MASSEY PRODUCTS IN DIFFERENTIAL COHOMOLOGY

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We generalize Massey products to differential cohomology and relate these products to trivializations of higher bundles equipped with connection. The construction presented here is analogous to the construction of matric Massey products given by Peter May, but generalized to sheaves. We also prove a theorem regarding the refinement of the singular Massey product to differential cohomology.

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DIAGRAMS AND FIGURES

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1.0 INTRODUCTION

Although there has been extensive work on the general theory of differential cohomology over the last 10 years, explicit calculations of differential cohomology groups remains out of reach for the vast majority of manifolds. There are many reasons for this difficulty. For example, differential cohomology is not a homotopy invariant and although one has a Mayer-Vietoris sequence, in practice the lack of this invariance makes the sequence close to useless. In fact, as a general rule, almost all the homological techniques one uses for usual cohomology work just as well for differential cohomology. It is the homotopy theoretic techniques which are lacking.

The previous observations have led us to consider developing differential cohomology operations — in particular, Massey products. Such operations could be utilized in spectral sequences [25] to pin down explicit differential cohomology groups. Moreover, the construction of these operations shed light on just how much homotopy theory can say about differential cohomology and offer a wide variety of applications to physics.

The main results of this dissertation are presented in chapters 5, 6 and 7. In these chapters, we define Massey products in differential cohomology via stacks and prove that it satisfies similar properties as classical Massey products [19] [23] [25]. To construct these products, we follow the formulation given by Peter May in [25]. We also prove a theorem relating the differential, de Rham and singular Massey products. In the last chapter we briefly discuss applications and expand on one of these applications. The rest of the applications can be found in [15]

1.1 MOTIVATION AND OVERVIEW

Differential cohomology was first introduced by Jeff Cheeger and James Simons in the form of differential characters [8]. Since its conception, it has been generalized in a number of ways [6][29][7][31]. The most interesting point of view for us cast differential cohomology as a theory represented by certain moduli n-stacks denoted

$$\mathbb{B}^n U(1)_{\text{conn}} .$$

This point of view has been developed by Hisham Sati, Urs Schreiber and Domenico Fiorenza [11][12][13][31] and has the advantage of providing a nice framework to describe differential cohomology from the point of view of homotopy theory. This perspective will allow us to use techniques in homotopy theory to construct differential Massey products via homotopy commuting diagrams. It is not clear from this definition how one can use such operations in explicit calculations. This motivates us to use the Dold-Kan machine to translate these homotopies into a place where they can be algebraically manipulated. More precisely, the normalized Moore functor

$$N : s\mathcal{A}\text{b} \to \mathcal{C}\text{h}^+$$

takes a simplicial abelian group to a positive graded chain complex. Applying this functor to certain abelian stacks allows us to apply the general machinery of May [25] to work with differential Massey products.

May's insight was to organize the cochain data involved in the Massey product in an upper triangular matrix A. Then if τ is a matrix with all zero entries except the top right corner, the simple equation

$$dA - \overline{A} \cdot A = \tau$$

gives exactly the list of defining equations one needs to define the Massey product. Here, $\overline{A}$ denotes the matrix with ij entry $(-1)^p a_{ij}$; a_{ij} is the ij entry of A, and p is the degree of a_{ij}.

In this thesis, we go one step further. We use the Dold-Kan correspondence to identify these cochain entries of the matrix with geometric objects called n-bundles or gerbes [6] and then refine these bundles to bundles *with connection*. Then passing back to cochains, we get the refined equation

$$d\hat{A} - \overline{\hat{A}} \cdot \hat{A} = \hat{\tau}$$

which we use to define the differential Massey product.

One may wonder if the Massey products defined this way refine the usual Massey products. It turns out that this is not necessarily the case. The problem is that a defining system for the singular Massey product does not necessarily refine to a defining system for the differential Massey product. In chapter 6, we prove the following theorem:

Theorem 1. *Let A be a formal connection on the algebra of singular cochains, and let $\hat{A}$ be a differential refinement of A. Then any differential refinement $\mu(\hat{A})$ satisfies the twisted Maurer-Cartan equation*

$$\mu(\hat{A}) = d\hat{A} - \overline{\hat{A}} \cdot \hat{A} \equiv B \mod \ker(\hat{A}) \ ,$$

where B is some matrix with entries in differential forms. Moreover, under the curvature map curv*, we have that the de Rham cohomology class* $[\mathrm{curv}(\mu(\hat{A}))] = [\mu(\mathrm{curv}(\hat{A}))]$ *is a de Rham Massey product.*

The previous theorem can be regarded as a refinement of a defining system for the singular Massey product. Moreover, the theorem shows that given a singular Massey product, one can refine the product to a bundle with connection whose curvature is the de Rham Massey product.

1.2 OVERVIEW OF CONTENTS

In chapter 2, we introduce the basic theory of Model categories which will be ubiquitous throughout this work. We begin with the basic definitions and quickly accelerate to the very specific subclass of combinatorial simplicial model categories. We also introduce the basic theory of homotopy limits and Bousfield localization, which will become essential in the definition of higher stacks.

In chapter 3, we present the basic definition and theory of smooth higher stacks. Most of the theory we present here has been developed by Daniel Dugger, Sharon Hollander and Daniel Isaksen in [10]. The general theory of higher topos (which we only refer to occasionally) has been developed by Jacob Lurie in [21].

In chapter 4, we discuss the Dold-Kan correspondence and its generalization to higher abelian stacks. We also discuss the monoidal structure of this correspondence as a weak monoidal Quillen equivalence which was developed in [26].

Chapters 5 and 6 present the main content of the paper. In chapter 5, we discuss a cup product morphism on stacks induced from the Deligne-Beilinson cup product. This cup product is the first step in constructing Massey products in differential cohomology and was developed by Hisham Sati, Urs Schreiber and Domenico Fiorenza in [11]. Here we also make explicit some of the properties that this cup product has as a refinement of singular and de Rham cohomology. In chapter 6, we discuss Massey products and the generalization of these products in stacks. We also discuss how these products refine usual cohomology and de Rham cohomology.

Finally, in chapter 7, we give a brief overview of possible applications which are developed in [15].

2.0 MODEL CATEGORIES

Roughly speaking, model category theory seeks to axiomatize and generalize the familiar homotopy theory of topological spaces. Surprisingly, homotopy theoretic techniques have interesting applications in categories other than $\mathcal{T}$op. In this section, we will introduce the basics of model category theory and provide some examples. For the most part, we follow [17] and encourage the reader to look there for further details.

2.1 DEFINITION

Before giving the definition, we need a little preperation.

Definition 2. *Let $\mathcal{C}$ be a category and let $f : X \to Y$ and $g : A \to B$ be morphisms in $\mathcal{C}$. We say that f has the* right lifting property *with respect to g and g has the* left lifting property *with respect to f if for every commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{k} & Y \end{array} \tag{2.1.1}$$

there is a morphism $l : B \to X$ such that $lg = h$ and $fl = k$.

One should compare this definition with the classical lifting property of a *Serre fibration* in $\mathcal{T}$op. We are now ready to define a Model category.

Definition 3. *Let $\mathcal{C}$ be a category which admits all small limits and colimits. A* model structure *on* $\mathcal{C}$ *is a choice of three subcategories* $\mathcal{W}$, $\mathcal{C}$ *and* $\mathcal{F}$ *(called weak equivalences, cofibrations and fibrations) such that:*

1. *(2-out-of-3) Given two composable morphisms* f *and* $g \in \mathcal{W}$*, if any two of the three morphisms* f, g, fg *are in* $\mathcal{W}$*, so is the third.*

2. *(Lifting) Fibration* $f \in \mathcal{F}$ *are precisely the right lifters against all* $g \in \mathcal{W} \cap \mathcal{C}$ *(called* acyclic *cofibrations). Cofibration* $f \in \mathcal{C}$ *are precisely the left lifters against all* $g \in \mathcal{W} \cap \mathcal{F}$ *(called* acyclic *fibrations).*

3. *(Factorization) There are two functorial factorizations* (Q, q) *and* (p, P) *such that for any morphism* f*,* $Q(f)$ *is a cofibration and* $q(f)$ *is an acyclic fibration;* $P(f)$ *is a fibration and* $p(f)$ *is an acyclic cofibration.*

We call such a category $\mathcal{C}$ *a* model *category.*

This definition differs slightly from that in [17] in that we do not require the retract axiom. If one adds the retract axiom, then one can weaken the lifting property slightly. In any case, the two definitions are equivalent.

Remark 1. *When referring to a model category* $\mathcal{C}$*, we will frequently leave the precise model structure implicit. We use the notation* $\rightarrowtail$, $\twoheadrightarrow$ *and* $\xrightarrow{\sim}$ *to mean cofibration, fibration and weak equivalence (resp.).*

Definition 4. *Let* $\mathcal{C}$ *be a category. We call a morphism* f *a* retract *of another morphism* g *if it is a retract of objects in* $Arr(\mathcal{C})$*. That is, the morphisms* f *and* g *fit into the following diagram:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "i"] & Y \arrow[r, "r"] & B \end{tikzcd}$$

where $ri = id$.

Proposition 5. *For a model category $\mathcal{C}$, all three subcategories $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts. That is, retracts of morphisms in each subcategory are again in that subcategory.*

Proof. We first show closure for fibrations. Let $g \in \mathcal{F}$ and let f be a retract of g. We want to show that f is a right lifter against acyclic cofibrations. Testing on the left with an acyclic cofibration gives

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail] & A \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r, "r"] \arrow[d, "g", two heads] & A \arrow[d, "f"] \\
T \arrow[r] & B \arrow[r, "i"] & Y \arrow[r, "r"] & B
\end{tikzcd}$$

we see have a lift $\tilde{h}$

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail] & A \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r, "r"] \arrow[d, "g", two heads] & A \arrow[d, "f"] \\
T \arrow[r] \arrow[urr, dotted] & B \arrow[r, "i"] & Y \arrow[r, "r"] & B
\end{tikzcd}$$

Then from the commutativity of the retract diagram, it follows that $r\tilde{h}$ is a lift of f. A similar argument shows closure for cofibrations (testing with an acyclic fibration on the right).

Now for weak equivalences, we observe that the above argument can be modified to show that acyclic fibrations and acyclic cofibrations are closed under retracts. If g is a weak equivalence and f a retract of g, we factorize $g = Q(g)q(g)$. Since $q(g)$ is a weak equivalence, 2-out-of-3 implies that $Q(g)$ is a weak equivalence. Now since the factorization is functorial, if we factorize $f = Q(f)q(f)$ similarly, we get a retract diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "Q(f)"] & X \arrow[r] \arrow[d, "Q(g)"] & A \arrow[d, "Q(f)"] \\
\tilde{B} \arrow[r] & \tilde{Y} \arrow[r] & \tilde{B}
\end{tikzcd}$$

Since $Q(g)$ is an acyclic cofibration, it follows that $Q(f)$ is an acyclic cofibration and therefore $f = Q(f)q(f)$ is a weak equivalence. $\square$

Model categories behave quite nice with respect to certain operations on categories. The following examples will be ubiquitous throughout this work.

Example 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. Then $\mathcal{C} \times \mathcal{D}$ is a model category with fibrations (cofibrations, weak equivalences) given by pairs of fibrations (cofibrations, weak equivalences) (f, g). It is a trivial exercise to verify the axioms.*

Example 2. *Let $\mathcal{C}$ be a model category. Then $\mathcal{C}^{op}$ is a model category with fibrations the cofibrations of $\mathcal{C}$, cofibrations the fibrations of $\mathcal{C}$, and weak equivalences are again the weak equivalences. Moreover, the functorial factorizations on $\mathcal{C}^{op}$ are reversed:*

$$(\bar{Q}, \bar{q}) = (P, p)$$

$$(\bar{p}, \bar{P}) = (q, Q)$$

Proposition 6. *Cofibrations (acyclic cofibrations) are closed under pushouts while fibrations (acyclic fibrations) are closed under pullback.*

Proof. We prove the claim only for cofibrations. The others are proved similarly and we leave the details to the reader. Let $i : A \to B$ be a cofibration and

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & X \arrow[d, "j"] \\ B \arrow[r] & Y \end{tikzcd}$$

be a pushout square. Testing on the right with an acyclic fibration, we have a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"] & X \arrow[r, "f"] \arrow[d, "j"] & C \arrow[d, two heads, "\sim"] \\
B \arrow[r] & Y \arrow[r] & D
\end{tikzcd}$$

By definition, the outer square can be filled with a lift h. The maps $h : B \to C$ and $f : X \to C$ give a cocone over the cospan and by the universal property of the pushout, we get a unique map (lift) $\tilde{h}$ from $\tilde{h} : Y \to C$ making the diagram commute. Since the acyclic fibration was arbitrary, the map j is a cofibration. □

The following lemma is extremely useful in calculation:

Lemma 7. *(Ken Brown's lemma) Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category of weak equivalences (subcategory satisfying 2-out-of-3). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor taking acyclic cofibrations (acyclic fibrations) between cofibrant (fibrant) objects to weak equivalences. Then F takes all weak equivalences between cofibrant (fibrant) objects to weak equivalences.*

Proof. Let $w : A \to B$ be a weak equivalence between cofibrant objects. We factor the map $\langle w, id \rangle : A \coprod B \to B$ into a cofibration $q : A \coprod B \to C$ followed by acyclic fibration $p : C \to B$. Since $\mathcal{C}$ has an initial object, $A \coprod B$ is a pushout. Since A and B are cofibrant, it follows that the maps $i_B : B \to A \coprod B$ and $i_A : A \to A \coprod B$ are cofibrations. By inspection of the maps and by (2-out-of-3) we have that qi_B and qi_A are weak equivalences, hence acyclic cofibrations.

Since A is cofibrant, the map

$$\varnothing \to A \to A \coprod B \to C$$

is a cofibration and C is cofibrant. By hypothesis, $F(qi_A)$ and $F(qi_B)$ are both weak equivalences. Since $F(pqi_B) = F(id) = id$, 2-out-of-3 gives that p is a weak equivalence. Therefore,

$F(w) = F(pqi_A)$ is a weak equivalence. $\square$

2.2 THE HOMOTOPY CATEGORY

In this section we describe the process of formally inverting the weak equivalences in a model category in order to study only the homotopy theory of the category and "forget" about the rest of the structure. The resulting category, after such a process, is called the *homotopy category.*

Definition 8. *Let $\mathcal{C}$ be a category of weak equivalences. We define the* homotopy *category* $\mathrm{Ho}\mathcal{C}$ *to be the free category $\mathcal{F}(\mathcal{C}, \mathcal{W}^{-1})$ ($\mathcal{W}^{-1}$ are formal inverses of $\mathcal{W}$) quotiented by the relations*

$$(f, g) \simeq (f \circ g)$$

$$id = (id)$$

and

$$id = (w, w^{-1}),\ id = (w^{-1}, w).$$

The homotopy category satisfies a universal property with respect to all functors $F : \mathcal{C} \to \mathcal{D}$ sending weak equivalences to isomorphisms. Since we will not use this property, we refer the reader to [17] for the details.

It is often preferable to work with a more concrete category as the homotopy category. In the case where $\mathcal{C}$ has a full model structure, there is a rather nice construction which turns out to be equivalent. In order to describe this point of view, we need a subclass of morphisms in $\mathcal{W}$ which act more like legitimate homotopies.

Definition 9. *For $\mathcal{C}$ a model category and an object $A \in \mathcal{C}$, we define a* cylinder object cyl(A) *to be factorization of the codiagonal*

$$\nabla : A \coprod A \to A.$$

We define a path object path(A) *of* A *similarly as a factorization of the diagonal*

$$\Delta : A \to A \times A.$$

Remark 2. *In the category of topological spaces* $\mathcal{T}\mathrm{op}$, *it is easy to see that the usual cylinder* $I \times A$ *and exponential* A^I *are cylinder and path space objects (respectively). It is helpful to keep this example in mind in the following definition.*

Definition 10. *A* left homotopy H *between two maps* $f, g : A \to X$ *is a map* $H : \mathrm{cyl}(A) \to X$ *making the diagram*

$$\begin{tikzcd}
A \arrow[d, "i_0"] \arrow[rrd, "f"] & & \\
A \coprod A \arrow[r] & \mathrm{cyl}(A) \arrow[r, "H"] & X \\
A \arrow[u, "i_1"'] \arrow[rru, "g"'] & &
\end{tikzcd}$$

commute. A right homotopy *between* f *and* g *is a map* $H : A \to \mathrm{path}(X)$ *making the diagram*

$$\begin{tikzcd}
X & & \\
X \times X \arrow[u, "p_0"] \arrow[d, "p_1"'] & \mathrm{path}(X) \arrow[l] & A \arrow[l, "H"'] \arrow[llu, "f"'] \arrow[lld, "g"] \\
X & &
\end{tikzcd}$$

commute.

Remark 3. *We will follow Hovey's notation and write $f \overset{l}{\sim} g$ when f is left homotopic to g; $f \overset{r}{\sim} g$ when f is right homotopic to g.*

The following proposition was taken from [17] and summarizes some of the nice properties of homotopies between fibrant and cofibrant objects. The proofs are straightforward exercises and can be found in [17].

Proposition 11. *Let $\mathcal{C}$ be a model category and let f and g be two maps. We have the following:*

1. *If $f \overset{l}{\sim} g$ and $h : X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h : A \to B$, then $fh \overset{l}{\sim} gh$.*

2. *If X is fibrant, $f \overset{l}{\sim} g$ and $h : A \to B$, then $fh \overset{l}{\sim} gh$. Dually, if B is cofibrant, $f \overset{l}{\sim} g$ and $h : X \to Y$, then $hf \overset{l}{\sim} hg$.*

3. *If B is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation on $\mathcal{C}(B, X)$.*

4. *If B is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence of fibrant objects, then h induces an isomorphism*

$$\mathcal{C}(B, X)/ \overset{l}{\sim} \simeq \mathcal{C}(B, Y)/ \overset{l}{\sim} .$$

Dually, if X is fibrant and $h : A \to B$ is an acyclic cofibration or weak equivalence of cofibrant objects, then h induces an isomorphism

$$\mathcal{C}(B, X)/ \overset{r}{\sim} \simeq \mathcal{C}(A, X)/ \overset{r}{\sim} .$$

5. *If B is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Moreover, there is a right homotopy for*

any given path object. Dually, if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$. Moreover, there is a left homotopy for any given cylinder object.

Notice that a direct consequence of the previous proposition is that, if B is cofibrant and X is fibrant, then the quotient of $\mathcal{C}(B, X)$ by both left and right homotopies is well defined and equal. Hence, there will be no ambiguity if we denote the quotient by $\mathcal{C}(B, X)/\sim$ and call the elements of the quotient *homotopy classes*.

Corollary 12. *Let $\mathcal{C}$ be a model category and let $\mathcal{C}^o$ denote the full subcategory on objects which are both fibrant and cofibrant. The homotopy relation on the morphisms is compatible with composition. Hence we have a well defined quotient $\mathcal{C}^o/\sim$ by this relation.*

The following proposition will provide a useful characterization of the homotopy category.

Proposition 13. *Let $\mathcal{C}$ be a model category. A morphism in $\mathcal{C}^o$ is a weak equivalence iff it is a homotopy equivalence.*

Proof. Let $f : X \to Y$ be a weak equivalence between cofibrant and fibrant objects. By the previous proposition, if B is also fibrant and cofibrant, we have an induced isomorphism

$$f_* \mathcal{C}(B, X)/\sim \quad \simeq \quad \mathcal{C}(B, Y)/\sim .$$

In particular, we can take $B = Y$ and there is a unique class $[g]$ such that $f_*[g] = [\mathrm{id}]$. Since the induced map is simply post composition, we can take a representative of $[g]$ giving $[fg] = [\mathrm{id}]$ or $fg \sim \mathrm{id}$. composing on the right with g and inverting f_* will give $gf \sim \mathrm{id}$.

Now suppose that $f : X \to Y$ is a homotopy equivalence. We can factor the map into an acyclic cofibration followed by a fibration. We want to show that this fibration is a weak equivalence. To this end, we show that it satisfies right lifting with respect to cofibrations. Let $i : V \rightarrowtail W$ be a cofibration and assume we have a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, tail, "i"] & \tilde{X} \arrow[d, two heads] \\ W \arrow[r] & Y \end{tikzcd}$$

where the right map is the fibration from the factorization. Since X and Y are fibrant and cofibrant, $\tilde{X}$ is fibrant and cofibrant, and the weak equivalence $X \to \tilde{X}$ is a homotopy equivalence. It follows that the fibration $\tilde{X} \to Y$ is a homotopy equivalence. □

Corollary 14. *Let $\mathcal{C}$ be a model category and $\mathrm{Ho}\mathcal{C}$ be its homotopy category. There is an isomorphism of categories*

$$\mathcal{C}^o/\sim \quad \simeq \ \mathrm{Ho}\mathcal{C}^o$$

Proof. Follows from the universal property for $\mathrm{Ho}\mathcal{C}$. See [17] for details. □

Theorem 15. *(Fundamental theorem) Let $\mathcal{C}$ be a model category and $\gamma : \mathcal{C} \to \mathrm{Ho}\mathcal{C}$ be the canonical functor. Let Q be the cofibrant replacement functor and R be the fibrant replacement.*

1. *The inclusion $\mathcal{C}^o \hookrightarrow \mathcal{C}$ induces an equivalence*

$$\mathcal{C}^o/\sim \quad \simeq \ \mathrm{Ho}\mathcal{C}^o \to \mathrm{Ho}\mathcal{C}$$

2. *There is a natural isomorphism*

$$\mathcal{C}(QX, RY)/\sim \quad \simeq \ \mathrm{Ho}(\gamma X, \gamma Y).$$

If X is already cofibrant and Y is fibrant, then we have a natural isomorphism

$$\mathcal{C}(X, Y)/\sim \quad \simeq \ \mathrm{Ho}(\gamma X, \gamma Y).$$

3. γ sends left or right homotopic maps to the same morphism in $\mathrm{Ho}\mathcal{C}$

4. If $f : X \to Y$ is such that γf is an isomorphism, then f is a weak equivalence.

Proof. To prove 1., observe that the composite functor QR induces the categorical equivalence. Indeed, using the universal property of the homotopy category, we see that we have induced functors $\mathrm{Ho}i : \mathrm{Ho}\mathcal{C}^o \to \mathrm{Ho}\mathcal{C}$ and $\mathrm{Ho}QR : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}^o$. Since the natural transformations $RQ \circ i \to \mathrm{id}$ and $\mathrm{id} \to i \circ RQ$ are componentwise weak equivalences, it follows that we have natural isomorphisms $\mathrm{Ho}RQ \circ \mathrm{Ho}i \to \mathrm{id}$ and $\mathrm{id} \to \mathrm{Ho}i \circ \mathrm{ho}QR$ and hence a categorical equivalence.

For part 2., We first observe that, using the previous corollary and part 1, we have a natural isomorphism $\mathcal{C}(QRX, QRY)/\sim \;\; \simeq \mathrm{Ho}\mathcal{C}(\gamma X, \gamma Y)$. Now since QRY is fibrant and $QRX \to RX$ is a weak equivalence we have, by the previous proposition,

$$\mathcal{C}(QX, QRY)/\sim \;\; \simeq \;\; \mathcal{C}(QRX, QRY)/\sim .$$

Similarly, using cofibrancy of QX and fibrancy of RY, we get an isomorphism

$$\mathcal{C}(QX, RY)/\sim \;\; \simeq \;\; \mathcal{C}(QX, QRY)/\sim .$$

3., follows immediately from 2. To prove 4., observe that if γf is an isomorphism, it follows that QRf is an isomorphism, hence a homotopy equivalence. Using the weak equivalences $X \to QX$ and $Q \to RX$ we see that f must be a weak equivalence. $\square$

2.3 QUILLEN FUNCTORS AND DERIVED FUNCTORS

In this section we introduce the morphisms between model categories and explore their properties.

Definition 16. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $(F \vdash G)$*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be an adjunction of categories

1. *We call F a* left Quillen functor *if F preserves cofibrations and acyclic cofibrations.*

2. *We call G a* right Quillen functor *is G preserves fibrations and acyclic fibrations.*

3. *In either of the above cases we call the adjunction $(F \vdash G)$ a* Quillen adjunction.

Actually, we can say more about the definition. Whenever one of the conditions on F and G holds, so does the other. This explains why we did not require both conditions in the definition of Quillen equivalence.

Proposition 17. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $(F \vdash G)$ be an adjunction. Then F is a left Quillen functor iff G is a right Quillen functor.*

Proof. Suppose F is a left Quillen functor. Let f be a fibration in G. Then, testing on the left with an acyclic cofibration and using the adjunction, we see immediately that Gf lifts. A similar argument shows that G preserves acyclic fibrations and the only if statement. □

Notice that it follows immediately by Ken Brown's lemma that left Quillen functors preserve weak equivalences between cofibrant objects and right Quillen functors preserve weak equivalences between fibrant objects. This implies, using the universal property of the homotopy category, that if $\mathcal{C}^c$ is the subcategory on cofibrant objects, a left Quillen functor $F : \mathcal{C} \to \mathcal{D}$ restricted to $\mathcal{C}^c$ will induce a functor

$$\mathrm{Ho}F : \mathrm{Ho}\mathcal{C}^c \to \mathrm{Ho}\mathcal{D}.$$

Similarly, a left Quillen functor $G : \mathcal{D} \to \mathcal{C}$ will induce a functor

$$\mathrm{Ho}G : \mathcal{D}^f \to \mathcal{C}$$

where $\mathcal{D}^f$ is the full subcategory on fibrant objects.

Definition 18. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories with cofibrant and fibrant replacements Q and R. A left Quillen functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor LF, called the* left derived functor, *between the homotopy categories and is defined as the composite*

$$LF : \mathrm{Ho}\mathcal{C} \xrightarrow{Q} \mathrm{Ho}\mathcal{C}^c \xrightarrow{\mathrm{Ho}F} \mathrm{Ho}\mathcal{D}.$$

Similarly, a right Quillen functor $G : \mathcal{D} \to \mathcal{C}$ induces a functor RG, called the right derived functor *between the homotopy categories*

$$RG : \mathrm{Ho}\mathcal{D} \xrightarrow{R} \mathrm{Ho}\mathcal{D}^f \xrightarrow{\mathrm{Ho}G} \mathrm{Ho}\mathcal{C}.$$

Note that it is straightforward from the definitions to show that composites of Quillen adjunctions are again Quillen adjunctions and that the induced derived functors are form composite adjunctions [17].

It is natural to ask whether the induced functors LF and RG form an adjunction at the level of homotopy categories. The following proposition shows that this is indeed the case.

Proposition 19. *Let $\mathcal{C}$ and $\mathcal{D}$ be model catgories and let $F : \mathcal{C} \to \mathcal{D}$ be a left Quillen functor with right Quillen adjoint G. Then we have an induced adjunction $(LF \vdash RG)$ between derived functors.*

Proof. Since F preserves cofibrant objects, it follows from theorem 15 that we have a natural isomorphism

$$\mathrm{Ho}\mathcal{D}(FQX, Y) \simeq \mathcal{D}(FQX, RY)/\sim$$

Similarly, we have a natural isomorphism

$$\mathrm{Ho}\mathcal{C}(X, GRY) \simeq \mathcal{C}(QX, GRY)/\sim .$$

Using the adjunction between $\mathcal{C}$ and $\mathcal{D}$, we have a natural isomorphism

$$\phi : \mathcal{D}(FQX, RY) \to \mathcal{C}(QX, GRY)$$

and it remains to show that ϕ respects the homotopy relation. To this end, choose a right homotopy r between f and g in $\mathcal{D}(FQX, RY)$. Since G is a right Quillen functor it preserves path objects and φr will define a right homotopy between φf and φg. Similarly, using the fact that F is a left Quillen functor, one shows that for a left homotopy l between f and g in $\mathcal{C}(QX, GRY)$, $\varphi^{-1} l$ defines a left homotopy between $\varphi^{-1} f$ and $\varphi^{-1} g$. □

2.4 EXAMPLES

In this section we give some basic examples of model categories and adjunctions. We will not provide proofs for any of the examples, but we will provide references for the interested reader.

We begin with *the* motivating example for model categories. The category of topological spaces [17].

Example 3. *The category* $\mathcal{T}\mathrm{op}$ *of topological spaces admits a model structure (Quillen model structure) with the following choices of subcategories:*

1. *Weak equivalences are* weak homotopy equivalences. *That is, continuous functions* $f : X \to Y$ *inducing isomorphisms*

$$f_* : \pi_n(X) \to \pi_n(Y)$$

on homotopy groups for all $n \in \mathbb{N}$.

2. *Fibrations are* Serre fibrations. *That is, continuous functions satisfying the right lifting*

property with respect to inclusions of faces

$$I^{n-1} \hookrightarrow I^n.$$

of cubes.

3. *Cofibrations are retracts of relative cell complexes.*

The next example is the most important example for us [14].

Example 4. *The category* $s\mathcal{S}\mathrm{et}$ *of simplicial sets (see appendix for the definition) admits a model structure (Quillen model structure) with the following choices of subcategories:*

1. *Weak equivalences are* weak equivalences of realizations*. That is, maps of simplicial sets* $f : X \to Y$ *inducing isomorphisms*

$$|f|_* : \pi_n(|X|) \to \pi_n(|Y|)$$

on homotopy groups for all $n \in \mathbb{N}$*. Here,* $|\cdot|$ *denotes the geometric realization functor.*

2. *Fibrations are* Kan fibrations*. That is, maps of simplicial sets satisfying the right lifting property with respect to all horn inclusions*

$$\Lambda_k[n-1] \hookrightarrow \Delta[n].$$

of the standard n *simplex.*

3. *Cofibrations are levelwise monomorphisms*

The next example can be found in [1].

Example 5. *The category of groupoids* $\mathcal{G}\mathrm{rpd}$ *admits a model structure with the following choices of subcategories*

1. *Weak equivalences are equivalences of categories*

2. *Let* $N : \mathcal{G}\mathrm{pd} \to s\mathcal{S}\mathrm{et}$ *denote the nerve functor. Fibrations are functors* $F : X \to Y$ *such that* $N(f) : N(X) \to N(Y)$ *are Kan fibrations*

3. *Cofibrations are functors which are monomorphisms on objects*

Recall that for a simplicial set X*, we have truncation functors (see appendix)* $\tau_n X$ *for each* n*. In particular, for* $n = 1$*, we have a* fundamental groupoid *functor* $\Pi_1 := \tau_1 : s\mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{rpd}$*. Now* Π_1 *is left adjoint to* N *(see appendix), and by construction, we easily see that* Π_1 *preserves cofibrations and weak equivalences. Hence,* Π_1 *is part of a Quillen adjunction*

$$s\mathcal{S}\mathrm{et} \underset{N}{\overset{\Pi_1}{\rightleftarrows}} \mathcal{G}\mathrm{pd} \tag{2.4.1}$$

Example 6. *The category of positively graded chain complexes* $\mathcal{C}\mathrm{h}^+$ *admits a model structure with the following choices of subcategories:*

1. *Weak equivalences are quasi-isomorphisms of chain complexes.*

2. *Fibrations are degree-wise epimorphisms*

3. *Cofibrations are degree-wise monomorphisms with projective cokernel.*

In particular, the cofibrant objects are those objects $A \in \mathcal{C}\mathrm{h}^+$ *such that the cokernel of the map*

$$\mathrm{coker}\{0 \to A\} \simeq A$$

is projective.

2.5 COMBINATORIAL AND SIMPLICIAL MODEL CATEGORIES

We now focus our attention on a subclass of model categories which behave particularly well in application. There are several adjectives we we need to define: *cofibrantly generated*, *simplicial* and *combinatorial*. We begin with cofibrantly generated model categories. These model categories arise naturally in the construction of model categories via the small object argument [17]. Since the small object argument requires several technical lemma's, we will not include the theorem and refer to [17].

Definition 20. *Let $\mathcal{C}$ be a model category and let I be a class of morphisms in $\mathcal{C}$. We call a morphism:*

1. I-injective *if it has the right lifting property with respect to all morphisms in I,*
2. I-projective *if it has the left lifting property with respect to all morphisms in I,*
3. I-cofibration *if it has the left lifting property with respect to all I-injectives,*
4. I-fibration *if it has the right lifting property with respect to all I-projectives.*

Definition 21. *Let $\mathcal{C}$ be a model category and let I and J be* sets *(not just proper classes) of morphisms. We say that $\mathcal{C}$ is* cofibrantly generated *if the set of I-cofibrations is the set of cofibrations of $\mathcal{C}$, the set of J-cofibrations is the set of acyclic cofibrations, and I and J admit the* small object argument *[17].*

In a cofibrantly generated model category the set I (J), along with the small object argument, gives a lot of control over the cofibrations (acyclic cofibrations); one can restrict attention to the generating set I (J) when proving claims about cofibrations (acyclic cofibrations). We will be most concerned with model categories which admit the stronger condition that every object in $\mathcal{C}$ is small and is the colimit over a set of generating objects. Such model categories are called *combinatorial*.

In order to define what it means for a model category to be simplicial, we need the following two definitions:

Definition 22. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a bifunctor which preserves colimits in each variable. We call $\otimes$ a* left Quillen bifunctor *if, in addition, it satisfies the following axiom:*

- *(Pushout product axiom) Let $A \rightarrowtail B$ and $C \rightarrowtail D$ be cofibrations (acyclic cofibrations) in $\mathcal{C}$, $\mathcal{D}$ (respectively). Then the map*

$$A \otimes D \coprod_{A \otimes C} B \otimes C \to B \otimes D$$

  is a cofibration (acyclic cofibration).

Definition 23. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $(-)^{(-)} : \mathcal{D} \times \mathcal{C} \to \mathcal{E}$ be a bifunctor which preserves limits in each variable. We call $(-)^{(-)}$ a* right Quillen bifunctor *if, in addition, it satisfies the following axiom:*

- *(corner axiom) Let $A \rightarrowtail B$ and $C \rightarrowtail D$ be cofibrations (acyclic cofibrations) in $\mathcal{C}$, $\mathcal{D}$ (respectively). Then the map*

$$C^B \to C^A \times_{D^A} D^B$$

  is a fibration (acyclic fibration).

Now suppose $\mathcal{C}$ is a model category which is enriched over $s\mathcal{S}\mathrm{et}$. Suppose in addition that there are actions on the left and the right

$$\otimes : s\mathcal{S}\mathrm{et} \times \mathcal{C} \to \mathcal{C}$$

and

$$(-)^{(-)} : \mathcal{C} \times s\mathcal{S}\mathrm{et} \to \mathcal{C}$$

that satisfies the adjunction conditions

$$\mathrm{Map}(X \otimes A, B) \simeq \mathrm{Map}(X, \mathrm{Map}(A, B))$$

and

$$\mathrm{Map}(A, B^X) \simeq \mathrm{Map}(\mathrm{Map}(A, B), X)$$

We call the left action the *tensor* and the right action the *power* and we say that $\mathcal{C}$ is powered and tensored over $s\mathcal{S}\mathrm{et}$.

Remark 4. *We will frequently need to distinguish the tensoring and powering over simplicial sets from closed monoidal structures on categories. For this reason we will denote the left tensoring*

$$\odot : s\mathcal{S}\mathrm{et} \times \mathcal{C} \to \mathcal{C}$$

We will still denote the right tensor by

$$(-)^{(-)} : \mathcal{C} \times s\mathcal{S}\mathrm{et} \to \mathcal{C}$$

and will reserve the notation $[-, -]$ *for the closed monoidal hom.*

The following condition ensures that the enrichment and copower behave well with respect to the model structure. We will not prove the claim, as it is a simple exercise using the adjunctions.

Proposition 24. *Let* $\mathcal{C}$ *be model category enriched, powered and tensored over* $s\mathcal{S}\mathrm{et}$*. We call* $\mathcal{C}$ *a* simplicial model category *if it satisfies any of the three equivalent conditions:*

1. *The tensor* $\odot$ *is a left Quillen bifunctor*

2. *The power* $(-)^{(-)}$ *is a right Quillen bifunctor*

3. For $A \rightarrowtail B$ and $C \rightarrowtail D$ be cofibrations (acyclic cofibrations) in $\mathcal{C}$, the the map

$$\mathrm{Map}(B, C) \to \mathrm{Map}(A, C) \times_{\mathrm{Map}(A,D)} \mathrm{Map}(B, D)$$

is a fibration (acyclic fibration)

Note that a simple consequence of the definitions is a natural choice of cylinder and path space object given by $\Delta[1] \otimes A$ and $A^{\Delta[1]}$. Using the tensoring, we have

$$\mathrm{Map}(\Delta[1] \times A, B) \simeq \mathrm{Map}(\Delta[1], \mathrm{Map}(A, B))$$

and a map out of the cylinder is equivalently an edge in the mapping space $\mathrm{Map}(A, B)$. It follows that homotopies in the mapping space correspond to homotopies defined using the model structure.

2.6 MODEL STRUCTURES ON FUNCTORS

Given a model category $\mathcal{C}$, there are several natural ways to define a model structure on the category of functors $[\mathcal{D}, \mathcal{C}]$ out of some category $\mathcal{D}$. We will briefly discuss the basic properties of these model structures and provide only necessary proofs. Most of these definitions and properties have been taken from [17] and [21]. The reader is encouraged to look there for further details.

2.6.1 Projective and injective model structures

Given a model category $\mathcal{C}$, it is natural to ask if there is a model structure on the functor category $[\mathcal{D}, \mathcal{C}]$, where the subcategories $\mathcal{W}, \mathcal{F}$ and $\mathcal{C}$ are somehow inherited from $\mathcal{C}$. The answer turns out to be slightly more nuanced than one might expect. The following definition and theorem can be found in [21].

Definition 25. *Let $\mathcal{C}$ be a model category and let $\mathcal{D}$ be a small category. Then we say that a natural transformation $F \to G$ between functors $F, G : \mathcal{D} \to \mathcal{C}$ is:*

- *An* injective cofibration *if the induced map $F(d) \to G(d)$ is a cofibration for all $d \in \mathcal{D}$.*
- *A* projective fibration *if the induced map $F(d) \to G(d)$ is a fibration for all $d \in \mathcal{D}$*
- *A* weak equivalence *if the induced map $F(d) \to G(d)$ is a weak equivalence for all $d \in \mathcal{D}$*
- *An* injective fibration *if the map lifts against all acyclic injective cofibrations*
- *An* projective cofibration *if the map lifts against all acyclic projective fibrations*

Theorem 26. *Let $\mathcal{C}$ be a combinatorial model category and let $\mathcal{D}$ be a small category. Then the functor category $[\mathcal{D}, \mathcal{C}]$ admits two combinatorial model structures: The projective*

1. *fibrations are* projective fibrations*;*
2. *weak equivalences are* weak equivalences *between functors;*
3. *cofibrations are* projective cofibrations*;*

and the injective

1. *fibrations are* injective fibrations*;*
2. *weak equivalences are* weak equivalences *between functors;*
3. *cofibrations are* injective cofibrations*;*

Remark 5. *If $\mathcal{C}$ is assumed to be proper or simplicial, then so are the model structures on $[\mathcal{D}, \mathcal{C}]$ [21].*

2.6.2 Reedy categories and Reedy model structures

We now discuss another model structure on functors over particular types of categories called *Reedy* categories. This model structure will be extremely useful in calculating homotopy limits and colimits and can be viewed as a sort of interpolation between the projective and injective model structures. The following definition was taken from [21].

Definition 27. *A* Reedy *category $\mathcal{R}$, is a small category equipped with two subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ that satisfy the following:*

1. *The pair $(\mathcal{R}^-, \mathcal{R}^+)$ forms a* factorization system*: every morphism $f \in \mathcal{R}$ can be factored $f = f^+ f^-$, with $f^- \in \mathcal{R}^-$ and $f^+ \in \mathcal{R}^+$.*

2. *Every isomorphism in $\mathcal{R}$ is identity.*

3. *Define the relation $X \leqslant Y$ iff there is a morphism $f : X \to Y$ in $\mathcal{R}^-$ or $g : Y \to X$ in $\mathcal{R}^+$. If $X \neq Y$, we write $X < Y$. Then there are no infinite descending chains*

$$\ldots < X_2 < X_1 < X_0.$$

Remark 6. *Note that the third condition guarantees that the category has a* good filtration*. That is, it has a transfinite sequence*

$$\{\mathcal{R}_\beta\}_{\beta<\alpha},$$

with the following properties:

1. *Every object of $\mathcal{R}$ belongs to $\mathcal{R}_\beta$ for a sufficiently large ordinal $\beta < \alpha$.*

2. *For each ordinal $\beta < \alpha$, $\mathcal{R}_\beta$ is obtained from $\mathcal{R}_{<\beta} := \bigcup_{\gamma<\beta} \mathcal{R}_\gamma$ by adjoining a single object X such that if $Y < X$, then $Y \in \mathcal{R}_{<\beta}$.*

Indeed, if no such filtration existed, then we can construct an infinite descending chain using the second condition.

Taking homotopy limits and colimits over cosimplicial and simplicial diagrams will be extremely useful for us in calculation. Keeping this in mind, we provide the following examples.

Example 7. *The category Δ of linearly ordered finite sets of integers is a Reedy category with the factorization system (Δ^-, Δ^+), given by the subcategories of epi and monomorphisms (respectively).*

Example 8. *If $\mathcal{R}$ is a Reedy category with factorization system $(\mathcal{R}^-, \mathcal{R}^+)$, then so is $\mathcal{R}^{op}$ with the factorization system $(\mathcal{R}^{+\ op}, \mathcal{R}^{-\ op})$.*

Definition 28. *Let $\mathcal{R}$ be a Reedy model category and $\mathcal{C}$ be any category admitting small limits and colimits. Let $F : \mathcal{R} \to \mathcal{C}$ be a functor. For each object $X \in \mathcal{R}$, we define the* latching *object to be the colimit*

$$L_X(F) := \lim_{X \in \mathcal{R}^+/Y, X \neq Y} F(X)$$

and the matching *object to be the limit*

$$M_X(F) := \lim_{X \in \mathcal{R}^-/Y, X \neq Y} F(X).$$

We have canonical maps

$$L_X(F) \to F(X) \to M_X(F).$$

It is instructive to think of the latching and matching objects over Δ^{op}. Functors over this category are simplicial objects. Since the monomorphisms in Δ can be written as a composition of coface maps and epimorphisms can be written as compositions of codegeneracies, we see that the latching objects are a limit over the degeneracies while the matching object is a colimit over the face maps.

We are now ready to describe the model structure on functors over a Reedy category. The following two propositions can be found in [21].

Proposition 29. *Let $\mathcal{R}$ be a Reedy category and let $\mathcal{C}$ be a model category. There is a model structure on the category of functors $[\mathcal{R}, \mathcal{C}]$ with the following properties:*

1. *A natural transformation $F \to G$ is a* Reedy cofibration *(acyclic cofibration) iff for every object $X \in \mathcal{R}$, the induced map*

$$F(X) \coprod_{L_X(F)} L_X(G) \to G(X)$$

is a cofibration (acyclic cofibration) in $\mathcal{C}$

2. *A natural transformation $F \to G$ is a* Reedy fibration *(acyclic fibration) iff for every object $X \in \mathcal{R}$, the map*

$$F(X) \to G(X) \times_{M_X(G)} M_X(F)$$

is a fibration (acyclic fibration) in $\mathcal{C}$.

3. *A natural transformation $F \to G$ is a weak equivalence iff it is an object-wise weak equivalence.*

Proposition 30. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be model categories and let $\mathcal{R}$ be a Reedy category. Let $[\mathcal{R}, \mathcal{C}]$ and $[\mathcal{R}, \mathcal{D}]$ denote the functor categories equipped with the Reedy model structure. Let $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor. Then the coend functor (see appendix for the definition)*

$$\int^{\mathcal{R}} : [\mathcal{R}^{op}, \mathcal{C}] \times [\mathcal{R}, \mathcal{D}] \to \mathcal{E}$$

is also a left Quillen bifunctor.

Proof. We want to show that given two Reedy cofibrations (acyclic cofibrations) $f : F \rightarrowtail F'$ and $g : G \rightarrowtail G'$, the induced map

$$\int^{\mathcal{R}} F \times G' \coprod_{\int^{\mathcal{R}} F \times G} \int^{\mathcal{R}} F' \times G \rightarrowtail \int^{\mathcal{R}} F' \times G' \tag{2.6.1}$$

is a cofibration (acyclic cofibration). Let $\{\mathcal{R}_\beta\}_{\beta<\alpha}$ be a good filtration (see remark 6). Let

$$R_\beta := \int^{\mathcal{R}_{<\beta}} F \times G' \coprod_{\int^{\mathcal{R}_{<\beta}} F\times G} \int^{\mathcal{R}_{<\beta}} F' \times G$$

and

$$R'_\beta := \int^{\mathcal{R}_{<\beta}} F' \times G'$$

denote the restrictions. Then it suffices to show that the map

$$R_\beta \coprod_{R_\gamma} R'_\gamma \to R_\beta \coprod_{R_\delta} R'_\delta \tag{2.6.2}$$

is a cofibration (acyclic cofibration) for every $\gamma \leqslant \delta \leqslant \beta \leqslant \alpha$. Indeed, if this is the case then

$$R_\beta \coprod_{R_0} R'_0 \to R_\alpha \coprod_{R_\alpha} R'_\alpha$$

is a cofibration (acyclic cofibration). But this map is exactly (2.6.1).

Now to prove the claim for (2.6.2), we proceed by induction on δ. To this end, observe that in the case that δ is a limit ordinal, we can write (2.6.2) as a composition

$$R_\beta \coprod_{R_\gamma} R'_\gamma \to R_\beta \coprod_{R_{\gamma+1}} R'_{\gamma+1} \to \ldots \to R_\beta \coprod_{R_\delta} R'_\delta.$$

and the claim follows from the induction hypothesis. Since all ordinals are either limit, zero, or successor ordinals, we are reduced to the case where $\delta = \delta_0 + 1$ is a successor. Writing

$$R_\beta \coprod_{R_\gamma} R'_\gamma \to R_\beta \coprod_{R_{\delta_0}} R'_{\delta_0} \to R_\beta \coprod_{R_\delta} R'_\delta$$

and using the induction hypothesis, we are further reduced to proving the claim for $\gamma = \delta_0$. Finally, using the fact that the map arises as a pushout we are reduced to proving that the map

$$h : R_{\delta_0+1} \coprod_{R_{\delta_0}} R'_{\delta_0} \to R'_{\delta_0}$$

is a cofibration (acyclic cofibration).

Now using the filtration we can choose $X \in \mathcal{R}_{\delta_0}$ such that $X \notin \mathcal{R}_{<\delta_0}$. We can form the pushout

$$\begin{tikzcd}
F(X) \coprod_{L_X(F)} L_X(F') \otimes G(X) \coprod_{L_X(G)} L_X(G') \arrow[r] \arrow[d] & F'(X) \otimes G(X) \coprod_{L_X(G)} L_X(G') \arrow[d] \\
F(X) \coprod_{L_X(F)} L_X(F') \otimes G'(X) \arrow[r] & P
\end{tikzcd} \tag{2.6.3}$$

where $L_X(F)$ denotes the latching object. The universal property produces a map

$$h' : P \to F'(X) \otimes G'(X)$$

which we see is a cofibration (acyclic cofibration), using the fact that $\otimes$ is a Quillen bifunctor along with the fact that f and g are Reedy cofibrations (acyclic cofibrations). Finally, the claim follows by observing that h is a pushout of h'. □

In the previous claim, we assumed that the categories $[\mathcal{R}, \mathcal{C}]$ and $[\mathcal{R}^{op}, \mathcal{D}]$ were equipped with the Reedy model structure. In practice, we will also need to endow these categories with the *projective* and *injective* model structures respectively. Luckily the claim is much easier in this case.

Proposition 31. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be combinatorial model categories and let $\mathcal{R}$ be a Reedy category. Let $[\mathcal{R}, \mathcal{C}]$ and $[\mathcal{R}, \mathcal{D}]$ denote the functor categories equipped with the projective and injective model structures (respectively). Let $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor. Then* *the coend functor*

$$\int^{\mathcal{R}} : [\mathcal{R}, \mathcal{C}] \times [\mathcal{R}^{op}, \mathcal{D}] \to \mathcal{E}$$

is also a left Quillen bifunctor.

Proof. Again we want to show that given two Reedy cofibrations (acyclic cofibrations) $f : F \rightarrowtail F'$ and $g : G \rightarrowtail G'$, the induced map

$$\int^{\mathcal{R}} F \times G' \coprod_{\int^{\mathcal{R}} F \times G} \int^{\mathcal{R}} F' \times G \to \int^{\mathcal{R}} F' \times G' \tag{2.6.4}$$

is a cofibration (acyclic cofibration). Since $\mathcal{C}$ is combinatorial, it suffices to prove the claim for generating cofibrations (acyclic cofibrations) of the form $\mathcal{F}^X_Y \rightarrowtail \mathcal{F}^X_{Y'}$, where $X \in \mathcal{R}$ and $i : Y \rightarrowtail Y'$ is a cofibration (acyclic cofibration). Using the definition of these generating cofibrations, we see that (2.6.4) is nothing but the map

$$Y \otimes G'(X) \coprod_{Y \otimes G(X)} Y' \otimes G(X) \to Y' \otimes G'(X).$$

Since $\otimes$ is a Quillen bifunctor, this is a cofibration (acyclic cofibration) and we are done. $\square$

The next example is the main motivation for us:

Example 9. *Let $\mathcal{C}$ be a combinatorial simplicial model category. Then by definition, we have a left Quillen bifunctor*

$$\otimes : \mathcal{C} \times s\mathcal{S}\mathrm{et} \to \mathcal{C}.$$

Since Δ is a Reedy category, the previous proposition gives that

$$\int^{\Delta} [\Delta^{op}, \mathcal{C}] \times [\Delta, s\mathcal{S}\mathrm{et}] \to \mathcal{C}$$

is a left Quillen bifunctor which takes a simplicial object $X_\bullet$ in $\mathcal{C}$ and a cosimplicial simplicial *set $Y^\bullet$ to the end*

$$\int^{[n]\in\Delta} X_n \times Y^n.$$

Dually, we have that

$$\int^{\Delta} [\Delta^{op}, \mathcal{C}] \times [\Delta, s\mathcal{S}\mathrm{et}] \to \mathcal{C}$$

is also a left Quillen bifunctor which takes a cosimplicial object $X^\bullet$ in $\mathcal{C}$ and a simplicial simplicial set $Y_\bullet$ to the end

$$\int^{[n]\in\Delta} X_n \times Y^n.$$

2.7 HOMOTOPY LIMITS AND COLIMITS

Calculations involving homotopy limits and colimits are crucial in many applications of model categories. These objects can be thought of as sort of "thickened" limits and colimits. This thickening allows one to include homotopy theoretic information in diagrams provided by the model structure on the category. We will begin by the abstract definition and then provide methods for explicit calculations using the Reedy model structure on diagram categories.

2.7.1 Definition and properties

Definition 32. *Let $\mathcal{C}$ be a category of weak equivalences and let $\mathcal{D}$ be a small category. Then the category of functors $[\mathcal{D}, \mathcal{C}]$ becomes a category of weak equivalences by taking those natural transformations which are object-wise weak equivalences. We define the* homotopy limit *and* colimit *to be the right and left derived functors of the limit and colimit (resp).*

From an abstract point of view, this definition is quite nice. However, in practice one wants a much more explicit description of what this functor does to a diagram $F \in [\mathcal{D}, \mathcal{C}]$.

We have already seen that to calculate the value of a derived functor on an object, one cofibrantly (or fibrantly) replaces the object and finds the value of the original functor on the resulting object. In particular, if $\mathcal{C}$ is a combinatorial simplicial model category, then we can calculate homotopy limits and colimits via the ordinary limits and colimits

$$\mathrm{hocolim}(F) = \mathrm{colim}(Q_{\mathrm{proj}}(F))$$

$$\mathrm{hocolim}(F) = \lim(R_{\mathrm{inj}}(F))$$

where Q_{proj} denotes cofibrant replacement in the projective model structure on functors. Although this may seem like a good way to calculate homotopy limits and colimits, in practice one typically finds that calculating the cofibrant replacement is the difficult. It is therefore advantageous to have a formula which more algorithmically computes the cofibrant replacement.

When the model category is a combinatorial simplicial model category, we have the following formula:

Proposition 33. *Let $\mathcal{C}$ be a combinatorial simplicial model category and let $X_\bullet : \Delta^{op} \to \mathcal{C}$ be a simplicial diagram. Then there is a canonical map*

$$h : \mathrm{hocolim} X_\bullet \to \int^{[n]\in\Delta^{op}} X_n \odot \Delta[n].$$

Moreover, this map is a weak equivalence if X_n is cofibrant for each n.

Proof. Recall from the previous section that in this context, the functor

$$\int^{[n]\in\Delta} [\Delta^{op}, \mathcal{C}]_{\mathrm{proj}} \times [\Delta, s\mathcal{S}\mathrm{et}]_{\mathrm{inj}} \to \mathcal{C}$$

is a left Quillen bifunctor. Let $QX_\bullet$ denote the cofibrant replacement in the projective model structure. We have that

$$\mathrm{hocolim} X_\bullet = \lim QX_\bullet \simeq \int^{[n]\in\Delta^{op}} QX_n \odot *$$

Now let $\Delta^\bullet : \Delta \to s\mathcal{S}\mathrm{et}$ denote the functor which assigns each linearly ordered set $[n] \to \Delta[n]$.

Then $\Delta^\bullet$ is the Reedy cofibrant replacement for the constant functor $1 : \Delta \to s\mathcal{S}\mathrm{et}$. Since Reedy cofibrant objects are injective cofibrant, and the functor $\int^{[n]\in\Delta^{op}} QX_\bullet \odot \bullet$ preserves weak equivalences between cofibrant objects, it follows that we have an equivalence

$$w : \int^{[n]\in\Delta^{op}} QX_n \odot \Delta[n] \to \int^{[n]\in\Delta^{op}} QX_n \odot *$$

The cofibrant replacement $QX_\bullet \to X_\bullet$ then induces a map

$$h : \mathrm{hocolim} X_\bullet \simeq \int^{[n]\in\Delta^{op}} QX_n \odot \Delta[n] \to \int^{[n]\in\Delta^{op}} X_n \odot \Delta[n].$$

Now if X_n is cofibrant in $\mathcal{C}$ for each n, it follows that $X_\bullet$ is Reedy cofibrant. Hence, by proposition 30, the functor $\int^{[n]\in\Delta^{op}} \bullet \odot \Delta_\bullet$ preserves weak equivalences between Reedy cofibrant objects and h is a weak equivalence. $\square$

It is clear from the proof that we can replace Δ with any Reedy category. In this case we get the following:

Proposition 34. *Let $\mathcal{C}$ be a combinatorial simplicial model category and let $\mathcal{R}$ be a Reedy category. Let $Q(1) : \mathcal{R} \to s\mathcal{S}\mathrm{et}$ be a Reedy cofibrant replacement of the constant functor 1. Let $F : \mathcal{R} \to \mathcal{C}$ be a functor taking values in cofibrant objects in $\mathcal{C}$, then the homotopy colimit over F is given by*

$$\mathrm{hocolim} F \simeq \int^{r\in\mathcal{R}} F(r) \odot Q(1).$$

2.7.2 Examples of homotopy colimits

We will now provide some examples of homotopy limits and colimits in various contexts. We begin with a few examples in the category of topological spaces.

Remark 7. *The category of spaces $\mathcal{T}\mathrm{op}$ is a cofibrantly generated, simplicial model category* *with simplicial enrichment given by*

$$\mathrm{Map}(X,Y) := \mathrm{sing}\ \mathrm{Map}(QX,Y)$$

where the mapping space on the right is given the compact open topology, QX is a cofibrant replacement for X (CW complex with same homotopy type). Since the standard model structure on $\mathcal{T}\mathrm{op}$ is created via the small object argument, it follows that this category is cofibrantly generated.

Proposition 35. *The tensor over $s\mathcal{S}\mathrm{et}$ is given by*

$$K \odot X = |K| \times X.$$

Proof. We have the enriched adjunction

$$\mathrm{Map}(K, \mathrm{Map}(X,Y)) = \mathrm{Map}(K, \mathrm{sing}\ \mathrm{Map}(QX,Y)) \simeq \mathrm{Map}(|K|, \mathrm{Map}(QX,Y))$$

Since $|K|$ is a CW complex, it is cofibrant. We have

$$\begin{aligned}\mathrm{Map}(|K|, \mathrm{Map}(QX,Y)) &= \mathrm{sing}\ \mathrm{Map}(|K|, \mathrm{Map}(QX,Y)) \\ &= \mathrm{sing}\ \mathrm{Map}(|K| \times QX, Y) \\ &= \mathrm{Map}(|K| \times X, Y).\end{aligned}$$

By definition of the tensor, we have proved the claim. $\square$

Example 10. *Consider the category*

$$\mathcal{C} := \left\{ \bullet \longleftarrow \bullet \longrightarrow \bullet \right\} \tag{2.7.1}$$

This category is Reedy, with the two outside bullets in the subcategory + and the middle bullet in −. Let $F : \mathcal{C} \to \mathcal{T}\mathrm{op}$ be a diagram of shape $\mathcal{C}$ taking values in CW complexes. Equivalently, this is a cospan

$$A \longleftarrow B \longrightarrow C \tag{2.7.2}$$

Let $1 : \mathcal{C} \to s\mathcal{S}\mathrm{et}$ denote the constant functor which sends everything to the point $*$. A Reedy cofibrant replacement for 1 is the functor $I : \mathcal{C} \to s\mathcal{S}\mathrm{et}$ which sends the outside bullets to $\Delta[0]$, the inside bullet to $\Delta[1]$ and the two maps to the face maps $\Delta[1] \to \Delta[0]$. Now we can calculate the homotopy colimit

$$hocolim(F) = \int^{\mathcal{C}} F \odot I \tag{2.7.3}$$

Explicitly, we can write this coend as the pushout

$$\begin{tikzcd} B \odot \Delta[1] \arrow[r] \arrow[d] & C \odot \Delta[0] \arrow[d] \\ A \odot \Delta[0] \arrow[r] & P \end{tikzcd} \tag{2.7.4}$$

Since the tensor over simplicial sets is given by the product with the geometric realization, this is equivalently the pushout

$$\begin{tikzcd} B \times \Delta^1 \arrow[r] \arrow[d] & C \times * \arrow[d] \\ A \times * \arrow[r] & P \end{tikzcd} \tag{2.7.5}$$

where Δ^1 is the topological 1 simplex. Hence we get the familiar definition of the homotopy pushout.

The next example shows how powerful the use of homotopy colimits can be. Here we assume some facts about homotopy limits and colimits not discussed here. We refer the reader to [21] for the details.

Example 11. *Consider the adjunction*

$$s\mathcal{S}\mathrm{et} \underset{N}{\overset{\Pi_1}{\rightleftarrows}} \mathcal{G}\mathrm{pd} \tag{2.7.6}$$

described in example 2.4. Post composing with the Quillen equivalence

$$\mathcal{T}\mathrm{op} \underset{|\cdot|}{\overset{\mathrm{sing}}{\rightleftarrows}} s\mathcal{S}\mathrm{et} \tag{2.7.7}$$

gives the left Quillen functor

$$\Pi_1 \mathrm{sing} : \mathcal{T}\mathrm{op} \to \mathcal{G}\mathrm{rpd} \ ,$$

which assigns to each topological space, its fundamental groupoid. Since a Quillen adjunction induces a Quillen adjunction on undercategories of the terminal object [17], we can pass to based spaces to get a left Quillen adjoint

$$\Pi_1 : \mathcal{T}\mathrm{op}_+ \to \mathcal{G}\mathrm{rpd}_+ \ ,$$

where we have omitted sing *for simplicity. Now the terminal object in* $\mathcal{G}\mathrm{rpd}$ *is the trivial groupoid* $\Delta[0]$ *with one object and identity morphism. Let* $\mathcal{G}$ *be a groupoid. Then a functor* $F : \Delta[0] \to \mathcal{G}$ *specifies an automorphism group* $Aut(g_0)$ *in* $\mathcal{G}$*, where* F *sends the unique object in* $\Delta[0]$ *to* g_0*.* F *sends the identity morphism in* $\Delta[0]$ *to the identity in* $Aut(g_0)$*. It is easy to see that this assignment is functorial and gives a functor*

$$Aut_+ : \mathcal{G}\mathrm{rpd}_+ \to \mathcal{G}\mathrm{rp} \ .$$

which is right adjoint to the delooping functor

$$B : \mathcal{G}\mathrm{rp} \to \mathcal{G}\mathrm{rpd}$$

which sends a group to its 1-object groupoid. In fact, the unit of this adjunction $\eta : \mathrm{id} \to Aut_+ \circ B$ *is the identity, while the counit* $\epsilon : B \circ Aut_+ \to \mathrm{id}$ *is a categorical equivalence. Now we can define the composite functor*

$$\pi_1 := Aut_+ \Pi_1 : \mathcal{T}\mathrm{op} \to \mathcal{G}\mathrm{rp} \; . \tag{2.7.8}$$

Notice what the functor (2.7.8) *is doing. It takes a based topological space* (X, x_0)*, and maps out of all based simplices giving a based simplicial set which in degree* n *is*

$$\hom(\Delta^n, X) \; .$$

Now Π_1 *truncates this simplicial set in degree 1, giving the groupoid*

$$\hom(\Delta^1, X) \rightrightarrows \hom(\Delta^0, X) \tag{2.7.9}$$

Notice that morphisms in this groupoid are paths in X *which start at* x_0*. Finally, taking the automorphism group at* x_0 *gives the fundamental group* $\pi_1(X, x_0)$*.*

So far, this has nothing to do with homotopy colimits. However, let us consider the Seifert-Van Kampen theorem, which states that for a space X *arising as the union of two open subspaces* U_1 *and* U_2 *(both containing* x_0*) such that* $U_1 \cap U_2$ *is path connected the fundamental group* $\pi_1(X, x_0)$ *is given by the pushout*

$$\pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \; .$$

In fact, the statement of this theorem is a statement about homotopy colimits. The conditions *on the space X ensure that X arises as a homotopy pushout*

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] & U_1 \arrow[d] \\ U_2 \arrow[r] & X \end{tikzcd} \tag{2.7.10}$$

Since both sing *and* Π_1 *are left Quillen, they preserve homotopy pushouts. We want to calculate the homotopy pushout in groupoids*

$$\begin{tikzcd} \Pi_1(U_1 \cap U_2) \arrow[r] \arrow[d] & \Pi_1(U_1) \arrow[d] \\ \Pi_1(U_2) \arrow[r] & P \end{tikzcd} \tag{2.7.11}$$

By virtue of the model structure on groupoids, we have that this diagram is weakly equivalent to the homotopy pushout diagram

$$\begin{tikzcd} B\pi_1(U_1 \cap U_2) \arrow[r] \arrow[d] & B\pi_1(U_1) \arrow[d] \\ B\pi_1(U_2) \arrow[r] & P \end{tikzcd} \tag{2.7.12}$$

Since the top and left maps are injective on objects, they are cofibrations. This implies that the homotopy pushout is presented by the strict pushout of this diagram in groupoids. This pushout is exactly the delooping of the pushout

$$\pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

in groups.

The next example will be of particular interest to us when dealing with higher stacks.

Example 12. *Let* $\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$ *denote the category whose objects are* $\mathbb{R}^n$ *for* $n \in \mathbb{N}$ *and whose morphisms are smooth maps. We can view a manifold* X *as an object in the functor category* $[\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}^{op}, s\mathcal{S}\mathrm{et}]$ *by assigning each test space* $\mathbb{R}^n$ *to the set*

$$\mathbb{R}^n \to C^\infty(\mathbb{R}^n, X)$$

and then viewing this set as a discrete simplicial set. Let $\mathcal{U} = \{U_i\}_{i \in I}$ *be a good open cover of* X *and let* $U_{i_1 i_2 \dots i_n}$ *denote the* n*-fold intersections of the* U_i'*s. Consider the simplicial diagram*

$$C(\mathcal{U}) = \left\{ \quad \dots \Rrightarrow \coprod_{i,j,k} y(U_{ijk}) \Rrightarrow \coprod_{i,j} y(U_{ij}) \rightrightarrows \coprod_i y(U_i) \quad \right\} \tag{2.7.13}$$

where y *denotes the Yoneda embedding and we identify* U_i *with the copy of* $\mathbb{R}^n$ *that it is diffeomorphic to. Here we have omitted the degeneracy maps for simplicity. The face maps are given by inclusion of subsets. Let us calculate the homotopy colimit over this diagram. Using the formula, we have*

$$\mathrm{hocolim} C(\mathcal{U}) = \int^{[n] \in \Delta} \coprod_{i_1 \dots i_n} y(U_{i_1 \dots i_n}) \odot \Delta[n] \tag{2.7.14}$$

2.8 LOCALIZATION

In this section we describe the process of localizing a model category by formally extending the class of weak equivalences. Although this definition seems quite simple at first, we will see many interesting interactions between the original model structure and that of the local model structure. We begin with the basic definition and properties of Bousfield localization.

2.8.1 Bousfield localization

Definition 36. *A* left Bousfield localization *of a model category* $(\mathcal{C}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ *is another model structure* $(\mathcal{C}, \mathcal{W}', \mathcal{C}', \mathcal{F}')$ *such that the subcategories* $\mathcal{W}^{prime}$ *and* $\mathcal{C}'$ *obey*

1. $\mathcal{C}' = \mathcal{C}$

2. $\mathcal{W} \subset \mathcal{W}'$.

Remark 8. *Since we will typically be dealing with categories with multiple model structures, we will often make the local model structure explicit, using the notation* $\mathcal{C}_{loc}$, $\mathcal{W}_{loc}$, $\mathcal{C}_{loc}$ *and* $\mathcal{F}_{loc}$ *for the category, weak equivalences, cofibrations and fibrations (respectively).*

Proposition 37. *Let* $\mathcal{C}$ *be a model category and let* $\mathcal{C}_{loc}$ *be a Bousfield localization. Then we have the following interactions between model structures.*

1. $\mathcal{F}_{loc} \subset \mathcal{F}$

2. $\mathcal{C}_{loc} = \mathcal{C}$

3. $\mathcal{W}_{loc} \supset \mathcal{W}$

4. $\mathcal{F}_{loc} \cap \mathcal{W}_{loc} = \mathcal{F} \cap \mathcal{W}$

Proof. Use the characterization of cofibrations and fibrations by lifting and the definition. □

The following shows that the two model structures are closely related.

Proposition 38. *The adjunction*

$$\mathcal{C}_{loc} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \mathcal{C}$$

is a Quillen adjunction.

Proof. By definition, the identity $id : \mathcal{C}_{loc} \to \mathcal{C}$ preserves fibrations and acyclic cofibrations and is therefore is a right Quillen adjoint. □

2.8.2 In combinatorial simplicial model categories

For particularly nice model categories, Bousfield localizations always exist and are characterized by a choice of subclass of cofibrations satisfying certain properties. We now want to investigate this characterization.

Definition 39. *Let $\mathcal{C}$ be a combinatorial simplicial model category. Fix a subclass $S \subset$ $\mathrm{Mor}(\mathcal{C})$ of morphisms with cofibrant domain.*

1. *We call a fibrant object $X \in \mathcal{C}$ an S-*local object* if for every $f : A \to B$ in S, the map*

   $$f^*\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

   is an acyclic Kan fibration.

2. *We call a cofibration f an S-*local weak equivalence* if for every S local object X, the morphism*

   $$f^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

   is a weak equivalence of Kan complexes.

One should think of the class S as being weak equivalences from the point of view of the *enrichment*. Although the previous definition works only for *fibrant* objects, we can extend the definition to all objects by requiring that the fibrant replacement satisfies the above conditions.

Definition 40. *Let S be a subclass of morphisms as above and let $\mathcal{C}_{loc}$ be a left Bousfield localization. We call this Bousfield localization an S-*localization* if the acyclic cofibrations are precisely the S-local weak equivalences and the fibrant objects are the S-local objects.*

Remark 9. *In the previous definition we can drop the second assumption on the S-local objects as it follows from the assumption on S-local weak equivalences [21].*

The following theorem will be essential in the definition of higher stacks. This theorem, along with its proof, appears in [21] as proposition A.3.7.3.

Theorem 41. *Let $\mathcal{C}$ be a combinatorial simplicial model category. Then if S is any set of morphisms with cofibrant domains, an S-localization exists. That is, there is a left Bousfield localization $\mathcal{C}_{loc}$ with the following properties:*

1. *Acyclic $\mathcal{C}$ cofibrations are precisely the S-local weak equivalences.*

2. *The fibrant objects are precisely the S-local objects.*

3.0 HIGHER STACKS

Before discussing the general theory of higher stacks, we will provide a bit of motivation from sheaf theory. Recall that a *presheaf* is nothing but a functor $\mathcal{F} \in [\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]$, where $\mathcal{C}$ is some small category. Usual $\mathcal{C}$ describes the local data for some object. For example, if X is a manifold, we could take $\mathcal{C} = \mathcal{O}\mathrm{pen}(X)$: the category of open sets on X with inclusion of open sets as morphisms. We could also take $\mathcal{C}$ to be the local data for *any* manifold. That is $\mathcal{C}$ has objects $\mathbb{R}^n$ and smooth maps as morphisms. The essential characteristic of a *sheaf* is that it glues together nicely with respect to certain local data. This data is described by the notion of covering.

Definition 42. *Let $\mathcal{C}$ be a small category. A* covering *is a set of morphisms $\{U_i \to U\}_{i \in I}$, where U is some fixed object in $\mathcal{C}$. A* coverage *on $\mathcal{C}$ is an assignment of such coverings to each object $U \in \mathcal{C}$.*

The notion of coverage provides the general framework to choose the local data we wish to glue along. However, in order for a coverage to behave well, we need to impose a few extra conditions.

Definition 43. *Let $\mathcal{C}$ be a small category. We call $\mathcal{C}$ a* site *if it comes equipped with a coverage satisfying the following:*

1. *Every isomorphism f gives a covering family with a single element $\{f\}$.*
2. *Given two coverings $\{U_i \to U\}_{i \in I}$ and $\{U_{ij} \to U_i\}_{j \in J}$, the set of composites $\{U_{ij} \to U\}_{i \in I, j \in J}$ is a covering.*

3. Let $\{U_i \to U\}_{i\in I}$ be a covering and and $V \to U$ be a morphism. Then the pullback $U_i \times_U V$ exists for each $i \in I$ and $\{U_i \times_U V\}_{i\in I}$ is a covering.

Example 13. *Let X be a topological manifold and let $\mathcal{O}\mathrm{pen}(X)$ be the category of open sets on X with inclusions as morphisms. then $\mathcal{O}\mathrm{pen}(X)$ be comes a site with coverage $\{U_i \to U : \bigcup U_i = U\}$.*

Example 14. *Let X be a smooth manifold. Then $\mathcal{O}\mathrm{pen}(X)$ becomes a site with coverage $\{U_i \to U : \bigcup U_i = U,$ contractible finite intersections$\}$*

Example 15. *Let $\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$ be the category with objects $\mathbb{R}^n$, $n \in \mathbb{N}$ and morphisms smooth maps. Then $\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$ becomes a site with coverage*
$\{U_i \to U : \bigcup U_i = U,$ contractible finite intersections$\}$.

With the definition of a site in hand, it is quite easy to define the gluing condition for a sheaf. We say that a presheaf $\mathcal{F} \in [\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]$, on a site $\mathcal{C}$, is a sheaf if its value at U is an equalizer

$$\mathcal{F}(U) \simeq \lim \left\{ \prod_i \mathcal{F}(U_i) \underset{\langle r_j \rangle}{\overset{\langle r_i \rangle}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right\} . \tag{3.0.1}$$

Notice that the condition that this diagram be an equalizer is equivalent to the requirement that whenever the restrictions of two elements in $\mathcal{F}(U_i)$ agree on intersections, there is a unique element in $\mathcal{F}(U)$ whose restriction to each U_i is the element in $F(U_i)$. One should think of the U_i is giving the local data which glues to give data on U.

We would now like to describe classical stacks. These objects are similar to sheaves, but they take values in *groupoids*: small categories in which every morphism is invertible. At first, such generality may seem intimidating. However, there are many canonical examples of well known geometric objects with defining data equivalent to specifying a stack. Indeed, even in the simple definition of a manifold the concept of a stack is implicit! Now our prestacks take the form of functors $\mathcal{F} \in [\mathcal{C}^{op}, \mathcal{G}\mathrm{pd}]$ where $\mathcal{G}\mathrm{pd}$ is the category of groupoids. If $\mathcal{C}$ is a site, we define a similar gluing condition, but this time we want to do so in a way which respects *equivalences* of groupoids and not just isomorphisms. One should really think about these equivalences as being like homotopy equivalences. In fact, the homotopy limit provides a natural way to define our new gluing condition. We say the a prestack $\mathcal{F}$ sf a *stack* if we have an equivalence of groupoids

$$\mathcal{F}(U) \simeq \operatorname{holim} \left\{ \prod_i \mathcal{F}(U_i) \underset{\langle r_j \rangle}{\overset{\langle r_i \rangle}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right\} \tag{3.0.2}$$

Notice that this diagram is Reedy, so we can calculate the homotopy limit explicitly, although we will not do this. We reserve the calculation for the general case in higher stacks.

Remark 10. *Note that we are assuming a particularly nice model structure on groupoids. We will not go into detail on this structure since we are using the calculation purely for the purpose of motivation. A detailed discussion can be found in [3]*

3.1 SIMPLICIAL SHEAVES

We want to generalize the previous discussion to include all higher stacks. The theory of ∞ categories says that one can think about higher categories as simplicial sets and functors between them as morphisms of simplicial sets. It is therefore of no surprise that we turn our attention to presheaves with values in $s\mathcal{S}\mathrm{et}$. For a small category $\mathcal{C}$, we define the category of *simplicial presheaves* on $\mathcal{C}$ to be the functor category $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]$.

Remark 11. *Note that since* $s\mathcal{S}\mathrm{et}$ *is a combinatorial simplicial model category, we can endow the category of presheaves with the projective (or injective) model structure.*

Now there is an natural equivalence of categories $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}] \simeq [\Delta^{op}, [\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]]$. Indeed, using the Cartesian closed monoidal structure on $\mathcal{C}\mathrm{at}$, we have

$$[\mathcal{C}^{op}, [\Delta^{op}, \mathcal{S}\mathrm{et}]] \simeq [\mathcal{C}^{op} \times \Delta^{op}, \mathcal{S}\mathrm{et}] \simeq [\Delta^{op}, [\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]].$$

In particular, we view a presheaf $\mathcal{F} \in [\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]$ as a simplicial presheaf by regarding it as a simplicial diagram with trivial higher degrees. Using this equivalence, there is also a natural way to embed a covering $\{U_i \to U\}_{i \in I}$ into simplicial presheaves. Indeed, let y denote the Yoneda embedding and suppose we have such a covering of U. Then we can form the simplicial object

$$\ldots \substack{\to\\\to\\\to\\\to} \coprod_{i,j,k} y(U_i) \times_{y(U)} \times y(U_j) \times_{y(U)} \times y(U_k) \substack{\to\\\to\\\to} \coprod_{i,j} y(U_i) \times_{y(U)} \times y(U_j) \rightrightarrows \coprod_i y(U_i) \tag{3.1.1}$$

where we have suppressed the degeneracy maps for simplicity. These simplicial objects will be important in defining the sheaf condition for simplicial sheaves. We call this simplicial object the Čech nerve of the covering $\{U_i \to U\}_{i \in I}$ and denote it $C(\{U_i\})$. Notice that since $\{U_i \to U\}_{i \in I}$ is a cover, we have a natural map

$$C(\{U_i\}) \to y(U) \ .$$

In general, these objects may not be a degree-wise coproduct of representables. However, we will always assume this is the case as the next example will be of central interest to us.

Example 16. *Let $\mathcal{C} = \mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$ be the site of Cartesian spaces (see example 15). Let $\{U_i \to U\}_{i \in I}$ be a covering and let y denote the Yoneda embedding. Then it is easy to verify directly that we have an isomorphism*

$$y(U_i) \times_{y(U)} y(U_j) \simeq y(U_{ij}).$$

where $U_{ij} := U_i \cap U_j$ denotes the double intersections. Hence, for $\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$, all Čech nerves are degree-wise representable presheaves, with degree n component

$$\coprod_{i_1 i_2 \ldots i_n} y(U_{i_1 i_2 \ldots i_n}) \ :$$

The coproduct of n-fold intersections.

For the purposes of calculation, we will often need to replace the Čech nerve by an object which is weak equivalent, but slightly larger. For the proof of the following proposition, we refer the reader to [10].

Proposition 44. *Let $\{U_i \to U\}_{i\in I}$ be a covering and let $C(\{U_i\})$ be the Čech nerve, regarded as a simplicial object in $[\mathcal{C}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj}}$ by embedding the presheaves in each degree into simplicial presheaves. Then*

$$\mathrm{hocolim} C(\{U_i\}) \simeq \int^{[n]\in\Delta^{op}} \coprod_{i_1\dots i_n} y(U_{i_1\dots i_n})\cdot\Delta[n]$$

is cofibrant and weak equivalent to $C(\{U_i\})$, regarded as a simplicial presheaf.

Since $s\mathcal{S}\mathrm{et}$ is a combinatorial and simplicial model category, it follows from proposition 26 that the projective model structure on simplicial presheaves $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{proj}$ is also combinatorial and simplicial. We can therefore perform Bousfield localization at a set of morphisms S. In particular, we define the following *local* model structure on $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj}}$.

Definition 45. *Let S be the set of morphisms of the form*

$$\mathrm{holim}\ C(\mathcal{U}_i) \to y(U).$$

We define the local *model structure on $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj}}$ to be the Bousfield localization at S and denote this model category $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}}$.*

We are now ready to describe the gluing condition for higher stacks.

Definition 46. *We say that a simplicial presheaf $\mathcal{F}$ satisfies* descent *if we have a weak equivalence*

$$\mathcal{F}(U) \simeq \mathrm{holim}\left\{\ \dots \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod_{i,j,k}\mathcal{F}(U_{ijk}) \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \prod_{i,j}\mathcal{F}(U_{ij}) \ \substack{\longrightarrow\\[-1ex]\longrightarrow} \prod_i \mathcal{F}(U_i)\ \right\} \tag{3.1.2}$$

Proposition 47. *Let $\mathcal{F}$ be a locally fibrant simplicial presheaf (fibrant in the local model structure). Then we have the following:*

1. *There is a weak equivalence of simplicial sets*

$$\mathrm{Map}(C(\{U_i\}), \mathcal{F}) \simeq \mathrm{Map}(y(U), \mathcal{F}).$$

2. *$\mathcal{F}$ satisfies descent.*

Proof. Suppose $\mathcal{F}$ is fibrant. By definition, every map

$$\text{hocolim } C(\{U\}_i) \to y(U)$$

is a local weak equivalence. Since $\{\text{hocolim } C(\{U_i\})\}$, $C(\{U_i\})$ and $y(U)$ are locally weak equivalent and cofibrant, and the functor $\mathrm{Map}(-, \mathcal{F})$ is left Quillen for fibrant $\mathcal{F}$, we have a weak equivalence $\mathrm{Map}(\text{hocolim } C(\{U\}_i), \mathcal{F}) \simeq \mathrm{Map}(C(\{U\}_i), \mathcal{F})$ of Kan complexes. Combining this with the local weak equivalence hocolim $C(\{U_i\}) \simeq y(U)$, we get weak equivalences

$$\mathrm{Map}(y(U), \mathcal{F}) \to \mathrm{Map}(\text{hocolim } C(\{U\}_i), \mathcal{F}) \simeq \mathrm{Map}(C(\{U\}_i), \mathcal{F})\ .$$

This gives 1. To see 2., observe that

$$\mathrm{Map}(\text{hocolim } C(\{U\}_i), \mathcal{F}) \simeq \text{hocolim } \mathrm{Map}(C(\{U\}_i), \mathcal{F}) \simeq \text{hocolim } \prod_i \mathcal{F}(U_i)$$

where the last homotopy limit is shorthand for the homotopy limit in definition 46. □

Let $\mathcal{F}$ be a simplicial presheaf. We call $\mathcal{F}$ a *higher stack*, *simplicial sheaf* or simply *stack* if it satisfies descent. By the proposition, every locally fibrant object is a stack.

Just as with classical sheaves, there is a "stackification" functor which turns any prestack into a stack. The functor is quite easy to describe in the language of model categories. Using the technology of ∞ categories, one can show that L preserves homotopy colimits and finite homotopy limits [21]. In this language, one would say that we have an adjunction of ∞ categories

$$[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}} \underset{i}{\overset{L}{\rightleftarrows}} [\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]^{f}_{\mathrm{proj,loc}} \tag{3.1.3}$$

where $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]^{f}_{\mathrm{proj,loc}}$ denotes the full subcategory on fibrant objects (although we will not need this perspective).

3.2 SMOOTH HIGHER STACKS AND COHOMOLOGY

We now restrict our attention to stacks on the site $\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$ defined in example 15. These stacks provide a natural home for differentiable structures (e.g. manifolds). In fact, every manifold naturally gives a presheaf by assigning to each test object $U \in \mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}$ the set of plots $C^{\infty}(U, X)$. To check that this is a sheaf, we observe that the diagram

$$C^{\infty}(U,X) \simeq \lim \left\{ \prod_i C^{\infty}(U_i, X) \underset{\langle r_j \rangle}{\overset{\langle r_i \rangle}{\rightrightarrows}} \prod_{i,j} C^{\infty}(U_i \cap U_j, X) \right\} \tag{3.2.1}$$

is an equalizer by definition of a manifold. That is two local plots which agree on intersections can be glued together. We can then embed this sheaf into higher stacks as a discrete stack.

Remark 12. *We will typically denote the above stack by X, identifying it with the manifold it represents. When further clarification is needed, we will make the definition manifest.*

The following proposition will be used frequently in computation.

Proposition 48. *Let X be a paracompact manifold in $[\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}^{op}, s\mathcal{S}\mathrm{et}]^{f}_{\mathrm{proj,loc}}$. Then if $\{U_i \to X\}_{i \in I}$ is a good open cover of X, the Čech nerve* hocolim $C(\{U_i\})$ *is weak equivalent to X.*

Proof. First observe that we have a natural map

$$\dots \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \coprod_{i,j,k} y(U_{ijk}) \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \coprod_{i,j} y(U_{ij}) \rightrightarrows \coprod_i y(U_i) \longrightarrow X \tag{3.2.2}$$

Induced by the covering $\{U_i \to X\}_{i\in I}$. This induces a map (unique up to equivalence) $\mathrm{hocolim} C(\{U_i\}) \to X$. We claim that this map is a weak equivalence. Since X is a stack, it suffices to prove that the map is an object-wise weak equivalence (Since the stackification functor sends precisely the weak equivalences to object-wise weak equivalence). But this follows immediately from the decent condition for X. □

With the previous proposition in hand, it is now quite easy to define cohomology in stacks.

Definition 49. *Let X be a paracompact manifold and let $\mathcal{F}$ be a stack. We define the cohomology of X with coefficients in $\mathcal{F}$ to be the set*

$$\mathrm{Ho}[\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}}(X, \mathcal{F}) \simeq \pi_0 \mathrm{Map}(\mathrm{hocolim} C(\{U_i\}), \mathcal{F})$$

where $\{U_i \to X\}_{i\in I}$ is any good open cover of X.

Notice that in the previous definition, the natural isomorphism is really the definition of the set of morphisms in the homotopy category. In fact, $\mathrm{holim} C(\{U_i\})$ is a cofibrant replacement of X in the local model structure [13], [10]. It may be difficult to reconcile this definition with that of sheaf cohomology. However, we will see that these two definitions really consist of the same idea.

Example 17. *Let G be a lie group. Consider the stack which assigns to each test space U the groupoid*

$$\mathbb{B}G := \quad C^\infty(U, G) \rightrightarrows * \tag{3.2.3}$$

where the vertical arrows on the left represent the face inclusions and those on the right represent restrictions. We claim that this stack represents the moduli stack of principal G bundles. Indeed, let X be a manifold. We want to calculate the set of vertices of the mapping space $\mathrm{Map}(X, \mathbb{B}G)$*. To calculate this set, we first take the cofibrant replacement* $\mathrm{hocolim} C(\{U_i\})$ *of X. Then using the Bousfield-Kan formula for the homotopy colimit, we* *have*

$$\begin{aligned}
\hom(\mathrm{hocolim} C(\{U_i\}), \mathbb{B}G) &\simeq \hom\left(\int^{[n]\in\Delta} \coprod_{i_1,\dots,i_n} U_{i_1\dots i_n} \times \Delta[n], \mathbb{B}G\right) \\
&\simeq \int_{[n]\in\Delta} \prod_{i_1,\dots,i_n} \hom(U_{i_1\dots i_n} \times \Delta[n], \mathbb{B}G) \\
&\simeq \int_{[n]\in\Delta} \prod_{i_1,\dots,i_n} \hom(\Delta[n], \mathbb{B}G(U_{i_1\dots i_n}))
\end{aligned}$$

Since there are no nondegenerate simplices in $\mathbb{B}G$ of degree $n > 1$, the end in the last line can be represented as a commuting diagram

$$\begin{tikzcd}
\Delta[1] \arrow[r] & \prod_{ij} \mathbb{B}G(U_i \cap U_j) \\
\Delta[0] \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & \prod_i \mathbb{B}G(U_i) \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd} \tag{3.2.4}$$

Since the 0 *simplices in $\mathbb{B}G$ are trivial the data contained in this diagram is simply a choice of smooth G valued function on intersections. That is, the transition functions for a bundle on X.*

In the previous example, the stack $\mathbb{B}G$ provides the data for locally trivial *smooth G* bundles on a manifold. This should be distinguished from what the classifying space BG represents. That is, topological G bundles. The next example is one which has no analogue in spaces.

Example 18. *Let G be a lie group. Consider the stack which assigns to each test space U the groupoid*

$$\mathbb{B}G_{\mathrm{conn}} := \quad C^\infty(U, G) \rightrightarrows \Omega^1(U, \mathfrak{g}) \tag{3.2.5}$$

where $\Omega^1(U, \mathfrak{g})$ are lie algebra valued 1-forms. The morphisms in this groupoid are given by *gauge transformations. That is, for a 1-form A, the transformation*

$$A \to gAg^{-1} + g^{-1}dg$$

$g \in C^\infty(U, G)$. We claim that this stack represents the moduli stack of principal G bundles with connection. Indeed, we calculate the set of vertices as in the previous example using the Bousfield-Kan formula for the homotopy colimit, we have

$$\begin{aligned}
\hom(\mathrm{hocolim} C(\{U_i\}), \mathbb{B}G) &\simeq \hom\left(\int^{[n]\in\Delta} \coprod_{i_1,\ldots,i_n} U_{i_1\ldots i_n} \times \Delta[n], \mathbb{B}G_{\mathrm{conn}}\right) \\
&\simeq \int_{[n]\in\Delta} \prod_{i_1,\ldots,i_n} \hom(U_{i_1\ldots i_n} \times \Delta[n], \mathbb{B}G_{\mathrm{conn}}) \\
&\simeq \int_{[n]\in\Delta} \prod_{i_1,\ldots,i_n} \hom(\Delta[n], \mathbb{B}G_{\mathrm{conn}}(U_{i_1\ldots i_n}))
\end{aligned}$$

Since there are no nondegenerate simplices in $\mathbb{B}G_{\mathrm{conn}}$ of degree $n > 1$, the end in the last line can be represented as a commuting diagram

$$\begin{tikzcd}
\Delta[1] \arrow[r] & \prod_{ij} \mathbb{B}G_{\mathrm{conn}}(U_i \cap U_j) \,. \\
\Delta[0] \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & \prod_i \mathbb{B}G_{\mathrm{conn}}(U_i) \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd} \tag{3.2.6}$$

This gives exactly the data which assigns each open set a lie algebra valued form A_i such that when two such forms are restricted to intersections, they differ by a gauge transformation

$$A_i - A_j = g_{ij}^{-1} dg_{ij}$$

for some $g_{ij} \in C^\infty(U_i \cap U_j, G)$. This is exactly the data required to define a smooth G bundle with connection.

Notice that in each of the previous examples, there was no higher simplicial data beyond

$n = 1$. In order to construct stacks with nontrivial data beyond this point, we will use the Dold-Kan correspondence discussed in the section 4.0. The stacks which result from this process will be of particular interest to us. Before discussing these types of stacks we will need a few properties which are specific to *smooth stacks*.

3.3 COHESION

For the category of smooth stacks $[\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}}$, there are several adjoint functors which will be of interest to us. Categories of stacks which exhibit such adjoints are called *cohesive* and have particularly nice properties [31].

Definition 50. *Let $\mathcal{C}$ be a site. The category $[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}}$ is called* cohesive *if there exists Quillen adjoints*

$$[\mathcal{C}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}} \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} s\mathcal{S}\mathrm{et} \tag{3.3.1}$$

where, reading from top to bottom, we have $(\Pi \vdash \mathrm{disc} \vdash \Gamma \vdash \mathrm{codisc})$*: Γ is the "global sections" functor which evaluates on the terminal object $*$,* disc *is the discrete functor which assigns a simplicial set X to the locally constant stack assigning each element of a cover U_i to X,* Π *takes the colimit over $\mathcal{C}$ and* codisc *is simply defined as the right adjoint of* Γ.

Notice that if $\mathcal{C}$ has a terminal object, it follows that Γ is simply the limit operation and hence const is the left adjoint. In this case one can easily show that the adjunction is in fact Quillen [31]. The existence of the other adjoints depends on more technical properties of $\mathcal{C}$ and we refer the reader to [31] for the details. We simply note the following:

Proposition 51. *The category $[\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}}$ is cohesive.*

Recall that we have previously defined cohomology of a manifold X with coefficients in a stack $\mathcal{F}$ as the set of morphisms

$$\pi_0 \mathrm{Map}(C(\{U_i\}), \mathcal{F})$$

where $\{U_i \to X\}_{i\in I}$ is some good open cover of X. Usually one defines sheaf cohomology by taking some resolution and applying the global sections functor. In fact, the two definitions are related. Let $\mathcal{G}$ and $\mathcal{F}$ be stacks and let $[\mathcal{G}, \mathcal{F}]$ denote the internal hom in stacks given by the prestack which assigns to each test object the simplicial set

$$U \to \mathrm{Map}(\mathcal{G} \times y(U), \mathcal{F}). \tag{3.3.2}$$

In particular, if $\mathcal{G} = C(\{U_i\})$, then the internal hom $[C(\{U_i\}), \mathcal{F}]$ is nothing but the Čech resolution of $\mathcal{F}$. Then applying the global sections functor and taking connected components gives

$$\pi_0\Gamma[C(\{U_i\}), \mathcal{F}] = \pi_0\mathrm{Map}(C(\{U_i\} \times *, \mathcal{F}) \simeq \pi_0\mathrm{Map}(C(\{U_i\} \times *, \mathcal{F})$$

and we recover our previous definition.

4.0 THE DOLD-KAN CORRESPONDENCE

The Dold-Kan correspondence provides a beautiful link between chain complexes and spaces. Surprisingly, much of the structure is preserved by this correspondence, as we shall see. This correspondence first appeared in [9], but a more modern approach can be found in [14].

4.1 CLASSICAL DOLD-KAN

We begin by describing three functors $C : s\mathcal{A}\mathrm{b} \to \mathcal{C}\mathrm{h}^+$, $N : s\mathcal{A}\mathrm{b} \to \mathcal{C}\mathrm{h}^+$ and $DK : \mathcal{C}\mathrm{h}^+ \to s\mathcal{A}\mathrm{b}$.

Definition 52. *(Moore functor) We define C to be the functor which takes a simplicial abelian group $A_\bullet$ to the chain complex (A_*, ∂) which is degree-wise identical to $A_\bullet$ and has differential in degree n:*

$$\partial = \sum_{i=0}^{n} (-1)^i d_i \ ;$$

d_i is the i'th face map.

Notice that for a morphism f of simplicial abelian groups, the morphism $C(f)$ does indeed define a chain map, since the commutativity with the face and degeneracy maps will ensure commutativity with ∂. It also follows from the simplicial identities that $\partial^2 = 0$.

This functor is, in some ways, the most natural way to get a chain complex from a simplicial abelian group. Moreover, it has the pleasant property:

$$H_n A_* = \pi_n A_\bullet$$

by definition.

However, we will see that there is a slightly better, but related, complex which will provide us with an equivalence of categories.

Definition 53. *(Normalized Moore functor) Let $A_\bullet$ be a simplicial abelian group. The functor N takes $A_\bullet$ to the chain complex (A_*, d) which in degree n is the intersection of the kernel of the first $n-1$ face maps*

$$A_n = \bigcap_{i=1}^{n-1} ker(d_i)$$

the differential in degree n is the last face map d_n.

Using the simplicial identities, one can easily show that $d^2 = 0$. One may wonder how this chain complex is related to the Moore complex defined above. It is easy to show, again using the simplicial identities, that there is a subcomplex $D(A)_*$ of the Moore complex A_* which in degree n is given by

$$D(A)_n = \bigoplus_{i=1}^{n} s_i A_{n-1},$$

where s_i are the degeneracy maps. One then has an isomorphism $A_*/D(A)_* \simeq N(A)_*$, where $N(A)_*$ is the normalized Moore complex defined above.

Now we want to describe the functor in the opposite direction. Recall that a simplicial abelian group is a functor $A : \Delta \to \mathcal{A}\mathrm{b}$, where Δ is the category with objects ordered sets $\{0, 1, ..., n\}$ and morphisms order preserving maps. That means, given a chain complex A_*, the adjoint should be a functorial assignment $\Gamma(A_*) : \Delta \to \mathcal{A}\mathrm{b}$. We now describe this assignment.

Definition 54. *Let A_* be a chain complex. We define the functor*

$$\Gamma(A_*) : [k] \to \hom_{\mathcal{C}\mathrm{h}^+}(N(\mathbb{Z}(\Delta[k])), A_*)$$

where $\mathbb{Z}(\Delta[k])$ is the free simplicial abelian group on the simplicial set $\Delta[k]$ (Yoneda em-

bedding of $[k]$). We define Γ on morphisms to be the natural transformation given by post composing.

This formula looks quite complicated, however it has the advantage of being intuitively clear. If one starts with a category and one wants to replace an object by a combinatorial (simplicial) object in order to study it. One looks at maps from internal simplices into the object. In $\mathcal{T}$op, this is exactly the singular nerve of the topological space. One could say that the Dold-Kan functor Γ is just the manifestation of this idea in chain complexes. We now turn to a definition which is computationally much better.

Proposition 55. *The degree n component of the simplicial abelian group $\Gamma(A_*)$ is given by*

$$\Gamma(A_*)_n = \bigoplus_{[n]\twoheadrightarrow[k]} A_k.$$

Proof. Follows from an explicit calculation of $N(\mathbb{Z}(\Delta[n]))$ and the formula

$$\hom_{\mathcal{C}\mathrm{h}^+}(B_*, A_*)_k = \bigoplus_{i=1}^{\infty} \hom(B_i, A_{i+k})$$

for the degree k component of the internal hom. $\square$

It is a bit trickier to describe the face and degeneracy maps. Let $d^i : [n-1] \hookrightarrow [n]$ be a coface map in Δ. We want to define the corresponding face map. To get a map out of the direct sum, it suffices to describe the map on each factor. Therefore, we need only define the face map on a term C_k given by a surjection $\sigma : [n] \twoheadrightarrow [k]$. To see where to send this term, we form the composite $\sigma d^i [n-1] \hookrightarrow [n] \twoheadrightarrow [k]$. Now this morphism need not be surjective, so we factorize $\mu\sigma'[n-1] \twoheadrightarrow [m] \hookrightarrow [k]$ where the first map is a surjection and the second map is an injection. Then σ' corresponds to a term $C_m \hookrightarrow \bigoplus_{[n-1]\to[m]} A_m = \Gamma(C_\bullet)_{n-1}$. We send the factor C_k to the factor C_m by a map $\mu' : C_k \to C_m$. This map is given by

$$\mu' = \begin{cases} id & \mu = id \\ (-1)^k d & \mu = d^k \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.1}$$

A similar construction is used to define the degeneracy maps. The following example illustrates the point quite well:

Example 19. *Consider the chain complex $A[1]$, with the abelian group A in degree 1 and 0's in all other degrees. We want to compute $DK(A[1])$. Using the above formula, we see that the only nonzero terms in degree n are given by the surjections $[n] \twoheadrightarrow [1]$ Each surjection can be thought of as being given by an element $i \in [n]$ which divides the set into two subsets: those that go to 0 and those that go to 1. We therefore have n surjections and*

$$\Gamma(A[1])_n = \bigoplus_{i=1}^{n} A.$$

For a coface map $d^j : [n-1] \to [n]$, the corresponding face map d^j is given as follows: Let A_i denote the copy of A corresponding to the ith surjection. Then

$$d_j(A_i) = \begin{cases} A_{i-1} & \text{if } \ i > j \\ A_i & \text{if } \ i \leqslant j \end{cases}$$

if $j \neq 0, n$.

$$d_0(A_i) = \begin{cases} A_{i-1} & \text{if } \ i \neq 0 \\ 0 & \text{if } \ i = 0 \end{cases}$$

$$d_n(A_i) = \begin{cases} A_i & \text{if } \ i \neq n \\ 0 & \text{if } \ i = n \end{cases}$$

Notice that for $j \neq 0, n$, the term corresponding to $i = j$ and $i = j+1$ both go to the same *copy of A. We therefore have a map $A \times A \to A$ extending the identity on each component. Hence, this morphism is just group multiplication. From this, one can see that this simplicial abelian group is just the delooping group BA.*

This functor is indeed both a left and right adjoint to N and provides a categorical equivalence (see [18]).

Theorem 56. *(Dold-Kan) The functors Γ and N form an adjoint pair $(\Gamma \dashv N)$ exhibiting a categorical equivalence.*

4.2 AS A WEAK MONOIDAL QUILLEN EQUIVALENCE

It is natural to ask just how much structure the Dold-Kan equivalence preserves. In particular, there is a monoidal product on both chain complexes and simplicial abelian groups: tensor product of chain complexes and degree wise tensor product (resp.) One can ask whether or not monoids are preserved by the correspondence. It turns out that both functors Γ and N are lax monoidal, however the adjunction fails to be a monoidal adjunction, meaning that either the unit or counit, $\eta : 1 \Rightarrow \Gamma \circ N$ or $\epsilon : N \circ \Gamma \Rightarrow 1$ fails to be monoidal. Therefore, the adjunction does not descend to an adjunction on the full subcategories of monoids.

4.2.1 Shuffle and Alexander-Whitney maps

We now take a closer look at the natural transformations for the monoidal functors N and Γ.

Definition 57. *Let C denote the Moore functor defined above. The* shuffle *map is the natural transformation*

$$\nabla_{AB} : CA \otimes CB \to C(A \otimes B)$$

defined on elements by

$$\nabla(a \otimes b) = \sum_{(\mu,\nu)} sgn(\mu,\nu) \cdot s_\nu a \otimes s_\mu b$$

where the sum is taken over all (p,q) *shuffles: permutations of the set* $\{0,...,p+q-1\}$ *which preserve the order of the first p and last q elements. Shuffles are of the form* $(\mu,\nu) = (\mu_1,...,\mu_p,\nu_1,...,\nu_q)$ *with* $\mu_1 < \mu_2 < ... < \mu_p$ *and* $\nu_1 < ... < \nu_q$*. The operators* s_μ *and* s_ν *are given by the composition of degeneracies*

$$s_\mu = s_{\mu_1} s_{\mu_2} ... s_{\mu_p}$$

$$s_\nu = s_{\nu 1} s_{\nu_2} ... s_{\nu_q}$$

Since the definition clearly relies on the elements of the groups, it is not at all obvious that this map is natural. For the proof that this transformation is natural, associative and unital, see [24]. This transformation gives C a lax monoidal structure.

Definition 58. *The* Alexander-Whitney *map*

$$AW : C(A \otimes B) \to CA \otimes CB$$

goes in the opposite direction, and is given on elements by

$$AW(a \otimes b) = \bigoplus_{p+q=n} d_p a \otimes d_0 b$$

where d_p *is the "front-face" map* $A_{p+q} \to A_p$ *induced by the inclusion* $[p] \to [p+q]$ *which sends* $i \in [p]$ *to* $i \in [p+q]$ *and* d_0 *is the "back-face" map induced by the inclusion* $[q] \to [p+q]$ *which sends* $i \in [q]$ *to* $i + p \in [p+q]$*.*

Again one needs to check naturality, associativity and unitality. For the details, we again refer the reader to [24]. This transformation gives C an op-lax monoidal structure. We now turn to the monoidal structure on the adjoint inverse Γ. We can use the op-lax structure given by the AW map along with the the unit and counit $\eta : 1 \Rightarrow \Gamma \circ N$, $\epsilon : N \circ \Gamma \Rightarrow 1$ to get a lax monoidal structure on DK.

$$\begin{aligned} \Gamma(A) \otimes \Gamma(B) \xrightarrow{\eta_{\Gamma(A)\otimes\Gamma(B)}} \Gamma \circ N(\Gamma(A) \otimes \Gamma(B)) \xrightarrow{\Gamma(AW_{\Gamma(A),\Gamma(B)})} \\ \Gamma(N(\Gamma(A)) \otimes N(\Gamma(B))) \xrightarrow{\Gamma(\epsilon_A \otimes \epsilon_B)} \Gamma(A \otimes B) \end{aligned} \tag{4.2.1}$$

We now want to investigate whether or not the adjunction is a monoidal adjunction. We will see that the counit $\epsilon : N \circ \Gamma \Rightarrow 1$ is a monoidal transformation, but that the unit η is not. The following proposition can be found in [26]:

Proposition 59. *The counit ϵ of the Dold-Kan adjunction is a monoidal transformation. That is, the diagram*

$$\begin{tikzcd} \Gamma \circ N(A) \otimes \Gamma \circ N(B) \arrow[r, "\nabla_{\Gamma(A),\Gamma(B)}"] \arrow[drr, "\epsilon_A \otimes \epsilon_B"'] & N(\Gamma(A) \otimes \Gamma(B)) \arrow[r, "N(\varphi_{A,B})"] & \Gamma \circ N(A \otimes B) \arrow[d, "\epsilon_{A\otimes B}"] \\ & & A \otimes B \end{tikzcd} \tag{4.2.2}$$

commutes.

The unit η is *not* monoidal. Indeed, if it were, the diagram

$$\begin{tikzcd} A \otimes B \arrow[r, "\eta_A \otimes \eta B"] \arrow[drr, "\eta_{A\otimes B}"'] & \Gamma(N(A)) \otimes \Gamma(N(B)) \arrow[r, "\varphi_{NA,NB}"] & \Gamma(N(A) \otimes N(B)) \arrow[d, "\Gamma(\nabla_{A,B})"] \\ & & \Gamma \circ N(A \otimes B) \end{tikzcd} \tag{4.2.3}$$

would have to commute for every A and $B \in \mathcal{C}h^+$. Consider the simplicial abelian group $\Gamma(\mathbb{Z}[1])$. A quick calculation shows that $\mathbb{Z}[1] \otimes \mathbb{Z}[1] = \mathbb{Z}[2]$. Now in degree 1, the top composite map must be 0 since

$$\Gamma(N(\Gamma(\mathbb{Z}[1]))) \otimes N(\Gamma(\mathbb{Z}[1]))) \simeq \Gamma(\mathbb{Z}[1] \otimes \mathbb{Z}[1]) \simeq \Gamma(\mathbb{Z}[2])$$

is 0 in degree 1. However, the bottom map $\eta_{A\otimes B}$ cannot be 0 in degree 1 since it is an isomorphism of simplicial abelian groups (hence an iso at each level). It turns out however, that although the unit is not strictly monoidal, it is monoidal up to homotopy in some sense. We will describe this in detail in the next section.

4.2.2 Weak monoidal Quillen equivalences

We first introduce some very general theory and then specialize to the above case. This definition can be found in [26].

Definition 60. *Let $\mathcal{C}$ be a model category. Suppose $\mathcal{C}$ is closed symmetric monoidal with product $\otimes$ and unit 1. Suppose in addition, $\mathcal{C}$ satisfies the following:*

1. *(Pushout product axiom) Let $A \hookrightarrow B$ and $C \hookrightarrow D$ be cofibrations (acyclic cofibrations) in $\mathcal{C}$. Then the map*
$$A \otimes D \coprod_{A\otimes C} B \otimes C \to B \otimes D$$
is a cofibration (acyclic cofibration).

2. *(unit axiom) Let $q : 1^c \to 1$ be a cofibrant replacement of the unit object. Then for every cofibrant object A, the morphism $q \otimes id : 1^c \otimes A \to 1 \otimes A$ is a weak equivalence.*

Then we call $\mathcal{C}$ a monoidal model category.

This definition was taken from [26]. There, the author explains that it deviates slightly from the definition in [17] in the unit axiom and motivates the slight change. However, in our examples, the unit object will be cofibrant and both definitions are equivalent.

Remark 13. *Notice that the first statement simply says that $\otimes$ is a Quillen bifunctor. Hence, a monoidal model category is simply a monoidal category with a model structure such that the monoidal product $\otimes$ is a left Quillen bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is appropriately unital.*

Suppose $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories and $R : \mathcal{C} \to \mathcal{D}$ a lax monoidal functor. If R has a left adjoint λ, then λ inherits an op-lax monoidal structure in the following way: Let ν and $\varphi_{A,B}$ be the coherence maps in for the monoidal functor R. Then the maps

$$\nu : 1_{\mathcal{D}} \to R(\mathcal{C})$$

and

$$\varphi_{\lambda A, \lambda B} \circ \eta_A \otimes \eta_B : A \otimes B \to R\lambda A \otimes R\lambda B \to R(\lambda A \otimes \lambda B)$$

have adjoints

$$\bar{\nu} : \lambda(1_{\mathcal{D}}) \to 1_{\mathcal{C}}$$

and

$$\bar{\varphi}_{A,B} : \lambda(A \otimes B) \to \lambda(A) \otimes \lambda(B).$$

These maps define the op-lax structure for λ.

We are now ready to define a weak monoidal Quillen adjunction.

Definition 61. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal model categories. A Quillen adjoint pair $(\lambda \dashv R)$ is called a* weak monoidal Quillen *adjunction if R is lax monoidal and the induced op-lax coherence maps on λ obey the following:*

1. *For all cofibrant objects A and B, the map*

$$\bar{\varphi}_{A,B} : \lambda(A \otimes B) \to \lambda(A) \otimes \lambda(B)$$

is a weak equivalence.

2. *For some cofibrant replacement $q : 1_{\mathcal{D}}^c \to 1_{\mathcal{D}}$ of the unit object in $\mathcal{D}$, the composite*

$$\bar{\nu} \circ \lambda(q) : \lambda(1_{\mathcal{D}}^c) \to \lambda(1_{\mathcal{D}}) \to 1_{\mathcal{C}}$$

is a weak equivalence.

If the op-lax coherence maps are isomorphisms in the above definition, we say that the adjunction is a *strong* monoidal Quillen adjunction. A weak monoidal Quillen *equivalence* is just a weak monoidal Quillen adjunction which is, in particular, a Quillen equivalence. The following theorem is due to Stefan Schwede and Brooke Shipley [26].

Theorem 62. *Under both equivalences* $(N \vdash \Gamma)$ *and* $(\Gamma \vdash N)$*, the Dold-Kan correspondence exhibits a weak monoidal Quillen equivalence.*

In fact, in this case one can show that the weak equivalence op-lax structure map is a *homotopy* equivalence [24].

4.3 SHEAVES OF CHAIN COMPLEXES

In order to describe the Dold-Kan correspondence at the level of sheaves, we will need to work with sheaves of chain complexes. We therefore recall some basic definitions surrounding sheaves of chain complexes. These definitions are quite classical and can be found in [6], [5].

Let $\mathcal{C}$ be a small category. We define the categories of *abelian presheaves* and *presheaves of chain complexes* to be the functor categories $[\mathcal{C}^{op}, \mathcal{A}\mathrm{b}]$ and $[\mathcal{C}^{op}, \mathrm{ch}]$ respectively. One can think of these presheaf categories as the algebraic analogues of presheaves of sets and simplicial sets (respectively). We have a natural inclusion $[\mathcal{C}^{op}, \mathcal{A}\mathrm{b}] \hookrightarrow [\mathcal{C}^{op}, \mathrm{ch}]$ given by regarding a presheaf of abelian groups as a presheaf of chain complexes with 0's in all degrees but zero. In the case that $\mathcal{C}$ is a site, we can define a sheaf of abelian groups in the same way we define a sheaf of sets. We define a *sheaf of chain complexes* to be a presheaf of chain complexes which is a sheaf in each degree.

Definition 63. *Let* $A_\bullet$ *be a sheaf of unbounded chain complexes. An* injective *resolution of* $A_\bullet$ *is a presheaf of chain complexes* $I_\bullet$*, where each* I_n *is injective along with a quasi-isomorphism of chain complexes*

$$A^\bullet \xrightarrow{\sim} I^\bullet \twoheadrightarrow 0$$

We can define the global sections functor as in the case of simplicial sheaves. That is, the evaluation functor which takes $I^\bullet$ to its value at the terminal object X.

$$I_0(X) \to I_1(X) \to ...I_n(X) \to ...$$

Definition 64. *Let $A_\bullet$ be a sheaf of chain complexes and $I_\bullet$ be an injective resolution of $A_\bullet$. We define the degree n* hypercohomology *of an object U with coefficients in $A_\bullet$ to be the cohomology group*

$$H_n(X; A^\bullet) := ker(d_n)/im(d_{n-1})$$

where d_n is the nth differential of the chain complex

$$\Gamma(X, I_\bullet) = I_0(X) \to I_1(X) \to ...$$

Example 20. *Let $\mathcal{A}$ be an abelian sheaf. Then if we regard $\mathcal{A}$ as a sheaf of chain complexes concentrated at zero, the sheaf* hypercohomology *of $\mathcal{A}$ is naturally isomorphic to the sheaf cohomology. Indeed, an injective resolution of $\mathcal{A}$, regarded as sheaf of abelian groups, is the same as an injective resolution of the corresponding sheaf of chain complexes.*

It is typically difficult to find injective resolutions. Really, the only reason one defines sheaf cohomology using injectives is because one always exists. In practice, it is usually much easier to use *acyclic* resolutions.

Definition 65. *Let $A_\bullet$ be a sheaf of unbounded chain complexes. An* acyclic *resolution of $A_\bullet$ is a presheaf of chain complexes $C_\bullet$, such that each sheaf C_n has vanishing sheaf cohomology for $k > 0$.*

Proposition 66. *The hypercohomology of a sheaf of chain complexes can be computed using any acyclic resolution.*

There are various adjectives describing different types of sheaves: fine, flabby, soft, flasque, ect. Each type of sheaf can be helpful in calculating sheaf cohomology. For us, since we are concerned with smooth manifolds, sheaves whose endomorphism group admit a partition of unity are essential. These sheaves are called *fine* sheaves and they are acyclic [6].

Example 21. *Let X be a smooth manifold. Consider the site* $\mathcal{O}\mathrm{pen}(X)$ *equipped with coverage as in* (14)*. The sheaf of differential n-forms Ω^n on this site is fine. Indeed, the sheaf of smooth $\mathbb{R}$ valued functions acts on Ω^n on the left via $\omega \to f\omega$. Since a partition of unity exists in $C^\infty(U, \mathbb{R})$ for each U, the sheaf is fine by definition.*

Example 22. *It follows from the previous example, that the de Rham complex*

$$\Omega^0 \to \Omega^1 \to \ldots \to \Omega^n \to \ldots$$

is an acyclic resolution of the discrete sheaf $\mathrm{disc}(\mathbb{R})$ *if locally constant $\mathbb{R}$ valued functions. We therefore have de Rham's theorem*

$$H^*(X; \mathbb{R}) \simeq H^*_{dR}(X)$$

4.4 DOLD-KAN CORRESPONDENCE FOR SHEAVES

Having defined both sheaves of chain complexes and higher stacks, one could ask whether the classical Dold-Kan correspondence lifts to an adjunction at the level of presheaves. The following proposition show that this is indeed the case.

Proposition 67. *The Dold-Kan equivalence lifts to a Quillen equivalence of model categories:*

$$\Gamma : [\mathcal{C}^{op}; \mathcal{C}\mathrm{h}^+]_{\mathrm{proj}} \rightleftarrows [\mathcal{C}^{op}; s\mathcal{A}\mathrm{b}]_{\mathrm{proj}} : N \tag{4.4.1}$$

Proof. Recall that the classical Dold-Kan is a Quillen equivalence. Since N is a right Quillen functor and we are taking the projective model structure, it immediately follows that N takes fibrations (acyclic fibrations) to fibrations (acyclic fibrations). Let $X \rightarrowtail Y$ be a cofibration (acyclic cofibration) in $[\mathcal{C}^{op};\mathcal{C}\mathrm{h}^+]_{\mathrm{proj}}$. We want to show that the induced map $\Gamma X \to \Gamma Y$ is a cofibration (acyclic cofibration). Testing with a fibration (acyclic fibration) on the right and evaluating at a test object U, we get a diagram

$$\begin{tikzcd} \Gamma X(U) \arrow[r] \arrow[d] & A(U) \arrow[d, two heads] \\ \Gamma Y(U) \arrow[r] & B(U) \end{tikzcd} \tag{4.4.2}$$

By the classical Dold-Kan equivalence (and Quillen equivalence), we can apply N and solve the lifing problem

$$\begin{tikzcd} X(U) \arrow[r] \arrow[d, tail] & NA(U) \arrow[d, two heads] \\ Y(U) \arrow[r] \arrow[ur, dotted] & NB(U) \end{tikzcd} . \tag{4.4.3}$$

This lift is natural in U since the map $X(U) \rightarrowtail Y(U)$ was assumed to be induced from a cofibration $X \rightarrowtail Y$. Again using the equivalence, we can apply Γ to get the required lift. We therefore have a Quillen adjunction. Io prove that it is an equivalence, we simply recall that the weak equivalences in the projective model structure are defined object-wise and the classical Dold-Kan Quillen equivalence gives the result. □

Unfortunately, there is no way to talk about a *local* Quillen equivalence between these presheaf categories. The problem is not that the Dold-Kan correspondence fails to preserve local weak equivalences, it is that the category $[\mathcal{C}^{op},\mathcal{C}\mathrm{h}^+]_{\mathrm{proj}}$ does not admit a local model structure. In fact as ∞ categories, the corresponding categories of sheaves are equivalent. Since we have chosen to not take this perspective, we will simply state the properties of such an equivalence. Although we will not provide the proof, it is not difficult. The main idea is to show that both Γ and N preserve local weak equivalences. In fact, this follows almost immediately from the definition of the coverage on the site $\mathcal{C}$ [5].

Proposition 68. *The Dold-Kan equivalence of presheaves preserves objects which satisfy descent. That is, if* $\mathcal{F} \in [\mathcal{C}^{op}, s\mathcal{A}\mathrm{b}]_{\mathrm{proj}}$ *satisfies descent, so does* $N(\mathcal{F})$*. Similarly, if* $\mathcal{G} \in [\mathcal{C}^{op}, \mathcal{C}\mathrm{h}^+]_{\mathrm{proj}}$ *satisfies descent, so does* $\Gamma(\mathcal{G})$*. Moreover, we have a weak equivalence of Kan complexes for all* $A \in [\mathcal{C}^{op}, \mathcal{C}\mathrm{h}^+]_{\mathrm{proj}}$

$$\Gamma(\hom_{\mathcal{C}\mathrm{h}^+}(A, N(\mathcal{F}))) =: \mathrm{Map}(A, N(\mathcal{F})) \simeq \mathrm{Map}(\Gamma(A), \mathcal{F})$$

and

$$\mathrm{Map}(B, \Gamma(\mathcal{G})) \simeq \mathrm{Map}(N(B), \mathcal{G})$$

for all cofibrant $B \in [\mathcal{C}^{op}, s\mathcal{A}\mathrm{b}]_{\mathrm{proj,loc}}$.

5.0 DIFFERENTIAL COHOMOLOGY

In this section we introduce the basic properties of differential cohomology. We begin with the definition as smooth Deligne cohomology [6] and then move to the stacky perspective [31], [11], [12], [13]. We highlight the connection between these two perspectives and provide motivating examples throughout.

5.1 AS SMOOTH DELIGNE COHOMOLOGY

For $n \in \mathcal{N}$, let $\mathbb{Z}^\infty_\mathcal{D}[n+1]$ be the sheaf of chain complexes given by

$$\mathbb{Z}^\infty_\mathcal{D}[n+1] := [\ldots \to 0 \to \underline{\mathbb{Z}} \hookrightarrow \Omega^0 \to \Omega^1 \to \ldots \to \Omega^{n-1}] \ ,$$

where $\underline{\mathbb{Z}}$ is in degree n and Ω^{n-1} is the sheaf of real valued $n-1$ forms in degree 0. Given a manifold X, the degree n sheaf hypercohomology with coefficients in $\mathbb{Z}^\infty_\mathcal{D}[n+1]$ is defined to be the degree n differential cohomology of X:

$$\hat{H}^n(X;\mathbb{Z}) := H^n(X;\mathbb{Z}^\infty_\mathcal{D}[n+1]) \ . \tag{5.1.1}$$

If X is paracompact, then these cohomology groups are given by the cohomology of the total complex of the ˇČech-Deligne double complex corresponding to a good open cover $\mathcal{U}$ of X. In what follows, we will always assume that X is paracompact, so that the hypercohomology groups can be calculated by either taking arbitrary injective resolutions, or via this more explicit Čech approach.

In [29] [7], it was observed that these cohomology groups fit nicely into an exact hexagon

$$\begin{tikzcd}
 & \Omega^{n-1}(X)/\mathrm{im}(d) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^n_{\mathrm{cl}}(X) \arrow[dr] & \\
H^{n-1}_{\mathrm{dR}}(X) \arrow[ur] \arrow[dr] & & \hat{H}^n(X;\mathbb{Z}) \arrow[ur, "\mathrm{R}"] \arrow[dr, "\mathrm{I}"] & & H^n_{dR}(X) \\
 & H^{n-1}(X;U(1)) \arrow[rr] \arrow[ur] & & H^n(X,\mathbb{Z}) \arrow[ur] &
\end{tikzcd} \tag{5.1.2}$$

where the bottom row is the Bockstein sequence and the diagonals are exact. The map R is called the *curvature* map and I is called the *integration* map. Notice that, by exactness, in the case that the curvature of a differential cohomology class vanishes, the class lies in the image of the inclusion $H^{n-1}(X;U(1)) \hookrightarrow \hat{H}^n(X;\mathbb{Z})$. We call these classes *flat*. Differential cohomology therefore detects the topological information when the class is flat and the differential geometric information encoded by the curvature.

5.2 DIFFERENTIAL COHOMOLOGY IN STACKS

In [12], the moduli stack of n-gerbes with connection, $\mathbb{B}^n U(1)_{\mathrm{conn}}$, was introduced. This stack was obtained as the stackification of the n-prestack obtained by applying the Dold-Kan map to the Deligne presheaf of chain complexes

$$\mathbb{Z}^\infty_{\mathcal{D}}[n+1] := [\ldots 0 \to \underline{\mathbb{Z}} \hookrightarrow \Omega^1 \to \Omega^2 \to \ldots \to \Omega^n] \ .$$

These stacks are the differential analogues of the Eilenberg-MacLane spaces, and there is a bijective correspondence (a 'representation')

$$\hat{H}_{\mathcal{D}}(X;\mathbb{Z}) \simeq \pi_0 \mathrm{Map}(X, \mathbb{B}^n U(1)_{\mathrm{conn}}) \ ,$$

where the right hand side is the set of morphisms in the homotopy category of stacks.

Remark 14. *Notice that, by definition, $\mathbb{B}^n U(1)_{\mathrm{conn}}$ is fibrant. Moreover, if X is a paracompact manifold, then the Čech nerve $C(\mathcal{U})$ of X is a cofibrant replacement of X in $[\mathcal{C}\mathrm{art}\mathcal{S}\mathrm{p}^{op}, s\mathcal{S}\mathrm{et}]_{\mathrm{proj,loc}}$. We, therefore, have*

$$\mathrm{ho}\mathcal{S}\mathrm{h}_\infty(X, \mathbb{B}^n U(1)_{\mathrm{conn}}) \simeq \pi_0 \mathrm{Map}(C(\mathcal{U}), \mathbb{B}^n U(1)_{\mathrm{conn}}) \ .$$

In fact, it follows from the properties of the Dold-Kan correspondence that we have a bijection

$$\pi_0 \mathrm{Map}(C(\mathcal{U}), \mathbb{B}^n U(1)_{\mathrm{conn}}) \simeq H_0 \, \mathrm{hom}_{\mathcal{C}\mathrm{h}^+}(C(\{U_i\}), \mathbb{Z}^\infty_{\mathcal{D}}[n+1]) \simeq \hat{H}^n_{\mathcal{D}}(X; \mathbb{Z}) \ ,$$

and the stacks $\mathbb{B}^n U(1)_{\mathrm{conn}}$ do indeed represent the differential cohomology functors $\hat{H}^n_{\mathcal{D}}(-; \mathbb{Z})$.

As explained in [12] and [13], these stacks have a nice geometric interpretation as well. To see this, we calculate the set of vertices in the mapping space $\mathrm{Map}(C(\mathcal{U}), \mathbb{B}^2 U(1)_{\mathrm{conn}})$. Since the Čech nerve is given by the homotopy colimit over coproduct of the representables $\coprod_{\alpha_1,..,\alpha_k} U_{\alpha_1,\alpha_2,\ldots,\alpha_k}$, we have

$$\begin{aligned} \mathrm{hom}(C(\mathcal{U}), \mathbb{B}^2 U(1)_{\mathrm{conn}}) &= \mathrm{hom}\left(\int^{k \in \Delta} \Delta[k] \cdot \coprod_{\alpha_1,..,\alpha_k} U_{\alpha_1,..,\alpha_k}, \mathbb{B}^2 U(1)_{\mathrm{conn}} \right) \\ &= \int_{k\in\Delta} \prod_{\alpha_1,..,\alpha_k} \mathrm{hom}(\Delta[k] \cdot U_{\alpha_1,..,\alpha_k}, \mathbb{B}^2 U(1)_{\mathrm{conn}}) \\ &= \prod_{\alpha_1,..,\alpha_k} \int_{k\in\Delta} \mathrm{hom}(\Delta[k], \mathbb{B}^2 U(1)_{\mathrm{conn}}(U_{\alpha_1,..,\alpha_k})) \ . \qquad (5.2.1) \end{aligned}$$

An element of the end in the last line can be written out explicitly as a choice maps

$$
\begin{aligned}
B_\alpha &: \Delta[0] \to \prod_\alpha \mathbb{B}^2U(1)(U_\alpha) \\
A_{\alpha\beta} &: \Delta[1] \to \prod_{\alpha\beta} \mathbb{B}^2U(1)(U_{\alpha\beta}) \\
g_{\alpha\beta\gamma} &: \Delta[2] \to \prod_{\alpha\beta\gamma} \mathbb{B}^2U(1)(U_{\alpha\beta\gamma}) \ ,
\end{aligned}
\tag{5.2.2}
$$

such that the face inclusions of each map are equal to their corresponding restrictions to higher intersections. Since $\mathbb{B}^2U(1)_{\mathrm{conn}}$ can equivalently be defined to be the stackification of the prestack given by applying the Dold-Kan functor to the presheaf of chain complexes

$$[0 \to \ldots \to \underline{U(1)} \xrightarrow{d\log} \Omega^1 \to \Omega^2]$$

using the obvious quasi-isomorphism. We can describe the choices of B_α,$A_{\alpha\beta}$ and $g_{\alpha,\beta\gamma}$ via the 2-simplex

$$
\begin{tikzcd}
 & B_\delta \arrow[dl, dash, "A_{\delta\alpha}"'] \arrow[dr, dash, "A_{\gamma\delta}"] \arrow[d, phantom, "g_{\alpha\beta\gamma}"] & \\
B_\alpha \arrow[rr, dash, "A_{\alpha\beta}"'] & {} & B_\gamma
\end{tikzcd}
\tag{5.2.3}
$$

Here, $g_{\alpha\beta\gamma}$ is a choice of smooth $U(1)$-valued function on triple intersections, $A_{\alpha\beta}$ is a choice of 1-form on double intersections and B_α is a choice of 2-form on open sets. Moreover, we have that these assignments must satisfy

1. $g_{\alpha\beta}g^{-1}_{\gamma\beta}g_{\gamma\alpha} = 1$;

2. $-ig^{-1}_{\alpha\beta\gamma}dg_{\alpha\beta\gamma} = d\log(g)_{\alpha\beta\gamma} = A_{\alpha\beta} - A_{\gamma\beta} + A_{\gamma\alpha}$;

3. $B_\beta - B_\alpha = dA_{\alpha\beta}$.

We identify this data as precisely giving a gerbe with connection. Moreover, the fact that $\mathbb{B}^n U(1)_{\mathrm{conn}}$ is a stack ensures that $F_\alpha = dB_\alpha$ is a globally defined 3-form: the curvature of the gerbe. Notice that these are only the vertices in the mapping space. The entire mapping space keeps track of more information, namely the homotopies and higher homotopies.

Example 23. *Let X be a paracompact manifold and $C(\mathcal{U})$ the Čech nerve of some good open cover. The maps*

$$P : C(\mathcal{U}) \to \mathbb{B}U(1)_{\mathrm{conn}}$$

are in bijective correspondence with circle bundles equipped with a connection. In fact, using the above calculation shows that such a morphism gives the data $U(1)$-valued functions $g_{\alpha\beta}$ on intersections satisfying

$$g_{\alpha\beta} g_{\beta\gamma}^{-1} g_{\gamma\delta} = 1$$

on triple intersections, along with 1-forms A_α on open sets satisfying

$$A_\alpha - A_\beta = d\log(g)_{\alpha\beta}$$

on double intersections.

If the homotopy class of P is trivial, then the circle bundle is trivializable*. In fact, the trivializing map ϕ is nothing but a homotopy $\phi : P \to 0$. To identify this homotopy, we use the proposition 68. In particular, an edge in $\mathrm{Map}(C(\mathcal{U}), \mathbb{B}U(1)_{\mathrm{conn}})$ is an edge in the stack*

$$\mathrm{Map}(C(\mathcal{U}), \mathbb{B}^n U(1)_{\mathrm{conn}}) \simeq \Gamma(\hom_{\mathrm{Ch}^+}(N(C(\mathcal{U})), \mathbb{Z}_{\mathcal{D}}^\infty[2])) \; . \tag{5.2.4}$$

The hom in positively graded chain complexes is the truncated total complex of the Čech-Deligne double complex

$$[\ldots \to \mathrm{tot}^1 C(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}^\infty[2]) \to Z\left(\mathrm{tot}^2 C(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}^\infty[2])\right)] \; ,$$

where Z denoted the group of cocycles in that degree. Recalling that the differential is given by $d + (-1)^k \delta$, where δ takes the alternating sum of restrictions, we identify an edge connecting P and 0 as an assignment of Čech-Deligne cochain h of degree one such that $(d - \delta)h = P$ Explicitly, this means a choice of $U(1)$-valued function h_α on open sets such that

1. $h_\alpha h_\beta^{-1} = g_{\alpha\beta}$;
2. $-ih_\alpha^{-1} dh_\alpha = d\log(h_\alpha) = A_\alpha$.

A straightforward calculation shows that the pattern continues and that null homotopies of n-gerbes (maps into $\mathbb{B}^n U(1)_{\mathrm{conn}}$) can again be identified with trivializations.

Motivated by this example, we will often refer to null homotopies as *trivializations*. To summarize, the mapping space $\mathrm{Map}(C(\mathcal{U}, \mathbb{B}^n U(1)_{\mathrm{conn}})$ can be identified with the set of all n-gerbes with connection, along with isomorphisms between these and higher homotopies between these isomorphisms.

Definition 69. *There are several other stacks related to $\mathbb{B}^n U(1)_{\mathrm{conn}}$ defined as follows (see [12]):*

1. *If we forget about the connection on the these n-bundles, we obtain the bare moduli stack of n-gerbes $\mathbb{B}^n U(1)$. Explicitly, this stack is obtained by applying the Dold-Kan functor to the sheaf of chain complexes $C^\infty(-, U(1))[n]$: the sheaf of smooth $U(1)$-valued functions in degree n.*
2. *The stack which represents* flat *n-bundles with connection, $\flat \mathbb{B}^n U(1)$. This stack is obtained by applying Dold-Kan to the sheaf of chain complexes* $\mathrm{disc} U(1)[n]$*: the sheaf of* locally constant $U(1)$ *valued functions in degree n.*
3. *The stack representing the truncated de Rham complex $\flat_{dR} \mathbb{B}^n U(1)$ obtained by applying Dold-Kan to the truncated de Rham sheaf of chain complexes*

$$\Omega_{cl}^{\leqslant n} := [\ldots \to \Omega^0 \to \Omega^1 \to \ldots \Omega_{cl}^n] \,.$$

4. *The stack representing the truncated de Rham complex with* all $n-1$ *forms in degree 0,* $\Omega^{\leqslant n-1}$*, obtained by applying Dold-Kan to the sheaf of chain complexes*

$$\Omega^{\leqslant n-1} := [\ldots \to \Omega^0 \to \Omega^1 \to \ldots \Omega^{n-1}] .$$

5. *The stack of closed n-forms* Ω^n_{cl} *to be the stack obtained by applying Dold-Kan to the sheaf of closed n-forms.*

One way to see that the second stack really does detect *flat* n-gerbes with connection is to observe that, by Poincaré lemma, one has a quasi-isomorphism of sheaves

$$\mathrm{disc}(U(1))[n] \simeq [0 \to \ldots \to \underline{U(1)} \xrightarrow{d\log} \Omega^1 \to \ldots \to \Omega^{n-1}_{cl}]$$

where on the right, we have *closed* $(n-1)$-forms in degree 0. These $(n-1)$-forms are to be interpreted as giving the connection on the corresponding bundle. Hence, if the form is closed, the bundle is flat.

The moduli stacks $\mathbb{B}^n U(1)_{conn}$ are related to the stacks in definition 69 in various ways. In [12][31], it was observed that $\mathbb{B}^n U(1)_{conn}$ is the homotopy pullback

$$\begin{tikzcd}
\mathbb{B}^n U(1)_{\mathrm{conn}} \arrow[r, "\mathrm{curv}"] \arrow[d] & \Omega^n_{cl} \arrow[d, "\iota"] \\
\mathbb{B}^n U(1) \arrow[r, "\theta"] & \flat_{dR}\mathbb{B}^{n+1}U(1)
\end{tikzcd} \tag{5.2.5}$$

where the composite $\mathbb{B}^n U(1)_{conn} \to \mathbb{B}^n U(1) \xrightarrow{\theta} \flat_{dR}\mathbb{B}^n U(1)$ is homotopic the map

$$\mathrm{curv} : \mathbb{B}^n U(1)_{\mathrm{conn}} \to \flat_{dR}\mathbb{B}^{n+1}U(1) \tag{5.2.6}$$

induced by the morphism of sheaves of chain complexes

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, "i"] \arrow[d, "i"] & \Omega^0 \arrow[d, "d"] \\
\Omega^0 \arrow[r, "d"] \arrow[d] & \Omega^1 \arrow[d] \\
\vdots \arrow[r, "d"] \arrow[d] & \vdots \arrow[d] \\
\Omega^{n-1} \arrow[r, "d"] & \Omega^n_{cl}
\end{tikzcd}
\tag{5.2.7}
$$

This map gives the full de Rham data for the curvature of a bundle with connection. In fact, since each degree of the right side is a fine sheaf, the sheaf hypercohomology in degree 0 is $H^n_{dR}(X)$ and the curv induces a map

$$\mathrm{curv} : \pi_0 \mathrm{Map}(X, \mathbb{B}^n U(1)_{conn}) \to H^n_{dR}(X) \tag{5.2.8}$$

which sends an n-bundle with connection to its de Rham class of its curvature. The curvature form itself is given by the pullback $R(\mathcal{G}) = \iota^* \mathrm{curv}(\mathcal{G})$, where $\mathcal{G}$ is a map

$$\mathcal{G} : X \to \mathbb{B}^n U(1)_{conn},$$

and ι is the inclusion map in (5.2.5). The following proposition can be found in [31], but we include a proof for completeness.

Proposition 70. *The homotopy fiber of the map*

$$\mathrm{curv} : \mathbb{B}^n U(1)_{conn} \twoheadrightarrow \Omega^n_{cl}$$

can be identified with $\flat\mathbb{B}^{n-1}U(1)$.

Proof. The map R is induced by the morphism of sheaves of chain complexes

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] \arrow[d, "i"] & 0 \arrow[d, "d"] \\
\Omega^0 \arrow[r, "0"] \arrow[d] & 0 \arrow[d] \\
\vdots \arrow[r, "0"] \arrow[d] & \vdots \arrow[d] \\
\Omega^{n-1} \arrow[r, "d"] & \Omega^n_{cl}
\end{tikzcd} \tag{5.2.9}$$

Since this map is degree-wise surjective by Poincaré lemma, it is a fibration in the projective model structure on presheaves. We can therefore calculate the homotopy fiber as the kernel of the map. By inspection, the kernel is

$$[\mathbb{Z} \to \Omega^0 \to \Omega^1 \to \dots \Omega^{n-1}_{cl}].$$

which is quasi-isomorphic to

$$[C^\infty(-, U(1)) \overset{d\log}{\to} \Omega^1 \to \dots \Omega^{n-1}_{cl}].$$

Again, by Poincaré lemma, this sheaf of chain complex is quasi-isomorphic to $\mathrm{disc}(U(1))[n]$. Since the Dold-Kan functor is a right Quillen adjoint and preserves weak equivalences, it takes fibration sequences to fibration sequences. Since stackification preserves finite homotopy limits, we have the result. $\square$

Using the previous proposition along with 5.2.5 and the pasting lemma for homotopy pullbacks, we observe that we have the following iteration of homotopy pullbacks [31].

$$
\begin{tikzcd}
\flat\mathbb{B}^{n-1}U(1) \arrow[r] \arrow[d] & \mathbb{B}^nU(1) \arrow[r] \arrow[d] & * \arrow[d] & \\
* \arrow[r] \arrow[d] & \flat_{dR}\mathbb{B}^{n-1}U(1) \arrow[r] \arrow[d] & \flat\mathbb{B}^nU(1) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Omega^{\leqslant n-1} \arrow[r] \arrow[d] & \mathbb{B}^nU(1)_{conn} \arrow[r, two heads] \arrow[d, two heads] & \Omega^n_{cl} \arrow[d] \\
& * \arrow[r] & \mathbb{B}^nU(1) \arrow[r] & \flat_{dR}\mathbb{B}^nU(1)
\end{tikzcd}
\tag{5.2.10}
$$

Corollary 71. *The loop space $\Omega\mathbb{B}^nU(1)_{\mathrm{conn}}$ can be identified with $\flat\mathbb{B}^{n-1}U(1)$.*

Corollary 72. *The differential cohomology diagram* (5.1.2) *lifts to a diagram of stacks*

$$
\begin{tikzcd}
& \Omega^{\leqslant n-1} \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^n_{cl} \arrow[dr] & \\
\flat_{dR}\mathbb{B}^{n-1}U(1) \arrow[ur] \arrow[dr] & & \mathbb{B}^nU(1)_{\mathrm{conn}} \arrow[ur, "R"] \arrow[dr, "I"] & & \flat_{dR}\mathbb{B}^nU(1) \\
& \flat\mathbb{B}^nU(1) \arrow[rr] \arrow[ur] & & \mathbb{B}^nU(1) \arrow[ur] &
\end{tikzcd}
\tag{5.2.11}
$$

where the diagonals are fibration sequences.

5.3 CUP PRODUCT STRUCTURE

Deligne and Beilinson showed that differential cohomology admits a distinguished cup product, refining the usual cup product on singular cohomology, defined on sections of $\mathbb{Z}^\infty_{\mathcal{D}}[n+1]$

by

$$\alpha \cup \beta = \begin{cases} \alpha\beta & \deg(\alpha) = n \\ \alpha \wedge d\beta & \deg(\alpha) = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.1}$$

Equipped with this cup product, $\hat{H}^*(X;\mathbb{Z})$ becomes an associative and graded commutative ring [6]. This cup product structure also refines the wedge product of forms in the sense that the curvature map $R : \hat{H}^*(X;\mathbb{Z}) \to \Omega^*_{cl}(X)$ defines a homomorphism of graded commutative rings [7]. In particular this implies that the cup product of two classes of odd degree is flat. It can also be shown [7] that the cup product of a flat class with any other class is again flat.

In [11] it was observed that the Lax-monoidal structure of the Dold-Kan map gives rise to a cup product, exhibited as a morphism

$$\cup : \mathbb{B}^m U(1)_{conn} \times \mathbb{B}^n U(1)_{conn} \to \mathbb{B}^{n+m+1} U(1)_{conn} \tag{5.3.2}$$

of stacks. This map is obtained by simply taking the Deligne-Beilinson cup product (5.3.1),

$$\cup_{DB} : \mathbb{Z}^\infty_{\mathcal{D}}[n+1] \otimes \mathbb{Z}^\infty_{\mathcal{D}}[m+1] \to \mathbb{Z}^\infty_{\mathcal{D}}[n+m+2]$$

applying the Dold-Kan map

$$\Gamma(\cup_{DB}) : \Gamma(\mathbb{Z}^\infty_{\mathcal{D}}[n+1] \otimes \mathbb{Z}^\infty_{\mathcal{D}}[m+1]) \to \Gamma(\mathbb{Z}^\infty_{\mathcal{D}}[n+m+2])$$

and using the lax monoidal structure φ of Γ to get a map

$$\cup = \Gamma(\cup_{DB}) \circ \varphi : \Gamma(\mathbb{Z}^\infty_{\mathcal{D}}[n+1]) \times \Gamma(\mathbb{Z}^\infty_{\mathcal{D}}[n+1]) \to \Gamma(\mathbb{Z}^\infty_{\mathcal{D}}[n+1] \otimes \mathbb{Z}^\infty_{\mathcal{D}}[m+1]) \to \Gamma(\mathbb{Z}^\infty_{\mathcal{D}}[n+m+2])\ .$$

Applying the stackification functor then gives the desired map. This map then induces a map of stacks (which we also denote as $\cup$)

$$\cup : [X, \mathbb{B}^n U(1)_{\mathrm{conn}}] \times [X, \mathbb{B}^m U(1)_{\mathrm{conn}}] \to [X, \mathbb{B}^{n+m+1} U(1)_{\mathrm{conn}}] \,. \tag{5.3.3}$$

on mapping stacks defined in 3.3.2. The following two propositions are implicit in [11].

Proposition 73. *The DB cup product refines the singular cup product. That is, we have a commutative diagram*

$$\begin{tikzcd}
\mathbb{B}^n U(1)_{\mathrm{conn}} \times \mathbb{B}^m U(1)_{\mathrm{conn}} \arrow[r, "\cup_{DB}"] \arrow[d, "I \times I"] & \mathbb{B}^{n+m+1} U(1)_{\mathrm{conn}} \arrow[d, "I"] \\
\mathbb{B}^{n+1}\mathbb{Z} \times \mathbb{B}^{m+1}\mathbb{Z} \arrow[r, "\cup"] & \mathbb{B}^{n+m+2}\mathbb{Z} \,.
\end{tikzcd}$$

Proof. The diagram

$$\begin{tikzcd}
\mathbb{Z}_{\mathcal{D}}^{\infty}[n+1] \otimes \mathbb{Z}_{\mathcal{D}}^{\infty}[m+1] \arrow[r, "\cup_{DB}"] \arrow[d, "p"] & \mathbb{Z}_{\mathcal{D}}^{\infty}[n+m+2] \arrow[d, "p"] \\
\underline{\mathbb{Z}}[n+1] \otimes \underline{\mathbb{Z}}[m+1] \arrow[r, "\cup"] & \underline{\mathbb{Z}}[n+m+2]
\end{tikzcd}$$

commutes in chain complexes. Applying the Dold-Kan functor and using naturality of the lax-monoidal structure map gives the result. $\square$

Proposition 74. *The cup product refines the wedge product and we have a commutative diagram*

$$\begin{tikzcd}
\mathbb{B}^n U(1)_{\mathrm{conn}} \times \mathbb{B}^m U(1)_{\mathrm{conn}} \arrow[r, "\cup_{DB}"] \arrow[d, "\mathrm{curv} \times \mathrm{curv}"] & \mathbb{B}^{n+m+1} U(1)_{\mathrm{conn}} \arrow[d, "\mathrm{curv}"] \\
\flat_{dR}\mathbb{B}^{n+1} U(1) \times \flat_{dR}\mathbb{B}^{m+1} U(1) \arrow[r, "\wedge"] & \flat_{dR}\mathbb{B}^{n+m+2} U(1) \,.
\end{tikzcd}$$

Proof. Let α and β be sections of $\mathbb{Z}^\infty_{\mathcal{D}}[n+1]$ and $\mathbb{Z}^\infty_{\mathcal{D}}[m+1]$, respectively. Then the DB cup product is given by

$$\alpha \cup_{DB} \beta = \begin{cases} \alpha\beta & \text{if } \deg(\alpha) = n \\ \alpha \wedge d\beta & \text{if } \deg(\beta) = 0\ . \end{cases} \tag{5.3.4}$$

Applying the curvature curv gives

$$\mathrm{curv}(\alpha \cup_{DB} \beta) = \begin{cases} \alpha d(\beta) & \text{if } \deg(\alpha) = n \\ d(\alpha) \wedge d(\beta) & \text{if } \deg(\beta) = 0\ , \end{cases}$$

which is $\mathrm{curv}(\alpha) \wedge \mathrm{curv}(\beta)$. We therefore have a commuting diagram

$$\begin{tikzcd} \mathbb{Z}^\infty_{\mathcal{D}}[n+1] \otimes \mathbb{Z}^\infty_{\mathcal{D}}[m+1] \arrow[r, "\cup_{DB}"] \arrow[d, "p"] & \mathbb{Z}^\infty_{\mathcal{D}}[n+m+2] \arrow[d, "p"] \\ \Omega^{\leqslant n}_{\mathrm{cl}} \otimes \Omega^{\leqslant m}_{\mathrm{cl}} \arrow[r, "\wedge"] & \Omega^{\leqslant n+m}_{\mathrm{cl}}\ , \end{tikzcd}$$

where $\Omega^{\leqslant r}_{\mathrm{cl}}$ are differential forms of degree $\leqslant r$. Applying Dold-Kan gives the result in stacks.

□

6.0 DIFFERENTIAL MASSEY PRODUCTS

We are now ready to present the main results of this thesis, which can be regarded as a refinement of classic Massey products to the differential setting. We first define these products in the differential setting and then show how they are related to Massey products in singular cohomology.

6.1 CLASSICAL (GENERALIZED) MASSEY PRODUCTS

We recall some notions from [25] [2]. This will be useful for the applications that we will consider later as well as a starting point for comparison with our stacky constructions.

Let $(\mathcal{A}, d)$ be a differential graded algebra over $\mathbb{R}$ endowed with augmentation. Let $M(\mathcal{A})$ be the set of all upper triangular half-infinite matrices with entries in $\mathcal{A}$, zeroes on the diagonal and finitely many nonzero entries, i.e.

$$M(\mathcal{A}) = \{A = (a_{ij}), a_{ij} \in \mathcal{A}, a_{ij} = 0 \text{ for } j \leqslant i \text{ and } i, j \geqslant n+1 \text{ for some } n\}. \tag{6.1.1}$$

The last condition distinguishes in $M(\mathcal{A})$ a subset (which is in fact a subalgebra) $M_n(\mathcal{A})$ consisting of all $(n \times n)$-matrices with entries in $\mathcal{A}$. The algebra $M(\mathcal{A})$ is bigraded and endowed with a bigraded Lie bracket. We introduce the differential d on $M(\mathcal{A})$ as $dA = (da_{ij})_{i,j \geqslant 1}$.

The algebra $\mathcal{A}$ admits an involution given by

$$a \mapsto \overline{a} = (-1)^k a, \tag{6.1.2}$$

which can be extended to an automorphism of $M(\mathcal{A})$ as $\overline{A} = (\overline{a}_{ij})_{i,j\geqslant 1}$, with the differential d satisfying the generalized Leibnitz rule

$$d(AB) = (dA)B + \overline{A}(dB). \tag{6.1.3}$$

In [2], the Maurer-Cartan operator $\mu : \; M \longrightarrow M$ was defined as

$$\mu(A) = dA - \overline{A} \cdot A. \tag{6.1.4}$$

Then a matrix $A \in M$ is said to be a matrix of formal connection if it satisfies the Maurer-Cartan equation in $\mathcal{A}$,

$$dA - \overline{A} \cdot A \equiv 0 \mod \ker A, \tag{6.1.5}$$

i.e. A is a formal connection if $\mu(A) \in \ker A$, where $\ker A$ is a $\mathcal{A}$-module generated by matrices 1 such that $A \cdot 1 = 1 \cdot A$ (which implies that $AB = BA$ for any matrix $B \in \ker A$). And then $\mu(A)$ is called the curvature of the formal connection A. The following proposition can be found in [2].

Proposition 75. *Let A be a formal connection on a DGA $\mathcal{A}$. Then the curvature $\mu(A)$ is closed.*

Proof. We simply compute $d\mu(A)$:

$$\begin{aligned} d\mu(A) &= d\left(dA - \overline{A} \cdot A\right) \\ &= -d\overline{A} \cdot A - A \cdot dA \\ &= \overline{(\mu(A) + \overline{A} \cdot A)} \cdot A - A \cdot (\mu(A) + \overline{A} \cdot A) \\ &= \overline{\mu(A)} \cdot A - A \cdot \mu(A) \end{aligned}$$

Since $\mu(A) \in \ker(A)$ so is $\overline{\mu(A)}$ and the last line is 0. $\square$

The following example will be the most important for us. In this case, we have an easy characterization of $\ker(A)$ [2].

Example 24. *Let A be an $n \times n$ matrix*

$$A = \begin{pmatrix} 0 & a_{12} & * & \dots & * \\ 0 & 0 & a_{23} & \dots & * \\ & & & \dots & \\ 0 & 0 & 0 & \dots & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with entries in a DGA $\mathcal{A}$, such that none of the $a_{i,i+1}$ vanish. Then $\ker(A) \cap M_n(\mathcal{A})$ is the set of matrices of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & * \\ 0 & 0 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with one nonzero entry in the upper right corner.

Remark 15. *For our purposes, the previous example of $\ker(A)$ is really the only important example. For this reason, given a matrix $A \in M(\mathcal{A})_n$, we define the kernel of A to be the $\mathcal{A}$-module given by elements of the form*

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & * \\ 0 & 0 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6.1.6}$$

By the previous example, this definition is equivalent when restricted to $M(\mathcal{A})_n$ (and when A has nonzero entries on the off-diagonal). In fact, the original matric Massey products defined by May in [25] were defined as solutions to

$$dA - \overline{A} \cdot A \equiv 0 \tag{6.1.7}$$

modulo elements in (6.1.6). *We will need to work with this definition in order to generalize to stacks, as it will not make sense to talk about the multiplicative identity in this context.*

Now comes the relation of the Maurer-Cartan to the Massey products. The generalized Massey products are the cohomology classes of the curvature matrices of the formal connection A, i.e. if A is a solution to the Maurer-Cartan equation then the entries of the matrix $[\mu(A)]$ are the generalized Massey products [2]. Geometrically, this means that the latter measure the deviation of connections from flat ones, so that the connection is flat if they vanish.

Classical Massey products in integral cohomology $H^*(X;\mathbb{Z})$ arise by taking $\mathcal{A}$ to be an algebra over the commutative ring $\mathbb{Z}$, with the multiplication being associative but not necessarily graded-commutative. Now let α, β, γ be the cohomology classes of closed elements $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$, and $c \in \mathcal{A}^r$. The triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined if one can solve the Maurer-Cartan equation with the formal connection

$$A = \begin{pmatrix} 0 & a & \tilde{f} & h \\ 0 & 0 & b & \tilde{g} \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is equivalent to

$$d\tilde{f} = (-1)^p a \wedge b \quad \text{and} \quad d\tilde{g} = (-1)^q a \wedge c \tag{6.1.8}$$

and that implies that the Massey product is defined if and only if

$$\alpha \cup \beta = \beta \cup \gamma = 0 \in H^*(\mathcal{A}) \ . \tag{6.1.9}$$

The matrix $\mu(A)$ has the form

$$dA - \overline{A} \cdot A = \begin{pmatrix} 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and defines the Massey product $[\mu(A)]$ which is equal to

$$\langle \alpha, \beta, \gamma \rangle = [\tau] = \left[(-1)^{p+1} a \wedge \tilde{g} + (-1)^{p+q} \tilde{f} \wedge c \right] \ . \tag{6.1.10}$$

Since $\tilde{f}$ and $\tilde{g}$ are defined by (6.1.8) up to closed elements from $\mathcal{A}$, the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined modulo $\alpha \cdot H^{q+r}(\mathcal{A}) + \gamma \cdot H^{p+q}(\mathcal{A})$.

The previous construction of Massey products was originally presented by Peter May in [25] and then further developed in [2]. May's original definition was far more general as he not only considered matrices with entries in a DGA, but also matrices with entries in graded modules $\mathcal{R}_{ij}$. More precisely, suppose we have a collection $\{\mathcal{R}_{ij}\}$ of modules indexed by pairs of integers $i, j \in \mathbb{Z}$. Suppose moreover that these graded modules are equipped with maps

$$\cup : \mathcal{R}_{ij} \otimes \mathcal{R}_{jk} \to \mathcal{R}_{ik}$$

which are associative in the sense that

$$\cup \circ (\mathrm{id} \circ \cup) = \cup \circ (\cup \circ \mathrm{id}).$$

Then we can consider the set of matrices

$$M(\mathcal{R}) = \{A = (a_{ij}), a_{ij} \in \mathcal{R}_{ij}, a_{ij} = 0 \text{ for } j \leqslant i \text{ and } i, j \geqslant n+1 \text{ for some } n\}\,. \tag{6.1.11}$$

Then $M(\mathcal{R})$ becomes a DGA under matrix multiplication, and the entirety of the previous discussion applies. We point out this more general context as it will be useful to us in defining stacky Massey products.

6.2 THE STACKY TRIPLE PRODUCT

We begin with a discussion on Massey triple products and then generalize to arbitrary Massey products.

The Massey triple product can be viewed as a homotopy built out of the associativity diagram of thee cup product. In fact, suppose one is given a triple of n-bundles with connection on a manifold X. These bundles are given by the data $\mathcal{G}_i : C(\mathcal{U}) \to \mathbb{B}^{n_i}U(1)$, $i = 1, 2, 3$, where $C(\mathcal{U})$ is the Čech nerve of some good open cover. Suppose moreover that these bundles are chosen so that $\mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{G}_2 \cup \mathcal{G}_3$ are homotopic to 0, with trivializing homotopies $\phi_{1,2}$ and $\phi_{2,3}$. In this case, we can build a *loop* trivializing the triple product. To see this, consider the associativity diagram

$$
\begin{tikzcd}
 & & \mathbb{B}^{n_1+n_2+1}U(1)_{\text{conn}} \arrow[dr, "id \times 0"] & & \\
 & \mathbb{B}^{n_1}U(1)_{\text{conn}} \times \mathbb{B}^{n_2}U(1)_{\text{conn}} \arrow[ur, "\mathcal{G}_1 \cup \mathcal{G}_2"] \arrow[dr, "id \times 0"] & & \mathbb{B}^{n_1+n_2+1}U(1)_{\text{conn}} \times \mathbb{B}^{n_3}U(1)_{\text{conn}} \arrow[dr, "\cup"] & \\
C(\mathcal{U}) \arrow[ur, "\mathcal{G}_1 \times \mathcal{G}_2"] \arrow[rr, "\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3"] \arrow[dr, "\mathcal{G}_2 \times \mathcal{G}_3"'] & & \mathbb{B}^{n_1}U(1)_{\text{conn}} \times \mathbb{B}^{n_2}U(1)_{\text{conn}} \times \mathbb{B}^{n_3}U(1)_{\text{conn}} \arrow[ur, "(\cup \times id)"] \arrow[dr, "(id \times \cup)"] & & \mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\text{conn}} \\
 & \mathbb{B}^{n_2}U(1)_{\text{conn}} \times \mathbb{B}^{n_3}U(1)_{\text{conn}} \arrow[ur, "0 \times id"] \arrow[dr, "\mathcal{G}_2 \cup \mathcal{G}_3"'] & & \mathbb{B}^{n_1}U(1)_{\text{conn}} \times \mathbb{B}^{n_2+n_3+1}U(1)_{\text{conn}} \arrow[ur, "\cup"'] & \\
 & & \mathbb{B}^{n_2+n_3+1}U(1)_{\text{conn}} \arrow[ur, "0 \times id"'] & &
\end{tikzcd}
$$

Figure 6.0.1: Associativity of the DB cup product

Although the outer two maps agree, there is still nontrivial homotopy theoretic information contained in the diagram. To see this, suppose $\mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{G}_2 \cup \mathcal{G}_3$ are trivializable. Since the cup product of a bundle with the 0 bundle is again 0, we can describe this nontrivial homotopy data via the diagram

$$
\begin{tikzcd}[column sep=small]
 & & \mathbb{B}^{n_1+n_2+1}U(1)_{\mathrm{conn}} \arrow[dr, "id \times 0"] & & \\
 & \mathbb{B}^{n_1}U(1)_{\mathrm{conn}} \times \mathbb{B}^{n_2}U(1)_{\mathrm{conn}} \arrow[ur, "\mathcal{G}_1 \cup \mathcal{G}_2"] \arrow[dr, "id \times 0"] \arrow[d, phantom, "\phi_{1,2}"] & & \mathbb{B}^{n_1+n_2+1}U(1)_{\mathrm{conn}} \times \mathbb{B}^{n_3}U(1)_{\mathrm{conn}} \arrow[dr, "\cup"] \arrow[d, phantom, "\phi_{1,2} \cup \mathcal{G}_3"] & \\
C(\mathcal{U}) \arrow[ur, "\mathcal{G}_1 \times \mathcal{G}_2"] \arrow[rr, "\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3"] \arrow[dr, "\mathcal{G}_2 \times \mathcal{G}_3"'] \arrow[uurr, dotted, bend left, "0"] \arrow[ddrr, dotted, bend right, "0"'] & & \mathbb{B}^{n_1}U(1)_{\mathrm{conn}} \times \mathbb{B}^{n_2}U(1)_{\mathrm{conn}} \times \mathbb{B}^{n_3}U(1)_{\mathrm{conn}} \arrow[ur, "(\cup \times id)"] \arrow[dr, "(id \times \cup)"'] \arrow[rr, dotted, "0"] & & \mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}} \\
 & \mathbb{B}^{n_2}U(1)_{\mathrm{conn}} \times \mathbb{B}^{n_3}U(1)_{\mathrm{conn}} \arrow[ur, "0 \times id"'] \arrow[dr, "\mathcal{G}_2 \cup \mathcal{G}_3"'] \arrow[u, phantom, "\phi_{2,3}"] & & \mathbb{B}^{n_1}U(1)_{\mathrm{conn}} \times \mathbb{B}^{n_2+n_3+1}U(1)_{\mathrm{conn}} \arrow[ur, "\cup"'] \arrow[u, phantom, "\mathcal{G}_1 \cup \phi_{2,3}"] & \\
 & & \mathbb{B}^{n_2+n_3+1}U(1)_{\mathrm{conn}} \arrow[ur, "0 \times id"'] & &
\end{tikzcd}
$$

Figure 6.0.2: Associativity diagram with homotopies

where the homotopies fill the regions bounded by the dotted arrows. Since the top and bottom composites are equal, the homotopy $\phi_{1,2} \cup \mathcal{G}_3$ and inverse homotopy $-\mathcal{G}_1 \cup \phi_{2,3}$ together give a loop of maps to the moduli stack $\mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}}$.

$$
\begin{tikzcd}[column sep=huge]
C(\mathcal{U}) \arrow[rr, bend left=50, dotted, "0"] \arrow[rr, bend right=50, dotted, "0"'] \arrow[rr, "\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3" description] \arrow[rr, phantom, shift left=5ex, "{\Downarrow\, \phi_{1,2} \cup \mathcal{G}_3}"] \arrow[rr, phantom, shift right=5ex, "{\Downarrow\, -\mathcal{G}_1 \cup \phi_{2,3}}"] & & \mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}}
\end{tikzcd}
$$

.

Figure 6.0.3: A loop of maps to the moduli stack

By the universal property, this is equivalently given by a map

$$
C(\mathcal{U}) \to \Omega\mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}} \simeq \flat\mathbb{B}^{n_1+n_2+n_3+1}U(1) \ : \tag{6.2.1}
$$

A *flat* bundle with connection.

Proposition 76. *The homotopy class of the loop* (6.2.1) *is an element of*

$$
H^{n_1+n_2+n_3+1}(X; U(1)) \hookrightarrow \hat{H}^{n_1+n_2+n_3+2}(X; \mathbb{Z}) \ .
$$

Proof. the homotopy class of the map (6.2.1) is an element in $\pi_0\mathrm{Map}(X, \flat\mathbb{B}^{n_1+n_2+n_3+1}U(1))$. Using the Dold-Kan correspondence along with the definition of $\flat\mathbb{B}^nU(1)$, we have the isomorphisms

$$
\begin{aligned}
\pi_0\mathrm{Map}(X, \flat\mathbb{B}^{n_1+n_2+n_3+1}U(1)) &\simeq H_0 \hom_{\mathrm{Ch}^+}(C(\mathcal{U}), \mathrm{disc}(U(1))[n_1+n_2+n_3+1]) \\
&\simeq H^{n_1+n_2+n_3+1}C(\mathcal{U}, U(1)) \\
&\simeq H^{n_1+n_2+n_3+1}(X; U(1)) \ .
\end{aligned}
$$

where the complex $C(\mathcal{U}, U(1))$ denotes the Čech complex with coefficients in $U(1)$. $\square$

Remark 16. **(i)** *Notice that we could have equivalently taken the homotopy class of the loop directly to get an element*

$$\begin{aligned}[\mathcal{G}_1 \cup \phi_{2,3} - \phi_{1,2} \cup \mathcal{G}_1] &\in \pi_1 \mathrm{Map}(X, \mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}}) \\ &\simeq H_1 \hom_{\mathcal{C}\mathrm{h}^+}(C(\mathcal{U}), \mathbb{Z}_{\mathcal{D}}^{\infty}[n_1+n_2+n_3+2]) \\ &\simeq H^1 C(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}^{\infty}[n_1+n_2+n_3+2]) \\ &\simeq H^{n_1+n_2+n_3+1}(X; U(1)) \ .\end{aligned}$$

(ii) *The above observations allow us to recover the usual definition of the Massey product as an element in cohomology. In section 1, we observed that such a class is not completely well defined purely at the level of cohomology and there was some dependence on the chosen cochain representatives. Taking this point of view, one can see this dependence as a choice of trivializations $\phi_{1,2}$ and $\phi_{2,3}$ of the cup products.*

This definition works well for the triple product and gives a clear picture on how the triple product is built out of the homotopies. However, to describe the higher triple products this way would be cumbersome. Moreover, the algebraic nature of the products would not be transparent. For these reasons, we will use the language of simplicial homotopy theory to describe these homotopy commuting diagrams and the Dold-Kan correspondence to organize these homotopies in an algebraic way. To prepare the reader for this perspective, we first recast the triple product in this language.

Notice that the triple product was described by two homotopies $\phi_{1,2}$ and $\phi_{2,3}$ connecting the basepoint 0 to the double cup products. We can express this situation diagramatically via

$$\begin{tikzcd} \partial\Delta[1] \arrow[r, "{(0,\mathcal{G}_1 \cup \mathcal{G}_2)}"] \arrow[d] & \mathbb{B}^{n_1+n_2+1}U(1)_{\mathrm{conn}} \\ \Delta[1] \arrow[ur, dotted, "\phi_{1,2}"'] & \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta[1] \arrow[r, "{(0,\mathcal{G}_2 \cup \mathcal{G}_3)}"] \arrow[d] & \mathbb{B}^{n_2+n_3+1}U(1)_{\mathrm{conn}} \\ \Delta[1] \arrow[ur, dotted, "\phi_{2,3}"'] & \end{tikzcd}$$

Figure 6.0.4: A choice of trivialization of the cup products

Now we would like to use these homotopies to construct a loop. To do this, we need to algebraically manipulate these homotopies. This motivates us to take the Moore complex of these diagrams in order to translate the data into the language of sheaves of chain complexes. This gives the data

$$\begin{tikzcd} \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "{(0,\mathcal{G}_1 \cup \mathcal{G}_2)}"] & (\mathbb{B}^{n_1+n_2+1}U(1)_{\mathrm{conn}})_0 \\ \mathbb{Z} \arrow[u, "{(1,-1)}"'] \arrow[r, "\phi_{1,2}"] & (\mathbb{B}^{n_1+n_2+1}U(1)_{\mathrm{conn}})_1 \arrow[u, "\partial"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "{(0,\mathcal{G}_2 \cup \mathcal{G}_3)}"] & (\mathbb{B}^{n_2+n_3+1}U(1)_{\mathrm{conn}})_0 \\ \mathbb{Z} \arrow[u, "{(1,-1)}"'] \arrow[r, "\phi_{2,3}"] & (\mathbb{B}^{n_2+n_3+1}U(1)_{\mathrm{conn}})_1 \arrow[u, "\partial"'] \end{tikzcd}$$

Figure 6.0.5: The corresponding homotopies in chain complexes

where the subindices indicate the degree of the simplicial abelian stack. Now since we are in the category of presheaves of chain complexes, we take products of elements in different degrees. We can organize all this data succinctly as an upper triangular matrix

$$A = \begin{pmatrix} 0 & \mathcal{G}_1 & \phi_{1,2} & * \\ 0 & 0 & \mathcal{G}_2 & \phi_{2,3} \\ 0 & 0 & 0 & \mathcal{G}_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the entries of the matrix are sections of the sheaf of chain complexes $(\mathbb{B}^{n_{ij}}U(1)_{\mathrm{conn}}, \partial)$

in various degrees. By construction, this matrix satisfies the Maurer-Cartan equation

$$dA - \overline{A}A = \mu(A) \in \ker(A) \ .$$

Moreover, $\mu(A)$ is of the form

$$\mu(A) = \begin{pmatrix} 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

with boundary

$$\begin{aligned} \partial(\mu(A)) = \partial(\tau) &= \partial\left(\mathcal{G}_1 \cup \phi_{2,3} - \phi_{1,2} \cup \mathcal{G}_3\right) \\ &= \partial\left(\mathcal{G}_1 \cup \phi_{2,3}\right) - \left(\phi_{1,2} \cup \mathcal{G}_3\right) \\ &= \partial(\mathcal{G}_1) \cup \phi_{2,3} + \mathcal{G}_1 \cup \partial(\phi_{2,3}) - \partial(\phi_{1,2}) \cup \mathcal{G}_3 + \phi_{1,2} \cup \partial(\mathcal{G}_3) \\ &= \mathcal{G}_1 \cup (\mathcal{G}_2 \cup \mathcal{G}_3) - (\mathcal{G}_1 \cup \mathcal{G}_2) \cup \mathcal{G}_3 \\ &= 0 \ . \end{aligned}$$

Hence, $\mu(A)$ is a choice of local cocycle in the sheaf of chain complexes $(\mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}}, \partial)$. We would like to understand how this local cocycle glues with respect to local data on a manifold X. That is, we want to interpret this section as an element of sheaf hypercohomology of this chain complex.

Let X be a (paracompact) manifold, and let $\mathcal{U}$ be a good open cover of X. Then we can calculate the sheaf cohomology by forming Čech resolution of $(\mathbb{B}^{n_1+n_2+n_3+2}U(1)_{\mathrm{conn}}, \partial)$ and taking the cohomology groups. We have

Proposition 77. *The cohomology class of the matrix cocycle $\mu(A)$ is the element*

$$[\mu(A)] = [\mathcal{G}_1 \cup \phi_{2,3} - \phi_{1,2} \cup \mathcal{G}_3] \in H^{n_1+n_2+n_3+1}(X; U(1)) \ .$$

Proof. We have the following sequence of isomorphisms

$$\begin{aligned}
[\mu(A)] &= [\mathcal{G}_1 \cup \phi_{2,3} - \phi_{1,2} \cup \mathcal{G}_3] \\
&\in \pi_1 \mathrm{Map}\left(C(\mathcal{U}), \mathbb{B}^{n_1+n_2+n_3+2} U(1)_{\mathrm{conn}}\right) \\
&\simeq \pi_0 \mathrm{Map}\left(C(\mathcal{U}), \flat \mathbb{B}^{n_1+n_2+n_3+1} U(1)\right) \\
&\simeq H^{n_1+n_2+n_3+1}(X; U(1)) \ .
\end{aligned}$$

□

6.3 GENERAL STACKY MASSEY PRODUCTS

Let $\mathcal{R}_{ij}$, be simplicial abelian presheaves equipped with maps

$$\cup : \mathcal{R}_{ij} \otimes \mathcal{R}_{jk} \to \mathcal{R}_{ik}$$

which are associative in the sense that

$$\cup \circ (\cup \otimes \mathrm{id}) = \cup \circ (\mathrm{id} \otimes \cup).$$

Remark 17. *Let N denote the normalized Moore functor. Recall that this is a functor, in fact an equivalence of categories, from simplicial abelian groups $s\mathcal{A}\mathrm{b}$ to chain complexes in non-negative degree $\mathrm{Ch}_\bullet^+$ (see [14]). It follows from the definition of the differential of the* *tensor product that the induced product*

$$\tilde{\cup} : N(\mathcal{R}_{ij}) \otimes N(\mathcal{R}_{jk}) \to N(\mathcal{R}_{ij} \otimes \mathcal{R}_{jk}) \to N(\mathcal{R}_{ik})$$

must satisfy the Leibniz type rule

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^p \alpha \cup d(\beta)$$

on sections (where α is in degree p.)

We can now utilize an extension of the machinery of May [25] locally to define the refined Matric Massey products in this setting. To this end, we consider the set of all upper triangular half-infinite matrices

$$M(\mathcal{R}) = \{A = (a_{ij}), a_{ij} \in N(\mathcal{R}_{ij}), a_{ij} = 0 \text{ for } j \leqslant i \text{ and } i, j \geqslant n+1 \text{ for some } n, k \in \mathcal{N}\} \ . \tag{6.3.1}$$

Notice that, with our definition, this set possesses more structure. It becomes a sheaf of differentially graded rings (sheaf of DGA's) with product given by "matrix multiplication" and differential given by applying the differential on $N(\mathcal{R}_{ij})$ to each entry of the matrix. Just as in the case of classical Massey products, we have a filtration of sheaves of DGA's

$$M(\mathcal{R})_1 \subset M(\mathcal{R})_2 \subset \ldots \subset M(\mathcal{R})_n \subset \ldots \tag{6.3.2}$$

and a bigrading

$$M(\mathcal{R}) = \bigoplus_{p \geqslant 1, k \geqslant 0} M^{p,k} \ , \tag{6.3.3}$$

where

$$M^{p,k} = \operatorname{span}\left\{ \begin{pmatrix} & & 0 \\ & a_{i,i+p} & \\ & & \\ 0 & & \\ & & \end{pmatrix} ; a_{i,i+p} \in N(\mathcal{R}_{i,i+p}) \right\} . \tag{6.3.4}$$

We also have an involution as before given by

$$a_{i,i+p} \mapsto \overline{a}_{i,i+p} = (-1)^k a_{i,i+p}, \tag{6.3.5}$$

where $a_{i,i+p} \in N(\mathcal{R}_{i,i+p})_k$.

Remark 18. *Note that given abeian stacks $\mathcal{R}_{ij}$ and a choice of simplex*

$$a_{ij} : \Delta[k] \to \mathcal{R}_{ij}$$

Under the free-forgetful adjunction, we have an adjoint map

$$\overline{a}_{ij} : \mathbb{Z}(\Delta[k]) \to \mathcal{R}_{ij}$$

in simplicial abelian stacks. Then applying the normalized Moore functor gives a map

$$N(\overline{a})_{ij} : N(\mathbb{Z}(\Delta[k])) \to N(\mathcal{R}_{ij}).$$

Since the degree k component of $N(\Delta[k]) \simeq \mathbb{Z}$, such a map chooses an element in degree k. Since N is part of a categorical equivalence it is both full and faithful. We have bijections

$$\hom(\Delta[k], \mathcal{R}_{ij}) \simeq \hom(\mathbb{Z}(\Delta[k]), \mathcal{R}_{ij}) \simeq \hom(N(\mathbb{Z}(\Delta[k])), N(\mathcal{R}_{ij})) \ . \tag{6.3.6}$$

We will often use the same symbol to denote the corresponding maps under this bijection.

We can define the following notions similarly to the classical case.

Definition 78. *We define the (stacky version) of the Maurer-Cartan equation as*

$$dA - \overline{A} \cdot A \equiv 0 \mod \ker(A) \ ,$$

for $A \in M(\mathcal{R})$ and call a solution a formal *connection with* curvature

$$\mu(A) = dA - \overline{A} \cdot A \ .$$

We are now ready to define the stacky Massey product.

Definition 79. *Let $\mathcal{R} = \{\mathcal{R}_{ij}\}$ be a sequence of abelian stacks equipped with maps*

$$\cup : \mathcal{R}_{ij} \otimes \mathcal{R}_{jk} \to \mathcal{R}_{ik} \ ,$$

which satisfy

$$\cup \circ (\mathrm{id} \otimes \cup) = \cup \circ (\cup \otimes \mathrm{id}) \ .$$

Let A is be a formal connection with curvature $\mu(A)$. Then the entries of the hypercohomology class $[\mu(A)]$ are called stacky Massey products.

Remark 19. *The following examples of stacks satisfy the compatibility requirement of definition 79 and will be of particular interest to us: Fix a manifold X, a good open cover $C(\mathcal{U})$ and a sequence (n_{ij}), $i < j \leqslant n$, of integers satisfying $n_{ij} + n_{jk} = n_{ik}$;*

1. *The stacks $[C(\mathcal{U}), \mathbb{B}^{n_{ij}-1}U(1)_{\mathrm{conn}}]$ of higher bundles with connection, with the stacky cup product.*
2. *The stacks $[C(\mathcal{U}), \mathbb{B}^{n_{ij}}\mathbb{Z}]$ of higher bundles, with the usual cup product.*
3. *The stacks $[C(\mathcal{U}), \flat_{dR}\mathbb{B}^{n}U(1)]$ of differential forms of degrees $\leqslant n$, with the wedge product.*

For each example the product is induced from the cup product structure on coefficients along with the Čech product induced from the Alexander-Whitney map. Explicitly, for section on p and q fold intersections, $\alpha_{1_0,i_1,\ldots i_p}$ and $\beta_{i_1,1_2,\ldots i_q}$ (respectively), the product is given by

$$(\alpha \cup \beta)_{i_0,\ldots i_{p+q}} := \alpha_{i_0,\ldots,i_p} \cup \beta_{i_p,\ldots,i_{p+q}} \ .$$

To simplify notion, for the above stacks we denote the corresponding sheaf of matrices

1. M_{diff}
2. M_{sing}
3. M_{dR}

respectively. We also define the 2-sided ideal $M_{\mathrm{form}} \subset M_{\mathrm{diff}}$ generated by those matrices with entries in the sheaf $[C(\mathcal{U}), \Omega^{\leqslant n-1}]$. Where $\Omega^{\leqslant n-1}$ is the sheaf fitting into the diagram in corollary 72.

We highlight the power of the above definitions in the following examples, where we are able to describe all three of the differential, singular, and de Rham triple products.

Example 25. *(Differential triple product) Let $\mathcal{G}_i$, $i = 1, 2, 3$, be bundles*

$$\mathcal{G}_i : \Delta[0] \to [X, \mathbb{B}^{n_{i,i+1}-1}U(1)_{\mathrm{conn}}] \ .$$

Suppose $\mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{G}_2 \cup \mathcal{G}_3$ represent trivial classes in $\pi_0\mathrm{Map}(X, \mathbb{B}^{n_{ij}-1}U(1)_{\mathrm{conn}})$. Choose a defining system

$$A = \begin{pmatrix} 0 & \mathcal{G}_1 & \phi_{1,2} & * \\ 0 & 0 & \mathcal{G}_2 & \phi_{2,3} \\ 0 & 0 & 0 & \mathcal{G}_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

where $\phi_{1,2}$ and $\phi_{2,3}$ are nondegenerate 1-simplices trivializing the cup products. Then the curvature of A is

$$\mu(A) = \begin{pmatrix} 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

and the hypercohomology class of $[\tau]$ is

$$[\mathcal{G}_1 \cup \phi_{2,3} - \phi_{1,2} \cup \mathcal{G}_3] \ .$$

The latter is an element in

$$\pi_1 \mathrm{Map}(X, \mathbb{B}^{n_{33}-1} U(1)_{\mathrm{conn}}) \simeq \pi_0 \mathrm{Map}(X, \flat \mathbb{B}^{n_{14}-2} U(1)) \simeq H^{n_{14}-2}(X; U(1)) \ ,$$

where we have $n_{14} = n_{13} + n_{34} = n_{12} + n_{23} + n_{34}$.

Example 26. *(Singular triple product) Let X be a manifold, and let $|X|$ be the topological space denoting its geometric realization. Let $a_i : |X| \to K(\mathbb{Z}, n_{i,i+1}) \simeq B^{n_{i,i+1}}\mathbb{Z}$, $i = 1, 2, 3$, be singular cochains with cup products vanishing in cohomology. Choose a defining system*

$$A = \begin{pmatrix} 0 & a_1 & f_{1,2} & * \\ 0 & 0 & a_2 & f_{2,3} \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Since geometric realization is a left Quillen adjoint to the discrete stack functor disc *[31], these are equivalently given by maps of stacks*

$$\bar{a}_i : \Delta[0] \to [C(\mathcal{U}), \mathbb{B}^{n_{i,i+1}}\mathbb{Z}] \ ,$$

and homotopies

$$\bar{f}_{i,i+1} : \Delta[1] \to [C(\mathcal{U}), \mathbb{B}^{n_{i,i+2}}\mathbb{Z}]$$

trivializing the cup products: hence, a defining system

$$A = \begin{pmatrix} 0 & \bar{a}_1 & \bar{f}_{1,2} & * \\ 0 & 0 & \bar{a}_2 & \bar{f}_{2,3} \\ 0 & 0 & 0 & \bar{a}_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The hypercohomology class of the entry $\tau \in \mu(A)$ is given by

$$\left[\bar{a}_1 \cup \bar{f}_{2,3} - \bar{f}_{1,2} \cup \bar{a}_3\right] ,$$

which is an element in

$$\begin{aligned} \pi_1 \mathrm{Map}(X, \mathbb{B}^{n_{14}}\mathbb{Z}) &\simeq \pi_1 \mathrm{Map}(|X|, K(\mathbb{Z}, n_{14})) \\ &\simeq \pi_0 \mathrm{Map}(|X|, K(\mathbb{Z}, n_{14} - 1)) \\ &\simeq H^{n_{14}-1}(X, \mathbb{Z}) . \end{aligned}$$

Example 27. *(de Rham triple product) Let X be a manifold and let α_i, $i = 1, 2, 3$, be closed forms in different degrees. These forms are equivalently given by maps*

$$\alpha_i : \Delta[0] \to [C(\mathcal{U}), \flat_{dR}\mathbb{B}^{n_{i,i+1}}U(1)] ,$$

where $\flat_{dR}\mathbb{B}^{n_{i,i+1}}U(1)$ is the stack of differential forms of degree $\leqslant n_{i,i+1}$. Suppose that the wedge products $\alpha_1 \wedge \alpha_2$ and $\alpha_2 \wedge \alpha_3$ are trivial in cohomology. Then we can choose a defining *system via*

$$A = \begin{pmatrix} 0 & \alpha_1 & \eta_{1,2} & * \\ 0 & 0 & \alpha_2 & \eta_{2,3} \\ 0 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The hypercohomology class of the entry $\tau \in \mu(A)$ is given by

$$[\alpha_1 \wedge \eta_{2,3} - \eta_{1,2} \cup \alpha_3] \ .$$

Since the sheaf $\flat_{dR}\mathbb{B}^{n_{i,i+1}}U(1)$ is fine in each degree, we can calculate the hypercohomology as

$$\begin{aligned} \pi_1 \mathrm{Map}(X, \flat_{dR}\mathbb{B}^{n_{14}}U(1)) &\simeq H_1 \flat_{dR}\mathbb{B}^{n_{14}}(X) \\ &\simeq H^{n_{14}-1}_{dR}(X) \ . \end{aligned}$$

Just as in the case of usual Massey products, there is indeterminacy in the stacky products. Let $\mathcal{R}_{ij}$ be a collection of abelian stacks equipped with cup products as before. Let $\mathcal{G}_i$, $1 \leqslant i \leqslant n$ be $[C(\mathcal{U}), \mathcal{R}_{i,i+1}]$. Suppose the Massey product is defined. That is, there is a formal connection A, which has $\mathcal{G}_i$ on the off-diagonal. In this case, we define this indeterminacy of the product to be the set

$$\mathrm{In}\langle \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \rangle := \{x - y : x, y \in \langle \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \rangle\}$$

In particular, if the product $\langle \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \rangle$ contains 0, then

$$\langle \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \rangle \subset \mathrm{In}\langle \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \rangle$$

Example 28. *Let $\mathcal{G}_i$, $i = 1, 2, 3$, be bundles $\Delta[0] \to [X, \mathbb{B}^{n_{i,i+1}-1}U(1)_{\mathrm{conn}}]$ with trivial cup* *products. Take the defining system in example 25 and form the triple product*

$$[\mathcal{G}_1 \cup \phi_{2,3} - \phi_{1,2} \cup \mathcal{G}_3] \in \langle \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \rangle$$

Now choose another defining system

$$B = \begin{pmatrix} 0 & \mathcal{G}_1 & \psi_{1,2} & * \\ 0 & 0 & \mathcal{G}_2 & \psi_{2,3} \\ 0 & 0 & 0 & \mathcal{G}_3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and form the product

$$[\mu(B)] = [\mathcal{G}_1 \cup \psi_{2,3} - \psi_{1,2} \cup \mathcal{G}_3] \ .$$

Then the difference is

$$[\mu(A) - \mu(B)] = [\mathcal{G}_1 \cup (\phi_{2,3} - \psi_{2,3}) - (\phi_{1,2} - \psi_{1,2}) \cup \mathcal{G}_3] \ .$$

Since

$$d\phi_{2,3} = d\psi_{2,3} = \mathcal{G}_2 \cup \mathcal{G}_3$$

and

$$d\phi_{1,2} = d\psi_{1,2} = \mathcal{G}_1 \cup \mathcal{G}_2 \ ,$$

the difference is an element in

$$[\mu(A) - \mu(B)] \in \mathcal{G}_1 \cup \pi_1 \mathrm{Map}(X, \mathbb{B}^{n_{13}} U(1)_{conn}) + \pi_1 \mathrm{Map}(X, \mathbb{B}^{n_{24}} U(1)_{conn}) \cup \mathcal{G}_3 \ .$$

This set is precisely the indeterminacy of the triple product. In this case, we can describe *the Massey triple product as an element in the quotient*

$$\mu(A) \in \frac{\pi_0 \mathrm{Map}(X, \mathbb{B}^{n_{14}} U(1))}{\mathcal{G}_1 \cup \pi_1 \mathrm{Map}(X, \mathbb{B}^{n_{13}} U(1)_{conn}) + \pi_1 \mathrm{Map}(X, \mathbb{B}^{n_{24}} U(1)_{conn}) \cup \mathcal{G}_3} \ .$$

6.4 PROPERTIES OF STACKY MASSEY PRODUCTS

Among the properties that the classical Massey products satisfy are the following:

1. *Dimension:* The dimension of $\langle x_1, x_2, \cdots, x_n \rangle$ is $\sum \deg(x_i) - n + 2$.

2. *Naturality:* If $f : X \to Y$ is a continuous map and $y_1 \cdots, y_k \in H^*(Y; R)$ such that the k-fold Massey product $\langle y_1, y_2, \cdots, y_k \rangle$ is defined, then $\langle x_1, \cdots, x_k \rangle = \langle f^*(y_1), \cdots, f^*(y_k) \rangle$ is defined as a Massey product on the cohomology of X and

$$f^*(\langle y_1, \cdots, y_k \rangle) \subset \langle f^*(y_1), \cdots, f^*(y_k) \rangle \ .$$

We now would like to extend these to the stack version. Indeed this is the case.

Proposition 80. *The stacky Massey products satisfy the following properties:*

1. Dimension: *The dimension of* $\langle \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_l \rangle$ *is* $\sum \deg(\mathcal{G}_i) - l + 2$.

2. Naturality*: If* $f : X \to Y$ *is a smooth map between manifolds and* $\mathcal{G}_1 \cdots, \mathcal{G}_k \in \hat{H}^*_{\mathcal{D}}(X; \mathbb{Z})$ *such that the* k*-fold Massey product* $\langle \mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_k \rangle$ *is defined, then* $\langle \mathcal{G}_1, \cdots, \mathcal{G}_k \rangle = \langle f^*(\mathcal{G}_1), \cdots, f^*(\mathcal{G}_k) \rangle$ *is defined as a Massey product on the differential cohomology of* X *and*

$$f^*(\langle \mathcal{G}_1, \cdots, \mathcal{G}_k \rangle) \subset \langle f^*(\mathcal{G}_1), \cdots, f^*(\mathcal{G}_k) \rangle \ .$$

Proof. Part 1. follows from noticing that each element in the Massey product is the homotopy class in

$$\pi_{k-2} \mathrm{Map}(X, \mathcal{R}_{1,k-2}) \ .$$

To prove part 2., recall that the functor $\mathrm{Map}(-, \mathcal{R})$ is contravariant, sending a map $f : X \to Y$ to its pullback

$$f^* : \mathrm{Map}(Y, \mathcal{R}_{ij}) \to \mathrm{Map}(X, \mathcal{R}_{ij}) \ .$$

Moreover, since the cup product is natural and f^* preserves the basepoint of the mapping space, it follows that the pullback sends a defining system to a defining system and naturality follows immediately. $\square$

6.5 REFINEMENTS OF DE RHAM AND SINGULAR MASSEY PRODUCTS

Although the stacky Massey product is the correct generalization of the product to the differential setting, it has the unfortunate property that it does not (in general) refine the singular or de Rham Massey products. This may seem surprising at first since the DB cup refines both the singular and de Rham cup products. However, observe that the defining equation

$$\mu = d\hat{A} + \overline{\hat{A}} \cdot \hat{A} \in \ker(\hat{A})$$

requires that the cup products and higher triple products vanish as *differential* cohomology classes. That is, we not only trivialize the bundle but the connection as well. In particular, this forces the connection to be flat and hence we have no hope of constructing a bundle whose curvature is the de Rham Massey product.

There is a way to amend this difficulty, but one needs to replace the Maurer-Cartan equation with a "twisted" equation which does refine the singular Massey product. We will see that the curvature of solution to such an equation is the de Rham Massey product. This point of view illuminates the true nature of differential refinements of Massey products, but treats the de Rham Massey product as a *curvature*. In particular, this implies that the integral of such forms are quantized, which is an added restriction in application.

We begin with a useful lemma.

Lemma 81. *Let $\mathcal{F}_{ij} \to \mathcal{R}_{ij} \twoheadrightarrow \mathcal{S}_{ij}$ be a fibration sequence abelian prestacks for each i and j. Suppose, moreover, that we have commuting diagrams*

$$\begin{tikzcd}
\mathcal{F}_{ij} \otimes \mathcal{F}_{jk} \arrow[r, "\cup"] \arrow[d, "i \otimes i"] & \mathcal{F}_{ik} \arrow[d, "i"] \\
\mathcal{R}_{ij} \otimes \mathcal{R}_{jk} \arrow[r, "\cup"] \arrow[d, "p \otimes p"] & \mathcal{R}_{ik} \arrow[d, "p"] \\
\mathcal{S}_{ij} \otimes \mathcal{S}_{jk} \arrow[r, "\cup"] & \mathcal{S}_{ik} .
\end{tikzcd}$$

Then the induced sequence $0 \to M(\mathcal{F}) \to M(\mathcal{R}) \twoheadrightarrow M(\mathcal{S}) \to 0$ is a short exact sequence of sheaves of differentially graded rings.

Proof. Since the normalized Moore functor is right Quillen and preserves equivalences, it follows that it sends fiber sequences to fiber sequences. Hence, we have a diagram

$$\begin{tikzcd}
N(\mathcal{F}_{ij}) \otimes N(\mathcal{F}_{jk}) \arrow[r, "\cup"] \arrow[d, "i \otimes i"] & N(\mathcal{F}_{ik}) \arrow[d, "i"] \\
N(\mathcal{R}_{ij}) \otimes N(\mathcal{R}_{jk}) \arrow[r, "\cup"] \arrow[d, "p \otimes p"] & N(\mathcal{R}_{ik}) \arrow[d, "p"] \\
N(\mathcal{S}_{ij}) \otimes N(\mathcal{S}_{jk}) \arrow[r, "\cup"] & N(\mathcal{S}_{ik}) ,
\end{tikzcd}$$

where the right hand side is a short exact sequence of presheaves of chain complexes. By definition, it follows that we have a short exact sequence

$$0 \to M(\mathcal{F}) \to M(\mathcal{R}) \twoheadrightarrow M(\mathcal{S}) \to 0$$

of chain complexes. By commutativity of the above diagram, both maps are homomorphisms of presheaves of bigraded rings. □

It follows from the lemma along with corollary 72, that there is a short exact sequence of presheaves of bigraded rings

$$0 \to M_{\text{form}} \to M_{\text{diff}} \xrightarrow{p} M_{\text{sing}} \to 0 \tag{6.5.1}$$

Hence, M_{form} is a two-sided ideal in M_{diff} and we have an isomorphism of presheaves

$$I : M_{\text{diff}}/M_{\text{form}} \to M_{\text{sing}} \ . \tag{6.5.2}$$

Now recall that for a matrix $A \in M_{\text{sing}}$, we defined the kernel of A to be the set of matrices of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & * \\ 0 & 0 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} . \tag{6.5.3}$$

By the above short exact sequence, it is clear that for any $C \in \ker(A)$, $\hat{C} - B \in \ker(\hat{A})$ for some $B \in M_{\text{form}}$.

We are now ready to prove the following theorem:

Theorem 82. *Let A be a formal connection for M_{sing}, and let $\hat{A}$ be a differential refinement of A. Then any differential refinement $\widehat{\mu(A)}$ satisfies the twisted Maurer-Cartan equation*

$$\mu(\hat{A}) = d\hat{A} - \overline{\hat{A}} \cdot \hat{A} \equiv B \mod \ker(\hat{A}) \ ,$$

where B is some matrix in the ideal M_{form}. Moreover, under the curvature map curv*, we* *have that the de Rham cohomology class* $[\mathrm{curv}(\mu(\hat{A}))] = [\mu(\mathrm{curv}(\hat{A}))]$ *is a de Rham Massey product.*

Proof. Since A is formal, $\mu(A)$ satisfies

$$\mu(A) = dA - \overline{A} \cdot A \equiv 0 \mod \ker(A) \ .$$

Hence, by (6.5.2) any refinement must satisfy the twisted equation

$$\mu(\hat{A}) = d\hat{A} - \overline{\hat{A}} \cdot \hat{A} \equiv B \mod \widehat{\ker(A)} \ .$$

where B is a matrix in M_{form}. Now since any Čech de Rham form η can be written as $\eta = \omega + (d' + (-1)^p\delta)\alpha$, where ω is globally defined, $\deg(\alpha) = p$ and d' is the exterior derivative operator [4], we can write

$$B = C + dD$$

where d is the Čech-de Rham differential and C is a matrix of global forms. Now by definition, the curvature map curv simply applies the exterior derivative to each entry in the matrix (the vertical differential on the Čech de Rham double complex). We have

$$\begin{aligned}
\mathrm{curv}(B) &= \mathrm{curv}C + \mathrm{curv}dD \\
&= d'C + d'dD \\
&= d'C + d'\delta\overline{D} \\
&= d(C + \delta\overline{D})
\end{aligned}$$

and $\mathrm{curv}(B)$ is exact in M_{dR}. Now recall that curv is a homomorphism of bigraded rings $\mathrm{curv} : M_{\mathrm{diff}} \to M_{\mathrm{dR}}$ and we have that $\mathrm{curv}(\ker(\hat{A})) \subset \ker(\mathrm{curv}(\hat{A}))$. Hence, applying curv to $\mu(\hat{A})$ gives a matrix in M_{dR} cohomologous to $\mu(\mathrm{curv}(A))$. $\square$

It is interesting to note that the previous proposition states that the obstruction to lifting the singular Massey product to the differential Massey product is measured by the curvature of a bundle satisfying the twisted equation. Moreover, this curvature is precisely the de Rham Massey product.

Remark 20. *Note that in the previous theorem we have used the fact that the differential on M_{dR} is given by the differential on the Čech de Rham double complex. In fact, for a manifold X and $C(\mathcal{U})$ a Čech resolution corresponding to a fixed open cover, we have*

$$N[C(\mathcal{U}), \flat_{dR}\mathbb{B}^n U(1)] \simeq [N(C(\mathcal{U})), \Omega^{\leqslant n}_{cl}]$$

where the right hand side is the internal hom in presheaves of positively graded chain complexes. Using the formula for the internal hom, one immediately sees that the right hand side is exactly the truncated Čech de Rham double complex.

The following proposition shows that the strict definition of the differential Massey product forces the resulting class to be flat.

Proposition 83. *Let $\hat{A}$ be a formal connection in M_{diff}. Then the generalized differential Massey products are flat.*

Proof. Since $\hat{A}$ satisfies the Maurer-Cartan equation, the entries of curvature $\mu(\hat{A})$ are hypercohomology classes of some degree $n_j > j > 0$. But these are elements of

$$\pi_j \mathrm{Map}(X, \mathbb{B}^{n_j} U(1)_{conn}) \simeq \pi_0 \mathrm{Map}(X, \flat\mathbb{B}^{n_j - j} U(1)).$$

□

We now put the above results into an explicit construction.

Example 29. *(Refinement of triple product) Let a_i, $i = 1, 2, 3$, be singular cochains of degree*

$n_{i,i+1}$. *Suppose that the triple product is defined and choose a defining system*

$$A = \begin{pmatrix} 0 & a_1 & \phi_{1,2} & * \\ 0 & 0 & a_2 & \phi_{2,3} \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\hat{A} = \begin{pmatrix} 0 & \hat{a}_1 & \hat{\phi}_{1,2} & * \\ 0 & 0 & \hat{a}_2 & \hat{\phi}_{2,3} \\ 0 & 0 & 0 & \hat{a}_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be a refinement. Then we know that the refinement $\widehat{\mu(A)}$ *satisfies the equation* $d\hat{A} = \hat{A} \cdot \overline{\hat{A}} + B$ *up to some element in* $\ker(\hat{A})$. *Explicitly, letting* $B = (\eta_{ij})$, *we have*

$$\hat{A} = \begin{pmatrix} 0 & 0 & d\hat{\phi}_{1,2} & d* \\ 0 & 0 & 0 & d\hat{\phi}_{2,3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \eta_{12} & \hat{a}_1 \cup \hat{a}_2 + \hat{\eta}_{13} & \hat{a}_1 \cup \hat{\phi}_{2,3} - \hat{\phi}_{1,2} \cup \hat{a}_3 \\ 0 & 0 & \eta_{23} & \hat{a}_1 \cup \hat{a}_2 + \hat{\eta}_{24} \\ 0 & 0 & 0 & \eta_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The requirement that this equation hold up to an element in ker$(\hat{A})$ *forces the equations*

- $\eta_{12} = 0$,
- $\eta_{23} = 0$,
- $\eta_{34} = 0$
- $d\hat{\phi}_{1,2} = \hat{a}_1 \cup \hat{a}_2 + \eta_{13}$,
- $d\hat{\phi}_{2,3} = \hat{a}_2 \cup \hat{a}_3 + \eta_{14}$.

Forming $\widehat{\mu(A)}$ gives the matrix

$$\widehat{\mu(A)} = \begin{pmatrix} 0 & 0 & 0 & \hat{a}_1 \cup \hat{\phi}_{2,3} - \hat{\phi}_{1,2} \cup \hat{a}_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now set $G_i := \mathrm{curv}(\hat{a}_i)$ *and let* A_i *denote the connection of the bundle* $\hat{a}_i$. *Applying* curv *to the matrix* (29) *and using the definition of the DB cup product gives a matrix with only non zero term*

$$\begin{aligned} \mathrm{curv}\left(\hat{a}_1 \cup \hat{\phi}_{2,3} - \widehat{\phi_{1,2}} \cup \hat{a}_3\right) &= G_1 \wedge \mathrm{curv}(\hat{\phi}_{2,3}) - \mathrm{curv}(\widehat{\phi_{1,2}}) \wedge G_3 \\ &= G_1 \wedge (A_2 \wedge G_3 - \eta_{14}) - (A_1 \wedge G_2 - \eta_{13}) \wedge G_3 \\ &= d(A_1 \wedge A_2 \wedge G_3) - (G_1 \eta_{14} - \eta_{13} \wedge G_3) . \end{aligned}$$

and the class $[G_1 \eta_{14} - \eta_{13} \wedge G_3]$ *is a Massey product.*

Remark 21. *We can view Example 29 as a construction giving the refinement of a triple Massey product explicitly. This construction will, in fact, be very useful in applications next section, where we start with classical Massey product and then refine them.*

7.0 APPLICATIONS IN PHYSICS

In this section, we will discuss a wide variety of applications. These applications have been worked out in [15] and we refer the reader there for details. Here, we will present only one application in order to highlight the strength of differential Massey products. There are many more applications which are considered in [15]

Trivializations of string and fivebrane structures In application, it frequently happens that a certain characteristic form is trivial in cohomology, but still contains come important geometric data. When these characteristic forms arise as a cup product, the trivializations naturally provide a defining system for a Massey product.

Example 30. *(Geometric String structures) Recall that a string structure on a manifold X exists when the fractional second Pontryagin class*

$$\frac{1}{6}p_2(TX) = 0$$

in cohomology and the set of such classes is parametrized by the trivializations of this class. It is well known that trivializations of the refinement *of this class are in correspondence with* geometric *string structures. That is, string bundles equipped with connection [32]. Let us consider the case when the refinement*

$$\frac{1}{6}\hat{p}_2(TX) = 0$$

in differential cohomology. Moreover, suppose that

$$\frac{1}{6}\hat{p}_2(TX) = \mathcal{G} \cup \mathcal{H}$$

for a 2-bundle $\mathcal{G}$ *and a 0-bundle* $\mathcal{H}$*. Let* ϕ *be a trivialization of* $\mathcal{G} \cup \mathcal{H}$ *(a geometric string structure) and suppose* $\mathcal{H} \cup \mathcal{H}$*. Notice that by the anti-commutativity of the DB cup product, we necessarily have that*

$$2\mathcal{H} \cup \mathcal{H} = 0$$

and $\mathcal{H}$ *must be at least* 2*-torsion. It is therefore not too restrictive to consider the case where* $\mathcal{H} \cup \mathcal{H} = 0$*. Now form the defining system*

$$A = \begin{pmatrix} 0 & \mathcal{G} & \phi & * \\ 0 & 0 & \mathcal{H} & 0 \\ 0 & 0 & 0 & \mathcal{H} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

and form the cocycle

$$\mu(A) = \phi \cup \mathcal{H}$$

The class of this cocycle is an element of the Massey triple product $\langle \mathcal{G}, \mathcal{H}, \mathcal{H} \rangle$ *and is built out of the geometric string structure* ϕ*.*

Remark 22. *One can ask a similar question for fivebrane structures and beyond. We address not only the previous example, but these as well in [15]*

Massey product in Chern-Simons theories In [15] we consider a general class of theories of Chern-Simons type arising from supergravity and string theory. Such theories have not only the usual Chern-Simons topological terms usually considered in this context, but also have kinetic terms. As such, an instance of the Massey product description can be viewed as an extension of topological theories to include the kinetic terms.

Two such theories share several striking features from the geometric and topological point of view, namely theories in five and eleven dimensions. Their relationship to Massey products was first observed by Hisham Sati and Igor Kriz in [20].

Example 31. *(Supergravity fields) Supergravity theories in five and eleven dimensions can be described in dimension d by action functionals which include terms of the form (up to numeric factors)*

$$S_n = F_n \wedge *F_n + A_{n-1} \wedge F_n \wedge F_n \ . \tag{7.0.1}$$

Here F_n is a differential form of degree n, which is to be generically interpreted as a curvature of an $(n-2)$-gerbe with connection A_{n-1}, and $*$ is the Hodge duality operation in dimension d. The equations of motion, obtained by varying with respect to A_{n-1} via the variational principle, are then of the form

$$d(*F_n) = F_n \wedge F_n \ . \tag{7.0.2}$$

We observe that the cohomology class corresponding to $F \wedge F$ is trivial in cohomology, with trivialization given by $*F$. This then gives rise to a defining system for a Massey triple product involving three copies of F, i.e. $\langle F, F, F \rangle$.

Though this is a classical example of a Massey product, it is interesting in its own right. In [15], we consider not only this classical example but its refinement to differential cohomology.

These are only a few of the many applications we have found for differential Massey products. Others include refinements of formality, action functionals in supergravity, Ramond Ramond fields in type II string theory and Green-Schwarz anomaly cancelation. Although many of these applications are discussed in [15], this list is far from exhaustive. Ultimately we hope to understand the cohomology operations enough to use in the spectral sequences for differential cohomology theories, although this work is still in progress.

A SIMPLICIAL SETS

Let Δ denote the category of linearly ordered sets of n elements $\{0 < 1 < \dots, < n\}$ with order preserving maps as morphisms. The category of simplicial sets is the category of functors $s\mathcal{S}\mathrm{et} := [\Delta, \mathcal{S}\mathrm{et}]$. Explicitly, we can view an object $X \in [\Delta, \mathcal{S}\mathrm{et}]$ as a sequence of sets X_n, $n \in \mathbb{N}$, connected by maps

$$\dots X_n \dots \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X_2 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X_1 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X_0 \tag{1.0.1}$$

where at each stage, the maps pointing to the left are induced from order preserving surjections $[n] \twoheadrightarrow [n-1]$ (of which there are n), and the maps pointing to the right are induced from order preserving injections $[n-1] \to [n]$ (of which there are $n+1$). These maps are usually called the *degeneracy* and *face* maps (respectively).

Example 32. *Let $[n] \in \Delta$ be a linearly ordered set. Then we can apply the Yoneda embedding to obtain a simplicial set $y([n])$ which in degree $[m]$ is the set*

$$y([n])_m = \hom([m], [n])$$

Notice that in degree m there are $m+1$ face maps going out and m degeneracies going in. Moreover in degree n we have

$$y([n])_n = \hom([n], [n]) = \{\mathrm{id}\} .$$

It is not hard to see that this simplicial set is a combinatorial model for the standard n-

simplex. We usually denote the embedding $y([n]) =: \Delta[n]$ to make this explicit.

Simplicial sets can be viewed as a combinatorial model for topological spaces. In fact, we have a Quillen equivalence

$$|\cdot| : s\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{sing} \quad . \tag{1.0.2}$$

This is one version of the well known "homotopy hypothesis". This shows that to understand topological spaces up to homotopy equivalence, it is enough to understand Kan complexes up to homotopy equivalence.

Just as spaces have Postnikov systems and CW approximations, so do simplicial sets. In fact, by truncating the simplicial set, it is quite easy to produce such systems. More precisely, given a simplicial set $X \in s\mathcal{S}\mathrm{et} = [\Delta^{op}, \mathcal{S}\mathrm{et}]$, we can define the n-th truncation of X by restriction of the functor X to the full subcategory $\Delta^{\leqslant n}$ on objects $[m]$ such that $m \leqslant n$. That is,

$$\tau_n X : \Delta^{\leqslant n \ op} \hookrightarrow \Delta^{op} \xrightarrow{X} \mathcal{S}\mathrm{et}$$

Now τ_n admits both a left and right adjoint σ_n and σ^n given by the left and right Kan extensions (respectively). We define the n-th skeleton functor

$$\mathrm{sk}_n : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$$

as the composite $\sigma_n \circ \tau_n$ and the n-th coskeleton functor

$$\mathrm{cosk}_n : s\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$$

as the composite functor $\sigma^n \circ \tau_n$. These functors have the following properties [14]:

Proposition 84. *Let X be a simplicial set. Then the maps*

$$\pi_k(|\mathrm{sk}_n X|) \to \pi_k(|X|)$$

and

$$\pi_k(|\mathrm{cosk}_n X|) \leftarrow \pi_k(|X|) ,$$

induced by the unit and counits of the adjunctions $(\tau_n \vdash \sigma^n)$ *and* $(\sigma_n \vdash \tau_n)$ *(respectively), are isomorphisms for* $k \leqslant n-1$ *and surjective for* $k = n$.

One can think of the functors $\mathrm{sk_n}$ as giving a CW approximation for X and $\mathrm{cosk_n}$ as the Postnikov tower for X.

We can also define simplicial objects in categories other than $\mathcal{S}\mathrm{et}$. In particular the category of simplicial abelian groups will be useful to us. We define the category of simplicial abelian groups to be the functor category

$$s\mathcal{A}\mathrm{b} := [\Delta^{op}, \mathcal{A}\mathrm{b}] .$$

Explicitly, an object is a sequence of abelian groups A_n, $n \in \mathbb{N}$, connected by face and degeneracy maps

$$\dots A_n \dots \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; A_2 \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; A_1 \; \substack{\longleftarrow\\ \longrightarrow\\ \longrightarrow} \; A_0 \quad . \tag{1.0.3}$$

This is almost identical to the definition of a simplicial set. The only difference is that all of the above morphisms are homomorphisms of abelian groups. If we forget about the abelian group structure we recover a simplicial set. In fact, it is easy to see that the forgetful functor

$$U : s\mathcal{A}\mathrm{b} \to s\mathcal{S}\mathrm{et}$$

is a faithful embedding.

B KAN EXTENSIONS AND COENDS

Definition 85. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be functors. The* left Kan extension *of F along G, if it exists, is a functor $Lan_G(F) : \mathcal{E} \to \mathcal{D}$ and a natural transformation $\eta : F \Rightarrow Lan_G(F) \circ G$ that is universal in the sense that it is the unique functor filling the diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] \arrow[dr, phantom, shift left=4ex, "\Downarrow \eta"] & & \mathcal{D} \\
& \mathcal{E} \arrow[ur, dashed, "Lan_G(F)"'] &
\end{tikzcd}$$

.

Figure B.1: Universal property for Kan extensions

The right Kan extension *is defined similarly but with a reversed natural transformation $\epsilon : Ran_G(F) \Rightarrow F$.*

Example 33. *Let $\mathcal{C}, \mathcal{D}$ be a categories of weak equivalences and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the left and right derived functors of F are left and right Kan extensions (resp.) of the functor $Q_D F : \mathcal{C} \to \mathrm{Ho}\mathcal{D}$ along $Q_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}\mathcal{C}$.*

the previous example shows that, in particular, homotopy limits and colimits arise as left and right Kan extensions. It may not be immediately obvious to the reader viewing homotopy limits and colimits as Kan extensions is helpful in calculation. However, we will see that there is a particularly nice formula for the evaluation of such extensions on objects. This formula comes in the form of a coend, which we now introduce.

Definition 86. *Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a bifunctor and let $X : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a constant diagram $X(A, B) = X \in \mathcal{D}$. A* coend*, if it exists, is the universal such object and natural transformation $\eta : F \to X$. Equivalently, it is the coequalizer*

$$\coprod_{f:A\to B} F(B,A) \underset{F(f,id)}{\overset{F(id,f)}{\rightrightarrows}} \coprod_A F(A,A) \xrightarrow{\ \eta\ } X$$

We will use the notation

$$\int^{A\in\mathcal{C}} F(A,A)$$

for the universal object and call it the *coend*. We leave the natural transformation implicit unless otherwise stated.

The following proposition provides an explicit calculation of left Kan extensions in terms of coends.

Proposition 87. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be functors. Then if the left Kan extension $Lan_G(F)$ exists, it is given (on objects) by the formula*

$$Lan_G(F)(A) = \int^{A\in\mathcal{C}} \mathcal{E}(GA, B) \cdot F(A)$$

whenever the right hand coend and tensor makes sense.

Proof. See [22] □

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