ON THE STEADY STATES OF THIN FILM EQUATIONS

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Guoqing Liu

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University of Science and Technology of China, 2010

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This dissertation was presented

by

Guoqing Liu

It was defended on

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and approved by

Prof. Huiqiang Jiang, Dept. of Mathematics, University of Pittsburgh

Prof. Xinfu Chen, Dept. of Mathematics, University of Pittsburgh

Prof. Ming Chen, Dept. of Mathematics, University of Pittsburgh

Prof. Marta Lewicka, Dept. of Mathematics, University of Pittsburgh

Prof. Dejan Slepčev, Dept. of Mathematics, Carnegie Mellon University

Dissertation Advisors: Prof. Huiqiang Jiang, Dept. of Mathematics, University of Pittsburgh,

Prof. Xinfu Chen, Dept. of Mathematics, University of Pittsburgh

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Guoqing Liu, PhD

University of Pittsburgh, 2015

This dissertation studies the steady state of thin film type equations. Different considerations of physical forces give different formulations of differential equations. We start with generalized thin film evolution and derive the second order elliptic equation for steady states.

For the thin film driven by both van der Waals force and Born repulsion force, we define associated energy and obtain a classical energy minimizing problem by taking semi-limit. The solution has been proven to converge to a Dirac measure in the limit that repulsive force term tends to 0. Asymptotic analysis show that the location of the spike would be a point on the boundary with maximal curvature.

Furthermore, we neglect the Born repulsion force and study radial steady state solution for van der Waals force driven thin film equation. We link the volume constraint problem with a initial value ordinary different equation and analyze how radial steady state solution and associated energy depend on the average thickness.

Keywords: Thin Film Equations, Dirac Measure, Maximal Curvature, Van der Waals Force, Radial Steady States, Born Repulsion Force.

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PREFACE

First and foremost, I would like to express my most sincere gratitude to my advisor, Dr. Huiqiang Jiang, for his enthusiastic support and patient guidance through every step of my graduate studies. I am also deeply grateful to my coadvisor Dr. Xinfu Chen for sharing his incomparably profound knowledge with me and providing many valuable discussions and suggestions in my research. I have learned lots of lots from their rigorous mathematical methodology and expectation of excellence. Without their enormous help and encouragement, this thesis could not have been possible.

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1.0 INTRODUCTION AND SUMMARY OF RESULTS

Thin film type equation models the evolution of a thin film of viscous fluids on a solid surface. Due to the large number of applications in coatings industry, including painting and adhesives, more and more researchers from physics, mathematics and engineering departments are conducting experiments and analysis to understand the dynamics. It is also a very interesting and challenging problem in mathematics.

1.1 MATHEMATICAL BACKGROUND

The general form of thin-film equation is given by a fourth order, nonlinear partial differential equation :

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u + g(u)\nabla u). \tag{1.1.1}$$

Here $u \geq 0$ stands for the thickness of the thin film. The set of points where $u = 0$ is called rupture set and the corresponding solution is rupture solution. The highest order term containing linearized curvature Δu describes the effects of surface tension. The particular form of the function $f(u)$ depends on the boundary conditions between the fluid and the solid surface. Another function $g(u)$ can model additional practical physical force. For most of cases, we choose polynomials for f and g as following:

$$u_t = -\nabla \cdot (u^n\nabla\Delta u + u^m\nabla u). \tag{1.1.2}$$

When $f(u) = u$ and $g(u) = u$, it models the thin jet in a gravity driven Hele-Shaw cell [2, 17, 34, 40, 45, 46, 47]. When $f(u) = u^3$ and $g(u) = u^3$, it models fluid droplet hanging from a ceiling [41]. When $f(u) = u^3$ and $g(u) = u^m$ with typical negative m, it models the thin film driven by van der Waals force.

If we neglect the second order term in (1.1.1) which is derived from lubrication approximation, we have the degenerate parabolic equation,

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u). \tag{1.1.3}$$

The equation models the surface tension dominated thin films. For the one dimensional space, there exists a weak solution which preserves the nonnegativity in [14] for the case $f(u) = u^n$. A remarkable technique used there is to study the regularized problem

$$u_t = -\left(\left(u^n + \varepsilon\right) u_{xxx}\right)_x$$

and prove the boundedness of Hölder norm with respect to x and t which is uniform in ε. The solution satisfying periodic boundary conditions [19] tends to be strongly positive in a finite time T and approaches to its mean as $t \to \infty$ when $n > 0$. Physically, the thin film spreads and merge into a uniform layer. Moreover, Bertta, Bertsch and Dal Passon [10] describes in detail about the support of solution depending on the exponent n. If n is large, the support of the solution is more likely to expand. Regarding the expansion of its support, Bernis [12, 13] has proven the interface between the region $\{u > 0\}$ and $\{u = 0\}$ moves with finite velocity. In multi-dimensional space, the existence and long time convergence of weak solutions have been proven in [35, 48] and finite speed of propagation has also been verified for $\frac{1}{8} < n < 2$ in [27]. One interesting phenomena worthing mention here is about the existence of so called waiting time. The support for thickness function u expands and covers the bounded domain eventually. However, if appropriate initial states are given [36], there will be a time point called waiting time locally at which the support does not expand.

Back to (1.1.1) and (1.1.2), the existence of weak solution for (1.1.2) in one dimension and its asymptotic behavior in infinite time similarly as degenerate case have already been shown in [20]. Some other aspects on this thin film equation have been investigated.

For blow up analysis, Bertozzi and Pugh [22] have proven that for (1.1.1), the solution would be possible to grow without bound only if

$$\lim_{s\to\infty}\frac{s^2 f(s)}{g(s)} < \infty \text{ and } \lim_{s\to\infty}\frac{g(s)^2}{f(s)} = \infty.$$

That is for power law case, only the cases $m \geq n+2$ can allow to have blow up. Later in [21], They analyzed the case $n = 1$ and proved the existence of a solution which blows up in a finite time if $m \leq 3 = n+2$. Regarding the critical case $m = n+2$, solutions are bounded for all the time if the total mass is small enough and the critical mass has been explicitly calculated for $n = 1, m = 3$ in [15].

Another important mathematical issue is about the self-similar solution in the critical case $m = n+2$ starting from [15]. When $n = 1$ and $m = 3$, Bernoff, Bertozzi and Witelski consider the rescaled solution of the form

$$u(t,x) = \lambda(t)\rho(\lambda(t)x).$$

There are two main types of solutions, self-similar solution that is spreading and exists for all t and self-similar solution which blows up at a finite time. They obtained that self-similar spreading solutions and self-similar blowup solutions with single bump are linearly stable and self-similar blowup solutions with multi-bumps are linearly unstable by numerically computing the eigenvalues for the linearized operator. For the general critical case with various m and n, Beretta [9] provided the condition for the existence of self-similar spreading solution that $0 < n < 3$. In [72], Pugh and Slepčev showed self-similar finite time blowup solution exists only when $0 < n < \frac{3}{2}$. And the solution might have one or more local maximum, comparing to the self-similar infinite-time spreading solutions are allowed to have only one local maximum. In 2007, Slepčev [71] demonstrated the numerical calculation results in [15] with a rigorous mathematical analysis.

Extensive mathematical analysis about the steady states for one-dimensional space case ($n = 1$) has also been made in the past decades. R. Laugesen and M. Pugh analyzed comprehensively the linear stability of positive steady states and touchdown steady states and compared their energy level for a more general setting [58, 59, 60]. Later they summarized the stability and instability results under zero mean perturbation in a bifurcation digram for power law coefficients [61].

Here we may rewrite (1.1.1) as

$$u_t = \nabla \cdot (f(u)\nabla p) \tag{1.1.4}$$

where p can be viewed as the pressure and for power-law coefficients case,

$$p = -\frac{1}{m-n+1}u^{m-n+1} - \Delta u.$$

We consider the thin film of viscous fluid in a cylindrical container with a finite size. Denote the bottom of the container with bounded and smooth boundary to be $\Omega \subset R^n$ with $n \geq 1$. Assume that there is no flux through the boundary,

$$\mathbf{n} \cdot \nabla p = 0, \text{ on } \partial\Omega \tag{1.1.5}$$

where $\mathbf{n}$ is the unit outer normal vector. Furthermore, we neglect the wetting and nonwetting effect and assume the surface of fluid is perpendicular to the container walls,

$$\mathbf{n} \cdot \nabla u = 0, \text{ on } \partial\Omega. \tag{1.1.6}$$

For physical meaning, we will assume that the total volume is given by a constant $M > 0$ as is the average film thickness

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u.$$

Under the above setting, we are going to introduce two related problems in the next two sections. Later in Chapter 2, we will formalized the first energy minimizing problem, proved the main convergence theorem and explicitly located the boundary spike. In Chapter 3, the radial steady states of some type of thin film equations are investigated. Finally, we make some conclusions about my work and post my further directions in the last chapter.

1.2 ENERGY MINIMIZING PROBLEM

For (1.1.1), we will study the semi-limit of the steady state of thin film equation with $f(u) = u^3$ and $p = u^{1-\alpha} - \varepsilon^\beta u^{1-\alpha-\beta} - \Delta u$.

$$u_t = -\nabla \cdot \left(u^3 \nabla \left(\Delta u - u^{1-\alpha} + \varepsilon^\beta u^{1-\alpha-\beta}\right)\right) \tag{1.2.1}$$

Here p is the total contributions from van der Waals force ($u^{1-\alpha}$ with $\alpha > 0$), the Born Repulsion force ($\varepsilon^\beta u^{1-\alpha-\beta}$ with $\varepsilon > 0$, $\beta > 0$) and surface tension effect ($-\Delta u$). We define the associated energy

$$\mathcal{E}_\varepsilon [u] = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 - \frac{u^{-\alpha}}{\alpha} + \frac{\varepsilon^\beta u^{-\alpha-\beta}}{\alpha+\beta} \right\}. \tag{1.2.2}$$

We have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_\varepsilon [u] &= \int_\Omega \left\{-\Delta u + u^{1-\alpha} - \varepsilon^\beta u^{1-\alpha-\beta}\right\} u_t \\ &= \int_\Omega \operatorname{div}\left(\mathrm{M}(u)\nabla p\right) p \\ &= -\int_\Omega \mathrm{M}(u) |\nabla p|^2 . \end{aligned}$$

Hence, for a thin film fluid at steady state, p has to be a constant. This leads to the following elliptic problem with Neumann boundary condition:

$$\begin{cases} -\Delta u = p - u^{1-\alpha}\left(1 - \left(\frac{\varepsilon}{u}\right)^\beta\right) & \text{in} \quad \Omega, \\ \nabla p = 0 & \text{in} \quad \Omega, \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2.3}$$

Similar lubrication type equations have been researched for the existence of strictly positive, smooth solution in [49, 50, 54]. In one dimensional space, the existence of solution to (1.2.1) with mass constraint have been proven in [24]. Also the authors analyzed the stability of equilibrium and investigated the asymptotic convergence to a δ distribution as $\varepsilon \to 0$. For a higher dimensional space, Chen and Jiang [32] verified this phenomena and obtained the concentration on the boundary for its singular limit.

We rewrite the energy (1.2.2) as following,

$$\mathcal{E}_{\varepsilon}\left[u\right]=\int_{\Omega}\left\{\frac{1}{2}\left|\nabla u\right|^{2}+\frac{1}{\varepsilon^{\alpha}}F\left(\frac{u-\varepsilon}{\varepsilon}\right)\right\}dx-\frac{F_{*}}{\varepsilon^{\alpha}} \tag{1.2.4}$$

where

$$F\left(s\right)=\frac{\left(1+s\right)^{-\alpha-\beta}}{\alpha+\beta}-\frac{\left(1+s\right)^{-\alpha}}{\alpha}+F_{*},\qquad F_{*}=\frac{\beta}{\alpha\left(\alpha+\beta\right)}.$$

If we take ε approaching 0, $F\left(\frac{u-\varepsilon}{\varepsilon}\right)$ tends to $F_{*}\chi_{\{u>0\}}$. We will investigate its semi limit energy minimizing problem

$$\mathcal{E}_{\varepsilon}[u]:=\int_{\Omega}\left\{\frac{1}{2}|\nabla u|^{2}+\frac{1}{\varepsilon^{2}}\chi_{\{u>0\}}\right\}dy. \tag{1.2.5}$$

with u satisfying mass constraint

$$\mathcal{H}\left(M\right)=\left\{u\in H^{1}\left(\Omega\right):u\geq 0\text{ a.e. in }\Omega\text{ and }\int_{\Omega}udy=M\right\}$$

We will prove that for such an energy minimizer problem (1.2.5), as $\varepsilon\to 0$, energy minimizers similarly converge to a Dirac measure concentrated on the boundary. Then we apply the asymptotic analysis and derive the energy formula in terms of curvature. In order to minimize the associated energy, the boundary spike has to be located at the point with maximal curvatures.

1.3 THIN FILM EQUATION WITH ATTRACTIVE VAN DER WAALS FORCE

Taking $\varepsilon = 0$ and $\alpha = 3$ in (1.2.1), we only take van der Waals force driven thin film into account in this section. Then the pressure is given by

$$p = \frac{1}{3}u^{-3} - \Delta u.$$

Fitting into the power law equation (1.1.2), $m = -1$ and $n = 3$. For this particular type thin film evolution equation, lots of research has been done about rupture. The numerical experiments in [38] imply, perturbation of uniform layer thin film leads to a rupture in finite time. Later some numerical simulations prove the existence of self-similar solution with finite time rupture [25, 26, 63]. Chou and Kwong [33] gave some theoretical proofs for the condition of initial value to have a finite time rupture.

We consider the steady states of this type of thin film equations. Mathematically, we generalize to for $\alpha > 1$,

$$p = \frac{1}{\alpha}u^{-\alpha} - \Delta u.$$

We associate (1.2.1) with an energy functional

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{\alpha(\alpha-1)}u^{1-\alpha} \right),$$

and formally, using the boundary conditions (1.1.5), (1.1.6), we have

$$\begin{aligned} \frac{d}{dt}E(u) &= \int_\Omega \nabla u \nabla u_t + \frac{1}{\alpha}u^{-\alpha}u_t \\ &= \int_\Omega \left(-\Delta u + \frac{1}{\alpha}u^{-\alpha} \right) u_t \\ &= \int_\Omega p \nabla (u^n \nabla p) \\ &= -\int_\Omega u^n |\nabla p|^2 \le 0. \end{aligned}$$

Hence, for a thin film fluid at rest the pressure p has to be a constant and u satisfies the elliptic equation

$$-\Delta u + \frac{1}{\alpha}u^{-\alpha} = p \text{ in } \Omega$$

with the Neumann boundary condition (1.1.6) and volume constraint

$$\frac{1}{|\Omega|}\int_{\Omega} u = \bar{u} \text{ for given } \bar{u}.$$

We restrict to the radial case with $\Omega = B_1(0)$ and $p = \frac{1}{\alpha}$,

$$u_{rr} + \frac{1}{r}u_r = \frac{1}{\alpha}u^{-\alpha} - \frac{1}{\alpha}.$$

Take initial value $u(0) = \eta$ and solution with $\eta = 0$ is the rupture solution [57, 56]. Every radial solution to volume prescribed problem can be constructed by this initial value problem by choosing appropriate η and k-th critical value [57]. In my dissertation, we research the continuous dependence of the average thickness $\bar{u}$ and associated energy E on $\eta \in [0, \infty)$. Especially, we provide theoretical proof for the asymptotic behavior of $\bar{u}$ and E as $\eta \to \infty$ which was verified numerically in [64]. Moreover, the description of limiting profile will be given.

2.0 SINGULAR LIMIT OF AN ENERGY MINIMIZING PROBLEM

2.1 INTRODUCTION AND MAIN RESULTS

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth bounded domain. For any $\varepsilon > 0$, we consider the energy functional

$$\mathcal{E}_\varepsilon[u] := \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}\chi_{\{u>0\}} \right\} dy \tag{2.1.1}$$

in the space

$$\mathcal{H}(M) = \left\{ u \in H^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega \text{ and } \int_\Omega u dy = M \right\}$$

where $M > 0$ is a given constant.

Such energy functional without mass constraint has been extensively studied. Here is the brief history. Caffarelli and Alt [3] showed the Lipschitz continuity for the minima and proved singularities cannot occur for minimizer in two dimensional space. Later, Alt, Caffarelli and Friedman [4] extended the result to the case with two phases using monotonicity formula and developed the full regularity theory of the free boundary $\partial\{u > 0\}$ in dimension 2 and partial regularity theory in higher dimension. In 1999, Weiss [73] claimed the existence of critical dimension k such that the free boundary is smooth if $n < k$. Later, Caffarelli, Jerison and Kenig [30] proved the full regularity result in three-dimensional space which indicated $k > 3$. Moreover, the work completed by De Silva and Jerison pointed out $k < 7$ since in 7-dimensional space the singular axisymmetric critical point of the functional is an energy minimizer. Till now, the cases $4 \leq k \leq 6$ remain open. Also more general energy functional has been studied in [1, 5, 31, 37, 62].

Under current setting with mass constraint, the energy minimizing problem can be viewed as a semi-limit of a singular elliptic equation [32, 54] modeling steady states of thin film equation with both van der Waals force and Born repulsion force. In these papers, the energy is defined in the following form,

$$\mathcal{E}_\varepsilon [u] = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^\alpha} F\left(\frac{u-\varepsilon}{\varepsilon}\right) \right\} dx - \frac{F_*}{\varepsilon^\alpha}$$

where

$$F(s) = \frac{(1+s)^{-\alpha-\beta}}{\alpha+\beta} - \frac{(1+s)^{-\alpha}}{\alpha} + F_*, \qquad F_* = \frac{\beta}{\alpha(\alpha+\beta)}.$$

If we let ϵ approaching 0, $F\left(\frac{u-\varepsilon}{\varepsilon}\right)$ tends to $F_*\chi_{\{u>0\}}$ formally. It was shown by Chen and Jiang, for thin film equation with both van der Waals force and Born repulsion force, the energy minimizing solutions converge to the limiting profile which is a Dirac measure located on the boundary. Such behavior has also been verified in one dimensional space in [18]. Abundant research on some other aspects of thin film equations including the stability of the solutions has been done by lots of authors [8, 9, 18, 19, 23, 34, 27, 35, 55, 57, 58, 59, 60, 61, 71].

The existence of energy minimizers of $\mathcal{E}_\varepsilon$ in $\mathcal{H}(m)$ follows from the direct method of calculus of variation. Moreover, we have the following asymptotic behavior of the energy minimizers:

Theorem 1. *Let $M > 0$ and $\{\varepsilon_k\}_{k=1}^\infty$ be a positive sequence converging to 0. For each $k \geq 1$, let $u_{\varepsilon_k} \in \mathcal{H}(M)$ be an energy minimizer of $\mathcal{E}_{\varepsilon_k}$ in $\mathcal{H}(M)$. Then up to subsequence if necessary, $\{u_{\varepsilon_k}\}_{k=1}^\infty$ approaches a Dirac mass supported on the boundary; that is, there exists $p \in \partial\Omega$ such that*

$$\lim_{k\to\infty} \int_\Omega u_{\varepsilon_k}(x)\varphi(x)\,dx = M\,\varphi(p) \qquad \forall \varphi \in C(\bar{\Omega}).$$

Next, we want to understand the microscopic structure of the energy minimizer near its concentration point. Let $u_\varepsilon \in \mathcal{H}(M)$ be a minimizer of $\mathcal{E}_\varepsilon$. Let $x_\varepsilon \in \Omega$ be a point where u_ε attains its maximum and $p_\varepsilon \in \partial\Omega$ be such that

$$|p_\varepsilon - x_\varepsilon| = \inf_{p\in\partial\Omega} |p - x_\varepsilon|. \tag{2.1.2}$$

Let $\delta = \varepsilon^{1/(n+1)}$. We define

$$\Omega_\delta = \left\{ \frac{x - p_\varepsilon}{\delta} : x \in \Omega \right\} \tag{2.1.3}$$

and

$$v_\delta\left(y\right) = \delta^n u_\varepsilon\left(x\right) \text{ where } y = \frac{x - p_\varepsilon}{\delta} \in \Omega_\delta. \tag{2.1.4}$$

Then one can verify that v_δ is an energy minimizer of

$$\mathbf{E}_\delta[v] := \int_{\Omega_\delta} \left\{ \frac{1}{2}|\nabla v|^2 + \chi_{\{v>0\}} \right\} dy \tag{2.1.5}$$

in the space

$$\mathbf{H}_\delta\left(M\right) = \left\{ v \in H^1\left(\Omega_\delta\right) : v \geq 0 \text{ a.e. in } \Omega_\delta \text{ and } \int_{\Omega_\delta} v dy = M \right\}.$$

Theorem 2. *Under the assumption of Theorem 1, passing to a subsequence if necessary, as $k \to \infty$, $p_{\varepsilon_k} \to p$ for some point $p \in \partial\Omega$ and $v_{\delta_k} \to v^*$, locally uniformly in*

$$\mathbb{R}^+_{\nu(p)} := \{ y \in \mathbb{R}^n \mid y \cdot \nu(p) < 0 \}$$

where $\delta_k = \varepsilon_k^{1/(n+1)}$ and $\nu(p)$ is the unit exterior normal of $\partial\Omega$ at p. Here

$$v^*\left(y\right) = A^* \ \max\{0, R^{*2} - |y|^2\}$$

where

$$R^* := \left(\frac{1}{2}\right)^{-\frac{1}{2(n+1)}} \left(\frac{(n+2)\, M}{\omega_n}\right)^{\frac{1}{n+1}}, \ A^* := \left(\frac{1}{2}\right)^{\frac{n+2}{2(n+1)}} \left(\frac{(n+2)\, M}{\omega_n}\right)^{-\frac{1}{n+1}}.$$

and ω_n denotes the volume of unit ball in $\mathbb{R}^n$.

Note that v^* is the global minimizer of

$$\mathbf{E}^*[v] := \int_{\mathbb{R}^n_+} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v>0\}} \right\} dy$$

in the space

$$\mathbf{H}^*(M) := \left\{ v \in H^1\left(\mathbb{R}^n_+\right) : \int_{\mathbb{R}^n_+} v(y)\, dy = M \text{ and } v \geq 0 \right\}.$$

Now, we are about to investigate the location of the boundary spike. After translation and rotation if necessary, we suppose the concentration point p to be the origin point. Locally the boundary of $\partial\Omega$ can be written as

$$x_n = \psi(x'), \quad x' = (x_1, \cdots, x_{n-1}), \qquad |x'| \leq \eta$$

where $\psi(0') = 0, \nabla_{x'}\psi(0') = 0'$. Consequently, the boundary of Ω_δ near the origin can be expressed as

$$y_n = \frac{1}{\delta}\psi(\delta y'), \quad y' = (y_1, \cdots, y_{n-1}), \qquad |y'| \leq \frac{\eta}{\delta}.$$

Based on the limit profile of v^*, we apply the asymptotic analysis and assume the energy minimizer has the asymptotic expansion as follows,

$$\begin{cases} D = \left\{ y \in \mathbb{R}^n \ : \ y_n > \psi(\delta y')/\delta, |y| < R + \delta R_1\left(\frac{y}{|y|}\right) + O(\delta^2) \right\}, \\ v = \frac{\lambda}{2n}[R^2 - |y|^2] + \delta v_1(y) + O(\delta^2) \qquad \forall\, y \in \bar{D}. \end{cases}$$

Here $D = \{y : v(y) > 0\}$, v and R are some constants depending on δ, λ depending on δ through R is Lagrange multipliers corresponding to the mass constraint. We know, in general, the solution does not necessarily have its mass concentrated near original point. Then some additional constraints have been added.

$$\int_{\Omega_\delta} y_i v dy = 0 \quad \forall\, i = 1, \cdots, n-1.$$

Therefore, v_1, R_1 satisfy

$$\begin{cases} -\Delta v_1 = 0 & \text{in } B_R \cap \mathbb{R}^{n+} =: B_R^+, \\ v_1 = R\partial_n v_1 & \text{on } \partial B_R \cap \mathbb{R}^{n+} =: \Gamma_R, \\ \partial_{y_n} v_1 = -\frac{\lambda}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 & \text{on } B_R' \times \{0\}, \\ R_1(y/|y|) = n\partial_n v_1(y)/\lambda & \forall\, y \in \Gamma_R. \end{cases} \tag{2.1.6}$$

Analyzing the system leads to the following theorem,

Theorem 3. *The energy of the Quasi-stationary solution* (v, D) *has the asymptotic expansion*

$$\mathbf{E}_\delta[v] \equiv \int_D \left\{ \frac{1}{2} |\nabla v|^2 + 1 \right\} = \mathbf{E}^* \left[v^*\right] - c\,(n)\, M\kappa\delta + O(\delta^2)$$

where

$$c\,(n) = \frac{(n-1)\,(n+2)\,(n+7)\,\omega_{n-1}}{\sqrt{2}(n+1)\,(n+3)\,\omega_n}$$

is a positive constant.

The above formula implies the peak should be situated near the "most curved"part of $\partial\Omega$. This type of behavior has been seen before in [67, 68] where Ni and Takagi proved that a type of semilinear elliptic equation with homogeneous Neumann boundary condition admits a least energy solution using Mountain-Pass Lemma and the solution, approximated by ground state solution in R^n, attains exactly one peak on the boundary with the maximum of the peak uniformly bounded. We all have that the points where the minimizer takes its maximum tends to a point on the boundary as $\varepsilon \to 0$. However, the difference of the behavior of our solution with theirs is that maximum of our solutions tends to be unbounded. Later, related results for the semilinear Dirichlet problem have been obtained by Ni and Wei in [69].

This chapter is organized as follows: after introduction, we prove the existence of energy minimizing solution to (2.1.1). Later, we present some preliminary results about the energy bound for (2.1.5) obtained after scaling in section 3. We derive corresponding Euler equation and prove the some regularity results in the following three sections. And then the limit profiles in the theorem 1 and 2 are obtained in section 7. We perform asymptotic analysis in section 8 and prove the existence of the solution to the linearization problem (2.1.6). Finally in section 9, we end with the derivation of the energy expansion which is the theorem 3.

2.2 EXISTENCE OF ENERGY MINIMIZING SOLUTIONS

We will show the existence of minimizers for energy function (2.1.1) in the admissible space using standard direct method of calculus of variations.

Recall the energy functional,

$$\mathcal{E}_\varepsilon[u] := \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}\chi_{\{u>0\}} \right\} dy \tag{2.2.1}$$

in the space

$$\mathcal{H}(M) = \left\{ u \in H^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega \text{ and } \int_\Omega u dy = M \right\}$$

where $M > 0$ is a given constant.

Theorem 4. *For any $\varepsilon > 0$, there exists at least one global minimizer of (2.2.1) in $\mathcal{H}(M)$.*

Proof. Taking the constant function $u_0 = \frac{M}{|\Omega|}$, we have

$$\mathcal{E}_\varepsilon[u_0] = \frac{|\Omega|}{\varepsilon^2} < \infty.$$

Since $\mathcal{E}_\varepsilon[u]$ is nonnegative, there exists a minimizing sequence $\{u_k\}_{k=1}^\infty$ in $\mathcal{H}(M)$. Then, $|\nabla u_k|$ is bounded in $L^2(\Omega)$. Applying Poincaré Inequality and $M = \int_\Omega u_k$, we have,

$$\int_\Omega |u_k - \frac{M}{|\Omega|}|^2 \leq C \int_\Omega |\nabla u_k|^2.$$

It follows that u_k is bounded in $H^1(\Omega)$. Up to a subsequence, there exists $u \in H^1(\Omega)$ such that $\{u_k\}_{k=1}^\infty$ weakly converges in $H^1(\Omega)$ and strongly converges in $L^2(\Omega)$ to u. Then, $\{u_k\}_{k=1}^\infty$ converges to u almost everywhere in Ω.

$$\int_\Omega \chi_{\{u>0\}} \leq \int_\Omega \chi_{\{u_k>0\}} \text{ and } \int_\Omega u = \lim_{k\to\infty} \int_\Omega u_k = M$$

which indicates the limit function $u \in \mathcal{H}(M)$. On the other side,

$$\int_{\Omega} |\nabla u|^2 \le \liminf_{k\to\infty} \int_{\Omega} |\nabla u_k|^2.$$

Hence,

$$\begin{aligned}\mathcal{E}_\varepsilon[u] &= \int_{\Omega} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}\chi_{\{u>0\}} \right\} dy \\ &\le \liminf_{k\to\infty} \int_{\Omega} \left\{ \frac{1}{2}|\nabla u_k|^2 + \frac{1}{\varepsilon^2}\chi_{\{u_k>0\}} \right\} dy = \liminf_{k\to\infty} \mathcal{E}_\varepsilon[u_k].\end{aligned}$$

Therefore, we conclude that u is a global minimizer in $\mathcal{H}(M)$.

□

2.3 PROPERTIES OF ENERGY

First of all, let v^* be a minimizer of

$$\mathbf{E}^*[v] = \int_{\mathbb{R}^n_+} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v>0\}} \right\} dy$$

in the admissible class

$$\mathbf{H}^*(M) := \left\{ v \in H^1\left(\mathbb{R}^n_+\right) : \int_{\mathbb{R}^n_+} v(y)\, dy = M \text{ and } v \ge 0 \right\}.$$

Regarding this limit case problem, Chen and Jiang [32] have proved the following proposition:

Proposition 1. *Up to a translation, any global minimizer of* $\mathbf{E}^*$ *in* $\mathbf{H}^*(M)$ *is of the form*

$$v^*(y) = A^* \max\{0, R^{*2} - |y|^2\}$$

where

$$\begin{cases} R^* := \left(\frac{1}{2}\right)^{-\frac{1}{2(n+1)}} \left(\frac{(n+2)M}{\omega_n}\right)^{\frac{1}{n+1}}, \\ A^* := \left(\frac{1}{2}\right)^{\frac{n+2}{2(n+1)}} \left(\frac{(n+2)M}{\omega_n}\right)^{-\frac{1}{n+1}} \end{cases}$$

and ω_n *denotes the volume of unit ball in* $\mathbb{R}^n$*. The minimum energy is given as*

$$\mathbf{e}^*(M) = \inf_{v\in\mathbf{H}^*(M)} \mathbf{E}^*[v] = 2(n+1)(\frac{\omega_n}{n+2})^{\frac{1}{n+1}}(\frac{1}{2})^{\frac{n+2}{2(n+1)}} M^{\frac{n}{n+1}}.$$

Now let v_δ be a minimizer to

$$\mathbf{E}_\delta[v] := \int_{\Omega_\delta} \left\{ \frac{1}{2}|\nabla v|^2 + \chi_{\{v>0\}} \right\} dy$$

in the admissible class

$$\mathbf{H}_\delta\left(M\right) = \left\{ v \in H^1\left(\Omega_\delta\right) : v \geq 0 \text{ a.e. in } \Omega_\delta \text{ and } \int_{\Omega_\delta} v dy = M \right\}.$$

Denote

$$\mathbf{e}_\delta(M) = \inf_{v \in \mathbf{H}_\delta(M)} \mathbf{E}_\delta[v].$$

We start with the dependence of energy on M.

Lemma 1. *For $0 < \delta < 1$, $0 < M_1 \leq M_2$,*

$$\mathbf{e}_\delta(M_1) \leq \mathbf{e}_\delta(M_2) \leq (\frac{M_2}{M_1})^2 \mathbf{e}_\delta(M_1).$$

In particular, $\mathbf{e}_\delta(M)$ is continuous in M.

Proof. Assuming v_1 is a minimizer for $\mathbf{E}_\delta[v]$ in $\mathbf{H}_\delta(M_1)$, we have

$$\frac{M_2}{M_1} v_1 \in \mathbf{H}_\delta(M_2)$$

and

$$\begin{aligned}
\mathbf{e}_\delta(M_2) \leq \mathbf{E}_\delta[\frac{M_2}{M_1} v_1] &= \int_{\Omega_\delta} \left\{ \frac{1}{2} (\frac{M_2}{M_1})^2 \left|\nabla v_1\right|^2 + \chi_{\{v_1>0\}} \right\} \\
&\leq (\frac{M_2}{M_1})^2 \int_{\Omega_\delta} \left\{ \frac{1}{2} \left|\nabla v_1\right|^2 + \chi_{\{v_1>0\}} \right\} \\
&= (\frac{M_2}{M_1})^2 \mathbf{e}_\delta(M_1).
\end{aligned}$$

Assuming v_2 is a minimizer for $\mathbf{E}_\delta[v]$ in $\mathbf{H}_\delta(M_2)$, we define

$$v_1 = \begin{cases} v_2 & \text{if } v_2 \leq \eta, \\ \eta & \text{if } v_2 > \eta \end{cases}$$

where $\eta > 0$ is chosen so that

$$\int_{\Omega_\delta} v_1 = M_1.$$

Therefore,

$$v_1 \in \mathbf{H}_\delta(M_1)$$

and

$$\mathbf{e}_\delta(M_1) \leq \mathbf{E}_\delta[v_1] = \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_1|^2 + \chi_{\{v_1>0\}} \right\}$$
$$\leq \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_2|^2 + \chi_{\{v_2>0\}} \right\} = \mathbf{e}_\delta(M_2).$$

Now for any given M and t, then if $t > 1$,

$$\mathbf{e}_\delta(M) \leq \mathbf{e}_\delta(tM) \leq t^2 \mathbf{e}_\delta(M).$$

If $t < 1$,

$$t^2 \mathbf{e}_\delta(M) \leq \mathbf{e}_\delta(tM) \leq \mathbf{e}_\delta(M).$$

Hence,

$$\lim_{t \to 1} \mathbf{e}_\delta(tM) = \mathbf{e}_\delta(M).$$

□

Next, we establish an upper bound of $\mathbf{e}_\delta(M)$.

Lemma 2. *For small $\delta > 0$,*

$$\mathbf{e}_\delta(M) \leq \mathbf{e}^*(M)[1 + O(\delta)].$$

Proof. Up to a translation and rotation, we can assume $p \in \partial\Omega$ is the origin and the unit exterior normal of $\partial\Omega$ at p is $(0, \cdots, 0, -1)$. In a small neighborhood of p we express the boundary of Ω as

$$x_n = \psi(x'), \quad x' = (x_1, \cdots, x_{n-1}), \qquad |x'| \leq \eta$$

where $\psi(0') = 0, \nabla_{x'}\psi(0') = 0'$ and $\psi_{x_i x_j}(0') = \kappa_i \delta^{ij}$, $1 \leq i, j \leq n-1$. Here κ_i is the principal curvature and

$$\kappa = \sum_{i=1}^{n-1} \kappa_i / (n-1)$$

is the mean curvature of Ω at p. Consequently, the boundary of Ω_δ near the origin can be expressed as

$$y_n = \frac{1}{\delta}\psi\left(\delta y'\right), \quad y' = \left(y_1, \cdots, y_{n-1}\right), \qquad |y'| \le \frac{\eta}{\delta}.$$

Denoting by B_r the ball of radius r centered at the origin, using the Taylor expansion

$$\frac{1}{\delta}\psi\left(\delta y'\right) = \frac{\delta}{2}\kappa_i y_i^2 + O\left(\delta^2\right),$$

we can conclude, for $r \in (0, R]$ with fixed R independent of small δ,

$$\begin{aligned} \left|\partial B_r \cap \Omega_\delta\right| - \frac{1}{2}\left|\partial B_r\right| &= -\frac{(n-1)\,\omega_{n-1}}{2}\kappa r^n \delta + O\left(\delta^2\right), \\ \left|B_r \cap \Omega_\delta\right| - \frac{1}{2}\left|B_r\right| &= -\frac{(n-1)\,\omega_{n-1}}{2\,(n+1)}\kappa r^{n+1}\delta + O\left(\delta^2\right). \end{aligned}$$

Let $v = A\left(R^{*2} - |y|^2\right)_+$ where

$$A = \frac{A^* \int_{B_{R^*} \cap \mathbb{R}^n_+} \left(R^{*2} - |y|^2\right) dy}{\int_{B_{R^*} \cap \Omega_\delta} \left(R^{*2} - |y|^2\right) dy},$$

we have

$$A = A^*\left[1 + O\left(\delta\right)\right] \text{ and } \int_{\Omega_\delta} v = M.$$

Consequently, $v \in \mathbf{H}_\delta\left(M\right)$ implies

$$\begin{aligned} \mathbf{e}_\delta(M) \le \mathbf{E}_\delta\left[v\right] &= \int_{\Omega_\delta} \left\{\frac{1}{2}\left|\nabla v\right|^2 + \chi_{\{v>0\}}\right\} dy \\ &\le \int_{\Omega_\delta} \frac{1}{2}(\frac{A}{A^*})^2 \left|\nabla v\right|^2 dy + \left|B_{R^*} \cap \Omega_\delta\right| \\ &= \mathbf{e}^*(M)[1 + O(\delta)]. \end{aligned}$$

□

Remark 1. *The above estimate gives that the boundedness of* $\mathbf{e}_\delta(M)$ *is uniform in* δ. *We can pick up* δ *small such that* $\mathbf{e}_\delta(M) \le 2\mathbf{e}^*(M)$. *Note that*

$$\mathbf{e}_\delta\left(M\right) \ge \int_{\Omega_\delta} \chi_{\{v_\delta > 0\}} dy.$$

The upper bound for $\mathbf{e}_\delta\left(M\right)$ *implies that for* δ *small enough, the minimum for* $v_\delta(y)$ *is equal to* 0. *Meanwhile, the measure of set* $\{x : v_\delta(x) > 0\}$ *is bounded above by* $2\boldsymbol{e}^*(M)$.

Lemma 3. *When* $0 < \delta < \frac{M}{|\Omega|}$,

$$\max_{y \in \overline{\Omega_\delta}} v_\delta(y) \geq \frac{M}{\mathbf{e}_\delta(M)}.$$

Proof. Since

$$\mathbf{e}_\delta(M) \geq \int_{\{v_\delta > 0\}} 1 \geq \int_{\{v_\delta > 0\}} \frac{v_\delta(y)dy}{\max v_\delta} = \frac{M}{\max v_\delta},$$

we have

$$\max_{y \in \overline{\Omega_\delta}} v_\delta(y) \geq \frac{M}{\mathbf{e}_\delta(M)}.$$

□

Analogously to [32], we use a rearrangement argument to establish a lower bound for $\mathbf{e}_\delta(M)$ and then obtain the limit of $\mathbf{e}_\delta(M)$.

Theorem 5. *For any* $M > 0$*,*

$$\liminf_{\delta \to 0^+} \mathbf{e}_\delta(M) \geq \mathbf{e}^*(M).$$

Moreover,

$$\lim_{\delta \to 0^+} \mathbf{e}_\delta(M) = \mathbf{e}^*(M).$$

Proof. Let v be a minimizer of $\mathbf{E}_\delta$ in $\mathbf{H}_\delta(M)$ and

$$\bar{v} = \max_{y \in \overline{\Omega_\delta}} v(y) \quad \text{and} \quad \underline{v} = \min_{y \in \overline{\Omega_\delta}} v(y).$$

For δ small enough, from the above remark, we know $\underline{v} = 0$. We define for any $t \in [0, \infty)$,

$$D(t) = \{x \in \Omega_\delta : v(x) > t\}, \quad \Gamma(t) = \partial D(t) \cap \Omega_\delta$$

and

$$\mu(t) = |D(t)|, \qquad \ell(t) = |\Gamma(t)|.$$

For any open interval (a, b), we have from the coarea formula [42],

$$-\mu'(t) = \int_{\Gamma(t)} \frac{1}{|\nabla v(y)|} d\mathcal{H}^{n-1}(y) \tag{2.3.1}$$

and

$$\int_{\{x\in\Omega_\delta, a<v<b\}} |\nabla v|^2 \, dy = \int_a^b \int_{\Gamma(t)} |\nabla v(y)| \, d\mathcal{H}^{n-1}(y) \, dt.$$

Using

$$|\ell(t)|^2 = \left(\int_{\Gamma(t)} 1 d\mathcal{H}^{n-1}\right)^2 \leq \int_{\Gamma(t)} \frac{1}{|\nabla v(y)|} d\mathcal{H}^{n-1} \int_{\Gamma(t)} |\nabla v(y)| \, d\mathcal{H}^{n-1},$$

we derive from (2.3.1),

$$\int_{\Gamma(t)} |\nabla v(y)| \, d\mathcal{H}^{n-1}(y) \geq \frac{|\ell(t)|^2}{-\mu'(t)}.$$

Thus,

$$\int_{\{x\in\Omega_\delta, a<v<b\}} |\nabla v|^2 \, dy \geq \int_a^b \frac{|\ell(t)|^2}{-\mu'(t)} dt.$$

Let $P(\cdot)$ be the best constant of isometric inequality:

$$P(\alpha) := \inf_{D\subset\Omega_\delta, |D|\leq\alpha} \frac{|\partial D \cap \Omega_\delta| \frac{2}{n\omega_n}}{\left(|D| \frac{2}{\omega_n}\right)^{\frac{n-1}{n}}}.$$

$P(\alpha)$ is decreasing in α. Now for small $\epsilon > 0$, since

$$\mathbf{e}_\delta(M) \geq \int_{\Omega_\delta} \chi_{\{v>\epsilon\}} = \mu(\epsilon),$$

we have

$$\mu(\epsilon) \leq \mathbf{e}^*(M)\left[1 + O(\delta)\right].$$

Also as Ω_δ has almost flat and smooth boundary $\partial\Omega_\delta = \partial\Omega/\delta$, we see that

$$P(\epsilon) = 1 + O(\delta).$$

Hence,

$$\begin{aligned}
\int_{\Omega_\delta} |\nabla v|^2 \, dy &\geq \int_\epsilon^{\bar{v}} \frac{|\ell(t)|^2}{-\mu'(t)} dt \\
&\geq \int_\epsilon^{\bar{v}} \frac{[P(\mu(t))]^2 \left(\frac{2\mu(t)}{\omega_n}\right)^{2-\frac{2}{n}} n^2\omega_n^2}{-4\mu'(t)} dt \\
&\geq (P(\epsilon))^2 \left(\frac{\omega_n}{2}\right)^{\frac{2}{n}} n^2 \int_\epsilon^{\bar{v}} \frac{\mu(t)^{2-\frac{2}{n}}}{|\mu'(t)|} dt.
\end{aligned}$$

Now define the symmetric decreasing rearrangement function w by

$$r(t) = \left(\frac{2\mu(t)}{\omega_n}\right)^{\frac{1}{n}}, \qquad w(r(t)) = t, \qquad t \in [0, \bar{v}].$$

Then

$$\mu(t) = \frac{\omega_n r(t)^n}{2}, \qquad w'(r(t))\, r'(t) = 1,$$
$$\mu'(t) = \frac{n\omega_n r^{n-1} r'(t)}{2} = \frac{n\omega_n r_n^{n-1}}{2w_r}, \qquad dt = \frac{dr}{r'(t)} = w_r dr.$$

It then follows that

$$\begin{aligned}
\int_{\Omega_\delta} |\nabla v|^2 \, dx &\geq (P(\epsilon))^2 \left(\frac{\omega_n}{2}\right)^{\frac{2}{n}} n^2 \int_\epsilon^{\bar{v}} \frac{\mu(t)^{2-\frac{2}{n}}}{|\mu'(t)|} dt \\
&\geq (P(\epsilon))^2 \left(\frac{\omega_n}{2}\right)^{\frac{2}{n}} n^2 \int_0^{r(\epsilon)} \frac{\left(\frac{\omega_n r^n}{2}\right)^{2-\frac{2}{n}}}{\left|\frac{n\omega_n r^{n-1}}{2w_r}\right|} w_r dr \\
&= \frac{[1+O(\delta)]}{2} \int_0^{r(\epsilon)} w'^2 n\omega_n r^{n-1} dr.
\end{aligned}$$

And then,

$$\begin{aligned}
\mathbf{e}_\delta(M) &= \int_{\Omega_\delta} \left\{\frac{1}{2} |\nabla v|^2 + \chi_{\{v>0\}}\right\} \\
&\geq \frac{[1+O(\delta)]}{2} \int_0^{r(\epsilon)} \left\{\frac{1}{2} w'^2 + \right\} n\omega_n r^{n-1} dr.
\end{aligned}$$

Finally, we define

$$\hat{w}(r) = \begin{cases} w(r) & \text{if } r \in [0, r(\epsilon)], \\ \epsilon + r(\epsilon) - r & \text{if } r \in [r(\epsilon), r(\epsilon) + \epsilon], \\ 0 & \text{if } r \in [r(\epsilon) + \epsilon, \infty). \end{cases}$$

Then we have

$$\int_0^\infty \hat{w}'^2 r^{n-1}dr - \int_0^{r(\epsilon)} w'^2 r^{n-1}dr = \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} \hat{w}'^2 r^{n-1}dr$$
$$\leq \epsilon \left| r(\epsilon) + \epsilon \right|^{n-1} = \epsilon \left| \left(\frac{2\mu(\epsilon)}{\omega_n} \right)^{\frac{1}{n}} + \epsilon \right|^{n-1}$$
$$\leq \epsilon \left| \left(\frac{2\mathbf{e}_\delta(M)}{\omega_n} \right)^{\frac{1}{n}} + \epsilon \right|^{n-1} = O(\epsilon).$$

Meanwhile

$$\int_0^{r(\epsilon)+\epsilon} r^{n-1}dr - \int_0^{r(\epsilon)} r^{n-1}dr$$
$$\leq \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} r^{n-1}dr$$
$$\leq \epsilon \left| r(\epsilon) + \epsilon \right|^{n-1} = O(\epsilon)$$

and

$$\hat{M} := \frac{n\omega_n}{2} \int_0^\infty \hat{w} r^{n-1}dr = \frac{n\omega_n}{2} \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} \hat{w} r^{n-1}dr + \int_{\{v>\epsilon\}} v(x)\,dx$$
$$\geq M - \int_{\{v\leq\epsilon\}} v(x)\,dx \geq M - \epsilon \left|\Omega_\delta\right|.$$

Thus, we obtain

$$\mathbf{e}_\delta(M) \geq \frac{[1+O(\delta)]}{2} \int_0^{r(\epsilon)} \left\{ \frac{1}{2} w'^2 + 1 \right\} n\omega r^{n-1}dr$$
$$\geq [1+O(\delta)] \left\{ \int_{\mathbb{R}_+^n} \left\{ \frac{1}{2} \left|\nabla \hat{w}\right|^2 + \chi_{\{\hat{w}>0\}} \right\} - O(\epsilon) \right\}$$
$$\geq [1+O(\delta)] \left\{ \mathbf{e}^*\left(\hat{M}\right) - O(\epsilon) \right\}$$
$$\geq [1+O(\delta)] \left\{ \mathbf{e}^*\left(M - \epsilon\left|\Omega_\delta\right|\right) - O(\epsilon) \right\}.$$

Letting $\epsilon \to 0$, we obtain

$$\mathbf{e}_\delta(M) \geq [1+O(\delta)]\, \mathbf{e}^*(M).$$

Taking $\delta \to 0$,

$$\liminf_{\delta\to 0^+} \mathbf{e}_\delta(M) \geq \mathbf{e}^*(M).$$

The assertion is completely proved due to Lemma 2. □

2.4 EULER-LAGRANGE EQUATION

In this section we are going to derive the Euler-Lagrange equation for the minimizer v_δ. Firstly, we prove that v_δ is continuous inside Ω_δ.

Theorem 6. *For any compact set $K \subset \Omega_\delta$, there exists a constant C such that*

$$|v_\delta(x) - v_\delta(y)| \le C|x-y| \log(\frac{1}{|x-y|}) \tag{2.4.1}$$

if $x, y \in K, |x-y| < r_0$ with r_0 small.

Proof. Let $B_r(y) \subset \Omega_\delta$ be any ball of radius r with center y and $u \in H^1(\Omega_\delta)$ be the unique function satisfying

$$\Delta u = 0 \text{ in } B_r(y) \text{ and } u = v_\delta \text{ in } \Omega_\delta \backslash \bar{B}_r(y).$$

Then let $M_r = \int_{\Omega_\delta}(v_\delta - u)$ then $\int_{\Omega_\delta} u = M - M_r$ and

$$\begin{aligned} |M_r| &\le \int_{B_r(y)} |v_\delta - u| \\ &\le Cr^{n/2}(\int_{B_r(y)} |v_\delta - u|^2)^{1/2} \\ &\le Cr^{n/2+1}(\int_{B_r(y)} |\nabla v_\delta - \nabla u|^2)^{1/2} && (2.4.2) \\ &= Cr^{n/2+1}(\int_{B_r(y)} |\nabla v_\delta|^2 - \int_{B_r(y)} |\nabla u|^2)^{1/2} && (2.4.3) \\ &\le Cr^{n/2+1}(\mathbf{e}_\delta(M))^{1/2} \le Cr^{n/2+1} && (2.4.4) \end{aligned}$$

where $C = C(M, n)$ is some constant depending on the total mass M and dimension n. Here (2.4.2) follows from the Poincaré Inequality and the last step (2.4.4) holds according to Lemma 2. We choose r small such that $|M_r| \le \frac{M}{2}$. Define $\widetilde{u} = ku$, where $k = \frac{M}{M - M_r}$. Note that $\frac{2}{3} \le k \le 2$. Since $\widetilde{u} \in \mathbf{H}_\delta(M)$, we can derive,

$$\begin{aligned} &0 \le \mathbf{E}_\delta(\widetilde{u}) - \mathbf{E}_\delta(v_\delta) \\ &= \int_{\Omega_\delta} \left\{ \frac{1}{2} k^2 |\nabla u|^2 + \chi_{\{u>0\}} \right\} - \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_\delta|^2 + \chi_{\{v_\delta>0\}} \right\} \\ &= \int_{B_r(y)} (\chi_{\{u>0\}} - \chi_{\{v_\delta>0\}}) + \frac{k^2-1}{2} \int_{\Omega_\delta} |\nabla v_\delta|^2 + \frac{k^2}{2} \int_{B_r(y)} (|\nabla u|^2 - |\nabla v_\delta|^2) \\ &\le C_1 r^n + C_2(k^2 - 1) - \frac{1}{2} k^2 \int_{B_r(y)} |\nabla(u - v_\delta)|^2. \end{aligned}$$

C_1, $C_2 = C_2(M, n)$ are constants following Lemma 2. Therefore,

$$\begin{aligned}\int_{B_r(y)} |\nabla u - \nabla v_\delta|^2 &\leq C_1 \frac{2}{k^2} r^n + C_2 \frac{2(k^2-1)}{k^2} \\ &\leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 M_r.\end{aligned} \tag{2.4.5}$$

Plug (2.4.2) into (2.4.5), we obtain

$$\int_{B_r(y)} |\nabla(u - v_\delta)|^2 \leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 r^{n/2+1} (\int_{B_r(y)} |\nabla(u - v_\delta)|^2)^{1/2}.$$

Consequently, solving the above quadratic equation yields

$$\int_{B_r(y)} |\nabla v_\delta|^2 - \int_{B_r(y)} |\nabla u|^2 = \int_{B_r(y)} |\nabla u - \nabla v_\delta|^2 \leq C r^n$$

where $C = C(M, n)$ is a constant.. Proceeding as [4] Theorem 2.1, we have finished the proof of the above estimate. □

For convenience, we will suppress the subscript δ here and let $v = v_\delta$. $D = \{y \in \Omega_\delta : v > 0\}$ is an open set as a result of the continuity. By the standard calculus of variation, for $\forall \zeta \in C_0^\infty(D)$ with $\int_D \zeta dy = 0$ and ε sufficiently small so that $v + \varepsilon\zeta > 0$ in D,

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\mathbf{E}_\delta[v + \varepsilon\zeta] - \mathbf{E}_\delta[v]) = \int_D \nabla v \cdot \nabla \zeta dy.$$

We can derive that

$$\Delta v = -\lambda_\delta \text{ in } D$$

where λ_δ is the Lagrange multiplier.

The next theorem shows that, in a generalized sense, on the free boundary $\partial D \cap \Omega_\delta$,

$$\partial_\nu v = -\sqrt{2}.$$

Theorem 7. *If $v = v_\delta$ is a minimizer of $\mathbf{E}_\delta[v]$, then*

$$\lim_{\epsilon \to 0} \int_{\partial\{v > \epsilon\}} (|\nabla v|^2 - 2) \eta \cdot \nu d\mathcal{H}^{n-1} = 0$$

for every $\eta \in C_0^\infty(\Omega, \mathbb{R}^n)$ where ν is the outer normal vector.

Proof. For $\eta \in C_0^\infty(\Omega, \mathbb{R}^n)$, define $\tau_\epsilon(y) = x + \epsilon\eta(y)$ for $\epsilon > 0$ small. Then, it follows that

$$D\tau_\epsilon = I + \epsilon D\eta \ \text{ and } \ \det D\tau_\epsilon = 1 + \epsilon\nabla\cdot\eta + O(\epsilon^2).$$

Let $v_\epsilon(\tau_\epsilon(y)) = v(y)$. The mass M_ϵ of v_ϵ is obtained by

$$\begin{aligned} M_\epsilon &= \int_\Omega v_\epsilon(\tau_\epsilon(y)) \det D\tau_\epsilon dy = \int_D v(y) \det D\tau_\epsilon dy \\ &= M + \epsilon \int_D v\nabla\cdot\eta dy + O(\epsilon^2). \end{aligned}$$

Since $\frac{M}{M_\epsilon} v_\epsilon \in \mathbf{H}_\delta(M)$,

$$\begin{aligned} 0 &\le \mathbf{E}_\delta[\frac{M}{M_\epsilon} v_\epsilon] - \mathbf{E}_\delta[v] \\ &= \int_D \left[\frac{1}{2}\left(\frac{M}{M_\epsilon}\right)^2 |\nabla v (D\tau_\epsilon)^{-1}|^2 + 1\right] \det D\tau_\epsilon dy - \int_D \left[\frac{1}{2}|\nabla v|^2 + 1\right] dy \\ &= \int_D \left[\frac{1}{2}\left(\frac{M}{M_\epsilon}\right)^2 |\nabla v (I - \epsilon D\eta)|^2 + 1\right] (1 + \epsilon\nabla\cdot\eta) dy - \int_D \left[\frac{1}{2}|\nabla v|^2 + 1\right] dy \\ &= \frac{1}{2}\left(1 - \frac{\epsilon\int_D v\nabla\cdot\eta dy}{M}\right)^2 \int_D \left[|\nabla v|^2 - 2\epsilon\nabla v\cdot D\eta\cdot\nabla v\right](1 + \varepsilon\nabla\cdot\eta) dy \\ &- \int_D \left[\frac{1}{2}|\nabla v|^2 + 1\right] dy + \epsilon\int_D \nabla\cdot\eta dy \\ &= \epsilon\left(\frac{1}{2}\int_D \left[-2\nabla v\cdot D\eta\cdot\nabla v + |\nabla v|^2\nabla\cdot\eta + 2\nabla\cdot\eta - 2v\nabla\cdot\eta\frac{\int_D |\nabla v|^2 dy}{M}\right] dy\right). \end{aligned}$$

The linear term in ε must vanish, giving

$$\begin{aligned} 0 &= \int_D \left[-2\nabla v\cdot D\eta\cdot\nabla v + |\nabla v|^2\nabla\cdot\eta + 2\nabla\cdot\eta - 2v\nabla\cdot\eta\frac{\int_D |\nabla v|^2 dy}{M}\right] dy \\ &= \lim_{\epsilon\to 0}\left(\int_{\{v>\epsilon\}} 2(\Delta v + \lambda)\nabla v\cdot\eta dy - \int_{\partial\{v>\epsilon\}} (|\nabla v|^2 - 2 + 2\lambda v)\eta\cdot\nu d\mathcal{H}^{n-1}\right) \\ &= \lim_{\epsilon\to 0}\left(\int_{\{v>\epsilon\}} 2(\Delta v + \lambda)\nabla v\cdot\eta dy - \int_{\partial\{v>\epsilon\}} (|\nabla v|^2 - 2 + 2\lambda\epsilon)\eta\cdot\nu d\mathcal{H}^{n-1}\right) \end{aligned}$$

where $\lambda = \frac{\int_D |\nabla v|^2 dy}{M}$. Therefore,

$$\lim_{\epsilon\to 0}\int_{\partial\{v>\epsilon\}} (|\nabla v|^2 - 2)\eta\cdot\nu d\mathcal{H}^{n-1} = 0.$$

□

So far we have shown that in the sense of Theorem 7, $v = v_\delta$ is a weak solution of the Euler-Lagrange equation

$$\begin{cases} \Delta v = -\lambda_\delta & \text{in} \quad D = \{y \in \Omega_\delta : v > 0\} \\ v = 0 \text{ and } \partial_\nu v = -\sqrt{2} & \text{on} \quad \partial D \cap \Omega_\delta, \\ \partial_n v = 0 & \text{on} \quad \partial D \cap \partial\Omega_\delta, \\ \int_D v\,(y)\,dy = M, \end{cases} \tag{2.4.6}$$

where ν is the unit outer normal and the constant λ_δ is the Lagrange multiplier such that for δ small,

$$\lambda_\delta = \frac{\int_D |\nabla v|^2 dy}{M} \leq \frac{2\mathbf{e}_\delta(M)}{M} \leq \frac{4\mathbf{e}^*(M)}{M}.$$

2.5 UNIFORM HÖLDER CONTINUITY

In this section, we prove the Uniform Hölder Continuity for the minimizer v_δ. we need the following uniform Poincaré inequality.

Lemma 4. *For any open connected domain Ω in class Θ defined by*

$$\Theta = \{\Omega(a,f) \text{ with } 0 \leq a \leq 1, f \in C^2(B_1^{n-1}), \|f\|_{C^2} \leq \varepsilon < \frac{1}{2} \text{ with } f(0) = f'(0) = 0\}$$

where

$$\Omega(a,f) = \{\forall x = (x', x_n) \in B_1(0),\ x_n < a + f(x')\},$$

there exists a uniform C such that

$$(\int_{B_r(0)\cap\Omega} u^2)^{1/2} \leq Cr(\int_{B_r(0)\cap\Omega} |\nabla u|^2)^{1/2}$$

for $\forall u \in H^1(\Omega)$ with $u = 0$ on $\partial B_r(0) \cap \Omega$ and $r < 1$.

Proof. Apply the Corollary 3 in [29]. One can check that $B_1(0) \cap \Omega$ satisfies the ε-cone property. Then there exists a uniform C such that

$$(\int_{B_1(0)\cap\Omega} u^2)^{1/2} \leq C(\int_{B_1(0)\cap\Omega} |\nabla u|^2)^{1/2}$$

for all Ω in class Θ and $\forall u \in H^1(\Omega)$ with $u = 0$ on $\partial B_1(0) \cap \Omega$. Rescaling argument shows that

$$\begin{aligned}(\int_{B_r(0)\cap\Omega} u^2)^{1/2} &= (r^n \int_{B_1(0)\cap\widetilde{\Omega}} u(ry)^2)^{1/2} \\ &\leq C(r^{n+2} \int_{B_1(0)\cap\widetilde{\Omega}} |\nabla u(ry)|^2)^{1/2} = Cr(\int_{B_r(0)\cap\Omega} |\nabla u|^2)^{1/2}\end{aligned}$$

where $\widetilde{\Omega}$ is the transformation of Ω after scaling which still belongs to class Θ. □

It is ready to make a comparison with a harmonic function in any small ball and obtain the growth of local integrals.

Lemma 5. *For any $y \in \Omega_\delta$ and $r > 0$,*

$$\int_{B_r(y)\cap\Omega_\delta} |\nabla v_\delta|^2 \leq \int_{B_r(y)\cap\Omega_\delta} |\nabla v|^2 + Cr^n \tag{2.5.1}$$

holds for any $v \in H^1(\Omega_\delta)$ satisfying v is harmonic in $B_r(y) \cap \Omega_\delta$ and $v = u$ in $\Omega_\delta \backslash B_r(y)$. Here C is a constant depending on m.

Proof. Let $B_r(y)$ be any ball of radius r with center point y in Ω_δ and define function $v \in H^1(\Omega_\delta)$ satisfying

$$\Delta v = 0 \text{ in } B_r(y) \cap \Omega_\delta \text{ and } v = v_\delta \text{ in } \Omega_\delta \backslash B_r(y).$$

Then let $M_r = \int_{\Omega_\delta}(v_\delta - v)$ then $\int_{\Omega_\delta} v = M - M_r$ and

$$\begin{aligned}|m_r| &\leq \int_{\Omega_\delta} |v_\delta - v| \leq \int_{B_r(y)\cap\Omega_\delta} |v_\delta - v| \\ &\leq Cr^{n/2}(\int_{B_r(y)\cap\Omega_\delta} |v_\delta - v|^2)^{1/2} \leq Cr^{n/2+1}(\int_{B_r(y)\cap\Omega_\delta} |\nabla v_\delta - \nabla v|^2)^{1/2} &(2.5.2)\\ &\leq Cr^{n/2+1}(\int_{B_r(y)\cap\Omega_\delta} |\nabla v_\delta|^2)^{1/2} \leq Cr^{n/2+1}\mathbf{e}_\delta(M) \leq Cr^{n/2+1} &(2.5.3)\end{aligned}$$

where $C = C(M)$ is some constant. (2.5.2) follows from the uniform Poincaré Inequality (Lemma 4) due to the fact that the smooth boundary of Ω_δ is almost flat if we take r so small. The last step (2.5.3) holds according to Lemma 2. We choose δ to be small so that $|M_r| \leq \frac{M}{2}$. Define $\widetilde{v} = kv$, where $\frac{2}{3} \leq k = \frac{M}{M-M_r} \leq 2$. Since $\widetilde{v} \in \mathbf{H}_\delta(M)$, we can derive,

$$\begin{aligned}
&0 \leq \mathbf{E}_\delta(\widetilde{v}) - \mathbf{E}_\delta(v_\delta) \\
&= \int_{\Omega_\delta} \left\{ \frac{1}{2} k^2 |\nabla v|^2 + \chi_{\{\widetilde{v}>0\}} \right\} - \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_\delta|^2 + \chi_{\{v_\delta>0\}} \right\} \\
&= \int_{B_r(y)\cap\Omega_\delta} \left(\chi_{\{\widetilde{v}>0\}} - \chi_{\{v_\delta>0\}} \right) + \frac{k^2-1}{2} \int_{\Omega_\delta} |\nabla v_\delta|^2 + \frac{k^2}{2} \int_{B_r(y)\cap\Omega_\delta} \left(|\nabla v|^2 - |\nabla v_\delta|^2 \right) \\
&\leq C_1 r^n + C_2(k^2-1) - \frac{1}{2} k^2 \int_{B_r(y)\cap\Omega_\delta} |\nabla v - \nabla v_\delta|^2.
\end{aligned}$$

C_1, $C_2 = C_2(M)$ are constants following Lemma 2. Therefore,

$$\begin{aligned}
\int_{B_r(y)\cap\Omega_\delta} |\nabla v - \nabla v_\delta|^2 &\leq C_1 \frac{2}{k^2} r^n + C_2 \frac{2(k^2-1)}{k^2} \\
&\leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 M_r.
\end{aligned} \tag{2.5.4}$$

Plug (2.5.2) into (2.5.4), we obtain

$$\int_{B_r(y)\cap\Omega_\delta} |\nabla v - \nabla v_\delta|^2 \leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 r^{n/2+1} (\int_{B_r(y)\cap\Omega_\delta} |\nabla v - \nabla v_\delta|^2)^{1/2}.$$

Consequently, solving the above quadratic equation yields

$$\int_{B_r(y)\cap\Omega_\delta} |\nabla v_\delta|^2 - \int_{B_r(y)\cap\Omega_\delta} |\nabla v|^2 = \int_{B_r(y)\cap\Omega_\delta} |\nabla v - \nabla v_\delta|^2 \leq C r^n$$

where $C = C(M)$ is a constant only depending on M. □

Checking the harmonic function v in the right hand side of (2.5.1) which satisfies the Neumann boundary condition on the boundary, we quote Lemma 9 in [32] which gives the estimate.

Lemma 6. *Let $0 < \varepsilon \leq 1$. For any $\tilde{\alpha} \in (0,1)$, there exist $r_0 > 0$ and $K_{\tilde{\alpha}} > 1$ such that for any $y \in \Omega_\delta$ and $r \in (0, r_0]$ and for any v satisfying*

$$\Delta v = 0 \ in \ \Omega_\delta \cap B_r(y), \ \partial_\nu v = 0 \ on \ \partial\Omega_\delta \cap B_r(y),$$

we have for any $\sigma \in (0,1)$,

$$\int_{B_{\sigma r}(y)\cap\Omega_\delta} |\nabla v|^2 \leq K_{\tilde{\alpha}} \sigma^{n-2+2\tilde{\alpha}} \int_{B_r(y)\cap\Omega_\delta} |\nabla v|^2. \tag{2.5.5}$$

Combining (2.5.1) and (2.5.5) gives the core lemma regarding the growth of the Dirichlet integral for v_δ. This is the key step to show C^α continuity.

Lemma 7. *Let $0 < \delta \leq 1$ and $\alpha \in (0,1)$. There exists $r_0 > 0$ such that for any $y \in \Omega_\delta$ and $r \in (0, r_0]$,*

$$\int_{B_r(y)\cap\Omega_\delta} |\nabla v_\delta|^2 \leq C_3 r^{n-2+2\alpha}.$$

Here we can take

$$C_3 = \inf_{\tilde{\alpha}\in(\alpha,1),\ 0<\sigma<K_{\tilde{\alpha}}^{-\frac{1}{2(\tilde{\alpha}-\alpha)}}} \left\{ \frac{\sqrt{C_1} r_0^{1-\alpha} \sigma^{1-\alpha-n/2}}{1-\sqrt{K_{\tilde{\alpha}}}\sigma^{\tilde{\alpha}-\alpha}} + \left(\frac{2\mathbf{e}_\delta(M)}{(\sigma r_0)^{n-2+2\alpha}} \right)^{\frac{1}{2}} \right\}.$$

Proof. For any $\alpha \in (0,1)$, let $\tilde{\alpha} \in (\alpha, 1)$. Let r_0 be defined in Lemma 6. For any $y \in \Omega_\delta$ and $r \in (0, r_0]$, for simplicity we denote $\tilde{B}_r = B_r(y) \cap \Omega_\delta$. Let v be the unique harmonic function in $\tilde{B}_r$ satisfying

$$v = v_\delta \text{ in } \Omega_\delta \cap \partial B_r(y), \ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\delta \cap B_r(y).$$

We have from Lemma 5,

$$\int_{\tilde{B}_r} |\nabla(v_\delta - v)|^2 = \int_{\tilde{B}_r} |\nabla v_\delta|^2 - \int_{\tilde{B}_r} |\nabla v|^2 \leq C r^n.$$

For any $\sigma \in (0,1)$, we have

$$\begin{aligned}
\left(\int_{\tilde{B}_{\sigma r}} |\nabla v_\delta|^2 \right)^{1/2} &\leq \left(\int_{\tilde{B}_{\sigma r}} |\nabla(v_\delta - v)|^2 \right)^{1/2} + \left(\int_{\tilde{B}_{\sigma r}} |\nabla v|^2 \right)^{1/2} \\
&\leq \left(\int_{\tilde{B}_r} |\nabla(v_\delta - v)|^2 \right)^{1/2} + \left(K_{\tilde{\alpha}} \sigma^{n-2+2\tilde{\alpha}} \int_{\tilde{B}_r} |\nabla v|^2 \right)^{1/2} \\
&\leq (C r^n)^{1/2} + \left(K_{\tilde{\alpha}} \sigma^{n-2+2\tilde{\alpha}} \int_{\tilde{B}_r} |\nabla v_\delta|^2 \right)^{1/2}.
\end{aligned}$$

Here in the second inequality, we have applied Lemma 6 to the second term on the right-hand side. Divide both sides by $(\sigma r)^{n/2-1+\alpha}$ and define

$$\phi(r) = \left(\frac{1}{r^{n-2+2\alpha}} \int_{\tilde{B}_r} |\nabla v_\delta|^2\right)^{\frac{1}{2}}.$$

We have

$$\begin{aligned}\phi(\sigma r) &\leq \sqrt{C} r^{1-\alpha} \sigma^{1-\alpha-n/2} + \sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha} \phi(r) \\ &\leq \sqrt{C} r_0^{1-\alpha} \sigma^{1-\alpha-n/2} + \sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha} \phi(r).\end{aligned}$$

Choose σ so that $\sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha} < 1$. A simple induction then gives for any $r \in (\sigma r_0, r_0]$ and for any $k \in \mathbb{N}$.

$$\begin{aligned}\phi(r) &\leq \left(\frac{1}{(\sigma r_0)^{n-2+2\alpha}} \int_{\tilde{B}_{r_0}} |\nabla v_\delta|^2\right)^{\frac{1}{2}} \\ &\leq \left(\frac{2\mathbf{e}_\delta(m)}{(\sigma r_0)^{n-2+2\alpha}}\right)^{\frac{1}{2}}.\end{aligned}$$

and furthermore,

$$\begin{aligned}\phi(\sigma^k r) &\leq \frac{\sqrt{C} r_0^{1-\alpha} \sigma^{1-\alpha-n/2} \left[1 - \left(\sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha}\right)^k\right]}{1 - \sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha}} + \left(\sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha}\right)^{k+1} \phi(r) \\ &\leq \frac{\sqrt{C} r_0^{1-\alpha} \sigma^{1-\alpha-n/2}}{1 - \sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha}} + \max_{r \in [\sigma r_0, r_0]} \phi(r) \\ &\leq \frac{\sqrt{C} r_0^{1-\alpha} \sigma^{1-\alpha-n/2}}{1 - \sqrt{K_{\tilde{\alpha}}} \sigma^{\tilde{\alpha}-\alpha}} + \left(\frac{2\mathbf{e}_\delta(M)}{(\sigma r_0)^{n-2+2\alpha}}\right)^{\frac{1}{2}}.\end{aligned}$$

□

A uniform Hölder bound for v_δ follows from the decay estimate of Lemma 7:

Theorem 8. *Let $M > 0$, $\delta > 0$ and v_δ be a minimizer of $\mathbf{E}_\delta$ in $\mathbf{H}_\delta(M)$. There exists a constant C such that for given small δ_0 and any $0 < \delta \leq \delta_0$,*

$$\|v_\delta\|_{C^\alpha(\Omega_\delta)} \leq C.$$

Therefore, v_δ is uniformly bounded in δ.

Proof . Applying Poincaré's inequality, we have for any $y \in \Omega_\delta$ and $r \in (0, r_0]$,

$$\inf_{c\in\mathbb{R}} \int_{B_r(y)\cap\Omega_\delta} |v_\delta - c|^2 \le Cr^2 \int_{B_r(y)\cap\Omega_\delta} |\nabla v_\delta|^2 \le C_4 r^{n+2\alpha}.$$

Hence, v_δ is in Campanato space $L^{2,n+2\alpha}(\Omega_\delta)$. Following from Theorem 1.2 in page 70 of [43], v_δ is Hölder continuous with αth Hölder seminorm bounded by constant C_4 which is independent of δ. Since

$$2\frac{\mathbf{e}^*(M)}{F^*} \ge \frac{\mathbf{e}_\delta(M)}{F^*} \ge \int_{\Omega_\delta} \chi_{\{v_\delta>0\}} dy$$

we are able to choose $R \ge (\frac{2\mathbf{e}^*(M)}{\omega_n F^*})^{1/n}$ such that $\forall y \in D = \{y : v_\delta > 0\}$, there exists $z \in B_R(y) \cap D^c$,then $v_\delta(z) = 0$.

$$v_\delta(y) \le v_\delta(z) + C_4 R^\alpha = C_4 R^\alpha.$$

Therefore, $\sup v_\delta$ is uniformly bounded which ends the proof of the theorem. □

2.6 UNIFORM LIPSCHITZ CONTINUITY

The main goal of this section is to prove the uniform Lipschitz continuity of v_δ in order to obtain the convergence of minimizer sequence. The idea is based on the work by Caffarelli and H.W.Alt [3] with Dirichlet boundary setting. The mass constraint is the new technical difficulty here and we requires uniform global estimate involving boundary under Neumann boundary setting which is not adjusted in [3]. In this section, we assume δ is small and all the constants are independent of such uniformly small δ.

Theorem 9. *Let $M > 0$, $\delta > 0$ and v_δ be a minimizer of $\mathbf{E}_\delta$ in $\mathbf{H}_\delta(M)$. There exists a constant $C = C(M, n)$ such that, for δ_0 sufficiently small and any $0 < \delta \le \delta_0$,*

$$\|\nabla v_\delta\|_{L^\infty(\Omega_\delta)} \le C.$$

Firstly, we prove the following lemma.

Lemma 8. *Let* $v \in \mathbf{H}_\delta[M]$ *be a minimizer of* $\mathbf{E}_\delta[v]$*. Then for any small ball* $B_r \subset \Omega_\delta$*,*

$$\frac{1}{r} \fint_{\partial B_r} v \geq C \quad \textit{implies } v > 0 \textit{ in } B_r$$

where C *is positive constant independent of* δ*.*

Proof. Take harmonic function u such that

$$\Delta u = 0 \text{ in } B_r \text{ and } u = v \text{ in } \Omega_\delta \backslash B_r.$$

Case 1: $\int_{B_r} u \geq \int_{B_r} v$. Since $\mathbf{e}_\delta(M)$ is increasing in M, we have

$$\int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v>0\}} \right\} dy \leq \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla u|^2 + \chi_{\{u>0\}} \right\} dy$$

It follows that

$$\int_{B_r} \frac{1}{2} |\nabla (v-u)|^2 dy \leq \int_{B_r} \chi_{\{v=0\}} dy. \tag{2.6.1}$$

Since (2.6.1) is scaling invariant, we just take $B_r = B_1(0)$. For $|z| \leq \frac{1}{2}$, define

$$v_z(x) = v((1-|x|)z + x)$$

and

$$u_z(x) = u((1-|x|)z + x).$$

Note that the map $x \longmapsto (1-|x|)z + x$ is an isomorphism from $B_1(0)$ to itself. Also for $\forall \xi \in \partial B_1$, define

$$r_\xi := \inf\{r | \frac{1}{8} \leq r \leq 1 \text{ and } v_z(r\xi) = 0\}$$

if the set is nonempty and $r_\xi = 1$ if the set is empty. For almost all $\xi \in \partial B_1$,

$$u_z(r_\xi \xi) = \int_{r_\xi}^1 \frac{d}{dr}(v_z - u_z)(r\xi) dr \leq \sqrt{1-r_\xi} (\int_{r_\xi}^1 |\nabla (v_z - u_z)(r\xi)|^2 dr)^{\frac{1}{2}}. \tag{2.6.2}$$

Also using Green function $G(x,y)$ with $\Delta G(x,y) = \delta_x$ for $x \in B_1(0)$,

$$u_z(r_\xi \xi) = \int_{\partial B_1} \frac{\partial G(r_\xi \xi, y)}{\partial y} u_z(y) dS(y) \geq c(n)(1-r_\xi) \fint_{\partial B_1} v. \tag{2.6.3}$$

Combining (2.6.2) and (2.6.3), we have,

$$\int_{r_\xi}^{1} |\nabla(v_z - u_z)(r\xi)|^2 dr \geq C(n)(1 - r_\xi)(\fint_{\partial B_1} v)^2.$$

Integrating over ξ and then integrating over z,

$$C(n)\int_{B_1} |\nabla(v_z - u_z)|^2 \geq \int_{B_1} \chi_{\{v=0\}}(\fint_{\partial B_1} v)^2. \tag{2.6.4}$$

Together with (2.6.1), we obtain,

$$C(n)\int_{B_1} \chi_{\{v=0\}} dy \geq \int_{B_1} \chi_{\{v=0\}}(\fint_{\partial B_1} v)^2. \tag{2.6.5}$$

Case 2: $\int_{B_r} u < \int_{B_r} v$. We define

$$\tilde{u} = u + \frac{\lambda}{2n}(r^2 - |x|^2)$$

where $\lambda > 0$ is chosen such that $\int_{B_r} \tilde{u} = \int_{B_r} v$. It is easy to check that

$$\begin{aligned}
\int_{B_r} |\nabla v|^2 - |\nabla \tilde{u}|^2 &= \int_{B_r} |\nabla v - \nabla \tilde{u}|^2 + 2\int_{B_r} \nabla(v - \tilde{u})\nabla \tilde{u} \\
&= \int_{B_r} |\nabla v - \nabla \tilde{u}|^2 - 2\int_{B_r} (v - \tilde{u})\Delta \tilde{u} \\
&= \int_{B_r} |\nabla v - \nabla \tilde{u}|^2 + 2\lambda\int_{B_r} (v - \tilde{u}) = \int_{B_r} |\nabla v - \nabla \tilde{u}|^2.
\end{aligned} \tag{2.6.6}$$

Then (2.6.1) follows. Repeat the process in Case 1. It remains to check (2.6.3). By the definition of Green function, $G(x, y) < 0$ for $x, y \in B_1(0)$.

$$\begin{aligned}
\tilde{u}_z(r_\xi \xi) &= \int_{\partial B_1} \frac{\partial G(r_\xi \xi, y)}{\partial y} \tilde{u}_z dS(y) + \int_{B_1} \Delta \tilde{u}_z G(r_\xi \xi, y) \\
&\geq C(n)(1 - r_\xi)\fint_{\partial B_1} \tilde{u}_z - \lambda \int_{B_1} G(r_\xi \xi, y) \\
&\geq C(n)(1 - r_\xi)\fint_{\partial B_1} v.
\end{aligned} \tag{2.6.7}$$

Therefore for both cases, we have (2.6.5), that is,

$$C(n)\int_{B_1} \chi_{\{v=0\}} dy \geq \int_{B_1} \chi_{\{v=0\}}(\fint_{\partial B_1} v)^2. \tag{2.6.8}$$

If $\fint_{\partial B_1} v > C(n)$, then $\int_{B_1} \chi_{\{v=0\}} = 0$ which indicates $v > 0$ in B_1.

□

Recall that $D = \{x \in \Omega_\delta : v(x) > 0\}$ and the measure of D is bounded above by $2\mathbf{e}^*(M)$. Define $\Sigma = \{x \in \Omega_\delta : v(x) = 0\}$. We have the following interior estimate,

Lemma 9. *For any $x \in \Omega_\delta \backslash \Sigma$ such that dist$(x, \Sigma) \leq$ dist$(x, \partial\Omega_\delta)$, we have*

$$|\nabla v(x)| \leq C$$

where $C = C(M, n)$ is a positive constant.

Proof. For any $x \in \Omega_\delta$, take the maximal ball $B_r(x) \subset D = \Omega_\delta \backslash \Sigma$. Since the measure of D is bounded above, then $r \leq C(M, n)$. Since $\Delta v = -\lambda_\delta$ in $B_r(x)$, then we can rewrite v as

$$v = v_* + \lambda_\delta \frac{r^2 - |x|^2}{2n} \text{ where } \Delta v_* = 0 \text{ in } B_r(x). \tag{2.6.9}$$

For $\text{dist}(x, \Sigma) < \text{dist}(x, \partial\Omega_\delta)$, $\partial B_r(x)$ does not touch $\partial\Omega_\delta$. Then for arbitrary small ϵ, $B_{r+\epsilon}(x) \cap D$ is nonempty which follows,

$$\frac{1}{r+\epsilon} \fint_{\partial B_{r+\epsilon}(x)} v \leq C(n).$$

Take $\epsilon \to 0$, then

$$\frac{1}{r} \fint_{\partial B_r(x)} v_* = \frac{1}{r} \fint_{\partial B_r(x)} v \leq C(n).$$

Consequently,

$$|\nabla v| \leq |\nabla v_*| + \frac{\lambda}{n} r \leq \frac{1}{r} \fint_{B_r} v_* + \frac{\lambda}{n} r \leq C(n, M).$$

For $\text{dist}(x, \Sigma) = \text{dist}(x, \partial\Omega_\delta)$, we see x as the limit of a sequence of points $\{x_n\}_{n=1}^{\infty}$ with $\text{dist}(x_n, \Sigma) < \text{dist}(x_n, \partial\Omega_\delta)$. By applying the continuity of $|\nabla v|$ in D, we will be able to finish our proof for this lemma. $\square$

In order to prove the uniform Lipschitz continuity, we have to make some boundary estimates and prove the boundedness of $|\nabla v(x)|$ for $\text{dist}(x,\Sigma) > \text{dist}(x,\partial\Omega_\delta)$. So we divide into two cases $\text{dist}(x,\Sigma) \geq r_0$ and $r_0 > \text{dist}(x,\Sigma) > \text{dist}(x,\partial\Omega_\delta)$ for fixed small r_0.

For any given small $r_0 > 0$ and $x \in D$, if $\text{dist}(x,\Sigma) \geq r_0$, we consider the following elliptic problem in $B_{r_0}(x) \cap \Omega_\delta$,

$$\begin{cases} \Delta v = -\lambda_\delta & \text{in} \quad B_{r_0}(x) \cap \Omega_\delta, \\ \partial_n v = 0 & \text{on} \quad B_{r_0}(x) \cap \partial\Omega_\delta. \end{cases} \tag{2.6.10}$$

Since $\partial\Omega$ is smooth, Neumann boundary condition allows us to perform the standard even reflection of v. Denote $d(y) := \text{dist}(y,\partial\Omega) = \text{dist}(y,p_y)$ and $n(y) := n(p_y)$ where $p_y \in \partial\Omega$. Then the resulting function

$$\tilde{v}(y) = \begin{cases} v(y) & \text{for} \quad y \in B_{r_0}(x) \cap \bar{\Omega}_\delta \\ v(y - 2n(y)d(y)) & \text{for} \quad y \in B_{r_0}(x) \backslash Omega_\delta. \end{cases} \tag{2.6.11}$$

satisfies

$$\partial_{y_i}(a_{ij}(y)\partial_{y_j}\tilde{v}) = -\lambda_\delta \text{ in } B_{\frac{1}{3}r_0}$$

where $|a_{ij} - \delta_{ij}|$ is uniformly small. Applying Lemma 6.5 and Theorem 6.6 in [44],

$$|\nabla\tilde{v}(x)| \leq C(\|\tilde{v}\|_{L^\infty} + |\lambda_\delta|).$$

According to Theorem 8, v is uniform bounded by a constant depending on total mass M and dimension n. Moreover, λ_δ is bounded by $\frac{4\mathbf{e}(\mathbf{M})}{M}$. Therefore, there exists a constant $C = C(M,n,r_0)$ such that for $x \in D$ with $\text{dist}(x,\Sigma) \geq r_0$,

$$|\nabla v(x)| \leq C. \tag{2.6.12}$$

The remaining case is $r_0 > \text{dist}(x,\Sigma) > \text{dist}(x,\partial\Omega)$. For simplicity, denote $R = \text{dist}(x,\Sigma)$. Considering $B_R(x) \cap \Omega_\delta$, we define this rescaled function

$$\hat{v}(z) = \frac{1}{R}v(y) \tag{2.6.13}$$

where $z \in B_1(x)$ and $y = x + R(z - x)$. Hence,

$$|\nabla \hat{v}(z)| = |\nabla v(y)| \text{ and } \Delta \hat{v}(z) = R\Delta v(y) = -R\lambda_\delta.$$

Also, $\partial\hat{\Omega}_\delta = \{z : y \in \partial\Omega_\delta\}$ is almost flat. Take the ball $B_{\tilde{r}}(x_0)$ with center point $x_0 = p_x - \frac{1}{2}(1+\tilde{R})n_x$ and radius $\tilde{r} = \frac{1}{2}(1+\tilde{R})$ where $\tilde{R} = \operatorname{dist}(x, \partial\hat{\Omega}_\delta)$. Roll $B_{\tilde{r}}(x_0)$ along $\partial\hat{\Omega}_\delta$ towards Σ unitl it touches Σ which results a ball $B_{\tilde{r}}(x_1)$ satisfying,

$$\tilde{r} = \operatorname{dist}(x_1, \Sigma) \leq \operatorname{dist}(x_1, \tilde{\Omega}_\delta).$$

Since the boundary is almost flat,

$$\operatorname{dist}(x_1, x_0) < 1.$$

According to Lemma 9, there exists some positive constant C independent of δ, $|\nabla \hat{v}(x_1)| \leq C$. It yields that $|\hat{v}(x_1)| \leq C$. Similarly as above, we extend $\hat{v}$ to w by even reflection,

$$w(y) = \begin{cases} \hat{v}(y) & \text{for} \quad y \in B_1(x) \cap \hat{\Omega}_\delta \\ \hat{v}(y - 2n(y)d(y)) & \text{for} \quad y \in B_1(x) \backslash \hat{\Omega}_\delta. \end{cases} \tag{2.6.14}$$

w satisfies

$$\partial_{y_i}(a_{ij}(y)\partial_{y_j} w(y)) = -\lambda_\delta R,$$

where $|a_{ij} - \delta_{ij}|$ is small and $a_{ij} = \delta_{ij}$ for $y \in B_1(x) \cap \hat{\Omega}_\delta$. We take finite series of ball $B_{\tilde{r}}(y_i)$ with $1 \leq i \leq N$ such that $\operatorname{dist}(x_0, y_1) = \operatorname{dist}(y_1, y_2) = \cdots = \operatorname{dist}(y_N, x_1) \leq \frac{1}{4}$. Apply the Harnack inequality for $\tilde{v}$ in each ball,

$$|\tilde{v}(x_0)| \leq C|\hat{v}(y_1)| \leq C^2|\tilde{v}(y_2)| \leq \cdots C^{N+1}|\tilde{v}(x_1)|.$$

Note that the Harnack inequality we used here is for function $\Delta u = -\lambda_\delta R \leq 0$ in a ball $B_R(0)$. The proof is to take the classical the Harnack inequality on harmonic function $u^* = u - \frac{\lambda_\delta R}{2n}(R^2 - |x|^2)$. Then,

$$|\tilde{v}(x_0)| \leq C.$$

Now take ball $B_{\frac{1}{2}}(x)$ and then $x_0 \in B_{\frac{1}{2}}(x)$. Apply the Harnack inequality again in $B_{\frac{1}{2}}(x)$,

$$\|w\|_{L^\infty(B_{\frac{1}{2}}(x))} \leq C.$$

Therefore, apply the same estimate as above for $w(x)$ in $B_{\frac{1}{3}}(x)$

$$|\nabla v(x)| = |\nabla \tilde{v}(x)| = |\nabla w(x)| \leq C(\|w\|_{L^\infty(B_{\frac{1}{2}}(x))} + |\lambda_\delta R|) =: \tilde{C}.$$

where $\tilde{C}$ only depends on M, n and r_0.

2.7 SINGULAR LIMIT PROFILE

Given the total mass $M > 0$, let $\{\varepsilon_k\}_{k=1}^{\infty} \subset \left(0, \frac{M}{|\Omega|}\right)$ be a sequence such that $\lim_{k\to\infty} \varepsilon_k = 0$. Let $u_{\varepsilon_k} \in \mathcal{H}_M$ be an energy minimizer of $\mathcal{E}_{\varepsilon_k}$ in $\mathcal{H}_M$. For simplicity, we will suppress the k subscript whenever there is no confusion.

Let $x_\varepsilon \in \bar{\Omega}$ be a point where u_ε attains its maximum and $p_\varepsilon \in \partial\Omega$ be such that

$$|p_\varepsilon - x_\varepsilon| = \min_{p\in\partial\Omega} |p - x_\varepsilon| .$$

Passing to a subsequence if necessary, we can assume

$$\lim_{k\to\infty} p_{\varepsilon_k} = p^* \in \partial\Omega$$

and we denote $\nu^* = \nu(p^*)$, the unit outer normal of $\partial\Omega$ at p^*. Let Ω_δ and v_δ be defined in (2.1.3) and (2.1.4). Then v_δ is a minimizer of $\mathbf{E}_\delta$ in $\mathbf{H}(M, \Omega_\delta)$ and as $k \to \infty$,

$$\Omega_\delta \to \mathbb{R}^n_{\nu^*} := \{y \in \mathbb{R}^n \mid y \cdot \nu^* < 0\}.$$

For simplicity, after a rotation if necessary, we assume $\nu^* = (0, \cdots, 0, -1)$ and hence $\mathbb{R}^n_{\nu^*} = \mathbb{R}^n_+$.

Proposition 2. *There exist constants C_1,C_2,$C_3 > 0$ such that for any $k \in \mathbb{N}$*

$$\max_{y\in\overline{\Omega_{\delta_k}}} v_{\delta_k}(y) \geq C_1,\ \|v_{\delta_k}\|_{C^{0,\alpha}\left(\Omega_{\delta_k}\right)} \leq C_2 \text{ and } \|\nabla v_{\delta_k}\|_{L^\infty\left(\Omega_{\delta_k}\right)} \leq C_3.$$

Proof. When k is sufficiently large, we have $\delta_k \leq \frac{m}{2|\Omega|}$ and from Theorem 5, $\mathbf{e}_\delta(M) \leq 2\mathbf{e}^*(M)$, hence Lemma 3 implies

$$\max_{y\in\overline{\Omega_{\delta_k}}} v_{\delta_k}(y) \geq \frac{M}{\mathbf{e}_\delta(M)} \geq \frac{M}{2\mathbf{e}^*(M)}.$$

On the other hand, the uniform Hölder norm of v_{δ_k} follows from Theorem 8 and uniform Lipschitz continuity follows from Theorem 9. Note that the constant is independent of δ. □

Due to the uniform bound of Hölder continuity, passing to a subsequence if necessary, we can assume v_δ converges locally uniformly to a limit v^* in $\mathbb{R}^n_+$. The main goal in this section is to show that v^* is the unique energy minimizer of $\mathbf{E}^*$ with $\int_{\mathbb{R}^n_+} v^*(x)\, dx = M$.

Lemma 10. *There exists a constant $C>0$, such that for any $k\in\mathbb{N}$,*

$$\frac{|p_{\varepsilon_k}-x_{\varepsilon_k}|}{\delta_k}\le C.$$

Proof. If such constant doesn't exist, passing to a subsequence if necessary, we can assume

$$\lim_{k\to\infty}\frac{|p_{\varepsilon_k}-x_{\varepsilon_k}|}{\delta_k}=\infty.$$

For simplicity, we again suppress the k subscript. We define a blow up sequence along x_ε by

$$\tilde{\Omega}_\delta=\left\{\frac{x-x_\varepsilon}{\delta}:x\in\Omega\right\}. \tag{2.7.1}$$

Correspondingly,

$$\tilde{v}_\delta\left(y\right)=\delta^n u_\varepsilon\left(x\right)\ \text{ where } y=\frac{x-x_\varepsilon}{\delta}\in\tilde{\Omega}_\delta. \tag{2.7.2}$$

Then $\tilde{v}_\delta$ is a minimizer of $\mathbf{E}_\delta$ in the space

$$\mathbf{H}\left(M;\tilde{\Omega}_\delta\right):=\left\{v\in H^1\left(\tilde{\Omega}_\delta\right):v\ge 0 \text{ a.e. and } \int_{\tilde{\Omega}_\delta}vdy=M\right\}$$

where in the definition of energy $\mathbf{E}_\delta$, Ω_δ is replaced by $\tilde{\Omega}_\delta$. Since $\frac{|p_\varepsilon-x_\varepsilon|}{\delta}\to\infty$, we have $\tilde{\Omega}_\delta\to\mathbb{R}^n$ as $k\to\infty$. Noticing that for each k, $\tilde{v}_\delta$ is a translation of v_δ, the uniform bound of Hölder norms of v_δ implies that, passing to a subsequence if necessary, $\tilde{v}_\delta\to v^*$ locally uniformly in $\mathbb{R}^n$ as $k\to\infty$, which implies

$$M^*=\int_{\mathbb{R}^n}v^*dy\le M.$$

Since

$$\tilde{v}_\delta\left(0\right)=\max_{y\in\overline{\Omega_{\delta_k}}}v_{\delta_k}\left(y\right)\ge C_1,$$

the uniform Hölder continuity of $\tilde{v}_\delta$ implies

$$M^*=\int_{\mathbb{R}^n}v^*dy>0.$$

For any $\sigma>0$ sufficiently small, we can choose $R_0>0$, such that

$$\int_{B_{R_0}(0)}v^*dx\ge M^*-\sigma.$$

Let $N = \left[\frac{1}{\sigma}\right] + 1$. For small $\delta > 0$, we have $B_{R_0+N} \subset \tilde{\Omega}_\delta$. Since

$$\int_{B_{R_0+N}\setminus B_{R_0}} \left\{\frac{1}{2}\left|\nabla \tilde{v}_\delta\right|^2 + \chi_{\{\tilde{v}_\delta>0\}}\right\} + \int_{B_{R_0+N}\setminus B_{R_0}} \left\{\left|\tilde{v}_\delta\right|^2 + \left|\tilde{v}_\delta\right|\right\}$$
$$\leq \mathbf{e}_\delta(M) + M\max\tilde{v}_\delta + M \leq K$$

for some $K > 0$ which is independent of small ε, we can choose $1 \leq l \leq N$ so that

$$\int_{B_{R_0+l}\setminus B_{R_0+l-1}} \left\{\frac{1}{2}\left|\nabla \tilde{v}_\delta\right|^2 + \chi_{\{\tilde{v}_\delta>0\}} + \left|\tilde{v}_\delta\right|^2 + \left|\tilde{v}_\delta\right|\right\} \leq \frac{K}{N} \leq \sigma K.$$

Now let $\eta \in C^\infty(\mathbb{R}^n)$ be a cutoff function, such that

$$\eta(x) = 1 \text{ if } x \in B_{R_0+l}(0)\,;\; \eta(x) = 0 \text{ if } x \notin B_{R_0+l-1}(0)\,;$$
$$\eta \in [0,1] \text{ and } |\nabla\eta| \leq 2 \text{ for any } x \in \mathbb{R}^n.$$

We have

$$\tilde{v}_\delta = \eta\tilde{v}_\delta + (1-\eta)\tilde{v}_\delta \equiv \tilde{v}_\delta^1 + \tilde{v}_\delta^2.$$

Direct calculation yields

$$\begin{aligned}
&\frac{1}{2}\int_{\tilde{\Omega}_\delta}\left|\nabla\tilde{v}_\delta^1\right|^2 + \frac{1}{2}\int_{\tilde{\Omega}_\delta}\left|\nabla\tilde{v}_\delta^2\right|^2 - \frac{1}{2}\int_{\tilde{\Omega}_\delta}\left|\nabla\tilde{v}_\delta\right|^2 \\
=&\frac{1}{2}\int_{\tilde{\Omega}_\delta}\left\{\left(\eta^2 + (1-\eta)^2 - 1\right)\left|\nabla\tilde{v}_\delta\right|^2 + 2\left|\nabla\eta\right|^2\left|\tilde{v}_\delta\right|^2 + (-2+4\eta)\,\tilde{v}_\delta\nabla\tilde{v}_\delta\nabla\eta\right\} \\
\leq&\frac{1}{2}\int_{B_{R_0+l}(0)\setminus B_{R_0+l-1}(0)}\left\{8\left|\tilde{v}_\delta\right|^2 + 4\left|\tilde{v}_\delta\nabla\tilde{v}_\delta\right|\right\} \\
\leq&\int_{B_{R_0+l}(0)\setminus B_{R_0+l-1}(0)}\left(\left|\nabla\tilde{v}_\delta\right|^2 + 5\left|\tilde{v}_\delta\right|^2\right) \\
\leq&7\sigma K,
\end{aligned}$$

and ,

$$\int_{\tilde{\Omega}_\delta}\chi_{\left\{\tilde{v}_\delta^1>0\right\}} + \chi_{\left\{\tilde{v}_\delta^2>0\right\}}dy - \int_{\tilde{\Omega}_\delta}\chi_{\{\tilde{v}_\delta>0\}}$$
$$\leq \int_{B_{R_0+l}(0)\setminus B_{R_0+l-1}(0)}\chi_{\{\tilde{v}_\delta>0\}} < \sigma K.$$

Hence, we conclude

$$\mathbf{e}_\delta(m) \geq \mathbf{E}_\delta\left[\tilde{v}_\delta^1\right] + \mathbf{E}_\delta\left[\tilde{v}_\delta^2\right] - 8\sigma K.$$

Since

$$\lim_{k\to\infty}\int_{\tilde{\Omega}_\delta}\tilde{v}^1_{\delta_k}=\lim_{k\to\infty}\int_{\mathbb{R}^n}\eta v^*\in[M^*-\sigma,M^*],$$

when k sufficiently large, we have

$$\int_{\tilde{\Omega}_\delta}\tilde{v}^1_\delta\in[M^*-2\sigma,M^*+\sigma]\ \text{ and }\ \int_{\tilde{\Omega}_\delta}\tilde{v}^2_\delta\in[M-M^*-\sigma,M-M^*+2\sigma].$$

Letting $k\to\infty$, we have

$$\mathbf{e}^*(M)\geq 2\mathbf{e}^*\left(\frac{M^*-2\sigma}{2}\right)+\mathbf{e}^*(M-M^*-\sigma)-8\sigma K$$

where

$$\liminf_{k\to\infty}\mathbf{E}_{\delta_k}\left[\tilde{v}^1_{\delta_k}\right]\geq 2\mathbf{e}^*\left(\frac{M^*-2\sigma}{2}\right)$$

follows from the fact that $\tilde{v}^1_\delta$ is compactly supported. Letting $\sigma\to 0$, we have

$$\mathbf{e}^*(M)\geq 2\mathbf{e}^*\left(\frac{M^*}{2}\right)+\mathbf{e}^*(M-M^*)$$

Recall that $\mathbf{e}^*(M)$ is strictly convex. Therefore for $M^*\in(0,M]$, we have a contradiction. □

Next, we show there is no loss of mass in the limiting process.

Lemma 11. *$\int_{\mathbb{R}^n_+}v^*(x)\,dx=M$.*

Proof. Let

$$M^*=\int_{\mathbb{R}^n_+}v^*(y)\,dy.$$

Since $\frac{|p_\varepsilon-x_\varepsilon|}{\delta}$ is uniformly bounded, the uniform Hölder bound for v_δ and uniformly positive lower bounds for

$$v_\delta\left(\frac{x_\varepsilon-p_\varepsilon}{\delta}\right)=\max_{y\in\overline{\Omega_\delta}}v_\delta(y)$$

implies

$$m^*>0.$$

Similar argument as Lemma 10 will imply

$$\mathbf{e}^*(M)\geq\mathbf{e}^*(M^*)+e^*(M-M^*)$$

Now $M^*>0$ implies $M^*=M$. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Up to a subsequence, we also assume v_{δ_k} converges to v^* weakly in $H^1_{loc}\left(\mathbb{R}^n_+\right)$ as $k \to \infty$ and hence, the lower semi-continuity of norms implies

$$\frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla v^*|^2 \, dy \leq \liminf_{k\to\infty} \frac{1}{2}\int_{\Omega_{\delta_k}} |\nabla v_{\delta_k}|^2 \, dy. \tag{2.7.3}$$

On the other hand, let

$$|\{v^* > 0\}| = \mu^* > 0.$$

For each $\sigma > 0$, there exists $N > 0$ such that

$$\left|\left\{v^* > \frac{1}{N}\right\} \cap B_N(0)\right| \geq \mu^* - \sigma.$$

Now since v_δ converges to v^* uniformly on $\left|\left\{v^* > \frac{1}{N}\right\} \cap B_N(0)\right|$, we conclude

$$\lim_{k\to\infty} \int_{\left\{v^* > \frac{1}{N}\right\}\cap B_N(0)\cap\Omega_\delta} \chi_{\{v_\delta > 0\}} = \left|\left\{v^* > \frac{1}{N}\right\} \cap B_N(0)\right| \geq \mu^* - \sigma.$$

Since σ is arbitrary,

$$\liminf_{k\to\infty} \int_{\Omega_\delta} \chi_{\{v_\delta > 0\}} \geq \mu^*. \tag{2.7.4}$$

Combining (2.7.3) and (2.7.4), we have

$$\mathbf{E}^*[v^*] \leq \lim_{k\to\infty} \mathbf{e}_{\delta_k}(M) = \mathbf{e}^*(M).$$

On the other hand, since $\int_{\mathbb{R}^n_+} v^*(x)\,dx = M$, we have $\mathbf{E}^*[v^*] \geq \mathbf{e}^*(M)$ and hence $\mathbf{E}^*[v^*] = \mathbf{e}^*(M)$. Our choice of p_ε guarantees that $\max v^*$ is assumed on the vertical line passing through the origin. So the theorem follows from the uniqueness up to a translation of the global energy minimizer for $\mathbf{E}^*$. □

The convergence of the blow up sequence v_δ implies the convergence of u_ε.

Proof of Theorem 1. Since $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ is a sequence of positive function with total mass m, there exists a measure μ on $\bar{\Omega}$ such that passing to a subsequence if necessary

$$u_{\varepsilon_k} \overset{*}{\rightharpoonup} \mu$$

in the weak star topology as $k \to \infty$. Passing to a subsequence if necessary , we also have the blow up sequence $v_\delta \to v^*$ locally uniformly as $k \to \infty$ and $\int_\Omega v^* = M$. Hence

$$\lim_{k\to\infty} \int_{B_{R^*}(0)\cap\Omega_\delta} v_\delta\left(y\right) dy = M$$

which implies

$$\lim_{k\to\infty} \int_{B_{\delta R^*}(p_\varepsilon)\cap\Omega} u_\varepsilon\left(x\right) dx = M.$$

Since $\int_\Omega u_\varepsilon\left(x\right) dx = M$ and $p_\varepsilon \to p^*$, we conclude $u_\varepsilon \overset{*}{\rightharpoonup} \mu = M\delta_{p^*}$ as $k \to \infty$. □

The above theorem implies when ε approaches zero, the energy minimizer converges to a Dirac measure concentrated on the boundary. Later we are going to show that actually specify the location of the Dirac Measure which is with maximal mean curvature. On the contrary, when ε is sufficient large, we can the classical Poincaré inequality and obtain a simple proposition as following,

Proposition 3. *When ε is large enough, the energy minimizer of $\mathcal{E}_\varepsilon[u]$ is a constant function.*

Proof.

$$\begin{aligned}
\mathcal{E}_\varepsilon[u] &= \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}\chi_{\{u>0\}} \right\} dy \\
&\geq \int_\Omega \left\{ \frac{1}{2C(\Omega)}|u - \frac{M}{|\Omega|}|^2 + \frac{1}{\varepsilon^2}\chi_{\{u>0\}} \right\} dy \\
&\geq \int_\Omega \frac{1}{2C(\Omega)}|u - \frac{M}{|\Omega|}|^2 \chi_{\{u>0\}} dy + \frac{|\Omega|}{\varepsilon^2} + \int_\Omega \left[\frac{1}{2C(\Omega)}(\frac{M}{|\Omega|})^2 - \frac{1}{\varepsilon^2} \right] \chi_{\{u=0\}} dy
\end{aligned}$$

where $C(\Omega)$ is a constant depending on Ω. Therefore if we choose ϵ large enough such that

$$\frac{1}{2C(\Omega)}(\frac{M}{|\Omega|})^2 - \frac{1}{\varepsilon^2} > 0,$$

then the energy minimizer will be a constant function $u = \frac{M}{|\Omega|}$. □

2.8 LINEARIZATION

To understand the location of the boundary spike, we consider the free boundary problem (2.4.6) associated to the scaled energy minimizing problem:

$$\begin{cases} \Delta v = -\lambda & \text{in} \quad D = \{y \in \Omega_\delta : v > 0\} \\ v = 0 \text{ and } \partial_n v = -\sqrt{2} & \text{on} \quad \partial D \cap \Omega_\delta, \\ \partial_n v = 0 & \text{on} \quad \partial D \cap \partial\Omega_\delta, \\ \int_D v(y)\, dy = M. \end{cases} \tag{2.8.1}$$

Since true solution should have spikes near specific boundary points, here for a fixed point $p \in \partial\Omega$, we seek a pair (v, D) such that the "center" of D is the origin and that (v, D) only approximately solves the free boundary problem (2.8.1), e.g., having error $O(\delta^2)$. Then we compare the energy when p is moving around the boundary.

By shifting and rotation, we assume that $p = 0$ and the unit normal of $\partial\Omega$ at p is $(0', -1)$. The boundary near p is represented in local coordinates as

$$x_n = \psi(x_1, \cdots, x_{n-1}), \qquad \psi(0') = 0, \psi_{x_i}(0') = 0, \qquad \psi_{x_i x_j}(0') = \kappa_i \delta^{ij}.$$

We call κ_i the principal curvature of $\partial\Omega$ at p and denote by $\kappa = \sum_{i=1}^{n-1} \kappa_i/(n-1)$ the mean curvature of $\partial\Omega$ at p. Locally the boundary of $\partial\Omega_\delta$ near $q := p/\delta$ is expressed as

$$\delta y_n = \psi(\delta y') = \frac{\delta^2}{2} \sum_{i=1}^{n-1} \kappa_i y_i^2 + O\left(\delta^3 |y'|^3\right).$$

In general, (2.8.1) does not have a solution that has mass concentrated near q. To overcome this difficulty, we add an extra constraint in the class of minimization to ensure that the mass is near q. Hence we consider the minimization of $\mathbf{E}_\delta^*$ in the space

$$\mathbf{H}(q, M) = \left\{ v \in H^1(\Omega_\delta) : v \geq 0 \text{ a.e. in } \Omega_\delta, \int_{\Omega_\delta} v dy = M \text{ and } \int_{\Omega_\delta} (y - q) v dy \,//\, N(q) \right\}$$

where $N(q)$ is the normal direction of Ω_δ at q. In the current notation, the second set of constraints mean

$$\int_{\Omega_\delta} y_i v dy = 0 \quad \forall\, i = 1, \cdots, n-1. \tag{2.8.2}$$

The corresponding free boundary problem can be written as

$$\begin{cases} \Delta v = -\lambda - \sum_{i=1}^{n-1} \lambda_i y_i & \text{in} \quad D = \{y \in \Omega_\delta : v > 0\} \\ v = 0 \text{ and } \partial_n v = -\sqrt{2} & \text{on} \quad \partial D \cap \Omega_\delta, \\ \partial_n v = 0 & \text{on} \quad \partial D \cap \partial\Omega_\delta, \\ \int_D v\,(y)\,dy = M. \end{cases} \tag{2.8.3}$$

where $\lambda, \lambda_1, \cdots, \lambda_{n-1}$ are Lagrange multipliers.

We search a solution of (2.8.3) that can be expanded in the δ-power series as follows

$$\begin{cases} D = \left\{ y \in \mathbb{R}^n \ : \ y_n > \psi(\delta y')/\delta, |y| < R + \delta R_1 \left(\frac{y}{|y|}\right) + O(\delta^2) \right\}, \\ v = \frac{\lambda}{2n}[R^2 - |y|^2] + \delta v_1(y) + O(\delta^2) \ \forall\, y \in \bar{D}, \qquad v(y) = 0 \ \forall\, y \in \Omega_\delta \setminus D, \\ \lambda_i = O(\delta^2) \qquad \forall\, i = 1, \cdots, n-1. \end{cases}$$

where R and λ are constants depending on δ, R_1 and v_1 are unknown functions that depend on δ only through the constants λ and R.

We derive the equations for (R, λ, v_1, R_1) as follows.

(1) The free boundary condition $v = 0$ on the free boundary implies

$$\begin{aligned} 0 = v\,(y) &= \frac{\lambda}{2n}\left[R^2 - \left| R + \delta R_1 \left(\frac{y}{|y|}\right) + O(\delta^2) y \right|^2 \right] + \delta v_1\,(y) + O(\delta^2) \\ &= \delta \left[-\frac{\lambda R R_1 \left(\frac{y}{|y|}\right)}{n} + v_1\,(y) \right] + O(\delta^2) \end{aligned}$$

which is equivalent to

$$v_1(y) = \frac{\lambda R}{n} R_1 \left(\frac{y}{|y|}\right) \qquad \forall\, y \in \Gamma_R := \partial B_R \cap \mathbb{R}^{n+}.$$

(2) The normal of the free boundary is

$$N = (N^1, \cdots, N^n), \quad N^i = \frac{y^i}{|y|} - \delta \sum_{j=1}^{n} \frac{\partial R_1}{\partial y_j} \left(\frac{\delta^{ji}}{|y|} - \frac{y^i y^j}{|y|^3} \right) + O(\delta^2),$$

$$\|N\| = \sqrt{ \sum_{i=1}^{n} \left| \frac{y^i}{|y|} - \delta \sum_{j=1}^{n} \frac{\partial R_1}{\partial y_j} \left(\frac{\delta^{ji}}{|y|} - \frac{y^i y^j}{|y|^3} \right) \right|^2 } = \sqrt{1 + O(\delta^2)}.$$

The free boundary condition $\partial_n v = -\sqrt{2F_*}$ becomes

$$
\begin{aligned}
-\sqrt{2F_*} &= n\cdot\nabla v\\
&= \frac{1}{\sqrt{1+O(\delta^2)}}\sum_{i=1}^{n}\left[\frac{y^i}{|y|}-\delta\sum_{j=1}^{n}\frac{\partial R_1}{\partial y_j}\left(\frac{\delta^{ji}}{|y|}-\frac{y^iy^j}{|y|^3}\right)\right]\left(-\frac{\lambda y_i}{n}+\delta\partial_i v_1\right)\\
&= \sum_{i=1}^{n}\left[\frac{y^i}{|y|}-\delta\sum_{j=1}^{n}\frac{\partial R_1}{\partial y_j}\left(\frac{\delta^{ji}}{|y|}-\frac{y^iy^j}{|y|^3}\right)\right]\left(-\frac{\lambda y_i}{n}+\delta\partial_i v_1\right)\left(1+O(\delta^2)\right)\\
&= \left(-\frac{\lambda\,(R+\delta R_1)}{n}+\delta\frac{y}{|y|}\cdot\nabla v_1\right)+O(\delta^2)
\end{aligned}
$$

which can be achieved by setting

$$\lambda = \frac{\sqrt{2}n}{R},$$

and

$$\partial_n v_1 = \tfrac{\lambda R_1}{n} = \tfrac{v_1}{R} \quad \text{on } \Gamma_R.$$

(3) Finally, using

$$y_n = \frac{\delta}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2 + O\left(\delta^2\,|y'|^3\right),$$

we have

$$N = (\delta\kappa_1 y_1,\cdots,\delta\kappa_{n-1}y_{n-1},-1)+O\left(\delta^2\right),$$

and the boundary condition $\partial_n v = 0$ on $\partial\Omega_\delta$ can be written as

$$
\begin{aligned}
0 &= N\cdot\nabla v\\
&= \delta\sum_{i=1}^{n-1}\kappa_i y_i\left(-\frac{\lambda y_i}{n}+\delta\partial_i v_1\right)-\left(-\frac{\lambda y_n}{n}+\delta\partial_n v_1\right)+O(\delta^2)\\
&= \delta\sum_{i=1}^{n-1}\kappa_i y_i\left(-\frac{\lambda y_i}{n}+\delta\partial_i v_1\right)-\left(-\frac{\lambda}{n}\frac{\delta}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2+\delta\partial_n v_1\right)+O(\delta^2)\\
&= \delta\left(-\frac{\lambda}{2n}\sum_{i=1}^{n-1}\kappa_i y_i^2-\partial_n v_1\right)+O(\delta^2).
\end{aligned}
$$

This can be achieved only by setting

$$\frac{\partial v_1}{\partial y_n}(y',0) = -\frac{\lambda}{2n}\sum_{i=1}^{n-1}\kappa_i y_i^2 \qquad \forall\, y'\in B'_R := \{y'\in\mathbb{R}^{n-1}\mid |y'|<R\}.$$

Thus we see that (v_1, R_1) needs to be a solution of the linearized problem given by

$$\begin{cases} -\Delta v_1 = 0 & \text{in } B_R \cap \mathbb{R}^{n+} =: B_R^+, \\ v_1 = R\partial_n v_1 & \text{on } \partial B_R \cap \mathbb{R}^{n+} =: \Gamma_R, \\ \partial_{y_n} v_1 = -\frac{\lambda}{2n}\sum_{i=1}^{n-1} \kappa_i y_i^2 & \text{on } B_R' \times \{0\}, \\ R_1(y/|y|) = n\partial_n v_1(y)/\lambda & \forall\, y \in \Gamma_R. \end{cases} \tag{2.8.4}$$

It is sufficient to consider only the equation for v_1. Note that

$$\int_D y_i(R^2 - |y|^2)dy = O(\delta^2) \quad \forall\, i = 1, \cdots, n-1.$$

We derive from (2.8.1) that

$$\int_{B_R^+} y_i v_1(y)dy = 0 \qquad \forall\, i = 1, \cdots, n-1. \tag{2.8.5}$$

Theorem 10. *The mixed boundary condition problem 2.8.4 with the constraint 2.8.5 admits a unique solution.*

First we establish the lemma for Robin boundary condition problem on a ball.

Lemma 12. *Assume $f \in L^2(S_R)$, the Robin boundary condition problem on a ball with radius R given by*

$$\begin{cases} \Delta u = 0 & \text{in } B_R, \\ \frac{u}{R} - \partial_n u = f & \text{on } \partial B_R =: S_R. \end{cases} \tag{2.8.6}$$

admits a solution $u \in H^1(B_R)$ if and only if f satisfies the compatibility condition

$$\int_{\partial B_R} y_i f(y) d\mathcal{H}^{n-1} = 0, \qquad \forall\, i = 1, \cdots, n.$$

The solution is unique if we add the constraints

$$\int_{B_R} y_i u dy = 0, \forall\, i = 1, \cdots, n.$$

Proof. Firstly, let $u \in H^1(B_R)$ be a solution to (2.8.6). $\forall\, i = 1, \cdots, n$,

$$
\begin{aligned}
0 &= -\int_{B_R} \nabla y_i \nabla u dy + \int_{S_R} y_i \partial_n u d\mathcal{H}^{n-1} \\
&= -\int_{S_R} \partial_n y_i u d\mathcal{H}^{n-1} + \int_{B_R} \Delta y_i u dy + + \int_{S_R} y_i \partial_n u d\mathcal{H}^{n-1} \\
&= -\int_{S_R} y_i (\frac{u}{R} - \partial_n u) d\mathcal{H}^{n-1}.
\end{aligned}
$$

that is,

$$
\int_{S_R} y_i f(y) d\mathcal{H}^{n-1} = 0, \qquad \forall\, i = 1, \cdots, n.
$$

Secondly, suppose $f \in L^2(S_R)$ satisfying the compatibility condition. Let $H_m(\mathbb{R}^n)$ denote the subspace of all the homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree m. and $H_m(S_R)$ represent the subspace of all the homogeneous harmonic polynomials in $H_m(\mathbb{R}^n)$ with restriction to S_R of degree m. Since $L^2(S_R) = \oplus_{m=0}^{\infty} H_m(S_R)$ (Theorem 5.12 and Theorem 5.29 in [6]),

$$
f = \sum_{m=0}^{\infty} p_m(y)
$$

where $p_m(y) \in H_m(S_R)$ satisfying

$$
\int_{S_R} p_m(y) p_k(y) d\mathcal{H}^{n-1} = 0, \qquad \forall\, m \neq k.
$$

Using the homogeneity, we see,

$$
\int_{S_r} p_m(y) p_k(y) d\mathcal{H}^{n-1} = \int_{S_R} (\frac{R}{r})^{m+k+n-1} p_m(y) p_k(y) d\mathcal{H}^{n-1} = 0,\ \forall\, r > 0.
$$

Furthermore,

$$
\int_{B_R} p_m(y) p_k(y) dy = \int_0^R \int_{S_r} p_m(y) p_k(y) d\mathcal{H}^{n-1} dr = 0
$$

and each component in $\nabla p_m(y)$ belongs to $H_{m-1}(\mathbb{R}^n)$ implies

$$
\int_{B_R} \nabla p_m(y) \cdot \nabla p_k(y) dy = 0.
$$

Suppose solution u has expansion

$$
u = \sum_{m=0}^{\infty} d_m p_m(y)
$$

where d_m is to be determined. Formal calculation gives

$$\frac{u}{R} - \partial_n u = \sum_{m=0}^{\infty} \frac{d_m p_m(y)}{R} - \sum_{m=0}^{\infty} \frac{m d_m p_m(y)}{R} = \sum_{m=0}^{\infty} \frac{(1-m) d_m p_m(y)}{R}.$$

According to the Robin boundary condition, we define

$$d_m = \frac{R}{1-m} \text{ and } u_M = \sum_{m\geq 0, m\neq 1}^{M} \frac{R}{1-m} p_m(y).$$

Then u_M is harmonic is B_R and for $N > M > 1$,

$$\begin{aligned}
\|u_N - u_M\|_{L^2(B_R)} &= \left\| \sum_{m>M}^{N} \frac{R}{1-m} p_m(y) \right\|_{L^2(B_R)} \\
&= (\int_0^R \int_{S_r} (\sum_{m>M}^{N} \frac{R}{1-m} p_m(y))^2 d\mathcal{H}^{n-1} dr)^{1/2} \\
&= (\int_0^R \int_{S_R} \sum_{m>M}^{N} \frac{R^2}{(1-m)^2} p_m(y)^2 \frac{r^{2m+n-1}}{R^{2m+n-1}} d\mathcal{H}^{n-1} dr)^{1/2} \\
&= (\int_{S_R} \sum_{m>M}^{N} \frac{R^2}{(m-1)^2} p_m(y)^2 \frac{R^{2m+n}}{(2m+n) R^{2m+n-1}} d\mathcal{H}^{n-1})^{1/2} \\
&\leq \frac{R^3}{(M-1)^2} \left\| \sum_{m>M}^{N} p_m(y) \right\|_{L^2(S_R)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\nabla u_N - \nabla u_M\|_{L^2(B_R)} &= \left\| \sum_{m>M}^{N} \frac{R}{1-m} \nabla p_m(y) \right\|_{L^2(B_R)} \\
&= \sum_{m>M}^{N} \frac{R}{m-1} \|\nabla p_m(y)\|_{L^2(B_R)} \\
&= \sum_{m>M}^{N} \frac{R}{m-1} (\int_{S_R} p_m(y) \cdot \frac{\partial p_m(y)}{\partial n} d\mathcal{H}^{n-1})^{1/2} \\
&= \sum_{m>M}^{N} \frac{R}{m-1} (\frac{m}{R})^{1/2} \|p_m(y)\|_{L^2(S_R)} \\
&\leq \frac{\sqrt{R}}{\sqrt{M}-1} \left\| \sum_{m>M}^{N} p_m(y) \right\|_{L^2(S_R)}.
\end{aligned}$$

Therefore u_M is a Cauchy sequence in $L^2(B_R)$ and ∇u_M is a Cauchy sequence in $(L^2(B_R))^n$. Then let

$$u = \sum_{m \neq 1} \frac{R}{1-m} p_m(y)$$

we can obtain that as the limit of u_M in $H^1(B_R)$, $u \in H^1(B_R)$ is harmonic in B_R.

Regarding the uniqueness, we consider solution $u \in H^1(B_R)$ to the homogeneous system,

$$\begin{cases} \Delta u = 0 & \text{in } B_R, \\ \frac{u}{R} - \partial_n u = 0 & \text{on } S_R. \end{cases}$$

Using Spherical Harmonic Functions (for example corollary 5.34 of [6]),

$$u = \sum_{m=0}^{\infty} p_m(y)$$

where $p_m(y) \in H_m(\mathbb{R}^n)$. Applying the robin boundary condition, we obtain,

$$0 = \frac{u}{R} - \partial_n u = \sum_{m=0}^{\infty} \frac{p_m(y)}{R} - \sum_{m=0}^{\infty} \frac{m p_m(y)}{R} = \sum_{m=0}^{\infty} \frac{(1-m) p_m(y)}{R}.$$

Due to the orthogonality of $H_m(\mathbb{R}^n)$ on S_R in sense of L^2 inner product, then,

$$p_m(y) = 0, \ \forall \ m \neq 1.$$

We have,

$$u = \sum_{i=1}^{n} c_i y_i \text{ where } c_i \text{ are arbitrary constant.}$$

Therefore, the constraints $\int_{B_R} y_i u dy = 0$ implies $c_i = 0$ which is the uniqueness of the solution. □

Now we are ready to prove the existence of the solution to the non-homogeneous problem on half ball.

Theorem 11. *Given $f \in L^2(B_R^+), g \in L^2(\Gamma_R)$ and $h \in L^2(B'_R \times \{0\})$, the mixed boundary condition problem given by*

$$\begin{cases} \Delta w = f & \text{in } B_R^+, \\ \frac{w}{R} - \partial_n w = g & \text{on } \Gamma_R, \\ \partial_{y_n} w = h & \text{on } B'_R \times \{0\} \end{cases} \tag{2.8.7}$$

admits a solution $w \in H^1(B_R^+)$, if and only if (f, g, h) satisfies the compatibility conditions

$$\int_{B_R^+} y_i f(y) dy + \int_{\Gamma_R} y_i g(y) d\mathcal{H}^{n-1} + \int_{B'_R} y_i h(y') dy' = 0 \qquad \forall\, i = 1, \cdots, n-1. \tag{2.8.8}$$

If there is a solution w_{sp}, then the general solution is given by

$$w(y) = w_{\text{sp}}(y) + \sum_{i=1}^{n-1} c_i y_i$$

where $c_1, \cdots, c_{n-1}$ are arbitrary constants. The solution is unique if we require

$$\int_{B_R^+} y_i w(y) dy = 0 \quad \forall\, i = 1, \cdots, n-1.$$

Proof. Let $w \in H^1(B_R^+)$ be a solution to (2.8.7). For $i = 1, \cdots, n-1$,

$$\begin{aligned} \int_{B_R^+} y_i \Delta w dy &= \int_{\Gamma_R} y_i \partial_n w d\mathcal{H}^{n-1} - \int_{B'_R} y_i \partial_{y_n} w dy' - \int_{\Gamma_R} w(y) \partial_n y_i d\mathcal{H}^{n-1} \\ &\quad - \int_{B'_R} w(y', 0) \partial_{y_n} y_i dy' + \int_{B_R^+} w \Delta y_i dy \\ &= \int_{\Gamma_R} y_i \partial_n w d\mathcal{H}^{n-1} - \int_{B'_R} y_i \partial_{y_n} w dy' - \int_{\Gamma_R} w \frac{y_i}{R} d\mathcal{H}^{n-1} \\ &= \int_{\Gamma_R} y_i (\partial_n w - \frac{w}{R}) d\mathcal{H}^{n-1} - \int_{B'_R} y_i \partial_{y_n} w dy' \\ &= -\int_{\Gamma_R} y_i g(y) d\mathcal{H}^{n-1} - \int_{B'_R} y_i h(y') dy'. \end{aligned}$$

That is the compatibility condition (2.8.8).

We first consider the homogeneous system

$$\begin{cases} \Delta w = 0 & \text{in } B_R^+, \\ \frac{w}{R} - \partial_n w = 0 & \text{on } \Gamma_R, \\ \partial_{y_n} w = 0 & \text{on } B'_R \times \{0\}. \end{cases}$$

Due to the Neumann boundary condition on $B'_R \times \{0\}$, even reflection gives

$$\begin{cases} \Delta w = 0 & \text{in } B_R, \\ \frac{w}{R} - \partial_n w = 0 & \text{on } \partial B_R. \end{cases}$$

Applying Lemma 12 and the fact that w is even in y_n, we have the general solutions for the homogeneous system are given by

$$w = \sum_{i=1}^{n-1} c_i y_i.$$

Next, for the non-homogeneous problem, we choose functions $F \in H^2(B_R^+)$ and $H \in H^1(B_R^+)$ such that

$$\begin{cases} \Delta F = f & \text{in } B_R^+, \\ F = 0 & \text{on } \partial B_R^+ \end{cases}$$

and

$$\begin{cases} \Delta H = 0 & \text{in } B_R^+, \\ \partial_{y_n} H = h - \partial_{y_n} F & \text{on } B'_R \times \{0\}. \end{cases}$$

Set $u = w - F - H$,

$$\begin{cases} \Delta u = 0 & \text{in } B_R^+, \\ \frac{u}{R} - \partial_n u = G = g + \partial_n F - \frac{H}{R} + \partial_n H & \text{on } \Gamma_R, \\ \partial_{y_n} u = h - \partial_{y_n} F - (h - \partial_{y_n} F) = 0 & \text{on } B'_R \times \{0\}. \end{cases}$$

Here $G \in L^2(\Gamma_R)$. Similarly apply even reflection for u and make use of Lemma 12. Solution $u \in H^1(B_R)$ exists if and only if

$$\int_{\partial B_R} p_1(y) G dy = 0.$$

That is $\forall\, i = 1, \cdots, n-1$

$$
\begin{aligned}
0 &= \int_{\Gamma_R} y_i G dy \\
&= \int_{\Gamma_R} y_i (g + \partial_n F - \frac{H}{R} + \partial_n H) d\mathcal{H}^{n-1} \\
&= \int_{\Gamma_R} (y_i g + y_i \partial_n F - y_i \frac{H}{R} + y_i \partial_n H) d\mathcal{H}^{n-1} \\
&= \int_{\Gamma_R} y_i g d\mathcal{H}^{n-1} + (\int_{B_R^+} y_i \Delta F dy + \int_{B'_R} y_i \partial_{y_n} F(y', 0) dy') + \int_{\Gamma_R} -y_i \frac{H}{R} d\mathcal{H}^{n-1} \\
&+ (\int_{B'_R} y_i \partial_{y_n} H(y', 0) dy' + \int_{\Gamma_R} \frac{\partial y_i}{\partial n} H d\mathcal{H}^{n-1} + \int_{B'_R} \frac{\partial y_i}{\partial n} H(y', 0) dy') \\
&= \int_{\Gamma_R} y_i g d\mathcal{H}^{n-1} + \int_{B_R^+} y_i f dy + \int_{B'_R} y_i \partial_{y_n} F(y', 0) dy' + \int_{\Gamma_R} -y_i \frac{H}{R} d\mathcal{H}^{n-1} \\
&+ \int_{B'_R} y_i (h - \partial_{y_n} F) dy' + \int_{\Gamma_R} \frac{y_i}{R} H d\mathcal{H}^{n-1} + \int_{B'_R} \frac{\partial y_i}{\partial n} H(y', 0) dy' \\
&= \int_{\Gamma_R} y_i g d\mathcal{H}^{n-1} + \int_{B_R^+} y_i f dy + \int_{B'_R} y_i h dy'.
\end{aligned}
$$

In order to obtain the explicit solution, we can compute the basis for $H_m(\mathbb{R}^n)$ using zonal harmonics (See Chapter 5 in [6]). Then given compatibility condition, the special solution for u can be calculated using inner product. Therefore it gives the special solution $w_{sp} = u + F + H$.The general solution is given by

$$
w(y) = w_{\text{sp}}(y) + \sum_{i=1}^{n-1} c_i y_i.
$$

□

It is easy to see problem (2.8.4) together with the constraint (2.8.5) is a special case of (2.8.7). Then the proof of Theorem 10 naturally follows theorem 11. Hence, v_1 is uniquely solvable.

2.9 LOCATION OF SPIKE

Though it is hard to find explicit solution of (2.8.4) and (2.8.5), we can still proceed to find quantities of our interest. In this section, we will focus on the energy expansion which helps to locate the position of the spike. Applying the asymptotic analysis, we consider the effect of the mass constraint. Later as the theorem 3 stated, the energy of the Quasi-stationary solution (v, D) has the asymptotic expansion

$$\mathbf{E}_\delta[v] \equiv \int_D \left\{ \frac{1}{2} |\nabla v|^2 + 1 \right\} = \mathbf{E}^* [v^*] - c(n) M\kappa\delta + O(\delta^2)$$

where

$$c(n) = \frac{(n-1)(n+2)(n+7)\,\omega_{n-1}}{\sqrt{2}(n+1)(n+3)\,\omega_n}.$$

Hence, the spike should locate on the boundary point with the maximum curvature. We begin with computing

$$\begin{aligned}
\int_{B_R^+} v_1(y)dy &= \int_{B_R^+} v_1(y) \frac{\Delta(|y|^2 + R^2)}{2n} dy \\
&= \frac{1}{n} \int_{\partial B_R^+ \cap \mathbb{R}_n^+} \left\{ Rv_1 - R^2 \partial_n v_1 \right\} + \frac{1}{2n} \int_{B_R'} \left\{ \left(|y'|^2 + R^2 \right) \partial_{y_n} v_1 \right\} \\
&= \frac{1}{2n} \int_{B_R'} \left(|y'|^2 + R^2 \right) \left[-\frac{\lambda}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 \right] dy' \\
&= -\frac{\lambda}{4n^2} \sum_{i=1}^{n-1} \kappa_i \int_{B_R'} \left(|y'|^2 + R^2 \right) y_i^2 dy'.
\end{aligned}$$

Note that, by symmetry,

$$\begin{aligned}
\int_{B_R'} \left(|y'|^2 + R^2 \right) {y_i}^2 dy' &= \int_{B_R'} \left(|y'|^2 + R^2 \right) \frac{|y'|^2}{n-1} dy' \\
&= \int_0^R \left(r^2 + R^2 \right) \frac{r^2}{n-1} (n-1)\,\omega_{n-1} r^{n-2} dr \\
&= \frac{2(n+2)}{(n+1)(n+3)} \omega_{n-1} R^{n+3}
\end{aligned}$$

where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$. Recalling the definition of mean curvature, we then obtain

$$\begin{aligned}\int_{B_R^+} v_1(y)dy &= -\frac{\lambda(n-1)(n+2)}{2n^2(n+1)(n+3)}\omega_{n-1}\kappa R^{n+3} \\ &= -\frac{\sqrt{2F_*}(n-1)(n+2)R^{n+2}}{2n(n+1)(n+3)}\omega_{n-1}\kappa.\end{aligned}$$

Similarly, we can estimate

$$\begin{aligned}&\int_D \frac{\lambda}{2n}\left(R^2-|y|^2\right)dy \\ &= \int_{B_R^+}\frac{\lambda}{2n}\left(R^2-|y|^2\right)dy - \int_{B'_R}\left[\int_0^{\frac{\delta}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2}\frac{\lambda}{2n}\left(R^2-|y|^2\right)dy_n\right]dy' \\ &= \frac{\lambda\omega_n R^{n+2}}{2n(n+2)} - \int_{B'_R}\frac{\delta\lambda}{4n}\sum_{i=1}^{n-1}\kappa_i y_i^2\left(R^2-|y'|^2\right)dy' \\ &= \frac{\lambda\omega_n R^{n+2}}{2n(n+2)} - \sum_{i=1}^{n-1}\kappa_i\frac{\delta\lambda}{4n(n-1)}\int_{B'_R}|y'|^2\left(R^2-|y'|^2\right)dy' \\ &= \frac{\lambda\omega_n R^{n+2}}{2n(n+2)} - \sum_{i=1}^{n-1}\kappa_i\frac{\delta\lambda}{4n(n-1)}(n-1)\omega_{n-1}R^{n+3}\frac{2}{(n+1)(n+3)} \\ &= \frac{\lambda\omega_n R^{n+2}}{2n(n+2)} - \delta\sum_{i=1}^{n-1}\kappa_i\frac{\lambda}{2n(n+1)(n+3)}\omega_{n-1}R^{n+3} \\ &= \frac{\lambda\omega_n R^{n+2}}{2n(n+2)} - \delta\frac{\lambda(n-1)}{2n(n+1)(n+3)}\omega_{n-1}\kappa R^{n+3}.\end{aligned}$$

Here we used

$$\begin{aligned}&\int_0^{\frac{\delta}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2}\frac{\lambda}{2n}\left(R^2-|y|^2\right)dy_n \\ &= \frac{\lambda}{2n}\int_0^{\frac{\delta}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2}\left(R^2-|y_n|^2-|y'|^2\right)dy_n \\ &= \frac{\lambda}{2n}\left(R^2-|y'|^2\right)\frac{\delta}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2 + O\left(\delta^3\right).\end{aligned}$$

Now the mass constraint $\int_D v = M$ is equivalent to

$$
\begin{aligned}
M &= \int_D \frac{\lambda}{2n}\left(R^2 - |y|^2\right) dy + \delta \int_D v_1 + O(\delta^2) \\
&= \int_D \frac{\lambda}{2n}\left(R^2 - |y|^2\right) dy + \delta \int_{B_R^+} v_1 + O(\delta^2) \\
&= \frac{\lambda \omega_n R^{n+2}}{2n(n+2)} - \frac{\delta\lambda\,(n-1)}{2n\,(n+1)\,(n+3)}\omega_{n-1}\kappa R^{n+3} - \frac{\delta\lambda\,(n-1)\,(n+2)}{2n^2\,(n+1)\,(n+3)}\omega_{n-1}\kappa R^{n+3} + O(\delta^2) \\
&= \frac{\lambda \omega_n R^{n+2}}{2n(n+2)} - \delta\frac{\lambda\,(n-1)}{n^2\,(n+3)}\omega_{n-1}\kappa R^{n+3} + O(\delta^2) \\
&= \frac{\sqrt{2}n}{R}\left(\frac{\omega_n R^{n+2}}{2n(n+2)} - \delta\frac{(n-1)}{n^2\,(n+3)}\omega_{n-1}\kappa R^{n+3}\right) + O(\delta^2) \\
&= \sqrt{2}\left(\frac{\omega_n R^{n+1}}{2(n+2)} - \delta\frac{(n-1)}{n\,(n+3)}\omega_{n-1}\kappa R^{n+2}\right) + O(\delta^2).
\end{aligned}
$$

Hence, the mass constraint implies

$$
\begin{aligned}
R &= R^*\left\{1 + \frac{\frac{(n-1)}{n(n+3)}\omega_{n-1}\kappa R^*}{(n+1)\frac{\omega_n}{2(n+2)}}\delta + O\left(\delta^2\right)\right\} \\
&= R^*\left\{1 + \frac{2(n+2)\,(n-1)\,\omega_{n-1}}{n(n+1)\,(n+3)\,\omega_n}\kappa R^*\delta + O\left(\delta^2\right)\right\}
\end{aligned}
$$

where

$$
(R^*)^{n+1} := \frac{2\,(n+2)\,M}{\omega_n\sqrt{2}}.
$$

For the solution of (2.8.3),(2.8.2), we can compute its energy as follows:

$$
\begin{aligned}
e\,(q, M) &= \int_D \left\{\frac{1}{2}\left|\nabla v\right|^2 + 1\right\} = -\frac{1}{2}\int_D v\Delta v\; dy + |D| \\
&= \frac{\lambda}{2}\int_D v dy + |D| \\
&= \frac{\lambda M}{2} + \left\{\frac{\omega_n R^n}{2} + \delta\int_{\Gamma_R} R_1 - \delta\int_{B'_R}\sum_{i=1}^{n-1}\kappa_i {y_i}^2 dy' + O\left(\delta^2\right)\right\}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\int_{\Gamma_R} R_1 &= \frac{n}{\lambda}\int_{\Gamma_R}\partial_n v_1 = \frac{n}{\lambda}\int_{B_R^+}\Delta v_1 + \frac{n}{\lambda}\int_{B'_R}\partial_{y_n} v_1(y', 0) dy' \\
&= -\frac{1}{2}\int_{B'_R}\sum_{i=1}^{n-1}\kappa_i {y_i}^2 dy' = -\frac{(n-1)\,\kappa\omega_{n-1}R^{n+1}}{2(n+1)}.
\end{aligned}
$$

Thus, using

$$R = R^* \left\{ 1 + \frac{2(n+2)\,(n-1)\,\omega_{n-1}}{n(n+1)\,(n+3)\,\omega_n} \kappa R^* \delta + O\left(\delta^2\right) \right\} = R^* \left\{ 1 + A\delta + O\left(\delta^2\right) \right\},$$

we have

$$\begin{aligned}
& e(q, M) - e^*(M) \\
&= \frac{M\sqrt{2}n}{2R} + \frac{\omega_n R^n}{2} - \frac{3\,(n-1)\,\kappa \omega_{n-1} R^{n+1}}{2(n+1)} \delta + O(\delta^2) - e^*(M) \\
&= \frac{M\sqrt{2}n}{2R^*}\,(1 - A\delta) + \frac{\omega_n \left(R^*\right)^n}{2}\,(1 + nA\delta) - \frac{3\,(n-1)\,\kappa\omega_{n-1} \left(R^*\right)^{n+1}}{2(n+1)} \delta \\
&+ O(\delta^2) - e^*(M) \\
&= -\delta\Big(\frac{M\sqrt{2}n}{2R^*} A - \frac{\omega_n \left(R^*\right)^n}{2} nA + \frac{3\,(n-1)\,\kappa\omega_{n-1} \left(R^*\right)^{n+1}}{2(n+1)}\Big) + O(\delta^2) \\
&= -\delta M\sqrt{2} \frac{(n+2)(n-1)\omega_{n-1}}{(n+1)(n+3)\omega_n} \kappa + \delta \left(R^*\right)^n \frac{(n+2)\,(n-1)\,\omega_{n-1}}{(n+1)\,(n+3)} \kappa R^* \\
&- \delta \left(R^*\right)^{n+1} \frac{3\,(n-1)\,\kappa\omega_{n-1}}{2(n+1)} + O(\delta^2) \\
&= \delta \left(R^*\right)^{n+1} \omega_{n-1}\kappa \left[\frac{(n+2)(n-1)}{(n+1)(n+3)} - \frac{3\,(n-1)}{2(n+1)} \right] \\
&- \delta M\sqrt{2} \frac{(n+2)(n-1)\omega_{n-1}}{(n+1)\,(n+3)\,\omega_n} \kappa + O(\delta^2) \\
&= -\delta M\sqrt{2} \left[\frac{(n+2)(n-1)\omega_{n-1}}{(n+1)(n+3)\omega_n} \kappa + \frac{(n+2)}{\omega_n} \frac{(n-1)(n+5)}{2(n+1)(n+3)} \right] + O(\delta^2) \\
&= -\frac{\delta M\kappa\sqrt{2}\omega_{n-1}}{\omega_n} \left[\frac{(n+2)(n-1)}{(n+1)(n+3)} + \frac{(n+2)\,(n-1)\,(n+5)}{2(n+1)(n+3)} \right] + O(\delta^2) \\
&= -\frac{\delta M\kappa\sqrt{2}\omega_{n-1}}{\omega_n} \frac{(n+2)\,(n-1)\,(n+7)}{2(n+1)\,(n+3)} + O(\delta^2) \\
&= -c\,(n)\,M\kappa\delta + O(\delta^2)
\end{aligned}$$

where

$$c(n) = \frac{(n-1)\,(n+2)\,(n+7)\,\omega_{n-1}}{\sqrt{2}(n+1)\,(n+3)\,\omega_n}$$

is a positive constant. It then follows that energy minimizer should be concentrated near the point of maximal mean curvature. Theorem 3 has been proved.

3.0 RADIAL STEADY STATE SOLUTION FOR THIN FILM EQUATION

In this chapter, we will focus on the radial steady states of van der Waals force driven thin film. In particular, we want to understand the thin film configuration when the total liquid volume in a cylindrical container is prescribed. We present a theoretical proof of a result claimed by Miloua in his thesis [64] with a more precise description of the limiting profile used in the construction of stationary solutions.

3.1 INTRODUCTION

Recall that the thin film type equation driven by van der Waals force and surface tension is governed by the fourth order nonlinear partial differential equation

$$u_t = \nabla \left(u^n \nabla p \right) \tag{3.1.1}$$

where u is the thickness of the thin film and the pressure

$$p = -\Delta u + \frac{1}{\alpha} u^{-\alpha} \tag{3.1.2}$$

is a sum of linearized surface tension and van der Waals force. Here $n > 0$ and $\alpha > 1$ are physical constants. Physical experiments suggest $\alpha = 3$ for van der Waals force. Let Ω be a bounded smooth domain in $\mathbb{R}^2$ which represents the bottom of a cylindrical container containing thin film liquid. For physical meaning, the total volume is fixed, i.e.

$$\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx$$

is a given constant.

As previously mentioned, we assume that there is no flux across the boundary, which yields the boundary condition

$$\frac{\partial p}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.1.3}$$

Also we assume the following Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{3.1.4}$$

which means that we ignore the wetting effect and the fluid surface is orthogonal to the boundary of the container.

Define associated energy functional

$$E(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{\alpha(\alpha-1)} u^{1-\alpha} \right). \tag{3.1.5}$$

We compute the derivative with respect to time by integration by parts, using equation (3.1.1) and the boundary conditions (3.1.3) , (3.1.4),

$$\begin{aligned} \frac{d}{dt} E(u) &= \int_\Omega \nabla u \cdot \nabla u_t + \frac{1}{\alpha} u^{-\alpha} u_t \\ &= \int_\Omega -\Delta u u_t + \frac{1}{\alpha} u^{-\alpha} u_t \\ &= \int_\Omega p \nabla \cdot (u^n \nabla p) \\ &= -\int_\Omega u^n |\nabla p|^2 \leq 0. \end{aligned}$$

Hence, for a thin film fluid at rest, u satisfies the elliptic equation

$$-\Delta u + \frac{1}{\alpha} u^{-\alpha} = p \text{ in } \Omega$$

with the Neumann boundary condition (3.1.4) and pressure p being a constant.

Therefore for any given $\bar{u} > 0$, we need to find a function u and an unknown constant p satisfying

$$\begin{cases} \Delta u = \frac{1}{\alpha} u^{-\alpha} - p \ \text{ in } \ \Omega, \\ \\ \frac{1}{|\Omega|} \int_\Omega u(x)dx = \bar{u}, \\ \\ \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \ \ \partial\Omega. \end{cases} \tag{3.1.6}$$

Comparing to this, in the chapter 2, we consider semi limit case of the thin film with additional born repulsion force that leads to the following elliptic equation,

$$\begin{cases} \Delta u = \frac{1}{\alpha} u^{-\alpha} \left(1 - (\frac{\varepsilon}{u})^{\beta}\right) - p \ \text{ in } \ \Omega, \\ \\ \frac{1}{|\Omega|} \int_\Omega u(x)dx = \bar{u}, \\ \\ \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \ \ \partial\Omega. \end{cases}$$

For the above elliptic equation, A. L. Bertozzi, G. Grun and T.P. Witelski [24] proved the existence of the solution and obtained asymptotic behavior of solution in one dimensional space numerically and theoretically. Later Jiang [54] extended the result to multi-dimensional situation. Their variational approach relied on the boundedness of the associated energy

$$\mathcal{E}_\varepsilon [u] = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 - \frac{u^{-\alpha}}{\alpha} + \frac{\varepsilon^\beta u^{-\alpha-\beta}}{\alpha+\beta} \right\}.$$

Our case can be seen as the limiting case of the above equation with $\varepsilon = 0$. However, the associated energy defined by (3.1.5) is indefinite due to its singularity. Hence we will try to understand the radial solutions first.

3.2 SETTING AND RESULTS

Now we will consider the profile of thin film on a disk in $\mathbb{R}^2$. Take $\Omega = B_1(0)$. Equation (3.1.6) can be rewritten as

$$\begin{cases} u_{rr} + \frac{1}{r}u_r = \frac{1}{\alpha}u^{-\alpha} - p \ \text{ in } \ B_1(0), \\ 2\int_0^1 ru(r)dr = \bar{u}, \\ u'(1) = 0. \end{cases} \tag{3.2.1}$$

From the elliptic theory, u is smooth whenever it is positive, hence we also require that $u'(0) = 0$ if $u(0) > 0$.

We first ignore the volume constraint and the Neumann boundary condition. We consider the ordinary differential equation

$$\begin{cases} u_{rr} + \frac{1}{r}u_r = \frac{1}{\alpha}u^{-\alpha} - p, \\ u(0) = \eta, \\ u'(0) = 0 \end{cases} \tag{3.2.2}$$

defined on $[0, \infty)$. Jiang and Ni [57] gave a complete description of the radial solution to (3.2.2). For $\eta > 0$, there exists a unique positive solution u^η defined on $[0, \infty)$. And when $\eta = 0$, there exists a unique rupture solution u^0 which is continuous on $[0, \infty)$ such that $u(0) = 0$ and u is positive and satisfies the ordinary differential equation in (3.2.2) on $(0, \infty)$. Obviously $u^\eta \equiv (\alpha p)^{-\frac{1}{\alpha}}$ if $\eta = (\alpha p)^{-\frac{1}{\alpha}}$. For $\eta \geq 0$ and $\eta \neq (\alpha p)^{-\frac{1}{\alpha}}$, u^η oscillates around the constant $(\alpha p)^{-\frac{1}{\alpha}}$. There exists an increasing sequence of critical radius $r_k^\eta \to \infty$ such that $(u^\eta)'(r_k^\eta) = 0$. Moreover, $u^\eta(r_k^\eta)$ achieves local maximum and local minimum alternatively and approaches the constant $(\alpha p)^{-\frac{1}{\alpha}}$ eventually.

We will obtain the radial solutions to (3.2.1) by scaling u^η with $p = \frac{1}{\alpha}$. We remark here that different values of p will yield the same scaled solution. Given $\eta \geq 0$, $\eta \neq 1$ and a positive integer k, we have $u^\eta(r)$ satisfies the Neumann boundary condition at $r = r_k^\eta$. We now define

$$u^{\eta,k}(r) = (r_k^\eta)^{-\frac{2}{1+\alpha}} u^\eta(r_k^\eta r).$$

Then,

$$\begin{aligned}
\Delta u^{\eta,k} &= u^{\eta,k}_{rr} + \frac{1}{r} u^{\eta,k}_{r} \\
&= (r^{\eta}_{k})^{2-\frac{2}{1+\alpha}} \left[(u^{\eta})'' (r^{\eta}_{k} r) + \frac{1}{r^{\eta}_{k} r} (u^{\eta})' (r^{\eta}_{k} r) \right] \\
&= (r^{\eta}_{k})^{\frac{2\alpha}{1+\alpha}} (\frac{1}{\alpha} (u^{\eta})^{-\alpha} - p) \\
&= \frac{1}{\alpha} (u^{\eta,k})^{-\alpha} - \frac{1}{\alpha} (r^{\eta}_{k})^{\frac{2\alpha}{1+\alpha}} .
\end{aligned}$$

Therefore, $u^{\eta,k}(x)$ satisfies the elliptic equation

$$\Delta u = \frac{1}{\alpha} \cdot u^{-\alpha} - p^{\eta,k} \quad in \quad B_1(0)$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \; on \;\; \partial B_1(0),$$

where $p^{\eta,k}$ is defined by

$$p^{\eta,k} = \frac{1}{\alpha} (r^{\eta}_{k})^{\frac{2\alpha}{1+\alpha}} .$$

We can calculate the average thickness for $u^{\eta,k}$,

$$\begin{aligned}
\bar{u}^{\eta,k} &= \frac{1}{|B_1(0)|} \int_{B_1(0)} u^{\eta,k}(x) dx = \frac{(r^{\eta}_{k})^{-\frac{2}{1+\alpha}}}{|B_{r^{\eta}_{k}}(0)|} \int_{B_{r^{\eta}_{k}}(0)} u^{\eta}(r) dr \\
&= 2 (r^{\eta}_{k})^{-\frac{2}{1+\alpha}-2} \int_0^{r^{\eta}_{k}} r u^{\eta}(r) dr.
\end{aligned}$$

Hence, we constructed a nontrivial radial solution to (3.2.1) with the average thickness

$$\bar{u} = \bar{u}^{\eta,k} = 2 (r^{\eta}_{k})^{-\frac{2}{1+\alpha}-2} \int_0^{r^{\eta}_{k}} r u^{\eta}(r) dr. \tag{3.2.3}$$

Since any nontrivial solutions to (3.2.1) can be obtained in this manner, in order to solve (3.2.1) with prescribed volume, we need to understand the dependence of $\bar{u}^{\eta,k}$ on initial value η and number of bumps k.

The following result is claimed by Attou Miloua in [64] which we will give a rigorous proof.

Theorem 12. *For fixed integer $k \geq 1$, $\bar{u}(\eta, k)$ can be defined as a continuous function for $\eta \in [0, \infty)$. Moreover,*

$$\lim_{\eta \to \infty} \frac{\bar{u}(\eta, k)}{\eta^{\frac{\alpha}{1+\alpha}}} = C, \text{ where } C = C(k, \alpha).$$

As we know, classical theory in ordinary differential equation gives the continuous dependence on the initial data which indicates, $\bar{u}(\eta, k) = \bar{u}^{\eta,k}$ is a continuous function on η on $(0,1) \cup (1, \infty)$. We will just concentrate on the three cases in the next several sections as $\eta \to 0$, $\eta \to 1$ and $\eta \to \infty$.

Also we compute associated energy for (3.2.1) as following,

$$\begin{aligned} E^{\eta,k} &= \int_{B_1(0)} \left(\frac{1}{2} \left| \nabla u^{\eta,k} \right|^2 - \frac{1}{\alpha(\alpha - 1)} \left(u^{\eta,k} \right)^{1-\alpha} \right) \\ &= (r_k^\eta)^{-\frac{4}{1+\alpha}} \int_{B_{r_k^\eta}(0)} \left(\frac{1}{2} \left| \nabla u^\eta \right|^2 - \frac{1}{\alpha(\alpha - 1)} \left(u^\eta \right)^{1-\alpha} \right) \\ &= 2\pi (r_k^\eta)^{-\frac{4}{1+\alpha}} \int_0^{r_k^\eta} \left(\frac{1}{2} \left(\frac{du^\eta}{dr} \right)^2 - \frac{1}{\alpha(\alpha - 1)} \left(u^\eta \right)^{1-\alpha} \right) r dr. \end{aligned}$$

Similarly, we have the following theorem that

Theorem 13. *For fixed integer k, $E^{\eta,k}$ is a continuous function for $\eta \in [0, \infty)$ and*

$$\lim_{\eta \to \infty} \frac{E^{\eta,k}}{\eta^{\frac{2\alpha}{1+\alpha}}} = C \text{ where } C = C(k, \alpha).$$

Hence,

$$E^{\eta,k} \sim \bar{u}^2 \text{ as } \eta \to \infty.$$

Therefore, for volume constraint problem (3.2.1), we have the existence of radial solution for given volume.

Theorem 14. *For given average thickness $\bar{u}$, the equation (3.2.1) admits infinitely many radial solutions.*

Proof. For radial rupture solution, there exists a sequence of thickness $\bar{u_k}$ [57] satisfying

$$\lim_{k\to\infty} \sqrt{k\pi}\bar{u_{0k}} = 1$$

and rupture solution exists if and only if $\bar{u} = \bar{u}_{0k}$. Any rupture solution with $\bar{u} = \bar{u}_{0k}$ has exactly k critical values including 1. It implies that

$$\lim_{k\to\infty} \bar{u}_{0k} = 0.$$

For any given average thickness $\bar{u}$, these exists K_0 such that for any $k \geq K_0$,

$$\bar{u}_{0k} \leq \bar{u}.$$

According to Theorem 12, $\bar{u}(\eta, k)$ is continuously depending on η and as $\eta \to \infty$,

$$\bar{u}(\eta, k) \to \infty.$$

It follows, there exists a η_k such that

$$\bar{u}(\eta_k, k) = \bar{u}.$$

As a consequence, we construct a radial solution with k critical values satisfies (3.2.1). Since k can be arbitrary integer satisfying $k \geq K_0$, we have the infinite many radial solution with critical values equal to or more than K_0. □

Now we will consider there cases in the next several sections and gives the uniform convergence to finish the proof of Theorem 12 and Theorem 13. Moreover, we will have a description of the limiting profile.

3.3 RUPTURE SOLUTION BY TAKING LIMIT $\eta \to 0$

Rupture solution has been investigated to a large extent in [26, 57, 51, 56, 52]. Jiang and Ni [57] proved the existence and uniqueness of rupture solution for our equation. Moreover, they have given an accurate estimation for a small interval starting from 0,

$$u(r) \sim r^{\frac{2}{1+\alpha}}.$$

As $\eta \to 0^+$, u^η converges uniformly to the rupture solution u^0 on $[0, \infty)$. Hence, $\bar{u}(\eta, k)$ is continuous at $\eta = 0$.

Denote $\{r_k\}_{k=1}^\infty$ to be the increasing sequence where $(u^0)'(r) = 0$. They have proven for such rupture solution,

$$\lim_{k \to \infty} (r_{k+1} - r_k) = \pi.$$

We remark here that rupture solutions has been constructed by Jiang and Miloua in physical dimension 2 [56] and in higher dimensions by Guo, Ye and Zhou [51] for more general equations

$$\Delta u = f(u)$$

with f satisfying certain growth conditions.

3.4 LINEARIZATION WHEN $\eta \to 1$

For $\eta = 1$, we know that the radial solution to (3.2.1) is a trivial constant solution $u^\eta \equiv 1$. Here, the critical values $\{r_k\}_{k=1}^\infty$ is not well defined. So we need to understand the behavior of u^η as $\eta \to 1$ first. As u^η oscillates around 1, we can apply linearization by defining

$$\varepsilon = \eta - 1,$$

and

$$w^\eta(r) = \frac{u^\eta(r) - 1}{\varepsilon}.$$

Then w^η is a solution to the differential equation

$$w_{rr} + \frac{1}{r} w_r = \frac{1}{\varepsilon} \left[\frac{1}{\alpha} (1 + \varepsilon w)^{-\alpha} - \frac{1}{\alpha} \right] \tag{3.4.1}$$

with initial condition

$$w(0) = 1,\ w'(0) = 0.$$

Sending $\eta \to 1$ and so $\varepsilon \to 0$, (3.4.1) formally converges to the limiting problem,

$$\begin{cases} w^*_{rr} + \frac{1}{r} w^*_r + w^* = 0, \\ w^*(0) = 1, \\ (w^*)'(0) = 0 \end{cases} \tag{3.4.2}$$

Denote $\{r_k\}_{k=1}^\infty$ to be the increasing sequence of the critical point of w and $\{r^*_k\}_{k=1}^\infty$ to be the increasing sequence of the critical points of w^*. We establish the following theorem,

Theorem 15. *As $\eta \to 1$, the solution w^η to (3.4.1) converges uniformly to the solution w^* to (3.4.2). Furthermore, r_k converges to r^*_k.*

Proof. In order to apply the classical perturbation theory, we will prove w^η is bounded first. For simplicity, we will suppress script η here. Since u is the solution to (3.2.2), we define

$$e(r) = \frac{1}{2}(u'(r))^2 + F(u(r))$$

with

$$F(u) = \frac{1}{\alpha(\alpha - 1)} u^{1-\alpha} + \frac{1}{\alpha} u.$$

One can easily verify that $F'(u) > 0$ for $u > 1$ and $F'(u) < 0$ for $0 < u < 1$. $F(u)$ attains its minimum $\frac{1}{\alpha - 1}$ at $u = 1$. We have

$$\begin{aligned} \frac{d}{dr}[e(r)] &= u'(r)u''(r) - \frac{1}{\alpha} u^{-\alpha} u'(r) + \frac{1}{\alpha} u'(r) \\ &= -\frac{1}{r}(u'(r))^2 \leq 0. \end{aligned}$$

$e(r)$ is decreasing which yields that $F(u(r)) \leq e(r) \leq e(0) = F(\eta)$. Note that for any positive constant $C > 1$, as $\eta \to 1$,

$$
\begin{aligned}
&F(1 + C(1-\eta)) - F(\eta) \\
&= F(1 + C(1-\eta)) - F(1 + (\eta - 1)) \\
&= [F(1) + \frac{1}{2}F''(1)C^2(1-\eta)^2] - [F(1) + \frac{1}{2}F''(1)(\eta-1)^2] + O((\eta-1)^3) \\
&= \frac{1}{2}(C^2-1)(\eta-1)^2 + O((\eta-1)^3) \\
&\geq 0
\end{aligned}
$$

For $\eta > 1$, then $1 + C(1-\eta) \leq u(r) \leq \eta$. It follows that

$$-C \leq \frac{C(1-\eta)}{\varepsilon} \leq w = \frac{u-1}{\varepsilon} \leq \frac{\eta-1}{\varepsilon} = 1.$$

For $\eta < 1$, then $\eta \leq u(r) \leq 1 + C(1-\eta)$. It follows that

$$-C \leq \frac{C(1-\eta)}{\varepsilon} \leq w = \frac{u-1}{\varepsilon} \leq \frac{\eta-1}{\varepsilon} = 1.$$

Thus as $\eta \to 1$, $|w| \leq C$ for some constant $C > 1$.

Now check the equation (3.4.1),

$$w_{rr} + \frac{1}{r}w_r = \frac{1}{\varepsilon}\left[\frac{1}{\alpha}(1+\varepsilon w)^{-\alpha} - \frac{1}{\alpha}\right] = -w + (\alpha+1)\varepsilon w^2 + O(\varepsilon^2).$$

By perturbation theory, solution to the equation system $w(r)$ and $w'(r)$ is continuous in parameter ε. That is, $w(r)$ and $w'(r)$ uniformly converges to $w^*(r)$ and $(w^*)'(r)$. It is easy to have the convergence of the critical points which ends the proof. □

Actually, (3.4.2) is called the Bessel's differential equation of the first kind with order 0. The solution is uniquely given by

$$J_0(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{r}{2}\right)^{2n}.$$

Note that $J_0\left(r\right)$ is oscillating around 0. We are interested in the behavior of critical values r_k^*. r_k^* are the zeros of $(w^*)'(r)$ which satisfies the Bessel's equation of the first kind with order 1, i.e.

$$r^2 y'' + r y' + (r^2 - 1) y = 0.$$

When r is sufficiently large, the asymptotic formula is given by [28].,

$$\tilde{J}_1(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \frac{3}{4}\pi) + O(r^{-\frac{3}{2}}).$$

The difference between two successive zeros becomes approximately π. That is,

$$\lim_{k\to\infty} (r_{k+1}^* - r_k^*) = \pi.$$

Since $u^\eta \to 1$ uniformly, $r_k^\eta \to r_k^*$ as $\eta \to 1$, we have the average thickness $\bar{u}(\eta, k)$ and the associated energy $E^{\eta,k}$ are continuously defined at $\eta = 1$. Therefore, we can define $\bar{u}^{\eta,k}$ and $E^{\eta,k}$ so that they are both continuous functions on $[0, \infty)$. Moreover,

$$\bar{u}^{\eta,k} = \frac{(r_k^\eta)^{-\frac{2}{1+\alpha}}}{|B_{r_k^\eta}(0)|} \int_{B_{r_k^\eta}(0)} u^\eta(r) dr \to (r_k^*)^{-\frac{2}{1+\alpha}}$$

as $\eta \to 1$ and

$$\begin{aligned} E^{\eta,k} &= (r_k^\eta)^{-\frac{4}{1+\alpha}} \int_{B_{r_k^\eta}(0)} \left(\frac{1}{2} \left|\nabla u^\eta\right|^2 - \frac{1}{\alpha\left(\alpha - 1\right)} \left(u^\eta\right)^{1-\alpha} \right) \\ &\to -\frac{1}{\alpha\left(\alpha - 1\right)} (r_k^*)^{-\frac{4}{1+\alpha}} \left|B_{r_k^*}\left(0\right)\right| = -\frac{\pi}{\alpha\left(\alpha - 1\right)} (r_k^*)^{2-\frac{4}{1+\alpha}} \end{aligned}$$

as $\eta \to 1$.

3.5 LIMITING PROFILE WHEN $\eta \to \infty$

In this section, we will end the proof of Theorem 12 and Theorem 13 by investigating the behavior of $\bar{h}^{\eta,k}$ and $E^{\eta,k}$ as $\eta \to \infty$.

Let $\eta > 1$ and u^{η} be the solution to (3.2.2). we define the blow down solution z after scaling by

$$z(x) = \frac{1}{\eta} u(r)$$

with $x = \frac{r}{\sqrt{\alpha\eta}}$. Then we have

$$z_x = \frac{\sqrt{\alpha}}{\sqrt{\eta}} u_r(r),$$
$$z_{xx} = \alpha u_{rr}(r)$$

and hence

$$z'' + \frac{1}{x} z' = \alpha \left(u_{rr} + \frac{1}{r} u_r \right) = u^{-\alpha} - 1$$
$$= \frac{\eta^{-\alpha}}{z^{\alpha}} - 1.$$

Denoting $\varepsilon = \frac{1}{\eta}$, we have $\varepsilon \to 0$ as $\eta \to \infty$. The blow down function z satisfies the initial value problem

$$\begin{cases} z'' + \frac{1}{x} z' = \frac{\varepsilon^{\alpha}}{z^{\alpha}} - 1, \\ z(0) = 1, \quad \text{and} \quad z'(0) = 0. \end{cases} \tag{3.5.1}$$

Formally, as $\varepsilon \to 0$, (3.5.1) converges to the limiting equation

$$\begin{cases} z'' + \frac{1}{x} z' = -1, \\ z(0) = 1, \quad \text{and} \quad z'(0) = 0. \end{cases} \tag{3.5.2}$$

which has a unique global solution

$$z(x) = 1 - \frac{1}{4} x^2.$$

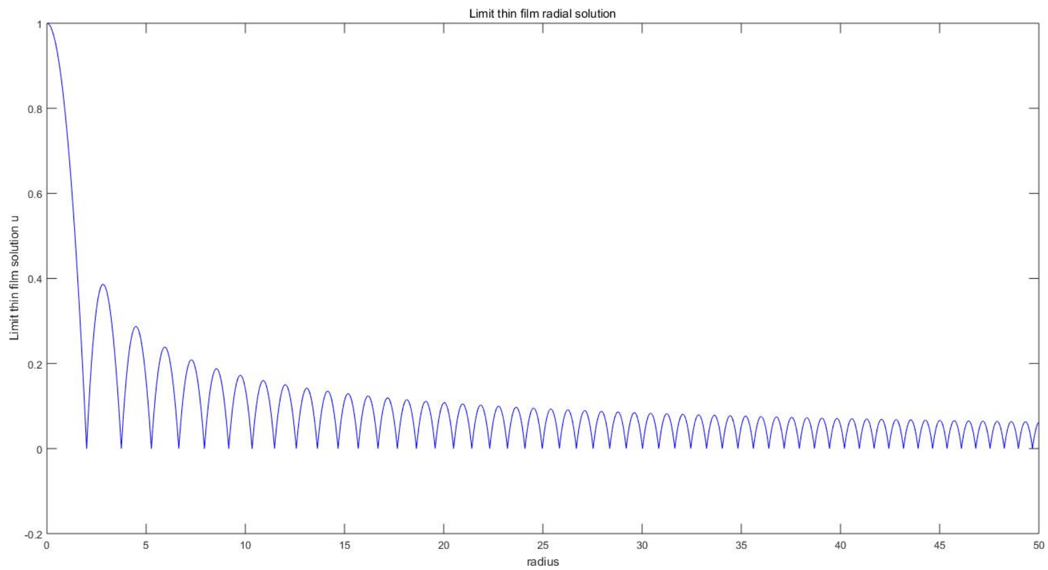

Figure 1: Limit radial solution z_*

However, we can't expect

$$\lim_{\varepsilon\to 0} z^{\varepsilon}(x) = 1 - \frac{1}{4}x^2$$

since the function $1 - \frac{1}{4}x^2$ becomes negative when $x > 2$.

Nonetheless, we can establish the following theorem:

Theorem 16. *For every $\varepsilon > 0$, let $z^{\varepsilon}(x)$ be the unique solution of the initial value problem (3.5.1). Then as ε tends to zero positively, $z^{\varepsilon}(x)$ converges uniformly to $z_*(x)$, the solution of the limiting initial value problem*

$$\begin{cases} z_*'' + \frac{1}{x} z_*' = -1, \quad z_* > 0 \ \ in \bigcup_{j=0}^{\infty} (a_j, a_{j+1}). \\ z_*(0) = 1, \quad and \ \ z_*'(0) = 0, \\ z_*(a_j) = 0, \quad z_*'(a_j+) = -z_*'(a_j-) \end{cases} \tag{3.5.3}$$

where $a_0 = 0$, $2 = a_1 < a_2 < \cdots$ are inductively computed by solving the IVP (3.5.3).

In the next sections we will prove this theorem and give a description of the limiting solution to equation (3.5.3) by constructing the estimate of a_j. From the above theorem, we have $z^{\varepsilon}(x)$ converges uniformly to $1-\frac{1}{4}x^2$ on $[0,2]$ as $\varepsilon\to 0$ and $a_1=2$. That is, $\frac{r_1^\eta}{\sqrt{\alpha\eta}}$ converges to 2 as $\eta\to\infty$. More generally, we have for $k=1,2,3,\cdots$,

$$\lim_{\eta\to\infty}\frac{r_{2k-1}^\eta}{\sqrt{\alpha\eta}}=a_k$$

and

$$\lim_{\eta\to\infty}\frac{r_{2k}^\eta}{\sqrt{\alpha\eta}}=b_k$$

where b_k is the maximum point of z^* in (a_{j-1},a_j).

Now we are ready to compute the average thickness. Given a positive integer k and η large enough, we have

$$\begin{aligned}\bar{u}^{\eta,k} &= 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\int_0^{r_k^\eta} ru^\eta(r)dr\\ &= 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\eta\int_0^{r_k^\eta} rz\left(\frac{r}{\sqrt{\alpha\eta}}\right)dr\\ &= 2(r_k^\eta)^{-\frac{2}{1+\alpha}-2}\alpha\eta^2\int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} sz(s)\,ds\\ &= 2\alpha^{-\frac{1}{1+\alpha}}\eta^{\frac{\alpha}{1+\alpha}}\left(\frac{r_k^\eta}{\sqrt{\alpha\eta}}\right)^{-\frac{2}{1+\alpha}-2}\int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} sz(s)\,ds.\end{aligned}$$

Hence, we have for $k=1,2,3,\cdots$,

$$\lim_{\eta\to\infty}\frac{\bar{u}^{\eta,2k-1}}{\eta^{\frac{\alpha}{1+\alpha}}}=2\alpha^{-\frac{1}{1+\alpha}}a_k^{-\frac{2}{1+\alpha}-2}\int_0^{a_k} sz^*(s)\,ds.$$

and

$$\lim_{\eta\to\infty}\frac{\bar{u}^{\eta,2k}}{\eta^{\frac{\alpha}{1+\alpha}}}=2\alpha^{-\frac{1}{1+\alpha}}b_k^{-\frac{2}{1+\alpha}-2}\int_0^{b_k} sz^*(s)\,ds.$$

For fixed integer k, $\bar{u}$ increases to infinity with order $\frac{\alpha}{\alpha+1}$ as η tends to infinity.

Similarly we investigate the energy of radial solutions as $\eta\to\infty$. Since

$$u(r)=\eta z(x),$$

$$\frac{du^\eta}{dr} = \eta z'(x)\frac{dx}{dr} = \frac{\sqrt{\eta}}{\sqrt{\alpha}} z'(x).$$

Hence

$$\begin{aligned} E^{\eta,k} &= 2\pi (r_k^\eta)^{-\frac{4}{1+\alpha}} \int_0^{r_k^\eta} \left(\frac{1}{2}\left(\frac{du^\eta}{dr}\right)^2 - \frac{1}{\alpha(\alpha-1)}(u^\eta)^{1-\alpha} \right) rdr \\ &= 2\pi (\frac{r_k^\eta}{\sqrt{\alpha\eta}})^{-\frac{4}{1+\alpha}} (\alpha\eta)^{1-\frac{2}{1+\alpha}} \int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} \left(\frac{1}{2}\frac{\eta}{\alpha}(z'(x))^2 - \frac{1}{\alpha(\alpha-1)}(\eta z(x))^{1-\alpha} \right) xdx \\ &= \pi\alpha^{-\frac{2}{1+\alpha}} (\frac{r_k^\eta}{\sqrt{\alpha\eta}})^{-\frac{4}{1+\alpha}} \eta^{2-\frac{2}{1+\alpha}} \int_0^{\frac{r_k^\eta}{\sqrt{\alpha\eta}}} x|z'|^2 dx + O\left(\eta^{2-\alpha-\frac{2}{1+\alpha}}\right). \end{aligned}$$

Hence, we have for $k = 1, 2, 3, \cdots$,

$$\lim_{\eta\to\infty} \frac{E^{\eta,2k-1}}{\eta^{2-\frac{2}{1+\alpha}}} = \pi\alpha^{-\frac{2}{1+\alpha}} a_k^{-\frac{4}{1+\alpha}} \int_0^{a_k} s\left|(z^*)'\right|^2 ds.$$

and

$$\lim_{\eta\to\infty} \frac{E^{\eta,2k}}{\eta^{2-\frac{2}{1+\alpha}}} = \pi\alpha^{-\frac{2}{1+\alpha}} b_k^{-\frac{4}{1+\alpha}} \int_0^{b_k} s\left|(z^*)'\right|^2 ds.$$

For fixed integer k, $E^{\eta,k}$ increases to infinity with order $\frac{2\alpha}{\alpha+1}$ as η tends to infinity. Therefore,

$$E^{\eta,k} \sim \bar{h}^2(\eta, k) \text{ as } \eta \to \infty.$$

3.6 PROOF OF THE MAIN CONVERGENCE THEOREM 15

Define energy as follows,

$$e(x) = \frac{1}{2}(z'(x))^2 + G(z) \tag{3.6.1}$$

where

$$G(z) = \frac{\varepsilon^\alpha}{\alpha-1} z^{1-\alpha} + z.$$

It is easy to check that G has the following properties:

$$\begin{cases} G(\varepsilon) = \min_{z\in(0,\infty)} G(z) = \frac{\alpha}{\alpha-1}\varepsilon \\ G'(z) > 0 \text{ for } v > \varepsilon \text{ and } G'(z) < 0 \text{ for } v < \varepsilon, \\ G''(z) \geq 0, \\ \lim_{z\to 0} G(z) = \infty. \end{cases}$$

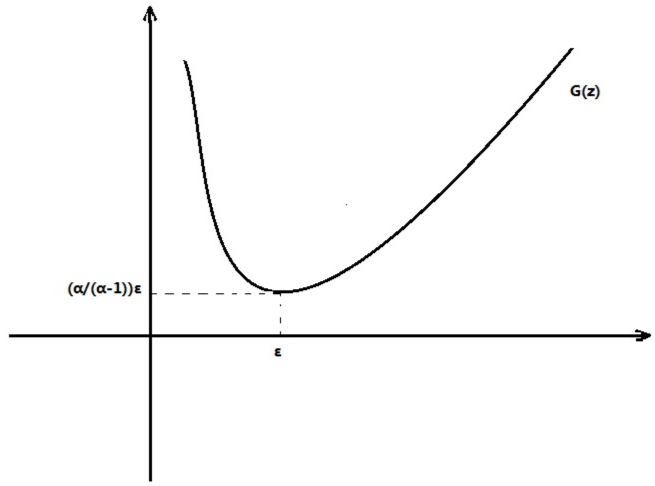

Figure 2: function G(z)

See the figure of function $G(z)$.

By integration by parts, we can rewrite energy as

$$e(x) = G(1) - \int_0^x \frac{2(z'(y))^2}{y} dy. \tag{3.6.2}$$

The above formula indicates the energy dissipation in x and therefore $e(x)$ is bounded above by $G(1) = 1 + \frac{\varepsilon^\alpha}{\alpha - 1}$.

We know that $z(x)$ oscillates around ε. Firstly, we will show an auxiliary problem for the above equation to analyze the part of the solution which is below $z = \varepsilon$.

Lemma 13. *Let $z(x)$ be the solution to*

$$\begin{cases} z'' + \frac{1}{x} z' = \frac{\varepsilon^\alpha}{z^\alpha} - 1, \\ z(x_m) = m, \quad \text{and} \quad z'(x_m) = 0. \end{cases} \tag{3.6.3}$$

Assume that:

(1) there exists an interval (a,b) *such that* $z<\varepsilon$ *in* (a,b) *and* $z(a)=z(b)=\varepsilon$.

(2) $x_m\in(a,b)$ *and* $x_m>a\geq 1$.

(3) $z(x)$ *attains its unique minimum* m *in* (a,b) *at* x_m *with* $0<m<\frac{\varepsilon}{2}$.

Then for ε *sufficiently small,*

(i)

$$b-a\leq\sqrt{2\varepsilon}(1+\frac{1}{1-\sqrt{2\varepsilon}}). \tag{3.6.4}$$

(ii)

$$\int_a^b\frac{(z'(x))^2}{x}dx\leq 3\varepsilon\sqrt{2G(m)} \tag{3.6.5}$$

Proof. Multiply both sides of (3.6.3) by $2x^2z'(x)$,

$$2x^2z'z''+2x(z')^2=2x^2(\frac{\varepsilon^\alpha z'}{z^\alpha}-z').$$

That is,

$$\frac{d}{dx}(x^2(z')^2)=-2x^2\frac{d}{dx}G(z),$$

Integrating from x_m to x gives

$$(z')^2=\frac{2}{x^2}\int_{x_m}^x y^2\frac{d}{dy}[G(m)-G(z(y))]dy.$$

Now it is necessary for us to compute

$$G(m)-G(z)=\frac{\varepsilon^\alpha}{\alpha-1}m^{1-\alpha}+m-\frac{\varepsilon^\alpha}{\alpha-1}z^{1-\alpha}-z=(z-m)(\frac{\varepsilon^\alpha}{\alpha-1}\frac{m^{1-\alpha}-z^{1-\alpha}}{z-m}-1).$$

Apply the mean value theorem, we have,

$$G(m)-G(z)=(z-m)(\frac{\varepsilon^\alpha}{\alpha-1}\frac{z^{\alpha-1}-m^{\alpha-1}}{(z-m)m^{\alpha-1}z^{\alpha-1}}-1)=(z-m)(\frac{\varepsilon^\alpha\xi^{\alpha-2}}{m^{\alpha-1}z^{\alpha-1}}-1).$$

For $0<m<\frac{\varepsilon}{2}$ and $m\leq\xi\leq z\leq\varepsilon$, it follows that

$$G(m)-G(z)\geq(z-m)(\frac{\varepsilon}{m}-1)\geq(z-m).$$

(1) For $x \in [a, x_m]$,

$$\frac{x_m}{x}\sqrt{2(G(m)-G(z))} \geq |\frac{dz}{dx}| \geq \sqrt{2(G(m)-G(z))}.$$

Then we have

$$x_m - a = \int_a^{x_m} dx \leq \int_m^{\varepsilon} \frac{dz}{\sqrt{2(G(m)-G(z))}} \leq \int_m^{\varepsilon} \frac{dz}{\sqrt{2(z-m)}} \leq \sqrt{2\varepsilon} \tag{3.6.6}$$

and

$$\int_a^{x_m} \frac{(z'(x))^2}{x} dx \leq \frac{x_m}{a^2}\int_m^{\varepsilon} \sqrt{2(G(m)-G(z))}dz \leq \frac{x_m}{a^2}\varepsilon\sqrt{2G(m)}. \tag{3.6.7}$$

(2) For $x \in [x_m, b]$,

$$\frac{x_m}{x}\sqrt{2(G(m)-G(z))} \leq |\frac{dz}{dx}| \leq \sqrt{2(G(m)-G(z))}.$$

Then we have,

$$\frac{x_m}{b}(b - x_m) \leq \int_{x_m}^{b} \frac{x_m}{x} dx \leq \int_m^{\varepsilon} \frac{dz}{\sqrt{2(G(m)-G(z))}} \leq \sqrt{2\varepsilon} \tag{3.6.8}$$

and

$$\int_{x_m}^{b} \frac{(z'(x))^2}{x} dx \leq \frac{1}{x_m}\int_m^{\varepsilon} \sqrt{2(G(m)-G(z))}dz \leq \frac{1}{x_m}\varepsilon\sqrt{2G(m)}. \tag{3.6.9}$$

We can derive from (3.6.8) ,

$$b - x_m \leq \sqrt{2\varepsilon}\frac{1}{1-\frac{\sqrt{2\varepsilon}}{x_m}}. \tag{3.6.10}$$

Consequently, adding (3.6.6) and (3.6.10), we obtain

$$b - a \leq \sqrt{2\varepsilon}(1 + \frac{1}{1-\frac{\sqrt{2\varepsilon}}{x_m}}) \leq \sqrt{2\varepsilon}(1 + \frac{1}{1-\sqrt{2\varepsilon}}).$$

Energy dissipation in (a, b) is given by adding (3.6.7) and (3.6.9),

$$\int_a^b \frac{(z'(x))^2}{x} dx \leq (\frac{x_m}{a^2} + \frac{1}{x_m})\varepsilon\sqrt{2G(m)}.$$

Use (3.6.6) and for ε small enough,

$$\int_a^b \frac{(z'(x))^2}{x} dx \leq 3\varepsilon\sqrt{2G(m)}.$$

□

Remark 2. *Since the solution to equation (3.5.1) in* $[0,2]$ *uniformly converges to* $1-\frac{x^2}{4}$*, the first point* a *such that* $u(a)=\varepsilon$ *is bounded below by* 1*. Then, (3.6.4) indicates that interval* (a,b) *tends to be shrinking to 0 as* ε *approaches* 0*. On the other side, the energy dissipation given by (3.6.5) converges to* 0*. This formally implies that for the limit solution* z_* *given by (3.5.3),* $z_*'(a_j+)=-z_*'(a_j-)$*.*

Now we are going to check the condition (3) in Lemma 13 for the first local minimum. As we know, starting from 0, z is decreasing in x. Suppose that z attains its first local minimum m_1 at x_{m_1}. Oscillation around ε gives that $m_1<\varepsilon$. We are able to prove the following estimate for m_1 as $\varepsilon\to 0$.

Lemma 14. *Let* z *be the solution to Equation (3.5.1) and* m_1 *be its first local minimum, then*

$$\lim_{\varepsilon\to 0}\frac{m_1}{\varepsilon}=0.$$

More accurately, $m_1=O(\varepsilon^{\frac{\alpha}{\alpha-1}})$ *and* $G(m_1)$ *is bounded below by a positive constant.*

Proof. Starting from the equation (3.5.1) , we obtain,

$$(xz'(x))'=x(\frac{\varepsilon^\alpha}{z^\alpha}-1).$$

Firstly, we derive the lower bound for $z(x)$. That is,

$$z'(x)\geq\frac{1}{x}\int_0^x -y dy=-\frac{x}{2}.$$

Thus,

$$z(x)\geq 1-\frac{x^2}{4}\text{ for arbitrary }x\in(0,\infty).$$

Now we restrict $x\in(0,2)$ and apply the above lower bound,

$$z'(x)\leq\frac{1}{x}\int_0^x y\left[\frac{\varepsilon^\alpha}{(1-\frac{y^2}{4})^\alpha}-1\right]dy=\frac{2\varepsilon^\alpha}{(\alpha-1)x}\left[\frac{1}{(1-\frac{x^2}{4})^{\alpha-1}}-1\right]-\frac{x}{2}.$$

Integrating from 0 to x again,

$$
\begin{aligned}
z(x) &\leq 1 - \frac{x^2}{4} + \frac{2\varepsilon^\alpha}{\alpha - 1}\int_0^x \frac{1}{y}\left[\frac{1}{(1-\frac{y^2}{4})^{\alpha-1}} - 1\right] dy \\
&= 1 - \frac{x^2}{4} + \frac{\varepsilon^\alpha}{\alpha - 1}\int_{1-\frac{x^2}{4}}^1 \frac{1-u^{\alpha-1}}{(1-u)u^{\alpha-1}} du \\
&\leq 1 - \frac{x^2}{4} + \varepsilon^\alpha \int_{1-\frac{x^2}{4}}^1 \frac{1}{u^{\alpha-1}} du.
\end{aligned}
$$

If $\alpha = 2$, then

$$z(x) \leq 1 - \frac{x^2}{4} - \varepsilon^\alpha \ln(1 - \frac{x^2}{4}).$$

Otherwise,

$$z(x) \leq 1 - \frac{x^2}{4} + \frac{\varepsilon^\alpha}{2-\alpha}(1 - (1 - \frac{x^2}{4})^{2-\alpha}).$$

Take appropriate x_* such that $1 - \frac{x_*^2}{4} = \varepsilon^{1+r}$ with $\frac{1}{\alpha-1} > r > 0$, therefore,

$$m_1 \leq z(x_*) \leq \varepsilon^{1+r} \tag{3.6.11}$$

which indicates that $\lim_{\varepsilon \to 0} \frac{m_1}{\varepsilon} = 0$. On the other side, if we denote a_1 to be the first value such that $z(a) = \varepsilon$, then $1 < a_1 < x_* < 2$.

$$
\begin{aligned}
e(a_1) &= G(1) - \int_0^{a_1} \frac{(z'(x))^2}{x} dx \\
&\geq 1 + \frac{\varepsilon^\alpha}{\alpha - 1} - \int_0^{a_1} \frac{x}{4} dx \geq \frac{1}{2} + \frac{\varepsilon^\alpha}{\alpha - 1}.
\end{aligned}
$$

Apply the estimate (3.6.7),

$$G(m_1) = e(a_1) - \int_{a_1}^{x_{m_1}} \frac{(z'(x))^2}{x} dx \geq \frac{1}{2} - \frac{a_1 + \sqrt{2\varepsilon}}{a_1^2}\varepsilon\sqrt{2G(m)}$$

For ε sufficiently small, $G(m_1)$ is bounded below and then

$$m_1 = O(\varepsilon^{\frac{\alpha}{\alpha-1}}).$$

□

Parallelly, we consider another auxiliary problem for the above equation to investigate the part of the solution which is above $z = \varepsilon$. Denote $z(x)$ be the solution to the following problem

$$\begin{cases} z'' + \frac{1}{x} z' = \frac{\varepsilon^\alpha}{z^\alpha} - 1, \\ z\left(x_M\right) = M, \quad \text{and} \quad z'\left(x_M\right) = 0. \end{cases} \tag{3.6.12}$$

Lemma 15. *Under the assumptions,*

(1) there exists an interval (a, b) such that $z > \varepsilon$ in (a, b) and $z(a) = z(b) = \varepsilon$.

(2) $e(a) \geq C$ for some positive constant C independent of ε.

(3) $z(x)$ attains its unique minimum M in (a, b) at x_M and $x_M > a > 1$.

we have,

$$M > \widetilde{C} \text{ where } \widetilde{C} \text{ is positive and independent of } \varepsilon.$$

Proof. To start, we make an estimate for $G(z)$ for $M \geq z \geq \varepsilon$.

$$\begin{aligned} G(M) - G(z) &= \frac{\varepsilon^\alpha}{\alpha - 1} M^{1-\alpha} + M - \frac{\varepsilon^\alpha}{\alpha - 1} z^{1-\alpha} - z \\ &= (M - z)(1 - \frac{\varepsilon^\alpha}{\alpha - 1} \frac{M^{\alpha-1} - z^{\alpha-1}}{M^{\alpha-1} z^{\alpha-1}(M - z)}) \\ &\geq (M - z)(1 - \frac{\varepsilon^\alpha}{M z^{\alpha-1}}) \geq (M - z)(1 - \frac{\varepsilon}{M}). \end{aligned}$$

Similarly as above, we have

$$(z')^2 = \frac{2}{x^2} \int_{x_M}^{x} y^2 \frac{d}{dy}[G(M) - G(z(y))] dy.$$

It follows that for $x \in [a, x_M]$,

$$\frac{x_M}{x} \sqrt{2(G(M) - G(z))} \geq |\frac{dz}{dx}| \geq \sqrt{2(G(M) - G(z))}. \tag{3.6.13}$$

Then we obtain,

$$x_M - a \leq \int_{\varepsilon}^{M} \frac{dz}{\sqrt{2(G(M) - G(z))}} \leq \int_{\varepsilon}^{M} \frac{dz}{\sqrt{2(M - z)(1 - \frac{\varepsilon}{M})}} = \sqrt{2M} \tag{3.6.14}$$

and

$$\int_{a}^{x_M} \frac{(z'(x))^2}{x} dx \leq \frac{x_M}{a^2} \int_{\varepsilon}^{M} \sqrt{2(G(M) - G(z))} dz \leq \frac{x_M}{a^2} M \sqrt{2G(M)}. \tag{3.6.15}$$

That is,

$$C \le e(a) \le G(M) + \frac{x_M}{a^2} M\sqrt{2G(M)} \le G(M) + \frac{a+\sqrt{2M}}{a^2} M\sqrt{2G(M)}.$$

Since $a > 1$ and $M < 1$,

$$C \le G(M) + (1+\sqrt{2})\sqrt{2G(M)}.$$

By proof of contradiction, it is easy to see that, there exists some positive constant D independent of ε such that

$$D \le G(M) = \frac{\varepsilon^\alpha}{\alpha - 1} M^{1-\alpha} + M.$$

Hence, we have finished proving the lemma. $\square$

As we know, the solution $z(x)$ is oscillating around $z = \varepsilon$. Define $x_m = \sup\{x > x_M : z'(y) < 0 \text{ for all } y \in (x_M, x)\}$ and $m = G(x_m)$. Then m is the local minimum of $z(x)$ and $m < \varepsilon$. The following lemma will give an estimate for m which is the same as m_1 proved in Lemma 14 for inductive purpose.

Lemma 16. *Under the assumptions in Lemma 15, then*

$$\lim_{\varepsilon \to 0} \frac{m}{\varepsilon} = 0.$$

Furthermore, $G(m)$ is bounded below by a positive constant independent of ε.

Proof. As shown above, $G(M) \ge D$ for some positive constant D independent of ε. For any $x \in [x_M, b]$,

$$\frac{x_M}{x}\sqrt{2(G(M) - G(z))} \le \left|\frac{dz}{dx}\right| \le \sqrt{2(G(M) - G(z))}. \tag{3.6.16}$$

Similarly, we calculate the upper bound for b and the energy dissipation in $[x_M, b]$.

$$\frac{x_M}{b}(b - x_M) \le \int_{x_M}^{b} \frac{x_M}{x} dx \le \int_{\varepsilon}^{M} \frac{dz}{\sqrt{2(G(M) - G(z))}} \le \sqrt{2M}.$$

Thus,

$$b - x_M \le \frac{\sqrt{2M}}{1 - \frac{\sqrt{2M}}{x_M}} \le \frac{\sqrt{2M}}{1 - \frac{\sqrt{2M}}{a+\sqrt{2M}}} \le 2\sqrt{2M}. \tag{3.6.17}$$

Meanwhile,

$$\int_{x_M}^{b} \frac{(z'(x))^2}{x} dx \leq \frac{1}{x_M} \int_{\varepsilon}^{M} \sqrt{2(G(M) - G(z))} dz \leq \frac{M}{x_M} \sqrt{2G(M)}. \tag{3.6.18}$$

It follows that

$$e(b) = G(M) - \int_{x_M}^{b} \frac{(z'(x))^2}{x} dx \geq G(M) - \frac{M}{x_M} \sqrt{2G(M)}.$$

By Lemma 15, $1 > M > \widetilde{C} > 0$. Note that $x_M > 2$. It is easy to check that for ε small enough, $e(b)$ is bounded below by a positive constant which is independent of ε. On the other side, for any $x \in [b, x_m]$, Apply the estimate

$$G(m) - G(z) \geq (z - m)(\frac{\varepsilon}{m} - 1).$$

into

$$x_m - b = \int_{b}^{x_m} dx \leq \int_{m}^{\varepsilon} \frac{dz}{\sqrt{2(G(m) - G(z))}}.$$

We obtain,

$$x_m - b \leq \int_{m}^{\varepsilon} \frac{1}{\sqrt{2(z-m)(\frac{\varepsilon}{m} - 1)}} dz = \sqrt{2m}.$$

Regarding the energy dissipation, same as Lemma (13),

$$\int_{b}^{x_m} \frac{(z'(x))^2}{x} dx \leq \frac{x_m}{b^2} \varepsilon \sqrt{G(m)} \leq \frac{b + \sqrt{2m}}{b^2} \varepsilon \sqrt{G(m)}.$$

Then,

$$e(b) \leq G(m) + \varepsilon \frac{\sqrt{G(m)}(b + \sqrt{2m})}{b^2} \leq G(m) + \varepsilon \sqrt{G(m)}.$$

In consequence, $G(m)$ is bounded below by a positive constant and bounded above by $G(1)$. That is, there exists positive constants C_1 and C_2 such that

$$C_1 \varepsilon^{\frac{\alpha}{\alpha-1}} \geq m \geq C_2 \varepsilon^{\frac{\alpha}{\alpha-1}}.$$

□

Now we obtain a global picture of the solution $z(x)$ to equation (3.5.1). $z(x)$ oscillates around ε which is its asymptotic limit. For ε sufficiently small, the local minimum m has the order $\varepsilon^{\frac{\alpha}{\alpha-1}}$ and the local maximum M is bounded below by a positive constant independent of ε. Each interval where $z(x) > \varepsilon$ is bounded above by $3\sqrt{2M}$ by adding (3.6.14) and (3.6.17) while each interval where $z(x) < \varepsilon$ is ε followed by (3.6.6). Before we process the proof of uniform convergence, we are about to show the core lemma as following,

Lemma 17. *For any given interval $[a, b]$ in the domain,*

$$\int_a^b \frac{\varepsilon^\alpha}{z^\alpha} dx \in L^1([a,b]) \text{ and } \lim_{\varepsilon \to 0} \int_a^b \frac{\varepsilon^\alpha}{z^\alpha} dx = 0.$$

Proof. Without loss of generality, we can restrict on the interval where $z(x)$ is above $z = \varepsilon$. We first assume that $z(a) = z(b) = \varepsilon$ and $z(x) > \varepsilon$ for $x \in (a, b)$. Then denote $M = z(x_M)$ as the maximum of $z(x)$ in $[a, b]$. Firstly, (3.6.13) implies

$$\int_a^{x_M} \frac{\varepsilon^\alpha}{z^\alpha} dx \le \varepsilon^\alpha \int_\varepsilon^M \frac{dz}{z^\alpha \sqrt{G(M) - G(z)}} \le \varepsilon^\alpha \int_\varepsilon^M \frac{dz}{z^\alpha \sqrt{2(M-z)(1-\frac{\varepsilon}{M})}}.$$

Direct calculation of the above integral shows,

$$\begin{aligned}
\varepsilon^\alpha \int_\varepsilon^M \frac{dz}{z^\alpha \sqrt{2(M-z)(1-\frac{\varepsilon}{M})}} &= \frac{\varepsilon^\alpha}{\sqrt{2(1-\frac{\varepsilon}{M})}} \frac{1}{\alpha - 1} \left[\int_\varepsilon^M \frac{(2\alpha-1)dz}{2Mz^{\alpha-1}\sqrt{(M-z)}} - \frac{\sqrt{M-z}}{Mz^{\alpha-1}}\Big|_\varepsilon^M \right] \\
&= \frac{\varepsilon^\alpha}{\sqrt{2M(M-\varepsilon)}} \frac{2\alpha-1}{2(\alpha-1)} \int_\varepsilon^M \frac{dz}{z^{\alpha-1}\sqrt{(M-z)}} + \frac{\varepsilon}{(\alpha-1)\sqrt{2M}} \\
&\le \frac{\varepsilon}{\sqrt{2M(M-\varepsilon)}} \frac{2\alpha-1}{2(\alpha-1)} \int_\varepsilon^M \frac{dz}{\sqrt{(M-z)}} + \frac{\varepsilon}{(\alpha-1)\sqrt{2M}} \\
&= \frac{\varepsilon}{\sqrt{2M(M-\varepsilon)}} \frac{2\alpha-1}{\alpha-1} \sqrt{(M-\varepsilon)} + \frac{\varepsilon}{(\alpha-1)\sqrt{2M}} \\
&= \frac{2\alpha\varepsilon}{(\alpha-1)\sqrt{2M}} \le \frac{2\alpha\varepsilon}{(\alpha-1)\sqrt{G(M)}}.
\end{aligned}$$

On the other side, from (3.6.16), we have

$$\int_{x_M}^b \frac{\varepsilon^\alpha}{z^\alpha} dx \le \frac{b}{x_M} \varepsilon^\alpha \int_\varepsilon^M \frac{dz}{z^\alpha \sqrt{G(M) - G(z)}} \le \frac{b}{x_M} \frac{2\alpha\varepsilon}{(\alpha-1)\sqrt{G(M)}}.$$

From (3.6.17), $b - x_M \le 2\sqrt{2M}$. Then $\frac{b}{x_M} \le 1 + \frac{2\sqrt{2M}}{x_M} < 5$, due to $x_M > 1$. Thus,

$$\int_a^b \frac{\varepsilon^\alpha}{z^\alpha} dx \le \frac{12\alpha\varepsilon}{(\alpha-1)\sqrt{G(M)}}.$$

If $x_M = 0$, then let $b = inf\{x > 0 : z(x) = \epsilon\}$. Then according to the estimate in Lemma 14, we simply compare with x_* where $z(x_*) < \varepsilon$ and

$$\int_0^b \frac{\varepsilon^\alpha}{z^\alpha}dx \le \int_0^{x_*} \frac{\varepsilon^\alpha}{(1-\frac{x^2}{4})^\alpha}dx \le \int_0^{2-\varepsilon^{1+r}} \frac{\varepsilon^\alpha}{(1-\frac{x}{2})^\alpha}dx < \frac{2^\alpha}{\alpha-1}\varepsilon^{1-(\alpha-1)r}.$$

□

Now we are ready to prove the main Theorem 16 for the case $\eta \to \infty$.

Proof. Firstly, denote $x_1 = \inf\{x > 0 : z(x) = \varepsilon\}$, $y_n = \inf\{x > x_n : z(x) = \varepsilon\}$ and $x_{n+1} = \inf\{x > y_n : z(x) = \varepsilon\}$. From lemma 13, we have $\lim_{\varepsilon\to 0}(y_n - x_n) = 0$. By lemma 14, $x_1 < 2 - \varepsilon^{1+r} < y_1$. Then it is easy to have $\lim_{\varepsilon\to 0} x_1 = \lim_{\varepsilon\to 0} y_1 = 2$.Now let us concentrate on the interval (a_0, a_1) with $a_1 = 2$. Then for $x \le x_1$,

$$|z'(x) - z'_*(x)| = |\frac{1}{x}\int_0^x y(\frac{\varepsilon^\alpha}{z^\alpha} - 1)dy - \frac{1}{x}\int_0^x -ydy| = \frac{1}{x}\int_0^x y\frac{\varepsilon^\alpha}{z^\alpha}dy < \frac{2^\alpha}{\alpha-1}\varepsilon^{1-(\alpha-1)r}.$$

It follows that

$$|z(x) - z_*(x)| \le \frac{2^\alpha}{\alpha-1}\varepsilon^{1-(\alpha-1)r}x \le \frac{2^{\alpha+1}}{\alpha-1}\varepsilon^{1-(\alpha-1)r}.$$

Since $x_1 \to a_1$ as $\varepsilon \to 0$, we have $z(x)$ uniformly converges to $z_*(x)$ in (a_0, a_1). Later, we consider on the interval (a_1, a_2). Due to the fact that $y_1 \to a_1$, we are able to pick ε small enough such that $y_1 < a_1 + \delta$. Here δ is to be determined so that $|z_*(x) - z_*(a_1)|$ and $|z'_*(x) - z'_*(a_1+)|$ sufficiently small for $x \in (y_1, a_1 + \delta)$. By lemma 13, $e(y_1) - e(x_1) \to 0$. Therefore,

$|z'(y_1) - z'_*(y_1)| \le ||z'(y_1)| - |z'(x_1)|| + |z'(x_1) - z'_*(a_1-)| + |z'_*(y_1) - z'_*(a_1+)|$ is small enough.

Hence for $x \in [a_1, y_1]$,

$$|z(x) - z_*(x)| \le |z_*(x)| + |z(x)| \le \varepsilon + |z_*(x) - z_*(a_1)|.$$

For $x \in [y_1, x_2]$, following Lemma 17,

$$\begin{aligned}|z'(x) - z'_*(x)| &\le |z'(y_1) - z'_*(y_1)| + |\frac{1}{x}\int_{y_1}^x y(\frac{\varepsilon^\alpha}{z^\alpha} - 1)dy - \frac{1}{x}\int_{y_1}^x -ydy| \\ &= |z'(y_1) - z'_*(y_1)| + \frac{1}{x}\int_{y_1}^x y\frac{\varepsilon^\alpha}{z^\alpha}dy \le C\varepsilon^{1-(\alpha-1)r}.\end{aligned}$$

It derives that

$$|z(x) - z_*(x)| \le C(a_2 - a_1)\varepsilon + |z(y_1) - z_*(y_1)|.$$

Meanwhile, we obtain that $|z(a_2)| < \varepsilon$ which indicates that $x_2 < a_2 < y_2$. Since $x_2 \to a_2$ as $\varepsilon \to 0$, we have $z(x)$ uniformly converges to $z_*(x)$ in (a_0, a_2). By inductive approach, we are able to show that on a fixed interval $[0, R]$, $z(x)$ converges to $z_*(x)$ uniformly. It is well known that as $R \to \infty$, $z(x)$ approaches ε while $z_*(x)$ approaches 0. Consequently, As ε goes to 0, $z(x)$ tends to 0 at infinity. Taking R sufficiently large, $|z(x) - z_*(x)| < \varepsilon$. Then it is easy to see that $z(x)$ uniformly converge to $z_*(x)$ in $(0, \infty)$. □

3.7 ASYMPTOTIC BEHAVIOR OF LIMIT SOLUTION

From the main result Theorem 16, we have $z(x)$ is uniformly convergent to limit $z_*(x)$ satisfying

$$\begin{cases} z_*'' + \frac{1}{x} z_*' = -1, \quad z_* > 0 \;\; in \bigcup_{j=0}^{\infty} (a_j, a_{j+1}). \\ z_*(0) = 1, \quad \text{and} \quad z_*'(0) = 0, \\ z_*(a_j) = 0, \quad z_*'(a_j+) = -z_*'(a_j-) \end{cases}$$

In this section, we are solving z_* in the following manner.

(i) In $[0, a_1]$,

$$(xz_*')' = -x \text{ and } z_*(0) = 1, \; z_*'(0) = 0.$$

Integrating twice gives

$$z_*(x) = 1 - \frac{x^2}{4}.$$

Hence,

$$a_1 = 2 \text{ and } e_1 := |z'(a_1)|^2 = 1.$$

(ii) In $[a_1, a_2]$,

$$(xz_*')' = -x \text{ and } z_*'(2+) = 1.$$

Then,

$$xz_*'(x) = 4 - \frac{x^2}{2}.$$

$$z_*(x) = 4\ln\frac{x}{2} + \frac{4-x^2}{4}.$$

Note that $z_*''(x) = -\frac{4}{x^2} - \frac{1}{2} < 0$, $z_*(x)$ is concave down. Therefore, there exists a unique solution $a_2 \in (2,\infty)$ to

$$4\ln\frac{a_2}{2} + \frac{4-a_2^2}{4} = 0.$$

That is,

$$a_2 = 3.74853 \text{ and } e_2 := |z'(a_2)|^2 = (\frac{a_2}{2} - \frac{4}{a_2})^2 = 0.6515.$$

(iii) In $[a_j, a_{j+1}]$,

$$(xz_*')' = -x \text{ and } z_*'(a_j+) = \sqrt{e_j}.$$

Integrating from a_j to x,

$$xz_*' = a_j\sqrt{e_j} + \frac{a_j^2}{2} - \frac{x^2}{2}.$$

$$z_*(x) = (a_j\sqrt{e_j} + \frac{a_j^2}{2})\ln\frac{x}{a_j} - \frac{x^2 - a_j^2}{4}.$$

Then similarly as above, a_{j+1} is the unique root in (a_j,∞) such that

$$(a_j\sqrt{e_j} + \frac{a_j^2}{2})\ln\frac{a_{j+1}}{a_j} - \frac{a_{j+1}^2 - a_j^2}{4} = 0. \tag{3.7.1}$$

And $\sqrt{e_{j+1}}$ is given by

$$\sqrt{e_{j+1}} = -z_*'(a_{j+1}) = -(a_j\sqrt{e_j} + \frac{a_j^2}{2})/a_{j+1} + \frac{a_{j+1}}{2}. \tag{3.7.2}$$

Theorem 17. *Let a_j and e_j be defined as above, we have as $j \to \infty$, for some constants A and B,*

$$a_j \sim Aj^{\frac{3}{4}} \text{ and } \sqrt{e_j} \sim Bj^{-\frac{1}{4}}.$$

Proof. Firstly, according to (3.7.2), we have,

$$\sqrt{e_{j+1}} + \frac{a_j}{a_{j+1}}\sqrt{e_j} = \frac{a_{j+1}^2 - a_j^2}{2a_{j+1}}$$

Since $\sqrt{e_j}$ is decreasing,

$$\frac{a_{j+1} + a_j}{a_{j+1}}\sqrt{e_{j+1}} \le \frac{a_{j+1}^2 - a_j^2}{2a_{j+1}} \le \frac{a_{j+1} + a_j}{a_{j+1}}\sqrt{e_j}.$$

It follows that

$$2\sqrt{e_{j+1}} \le a_{j+1} - a_j \le 2\sqrt{e_j}.$$

Then as $j \to \infty$, $\frac{a_{j+1}}{a_j} - 1 \to 0$. Denote $t = (\frac{a_{j+1}}{a_j})^2 - 1$. Now rewrite (3.7.1),

$$\frac{t}{\ln(t+1)} = \frac{2\sqrt{e_j}}{a_j} + 1.$$

For convenience, we take $b_j = \frac{\sqrt{e_j}}{a_j}$. By Taylor expansion, we have,

$$\frac{t}{2} - \frac{t^2}{12} \simeq 2b_j$$

which yields

$$t \simeq 4b_j + \frac{t^2}{6} = 4b_j + \frac{8}{3}b_j^2 + O(b_j^3).$$

Therefore,

$$\begin{aligned} \frac{a_{j+1}}{a_j} &= \sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + O(t^3) \qquad (3.7.3) \\ &= 1 + 2b_j - \frac{2}{3}b_j^2 + O(b_j^3). \qquad (3.7.4) \end{aligned}$$

Plug into (3.7.2),

$$\begin{aligned} \frac{\sqrt{e_{j+1}}}{\sqrt{e_j}} &= \frac{a_{j+1} - a_j}{2\sqrt{e_j}} - \frac{a_j}{a_{j+1}}\left[1 - \frac{a_{j+1} - a_j}{2\sqrt{e_j}}\right] \qquad (3.7.5) \\ &= \frac{1}{2b_j}(\frac{a_{j+1}}{a_j} - 1) - \frac{a_j}{a_{j+1}}(1 - \frac{1}{2b_j}(\frac{a_{j+1}}{a_j} - 1)) \qquad (3.7.6) \\ &= 1 - \frac{2}{3}b_j + O(b_j^2). \qquad (3.7.7) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{b_{j+1}}{b_j} = \frac{\sqrt{e_{j+1}}}{\sqrt{e_j}}(\frac{a_{j+1}}{a_j})^{-1} &= (1 - \frac{2}{3}b_j + O(b_j^2))(1 + 2b_j - \frac{2}{3}b_j^2 + O(b_j^3))^{-1} \\ &= (1 - \frac{2}{3}b_j + O(b_j^2))(1 - 2b_j + O(b_j^2)) \\ &= 1 - \frac{8}{3}b_j + O(b_j^2). \end{aligned}$$

As b_j is decreasing and converges to 0,

$$\frac{1}{b_{j+1}} = \frac{1}{b_j} + \frac{8}{3} + O(b_j).$$

We can obtain that for j large enough, $b_j = O(\frac{1}{j})$. Then,

$$\begin{aligned}\frac{\sqrt{e_{j+1}}}{\sqrt{e_j}}(\frac{a_{j+1}}{a_j})^{\frac{1}{3}} &= (1-\frac{2}{3}b_j + O(b_j^2))(1+2b_j - \frac{2}{3}b_j^2 + O(b_j^3))^{\frac{1}{3}} \\ &= (1-\frac{2}{3}b_j + O(b_j^2))(1+\frac{2}{3}b_j + O(b_j^2)) \\ &= 1 + O(b_j^2) = 1 + O(\frac{1}{j^2}).\end{aligned}$$

It follows that $\lim_{j\to\infty}\sqrt{e_{j+1}}a_{j+1}^{\frac{1}{3}}$ exists and assume the limit is γ. (3.7.3) implies ,

$$a_{j+1} = a_j + 2\gamma a_j^{-1/3} + O(\frac{1}{a_j^{4/3}}).$$

Hence,

$$a_j \sim Aj^{\frac{3}{4}} \text{ with } A = (\frac{8\gamma}{3})^{\frac{3}{4}}.$$

As a consequence,

$$\sqrt{e_j} \sim Bj^{-\frac{1}{4}}, \text{ with } B = (\frac{8\gamma}{3})^{-\frac{1}{4}}\gamma$$

and

$$a_{j+1} - a_j \sim j^{-\frac{1}{4}}.$$

□

4.0 CONCLUSIONS AND FUTURE WORK

In my dissertation, we have started with a thin film coupled with Born repulsion force and van der Waals forces. In [32, 54], they have proved the existence of positive, smooth solutions with prescribed volume and gave an rigorous proof for the asymptotic zero Born repulsion force limit. In order to investigate the location of boundary spike, we simplify the problem by taking semi limit. However, the solution admitted by new energy minimizing problem with volume constraint is no longer smooth with a jump for its Laplacian. It can be identified as a free boundary problem and regularity of the free boundary $\Omega \cap \partial\{u > 0\}$ attracts great attention in [3, 4]. We have proven the uniform convergence of the minimizer to a Dirac measure and located the concentration at the point on the boundary with maximal curvature by applying the asymptotic analysis.

Later in chapter 3, we performed the theoretical analysis for the radial steady states for thin film equation driven by van der Waals force only. We tried to understand the physical quantities, the average thickness and energy through the initial value $u(0)$. Based on the previous work by [57, 64], the construction of radial solutions has been given and numerical experiments has shown the convergence for $u(0) \to 0$, $u(0) \to 1$ and $u(0) \to \infty$. Making adequate use of the oscillation property, we scale the thickness function and derive the accurate estimate for the critical values. As a consequence, we show the dependence of thin film solution and its energy on the average thickness and give a description of limiting profile, especially when $u(0)$ is large.

So far, most of the rigorous mathematical work about thin film equation were done when the space dimension is one. For the physically realistic dimension, the dynamics is not well understood. Many questions and challenges are still open. The following are a few ideas for future research.

- **Location of boundary spike for thin film equation:** The energy minimizing problem we consider here is the semi-limit of the original thin film energy. Let us come back to original energy problem (1.2.4). It has been proven that the zero Born repulsion limit is a Dirac Measure located on the boundary. Regarding the location of boundary spike, the complexity of pressure function makes the problem still open.
- **Rigorous proof for asymptotic analysis** In my formal calculation for asymptotic analysis, we are seeking for a special type solution by perturbation to limiting profile. We need a rigorous proof to have the existence of this solution. Moreover, we have to prove that energy minimizer would only have one piece.
- **Comparison of energy level of radial solution:** We know that for thin film equation driven by van der Waals forces, the radial solution can be constructed for given initial value η and numbers of oscillations k. It is natural to ask, what is the optimal k to minimize the energy? Numerical experiment indicates that for a given volume, the total energy is increasing in k. However, there exists a critical value $\bar{u}_0$ for given k_0 such that no radial solution with $k < k_0$ exists with average thickness $\bar{u} < \bar{u}_0$. Comparing to constant solution, numerical experiment also shows that for $\bar{u}$ large, constant solution usually has a lower energy. We need more theoretical results about the dependence of average thickness and energy on k.
- **Stability property of these radial solution and rupture solution:** In dimension one, R. Laugesen and M. Pugh has concluded the linear stability and energy stability for a more general setting in [58, 60, 61]. The next step in my work would be to investigate the stability of radial solutions and rupture solution and give a bifurcation digram for multiple solutions.
- **Rupture solution for non radial case:** We know that nonunformities in coating industry are very undesirable. So people are very interesting in the rupture set $\Sigma = \{x \in \Omega : u(x) = 0\}$ where "dry spot" occurs. It has been proven that rupture can only occur at original point for radial rupture solution. For non radial case, Huiqiang Jiang and Fanghua Lin [55] has obtained an estimate on the Hausdorff dimension of the rupture set under the assumption that the total energy is finite. What is the optimal Hausdorff dimensions is still open? And how to construct a non trivial rupture solution in non

radial case?

- **Thin film evolution:** In a more general setting, we would like to come back to fourth order parabolic equation with nonnegative initial values for any given domain. A systematic theory in multi-dimensional space about the existence, uniqueness, long time behavior, blow up analysis and finite time rupture is still open.

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