive odd iterates in our sequence as

$$\left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{3}{4}\right)^{\frac{1}{4}} \left(\frac{3}{8}\right)^{\frac{1}{8}} \left(\frac{3}{16}\right)^{\frac{1}{16}} \left(\frac{3}{32}\right)^{\frac{1}{32}} \cdots$$

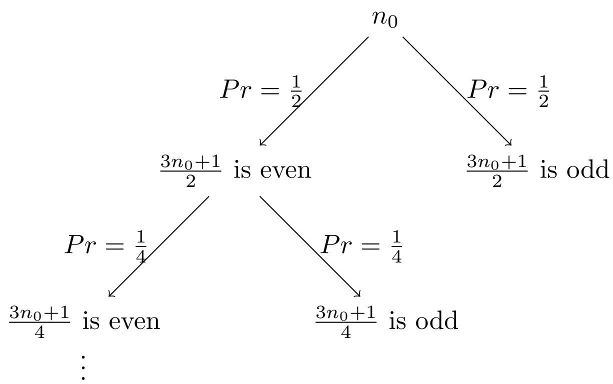

Figure 1: Probability tree

Lemma 31.
$$\prod_{i=1}^{\infty}\left(\frac{3}{2^i}\right)^{\frac{1}{2^i}} = \frac{3}{4}$$

Proof. We show $\prod_{i=1}^{\infty} 3^{\frac{1}{2^i}}$ and $\prod_{i=1}^{\infty}\left(\frac{1}{2^i}\right)^{\frac{1}{2^i}}$ both converge, and thus

$$\prod_{i=1}^{\infty}\left(\frac{3}{2^i}\right)^{\frac{1}{2^i}} = \prod_{i=1}^{\infty} 3^{\frac{1}{2^i}} \cdot \prod_{i=1}^{\infty}\left(\frac{1}{2^i}\right)^{\frac{1}{2^i}}$$

Note:

$$\begin{aligned}\prod_{i=1}^{\infty} 3^{\frac{1}{2^i}} &= 3^{\sum_{i=1}^{\infty}\frac{1}{2^i}} \\ &= 3\end{aligned}$$

For $\prod_{i=1}^{\infty}\left(\frac{1}{2^i}\right)^{\frac{1}{2^i}}$ we examine partial products, and after simplification we see:

$$P_n = \left(\frac{1}{2}\right)^{\sum_{i=1}^{n}\frac{i}{2^i}}$$

Thus

$$\prod_{i=1}^{\infty}\left(\frac{1}{2^i}\right)^{\frac{1}{2^i}} = \lim_{n\to\infty} P_n$$
$$= \frac{1}{4}$$

So our entire product is:

$$\prod_{i=1}^{\infty}\left(\frac{3}{2^i}\right)^{\frac{1}{2^i}} = 3\cdot\frac{1}{4} = \frac{3}{4}$$

$\square$

The significance is that this infinite product represents the expected growth between successive odd iterates of a Collatz sequence i.e., we expect successive odd iterates to shrink by a factor of $\frac{3}{4}$. In particular, divergent trajectories should not exist.

We can use this idea to give us an approximation on the total stopping time of a number n. If $\sigma_\infty(n) = k$ we recall we are under the assumption that $T(n)$ mixes odds and evens equally, so we should have as many evens as odds, thus in our sequence of k numbers (starting with n and ending with 1) we should have about $\frac{k}{2}$ evens and $\frac{k}{2}$ odds. Thus following our above calculation starting with n and ending at 1 should have us decrease by a factor of $\frac{3}{4}$ each odd iterate. That is:

$$\left(\frac{3}{4}\right)^{\frac{k}{2}} n = 1$$
$$\frac{k}{2}\log\left(\frac{3}{4}\right) = \log\left(\frac{1}{n}\right)$$
$$k = -\frac{2\log(n)}{\log\left(\frac{3}{4}\right)}$$

So the total stopping time for a number n should roughly be a constant multiple of $\log n$. From this approximation on the total stopping time we can create a linear model of where we expect our sequence to be after a given number of iterations. We will plot the iterations and our model on a semilog plot (x-axis is normal but we take take the natural log of the y-axis). We can create our linear model by letting the x-axis represent the number of iterations of our Collatz function, and the y-axis represent the natural log of the xth iterate of the Collatz function. Examining two points we know will agree with the value of Collatz sequence: we know the first entry of our sequence is n, so if we take the natural log and place it on the graph of our linear function we get the point $(0, \log n)$. If k (above) is the total stopping time of the Collatz sequence that means that after k iterations of T we arrive at 1 in our sequence. Taking the natural log of the y-coordinate we get the point $\left(\frac{-2\log n}{\log \frac{3}{4}}, 0\right)$, the slope between these points is:

$$
\begin{aligned}
m &= \frac{\log(n) - 0}{0 - -\frac{2\log n}{\log \frac{3}{4}}} \\
&= \frac{\log n}{\frac{2\log n}{\log \frac{3}{4}}} \\
&= \frac{\log \frac{3}{4}}{2}
\end{aligned}
$$

Using this slope we can look at how well this linear model approximates the Collatz iteration:

As shown in Figure 2 we can see that the model follows our sequence well. The overall slope is the same, and the model fits with the plot. However Figure 3 shows a different picture. The model appears to have the same general slope as the sequence, but there is a jump in the beginning that the model does not notice. Figure 5 also shows how the model does not follow the Collatz sequence well in all cases. Even the overall shape of the iteration plot is different than the slope of the model.

Examining total stopping times in Figure 3.1 we can see that the total stopping time $n = 2^{30} - 1$ is approximated well by the model. It takes 122 iterations of the Collatz function $T(n)$ to reach 1 and the model predicts 145 iterations. However looking at $n = 2^{30} + 1$ we get a much worse approximation. It takes $T(n)$ 288 iterations to reach 1, but the model still predicts 145. **This example characterizes the difficulty of the Collatz problem well: the function $T(n)$ is not well behaved on neighboring starting values.** Having looked at a sequence from one value of n it is easy to convince ourselves of this by watching the falling and rising of the iterations, but more than that, the behavior from two different starting values are seemingly unrelated. **This relates back to the problem of integer factorization. Knowing how T behaves on n does not lend insight to how T behaves on $n + 1$.**

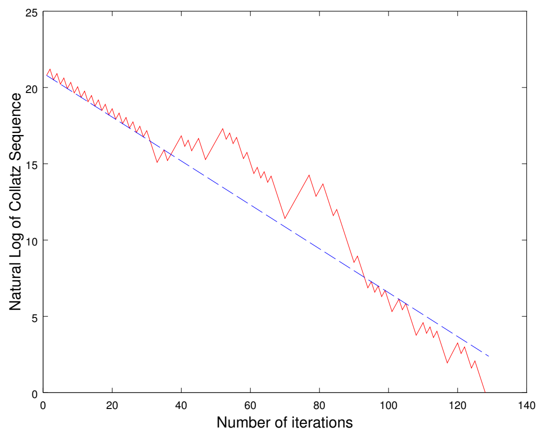

Figure 2: $n = 2^{30} + 1$

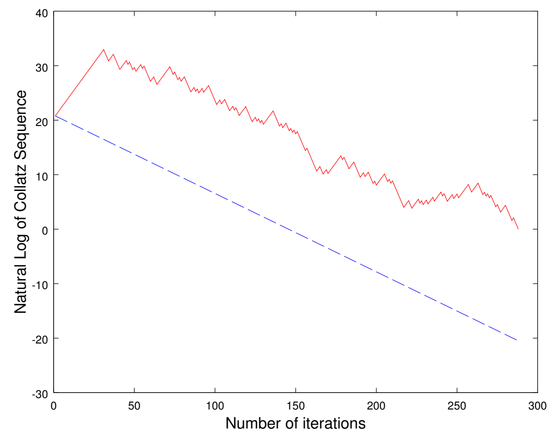

Figure 3: $n = 2^{30} - 1$

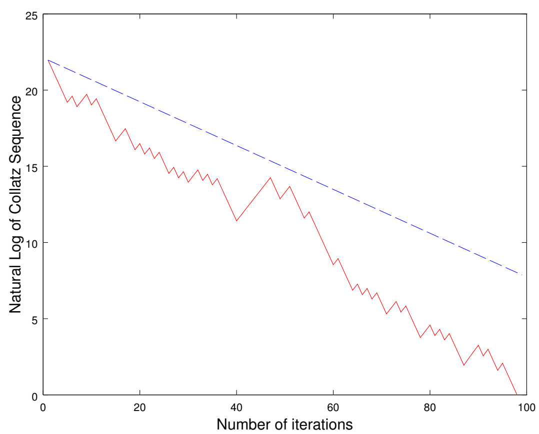

Figure 4: $n = 3^{20} - 1$

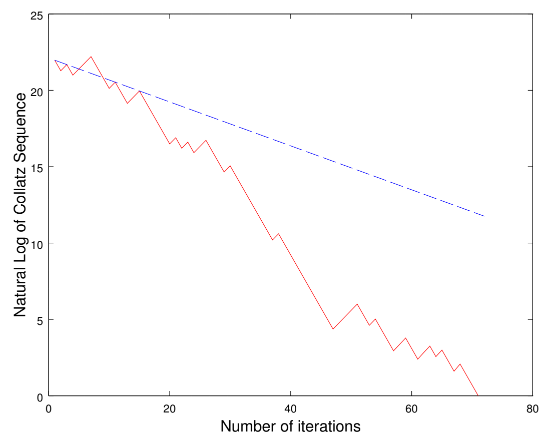

Figure 5: $n = 3^{20} + 1$

n	$\sigma_\infty(n)$	k
$2^{30}-1$	122	145
$2^{30}+1$	288	145
$3^{20}-1$	98	153
$3^{20}+1$	71	153

Figure 6: Table of total stopping times vs. model approximations

4.0 PARTIAL RESULTS

Though the conjecture as a whole does not yet have a proof, there are many partial results that have been shown. We cover some of these results now; in particular, we examine a more rigorous explanation of why we expect almost every positive integer to have a finite stopping time.

4.1 A DENSITY PROOF

First we define a few terms:

Definition 32 (Parity sequence)**.** Given a positive integer starting value, n, we can assign a **parity sequence**

$$n \to \{x_0, x_1, x_2, \ldots\}$$

where

$$x_m = \begin{cases} 0 & \text{if } T^m(n) \text{ is even} \\ 1 & \text{if } T^m(n) \text{ is odd} \end{cases}$$

Example 33. The parity sequence of 7 is:

$$\{1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, \ldots\}$$

Where $\{1, 0, 1, 0, \ldots\}$ corresponds to our sequence having reached one.

We show, following Everett's paper [3], that we can think of starting values instead by the parity sequence it creates. That is we will show that given a starting value $n \leq 2^N$, for some positive integer N, this number corresponds (in a one-to-one fashion) to a sequence $\{x_0, x_1, \ldots, x_{N-1}\}$ with $x_i \in \{0, 1\}$.

Of course there is the usual method of representing integers as a sequence of 0s and 1s, in binary. But here we instead want to represent them as parity sequences, where $x_n = T^n(m) \mod 2$. In fact what we show is that every finite binary sequence (of length N) corresponds to the first N terms of a parity sequence for a unique integer less than 2^N. Given this result we have an interesting consequence for infinite binary sequences: the following theorem gives that every *finite* binary sequence of length N corresponds to the first N terms of a parity sequence for a unique integer less than 2^N but if the Collatz Conjecture is true, then every parity sequence must end in $\{\ldots, 1, 0, 1, 0, 1, 0, \ldots\}$ since this is the parity sequence for the number 1. This means that not every infinite binary sequence is an integer's parity sequence, in particular sequences that end with any pattern other than $\{\ldots, 1, 0, 1, 0, 1, 0, \ldots\}$ cannot be a parity sequence for an integer since its Collatz sequence would not end in 1.

The fact that we can represent every integer $n \leq 2^N$ as a unique sequence like this requires proof. Another way of stating our immediate goal is: if we take all positive integers $n \leq 2^N$ and list their parity sequences no two sequences will have the exact same first N terms. Here all we are referencing is the first N terms of the parity sequence, we have no clue (or at this point interest) in how long our parity sequence is. We state the result rigorously as:

Theorem 34. *There is a one-to-one correspondence between integers less than 2^N and the first N terms of parity sequences. That is the integers $0, \ldots, 2^{N-1}$ have that the first N terms of their parity sequences are distinct.*

Before we prove this recall that a parity sequence of an integer, m, is a binary sequence, $\{x_0, x_1, \ldots\}$, such that $x_i = T^{(i)}(m) \mod 2$. Thus the first term of the sequence lists the parity of m. Further this correspondence gives that with a sequence $\{x_0, x_1, \ldots\}$ we get an integer m such that $m \mod 2 = x_0$ and in general $T^{(i)}(m) \mod 2 = x_i$.

Proof. We show that for any sequence of length N we can find a unique integer modulo 2^N

that corresponds. We do this via induction, by considering the binary sequence of length N as the first N terms of a parity sequence where $x_i = T^{(i)}(m) \mod 2$, our goal is to find this m.

Base case $N = 1$: here we deal with $\{x_0\}$ if $x_0 = 0$ then m is even so $m = 2k \equiv 0 \mod 2$. If $x_0 = 1$ then m is odd so $m = 2k + 1 \equiv 1 \mod 2$.

Induction Hypothesis: For $N = p$ we have a one-to-one correspondence between the integers $1, \ldots, 2^p - 1$ and the first N terms of their parity sequences.

Induction step $N = p + 1$: we deal with $\{x_0, \ldots, x_p\}$. First we ignore x_0 and examine the sequence $\{x_1, \ldots, x_p\}$ this sequence is length p and by our induction hypothesis we have this sequence corresponds uniquely to an integer less than 2^p.

Thus we can write $m' \equiv j \mod 2^p$. Where m' is in the same equivalence class as $T(m)$ modulo 2^p since $T(m)$ and m' have the same first p terms of their parity sequence agreeing, they must be in the same equivalence class modulo 2^p. Then based of x_0:

Case 1 ($x_0 = 0$): here we have that m is even. Since $T(m) = \frac{m}{2}$ and we know that m' and $T(m)$ are the same modulo 2^p we have that $m \equiv 2m' \mod 2^{p+1}$. Thus $m \equiv 2j \mod 2^{p+1}$. This corresponds to all the even integers less than 2^{p+1} since we have $0 \leq j < 2^p$.

Case 2 ($x_0 = 1$): here m is odd. Since $T(m) = \frac{3m+1}{2}$ and m' and $T(m)$ are the same modulo 2^p we have that $m \equiv \frac{2m'-1}{3} \mod 2^{p+1}$. Thus $m \equiv \frac{2j-1}{3} \mod 2^{p+1} \equiv (2j-1) \cdot 3^{-1} \mod 2^{p+1}$ which corresponds to all the odd integers less than 2^{p+1} since we have that

$$\begin{aligned}
0 \leq j < 2^p &\implies 0 \leq 2j < 2^{p+1} \\
&\implies -1 \leq 2j - 1 < 2^{p+1} - 1 \\
&\implies 0 \leq 2j - 1 \mod 2^{p+1} < 2^{p+1}
\end{aligned}$$

the $2j - 1$ corresponds to all the odd integers less than 2^{p+1} and multiplying by 3^{-1} is an invertible operation preserving our correspondence.

Thus in each case we have a correspondence between the first N terms of a binary sequences and integers less than 2^N. $\square$

It can be a bit more clarifying to work a few examples to help understand this correspondence.

$\{0,0,\ldots\} \leftrightarrow 0 \mod 4$

i	m_i	implication
1	0	$m_1 = 2k$
0	0	$m = 2m_1 = 4k \equiv 0 \mod 4$

$\{0,1,\ldots\} \leftrightarrow 2 \mod 4$

i	m_i	implication
1	1	$m_1 = 2k+1$
0	0	$m = 2m_1 = 4k+2 \equiv 2 \mod 4$

$\{1,0,\ldots\} \leftrightarrow 1 \mod 4$

i	m_i	implication
1	0	$m_1 = 2k$
0	1	$m = \frac{2m_1-1}{3} = \frac{4k-1}{3} \equiv (-1)(-1) \mod 4 = 1 \mod 4$

$\{1,1,\ldots\} \leftrightarrow 3 \mod 4$

i	m_i	implication
1	1	$m_1 = 2k+1$
0	1	$m = \frac{2m_1-1}{3} = \frac{4k+1}{3} \equiv 1(-1) \mod 4 \equiv 3 \mod 4$

Figure 7: Method for the equivalence between parity sequences and a number modulo 4.

Example 35. For $N = 2$ we have the four sequences: $\{0,0,\ldots\}$, $\{0,1,\ldots\}$, $\{1,0,\ldots\}$, $\{1,1,\ldots\}$. We adopt the notation $m_i = T^{(i)}(m)$. Thus our sequences give the following dependencies, with the understanding that once we know the parity of an iterate of T we know how the function behaves.

In the tables of Figure 7 we determine what m is by examining parity sequences indexed as $\{m_0 \mod 2, m_1 \mod 2, \ldots\}$ where $m_0 = m$. We work from the back of the sequence first, that is if we know $m_1 = 1$ then it is odd so $m_1 = 2k+1$ and then use knowledge of m_0 and knowing $m_1 = T(m_0)$.

Example 36. We can do the same for $N = 3$, and $N = 4$. We summarize the results in Figure 8.

Since we have shown that the parity sequences of length N and integers less than 2^N are in bijection, we can use the term parity sequence a bit more loosely for either category, as convenient. Now we introduce a new concept to help make our goal of almost every number

$N=2$	$0 \le m < 2^2$	$N=3$	$0 \le m < 2^3$	$N=4$	$0 \le m < 2^4$
$\{0,0,\ldots\}$	0	$\{0,0,0,\ldots\}$	0	$\{0,0,0,0,\ldots\}$	0
				$\{0,0,0,1,\ldots\}$	8
		$\{0,0,1,\ldots\}$	4	$\{0,0,1,0,\ldots\}$	4
				$\{0,0,1,1,\ldots\}$	12
$\{0,1,\ldots\}$	2	$\{0,1,0,\ldots\}$	2	$\{0,1,0,0,\ldots\}$	10
				$\{0,1,0,1,\ldots\}$	2
		$\{0,1,1,\ldots\}$	6	$\{0,1,1,0,\ldots\}$	6
				$\{0,1,1,1,\ldots\}$	14
$\{1,0,\ldots\}$	1	$\{1,0,0,\ldots\}$	5	$\{1,0,0,0,\ldots\}$	5
				$\{1,0,0,1,\ldots\}$	13
		$\{1,0,1,\ldots\}$	1	$\{1,0,1,0,\ldots\}$	1
				$\{1,0,1,1,\ldots\}$	9
$\{1,1,\ldots\}$	3	$\{1,1,0,\ldots\}$	3	$\{1,1,0,0,\ldots\}$	3
				$\{1,1,0,1,\ldots\}$	11
		$\{1,1,1,\ldots\}$	7	$\{1,1,1,0,\ldots\}$	7
				$\{1,1,1,1,\ldots\}$	15

Figure 8: Summary of Theorem 34 for $N = 2, 3, 4$.

has a finite stopping time more rigorous, that is we define what "almost every" means.

Definition 37 (Natural density)**.** A subset A of positive integers has **natural density** α where $0 \le \alpha \le 1$ if the proportion of elements of A among all natural numbers from 1 to n is asymptotic to α as n tends to infinity.

More explicitly if $A(n) = \#\{i : 1 \le i \le n \text{ and } i \in A\}$ (often called a **counting function**) then A has natural density α if

$$\lim_{n\to\infty} \frac{A(n)}{n} = \alpha$$

Example 38. A good way to examine the difference between natural density and cardinality is through perfect squares. A natural thought is that there are more positive integers than perfect squares, since every perfect square is a positive integer. Though of course both are countable sets and can be put in one-to-one correspondence.

However, the perfect squares become very sparse in the positive integers as we examine larger and larger numbers. Indeed in this case our counting function $A(n) = \lfloor \sqrt{n} \rfloor$ counts the number of perfect squares less than n. It satisfies $0 \leq \frac{A(n)}{n}$ and:

$$\begin{aligned}\lim_{n\to\infty}\frac{A(n)}{n} &= \lim_{n\to\infty}\frac{\lfloor\sqrt{n}\rfloor}{n}\\ &\leq \lim_{n\to\infty}\frac{\sqrt{n}}{n}\\ &= 0\end{aligned}$$

So by squeeze theorem, the natural density of $A = \{n^2 : n \in \mathbb{N}\}$ is 0. Which fits with our intuition that there are "less" perfect squares than natural numbers.

For the following theorem we need a standard result in statistics:

Theorem 39 (Bernoulli's Law of Large Numbers)**.** *In an experiment with probability of success P, after increasing the number of repeated independent trials, the ratio of successful trials to the total number of trials approaches P.*

Example 40. This can best be explained with coin flips. With a fair coin, which has a probability of 0.5 to land heads (or tails), as the number of coin flips increases the number of successes (heads) approaches the number of failures (tails). Stated colloquially, flipping a coin many times results in (roughly) the same number of the heads as tails.

Back to our problem:

Definition 41. Let $A(M)$ be the number of positive integers less than M that have a finite stopping time.

We want to show that $A(M)$ has natural density 1. That is we want to show that $A(M)/M$ approaches 1 as $M \to \infty$. More rigorously:

Theorem 42. *For the Collatz function T "almost every" integer m has some iterate $m_k = T^k(m) < m$, in the sense that the density, $A(M)/M$, of such integers approaches 1.*

Proof. We first show when $M = 2^N$ for some N a positive integer.

Our motivation is that for large N most parity sequences have roughly the same number of 0s and 1s, (Theorem 39) . We again use the notation above to define $x_i = T^{(i)}(m) \mod 2$. From such a parity sequence, which we know from above represents a unique integer $m \leq 2^N$, we see that for any $0 \leq n < N$ if $x_n = 0$ then the nth iterate of T is even so: $\frac{m_{n+1}}{m_n} = \frac{1}{2}$. and if $x_n = 1$ the nth iterate of T is odd so: $\frac{m_{n+1}}{m_n} = \frac{3Q+2}{2Q+1} < \frac{5}{3}$, for some Q. And since we have roughly the same number of 1s as 0s we can approximate: $m_N \approx \left(\frac{1}{2}\right)^{N/2} \left(\frac{5}{3}\right)^{N/2} m_0 < m_0$.

More rigorously: define H_N to be the set of all sequences of 1s and 0s of length N, $\{x_0, \dots, x_{N-1}\}$, that satisfy the relation:

$$\frac{1}{2} - \epsilon < \frac{X}{N} < \frac{1}{2} + \epsilon \tag{4.1}$$

where $X = \sum x_i$ and $\epsilon = \frac{\log 2}{\log(10/3)} - \frac{1}{2}$.

Since there are at most 2^N sequences of length N of 0s and 1s we have trivially that $\frac{\#H_N}{2^N} \leq 1$. From [15] we have that

$$1 - \frac{1}{4\epsilon^2 N} \leq \frac{\#H_N}{2^N} \tag{4.2}$$

Define D_N to be the set of all sequences of 1s and 0s of length N that satisfy only the upper bound above, i.e.,

$$\frac{X}{N} < \frac{1}{2} + \epsilon \tag{4.3}$$

this gives that $\#D_N \geq \#H_N$.

Of the numbers $m \leq 2^N$ represented as parity sequences in D_N we have exactly two groups, either m has a total stopping time less than N or it does not. That is for each sequence in D_N, which we have shown corresponds to a unique integer $m \leq 2^N$, either $T^n(m) = 1$ with $n \leq N - 1$ or no iterate $T^n(m) = 1$ for $n \leq N - 1$. For the second case we can still show that the Nth iterate of T is smaller than the starting location m. But we need an equivalence of Eq. (4.3) with a new identity:

$$\begin{aligned}
\frac{X}{N} < \frac{1}{2} + \epsilon \iff & \frac{X}{N} < \frac{\log 2}{\log(10/3)} \\
\iff & X \log(10/3) < N \log 2 \\
\iff & X(\log 2 + \log(5/3)) - N \log 2 < 0 \\
\iff & (X - N) \log 2 + X \log(5/3) < 0 \\
\iff & (N - X) \log(1/2) + X \log(5/3) < 0 \\
\iff & \log \left[\left(\frac{1}{2}\right)^{N-X} \right] + \log \left[\left(\frac{5}{3}\right)^{X} \right] < 0 \\
\iff & \left(\frac{1}{2}\right)^{N-X} \left(\frac{5}{3}\right)^{X} < 1
\end{aligned}$$

Thus this condition shows us that for each $m \in D_N$ we have that

$$m_N = m_0(m_1/m_0) \cdots (m_N/m_{N-1}) \leq \left(\frac{1}{2}\right)^{N-X} \left(\frac{5}{3}\right)^{X} m_0 < m_0 = m$$

where the inequality comes from $(m_i/m_{i-1}) = \frac{1}{2}$ if $x_{i-1} = 0$ and $(m_i/m_{i-1}) \leq \frac{5}{3}$ if $x_{i-1} = 1$ we have exactly X ones and $N - X$ zeros.

Thus we have that every element of D_N has a finite stopping time, and thus

$$A(2^N) \geq \#D_N - 1 \geq \#H_N - 1$$

we need to subtract an element from D_N and H_N since the sequence corresponding to the number 1 is in D_N and H_N but technically 1 does not have a finite stopping time (it never gets smaller) thus 1 is not counted in $A(M)$ for any M.

Since $A(2^N)$ is at most 2^N we have $1 \geq A(2^N)/2^N$ and by the condition on H_N, (4.2), we have:

$$\begin{aligned}
\lim_{N\to\infty} \frac{A(2^N)}{2^N} &\geq \lim_{N\to\infty} \frac{\#H_N - 1}{2^N} \\
&\geq \lim_{N\to\infty} 1 - \frac{1}{4\epsilon^2 N} - \frac{1}{2^N} \\
&= 1
\end{aligned}$$

So with Squeeze Theorem we have:

$$\lim_{N\to\infty} \frac{A(2^N)}{2^N} = 1$$

Thus it only remains to show our result when $2^N < M < 2^{N+1}$. For simplicity set $A_N = A(2^N)$ and define

$$n_N = A_N + (2^{N+1} - A_{N+1})$$

n_N represents the number of sequences of length 2^N which have a finite stopping time plus the number of sequences of length between 2^N and 2^{N+1} which do not have finite stopping time. We can recognize that $n_N \geq 2^N$ by adding and subtracting 2^N as follows:

$$n_N = 2^N + \left[(2^{N+1} - 2^N) - (A_{N+1} - A_N)\right]$$

which is greater than 2^N since $(2^{N+1} - 2^N) - (A_{N+1} - A_N)$ reprsents the numbers between 2^N and 2^{N+1} which do not have finite stopping time, a non-negative number.

Thus for $M = 2^N + 1, 2^N + 2, \ldots, n_N$ we have that $A(M) \geq A_N$ and that $M \leq n_N$ thus

$$\frac{A(M)}{M} \geq \frac{A_N}{n_N}$$

Now for $M = n_N + k$ where $k = 1, 2, \ldots, 2^{N+1} - 1 - n_N$ we can say that

$$\frac{A(M)}{M} \geq \frac{A_N + k}{n_N + k} \geq \frac{A_N}{n_N}$$

For the first inequality we can see this is true conceptually. Since $M = n_N + k$ we need to show $A(M) \geq A_N + k$, we think about this for each k. When $k = 1$, $M = n_N + 1$ or conceptually it is 2^N plus how many numbers between 2^N and 2^{N+1} do not converge to one under the Collatz iteration, plus one. Thus if we look at $A(M)$ in the worse case scenario all of the numbers which do not converge are first, that is all the numbers between 2^N and 2^{N+1} which do not converge are immediately after 2^N. In this case we would have $A(n_N) = A_N$, so when we go one further it MUST be that the next number converges, since all of the non-convergent numbers were first; we have $A(M) = A_N + 1$. Again this is the worst case scenario, in generally the numbers which do not converge (if they exist) would be spread out in which case $A(M) \geq A_N + 1$. The same idea holds true for every k.

For the second inequality since $n_N \geq 2^N \geq A_N \geq 0$ we can add a positive constant to the numerator and denominator of $\frac{A_N}{n_N}$ and our result is bigger:

$$\begin{aligned}
n_N \geq A_N &\iff kn_N \geq A_N k \\
&\qquad (k \text{ is greater than } 0) \\
&\iff A_N n_N + kn_N \geq A_N n_N + A_N k \\
&\iff (A_N + k)n_N \geq A_N(n_N + k) \\
&\iff \frac{A_N + k}{n_N + k} \geq \frac{A_N}{n_N}
\end{aligned}$$

Thus regardless of where M falls between 2^N and 2^{N+1} we have that:

$$\frac{A(M)}{M} \geq \frac{A_N}{n_N}$$

We still have that $1 \geq \frac{A(M)}{M}$ for all M and by our inequalities here we have:

$$\begin{aligned}
\lim_{M\to\infty} \frac{A(M)}{M} &\geq \lim_{N\to\infty} \frac{A_N}{n_N} \\
&= \lim_{N\to\infty} \frac{A_N}{A_N + (2^{N+1} - A_{N+1})} \\
&= \lim_{N\to\infty} \frac{\frac{A_N}{2^N}}{\frac{A_N}{2^N} + \left(\frac{2^{N+1}}{2^N} - \frac{A_{N+1}}{2^N}\right)} \\
&= \lim_{N\to\infty} \frac{\frac{A_N}{2^N}}{\frac{A_N}{2^N} + 2\left(1 - \frac{A_{N+1}}{2^{N+1}}\right)} \\
&= \frac{1}{1 + 2(0)} \\
&= 1
\end{aligned}$$

Thus by Squeeze Theorem we have

$$\lim_{M\to\infty} \frac{A(M)}{M} = 1$$

as desired. □

This concludes our section on natural density. The above result shows that we expect almost every integer to eventually converge; if there is a number that does not converge under the Collatz iteration then it is not part of a “large” class of number, in the sense of natural density.

4.2 CYCLE LENGTH

As shown above we would expect most numbers to eventually converge to 1 under the Collatz iteration. If there is a number that does not converge then the iteration will either shoot towards infinity or will enter a cycle. We now demonstrate that if a cycle exists its length must be very, very large. We demonstrate a lower bound on the length of a cycle (that is not the trivial $\{4, 2, 1, \ldots\}$ cycle) using Shalom Eliahou's paper "The $3x+1$ problem: new lower bounds on nontrivial cycle lengths". The method of determining the minimal length of a cycle here is particularly helpful because as the Collatz conjecture is verified for higher and higher numbers this method provides larger and larger lower bounds on cycle length. Indeed presented in Eliahou's paper, [2], is the lower bound of $17,087,915$ for cycle length, using Eliahou's method Tempkin and Arteaga in [14] improve the lower bound to $272,500,658$; we present an even larger lower bound of $10,439,860,591$.

Recall the trajectory of a number n is the set of iterates:

$$\Omega(n) = \{n, T(n), T^{(2)}(n), \ldots\}$$

A trajectory Ω is called a **cycle** of length k if $T^{(k)}(x) = x$ for all $x \in \Omega$. We call k the **length** or cardinality of the cycle. We consider the trajectory to be the finite number of elements where no terms are repeated. A simple (and the only so far) example is $\Omega(1) = \{1, 2\}$, which is a cycle of length 2, called the trivial cycle.

Example 43. A less trivial example cannot be produced with T as defined throughout this paper, no other cycle has been found; the existence of one would disprove the conjecture. However if we extend T to allow negative integers as input as well then we can see additional cycles, such as $\{0\}$, $\{-5, -7, -10\}$ and $\{-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34\}$. There is more on extensions of the problem discussed in Section 5.1.

We now wish to present bounds on cycle length, to do so we need some intermediate results first:

Lemma 44. *Let Ω be a cycle of T and let $\Omega_1 \subset \Omega$ denote the subset of its odd elements and let $k = \#\Omega$, then*

$$\prod_{n\in\Omega_1} (3 + n^{-1}) = 2^k$$

Proof. Since Ω is a cycle then evaluating T on each element of Ω does not change the elements (other than their order), thus looking at the product of elements in Ω:

$$\prod_{n\in\Omega} n = \prod_{n\in\Omega} T(n)$$

which of course gives:

$$\prod_{n\in\Omega} \frac{T(n)}{n} = 1$$

Now recalling the definition of $T(n)$ we can evaluate $\frac{T(n)}{n}$:

$$\frac{T(n)}{n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{3+n^{-1}}{2} & \text{if } n \text{ is odd} \end{cases}$$

Then our product turns into:

$$\begin{aligned} 1 &= \prod_{n\in\Omega} \frac{T(n)}{n} \\ &= \left(\prod_{n\in\Omega_1} \frac{T(n)}{n}\right)\left(\prod_{n\in\Omega\setminus\Omega_1} \frac{T(n)}{n}\right) \\ &= \left(\prod_{n\in\Omega_1} \frac{3+n^{-1}}{2}\right)\left(\prod_{n\in\Omega\setminus\Omega_1} \frac{1}{2}\right) \\ &= \left(\prod_{n\in\Omega_1} (3+n^{-1})\right)\left(\prod_{n\in\Omega} \frac{1}{2}\right) \\ 2^k &= \prod_{n\in\Omega_1} (3+n^{-1}) \end{aligned}$$

□

Using the above lemma we can prove a slightly stronger theorem which bounds the ratio of $\dfrac{\#\Omega}{\#\Omega_1}$:

Theorem 45. *Let Ω be a cycle of T and let $\Omega_1 \subset \Omega$ be the odd elements of Ω. Then:*

$$\log_2(3 + M^{-1}) < \frac{\#\Omega}{\#\Omega_1} \leq \log_2(3 + m^{-1})$$

where $M = \max\Omega$, $m = \min\Omega$. We can have a stronger right inequality of:

$$\frac{\#\Omega}{\#\Omega_1} \leq \log_2(3 + \mu)$$

where $\mu = (1/\#\Omega_1)\left(\sum_{n\in\Omega_1} n^{-1}\right)$

Proof. First note that $\max\Omega > \max\Omega_1$, that is the largest element of a trajectory will be even. This is simple to see: if the largest, M, were odd then the cycle contains $T(M) = \frac{3M+1}{2}$ which is larger, a contradiction. Now notice that

$$3 + M^{-1} < 3 + n^{-1}$$

where $M = \max\Omega$ and $n \in \Omega_1$, similarly:

$$3 + n^{-1} \leq 3 + m^{-1}$$

where $m = \min\Omega$. Thus if we let $k_1 = \#\Omega_1$ then by Lemma 44 we get:

$$\left(3 + M^{-1}\right)^{k_1} = \prod_{n\in\Omega_1}\left(3 + M^{-1}\right) < 2^k \leq \prod_{n\in\Omega}\left(3 + m^{-1}\right) = \left(3 + m^{-1}\right)^{k_1}$$

algebra yields:

$$\log_2\left(3 + M^{-1}\right) < \frac{k}{k_1} \leq \log_2\left(3 + m^{-1}\right)$$

As for the the stronger inequality we call on the Arithmetic-Mean-Geometric-Mean (AMGM) inequality for a sequence of numbers $x_1, \ldots, x_r$:

$$\sqrt[r]{\prod_i x_i} \leq \frac{1}{r}\sum_i x_i$$

In our case we obtain:

$$
\begin{aligned}
2^k = \prod_{n\in\Omega_1}\left(3+n^{-1}\right) &\leq \left(\frac{1}{k_1}\sum_{n\in\Omega_1}\left(3+n^{-1}\right)\right)^{k_1} \\
&= \left(3+\frac{1}{k_1}\sum_{n\in\Omega_1} n^{-1}\right)^{k_1}
\end{aligned}
$$

Taking logarithms base 2 gets the stronger inequality:

$$
\frac{k}{k_1} \leq \log_2\left(3+\mu\right)
$$

□

One immediate comment is that for practical purposes the use of μ as a stronger bound is mostly useless. The computation of μ directly requires knowledge of the cycle, but if we can find a cycle then we immediately disprove the Collatz Conjecture. However, even though directly using μ is not possible we can still get an inequality on the size of μ that can be helpful:

Lemma 46. *Using the notation above we have that $\mu \leq \frac{8}{9}m^{-1}$, as long as $m = \min\Omega > 1$.*

Proof. First we show that if $n \in \Omega_1$ and $n < \frac{9}{7}m$ then we have $T(n) \in \Omega_1$ and $T(n) \geq \frac{9}{7}m$. For the inequality we see that

$$
\frac{3m+1}{2} > \frac{9}{7}m
$$

And that T is an increasing function on odd inputs. That is if $m \leq n$ and n, m both odd then $T(m) \leq T(n)$. Given our assumptions we have $m \leq n < \frac{9}{7}m$; m, n both odd. This gives that $T(m) \leq T(n)$ and by the above inequality we have that $\frac{9}{7}m < T(m) \leq T(n)$ which gives $T(n) \geq \frac{9}{7}m$.

Suppose for a contradiction that $T(n)$ is even. Then we have that $T^2(n) = \frac{3n+1}{4}$, notice:

$$
\begin{aligned}
T^2(n) &= \frac{3n+1}{4} \\
&< \frac{3\left(\frac{9}{7}m\right)+1}{4} \\
&= \frac{27}{28}m + \frac{3}{4}
\end{aligned}
$$

and for $m \geq 21$:

$$< m$$

This contradicts minimality of m.

Since $n < \dfrac{9}{7}m$ then $\dfrac{3n+1}{4} < \dfrac{3(\frac{9}{7}m)+1}{4}$. Also, it is acceptable to require $m \geq 21$ since we know the $3n+1$ problem holds for numbers less than 21; thus numbers less than 21 cannot be part of a cycle.

From the above it follows that at least half of Ω_1 lies in the interval $[\frac{9}{7}m, \infty)$, since for every number below $\frac{9}{7}m$ we have $T(n) \in [\frac{9}{7}, \infty)$. Thus we can compute, using $k_1 = \#\Omega_1$:

$$\begin{aligned}\mu &= \frac{1}{k_1}\sum_{n\in\Omega_1} n^{-1} \\ &\leq \frac{1}{k_1}\left(\frac{k_1}{2}m^{-1} + \frac{k_1}{2}\frac{7}{9}m^{-1}\right)\end{aligned}$$

This follows from at least half of Ω_1 is in $[\frac{9}{7}m, \infty)$, and the other "half" are in $[m, \frac{9}{7}m)$

$$= \frac{8}{9}m^{-1}$$

$\square$

For the rest of this result we do not use the stronger bounds on $\dfrac{\#\Omega}{\#\Omega_1}$, but we have included how it may be used for completeness. Back to proving our result on the length of a cycle we introduce another function:

We introduce two functions, $K : \mathbb{N} \to \mathbb{N}$, $L : \mathbb{N} \to \mathbb{N}$, defined as follows: for every $m \in \mathbb{N}$, $K(m)$ is the smallest positive integer k with:

$$\log_2(3) < \frac{k}{l} \leq \log_2(3 + m^{-1})$$

for some positive integer l. Analogously we define $L(m)$ to be the smallest positive l that satisfies the above inequality for some k. Alternatively, since $\log_2(3) < \log_2(3 + m^{-1})$ there must be a rational number between them, $K(m)$ is the numerator of the fraction with smallest numerator. Similarly $L(m)$ is the denominator of the fraction with the smallest denominator. We examine an example of how $K(n)$ can be determined for a given n.

With these definitions we get an easy Corollary of Theorem 45:

Corollary 47. *Let Ω be a cycle of T, and let $\Omega_1 \subset \Omega$ denote the subset of its odd elements. Then*

$$\#\Omega \geq K(\min \Omega)$$
$$\#\Omega_1 \geq L(\min \Omega)$$

To get a (meaningful) lower bound on the size of a cycle we consider the continued fraction expansion of $\theta = \log_2(3)$, $\theta = [a_0; a_1; a_2; \ldots]$ (for a review of the needed materials in continued fractions, see Section 2.4). We also need p_n, q_n which represent the rational number obtained by truncating the continued fraction as $[a_0, a_1; \ldots; a_n] = p_n/q_n$ with $\gcd(p_n, q_n) = 1$; defined by the recurrence relations in equations: (2.1) and (2.2) repeated for convenience below:

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

for $n \geq 0$ with $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, and $q_{-1} = 0$.

We also use the sequence of upper intermediate convergents to θ, recall that the upper intermediate convergents are steps in between p_n/q_n and p_{n+2}/q_{n+2} for n odd and are defined:

$$\frac{p_{n,i}}{q_{n,i}} = \frac{p_n + ip_{n+1}}{q_n + iq_{n+1}}$$

where $i = 0, 1, \ldots, a_{n+2} - 1$.

It will help to have an understanding of how the functions K and L behave. First, both K and L are increasing functions, as we increase n the distance between $\log_2(3)$ and $\log_2(3 + n^{-1})$ decreases, eliminating "more" rational numbers between them. Thus we must increase the denominator and numerator to stay in the correct range. More accurately the functions are not decreasing, they tend to be constant for many n. This should fit with intuition, after finding the smallest k (or l) for a given n, increasing it will not change the distance between $\log_2(3)$ and $\log_2(3 + n^{-1})$ much.

Definition 48 (Transition point)**.** A **transition point** for the function K is an integer n such that $K(n) > K(n-1)$.

Example 49. Notice that $\log_2(3) \approx 1.5850$. If we choose $n = 1$ then $\log_2(3+1) = 2$. So we look for a rational number between $\log_2(3)$ and 2 with smallest numerator. One method to determine K is to proceed from the smallest possible numerator, looking at all possible denominators until we have one in the correct range. The smallest possible numerator is 1. But no possible denominators put our fraction in the correct range (smallest possible is 1 which makes our fraction $1/1 = 1 < 1.5$).

The next possible numerator is 2. In this case with denominator 1 our fraction $2/1 = 2$ falls just in the region $\log_2(3) < 2 \leq 2$. Thus we have that $K(1) = 2$.

Example 50. A more interesting example is when we allow $m = 2$. In this case $\log_2\left(3 + \frac{1}{2}\right) \approx 1.8074$. Here we have 5 is the smallest numerator and 3 the smallest denominator. This can be seen by inspection, in the same process as above.

If our numerator were 1 then no denominator allows our fraction to be in the correct range. Similarly with a numerator of 2 a denominator of 1 makes our fraction too large but any larger denominator makes our fraction too small. For a numerator of 3 we have the same split, a denominator of 1 makes the fraction too big but any larger denominator makes the fraction too small. With a numerator of 4 a denominator of 1 and 2 makes the fraction too big but any larger denominator makes the fraction too small.

Finally with a numerator of 5, a denominator of 1 and 2 makes the fraction too large, but with a denominator of 3 we get the fraction $5/3 \approx 1.6667$ our fraction falls right into the desired range. Thus we have $K(2) = 5$. The above procedure can be summarized by the following:

1. $k = 1$, $l = 1$ too small
2. $k = 2$, $l = 1$ too large, $l = 2$ too small.
3. $k = 3$, $l = 1$ too large, $l = 2, 3$ too small.
4. $k = 4$, $l = 1, 2$ too large, $l = 3, 4$ too small.
5. $k = 5$, $l = 1, 2$ too small $l = 3$ fits!

So $K(2) = 5$, in fact we have the same value for K and L for $n = 2, 3, 4, 5$.

Recall Theorem 28 (from Section 2.4) that the range of K is exactly the $p_{n,i}$, the upper intermediate convergents to $\log_2(3)$. This means we have a transition point of K for each value of $p_{n,i}$, i.e., for each $p_{n,i}$ we get a transition point of K. Thus we can define the function:

$$tr(n, i) = \text{the least integer } m \text{ such that } K(m) \text{ equals } p_{n,i}$$

See [2] for how to compute the transition points of K, as well as for a very informative table with values $p_{n,i}$, $q_{n,i}$, $tr(n, i)$.

In order to give an explanation of how these numbers all fit together, recall that the $3n+1$ problem has been verified up to 5×2^{60} [9]. Thus if a cycle exists its smallest number must be bigger than 5×2^{60}. Let this number be m, it turns out there is a transition point near 2^{60}, at $1.08 \times 2^{60} = tr(19, 0)$. Thus this means that $K(5 \times 2^{60}) = p_{19,0} = 630, 138, 897$, since the next transition number is 1.46×2^{67}. Thus by Corollary 47 we have that $\#\Omega \geq 630, 138, 897$. The next transition number being 1.46×2^{67} means that to get a stronger lower bound on cycle lengths, using just this method, we would need to verify the $3n + 1$ conjecture past this number. Now we can prove a more explicit statement about the cardinality of a cycle.

Theorem 51. *Let Ω be a nontrivial cycle of T. Provided that $\min\Omega > 1.08 \times 2^{60}$ we have*

$$\#\Omega = 630\,138\,897a + 10\,439\,860\,591b + 103\,768\,467\,013c$$

where a, b, c are nonnegative integers, $b > 0$ and $ac = 0$. Specifically the smallest possible values for $\#\Omega$ are $10, 439, 860, 591$; $11, 069, 999, 488$; $11, 700, 138, 385$, etc.

Proof. As stated above the conjecture has been verified up to 5×2^{60}, thus if Ω is a nontrivial cycle of T we have $\min\Omega > 5 \times 2^{60} > 1.08 \times 2^{60}$. Using the notation above, with $\frac{p_n}{q_n}$ being the rational approximation of $\log_2(3)$ with continued fractions to n terms. Indeed we know that for n even $\frac{p_n}{q_n}$ under approximates $\log_2(3)$ and with n odd it over approximates $\log_2(3)$. With very precise computations[1] it can be observed that we have the following inequalities:

$$\frac{p_{22}}{q_{22}} < \log_2(3) < \frac{p_{21}}{q_{21}} < \log_2\left(3 + \frac{1}{5 \times 2^{60}}\right) < \frac{p_{19}}{q_{19}}$$

[1]Eliahou's paper [2] actually gives a method for fitting this inequality using the transition points. When going through this I numerically verified this computation using quadruple precision in C++, I have included the code in the Appendix.

If we continue with the notation $k = \#\Omega$, $l = \#\Omega_1$ where Ω_1 is the set of odd entries of Ω, Theorem 45 offers that $\frac{k}{l} \in (\log_2(3+M^{-1}), \log_2(3+m^{-1})]$. Where M and m are the largest and smallest elements in the cycle respectively; in particular we have:

$$\frac{k}{l} \in \left[\log_2(3), \log_2\left(3 + \frac{1}{5 \times 2^{60}}\right)\right]$$

It can be computed that there are no intermediate convergents between p_{19}/q_{19} and p_{21}/q_{121}. Then by Theorem 27 (from Section 2.4) p_{19}/q_{19}, and p_{21}/q_{21} form a Farey pair, that is $p_{19}q_{21} - p_{21}q_{19} = 1$ (in our case); similarly the pair p_{21}/q_{21} and p_{22}/q_{22} form a Farey pair (this time by Theorem 24).

All together we have three possibilities for the fraction $\frac{k}{l}$:

$$\frac{k}{l} \in \left(\frac{p_{22}}{q_{22}}, \frac{p_{21}}{q_{21}}\right)$$

or

$$\frac{k}{l} = \frac{p_{21}}{q_{21}}$$

or

$$\frac{k}{l} \in \left(\frac{p_{21}}{q_{21}}, \frac{p_{19}}{q_{19}}\right)$$

In the first we can use Lemma 25 to get that $k = p_{21}b + p_{22}c$, with $b, c \in \mathbb{N}$ (> 0); since $\frac{k}{l}$ is a fraction in between a pair of fractions which form a Farey Pair; the second case of course gives $k = p_{21}b$, with $b \in \mathbb{N}$ (in this case $b = 1$). The last case gives $k = p_{19}a + p_{21}b$ some $a, b \in \mathbb{N}$ (> 0) again using Lemma 25. It is worth noting that in this last paragraph we have kept the coefficient for p_{21} to be b and given p_{19} and p_{22} different coefficients.

Since $\frac{k}{l}$ must fall into exactly one of the cases above we have that at least one of a or c is zero, thus $ac = 0$. Since b shows in each of the cases and $b \in \mathbb{N}$ we have that $b > 0$. And we can state our result as:

$$\#\Omega = 630\,138\,897a + 10\,439\,860\,591b + 103\,768\,467\,013c$$

with $a, b, c \in \mathbb{N}$ and $ac = 0$ and $b > 0$. $\square$

This completes the finished work on cycle length. We have demonstrated that if a cycle exists it must have at the very least 10,439,860,591 elements, and as the conjecture is verified for larger and larger numbers this lower bound will also get larger (once we surpass another transition point of K).

4.3 FUTURE WORK ON CYCLE LENGTH

An interesting consequence of the result from the last section is that as we strengthen the claim of Theorem 51 to get larger lower bounds, we of course need to increase our supposition to insist the smallest element of the cycle increase to higher transition points of K. For example in [2] Eliahou uses that at the time the conjecture had been verified only up to 2^{40} to prove the result that

$$|\Omega| = 301\,994a + 17\,087\,915b + 85\,137\,581c$$

where the a, b, c are as in Theorem 51

The significance of this is that even though the conjecture has been verified for larger numbers the assumptions of his theorem are still satisfied. In fact we can play this game for every transition point of K we get an equation of the same form. Thus we must have the following equalities hold:

$$\begin{array}{rcrcrcl}
24\,727\ a_1 & + & 75\,235\ b_1 & + & 50\,508\ c_1 & = & |\Omega| \\
75\,235\ a_2 & + & 125\,743\ b_2 & + & 176\,251\ c_2 & = & |\Omega| \\
125\,743\ a_3 & + & 301\,994\ b_3 & + & 16\,785\,921\ c_3 & = & |\Omega| \\
301\,994\ a_4 & + & 17\,087\,915\ b_4 & + & 85\,137\,581\ c_4 & = & |\Omega| \\
17\,087\,915\ a_5 & + & 102\,225\,496\ b_5 & + & 85\,137\,581\ c_5 & = & |\Omega| \\
102\,225\,496\ a_6 & + & 187\,363\,077\ b_6 & + & 85\,137\,581\ c_6 & = & |\Omega| \\
187\,363\,077\ a_7 & + & 272\,500\,658\ b_7 & + & 357\,638\,239\ c_7 & = & |\Omega| \\
272\,500\,658\ a_8 & + & 630\,138\,897\ b_8 & + & 9\,809\,721\,694\ c_8 & = & |\Omega| \\
630\,138\,897\ a_9 & + & 10\,439\,860\,591\ b_9 & + & 103\,768\,467\,013\ c_9 & = & |\Omega|
\end{array}$$

among other equations outside the transition points of K which are listed in Eliahou's paper, that is transition points less than 2^{28}. The point is that these equations must all be simultaneously satisfied with non-negative integers ($b_i > 0$) and at least one of a_i and c_i needs to be zero for each i. It is important to note that each equation has its own a_i, b_i, c_i so that for each equation we get 3 variables, but each equation needs to evaluate to the same positive integer.

If it is possible to solve this system of simultaneous linear Diophantine equations (with conditions), I believe it will result in a much larger lower bound on cycle length, requiring no further assumptions.

5.0 OTHER VIEWS OF THE PROBLEM

Above we have seen much about the $3n+1$ problem. However there is no hope of portraying all the recent or important results all in one paper; and certainly not in the depth that is presented above. In order to paint a more complete picture of the $3n+1$ problem we will now present a few different methods of working on the problem, but not in as much detail.

5.1 GENERALIZATIONS

Often in the realm of mathematics a specific problem can be difficult to solve on its own but be a direct result from a more general theorem. In hopes that this is the case for the $3n+1$ problem many mathematicians have examined various generalizations and extension of the problem. One such generalization is to consider the class of functions:

$$T_{3,k}(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{3n+k}{2} & \text{if } n \text{ is odd} \end{cases}$$

for $k \equiv 1$ or $5 \mod 6$. In this case our $3n+1$ problem becomes a consequence of showing that for any allowed k every iteration of $T_{3,k}$ becomes periodic (hits a cycle) and that there are only a finite number of distinct cycles. [6]

This generalization of the problem is actually equivalent to an extension of this problem to the rationals. The Collatz function T can be extended to take all rational numbers $\frac{n}{k}$ with odd denominator with the exact same definition (based on parity of the numerator). However a simple rearranging of terms turns this function into exactly $T_{3,k}$ as above. There are extensions to the real and complex numbers in an effort that continuous extensions can lend insight to their discrete counterparts.

This problem can also be generalized to use different coefficients of n when n is odd. The $5n+1$ problem, for example, is one of interest which is thought to have a divergent sequence. This problem is analyzed in [4]

There is also an extension of this problem to the d-adic integers, $\mathbb{Z}_d$; specifically of interest are the 2-adic integers, $\mathbb{Z}_2$ (see review section for a brief introduction to p-adic integers). These extensions are of interest to those working in ergodic theory; an area of math which studies iterations of functions which preserve a measure. Lagarias among others has published their work on the Collatz problem using ergodic theory [5]. Terry Tao has also done a bit of investigation into this problem using the 2-adic numbers [13].

6.0 CONCLUSIONS

As we have seen the Collatz conjecture is very easy to approach; namely it allows a very quick understanding of the premise. But in order to get results we needed to delve deeper into more advanced mathematics. We approached this problem with using probability theory, number theory, and combinatorics; though this only scratches the surface of the various ways this problem can be attacked.

We now have a better understanding of the problem and some of the major results on the problem; including justification that almost every positive integer should converge (eventually), and that cycles of our iteration should not exist (more precisely, if they exist they will be extremely large).

The question some mathematicians (and every mathematician's parents) ask about this problem is why? What is the importance of this problem and why do we care? Shortly after this problem was proposed, when none could find a solution, the same questions were asked of it. At the time, aspiring and settled mathematicians alike were focused on getting strong results and solving the problem, things the $3n + 1$ problem does not seem to allow. This lead to a period of time where very few worked on the problem.

Now the same questions can be asked of the problem, and the problem itself has not been more lenient in providing results. But personally, the difficulty of this problem along with its easy-to-state nature makes this a very interesting problem; it also suggests that if a solution is to come along it will be the result of a very beautiful area of mathematics. Whether that area is already studied or still needs to be discovered, well, this is still an open problem.

APPENDIX

C++ CODE

As mentioned in Section 4.2, specifically the footnote in the proof of Theorem 51 the inequality to prove the Theorem can be explained with the transition number discussed in [2]. However, when going through his proof I attempted my own computations using the following C++ code to verify the inequalities. The following code uses quadmath.h which is included with most compilers (anything after GCC 4.6.0), this also requires an extra flag -lquadmath when linking the o-file.

```
//Theodore Ian Martiny
#include<quadmath.h>
#include<iostream>

using namespace std;

/*
    This code will compute the continued fraction approximations
    to log_2(3). Then it will determine the numerator of the last
    continued fraction that is larger than log_2(3+1/(5*2^{60})).
    This is used to create the rational inequality p(i+1)/q(i+1)
     < log_2(3) < p(i)/q(i) < log_2(3+1/(5*2^{60})) < p(i-2)/q(i-2).
```

```
    Note that finding this value for other values of n may not work
    as expected, since it is occasionally the case that we need
    intermediate upper convergents.
*/

int main()
{
    int j, n=60;
    __float128 pg, pm1, pm2, qm1,qm2,theta,a,pold,pnew,qold;
    __float128 qnew, pold2, qold2;

    // compute log_2(3+2^(-n)) for n = 60, scale by 5 since
    // min value is 5*2^60
    pg = log2q(3. + 1/(5*powq(2.,n)));

    // we start with computing the components of the
    // continued fraction approx to pg and then use this
    // to compute the p_n.

    // computes p and q using: where a_n nth term in continued fraction,
    // computed as well below.
    // p_n = a_n*p_{n-1} + p_{n-2}
    // q_n = a_n*q_{n-1} + p_{n-2}
    // using initial conditions:

    pm2 = 0; // p_{-2}
    pm1 = 1; // p_{-1}
    qm2 = 1; // q_{-2}
    qm1 = 0; // q_{-1}
```

```
// now we begin the recursive definition, until we get
// out of base cases.

theta = log2q(3.);
a = floorq(theta);
pold2 = a*pm1 + pm2;
qold2 = a*qm1 + qm2;

theta = 1./(theta -a);
a = floorq(theta);
pold = a*pold2 + pm1;
qold = a*qold2+qm1;

j= 2;

// now we loop and compute  updated a, p, q, theta for more j.
while(true)
{
    // we do this twice since we are only interested in upper
    // convergents.
    for (int k=0;k<2;k++)
    {
        //compute the next continued fraction coefficient
        theta = 1./(theta-a);
        a = floorq(theta);

        //compute the next continued fraction num/denom
        pnew = a*pold + pold2;
        qnew = a*qold + qold2;
```

```
            j = j+k;

            pold2 = pold;
            qold2 = qold;

            pold = pnew;
            qold = qnew;
        }//for

        // now j is odd. Thus log2(3)<p(j)/q(j) we want to know if
        // p(j)/q(j) < log2(3 + 2^(-n)) or not, if so then the index we want
        // is i = j-2, otherwise we carry on
        if (pold/qold < pg)
            break;

        j = j+1;
    }//while

    // we've broken out of the loop so print the desired index.
    cout<<"n="<<n<<": Value of i: "<<j-2<<endl;

    return 0;
}//main
```

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