A COUNTEREXAMPLE TO THE SIMPLE LOOP CONJECTURE FOR THE PUNCTURED KLEIN BOTTLE IN PGL$(2, \mathbb{R})$

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In this paper, inspired by the work of Katherine Mann [10], we show a counterexample to the simple loop conjecture for a representation of the surface group of the punctured Klein bottle, K, in **PGL(2,$\mathbb{R}$)**. In the process, we provide an explicit description of the mapping class group, as well as a classification of possible simple closed curves on K.

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1.0 INTRODUCTION

The simple loop conjecture states, any non-injective homomorphism from a closed orientable surface group to another closed orientable surface group must contain an element representing a simple closed curve in the kernel; The result was proved true in 1985 by Gabai [3]. The conjecture was reformulated by Kirby in 1993 , considering the fundamental group of an orientable 3-manifold as the target [7], which was verified under certain circumstances by Hass and Wang [5] [13] . In 2000 Minsky posed the question mapping into $\mathrm{SL}(2, \mathbb{C})$ [12]. This was addressed and proved false by Cooper and Manning in 2011 [1]. Recently, in 2014, Katherine Mann improved upon their result by replacing $\mathrm{SL}(2, \mathbb{C})$ with $\mathrm{SL}(2, \mathbb{R})$ using the representation explicitly constructed in Deblois and Kent's paper [2]. Mann shows that on all compact orientable surfaces with boundary and genera at least 1 the representation is non-injective and kills no simple closed curve [10, Theorem1.1]. Her result precisely stated says,

Theorem 1.0.1. *Let Σ be an orientable surface of negative Euler characteristic and genus $g \geq 1$, possibly with boundary. Then there is a homomorphism $\rho : \pi_1(\Sigma) \to SL(2, \mathbb{R})$ such that*

1. *ρ is not injective.*
2. *If $\rho(\alpha) = \pm I$, then α is not represented by a simple closed curve.*
3. *If α is represented by a simple closed curve, then $\rho(\alpha^k) \neq I$ for any $k \in \mathbb{Z} \not\{0\}$*

. An equivalent result extends to non-orientable surfaces of negative Euler characteristic and genus $g \geq 2$ excluding the punctured Klein bottle and the non-orientable surface of genus three by changing the target group from projective special linear group to the general linear group $\mathrm{GL}(2, \mathbb{R})$ [10, Theorem 1.2].

Theorem 1.0.2. *Let Σ be a non-orientable surfaceof negative Euler characteristic and genus $g \geq 2$ not the punctured Klein bottle or the non-orientable surface of genus 3, then there are uncountably many representations $\rho : \pi_1(\Sigma) \to PGL(2, \mathbb{R})$ satisfying 1 through 3 of theorem 1.1.*

In the following discussion, we extend this result to the punctured Klein bottle:

Theorem 1.0.3. *Let K be the punctured Klein bottle; there exists a homomorphism $\rho : \pi_1(K) \to PGL(2, \mathbb{R})$ such that*

1. *ρ is not injective.*
2. *If $\rho(\alpha) = \pm I$, then α is not represented by a simple closed curve.*
3. *If α is represented by a simple closed curve, then $\rho(\alpha^k) \neq I$ for any $k \in \mathbb{Z} \not\{0\}$*

The strategy of our paper is to create an explicit description of the generators mapping class group of K, using work of Korkmaz [8]. In Chapter 4, we categorize all possible simple closed curves on K and express them as products of our generators of the fundamental group. We then classify the actions of the Mapping class group on the possible simple closed curves (this result may be of independent interest). Finally, using a representation in $\mathrm{PGL}(2, \mathbb{R})$, we can prove Theorem 1.0.3.

2.0 PRELIMINARIES

It is helpful to consider K the unit square after removing its intersections with open circles of radius $\frac{1}{4}$ centered at each vertex and the identification $(x, 0) \to (x, 1)$ and $(0, y) \to (1, 1-y)$. This model allows for a particularly accessible way of envisioning curves on the surface of K.

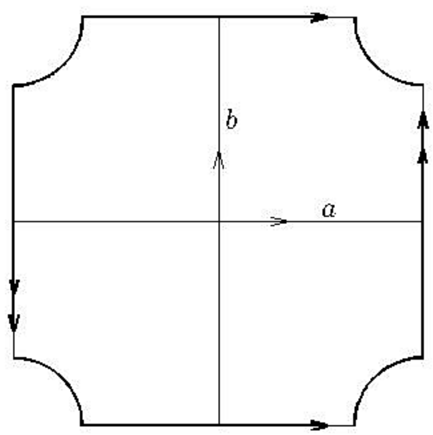

Figure 2.1: A depiction of K, and the generators of $\pi_1(K) = \langle a, b \rangle$

Since we rely on the classification of simple closed curves on K, we first define the notion of closure and simplicity. A *closed* curve c on K is a continuous map $c : I \to K$ such that $c(0) = c(1)$; we say, the point $c(0) = c(1)$ is the *base point* of curve c. A closed curve c is *simple* if the map $S^1 \to K$ induced by c is injective. Two closed curves c_0 and c_1 are said to be *freely homotopic* on K if there exists a continuous map $C : I \times I \to K$ such that $C(t, 0) = c_0(t)$ and $C(t, 1) = c_1(t)$ for all $t \in I$, and $C(0, s) = C(1, s)$ for all $s \in I$. If

$C(0,s) = C(1,s) = C(0,0)$ for all $s \in I$, we say the homotopy C is *base point preserving.*

With this description of closed curves on the surface of K, we can define the *fundamental group* of K based at x:

$$\pi_1(K,x) = \{[\gamma] \mid \gamma : I \to K \textit{ is a closed curve with base point } x\}$$

. Here, $[\gamma]$ is the set of all curves homotopic to γ by a base point preserving homotopy.

Theorem 2.0.4. *The fundamental group $\pi_1(K) \cong F_2$ the free group on two generators.*

Proof. The punctured Klein bottle is retractable to the union of an annulus and Mobius band, both of which are retractable to their center lines. Therefore, $\pi_1(K) \cong \pi_1(S^1 \bigvee S^1) \cong F_2$. $\square$

It is important to note that two elements $c \in [\gamma]$ and $d \in [\delta]$ of the fundamental group $\pi_1(K,x)$ represent the same closed curve if and only if $c = gdg^{-1}$ for some $g \in \pi_1(K,x)$ [6, Exercise 6, Section 1.1].

The *mapping class group* of the punctured Klein bottle K is the group of isotopy classes of all diffeomorphisms mapping K to itself. Two diffeomorphisms d_0 and d_1 on K are said to be *isotopic* if there exists a map $D : K \times I \to K$ such that $D(x,0) = d_0(x)$ and $D(x,1) = d_1(x)$ for all $x \in K$ and for any $t \in I$, $d_t : K \to K$ defined by $D(x,t)$ is a homeomorphism. The notion is equivalent to a homotopy through homeomorphisms on K.

3.0 THE MAPPING CLASS GROUP OF K

To give a description of the mapping class group of the punctured Klein Bottle, we define the necessary diffeomorphisms to generate the mapping class group and provide notation for each. We then prove these particular maps are indeed the elements that generate the group.

The first is a diffeomorphism called the *Dehn twist*. It is supported in an oriented regular neighborhood of any simple closed curve. In the case of the punctured Klein bottle, the necessary Dehn twist is defined about the simple closed curve represented by $b \in \pi_1(K)$, which can be written explicitly as the map $S^1 \times I$ defined by $(\theta, t) \to (\theta + 2\pi t, t)$, since the curve represented by b is simple. Intuitively, this can be thought of by separating the surface along b, rotating one end by 2π and gluing the surface back together. We borrow notation from Korkmaz and denote the Dehn twist about b and its isotopy class t_b [8]. The Dehn twist acts on the generators of $\pi_1(K)$ in the following way:

$$t_b = \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases}$$

. The *cross cap slide*, or $Y-homeomorphism$ is a map defined on non-orientable surface with genus at least two. Consider the punctured Klein bottle as the connected sum of a Mobius band M with two disjoint punctures and another Mobius band N, by gluing the boundary of N to one of the punctures. The map is constructed by sliding the puncture to which N is attached around the neighborhood of a Mobius band, then identifying antipodal points of the boundary [8]. Alternatively, If we consider K to be the annulus, we can describe the $Y-heomomorphism$ as induced by the reflection about the diameter gluing two Mobius bands, one with a puncture, to each boundary in opposite orientation[9]. Another description regards the punctured Klein Bottle as the cylinder $S^1 \times I$ with one puncture after the proper identification of its ends, in which the $Y-homeomorphism$ is seen by a reflection of the cylinder about $S^2 \times I$ [9]. We can extend this map to a diffeomorphism of any surface containing a non-orientable genus two surface by the identity map [9]. We reserve y to represent the y-homeomorphism and the associated isotopy class:

$$y = \begin{cases} a \mapsto a^{-1} \\ b \mapsto b \end{cases}$$

.

The last of the mapping class group generators is the diffeomorphism supported in a neighborhood of an orientation reversing simple closed curve defined by pushing the puncture once along the curve[8]. This action is called the *boundary slide*, and acts trivially along the boundary of any puncture [8]. We will reserve w_1 to represent the *boundary slide* on K.

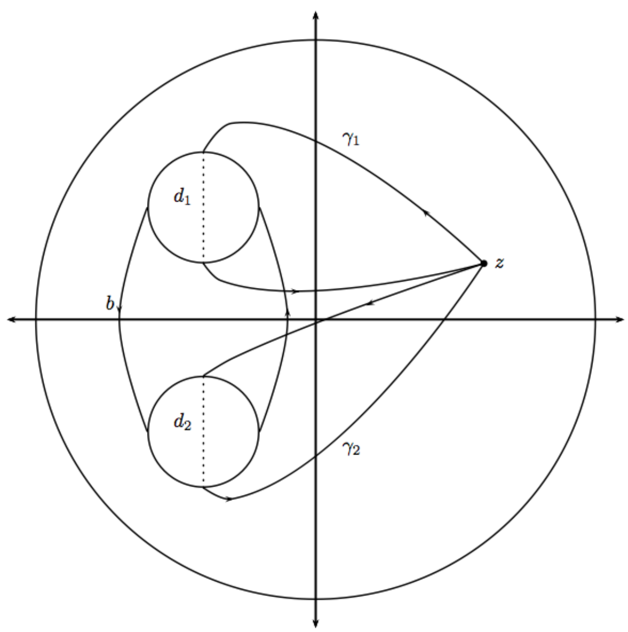

Figure 3.1: Second useful model of K in the plane

Our description of the mapping class group generators is a corollary to the following result of Korkmaz [8, Theorem 4.9]:

Theorem 3.0.5. *Let S be a non orientable surface of genus 2 with 1 puncture. The Pure mapping class group $\mathcal{MP}_{2,1}$ of S is generated by $\{t_b, y, w_1, v_1\}$.*

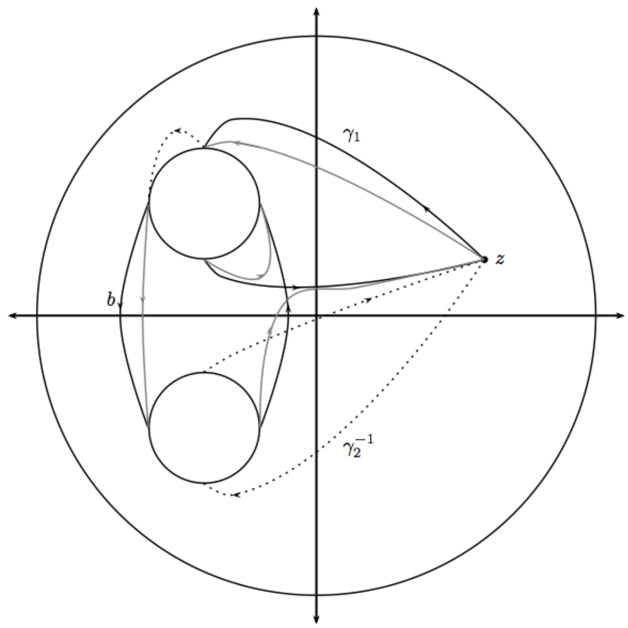

Figure 3.2: Depiction of closed curve $\gamma_2^{-1} = t_b(\gamma_1)$ after isotopy.

Here, it is helpful to think of the punctured Klein bottle as the 2-sphere after the removal of 2 interior open discs $d1$ and $d2$ and identifying antipodal points on each boundary; The 2-sphere is the compactification of the cartesian plane. Let the puncture be the removal of the point z . Let γ_1 be the simple closed curve based at z traveling through $d1$ and γ_2 be another simple closed curve based at z traveling through $d2$, see Figure 3.1. In this picture, a is represented by the boundary of $d2$, and b is a path connecting $d1$ and $d2$. Let w_1 denote the boundary slide around γ_1 and let v_1 be the boundary slide around γ_2.

Lemma 3.0.6. *The mapping class of K is generated by $\{t_b, y, w_1\}$*

Proof. In Figure 3.2 we show $tb(\gamma_1) = \gamma_2^{-1}$ after isotopy. It follows that the boundary slide around γ_1^{-1} is $t_b(\gamma_1)w_1t_b(\gamma_1)^{-1} = v_1^{-1}$. Thus there are three generators of the mapping class group $\{t_b, y, w_1\}$. □

The boundary slide along γ_1 operates on the generators of $\pi_1(K)$ as follows:

$$w_1 = \begin{cases} a \mapsto a \\ b \mapsto b^{-1} \end{cases}$$

. This concludes the description of generators of the Mapping class group on K.

4.0 CLASSIFICATION OF SIMPLE CLOSED CURVES ON K

To classify simple closed curves on the surface of K, we rely heavily on the classification theorem for compact 2-manifolds [11, Theorem 5.1].

Theorem 4.0.7. *Any compact 2-manifold is homeomorphic to either a sphere, connected sum of tori, or a connected sum of projective planes* $\mathbb{R}\mathbf{P}^2$.

This result allows to distinguish possible simple closed curves γ by Identifying the resultant surfaces after cutting along the specified curve.

Lemma 4.0.8. *Every simple closed curve,* γ*, on* K *is mapping class group equivalent to either* $a, a^2, b,$ *or* $ab^{-1}a^{-1}b^{-1}$.

We consider the following cases:

(i) γ *separates* K *into two orientable surfaces* S_1 *and* S_2

(ii) γ *separates* K *into two non-orientable surfaces* S_1 *and* S_2

(iii) γ *separates* K *into an orientable surface* S_1 *and a non-orientable surface* S_2.

(a) S_1 *is punctured*

(b) S_2 *is punctured*

(iv) γ *is non-separating the resulting surface is orientable.*

(a) S *has 2 boundary components*

(b) S *has 3 boundary components*

(v) γ *is non-separating the resulting surface is orientable.*

(a) S *has 2 boundary components*

(b) S *has 3 boundary components*

It is useful to recall the genus of a non-orientable surface is the number of connected summands of $\mathbb{R}\mathbf{P^2}$. We show Lemma 4.2 arguing case by case.

Proof.

(i) γ *separates* K *into two orientable surfaces* S_1 *and* S_2

Without loss of generality, let S_i have i boundary components (i.e. the puncture is contained in S_2). The Euler characteristic of K is $\chi = -1$. Then $\chi = \chi_1 + \chi_2 = -1$, which restricts the genera of the surfaces S_1 and S_2 to satisfy $g_1 + g_2 = 1$ where g_i is a natural number greater than or equal to zero. Taking $g_1 = 0$ forces S_1 to be the disk, which implies γ is trivial in $\pi_1(K)$, and thus not considered.

Taking $g_1 = 1$ identifies S_1 as the punctured torus and forces $g_2 = 0$ identifying S_2 with the annulus. However it is not possible to identify boundary components of S_1 and S_2 to produce a non-orientable surface, since an orientation on (say) S_2 can always be chosen so that boundary components to be identified have opposite orientations.

(ii) γ *separates* K *into two* $non\!-\!orientable$ *surfaces* S_1 *and* S_2

Again, we let S_i have i boundary components which restricts the genera to the relation $g_1 + g_2 = 2$, where g_i is a natural number. As such, $g_1 = g_2 = 1$ is the only possibility. This implies S_1 is a Mobius band and S_2 is the Mobius band with one puncture. This curve is produced by cutting K along the closed curve represented by a^2 in the fundamental group.

(iii) γ *separates* K *into an orientable surface* S_1 *and a* $non\!-\!oientable$ *surface* S_2.

(a) S_1 *is punctured*

Again the relation $\chi_1 + \chi_2 = -1$ restricts the genera to satisfy $2g_1 + g_2 = 2$ with $g_1 \geq 0$ and $g_2 \geq 1$. When $g_1 = 0$, $g_2 = 2$, which implies S_1 is the surface of Euler characteristic zero with two boundary components, i.e. the Annulus. S_2 is the surface of genus 2 with Euler characteristic -1 and a single boundary component, i.e. the punctured Klein bottle. The disjoint union of the annulus and punctured Klein bottle arise by cutting K along $ab^{-1}a^{-1}b^{-1}$ in our representation of $\pi_1(K)$.

(b) S_2 *is punctured*

When the puncture is on the non-orientable surface, the same relation on the genera holds. When $g_1 = 0$, S_1 is a disk, so the case is not considered.

(iv) γ *is non−separating and the resulting surface is orientable.*

(a) *S has 2 boundary components*

The Euler characteristic is given by $\chi = 2 - 2g - 2 = -1$. There is not natural number satisfying the equation, so this case is not possible.

(b) *S has 3 boundary components*

γ cuts K such that the resulting surface has 3 boundary components, and satisfies $\chi = 2 - 2g - 3 = -1$ which forces g to be zero, and the surface to be the 3-punctured sphere, attained by cutting along the generator b of $\pi_1(K)$.

(v) γ *is non−separating and the resulting surface is non−orientable.*

(a) *S has 2 boundary components*

γ cuts K such that the resulting surface has 2 boundary components, and satisfies $\chi = 2 - g - 2 = -1$, which forces g to be 1, so the surface to be the once punctured Mobius band and is produced by Cutting K along the curve represented by $a \in \pi_1(K)$.

(b) *S has 3 boundary components*

The Euler characteristic is $\chi = 2 - g - 2 = -1$, which implies $g = 0$. The genus of a non-orientable surface must be greater than or equal to 1, so this case is disqualified.

□

Theorem 4.0.9. *Let $\mathcal{C}$ denote words of the following following form up to exchanging positive with negative exponents:*

(i) $a^{\pm 1}$, $a^{\pm 2}$, *or* $b^{\pm 1}$

(ii) $ab^{-1}a^{-1}b^{-1}$

(iii) ab^n *or* ab^nab^n

Every simple closed curve on K is represented by a word in $\mathcal{C}$, and every word in $\mathcal{C}$ represents a simple closed curve.

We use the following lemmas to prove Theorem 4.0.9.

Lemma 4.0.10. *The the mapping class group preserves the structure of $\mathcal{C}$, i.e. for any element $[\phi]$ in the mapping class group, and any word form $c \in \mathcal{C}$, $\phi(c) \in \mathcal{C}$.*

Proof.

Case 1: $t_b(c) \in \mathcal{C}$ for any $c \in \mathcal{C}$.

By definition of the Dehn twist around b, $t_b(a) = ab$, $t_b(a^2) = abab$, and $tb(b) = b$. On the boundary curves beginning with a, t_b acts trivially; $t_b(a^{-1}b^{-1}ab^{-1}) = b^{-1}a^{-1}b^{-1}a$, which is conjugate to $a^{-1}b^{-1}ab$; the last boundary curve, $a^{-1}bab$ is mapped to $b^{-1}a^{-1}bab^2$ which is conjugate to $a^{-1}bab$. Applying t_b to $a^{-1}b^n$ gives $b^{-1}a^{-1}b^n$, which is conjugate to $a^{-1}b^{n-1}$, and t_b applied to $a^{-1}b^n a^{-1}b^n$ equals $b^{-1}a^{-1}b^{n-1}a^{-1}b^n$ which is conjugate to $a^{-1}b^{n-1}a^{-1}b^{n-1}$. Finally, t_b maps the elements ab^n and $ab^n ab^n$ to ab^{n+1} and $ab^{n+1}ab^{n+1}$ respectively.

Case 2: $w_1(c) \in \mathcal{C}$ for any $c \in \mathcal{C}$.

Our boundary slide fixes a and takes b to its inverse, so acting on any word in $\mathcal{C}$ will preserve the structure of the product exchanging powers of b for their negative.

Case 3: $y(c) \in \mathcal{C}$ for any $c \in \mathcal{C}$.

Similarly, the Y-homeomorphism fixes b and takes a to its inverse, so the argument in the previous case applies, which concludes this Lemma. □

Lemma 4.0.11. *Let $\mathcal{C}_0 = \{a, b, a^2, ab^{-1}a^{-1}b^{-1}\}$ for any $c \in \mathcal{C}$ there is a mapping class group element $[\phi]$ such that $\phi(c)$ is conjugate $c_0 \in \mathcal{C}_0$ contained in the fundamental group.*

Proof. Clearly $y : a \mapsto a^{-1}$, $y : a^2 \mapsto a^{-2}$, and $w_1 : b \mapsto b^{-1}$, and it is obvious the maps hold in the other direction as well. The boundary slide w_1 takes $ab^{-1}a^{-1}b^{-1}$ to $aba^{-1}b$ and again can be applied in the opposite direction. Acting on $ab^{-1}a^{-1}b^{-1}$ with w_1 maps to $a^{-1}b^{-1}ab^{-1}$. w_1 applied to $aba^{-1}b$ is $a^{-1}bab$.

The Dehn twist around b maps $a \mapsto ab$ and by induction, $t_b^n(a) = ab^n$. and similarly, $t_b : a^2 \mapsto abab$ and again, $t_b^n(a^2) = ab^n ab^n$ follows by induction. Applying y to $ab^n ab^n$ and ab^n yields $a^{-1}b^n a^{-1}b^n$ and $a^{-1}b^n$ respectively. Acting on $ab^n ab^n$, ab^n, $a^{-1}b^n a^{-1}b^n$ and $a^{-1}b^n$ with the boundary slide produces a representation of the remaining classes of words in $\mathcal{C}$. □

Proof of Theorem 4.0.9.

Let c be a simple closed curve on K. Lemma 4.0.8 implies there is a mapping class group element such that $\phi(c)$ is conjugate to some $c_0 \in \mathcal{C}_0$. It follows by Lemma 4.0.10 that $\phi^{-1}(c_0)$ which represents c is conjugate to a word in $\mathcal{C}$. Since every word in $\mathcal{C}_0$ represents a simple closed curve, and mapping classes take simple closed curves to simple closed curves, Lemma 4.0.11 impies every word in $\mathcal{C}$ represents a simple closed curve. $\square$

Thus, we have an explicit description of Simple closed curves on K up to inner automorphism, which corresponds to words permitted in $\mathcal{C}$.

5.0 PROOF OF THEOREM 1.0.3

In this section, we use a construction of Goldman to define a solvable representation on the fundamental group of the punctured Klein bottle $\pi_1(K)$ [4]. Since our simple closed curves represented by a and b generate $\pi_1(K)$, the representation $\phi_K : \pi_1(K) \to PGL(2, \mathbb{R})$ can be defined by,

$$\phi_K(a) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad \phi_K(b) = \begin{pmatrix} \beta & 1 \\ 0 & \beta^{-1} \end{pmatrix},$$

where α and β satisfy $\alpha^k \beta^l \neq \pm 1$ for any integers $k, l \in \mathbb{Z} \backslash \{0\}$. Note that it is sufficient to choose α and β to be relatively prime.

Proof of Theorem 1.0.3. To show The representation is non-injective, we first show $\pi_1(K)$ is not solvable since it contains a subgroup which is not solvable. This fact follows since there exists a homomorphism $f : pi_1(K) \to A_5$ where $A_5 = \langle (12345), (123) \rangle$ the alternating group of even permutations on 5 elements defined by $a \mapsto (12345)$ and $b \mapsto (123)$. The group A_5 is non commutative and simple. If we assume for a contradiction that A5 is solvable, by simplicity, the only composition series is $A_5 \trianglerighteq 1$, which implies the quotient group $A_5/1 = A_5$ is abelian since quotient groups of consecutive terms of a composition series are abelian. This is our contradiction, thus A_5 is not solvable. It is a common fact that a subgroup of a solvable group is solvable, which by contrapositive, shows $\pi_1(K)$ is not solvable since there exists $N \trianglelefteq \pi_1(K)$ such that $N \cong A_5$.

Our representation ϕ_K of $\pi_1(K)$ in contrast is 2 step solvable since any element X in

$G^1 = [\phi_K, \phi_K]$ is of the form

$$X = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad X^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}.$$

And by direct computation, we see that any element in $G^2 = [G^1, G^1]$ is given by a product

$$XYX^{-1}Y^{-1} = \begin{pmatrix} 1 & x+y-x-y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, ϕ_K has a composition series terminating in the trivial group. This implies ϕ_K is solvable and non-injective proving 1 from Theorem 1.0.2.

To show consequences 2 and 3 of Theorem 1.0.2, it is enough to note that products of upper triangular matrices are upper triangular, and as a consequence have $1,1$-entries in the form of words in $\mathcal{C}$ prior to reduction in $\mathbb{R}$. Since no word form in $\mathcal{C}$ reduces to a nontrivial product of α and β or their inverses in $\mathbb{R}$, the top right left entry of any matrix representing a simple closed curve in $\mathrm{PGL}(2, \mathbb{R})$ is never of the form $\alpha^k \beta^l$ with k and l simultaneously 0. Thus, no matrix representation or any power of a simple closed curve can be the identity. No power of simple closed curve is in the kernel of our representation ϕ_K, which concludes Theorem 1.0.3. $\square$

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