ON THE DYNAMICS OF A RIGID BODY WITH CAVITIES COMPLETELY FILLED BY A VISCOUS LIQUID

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This thesis deals with the dynamics of a coupled system comprised of a rigid body containing one or more cavities entirely filled with a viscous liquid. We will present a rigorous mathematical analysis of the motions about a fixed point of this system, with special regard to their asymptotic behavior in time.

In the case of inertial motions and motions under the action of gravity, we will show that viscous liquids have a stabilizing effect on the motion of the solid. The long-time behavior of the coupled is characterized by a rigid body motion, and in particular a permanent rotation in the case of inertial motions, and the rest state in the case of a liquid-filled heavy pendulum. Some questions about the attainability and stability of the equilibrium configurations are also answered.

Furthermore, we will investigate the time-periodic motions performed by the coupled system liquid-filled rigid body when a time-periodic torque is applied on the solid.

Keywords: Liquid-filled cavity, Navier-Stokes equations, rigid body, inertial motions, pendulum, attainability, stability, periodic motions.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

PREFACE

I would like to express my gratitude to Professor Giovanni P. Galdi for accepting me as his graduate student, for his precious academic advising and mentorship in these five years of collaboration. I owe him all my knowledge on the mathematical theory of the Navier-Stokes equations, fluid/solid interaction problems and many other things of life.

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Many thanks go to my friends and colleagues here and spread around the world, and in a special way to Annalisa Pellegrini for her twenty-five years long friendship.

This thesis is dedicated to my parents: to my mother, the strongest woman I know, and the role model I will always tend to, and to my father, the first man I loved in my life, no man will ever take his place.

1.0 INTRODUCTION

In this thesis, I will present a comprehensive study of the dynamics of a rigid body containing one or more cavities entirely filled by a viscous fluid (simply called *liquid*), whose motion is governed by the Navier-Stokes equations.

The main focus of this work is centered around a number of problems that I have investigated in collaboration and under the supervision of Dr. G. P. Galdi in the past years.

Specifically, consider a rigid body $\mathcal{B}$, which is constrained to move (without friction) about a fixed point in the physical space. Assume that, inside $\mathcal{B}$, there is a hollow cavity, $\mathcal{C}$, completely contained in the solid, and entirely filled by a viscous liquid. The assumption of only one cavity within $\mathcal{B}$ is not restrictive. In fact, all the results shown in this thesis still apply in the case of more than one cavity.

The long-time behavior of this coupled system liquid-filled rigid body is expected to be peculiar. In fact, after an initial interval of time, whose length depends on the size of initial data and viscosity of the liquid, where the motion has a typically "chaotic" nature, the coupled system eventually reaches a more orderly configuration, due to the combined effect of viscosity and incompressibility (see Chapter 3 and Chapter 4). This feature was first pointed out in 1885, for the simple case of *inertial motions* (i.e. when no external forces are applied on the system), by N. Ye. Zhukovskii, who stated that the asymptotic behavior of a liquid-filled rigid body is characterized by a rigid body motion, and precisely a uniform rotation (here called *permanent rotations*) with the liquid at rest relatively to the solid, no matter the physical and geometrical properties of both liquid and solid, and the initial movement imparted on the system (Theorem, 3.0.6). This property has to contrasted with the case of an empty cavity. Indeed, the dynamics of a rigid body undergoing inertial motions are quite rich: permanent rotations can occur if and only if the (uniform) angular velocity is directed along one of the principal axes of inertia of $\mathcal{B}$. Whereas, time-dependent motions can be very involved depending on the mass distribution of the solid. *Regular precessions* furnish an example of these time-dependent motions which fall in the more general class of *motions à la Poinsot* (see [35]).

In a similar fashion, experimental and theoretical studies show that a physical pendulum with a cavity completely filled with a liquid, that is initially at rest, eventually reaches the equilibrium configuration which is the *rest state*, with the center of mass at its lowest position (see Chapter 3 and Section 5.3). The behavior is the same as for a classical pendulum immersed in a liquid, although the global dynamics is quite different in both cases (see Section 5.3). The conclusion is that the liquid is able to reduce the (otherwise undamped) oscillations that a heavy pendulum executes when it rotates frictionless about a fixed axis. Summarizing, a liquid which completely fills a cavity within a rigid body has the property of stabilizing the motion of the solid. In the particular case of a pendulum, this feature can be viewed as an *internal damping* for the system. This is a remarkable property if one thinks about possible uses in the Applied Sciences. In fact, this stabilizing effect has been largely used (even when a rigorous mathematical proof was still lacking) in the Applied Sciences, and in particular for geophysical problems in the research of sources generating the geomagnetic field (see Subsection 2.1.2.1, more details can be found in [50]), for military purposes in the study of the dynamics of fuel-filled projectiles ([46]), and for applications in Space Engineering where, just as example, *passive dampers* constituted by a circular ring entirely filled by a liquid are analyzed in order to stabilize the motion of spacecrafts when they undergo some "wobbling" motions, i.e. precessional motions of the spin axis caused either by mechanical maneuvers or by external disturbance (like the presence of a magnetic field), we refer the reader to the Subsection 2.1.2.2 and to [5] for the details. Besides the interests for the applications, the problem of the motions of liquid-filled rigid bodies have caught the attentions of many mathematicians too. In this respect, I would like to mention the works by Stokes, [51], Zhukovskii, [55], Hough, [24], and the most recent contributions by Sobolev, [49], Rumyantsev, [41, 40, 42, 43], Chernousko, [12], Smirnova, [48], and Kopachevsky and Krein, [26]. An account of these results is presented in Chapter 3; one can see there that these results are rarely of an exact nature, either due to the approximate models or else due to an approximate mathematical treatment. The primary goal of this thesis is to perform a rigorous mathematical analysis of the motions about a fixed point of a rigid body with an interior cavity completely filled by a viscous liquid, with special regard to their asymptotic behavior in time. I have studied the following problems:

a) Motions about a fixed point of rigid bodies with liquid-filled cavities under given constant forces (Chapter 5). In particular, I have considered:

   a1) *Inertial motions* about the center of mass of a rigid body with a hollow cavity completely filled by a Navier-Stokes liquid (Section 5.2).

   a2) Motions of a *physical pendulum* containing a cavity completely filled by a viscous liquid (Section 5.3).

b) Motion of the coupled system liquid-filled rigid body under the action of a *time-periodic torque* applied on the solid (Chapter 6).

Concerning the inertial motions about the center of mass of the whole system, we have rigorously shown that Navier-Stokes liquids indeed have a *stabilizing effect* on the motion of the solid. More precisely, we have proved that *weak solutions* (*à la Leray-Hopf*) corresponding to initial data having arbitrary but finite total energy, as time approaches to infinity, converge (in a proper topology) to a state of motion characterized by zero relative velocity of the fluid with respect to the rigid body and constant angular velocity. This means that, eventually, the coupled system fluid-filled rigid body moves as a single rigid body with constant rotation, thus proving Theorem 3.0.6 conjectured by Zhukovskii in 1885. The method we use to show the above results utilizes tools from classical Dynamical System theory. The main difficulty here is the convergence of weak solutions to the corresponding Ω-limit set, which is characterized by zero relative velocity of the liquid and a constant angular velocity about the principal axes of inertia of the whole system. This is achieved by proving that the Ω-limit set is positively invariant in the class of weak solutions. From the Dynamical Systems theory, it is known that the invariance property requires the uniqueness (and more generally, continuous dependence upon initial data) of the solutions. This latter property is not yet available in my case, as the weak solutions posses (in their liquid variable) the same peculiarities as the three-dimensional weak solutions to the Navier-Stokes equations. Nevertheless, since the relative velocity decays to zero as time approaches to infinity (see (5.12) and (5.17)), I have recovered the invariance property by demonstrating that weak solutions become strong (and therefore unique) for sufficiently "large" times, with no restriction on the size of initial data (Proposition 5.1.5). This is by no means a trivial property in my case, due to the presence of a, in principle, large "conservative" components of my equations, given by the conservation of the total angular momentum of the whole system. Theorem 5.2.4 furnish a rather complete description of the asymptotic behavior in time of the coupled system solid-liquid. However, this theorem is silent about which axis the "final" permanent rotation is attained. We have then investigated the *attainability of permanent rotations*. This is not an obvious problem. In fact, due to the coupling with the Navier-Stokes equations, our weak solutions may lack uniqueness; in principle, we may have two different solutions with the same initial data generating, asymptotically, two permanent rotations around different axes. In this regards, I have have shown that, for initial data in a suitable range, the coupled system tends eventually to reach the state of *minimal motion*; in other words, it chooses to rotate around the axis where the spin is a (non-zero) minimum (Theorem 5.2.7). Using these attainability results, I furnish necessary and sufficient conditions for the stability (in the sense of Lyapunov) for the full nonlinear problem without any approximation or assumptions on the shape of the cavity (Theorem 5.2.10). This analytical study has been enriched by targeted numerical and experimental tests. The numerical ones simulate the inertial motions of rigid body with a *quasi-ellipsoidal* cavity entirely filled by a viscous liquid. The physical experiments investigate the motions of a liquid-filled gyroscope (Chapter 4). Both numerical and physical experiments agree with the theoretical findings and furnish useful insights for questions which are still open. In particular, accordingly to Theorem 5.2.7, the final permanent rotation occurs around the principal axis corresponding to the largest moment of inertia, and, at least under suitable "smallness" assumptions, having same or opposite orientation as the initial angular momentum $\boldsymbol{K}_G$ given to the system. Numerical tests show a *flip-over* phenomenon triggered by the viscosity of the liquid. Precisely, as the kinematic viscosity coefficient ν is decreased from a "sufficiently large" value -all other data being kept fixed- there is a critical value ν_c such that the orientation of the final rotation and $\boldsymbol{K}_G$ are the same or opposite according to whether $\nu > \nu_c$ or $\nu < \nu_c$.

Next, I have considered the motions of a liquid-filled physical pendulum, i.e. a *heavy* rigid body constrained to rotate (frictionless) around a horizontal axis, in such a way that the center of mass of the whole system moves on a plane orthogonal to the axis of rotation, and containing a cavity completely filled by a viscous liquid. For the long-time behavior of this coupled system, one can show the convergence, for large times, of the weak solutions to the *rest state* (zero velocities for both solid and liquid), no matter the shape of the cavity, the physical properties of the solid and the liquid, and the initial conditions imposed on the whole system (Theorem 5.3.3). Moreover, I show the existence of a broad class of initial data, corresponding to which the rest state with the center of mass at its lower position is attained (Theorem 5.3.5). Concerning the stability, Theorem 5.3.6 infers that the rest state with the centre of mass of the system occupying its lowest position is always stable (in the sense of Lyapunov). Whereas, the rest state with the center of mass at its highest position is always unstable (in the sense of Lyapunov). All the results concerning the attainability and the stability of the equilibrium configurations hold with no further assumption and/or restriction on the shape of the cavity.

Finally, I will present a rigorous study of the motions under the action of a time-periodic torque applied on the solid. Specifically, I have investigated whether the coupled system executes a time-periodic motion in a *moving frame* with origin at the center of mass of the whole system and axes directed along the principal axes of inertia of the system. This kind of motion is possible if the torque is directed along a constant direction with respect to an inertial frame, and has a time-periodic magnitude with zero average. The existence of time-periodic weak solutions to the relevant equations of motion can be proved (Theorem 6.1.1). Moreover, if the magnitude of the torque is essentially bounded by a sufficiently small norm, then the solution is strong and the equations of motion are satisfied almost everywhere in space-time (Theorem 6.2.1). The proof of existence of weak and strong solutions can be achieved by an appropriate combination of the Galerkin method with a fixed point argument for triangulable manifolds based on *Lefschetz-Hopf Theorem*.

Here is the plan of my thesis. In Chapter 2, the problem of motions of a liquid-filled rigid body about a fixed point is introduced in both an inertial and a moving frame. Some applications in the Applied Sciences are presented in order to motivate our interest in this kind of problems. Moreover, notation and well-known inequalities are recalled to create a self-contained treatment. In Chapter 3, I give an account of the historical background and some of the previous mathematical contributions. In Chapter 4, numerical and physical experiments are presented. In Chapter 5, I exploit a dynamical system approach to investigate the motions of a liquid-filled rigid body which undergoes inertial motions about its center of mass and motions around a fixed axis when gravity is applied on the system. The numerical and analytical results presented in Chapters 4 and 5 first appeared in [20], a detailed account of them can be found in [13]. For the motions of a liquid-filled physical pendulum we refer to [17]. Some preliminary results as well as technicalities can also be found in my thesis [33]. In Chapter 6, I investigate the periodic motions of a rigid body with a liquid-filled cavity in a moving frame, when a time-periodic torque is applied on the body. These results have been published in [18].

2.0 MOTIONS OF A LIQUID-FILLED RIGID BODY ABOUT A FIXED POINT

In this chapter, the problem of motions about a fixed point of a rigid body with a cavity completely filled by a viscous liquid is introduced. The mathematical formulation of the problem is given in an inertial and a moving frame, respectively. Moreover, we present some applications in the Applied Sciences, in particular for geophysical and engineering problems.

We will conclude this chapter by introducing the mathematical notations used throughout this thesis, and some well-known results of the Mathematical Analysis aimed to make this thesis self-contained.

2.1 MATHEMATICAL FORMULATION AND RELEVANT APPLICATIONS

In this section, we introduce the equations governing the motions about a fixed point of a rigid body with a liquid-filled cavity when external forces and torques are applied on the system. Moreover, applications in Geophysics and Space Engineering are presented.

2.1.1 Mathematical formulation

Consider a rigid body, $\mathcal{B}$, with a cavity, $\mathcal{C}$, completely filled with a viscous liquid of constant density, ρ. In mathematical terms, $\mathcal{B} := \Omega_1 \setminus \overline{\Omega_2}$ and $\mathcal{C} := \Omega_2$, where, for $i = 1, 2$, Ω_i are simply connected, bounded domains in $\mathbb{R}^3$. Throughout this thesis, we shall make the following two assumptions:

H1. The coupled system $\mathcal{S} := \mathcal{B} \cup \mathcal{C}$ is constrained to move (without friction) about a fixed point $O \in \mathcal{B}$, which is at rest with respect to an inertial frame at all times.

H2. The center of mass, G, of the coupled system $\mathcal{S}$ belongs to one of the principal axes of inertia, $\mathbf{a}$, of $\mathcal{S}$ with respect to O [1].

The equations of motion for the coupled system $\mathcal{S}$ with respect to an inertial frame of reference $\mathcal{I} \equiv \{O; \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$, with its origin at the fixed point O and coordinate axes $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$ and $\tilde{\boldsymbol{e}}_3$, are given by

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{w} = 0 \\ &\rho \frac{d\boldsymbol{w}}{dt} = \operatorname{div} \boldsymbol{\mathcal{T}} + \rho \boldsymbol{\mathcal{F}} \\ &\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}^T \end{aligned}\right\} (\boldsymbol{x}, t) \in \bigcup_{t>0} \mathcal{C}(t) \times \{t\},$$
$$\frac{d(\boldsymbol{J}_{\mathcal{B}} \cdot \boldsymbol{\Omega})}{dt} = \mathbf{M} - \int_{\partial \mathcal{C}(t)} (\boldsymbol{x} - \boldsymbol{x}_O) \times \boldsymbol{\mathcal{T}} \cdot \mathbf{n} \, d\sigma. \tag{2.1}$$

Here $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, t)$ denotes the Eulerian absolute velocity of the liquid and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(t)$ is the angular velocity of the body. $\boldsymbol{J}_{\mathcal{B}} = \boldsymbol{J}_{\mathcal{B}}(t)$ is a positive definite, symmetric tensor, denoting the inertial tensor of the rigid body written with respect to O. $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{C}$, and $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}(\boldsymbol{w}, \Pi)$ is the Cauchy stress tensor with $\Pi = \Pi(\boldsymbol{x}, t)$ the Eulerian pressure of the liquid. Furthermore, $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\boldsymbol{x}, t)$ and $\mathbf{M} = \mathbf{M}(t)$ are the total *body force* (i.e. the total external force per unit volume) acting on (an infinitesimal volume of) the liquid and the total external torque with respect to O applied on the body. It is worth emphasizing that, when the equations of motion are written with respect to an inertial frame, also the volumes $\mathcal{B}$ and $\mathcal{C}$ are time dependent. Finally,

$$\operatorname{div} \boldsymbol{u} := \frac{\partial u_i}{\partial x_i}, \quad \operatorname{div} \boldsymbol{\mathcal{T}} := \frac{\partial T_{ij}}{\partial x_i} \tilde{\boldsymbol{e}}_{\boldsymbol{j}},$$

where the components are taken with respect to the coordinate axes $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$ and $\tilde{\boldsymbol{e}}_3$ of $\mathcal{I}$.

From the physical point of view, $(2.1)_1$ represents the conservation of mass for the liquid; it reduces to the *incompressibility constraint* since the density of the liquid is assumed to be constant. Moreover, $(2.1)_2$ and $(2.1)_3$ express the balance of linear and angular momentum for the liquid, respectively; here, the pressure field, Π, is the Lagrange multiplier due to

[1] In the case of motions about the center of mass, i.e. in the case $G \equiv O$, hypothesis H2. is automatically satisfied.

the incompressibility constraint. Finally, equations $(2.1)_4$ represent the balance of angular momentum of the body with respect to O. In particular, the integral term in $(2.1)_4$ represents the total torque exerted by the liquid on the body.

We notice that (2.1) is a system of 10 scalar equations in 13 scalar unknowns given by $\{\boldsymbol{w}, \Pi, \boldsymbol{\Omega}, \boldsymbol{\mathcal{T}}\}$. This mismatch between number of equations and number of unknowns can be easily resolved by the following simple physical fact: we have not yet introduced the class of liquids that we want to work with. In the language of Continuum Mechanics, this means that we have to add a *constitutive equation* for the stress. We adopt the *Newtonian incompressible stress model*

$$\boldsymbol{\mathcal{T}} := -\Pi\mathbf{1} + 2\mu\boldsymbol{D}(\boldsymbol{w}), \tag{2.2}$$

where μ is the (constant) *shear viscosity coefficient* of the liquid and

$$\boldsymbol{D}(\boldsymbol{w}) := \frac{1}{2}\left(\nabla\boldsymbol{w} + (\nabla\boldsymbol{w})^T\right),$$

is the *stretching tensor*. Here, $\mathbf{1}$ is the identity tensor, $\nabla\cdot$ denotes the gradient operator,

$$\nabla\cdot \equiv \left(\frac{\partial\,\cdot}{\partial x_1}, \frac{\partial\,\cdot}{\partial x_2}, \frac{\partial\,\cdot}{\partial x_3}\right),$$

and ^{T} denotes the transpose operator. We note that $(2.1)_{1,2,3}$ together with (2.2) reduce to the well known *Navier-Stokes equations*. Finally, for what concerns the boundary conditions, we append the *no-slip* condition of the liquid at the bounding wall of the cavity

$$\boldsymbol{w} = \boldsymbol{\Omega}\times(\boldsymbol{x}-\boldsymbol{x}_O) \quad \text{on } \bigcup_{t>0}\partial\mathcal{C}(t)\times\{t\}. \tag{2.3}$$

Taking into account (2.2) and (2.3), the equations of motion for the coupled system $\mathcal{S}$, in the inertial frame $\mathcal{I}$, read as follows

$$\begin{aligned}
&\left.\begin{aligned}
&\operatorname{div}\boldsymbol{w} = 0\\
&\rho\frac{d\boldsymbol{w}}{dt} = -\nabla\Pi + \mu\Delta\boldsymbol{w} + \rho\boldsymbol{\mathcal{F}}
\end{aligned}\right\} (\boldsymbol{x},t)\in\bigcup_{t>0}\mathcal{C}(t)\times\{t\},\\
&\frac{d(\boldsymbol{J}_{\mathcal{B}}\cdot\boldsymbol{\Omega})}{dt} = \mathsf{M} - \int_{\partial\mathcal{C}(t)}(\boldsymbol{x}-\boldsymbol{x}_O)\times\boldsymbol{\mathcal{T}}\cdot\mathbf{n}\,d\sigma,\\
&\boldsymbol{w} = \boldsymbol{\Omega}\times(\boldsymbol{x}-\boldsymbol{x}_O) \qquad \text{on } \bigcup_{t>0}\partial\mathcal{C}(t)\times\{t\}.
\end{aligned} \tag{2.4}$$

Here, $\Delta\cdot$ denotes the Laplace operator

$$\Delta\cdot := \sum_{i=1}^{3}\frac{\partial\,\cdot}{\partial x_i}.$$

2.1.2 Some relevant applications

Equations (2.4) have been largely used for several problems arising in Mechanics. Gyroscopes or pendulums with an interior cavity entirely filled with a liquid are just two examples of mechanical systems modeled by (2.4). More complicated liquid-filled rigid bodies have been studied for several applications in different fields of the Applied Sciences spanning from Geophysics to Space Engineering, and also for military research. In the following, a couple of applications are presented.

2.1.2.1 An application in Geophysics. In [50], Stewartson and Roberts consider the motion of an incompressible liquid which fills completely an oblate cavity of a precessing rigid body. More specifically, here the motion of the body is prescribed, and initially $\mathcal{S}$ is rotating as a whole rigid body about its center of mass, G, with a constant angular velocity $\bar{\boldsymbol{\omega}}$. At time $t = 0$, the symmetry axis is impulsively set in a rotation, with a "small" constant angular velocity $\bar{\boldsymbol{\Omega}}$, around a fixed axis in the space. Let a and b be the semi-axes of the cavity, and α be the angle between the precessional axis and the symmetry axis. Consider the frame of reference $\{G, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ with $\boldsymbol{e}_2$ parallel to $\bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{\Omega}}$ and $\boldsymbol{e}_3$ parallel to $\bar{\boldsymbol{\omega}}$. The authors show that, if the following quantities are "large" enough

$$R_1 := \frac{a^2 - b^2}{\bar{\Omega} a^2} \bar{\omega}, \quad R_1 R_2 = \frac{\bar{\omega}\rho(a^2 - b^2)}{\mu}, \quad R_3 = \frac{\bar{\omega}\rho a^2}{\mu},$$

then, the long-time behavior of the liquid is described by a steady state which is composed of a primary rigid body motion of the whole system,

$$\boldsymbol{w}_1 = \frac{2\bar{\Omega} a^2 b^2 \sin\alpha}{a^2 - b^2} \left(\frac{z}{b^2} \boldsymbol{e}_2 - \frac{y}{a^2} \boldsymbol{e}_3 \right), \text{ all } \boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3 \in \mathcal{C},$$

and a secondary motion which, outside a boundary layer of thickness of order $\sqrt{\mu}/\sqrt{\rho\bar{\omega}}$, is characterized by closed elliptical streamlines lying on a plane orthogonal to $\bar{\boldsymbol{\omega}} \times (\bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{\Omega}})$, and with uniform vorticity. For this specific problem, the Eulerian absolute velocity of the liquid is given by

$$\boldsymbol{w} = \boldsymbol{V} + (\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\Omega}}) \times \boldsymbol{x}, \quad \text{all } x \in \mathcal{C},$$

where $\boldsymbol{V} := \boldsymbol{w} - (\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\Omega}}) \times \boldsymbol{x}$ denotes the relative velocity of the liquid with respect to the solid. In a non-inertial frame rotating with precessional angular velocity $\bar{\boldsymbol{\Omega}}$ and with respect to which both the precessional axis and the symmetry axis are fixed, the equations of motion (2.4) become (see [33] Chapter 1, or next section for more details on its derivation)

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{w} = 0, \\ &\rho\left(\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{V} + \bar{\boldsymbol{\omega}} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{w} + \bar{\boldsymbol{\Omega}} \times \boldsymbol{w}\right) = \mu \Delta \boldsymbol{w} - \nabla \Pi, \end{aligned}\right\} \text{in } \mathcal{C} \times (0, \infty),$$
$$\boldsymbol{w} = (\bar{\boldsymbol{\Omega}} + \bar{\boldsymbol{\omega}}) \times \boldsymbol{x}, \text{ on } \partial\mathcal{C}.$$

The treatment relies on finding exact solutions to the corresponding linearized equations of motion for the inviscid case. These equations read as follows

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{V} = 0, \\ &\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + 2\bar{\boldsymbol{\omega}} \times \boldsymbol{V} - (\bar{\boldsymbol{\omega}} \times \boldsymbol{x}) \times \operatorname{curl} \boldsymbol{V} + \boldsymbol{x} \times (\bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{\Omega}})\right) = -\nabla \tilde{\Pi}, \end{aligned}\right\} \text{in } \mathcal{C} \times (0, \infty)$$
$$\boldsymbol{V} \cdot \mathbf{n} = \mathbf{0}, \text{ on } \partial\mathcal{C},$$

where $\tilde{\Pi}$ is the "generalized pressure" [2]. Then, the authors notice that their "inviscid solution" satisfies also the equations for the viscous case, with the exception of the no-slip boundary conditions on the cavity surface. This leads them to calculate the boundary layer where possible "adjustments" in the tangential components of velocity can happen. The results found in [50] have great significance because of their applications in Geophysics. Direct calculations made by the authors show that, for the Earth, the quantities R_1, $R_1 R_2$ and R_3 given above satisfy the conditions stated in their paper, and the consequent results can have great relevance in theories about the source generating the geomagnetic field. Some of these theories indeed suggest that the motion of the Earth core (which is also due to the luni-solar precession) might produce the geodynamo (see [9]). Sharper bounds for R_1, $R_1 R_2$ and R_3 are found by the same authors in [38] by assuming that $\alpha \ll 1$.

Other applications to geophysical problems can be found also in [10], [11] and [52].

[2] All the details on how to get these equations from the previous ones are given in [50]; in particular, see equations (2.5) and (2.6) therein contained.

2.1.2.2 An application in Space Engineering. Bhuta and Koval, in [5], present and analyze a *passive damper* constituted by a circular ring with circular cross-section, entirely filled by a liquid and installed on a plane parallel to the spin axis of a satellite. The aim is to stabilize the motion of spacecrafts when they undergo some "wobbling" motions, i.e. precessional motions of the spin axis. Separation from the booster, maneuvers to reorient satellites or even external disturbance (like the presence of a magnetic field) are some of the possible causes for "wobbles". The advantages of having internal passive dampers are manifold. Most importantly, they are efficient and reliable, since they eliminate the need of sensors, power sources and extra moving parts within the spacecraft.

The authors analyze the damping properties of the damper by determining the *energy dissipation* in the case of a liquid contained in an infinite long straight tube with "small" circular cross-section, when the motion is driven only by a periodic motion of the boundary. The motion of the liquid is considered 2-dimensional and fully developed along the axis of the tube. Specifically, in a cylindrical coordinate system, $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$, with $\boldsymbol{e}_z$ along the tube axis, they consider $\boldsymbol{w} = u(r,t)\boldsymbol{e}_z$ for the absolute velocity of the liquid. Within these assumptions, the pressure gradient is independent of the z-coordinate, and the Navier-Stokes equations reduce to

$$\begin{aligned} &\rho\frac{\partial u}{\partial t} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \qquad \text{for } 0 \le r < a,\ t > 0, \\ &u(a,t) = U_1\cos(\bar{\omega}t), \qquad t > 0, \\ &u(r,0) = 0, \qquad 0 < r \le a, \end{aligned} \tag{2.5}$$

where a is the radius of the tube, U_1 is the magnitude of the liquid fluctuation and $\bar{\omega}$ is the frequency of the prescribed periodic motion of the boundary. Multiplying the first equation by u and integrating the resulting equation over each cross-section of the tube, one can then find the rate of change of the energy per unit length, E, of the liquid

$$\frac{dE}{dt} = 2\pi\mu a\frac{\partial u}{\partial r}(a,t)u(a,t) - 2\pi\mu\int_0^a r\left(\frac{\partial u}{\partial r}\right)^2 dr.$$

The last term in the latter displayed equation is considered to be the rate of change of the "energy dissipated per unit length along the axis of the tube". This energy dissipation can be explicitly calculated from the solution to the equations of motion (2.5), and computed as function of what the author call "wobble Reynolds number", $Re := \bar{\omega}a^2\rho/\mu$. The authors performed some numerical and experimental tests to validate their theoretical computations; this allowed them to apply their results to the realistic situation of the wobble damping of a satellite and to present a preliminary design of the damper. The performance of the damper can be analyzed by the "energy sink approximation", for which the dissipation of the energy of the liquid is considered as a "sink" for the total kinetic energy of the satellite. The authors' results show that the "wobbling" angle eventually decays to zero with an exponential rate.

For other applications in Space Engineering, we refer the interested reader to the work of Abramson in [1], where the dynamics of the propellant in spacecrafts are examined. Moreover, the papers by Boyevkin et al. ([6]), Sarychev ([44]), and Alfriend and Spencer ([3]) offer further applications in Space Technology. Finally, in [46], Scott investigates the free flight of liquid-filled shells with applications to fuel-filled projectiles. In this direction, we would like to cite also the experimental results obtained by Karpov in [25].

Besides the interests for the applications in Applied Sciences, there have been numerous contributions also aimed at furnishing a rigorous mathematical analysis of the motions of liquid-filled rigid bodies. These contributions span from the early works by Stokes, [51], Zhukovskii, [55], and Hough, [24], to more recent ones by Sobolev, [49], Chernousko, [12], and Kopachevsky and Krein, [26]. In Chapter 3, we will present an overview of some of these previous mathematical results.

Next, we present more details about the mathematical model that we are going to work with and some of its related features.

2.2 THE EQUATIONS OF MOTION IN A MOVING FRAME

In order to better understand the dynamics of the coupled system $\mathcal{S}$, it turns out to be useful to consider the time evolution of the *total angular momentum*, $\boldsymbol{a}_O$, of the whole system calculated with respect to the fixed point O. To this end, let us consider

$$\boldsymbol{a}_O := \boldsymbol{J}_{\mathcal{B}} \cdot \boldsymbol{\Omega} + \int_{\mathcal{C}} \rho(\boldsymbol{x} - \boldsymbol{x}_O) \times \boldsymbol{w}. \tag{2.6}$$

If $\{\boldsymbol{w}, \boldsymbol{\Omega}, \Pi\}$ is a solution to (2.4), by the Leibniz-Reynolds transport theorem, we find that $\{\boldsymbol{w}, \boldsymbol{a}_O, \Pi\}$ has to satisfy the following equation

$$\begin{aligned}\frac{d\boldsymbol{a}_O}{dt} &= \frac{d}{dt}(\boldsymbol{J}_{\mathcal{B}} \cdot \boldsymbol{\Omega}) + \int_{\mathcal{C}} \frac{d}{dt}\left[\rho(\boldsymbol{x}-\boldsymbol{x}_O)\times\boldsymbol{w}\right]\\ &= \mathbf{M} - \int_{\partial\mathcal{C}} (\boldsymbol{x}-\boldsymbol{x}_O)\times\boldsymbol{\mathcal{T}}\cdot\mathbf{n}\, d\sigma + \int_{\mathcal{C}} \rho(\boldsymbol{x}-\boldsymbol{x}_O)\times \operatorname{div}\boldsymbol{\mathcal{T}} + \int_{\mathcal{C}} (\boldsymbol{x}-\boldsymbol{x}_O)\times(\rho\boldsymbol{\mathcal{F}}).\end{aligned}$$

Next, we notice that the first two integrals on the right-hand side of the latter displayed equation cancel out since $\boldsymbol{\mathcal{T}}$ is a symmetric tensor. In fact, with respect to the inertial frame $\mathcal{I}$

$$\begin{aligned}\tilde{\boldsymbol{e}}_i \cdot \int_{\mathcal{C}} \rho(\boldsymbol{x}-\boldsymbol{x}_O)\times\operatorname{div}\boldsymbol{\mathcal{T}} &= \int_{\mathcal{C}} \varepsilon_{ijk}(x_j - x_{Oj})\frac{\partial T_{\ell k}}{\partial x_\ell}\\ &= \int_{\partial\mathcal{C}} \varepsilon_{ijk}(x_j - x_{Oj})T_{k\ell}n_\ell\, d\sigma - \int_{\mathcal{C}} \varepsilon_{i\ell k}T_{\ell k}\\ &= \tilde{\boldsymbol{e}}_i \cdot \int_{\partial\mathcal{C}} (\boldsymbol{x}-\boldsymbol{x}_O)\times\boldsymbol{\mathcal{T}}\cdot\mathbf{n}\, d\sigma.\end{aligned}$$

In the previous calculations, we have used the Einstein notation for the summation over dummy indexes and an integration by parts. Moreover, ε_{ijk} denotes the permutation symbol; it is then clear that $\varepsilon_{i\ell k}T_{\ell k} = 0$ for all $i = 1, 2, 3$, since $\boldsymbol{\mathcal{T}}$ is a symmetric tensor. In conclusion, we have just shown that if $\{\boldsymbol{w}, \boldsymbol{\Omega}, \Pi\}$ is a solution to (2.4), then $\{\boldsymbol{w}, \boldsymbol{a}_O, \Pi\}$ is a solution to

$$\left.\begin{aligned}&\operatorname{div}\boldsymbol{w} = 0\\ &\rho\frac{d\boldsymbol{w}}{dt} = -\nabla\Pi + \mu\Delta\boldsymbol{w} + \rho\boldsymbol{\mathcal{F}}\end{aligned}\right\} (\boldsymbol{x}, t) \in \bigcup_{t>0}\mathcal{C}(t)\times\{t\},$$
$$\frac{d\boldsymbol{a}_O}{dt} = \boldsymbol{m}_O, \tag{2.7}$$
$$\boldsymbol{w} = \boldsymbol{\Omega}\times(\boldsymbol{x}-\boldsymbol{x}_O) \qquad \text{on } \bigcup_{t>0}\partial\mathcal{C}(t)\times\{t\},$$
$$\boldsymbol{a}_O = \boldsymbol{J}_{\mathcal{B}}\cdot\boldsymbol{\Omega} + \textstyle\int_{\mathcal{C}} \rho(\boldsymbol{x}-\boldsymbol{x}_O)\times\boldsymbol{w},$$

where

$$\boldsymbol{m}_O := \mathbf{M} + \int_{\mathcal{C}} (\boldsymbol{x}-\boldsymbol{x}_O)\times(\rho\boldsymbol{\mathcal{F}})$$

denotes the *total external torque* applied on $\mathcal{S}$ and calculated with respect to O. Retracing backward the calculations just performed, we can then conclude that (2.4) and (2.7) are two equivalent formulations for the equations of motion of the system liquid-filled rigid body.

In this thesis we will focus on the following types of motions that the coupled system $\mathcal{S}$ can perform about a fixed point O.

a) *Inertial motions*: neither external forces nor torques are applied on $\mathcal{S}$, which moves about its center of mass G driven by its inertia, after an initial angular momentum is imparted on the whole system. Thus, in this case, $O \equiv G$ and $\boldsymbol{\mathcal{F}} = \mathbf{M} \equiv \mathbf{0}$.

b) *Motions of liquid-filled heavy rigid bodies*: the total body force applied on $\mathcal{S}$ is constant in space and time. In particular, we will consider the case of motions under the action of the *gravity* force. Thus, $\boldsymbol{\mathcal{F}} \equiv \boldsymbol{g}$ and $\boldsymbol{m}_O \equiv (G-O)\times(M\boldsymbol{g})$, where M is the mass of $\mathcal{S}$ and $\boldsymbol{g}$ is the acceleration of gravity vector.

c) *Time periodic motions*: the total external torque applied on $\mathcal{S}$ is of the type $\boldsymbol{m}_O \equiv f(t)\boldsymbol{h}$, where $f = f(t)$ is a time periodic function with period T (i.e. $f(t+T) = f(t)$ for all $t \geq 0$), and $\boldsymbol{h}$ is a time-independent vector.

One of the features of (2.7) is that the volumes $\mathcal{B}$ and $\mathcal{C}$ are time dependent, thus making the mathematical treatment more involved. This can be overcome by rewriting the equations of motion in a non-inertial frame of reference, $\mathsf{F} := \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, with origin at the fixed point O, $\boldsymbol{e}_1 \equiv \mathbf{a}$ (from hypothesis H2.), and $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ directed along the remaining principal axes of $\mathcal{S}$ with respect to O. In mathematical terms, we introduce a proper orthogonal transformation $\boldsymbol{Q} = \boldsymbol{Q}(t)$, $\boldsymbol{Q}\cdot\boldsymbol{Q}^T = \boldsymbol{Q}^T\cdot\boldsymbol{Q} = \mathbf{1}$, $\det\boldsymbol{Q} = 1$, such that

$$\boldsymbol{y} := \boldsymbol{Q}^T\cdot(\boldsymbol{x}-\boldsymbol{x}_O)$$

denotes the position vector of a generic point, $P \in \mathbb{R}^3$, with respect to the new frame F. Moreover, we assume that $\boldsymbol{Q}(0) = \mathbf{1}$. In a similar fashion as in [33] (Section 1.2), it can be easily shown that $\boldsymbol{\Omega}$ is the adjoint vector of the tensor $\dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^T$, i.e.

$$\dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{a} = \boldsymbol{\Omega}\times\boldsymbol{a}, \qquad \text{for all } \boldsymbol{a}\in\mathbb{R}^3.$$

Since $\mathbf{e}_i = \boldsymbol{Q}^T\cdot\tilde{\boldsymbol{e}}_i$, $i=1,2,3$, for all $\boldsymbol{a}$, $\boldsymbol{b}\in\mathbb{R}^3$, their representation in F is given by $\boldsymbol{Q}^T\cdot\boldsymbol{a}$ and $\boldsymbol{Q}^T\cdot\boldsymbol{b}$, respectively. Since $\det\boldsymbol{Q} = 1$, it then follows that $\boldsymbol{Q}^T\cdot(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{Q}^T\cdot\boldsymbol{a})\times(\boldsymbol{Q}^T\cdot\boldsymbol{b})$.

Let us consider the following fields

$$\begin{aligned}
&\boldsymbol{u}(\boldsymbol{y},t) := \boldsymbol{Q}^T(t)\boldsymbol{w}(\boldsymbol{Q}(t)\cdot\boldsymbol{y}+\boldsymbol{x}_O,t), \qquad \boldsymbol{\omega}(t) := \boldsymbol{Q}^T(t)\cdot\boldsymbol{\Omega}(t),\\
&\boldsymbol{v}(\boldsymbol{y},t) := \boldsymbol{u}-\boldsymbol{\omega}\times\boldsymbol{y}, \qquad \boldsymbol{A}_O := \boldsymbol{Q}^T(t)\cdot\boldsymbol{a}_O(t),\\
&p(\boldsymbol{y},t) := \tilde{p}(\boldsymbol{y},t)-\frac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{y})^2, \qquad \tilde{p}(\boldsymbol{y},t) := \Pi(\boldsymbol{Q}(t)\cdot\boldsymbol{y}+\boldsymbol{x}_O,t).
\end{aligned} \tag{2.8}$$

In particular, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t)$ represents the *relative velocity* of liquid in the frame F, while $p = p(\boldsymbol{y}, t)$ is the *generalized pressure field*. With respect to the non-inertial frame F, equations (2.4) become

$$
\left.\begin{aligned}
&\operatorname{div} \boldsymbol{u} = 0, \\
&\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u}\right) = \mu\Delta\boldsymbol{u} - \nabla\tilde{p} + \rho\boldsymbol{Q}^T\cdot\boldsymbol{\mathcal{F}}
\end{aligned}\right\} \text{ in } \mathcal{C}\times(0,\infty),
$$
$$
\begin{aligned}
&\boldsymbol{I}_{\mathcal{B}}\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\boldsymbol{I}_{\mathcal{B}}\cdot\boldsymbol{\omega} = -\int_{\partial\mathcal{C}} \boldsymbol{y}\times\boldsymbol{T}(\boldsymbol{u},\tilde{p})\, d\sigma + \boldsymbol{Q}^T\cdot\mathsf{M}, \\
&\dot{\boldsymbol{Q}} = \boldsymbol{Q}\cdot\mathsf{A}(\boldsymbol{\omega}), \\
&\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{y} \qquad \text{on } \partial\mathcal{C},
\end{aligned}
\tag{2.9}
$$

where $\boldsymbol{I}_{\mathcal{B}} := \boldsymbol{Q}^T\cdot\boldsymbol{J}_{\mathcal{B}}\cdot\boldsymbol{Q}$ is the inertial tensor of the solid calculated with respect to O, and $\boldsymbol{T} := \boldsymbol{Q}^T\cdot\boldsymbol{\mathcal{T}}\cdot\boldsymbol{Q}$ is the Cauchy stress tensor in the frame F. Moreover, A is a (bijective) map from $\mathbb{R}^3$ to the space of all skew-symmetric 3×3 matrices such that, for each $\boldsymbol{b}\in\mathbb{R}^3$, $\mathsf{A}(\boldsymbol{b})$ is the skew-symmetric matrix having $\boldsymbol{b}$ as its adjoint vector. The need of an equation for $\dot{\boldsymbol{Q}}$ is due to the fact that, although external forces and torques are given data in the inertial frame, they become unknown when they are rewritten in the non-inertial frame F, as the orthogonal transformation $\boldsymbol{Q}$ is an unknown function of time.

In a similar fashion, equations (2.7) can be rewritten as follows in F (see also [33] and [13])

$$
\left.\begin{aligned}
&\operatorname{div} \boldsymbol{v} = 0 \\
&\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \dot{\boldsymbol{\omega}}\times\boldsymbol{y} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + 2\boldsymbol{\omega}\times\boldsymbol{v}\right) = -\nabla p + \mu\Delta\boldsymbol{v} + \rho\boldsymbol{Q}^T\cdot\boldsymbol{\mathcal{F}}
\end{aligned}\right\} \text{ in } \mathcal{C}\times(0,\infty),
$$
$$
\begin{aligned}
&\frac{d\boldsymbol{A}_O}{dt} + \boldsymbol{\omega}\times\boldsymbol{A}_O = \boldsymbol{Q}^T\cdot\boldsymbol{m}_O, \\
&\dot{\boldsymbol{Q}} = \boldsymbol{Q}\cdot\mathsf{A}(\boldsymbol{\omega}), \\
&\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{C}.
\end{aligned}
\tag{2.10}
$$

From (2.6) and (2.8), we notice that

$$
\boldsymbol{A}_O = \boldsymbol{I}\cdot\boldsymbol{\omega} + \int_{\mathcal{C}} \rho\boldsymbol{y}\times\boldsymbol{v}, \tag{2.11}
$$

where $\boldsymbol{I}$ is the *total inertial tensor* of $\mathcal{S}$ with respect to O; it is time-independent, and

$$
\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{I}_{\mathcal{B}}\cdot\boldsymbol{b} + \int_{\mathcal{C}} \rho(\boldsymbol{y}\times\boldsymbol{a})\cdot(\boldsymbol{y}\times\boldsymbol{b}), \quad \text{all } \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3. \tag{2.12}
$$

2.3 NOTATION AND USEFUL INEQUALITIES

In this section, we will present the basic functional spaces and related inequalities that will be used throughout this thesis.

Given a vector $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, the *modulus* of $\boldsymbol{a}$ is indicated by

$$|\boldsymbol{a}| := \left(\sum_{i=1}^{3} a_i^2\right)^{1/2}.$$

Moreover, S^1 and S^2 denote the unit sphere in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Let $\mathcal{V} \subset \mathbb{R}^3$. For any $k \in \mathbb{N}$, the *partial derivative of order* k of a vector function $\boldsymbol{w} : \mathcal{V} \to \mathbb{R}^3$ is denoted by $D^k\boldsymbol{w}$; it is a tensor of order $k+1$ with components

$$\frac{\partial^{|\alpha|} w_i}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_2}},$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| = k$, and all $i = 1, 2, 3$. In particular, when $k = 1$, with respect a Cartesian coordinate system $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the *gradient* of $\boldsymbol{w}$ is

$$\nabla \boldsymbol{w} := D(\boldsymbol{w}) = \frac{\partial w_j}{\partial x_i} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \operatorname{div} \boldsymbol{w} := \frac{\partial w_i}{\partial x_i},$$

where $\otimes$ stays for the tensor product. Moreover, when $k = 2$, the second partial derivatives of $\boldsymbol{w}$ are given by

$$\frac{\partial^2 w_k}{\partial x_i \partial x_j}, \quad \text{all } i, j, k = 1, 2, 3,$$

and the *Laplacian* of $\boldsymbol{w}$ is then defined as

$$\Delta \boldsymbol{w} = \frac{\partial^2 w_k}{\partial x_i \partial x_i} \boldsymbol{e}_k.$$

$C^k(\mathcal{V})$ denotes the linear space of all vector fields $\boldsymbol{w}$ defined on $\mathcal{V}$ which are k-times continuously differentiable. Whereas, $C^\infty(\mathcal{V}) := \bigcap_{k \geq 0} C^k(\mathcal{V})$. We define $C^k(\overline{\mathcal{V}})$ to be the linear space of all functions $\boldsymbol{\phi} \in C^k(\mathcal{V})$ such that $D^\alpha \boldsymbol{\phi}$ is bounded and uniformly continuous on $\mathcal{V}$ for $0 \leq |\alpha| \leq k$. $C^k(\overline{\mathcal{V}})$ is a Banach space if equipped with the following norm

$$\|\cdot\|_{C^k} := \max_{0 \leq |\alpha| \leq k} \sup_{\boldsymbol{y} \in \mathcal{V}} |D^\alpha \cdot |.$$

Moreover, $C_0^\infty(\mathcal{V})$ is the linear subspace of $C^\infty(\mathcal{V})$ of all vector fields having compact support in $\mathcal{V}$. In a similar fashion, $L^p(\mathcal{V})$, and $W^{k,p}(\mathcal{V})$, $W_0^{k,p}(\mathcal{V})$, $k \in \mathbb{N}$, $p \in [1,\infty]$ denote the usual Lebesgue and Sobolev spaces for vector fields defined on $\mathcal{V}$ [3]. Moreover,

$$(\cdot,\cdot): \ \boldsymbol{a},\ \boldsymbol{b} \in L^2(\mathcal{V}) \mapsto (\boldsymbol{a},\boldsymbol{b}) := \int_{\mathcal{C}} \boldsymbol{a}\cdot\boldsymbol{b} \quad \in \mathbb{R}$$

is the usual L^2-inner product, and

$$\|\cdot\|_2 := \sqrt{(\cdot,\cdot)} = \left(\int_{\mathcal{C}} |\cdot|^2\right)^{1/2}.$$

Finally,

$$\|\cdot\|_p := \left(\int_{\mathcal{C}} |\cdot|^p\right)^{1/p} \text{ for } 1 \le p < \infty, \quad \|\cdot\|_\infty := \operatorname*{ess\,sup}_{\mathcal{V}} |\boldsymbol{w}|$$

and

$$\|\cdot\|_{k,p} := \left(\sum_{0\le|\alpha|\le k} \|D^\alpha\cdot\|_p^p\right)^{1/p} \quad \text{for } 1 \le p < \infty, \quad \|\cdot\|_{k,\infty} := \max_{0\le|\alpha|\le k} \|D^\alpha\cdot\|_\infty .$$

In the last norms, D^α has to be understood in the distributional sense.

If $(X, \|\cdot\|_X)$ is a Banach space and $I \subset \mathbb{R}$ is an interval, we write $f \in C^k(I;X)$ if f is k-times differentiable with values in X and

$$\max_{t\in I} \left\|\frac{\partial^\ell f(t)}{\partial t^\ell}\right\|_X < \infty, \quad \text{all } \ell = 0, 1, ..., k.$$

Moreover, $f \in C_w(I;X)$ means that the map $t \in I \mapsto \Phi(f(t)) \in \mathbb{R}$ is continuous for all bounded linear functionals Φ defined on X.

Let $q \in [1,\infty)$, we denote by $L^q(I;X)$ (respectively $W^{k,q}(I;X)$, $k \in \mathbb{N}$), the space of functions $f: \ I \to X$, such that

$$\left(\int_I \|f(t)\|_X^q \ dt\right)^{1/q} < \infty \qquad \left(\text{resp. } \sum_{\ell=0}^{k} \left(\int_I \left\|\frac{\partial^\ell f(t)}{\partial t^\ell}\right\|_X^q \ dt\right)^{1/q} < \infty\right).$$

[3]We will use the same symbol for spaces of scalar, vector and tensor functions. Moreover, in the integrals we usually omit the infinitesimal element of integration.

When concerned with *periodic functions*, we will use the following spaces defined for $T > 0$, $q \in [1, \infty]$, and $k \in \mathbb{N}$:

$$L^q_T(\mathbb{R}) := \{u \in L^q_{\text{loc}}(\mathbb{R}) | \; u(t) = u(t+T), \text{ for a.a. } t \in \mathbb{R}\}$$

$$C^k_T(\mathbb{R}) := \{\xi \in C^k(\mathbb{R}) | \; \xi(t) = \xi(t+T), \text{ for all } t \in \mathbb{R}\}\,.$$

Let us now introduce the basic function spaces of Hydrodynamics. We set

$$\mathcal{D}(\mathcal{V}) := \{\boldsymbol{w} \in C^\infty_0(\mathcal{V}) : \; \operatorname{div} \boldsymbol{w} = 0\}.$$

We denote by $H(\mathcal{V})$ the completion of $\mathcal{D}(\mathcal{V})$ in the L^2-norm, and

$$G(\mathcal{V}) := \{\boldsymbol{w} \in L^2(\mathcal{V}) : \; \boldsymbol{w} = \nabla p, \text{ for some } p \in W^{1,2}(\mathcal{V})\}.$$

Moreover, $\mathcal{D}^{1,2}_0(\mathcal{V}) := H(\mathcal{V}) \cap W^{1,2}_0(\mathcal{V})$. The following decomposition of $L^2(\mathcal{V})$ holds.

Theorem 2.3.1 (Helmholtz-Weyl decomposition)**.** *$H(\mathcal{V})$ and $G(\mathcal{V})$ are orthogonal spaces in $L^2(\mathcal{V})$, and*

$$L^2(\mathcal{V}) = H(\mathcal{V}) \oplus G(\mathcal{V}). \tag{2.13}$$

Moreover, if $\mathcal{V}$ is a bounded, locally Lipschitz domain in $\mathbb{R}^3$ with outer normal $\boldsymbol{n}$, then

$$H(\mathcal{V}) \equiv \{\boldsymbol{u} \in L^2(\mathcal{V}) : \; \operatorname{div} \boldsymbol{u} = 0 \textit{ in } \mathcal{V}, \; \boldsymbol{u} \cdot \boldsymbol{n} = 0 \textit{ on } \partial\mathcal{V}\}. \tag{2.14}$$

A proof of the previous theorem can be found in [16], Theorem III.1.1 and Theorem III.2.3. Finally, the decomposition (2.13) implies the existence of a *projection operator*, i.e. a unique bounded, linear, idempotent operator

$$\mathcal{P} : \; L^2(\mathcal{V}) \to H(\mathcal{V}) \tag{2.15}$$

such that $\text{Range}(\mathcal{P}) = \text{H}(\mathcal{V})$ and $\text{Ker}(\mathcal{P}) = \text{G}(\mathcal{V})$.

In the following, we will collect some well-known inequalities that will be frequently used in this thesis. Let $a, b \in \mathbb{R}$, and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Then, for all $\varepsilon > 0$

$$ab \le \frac{\varepsilon a^p}{p} + \varepsilon^{-q/p} \frac{b^q}{q}, \tag{2.16}$$

this is the *Young inequality*. From this, the following important inequalities can be easily derived (see Chapter 2 in [2] for their proofs). Let $q \in [1, \infty]$ and consider $p \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, the *Hölder inequality* holds:

$$\int_{\mathcal{V}} |uw| \leq \|u\|_q \|w\|_p , \tag{2.17}$$

for all real-valued functions $u \in L^q(\mathcal{V})$ and $w \in L^p(\mathcal{V})$. A corollary to this inequality is given by the following

$$\|uw\|_r \leq \|u\|_p \|w\|_q \tag{2.18}$$

which holds for all $u \in L^p(\mathcal{V})$, $w \in L^q(\mathcal{V})$, $1 \leq p, q, r \leq \infty$ such that $1/p + 1/q = 1/r$. In the case $q = p = 2$, (2.17) is often referred as *Cauchy-Schwarz inequality*. The Hölder inequality can be generalized as follows

$$\int_{\mathcal{V}} |u_1 u_2 \dots u_N| \leq \|u_1\|_{q_1} \|u_2\|_{q_2} \dots \|u_N\|_{q_N} \tag{2.19}$$

where $u_i \in L^{q_i}(\mathcal{V})$, $1 \leq q_i \leq \infty$ for all $i = 1, \dots, N$, and $\sum_{i=1}^N q_i^{-1} = 1$. Moreover, we recall the *interpolation inequality*:

$$\|u\|_q \leq \|u\|_r^{\theta} \|u\|_p^{1-\theta} , \tag{2.20}$$

which holds for all $u \in L^p(\mathcal{V}) \cap L^r(\mathcal{V})$ with $1 \leq p \leq q \leq r \leq \infty$ and $\theta \in [0, 1]$ such that

$$\frac{\theta}{r} + \frac{1 - \theta}{p} = \frac{1}{q}.$$

Next, we recall some *embedding theorems* referring to [2], Chapter 2 and Chapter 4, for the proofs. If $1 \leq p \leq q \leq \infty$ and $\mathcal{V}$ is a bounded domain in $\mathbb{R}^3$, then

$$L^q(\mathcal{V}) \hookrightarrow L^p(\mathcal{V}). \tag{2.21}$$

For what concerns the Sobolev spaces, we have the following theorem.

Theorem 2.3.2 (The Sobolev embedding Theorem)**.** *Let $\mathcal{V} \subset \mathbb{R}^3$ be a bounded domain, and consider $1 \leq p < \infty$ and an integer $m \geq 1$. We have the following cases.*

1. *If $mp > 3$ or $m = 3$ and $p = 1$, then $W^{m,p}(\mathcal{V}) \hookrightarrow L^q(\mathcal{V})$ for all $p \leq q \leq \infty$.*

2. *If* $mp = 3$, *then* $W^{m,p}(\mathcal{V}) \hookrightarrow L^q(\mathcal{V})$ *and* $W_0^{m,p}(\mathcal{V}) \hookrightarrow L^q(\mathcal{V})$ *for all* $p \le q < \infty$.
3. *If* $mp < 3$, *then* $W^{m,p}(\mathcal{V}) \hookrightarrow L^q(\mathcal{V})$ *and* $W_0^{m,p}(\mathcal{V}) \hookrightarrow L^q(\mathcal{V})$ *for all* $p \le q \le p^*$, *where*

$$p^* := \frac{3p}{3 - mp}.$$

We will also use the *Sobolev inequality*:

$$\|\boldsymbol{u}\|_r \le \frac{p}{\sqrt{3}(3-p)} \|\nabla \boldsymbol{u}\|_p \,, \quad \text{for } 1 \le p < 3,\ r = \frac{3p}{3-p}. \tag{2.22}$$

Finally, we would like to recall the following *compact embeddings*

1. If $p > 3$, then $W_0^{1,p}(\mathcal{V}) \hookrightarrow\hookrightarrow C(\bar{\mathcal{V}})$.
2. If $p = 3$, then $W_0^{1,p}(\mathcal{V}) \hookrightarrow\hookrightarrow L^q$ for $1 \le q < \infty$.
3. If $1 \le p < 3$, then $W_0^{1,p}(\mathcal{V}) \hookrightarrow\hookrightarrow L^q$ for $1 \le q < 3p/(3-p)$.

which hold for every bounded domain, $\mathcal{V}$, in $\mathbb{R}^3$.

The *Poincaré inequality* is also very useful in hydrodynamic problems; for completeness we recall it here

$$\|\boldsymbol{u}\|_2 \le C_p \|\nabla \boldsymbol{u}\|_2 \,, \qquad \text{for all } \boldsymbol{u} \in W_0^{1,2}(\mathcal{V}), \tag{2.23}$$

where C_p is a positive constant depending only on the bounded domain $\mathcal{V}$.

Consider the *Stokes problem* on a bounded domain

$$\begin{cases} \operatorname{div} \boldsymbol{w} = 0, \\ \Delta \boldsymbol{w} = \nabla p + \boldsymbol{f}, \\ \boldsymbol{w} = \boldsymbol{0}, \quad \text{on } \partial\mathcal{V}. \end{cases}$$

Interior and boundary estimates for the previous problem imply the following inequality (see [16], Section IV.6)

$$\|\boldsymbol{w}\|_{2,2} \le C \|\mathcal{P}\Delta \boldsymbol{w}\|_2 \tag{2.24}$$

which holds for all $\boldsymbol{w} \in W^{2,2}(\mathcal{V}) \cap \mathcal{D}_0^{1,2}(\mathcal{V})$, and where C is a positive constant independent of $\boldsymbol{w}$.

For the study of problems concerning liquid-filled rigid bodies, the equation of balance for the kinetic energy of the whole system $\mathcal{S}$ plays a fundamental role. We then premise the following results which will be useful to identify an appropriate energy functional for our problem (see [26], Chapter 1, Sections 7.2.2 and 7.2.3, for their proofs). We start by considering the linear space $\mathsf{E}_3 := \{\boldsymbol{\varphi} \in L^2(\mathcal{C}) : \ \boldsymbol{\varphi}(\boldsymbol{y}) = \boldsymbol{c}\times\boldsymbol{y},$ for some constant vector $\boldsymbol{c} \in \mathbb{R}^3\}$. E_3 is a finite dimensional subspace of $L^2(\mathcal{C})$ with basis fields $\boldsymbol{b}_i := \boldsymbol{e}_i \times \boldsymbol{y}$, for all $i = 1, 2, 3$. We then consider the operator

$$\boldsymbol{B} : \ \boldsymbol{\psi} \in H(\mathcal{C}) \mapsto \boldsymbol{B}\cdot\boldsymbol{\psi} \in \mathsf{E}_3 \subset L^2(\mathcal{C}) \tag{2.25}$$

such that

$$(\boldsymbol{B}\cdot\boldsymbol{\psi})(\boldsymbol{y}) = \rho\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{\psi}\right)\times\boldsymbol{y}. \tag{2.26}$$

The following lemma holds.

Lemma 2.3.3. *The operator $\boldsymbol{B}$ defined by (2.25) and (2.26) is non-negative and self-adjoint. Moreover, $\mathbf{1}-\boldsymbol{B}$ is a non-negative operator with a bounded inverse.*

Proof. The proof of this lemma can be found in [26], Section 7.2.3, we include it here for completeness.

By (2.26), the symmetry of $\boldsymbol{I}^{-1}$, and the property

$$[(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}]\cdot\boldsymbol{d} = (\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}), \quad \text{for all } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\in\mathbb{R}^3,$$

we infer that

$$\begin{aligned}(\boldsymbol{B}\cdot\boldsymbol{u},\boldsymbol{v}) &= \int_{\mathcal{C}}\left\{\left[\rho\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{V}}\boldsymbol{x}\times\boldsymbol{u}\right)\times\boldsymbol{y}\right]\cdot\boldsymbol{v}\right\} = \rho\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{V}}\boldsymbol{x}\times\boldsymbol{u}\right)\cdot\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}\right)\\ &= \rho\left(\int_{\mathcal{V}}\boldsymbol{y}\times\boldsymbol{u}\right)\cdot\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}\right), \qquad \text{all } \boldsymbol{u},\boldsymbol{w}\in H(\mathcal{C}).\end{aligned}$$

Using again the symmetry of $\boldsymbol{I}^{-1}$, we find that $\boldsymbol{B}$ is self-adjoint. Moreover, since $\boldsymbol{I}^{-1}$ is positive definite,

$$(\boldsymbol{B}\cdot\boldsymbol{u},\boldsymbol{u}) = \rho\left(\int_{\mathcal{V}}\boldsymbol{y}\times\boldsymbol{u}\right)\cdot\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{u}\right)\geq 0, \qquad \text{all } \boldsymbol{u}\in H(\mathcal{C}),$$

that is, $\boldsymbol{B}$ is a nonnegative operator.

For all $\boldsymbol{u}\in H(\mathcal{C})$, let us denote

$$\boldsymbol{a} := -\rho\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{u}\right).$$

Then,

$$\begin{aligned}\rho((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{u},\boldsymbol{u}) &= \rho(\boldsymbol{u},\boldsymbol{u})-\rho(\boldsymbol{B}\cdot\boldsymbol{u},\boldsymbol{u}) = \rho(\boldsymbol{u},\boldsymbol{u})-\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}\\ &= \rho(\boldsymbol{u}+\boldsymbol{a}\times\boldsymbol{y},\boldsymbol{u}+\boldsymbol{a}\times\boldsymbol{y})-\rho(\boldsymbol{a}\times\boldsymbol{y},\boldsymbol{a}\times\boldsymbol{y})-2\rho(\boldsymbol{u},\boldsymbol{a}\times\boldsymbol{y})-\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}\\ &= \rho\left\|\boldsymbol{u}+\boldsymbol{a}\times\boldsymbol{y}\right\|_2^2-\rho(\boldsymbol{a}\times\boldsymbol{y},\boldsymbol{a}\times\boldsymbol{y})+2\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}-\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}\\ &= \rho\left\|\boldsymbol{u}+\boldsymbol{a}\times\boldsymbol{y}\right\|_2^2-\rho(\boldsymbol{a}\times\boldsymbol{y},\boldsymbol{a}\times\boldsymbol{y})+\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}\\ &= \rho\left\|\boldsymbol{u}+\boldsymbol{a}\times\boldsymbol{y}\right\|_2^2+\boldsymbol{a}\cdot\boldsymbol{I}_{\mathcal{B}}\cdot\boldsymbol{a}\geq 0.\end{aligned}$$

We have used (2.12) to get last equality, and the positive definiteness of $\boldsymbol{I}_{\mathcal{B}}$ to obtain the displayed inequality. Thus, the operator $\mathbf{1}-\boldsymbol{B}$ is nonnegative. Furthermore, we claim that $((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{u},\boldsymbol{u})=0$ iff $\boldsymbol{u}\equiv\mathbf{0}$. The "if" part is obviously true. To show the "only if" part of the claim, we observe that

$$\rho((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{u},\boldsymbol{u}) = \rho\left\|\boldsymbol{u}+\boldsymbol{a}\times\boldsymbol{y}\right\|_2^2+\boldsymbol{a}\cdot\boldsymbol{I}_{\mathcal{B}}\cdot\boldsymbol{a}\geq\boldsymbol{a}\cdot\boldsymbol{I}_{\mathcal{B}}\cdot\boldsymbol{a}.$$

If $((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{u},\boldsymbol{u})=0$, necessarily $\boldsymbol{a}\cdot\boldsymbol{I}_{\mathcal{B}}\cdot\boldsymbol{a}=0$, and so $\boldsymbol{a}=\mathbf{0}$ since $\boldsymbol{I}_{\mathcal{B}}$ is positive definite. By (2.26), the latter implies that $\boldsymbol{B}\cdot\boldsymbol{u}=\mathbf{0}$. Thus, from the above calculations,

$$0=\rho((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{u},\boldsymbol{u})=\rho(\boldsymbol{u},\boldsymbol{u})-\rho(\boldsymbol{B}\cdot\boldsymbol{u},\boldsymbol{u})=\rho\left\|\boldsymbol{u}\right\|_2^2$$

implies that $\boldsymbol{u}\equiv\mathbf{0}$.

Summarizing, $\mathbf{1}-\boldsymbol{B}$ is a nonnegative operator that is zero at zero only, and $\boldsymbol{B}$ is a bounded operator with image contained in a finite dimensional space. Therefore, $\boldsymbol{B}$ is a linear, continuous, compact operator and $\lambda=1$ is not an eigenvalue of $\boldsymbol{B}$. We can then apply Fredholm Alternative Theorem (see Proposition 19.16 in [54]) to conclude that $\mathbf{1}-\boldsymbol{B}$ has a bounded inverse operator, and this concludes the proof of the lemma. ■

The functional

$$\langle\cdot,\cdot\rangle : \ (\boldsymbol{u},\boldsymbol{v}) \in L^2(\mathcal{C}) \times L^2(\mathcal{C}) \mapsto \langle \boldsymbol{u},\boldsymbol{v}\rangle := ((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{u},\boldsymbol{v}) \ \in \mathbb{R}, \tag{2.27}$$

defines a scalar product in $L^2(\mathcal{C})$ with the associate norm

$$\|\boldsymbol{w}\|_B := \langle \boldsymbol{w},\boldsymbol{w}\rangle^{1/2} = ((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{w},\boldsymbol{w})^{1/2},$$

which is equivalent to the norm $\|\cdot\|_2$. Indeed, since, by Lemma 2.3.3, $\boldsymbol{B}$ is non-negative and $\mathbf{1}-\boldsymbol{B}$ admits a bounded inverse, we find

$$((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{w},\boldsymbol{w}) = \|(\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{w}\|_2^2 + (\boldsymbol{B}\cdot\boldsymbol{w},\boldsymbol{w}) \geq c^2\,\|\boldsymbol{w}\|_2^2$$

where, $c = c(\mathcal{C}) > 0$. Furthermore, again using the fact that $\boldsymbol{B}$ is non-negative, we deduce

$$((\mathbf{1}-\boldsymbol{B})\cdot\boldsymbol{w},\boldsymbol{w}) \leq (\boldsymbol{w},\boldsymbol{w}),$$

so that

$$c\,\|\boldsymbol{w}\|_2 \leq \|\boldsymbol{w}\|_B \leq \|\boldsymbol{w}\|_2\,, \tag{2.28}$$

or in a more explicit form:

$$c\,\|\boldsymbol{w}\|_2^2 \leq \|\boldsymbol{w}\|_2^2 - \left(\int_{\mathcal{C}} \rho\,\boldsymbol{y}\times\boldsymbol{w}\right)\cdot\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}} \boldsymbol{y}\times\boldsymbol{w}\right) \leq \|\boldsymbol{w}\|_2^2 \tag{2.29}$$

for all $\boldsymbol{w} \in H(\mathcal{C})$.

We conclude this chapter by proving the following *Gronwall-type Lemmas.*

Lemma 2.3.4. *Consider $y \in L^\infty(0,\infty)$, $y \geq 0$, and satisfying the following inequality*

$$y(t) \leq y(s) - k \int_s^t y(\tau)\, d\tau + \int_s^t F(\tau, y)\, d\tau$$

for a.a. $s \geq 0$, including $s = 0$, and all $t \geq s$. Here, $k > 0$ is a constant and $F(t,w)$ is continuous in t and Lipschitz continuous in w, $F(t,w) \geq 0$ for a.a $t \geq 0$ and all $w \in L^\infty(0,\infty)$. Moreover,

$$\int_a^\infty |F(t,w)|^q\, dt < \infty,$$

for some $a > 0$ and $q \in [1,\infty)$, and for all $w \in L^\infty(0,\infty)$. Then,

$$\lim_{t\to\infty} y(t) = 0.$$

Moreover, if in particular $F \equiv 0$, then

$$y(t) \leq y(s)\mathrm{e}^{-k(t-s)}, \qquad \textit{for all } t \geq s.$$

Proof. By assumption, $F(t,y)$ is continuous in t and Lipschitz continuous in y; by Lemma 2.1 in [45] and subsequent remarks, this implies that $y(t) \leq z(t)$ where $z(t)$ satisfies the ODE

$$\frac{dz}{dt} = -kz(t) + F(t,z), \quad y(0) = z(0).$$

If $F \equiv 0$, the last part of the statement immediately follows by integrating the latter displayed equation.

If $F(t,z) > 0$ for a.a $t \geq 0$, using the hypotheses

$$\int_a^\infty |F(t,z)|^q\, dt < \infty,$$

we can then apply Lemma 5.2.1 in [33] to conclude that necessarily $z(t) \to 0$ as $t \to \infty$. Since $0 \leq y(t) \leq z(t)$, the statement of the lemma is completely proved by passing to the limit as $t \to \infty$ in the latter displayed inequalities. ∎

Lemma 2.3.5. *Let $y : [t_0, t_1) \to [0,\infty)$, $t_1 > t_0 \geq 0$, be an absolutely continuous function satisfying for some $a, b, c, \delta > 0$ and $\alpha > 1$,*

(i) $y' \leq -a\, y + b\, y^\alpha + c$ *in* (t_0, t_1) *;*

(ii) $\displaystyle\int_{t_0}^{t_1} y(\tau)\,d\tau < \frac{\delta^2}{4c}\,, \quad y(t_0) < \frac{\delta}{\sqrt{2}}\,.$

Then, if $k := -a + b\,\delta^{\alpha-1} < 0$, we have

$$y(t) < \delta\,, \quad \text{for all } t \in [t_0, t_1)\,. \tag{2.30}$$

Moreover, if $t_1 = \infty$ we have also

$$\lim_{t\to\infty} y(t) = 0\,. \tag{2.31}$$

Proof. Setting $Y := y^2$, from (i) we get

$$Y' \le -2a\,Y + 2b\,Y^{\beta} + F\,, \quad t \in [t_0, t_1)\,, \tag{2.32}$$

where $\beta := (\alpha+1)/2$, $F := 2c\,y$. In view of the second condition in (ii), contradicting (2.30) means that there exists $t^* \in (t_0, t_1)$ such that

$$Y(t) < \delta^2\,, \quad \text{for all } t \in [t_0, t^*)\,; \quad Y(t^*) = \delta^2\,. \tag{2.33}$$

Using this information back in (2.32) we find for all $t \in [t_0, t^*)$

$$Y'(t) \le 2(-a + b\delta^{\alpha-1})Y(t) + F(t)\,,$$

which in view of the assumptions, after integration from t_0 to t^*, furnishes

$$Y(t^*) < \frac{\delta^2}{2} + \int_{t_0}^{t_1} F(t)\,dt < \delta^2\,.$$

However, the latter is at odds with (2.33), and we thus conclude the proof of the first part of the lemma. In order to show the second part, we observe that from (2.30) and (2.32) we deduce

$$Y' \le -2a\,Y + 2(b\delta^{\alpha} + c)y\,,$$

so that (2.31) follows from the first condition in (ii) and Lemma 2.3.4. ■

3.0 HISTORICAL BACKGROUND AND PREVIOUS CONTRIBUTIONS

In Chaper 2, we have presented some applications of the study of liquid-filled rigid bodies in the Applied Sciences, and precisely in Space Engineering and Geophysics (Subsections 2.1.2.2 and 2.1.2.1, respectively). Besides, these practical applications, in the past century and decades, the problem of rigid bodies with cavities containing liquids have gained a lot of attention also from the theoretical point of view. There is a vast mathematical literature on this subject which goes back to the early contributions of Stokes in [51], and Zhukovskii in [55], at the end of the 19th century. In particular, Zhukovskii focused his attention on the motions of a rigid body having a cavity entirely filled with an ideal, irrotational, incompressible liquid. Under these assumptions, Zhukovskii looked for *potential-like* solutions for the liquid absolute velocity, $\boldsymbol{u}$. In fact, assuming that $\boldsymbol{u} \equiv \nabla\phi$, then $\phi = \phi(\boldsymbol{y}, t)$ can be found by solving

$$\Delta\phi = 0, \qquad \text{in } \mathcal{C} \times (0,\infty), \quad \nabla\phi \cdot \boldsymbol{n} = (\boldsymbol{\omega} \times \boldsymbol{y}) \cdot \boldsymbol{n}, \qquad \text{on } \partial\mathcal{C},$$

which correspond to incompressibility together with boundary conditions. The function $\phi = \phi(\boldsymbol{y}, t)$ can be found as a linear combination of the components of $\boldsymbol{\omega}$. Specifically,

$$\phi(\boldsymbol{y}, t) = \sum_{i=1}^{3} \omega_i(t)\varphi_i(\boldsymbol{y}),$$

where $\varphi_i = \varphi_i(\boldsymbol{y})$ satisfy

$$\begin{aligned} &\Delta\varphi_1 = 0 \text{ in } \mathcal{C},\ \nabla\varphi_1 \cdot \boldsymbol{n} = (\boldsymbol{y} \times \boldsymbol{n}) \cdot \boldsymbol{e}_1 \text{ on } \partial\mathcal{C}, \\ &\Delta\varphi_2 = 0 \text{ in } \mathcal{C},\ \nabla\varphi_2 \cdot \boldsymbol{n} = (\boldsymbol{y} \times \boldsymbol{n}) \cdot \boldsymbol{e}_2 \text{ on } \partial\mathcal{C}, \\ &\Delta\varphi_3 = 0 \text{ in } \mathcal{C},\ \nabla\varphi_3 \cdot \boldsymbol{n} = (\boldsymbol{y} \times \boldsymbol{n}) \cdot \boldsymbol{e}_3 \text{ on } \partial\mathcal{C}. \end{aligned} \tag{3.1}$$

The functions φ_i depend only on the geometric properties of the cavity, they are called *Zhukovskii potentials* (see [55] and [14]). Once these potentials are found, in the same fashion as for the case of a body immersed in an ideal, irrotational, incompressible liquid, one can then evaluate the total torque exerted by the liquid on the body, for each given cavity, as

$$-\int_{\partial\mathcal{C}} \boldsymbol{y} \times (\tilde{p}\boldsymbol{n}) d\sigma \equiv \boldsymbol{I}_a \cdot \dot{\boldsymbol{\omega}},$$

where $\boldsymbol{I}_a$ is the *tensor of virtual mass*, it is a symmetric tensor depending only on the liquid density and on the shape of the cavity; it takes into account the inertia added to the liquid due to the acceleration of the body. The components of $\boldsymbol{I}_a$ are given by

$$I_{a,ij} = -\rho \int_{\partial\mathcal{C}} \varphi_j \nabla\varphi_i \cdot \boldsymbol{n}\, d\sigma, \quad \text{all } i, j = 1, 2, 3.$$

Thus, the system of equations (2.10) governing the motion of the whole system $\mathcal{S}$ can be decoupled by solving the three boundary value problems (3.1) first, and then, after evaluating the tensor of virtual mass, by solving a system of ODEs, which reads as follows

$$(\boldsymbol{I}_a + \boldsymbol{I}_B) \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}_{\mathcal{B}} \cdot \boldsymbol{\omega} = \boldsymbol{m},$$

where $\boldsymbol{m}$ is the total external torque applied on the solid in the moving frame F.

Other results concerning ideal liquid contained in a cavity within a solid were obtained by Hough ([24]), Poincaré ([36]) and Sobolev ([49]). Some of these authors analyzed the stability properties of the coupled system liquid-filled rigid body either by considering small oscillations in the case of a solid with an ellipsoidal cavity filled by an ideal liquid in approximately uniform rotation ([24]) or by examining the linearized equations of motion for a heavy symmetric top containing an ideal liquid ([49]).

The case of a viscous fluid turns out to be more delicate than the ideal one. In the simplest case of inertial motions, Zhukovskii conjectured the following concerning the long-time behavior of the solutions to (2.10).

Theorem 3.0.6 (Zhukovskii's Conjecture, [55], p. 152)**.** *The motions of $\mathcal{S}$ (about its center of mass) will eventually be rigid motions and, precisely, permanent rotations, no matter the size and shape of the cavity, the viscosity of liquid, and the initial movement imparted on the system.*

This intriguing property shown by liquid-filled rigid bodies has to be contrasted with the case of no liquid contained in the cavity. In this latter case, the equations of motion (2.10) reduce to the well-known *Euler equations* for a rigid body:

$$\boldsymbol{I}_B \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}_B \cdot \boldsymbol{\omega} = \mathbf{0}. \tag{3.2}$$

The dynamics of $\mathcal{B}$ described by the latter displayed equations is very rich (see [35], Chpater VII and Chapter VIII). Permanent rotations (i.e. rigid body motions with constant angular velocities) are solutions to (3.2); they may occur if and only if the angular velocity $\boldsymbol{\omega}$ is directed along one of the principal axes of inertia, and the latter must align with the given initial total angular momentum. Time-dependent motions may be very complicated depending on the mass distribution of $\mathcal{B}$. In mathematical terms, this can be expressed by conditions on the principal moment of inertia of $\mathcal{B}$, A, B, and C (i.e. the eigenvalues of $\boldsymbol{I}_B$ with corresponding eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively). For example, if $A = B \neq C$, then the most general motion of $\mathcal{B}$ about G is a *regular precession*, where $\mathcal{B}$ performs a permanent rotation around the axis parallel to $\boldsymbol{e}_3$ and passing through G, while the latter rotates uniformly around the direction of the initial (given) angular momentum. Therefore, what Zhukovskii suggested is that the liquid has a *stabilizing effect* on the motion of the rigid body when no external forces and torques are applied on the coupled system ($\boldsymbol{\mathcal{F}} = \boldsymbol{m}_O \equiv \mathbf{0}$). This property has, indeed, a simple heuristic explanation. Because of the liquid viscosity, the velocity field of fluid relative to solid must eventually vanish, so that the coupled system $\mathcal{S}$ will eventually move by rigid motion. Under this condition, from the equation $(2.10)_2$ and by $(2.8)_2$, we derive that the pressure gradient of the liquid, $\nabla\tilde{p}$, must balance the centrifugal forces:

$$\rho(\dot{\boldsymbol{\omega}} \times \boldsymbol{y} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y})) = \nabla\tilde{p}. \tag{3.3}$$

Thus, taking the curl of both sides of the equation we get $\dot{\boldsymbol{\omega}} = \mathbf{0}$, which shows that only permanent rotations may occur. The previous argument is purely heuristic and takes for granted the following two facts: the liquid goes to rest (relative to a moving frame) as time goes to infinity, and the long-time dynamics of $\mathcal{S}$ are governed by (3.3). Unfortunately, neither of these properties is obvious from a rigorous viewpoint. As we will see later in this thesis, even though the total energy of $\mathcal{S}$ is a decreasing function of time, the dissipation is in terms of the liquid relative velocity only, and there is a (in principle large) conservative component in the motion of $\mathcal{S}$ given by its total angular momentum. In fact, let us formally dot-multiplying (2.10)$_3$ by $\boldsymbol{A}_O$, we find that

$$\frac{d|\boldsymbol{A}_O|^2}{dt} = 0.$$

Thus, the fact that the velocity of the liquid relative to the body must eventually vanish, especially when the initial data are such that the magnitude of the total angular momentum is arbitrary large, is not so evident. Furthermore, admitting that the asymptotic dynamics of $\mathcal{S}$ are governed by (3.3) implies, in terms of Dynamical Systems, that the Ω-limit set of a generic trajectory is not empty and invariant in the class of solutions to (2.10). This latter statement is not so clear since the equations (2.10) involve the Navier-Stokes equations. Finally, even assuming the validity of both of the above statements, it is still obscure why the angular velocity of the coupled system should tend to a specific constant value. Even though never rigorously proved right or wrong, Zhukovskii's statement is often presented as a theorem mainly by Russian authors (see e.g. [12], [34]).

We are not aware of any new or substantial contribution on the problem of motions of rigid bodies with liquid-filled cavities in the years immediately after Zhukovskii. This lack in the mathematical and engineering literature lasted till the beginning of the Cold War, when the United Staes and the Soviet Union intensified their missiles race and began the so called *space race*, a competition aimed to show their technological superiority. The race to space foresaw the launch of artificial satellites and space probes to explore other planets or the Moon, and the attempt to perform the first human space flights. It is in this scenario that we find some of the contributions in Space Engineering cited in Chapter 2 (e.g. [5], [1], [6], [44], [3], [25] and [46]). From the mathematical point of view, Rumyantsev has extensively studied the stability of motion of a rigid body with a cavity partially or entirely filled by ideal and viscous liquids (see [41]–[43], [14], and also [34]). The author investigates the problem of stability with respect to a finite number of variables: the components, ω_i, of the angular velocity and some integral functions of the liquid absolute velocity, $\boldsymbol{u}$,

$$G_s(t) = \int_{\mathcal{C}} \Phi(t, \boldsymbol{u}), \quad \text{for } i \leq s \leq N,$$

for some $N \in \mathbb{N}$. If one considers $N = 3$, a possible choice for the functions G_s is given by

$$G_s(t) = \boldsymbol{e}_s \cdot \int_{\mathcal{C}} \rho(\boldsymbol{y} \times \boldsymbol{u}), \quad \text{for } s = 1, 2, 3,$$

i.e. the projections of the liquid angular momentum with respect to O, in the moving frame F. The idea is to construct suitable Lyapunov functionals, $V = V(t)$, which depend only on $(\boldsymbol{G}, \boldsymbol{\omega})$. The stability properties are then found by a suitable modification of Lyapunov Stability Theorem for V (see [41]). For the case of inertial motions of $\mathcal{S}$ about its center of mass, G, Rumyantsev shows that any permanent rotation about the principal axes of inertia corresponding to the largest moment of inertia of $\mathcal{S}$ is stable with respect to the variables $(\boldsymbol{G}, \boldsymbol{\omega})$. Stability conditions are obtained also for a heavy rigid body filled with a viscous liquid about a fixed point O ([40], [34]). It has to be noticed that the quantities G_s do not completely characterize the motion of the liquid, as the liquid velocity belongs to an infinite dimensional space. So, Rumyantsev's results concern only *conditional* stability of the liquid variable. Moreover, Rumyantsev's analysis is still formal, because it lacks of a suitable corresponding existence theorem for the relevant equations.

Also Chernousko investigated the motion of a rigid body with a cavity entirely filled by a viscous liquid ([12]). For certain type of motions, corresponding to total external torques and angular momentum of the liquid "small" compared to $\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}$, the author is able to separate the problem of the motion of $\mathcal{S}$ into two parts that can be solved independently. The first part consists of solving a hydrodynamic problem described by boundary value problems which depend only on the shape of the cavity; moreover, they are independent of the motion of the solid. This enables Chernousko to calculate suitable "coefficients" which determine the effect that the liquid has on the body. The second part of the problem deals with a system of ODEs governing the dynamics of the rigid body. Let us develop Chernousko's arguments in some details. For the problem in hand, a suitable choice for a Reynolds number is given by

$$Re := \frac{\ell}{T\nu},$$

where ℓ is a characteristic length scale for the cavity, T is a characteristic time scale (say, T is of order $|\boldsymbol{\omega}|^{-1}$), and ν is the coefficient of kinematic viscosity for the liquid ($\nu := \mu/\rho$).

Assume that $Re << 1$, and without loss of generality, $\ell \equiv 1$ and $T \equiv 1$. Chernousko looks for a solution to $(2.10)_{1,2,5}$ (augmented with suitable initial conditions) of the form

$$\boldsymbol{v} = \boldsymbol{v}^* + \tilde{\boldsymbol{v}}, \quad p = p^* + \tilde{p},$$

where

$$\boldsymbol{v}^*(\boldsymbol{y}, t) = \sum_{n=1}^{\infty} \nu^{-n} \boldsymbol{v}_n^*(\boldsymbol{y}, t), \qquad p^*(\boldsymbol{y}, t) = \sum_{n=0}^{\infty} \nu^{-n} p_n^*(\boldsymbol{y}, t)$$

and, for $\tau = \nu t$,

$$\tilde{\boldsymbol{v}}(\boldsymbol{y}, \tau) = \sum_{n=1}^{\infty} \nu^{-n} \tilde{\boldsymbol{v}}_n(\boldsymbol{y}, \tau), \qquad \tilde{p}(\boldsymbol{y}, \tau) = \sum_{n=0}^{\infty} \nu^{-n} \tilde{p}_n(\boldsymbol{y}, \tau).$$

The field $\boldsymbol{v}^*$ will eventually converge to a steady solution. Whereas, $\tilde{\boldsymbol{v}}$ represents the "transient" part of the liquid relative velocity, it will decay to zero as time approaches to infinity. Replacing $(\boldsymbol{v}^*, p^*)$ in $(2.10)_{1,2,5}$, one finds that the $n-$th approximation $(\boldsymbol{v}_n^*, p_n^*)$ has to satisfy

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{v}_n^* = 0, \\ &\Delta \boldsymbol{v}_n^* = \nabla p_n^* - \boldsymbol{F}_n, \end{aligned}\right\} \text{ in } \mathcal{C} \times (0, \infty), \\ \boldsymbol{v}_n^* = \boldsymbol{0} \text{ on } \partial\mathcal{C}, \tag{3.4}$$

where $\boldsymbol{F}_n$ depends on the previous approximations $(\boldsymbol{v}_0^*, p_0^*), \ldots (\boldsymbol{v}_{n-1}^*, p_{n-1}^*)$. In particular, $\boldsymbol{F}_0 \equiv \boldsymbol{0}$, thus the following system of PDEs

$$\operatorname{div} \boldsymbol{v}_0^* = 0, \ \ \Delta \boldsymbol{v}_0^* = \nabla p_0^* \ \text{ in } \mathcal{C} \times (0, \infty), \quad \boldsymbol{v}_0^* = \boldsymbol{0} \text{ on } \partial\mathcal{C}$$

admits $\boldsymbol{v}_0^* \equiv \boldsymbol{0}$ as unique solution (see [28]). Moreover,

$$\operatorname{div} \boldsymbol{v}_1^* = 0, \ \ \Delta \boldsymbol{v}_1^* = \nabla p_1^* - \boldsymbol{F}_1 \ \text{ in } \mathcal{C} \times (0, \infty), \quad \boldsymbol{v}_1^* = \boldsymbol{0} \text{ on } \partial\mathcal{C}, \tag{3.5}$$

where

$$\boldsymbol{F}_1 = \dot{\boldsymbol{\omega}} \times \boldsymbol{y} + 2\boldsymbol{\omega} \times \boldsymbol{v}_0^* + \frac{\partial \boldsymbol{v}_0^*}{\partial t} + \boldsymbol{v}_0^* \cdot \nabla \boldsymbol{v}_0^* + \boldsymbol{Q}^T \cdot \boldsymbol{\mathcal{F}} = \dot{\boldsymbol{\omega}} \times \boldsymbol{y} + \boldsymbol{Q}^T \cdot \boldsymbol{\mathcal{F}}.$$

Replacing $(\boldsymbol{v}^* + \tilde{\boldsymbol{v}}, p^* + \tilde{p})$ in (2.10)$_{1,2,5}$, we find that $n-$th approximation $(\tilde{\boldsymbol{v}}_n, \tilde{p}_n)$ has to satisfy

$$
\left.\begin{aligned}
&\operatorname{div} \tilde{\boldsymbol{v}}_{\boldsymbol{n}} = 0, \\
&\frac{\partial \tilde{\boldsymbol{v}}_n}{\partial t} = \Delta \tilde{\boldsymbol{v}}_n - \nabla \tilde{p}_n + \boldsymbol{G}_n,
\end{aligned}\right\} \text{ in } \mathcal{C} \times (0, \infty),
$$
$$
\tilde{\boldsymbol{v}}_n = \boldsymbol{0} \text{ on } \partial\mathcal{C},
$$
$$
\tilde{\boldsymbol{v}}_0(\boldsymbol{y}, 0) = \boldsymbol{v}_0, \qquad \tilde{\boldsymbol{v}}_n(\boldsymbol{y}, 0) = -\boldsymbol{v}_n^*(\boldsymbol{y}, 0), \tag{3.6}
$$

where $\boldsymbol{v}_0$ is the (given) initial relative velocity of the liquid, and $\boldsymbol{G}_n$ depends on the previous approximations $(\tilde{\boldsymbol{v}}_0, \tilde{p}_0), \dots (\tilde{\boldsymbol{v}}_{n-1}, \tilde{p}_{n-1})$. In particular, $\boldsymbol{G}_0 \equiv \boldsymbol{0}$. The existence and uniqueness of solutions to (3.4) and (3.6) for given $\boldsymbol{F}_n$ and $\boldsymbol{G}_n$ is guaranteed by classical results in the mathematical theory of viscous fluids (see [28]). In particular, one can show that

$$
\|\tilde{\boldsymbol{v}}_0\|_2 , \quad \|\tilde{\boldsymbol{v}}_1\|_2 \approx \exp(-c\nu t)
$$

for some positive constant c. Therefore, for sufficiently "large" times, and with an accuracy of order ν^{-2}, the relative velocity of the liquid is "small" (it is of order ν^{-1}), and

$$
\boldsymbol{v} \approx \nu^{-1} \boldsymbol{v}_1^*,
$$

where $\boldsymbol{v}_1^*$ satisfies the linear stationary problem (3.5). If $\boldsymbol{\mathcal{F}} \equiv \boldsymbol{0}$, then Chernousko shows that

$$
\boldsymbol{v}(\boldsymbol{y}, t) \approx \nu^{-1} \dot{\omega}_i(t) \boldsymbol{V}_i(\boldsymbol{y}),
$$

where, for all $i = 1, 2, 3$, $\boldsymbol{V}_i = \boldsymbol{V}_i(\boldsymbol{y})$ with the corresponding pressure fields $q_i = q_i(\boldsymbol{y})$ satisfy the following boundary value problems which depend only on the shape of the cavity, and they are independent of the motion of the solid:

$$
\operatorname{div} \boldsymbol{V}_i = 0, \quad \Delta \boldsymbol{V}_i = \nabla q_i + \boldsymbol{e}_{\boldsymbol{i}} \times \boldsymbol{y} \ \text{ in } \mathcal{C}, \quad \boldsymbol{V}_i = \boldsymbol{0} \quad \text{on } \partial\mathcal{C}.
$$

Thus, the kinematic angular momentum of the liquid can be approximated as follows

$$
\int_{\mathcal{C}} \rho \, \boldsymbol{y} \times \boldsymbol{v} \approx \rho \nu^{-1} \boldsymbol{P} \cdot \boldsymbol{\omega},
$$

$\boldsymbol{P}$ is a symmetric, positive definite tensor, which depends only on the geometric properties of cavity. The equation $(2.10)_3$ is then properly modified to obtain a system of ODEs of the same order as the corresponding equations governing the dynamics of a frozen liquid. Chernousko also proves that, under all the above mentioned assumptions, permanent rotations about the principal axis of inertia of $\mathcal{S}$ corresponding to the largest moment of inertia are stable in the sense of Lyapunov. Moreover, permanent rotations of $\mathcal{S}$ about the principal axes of inertia of $\mathcal{S}$ corresponding to the mean and least moment of inertia are both unstable. The case of high Reynolds numbers is also studied in [12]; the techniques involve the linearization of the Navier-Stokes equations and the construction of a solution by the boundary layer method. We will not go further into the details of this latter case, as it goes far beyond the purpose of this thesis. A detailed account of the previous results and their applications can be found in [34].

In the more recent years, the mathematical literature on the problem of rigid bodies with liquid-filled cavities has been focused on the stability and instability properties of the motions of $\mathcal{S}$. In this direction, we would like to mention the work by Smirnova (in [48]), in which the author confirms the stability and instability properties obtained by Chernousko, but with less assumptions. We would like to cite also the papers by Lyashenko ([32]) and Kostyuchenko et al. ([27]), and the book by Kopachevsky and Krein ([26]) in which the authors consider the linearized equations for the perturbed motion around an equilibrium configuration, and analyze the spectrum of the corresponding evolution operator. Even though these results are interesting from the mathematical point of view, they need not be valid for the original nonlinear problem due to the lack, to date, of a *linearization principle* that may validate the above findings at the nonlinear level. In Chapter 5, we will present *necessary and sufficient conditions for stability* (in the sense of Lyapunov) for the full nonlinear problem, without any approximation or assumptions on the shape of the cavity. These conditions contain those of [43] as a particular case, and extend those of [12],[27] and [48] to the nonlinear level.

Finally, in [47] and [33], the problem of liquid-filled rigid body is treated with the more modern techniques of energy methods. [33] represents the starting point of the mathematical analysis of the motions of solids with liquid-filled cavities presented in Chapter 5. Concerning the inertial motions, in [33], the existence of a dynamical system in the 2D case is proved.

In particular, the existence of weak solutions is shown in detail; the proof is given both in the 2D and 3D case along with all the properties of weak solutions and the uniqueness results. Moreover, the existence of strong solutions for the coupled system in two and three dimensions is also studied. Local strong solutions for any initial data (provided that the initial motion has finite kinetic energy) and global strong solutions for initial data which are "sufficiently small" are proved to exist. Moreover, it is shown that, within the class of global strong solutions corresponding to "small" initial data, the long-time dynamics of a liquid-filled rigid body is completely characterized by a rigid body motion with the liquid at rest relatively to the solid. This rigid body motion is a permanent rotation about one of the principal axis of inertia if $\mathcal{S}$ has a "symmetric mass distribution", that is when the inertial tensor of the whole system is a multiple of the identity tensor.

4.0 NUMERICAL SIMULATIONS AND PHYSICAL EXPERIMENTS

In this Chapter, we would like to present some numerical simulations and physical experiments that we have performed in order to obtain insights and further informations on the behavior of coupled system $\mathcal{S}$. We are mostly interested in the long-time behavior of solutions to (2.9) in the case of *inertial motions* of $\mathcal{S}$ about its center of mass G ($\boldsymbol{\mathcal{F}} \equiv \mathbf{M} \equiv \mathbf{0}$). These tests represent the starting point for some ongoing numerical and physical experiments, and they also complement some of the analytical results described in Chapter 5.

We will start by presenting some results of numerical tests simulating the inertial motions of a liquid-filled rigid body. These simulations have been performed in collaboration with Professor P. Zunino[1]. The results first appeared in [20]; an account of them can be found in our joint paper [13].

4.1 NUMERICAL SIMULATIONS

As we will see in Chapter 5, conservation of angular momentum and energy balance will play a fundamental role in the analysis of the system at hand. It is, therefore, essential that the time discretization method accurately preserves the invariants of the system. At the level of numerical approximation this is not a trivial task, even when we restrict ourselves to the analysis of the motion of the body solely, i.e. equations $(2.9)_3$. We have employed the following time discretization algorithm for the coupled system liquid-filled rigid body whose motion is governed by (2.9). For the time integration of the body dynamics, we use the θ-method. In particular, we adopt the implicit midpoint integration rule, $\theta =$

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$\frac{1}{2}$, because of its good properties as a geometric integrator. This turns out particularly useful since the midpoint rule exactly satisfies the conservation of momentum for simple Hamiltonian systems. For the Navier-Stokes equations, we apply the implicit Euler time advancing scheme in order to guarantee the stability of the algorithm. At each time step, we use sub-iterations to uncouple the solution of the discrete body and liquid problems and to linearize the corresponding equations. The convergence to the approximated solution is achieved through a fixed point argument. The combination of these techniques gives rise to the following algorithm:

Given $\boldsymbol{u}_0$, $\boldsymbol{\omega}_0$ and a partition of the interval $(0,T]$ in evenly distributed time steps $t_n = n\tau$ with $\tau > 0$, for $n = 1,2,3,\ldots$ find $\boldsymbol{u}_n$, $\tilde{p}_n$, $\boldsymbol{\omega}_n$ in the following way:

Set $\boldsymbol{u}_n^0 = \boldsymbol{u}_{n-1}$, $\boldsymbol{\omega}_n^0 = \boldsymbol{\omega}_{n-1}$. For $k = 1,2,3,\ldots$ solve the sub-problems:

<u>Body problem</u>: find $\boldsymbol{\omega}_n^*$ such that,

$$\tau^{-1}\boldsymbol{I}_{\mathcal{B}}\cdot\big(\boldsymbol{\omega}_n^* - \boldsymbol{\omega}_{n-1}\big) + \theta\big[\boldsymbol{\omega}_n^{k-1} \times \big(\boldsymbol{I}_{\mathcal{B}} \cdot \boldsymbol{\omega}_n^{k-1}\big) + \int_{\partial\mathcal{C}} \boldsymbol{y} \times \boldsymbol{T}(\boldsymbol{u}_n^{k-1}, \tilde{p}_n^{k-1}) \cdot \boldsymbol{n}\big] + (1-\theta)\big[\boldsymbol{\omega}_{n-1} \times \big(\boldsymbol{I}_{\mathcal{B}} \cdot \boldsymbol{\omega}_{n-1}\big) + \int_{\partial\mathcal{C}} \boldsymbol{y} \times \boldsymbol{T}(\boldsymbol{u}_{n-1}, \tilde{p}_{n-1}) \cdot \boldsymbol{n}\big] = \boldsymbol{0}. \quad (4.1)$$

Relaxation: given $\sigma \in (0,1]$, set $\boldsymbol{\omega}_n^k = \sigma\boldsymbol{\omega}_n^* + (1-\sigma)\boldsymbol{\omega}_{n-1}$.

<u>Liquid problem</u>: find $\boldsymbol{u}_n^k$, $\tilde{p}_n^k$ such that,

$$\begin{cases} \rho\big(\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}\big) + \rho\boldsymbol{\omega}_n^k \times \boldsymbol{u}_n^k + \rho\,\boldsymbol{v}(\boldsymbol{u}_n^{k-1}, \boldsymbol{\omega}_n^k) \cdot \nabla\boldsymbol{u}_n^k \\ \qquad - \operatorname{div}\boldsymbol{T}(\boldsymbol{u}_n^k, \tilde{p}_n^k) = \boldsymbol{0} \quad \text{in } \mathcal{C}, \\ \operatorname{div}\boldsymbol{u}_n^k = 0, \quad \text{in } \mathcal{C}, \\ \boldsymbol{u}_n^k = \boldsymbol{\omega}_n^k \times \boldsymbol{y}, \quad \text{on } \partial\mathcal{C}, \end{cases} \quad (4.2)$$

where we recall that $\boldsymbol{v}(\boldsymbol{u}_n^{k-1}, \boldsymbol{\omega}_n^k) = \boldsymbol{u}_n^{k-1} - \boldsymbol{\omega}_n^k \times \boldsymbol{y}$.

Convergence test: given ϵ small enough, if $\|\boldsymbol{\omega}_n^k - \boldsymbol{\omega}_n^{k-1}\| < \epsilon$ then set

$$\boldsymbol{u}_n = \boldsymbol{u}_n^k, \quad \tilde{p}_n = \tilde{p}_n^k, \quad \boldsymbol{\omega}_n = \boldsymbol{\omega}_n^k.$$

Since our numerical tests involve relatively simple geometrical configurations, moderately refined meshes will be applied, see for instance Figure 1 that shows the geometrical model for the cavity and the corresponding computational mesh, $\mathcal{C}_h$. A mesh sensitivity analysis, not reported here, confirms that the qualitative behavior of the system does not change when the computational mesh of the liquid cavity is refined. Furthermore, we point out that the simulations are insensitive to the geometric representation and discretization of the solid shell, because the tensor of inertia $\boldsymbol{I}_{\mathcal{B}}$ and consequently its eigenvalues A, B, C are prescribed as parameters of the numerical algorithm.

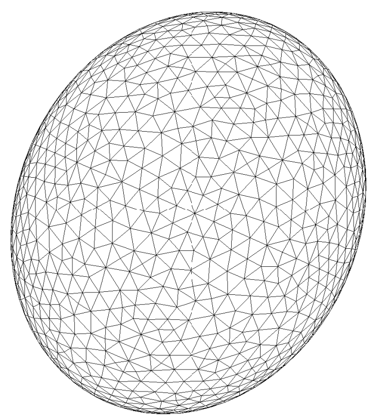


Figure 1: Geometrical configuration of the cavity $\mathcal{C}$ used for the numerical experiments.

The liquid problem (4.2) is solved by the finite element method ([22, 23, 39]). In order to achieve a stable discretization of the divergence-free constraint, we use *inf-sup* stable mixed finite elements, such as $\mathbb{P}^2-\mathbb{P}^1$ approximation of the velocity and pressure fields, respectively. Here $\mathbb{P}^k$ denotes the space of all polynomials in $\mathbb{R}^3$ of degree less or equal to k, for $k = 1, 2$ (see [22, 8]). The variational formulation of the problem in terms of velocity and pressure variables reads as follows

$$\int_{\mathcal{C}} \rho \left[\tau^{-1}(\boldsymbol{u}_n^k - \boldsymbol{u}_n) \cdot \boldsymbol{v}_h + (\boldsymbol{\omega}_n^k \times \boldsymbol{u}_n^k) \cdot \boldsymbol{v}_h + \big(\boldsymbol{v}(\boldsymbol{u}_n^{k-1}, \boldsymbol{\omega}_n^k) \cdot \nabla \boldsymbol{u}_n^k\big) \cdot \boldsymbol{v}_h\right] + \mu \nabla \boldsymbol{u}_n^k : \nabla \boldsymbol{v}_h \\ - \int_{\mathcal{C}} [q_h \operatorname{div} \boldsymbol{u}_n^k + \tilde{p}_n^k \operatorname{div} \boldsymbol{v}_h] = 0,$$

for all $\boldsymbol{v}_h \in V_h$, $q_h \in P_h$, where

$$V_h := \{\boldsymbol{v}_h \in C^0(\overline{\mathcal{C}}) : \boldsymbol{v}_h|_T \in \mathbb{P}^2 \text{ for all } T \in \mathcal{C}_h\},$$

$$P_h := \{q_h \in C^0(\overline{\mathcal{C}}) : q_h|_T \in \mathbb{P}^1 \text{ for all } T \in \mathcal{C}_h\}.$$

The system of algebraic equations arising from the discretization scheme is solved by means of direct techniques, which turn out to be an effective option since the number of degrees of freedom is not excessively large.

4.1.1 Effect of the viscosity on the long-time behavior of the coupled system

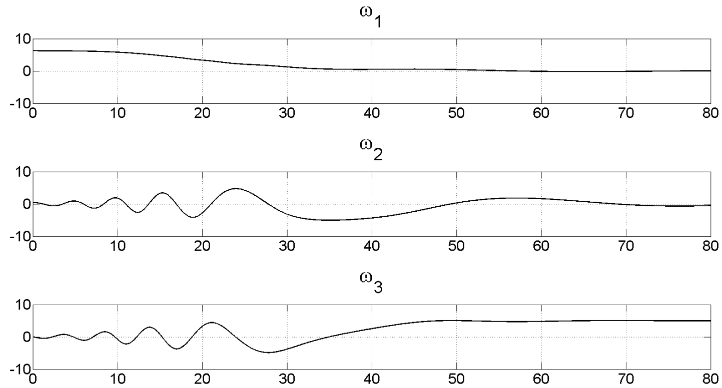
We study the dynamics of a system where the cavity has the (*quasi-ellipsoidal*) shape shown in Figure 1, and the rigid body is characterized by an inertial tensor of inertia with eigenvalues $A = 5.54$, $B = 6.73$, $C = 6.76$ (which corresponds to depositing a layer of uniform material of constant thickness around the cavity). At the initial time, the motion of $\mathcal{B}$ is identified by the angular velocity $\boldsymbol{\omega}_0 = 2\pi(\cos(\theta), \cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta))$ with $\theta = \pi/48$, $\phi = 0$, while the relative velocity of the liquid is $\boldsymbol{v}_0 = \mathbf{0}$ everywhere in $\mathcal{C}$. In Figure 2 we visualize the plots of $\boldsymbol{\omega}(t) = (p(t), q(t), r(t))$ for decreasing values of the kinematic viscosity of the liquid, namely $\nu = \mu/\rho$. The numerical simulations show that, for moderately large values of the viscosity ($\nu = 0.1$), the system quickly reaches a steady state which is a permanent rotation around the central axis of inertia corresponding to the largest moment of inertia. As expected, the trend through which the rotational equilibrium is reached is extremely sensitive to the viscosity. Indeed, for $\nu = 0.001$ the rotation of the liquid-solid system is "chaotic", at least for the timescale used in the case of large viscosity. Only when the timespan of simulation is significantly extended, the numerical experiments show that the steady rotation is eventually recovered (Figure 2, bottom panel).

The numerical simulations enable a more quantitative analysis of the effect of the viscosity on the time required to reach equilibrium. Let us denote by t_c the instant at which the following condition is satisfied for the first time,

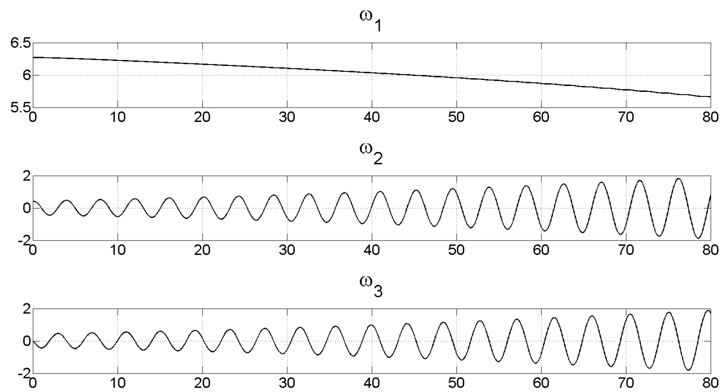
$$t_c : \quad \frac{\|\boldsymbol{\omega}(t) - \boldsymbol{\omega}(\infty)\|}{\|\boldsymbol{\omega}_0 - \boldsymbol{\omega}(\infty)\|} < 0.1$$

For the viscosities $\nu = 0.1, 0.05, 0.02, 0.01$ we have calculated t_c and the corresponding $\boldsymbol{\omega}(t_c)$. The results are reported in Table 1. From these data, it is possible to estimate how t_c depends on ν. We begin by postulating a power law dependence such as $t_c \simeq \nu^{\alpha}$. Then, in the range $\nu \in (0.01, 0.1)$ the value of α that best fits the data is $\alpha = -0.305$. This result

$\nu = 0.1,\ \boldsymbol{\omega}_0 = [6.2697, 0.4109, 0],\ \boldsymbol{v}_0 = \mathbf{0},\ t \in (0, 80)$



$\nu = 0.001,\ \boldsymbol{\omega}_0 = [6.2697, 0.4109, 0],\ \boldsymbol{v}_0 = \mathbf{0},\ t \in (0, 80)$



$\nu = 0.001,\ \boldsymbol{\omega}_0 = [6.2697, 0.4109, 0],\ \boldsymbol{v}_0 = \mathbf{0},\ t \in (0, 1200)$

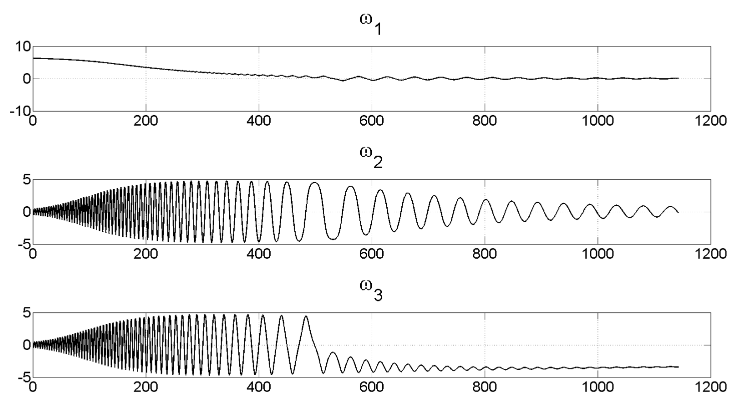


Figure 2: Dynamics of the liquid-solid system for decreasing values of the viscosity.

confirms the inverse dependence of the time required to reach equilibrium on the magnitude of the viscosity.

ν	$p(t_c)$	$q(t_c)$	$r(t_c)$	t_c
0.1	-0.4018	0.4558	4.8682	50.8
0.05	0.4908	-0.568	4.7584	63.3
0.02	-0.4887	-0.4171	-4.5768	75.2
0.01	0.6319	0.4658	-4.3192	99.8

Table 1: Dependence of the (numerically estimated) time to reach equilibrium on the liquid kinematic viscosity. The initial rotation is $\boldsymbol{\omega}_0 = (6.2697, 0.4109, 0)$.

The numerical results presented here provide a rather complete description of the asymptotic behavior in time of a liquid-filled rigid body for the cases $A \leq B < C$, and $A = B = C$. Moreover, Figure 3 shows the dynamics of $\mathcal{S}$ when $A = 5.54$ and $B = C = 6.76$. We see that, also in this case, the motion of coupled system $\mathcal{S}$ will reach a steady state which is a permanent rigid rotation about the central axis of inertia corresponding to the large moment of inertia. We will see in Chapter 5, specifically Theorem 5.2.4 and Remark 5.2.5, that the analytical proof of the latter numerical results is still open.

4.1.2 Effect of the initial rotation on the final angular velocity

We have also found some numerical verifications of the analytical results about the attainability and the stability of permanent rotations, for which the main analytical result are Theorem 5.2.7 and Theorem 5.2.10, respectively. The numerical experiments are particularly helpful to test the validity of the analysis beyond the restrictions on the initial data stated in Theorem 5.2.7.

In these cases, the liquid kinematic viscosity is set to $\nu = 0.1$. We begin with a problem configuration where the condition (5.76) is not satisfied because of large initial rotational speed,

$$9.6860 = \frac{A}{2B}(B - A)p^2(0) > \frac{C}{2B}(C - B)r^2(0) = 0.1310.$$

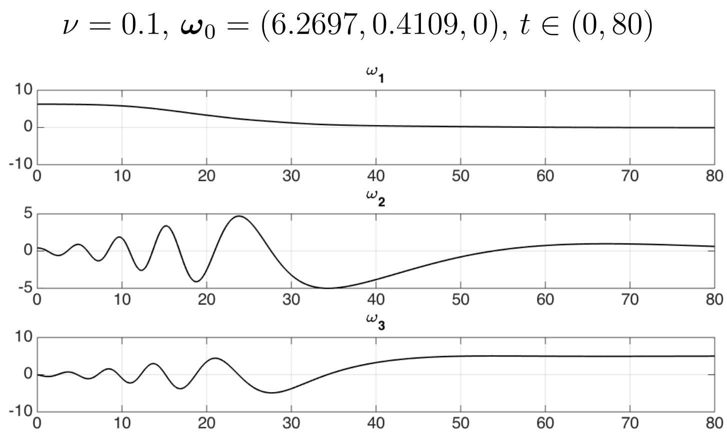


Figure 3: Dynamics of the liquid-solid system in the case $5.54 = A < B = C = 6.76$.

The computed plots of $\boldsymbol{\omega}(t)$ reported in Figure 4 (top panel) show that, for "large" initial data, the conclusions of Remark 5.2.11 are no longer valid. In particular, we observe that in this case $r(0) > 0 > \overline{r}$, which differs from the predicted behavior for small data, while $\overline{p} = \overline{q} = 0$ as proved in the analysis. In other words, sufficiently large $p(0), q(0)$ may trigger a *flip-over* effect. In Figure 4 (bottom panel) we investigate a similar situation, where $p(0)$ and $q(0)$ are sufficiently small. In this case, the validity of condition (5.76) is restored and the results of Remark 5.2.11 (i.e. $r(0)$ and $\overline{r}$ share the same sign) are reproduced, as expected, by the numerical simulation.

Numerical experiments also elucidate the behavior of the system when the initial relative velocity of the liquid with respect to the rigid body is varied. More precisely, we compare two cases that only differ in the initial liquid energy, $\mathcal{E}_F(0)$. In one case the initial relative velocity of the liquid is initialized to $\boldsymbol{v} = \mathbf{0}$ in $\mathcal{C}$, as a result $\mathcal{E}_F(0) = 0$. In the other case we define $\boldsymbol{v}$ as a nonzero compatible velocity field, such that $\boldsymbol{v} = \mathbf{0}$ on $\partial\mathcal{C}$ and $\operatorname{div}\boldsymbol{v} = 0$ in $\mathcal{C}$, such that $\mathcal{E}_F(0) \gg 1$. In particular, at the initial time the (absolute) liquid velocity can be expressed in the following form $\boldsymbol{u}_0 = f(\|\boldsymbol{x}\|)\boldsymbol{\omega}_0 \times \boldsymbol{x}$. Since $f(\|\boldsymbol{x}\|) \neq 1$, then $\boldsymbol{v} \neq \mathbf{0}$. For this numerical experiment we consider a different tensor of inertia, $A = 4.99$, $B = 4.99$, $C = 5.54$.

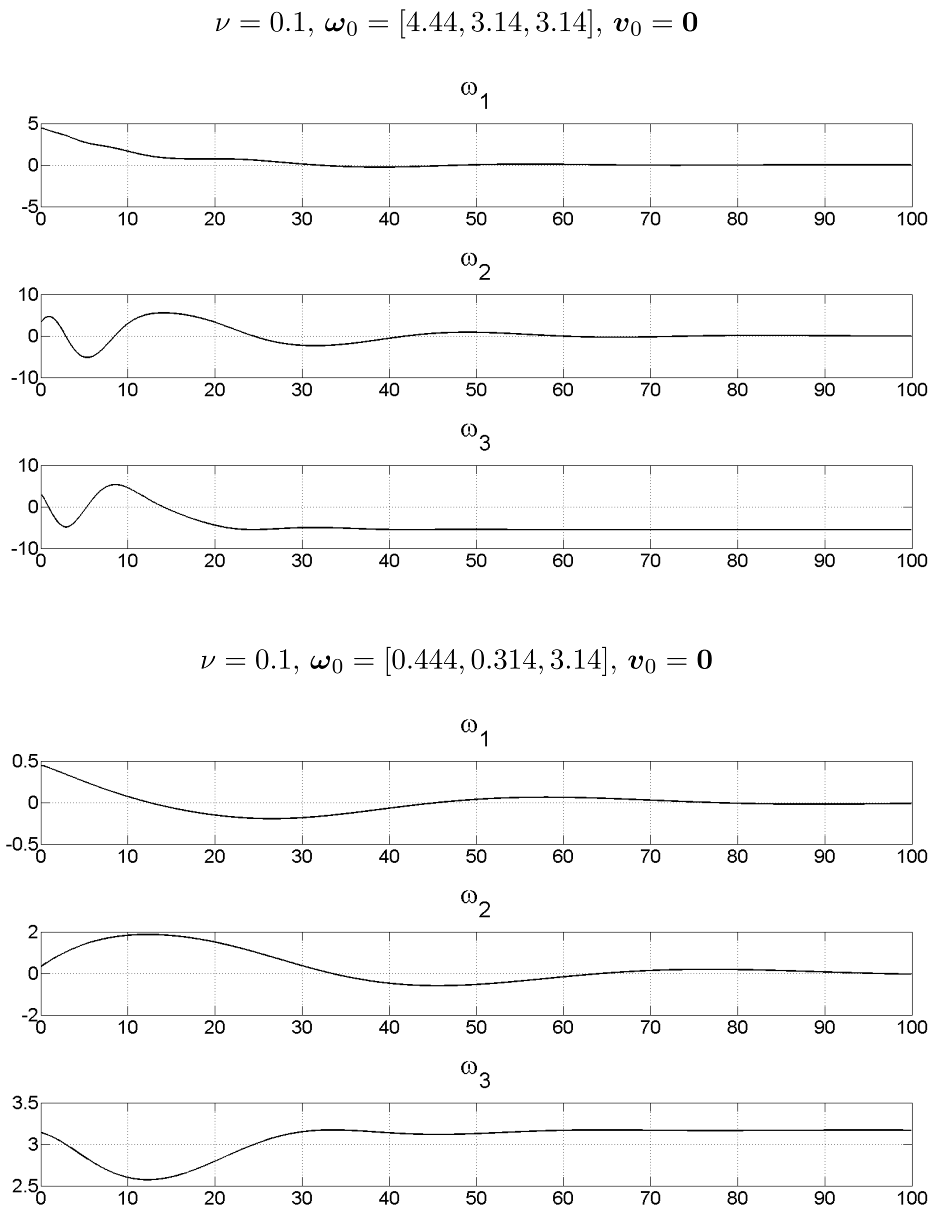


Figure 4: Dynamics of the liquid-solid system for different initial angular velocity.

For these initial data, we observe that (5.74) is not satisfied because

$$3.0393 = \frac{C}{A}(C-A)r^2(0) \ll \mathcal{E}_F(0) = 170.8$$

The corresponding results are shown in Figure 5. We see that the qualitative behavior of the system is substantially unaffected. This test shows that the thesis of Theorem 5.2.7, case (a), may be verified also when condition (5.74) is violated. This fact may be interpreted observing that for large viscosities, the liquid quickly adjusts its motion to satisfy $\boldsymbol{v} = \mathbf{0}$ everywhere in the cavity and in the same process, the initial angular momentum of the liquid is transferred to the solid. After this transition, the previous considerations relative to the sensitivity on the initial angular velocity apply.

4.1.3 The "flip-over" effect

We conclude this section by discussing how, besides the magnitude of the initial rotation along the unstable axes, the viscosity can also trigger the *flip-over* effect, in some particular configurations. As is discussed in Remark 5.2.11, if $A \le B < C$ and $r(0) = 0$, the analysis is not sufficient to determine the orientation of the final rotation along the stable axis, namely $\boldsymbol{e}_3$. The numerical simulations reflect this type of uncertainty and suggest that the determining factor is the liquid viscosity. More precisely, a careful analysis of Table 1 shows that, when $r(0) = 0$, changing the viscosity of the liquid not only affects the time to reach equilibrium, but also the orientation of the final rotation, namely sign$(\bar{r})$. Indeed, jumping from $\nu = 0.05$ to $\nu = 0.02$, the component $r(t_c)$ changes its sign, while the modulus is almost invariant. Figure 6 illustrates this effect with more details. For these tests, we consider the case $A = 5.54$, $B = 6.73$, $C = 6.76$ and the initial rotation $\boldsymbol{\omega}_0 = [6.2697, 0.4109, 0]$. Figure 6 shows that, for fixed initial conditions, the sign of r is sensitive to the liquid viscosity. On the basis of numerical experiments, we believe there exists a precise transition point at which the orientation of the rotation is flipped. For the particular configuration considered here, using Figure 6, we estimate that the transition point is $\nu^* \in (0.035, 0.0375)$.

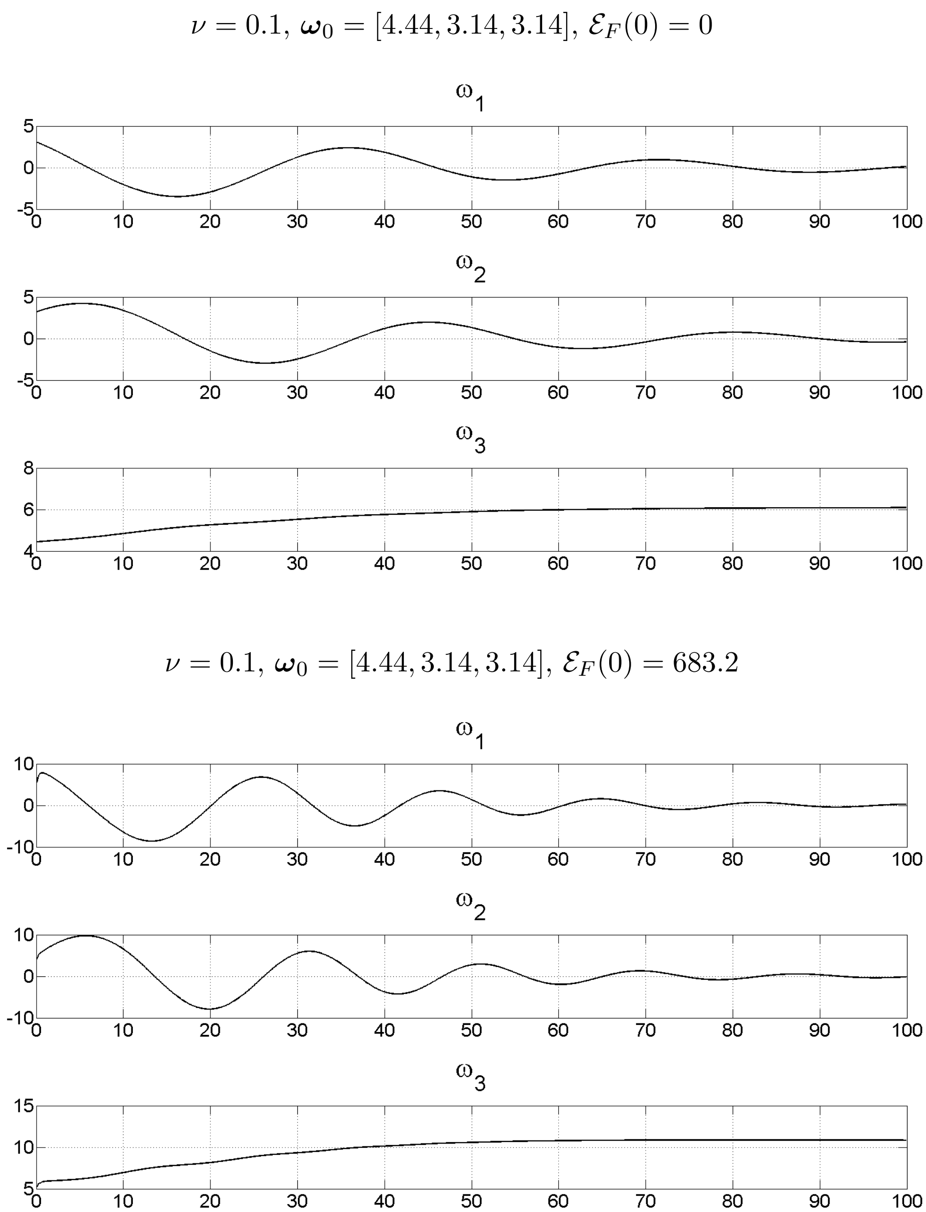
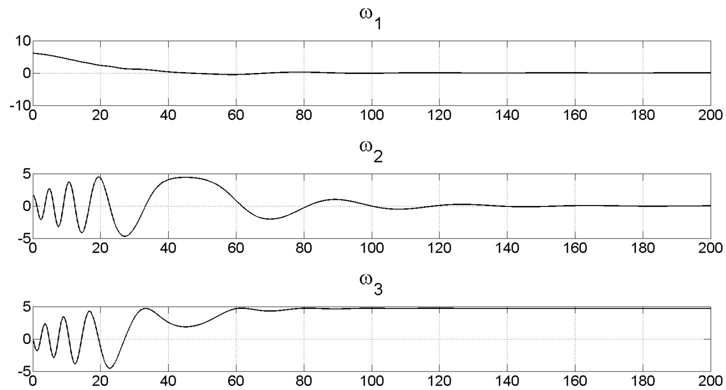
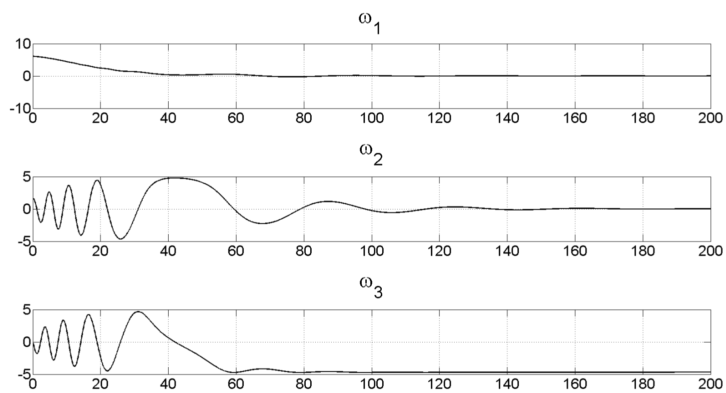


Figure 5: Dynamics of the liquid-solid system for different values of initial kinetic energy for the liquid.

$$\nu = 0.0375,\ \boldsymbol{\omega}_0 = [6.2697, 0.4109, 0],\ \mathcal{E}_F(0) = 0$$



$$\nu = 0.035,\ \boldsymbol{\omega}_0 = [6.2697, 0.4109, 0],\ \mathcal{E}_F(0) = 0$$



$$\nu = 0.0325,\ \boldsymbol{\omega}_0 = [6.2697, 0.4109, 0],\ \mathcal{E}_F(0) = 0$$

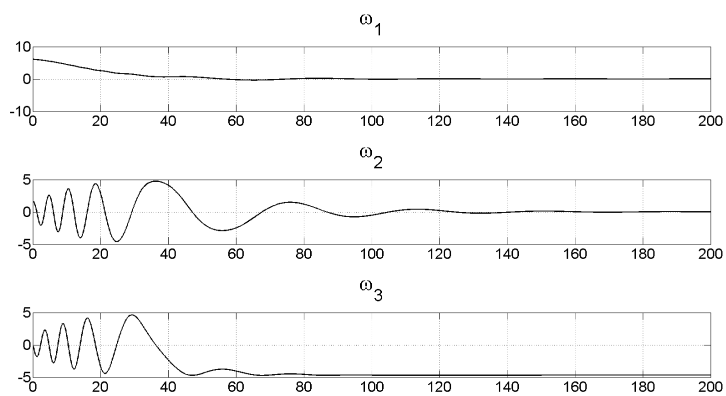


Figure 6: Visualization of the "flip-over" effect. The orientation of the final rotation changes, when moving form $\nu = 0.0375$ to $\nu = 0.035$.

4.2 EXPERIMENTAL TESTS: A LIQUID-FILLED GYROSCOPE

We have supervised some physical experiments for a gyroscope with a cavity entirely filled with a viscous liquid. These experiments have been conducted by several groups of undergraduates students for their Senior Design Project at the Department of Mechanical Engineering and Materials Science, University of Pittsburgh.

The device is the one shown in Figure 4.2.

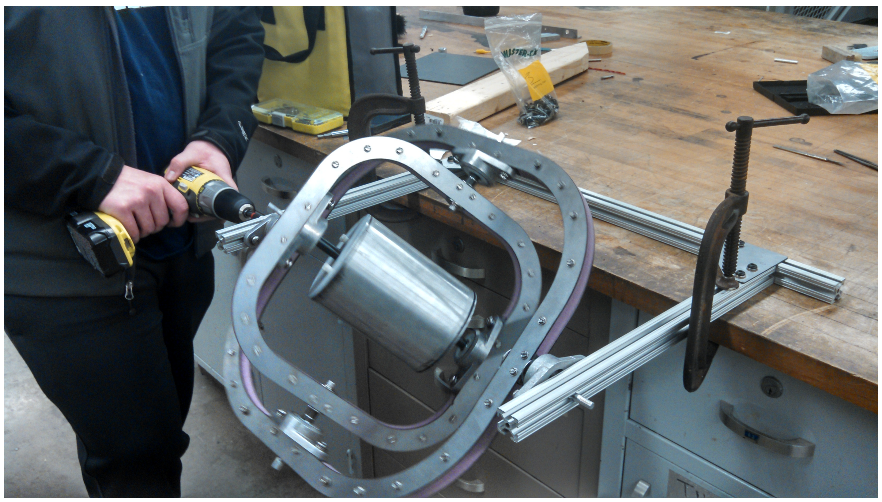


Figure 7: A liquid-filled gyroscope

The liquid container has a cylindrical shape, and it is machined from 6061 T6 aluminum stock: the end caps from a 0.250 inches-thick plate, and the sleeve from a 4.5 inches outer diameter pipe (0.120 inches wall thickness). Between stainless steel and aluminum as possible choices for the materials, aluminum was ultimately selected because of its significantly lower density in comparison to stainless steel. The axles of the liquid container were machined out of anodized aluminum precision shafting. Hex-shaped heads machined onto the ends of the axles provide a mechanism for torque transmission.

The frame brackets are made of low carbon steel. The inner frame has a dimeter of 12 inches and a weight of 2.38 pounds; whereas the outer frame is 17 inches in diameter and 3.21 pounds of weight. Finally, in order to reduce the friction during the motion, we chose acetyl polymer bearings with glass rolling elements. They are constructed of very lightweight plastic, featured an open design (with no dust shields impinging upon rolling elements), and required no viscous lubricant. These attributes brought substantial reductions in mass and friction. Lightweight cast aluminum housings were selected to attach the bearings to the frames.

The system was mainly tested with a viscous solution made of 20% water and 80% glycerine. The experiments plan was to accelerate the liquid container for about 60 seconds in the direction of the principal axis of the container corresponding to smallest moment of inertia. The acceleration was accomplished by applying a hand drill to the hex head machined into the end of the axle. An experiment sample can be viewed on https://www.youtube.com/watch?v=wXlD_yPbla8&feature=youtu.be.

The experiments show that after an interval of time where the motion of the system appear to be "chaotic", the liquid container attains a rotation about the axis corresponding to the largest moment of inertia that eventually decreases to zero, taking the whole system to the rest state. We notice that these experiments were not performed in the vacuum in contrast with the general settings for the numerical and analytical investigations. Nevertheless, the behavior of the liquid-filled gyroscope is in agreement with the numerical tests presented in this chapter and the analytical findings reported in Section 5.2, regarding the stabilizing effect that a liquid has on the motion of a rigid body: due to viscous effects, the velocity of the liquid relative to solid eventually vanishes, so the pressure gradient in liquid balances the centrifugal forces, and the system reaches a steady state which is a permanent rotation around the axis where the spin is a (non-zero) minimum (at least for suitable initial conditions). When initial experiments were made with cavity filled with air only, the system rotated quickly about all axes for an extended period of time - spin times in excess of 90 seconds were attained. The system had a long-lasting "chaotic" motion, and it did not attain any rotation about one axis before going to rest. Moreover, given the choice of the physical and geometric properties of the liquid-filled gyroscope, and the initial motion imparted on the system, these experiments are in agreement with the numerical results reported in Figure 3 for the dynamics of $\mathcal{S}$ when $A < B = C$, and for which an analytical proof is still missing (see Theorem 5.2.4 and Remark 5.2.5).

5.0 A DYNAMICAL SYSTEMS APPROACH

In this chapter, we will present a comprehensive and rigorous mathematical analysis of the inertial motions and the motions under the action of gravity of a rigid body with a liquid-filled cavity. We are mainly interested in the long-time behavior of the coupled system $\mathcal{S}$. In particular, we shall show that, provided $\mathcal{C}$ is sufficiently regular, all motions of $\mathcal{S}$ described within a very general class of solutions to the relevant equations (*weak solutions*), must tend to a steady state for large times, with the liquid at rest relatively to the solid, and $\mathcal{S}$ behaving as a whole rigid body, no matter the shape of $\mathcal{C}$, the physical characteristics of the body and the liquid, and the initial conditions imparted to $\mathcal{S}$. Moreover, we will answer some question regarding the attainability and the stability of some equilibrium configurations.

The method we use borrows tools from classical Dynamical System theory. The adaptation of these tools to our problem is not trivial in that we deal with *weak solutions* (*à la Leray-Hopf*) where the uniqueness property is not guaranteed. Some preliminary results which include the dynamical system approach for the problem at hand will be presented in the next section.

5.1 SOME PRELIMINARY RESULTS

Throughout this section, we will consider the motions of the coupled system $\mathcal{S}$ about a fixed point O, and satisfying the hypotheses H1. and H2. when an external force per unit volume $g_0\boldsymbol{h}$ is acting on the center of mass of both liquid and solid. Here, g_0 is a constant and $\boldsymbol{h}$ is a (given) time-independent unit vector in the inertial frame $\mathcal{I}$. From the physical point of view, in the case $G \equiv O$ and $g_0 \equiv 0$, the coupled system performs *inertial motions* about its center of mass. If $G \neq O$ and $g_0 \equiv g$, where g is the acceleration of gravity, then we are considering the motions of a heavy rigid body with a liquid-filled cavity about a fixed point, this includes the case of motions about a fixed axis, like in the case of a liquid-filled physical pendulum. These cases will be treated in details in the sections 5.2, and 5.3.

The external force per unit volume acting on the fluid is given by $\boldsymbol{\mathcal{F}} = g_0\boldsymbol{h}$. Whereas, the total external torque applied on $\mathcal{S}$, and calculated with respect to O in the inertial frame $\mathcal{I}$, is given by

$$\boldsymbol{m}_O = (G - O) \times (Mg_0\boldsymbol{h}),$$

where M is the total mass of $\mathcal{S}$. Following Section 2.2, we introduce

$$\boldsymbol{\gamma}(t) := \boldsymbol{Q}^T(t) \cdot \boldsymbol{h}.$$

$\boldsymbol{\gamma}$ is a unit vector denoting the direction of the external force applied on $\mathcal{S}$; it is an unknown function of time since the equations of motion are written with respect to the non-inertial frame of reference $\mathsf{F} := \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, with origin at the fixed point O, $\boldsymbol{e}_1 \equiv \overrightarrow{OG}/|\overrightarrow{OG}|$ (from hypothesis H2.), and $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ directed along the remaining principal axes of $\mathcal{S}$ with respect to O. In F, the total external torque $\boldsymbol{m}_O$ becomes

$$\boldsymbol{Q}^T \cdot \boldsymbol{m}_O = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma},$$

where $\beta^2 := Mg_0\ell$ and $\ell := |\overrightarrow{OG}|$. Moreover,

$$0 = \frac{d\boldsymbol{h}}{dt} = \frac{d(\boldsymbol{Q} \cdot \boldsymbol{\gamma})}{dt} = \dot{\boldsymbol{Q}} \cdot \boldsymbol{\gamma} + \boldsymbol{Q} \cdot \dot{\boldsymbol{\gamma}}.$$

Thus, $(2.10)_4$ can be replaced by

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \boldsymbol{0}.$$

We introduce the following vector fields

$$\boldsymbol{a} := -\rho \boldsymbol{I}^{-1} \cdot \int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}, \quad \boldsymbol{\omega}_\infty := \boldsymbol{\omega} - \boldsymbol{a}. \tag{5.1}$$

So, $\boldsymbol{A}_0 = \boldsymbol{I}\cdot\boldsymbol{\omega}_\infty$, and (2.10) can be equivalently rewritten as follows

$$
\left.\begin{aligned}
&\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{y} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{v}\right)\\
&\qquad\qquad\qquad\qquad\qquad = \mu\Delta\boldsymbol{v} - \nabla p + \rho g_0\boldsymbol{\gamma}\\
&\operatorname{div}\boldsymbol{v} = 0
\end{aligned}\right\} \text{ on } \mathcal{C}\times(0,\infty),
$$

$$
\begin{aligned}
&\boldsymbol{I}\cdot\dot{\boldsymbol{\omega}}_\infty + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty = \beta^2\boldsymbol{e}_1\times\boldsymbol{\gamma} \qquad \text{in } (0,\infty),\\
&\dot{\boldsymbol{\gamma}} + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{\gamma} = \boldsymbol{0},\\
&\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{C}.
\end{aligned} \tag{5.2}
$$

Our investigation on the motions of a liquid-filled rigid body constrained to move around a fixed point is carried out in a considerably large class of solutions to (5.2) having finite total energy. Before introducing this class of solutions, let us formally derive the balance of the total energy.

The *energy balance* is given by

$$
\frac{d}{dt}(\mathcal{E}+\mathcal{U}) + 2\mu\|\nabla\boldsymbol{v}\|_2^2 = 0 \tag{5.3}
$$

where we have denoted by

$$
\mathcal{E}(t) := \mathcal{E}_F(t) + \boldsymbol{\omega}_\infty\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty \quad \text{and} \quad \mathcal{U}(t) := -2\beta^2\boldsymbol{\gamma}\cdot\boldsymbol{e}_3,
$$

the *kinetic* and *potential energy* of $\mathcal{S}$, respectively. Specifically, we have defined

$$
\mathcal{E}_F(t) := \rho\|\boldsymbol{v}\|_2^2 - \boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}, \tag{5.4}
$$

and by (2.29), it satisfies

$$
c_1\|\boldsymbol{v}\|_2^2 \le \mathcal{E}_F(t) \le c_2\|\boldsymbol{v}\|_2^2, \tag{5.5}
$$

for some positive constants c_1 and c_2. In order to formally obtain (5.3), let us first take the inner product in $L^2(\mathcal{C})$ of $(5.2)_1$ by $\boldsymbol{v}$, we get

$$
\frac{1}{2}\rho\frac{d}{dt}\|\boldsymbol{v}\|_2^2 + \rho\int_{\mathcal{L}}[(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{y}]\cdot\boldsymbol{v} + \mu\|\nabla\boldsymbol{v}\|_2^2 = 0. \tag{5.6}
$$

Next, we notice that

$$\rho \int_{\mathcal{L}} [(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}}) \times \boldsymbol{y}] \cdot \boldsymbol{v} = \rho(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}}) \cdot \int_{\mathcal{L}} \boldsymbol{y} \times \boldsymbol{v}$$
$$= -\dot{\boldsymbol{\omega}}_\infty \cdot \boldsymbol{I} \cdot \boldsymbol{a} - \frac{1}{2}\frac{d}{dt}(\boldsymbol{a} \cdot \boldsymbol{I} \cdot \boldsymbol{a}) = (\boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) \cdot \boldsymbol{a} - \beta^2(\boldsymbol{e}_1 \times \boldsymbol{\gamma}) \cdot \boldsymbol{a} - \frac{1}{2}\frac{d}{dt}(\boldsymbol{a} \cdot \boldsymbol{I} \cdot \boldsymbol{a}),$$

where, in the last equality, we have used $(5.2)_3$ dot multiplied by $\boldsymbol{a}$. Moreover, let us take the dot product of $(5.2)_3$ by $\boldsymbol{\omega}_\infty$, we have

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}_\infty \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) = -\boldsymbol{\omega}_\infty \cdot (\boldsymbol{a} \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) + \beta^2(\boldsymbol{e}_1 \times \boldsymbol{\gamma}) \cdot \boldsymbol{\omega}_\infty.$$

Therefore,

$$\rho \int_{\mathcal{L}} [(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}}) \times \boldsymbol{y}] \cdot \boldsymbol{v} = -\boldsymbol{\omega}_\infty \cdot (\boldsymbol{a} \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) - \beta^2(\boldsymbol{e}_1 \times \boldsymbol{\gamma}) \cdot \boldsymbol{a} - \frac{1}{2}\frac{d}{dt}(\boldsymbol{a} \cdot \boldsymbol{I} \cdot \boldsymbol{a})$$
$$= \frac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}_\infty \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) - \beta^2(\boldsymbol{e}_1 \times \boldsymbol{\gamma}) \cdot (\boldsymbol{\omega}_\infty + \boldsymbol{a}) - \frac{1}{2}\frac{d}{dt}(\boldsymbol{a} \cdot \boldsymbol{I} \cdot \boldsymbol{a})$$

Finally, by taking the scalar product of $(5.2)_4$ by $\boldsymbol{e}_3$, we get

$$\frac{d}{dt}(\boldsymbol{\gamma} \cdot \boldsymbol{e}_3) - (\boldsymbol{\omega}_\infty + \boldsymbol{a}) \cdot (\boldsymbol{e}_1 \times \boldsymbol{\gamma}) = 0,$$

and then

$$\rho \int_{\mathcal{L}} [(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}}) \times \boldsymbol{y}] \cdot \boldsymbol{v} = \frac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}_\infty \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) - \beta^2 \frac{d}{dt}(\boldsymbol{\gamma} \cdot \boldsymbol{e}_3) - \frac{1}{2}\frac{d}{dt}(\boldsymbol{a} \cdot \boldsymbol{I} \cdot \boldsymbol{a}). \tag{5.7}$$

Taking into account (5.6) and (5.7), we finally obtain (5.3).

In order to give a *weak formulation* to the problem (5.2), let us dot-multiply both sides of $(5.2)_1$ by $\boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\mathcal{C})$, and integrate by parts over $\mathcal{C} \times (0,t)$, we deduce

$$(\rho\, \boldsymbol{v}(t), \boldsymbol{\psi}) + \rho(\boldsymbol{\omega}_\infty(t) + \boldsymbol{a}(t)) \cdot \int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi} + \int_0^t \{\rho(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) + 2\rho((\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v}, \boldsymbol{\psi})\}$$
$$+ \int_0^t \mu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = (\rho\, \boldsymbol{v}(0), \boldsymbol{\psi}) + \rho(\boldsymbol{\omega}_\infty(0) + \boldsymbol{a}(0)) \cdot \int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{\psi}\,, \tag{5.8}$$

for all $\boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\mathcal{C})$ and all $t \in (0,\infty)$. Moreover, integrating $(5.2)_3$ and $(5.2)_4$ over $(0,t)$ we get

$$\boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(t) = \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(0) - \int_0^t \left[(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times (\boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) - \beta^2(\boldsymbol{e}_1 \times \boldsymbol{\gamma})\right] \tag{5.9}$$

and

$$\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(0) - \int_0^t (\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{\gamma}\,, \qquad \text{for all } t \in (0,\infty). \tag{5.10}$$

Definition 5.1.1. *We will say that the triple* $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ *is a* weak solution *to* (5.2) *if it satisfies the following conditions:*

(a) $\boldsymbol{v} \in C_w([0,\infty); H(\mathcal{C})) \cap L^\infty(0,\infty; H(\mathcal{C})) \cap L^2(0,\infty; W_0^{1,2}(\mathcal{C}))$*;*

(b) $\boldsymbol{\omega}_\infty \in C([0,\infty)) \cap C^1(0,\infty)$, $\boldsymbol{\gamma} \in C^1([0,\infty); \mathsf{S}^2)$*;*

(c) $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ *satisfies* (5.8), (5.9) *and* (5.10)*;*

(d) the Strong Energy Inequality:

$$\mathcal{E}(t) + \mathcal{U}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \, d\tau \leq \mathcal{E}(s) + \mathcal{U}(s) \tag{5.11}$$

holds for all $t \geq s$ *and a.a.* $s \geq 0$ *including* $s = 0$.

In the next proposition, we show that this class is, in fact, not empty, provided that the initial motion imparted to the system has finite total energy.

Proposition 5.1.2. *Let* $\mathcal{C}$ *be a bounded domain in* $\mathbb{R}^3$*. Then, for every* $\boldsymbol{v}_0 \in H(\mathcal{C}), \boldsymbol{\omega}_{\infty 0} \in \mathbb{R}^3$ *and* $\boldsymbol{\gamma}_0 \in \mathsf{S}^2$*, there exists at least one weak solution,* $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$*, to* (5.2) *such that*

$$\lim_{t\to 0^+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 = \lim_{t\to 0^+} |\boldsymbol{\omega}_\infty(t) - \boldsymbol{\omega}_{\infty 0}| = \lim_{t\to 0^+} |\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}_0| = 0.$$

Moreover,

$$\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_2 = 0. \tag{5.12}$$

Proof. The proof of existence of at least one weak solution can be accomplished with a combination of the classical Galerkin method with a priori estimates of the energy. This proof is analogous (up to some minor changes and adaptations) to the one given in [33], Chapter 3. So, we will omit its proof.

To show (5.12), let us notice that, from the strong energy inequality (5.11), for all $t \geq s$ and a.e. $s \geq 0$, including $s = 0$ one has

$$\mathcal{E}_F(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}(\tau)\|_2 \, d\tau \leq \mathcal{E}_F(s) + F(t,s),$$

where

$$F(t,s) = \boldsymbol{\omega}_\infty(s) \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(s) - \boldsymbol{\omega}_\infty(t) \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(t) + 2\beta^2(\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(s)) \cdot \boldsymbol{e}_1.$$

By Poincaré inequality and (5.5), we find that

$$\mathcal{E}_F(t) + \mu C_0 \int_s^t \mathcal{E}_F(\tau)\, d\tau \le \mathcal{E}_F(s) + F(t,s),$$

for all $t \ge s$ and a.e. $s \ge 0$, including $s = 0$. Let us estimate F. By (5.9), we find that

$$\boldsymbol{\omega}_\infty(s) \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(s) - \boldsymbol{\omega}_\infty(t) \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(t) = -2 \int_t^s \left[\boldsymbol{\omega}_\infty \cdot (\boldsymbol{a} \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) - \beta^2 \boldsymbol{\omega}_\infty \cdot (\boldsymbol{e}_1 \times \boldsymbol{\gamma}) \right].$$

Moreover, from (5.10),

$$2\beta^2 \int_t^s \boldsymbol{\omega}_\infty \cdot (\boldsymbol{e}_1 \times \boldsymbol{\gamma}) = 2\beta^2 (\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(t)) \cdot \boldsymbol{e}_1 - 2\beta^2 \int_t^s \boldsymbol{a} \cdot (\boldsymbol{e}_1 \times \boldsymbol{\gamma}).$$

Taking into account the last two displayed equations, we can estimate F as follows

$$|F(t,s)| = \left| \int_s^t \boldsymbol{\omega}_\infty \cdot (\boldsymbol{a} \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) + 2\beta^2 \int_s^t \boldsymbol{a} \cdot (\boldsymbol{e}_1 \times \boldsymbol{\gamma}) \right| \le C_1 \int_s^t \|\nabla \boldsymbol{v}\|_2,$$

where, in the last inequality, we have used $(5.1)_1$ and Poincaré inequality. Moreover, C_1 is a positive constant depending on the initial conditions. In fact, by the strong energy inequality (5.11) with $s = 0$, by (5.5), and since $|\boldsymbol{\gamma}| = 1$ at all times, it follows that

$$|\boldsymbol{a}(t)|, \quad |\boldsymbol{\omega}_\infty(t)|, \quad \|\boldsymbol{v}(t)\|_2 \le k(|\boldsymbol{\omega}_{\infty 0}| + \|\boldsymbol{v}_0\|_2), \tag{5.13}$$

where k is positive constant depending only on the physical and geometric properties characterizing the body and the liquid, and not on the initial motion imparted on the system. Therefore,

$$\mathcal{E}_F(t) + \mu C_0 \int_s^t \mathcal{E}_F(\tau)\, d\tau \le \mathcal{E}_F(s) + C_1 \int_s^t \|\nabla \boldsymbol{v}\|_2, \quad \text{all } t > 0.$$

Since, by the strong energy inequality,

$$\int_0^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \, d\tau < \infty, \qquad \text{all } t > 0,$$

we can then apply Lemma 2.3.4 and find that $\mathcal{E}_F(t) \to 0$ as $t \to \infty$. Thus, (5.12) follows from (5.5). ■

Remark 5.1.3. With standard arguments, one can show that if $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ is sufficiently smooth to allow for integration by parts on $\mathcal{C} \times (0, \infty)$, then there exists a *pressure field* $p = p(\boldsymbol{x}, t)$ such that $(5.2)_{1,2,3,4}$ are satisfied a.e. in space and time.

Due to the coupling with the Navier-Stokes equations, also for this problem, it is an open problem whether weak solutions constructed in Proposition 5.1.2 are unique, or more if they continuously depend upon the initial data. Nevertheless, as in the classical Navier-Stokes case, one can prove that the above property holds for any weak solution possessing a further regularity.

Proposition 5.1.4. *Let $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$, $(\boldsymbol{v}^*, \boldsymbol{\omega}_\infty^*, \boldsymbol{\gamma}^*)$ be two weak solutions corresponding to initial data $(\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0)$ and $(\boldsymbol{v}_0^*, \boldsymbol{\omega}_0^*, \boldsymbol{\gamma}_0^*)$, respectively. Suppose that there exists a time $T > 0$ such that*

$$\boldsymbol{v}^* \in L^p(0, T; L^q(\mathcal{C})), \quad \frac{2}{p} + \frac{3}{q} = 1, \text{ for some } q > 3.$$

Then, the following two statements hold:

1. *There exists a constant $c > 0$, depending only on $\max_{t \in [0,T]} |\boldsymbol{\omega}_\infty^*(t)|$, $\|\boldsymbol{v}^*(t)\|_{L^\infty(0,T;L^2(\mathcal{C}))}$, and $\|\boldsymbol{v}^*\|_{L^p(0,T;L^q(\mathcal{C}))}$, such that for all $t \in [0, T]$*

$$\begin{aligned} &\|\boldsymbol{v}(t) - \boldsymbol{v}^*(t)\|_2 + |\boldsymbol{\omega}_\infty(t) - \boldsymbol{\omega}_\infty^*(t)| + |\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}^*(t)| \\ &\qquad \le c\left(\|\boldsymbol{v}_0 - \boldsymbol{v}_0^*\|_2 + |\boldsymbol{\omega}_0 - \boldsymbol{\omega}_0^*| + |\boldsymbol{\gamma}_0 - \boldsymbol{\gamma}_0^*|\right). \qquad (5.14) \end{aligned}$$

2. *If $(\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0) \equiv (\boldsymbol{v}_0^*, \boldsymbol{\omega}_0^*, \boldsymbol{\gamma}_0^*)$, then $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma}) \equiv (\boldsymbol{v}^*, \boldsymbol{\omega}_\infty^*, \boldsymbol{\gamma}^*)$ a.e. in $[0, T] \times \mathcal{C}$. Moreover, the* energy equality *holds:*

$$\mathcal{E}(t) + \mathcal{U}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}\|_2^2 = \mathcal{E}(s) + \mathcal{U}(s) \quad \text{for all } 0 \le s \le t \le T.$$

Proof. We show here the formal estimates that lead to (5.14), a rigorous proof of the above statements can be obtained using similar techniques as in the proof of Theorem 3.4.2 in [33].

Let $\tilde{\boldsymbol{v}} := \boldsymbol{v} - \boldsymbol{v}^*$, $\tilde{\boldsymbol{\omega}}_\infty := \boldsymbol{\omega}_\infty - \boldsymbol{\omega}_\infty^*$, $\tilde{\boldsymbol{\gamma}} := \boldsymbol{\gamma} - \boldsymbol{\gamma}^*$, and correspondingly $\tilde{\boldsymbol{a}} := \boldsymbol{a} - \boldsymbol{a}^*$. Let

$$\tilde{\mathcal{E}} = \rho\,\|\tilde{\boldsymbol{v}}\|_2^2 - \tilde{\boldsymbol{a}} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{a}} + \tilde{\boldsymbol{\omega}}_\infty \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty,$$

which is positive definite by (5.5). Then, the triple $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}_\infty, \tilde{\boldsymbol{\gamma}})$ has to satisfy (formally)

$$
\begin{aligned}
\tilde{\mathcal{E}}(t) + |\tilde{\boldsymbol{\gamma}}(t)|^2 + 2\mu \int_0^t \|\nabla \tilde{\boldsymbol{v}}(\tau)\|_2^2 \leq \tilde{\mathcal{E}}(0) + |\tilde{\boldsymbol{\gamma}}(0)|^2 \\
+ 2\rho \int_0^t \left\{ \int_{\mathcal{C}} [(\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\boldsymbol{v}}) \cdot \boldsymbol{v}^* + 2[(\tilde{\boldsymbol{\omega}}_\infty + \tilde{\boldsymbol{a}}) \times \tilde{\boldsymbol{v}}] \cdot \boldsymbol{v}^*] \right\} \\
+ 2 \int_0^t \{(\boldsymbol{\omega}^*_\infty + \boldsymbol{a}^*) \cdot [(\tilde{\boldsymbol{\omega}}_\infty + \tilde{\boldsymbol{a}}) \times \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty] \\
+\beta^2 (\boldsymbol{e}_3 \times \tilde{\boldsymbol{\gamma}}) \cdot (\tilde{\boldsymbol{\omega}}_\infty + \tilde{\boldsymbol{a}}) + \beta^2 [(\tilde{\boldsymbol{\omega}}_\infty + \tilde{\boldsymbol{a}}) \times \tilde{\boldsymbol{\gamma}}] \cdot \boldsymbol{\gamma}^* \} .
\end{aligned}
\tag{5.15}
$$

Let us estimate the nonlinear term. By the generalized Hölder inequality (2.19), the interpolation inequality (2.20) with $\theta = 3/q$, the Sobolev inequality (2.22) and Young inequality (2.16), we find that

$$
\begin{aligned}
\int_{\mathcal{C}} (\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\boldsymbol{v}}) \cdot \boldsymbol{v}^* &\leq \|\tilde{\boldsymbol{v}}\|_{2q/(q-2)} \|\nabla \tilde{\boldsymbol{v}}\|_2 \|\boldsymbol{v}^*\|_q \leq \|\tilde{\boldsymbol{v}}\|_2^{1-\theta} \|\tilde{\boldsymbol{v}}\|_6^{\theta} \|\nabla \tilde{\boldsymbol{v}}\|_2 \|\boldsymbol{v}^*\|_q \\
&\leq c_1 \|\tilde{\boldsymbol{v}}\|_2^{1-\theta} \|\nabla \tilde{\boldsymbol{v}}\|_2^{1+\theta} \|\boldsymbol{v}^*\|_q \leq \frac{\mu}{2\rho} \|\nabla \tilde{\boldsymbol{v}}\|_2^2 + c_2 \|\tilde{\boldsymbol{v}}\|_2^2 \|\boldsymbol{v}^*\|_q^p ,
\end{aligned}
$$

where c_1 and c_2 are positive constants. Using these estimates and (5.5) in (5.15), we get

$$
\begin{aligned}
\tilde{\mathcal{E}}(t) + |\tilde{\boldsymbol{\gamma}}(t)|^2 + \mu \int_0^t \|\nabla \tilde{\boldsymbol{v}}(\tau)\|_2^2 \leq \tilde{\mathcal{E}}(0) + |\tilde{\boldsymbol{\gamma}}(0)|^2 \\
+c_3 \int_0^t \left[\|\boldsymbol{v}^*(\tau)\|_q^p + \|\boldsymbol{v}^*(\tau)\|_2 + |\boldsymbol{\omega}^*_\infty(\tau)| + 1 \right] (\tilde{\mathcal{E}}(\tau) + |\tilde{\boldsymbol{\gamma}}(\tau)|^2).
\end{aligned}
$$

(5.14) then follows from the latter displayed inequality, Gronwall Lemma, (5.5) and the fact that $\boldsymbol{I}$ is a positive definite, symmetric tensor. ■

5.1.1 Large-time properties of weak solutions

Weak solutions constructed in Proposition 5.1.2 satisfy the additional property of becoming *strong* after a *sufficiently large time*, in the sense of next proposition.

Proposition 5.1.5. *Let $\mathcal{C} \subset \mathbb{R}^3$ be a bounded domain of class C^2, and $s := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution corresponding to some initial data of finite energy, in the sense of Proposition 5.1.2. Then, there exists $t_0 = t_0(s) > 0$ such that for all $T > 0$*

$$\begin{aligned}&\boldsymbol{v} \in C^0([t_0, t_0+T]; W_0^{1,2}(\mathcal{C})) \cap L^2(t_0, t_0+T; W^{2,2}(\mathcal{C})),\\ &\qquad\boldsymbol{v}_t \in L^2(t_0, t_0+T; H(\mathcal{C})),\ \boldsymbol{\omega}_\infty \in W^{1,\infty}(t_0, t_0+T),\ \boldsymbol{\gamma} \in W^{2,\infty}(t_0, t_0+T; \mathsf{S}^2). \quad (5.16)\end{aligned}$$

Moreover, there exists $p \in L^2(t_0, t_0+T; W^{1,2}(\mathcal{C}))$, all $T > 0$; such that $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma}, p)$ satisfies (5.2) a.e. in (t_0, ∞). Finally,

$$\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{1,2} = 0. \qquad (5.17)$$

Proof. Let S be the set of all times $\tau \in [0, \infty)$ at which the strong energy inequality (5.11) holds. Since $\boldsymbol{v} \in L^2(0, \infty; W_0^{1,2}(\mathcal{C}))$, we can find an increasing, unbounded sequence $\{t_m\}_{m\in\mathbb{N}} \subset S$ such that for any $\varepsilon > 0$, there exists $\bar{k} \in \mathbb{N}$ such that

$$\|\nabla \boldsymbol{v}(t_k)\|_2 < \varepsilon, \qquad \text{for all } k \geq \bar{k}.$$

Moreover, for the same reason as above, for any $\eta > 0$ there exists $\bar{t} > 0$ such that

$$\int_{\bar{t}}^{\infty} \|\nabla \boldsymbol{v}\|_2^2 < \eta.$$

Thus, for any $\varepsilon, \eta > 0$ there exists $t_0 = t_0(\varepsilon, \eta, s) > 0$ (by considering $t_0 \equiv t_{k^*}$ where $k^* \geq \bar{k}$ and $t_{k^*} \geq \bar{t}$) such that

$$\|\nabla \boldsymbol{v}(t_0)\|_2 < \varepsilon, \qquad \int_{t_0}^{\infty} \|\nabla \boldsymbol{v}\|_2^2 < \eta. \qquad (5.18)$$

Next, we consider $(\boldsymbol{v}(t_0), \boldsymbol{\omega}_\infty(t_0), \boldsymbol{\gamma}(t_0))$ as initial condition for a local strong solution $\tilde{s} \equiv (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}_\infty, \tilde{\boldsymbol{\gamma}})$ in the interval $[t_0, t_0 + T^*)$, for some $T^* > 0$. The existence of such local strong solution can be accomplished again by a combination of the Galerkin method with suitable energy estimates. We refer to [33] for a rigorous proof and all technical details for similar results; here, we will only formally derive the main estimates.

Let us dot-multiply $(5.2)_1$ by $\partial \boldsymbol{v}/\partial t$ and integrate by parts over $\mathcal{C}$. Using $(5.2)_3$, we find

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \rho\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 - \dot{\boldsymbol{a}}\cdot \boldsymbol{I}\cdot\dot{\boldsymbol{a}} = -\rho\int_{\mathcal{C}}(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\frac{\partial \boldsymbol{v}}{\partial t}$$
$$-\dot{\boldsymbol{a}}\cdot\left[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times \boldsymbol{I}\cdot\boldsymbol{\omega}_\infty - \beta^2\boldsymbol{e}_1\times\boldsymbol{\gamma}\right] - 2\rho\int_{\mathcal{C}}[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{v}]\cdot\frac{\partial \boldsymbol{v}}{\partial t}.$$

Using (5.5), Cauchy-Schwarz and Young inequalities, from the previous equality, we get

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + C_1\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 \leq C_2(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2+1), \tag{5.19}$$

where C_2 is a positive constant depending on the physical properties of $\mathcal{S}$ and on the initial data of the weak solution s by (5.13).

Moreover, let us take the L^2-inner product of $(5.2)_1$ with $\mathcal{P}\Delta\boldsymbol{v}$, where the projection operator $\mathcal{P}$ has been defined in (2.15), and use $(5.2)_3$,

$$\mu\|\mathcal{P}\Delta\boldsymbol{v}\|_2^2 = \Bigg(\rho\Bigg(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \dot{\boldsymbol{a}}\times\boldsymbol{y} - \boldsymbol{I}^{-1}\cdot[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty]\times\boldsymbol{y}$$
$$+\beta^2\boldsymbol{I}^{-1}\cdot(\boldsymbol{e}_1\times\boldsymbol{\gamma})\times\boldsymbol{y} + 2(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{v}\Bigg), \mathcal{P}\Delta\boldsymbol{v}\Bigg).$$

By Cauchy-Schwarz and Young inequalities, we have the following estimate

$$\mu\|\mathcal{P}\Delta\boldsymbol{v}\|_2^2 \leq C_3\left(\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 + 1\right) \tag{5.20}$$

where also C_3 is a positive constant depending on the physical properties of $\mathcal{S}$ and on the initial data of the weak solution s by (5.13).

Since $\mathcal{C}$ is a bounded domain of class C^2, by (2.24),

$$C_4\|\boldsymbol{v}\|_{2,2}^2 \leq \|\mathcal{P}\Delta\boldsymbol{v}\|_2^2. \tag{5.21}$$

Multiplying both sides of (5.20) by $C_1/(2C_3)$ and adding the resulting equation to (5.19), then using (5.21), we find that $\|\nabla\boldsymbol{v}\|_2^2$ has to satisfy the following differential inequality

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + C_5\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 + C_6\|\boldsymbol{v}\|_{2,2}^2 \leq C_7(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2+1). \tag{5.22}$$

Thus, it remains to estimate the nonlinear term. By Hölder inequality (2.17), Sobolev embedding Theorem (2.3.2), interpolation and Young inequalities, we get

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 \le \|\boldsymbol{v}\|_6^2\,\|\nabla\boldsymbol{v}\|_3^2 \le C_8\,\|\nabla\boldsymbol{v}\|_2^3\,\|\boldsymbol{v}\|_{2,2} \le C_9\,\|\nabla\boldsymbol{v}\|_2^6 + \lambda\,\|\boldsymbol{v}\|_{2,2}^2 \tag{5.23}$$

for arbitrary $\lambda > 0$ and with $C_9 \to 0$ as $\lambda \to \infty$. Considering $\lambda = C_6/(2C_7)$, we conclude that

$$\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 + C_5\left\|\frac{\partial\boldsymbol{v}}{\partial t}\right\|_2^2 + C_{10}\,\|\boldsymbol{v}\|_{2,2}^2 \le C_{11}(\|\nabla\boldsymbol{v}\|_2^6 + 1). \tag{5.24}$$

The last displayed equation guarantees the existence of a time $T^* > 0$ and continuous functions G_1 and G_2 defined on $[t_0, t_0 + T^*)$, such that

$$\|\boldsymbol{v}(t)\|_{1,2} \le G_1(t), \qquad \int_{t_0}^{t}\left(\left\|\frac{\partial\boldsymbol{v}}{\partial\tau}\right\|_2^2 + \|\boldsymbol{v}\|_{2,2}^2\right) \le G_2(t). \tag{5.25}$$

These estimates, combined with the Galerkin method, ensure the existence of a strong solution $\tilde{s} \equiv (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{\omega}}_\infty, \tilde{\boldsymbol{\gamma}})$ corresponding to initial data $(\boldsymbol{v}(t_0), \boldsymbol{\omega}_\infty(t_0), \boldsymbol{\gamma}(t_0))$, such that

$$\begin{aligned}
&\tilde{\boldsymbol{v}} \in C^0([t_0, t_0+\tau]; W_0^{1,2}(\mathcal{C})) \cap L^\infty(t_0, t_0+\tau; W_0^{1,2}(\mathcal{C})) \cap L^2(t_0, t_0+\tau; W^{2,2}(\mathcal{C}))\\
&\frac{\partial\tilde{\boldsymbol{v}}}{\partial t} \in L^2(t_0, t_0+\tau; H(\mathcal{C})),\ \tilde{\boldsymbol{\omega}}_\infty \in W^{1,\infty}(t_0, t_0+\tau)\\
&\tilde{\boldsymbol{\gamma}} \in W^{2,\infty}(t_0, t_0+\tau), \qquad \text{for all } \tau \in (0, T^*).
\end{aligned} \tag{5.26}$$

Moreover, by Sobolev embeddings, $\tilde{\boldsymbol{v}} \in L^2(t_0, t_0+T^*; L^\infty(\mathcal{C}))$; thus, we can apply Proposition 5.1.4 to conclude that $\tilde{s} \equiv s$ on $[t_0, t_0 + T^*)$.

Let T^* be relabeled as the maximal time for $\tilde{s}$ to exist. We have the following blow-up criterion: either $T^* = +\infty$ or

$$\lim_{t\to T_1^+}\|\nabla\boldsymbol{v}(t)\|_2 = +\infty, \qquad T_1 := t_0 + T^*. \tag{5.27}$$

The latter is shown by a classical argument. In fact, suppose there is a sequence $\{t_m\} \subset [t_0, T_1)$ with $t_m \to T_1^+$ and such that

$$\|\nabla\tilde{\boldsymbol{v}}(t_m)\|_2 \le M\,, \tag{5.28}$$

where M independent of m. Setting $z := \|\nabla\tilde{\boldsymbol{v}}\|_2^2 + 1$, from (5.24) one shows

$$z'(t) \le C_{12}\,z^3(t)\,, \tag{5.29}$$

which in turn furnishes

$$z^2(t) \leq \frac{z^2(t_m)}{1 - C_{12} z^2(t_m)(t - t_m)}\,.$$

Using this inequality and (5.28), it immediately follows that $z^2(t) \leq C_{13}$, for $t \in [t_m, t_m + M_1]$, where M_1 independent of m, which, by taking m sufficiently large, proves $\|\nabla\tilde{\boldsymbol{v}}(t)\| \leq C_{14}$ for all $t \in [t_m, T_2]$ with $T_2 > T_1$. By employing the method previously described we can then extend the solution in the class (5.26) to a time interval $[0, T_2]$, with $T_2 > t_0 + T^*$, which contradicts the assumption that $[t_0, t_0 + T^*)$ is maximal with $T^* < \infty$.

We shall now show that, by choosing t_0 appropriately, (5.27) does not hold for s, thus implying $T^* = \infty$, which completes the proof of the proposition. In fact, by (5.24), $\|\nabla\boldsymbol{v}\|_2^2$ satisfies

$$\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 \leq -C_{15}\|\nabla\boldsymbol{v}\|_2^2 + C_{11}(\|\nabla\boldsymbol{v}\|_2^6 + 1). \tag{5.30}$$

Choosing ε and η in (5.18), in such a way that condition (ii) of Lemma 2.3.5 is satisfied, then (5.27) can not occur, and $T^* = +\infty$. Moreover, Lemma 2.3.5 and Poincaré inequality imply also (5.17). ∎

From the previous propositions, next corollary immediately follows.

Corollary 5.1.6. *Let $\mathcal{C}$ be a bounded domain of class C^2 in $\mathbb{R}^3$. Let $\mathsf{s} = (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (5.2). Then, there exists $t_0 > 0$ such that*

1. *s is unique in the class of weak solutions to (5.2) in $[t_0, \infty)$;*
2. *s depends continuously upon the data in $[t_0, \infty)$, in the class of weak solutions, in the sense of Proposition 5.1.4.*

5.1.2 Existence of the $\Omega-$limit set and its preliminary characterization

We are now in positions to introduce the main tools from Dynamical System theory.

Let $\mathsf{s} = (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (5.2), and set $\mathcal{H} := H(\mathcal{C}) \times \mathbb{R}^3 \times \mathsf{S}^2$, endowed with its natural topology. We define the Ω-*limit set* of s:

$$\begin{aligned}\Omega(\mathsf{s}) := \{(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{H} :\ & \text{there exists } t_k \geq 0,\ t_k \nearrow \infty \text{ s.t.} \\ & \lim_{k\to\infty}\|\boldsymbol{v}(t_k) - \boldsymbol{u}\|_2 = \lim_{k\to\infty}|\boldsymbol{\omega}_\infty(t_k) - \boldsymbol{\omega}| = \lim_{k\to\infty}|\boldsymbol{\gamma}(t_k) - \boldsymbol{q}| = 0\}.\end{aligned}$$

For all $t \geq 0$, we denote by $\mathsf{w}(t;\mathsf{z})$ a weak solution to (5.2) corresponding to the initial data $\mathsf{z} \in \mathcal{H}$, in the sense of Proposition 5.1.2.

Definition 5.1.7. $\Omega(\mathsf{s})$ *is positively invariant if the following implication holds:*

$$\mathsf{y} \in \Omega(\mathsf{s}) \quad \Rightarrow \quad \mathsf{w}(t;\mathsf{y}) \in \Omega(\mathsf{s}), \qquad \textit{all } t \geq 0,$$

and for all weak solutions $\mathsf{w}(t;\mathsf{y})$*.*

It is well known that invariance typically requires (at least) the uniqueness of the solution, a feature that, in the present case, is not available due to the coupling with the Navier-Stokes equations [1]. Nevertheless, using the fact that the velocity field of the liquid decays asymptotically to zero, to infer invariance we only need a sort of "asymptotic uniqueness" and "asymptotic continuous data dependence", properties that are ensured by Proposition 5.1.5. Intact, in [17] (Proposition 1.4.2), we have proved that if a weak solution $\mathsf{s}(t;\mathsf{s}_0)$ is *asymptotically regular*, then $\Omega(\mathsf{s})$ is positively invariant in the class of weak solutions.

Proposition 5.1.8. *Let* $\mathsf{s}(t;\mathsf{s}_0)$ *be a weak solution to* (5.2)*. Suppose that there exists* $t_0 > 0$ *such that the following properties hold.*

(i) Asymptotic Uniqueness*:*

$$\mathsf{s}(t+\tau;\mathsf{s}_0) = \mathsf{s}(t;\mathsf{s}(\tau;\mathsf{s}_0)), \qquad \textit{for all } \tau \geq t_0 \textit{ and } t \geq 0.$$

(ii) Asymptotic Continuous Data Dependence*:*

$$\{t_k\}_{k\in\mathbb{N}} \subset [t_0, +\infty) \quad \textit{with} \quad \mathsf{s}(t_k;\mathsf{s}_0) \to \mathsf{y} \quad \textit{in } \mathcal{H}$$
$$\Rightarrow \mathsf{s}(t;\mathsf{s}(t_k;\mathsf{s}_0)) \to \mathsf{w}(t;\mathsf{y}) \quad \textit{in } \mathcal{H}, \quad \textit{all } t \geq 0.$$

Then, $\Omega(\mathsf{s})$ *is positively invariant.*

[1] In the specific case of the Navier–Stokes equations, the uniqueness request can be relaxed to an a priori weaker condition like, for example, continuity in the "energy" norm [4], which, however, it is still an unproved property for weak solutions.

Proof. Let $\mathsf{y} \in \Omega(\mathsf{s})$ and let $\mathsf{w}(t;\mathsf{y})$ be a corresponding weak solution. We have to show that for each $t \geq 0$ there is $\{\tau_n\} \subset \mathbb{R}_+$ unbounded and such that

$$\mathsf{s}(\tau_n;\mathsf{s}_0) \to \mathsf{w}(t;\mathsf{y}) \quad \text{in } \mathcal{H}\,. \tag{5.31}$$

We observe that, by definition,

$$\mathsf{s}(t_n;\mathsf{s}_0) \to \mathsf{y} \quad \text{in } \mathcal{H}\,, \tag{5.32}$$

for some unbounded sequence $\{t_n\} \subset \mathbb{R}_+$. Now, let $\bar{n}$ be such that $t_n \geq t_0$, for all $n \geq \bar{n}$ and set $\tau_n := t + t_n$, for all $n \geq \bar{n}$, and $t \geq 0$. By (i) we thus have

$$\mathsf{s}(\tau_n;\mathsf{s}_0) = \mathsf{s}(t_n + t;\mathsf{s}_0) = \mathsf{s}(t;\mathsf{s}(t_n;\mathsf{s}_0))\,, \tag{5.33}$$

whereas, by (ii) and (5.32) we also have

$$\mathsf{s}(t;\mathsf{s}(t_n;\mathsf{s}_0)) \to \mathsf{w}(t;\mathsf{y}) \quad \text{in } \mathcal{H}\,.$$

Consequently, (5.31) follows from the latter and (5.33). ■

We are now in a position to give the following characterization of the Ω-limit set of any weak solution to (5.2)

Proposition 5.1.9. *Let $\mathsf{s} = \mathsf{s}(t;\mathsf{s}_0) := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (5.2), with $\mathcal{C}$ of class C^2, and initial data, s_0, of finite total energy in the sense of Proposition 5.1.2. Then, $\Omega(\mathsf{s})$ is non-empty, compact, connected, and it is positively invariant in the class of weak solutions to (5.2). Moreover,*

$$\Omega(\mathsf{s}) \subset \{(\bar{\boldsymbol{v}}, \bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\gamma}}) \in \mathcal{H} : \;\; \bar{\boldsymbol{v}} \equiv \boldsymbol{0}, \;\; \bar{\boldsymbol{\omega}} \times \boldsymbol{I} \cdot \bar{\boldsymbol{\omega}} = \beta^2 \boldsymbol{e}_1 \times \bar{\boldsymbol{\gamma}}, \;\; \bar{\boldsymbol{\omega}} \times \bar{\boldsymbol{\gamma}} = \boldsymbol{0}\}. \tag{5.34}$$

Proof. The strong energy inequality (5.11) and Proposition 5.1.5 ensure the uniform boundedness and continuity of the trajectory, implying that $\Omega(\mathsf{s})$ is non-empty, connected and compact.

By Corollary 5.1.6 and Proposition 5.1.8, it immediately follows that $\Omega(\mathsf{s})$ is left invariant in the class of weak solutions to (5.2).

Let us show (5.34). By Proposition 5.1.5, in particular by (5.17), and from (5.8)–(5.10), we can infer that the dynamics on $\Omega(\mathsf{s})$ is governed by the following set of equations

$$\begin{aligned}
&\boldsymbol{v} \equiv \mathbf{0}, \quad \rho \int_{\mathcal{C}} (\dot{\boldsymbol{\omega}}_\infty \times \boldsymbol{y}) \cdot \boldsymbol{\psi} = 0 \text{ for all } \boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\mathcal{C}), \\
&\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}}_\infty + \boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma}, \\
&\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \mathbf{0}.
\end{aligned} \tag{5.35}$$

From $(5.35)_2$, it easily follows that

$$\dot{\boldsymbol{\omega}}_\infty \times \boldsymbol{y} = \nabla \phi, \tag{5.36}$$

with ϕ a suitable smooth scalar field. Thus, operating the curl $\cdot$ on both sides of (5.36), we get that $\dot{\boldsymbol{\omega}}_\infty = \mathbf{0}$. Thus, $(5.35)_3$ becomes $\boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_1 \times \boldsymbol{\gamma}$, and it also implies that

$$\boldsymbol{e}_1 \times \dot{\boldsymbol{\gamma}} = \mathbf{0}, \quad (\boldsymbol{\omega}_\infty \times \boldsymbol{\gamma}) \cdot \boldsymbol{e}_1 = 0. \tag{5.37}$$

Dot-multiplying $(5.35)_4$ by $\boldsymbol{e}_1$, and using the latter displayed equation, we can conclude that $\boldsymbol{e}_1 \cdot \dot{\boldsymbol{\gamma}} = 0$. Therefore, $\dot{\boldsymbol{\gamma}} \equiv \mathbf{0}$, and the proof of the proposition is then complete. ■

With this result in hand, we are now able to provide a further refinement of the structure of the $\Omega-$limit set. This refinement depends on the physical problems at hand. In order to accomplish it, we will specialize our results to the problems of inertial motions of liquid-filled rigid body, and motions of a liquid-filled physical pendulum.

5.2 INERTIAL MOTIONS

In this section, we will focus on the inertial motions of a rigid body with a cavity entirely filled by a viscous fluid. Here, no external forces and torques are applied on the coupled system $\mathcal{S}$, which moves by inertia after an initial angular momentum is imparted on the whole system.

With the same notations as in the previous section, taking $g_0 = \beta^2 \equiv 0$, and noticing that, in this case, $(5.2)_4$ is redundant as no forces are applied on the coupled system, (5.2) now reads as follows

$$\left.\begin{aligned}&\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{y} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{v}\right) = \mu\Delta\boldsymbol{v} - \nabla p\\ &\operatorname{div}\boldsymbol{v} = 0\end{aligned}\right\}\text{ on } \mathcal{C}\times(0,\infty),$$
$$\boldsymbol{I}\cdot\dot{\boldsymbol{\omega}}_\infty + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty = \boldsymbol{0} \qquad \text{in } (0,\infty),$$
$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{C}. \tag{5.38}$$

The *energy balance* is given by

$$\frac{d}{dt}(\mathcal{E}_F + \boldsymbol{\omega}_\infty\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty) + 2\mu\,\|\nabla\boldsymbol{v}\|_2^2 = 0. \tag{5.39}$$

We recall here the definition of weak solutions.

Definition 5.2.1. *We will say that the couple* $(\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ *is a* weak solution *to* (5.38) *if it satisfies the following conditions:*

(a) $\boldsymbol{v} \in C_w([0,\infty); H(\mathcal{C})) \cap L^\infty(0,\infty; H(\mathcal{C})) \cap L^2(0,\infty; W_0^{1,2}(\mathcal{C}))$*;*

(b) $\boldsymbol{\omega}_\infty \in C([0,\infty)) \cap C^1(0,\infty)$*;*

(c) $(\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ *satisfies the following equations:*

$$\begin{aligned}(\rho\,\boldsymbol{v}(t),\boldsymbol{\psi}) + \rho(\boldsymbol{\omega}_\infty(t) + \boldsymbol{a}(t))\cdot\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{\psi} + \int_0^t\{\rho(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\psi}) + 2\rho((\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{v},\boldsymbol{\psi})\}\\ + \int_0^t\mu(\nabla\boldsymbol{v},\nabla\boldsymbol{\psi}) = (\rho\,\boldsymbol{v}(0),\boldsymbol{\psi}) + \rho(\boldsymbol{\omega}_\infty(0) + \boldsymbol{a}(0))\cdot\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{\psi}\,,\end{aligned} \tag{5.40}$$

for all $\boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\mathcal{C})$ *and all* $t \in (0,\infty)$*, and*

$$\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty(t) = \boldsymbol{I}\cdot\boldsymbol{\omega}_\infty(0) - \int_0^t[(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times(\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty)]\,. \qquad \textit{for all } t\in(0,\infty). \tag{5.41}$$

(d) the Strong Energy Inequality:

$$\mathcal{E}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \; d\tau \leq \mathcal{E}(s) \tag{5.42}$$

holds for all $t \geq s$ and a.a. $s \geq 0$ including $s = 0$.

All the results in Section 5.1 continue to hold for this problem. Moreover, $(\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ is a weak solution to (5.38), then the following invariant is satisfied at all times:

$$|\boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(t)| = |\boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(0)|, \qquad \text{all } t \in (0, \infty), \tag{5.43}$$

it represents the *conservation of the magnitude of the total angular momentum* of $\mathcal{S}$.

5.2.1 The Ω-limit set for inertial motions

We denote, as customary, by A, B, and C the eigenvalues of $\boldsymbol{I}$, and by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ the corresponding (orthonormalized) eigenvectors, and set

$$\boldsymbol{\omega}_\infty = p\,\boldsymbol{e}_1 + q\,\boldsymbol{e}_2 + r\,\boldsymbol{e}_3\,.$$

The following property holds.

Lemma 5.2.2. *Let $(\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ be a weak solution to (5.38). Suppose $A = B < C$ and $r(t) \to 0$ as $t \to \infty$. Then, there are constants $c_1, c_2 > 0$ such that*

$$\|\boldsymbol{v}(t)\|_2 \leq c_1\, \mathrm{e}^{-c_2\, t}\,,$$

for all sufficiently large $t > 0$.

Proof. Let us begin to show that, under the given assumptions,

$$\mathcal{E}_F(t) > \frac{C(C-A)}{A} r^2(t)\,, \text{ all } t \text{ large enough.} \tag{5.44}$$

In fact, assume by contradiction that there is a (sufficiently large) $\bar{t}$ such that

$$\mathcal{E}_F(\bar{t}) \leq \frac{C(C-A)}{A} r^2(\bar{t})\,. \tag{5.45}$$

By Proposition 5.1.4 (2.), we have that $(\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ obeys the energy equation

$$\begin{aligned} \mathcal{E}_F(t) + A(p^2(t) + q^2(t)) + Cr^2(t) + 2\mu \int_{\bar{t}}^{t} \|\nabla \boldsymbol{v}(s)\|_2^2\, ds \\ = \mathcal{E}_F(\bar{t}) + A(p^2(\bar{t}) + q^2(\bar{t})) + Cr^2(\bar{t})\,. \end{aligned} \tag{5.46}$$

Moreover, in view of (5.43), we deduce the following

$$A^2(p^2(t) + q^2(t)) + C^2 r^2(t) = A^2(p^2(\bar{t}) + q^2(\bar{t})) + C^2 r^2(\bar{t})\,. \tag{5.47}$$

So that from (5.46) and (5.47) we conclude

$$A\mathcal{E}_F(t) + C(A-C)\, r^2(t) + 2\mu\, A \int_{\bar{t}}^{t} \|\nabla \boldsymbol{v}(s)\|_2^2 ds = A\mathcal{E}_F(\bar{t}) + C(A-C)\, r^2(\bar{t}) \tag{5.48}$$

As a consequence, passing to the limit $t \to \infty$ in the latter relation and assuming (5.45) would lead to a contradiction. We next set

$$G(t) := A\mathcal{E}_F(t) + C(A-C) r^2(t)$$

and observe that, by what we have just proved, $G(t) > 0$ for all sufficiently large t. We also notice that, by Poincaré inequality and (5.5),

$$2A\|\nabla \boldsymbol{v}(s)\|_2^2 \geq 2\, C_p A\, \mathcal{E}_F(s) \geq 2C_p\, G(s)\,.$$

Employing this inequality back in (5.48), with the help of Gronwall Lemma we thus conclude, in particular,

$$G(t/2) \leq G(t/4)\, \mathrm{e}^{-\frac{C_p\, \mu}{2}\, t} \leq M\, \mathrm{e}^{-\frac{C_p\, \mu}{2}\, t}\,, \text{ all large } t\,, \tag{5.49}$$

where we have used the uniform boundedness of G in time. We now go back to (5.48) with $\bar{t} = t/2$ (keep in mind that (5.48) holds for all $t \geq \bar{t}$, for all large $\bar{t}$) and show the following inequality

$$\int_{t/2}^{t} \|\nabla \boldsymbol{v}(s)\|_2^2 ds \leq C\, \mathrm{e}^{-\frac{C_p\,\mu}{2}\,t}\,. \tag{5.50}$$

At this point, we recall (5.17) and choose t so large as

$$\|\nabla \boldsymbol{v}(t)\|_2^4 < \frac{C_{15}}{2C_{11}},$$

so that (5.30) furnishes

$$\frac{d}{dt}\|\nabla \boldsymbol{v}(t)\|_2^2 \leq -k_1\|\nabla \boldsymbol{v}(t)\|_2^2 + k_2\,. \tag{5.51}$$

Putting $y := \|\nabla \boldsymbol{v}\|_2^4$ from (5.51) it follows that

$$\frac{dy}{dt} \leq -2k_1\, y + 2k_2\|\nabla \boldsymbol{v}\|_2^2\,,$$

for some $k_1, k_2 > 0$. Multiplying both sides of the latter displayed equation by $\mathrm{e}^{2k_1 t}$ and integrating the resulting equation over $(t/2, t)$, we show

$$y(t) \leq y(t/2)\,\mathrm{e}^{-k_1 t} + 2k_2\int_{t/2}^{t} \mathrm{e}^{-2k_1(t-s)}\|\nabla \boldsymbol{v}(s)\|_2^2 ds \leq y(t/2)\,\mathrm{e}^{-k_1 t} + 2k_2\int_{t/2}^{t}\|\nabla \boldsymbol{v}(s)\|_2^2 ds\,.$$

Since, by Proposition 5.1.5, $\|\nabla \boldsymbol{v}(t)\|_2$ is uniformly bounded in t for all large t, by (5.50) and the latter displayed equation we infer

$$\|\nabla \boldsymbol{v}(t)\|_2 \leq k_3\,\mathrm{e}^{-k_4 t}\,,\ \text{all large } t\,,$$

and the lemma follows from this and the Poincaré inequality. ■

We are now in position to prove the following result.

Proposition 5.2.3. *Let* $\mathsf{s} = (\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ *be a weak solution corresponding to initial conditions* $\mathsf{s}_0 = (\boldsymbol{v}_0, \boldsymbol{\omega}_{\infty 0}) \in \mathcal{H}$*, and let* $\Omega(\mathsf{s})$ *be the corresponding* $\Omega-$*limit set. Moreover, let* A, B *and* C *be the eigenvalues of the inertia tensor* $\boldsymbol{I}$*, with corresponding (orthonormalized) eigenvectors* $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$*, respectively, and suppose, without loss of generality,* $A \leq B \leq C$*. Then*

$$\Omega(\mathsf{s}) = \{\mathbf{0}\} \times \mathcal{A}$$

where $\mathcal{A}$ *satisfies the following properties.*

(a) $\mathcal{A} = \{\mathbf{0}\}$ *if and only if* $\boldsymbol{\omega}_{\infty 0} = \mathbf{0}$*;*

(b) *If* $A \le B < C$*, or* $A = B = C$*, then* $\mathcal{A} = \{\bar{\boldsymbol{\omega}}\}$*, for some* $\bar{\boldsymbol{\omega}} \in \mathbb{R}^3$*;*

(c) *If* $A < B = C$*, then either*

$$\mathcal{A} = \{p_0 \boldsymbol{e}_1\}\,,$$

or

$$\mathcal{A} \subseteq \{q_0 \boldsymbol{e}_2 + r_0 \boldsymbol{e}_3\}$$

where

$$\begin{aligned} |p_0| &= \frac{|\boldsymbol{I} \cdot \boldsymbol{\omega}_{\infty 0}|}{A} \\ (q_0^2 + r_0^2)^{\frac{1}{2}} &= \frac{|\boldsymbol{I} \cdot \boldsymbol{\omega}_{\infty 0}|}{C}\,. \end{aligned} \tag{5.52}$$

Proof. We begin to observe that in view of Proposition 5.1.9, and by (5.43),

$$\Omega(\mathsf{s}) = \{\mathbf{0}\} \times \mathcal{A},$$

where $\mathcal{A}$ is a non-empty, compact, connected subset of $\mathbb{R}^3$ such that

$$\mathcal{A} \subseteq \{\bar{\boldsymbol{\omega}} \in \mathbb{R}^3 : \; |\boldsymbol{I} \cdot \bar{\boldsymbol{\omega}}| = |\boldsymbol{I} \cdot \boldsymbol{\omega}_{\infty 0}| =: M_0, \; \bar{\boldsymbol{\omega}} \times \boldsymbol{I} \cdot \bar{\boldsymbol{\omega}} = \mathbf{0}\}. \tag{5.53}$$

From (5.53) we at once deduce that $\bar{\boldsymbol{\omega}} = \mathbf{0}$ if and only if $\boldsymbol{\omega}_{\infty 0} = \mathbf{0}$, and property (a) is demonstrated. In the following, we can then assume $\bar{\boldsymbol{\omega}}$ is an eigenvector of $\boldsymbol{I}$, which is equivalent to $M_0 \neq 0$. To show (b), we suppose first $A < B < C$. The above then implies that

$$\mathcal{A} \subset \{\pm p_0\, \boldsymbol{e}_1\} \cup \{\pm q_0\, \boldsymbol{e}_2\} \cup \{\pm r_0\, \boldsymbol{e}_3\}$$

where $p_0 = M_0/A$, $q_0 = M_0/B$, and $r_0 = M_0/C$. However, $\mathcal{A}$ is connected, so that (b) follows when $A < B < C$. Next, suppose $A = B < C$. In that case, also using (5.53), we deduce

$$\mathcal{A} \subset \{p_0\, \boldsymbol{e}_1 + q_0\, \boldsymbol{e}_2\} \cup \{r_0\, \boldsymbol{e}_3\} \cup \{-r_0\, \boldsymbol{e}_3\} =: \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3\,,$$

where

$$r_0 = \frac{|\boldsymbol{I} \cdot \boldsymbol{\omega}_{\infty 0}|}{C}\,, \quad (p_0^2 + q_0^2)^{\frac{1}{2}} = \frac{M_0}{A}\,.$$

However, since $\mathcal{A}$ is connected, we must have

$$\mathcal{A} \subset \mathcal{A}_i \,, \text{ for some } i = 1, 2, 3.$$

If $\mathcal{A} \subset \mathcal{A}_i$, $i = 2, 3$, the proof is completed. So, assume

$$\mathcal{A} \subset \mathcal{A}_1 \,, \tag{5.54}$$

and let $\boldsymbol{\omega}_* \in \mathcal{A}$. This is equivalent to say that for *any* unbounded sequence $\{t_n\}$, we may select a subsequence (still denoted by $\{t_n\}$) and find $p_*, q_* \in \mathbb{R}$ (in principle, depending on the particular sequence) such that $\boldsymbol{\omega}_* = p_* \boldsymbol{e}_1 + q_* \boldsymbol{e}_2$, namely,

$$\begin{gathered} \lim_{n\to\infty} p(t_n) = p_* \,, \quad \lim_{n\to\infty} q(t_n) = q_*, \\ \lim_{n\to\infty} r(t_n) = 0 \,. \end{gathered} \tag{5.55}$$

Our objective is to show that, in fact, p_* and q_* are *independent* of the particular sequence. In the first place we notice that, by the arbitrariness of the sequence $\{t_n\}$, from (5.55) it follows that $r(t) \to 0$ as $t \to \infty$. Thus, Lemma 5.2.2 ensures that there are $c_1, c_2 > 0$ such that

$$\|\boldsymbol{v}(t)\|_2 \leq c_1 \mathrm{e}^{-c_2 t} \,, \quad \text{for all } t \text{ large enough.} \tag{5.56}$$

Pick $\varphi \in C_0^\infty(\mathcal{C})$ with $\int_{\mathcal{C}} \varphi = 1$, and set $\boldsymbol{\phi}_i = \varphi\, \boldsymbol{e}_i$, $i = 1, 2, 3$. Proceeding as in [33, p. 129], we dot multiply both sides of $(5.38)_1$ by $\operatorname{curl} \boldsymbol{\phi}_i$ and integrate by parts over $\mathcal{C}$ to obtain for t sufficiently large and $i = 1, 2, 3$, [2]

$$\begin{aligned} 2\dot{\omega}_{\infty\, i} = &- (\boldsymbol{v}_t, \operatorname{curl} \boldsymbol{\phi}_i) + (\boldsymbol{v} \otimes \boldsymbol{v}, \nabla(\operatorname{curl} \boldsymbol{\phi}_i)) - (\dot{\boldsymbol{a}} \times \boldsymbol{y}, \operatorname{curl} \boldsymbol{\phi}_i) \\ &- 2((\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v}, \operatorname{curl} \boldsymbol{\phi}_i) + \mu(\boldsymbol{v}, \Delta(\operatorname{curl} \boldsymbol{\phi}_i)) \,. \end{aligned} \tag{5.57}$$

Integrating both sides of (5.57) between two *arbitrary* instants of time, t_1 and t_2 sufficiently large and employing Cauchy-Schwartz inequality, we show with $\tau_1 = \min\{t_1, t_2\}$, $\tau_2 = \max\{t_1, t_2\}$,

$$|\omega_{\infty\, i}(t_2) - \omega_{\infty\, i}(t_1)| \leq C \left[\|\boldsymbol{v}(t_2)\|_2 + \|\boldsymbol{v}(t_1)\|_2 + \int_{\tau_1}^{\tau_2} (\|\boldsymbol{v}(s)\|_2^2 + \|\boldsymbol{v}(s)\|_2) \, ds \right],$$

[2] Recall that, by Proposition 5.1.5, the weak solution is regular for all large times.

where we also have used (5.13). In view of (5.56), the right-hand side of the latter equation can be made small for $i = 1, 2, 3$, by taking t_1, t_2 large enough, which proves that

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t)$$

exists and concludes the proof of (b) when $A = B < C$. To complete the proof of the statement in (b), it remains to show it in the case $A = B = C =: \lambda$. From (5.9) (recall that here $g_0 = \beta^2 \equiv 0$), and Definition 5.2.1 (b), we have that

$$\frac{d}{dt}(\boldsymbol{\omega}_\infty \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty) = 0.$$

Using the latter together with (5.5), the strong energy inequality (5.42) becomes

$$\mathcal{E}_F(t) + c_0\mu \int_s^t \mathcal{E}_F(\tau)\, d\tau \leq \mathcal{E}_F(s),$$

for all $t \geq s$ and a.a. $s \geq 0$ including $s = 0$. The Gronwall-type Lemma 2.3.4 and (5.5) then imply that there exist two positive constants, c_1 and c_2, such that

$$\|\boldsymbol{v}(t)\|_2 \leq c_1 \|\boldsymbol{v}(0)\|_2\, \mathrm{e}^{-c_2\mu t} \text{ for all } t > 0. \tag{5.58}$$

Furthermore, from the strong energy inequality (5.42) with $s = 0$, we derive

$$|\boldsymbol{\omega}_\infty(t)| \leq M, \tag{5.59}$$

with M depending only on the initial data and physical and geometric properties of $\mathcal{S}$. Therefore, under the above assumption on $\boldsymbol{I}$, by (5.59) and (5.41) we show that

$$|\boldsymbol{\omega}_\infty(t) - \boldsymbol{\omega}_\infty(s)| \leq M \int_s^t |\boldsymbol{a}(\tau)|\,, \quad \text{for all } t \geq s \geq 0\,.$$

From the latter inequality, (5.59) and (5.58), it follows that there exists $\bar{\boldsymbol{\omega}} \in \mathbb{R}^3$ such that

$$|\boldsymbol{\omega}_\infty(t) - \bar{\boldsymbol{\omega}}| \leq c_3\, \mathrm{e}^{-c_2\,\mu\, t}\,, \quad \text{all } t > 0\,. \tag{5.60}$$

Finally, to prove the property (c), it is enough to observe that, by (5.53), the eigenvector $\bar{\boldsymbol{\omega}}$ is either of the form $\pm p_*\, \boldsymbol{e}_1$, with $|p_*| = M_0/A$, or else $\bar{\boldsymbol{\omega}} = q_0\, \boldsymbol{e}_2 + r_0\, \boldsymbol{e}_3$, with q_0, r_0 satisfying (5.52). The statement in (c) then follows from this and from the fact that $\mathcal{A}$ is connected.
■

We are now in a position to provide a rather complete description of the asymptotic behavior in time of the coupled system solid-liquid.

Theorem 5.2.4. *Let $\mathcal{S}$ be the coupled system constituted by a rigid body with an interior cavity $\mathcal{C}$ of class C^2 completely filled with a Navier-Stokes liquid. Suppose that no external forces act on $\mathcal{S}$.*

Let $(\boldsymbol{v}, \boldsymbol{\omega}_\infty)$ be any weak solution, in the sense of Definition 5.2.1, to the initial-boundary value problem (5.38), governing the motion of $\mathcal{S}$, and corresponding to initial data $(\boldsymbol{v}_0, \boldsymbol{\omega}_{\infty 0})$. Also, let A, B, C, and $\{\boldsymbol{e}_i\}$ be as in Proposition 5.2.3.

Then,

$$\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{1,2} = 0\,, \tag{5.61}$$

whereas

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = 0\,, \tag{5.62}$$

if and only if $\boldsymbol{\omega}_{\infty 0} = \mathbf{0}$.

Moreover, if $\boldsymbol{\omega}_{\infty 0} \neq \mathbf{0}$, the following holds. If $A \le B < C$ or $A = B = C$, there exists $\bar{\boldsymbol{\omega}} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \bar{\boldsymbol{\omega}}\,. \tag{5.63}$$

In particular, when $A = B = C$,

$$\begin{aligned} \|\boldsymbol{v}(t)\|_2 + |\boldsymbol{\omega}_\infty(t) - \bar{\boldsymbol{\omega}}| &\le c_1\,\mathrm{e}^{-c_2 t}\,, \quad \text{all } t > 0\,, \\ \|\nabla \boldsymbol{v}(t)\|_2 &\le c_1\,\mathrm{e}^{-c_2 t}\,, \quad \text{all sufficiently large } t > 0\,, \end{aligned} \tag{5.64}$$

for some $c_1, c_2 > 0$.

In any case, the vector $\bar{\boldsymbol{\omega}}$, is parallel to $\boldsymbol{e} \in \{\boldsymbol{e}_i\}$. Also,

$$\bar{\boldsymbol{\omega}} = \frac{1}{\lambda}\,\boldsymbol{K}_G \tag{5.65}$$

where λ is the eigenvalue of $\boldsymbol{I}$ associated with $\boldsymbol{e}$, representing the moment of inertia of $\mathcal{S}$ with respect to $\boldsymbol{e}$, and $\boldsymbol{K}_G \equiv \boldsymbol{I} \cdot \boldsymbol{\omega}_{\infty 0}$ is the (constant) angular momentum of $\mathcal{S}$ with respect to G.

Therefore, under the stated assumptions on A, B, and C the asymptotic motion of $\mathcal{S}$ is a constant rigid rotation around a central axis of inertia of $\mathcal{S}$ that aligns with the direction of the constant total angular momentum.

Finally, if $A < B = C$, then either (5.63) holds with $\bar{\boldsymbol{\omega}} = p_0\, \boldsymbol{e}_1$, $|p_0| = |\boldsymbol{K}_G|/A$ or else

$$\lim_{t\to\infty} \operatorname{dist}(\boldsymbol{\omega}_\infty(t), \mathcal{R}) = 0$$

where

$$\mathcal{R} = \left\{ q_0\, \boldsymbol{e}_2 + r_0\, \boldsymbol{e}_3 : (q_0^2 + r_0^2)^{\frac{1}{2}} = |\boldsymbol{K}_G|/C \right\}.$$

Proof. In view of the results proved in Proposition 5.1.5 and Proposition 5.2.3 we only have to show the asymptotic property (5.64), and (5.65). As for the latter, we observe that, by the conservation of angular momentum,

$$\boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(t) = \sum_{i=1}^{3} \lambda_i \omega_{\infty i}(t) \boldsymbol{e}_i(t) = \boldsymbol{K}_G\,, \ \text{all } t \geq 0\,,$$

where, we recall, λ_i is an eigenvalue of $\boldsymbol{I}$ and $\boldsymbol{e}_i$ the corresponding eigenvector, $i = 1, 2, 3$. Consequently, by passing to the limit $t \to \infty$ in the latter relation and taking into account (5.63) and that $\bar{\boldsymbol{\omega}}$ is an eigenvector of $\boldsymbol{I}$, we show the validity of (5.65). It remains to prove the exponential decay, under the assumption that $\boldsymbol{I} = \lambda \mathbf{1}$, $\lambda > 0$. To this end, we notice that we have proved the validity of $(5.64)_1$ in the proof of Proposition 5.2.3 (see (5.58) and (5.60)). Moreover, by Cauchy-Schwartz inequality and (5.58),

$$|\boldsymbol{a}(t)| + \|\boldsymbol{v}(t)\|_2 \leq c_1\, \mathrm{e}^{-c_2 t}\,. \tag{5.66}$$

By (5.66) we also obtain

$$|\dot{\boldsymbol{\omega}}_\infty(t)| \leq c_3\, \mathrm{e}^{-c_2 t}\,, \quad \text{all } t > 0\,. \tag{5.67}$$

Let us dot-multiply $(5.38)_1$ by $\partial \boldsymbol{v}/\partial t$ and integrate by parts over $\mathcal{C}$. Using $(5.38)_3$, we find

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \rho \left\| \frac{\partial \boldsymbol{v}}{\partial t} \right\|_2^2 - \dot{\boldsymbol{a}} \cdot \boldsymbol{I} \cdot \dot{\boldsymbol{a}} = -\rho \int_{\mathcal{C}} (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) \cdot \frac{\partial \boldsymbol{v}}{\partial t} - \dot{\boldsymbol{a}} \cdot [\boldsymbol{a} \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty] - 2\rho \int_{\mathcal{C}} [(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v}] \cdot \frac{\partial \boldsymbol{v}}{\partial t}.$$

Using (5.5), Young inequality and (5.66) on the right-hand side of the previous equality, we get

$$\frac{\mu}{2}\frac{d}{dt}\,\|\nabla \boldsymbol{v}\|_2^2 + c_4 \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 \le c_5(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 + \mathrm{e}^{-2c_2 t}). \tag{5.68}$$

Moreover, let us take the L^2-inner product of $(5.38)_1$ with $\mathcal{P}\Delta\boldsymbol{v}$, and use $(5.38)_3$,

$$\mu\,\|\mathcal{P}\Delta\boldsymbol{v}\|_2^2 = \Big(\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \dot{\boldsymbol{a}}\times\boldsymbol{y} - \boldsymbol{I}^{-1}\cdot[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty]\times\boldsymbol{y}$$
$$+\beta^2\boldsymbol{I}^{-1}\cdot(\boldsymbol{e}_1\times\boldsymbol{\gamma})\times\boldsymbol{y} + 2(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{v}\Big), \mathcal{P}\Delta\boldsymbol{v}\Big).$$

By Young inequality and (5.66), we have the following estimate

$$\mu\,\|\mathcal{P}\Delta\boldsymbol{v}\|_2^2 \le c_6\left(\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|_2^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 + \mathrm{e}^{-2c_2 t}\right). \tag{5.69}$$

Multiplying both sides of (5.69) by $c_4/(2c_6)$ and adding the resulting equation to (5.68), then using (2.24) we deduce, for all sufficiently large t,

$$\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 + c_7\|\boldsymbol{v}\|_{2,2}^2 \le c_8\left(\mathrm{e}^{-2c_2 t} + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2\right). \tag{5.70}$$

Next, by Hölder and Sobolev inequalities,

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2 \le c_9\,\|\boldsymbol{v}\|_2^{\frac14}\,\|\nabla\boldsymbol{v}\|_2^{\frac34}\,\|\boldsymbol{v}\|_{1,2}^{\frac14}\|\boldsymbol{v}\|_{2,2}^{\frac34},$$

so that using Poincaré inequality, and recalling that by Proposition 5.1.5, $\|\nabla\boldsymbol{v}(t)\|_2$ is uniformly bounded for sufficiently large t we infer for all such times

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 \le c_{10}\|\boldsymbol{v}\|_2^{\frac12}\|\boldsymbol{v}\|_{2,2}^{\frac32} \le c_{11}\,\|\boldsymbol{v}\|_2^2 + \frac{c_7}{2c_8}\|\boldsymbol{v}\|_{2,2}^2,$$

where, in the last step, we made use of the Young inequality. From the latter relation, (5.66), and (5.70) we thus derive, in particular,

$$\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 \le c_{12}\,\mathrm{e}^{-2c_2 t}. \tag{5.71}$$

Integrating (5.71) over (t,∞) and taking into account (5.61) we show

$$\|\nabla\boldsymbol{v}(t)\|_2^2 \le c_{13}\,\mathrm{e}^{-2c_2 t},\ \text{all sufficiently large } t>0,$$

which, once combined with (5.66) and (5.60) completes the proof of the property. ■

Remark 5.2.5. The previous theorem gives a full rigorous proof of Zhukovskii's conjecture (Theorem 3.0.6) in a very general class of solutions, and for sufficiently smooth cavities, under either assumption $A \leq B < C$ or $A = B = C$. The proof of the conjecture when $A < B = C$ remains thus *open*. Numerical tests suggest that Zhukovskii's conjecture is true also in this latter case. Figure 3 shows the dynamics of $\mathcal{S}$ when $A = 5.54$ and $B = C = 6.76$; also in this case, the motion of coupled system $\mathcal{S}$ will reach a steady state which is a permanent rigid rotation.

Remark 5.2.6. From the previous theorem it follows, in particular, that (5.62) holds if and only if $\boldsymbol{K}_G = \boldsymbol{0}$. Notice that the latter condition is not physically relevant. Actually, it is satisfied either by identically vanishing initial data, in which case the rest is the only corresponding weak solution, or else, more generally, for initial data able to produce, at time $t = 0$ an angular momentum of the liquid (relative to the rigid body) that is *exactly* the opposite of that of the rigid body, a circumstance that is very unlikely to happen. We also recall that, by (5.43), if $\boldsymbol{K}_G = \boldsymbol{0}$ *every* weak solution must have $\boldsymbol{\omega}_\infty(t) = \boldsymbol{0}$ for all $t \geq 0$. With the help of (5.5) and Gronwall Lemma, the strong energy inequality implies, in turn, $\|\boldsymbol{v}(t)\|_2 \leq c_1 \|\boldsymbol{v}(0)\|_2 \, \mathrm{e}^{-c_2 t}$, for some c_1, $c_2 > 0$ and all $t \geq 0$, thus re-obtaining, in a simpler way, the result of [47, Theorem 5.6].

5.2.2 Attainability and stability of permanent rotations

By Theorem 5.2.4, the system $\mathcal{S}$, under the stated assumptions on A, B, and C, will eventually perform a permanent rotation, as a single rigid body, around one of the central axes of inertia. However, our result does not specify around which axis this rotation will be attained. This issue assumes even more significance if we keep in mind that weak solutions may lack of uniqueness and therefore, in principle, we may have two different solutions with the same initial data generating, asymptotically, two permanent rotations around different axes. One of our next objectives is therefore to analyze this problem in some details. In particular, we shall prove that, if the initial data satisfy certain sufficiently general conditions, the permanent rotation will always occur along that central axis with the largest moment of inertia; see Theorem 5.2.7.

The other related objective concerns the stability of such permanent rotations. We will show *necessary and sufficient* conditions for stability for the full nonlinear problem, without any mathematical approximation or assumptions of the shape of the cavity. These conditions contain those of [43] as a particular case, and extend those of [27, 12, 48] to the nonlinear level.

In order to show all the above, we recall that from the physical viewpoint, the eigenvalues A, B and C of $\boldsymbol{I}$ are the moments of inertia of $\mathcal{S}$ around the axes passing through G and parallel to their corresponding eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, respectively (*central moment of inertia*). As before, we set

$$\boldsymbol{\omega}_\infty = p\,\boldsymbol{e}_1 + q\,\boldsymbol{e}_2 + r\,\boldsymbol{e}_3\,.$$

Our approach to attainability and stability is quite straightforward and relies upon the following three ingredients: (i) Theorem 5.2.4, (ii) balance of energy, and (iii) conservation of angular momentum. To this end, we begin to observe that the strong energy inequality (5.11) can be written as follows

$$\begin{aligned}\mathcal{E}_F(t) + Ap^2(t) + Bq^2(t) + Cr^2(t) + 2\mu\int_0^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \\ \le \mathcal{E}_F(0) + Ap^2(0) + Bq^2(0) + Cr^2(0)\,, \quad \text{all } t\ge 0\,,\end{aligned} \tag{5.72}$$

where the "energy" $\mathcal{E}_F$ is defined in (5.4). Furthermore, by dot-multiplying both sides of $(5.38)_3$ by $\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty$ we obtain the following equation representing the conservation of (the magnitude of) angular momentum

$$A^2p^2(t) + B^2q^2(t) + C^2r^2(t) = A^2p^2(0) + B^2q^2(0) + C^2r^2(0)\,. \tag{5.73}$$

The next result concerns the attainability of permanent rotations. Without loss of generality, we continue to assume throughout $A \le B \le C$.

Theorem 5.2.7. *The following statements hold.*[3]

[3] We assume $\boldsymbol{\omega}_\infty(0) \ne \mathbf{0}$, otherwise the motion of the coupled system is physically irrelevant; see Remark 5.2.6. Moreover, we also exclude that the initial data $(\boldsymbol{v}_0, \boldsymbol{\omega}_{\infty 0})$ are of the type $(\mathbf{0}, p_0\boldsymbol{e}_1)$, $(\mathbf{0}, q_0\boldsymbol{e}_2)$, or $(\mathbf{0}, r_0\boldsymbol{e}_3)$, $(p_0, q_0, r_0) \in \mathbb{R}^3$, since the corresponding motion (weak solution) will then reduce simply to a rigid rotation of $\mathcal{S}$ around one of the central axes.

(a) *Suppose $A = B < C$. Then, if*

$$\mathcal{E}_F(0) \leq \frac{(C-A)\,C}{A} r^2(0)\,, \tag{5.74}$$

necessarily

$$\begin{aligned} &\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t) = 0 \\ &\lim_{t\to\infty} r(t) = \bar{r} \neq 0\,. \end{aligned} \tag{5.75}$$

(b) *Suppose $A < B < C$. Then, if*

$$\begin{aligned} &\mathcal{E}_F(0) + \frac{A}{B}(B-A)p^2(0) \leq \frac{C}{B}(C-B)r^2(0)\,, \\ &0 < \mathcal{E}_F(0) \leq \frac{B}{A}(B-A)q^2(0) + \frac{C}{A}(C-A)r^2(0)\,, \end{aligned} \tag{5.76}$$

necessarily (5.75) *follows.*

(c) *Suppose $A < B = C$. Then, if*

$$\mathcal{E}_F(0) \leq \frac{B(B-A)}{A}(q^2(0) + r^2(0))\,, \tag{5.77}$$

necessarily

$$\lim_{t\to\infty} p(t) = 0\,. \tag{5.78}$$

Proof. We commence by proving the properties stated in (a) and (b). To this end, we notice that from Theorem 5.2.4 we know that

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \bar{p}\boldsymbol{e}_1 + \bar{q}\boldsymbol{e}_2 + \bar{r}\boldsymbol{e}_3\,, \quad \lim_{t\to\infty} \mathcal{E}_F(t) = 0\,, \tag{5.79}$$

for some $\bar{p}, \bar{q}, \bar{r} \in \mathbb{R}$. Thus, passing to the limit $t \to \infty$ on both sides of (5.72) and (5.73) we deduce

$$\begin{aligned} A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2 + 2\mu \int_0^\infty \|\nabla \boldsymbol{v}(t)\|_2^2 &\leq \mathcal{E}_F(0) + Ap^2(0) + Bq^2(0) + Cr^2(0) \\ A^2\bar{p}^2 + B^2\bar{q}^2 + C^2\bar{r}^2 &= A^2p^2(0) + B^2q^2(0) + C^2r^2(0)\,. \end{aligned} \tag{5.80}$$

In order to prove (a), we observe that again by Theorem 5.2.4 either $\bar{p} = \bar{q} = 0$ or $\bar{r} = 0$. Let us show that the latter cannot occur. In fact, multiplying both sides of $(5.80)_1$ by A $(=B)$, subtracting $(5.80)_2$, side by side, to the resulting inequality and taking $\bar{r} = 0$, we deduce

$$2A\mu \int_0^\infty \|\nabla \boldsymbol{v}(\tau)\|_2^2 \leq A\mathcal{E}_F(0) + C(A - C)r^2(0)\,,$$

which cannot hold under the assumption (5.74). We next demonstrate (b), namely, $\bar{p} = \bar{q} = 0$. Suppose $\bar{p} \neq 0$. Then by Theorem 5.2.4, $\bar{q} = \bar{r} = 0$. We thus multiply both sides of $(5.80)_1$ by A and subtract to the resulting inequality equation $(5.80)_2$, side by side, to get

$$2A\mu \int_0^\infty \|\nabla \boldsymbol{v}(t)\|_2^2 \leq A\mathcal{E}_F(0) + B(A - B)q^2(0) + C(A - C)r^2(0)\,, \tag{5.81}$$

which is contradicted by $(5.76)_2$. Suppose, instead, $\bar{q} \neq 0$. Then, again by Theorem 5.2.4, $\bar{p} = \bar{r} = 0$. Thus, multiplying both sides of $(5.80)_1$ by B and subtracting to the resulting inequality equation $(5.80)_2$, side by side, we infer

$$2B\mu \int_0^\infty \|\nabla \boldsymbol{v}(t)\|_2^2 \leq B\mathcal{E}_F(0) + A(B - A)p^2(0) + C(B - C)r^2(0)\,. \tag{5.82}$$

However, (5.82) is in contrast with $(5.76)_1$, and the proof of (b) is completed. It remains to show statement (c). By Theorem 5.2.4, we know that the limit in (5.78) is either 0, as claimed, or it is not. In the latter case, again by Theorem 5.2.4, we must have

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} r(t) = 0\,.$$

We may then use again (5.81) which is at odds with (5.77), and the proof of the theorem is completed. ■

Remark 5.2.8. As we mentioned earlier on, the above theorem assumes great relevance when the coupled system $\mathcal{S}$ has gyroscopic structure around the $\boldsymbol{e}_3$ axis (say), that is, $A = B \neq C$. In such a case, our result ensures, in particular, that if the liquid is initially at rest with respect to the rigid body (that is, the relative velocity field of the liquid is zero at $t = 0$), eventually, the final motion of $\mathcal{S}$ will be a permanent rotation occurring along the axis of the gyroscope, $\{G, \boldsymbol{e}_3\} := \mathsf{a}$, *if and only if* the moment of inertia with respect to that axis is larger than those around the other two. As an illustration of this fact, consider the case where the body $\mathcal{B}$ is a hollow cylinder (like a metal can), completely filled with a viscous liquid. In this situation, a coincides with the axis of the cylinder. We assume that the central moments of inertia of $\mathcal{B}$ are negligible compared to those of the liquid. Then combining Theorem 5.2.4 and Theorem 5.2.7, we may state that for any rigid motion impressed initially to the coupled system $\mathcal{S}$, the asymptotic motion will be a permanent rotation around a *if and only if the cylinder is "flattened" enough.* More precisely, let h and R be height and radius of $\mathcal{B}$, respectively. Taking into account that

$$A = B = \frac{M}{12}(3R^2 + h^2)\,, \quad C = \frac{1}{2}MR^2\,,$$

with M mass of the liquid, the final motion will be a rotation around a if and only if $h < \sqrt{3}R$.

Remark 5.2.9. Results proved in Theorem 5.2.7 require the initial data to be in a certain range (see (5.74), (5.77), and (5.76)). However, the numerical tests reported in Section 4.1, suggest that such a requirement might be unnecessary. The question of whether this restriction can be removed analytically is at the moment open.

With the help of Theorem 5.2.7 we are now able to derive the following results, which ensure *stability of permanent rotations of the coupled system* around the central axis with the largest moment of inertia, and instability in the other cases.

Theorem 5.2.10. *Let $\mathcal{S}$ perform a permanent rotation around the central axis $\{G, \boldsymbol{e}\}$, say, $\boldsymbol{v} \equiv \boldsymbol{0}$, $\boldsymbol{\omega}_\infty \equiv \omega_0 \boldsymbol{e}$, $\boldsymbol{e} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Assume that, at time $t = 0$, this state is perturbed, and denote by $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t)$, $\tilde{\boldsymbol{\omega}}_\infty(t) = (\tilde{p}(t), \tilde{q}(t), \tilde{r}(t))$ the corresponding perturbation fields.*

The following properties hold.

(a) *If $A < B = C$, then the permanent rotation with $\boldsymbol{e} \equiv \boldsymbol{e}_1$ is unstable in the sense of Lyapunov, i.e. there exists $\varepsilon > 0$ such that for every $\delta > 0$, if $\mathcal{E}_F(0) + \tilde{p}^2(0) + \tilde{q}^2(0) + \tilde{r}^2(0) < \delta$, then there exists $\bar{t} > 0$ such that*

$$\mathcal{E}_F(\bar{t}) + \tilde{p}^2(\bar{t}) + \tilde{q}^2(\bar{t}) + \tilde{r}^2(\bar{t}) > \varepsilon.$$

(b) *If $A \leq B < C$, then the permanent rotation with $\boldsymbol{e}$ being either $\boldsymbol{e}_1$ or $\boldsymbol{e}_2$ is unstable in the sense of Lyapunov. If, however, $\boldsymbol{e} \equiv \boldsymbol{e}_3$, then the corresponding permanent rotation is stable. Precisely, for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\mathcal{E}_F(0) + \tilde{p}^2(0) + \tilde{q}^2(0) + \tilde{r}^2(0) < \delta \quad \Longrightarrow \quad \mathcal{E}_F(t) + \tilde{p}^2(t) + \tilde{q}^2(t) + \tilde{r}^2(t) < \varepsilon\,, \tag{5.83}$$

for all $t \geq 0$. Moreover, there is $\gamma = \gamma(A, B, C, \omega_0) > 0$ such that if

$$\mathcal{E}_F(0) + \tilde{p}^2(0) + \tilde{q}^2(0) + \tilde{r}^2(0) \leq \gamma\,, \tag{5.84}$$

it results[4]

$$\tilde{p}(t), \tilde{q}(t) \to 0\,, \quad \tilde{r}(t) \to r^*, \quad \text{as } t \to \infty\,, \tag{5.85}$$

where

$$r^* = -\omega_0 \pm \sqrt{\frac{1}{C^2}(A^2\tilde{p}^2(0) + B^2\tilde{q}^2(0)) + (\tilde{r}(0) + \omega_0)^2}\,, \tag{5.86}$$

and where we take $+$ or $-$ according to whether $\omega_0 > 0$ or $\omega_0 < 0$.

(c) *If $A = B = C$, the permanent rotation corresponding to arbitrary $\boldsymbol{e}$ is stable in the sense of Lyapunov, namely, (5.83) holds for all $t \geq 0$.*

Proof. We begin to notice that $(\boldsymbol{v}, \boldsymbol{\omega}_\infty \equiv \tilde{\boldsymbol{\omega}}_\infty + \omega_0 \boldsymbol{e})$ must satisfy (5.72) and (5.73), and, consequently, we may apply Theorem 5.2.7. In order to show the property in (a), take initial conditions for the perturbed field satisfying $E(0) = \tilde{q}(0) = \tilde{r}(0) = 0$, and $\tilde{p}(0)$ non-zero and as small as we please. Thus, in particular, (5.77) is satisfied. As a consequence, by (c) of Theorem 5.2.7,

$$\lim_{t\to\infty} p(t) = 0, \tag{5.87}$$

[4] Recall that, by Theorem 5.2.4, $\mathcal{E}_F(t) \to 0$, as $t \to \infty$ regardless of the “size” of the initial conditions.

and by Theorem 5.2.4, there is a positive, unbounded sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} q(t_k) = \bar{q}\,, \quad \lim_{k\to\infty} r(t_k) = \bar{r}\,,$$

with at least one of $\bar{q}, \bar{r}$ being nonzero. Thus, evaluating (5.73) along this sequence, passing to the limit $k \to \infty$, and using (5.87) we must have

$$B^2(\bar{q}^2 + \bar{r}^2) = A^2(\omega_0 + \tilde{p}(0))^2\,.$$

Since $(\omega_0+\tilde{p}(0))^2 \geq \frac{1}{2}\omega_0^2 - \tilde{p}^2(0)$, then, for all $\tilde{p}(0)$ sufficiently small, there exists $\bar{k}$ such that

$$B^2(\tilde{q}^2(t_{\bar{k}}) + \tilde{r}^2(t_{\bar{k}})) > A^2(\omega_0 + \tilde{p}(0))^2 - A^2\tilde{p}^2(0) \,\geq A^2\left(\frac{1}{2}\omega_0^2 - 2\tilde{p}^2(0)\right).$$

Thus, for all $\tilde{p}(0)$ sufficiently small and satisfying $\tilde{p}^2 < \frac{1}{8}\omega_0^2$, there is $\bar{t} > 0$, such that $(\tilde{q}^2(\bar{t}) + \tilde{r}^2(\bar{t})) > \varepsilon$, where $\varepsilon := \frac{1}{4}(A^2\omega_0^2/B^2)$, and this furnishes the desired instability result.

Next, to prove the first property stated in (b), we take (in both cases $\boldsymbol{e} = \boldsymbol{e}_1, \boldsymbol{e}_2$) $E(0) = \tilde{p}(0) = \tilde{q}(0) = 0$ and $\tilde{r}(0)$ arbitrarily small, and notice that (5.74) and (5.76) is satisfied. By a completely analogous reasoning to the one employed previously we then show $\tilde{r}^2(t) \geq \frac{1}{2}(A^2\omega_0^2/C^2)$ for all sufficiently large t, thus proving instability.

To show the other statement in (b), we multiply both sides of (5.72) by C, and subtract to the resulting inequality (5.73), side by side. We deduce, in particular,

$$C\mathcal{E}_F(t) + A(C-A)\tilde{p}^2(t) + B(C-B)\tilde{q}^2(t) \leq C\mathcal{E}_F(0) + A(C-A)\tilde{p}^2(0) + B(C-B)\tilde{q}^2(0)\,,$$

which, in turn, implies

$$\mathcal{E}_F(t)+\tilde{p}^2(t)+\tilde{q}^2(t) \leq m\big[\mathcal{E}_F(0)+\tilde{p}^2(0)+\tilde{q}^2(0)\big]\,, \quad m := \frac{\max\{C, A(C-A), B(C-B)\}}{\min\{C, A(C-A), B(C-B)\}}\,. \tag{5.88}$$

Thus, given $\varepsilon > 0$, we have

$$\mathcal{E}_F(0) + \tilde{p}^2(0) + \tilde{q}^2(0) < \frac{\delta_1}{m} \implies \mathcal{E}_F(t) + \tilde{p}^2(t) + \tilde{q}^2(t) < \delta_1\,, \quad \text{for all } \delta_1 \in (0, \varepsilon/2)\,. \tag{5.89}$$

Next, we want to show that for a suitable choice of $\delta_2 > 0$, the following property holds

$$\tilde{r}^2(0) < \delta_2 \implies \tilde{r}^2(t) < \frac{\varepsilon}{2}\,. \tag{5.90}$$

Without loss of generality, we take

$$\varepsilon = 2\eta^2\omega_0^2\,, \tag{5.91}$$

with η arbitrarily fixed in $(0,1)$, and choose $\delta_2 < \varepsilon/2$. Assume (5.90) is not true. In view of the continuity of $r(t)$, let $\bar{t} > 0$ be the *first* instant of time such that $\tilde{r}^2(\bar{t}) = \varepsilon/2$. Thus, by (5.73) and (5.91) we deduce

$$\pm C^2\omega_0^2\eta(2\pm\eta) = A^2\tilde{p}^2(0) + B^2\tilde{q}^2(0) + C^2\tilde{r}(0)(\tilde{r}(0)+2\omega_0) - A^2p^2(\bar{t}) - B^2\tilde{q}^2(\bar{t})\,. \tag{5.92}$$

Recalling that $A \le B$, $\eta \in (0,1)$ and using (5.89), from (5.92) we show

$$C^2\omega_0^2\eta \le B^2\left(\frac{m+1}{m}\right)\delta_1 + C^2\sqrt{\delta_2}\big(\sqrt{\delta_2} + 2|\omega_0|\big)\,.$$

Employing in the latter relation the inequality $2\sqrt{\delta_2}|\omega_0| \le 2\delta_2/\eta + \eta\omega_0^2/2$, and recalling again that $\eta \in (0,1)$, we get

$$C^2\omega_0^2\eta \le 2B^2\left(\frac{m+1}{m}\right)\delta_1 + 6C^2\frac{\delta_2}{\eta}\,. \tag{5.93}$$

However, (5.93) cannot be true as long as we pick δ_1, δ_2 such that (for instance)

$$0 < \delta_1 < \frac{m\,C^2\omega_0^2}{2(m+1)B^2}\frac{\eta}{4} \equiv \frac{m\,C^2|\omega_0|}{8(m+1)B^2}\frac{\sqrt{\varepsilon}}{\sqrt{2}}, \qquad 0 < \delta_2 < \omega_0^2\frac{\eta^2}{24} \equiv \frac{\varepsilon}{48}\,.$$

As a consequence, (5.83) follows, provided we choose

$$\delta < \min\left\{\frac{\varepsilon}{48}\,, \frac{\varepsilon}{2m}\,, \frac{m\,C^2|\omega_0|}{8(m+1)B^2}\frac{\sqrt{\varepsilon}}{\sqrt{2}}\right\}.$$

Let us now show the last property stated in (b). From Theorem 5.2.7, we know that the asymptotic property (5.85) is valid whenever the initial conditions of the motion $(\boldsymbol{v}, \tilde{\boldsymbol{\omega}}_\infty + \omega_0\boldsymbol{e})$ satisfy (5.74) and (5.76). Recalling that $A \le B < C$, one shows that both conditions are certainly met if

$$\mathcal{E}_F(0) + \frac{A}{B}(B-A)(\tilde{q}^2(0) + \tilde{p}^2(0)) \le \frac{C}{B}(C-B)(\omega_0 + \tilde{r}(0))^2\,.$$

However, since $(\tilde{r}(0)+\omega_0)^2 \ge \frac{1}{2}\omega_0^2 - \tilde{r}^2(0)$, we see that the latter is satisfied provided $\mathcal{E}_F(0)$, and $\tilde{p}(0), \tilde{q}(0)$, and $\tilde{r}(0)$ obey (5.84), for a suitable definition of γ. However, by taking γ even smaller if necessary, from the stability property proved above we know that $|\tilde{r}(t)| < |\omega_0|$, for all $t \ge 0$. Consequently, (5.86) follows from this consideration, by passing to the limit

$t \to \infty$ in (5.73). It remains to show property (c). In this regard, we observe that from our hypothesis and $(5.38)_3$ we deduce

$$\dot{\tilde{\boldsymbol{\omega}}}_\infty + \boldsymbol{a} \times (\tilde{\boldsymbol{\omega}}_\infty + \omega_0 \boldsymbol{e}) = \boldsymbol{0}\,, \tag{5.94}$$

from which it follows that

$$|\tilde{\boldsymbol{\omega}}_\infty(t) + \omega_0 \boldsymbol{e}| = |\tilde{\boldsymbol{\omega}}_\infty(0) + \omega_0 \boldsymbol{e}|\,. \tag{5.95}$$

From (5.94) and (5.95) we thus obtain

$$|\tilde{\boldsymbol{\omega}}_\infty(t)| \le |\tilde{\boldsymbol{\omega}}_\infty(0)| + |\tilde{\boldsymbol{\omega}}_\infty(0) + \omega_0 \boldsymbol{e}| \int_0^\infty |\boldsymbol{a}(t)|\,. \tag{5.96}$$

Using Schwartz inequality and (5.58) in (5.96) allow us to conclude

$$|\tilde{\boldsymbol{\omega}}_\infty(t)| \le |\tilde{\boldsymbol{\omega}}_\infty(0)| + c\,|\tilde{\boldsymbol{\omega}}_\infty(0) + \omega_0 \boldsymbol{e}|\, \|\boldsymbol{v}(0)\|_2\,,$$

and the property stated in (c) immediately follows from this last inequality and (5.58). ■

Remark 5.2.11. Combining Theorem 5.2.7 and Theorem 5.2.10 we derive the following interesting consequence. Suppose $C > A \ge B$, $\mathcal{E}_F(0) = 0$, $p(0), q(0)$ sufficiently "small", and $r(0) \neq 0$. Then, the asymptotic motion of the coupled system - which we know is a permanent rotation around $\boldsymbol{e}_3$ - will have angular velocity $\bar{\boldsymbol{\omega}} = \bar{r}\boldsymbol{e}_3$, where $\bar{r}$ has the *same sign as* $r(0)$. Observing that $\bar{\boldsymbol{\omega}} = \kappa\, \boldsymbol{K}_G$, $\kappa > 0$, this property implies that $\{G, \boldsymbol{e}_3\}$ has to keep (asymptotically) the same *orientation* with $\boldsymbol{K}_G$ that it had at time $t = 0$; see Figure 5.2.2. Stated differently, this means that, at least under the above conditions, the axis $\boldsymbol{e}_3$ cannot (eventually) *flip-over*. This property is confirmed by the numerical tests presented in Section 4.1; see Figure 4, bottom panel. These tests also show, however, that the above property is no longer valid for initial data of *finite* size; see Figure 4, top panel. Similar experiments prove, in addition, that if $r(0) = 0$, a change of viscosity of the liquid may trigger such an effect as well. In other words, it is found that in some range of viscosities the orientation of $\boldsymbol{e}_3$ and $\boldsymbol{K}_G$ is the same, whereas in another range it is opposite; see Section 4.1. It will be the object of future work to investigate the analytical aspect of this interesting phenomenon.

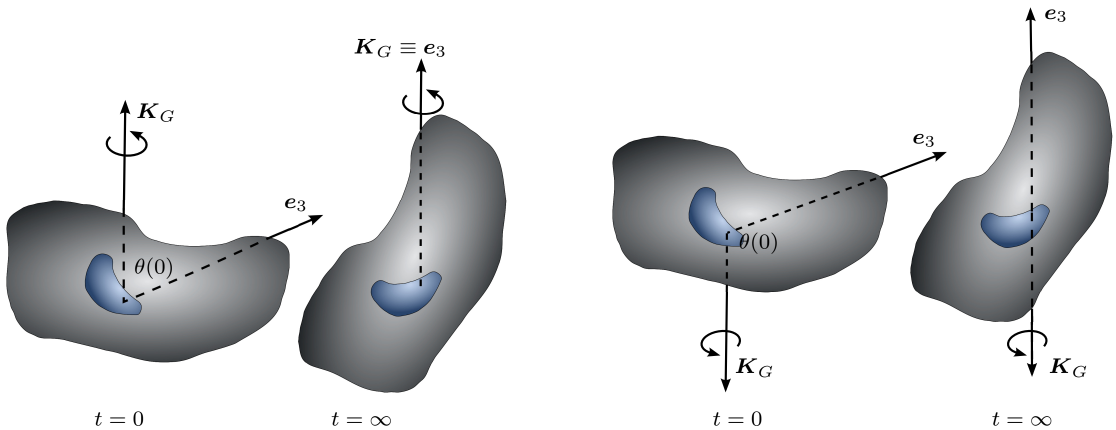


Figure 8: Dependence of the orientation of a body filled with a viscous liquid on the initial angle $\theta(0)$ between $\boldsymbol{e}_3$ and $\boldsymbol{K}_G$: $\theta(0) < \pi/2$ (left); $\theta(0) > \pi/2$ (right).

Remark 5.2.12. The instability result in Theorem 5.2.10 should be contrasted with their "classical" counterpart when the cavity in the gyroscope is empty. In fact, in such a situation, as is well known, permanent rotations about the gyroscope axis are stable in *both* cases $A, B \lessgtr C$; see [35]. Whereas, if the cavity is filled with a viscous liquid, this permanent rotation is (axially) stable if and only if $C > A, B$.

5.3 LIQUID-FILLED PHYSICAL PENDULUM

In this section, we will consider a liquid-filled (physical) pendulum, i.e. a coupled system, $\mathcal{S}$, characterized by a heavy rigid body, $\mathcal{B}$, containing a cavity, $\mathcal{C}$ entirely filled with a viscous liquid, and constrained to rotate (without friction) around a horizontal axis, a, so that its center of mass G satisfies the following properties:

(i) the distance, ℓ, between G and its orthogonal projection O on a (*point of suspension*), does not depend on time;

(ii) G always moves in a plane orthogonal to a.

Experimental evidence shows that the liquid will have a stabilizing effect on the motion of the pendulum, by reducing the amplitude of oscillations. A most remarkable application of this property occurs in Space Engineering, where tube dampers filled with a viscous liquid are used to suppress oscillations in spacecraft and artificial satellites; see Subsection 2.1.2.2, and also [5, 1, 6, 44, 3] with the literature there cited.

Objective of this section is to provide a rigorous analysis of the motion of the coupled system $\mathcal{S}$. In particular, we shall show that, provided $\mathcal{C}$ is sufficiently regular, all motions of $\mathcal{S}$ described within a very general class of solutions to the relevant equations (*weak solutions*), must tend to a rest state for large times, no matter the shape of $\mathcal{C}$, the physical characteristics of $\mathcal{B}$ and the liquid, and the initial motions imparted to $\mathcal{S}$. We show that, as expected, the rest state is realized by only two equilibrium configurations of $\mathcal{S}$, namely, those where the velocity field of the liquid is zero, and the center of mass G of $\mathcal{S}$ is in its lowest, G_l, or highest, G_h, position; see Theorem 5.3.3.

We then further prove that for a broad set of initial data, the final state must be the one with $G \equiv G_l$. This set includes the case when the system $\mathcal{S}$ is released from rest; see Theorem 5.3.5. In physical terms, the latter translates into the following interesting property, namely, that *a pendulum with a cavity filled with a viscous liquid that is initially at rest eventually reaches the equilibrium configuration where the center of mass is at its lowest point, exactly like it happens to a classical pendulum immersed in a viscous liquid.* However, it must be also observed that the *global* dynamics can be quite different in the two cases. In fact, while in the latter the amplitude of oscillations may gradually decrease from the outset till it reduces to zero, in the former, in analogy to similar problems of solids with liquid-filled cavity, see Sections 5.2 and 4.1 (and also [29, 20, 13]), the damping of the oscillations may take place only after an interval of time $[0, T]$, say, where, possibly, a motion of "chaotic" nature occurs, with T depending on the magnitude of the kinematic viscosity ν.

5.3.1 Long-time behavior of a liquid-filled pendulum

Let $\mathsf{F} \equiv \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a frame attached to $\mathcal{B}$, with the origin at O, $\boldsymbol{e}_1 \equiv \overrightarrow{OG}/|\overrightarrow{OG}|$ and $\boldsymbol{e}_3$ directed along a. Then, the motion of $\mathcal{S}$ in F is governed by the following set of equations (as from (5.2))

$$
\left.\begin{aligned}
&\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\left(\dot{a}+\beta^2\gamma_2\right)\boldsymbol{e}_3\times\boldsymbol{x}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+2\omega\,\boldsymbol{e}_3\times\boldsymbol{v}\right)=\mu\Delta\boldsymbol{v}-\nabla p\\
&\operatorname{div}\boldsymbol{v}=0
\end{aligned}\right\}\ \text{in } \mathcal{C}\times(0,\infty),
$$

$$
\begin{aligned}
&\boldsymbol{v}(\boldsymbol{x},t)|_{\partial\mathcal{C}}=\mathbf{0},\\
&\dot{\omega}-\dot{a}=\beta^2\gamma_2\,,\\
&\dot{\boldsymbol{\gamma}}+\omega\,\boldsymbol{e}_3\times\boldsymbol{\gamma}=\mathbf{0}\,.
\end{aligned}
\tag{5.97}
$$

Here, the angular velocity of the solid is given by $\omega\,\boldsymbol{e}_3$; whereas $\boldsymbol{\gamma}=(\gamma_1\equiv\cos\varphi,\gamma_2\equiv-\sin\varphi,0)$ denotes the direction of the gravity in the non-inertial frame F, φ is the angle between $\boldsymbol{e}_1$ and the gravity $\boldsymbol{g}$. Furthermore,

$$
a:=-\frac{\rho}{C}\,\boldsymbol{e}_3\cdot\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{v}\,,
\tag{5.98}
$$

where C is the moment of inertia of $\mathcal{S}$ with respect to a, and

$$
\beta^2=M\,g\,|\overrightarrow{OG}|/C\,,
$$

with M mass of $\mathcal{S}$.

The energy balance (5.3) now reads as follows

$$
\frac{d}{dt}\left[\rho\,\|\boldsymbol{v}\|_2^2-C\,a^2+C\,(\omega-a)^2-2C\beta^2\gamma_1\right]+2\mu\,\|\nabla\boldsymbol{v}\|_2^2=0\,.
\tag{5.99}
$$

In this equation, the quantity

$$
\mathcal{E}=\rho\,\|\boldsymbol{v}\|_2^2-C\,a^2+C\,(\omega-a)^2
\tag{5.100}
$$

represents the *total kinetic energy* of $\mathcal{S}$, while

$$
\mathcal{U}=-2C\beta^2\gamma_1
\tag{5.101}
$$

is its *potential energy*. By (2.29), it follows that there is a positive constant $c_0\le 1$, such that

$$
c_0\,\left(\rho\,\|\boldsymbol{v}\|_2^2+C\,(\omega-a)^2\right)\le\mathcal{E}\le\left(\rho\,\|\boldsymbol{v}\|_2^2+C\,(\omega-a)^2\right)\,.
\tag{5.102}
$$

As for the inertial motions, our study on the asymptotic behavior in time of the coupled system $\mathcal{S}$ is carried out in the very general class constituted by weak solutions (à la Leray-Hopf) to (5.97). To this end, we specialize Definition 5.1.1 to the problem at hand. All the results in Section 5.1 continue to hold for this problem.

Definition 5.3.1. *The triple* $(\boldsymbol{v}, \omega, \boldsymbol{\gamma})$ *is a* weak solution *to* (5.97) *if it meets the following requirements:*

(a) $\boldsymbol{v} \in C_w([0,\infty); H(\mathcal{C})) \cap L^\infty(0,\infty; H(\mathcal{C})) \cap L^2(0,\infty; W_0^{1,2}(\mathcal{C}))$ *;*

(b) $\omega \in C^0([0,\infty)) \cap C^1(0,\infty)$, $\boldsymbol{\gamma} \in C^1([0,\infty); \mathsf{S}^1)$;

(c) Strong Energy Inequality*:*

$$\mathcal{E}(t) + \mathcal{U}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \, d\tau \leq \mathcal{E}(s) + \mathcal{U}(s) \tag{5.103}$$

for all $t \geq s$ *and a.a.* $s \geq 0$ *including* $s = 0$ *;*

(d) $(\boldsymbol{v}, \omega, \boldsymbol{\gamma})$ *satisfies the following equations:*

$$\begin{aligned} \rho(\boldsymbol{v}(t), \boldsymbol{\psi}) + \rho a(t) \int_{\mathcal{C}} (\boldsymbol{e}_3 \times \boldsymbol{x}) \cdot \boldsymbol{\psi} + \int_0^t \left\{ \rho[(\beta^2 \gamma_2 \boldsymbol{e}_3 \times \boldsymbol{x} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}) + 2\rho(\omega \boldsymbol{e}_3 \times \boldsymbol{v}, \boldsymbol{\psi}) \right\} \\ + \int_0^t \mu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) = (\rho\, \boldsymbol{v}(0), \boldsymbol{\psi}) + \rho a(t) \int_{\mathcal{C}} (\boldsymbol{e}_3 \times \boldsymbol{x}) \cdot \boldsymbol{\psi} \end{aligned} \tag{5.104}$$

for all $\boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\mathcal{C})$ *and all* $t \in (0,\infty)$*. Moreover,*

$$\omega(t) - a(t) = \omega(0) - a(0) + \beta^2 \int_0^t \gamma_2(\tau)\, d\tau \tag{5.105}$$

and

$$\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(0) - \int_0^t \omega \boldsymbol{e}_3 \times \boldsymbol{\gamma} \,, \qquad \textit{for all } t \in (0,\infty). \tag{5.106}$$

We are now in a position to give the following characterization of the Ω-limit set of any weak solution to (5.97) following Proposition 5.1.9.

Proposition 5.3.2. *Let* $\mathsf{s} \equiv (\boldsymbol{v}, \omega, \boldsymbol{\gamma})$ *be a weak solution to* (5.97)*, with* $\mathcal{C}$ *of class* C^2*, and initial data of finite energy in the sense of Proposition 5.1.2. Then, the corresponding* Ω*-limit set admits the following characterization: either*

$$\Omega(\mathsf{s}) = \{(\boldsymbol{0}, 0, \boldsymbol{e}_1)\}\ ,$$

or

$$\Omega(\mathsf{s}) = \{(\boldsymbol{0}, 0, -\boldsymbol{e}_1)\}\ .$$

Proof. In view of Proposition 5.1.9, the dynamics on $\Omega(\mathsf{s})$ is governed by the following set of equations

$$\boldsymbol{v} \equiv \boldsymbol{0}\,,\ \ \gamma_2 = 0,\ \ \omega\,\boldsymbol{e}_3 \times \boldsymbol{\gamma} = \boldsymbol{0}\,. \tag{5.107}$$

The latter two equations imply that $\omega\,\gamma_1 = 0$. Taking into account that $|\boldsymbol{\gamma}| \equiv 1$, from all the above we then conclude $\omega = 0$ and $\gamma_1 = \pm 1$. Consequently,

$$\Omega(\mathsf{s}) \subset \{(\boldsymbol{0}, 0, \boldsymbol{e}_1)\} \cup \{(\boldsymbol{0}, 0, -\boldsymbol{e}_1)\}\ .$$

However, again by Proposition 5.1.9, $\Omega(\mathsf{s})$ is connected, and the proof of the proposition is therefore completed. ■

We are now ready to give a complete description of the asymptotic behavior of weak solutions to (5.97).

Theorem 5.3.3. *Let $(\boldsymbol{v}, \omega, \boldsymbol{\gamma})$ be a weak solution to (5.97) with $\mathcal{C}$ of class C^2, and initial data of finite energy in the sense of Proposition 5.1.2. Then,*

$$\lim_{t\to\infty}\left(\|\boldsymbol{v}(t)\|_{2,2} + \left\|\frac{\partial \boldsymbol{v}(t)}{\partial t}\right\|_2\right) = 0\,, \tag{5.108}$$

so that, in particular,

$$\lim_{t\to\infty}\,(\max_{x\in\mathcal{C}} |\boldsymbol{v}(\boldsymbol{x}, t)|) = 0\,. \tag{5.109}$$

Moreover,

$$\lim_{t\to\infty} |\omega(t)| = 0\,, \quad \lim_{t\to\infty} |\boldsymbol{\gamma}(t) - \alpha\,\boldsymbol{e}_1| = 0\,, \tag{5.110}$$

where $\alpha = 1$ or $\alpha = -1$.

Proof. We commence by observing that, as a result of the classical embedding inequality

$$\max_{x\in\mathcal{C}} |\boldsymbol{w}(\boldsymbol{x})| \le c_1 \|\boldsymbol{w}\|_{2,2}\,, \quad \text{all } \boldsymbol{w} \in W^{2,2}(\mathcal{C})\,,$$

property (5.109) follows from (5.108). Next, we notice that in view of this and Proposition 5.3.2, to prove the theorem completely we only have to prove the validity of (5.108). This can be achieved by the following procedure. By virtue of Proposition 5.1.5, our weak solution must have $\boldsymbol{v}(t) \in W^{2,2}(\mathcal{C})$, for a.a. $t \in [t_0, \infty)$. For simplicity, and without loss of generality, we assume

$$\boldsymbol{v}(t_0) \in W^{2,2}(\mathcal{C})\,. \tag{5.111}$$

Next, we formally take the time derivative of both sides of (5.97)$_1$, dot-multiply both sides of the resulting equation by $\partial \boldsymbol{v}/\partial t$ and integrate by parts over $\mathcal{C}$. By taking into account (5.97)$_{2,3,5}$ and (5.98), we easily show that ($\nu := \mu/\rho$ and $\boldsymbol{v}_t := \partial \boldsymbol{v}/\partial t$)

$$\frac{1}{2}\frac{dE_1}{dt} = -C\,\beta^2 \dot{a}\,\omega\,\gamma_1 - 2\dot{\omega}\,(\boldsymbol{e}_3 \times \boldsymbol{v}, \boldsymbol{v}_t) - (\boldsymbol{v}_t \cdot \nabla \boldsymbol{v}, \boldsymbol{v}_t) - \nu \|\nabla \boldsymbol{v}_t\|_2^2\,, \tag{5.112}$$

where

$$E_1 := \|\boldsymbol{v}_t\|_2^2 - \frac{C}{\rho}\,\dot{a}^2\,.$$

Notice that, by (2.29),

$$c_0\,\|\boldsymbol{v}_t\|_2^2 \le E_1 \le \|\boldsymbol{v}_t\|_2^2\,. \tag{5.113}$$

Employing Young inequality (2.16) and Poincaré inequality (2.23), and recalling that $|\boldsymbol{\gamma}| = 1$ together with (5.97)$_4$, we deduce

$$\frac{dE_1}{dt} + C_1 \|\nabla \boldsymbol{v}_t\|_2^2 \le C_2 \left(-(\boldsymbol{v}_t \cdot \nabla \boldsymbol{v}, \boldsymbol{v}_t) + \omega^2 + \|\boldsymbol{v}\|_2^2 + \|\boldsymbol{v}\|_2\, \|\boldsymbol{v}_t\|_2^2 \right)\,. \tag{5.114}$$

Applying Hölder inequality (2.17), the interpolation inequality (2.20), Sobolev inequality (2.22), and Young inequality (2.16), in the order, we infer

$$\begin{aligned} |(\boldsymbol{v}_t \cdot \nabla \boldsymbol{v}, \boldsymbol{v}_t)| &\le \|\boldsymbol{v}_t\|_4^2 \|\nabla \boldsymbol{v}\|_2 \le \|\boldsymbol{v}_t\|_6^{\frac{3}{2}} \|\boldsymbol{v}_t\|_2^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_2 \le C_3\, \|\nabla \boldsymbol{v}_t\|_2^{\frac{3}{2}} \|\boldsymbol{v}_t\|_2^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_2 \\ &\le \frac{C_1}{2C_2} \|\nabla \boldsymbol{v}_t\|_2^2 + C_4 \left(\|\boldsymbol{v}_t\|_2^6 + \|\nabla \boldsymbol{v}\|_2^6 \right)\,. \end{aligned} \tag{5.115}$$

Taking into account (5.13), we may combine the latter displayed equation with (5.114), to get

$$\frac{dE_1}{dt} + \frac{C_1}{2} \|\nabla \boldsymbol{v}_t\|_2^2 \le C_5 + C_6 \left(\|\boldsymbol{v}_t\|_2^6 + \|\nabla \boldsymbol{v}\|_2^6 \right)\,. \tag{5.116}$$

From (5.116), (5.113), and (5.24) to show, in particular, the validity of (5.29) with $z := \|\nabla \boldsymbol{v}\|_2^2 + E_1 + 1$. Integrating the differential inequality thus obtained and using again (5.24), (5.113), and (5.116), we prove, in addition to the bounds (5.25), the following ones:

$$\|\boldsymbol{v}_t(t)\|_2 \le G_3(t)\,, \quad \int_{t_0}^{t} \|\nabla \boldsymbol{v}_\tau(\tau)\|^2\, d\tau \le G_4(t)\,, \tag{5.117}$$

with G_i, $i=3,4$, continuous functions in the interval $[t_0, t_0+T^*)$, where

$$T^* \geq \frac{C_7}{\|\nabla \boldsymbol{v}(t_0)\|_2^4 + \|\boldsymbol{v}_t(t_0)\|_2^4 + 1}\,,$$

and $C_7>0$ independent of t_0. We now go back to $(5.97)_1$, dot-multiply both sides by $\boldsymbol{v}_t$ and integrate over $\mathcal{C}$. We get

$$\rho\,\|\boldsymbol{v}_t\|_2^2 - C\,\dot{a}^2 = \frac{C\,\beta^2}{\rho}\,\gamma_2\,\dot{a} - \rho\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}_t) - 2\omega(\boldsymbol{e}_3\times\boldsymbol{v},\boldsymbol{v}_t) + \mu(\Delta\boldsymbol{v},\boldsymbol{v}_t)\,.$$

Exploiting in this relation Young inequality, (5.113), (5.13), and (5.23) we show (formally)

$$\|\boldsymbol{v}_t(t_0)\|_2 \leq C_8(\|\boldsymbol{v}(t_0)\|_{2,2}^3 + \|\boldsymbol{v}(t_0)\|_{2,2} + 1)\,, \tag{5.118}$$

which implies, on the one hand, by (5.111) that $\|\boldsymbol{v}_t(t_0)\|_2$ is well-defined, and, on the other hand, that

$$T^* \geq \frac{C_9}{D(\|\boldsymbol{v}(t_0)\|_{2,2})+1}$$

where $D=D(\sigma)$ is a polynomial satisfying $D(0)=0$. Collecting all the above informations, we may thus employ the standard Galerkin method and show the existence of a solution $(\tilde{\boldsymbol{v}},\tilde{\omega},\tilde{\boldsymbol{\gamma}})$ with data $(\boldsymbol{v}(t_0),\omega(t_0),\boldsymbol{\gamma}(t_0))$ that, besides (5.26), satisfies also

$$\tilde{\boldsymbol{v}}_t \in L^\infty(t_0,t_0+\tau;H(\mathcal{C}))\cap L^2(t_0,t_0+\tau;W_0^{1,2}(\mathcal{C}))\,,\quad \text{all } \tau\in(0,T^*)\,;$$

see, e.g., [33, Chapter 4] for technical details. However, by the uniqueness property of Proposition 5.1.4, this solution must coincide with the given weak solution on $[t_0,t_0+T^*)$. We can then show that, in fact, $T^*=\infty$. Actually, if $T^*<\infty$, it easily follows that necessarily $\|\boldsymbol{v}(t)\|_{2,2}$ must become unbounded in a left-neighborhood of t_0+T^*. Let us show that such a situation cannot occur. To this end, we begin to observe that, by what just shown, the given weak solution satisfies (5.114) in (t_0,t_0+T^*). Now, by Hölder inequality (2.17), interpolation inequality (2.20), Sobolev embedding Theorem 2.3.2, and Young inequality (2.16), in the order, we obtain

$$\begin{aligned}|(\boldsymbol{v}_t\cdot\nabla\boldsymbol{v},\boldsymbol{v}_t)| = |(\boldsymbol{v}_t\cdot\nabla\boldsymbol{v}_t,\boldsymbol{v})| &\leq \|\boldsymbol{v}_t\|_4\|\boldsymbol{v}\|_4\|\nabla\boldsymbol{v}_t\|_2 \leq \|\boldsymbol{v}_t\|_6^{\frac{3}{4}}\|\boldsymbol{v}_t\|_2^{\frac{1}{4}}\|\boldsymbol{v}\|_4\|\nabla\boldsymbol{v}_t\|_2\\ &\leq C_{10}\,\|\nabla\boldsymbol{v}_t\|_2^{\frac{7}{4}}\|\boldsymbol{v}_t\|_2^{\frac{1}{4}}\|\nabla\boldsymbol{v}\|_2 \leq \frac{C_1}{2C_2}\|\nabla\boldsymbol{v}_t\|_2^2 + C_{11}\|\boldsymbol{v}_t\|_2^2\|\nabla\boldsymbol{v}\|_2^8\,.\end{aligned}$$

Replacing this inequality back into (5.114) we deduce

$$\frac{\partial E_1}{\partial t}+\frac{C_1}{2}\|\nabla \boldsymbol{v}_t\|_2^2\le C_{12}\left[\omega^2+\|\boldsymbol{v}\|_2^2+\left(\|\boldsymbol{v}\|_2+\|\nabla\boldsymbol{v}\|_2^8\right)\|\boldsymbol{v}_t\|_2^2\right]\,, \tag{5.119}$$

However, $\boldsymbol{v}$ and ω must satisfy (5.16), so that, integrating both sides of (5.119) over (t_0, t_0+T^*) and using (5.111), (5.113), and (5.117) we infer

$$\boldsymbol{v}_t\in L^\infty(t_0,t_0+T^*;H(\mathcal{C}))\,. \tag{5.120}$$

Furthermore, as a consequence of (2.24), (5.20), and (5.23) with $\lambda=\mu/(2C\,C_3)$, we have

$$\|\boldsymbol{v}(t)\|_{2,2}\le C_{13}\left(\|\nabla\boldsymbol{v}(t)\|_2^3+\|\boldsymbol{v}_t(t)\|_2+1\right),\ \text{ for a.a. } t\in[t_0,t_0+T^*)\,. \tag{5.121}$$

Therefore, this inequality along with (5.120) allows us to conclude

$$\boldsymbol{v}\in L^\infty(t_0,t_0+T^*;W^{2,2}(\mathcal{C}))\,,$$

which, in turn, implies $T^*=\infty$. We now go back to (5.119) - valid for all $t\in(t_0,\infty)$ - and use (2.23) to show

$$\frac{\partial E_1}{\partial t}+C_{14}\left[1-\left(\|\boldsymbol{v}\|_2+\|\nabla\boldsymbol{v}\|_2^8\right)\right]\,\|\boldsymbol{v}_t\|_2^2\le C_{12}\left(\omega^2+\|\boldsymbol{v}\|_2^2\right)\,. \tag{5.122}$$

By (5.17) we may find $t_1\ge t_0$ such that

$$C_{14}\left(1-\|\boldsymbol{v}\|_2-\|\nabla\boldsymbol{v}\|_2^8\right)\ge C_{15}\,,\quad \text{for all } t\ge t_1\,,$$

which, once replaced into (5.122), with the help of (5.113) delivers

$$\frac{\partial E_1}{\partial t}+C_{15}\,E_1\le F(t)\,, \tag{5.123}$$

where $F(t):=C_{12}\left(\omega^2(t)+\|\boldsymbol{v}(t)\|_2^2\right)$. Notice that, by Proposition 5.1.5 and Proposition 5.3.2

$$\lim_{t\to\infty}F(t)=0\,. \tag{5.124}$$

Multiplying both sides of (5.123) by $\mathrm{e}^{C_{15}t}$ and integrating the resulting equation over $(t/2,t)$, using again (5.113) we get, for all $t \geq 2t_1$, [5]

$$\|\boldsymbol{v}_t(t)\|_2^2 \leq C_{16}\left(\|\boldsymbol{v}_t(t/2)\|_2^2\,\mathrm{e}^{-C_{15}t/2} + \int_{t/2}^{t} \mathrm{e}^{-C_{15}(t-s)}F(s)ds\right).$$

Employing in this relation (5.120) (valid with $T^* = \infty$) and (5.124) we then show that for any $\varepsilon > 0$ there is $\bar{t} > 0$ such that

$$\|\boldsymbol{v}_t(t)\|_2^2 \leq C_{17}\,\mathrm{e}^{-C_{15}t/2} + \varepsilon\,\frac{C_{16}}{C_{15}}\,, \quad \text{for all } t \geq \bar{t}\,,$$

namely,

$$\lim_{t\to\infty} \|\boldsymbol{v}_t(t)\|_2 = 0\,. \tag{5.125}$$

As a result, (5.108) follows from (5.17), (5.121) (valid with $T^* = \infty$), (5.110), and (5.125). This concludes the proof of the theorem. ■

Remark 5.3.4. From Theorem 5.3.3 and $(5.97)_{4,5}$ it also follows that

$$\lim_{t\to\infty} |\dot{\omega}(t)| = \lim_{t\to\infty} |\dot{\boldsymbol{\gamma}}(t)| = 0\,.$$

5.3.2 Attainability and stability of the equilibrium configurations.

The results proved in Theorem 5.3.3 imply that the coupled system solid-liquid $\mathcal{S}$ will eventually reach an equilibrium configuration where the liquid is at rest, and the center of mass G of $\mathcal{S}$ is on the vertical axis passing through the point of suspension O. However, the theorem does not specify whether G lies above O (i.e., $\boldsymbol{\gamma} = -\boldsymbol{e}_1$), or below O (i.e., $\boldsymbol{\gamma} = \boldsymbol{e}_1$). The objective of this subsection is to show that, under suitable conditions on the initial data, $\mathcal{S}$ will reach the equilibrium configuration where G is in its lowest position (i.e., $\boldsymbol{\gamma} = \boldsymbol{e}_1$). It is worth observing that if $\mathcal{S}$ is initially released from rest, the above conditions are certainly satisfied. More specifically, we have the following.

[5] Observe that by (5.112) and the property just shown, it follows that the function $t \to \|\boldsymbol{v}_t(t)\|_2$ is absolutely continuous for all "large" t.

Theorem 5.3.5. *Let $\mathcal{C}$ be of class C^2, and let $(\boldsymbol{v}_0, \omega_0, \boldsymbol{\gamma}_0) \in H(\mathcal{C}) \times \mathbb{R} \times \mathsf{S}^1$ be given with*

$$\rho \, \|\boldsymbol{v}_0\|_2^2 + C\,(\omega_0 - a(0))^2 < 2\,C\,\beta^2\,(1 + \gamma_{1,0})\,. \tag{5.126}$$

Then all weak solutions corresponding to initial data $(\boldsymbol{v}_0, \omega_0, \boldsymbol{\gamma}_0)$ tend to the equilibrium configuration $(\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv \boldsymbol{e}_1)$, namely, the one where the center of mass lies in its lowest position.

Proof. Suppose, by contradiction, that the final equilibrium position is, instead, $(\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv -\boldsymbol{e}_1)$. Then, passing to the limit $t \to \infty$ in the energy inequality (5.103) with $s = 0$, and taking into account Theorem 5.3.3, we find, in particular,

$$2C\beta^2 + 2\mu \int_0^\infty \|\nabla \boldsymbol{v}(t)\|_2^2 dt \le \rho \|\boldsymbol{v}_0\|_2^2 + C(\omega_0 - a(0))^2 - 2C\beta^2\gamma_{1,0}\,,$$

which cannot be true whenever the initial data satisfy (5.126). ■

Also with the help of the previous result, we may prove the following one.

Theorem 5.3.6. *Suppose $\mathcal{C}$ of class C^2. Then the equilibrium configuration $\mathsf{c}_1 := (\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv -\boldsymbol{e}_1)$, namely, the one where the center of mass lies in its highest position, is unstable in the sense of Lyapunov in the class of weak solutions, whereas the configuration $\mathsf{c}_2 := (\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv \boldsymbol{e}_1)$, where the center of mass lies in its lowest positionis stable.*

Proof. Consider a weak solution corresponding to the initial data $\boldsymbol{v}(0) = \boldsymbol{0}$, $\omega(0) = 0$ and $\boldsymbol{\gamma}(0) = -\cos\delta \boldsymbol{e}_1 + \sin\delta \boldsymbol{e}_2$, $\delta \neq 0$. Since these data satisfy (5.126), any corresponding weak solution will tend to the equilibrium $(\boldsymbol{v} \equiv \boldsymbol{0}, \omega \equiv 0, \boldsymbol{\gamma} \equiv \boldsymbol{e}_1)$, no matter how close δ to zero, namely, no matter how close the initial conditions to the configuration c_1. This shows the claimed instability property.

Next, let $(\boldsymbol{v}, \omega, \boldsymbol{\gamma}')$ denote a perturbation to the configuration c_2 in the class of weak solutions. This means that $(\boldsymbol{v}, \omega, \boldsymbol{e}_1 + \boldsymbol{\gamma}')$ is a weak solution to (5.97) corresponding to initial data, say, $(\boldsymbol{v}_0, \omega_0, \boldsymbol{e}_1 + \boldsymbol{\gamma}'_0)$. From the strong energy inequality (5.103) and (5.102) we at once deduce that, for all $t \ge 0$,

$$\begin{aligned} c_0\, \left[\rho \|\boldsymbol{v}(t)\|_2^2 + C(\omega(t) - a(t))^2\right] - 2C\beta^2\gamma'_1(t) \\ \le \left[\rho \|\boldsymbol{v}_0\|_2^2 + C(\omega_0 - a(0))^2\right] - 2C\beta^2\gamma'_{1,0}\,. \end{aligned} \tag{5.127}$$

Moreover, from the condition $|\boldsymbol{e}_1 + \boldsymbol{\gamma}'(t)|^2 = 1$, all $t \geq 0$, we find

$$-2\gamma_1'(t) = (\gamma_1'(t))^2 + (\gamma_2'(t))^2\,, \quad \text{all } t \geq 0\,. \tag{5.128}$$

From (5.127) and (5.128), and recalling (5.98), we immediately deduce that c_2 is stable in the sense of Lyapunov, namely, for any given $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\|\boldsymbol{v}_0\|_2 + |\omega_0| + |\boldsymbol{\gamma}_0'| < \delta \implies \|\boldsymbol{v}(t)\|_2 + |\omega(t)| + |\boldsymbol{\gamma}'(t)| < \varepsilon\,, \quad \text{for all } t > 0\,.$$

The proof of the theorem is completed. ■

We have shown that a physical pendulum containing an interior cavity entirely filled with a viscous (Navier-Stokes) liquid must eventually go to an equilibrium state where the liquid is at rest and the center of mass of the system occupies its highest (configuration c_1) or lowest (configuration c_2) position. Moreover, we have proved that the former is unstable, while the latter is stable, and also attainable provided the initial data satisfy (5.126).

The following two interesting questions are, however, left open.

(i) We do not know the rate at which the equilibrium configuration c_2 will be reached, at least for sufficiently large times. In fact, in analogy with similar problems of rigid bodies with a liquid-filled cavity, it is expected that the motion would be "chaotic" for some interval of time, but then, once the velocity of the liquid becomes "sufficiently small" (the latter, all other parameters kept fixed, depending on the magnitude of the viscosity), it is conjectured that the system should go to the equilibrium configuration at a very fast pace, possibly, even of exponential type.

(ii) The second open question regards whether condition (5.126) on the initial data is indeed necessary for the proof of attainability of the equilibrium configuration c_2. Actually, given the instability property of c_1, we conjecture that c_2 should be reached from "almost all" initial data (of finite energy).

6.0 FURTHER RESULTS: THE TIME-PERIODIC MOTIONS

Objective of chapter is to give a detailed analytical study of the motions of the coupled system liquid-filled rigid body about its center of mass under the action of a time-periodic torque.

With the same notations used in Chapter 2, here we suppose that $O \equiv G$ and, with respect to an inertial frame $\mathcal{I} \equiv \{G, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$, a *time-periodic torque*, M, acts on $\mathcal{B}$:

$$\mathsf{M} = f_i(t)\, \tilde{\boldsymbol{e}}_i(t)\,, \tag{6.1}$$

where f_i, $i = 1, 2, 3$, are given T-periodic scalar functions of time t, i.e. $f_i(t+T) = f_i(t)$, for all $t \in \mathbb{R}$.

In the wake of analogous classical problems formulated in absence of liquid, we propose to investigate whether, under the given assumptions, the coupled system $\mathcal{S}$ will execute a T-periodic motion in the non-inertial frame F introduced in Chapter 2, Section 2.2.

In order to handle the above question, it appears necessary to impose some restrictions on the functions f_i in (6.1), as we shall show next.

We begin to observe that, as we have seen in Section 2.2, in the frame F, the torque M can be rewritten as follows

$$\mathsf{m} := \boldsymbol{Q}^T(t) \cdot \mathsf{M} = f_i(t)\, \boldsymbol{Q}^\top(t) \cdot \tilde{\boldsymbol{e}}_i\,, \tag{6.2}$$

where $\boldsymbol{Q} = \boldsymbol{Q}(t)$ is the (unknown) one-parameter family of elements of the special orthogonal group, $\mathsf{SO}(3)$, associated with the change of frame $\mathcal{I} \to \mathsf{F}$, and introduced in Section 2.2.

We recall that $\boldsymbol{Q}$ has to satisfy (2.10)$_4$:

$$\frac{d\boldsymbol{Q}^T}{dt} = \mathsf{A}(\boldsymbol{\omega})\cdot\boldsymbol{Q}^T \;\;,\; \boldsymbol{Q}^T(0) = \boldsymbol{1}\,, \quad \mathsf{A}(\boldsymbol{\omega}) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \tag{6.3}$$

Assuming the motion of $\mathcal{S}$ in the frame F to be T-periodic implies, in particular, that both $\boldsymbol{\omega}$ and $\mathbf{m}$ have to be T-periodic as well. From (6.3) and standard Floquet theory it follows that $\boldsymbol{Q}(t)$ has the following representation [53, Theorem 1]

$$\boldsymbol{Q}^T(t) = \boldsymbol{P}(t)\cdot e^{t\boldsymbol{S}}\,,\quad t\in\mathbb{R}\,, \tag{6.4}$$

where $\boldsymbol{S}$ is a real, skew-symmetric matrix, and

$$\boldsymbol{P}(t)\in\mathsf{SO(3)}\,,\quad \boldsymbol{P}(t+T) = \boldsymbol{P}(t)\,,\ \text{for all } t\in\mathbb{R}\,. \tag{6.5}$$

From the latter and from (6.2) (taken component by component), we then deduce that for $\mathbf{m}$ to be a T-periodic function we must have

$$e^{T\boldsymbol{S}}\cdot\tilde{\mathbf{e}}_i = \tilde{\boldsymbol{e}}_i\,,\quad i=1,2,3\,,$$

namely, $\tilde{\boldsymbol{e}}_i$ must be parallel to the eigenvectors corresponding to the eigenvalue $\lambda=0$ of $\boldsymbol{S}$. Since, in general, $\lambda=0$ is simple, the existence of a T-periodic solution in the moving frame F requires, in general, that $\mathbf{M}$ is directed along a *constant* direction. We thus have

$$\mathbf{M} = f(t)\,\boldsymbol{h}\,, \tag{6.6}$$

where $\boldsymbol{h}$ is a unit, time-independent vector in $\mathcal{I}$, and f is a T-periodic function.

We next observe that, denoting by $\boldsymbol{K}_G$ the total angular momentum of $\mathcal{S}$ with respect to G, the balance of angular momentum in the frame $\mathcal{I}$ requires

$$\frac{d}{dt}\boldsymbol{K}_G = f(t)\,\boldsymbol{h}\,, \tag{6.7}$$

from which we at once deduce that $|\boldsymbol{K}_G(t)|$ is T-periodic if and only if f has a zero average over a period:

$$\int_0^T f(t)\,dt = 0\,. \tag{6.8}$$

In fact, if f has a zero average over a period, from the balance of the total angular momentum, it follows that $K_G(t)$ is T-periodic, thus implying that also its modulus is T-periodic. To show that (6.8) is a necessary condition for the T-periodicity of $|\boldsymbol{K}_G(t)|$, let us argue by contradiction, and assume that

$$\int_0^T f(t)\,dt = c \neq 0.$$

Integrating (6.7) over $[0, nT]$, with $n \in \mathbb{N}$, and taking the modulus of the resulting equation, using the fact that $\boldsymbol{h}$ is a unit, time-independent vector together with the Triangle Inequality, we find that

$$n|c| = \left|\boldsymbol{h}\int_0^{nT} f(t)\;dt\right| = |\boldsymbol{K}_G(nT) - \boldsymbol{K}_G(0)| \leq |\boldsymbol{K}_G(nT)| + |\boldsymbol{K}_G(0)|.$$

The contradiction then arises by taking the limit as $n \to \infty$ in the latter displayed inequality. However, $|\boldsymbol{K}_G(t)|$ is invariant by the frame change $\mathcal{I} \to \mathsf{F}$, so that the searched T-periodicity of the motion of $\mathcal{S}$ with respect to the frame F requires that f obeys (6.8). As a consequence of what just shown, we shall then suppose that the torque M acting on $\mathcal{B}$ satisfies (6.6)–(6.8).

Under these assumptions, the main goal of this chapter consists in proving the existence of a motion of the coupled system $\mathcal{S}$ that is time-periodic with respect to the moving frame F. It is worth remarking that in F the direction of the torque becomes a function of time given by $\boldsymbol{H}(t) := \boldsymbol{Q}^T(t) \cdot \boldsymbol{h}$, and since $\boldsymbol{Q}$ is not known, $\boldsymbol{H}$ becomes a further unknown of the problem at hand. From the physical viewpoint, the latter circumstance means that, in order to perform such a periodic motion, the body has to find an "appropriate orientation" with respect to the direction of the given torque M.

We thus show that, under the hypothesis that f is T-periodic and square-summable over a period, the problem admits a corresponding (suitably defined) T-periodic *weak* solution. If, moreover, f is essentially bounded with a sufficiently small norm, then the solution is *strong* and the relevant equations are satisfied almost everywhere in space-time.

Under the above mentioned hypotheses, the equations of motions of $\mathcal{S}$, (2.10), now read as follows

$$\left.\begin{aligned} &\operatorname{div} \boldsymbol{v} = 0 \\ &\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \dot{\boldsymbol{\omega}} \times \boldsymbol{y} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\boldsymbol{\omega} \times \boldsymbol{v}\right) = -\nabla p + \mu \Delta \boldsymbol{v} \end{aligned}\right\} \text{ in } \mathcal{C} \times (0, \infty),$$

$$\begin{aligned} &\frac{d\boldsymbol{A}}{dt} + \boldsymbol{\omega} \times \boldsymbol{A} = f(t)\boldsymbol{H}, \\ &\frac{d\boldsymbol{H}}{dt} + \boldsymbol{\omega} \times \boldsymbol{H} = \boldsymbol{0}, \\ &\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{C}, \\ &\boldsymbol{v}(t+T) = \boldsymbol{v}(t), \quad \boldsymbol{\omega}(t+T) = \boldsymbol{\omega}(t), \quad \boldsymbol{H}(t+T) = \boldsymbol{H}(t), \quad \text{all } t \geq 0. \end{aligned} \tag{6.9}$$

We recall that $\boldsymbol{A}$ has been defined in (2.11) (in this case $O \equiv G$, and we have dropped the dependence on the pole in the notation):

$$\boldsymbol{A} = \boldsymbol{I} \cdot \boldsymbol{\omega} + \int_{\mathcal{C}} \rho \boldsymbol{y} \times \boldsymbol{v}, \tag{6.10}$$

and $\boldsymbol{I}$ is the *total inertial tensor* of $\mathcal{S}$ with respect to G.

Using (6.10) we can then eliminate $\boldsymbol{\omega}$ and write the relevant equations only in terms of the unknowns $\boldsymbol{v}, p, \boldsymbol{A}$, and $\boldsymbol{H}$. Thus, observing that

$$\boldsymbol{\omega} = \boldsymbol{I}^{-1} \cdot \boldsymbol{A} - \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \rho \boldsymbol{y} \times \boldsymbol{v}\right), \tag{6.11}$$

the system of equations (6.9) with (6.10) becomes (see also [26]):

$$\left.\begin{aligned} &(\boldsymbol{1} - \boldsymbol{B}) \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{I}^{-1} \cdot \frac{d\boldsymbol{A}}{dt}\right) \times \boldsymbol{y} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\left(\boldsymbol{I}^{-1} \cdot \boldsymbol{A}\right) \times \boldsymbol{v} \\ &\qquad -2\left[\boldsymbol{I}^{-1} \cdot \left(\rho \int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}\right)\right] \times \boldsymbol{v} + \tfrac{1}{\rho} \nabla p - \nu \Delta \boldsymbol{v} = \boldsymbol{0} \\ &\operatorname{div} \boldsymbol{v} = 0, \end{aligned}\right\} \text{ in } \mathcal{C} \times [0, T],$$

$$\boldsymbol{v}(x, t) = \boldsymbol{0} \quad \text{on } \partial\mathcal{C} \times [0, T],$$

$$\left.\begin{aligned} &\frac{d\boldsymbol{A}}{dt} + (\boldsymbol{I}^{-1} \cdot \boldsymbol{A}) \times \boldsymbol{A} - \rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}\right) \times \boldsymbol{A} = f(t)\boldsymbol{H} \\ &\frac{d\boldsymbol{H}}{dt} + (\boldsymbol{I}^{-1} \cdot \boldsymbol{A}) \times \boldsymbol{H} - \rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}\right) \times \boldsymbol{H} = 0 \end{aligned}\right\} \text{ in } [0, T], \tag{6.12}$$

where the operator $\boldsymbol{B}$ has been defined in (2.25)-(2.26), it satisfies Lemma 2.3.3 along with the properties following it. We recall that $\nu = \mu/\rho$ is the coefficient of kinematic viscosity of the liquid.

Our problem can be then formulated as follows: *given a sufficiently smooth T-periodic function f, find a corresponding T-periodic solution to* (6.12). Our investigation will be carried out in the very general class of *weak solutions* to (6.12), defined in the following definition.

Definition 6.0.7. *A triple* $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ *is a* T-periodic*, or simply* periodic weak solution *to the problem* (6.12)*,* (2.25)*–*(2.26) *if it satisfies the following conditions.*

1. $\boldsymbol{v} \in L^2(0,T;H^1(\mathcal{C})) \cap L^\infty(0,T;H(\mathcal{C}))$*,* $\boldsymbol{A}, \boldsymbol{H} \in C_T(\mathbb{R})$*;*
2. $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ *satisfies the following equations*

$$\begin{aligned}\int_0^T \left[\langle \boldsymbol{v}, \boldsymbol{\psi}\rangle + \boldsymbol{A}\cdot \boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}} \boldsymbol{y}\times\boldsymbol{\psi}\right)\right]\frac{d\xi(t)}{dt}\,dt \\ = -\Bigg[-2\rho\int_0^T\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}\right)\cdot\boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}}\boldsymbol{v}\times\boldsymbol{\psi}\right)\xi(t)\,dt \\ + 2\int_0^T\left(\int_{\mathcal{C}}\boldsymbol{A}\cdot\boldsymbol{I}^{-1}\cdot(\boldsymbol{v}\times\boldsymbol{\psi})\right)\xi(t)\,dt \\ + \int_0^T(\boldsymbol{v}\cdot\nabla\boldsymbol{\psi},\boldsymbol{v})\xi(t)\,dt - \nu\int_0^T(\nabla\boldsymbol{v},\nabla\boldsymbol{\psi})\xi(t)\,dt\Bigg],\end{aligned} \tag{6.13}$$

for all $\boldsymbol{\psi}\in\mathcal{D}(\mathcal{C})$*,* $\xi\in C_T^\infty(\mathbb{R})$ *; and for all* $t\in[0,T]$

$$\boldsymbol{A}(t) = \boldsymbol{A}(0) - \int_0^t(\boldsymbol{I}^{-1}\cdot\boldsymbol{A})\times\boldsymbol{A}\,d\tau + \rho\boldsymbol{I}^{-1}\cdot\int_0^t\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}\right)\times\boldsymbol{A}\,d\tau + \int_0^t f(\tau)\boldsymbol{H}\,d\tau, \tag{6.14}$$

$$\boldsymbol{H}(t) = \boldsymbol{H}(0) - \int_0^t(\boldsymbol{I}^{-1}\cdot\boldsymbol{A})\times\boldsymbol{H}\,d\tau + \rho\boldsymbol{I}^{-1}\cdot\int_0^t\left(\int_{\mathcal{C}}\boldsymbol{y}\times\boldsymbol{v}\right)\times\boldsymbol{H}\,d\tau. \tag{6.15}$$

Remark 6.0.8. A weak solution has, in fact, more regularity in time than the one stated in the above definition. In fact, on the one hand, from (6.14) and (6.15) we deduce $\boldsymbol{A},\ \boldsymbol{H}\in W^{1,r}(0,T)$, provided $f\in L^r_T(\mathbb{R})$, $r\in[1,\infty]$. On the other hand, proceeding in a similar fashion as in [15, Lemma 2.2], and [21], one can show that $\boldsymbol{v}(\cdot,t)$ is continuous in $[0,T]$ weakly in $L^2(\mathcal{C})$, and strongly in $H^{-1}(\mathcal{C})$. As a consequence, with the help of (6.13) it follows that $\boldsymbol{v}$ is indeed periodic in time in the sense of the above topologies.

Remark 6.0.9. If $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ is a periodic weak solution to (6.12), then the corresponding angular velocity is defined via (6.11) and belongs to $C_T(\mathbb{R})$; see also Remark 6.0.8.

Remark 6.0.10. If $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ is a periodic weak solution to (6.12) and is sufficiently regular then, by a standard procedure one shows that there exists a scalar field $p = p(\boldsymbol{y}, t)$ such that $\boldsymbol{v}, p$, and $\boldsymbol{A}$ satisfy $(6.12)_1$ a.e. in space and time.

6.1 EXISTENCE OF PERIODIC WEAK SOLUTIONS

Objective of this section is to show the existence of a periodic weak solution to (6.12) under suitable assumptions on f. This will be achieved by combining the Faedo-Galerkin method with a fixed point argument. Specifically, we have the following.

Theorem 6.1.1. *Let $f \in L^2_T(\mathbb{R})$ satisfy (6.8), and let $\mathcal{C}$ be a domain of $\mathbb{R}^3$. Then, there exists at least one periodic weak solution, $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$, to (6.12).*

Proof. Let $\{\boldsymbol{\psi}_n\}_{n\in\mathbb{N}}$ be a denumerable subset of $\mathcal{D}(\mathcal{C})$ whose linear hull is dense in $H^1(\mathcal{C})$, and let us normalize it as $\langle \boldsymbol{\psi}_n, \boldsymbol{\psi}_m \rangle = \delta_{nm}$ [1]. We look for "approximate solutions" of the type

$$\boldsymbol{v}_n(\boldsymbol{y}, t) := \sum_{k=1}^{n} c_{nk}(t)\boldsymbol{\psi}_k(\boldsymbol{y}), \quad \boldsymbol{A}_n := \sum_{i=1}^{3} \tilde{c}_{ni}(t)\boldsymbol{e}_i, \quad \boldsymbol{H}_n := \sum_{j=1}^{3} \hat{c}_{nj}(t)\boldsymbol{e}_k.$$

The coefficients c_{nk}, $\tilde{c}_{ni}$ and $\hat{c}_{nj}$ are found by solving the following system of ordinary differential equations:

$$\begin{aligned}
&\frac{d}{dt}\left[\langle \boldsymbol{v}_n, \boldsymbol{\psi}_r \rangle + \boldsymbol{A}_n \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r\right)\right] = -2\rho\left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n\right) \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{v}_n \times \boldsymbol{\psi}_r\right) \\
&\qquad\qquad + 2\int_{\mathcal{C}} \boldsymbol{A}_n \cdot \boldsymbol{I}^{-1} \cdot (\boldsymbol{v}_n \times \boldsymbol{\psi}_r) \; + (\boldsymbol{v}_n \cdot \nabla \boldsymbol{\psi}_r, \boldsymbol{v}_n) - \nu(\nabla \boldsymbol{v}_n, \nabla \boldsymbol{\psi}_r), \\
&\frac{d\boldsymbol{A}_n}{dt} = -(\boldsymbol{I}^{-1} \cdot \boldsymbol{A}_n) \times \boldsymbol{A}_n + \rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n\right) \times \boldsymbol{A}_n + f(t)\boldsymbol{H}_n, \\
&\frac{d\boldsymbol{H}_n}{dt} = -(\boldsymbol{I}^{-1} \cdot \boldsymbol{A}_n) \times \boldsymbol{H}_n + \rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n\right) \times \boldsymbol{H}_n,
\end{aligned} \tag{6.16}$$

[1] The scalar product $\langle \cdot, \cdot \rangle$ has been defined in (2.27).

which, in terms of c_{nk}, $\tilde{c}_{ni}$ and $\hat{c}_{nj}$ reads as follows

$$
\begin{aligned}
&\frac{dc_{nr}}{dt} + \frac{d\tilde{c}_{ni}}{dt}\boldsymbol{e}_i \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right) = -2\rho \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{\psi}_m \times \boldsymbol{\psi}_r \right) c_{nk} c_{nm} \\
&+ 2 \left[\int_{\mathcal{C}} \boldsymbol{e}_i \cdot \boldsymbol{I}^{-1} \cdot (\boldsymbol{\psi}_k \times \boldsymbol{\psi}_r) \right] \tilde{c}_{ni} c_{nk} + (\boldsymbol{\psi}_k \cdot \nabla \boldsymbol{\psi}_r, \boldsymbol{\psi}_m) c_{nk} c_{nm} - \nu (\nabla \boldsymbol{\psi}_k, \nabla \boldsymbol{\psi}_r) c_{nk}, \\
&\frac{d\tilde{c}_{ni}}{dt}\boldsymbol{e}_i = -\tilde{c}_{ni}\tilde{c}_{nj}(\boldsymbol{I}^{-1} \cdot \boldsymbol{e}_i) \times \boldsymbol{e}_j + c_{nk}\tilde{c}_{ni}\rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \times \boldsymbol{e}_i + f(t)\hat{c}_{nj}\boldsymbol{e}_j, \\
&\frac{d\hat{c}_{nj}}{dt}\boldsymbol{e}_j = -\tilde{c}_{ni}\hat{c}_{nj}(\boldsymbol{I}^{-1} \cdot \boldsymbol{e}_i) \times \boldsymbol{e}_j + c_{nk}\hat{c}_{nj}\rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \times \boldsymbol{e}_j.
\end{aligned}
\tag{6.17}
$$

By replacing the second equation in the first one, we get

$$
\begin{aligned}
\frac{dc_{nr}}{dt} = &\left[(\boldsymbol{I}^{-1} \cdot \boldsymbol{e}_i) \times \boldsymbol{e}_j \right] \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right) \tilde{c}_{ni}\tilde{c}_{nj} \\
&- \rho \left[\boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \times \boldsymbol{e}_i \right] \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right) c_{nk}\tilde{c}_{ni} \\
&- f(t)\boldsymbol{e}_j \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right) \hat{c}_{nj} \\
&- 2\rho \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{\psi}_m \times \boldsymbol{\psi}_r \right) c_{nk} c_{nm} \\
&+ 2 \left[\int_{\mathcal{C}} \boldsymbol{e}_i \cdot \boldsymbol{I}^{-1} \cdot (\boldsymbol{\psi}_k \times \boldsymbol{\psi}_r) \right] \tilde{c}_{ni} c_{nk} \\
&+ (\boldsymbol{\psi}_k \cdot \nabla \boldsymbol{\psi}_r, \boldsymbol{\psi}_m) c_{nk} c_{nm} - \nu (\nabla \boldsymbol{\psi}_k, \nabla \boldsymbol{\psi}_r) c_{nk}.
\end{aligned}
$$

Setting

$$
\begin{aligned}
b_{rij} &:= \left[(\boldsymbol{I}^{-1} \cdot \boldsymbol{e}_i) \times \boldsymbol{e}_j \right] \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right), \\
d_{rki} &:= -\rho \left[\boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \times \boldsymbol{e}_i \right] \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right), \\
f_{rj}(t) &:= -f(t)\boldsymbol{e}_j \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right), \\
g_{rkm} &:= -2\rho \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{\psi}_m \times \boldsymbol{\psi}_r \right), \\
s_{rki} &:= 2 \left[\int_{\mathcal{C}} \boldsymbol{e}_i \cdot \boldsymbol{I}^{-1} \cdot (\boldsymbol{\psi}_k \times \boldsymbol{\psi}_r) \right], \\
p_{rkm} &:= (\boldsymbol{\psi}_k \cdot \nabla \boldsymbol{\psi}_r, \boldsymbol{\psi}_m), \qquad p_{rk} := -\nu (\nabla \boldsymbol{\psi}_k, \nabla \boldsymbol{\psi}_r) c_{nk},
\end{aligned}
$$

equation $(6.17)_1$ becomes

$$\frac{dc_{nr}}{dt} = b_{rij}\tilde{c}_{ni}\tilde{c}_{nj} + d_{rki}c_{nk}\tilde{c}_{ni} + f_{rj}(t)\hat{c}_{nj} + g_{rkm}c_{nk}c_{nm} + s_{rik}\tilde{c}_{ni}c_{nk} + p_{rkm}c_{nk}c_{nm} + p_{rk}c_{nk}, \tag{6.18}$$

where here and in the rest of the proof i, j, ℓ vary in the set $\{1, 2, 3\}$, whereas r, k, m in the set $\{1, \ldots, n\}$. Concerning $(6.17)_2$ and $(6.17)_3$, taking the dot product of each side of both with $\boldsymbol{e}_\ell$, and setting

$$u_{\ell ij} := -\left[(\boldsymbol{I}^{-1} \cdot \boldsymbol{e}_i) \times \boldsymbol{e}_j\right] \cdot \boldsymbol{e}_\ell, \qquad w_{\ell kj} := \rho \boldsymbol{e}_\ell \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_k \right) \times \boldsymbol{e}_j,$$

we deduce

$$\frac{d\tilde{c}_{n\ell}}{dt} = u_{\ell ij}\tilde{c}_{ni}\tilde{c}_{nj} + w_{\ell ki}c_{nk}\tilde{c}_{ni} + f(t)\hat{c}_{n\ell}, \tag{6.19}$$

and

$$\frac{d\hat{c}_{n\ell}}{dt} = u_{\ell ij}\tilde{c}_{ni}\hat{c}_{nj} + w_{\ell kj}c_{nk}\hat{c}_{nj}, \tag{6.20}$$

respectively. Following [37] we shall next prove that there exist initial data such that the system of ordinary differential equations (6.18), (6.19), and (6.20) admits a corresponding solution $(c_{nr}, \tilde{c}_{n\ell}, \hat{c}_{n\ell})$ such that $c_{nr}(0) = c_{nr}(T)$, $\tilde{c}_{nr}(0) = \tilde{c}_{nr}(T)$ and $\hat{c}_{nr}(0) = \hat{c}_{nr}(T)$. To reach this goal, let

$$\boldsymbol{v}_{n,0} \in span\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n\}\,,\ \boldsymbol{A}_{n,0} \in \mathbb{R}^3\,,\ \boldsymbol{H}_{n,0} \in \mathsf{S}^2\,,$$

and set $c_{nr}(0) = c_{nr,0} := \langle \boldsymbol{v}_{n,0}, \boldsymbol{\psi}_r \rangle$, $\tilde{c}_{n\ell}(0) = \tilde{c}_{n\ell,0} := \boldsymbol{A}_{n,0} \cdot \boldsymbol{e}_\ell$, and $\hat{c}_{n\ell}(0) = \hat{c}_{n\ell,0} := \boldsymbol{H}_{n,0} \cdot \boldsymbol{e}_\ell$. Since $f, f_{r,j} \in C_T(\mathbb{R})$, by Picard theorem, there exists a unique solution, $(c_{nr}, \tilde{c}_{n\ell}, \hat{c}_{n\ell})$, to the Cauchy problem associated to (6.18), (6.19) and (6.20) with $c_{nr}, \tilde{c}_{n\ell}, \hat{c}_{n\ell} \in C^1(0, T')$, $r = 1, \ldots, n$, $\ell = 1, 2, 3$, where $0 < T' \leq T$. Multiplying both sides of (6.18) by c_{nr}, summing over $r = 1, \ldots, n$, and noticing that the terms corresponding to $d_{rki}, g_{rkm}, s_{rki}$ and p_{rkm} vanish, we get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_B^2 + \nu\|\nabla \boldsymbol{v}_n\|_2^2 = \rho\left[(\boldsymbol{I}^{-1} \cdot \boldsymbol{A}_n) \times \boldsymbol{A}_n\right] \cdot \boldsymbol{I}^{-1} \cdot \int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n - f(t)\boldsymbol{H}_n \cdot \boldsymbol{I}^{-1} \cdot \int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n\,. \tag{6.21}$$

Next, multiplying both sides of (6.19) by $\tilde{c}_{n\ell}$, summing over $\ell = 1, 2, 3$, and taking into account that the terms corresponding to $u_{\ell ij}$ and $w_{\ell kj}$ vanish, it follows that

$$\frac{1}{2}\frac{d|\boldsymbol{A}_n|^2}{dt} = f(t)\boldsymbol{H}_n \cdot \boldsymbol{A}_n. \tag{6.22}$$

Finally, multiplying both sides of (6.20) by $\hat{c}_{n\ell}$ and summing over $\ell = 1, 2, 3$, similarly as above, we get

$$\frac{d|\boldsymbol{H}_n|^2}{dt} = 0\,, \tag{6.23}$$

which implies $|\boldsymbol{H}_n(t)| = |\boldsymbol{H}_{n,0}| = 1$. Thus, from (6.22) and (6.23) we infer

$$\frac{d|\boldsymbol{A}_n|}{dt} \leq |f(t)| \tag{6.24}$$

and

$$|\boldsymbol{A}_n(t)| \leq |\boldsymbol{A}_{n,0}| + \int_0^T |f(\tau)|\, d\tau, \qquad \text{for all } t \in [0, T'), n \in \mathbb{N}\,. \tag{6.25}$$

Moreover, by Poincaré, Schwarz, and Young inequalities, from (6.21) we deduce, on the one hand,

$$\frac{1}{2}\frac{d\,\|\boldsymbol{v}_n\|_B^2}{dt} + C_{1p}\,\|\nabla\boldsymbol{v}_n\|_2^2 \leq C_1|\boldsymbol{A}_n|^4 + C_2|f(t)|^2, \tag{6.26}$$

and, on the other hand, using one more time Poincaré inequality in conjunction with (2.28),

$$\frac{1}{2}\frac{d\,\|\boldsymbol{v}_n\|_B^2}{dt} + C_{2p}\,\|\boldsymbol{v}_n\|_B^2 \leq C_1|\boldsymbol{A}_n|^4 + C_2|f(t)|^2, \tag{6.27}$$

where $C_{ip} = C_{ip}(\mathcal{C}, \nu) > 0$, while $C_i = C_i(\mathcal{C}, \mathcal{B}, \nu) > 0$, $i = 1, 2$. Using Gronwall Lemma in (6.27) furnishes,

$$\exp(C_3 t)\,\|\boldsymbol{v}_n(t)\|_B^2 \leq \|\boldsymbol{v}_{n,0}\|_B^2 + C_1\int_0^t \exp(C_3\tau)|\boldsymbol{A}_n(\tau)|^4\, d\tau + C_2\int_0^t \exp(C_3\tau)|f(\tau)|^2\, d\tau$$

which, by (6.25), implies

$$\begin{aligned}
|c_{nr}(t)|^2 = \|\boldsymbol{v}_n(t)\|_B^2 &\leq \exp(-C_3 t)\,\|\boldsymbol{v}_{n,0}\|_B^2 + C_1\exp(-C_3 t)\int_0^t \exp(C_3\tau)|\boldsymbol{A}_n(\tau)|^4\, d\tau \\
&\quad + C_2\exp(-C_3 t)\int_0^t \exp(C_3\tau)|f(\tau)|^2\, d\tau \\
&\leq \exp(-C_3 t)\,\|\boldsymbol{v}_{n,0}\|_B^2 + C_4\sup_{t\in[0,T]}|\boldsymbol{A}_n(t)|^4 + C_2\int_0^T |f(\tau)|^2\, d\tau,
\end{aligned} \tag{6.28}$$

with C_3 and C_4 positive constants depending, at most, on $\mathcal{C}$, $\mathcal{B}$ and ν. As a result, from (6.23), (6.25), and (6.28) we conclude $T' = T$. In order to build our periodic solution, we will use a suitable fixed point argument. Let us multiply both sides of (6.19) by $\hat{c}_{n\ell}$ and sum over $\ell = 1, 2, 3$, then multiply (6.20) by $\tilde{c}_{n\ell}$ and sum over $\ell = 1, 2, 3$, and finally add the two resulting equations. We get

$$\frac{d(\boldsymbol{H}_n \cdot \boldsymbol{A}_n)}{dt} = f(t)|\boldsymbol{H}_n|^2 = f(t).$$

Since f has zero average by assumption, it follows that

$$\boldsymbol{H}_{n,0} \cdot \boldsymbol{A}_{n,0} = \boldsymbol{H}_n(T) \cdot \boldsymbol{A}_n(T). \tag{6.29}$$

Taking the cross product of $(6.16)_2$ by $\boldsymbol{H}_n$ on the left, and then that of $(6.16)_3$ by $\boldsymbol{A}_n$ on the left, and summing the two equation so obtained, we deduce

$$\frac{d(\boldsymbol{H}_n \times \boldsymbol{A}_n)}{dt} = \boldsymbol{\omega}_n \times (\boldsymbol{H}_n \times \boldsymbol{A}_n),$$

where $\boldsymbol{\omega}_n$ is given by (6.11). It then follows that

$$\frac{1}{2}\frac{d|\boldsymbol{H}_n \times \boldsymbol{A}_n|^2}{dt} = \frac{1}{2}\left(\frac{d|\boldsymbol{A}_n|^2}{dt} - \frac{d(\boldsymbol{H}_n \cdot \boldsymbol{A}_n)^2}{dt}\right) = 0$$

where we have also used the fact that $|\boldsymbol{H}_n| = 1$. From the last displayed equation and (6.29), we conclude that

$$|\boldsymbol{A}_{n,0}| = |\boldsymbol{A}_n(T)|. \tag{6.30}$$

We next fix $R_1 > 0$ and take $|\boldsymbol{A}_{n,0}| \le R_1$. By (6.30), we obtain $|\boldsymbol{A}_n(T)| \le R_1$. Combining (6.28) and (6.25), we infer

$$\begin{aligned}\|\boldsymbol{v}_n(T)\|_B^2 &\le \exp(-C_3 T)\,\|\boldsymbol{v}_{n,0}\|_B^2 + C_5\left(|\boldsymbol{A}_{n,0}|^4 + \|f\|^4_{L^1(0,T)} + \|f\|^2_{L^2(0,T)}\right)\\ &\le \exp(-C_3 T)\,\|\boldsymbol{v}_{n,0}\|_B^2 + C_5 R_1^4 + C_6,\end{aligned} \tag{6.31}$$

where $C_5 = C_5(\mathcal{C}, \mathcal{B}, \nu) > 0$, and

$$C_6 := C_5\big(\|f\|^4_{L^1(0,T)} + \|f\|^2_{L^2(0,T)}\big)\,.$$

Thus, choosing

$$R_2^2 \geq \frac{C_5 R_1^4 + C_6}{1 - \exp(-C_3 T)}, \tag{6.32}$$

from (6.31) we show that if $\|\boldsymbol{v}_{n,0}\|_B^2 \leq R_2^2$, then $\|\boldsymbol{v}_n(T)\|_B^2 \leq R_2^2$. Set $\mathbb{B} := \mathbb{B}_{R_2} \times \mathbb{B}_{R_1} \times \mathsf{S}^2$, where $\mathbb{B}_{R_i}$ denotes the ball of radius R_i in $\mathbb{R}^3$, $i = 1, 2$. Let $\Phi : \mathbb{B} \to \mathbb{B}$ be the map that takes any $(\boldsymbol{v}_{n,0}, \boldsymbol{A}_{n,0}, \boldsymbol{H}_{n,0}) \in \mathbb{B}$ to $(\boldsymbol{v}_n(T), \boldsymbol{A}_n(T), \boldsymbol{H}_n(T)) \in \mathbb{B}$, where $(\boldsymbol{v}_n(T), \boldsymbol{A}_n(T), \boldsymbol{H}_n(T))$ is the solution to (6.16) at time T. By a straightforward calculation (see e.g. [37]), one shows that Φ is continuous. Moreover, Φ is homotopic to the identity by the following homotopy

$$\mathsf{H} : \mathbb{B} \times [0,1] \to \mathbb{B}$$
$$\mathsf{H}(\boldsymbol{v}_{n,0}, \boldsymbol{A}_{n,0}, \boldsymbol{H}_{n,0}, s) := (\boldsymbol{v}_n(sT), s\boldsymbol{A}_n(T) + (1-s)\boldsymbol{A}_n(0), \boldsymbol{H}_n(sT)).$$

Notice that H is well defined, since by similar calculations which lead to the estimate (6.32), we show that $\boldsymbol{v}_n(sT) \in \mathbb{B}_{R_2}$ for all $s \in [0,1]$.[2] Since the Euler characteristic of $\mathbb{B}$ is given by

$$\chi(\mathbb{B}) = \chi(\mathbb{B}_{R_2})\chi(\mathbb{B}_{R_1})\chi(\mathsf{S}^2) = 1 \cdot 1 \cdot 2 \neq 0$$

(see [7]), by the Lefschetz-Hopf fixed-point theorem (see [7, Chapter IV, Section 23],), Φ has at least one fixed point, from which it follows that there exist $(\boldsymbol{v}_{n,0}, \boldsymbol{A}_{n,0}, \boldsymbol{H}_{n,0})$ such that the solution to (6.16), $(\boldsymbol{v}_n, \boldsymbol{A}_n, \boldsymbol{H}_n)$, starting from $(\boldsymbol{v}_{n,0}, \boldsymbol{A}_{n,0}, \boldsymbol{H}_{n,0})$ satisfies

$$\boldsymbol{v}_n(\cdot, T) = \boldsymbol{v}_{n,0}(\cdot), \quad \boldsymbol{A}_n(T) = \boldsymbol{A}_{n,0}, \quad \boldsymbol{H}_n(T) = \boldsymbol{H}_{n,0}.$$

Now, multiplying $(6.16)_1$ by $\xi \in C_T^\infty(\mathbb{R})$ and integrating over $[0,T]$, we show that $\boldsymbol{v}_n$ satisfies the following

$$\begin{aligned}
\int_0^T \left[\langle \boldsymbol{v}_n, \boldsymbol{\psi}_r \rangle + \boldsymbol{A}_n \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{\psi}_r \right) \right] \frac{d\xi(t)}{dt}\, dt& \\
= \Bigg[2\rho \int_0^T \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n \right) \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{v}_n \times \boldsymbol{\psi}_r \right) \xi(t)\, dt& \\
- 2 \int_0^T \left(\int_{\mathcal{C}} \boldsymbol{A}_n \cdot \boldsymbol{I}^{-1} \cdot (\boldsymbol{v}_n \times \boldsymbol{\psi}_r) \right) \xi(t)\, dt& \\
- \int_0^T (\boldsymbol{v}_n \cdot \nabla \boldsymbol{\psi}_r, \boldsymbol{v}_n) \xi(t)\, dt - \nu \int_0^T (\nabla \boldsymbol{v}_n, \nabla \boldsymbol{\psi}_r) \xi(t)\, dt \Bigg],&
\end{aligned} \tag{6.33}$$

[2]The same argument does not work for $\boldsymbol{A}_n$, i.e. the solution map does not necessarily lie in $\mathbb{B}_{R_1}$ for all times $t \in [0,T]$. This is the reason for which we have used the linear homotopy for the $\boldsymbol{A}_n$ component.

for all $r = 1, \dots, n$, and all $\xi \in C_T^\infty(\mathbb{R})$. Likewise, integrating $(6.16)_{2,3}$ over $[0,t]$, $t \in [0,T]$, we deduce that $\boldsymbol{A}_n$ and $\boldsymbol{H}_n$ satisfy

$$\boldsymbol{A}_n(t) = \boldsymbol{A}_n(0) - \int_0^t (\boldsymbol{I}^{-1} \cdot \boldsymbol{A}_n) \times \boldsymbol{A}_n \, d\tau + \rho \boldsymbol{I}^{-1} \cdot \int_0^t \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n \right) \times \boldsymbol{A}_n \, d\tau + \int_0^t f(\tau) \boldsymbol{H}_n \, d\tau, \tag{6.34}$$

and

$$\boldsymbol{H}_n(t) = \boldsymbol{H}_n(0) - \int_0^t (\boldsymbol{I}^{-1} \cdot \boldsymbol{A}_n) \times \boldsymbol{H}_n \, d\tau + \rho \boldsymbol{I}^{-1} \cdot \int_0^t \left(\int_{\mathcal{C}} \boldsymbol{y} \times \boldsymbol{v}_n \right) \times \boldsymbol{H}_n \, d\tau, \tag{6.35}$$

respectively. With standard techniques (see [19, Theorem 4.1] for all the details), one can finally show the existence of subsequences of $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}}$, $\{\boldsymbol{A}_n\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{H}_n\}_{n\in\mathbb{N}}$ (still denoted by $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}}$, $\{\boldsymbol{A}_n\}_{n\in\mathbb{N}}$, $\{\boldsymbol{H}_n\}_{n\in\mathbb{N}}$), and functions $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ such that

$$\begin{aligned}
&\boldsymbol{v} \in L^2(0,T;H^1(\mathcal{C})) \cap L^\infty(0,T;H(\mathcal{C})), \;\; \boldsymbol{A}, \boldsymbol{H} \in C([0,T]), \\
&\boldsymbol{v}_n \to \boldsymbol{v} \quad \text{strongly in } L^2(0,T;H(\mathcal{C})), \\
&\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^2(0,T;H^1(\mathcal{C})), \\
&\boldsymbol{v}_n(t) \rightharpoonup \boldsymbol{v}(t) \quad \text{weakly in } L^2(\mathcal{C}), \;\text{uniformly in } t \in [0,T], \\
&\boldsymbol{A}_n \to \boldsymbol{A} \quad \text{uniformly in } t \in [0,T], \\
&\boldsymbol{H}_n \to \boldsymbol{H} \quad \text{uniformly in } t \in [0,T].
\end{aligned} \tag{6.36}$$

Since $\boldsymbol{A}_n(0) = \boldsymbol{A}_n(T)$ and $\boldsymbol{H}_n(0) = \boldsymbol{H}_n(T)$, from (6.36) we can conclude also that $\boldsymbol{A}, \boldsymbol{H} \in C_T$. Finally, again employing (6.36) and taking into account the properties of $\{\boldsymbol{\psi}_n\}_{n\in\mathbb{N}}$, we can then pass to the limit in (6.33), (6.34), and (6.35) and conclude that $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ possesses all the properties of a periodic weak solution to (6.12). ■

Remark 6.1.2. For future reference, we observe that the choice of the radius R_1 in the proof of the previous theorem is completely arbitrary, provided, of course, R_2 is taken appropriately according to (6.32). This follows from the fact that $|\boldsymbol{A}_n(t)|$ is T-periodic regardless of the choice of R_1; see (6.30).

6.2 EXISTENCE OF STRONG PERIODIC SOLUTIONS

In this section we will show existence of a strong solution, provided that the data are sufficiently small.

Theorem 6.2.1. *Let $f \in L^\infty_T(\mathbb{R})$ satisfy (6.8), and assume $\mathcal{C}$ of class C^2. There is a positive constant $\kappa = \kappa(\mathcal{C}, \mathcal{B}, \mathcal{C}, T)$ such that if*

$$\|f\|_{L^\infty(0,T)} \leq \kappa \tag{6.37}$$

then there exists a periodic weak solution $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ that, in addition, enjoys the following properties

$$\boldsymbol{v} \in C([0,T], L^2(\mathcal{C})), \quad \frac{\partial \boldsymbol{v}}{\partial t} \in L^2(0,T,L^2(\mathcal{C})), \quad \nabla \boldsymbol{v} \in L^2(0,T;W^{1,2}(\mathcal{C})) \cap C([0,T],L^2(\mathcal{C})),$$
$$\boldsymbol{A} \in W^{1,\infty}(0,T;\mathbb{R}^3)\,, \quad \boldsymbol{H} \in C([0,T];\mathsf{S}^2) \cap W^{1,\infty}(0,T;\mathbb{R}^3).$$

Finally, there is $p \in L^2(0,T,W^{1,2}(\mathcal{C}))$, such that $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H}, p)$ satisfies (6.12) almost everywhere in $\mathcal{C} \times [0,T]$.

Proof. The last statement is an immediate consequence of classical results about the existence of the pressure field for Navier-Stokes equations once the above properties of $\boldsymbol{v}$ and $\boldsymbol{A}$ have been established; see, e.g., [15]. To show the latter we will use the same Galerkin method employed in the proof of Theorem 6.1.1, the only difference being that this time we choose as orthonormal base of $H(\mathcal{C})$, again denoted by $\{\boldsymbol{\psi}_n\}_{n\in\mathbb{N}}$, the one constituted by the eigenfunctions of Stokes operator, with corresponding eigenvalues denoted by $\{\lambda_n\}_{n\in\mathbb{N}}$; see, e.g., [28, Chapter 2., Section 4]. As in the proof of Theorem 6.1.1, we can thus prove the existence of approximate periodic solutions $(\boldsymbol{v}_n, \boldsymbol{A}_n, \boldsymbol{H}_n)$ satisfying (6.28), (6.25) and $\boldsymbol{H}_n(t) \in \mathsf{S}^2$ for all $t \in [0,T]$. Next, we shall show some further estimates that will lead to the improved regularity properties of the weak solution stated in the theorem. In the rest of the proof we shall denote by C_i, $i \in \mathbb{N}$, positive constants depending at most on $\mathcal{C}$, $\mathcal{B}$, R_2 and T, where R_2 is defined in (6.32). Moreover, we set $F := \|f\|_{L^\infty(0,T)}$.

We begin by considering the magnitude of both sides of $(6.17)_2$ and then use Hölder, Schwarz, and Young inequalities. Taking into account (2.28), estimates (6.28), (6.25), the fact that $|\boldsymbol{H}_n(t)| = 1$, and that $|\boldsymbol{A}_{n,0}| \le R_1$, we thus obtain

$$\left|\frac{d\boldsymbol{A}_n}{dt}\right| \le C_1\,(R_1^2 + R_1 + F^2 + F) =: C_1\,D_{R_1,F}\,. \tag{6.38}$$

Similarly, considering the magnitude of $(6.17)_3$, we show that

$$\left|\frac{d\boldsymbol{H}_n}{dt}\right| \le C_2\,(R_1 + F + 1). \tag{6.39}$$

We next multiply $(6.17)_1$ by dc_{nr}/dt and sum over $r = 1, \dots, n$, to get

$$\begin{aligned}\left\|\frac{\partial \boldsymbol{v}_n}{\partial t}\right\|_B^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}_n\|_2^2 = -\frac{d\boldsymbol{A}_n}{dt}\cdot \boldsymbol{I}^{-1}\cdot \int_{\mathcal{C}} \boldsymbol{y}\times \frac{\partial \boldsymbol{v}_n}{\partial t} - 2\boldsymbol{A}_n \cdot \boldsymbol{I}^{-1}\cdot \int_{\mathcal{C}} \boldsymbol{v}_n \times \frac{\partial \boldsymbol{v}_n}{\partial t} \\ + 2\rho\left(\int_{\mathcal{C}} \boldsymbol{y}\times \boldsymbol{v}_n\right)\cdot \boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}} \boldsymbol{v}_n \times \frac{\partial \boldsymbol{v}_n}{\partial t}\right) - \int_{\mathcal{C}} (\boldsymbol{v}_n\cdot \nabla \boldsymbol{v}_n)\cdot \frac{\partial \boldsymbol{v}_n}{\partial t}.\end{aligned} \tag{6.40}$$

Let us estimate each term on the right-hand side of this equation. By (6.38), and Cauchy-Schwarz inequality we have

$$-\frac{d\boldsymbol{A}_n}{dt}\cdot \boldsymbol{I}^{-1}\cdot \int_{\mathcal{C}} \boldsymbol{y}\times \frac{\partial \boldsymbol{v}_n}{\partial t} \le C_3 + D_{R_1,F}^2 \left\|\frac{\partial \boldsymbol{v}_n}{\partial t}\right\|_2^2 . \tag{6.41}$$

Again by Cauchy-Schwarz inequality and (6.25) we get

$$-2\boldsymbol{A}_n \cdot \boldsymbol{I}^{-1}\cdot \int_{\mathcal{C}} \boldsymbol{v}_n \times \frac{\partial \boldsymbol{v}_n}{\partial t} \le C_4 + (R_1 + F)^2 \left\|\frac{\partial \boldsymbol{v}_n}{\partial t}\right\|_2^2 \tag{6.42}$$

By the same token, we show

$$2\rho\left(\int_{\mathcal{C}} \boldsymbol{y}\times \boldsymbol{v}_n\right)\cdot \boldsymbol{I}^{-1}\cdot\left(\int_{\mathcal{C}} \boldsymbol{v}_n \times \frac{\partial \boldsymbol{v}_n}{\partial t}\right) \le \frac{C_5}{2\varepsilon} + \varepsilon\left\|\frac{\partial \boldsymbol{v}_n}{\partial t}\right\|_2^2 , \tag{6.43}$$

and

$$-\int_{\mathcal{C}} (\boldsymbol{v}_n\cdot \nabla \boldsymbol{v}_n)\cdot \frac{\partial \boldsymbol{v}_n}{\partial t} \le \frac{1}{2\varepsilon}\|\boldsymbol{v}_n\cdot \nabla \boldsymbol{v}_n\|_2^2 + \varepsilon\left\|\frac{\partial \boldsymbol{v}_n}{\partial t}\right\|_2^2 , \tag{6.44}$$

where ε is an arbitrary positive number. As for the first term on the right-hand side of the previous inequality, by (5.23),

$$\|\boldsymbol{v}_n\cdot \nabla \boldsymbol{v}_n\|_2^2 \le K\,\|\nabla \boldsymbol{v}_n\|_2^6 + \varepsilon\,\|\boldsymbol{v}_n\|_{2,2}^2\ ,$$

for arbitrary $\varepsilon > 0$, $K = K(\mathcal{C}, \varepsilon) > 0$, and with $K \to 0$ as $\varepsilon \to \infty$. Combining the latter with (6.44) we then conclude

$$-\int_{\mathcal{C}} (\boldsymbol{v}_n \cdot \nabla \boldsymbol{v}_n) \cdot \frac{\partial \boldsymbol{v}_n}{\partial t} \le K_1 \left\| \nabla \boldsymbol{v}_n \right\|_2^6 + \varepsilon \Big(\left\| \boldsymbol{v}_n \right\|_{2,2}^2 + \left\| \frac{\partial \boldsymbol{v}_n}{\partial t} \right\|_2^2 \Big), \tag{6.45}$$

where $K_1 = K_1(\mathcal{C}, \varepsilon) > 0$. Taking into account (6.40)–(6.43), the last displayed equation, and (2.28), from (6.40) we deduce that

$$\left[C - D_{R_1,F}^2 - (R_1 + F)^2 - 3\varepsilon \right] \left\| \frac{\partial \boldsymbol{v}_n}{\partial t} \right\|_2^2 + \frac{\nu}{2} \frac{d}{dt} \left\| \nabla \boldsymbol{v}_n \right\|_2^2 \le C_6 + K_1 \left\| \nabla \boldsymbol{v}_n \right\|_2^6 + \varepsilon \left\| \boldsymbol{v}_n \right\|_{2,2}^2 . \tag{6.46}$$

As a result, if we choose $\varepsilon < C/6$ and take $R_1 \equiv F$ (see Remark 6.1.2), from (6.46) it follows that there is a constant $\kappa_1 = \kappa_1(\mathcal{C}) > 0$ such that if

$$\|f\|_{L^\infty(0,T)} \le \kappa_1 , \tag{6.47}$$

then

$$C \left\| \frac{\partial \boldsymbol{v}_n}{\partial t} \right\|_2^2 + \nu \frac{d}{dt} \left\| \nabla \boldsymbol{v}_n \right\|_2^2 \le C_7 + 2K_1 \left\| \nabla \boldsymbol{v}_n \right\|_2^6 + 2\varepsilon \left\| \boldsymbol{v}_n \right\|_{2,2}^2 . \tag{6.48}$$

Let us handle the term $\varepsilon \left\| \boldsymbol{v}_n \right\|_{2,2}^2$. Let us multiply $(6.17)_1$ by $\lambda_r c_{nr}$ and sum over $r = 1, \dots, n$. Proceding in a way completely analogous to that leading to (6.41)–(6.43), and (6.45), using also (2.24), we can show that, under an assumption similar to (6.47), the following estimate holds

$$\left\| \boldsymbol{v}_n \right\|_{2,2}^2 \le C \left\| P\Delta \boldsymbol{v}_n \right\|_2^2 \le C_8 + K_2 \Big(\left\| \frac{\partial \boldsymbol{v}_n}{\partial t} \right\|_2^2 + \left\| \nabla \boldsymbol{v}_n \right\|_2^6 \Big) , \tag{6.49}$$

where $K_2 = K_2(\mathcal{C}, \mu, \rho) > 0$ and C_8 depends also on μ. Let us multiply both sides of (6.49) by 4ε, and then add side by side the resulting inequality and inequality (6.48). If we take ε sufficiently small we thus arrive at

$$K_3 \Big(\left\| \boldsymbol{v}_n \right\|_{2,2}^2 + \left\| \frac{\partial \boldsymbol{v}_n}{\partial t} \right\|_2^2 \Big) + \frac{d}{dt} \left\| \nabla \boldsymbol{v}_n \right\|_2^2 \le C_9 + K_4 \left\| \nabla \boldsymbol{v}_n \right\|_2^6 \tag{6.50}$$

where, $K_i = K_i(\mathcal{C}, \mu, \rho) > 0$, $i = 3, 4$, and C_9 depends also on μ. At this point, we observe the following facts. Recalling that $R_1 \equiv \|f\|_{L^\infty(0,T)}$ and that f satisfies (6.47), from (6.32) we deduce that the radius R_2 may be chosen as a function of $\mathcal{C}$, $\mathcal{B}$, $\mathcal{C}$, and T only. As a consequence, the constant C_9 in (6.50) depends only on the same quantities and ρ. In view of all the above, from (6.50), we conclude that $y := \|\nabla \boldsymbol{v}_n\|_2^2$ satisfies the following differential inequality

$$y' \le -K_3 y + K_4 y^3 + C_9$$

Our next objective is to show that $y = y(t)$ obeys the hypothesis of Lemma 2.3.5 provided f satisfies a restriction of the type (6.37). To this end, we observe that integrating both sides of (6.26) between 0 and T and using the T-periodicity of $\boldsymbol{v}_n$, along with (6.25) we show

$$\begin{aligned}\int_0^T \|\nabla \boldsymbol{v}_n(t)\|_2^2 \; dt &\le k_1 \left(|\boldsymbol{A}_{n,0}|^4 + \|f\|_{L^1(0,T)}^4 + \|f\|_{L^2(0,T)}^2 \right) \\ &\le k_2 \left(R_1^4 + \|f\|_{L^\infty(0,T)}^4 + \|f\|_{L^\infty(0,T)}^2 \right) \le k_3 \left(\|f\|_{L^\infty(0,T)}^4 + \|f\|_{L^\infty(0,T)}^2 \right) ,\end{aligned} \tag{6.51}$$

where $k_i = k_i(\mathcal{C}, \mathcal{B}, \mathcal{C}, T) > 0$, $i = 1, 2, 3$, and in the last step we used the fact that $R_1 = \|f\|_{L^\infty(0,T)}$. From (6.51), the integral mean-value theorem, and the T-periodicity of $\boldsymbol{v}_n$ we deduce

$$\|\nabla \boldsymbol{v}_n(\bar{t})\|_2^2 + \int_{\bar{t}}^{\bar{t}+T} \|\nabla \boldsymbol{v}_n(t)\|_2^2 \; dt \le 2k_3 \left(\|f\|_{L^\infty(0,T)}^4 + \|f\|_{L^\infty(0,T)}^2 \right) , \tag{6.52}$$

for some $\bar{t} \in (0, T)$. From (6.52), Lemma 2.3.5 and again the T-periodicity of $\boldsymbol{v}_n$ we then derive that there is $\kappa_2 = \kappa_2(\mathcal{C}, \mathcal{B}, \mathcal{C}, T) > 0$ such that if

$$\|f\|_{L^\infty(0,T)} \le \kappa_2 \tag{6.53}$$

it follows

$$\|\nabla \boldsymbol{v}_n(t)\|_2 < \delta \quad \text{for all } t \in [0, T], \tag{6.54}$$

where $\delta > 0$ depends also on $\|f\|_{L^\infty(0,T)}$. Moreover, integrating (6.50) over a period, and taking into account (6.54) we also conclude

$$\int_0^T \|\boldsymbol{v}_n(t)\|_{2,2}^2 \; dt + \int_0^T \left\| \frac{\partial \boldsymbol{v}_n(t)}{\partial t} \right\|_2^2 \; dt \le k_4. \tag{6.55}$$

where $k_4 = \kappa_4(\mathcal{C}, \mathcal{B}, \mathcal{C}, T) > 0$. Therefore, we conclude that setting $\kappa = \min\{\kappa_1, \kappa_2\}$, κ_1, κ_2 defined in (6.47) and (6.53), respectively, under the hypothesis (6.37) the approximating T-periodic solutions $(\boldsymbol{v}_n, \boldsymbol{A}_n, \boldsymbol{H}_n)$ constructed in the proof of Theorem 6.1.1 satisfy, in addition, the uniform bounds (6.38), (6.39), (6.54) and (6.55).

As a consequence, the limiting fields $(\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{H})$ defined through (6.36) satisfy all the properties stated in the theorem (in particular, the continuity property of $\nabla \boldsymbol{v}$ follows from classical interpolation results; see, e.g., [30, Théorème 2.1]). The theorem is thus completely proved. ■

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