MATHEMATICAL ANALYSIS OF CREDIT DEFAULT SWAPS

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In this thesis, we establish a financial credit derivative pricing model for a credit default swap (CDS) contract which is subject to counterparty risks. A credit default swap is an agreement on exchange of cash flows between two parties, the buyer and the seller, about the occurrence of a credit event. The buyer makes a series of payments to the seller before the event and before the expiration date. The seller pays the buyer a fixed compensation at the moment when the event occurs, if it is before the expiry. The model arises a linear partial differential equation problem. We study this model, i.e. differential equation and show that a solution of the PDE problem from structure model can be obtained as the limit of a sequence of PDE problems which comes from intensity model. In addition, we study the infinite horizon problem of the pricing model which leads to a nonlinear ordinary differential equation problem. We obtain a implicit solution of the ODE problem and prove the solution can be converged by the solution of the PDE problem exponentially. Furthermore, the models and theoretical methods in this study get connected between two main risk frameworks: term structure model and intensity model, which greatly extend the area of applicability of structure models in financial problems. Moreover, We obtain the uniqueness, existence, and properties of the solutions of the PDE and ODE problems. Nevertheless, we implement numerical methods to calibrate the parameters of stochastic interest rate model and analyze the numerical solutions of the pricing model.

Keywords: Structure model, counterparty risk, linear PDE, infinite horizon, numerical analysis.

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PREFACE

This dissertation is submitted for the degree of Doctor of Philosophy at the University of Pittsburgh. The research described herein was conducted under the supervision of Professor Xinfu Chen in the Department of Mathematics, University of Pittsburgh, between August 2012 and April 2016.

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1.0 INTRODUCTION

We established a financial pricing model for a credit default swap contract which is subject to counterparty risks in this thesis. A credit default swap (CDS) is a popular and highly liquid financial swap agreement that the seller of the CDS will compensate the buyer (usually the creditor of the reference loan) in the event of a loan default (by the debtor) or other credit event. This is to say that the seller of the CDS insures the buyer against some reference loan defaulting. The buyer of the CDS makes a series of payments (the CDS "fee" or "spread") to the seller and, in exchange, receives a payoff if the loan defaults. It was invented by Blythe Masters from JP Morgan in 1994.

Due to financial difficulties, during the life of the contract, either the buyer or the seller may default and therefore the contract is terminated before the regular expiration date. Hence, the price of CDS should include this so called counterparty risk. Furthermore, if the counterparty risk has some positive correlation with the reference risk, we also refer the risk as a "wrong-way risk".

Structural and intensity models are two primary types of models among papers dealing with default risks. A structural model is described in Black-Cox's work ([6],1976) and Longstaff-Schwartz's work ([13],1995),which bringing to first time arrival model. A second structural model is introduced by Merton's paper ([14],1974), which presents a theory associated with the risk structure of interest rates, the use of the term "risk" (default time) is restricted to the possible gains or losses to bond-holders. However, for an intensity model, the default is not determined by the value of the company, but controlled by a default hazard rate with parameters inferred from market data and macroeconomic variables.One can find such examples of research following this method from Duffie-Singleeton's ([9], 1999) and Lando's works ([12], 1998). The biggest difference between structural and intensity frameworks is they employed two quite different ways to describe the default time and default properties. It is hard to compare which one is better than another one, but the relationship of the two is always attractive. In this thesis, we build a bridge between the two by considering a fully non-linear partial differential equation (PDE) model, which arises from real financial market.

Considering the correlated reference and counterparty risks in a CDS contract, we arise a mathematical pricing model in this study. One is introduced by an intensity model while the other is described by a structural one. For completeness, a wrong-way risk is also considered. The three correlated risks are related to a common factor of stochastic interest rate ([1],1985). Both models can be expressed as partial differential equation problems. The structure model is reduced to a new fully nonlinear PDE boundary problem. In this thesis we approximate the structure model by an intensity one, where the function describing the intensity rate has several jumps at some predetermined level. When the amplitudes of the jump go to infinite, the problem becomes the structure

model. In another words, we construct a connection between the structure model and intensity model,i.e the structure model can be approached by a series of intensity models, where the intensity rates are piece-wise functions, which are considered as "default impulsions" in financial sense. More importantly, this connection not only links the relationship between two financial default frameworks but also provides a method to solve some hard mathematical boundary problems. In fact, it is very difficult to deal with a PDE problem of structure model since it usually brings a complicated boundary condition. To simplify and approximate the structural one, a sequence of intensity models, which are initial PDE problems with low nonlinear terms can be applied based on the method in this study. It then greatly extends the area of applicability of structure models in finance problems since it makes possible to deal with some complicated default barriers from structure models. Furthermore, we also study an infinite horizon problem of the intensity model, which leads to a fully nonlinear ordinary differential equation problem. By obtaining the theoretical implicit solution of the ODE problem, we show that the solution can be approximated exponential uniformly by the solution of PDE problem, comes from the intensity model as time factor approaching to infinite. Moreover, because of involving ODE and PDE problems, we are also naturally interested in the existence, uniqueness and properties of the solution of the problems.

In this study, we formulate the stochastic interest rate by the CIR model ([1], 1985). To distinguish the study ([2], 2012) and to ensure the application of the CIR model robust, we make a improvement to consider the model under some general assumptions. The change is significant and it makes two studies completely distinctive and thus the mathematical analysis in these two works are remarkably different. One goal of this thesis is to introduce this new theoretical development of the CIR model to the public.

The structure of the thesis is as follows: In Chapter 2, we derive the CDS intensity model from real financial instrument by probabilistic and stochastic point of view. In Chapter 3, we show the well-posedness of the CDS intensity model. Infinite horizon problem and asymptotic behavior of the CDS intensity model are discussed in Chapter 4 and Chapter 5. In Chapter 6, we apply numerical methods to calibrate the parameters in the CIR model and perform the simulation of numerical solution of the CDS intensity model by Monte-Carlo method and Finite Difference method. Chapter 7 is a conclusion.

2.0 MATHEMATICAL FORMULATION OF CDS MODEL

2.1 THE CDS MODEL AND MATHEMATICAL ASSUMPTIONS

In this section we model a credit default swap contract.

The CDS Contract. We describe the key ingredients of the contract as follows:

<u>The CDS Contract</u>

This contract bonds the exchange of financial services between two parties, the buyer and the seller, against credit and default events. Let h, K, T, τ, τ_1 and τ_2 be the agreed insurance premium, insurance compensation, expiry, time of occurrence of the designated credit event, default times of the seller and the buyer, respectively. The seller and the buyer agree on the following:

1. *The seller's right and the buyer's obligation: Before $\tau_1 \wedge \tau_2 \wedge \tau \wedge T$, the buyer pays the seller the insurance premium being a cash flow of continuous rate h (\$/year).*
2. *The buyer's right and the seller's obligation: If $\tau \leq \tau_1 \wedge \tau_2 \wedge T$, the seller pays the buyer at time τ the insurance compensation being a lump sum of (\$) K.*
3. *If $\tau_1 \wedge \tau_2 < \tau \wedge T$, the contract terminates at $\tau_1 \wedge \tau_2$, with no further rights and obligations between the buyer and the seller.*

We implicitly assume that the current time is $t = 0$, so T, K, and h are positive constants, τ, τ_1 and τ_2 are assumed to be non-negative stopping time. The contract terminates at $\tau \wedge \tau_1 \wedge \tau_2 \wedge T$.

To evaluate the CDS contract, we need to model the following:

1. Short interest rate $r(t)$.
2. Time (τ) of occurrence of the designated credit event and default times (τ_1 and τ_2) of the seller and the buyer.

We described above two models as follows.

2.2 THE TERM STRUCTURE MODELS FOR INTEREST RATE

We model the risky-free short term interest rate by a stochastic process $\{r_t\}_{t\geq 0}$ defined on a filtrated probability space $(\mathcal{D}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. In the literature, there are two primary types of models to describe short term interest rate, i.e. Vasicek ([3], 1977) and CIR ([1], 1985) interest rate models. Here we will briefly introduce the basic backgrounds for these two interest rate models.

2.2.1 Vasicek Interest Rate Model

Let $\{W_t\}_{t\geq 0}$ be a Brownian motion. The Vasicek model for the interest rate process $r_t := r(t)$ is

$$d\,r_t = \big(\kappa - \beta\, r_t\big)\, dt + \sigma\, d\,W_t, \tag{2.1}$$

where κ, β, and σ are positive constants. It defines a random process, $r(t)$ in this case, by giving a formula for its differential and the formula involves the random process itself and the differential of a Brownian motion.

The solution to the stochastic differential equation (2.1) can be determined in the closed form and is represented as

$$r(t) = e^{-\beta\, t}\, r(0) + \frac{\kappa}{\beta}\left(1 - e^{-\beta\, t}\right) + \sigma\, e^{-\beta\, t} \int_0^t e^{\beta\, s}\, d\,W_s \tag{2.2}$$

We briefly verify the above expression indeed satisfied the equation (2.1). We first can rewrite the expression (2.2) as follows,

$$r(t) = f\big(t, X(t)\big) = e^{-\beta\, t}\, r(0) + \frac{\kappa}{\beta}\left(1 - e^{-\beta\, t}\right) + \sigma\, e^{-\beta\, t}\, X(t),$$

where $X(t) = \int_0^t e^{\beta\, s}\, d\,W_s$. With the following calculations

$$\begin{cases} dX(t) = e^{\beta\, t}\, d\,W_t \\ \\ f_t(t,x) = \kappa - \beta\, f(t,x) = \kappa - \beta\, r_t \\ \\ f_x(t,x) = \sigma\, e^{-\beta\, t} \\ \\ f_{xx}(t,x) = 0 \end{cases}$$

and by using the Itô lemma,

$$\begin{aligned} d\,r_t &= d\,f\big(t, X(t)\big) \\ &= f_t\left(t, X(t)\right) dt + f_X\left(t, X(t)\right) dX(t) + \frac{1}{2}\, f_{XX}\left(t, X(t)\right) dX(t)\, dX(t) \\ &= (\kappa - \beta\, r_t)\, dt + \sigma\, e^{-\beta\, t}\, e^{\beta\, t}\, dW_t \\ &= (\kappa - \beta\, r_t)\, dt + \sigma\, dW_t \end{aligned}$$

Also by the property of Brownian motion, one can show that the random variable

$$\begin{aligned} \int_0^t e^{\beta\, s}\, dW_s &\sim N\Big(0, \int_0^t e^{2\,\beta\, s}\, ds\Big) \\ &\sim N\Big(0, \frac{1}{2\,\beta}\,(e^{2\,\beta\, t} - 1)\Big). \end{aligned}$$

Therefore, the Vasicek model for the interest rate process $r(t)$ is normally distributed with mean

$$e^{-\beta\, t}\, r(0) + \frac{\kappa}{\beta}\left(1 - e^{-\beta\, t}\right)$$

and variance

$$\sigma^2\, e^{-2\,\beta\, t} \cdot \frac{1}{2\,\beta}\,(e^{2\,\beta\, t} - 1) = \frac{\sigma^2}{2\,\beta}(1 - e^{-2\,\beta\, t}).$$

In particular, no matter how the parameters $\kappa > 0$, $\beta > 0$, and $\sigma > 0$ are chosen, there is positive probability that $r(t)$ is negative, an undesirable (it now seems a desirable) property for an interest rate model.

But the Vasicek model has the desirable property that the interest rate is *mean-reverting*. When $r(t) = \frac{\kappa}{\beta}$, the drift term (the dt term) in (2.1) is zero. When $r(t) > \frac{\kappa}{\beta}$, this term is negative, which pushes $r(t)$ back toward to $\frac{\kappa}{\beta}$. When $r(t) < \frac{\kappa}{\beta}$, this term is positive, which again pushes $r(t)$ back toward to $\frac{\kappa}{\beta}$. If $r(0) = \frac{\kappa}{\beta}$, then $\mathbb{E}\,[r(t)] = \frac{\kappa}{\beta}$ for all $t \geq 0$. If $r(0) \neq \frac{\kappa}{\beta}$, then $\lim\limits_{t\to\infty} \mathbb{E}\,[r(t)] = \frac{\kappa}{\beta}$.

2.2.2 Cox-Ingersoll-Ross (CIR) Interest Rate Model

Let $\{W_t\}_{t\geq 0}$ be a Brownian motion. The Cox-Ingersoll-Ross model for the interest rate process $r_t := r(t)$ is

$$d\,r_t = \big(\kappa - \beta\, r_t\big)\,dt + \sigma\,\sqrt{r_t}\,dW_t, \tag{2.3}$$

where κ, β, and σ are positive constants. Unlike the Vasicek equation (2.1), the CIR equation (2.3) does not have a closed form solution. The advantage of (2.3) over the Vasicek model is that the interest rate in the CIR model does not become negative. If $r(t)$ reaches zero, the term multiplying dW_t vanishes and the positive drift term $\kappa\,dt$ in the equation (2.3) drives the interest rate back into positive territory. Like the Vasicek model, the CIR model is *mean-reverting*.

Although one cannot derive a closed form solution for (2.3), the distribution of $r(t)$ for each positive t can be determined. That computation would take us too far afield. We instead content ourselves with the derivation of the expected value and variance of $r(t)$. To do this, we use the function $f(t,x) = e^{\beta\,t}\,x$ and the Itô-Doeblin formula to compute

$$\begin{aligned}
d(e^{\beta\,t}\,r_t) &= d\,f(t,r_t) \\
&= f_t(t,r_t)\,dt + f_x(t,r_t)\,dr_t + \frac{1}{2}\,f_{xx}(t,r_t)\,dr_t\,dr_t \\
&= \beta\,e^{\beta\,t}\,r_t\,dt + e^{\beta\,t}(\kappa - \beta\,r_t)\,dt + e^{\beta\,t}\,\sigma\,\sqrt{r_t}\,dW_t \\
&= \kappa\,e^{\beta\,t}\,dt + \sigma\,e^{\beta\,t}\,\sqrt{r_t}\,dW_t
\end{aligned}$$

Integration of both sides of above expression yields

$$\begin{aligned}
e^{\beta\,t}\,r(t) &= r(0) + \kappa\int_0^t e^{\beta\,\mu}\,d\mu + \sigma\int_0^t e^{\beta\,\mu}\,\sqrt{r_\mu}\,dW_\mu \\
&= r(0) + \frac{\kappa}{\beta}(e^{\beta\,t} - 1) + \sigma\int_0^t e^{\beta\,\mu}\,\sqrt{r_\mu}\,dW_\mu.
\end{aligned}$$

Recalling that the expectation of an Itô integral is zero, we obtain

$$e^{\beta\,t}\,\mathbb{E}\,[r(t)] = r(0) + \frac{\kappa}{\beta}(e^{\beta\,t} - 1)$$

or, equivalently,

$$\mathbb{E}\,[r(t)] = e^{-\beta\,t}\,r(0) + \frac{\kappa}{\beta}(1 - e^{-\beta\,t})$$

This is the same expectation as in the Vasicek model.

To compute the variance of $r(t)$, we set $X(t) = e^{\beta\, t}\, r_t$, for which we have already computed

$$\begin{aligned} dX(t) &= \kappa\, e^{\beta\, t}\, dt + \sigma\, e^{\beta\, t}\, \sqrt{r_t}\, dW_t \\ &= \kappa\, e^{\beta\, t}\, dt + \sigma\, e^{\frac{\beta\, t}{2}}\, \sqrt{X(t)}\, dW_t \end{aligned}$$

and

$$\mathbb{E}\,[X(t)] = e^{\beta}\, \mathbb{E}\,[r(t)] = r(0) + \frac{\kappa}{\beta}(e^{\beta\, t} - 1).$$

According to the Itô-Doeblin formula (with $f(x) = x^2$, $f'(x) = 2\,x$, and $f''(x) = 2$),

$$\begin{aligned} d\big(X^2(t)\big) &= 2\, X(t)\, dX(t) + dX(t)\, dX(t) \\ &= 2\kappa\, e^{\beta\, t}\, X(t)\, dt + 2\sigma\, e^{\frac{\beta\, t}{2}}\, X^{\frac{3}{2}}(t)\, dW_t + \sigma^2\, e^{\beta\, t}\, X(t)\, dt \end{aligned}$$

Integration of above expression yields

$$X^2(t) = X^2(0) + (2\kappa + \sigma^2) \int_0^t e^{\beta\, \mu}\, X(\mu)\, d\mu + 2\sigma \int_0^t e^{\frac{\beta\, \mu}{2}}\, X^{\frac{3}{2}}(\mu)\, dW_\mu$$

Taking expectations, using the fact that the expectation of an Itô integral is zero and the formula already derived for $\mathbb{E}\,[X(t)]$, we obtain

$$\begin{aligned} \mathbb{E}\,[X^2(t)] &= X^2(0) + (2\kappa + \sigma^2) \int_0^t e^{\beta\, \mu}\, \mathbb{E}\,[X(\mu)]\, d\mu \\ &= r^2(0) + (2\kappa + \sigma^2) \int_0^t e^{\beta\, \mu}\, \big(r(0) + \frac{\kappa}{\beta}(e^{\beta\, t} - 1)\big)\, d\mu \\ &= r^2(0) + \frac{2\kappa + \sigma^2}{\beta}\, \big(r(0) - \frac{\kappa}{\beta}\big)\, \big(e^{\beta\, t} - 1\big) + \frac{2\kappa + \sigma^2}{2\beta} \cdot \frac{\kappa}{\beta}\, \big(e^{2\,\beta\, t} - 1\big). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\,[r^2(t)] &= e^{-2\,\beta\, t}\, \mathbb{E}\,[X^2(t)] \\ &= e^{-2\,\beta\, t}\, r^2(0) + \frac{2\kappa + \sigma^2}{\beta}\, \big(r(0) - \frac{\kappa}{\beta}\big)\, \big(e^{-\beta\, t} - e^{-2\,\beta\, t}\big) + \frac{2\kappa + \sigma^2}{2\beta} \cdot \frac{\kappa}{\beta}\, \big(1 - e^{-2\,\beta\, t}\big). \end{aligned}$$

Finally,

$$\begin{aligned} \mathrm{Var}\,[r(t)] &= \mathbb{E}\,[r^2(t)] - \Big(\mathbb{E}\,[r(t)]\Big)^2 \\ &= e^{-2\,\beta\, t}\, r^2(0) + \frac{2\kappa + \sigma^2}{\beta}\, \big(r(0) - \frac{\kappa}{\beta}\big)\, \big(e^{-\beta\, t} - e^{-2\,\beta\, t}\big) + \frac{\kappa(2\kappa + \sigma^2)}{2\beta^2}\, \big(1 - e^{-2\,\beta\, t}\big) \\ &\quad - e^{-2\,\beta\, t}\, r^2(0) - \frac{2\kappa}{\beta}\, r(0)\, (e^{-\beta\, t} - e^{-2\,\beta\, t}) - \frac{\kappa^2}{\beta^2}\, (1 - e^{-\beta\, t})^2 \\ &= \frac{\kappa^2}{\beta}\, r(0)\, (e^{-\beta\, t} - e^{-2\,\beta\, t}) + \frac{\kappa\, \sigma^2}{2\, \beta^2}\, (1 - e^{-\beta\, t})^2. \end{aligned}$$

In particular,

$$\lim_{t \to \infty} \mathrm{Var}\,[\, r(t)\,] = \frac{\kappa\, \sigma^2}{2\, \beta^2}.$$

One thing need to be pointed out is that the original CIR model (2.3) is only well-defined for the assumption

$$\sigma > 0, \;\; \kappa > \frac{\sigma^2}{2} > 0, \;\; \beta > 0. \tag{2.4}$$

Under this assumption, one can show that $r(t)$ is a positive process, i.e. $\mathbb{P}\,\big(r(t) > 0\big) = 1$, for $\forall\, t > 0$.

2.2.3 Modified Cox-Ingersoll-Ross (CIR) Interest Rate Model

From financial point of view, to ensure the application of the CIR model robust, we make a improvement to the original CIR model (2.3) to obtain a new form of CIR model, which is not only well-defined under the condition (2.4) but also well-defined under some general assumptions.

The modified CIR model is as follows:

$$dr_t = \mu_t\, dt + \varsigma_t\, dW_t, \quad \mu_t = \mu(r_t), \quad \varsigma_t = \varsigma(r_t), \tag{2.5}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion and $\mu(\cdot)$ and $\varsigma(\cdot)$ are functions defined by

$$\mu(r) = \kappa - \beta\, r, \quad \varsigma(r) = \sigma\, \sqrt{\max\{r, 0\}}, \tag{2.6}$$

where κ, β and σ are positive constants. The stochastic differential equation (2.5) is well-defined under the following general assumption

$$\kappa > 0,\ \beta > 0,\ \sigma > 0, \tag{2.7}$$

and it is indeed the limit of

$$dr_t^\epsilon = (\kappa - \beta\, r_t^\epsilon)\, dt + \sigma\, \sqrt{\epsilon^2 + \max\{r_t^\epsilon, 0\}}\, dW_t.$$

2.2.4 Discount Factor in Cox-Ingersoll-Ross (CIR) Model

In the CIR model, the discount factor of a time t ($\geqslant 0$) payment is

$$e^{-\int_0^t r_\theta\, d\theta}.$$

In particular, the price of a T-bond (i.e., a unit payment at time T) is

$$B(r, T) = \mathbb{E}\Big[e^{-\int_0^T r_\theta d\theta}\ \Big|\ r_0 = r\Big].$$

To evaluate B, we introduce the Black-Scholes operator $\mathcal{L}_1$ associated with the model by

$$\mathcal{L}_1\phi(r, T) = \Big\{\frac{\partial}{\partial T} - \frac{\sigma^2}{2}\, r\, \frac{\partial^2}{\partial r^2} - (\kappa - \beta\, r)\frac{\partial}{\partial r} + r\Big\}\phi(r, T) \tag{2.8}$$

for $T > 0$ and $r \in \Omega$. Here $\Omega = (0, \infty)$ is the state space for the interest rate in the CIR model.

Then by Feynman-Kac formula, $B(r, T)$ is the solution of

$$\begin{cases} \mathcal{L}_1 B(r, T) = 0 & \forall\, r \in \Omega, T > 0, \\ \\ B(r, 0) = 1 & \forall\, r \in \bar{\Omega}, T = 0. \end{cases}$$

According to the theory of affine term structure model, $B(r, T) = e^{-C(T)r - \kappa \int_0^T C(\tau) d\tau}$, where C is the solution of the Ricarti equation

$$\begin{cases} C'(T) = -\frac{\sigma^2}{2} C^2(T) - \beta\, C(T) + 1 \quad \forall\, T > 0 \\ \\ C(0) = 0. \end{cases}$$

Namely,

$$C(T) = \frac{2(1 - e^{-\sqrt{\beta^2 + 2\sigma^2}T})}{(\sqrt{\beta^2 + 2\sigma^2} + \beta) + (\sqrt{\beta^2 + 2\sigma^2} - \beta)e^{-\sqrt{\beta^2 + 2\sigma^2}T}}.$$

2.3 THE CREDIT EVENT AND DEFAULT TIMES

We model the time τ, τ_1 and τ_2 of the credit event and default times by the first arrival time of inhomogeneous Poisson processes with variable rates $\{\lambda_t\}_{t\geq 0}$, $\{\lambda_{1t}\}_{t\geq 0}$ and $\{\lambda_{2t}\}_{t\geq 0}$, respectively. For definiteness, we suppose that the default rates of the credit event, the seller and the buyer all depend on the observable interest rate, but in different ways as follows:

$$\lambda_t = \Lambda(r_t), \quad \Lambda(r) = ar + b, \tag{2.9}$$

$$\lambda_{1t} = \Lambda_1(r_t), \quad \Lambda_1(r) = p\, H(r - B_2), \tag{2.10}$$

$$\lambda_{2t} = \Lambda_2(r_t), \quad \Lambda_2(r) = q\, H(B_1 - r), \tag{2.11}$$

where a, b, B_1 and B_2 are positive constants with $B_1 < B_2$ and p and q are non-negative constants. $H(\cdot)$ is the Heaviside function. For simplicity, we further assume that τ, τ_1, and τ_2 are conditionally independent; this implies that, when $t \geqslant s \geqslant 0$,

$$\begin{aligned}
& \mathbb{P}(\tau \wedge \tau_1 \wedge \tau_2 > t \mid \mathcal{F}_t,\, \tau \wedge \tau_1 \wedge \tau_2 > s) \\
= \;& \mathbb{P}(\tau > t,\ \tau_1 > t,\ \tau_2 > t \mid \mathcal{F}_t,\, \tau \wedge \tau_1 \wedge \tau_2 > s) \\
= \;& \mathbb{P}(\tau > t \mid \mathcal{F}_t,\ \tau > s)\, \mathbb{P}(\tau_1 > t \mid \mathcal{F}_t,\ \tau_1 > s)\, \mathbb{P}(\tau_2 > t \mid \mathcal{F}_t,\ \tau_2 > s) \\
= \;& e^{-\int_s^t (\lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta}) d\theta},
\end{aligned}$$

and

$$\mathbb{P}\big(\tau \in [t, t+dt), \tau_1 \wedge \tau_2 > t \big| \mathcal{F}_t, \tau \wedge \tau_1 \wedge \tau_2 > s\big) = \lambda_t e^{-\int_s^t (\lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta}) d\theta} dt.$$

The assumption of (2.9) indicates that the default of the credit event depends on the interest rate in a linear combination. The assumptions of (2.10) and (2.11) explain the default of the seller and the buyer are in impulsion from with respect to the interest rate at specific level B_1 and B_2. More specifically, if $B_1 < r < B_2$, there is no default possibility for both the seller and the buyer; if r reaches or is over the level B_2, there is no default possibility for the buyer but the default intensity rate of the seller suddenly jump to p; similarly if r reaches or is below the level B_1, there is no default possibility for the seller but the default intensity rate of the buyer suddenly jump to q.

2.4 VALUATION OF CDS MODEL

Assume that the current time is $t = 0$. For each $x \in \mathcal{D}$, from the buyer's point of view, the present value of all payments from the seller to the buyer is

$$p = K\, e^{-\int_0^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_1 \wedge \tau_2 \wedge T\}} - \int_0^{\tau \wedge \tau_1 \wedge \tau_2 \wedge T} e^{-\int_0^t r_\theta d\theta}\, h\, dt. \tag{2.12}$$

We define the value of the CDS, denoted as $u(r,T)$, from buyer's point of view by the expectation of the present value of all payments received by the buyer from the seller:

$$\begin{aligned} u(r,T) &:= \mathbb{E}\Big[p \;\Big|\; r_0 = r, \tau\wedge\tau_1\wedge\tau_2 > 0\Big] \\ &= \mathbb{E}\left[\int_0^T \mathbb{P}\Big(\tau\in[s,s+ds), \tau_1\wedge\tau_2 > s \Big| \mathcal{F}_s, \tau\wedge\tau_1\wedge\tau_2 > 0\Big)\, p \Big| r_0 = r\right] \\ &= K\, u_1(r,T) - h\, u_2(r,T), \end{aligned}$$

where

$$\begin{aligned} u_1(r,T) &:= \mathbb{E}\left[e^{-\int_0^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}} \;\Big|\; r_0 = r, \tau\wedge\tau_1\wedge\tau_2 > 0\right], \\ u_2(r,T) &:= \mathbb{E}\left[\int_0^{\tau\wedge\tau_1\wedge\tau_2\wedge T} e^{-\int_0^t r_\theta d\theta} dt \;\Big|\; r_0 = r, \tau\wedge\tau_1\wedge\tau_2 > 0\right]. \end{aligned}$$

Using the assumptions of τ, τ_1 and τ_2, we can express $u_1(r,T)$ and $u_2(r,T)$ as follows:

$$\begin{aligned} u_1(r,T) &= \mathbb{E}\left[\int_0^T \mathbb{P}\Big(\tau\in[s,s+ds),\, \tau_1\wedge\tau_2 > s \Big| \mathcal{F}_s, \tau\wedge\tau_1\wedge\tau_2 > 0\Big) e^{-\int_0^s r_\theta d\theta} \;\Big|\; r_0 = r\right] \\ &= \mathbb{E}\left[\int_0^T \lambda_s e^{-\int_0^s (r_\theta+\lambda_\theta+\lambda_{1\,\theta}+\lambda_{2\,\theta})d\theta} ds \;\Big|\; r_0 = r\right], \\ u_2(r,T) &= \mathbb{E}\left[\int_0^\infty \mathbb{P}\Big(\tau\in[s,s+ds),\, \tau_1\wedge\tau_2 > s \Big| \mathcal{F}_s, \tau\wedge\tau_1\wedge\tau_2 > 0\Big) \int_0^{s\wedge T} e^{-\int_0^t r_\theta d\theta} dt \;\Big|\; r_0 = r\right] \\ &= \mathbb{E}\left[\int_0^\infty \Big(\int_0^{s\wedge T} e^{-\int_0^t r_\theta d\theta} dt\Big) d\Big(-e^{-\int_0^s (\lambda_\theta+\lambda_{1\,\theta}+\lambda_{2\,\theta})\, d\theta}\Big) \;\Big|\; r_0 = r\right] \\ &= \mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta+\lambda_\theta+\lambda_{1\,\theta}+\lambda_{2\,\theta})d\theta} ds \;\Big|\; r_0 = r\right], \end{aligned}$$

by integration by parts. It then follows by the Feynman-Kac formula that

$$\begin{cases} \Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big) u_1 = \Lambda & \forall\, r\in\Omega,\ T>0, \\ u_1(r,0) = 0 & \forall\, r\in\bar{\Omega}, T=0; \end{cases} \tag{2.13}$$

Similarly, we find that u_2 is the solution of

$$\begin{cases} \Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big) u_2 = 1 & \forall\, r\in\Omega,\ T>0, \\ u_2(r,0) = 0 & \forall\, r\in\bar{\Omega}, T=0. \end{cases} \tag{2.14}$$

Hence, the value, $u(r,T)$, of the CDS model from the buyer's point of view at time $t=0$ with $r_0 = r$ is the solution of

$$\begin{cases} \Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big) u = K\Lambda - h & \forall\, r\in\Omega,\ T>0, \\ u(r,0) = 0 & \forall\, r\in\bar{\Omega}, T=0. \end{cases} \tag{2.15}$$

Here $\Omega = (0, \infty)$ is the state space for the interest rate and $\mathcal{L}$ is the Black-Scholes operator. For the different kinds of short interest rate models, the expressions of $\mathcal{L}$ are slightly different. For Vasicek interest rate model, the $\mathcal{L}$ is defined as

$$\mathcal{L} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r \ . \tag{2.16}$$

For Cox-Ingersoll-Ross (CIR) interest rate, the $\mathcal{L}$ is defined as

$$\mathcal{L} = -\frac{\sigma^2}{2}\, r\, \frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r \ . \tag{2.17}$$

If the current time is any $t \in [0, T)$, and $\tau \wedge \tau_1 \wedge \tau_2 > t$, then the value of the contract is

$$\mathbb{E}\left[K e^{-\int_t^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_1 \wedge \tau_2 \wedge T\}} - \int_t^{\tau \wedge \tau_1 \wedge \tau_2 \wedge T} e^{-\int_t^s r_\theta d\theta}\, h\, ds \Big| \mathcal{F}_t, \tau \wedge \tau_1 \wedge \tau_2 > t \right] = u(r_t, T - t).$$

2.5 THE PDE PROBLEM

Since for the different interest rate models used, the expressions of the Black-Scholes operator are different, which lead to the different PDE problems. The difference seems very slight, but actually this change is significant and it makes all studies totally distinct. For the PDE problem under the Vasicek model, please refer to the work ([4], 2015). Therefore in this thesis, we mainly study the PDE problem under using CIR interest rate model. Thus through the thesis, we only consider the Black-Scholes operator $\mathcal{L}$ defined as in (2.17). Recall that for Cox-Ingersoll-Ross (CIR) model, we have the original CIR model (2.3) only well-defined under the condition $\kappa > \frac{\sigma^2}{2} > 0$. The paper ([2], 2012) studied the PDE problem under this case. Different from the problem where it is assume that $\kappa > \frac{\sigma^2}{2}$, here we shall drop this condition by the improved CIR model introduced in section 2.2. In this study, we shall consider the general case only:

$$\kappa > 0, \ \beta > 0, \ \sigma > 0.$$

To treat the case that $0 < 2\kappa \leq \sigma^2$, instead we introduce the following restriction, the 'boundary condition ' on the solution space to ensure the PDE problem (2.15) be well-defined:

$$u_r \in L^\infty((0, \infty)^2) \tag{2.18}$$

We then summarize the mathematical **PDE problem** for u as follows:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u = f & \text{in } (0, \infty)^2, \\ u(\cdot, 0) = 0 & \text{in } (0, \infty), \\ u_r \in L^\infty((0, \infty)^2) \end{cases} \tag{2.19}$$

where Λ, Λ_1, Λ_2 and $\mathcal{L}$ are defined as in (2.9), (2.10), (2.11) and (2.17) respectively; $f = K\Lambda - h$ and $\sigma > 0, \kappa > 0, \beta > 0, a > 0, b > 0, 0 < B_1 < B_2$, $p \geq 0, q \geq 0, K > 0, h > 0$.

3.0 WELL-POSEDNESS OF CDS MODEL

In this chapter we study and show the well-posedness of PDE problem (2.19).

Theorem 3.1. *Let Λ, Λ_1, Λ_2 and $\mathcal{L}$ be defined as in (2.9), (2.10), (2.11) and (2.17), respectively; $f = K\Lambda - h$ and $\sigma > 0, \kappa > 0, \beta > 0, a > 0, b > 0, 0 < B_1 < B_2$, $p \geq 0, q \geq 0, K > 0, h > 0$. Then problem (2.19) admits a unique solution $u \in \mathbb{X}$, where,*

$$\mathbb{X} = C^{1,\frac{1}{2}}((0,\infty) \times [0,\infty)) \cap C^{2,1}(Q_1) \cap C^{2,1}(Q_2) \cap C^{2,1}(Q_3), \tag{3.1}$$

and

$$\begin{cases} Q_1 = (0, B_1] \times [0,\infty) \\ \\ Q_2 = [B_1, B_2] \times [0,\infty) \\ \\ Q_3 = [B_2, \infty) \times [0,\infty). \end{cases}$$

We will show above theorem by the following two sections. The first section is about uniqueness. In this section, we first construct the auxiliary function and then use the function to show the uniqueness. The second section is about existence. We first regularize the above PDE problem and then perform the apriori estimations to get the existence. Please refer to the followings for more details.

3.1 UNIQUENESS OF SOLUTION OF PDE PROBLEM

We first construct the below auxiliary function.

Lemma 3.1. *Let a^* and b^* be constants satisfies*

$$a^* > -1, \;\; 1 - \sqrt{1 + \frac{2\sigma^2(a^*+1)}{\beta^2}} < \frac{b^*\sigma^2}{\kappa\beta} < 1 + \sqrt{1 + \frac{2\sigma^2(a^*+1)}{\beta^2}} \tag{3.2}$$

Then there exists $\psi(r) \in C^\infty((0,\infty))$ such that

$$\begin{cases} \big(\mathcal{L} + a^* r + b^*\big)\psi = 0 & \text{in } (0,\infty), \\ \\ \psi > 0 & \text{in } (0,\infty), \\ \\ \lim\limits_{r\to\infty} \psi'(r) = \infty, \;\; \lim\limits_{r\searrow 0} \psi'(r) = -\infty. \end{cases} \tag{3.3}$$

Proof. Consider the function ψ defined by

$$\psi(r) = \int_{-\infty}^{\lambda_2} |\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1} e^{sr}\, ds \quad \forall\, r > 0, \tag{3.4}$$

where

$$\begin{cases} \lambda_1 = \frac{\beta - \sqrt{\beta^2 + 2(a^*+1)\sigma^2}}{\sigma^2}, \\ \lambda_2 = \frac{\beta + \sqrt{\beta^2 + 2(a^*+1)\sigma^2}}{\sigma^2}, \\ \alpha_1 = \frac{1}{\sqrt{1 + \frac{2\sigma^2(a^*+1)}{\beta^2}}} \Big\{ \frac{b^*}{\beta} + \frac{\kappa}{\sigma^2}\Big(\sqrt{1 + \frac{2\sigma^2(a^*+1)}{\beta^2}} - 1\Big)\Big\}, \\ \alpha_2 = \frac{1}{\sqrt{1 + \frac{2\sigma^2(a^*+1)}{\beta^2}}} \Big\{ -\frac{b^*}{\beta} + \frac{\kappa}{\sigma^2}\Big(\sqrt{1 + \frac{2\sigma^2(a^*+1)}{\beta^2}} + 1\Big)\Big\}. \end{cases} \tag{3.5}$$

One can easily check that $\psi(r)$ defined as in (3.4) satisfies the last two conditions in (3.3). We will only focus on the proof of the first condition:

$$\big(\mathcal{L} + a^* r + b^*\big)\psi = 0.$$

By the definition of Black-Scholes operator, we have

$$\begin{aligned} \big(\mathcal{L} + a^* r + b^*\big)\psi \quad = \quad & -\frac{\sigma^2}{2} r \int_{-\infty}^{\lambda_2} s^2\, |\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1} e^{sr}\, ds \\ & -(\kappa - \beta r) \int_{-\infty}^{\lambda_2} s\, |\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1} e^{sr}\, ds \\ & +\big[(a^* + 1)r + b^*\big] \int_{-\infty}^{\lambda_2} |\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1} e^{sr}\, ds. \end{aligned}$$

After simplification and integration by parts, the above expression can be written as

$$\begin{aligned} & \big(\mathcal{L} + a^* r + b^*\big)\,\psi \\ = \quad & -\int_{-\infty}^{\lambda_2} \Big[\frac{\sigma^2}{2} s^2 - \beta\, s - (a^* + 1)\Big]\, |\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1}\, d\, e^{sr} \\ & -\int_{-\infty}^{\lambda_2} (\kappa\, s - b^*)|\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1}\, e^{sr}\, ds \\ = \quad & -\int_{-\infty}^{\lambda_2} \frac{\sigma^2}{2} |\lambda_1 - s|^{\alpha_1}(\lambda_2 - s)^{\alpha_2}\, d\, e^{sr} - \int_{-\infty}^{\lambda_2} (\kappa\, s - b^*)|\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1}\, e^{sr}\, ds \\ = \quad & \int_{-\infty}^{\lambda_2} \frac{\sigma^2}{2}\, e^{s\,r}\big[-\alpha_1\, |\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2} - \alpha_2\, |\lambda_1 - s|^{\alpha_1}(\lambda_2 - s)^{\alpha_2 - 1}\big]\, ds \\ & -\int_{-\infty}^{\lambda_2} (\kappa\, s - b^*)|\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1}\, e^{sr}\, ds \\ = \quad & \int_{-\infty}^{\lambda_2} e^{s\,r}|\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1}\Big[-\alpha_1 \frac{\sigma^2}{2}(\lambda_2 - s) - \alpha_2 \frac{\sigma^2}{2}(\lambda_1 - s) - \kappa\, s + b^*\Big]\, ds \\ = \quad & \int_{-\infty}^{\lambda_2} e^{s\,r}|\lambda_1 - s|^{\alpha_1 - 1}(\lambda_2 - s)^{\alpha_2 - 1}\Big[\big((\alpha_1 + \alpha_2)\frac{\sigma^2}{2} - \kappa\big)\, s + b^* - \frac{\sigma^2}{2}\big(\alpha_1\, \lambda_2 + \alpha_2\, \lambda_1\big)\Big]\, ds. \end{aligned}$$

By (3.5), we have

$$\alpha_1 + \alpha_2 = \frac{2\kappa}{\sigma^2} \quad \text{and} \quad \alpha_1\, \lambda_2 + \alpha_2\, \lambda_1 = \frac{2\, b^*}{\sigma^2}.$$

Therefore, we showed the Lemma 3.1. □

By using the auxiliary function constructed above, we have the following lemma to show the uniqueness of problem (2.19).

Lemma 3.2. *Let $\mathbb{X}$ be defined as in (3.1). Then in $\mathbb{X}$, problem (2.19) admits at most one solution.*

Proof. Suppose the PDE problem (2.19), for u, admits two solutions, say, u_1 and u_2 in $\mathbb{X}$. Set

$$w := (u_1 - u_2)\, e^{-T}.$$

Then w satisfies the following equations:

$$\begin{cases} \Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda(r) + 1\Big) w = 0 & \text{in } (0,\infty)^2, \\ w(\cdot, 0) = 0 & \text{in } (0,\infty), \\ w_r \in L^\infty((0,\infty)^2) \end{cases} \tag{3.6}$$

where

$$\lambda(r) = \Lambda(r) + \Lambda_1(r) + \Lambda_2(r) = a\, r + b + p\, H(r - B_2) + q\, H(B_1 - r).$$

We are going to show that $w(r,T) \equiv 0$. Let ψ be the function defined as in (3.4) with $a^* = a > 0$ and $b^* = 0$. Fix $\epsilon > 0$. Set $\psi_\epsilon = \epsilon\psi \pm w$.

Claim: $\psi_\epsilon \geq 0$.

Proof of claim: Suppose the claim is not true. Then there exists $(x_0, t_0) \in (0,\infty)^2$ such that $\psi_\epsilon(x_0, t_0) < 0$. From lemma 3.1, we know that $\lim\limits_{r\to\infty} \psi'(r) = \infty$ and $\lim\limits_{r\searrow 0} \psi'(r) = -\infty$, also $w_r \in L^\infty((0,\infty)^2)$, thus there exists $\delta > 0, R > 0$ such that

$$\begin{cases} \frac{\partial \psi_\epsilon}{\partial r} > 1 & \text{on } [R,\infty) \times [0,\infty), \\ \frac{\partial \psi_\epsilon}{\partial r} < -1 & \text{on } (0,\delta] \times [0,\infty). \end{cases}$$

Since $\psi_\epsilon \in C([\delta, R] \times [0, t_0])$, there exits $(x^*, t^*) \in [\delta, R] \times [0, t_0]$ such that

$$\psi_\epsilon(x^*, t^*) = \min_{[\delta,R]\times[0,t_0]} \psi_\epsilon < 0.$$

Hence, (x^*, t^*) is a point of global minimum of ψ_ϵ on $(0,\infty) \times [0,\infty)$. There are two cases may happen here: (1) $x^* \neq B_1$ and $x^* \neq B_2$, (2) $x^* = B_1$ or $x^* = B_2$.
(1) Suppose $x^* \neq B_1$ and $x^* \neq B_2$. Obviously, $t^* > 0, \delta < x^* < R$. Then

$$\frac{\partial \psi_\epsilon}{\partial T}(x^*, t^*) \leq 0, \quad \frac{\partial \psi_\epsilon}{\partial r}(x^*, t^*) = 0, \quad \frac{\partial^2 \psi_\epsilon}{\partial r^2}(x^*, t^*) \geq 0.$$

Therefore,

$$\Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda(r) + 1\Big)\psi_\epsilon\Big|_{(x^*,t^*)} = \frac{\partial \psi_\epsilon}{\partial T} - \frac{\sigma^2}{2} r \frac{\partial^2 \psi_\epsilon}{\partial r^2} - (\kappa - \beta\, r)\frac{\partial \psi_\epsilon}{\partial r} + (r + \lambda(r) + 1)\psi_\epsilon\Big|_{(x^*,t^*)} \leq 0. \tag{3.7}$$

On the other hand,

$$\begin{aligned}\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)\psi_\epsilon\Big|_{(x^*,t^*)} &= \Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)(\epsilon\psi\pm w)\Big|_{(x^*,t^*)}\\ &= \epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)\psi\Big|_{(x^*,t^*)}\\ &= \epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}+ar\Big)\psi\Big|_{(x^*,t^*)}+\epsilon\Big(\lambda(r)-ar+1\Big)\psi\Big|_{(x^*,t^*)}>0,\end{aligned}$$

which contradicts (3.7).

(2) Suppose $x^*=B_1$ or $x^*=B_2$. For simplicity, here we only consider the case $x^*=B_1$, similarly to the case $x^*=B_2$. Now we obtain two sub-cases, one from the left of B_1 and another one from the right of B_1:

$$\begin{cases}\psi_{\epsilon T}(B_1^-,t^*)\le 0,\\ \\ \psi_{\epsilon r}(B_1,t^*)=0,\ (\because \psi_\epsilon(\cdot,t^*)\in C^1((0,\infty)) \text{ and at } B_1 \text{ obtain of min})\\ \\ \psi_{\epsilon rr}(B_1^-,t^*)\ge 0.\ (\because \psi_\epsilon(\cdot,t^*)\in C^2((0,B_1]) \text{ and at } B_1 \text{ obtain of min})\end{cases}$$

The third argument above is by L'Hospital's rule. Similarly,

$$\begin{cases}\psi_{\epsilon T}(B_1^+,t^*)\le 0,\\ \\ \psi_{\epsilon r}(B_1,t^*)=0,\ (\because \psi_\epsilon(\cdot,t^*)\in C^1((0,\infty)) \text{ and at } B_1 \text{ obtain of min})\\ \\ \psi_{\epsilon rr}(B_1^+,t^*)\ge 0.\ (\because \psi_\epsilon(\cdot,t^*)\in C^2([B_1,B_2])) \text{ and at } B_1 \text{ obtain of min})\end{cases}$$

Therefore,

$$\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)\psi_\epsilon\Big|_{(B_1^\pm,t^*)}\le 0 \tag{3.8}$$

However,

$$\begin{aligned}\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)\psi_\epsilon\Big|_{(B_1^\pm,t^*)} &= \Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)(\epsilon\psi\pm w)\Big|_{(B_1^\pm,t^*)}\\ &= \epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda(r)+1\Big)\psi\Big|_{(B_1^\pm,t^*)}\\ &= \epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}+ar\Big)\psi\Big|_{(B_1^\pm,t^*)}+\epsilon\big(\lambda(r)-ar+1\big)\psi\Big|_{(B_1^\pm,t^*)}>0,\end{aligned}$$

which contradicts with (3.8). Therefore, the claim holds and $\psi_\epsilon\ge 0$. Thus,

$$|w|\le\epsilon\psi,\ \forall(r,T)\in(0,\infty)\times[0,\infty).$$

Sending $\epsilon\searrow 0$, we have $w\equiv 0$, i.e., $u_1=u_2$. Thus we showed the uniqueness of problem (2.19). □

3.2 EXISTENCE OF SOLUTION OF PDE PROBLEM

We first need to regularize the PDE problem (2.19) due to the degeneracy $\mathcal{L}$ at $r = 0$ and unboundedness of Λ and f near $r = \infty$.

3.2.1 Regularization of PDE problem

Fix $\epsilon \in (0, 1)$. We consider the problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u^\epsilon = f(r) & \text{in } (\epsilon, \frac{1}{\epsilon}) \times (0, \frac{1}{\epsilon^2}) := Q_\epsilon, \\ u^\epsilon(\cdot, 0) = 0 & \text{in } (\epsilon, \frac{1}{\epsilon}), \\ u^\epsilon_r = 0 & \text{on } \{\epsilon, \frac{1}{\epsilon}\} \times (0, \frac{1}{\epsilon^2}). \end{cases} \tag{3.9}$$

By standard parabolic PDE theorem, problem (3.9) admits a unique solution

$$u^\epsilon \in \mathbb{W} := W^{2,1}_p(Q_\epsilon) \cap C^{2,1}((\epsilon, B_1] \times (0, \frac{1}{\epsilon^2})) \cap C^{2,1}([B_1, B_2) \times (0, \frac{1}{\epsilon^2})) \cap C^{2,1}([B_2, \frac{1}{\epsilon}) \times (0, \frac{1}{\epsilon^2}))$$

3.2.2 Apriori Estimate of Regularized PDE Problem

Lemma 3.3. (L^∞*-estimate of* u^ϵ) *There exists a constant* C_0*, which does not depend on* ϵ *such that* $|u^\epsilon| \le C_0$.

Proof. Set $\Lambda_0(r) = r$ and $C_0 = \|\frac{f}{\Lambda_0+\Lambda+\Lambda_1+\Lambda_2}\|_\infty$. Based on the following relation

$$-\left\|\frac{f}{\Lambda_0 + \Lambda + \Lambda_1 + \Lambda_2}\right\|_\infty \le \frac{f}{\Lambda_0 + \Lambda + \Lambda_1 + \Lambda_2} \le \left\|\frac{f}{\Lambda_0 + \Lambda + \Lambda_1 + \Lambda_2}\right\|_\infty$$

one can check that C_0 is a super solution and $-C_0$ is a sub solution of problem (3.9). Hence, by comparison principle, $|u^\epsilon| \le C_0$ holds. □

The most difficult part of the apriori estimate is the Lipschitz continuity in r. We shall first estimate u^ϵ_T, and then regard the equation as a boundary value problem for the ODE to estimate u^ϵ_r.

Lemma 3.4. (L^∞-estimate of u^ϵ_T) There exists constants C_1 and C_2, which does not depend on ϵ such that $|u^\epsilon_T| \le C_1 + C_2\, r$.

Proof. Set $w^\epsilon = u^\epsilon_T$. Then w^ϵ satisfies the following problem:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) w^\epsilon = 0 & \text{in } (\epsilon, \frac{1}{\epsilon}) \times (0, \frac{1}{\epsilon^2}) =: Q_\epsilon \\ w^\epsilon(\cdot, 0) = f & \text{in } (\epsilon, \frac{1}{\epsilon}) \\ w^\epsilon_r = 0 & \text{on } \{\epsilon, \frac{1}{\epsilon}\} \times (0, \frac{1}{\epsilon^2}) \end{cases} \tag{3.10}$$

Consider $W_1(r,T) = \sqrt{\kappa + \frac{\sigma^2}{2} + (r-\epsilon)^2}$. By the definitions of W_1 and f, $\|\frac{f}{W_1}\|_\infty$ is well-defined. Denote

$$W_2(r,T) := \Big\|\frac{f}{W_1}\Big\|_\infty W_1.$$

One can check that

$$\begin{cases} \Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big) W_2 \geq 0 & \text{in } (\epsilon, \frac{1}{\epsilon}) \times (0, \frac{1}{\epsilon^2}) =: Q_\epsilon \\ W_2(r,0) \geq f(r) & \text{in } (\epsilon, \frac{1}{\epsilon}) \\ \frac{\partial W_2}{\partial \mathbf{n}} \geq 0 & \text{on } \{\epsilon, \frac{1}{\epsilon}\} \times (0, \frac{1}{\epsilon^2}). \end{cases}$$

Hence, W_2 is a super solution and $-W_2$ is a sub solution of problem (3.10). By comparison principle, we have

$$|w^\epsilon| = |u_T^\epsilon| \leq W_2 = \Big\|\frac{f}{W_1}\Big\|_\infty W_1 = \Big\|\frac{f}{W_1}\Big\|_\infty \sqrt{\kappa + \frac{\sigma^2}{2} + (r-\epsilon)^2}$$

Let $C_1 = \|\frac{f}{W^*}\|_\infty (1 + \sqrt{\kappa + \frac{\sigma^2}{2}})$ and $C_2 = 1 + \|\frac{f}{W^*}\|_\infty$, Then

$$|u_T^\epsilon| \leq W_2 \leq C_1 + C_2 r, \quad \forall (r,T) \in Q_\epsilon$$

Therefore we showed the Lemma 3.4. □

Lemma 3.5. (L^∞-estimate of u_r^ϵ) There exists a constant C_3, which does not depend on ϵ such that $|u_r^\epsilon| \leq C_3$.

Proof. We can write the first equation in (3.9) as

$$-\frac{\sigma^2}{2} r u_{rr}^\epsilon - (\kappa - \beta r) u_r^\epsilon = F := f(r) - u_T^\epsilon - (r + \Lambda + \Lambda_1 + \Lambda_2) u^\epsilon \tag{3.11}$$

Set $\mu = \frac{2\kappa}{\sigma^2}$ and $\nu = \frac{2\beta}{\sigma^2}$. Multiplying $r^{\mu-1}e^{-\nu r}$ on both sides of(3.11), gives

$$-\Big(r^\mu e^{-\nu r} u_r^\epsilon\Big)_r = r^{\mu-1} e^{-\nu r} \bar{F} \tag{3.12}$$

where $\bar{F} = \frac{2}{\sigma^2} F$. Notice that for any $r \in (\epsilon, \frac{1}{\epsilon})$, based on the previous estimates on u^ϵ and u_T^ϵ, there exists a constant C_4 such that

$$|\bar{F}| \leq \frac{2}{\sigma^2}\Big(|f(r)| + |u_T^\epsilon| + |(r + \Lambda + \Lambda_1 + \Lambda_2)|\,|u^\epsilon|\Big) \leq C_4\,(1 + r)$$

Denote $C_5 = \max\{1, \frac{2\mu}{\nu}\}$. Therefore, when $r \in (\epsilon, C_5)$, integrating (3.12) over $[\epsilon, r]$ gives,

$$|u_r^\epsilon| \leq r^{-\mu} e^{\nu r} \int_\epsilon^r \rho^{\mu-1} e^{-\nu\rho} |\bar{F}|\, d\rho \leq C_4 e^{\nu r} r^{-\mu} \int_0^r (\rho^{\mu-1} + \rho^\mu) d\rho \leq C_4 e^{\nu C_5} (\frac{1}{\mu} + \frac{C_5}{\mu+1}) =: C_6$$

When $r \in (C_5, \frac{1}{\epsilon})$, by using integration by parts, we have the following estimation:

$$\int_r^\infty e^{-\nu\rho} \rho^\mu d\rho = \frac{1}{\nu} e^{-\nu r} r^\mu + \frac{\mu}{\nu} \int_r^\infty e^{-\nu\rho} \rho^{\mu-1} d\rho \leq \frac{1}{\nu} e^{-\nu r} r^\mu + \frac{\mu}{\nu r} \int_r^\infty e^{-\nu\rho} \rho^\mu d\rho.$$

Therefore,

$$\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho \le \frac{r^\mu e^{-\nu r}}{\nu - \frac{\mu}{r}}$$

Then integrating (3.12) over $[r, \frac{1}{\epsilon}]$ gives,

$$\begin{aligned}|u_r^\epsilon| &\le r^{-\mu}e^{\nu r}\int_r^{\frac{1}{\epsilon}} \rho^{\mu-1}e^{-\nu\rho}|\bar{F}|d\rho \le C_4 r^{-\mu}e^{\nu r}\int_r^\infty e^{-\nu\rho}(\rho^{\mu-1}+\rho^\mu)\,d\rho \le 2C_4 r^{-\mu}e^{\nu r}\int_r^\infty e^{-\nu\rho}\rho^\mu\,d\rho \\ &\le 2C_4 r^{-\mu}e^{\nu r}\frac{r^\mu e^{-\nu r}}{\nu - \frac{\mu}{r}} = \frac{4C_4}{\nu} := C_7\end{aligned}$$

Therefore, there exists some positive constant $C_3 := \max\{C_6, C_7\}$ such that $|u_r^\epsilon| \le C_3$, $\forall (r,T) \in Q_\epsilon$. □

Based on above apriori estimates of regularized PDE problem (3.9), one can easily show the existence of problem (2.19).

Lemma 3.6. *Let $\mathbb{X}$ be defined as in (3.1). Then in $\mathbb{X}$, problem (2.19) admits at least one solution.*

Proof. By L^p interior estimation, $\forall\delta > 0$, $\forall p > 1$, $\exists\, C(\delta,p) > 0$, s.t. $\forall\, 0 < \epsilon < \frac{\delta}{2}$, we have,

$$\|u^\epsilon\|_{W_p^{2,1}([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}])} \le C(\delta,p)\Big\{\|f\|_{L^p([\frac{\delta}{2},\frac{2}{\delta}])} + \|u^\epsilon\|_{L^\infty([\frac{\delta}{2},\frac{2}{\delta}]\times[0,(\frac{2}{\delta})^2])}\Big\} \le C(\delta,p)$$

Also,

$$W_p^{2,1}\Big(\Big[\delta,\frac{1}{\delta}\Big]\times\Big[0,\frac{1}{\delta^2}\Big]\Big) \hookrightarrow\hookrightarrow C^{1,\frac{1}{2}}\Big(\Big[\delta,\frac{1}{\delta}\Big]\times\Big[0,\frac{1}{\delta^2}\Big]\Big)$$

Therefore, by using the routing argument, there exists $\{\epsilon_n\}_{n=1}^\infty \subseteq (0,\frac{\delta}{2})$ and $u \in \mathbb{X}$ such that

$$\begin{cases}\lim\limits_{n\to\infty}\|u^{\epsilon_n} - u\|_{C^{1,\frac{1}{2}}([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}])} = 0 \\ \\ \lim\limits_{n\to\infty}\epsilon_n = 0\end{cases}$$

Also, by lemma 3.5, $\{u_r^{\epsilon_n}\}_{n=1}^\infty$ is uniformly bounded. By sending $\delta \searrow 0$, u is a solution of problem (2.19). □

Combine Lemma 3.2 and Lemma 3.6, we proved Theorem 3.1.

4.0 INFINITE HORIZON PROBLEM

In this chapter we study an infinite horizon problem of PDE problem (2.19).

Theorem 4.1. *Let* $p = \nu - \frac{\mu}{r}$, $q = \frac{2}{\sigma^2}\left(1 + \frac{C(r)}{r}\right)$, $|\bar{f}| \leq \frac{C(1+r)}{r}$ *for some constant* C, *where* $\nu = \frac{2\beta}{\sigma^2}$, $\mu = \frac{2\kappa}{\sigma^2}$, $C(r) \in L^\infty_{loc}([0,\infty])$, $C(r) \geq 0$, $\beta > 0$, $\kappa > 0$ *and* $\sigma > 0$. *Then the following ODE problem*

$$\begin{cases} -\varphi'' + p\,\varphi' + q\,\varphi = \bar{f} & \text{in } (0,\infty) \\ \varphi' \in L^\infty(0,\infty) \end{cases} \tag{4.1}$$

admits a unique solution.

Theorem 4.2. *Let* u *be the solution of problem* (2.19) *given by Theorem 3.1. Then as* $T \to \infty$, $u(\cdot, T)$ *approaches exponentially uniformly to* $\varphi(\cdot)$, *the unique solution of problem* (4.1).

Remark 4.1. *Let* Λ, Λ_1, Λ_2 *and* $\mathcal{L}$ *are defined as in* (2.9), (2.10), (2.11) *and* (2.17), *respectively;* $f = K\Lambda - h$ *and* $\sigma > 0, \kappa > 0, \beta > 0, a > 0, b > 0, 0 < B_1 < B_2, p \geq 0, q \geq 0, K > 0, h > 0$. *Theorem 4.1 covers the case when we divide* $\frac{\sigma^2}{2}r$ *both sides of the first equation of problem* (2.19) *with taking* $T \to \infty$.

We will first study the properties of solutions of corresponding homogeneous ODE problem. Then we use the boundary conditions and variation of constants method to show the well-posedness of inhomogeneous ODE problem (4.1), i.e. to show the Theorem 4.1. For the proof of Theorem 4.2, please refer to section 4.4 for details.

4.1 THE HOMOGENEOUS ODE PROBLEM

To solve the inhomogeneous ODE problem (4.1), we first consider the corresponding homogeneous equation:

$$-\phi'' + p\,\phi' + q\,\phi = 0 \quad \text{in } (0,\infty) \tag{4.2}$$

Lemma 4.1. *Assume* p *and* q *are defined as in Theorem 4.1. Then the homogeneous ODE problem* (4.2) *admits two independent solutions* ϕ_1 *and* ϕ_2 *satisfying:*

$$\phi_1' < 0,\ \phi_2' > 0,\ \phi_1 > 0,\ \phi_2 > 0 \quad \text{in } (0,\infty) \tag{4.3}$$

$$\phi_1(r) \to 0 \quad \text{as } r \to \infty \tag{4.4}$$

$$\phi_1'(r) \to -r^{-\mu} \quad \text{as } r \to 0 \tag{4.5}$$

$$\phi_1'(r) \to 0 \quad \text{as } r \to \infty \tag{4.6}$$

$$\phi_2'(r) = O(1) \quad \text{as } r \to 0 \tag{4.7}$$

$$\frac{\phi_2'(r)}{\phi_2(r)} > \nu \quad \text{as } r \to \infty \tag{4.8}$$

Proof. In order to solve the homogeneous ODE problem (4.2), we need to construct two linearly independent general solutions ϕ_1 and ϕ_2. We will first construct one solution and study its asymptotic behaviour, then we use the Wronskian method to construct another solution. Denote

$$W(r) = \phi_1(r)\,\phi_2'(r) - \phi_1'(r)\,\phi_2(r)$$

as the Wronskian of ϕ_1 and ϕ_2. Since ϕ_1 and ϕ_2 are linearly independent and $W(r)$ satisfies the first order differential equation:

$$W'(r) = p\,W(r),$$

there exists a nonzero constant C_1 such that

$$W(r) = C_1 e^{\nu r} r^{-\mu} \tag{4.9}$$

For convenience, we set $C_1 = 1$. We define ϕ_2 as the solution of the ODE problem with the initial conditions

$$\begin{cases} \phi_2(0) = 1, \\ \phi_2'(0) = 0. \end{cases}$$

Set $A(r) = \nu r - \mu \ln(r)$. Then $e^{-A(r)} = r^{\mu} e^{-\nu r}$ and $A'(r) = \nu - \frac{\mu}{r} = p$. Since ϕ_2 satisfies the equation (4.2), we have

$$\left(e^{-A}\,\phi_2'\right)' = e^{-A}\left(-p\,\phi_2' + \phi_2''\right) = e^{-A}\,q\,\phi_2 \tag{4.10}$$

Integrating both sides of above equation from 0 to r and use initial conditions, we have

$$\begin{aligned} & e^{-A(r)}\,\phi_2'(r) = \int_0^r e^{-A(\rho)}\,q\,\phi_2(\rho)\,d\rho \\ \Longrightarrow\quad & r^{\mu}\,e^{-\nu r}\,\phi_2'(r) = \int_0^r \rho^{\mu}\,e^{-\nu\rho}\left[1 + \frac{C(\rho)}{\rho}\right]\phi_2(\rho)\,d\rho \\ \Longrightarrow\quad & \phi_2'(r) = \frac{e^{\nu r}}{r^{\mu}} \int_0^r \rho^{\mu}\,e^{-\nu\rho}\left[1 + \frac{C(\rho)}{\rho}\right]\phi_2(\rho)\,d\rho \\ \Longrightarrow\quad & \phi_2(r) = 1 + \int_0^r \frac{\int_0^s \rho^{\mu}\,e^{-\nu\rho}\left[1 + \frac{C(\rho)}{\rho}\right]\phi_2(\rho)\,d\rho}{s^{\mu}\,e^{-\nu s}}\,ds \end{aligned}$$

By Picard's method and exchanging integration, we have the expression of $\phi_2(r)$ as:

$$\phi_2(r) = 1 + \int_0^r \phi_2(\rho)\,\mathcal{K}(r,\rho)\,d\rho > 0$$

where

$$\mathcal{K}(r,\rho) = \rho^{\mu}\, e^{-\nu\rho}\left[1+\frac{C(\rho)}{\rho}\right]\int_{\rho}^{r} s^{-\mu}\, e^{\nu s}\, ds$$

and

$$\phi_2'(r) = \phi_2(r)\,\mathcal{K}(r,r) + \int_0^r \phi_2(\rho)\,\frac{\partial \mathcal{K}(r,\rho)}{\partial r}\, d\rho = 0 + r^{-\mu}\, e^{\nu r}\int_0^r \phi_2(\rho)\,\rho^{\mu}\, e^{-\nu\rho}\left[1+\frac{C(\rho)}{\rho}\right] d\rho > 0$$

Now let's study the asymptotic behaviour of $\phi_2'(r)$. When $r \to 0$,

$$\phi_2'(r) \approx \frac{1}{r^{\mu}}\int_0^r \left[\rho^{\mu} + \rho^{\mu-1}\, C(\rho)\right] d\rho = O(r) + \frac{1}{r^{\mu}}\int_0^r \rho^{\mu-1}\, C(\rho)\, d\rho \le O(r) + \|C(r)\|_{L^{\infty}([0,1])} = O(1)$$

When $r \to \infty$. Since $\phi_2'(r) > 0$, for some positive $k(\rho)$, we can write

$$\phi_2(r) = e^{\int_0^r k(\rho)\, d\rho}$$

Therefore,

$$\begin{cases} \phi_2'(r) = \phi_2(r)\, k(r) \\ \\ \phi_2''(r) = k'(r)\,\phi_2(r) + k^2(r)\,\phi_2(r) \end{cases}$$

Plugging $\phi_2''(r)$ and $\phi_2'(r)$ into equation (4.2) to obtain:

$$\begin{aligned} & k'(r)\,\phi_2(r) + k^2(r)\,\phi_2(r) = p\, k(r)\,\phi_2(r) + q\,\phi_2(r) \\ \Rightarrow\quad & k'(r) = -k^2(r) + p\, k(r) + q \\ \Rightarrow\quad & k'(r) \ge -k^2(r) + (\nu-\varepsilon)\, k(r) + \frac{2}{\sigma^2} \qquad \text{where } \lim_{r\to\infty}\varepsilon = 0 \\ \Rightarrow\quad & \liminf_{r\to\infty} k(r) = \liminf_{r\to\infty}\frac{\phi_2'(r)}{\phi_2(r)} \ge k^* = \frac{\nu + \sqrt{\nu^2 + \frac{8}{\sigma^2}}}{2} > \nu \end{aligned}$$

Based on above estimation, as $r \to \infty$, we have

$$\Big(\ln\phi_2(r)\Big)' > \nu > 0 \Rightarrow \ln\,\phi_2(r) - \ln\,\phi_2(0) > \nu\, r \Rightarrow \phi_2(r) > e^{\nu\, r} \text{ and } \phi_2'(r) > \nu e^{\nu\, r} \tag{4.11}$$

Now we construct another solution $\phi_1(r)$. By the Wronskian of ϕ_1 and ϕ_2,

$$W(r) = \phi_1(r)\,\phi_2'(r) - \phi_1'(r)\,\phi_2(r) = -\phi_2^2(r)\left(\frac{\phi_1(r)}{\phi_2(r)}\right)' = e^{\nu\, r}\, r^{-\mu} \tag{4.12}$$

We have the expression of $\phi_1(r)$:

$$\phi_1(r) = \phi_2(r)\int_r^{\infty}\frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)}\, d\rho > 0$$

Now let's prove (4.4). Since

$$\phi_2(r) > e^{\nu r} > \frac{e^{\nu r}}{r^{\mu}} > 0 \quad \text{as } r \to \infty$$

It is easy to get the following estimate as $r \to \infty$,

$$0 < \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)} < \frac{1}{\phi_2(\rho)} < e^{-\nu\rho}$$

Then the following argument is easy to see

$$0 \leq \lim_{r\to\infty} \int_r^\infty \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)}\, d\rho \leq \lim_{r\to\infty} \int_r^\infty e^{-\nu\rho}\, d\rho = \lim_{r\to\infty} \frac{1}{\nu} e^{-\nu r} = 0 \tag{4.13}$$

Therefore, based on (4.11), (4.13) and L'Hospital rule,

$$\lim_{r\to\infty} \phi_1(r) = \lim_{r\to\infty} \frac{\int_r^\infty \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)}\, d\rho}{\frac{1}{\phi_2(r)}} = \lim_{r\to\infty} \frac{-\frac{e^{\nu r} r^{-\mu}}{\phi_2^2(r)}}{-\frac{\phi_2'(r)}{\phi_2^2(r)}} = \lim_{r\to\infty} \frac{e^{\nu r} r^{-\mu}}{\phi_2'(r)} = 0$$

Hence, by Mean Value theorem, there exists ϱ_n such that

$$\begin{cases} \lim\limits_{n\to\infty} \Big[\phi_1(n+1) - \phi_1(n)\Big] = \lim\limits_{n\to\infty} \phi_1'(\varrho_n) = 0 \\ \\ \lim\limits_{n\to\infty} \varrho_n = \infty \end{cases}$$

Recall that (4.10), we have

$$\int_r^{\varrho_n} \left(e^{-A}\phi_1'\right)' ds = \int_r^{\varrho_n} e^{-A(s)}\, q\, \phi_1(s)\, ds > 0 \Rightarrow e^{-A(\varrho_n)}\, \phi_1'(\varrho_n) - e^{-A(r)}\, \phi_1'(r) > 0$$

Sending $n \to \infty$, we have

$$e^{-A(\varrho_n)}\, \phi_1'(\varrho_n) \to 0 \Rightarrow -e^{-A(r)}\, \phi_1'(r) > 0 \Rightarrow \phi_1'(r) < 0$$

Next, we are going to show (4.5). As $r \to 0$, we have two cases. If $\mu \neq 1$, plugging in $\phi_1 = r^n$ into (4.2), we obtain $\phi_1 = r^{1-\mu}$. If $\mu = 1$, plugging in $\phi_1 = r^n$ into (4.2), we obtain $\phi_1 = \ln r$. Plug $\phi_1 = r^{1-\mu}$ or $\phi_1 = \ln r$ into (4.12). Since as $r \to 0$, we have $\phi_2'(r) \to O(1)$, $\phi_2(r) \to 1$ and $\phi_1'(r) < 0$, it's easy to see that

$$\lim_{r\to 0} \phi_1'(r) = -r^{-\mu}$$

Lastly, we need show (4.6). Since

$$\phi_1 > 0, \quad \phi_2' > 0 \text{ and } W(r) = \phi_1(r)\, \phi_2'(r) - \phi_1'(r)\, \phi_2(r) = e^{\nu r} r^{-\mu}$$

we know that

$$-\phi_1'(r)\, \phi_2(r) \leq e^{\nu r} r^{-\mu} \Rightarrow -\frac{e^{\nu r} r^{-\mu}}{\phi_2(r)} \leq \phi_1'(r) \leq 0$$

Since $\lim\limits_{r\to\infty} \frac{\phi_2'(r)}{\phi_2(r)} \geq k^* > \nu$, by squeeze theorem, we know that

$$\lim_{r\to\infty} \phi_1'(r) = 0$$

Therefore, we complete the proof of Lemma 4.1. □

By applying above lemma with boundary condition, one can easily show the uniqueness of inhomogeneous ODE problem (4.1).

4.2 UNIQUENESS OF SOLUTION OF INHOMOGENEOUS ODE PROBLEM

Lemma 4.2. *$\phi \equiv 0$ is the only solution of*

$$\begin{cases} -\phi'' + p\,\phi' + q\,\phi = 0 \quad \text{in } (0,\infty) \\ \phi' \in L^\infty(0,\infty) \end{cases} \tag{4.14}$$

Proof. The general solution of above ODE problem (4.14) is

$$\phi(r) = C_1\phi_1(r) + C_2\phi_2(r) \tag{4.15}$$

for some constants C_1 and C_2. From (4.6) and (4.8), we know that $\phi_1'(r) = 0$ and $\phi_2'(r) \to \infty$ as $r \to \infty$. Since $\phi'(r) \in L^\infty(0,\infty)$, we then have $C_2 = 0$. From (4.5), we know that $|\phi_1'(r)| \to \infty$ as $r \to 0$. Since $\phi'(r) \in L^\infty(0,\infty)$, we then have $C_1 = 0$. Therefore, $\phi \equiv 0$ is the only solution of ODE problem (4.14). □

By applying the above lemma, we obtained the following corollary easily.

Corollary 4.3. *The inhomogeneous ODE problem (4.1) admits at most one solution.*

Proof. Suppose the inhomogeneous ODE problem (4.1) admits two solutions $\varphi_1(r)$ and $\varphi_2(r)$, then $\phi(r) = \varphi_1(r) - \varphi_2(r)$ satisfies

$$\begin{cases} -\phi'' + p\,\phi' + q\,\phi = 0 \quad \text{in } (0,\infty) \\ \phi' \in L^\infty(0,\infty) \end{cases}$$

By using lemma 4.2,

$$\phi(r) \equiv 0 \Rightarrow \varphi_1(r) \equiv \varphi_2(r) \tag{4.16}$$

Therefore, we completed the proof of uniqueness of ODE problem (4.1). □

4.3 EXISTENCE OF SOLUTION OF INHOMOGENEOUS ODE PROBLEM

By variation of constants, the following lemma showed the existence of ODE problem (4.1).

Lemma 4.4. *Let $\phi_1(r)$ and $\phi_2(r)$ be defined as in Lemma 4.1, and $C_1(r)$, $C_2(r)$ be defined by*

$$\begin{cases} C_1(r) := \int_0^r \frac{\phi_2(\rho)\bar{f}(\rho)}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho, \\ C_2(r) := \int_r^\infty \frac{\phi_1(\rho)\bar{f}(\rho)}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho. \end{cases} \tag{4.17}$$

Then

$$\varphi(r) = C_1(r)\phi_1(r) + C_2(r)\phi_2(r) \tag{4.18}$$

is a solution of inhomogeneous ODE problem (4.1).

Proof. In order to show Lemma 4.4, we will show the following three steps.

Step 1: First we will show that $C_1(r)$ and $C_2(r)$ are well-defined; i.e, the improper integrals are convergent.

As $r \to 0$, ϕ_2 is bounded. We have,

$$\Big|\frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\Big| = O(1)\, r^{\mu-1} \tag{4.19}$$

Since $\mu > 0$, the integral defining $C_1(r)$ is convergent; i.e, $C_1(r)$ is well-defined and is a C^1 function.

As $r \to \infty$,

$$\Big|\frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2}\Big| = O(1)e^{-\nu r} r^{\mu} \tag{4.20}$$

thus the integral defining $C_2(r)$ is convergent; i.e, $C_2(r)$ is well-defined and is a C^1 function.

Step 2: By variation of parameter, it is straightforward to verify that $\varphi(r)$ is a solution of

$$-\varphi'' + p\,\varphi' + q\,\varphi = \bar{f}$$

Step 3: We will lastly show that $\varphi(r)$ defined as in (4.18) satisfies $\varphi' \in L^\infty(0,\infty)$. By differentiation, we obtain,

$$\begin{aligned}
\varphi'(r) &= C_1'(r)\,\phi_1(r) + \phi_1'(r)\,C_1(r) + C_2'(r)\,\phi_2(r) + \phi_2'(r)\,C_2(r) \\
&= \frac{\phi_2\bar{f}}{\phi_1\phi_2' - \phi_1'\phi_2}\phi_1 + \phi_1'(r)\,C_1(r) - \frac{\phi_1\bar{f}}{\phi_1\phi_2' - \phi_1'\phi_2}\phi_2 + \phi_2'(r)\,C_2(r) \\
&= \phi_1'(r)\,C_1(r) + \phi_2'(r)\,C_2(r)
\end{aligned}$$

As $r \to 0$, by (4.5) and (4.19), we have

$$C_1(r) = \int_0^r O(1)\rho^{\mu-1}\,d\rho = O(1)r^{\mu} \qquad \text{and} \quad \phi_1'(r) \to -r^{-\mu}.$$

Thus

$$\phi_1'(r)\,C_1(r) = O(1)r^{-\mu}r^{\mu} = O(1) \tag{4.21}$$

Then we consider about $\phi_2'(r)\,C_2(r)$ as $r \to 0$.

$$C_2(r) = \int_1^\infty \frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2}\,d\rho + \int_r^1 \frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2}\,d\rho = O(1) + \int_r^1 O(1)\,\phi_1(\rho)\,\rho^{\mu-1}\,d\rho$$

Case 1: If $\mu = 1$, by (4.5) we have

$$\phi_1'(r) \approx -\frac{1}{r} \Rightarrow \phi_1(r) \approx -\ln r$$

Therefore,

$$C_2(r) = O(1) + O(1)\int_r^1 \ln\rho\,d\rho = O(1)$$

Case 2: If $\mu \neq 1$, by (4.5) we have

$$\phi_1'(r) \approx -r^{-\mu} \Rightarrow \phi_1(r) \approx O(1)\,r^{1-\mu}$$

Therefore,

$$C_2(r) = O(1) + O(1) \int_r^1 \rho^{1-\mu}\, \rho^{\mu-1}\, d\rho = O(1)$$

Thus by (4.7),

$$\phi_2'(r)\, C_2(r) = O(1). \tag{4.22}$$

By (4.21) and (4.22), hence

$$\varphi'(r) = \phi_1'(r)\, C_1(r) + \phi_2'(r)\, C_2(r) = O(1) \qquad \text{as } r \to 0 \tag{4.23}$$

As $r \to \infty$,

$$\begin{aligned} C_1(r) &= \int_0^r \frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho = \int_0^R \frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho + \int_R^r \frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho \\ &= O(1) + \frac{\phi_2(r)}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_R^r \frac{r^{-\mu}e^{\nu r}}{\rho^{-\mu}e^{\nu\rho}} \frac{\phi_2(\rho)}{\phi_2(r)}\, d\rho \end{aligned}$$

and by (4.8), we have

$$\frac{\phi_2'(r)}{\phi_2(r)} \geq k^* > \nu \Rightarrow \ln \frac{\phi_2(\rho)}{\phi_2(r)} \leq -k^*(r-\rho) \quad \text{and} \quad k^* > \nu.$$

Then,

$$\begin{aligned} \left|\phi_1'(r)C_1(r)\right| &= O(1) + \frac{\left|\phi_1'(r)\phi_2(r)\right|}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_R^r \left(\frac{\rho}{r}\right)^{\mu} e^{\nu(r-\rho) + \ln\frac{\phi_2(\rho)}{\phi_2(r)}}\, d\rho \\ &\leq O(1) + O(1) \int_R^r e^{(\nu-k^*)(r-\rho)}\, d\rho = O(1) + O(1) \int_0^\infty e^{(\nu-k^*)t}\, dt \end{aligned}$$

Therefore,

$$\left|\phi_1'(r)C_1(r)\right| = O(1) \qquad \text{as } r \to \infty \tag{4.24}$$

Then we consider about $\phi_2'(r)\, C_2(r)$ as $r \to \infty$:

$$C_2(r) = \int_r^\infty \frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho = \frac{\phi_1(r)}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_r^\infty \frac{r^{-\mu}e^{\nu r}}{\rho^{-\mu}e^{\nu\rho}} \frac{\phi_1(\rho)}{\phi_1(r)}\, d\rho$$

and since $\phi_1'(r) < 0$, as $\rho > r$,

$$0 < \phi_1(\rho) < \phi_1(r) \Rightarrow \ln \frac{\phi_1(\rho)}{\phi_1(r)} < 0.$$

Therefore,

$$\left|\phi_2'(r)C_2(r)\right| = \frac{\phi_2'(r)\phi_1(r)}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_r^\infty \left(\frac{\rho}{r}\right)^{\mu} e^{-\nu(\rho-r) + \ln\frac{\phi_1(\rho)}{\phi_1(r)}}\, d\rho \leq O(1) \int_r^\infty e^{-\nu(\rho-r)}\, d\rho$$

Hence,

$$\left|\phi_2'(r)C_2(r)\right| = O(1) \qquad \text{as } r \to \infty \tag{4.25}$$

Based on (4.24) and (4.25), we have

$$\varphi'(r) = \phi_1'(r)\, C_1(r) + \phi_2'(r)\, C_2(r) = O(1) \qquad \text{as } r \to \infty \tag{4.26}$$

Combine (4.23) and (4.26) together, we showed that $\varphi' \in L^\infty(0,\infty)$. Therefore we complete the proof of Lemma 4.4. □

Combine Corollary 4.3 and Lemma 4.4, we complete the proof of Theorem 4.1.

4.4 ASYMPTOTIC BEHAVIOUR OF SOLUTION OF PDE PROBLEM AS $T \to \infty$

Let's prove Theorem 4.2.

Proof. Set $v(r,T) = u(r,T) - \varphi(r)$. Then v satisfies the following equations:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) v = 0 & \text{in } (0,\infty)^2 \\ v(\cdot,0) = -\varphi(\cdot) & \text{in } (0,\infty) \\ v_r \in L^\infty((0,\infty)^2) \end{cases} \tag{4.27}$$

Claim: $\|\varphi\|_\infty e^{-bT}$ and $-\|\varphi\|_\infty e^{-bT}$ are the super and sub solutions of problem (4.27).

Proof of Claim: Plugging $\|\varphi\|_\infty e^{-bT}$ and $-\|\varphi\|_\infty e^{-bT}$ into above equation, we have

$$\begin{cases} -b\,\|\varphi\|_\infty e^{-b\,T} + \Big[(a+1)\,r + b + p\,H(r-B_2) + q\,H(B_1-r)\Big]\,||\varphi||_\infty e^{-b\,T} \geq 0 \\ ||\varphi||_\infty e^{-b\,T}\Big|_{T=0} = ||\varphi||_\infty \geq -\varphi(r) \end{cases}$$

and

$$\begin{cases} b\,||\varphi||_\infty e^{-b\,T} - \Big[(a+1)\,r + b + p\,H(r-B_2) + q\,H(B_1-r)\Big]\,||\varphi||_\infty e^{-b\,T} \leq 0 \\ -||\varphi||_\infty e^{-b\,T}\Big|_{T=0} = -||\varphi||_\infty \leq -\varphi(r) \end{cases}$$

By comparison principle,

$$|v| \leq ||\varphi||_\infty e^{-b\,T} \tag{4.28}$$

As $T \to \infty$, it is easily to see that $v \to 0$, thus

$$u(r,T) \to \varphi(r)$$

Therefore we proved Theorem 4.2. □

5.0 ASYMPTOTIC BEHAVIOUR OF SOLUTION OF CDS MODEL

In this chapter, we study the asymptotic behavior of solution u_{pq} of problem (2.19) as $p,\ q \to \infty$.

5.1 THE LIMITING PROBLEM OF CDS MODEL

First we consider about the following PDE problem comes from structure financial model:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) u = f & \text{in } (B_1, B_2) \times (0, \infty) \\ u(\cdot, 0) = 0 & \text{in } (0, \infty) \\ u(r, t) = 0 & \text{for } r \in (0, B_1] \cup [B_2, \infty), \quad t \geq 0 \\ u_r \in L^\infty((0, \infty)^2) \end{cases} \tag{5.1}$$

where Λ and $\mathcal{L}$ are defined as in (2.9) and (2.17), respectively; $f = K\Lambda - q$ and $\sigma > 0, \kappa > 0, \beta > 0, a > 0, b > 0, 0 < B_1 < B_2, K > 0, q > 0$.

It is easy to check that the well-posedness of above PDE problem (5.1). Specifically, there exists a unique solution u of problem (5.1), and

$$u \in C^{1,\frac{1}{2}}((0, \infty) \times [0, \infty)) \cap C^{2,1}((B_1, B_2) \times [0, \infty)) \tag{5.2}$$

5.2 ASYMPTOTIC BEHAVIOR OF PDE PROBLEM AS $P,\ Q \to \infty$

Recall that $\lambda_{1\,t} = p\,H(r_t - B_2)$ and $\lambda_{2\,t} = q\,H(B_1 - r_t)$. As $p,\ q \to \infty$,

$$\tau_1 \to \tau_1^* = \inf\{t \,|\, r_t \geq B_2\} \quad \textit{and} \quad \tau_2 \to \tau_2^* = \inf\{t \,|\, r_t \leq B_1\}$$

We consider the asymptotic behavior, as $p,\ q \to \infty$, of the solution, u_{pq} of problem (2.19).

Theorem 5.1. *As $p \to \infty$ and $q \to \infty$, the solution u_{pq} of problem (2.19) given by Theorem 3.1, converges to u, the solution of problem (5.1).*

In order to show above Theorem, we start with the following lemmas.

Lemma 5.1. *There exists a continuous bounded function $m_1(\cdot)$ in $(0,\infty)$ such that*

$$\Big|u_{pq}(B_2+\rho,t)\Big| \le \frac{m_1(\rho)}{p}, \quad \forall\ \rho>0,\ t\ge 0 \tag{5.3}$$

Proof. Set $\varphi_1(r) = M_1 + \phi_1(r)$, where M_1 is a positive constant and $\phi_1(r) = \frac{(r-B_2-\rho)^2}{\rho^2}$. Suppose we want $\pm\varphi_1(r)$ to be the super\sub solution of problem (2.19). By comparison principle, we then need

$$\Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big)\,\varphi_1 - f \ge 0 \Longleftrightarrow (\mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2)\,\varphi_1 \ge f$$

Also

$$-\Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big)\,\varphi_1 - f \le 0 \Longleftrightarrow -(\mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2)\,\varphi_1 \le f$$

Therefore, we actually need $\varphi_1(r)$ satisfy

$$\begin{aligned}\Big(\mathcal{L} + \Lambda + p\,H(r-B_2) + q\,H(B_1-r)\Big)\,\varphi_1(r) &= \Big(\mathcal{L} + \Lambda + p\,H(r-B_2) + q\,H(B_1-r)\Big)\big(M_1+\phi_1(r)\big)\\ &\ge \big|f(r)\big|.\end{aligned}$$

Thus we require the following to ensure above statement holds,

$$p\,M_1 + \mathcal{L}\,\phi_1(r) \ge \big|f(r)\big|$$

which equivalent to

$$M_1 \ge \frac{1}{p}\Big[\big|f(r)\big| - \mathcal{L}\,\phi_1(r)\Big]$$

Set

$$M_1 = \frac{m_1(\rho)}{p},$$

where $m_1(\rho) = \sup\limits_{B_2\le r\le B_2+2\rho}\Big\{\big|f(r)\big| + \big|\mathcal{L}\,\phi_1(r)\big|\Big\}$. Therefore by comparison principle and in particular that set $r = B_2+\rho$, we have the following holds:

$$\Big|u_{pq}(B_2+\rho,t)\Big| \le M_1 = \frac{m_1(\rho)}{p}, \quad \forall\ \rho>0,\ t\ge 0$$

□

Lemma 5.2. *There exists a continuous bounded function $m_2(\cdot)$ in $(0,\infty)$ such that*

$$\Big|u_{pq}(B_1-\rho,t)\Big| \le \frac{m_2(\rho)}{p}, \quad \forall\ \rho>0,\ t\ge 0 \tag{5.4}$$

Proof. Set $\varphi_2(r) = M_2 + \phi_2(r)$, where M_2 is a positive constant and $\phi_2(r) = \frac{(r-B_1+\rho)^2}{\rho^2}$. Suppose we want $\pm\varphi_2(r)$ to be the super\sub solution of problem (2.19). By comparison principle, we then need

$$\Big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\Big)\,\varphi_2 - f \ge 0 \Longleftrightarrow (\mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2)\,\varphi_2 \ge f$$

Also

$$-\Big(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda+\Lambda_1+\Lambda_2\Big)\,\varphi_2-f\le 0\Longleftrightarrow -(\mathcal{L}+\Lambda+\Lambda_1+\Lambda_2)\,\varphi_2\le f$$

Therefore, we actually need $\varphi_2(r)$ satisfy

$$\begin{aligned}\Big(\mathcal{L}+\Lambda+p\,H(r-B_2)+q\,H(B_1-r)\Big)\,\varphi_2(r) &= \Big(\mathcal{L}+\Lambda+p\,H(r-B_2)+q\,H(B_1-r)\Big)\big(M_2+\phi_2(r)\big)\\ &\ge \big|f(r)\big|.\end{aligned}$$

Thus we require the following to ensure above statement holds,

$$q\,M_2+\mathcal{L}\,\phi_2(r)\ge\big|f(r)\big|$$

which equivalent to

$$M_2\ge\frac{1}{q}\Big[\big|f(r)\big|-\mathcal{L}\,\phi_2(r)\Big]$$

Set

$$M_2=\frac{m_2(\rho)}{q},$$

where $m_2(\rho)=\sup\limits_{B_1-2\,\rho\le r\le B_1}\Big\{\big|f(r)\big|+\big|\mathcal{L}\,\phi_2(r)\big|\Big\}$. Therefore by comparison principle and in particular that set $r=B_1-\rho$, we have the following holds:

$$\Big|u_{pq}(B_1-\rho,t)\Big|\le M_2=\frac{m_2(\rho)}{p},\quad\forall\ \rho>0,\ t\ge 0$$

□

Lemma 5.3. *Let u_{pq} be the solution of problem (2.19). Then the following:*

$$\big\|u_{pq}\big\|_{C^{\frac{1}{2},\frac{1}{4}}([L_1,L_2]\times[0,T])}\le C(L_1,L_2,T)\tag{5.5}$$

holds for $\forall\,p>0,q>0,0<L_1<B_1<B_2<L_2,T>0$.

Proof. For simplicity, we use $\bar{u}$, $\bar{a}$, $\bar{b}$, $\bar{c}$ to replace u_{pq}, $\frac{\sigma^2}{2}r$, $\kappa-\beta r$, $(a+1)r+b$. Then we first multiply $\eta^2\bar{u}$ on both sides of the first equation in (2.19) and integrate it on $\Omega\times(0,t)$, where η is a cut-off function and $\Omega=(L_1,L_2)$. We obtain:

$$\begin{aligned}0 &= \int_0^t\int_\Omega\eta^2\bar{u}\Big[\bar{u}_t-\bar{a}\bar{u}_{rr}-\bar{b}\bar{u}_r+\bar{c}\bar{u}+\big(pH(r-B_2)+qH(B_1-r)\big)\bar{u}-f\Big]dr\,ds\\ &= \int_0^t\Big\{\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega\eta^2\bar{u}^2\,dr-\int_\Omega\bar{a}\,\eta^2\,\bar{u}\,\bar{u}_{rr}\,dr-\int_\Omega\bar{b}\,\eta^2\,\bar{u}\,\bar{u}_r\,dr+\int_\Omega\bar{c}\,\eta^2\,\bar{u}^2\,dr\\ &\quad+\int_\Omega\big(p\,H(r-B_2)+q\,H(B_1-r)\big)\eta^2\,\bar{u}^2\,dr-\int_\Omega f\,\eta^2\,\bar{u}\,dr\Big\}\,ds\end{aligned}$$

Since by integration by parts, we have

$$\begin{aligned}-\int_\Omega\bar{a}\,\eta^2\,\bar{u}\,\bar{u}_{rr}\,dr &= -\int_\Omega\bar{a}\,\eta^2\,\bar{u}\,d\,\bar{u}_r\\ &= \int_\Omega\big(\bar{a}\,\eta^2\,\bar{u}\big)_r\,\bar{u}_r\,dr\\ &= \int_\Omega\big(\bar{a}\,\eta^2\big)_r\,\bar{u}\,\bar{u}_r\,dr+\int_\Omega\bar{a}\,\eta^2\,\bar{u}_r^2\,dr\end{aligned}$$

Then above expression can be summarized as follows:

$$
\begin{aligned}
0 \;=\; & \int_0^t \Big\{ \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \eta^2 \bar{u}^2\, dr + \int_\Omega \bar{a}\,\eta^2\,\bar{u}_r^2\, dr + \int_\Omega \big\{ \bar{a}_r\,\eta^2\,\bar{u}\,\bar{u}_r + 2\,\bar{a}\,\eta\,\eta_r\,\bar{u}\,\bar{u}_r - \bar{b}\,\eta^2\,\bar{u}\,\bar{u}_r \\
& + \bar{c}\,\eta^2\,\bar{u}^2 + \big(p\,H(r-B_2) + q\,H(B_1-r)\big)\eta^2\,\bar{u}^2 - f\,\eta^2\,\bar{u}\big\}\, dr \Big\}\, ds
\end{aligned}
$$

Since by Cauchy Inequality,the following estimates are hold:

$$
\begin{aligned}
\big|f\,\bar{u}\,\eta^2\big| &= \big|\sqrt{\bar{c}}\,\bar{u}\,\eta\cdot\frac{f\,\eta}{\sqrt{\bar{c}}}\big| \le \bar{c}\,\bar{u}^2\,\eta^2 + \frac{f^2\,\eta^2}{4\,\bar{c}} \\
\big|2\,\bar{a}\eta\,\eta_r\,\bar{u}\,\bar{u}_r\big| &= \big|2\sqrt{\bar{a}}\,\eta_r\bar{u}\cdot\sqrt{\bar{a}}\,\eta\,\bar{u}_r\big| \le \frac{1}{4}\,\bar{a}\,\eta^2\,\bar{u}_r^2 + 4\,\bar{a}\,\eta_r^2\,\bar{u}^2 \\
\big|\bar{a}_r\,\eta^2\,\bar{u}\,\bar{u}_r\big| &= \big|\sqrt{\bar{a}}\,\eta\,\bar{u}_r\cdot\frac{\bar{a}_r\,\eta\,\bar{u}}{\sqrt{\bar{a}}}\big| \le \frac{1}{4}\bar{a}\,\eta^2\,\bar{u}_r^2 + \frac{\bar{a}_r^2\,\eta^2\,\bar{u}^2}{\bar{a}} \\
\big|\bar{b}\,\bar{u}\,\bar{u}_r\,\eta^2\big| &= \big|\sqrt{\bar{a}}\,\eta\,\bar{u}_r\cdot\frac{\bar{b}\,\eta\,\bar{u}}{\sqrt{\bar{a}}}\big| \le \frac{1}{4}\bar{a}\,\eta^2\,\bar{u}_r^2 + \frac{\bar{b}^2\,\eta^2\,\bar{u}^2}{\bar{a}}
\end{aligned}
$$

Applying all above estimations,

$$
\begin{aligned}
0 \;\ge\; & \frac{1}{2}\int_\Omega \eta^2\,\bar{u}^2 dr + \int_0^t \Big\{ \int_\Omega \Big[\bar{a}\,\eta^2\,\bar{u}_r^2 - \frac{f^2\,\eta^2}{4\bar{c}} - \frac{\bar{a}\,\eta^2\,\bar{u}_r^2}{4} - \frac{\bar{a}_r^2\,\eta^2\,\bar{u}^2}{\bar{a}} - \frac{\bar{a}\,\eta^2\,\bar{u}_r^2}{4} \\
& -4\bar{a}\,\eta_r^2\,\bar{u}^2 - \frac{\bar{a}\,\eta^2\,\bar{u}_r^2}{4} - \frac{\bar{b}^2\,\eta^2\,\bar{u}^2}{\bar{a}}\Big] dr \Big\} ds
\end{aligned}
$$

After simplification, we obtain

$$
\frac{1}{2}\int_\Omega \eta^2\,\bar{u}^2\, dr + \frac{1}{4}\int_0^t\int_\Omega \bar{a}\,\eta^2\,\bar{u}_r^2\, dr\, ds \le \int_0^t\int_\Omega \Big[\frac{f^2\,\eta^2}{4\,\bar{c}} + \frac{\bar{a}_r^2\,\eta^2\,\bar{u}^2}{\bar{a}} + 4\,\bar{a}\,\eta_r^2\,\bar{u}^2 + \frac{\bar{b}^2\,\eta^2\,\bar{u}^2}{\bar{a}}\Big]\, dr\, ds \quad (5.6)
$$

Next, we multiply $\zeta^2\,\bar{u}_t$ on both sides of the first equation in (2.19) and integrate it by parts on Ω, where ζ is another cut-off function included in η. Therefore, we have that

$$
0 = \int_\Omega \zeta^2 \Big\{ \bar{u}_t^2 - \bar{a}\,\bar{u}_{rr}\,\bar{u}_t - \bar{b}\,\bar{u}_r\,\bar{u}_t + \bar{c}\,\bar{u}\,\bar{u}_t + \big(p\,H(r-B_2) + q\,H(B_1-r)\big)\,\bar{u}\,\bar{u}_t - f\,\bar{u}_t \Big\}\, dr \quad (5.7)
$$

Since by integration by parts,

$$
\begin{aligned}
-\int_\Omega \bar{a}\,\zeta^2\,\bar{u}_{rr}\,\bar{u}_t\, dr &= -\int_\Omega \bar{a}\,\zeta^2\,\bar{u}_t\, d\,\bar{u}_r \\
&= \int_\Omega \big(\bar{a}\,\zeta^2\,\bar{u}_t\big)_r\,\bar{u}_r\, dr \\
&= \int_\Omega \bar{a}\,\zeta^2\,\bar{u}_{tr}\,\bar{u}_r\, dr + \int_\Omega \big(\bar{a}\,\zeta^2\big)_r\,\bar{u}_t\,\bar{u}_r\, dr \\
&= \int_\Omega \zeta^2\Big(\frac{\bar{a}\,\bar{u}_r^2}{2}\Big)_t dr + \int_\Omega \big(\bar{a}_r\,\zeta^2\,\bar{u}_t\,\bar{u}_r + 2\,\bar{a}\,\zeta\,\zeta_r\,\bar{u}_t\,\bar{u}_r\big)\, dr
\end{aligned}
$$

then the above expression (5.7) can be summarized as follows:

$$
\begin{aligned}
0 \;=\; & \int_\Omega \zeta^2\,\bar{u}_t^2\, dr + \int_\Omega \zeta^2\Big(\frac{\bar{a}\,\bar{u}_r^2}{2}\Big)_t dr + \int_\Omega \big(\bar{a}\,\zeta^2\big)_r\,\bar{u}_t\,\bar{u}_r\, dr \\
& + \int_\Omega \zeta^2\,\bar{u}_t\big\{ -\bar{b}\,\bar{u}_r + \big(\bar{c} + p\,H(r-B_2) + q\,H(B_1-r)\big)\,\bar{u} - f\big\}\, dr \\
= \; & \int_\Omega \zeta^2\,\big\{\bar{u}_t^2 + (\frac{\bar{a}\,\bar{u}_r^2}{2})_t\big\}\, dr + \int_\Omega \bar{u}_t\big\{ -\zeta^2\,\bar{b}\,\bar{u}_r + \zeta^2\,\big(\bar{c} + p\,H(r-B_2) + q\,H(B_1-r)\big)\,\bar{u} \\
& + \bar{a}_r\,\zeta^2\,\bar{u}_r + 2\,\bar{a}\,\zeta\,\zeta_r\,\bar{u}_r - f\,\zeta^2\big\}\, dr
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \int_\Omega \zeta^2\, \bar{u}_t^2\, dr + \frac{d}{dt}\int_\Omega \frac{\zeta^2\left\{\bar{a}\,\bar{u}_r^2 + \big(\bar{c} + p\,H(r-B_2) + q\,H(B_1-r)\big)\,\bar{u}^2\right\}}{2}\, dr \\
= & \int_\Omega \Big\{(-\bar{a}_r\,\zeta^2 - 2\,\bar{a}\,\zeta\,\zeta_r + \bar{b}\,\zeta^2)\,\bar{u}_t\,\bar{u}_r + \bar{u}_t\, f\,\zeta^2\Big\}\, dr \\
= & \int_\Omega (\zeta\,\bar{u}_t)\Big\{(\bar{b}-\bar{a}_r)\,\zeta\,\bar{u}_r - 2\,\bar{a}\,\zeta_r\,\bar{u}_r\Big\}\, dr + \int_\Omega (\zeta\,\bar{u}_t)\cdot(\zeta\, f)\, dr \\
\le & \frac{1}{4}\int_\Omega \zeta^2\,\bar{u}_t^2\, dr + \int_\Omega \Big\{(\bar{a}_r-\bar{b})\,\zeta + 2\,\bar{a}\,\zeta_r\Big\}^2\,\bar{u}_r^2\, dr + \frac{1}{4}\int_\Omega \zeta^2\,\bar{u}_t^2\, dr + \int_\Omega \zeta^2\, f^2\, dr \\
= & \frac{1}{2}\int_\Omega \zeta^2\,\bar{u}_t^2\, dr + \int_\Omega \Big\{\big[(\bar{a}_r-\bar{b})\,\zeta + 2\,\bar{a}\,\zeta_r\big]^2\,\bar{u}_r^2 + f^2\,\zeta^2\Big\}\, dr
\end{aligned}
$$

Simplify the above inequality, we have

$$
\begin{aligned}
& \frac{1}{2}\int_\Omega \zeta^2\,\bar{u}_t^2\, dr + \frac{d}{dt}\int_\Omega \frac{\zeta^2\left\{\bar{a}\,\bar{u}_r^2 + \big(\bar{c} + p\,H(r-B_2) + q\,H(B_1-r)\big)\,\bar{u}^2\right\}}{2}\, dr \\
\le & \int_\Omega \Big\{f^2\,\zeta^2 + \big(\bar{a}_r-\bar{b}\big)^2\,\zeta^2\,\bar{u}_r^2 + 4\,\bar{a}\,\big(\bar{a}_r-\bar{b}\big)\,\zeta\,\zeta_r\,\bar{u}_r^2 + 4\,\bar{a}^2\,\zeta_r^2\,\bar{u}_r^2\Big\}\, dr \\
= & \Big\|f\,\zeta\Big\|_{L^2(\Omega)}^2 + \int_\Omega \Big\{\Big(\frac{\bar{a}_r-\bar{b}}{\sqrt{\bar{a}}}\Big)^2 + 4(\bar{a}_r-\bar{b})\,\frac{\zeta_r}{\zeta}\Big\}\,\bar{a}\,\zeta^2\,\bar{u}_r^2\, dr + \int_\Omega 4\,\bar{a}^2\,\zeta_r^2\,\bar{u}_r^2\, dr \\
\le & \Big\|f\,\zeta\Big\|_{L^2(\Omega)}^2 + \Big\|\Big\{\Big(\frac{\bar{a}_r-\bar{b}}{\sqrt{\bar{a}}}\Big)^2 + 4(\bar{a}_r-\bar{b})\Big\}\cdot\ \mathbf{1}_{\{\zeta>0\}}\Big\|_\infty \int_\Omega \bar{a}\,\zeta^2\,\bar{u}_r^2\, dr + \int_\Omega 4\,\bar{a}^2\,\zeta_r^2\,\bar{u}_r^2\, dr
\end{aligned}
$$

Then if we set

$$
E(t) = \int_\Omega \frac{1}{2}\,\zeta^2\,\bar{a}\,\bar{u}_r^2\, dr, \qquad \psi(t) = \int_\Omega 4\,\bar{a}^2\,\zeta_r^2\,\bar{u}_r^2\, dr,
$$

from above estimation, we have that

$$
\frac{1}{2}\int_\Omega \zeta^2\,\bar{u}_t^2\, dr + \frac{dE(t)}{dt} \le \Big\|f\,\zeta\Big\|_{L^2(\Omega)}^2 + C\,E(t) + \psi(t), \tag{5.8}
$$

where $C > 0$. Hence,

$$
\begin{aligned}
& E'(t) \le \Big\|f\,\zeta\Big\|_{L^2(\Omega)}^2 + C\,E(t) + \psi(t) \\
\Longrightarrow\ & E(t) \le \int_0^t \Big\{\big\|f(s)\,\zeta\big\|_{L^2(\Omega)}^2 + \psi(s)\Big\}\, ds + \int_0^t C\,E(s)\, ds
\end{aligned}
$$

By applying for Gronwall Inequality, we obtain that

$$
\begin{aligned}
E(t) \le & \int_0^t \Big\{\big\|f(s)\,\zeta\big\|_{L^2(\Omega)}^2 + \psi(s)\Big\}\, ds + \int_0^t C\int_0^s \Big\{\big\|f(\mu)\,\zeta\big\|_{L^2(\Omega)}^2 + \psi(\mu)\Big\}\, d\mu\, e^{C\,(t-s)}\, ds \\
\le & \Big(1 + \int_0^t C\,e^{C\,(t-s)}\, ds\Big)\cdot \int_0^t \Big\{\big\|f(s)\,\zeta\big\|_{L^2(\Omega)}^2 + \psi(s)\Big\}\, ds \\
= & e^{Ct}\int_0^t \Big\{\big\|f(s)\,\zeta\big\|_{L^2(\Omega)}^2 + \psi(s)\Big\}\, ds
\end{aligned}
$$

And by estimate (5.6), we have

$$
\int_0^t \psi(s)\, ds = \int_0^t \int_{L_1}^{L_2} 4\,\bar{a}^2\,\zeta_r^2\,\bar{u}_r^2\, dr\, ds \le C(L_1, L_2, t)
$$

Therefore,

$$\sup_{0\le t\le T}\int_{L_1}^{L_2}\bar{u}_r^2\,dr\le C(L_1,L_2,T) \tag{5.9}$$

Combine estimates (5.6), (5.8) and (5.9), we obtain

$$\int_0^T\int_{L_1}^{L_2}\bar{u}_t^2\,dr\,ds+\sup_{0\le t\le T}\int_{L_1}^{L_2}\bar{u}_r^2\,dr\le C(L_1,L_2,T) \tag{5.10}$$

Therefore $\forall t\in[0,T]$ and $L_1\le x\le y\le L_2$, we have

$$\begin{aligned}\Big|\bar{u}(x,t)-\bar{u}(y,t)\Big| &= \Big|\int_x^y\bar{u}_r(r,t)\,dr\Big|\\ &\le \sqrt{\int_x^y\bar{u}_r^2(r,t)\,dr}\cdot\sqrt{\int_x^y 1^2\,dr}\\ &\le \Big|x-y\Big|^{\frac12}\sup_{0\le t\le T}\Big\|\bar{u}_r(\cdot,t)\Big\|_{L^2(L_1,L_2)}\end{aligned}$$

Therefore by (5.10), we have

$$\big\|\bar{u}\big\|_{C^{\frac12}}=\big\|u_{pq}\big\|_{C^{\frac12}}:=\frac{\big|\bar{u}(x,t)-\bar{u}(y,t)\big|}{\big|x-y\big|^{\frac12}}\le\sup_{0\le t\le T}\Big\|\bar{u}_r(\cdot,t)\Big\|_{L^2(L_1,L_2)}\le C(L_1,L_2,T) \tag{5.11}$$

Next $\forall x\in[L_1,L_2]$ and $0\le\tau\le t\le T$, we have

$$\begin{aligned}\big|\bar{u}(x,t)-\bar{u}(x,\tau)\big| &\le \big|\bar{u}(x,t)-\frac{1}{|D|}\int_D\bar{u}(z,t)\,dz\big|+\big|\bar{u}(x,\tau)-\frac{1}{|D|}\int_D\bar{u}(z,\tau)\,dz\big|\\ &\quad+\frac{1}{|D|}\big|\int_D\big\{\bar{u}(z,t)-\bar{u}(z,\tau)\big\}\,dz\big|\\ &\le \frac{1}{|D|}\int_D\big|\bar{u}(x,t)-\bar{u}(z,t)\big|\,dz+\frac{1}{|D|}\int_D\big|\bar{u}(x,\tau)-\bar{u}(z,\tau)\big|\,dz\\ &\quad+\frac{1}{|D|}\big|\int_D\int_\tau^t\bar{u}_t(z,s)\,ds\,dz\big|\\ &\le \frac{2}{|D|}\int_D\|\bar{u}\|_{C^{\frac12}}|x-z|^{\frac12}\,dz+\frac{1}{|D|}\sqrt{\int_D\int_\tau^t\bar{u}_t^2(z,s)\,ds\,dz}\cdot\sqrt{\int_D\int_\tau^t 1^2\,ds\,dz}\\ &\le 2\|\bar{u}\|_{C^{\frac12}}|D|^{\frac12}+\frac{1}{|D|}\|\bar{u}_t\|_{L^2}\sqrt{|D|\,(t-\tau)}\\ &= 2\|\bar{u}\|_{C^{\frac12}}|D|^{\frac12}+\|\bar{u}_t\|_{L^2}\sqrt{\frac{t-\tau}{|D|}}\end{aligned}$$

Now just set $|D|=\sqrt{t-\tau}=|t-\tau|^{\frac12}$, then

$$\begin{aligned}\big|\bar{u}(x,t)-\bar{u}(x,\tau)\big| &\le 2\|\bar{u}\|_{C^{\frac12}}|D|^{\frac12}+\|\bar{u}_t\|_{L^2}\sqrt{\frac{t-\tau}{|D|}}\\ &\le 2\|\bar{u}\|_{C^{\frac12}}(t-\tau)^{\frac14}+\|\bar{u}_t\|_{L^2}(t-\tau)^{\frac14}\\ &\le \big\{2\sup_{0\le t\le T}\big\|\bar{u}_r(\cdot,t)\big\|_{L^2(L_1,L_2)}+\|\bar{u}_t\|_{L^2}\big\}(t-\tau)^{\frac14}\end{aligned}$$

Therefore by (5.10), we have

$$\big\|\bar{u}\big\|_{C^{\frac14}}=\big\|u_{pq}\big\|_{C^{\frac14}}:=\frac{\big|\bar{u}(x,t)-\bar{u}(x,\tau)\big|}{\big|t-\tau\big|^{\frac14}}\le C(L_1,L_2,T) \tag{5.12}$$

Combine (5.11)and (5.12), we prove inequality (5.5), thus complete the proof of Lemma 5.3. □

Combine Lemma 5.1, Lemma 5.2 and Lemma 5.3, and send $p, q \to \infty$, we prove Theorem 5.1.

6.0 NUMERICAL ANALYSIS OF CDS MODEL

After the completely mathematical analysis of the CDS model, in this chapter, we will mainly focus on the numerical analysis of the models which is set up previously. First part of this chapter is that we will use the historical time series data, USD short interest rate (Libor 1M and Libor 3M) and JPY short interest rate (Tibor 1Y) to calibrate the parameters (κ, β, σ) in the CIR interest rate model by using linear regression (Least Square) Method. Moreover we use the calibrated parameters in CIR model with other specific constants to simulate the numerical solution of the CDS model by Monte Carlo simulation and Finite Difference Method.

6.1 CALIBRATION OF CIR INTEREST RATE MODEL

We will first describe the background of numerical method for calibrating the CIR model. Furthermore based on the theoretical method described, we implement the programming in the *Mathematica* software to obtain the specific values of parameters and last make a conclusion.

6.1.1 Background of CIR Model Calibration

Recall the CIR interest rate model, we have the following expression:

$$dr_t = \big(\kappa - \beta\, r_t\big)\, dt + \sigma\, \sqrt{r_t}\, dW_t, \tag{6.1}$$

where κ, β, and σ are positive constants, W_t is a Standard Brownian motion. Numerically, we can rewrite the above SDE (6.1) as:

$$r_{j+1} - r_j = (\kappa - \beta\, r_j)\, \Delta\, t + \sigma\, \sqrt{r_j\, \Delta\, t}\, \xi_j, \tag{6.2}$$

where $\xi_j \sim N(0,1)$, for $j = 1, 2, ..., n$. Then we divide $\sqrt{r_j\, \Delta\, t}$ both sides of numerical equation (6.2) to obtain:

$$\frac{r_{j+1} - r_j}{\sqrt{r_j\, \Delta\, t}} = \frac{\kappa - \beta\, r_j}{\sqrt{r_j\, \Delta\, t}}\, \Delta\, t + \sigma\, \xi_j = \kappa \sqrt{\frac{\Delta\, t}{r_j}} - \beta\, \sqrt{r_j\, \Delta\, t} + \sigma\, \xi_j. \tag{6.3}$$

Based on above equation, we introduce the following notations:

$$\begin{cases} \vec{X} = \left\{ \frac{r_{j+1} - r_j}{\sqrt{r_j \, \Delta t}} \right\}_{j=1}^{n} \\ \\ \vec{Y} = \left\{ \sqrt{\frac{\Delta t}{r_j}} \right\}_{j=1}^{n} \\ \\ \vec{Z} = \left\{ \sqrt{r_j \, \Delta t} \right\}_{j=1}^{n} \\ \\ \vec{\xi} = \left\{ \xi_j \right\}_{j=1}^{n} \end{cases}$$

Then the equation (6.3) becomes:

$$\vec{X} = \kappa \, \vec{Y} - \beta \, \vec{Z} + \sigma \, \vec{\xi}. \tag{6.4}$$

Therefore we use linear regression, i.e. Least Square Method to calibrate the values of positive constants κ^*, β^*, and σ^* such that satisfying the follows,

$$\begin{cases} (\kappa^*, \beta^*) = \underset{(\kappa,\beta) \in \mathbb{R}^2}{\operatorname{argmin}} \, \|\vec{X} - \kappa \, \vec{Y} + \beta \, \vec{Z}\|^2 \\ \\ \sigma^* = \text{Standard Deviation of} \left(\vec{X} - \kappa^* \, \vec{Y} + \beta^* \, \vec{Z} \right). \end{cases} \tag{6.5}$$

To compute above equation, we set

$$f(\kappa, \beta) = \left(\vec{X} - \kappa \, \vec{Y} + \beta \, \vec{Z}, \vec{X} - \kappa \, \vec{Y} + \beta \, \vec{Z} \right).$$

To obtain the optimizer κ^* and β^*, we have the following calculation:

$$\begin{cases} \frac{\partial f}{\partial \kappa} = -2 \left(\vec{X} - \kappa \, \vec{Y} + \beta \, \vec{Z}, \, \vec{Y} \right) \\ \\ \frac{\partial f}{\partial \beta} = \quad 2 \left(\vec{X} - \kappa \, \vec{Y} + \beta \, \vec{Z}, \, \vec{Z} \right) \end{cases}$$

Then by setting

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial \kappa} \\ \\ \frac{\partial f}{\partial \beta} \end{pmatrix} = \vec{0},$$

we have

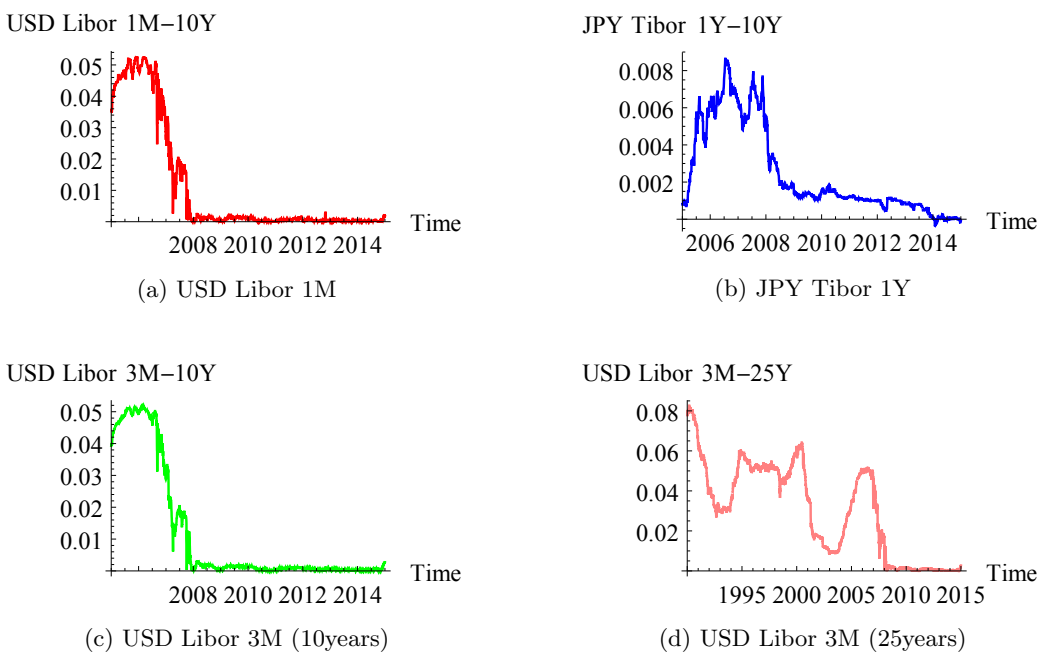
$$\begin{pmatrix} \vec{Y} \cdot \vec{Y} & -\vec{Y} \cdot \vec{Z} \\ \\ \vec{Y} \cdot \vec{Z} & -\vec{Z} \cdot \vec{Z} \end{pmatrix} \begin{pmatrix} \kappa^* \\ \\ \beta^* \end{pmatrix} = \begin{pmatrix} \vec{X} \cdot \vec{Y} \\ \\ \vec{X} \cdot \vec{Z} \end{pmatrix},$$

Therefore we have the values of κ^*, β^* and σ^* as follows:

$$\begin{cases} \begin{pmatrix} \kappa^* \\ \\ \beta^* \end{pmatrix} = \begin{pmatrix} \vec{Y} \cdot \vec{Y} & -\vec{Y} \cdot \vec{Z} \\ \\ \vec{Y} \cdot \vec{Z} & -\vec{Z} \cdot \vec{Z} \end{pmatrix}^{-1} \begin{pmatrix} \vec{X} \cdot \vec{Y} \\ \\ \vec{X} \cdot \vec{Z} \end{pmatrix}, \\ \\ \sigma^* = \text{Standard Deviation of } \left(\vec{X} - \kappa^* \vec{Y} + \beta^* \vec{Z}\right). \end{cases} \tag{6.6}$$

6.1.2 Results of CIR Model Calibration

We implement the above method by using the *Mathematica* platform to perform our programming. Here we just display the results of the three cases. One thing need to be pointed out is that, for USD Libor 3M yield curve, we calibrated the parameters twice by using long term data for latest 25 years and short term data for latest 10 years. All other two cases just applied the short term data for latest 10 years. For the detail programmings, please refer to Appendix A.



(a) USD Libor 1M (b) JPY Tibor 1Y

(c) USD Libor 3M (10years) (d) USD Libor 3M (25years)

Figure 1: Basic Tendency of Interest Rate

The figure 1 above describes the basic tendency of different interest rates with respect to time. We get those data information from some official sources, U.S Department of the Treasury and Ministry of Finance Japan, to ensure the accuracy of the input. Here we can see the interest rate dropped dramatically after 2007 financial crisis. This is also the major reason that we calibrated the USD Libor 3M twice by using long term and short term data. The following matrix represent

the mean, variance and standard deviation for each case.

$$\begin{pmatrix} Cases & Mean & Variance & StandardDeviation \\ USDLibor1M: & 0.0110297 & 0.000335483 & 0.0183162 \\ USDLibor3M(10Y): & 0.011472 & 0.000343989 & 0.0185469 \\ USDLibor3M(25Y): & 0.0300545 & 0.00055541 & 0.0235671 \\ JPYTibor1Y: & 0.00245664 & 5.68558 \times 10^{-}6 & 0.00238444 \end{pmatrix}$$

6.1.2.1 Calibration of USD Libor 1M. The calibrated parameters for USD Libor 1M based on latest 10 year data (2005-2015) are:

$$\begin{cases} \kappa^* = 0.00620318 \\ \beta^* = 0.866107 \\ \sigma^* = 0.161495 \end{cases}$$

Thus the calibrated parameters satisfied

$$\frac{2\,\kappa^*}{\sigma^{*2}} = 0.47569 < 1 \tag{6.7}$$

Besides, we also verified that

$$\vec{\xi} = \frac{\vec{X} - \kappa^*\,\vec{Y} + \beta^*\,\vec{Z}}{\sigma^*}$$

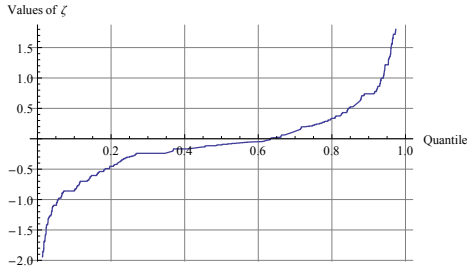
follows standard norm distribution.Therefore, the below matrix display the maximum, minimum, mean and standard deviation of $\vec{\xi}$.

$$\begin{pmatrix} Maximum & Minimum & Mean & StandardDeviation \\ 21.0026 & -8.42938 & -0.0232748 & 1. \end{pmatrix}$$

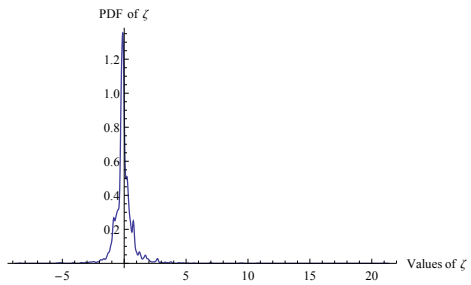
The figure 2 shows the Quantile, PDF and CDF of $\vec{\xi}$.

6.1.2.2 Calibration of USD Libor 3M with short term data. The calibrated parameters for USD Libor 3M based on latest 10 year data (2005-2015) are:

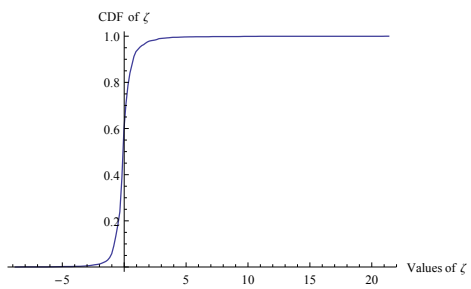
$$\begin{cases} \kappa^* = 0.0052817 \\ \beta^* = 0.779155 \\ \sigma^* = 0.121724 \end{cases}$$



(a) Quantile of $\vec{\xi}$



(b) pdf of $\vec{\xi}$



(c) cdf of $\vec{\xi}$

Figure 2: Verification of USD Libor 1M

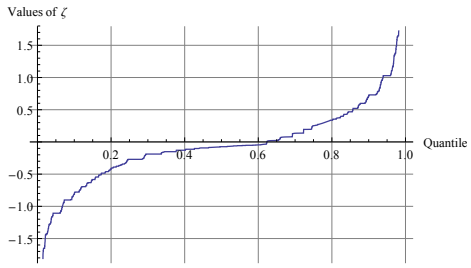
Thus the calibrated parameters satisfied

$$\frac{2\,\kappa^*}{\sigma^{*2}} = 0.712943 < 1 \tag{6.8}$$

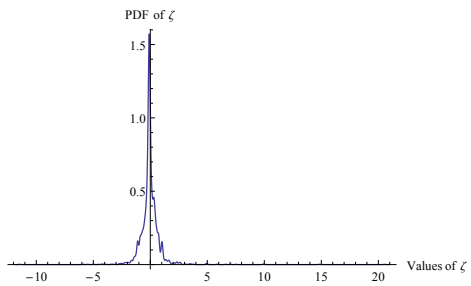
Besides, we also verified that $\vec{\xi} = \frac{\vec{X} - \kappa^*\,\vec{Y} + \beta^*\,\vec{Z}}{\sigma^*}$ follows standard norm distribution.Therefore, the below matrix display the maximum, minimum, mean and standard deviation of $\vec{\xi}$.

$$\begin{pmatrix} Maximum & Minimum & Mean & StandardDeviation \\ \\ 20.5469 & -11.4728 & -0.0303718 & 1. \end{pmatrix}$$

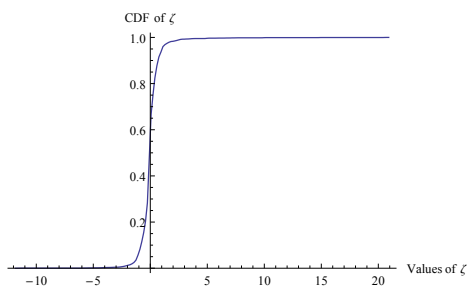
The figure 3 shows the Quantile, PDF and CDF of $\vec{\xi}$.



(a) Quantile of $\vec{\xi}$



(b) pdf of $\vec{\xi}$



(c) cdf of $\vec{\xi}$

Figure 3: Verification of USD Libor 3M (10Y)

6.1.2.3 Calibration of USD Libor 3M with long term data. The calibrated parameters for USD Libor 3M based on latest 25 year data (1990-2015) are:

$$\begin{cases} \kappa^* = 0.00507039 \\ \beta^* = 0.265533 \\ \sigma^* = 0.08086 \end{cases}$$

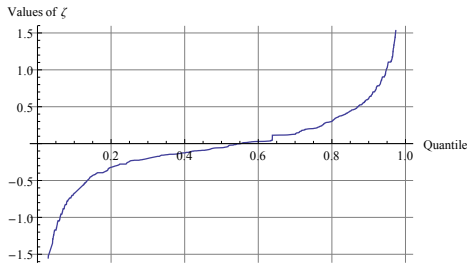
Thus the calibrated parameters satisfied

$$\frac{2\,\kappa^*}{\sigma^{*2}} = 1.55097 > 1 \tag{6.9}$$

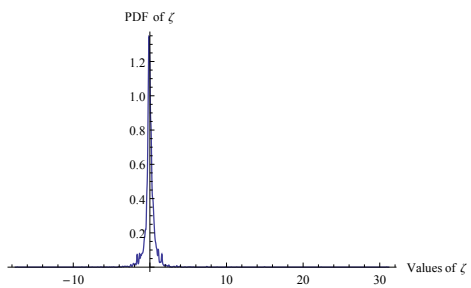
Besides, we also verified that $\vec{\xi} = \frac{\vec{X} - \kappa^*\,\vec{Y} + \beta^*\,\vec{Z}}{\sigma^*}$ follows standard norm distribution.Therefore, the below matrix display the maximum, minimum, mean and standard deviation of $\vec{\xi}$.

$$\begin{pmatrix} Maximum & Minimum & Mean & StandardDeviation \\ \\ 30.9147 & -17.3057 & -0.0188774 & 1. \end{pmatrix}$$

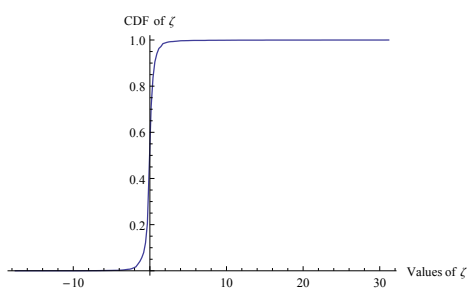
The figure 4 shows the Quantile, PDF and CDF of $\vec{\xi}$.



(a) Quantile of $\vec{\xi}$



(b) pdf of $\vec{\xi}$



(c) cdf of $\vec{\xi}$

Figure 4: Verification of USD Libor 3M (25Y)

6.1.2.4 Calibration of JPY Tibor 1Y. The calibrated parameters for JPY Tibor 1Y based on latest 10 year data (2005-2015) are:

$$\begin{cases} \kappa^* = 0.000188241 \\ \\ \beta^* = 0.105249 \\ \\ \sigma^* = 0.0245191 \end{cases}$$

Thus the calibrated parameters satisfied

$$\frac{2\,\kappa^*}{\sigma^{*2}} = 0.626233 < 1 \tag{6.10}$$

Besides, we also verified that

$$\vec{\xi} = \frac{\vec{X} - \kappa^*\,\vec{Y} + \beta^*\,\vec{Z}}{\sigma^*}$$

follows standard norm distribution.Therefore, the below matrix display the maximum, minimum, mean and standard deviation of $\vec{\xi}$.

$$\begin{pmatrix} Maximum & Minimum & Mean & StandardDeviation \\ \\ 7.72951 & -8.75618 & -0.00203466 & 1. \end{pmatrix}$$

The figure 5 shows the Quantile, PDF and CDF of $\vec{\xi}$.

6.1.3 Conclusions of CIR Model Calibration

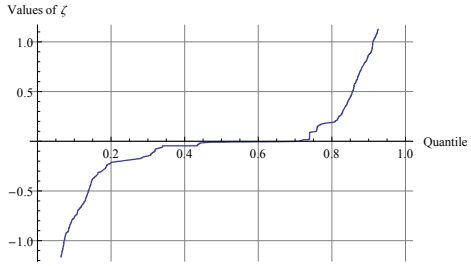
From above results, one can notice that if the data before 2007 financial crisis included, then based on result (6.9), the calibration parameters satisfied

$$\kappa > \frac{\sigma^2}{2} > 0. \tag{6.11}$$

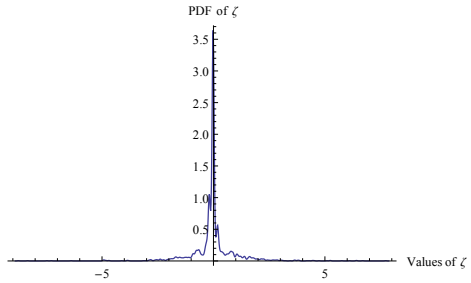
But if only we use the latest 10 years data or such kind of situation on interest rate continues, then based on results (6.7), (6.8) and (6.10), the calibration parameters satisfied only the general case:

$$\kappa > 0, \;\; \sigma > 0, \;\; \beta > 0. \tag{6.12}$$

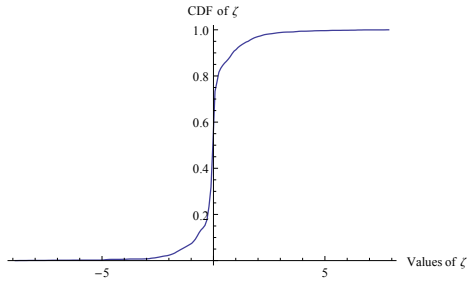
Usually people only study the PDE problem ([2], 2012) under the condition (6.11) as historical time series data for interest rate keeps stable in some sense, but after 2007 world wide financial crisis, interest rate dropped dramatically, which increased the volatility of interest rate to break the condition described in (6.11). Therefore, in order to extend the application of CIR model in mathematical analysis, we have to study the case when volatility, σ, become bigger, the case stated in (6.12), i.e the PDE problem studied in this thesis. In another word, the removal of the condition $\kappa > \frac{\sigma^2}{2}$ made the application of the CIR model robust and also one goal of this thesis is to introduce this new theoretical development of the CIR model to the public.



(a) Quantile of $\vec{\xi}$



(b) pdf of $\vec{\xi}$



(c) cdf of $\vec{\xi}$

Figure 5: Verification of JPY Tibor 1Y

6.2 NUMERICAL METHODS OF SIMULATING CDS MODEL

We will first describe the background of numerical methods for simulating the CDS model. Moreover based on the theoretical methods described, we implement the programmings in the *Matlab* software to obtain the specific solution of the CDS model and last make a conclusion.

6.2.1 Background of Numerical Methods for Simulating CDS Model

In numerical analysis, two methods are generally applied. One is called Monte-Carlo simulation and another method is called finite difference method. In this section, we applied both of the methods to

perform to our pricing model. First reason is to ensure each of the methods performed correctly and second reason is to check the difference between two methods. The followings are the background of the two customized methods based on the specific model which stated previously.

6.2.1.1 Monte-Carlo Simulation Method. Recall in the section 2.4, the value, $u(r,T)$, of the CDS model from the buyer's point of view at time $t=0$ with $r_0 = r$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u = K\Lambda - h & \forall\, r \in \Omega,\ T > 0, \\ u(r,0) = 0 & \forall\, r \in \bar{\Omega}, T = 0. \end{cases} \tag{6.13}$$

Here $\Omega = (0,\infty)$ is the state space for the interest rate and

$$\begin{cases} \mathcal{L} = -\frac{\sigma^2}{2}\, r\, \frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r \\ \Lambda = a\, r + b \\ \Lambda_1 = p\, H(r - B_2) \\ \Lambda_2 = q\, H(B_1 - r) \end{cases} \tag{6.14}$$

where $\sigma > 0, \kappa > 0, \beta > 0, a > 0, b > 0, 0 < B_1 < B_2,\ p \geq 0, q \geq 0, K > 0, h > 0$.

To apply Monte-Carlo simulation on above model, we need to rewrite the model (6.13) as of the expectation form. More specifically, we have

$$u(r,T) := K\, u_1(r,T) - h\, u_2(r,T), \tag{6.15}$$

where

$$\begin{cases} u_1(r,T) = \mathbb{E}\left[\int_0^T \lambda_s e^{-\int_0^s (r_\theta + \lambda_\theta + \lambda_{1\,\theta} + \lambda_{2\,\theta})d\theta} ds \;\middle|\; r_0 = r\right], \\ u_2(r,T) = \mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta + \lambda_\theta + \lambda_{1\,\theta} + \lambda_{2\,\theta})d\theta} ds \;\middle|\; r_0 = r\right]. \end{cases} \tag{6.16}$$

Therefore based on CIR model,

$$dr_t = \left(\kappa - \beta\, r_t\right) dt + \sigma\, \sqrt{r_t}\, dW_t,$$

where κ, β, and σ are positive constants, W_t is a Standard Brownian motion. Numerically, we can rewrite the above SDE as:

$$r_{j+1} - r_j = (\kappa - \beta\, r_j)\, \Delta\, t + \sigma\, \sqrt{r_j\, \Delta\, t}\, \xi_j,$$

where $\xi_j \sim N(0,1)$, for $j = 1, 2, ..., n$. Then we set

$$u_0 = \int_0^s (r_\theta + \lambda_\theta + \lambda_{1\,\theta} + \lambda_{2\,\theta})\, d\,\theta \tag{6.17}$$

Numerically, for $j = 1, 2, ..., n$, we can rewrite above expression (6.17) as

$$u_{0\,j+1} = \begin{cases} u_{0\,j} + \left((a+1)\, r_j + b + p\right) \Delta\, t & if\ \ r_j > B_2 \\ u_{0\,j} + \left((a+1)\, r_j + b + q\right) \Delta\, t & if\ \ r_j < B_1 \\ u_{0\,j} + \left((a+1)\, r_j + b\right) \Delta\, t & if\ \ B_1 \leq r_j \leq B_2 \end{cases} \tag{6.18}$$

Hence by equation (6.16), we have the estimations of $u_1(r,T)$ and $u_2(r,T)$ as

$$\begin{cases} u_{1\,j+1} = u_{1\,j} + (a\,r_j + b)\,e^{-u_{0\,j}}\,\Delta\,t \\ \\ u_{2\,j+1} = u_{2\,j} + e^{-u_{0\,j}}\,\Delta\,t \end{cases} \tag{6.19}$$

Last based on equation (6.15), we have

$$u_j = K\,u_{1\,j} - h\,u_{2\,j}, \tag{6.20}$$

for $j = 1, 2, ..., n$, where $n\,\Delta\,t = T$ and given the initial interest rate r_0. Follow above procedures, we obtain a path of solution. Then repeat the procedures many times (say 10000 times) and average the results to get the Monte-Carlo numerical solution of the CDS model.

6.2.1.2 Finite Difference Method. Recall the first equation of (6.13), we can rewrite it as follows.

$$u_T = \frac{\sigma^2}{2}\,r\,u_{rr} + (\kappa - \beta\,r)u_r - C(r)\,u + K\,(a\,r + b) - h, \tag{6.21}$$

where

$$C(r) = \begin{cases} (a+1)\,r + b + p & if\ \ r > B_2 \\ \\ (a+1)\,r + b + q & if\ \ r < B_1 \\ \\ (a+1)\,r + b & if\ \ B_1 \le r_j \le B_2 \end{cases} \tag{6.22}$$

Therefore, we keep the time space (T) the same and divide the interest rate space (r) by m sub-intervals from 0 to 1. Set

$$u_i(t) = u(r_i, t) \quad \text{and} \quad u_i'(t) = \frac{\partial\,u}{\partial\,T}\,(r_i, t),$$

where $i = 0, 1, 2, ...m$, and $m \cdot\ dr = 1$.

Hence by finite difference method on interest rate (r) space, we have the following estimations:

$$\begin{cases} u_{rr}(r_i, t)\ \approx \frac{u_{i+1} + u_{i-1} - 2\,u_i}{dr^2} \\ \\ u_r(r_i, t)\ \approx \frac{u_{i+1} - u_{i-1}}{2\,dr} \end{cases} \tag{6.23}$$

Plugging above estimations into equation (6.21) to get the following $m+1$ equations:

$$\begin{cases} u_i'\,(t)\ \approx\ \frac{\sigma^2}{2}\,r_i\,\frac{u_{i+1} + u_{i-1} - 2\,u_i}{dr^2} + (\kappa - \beta\,r_i)\,\frac{u_{i+1} - u_{i-1}}{2\,dr} - C(r_i)\,u_i + K\,(a\,r_i + b) - h \\ \\ u_i(0) = 0 \\ \\ u_0(t) = u_1(t) \\ \\ u_m(t) = u_{m-1}(t) \end{cases} \tag{6.24}$$

where $i = 0, 1, 2, ...m$, and $m \cdot\ dr = 1$. Last we use the function *ODE45* in the *Matlab* to solve above $m+1$ ODE equations with $m+1$ unknowns $\{u_0(t),\ u_1(t),\ ...\,u_{m-1}(t),\ u_m(t)\}$.

6.2.2 Results of Numerical Methods for Simulating CDS Model

We implement the above two methods by using the *Matlab* platform to perform our programming. Here we just display the results of the two methods. For the detail programmings, please refer to Appendix B.

The table 1 represents the constants which appeared in the PDE problem with calibrated parameters in CIR interest rate model. Here we performed the numerical simulation for the problem twice based on two different conditions on calibrated parameters in CIR model. In the first simulation, we used values κ, β and σ calibrated in the case of USD Libor 3M for the latest 25 years data (1990-2015). In this case, the data included which before 2007 financial crisis. Therefore, the volatility in this case is small so the calibrated parameters satisfied the condition:

$$\frac{2\,\kappa}{\sigma^2} > 1.$$

We performed the numerical simulation with Monte-Carlo path $N = 10000$ and finite difference method $m = 1500$. It took around 20 seconds to obtain the numerical result.

Table 1: Constants Setting Up: $\frac{2\,\kappa}{\sigma^2} > 1$

Constant	Value
T	10 (year)
r_0	6% (/year)
κ	0.005 (/year^2)
β	0.266 (/year)
σ	8% (/year)
a	0.5 ($)
b	0.1 (/year)
B_1	4% (/year)
B_2	6% (/year)
p	0.5 (/year)
q	0.3 (/year)
K	1 ($)
h	0.125 ($/year)
n	360
m	1500
N	10000

Based on these constants setting up, we use *Matlab* to perform the program and get the results showed in figure 6, 7 and 8.

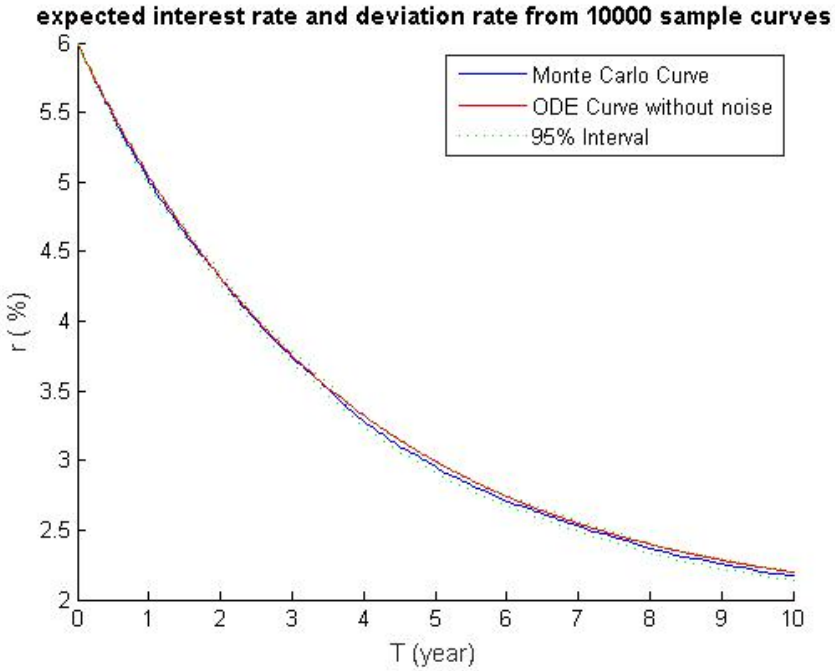


Figure 6: Interest Rate Simulation: $\frac{2\kappa}{\sigma^2} > 1$

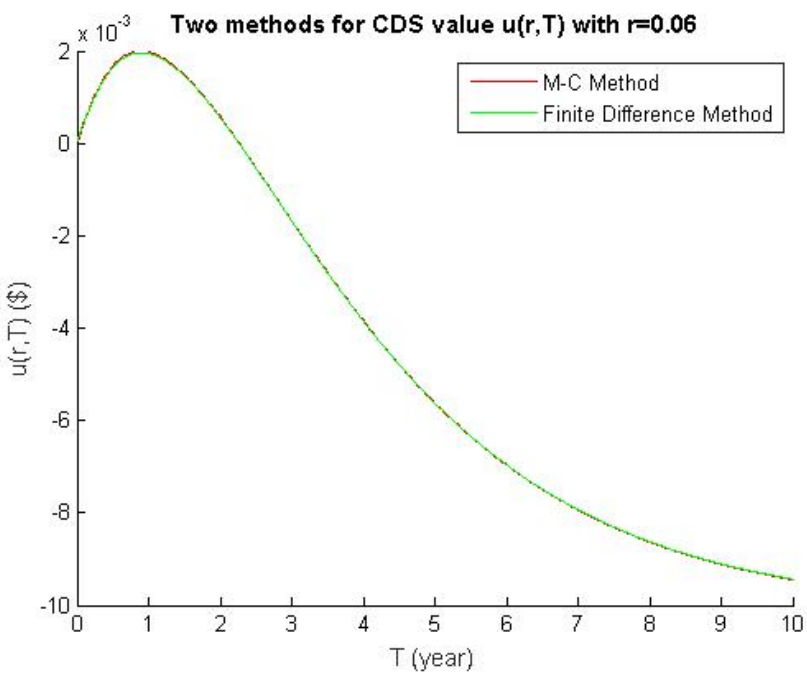


Figure 7: Monte-Carlo v.s Finite Difference Method: $\frac{2\kappa}{\sigma^2} > 1$

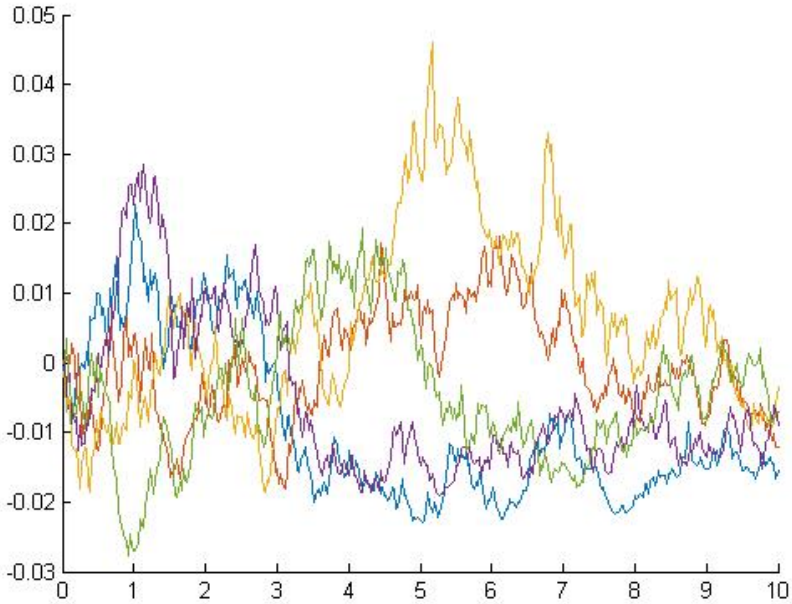


Figure 8: Sample Interest Rate Curves with Net Difference: $\frac{2\kappa}{\sigma^2} > 1$

The table 2 represents the constants and calibrated parameters in CIR model. For example, here we use values κ, β and σ in the case of USD Libor 3M for the latest 10 years data. In this case, the calibrated parameters have the relation:

$$\frac{2\kappa}{\sigma^2} < 1.$$

In this case, one can see that the volatility σ increased to 12.2%, which will result in a significant calculation. We initially performed the code in personal laptop with $m = 1500$ points, but the calculation is too much so that the process usually failed because out of memory. Based on this fact, we decided to decrease the step points from $m = 1500$ to $m = 1000$. Although the number of points decreased, it still took around 3 minutes to run the program and get the result. Based on these constants setting up, we use *Matlab* to perform the program and get the results showed in figure 9, 10 and 11.

One thing need to be pointed out. From figure 10, we can see that

$$\begin{cases} u(0.06, T) \geq 0 & \text{for } 0 \leq T \leq 1, \\ u(0.06, T) \leq 0 & \text{for } 1 \leq T \leq 10, \end{cases}$$

which means that, in the first year, the contract favors the buyer but after that, the contract favors the seller instead.

Table 2: Constants Setting Up: $\frac{2\kappa}{\sigma^2} < 1$

Constant	Value
T	10 (year)
r_0	6% (/year)
κ	0.005 (/year^2)
β	0.780 (/year)
σ	12.2% (/year)
a	0.5 ($)
b	0.1 (/year)
B_1	4% (/year)
B_2	6% (/year)
p	0.5 (/year)
q	0.3 (/year)
K	1 ($)
h	0.125 ($/year)
n	360
m	1000
N	10000

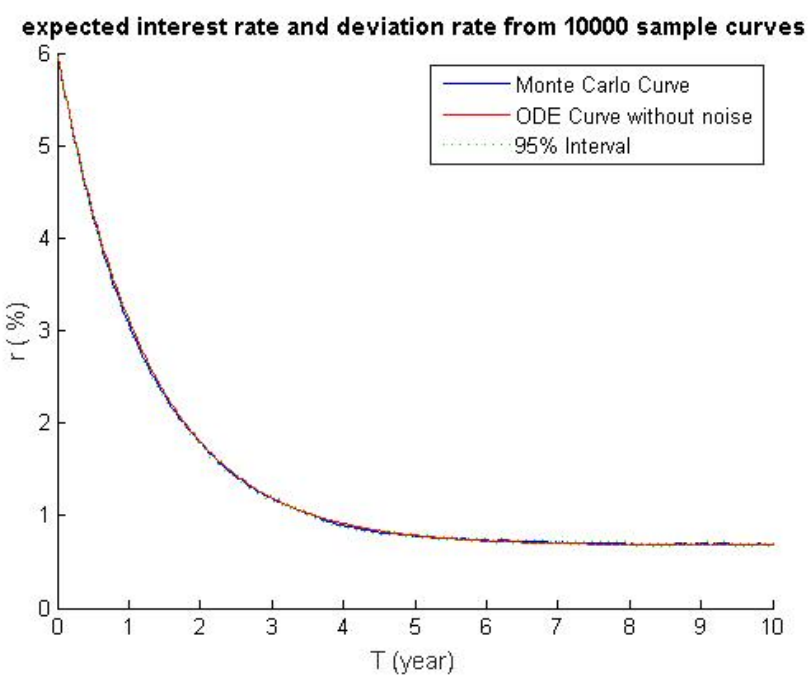


Figure 9: Interest Rate Simulation: $\frac{2\kappa}{\sigma^2} < 1$

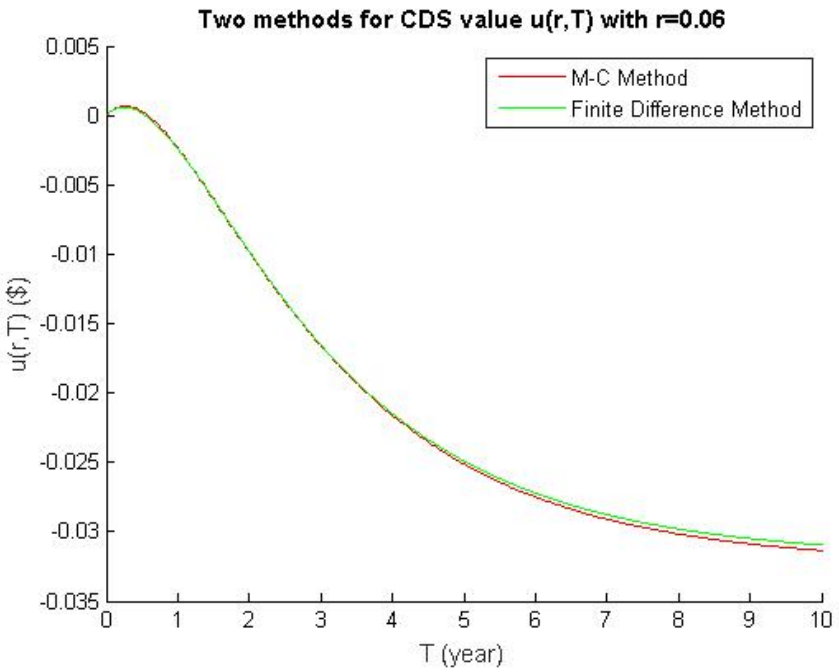


Figure 10: Monte-Carlo v.s Finite Difference Method: $\frac{2\kappa}{\sigma^2} < 1$

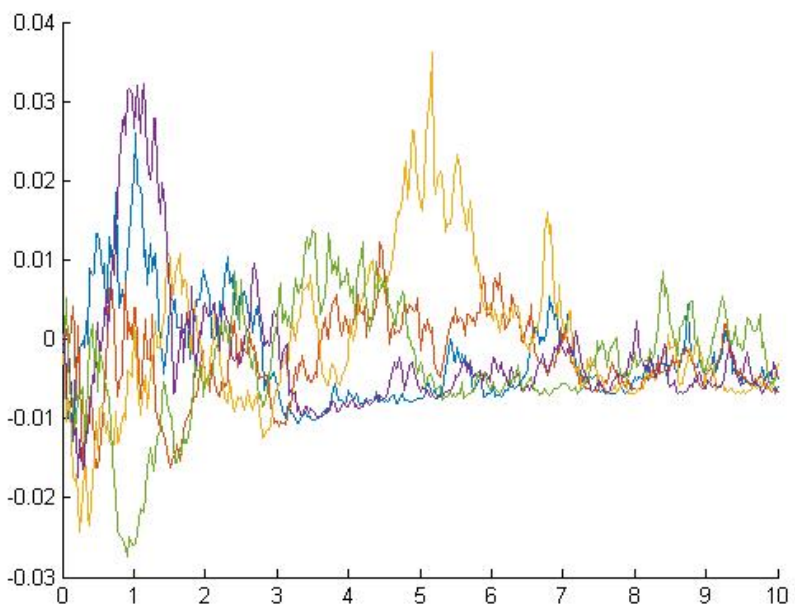


Figure 11: Sample Interest Rate Curves with Net Difference: $\frac{2\kappa}{\sigma^2} < 1$

6.2.3 Conclusions of Numerical Methods for Simulating CDS Model

1. From figure 6 and figure 9, we verified that the CIR model has the mean reverting property. The initial interest rate has been set as $r_0 = 0.06$, but the mean reverting values are $\frac{\kappa}{\beta} = 0.02$ and $\frac{\kappa}{\beta} = 0.0068$, respectively. Therefore, from figure 6 and figure 9, the interest rate decrease to $\frac{\kappa}{\beta}$ as $T \to 10$.
2. From figure 7 and figure 10, the simulated solutions of the PDE problem for two methods are matched perfectly. This means that the programming of the two methods is coded correctly and they have the same approximation. Furthermore, one can notice that the result of the first case matched better than the result of the second case.
3. From figure 7, one can see that

$$\begin{cases} u(0.06, T) \geq 0 & \text{for } 0 \leq T \leq 3, \\ \\ u(0.06, T) \leq 0 & \text{for } 3 \leq T \leq 10, \end{cases}$$

   which means that, in the first three years, the contract favors the buyer but after that, the contract favors the seller instead. The similar situation appeared in figure 10.
4. For the different constants setting up, we have the different numerical solutions and different level of calculation.
5. When calibrated parameters satisfied $\frac{2\kappa}{\sigma^2} > 1$, from figure 8, there is no interest rate become zero, which verified that the probability of positive interest rate is one. But when calibrated parameters satisfied $\frac{2\kappa}{\sigma^2} < 1$, in figure 11, one can notice that some interest rate become zero and then reflected, which verified that the probability of zero interest rate is positive. This phenomenon caused the bad performance of the CDS model, i.e. the solution of the PDE problem may diverge. Therefore, it is necessary and meaningful to study the behavior of the solution of the PDE problem theoretically under the case $\frac{2\kappa}{\sigma^2} < 1$.

7.0 CONCLUSIONS

In this thesis, we establish a financial credit derivative (credit default swap) pricing model for a contract which is subject to counterparty risks. Based on improved CIR model, we obtain a fully nonlinear partial differential equation problem with certain boundary condition under the general condition. We study the CDS model by showing the well-posedness of the PDE problem. In addition, we also study the infinite horizon problem of the corresponding pricing model which leads to a nonlinear ordinary differential equation problem. We obtain the uniqueness and existence of the ODE problem and prove the unique solution can be converged by the solution of the PDE problem. Furthermore, the models and theoretical analysis in this study get connection between two main risk frameworks: term structure model and intensity model. We show that a solution of the structure model can be obtained as the limit of a sequence of solution of intensity models. More importantly, this connection not only links the relationship between two financial default frameworks but also provides a method to solve some hard mathematical boundary problems. In fact, it is very difficult to deal with a PDE problem of structure model since it usually brings a complicated boundary condition. To simplify and approximate the structural one, a sequence of intensity models, which are initial PDE problems with low nonlinear terms can be applied based on the method in this study. It then greatly extends the area of applicability of structure models in finance problems since it makes possible to deal with some complicated default barriers from structure models.

Recall that two different parameter conditions in the CIR model, one is that we assume

$$\beta > 0,\ \sigma > 0,\ \kappa > \frac{\sigma^2}{2} > 0,$$

and the PDE problem is well-posed under the following boundary condition

$$u \in L^\infty.$$

For more details of mathematical analysis, please refer to the paper ([2], 2012). Another case, which is studied in this thesis, is that we only assume

$$\beta > 0,\ \sigma > 0,\ \kappa > 0,$$

and the PDE problem is well-posed under the following boundary condition instead

$$u_r \in L^\infty.$$

The first case can be obtained by using long term data from numerical calibration and the second case can be obtained by using recent data, especially the data after 2007 financial crisis. Thus it

is necessary and also meaningful for us to study the both cases in the study. Nevertheless, from our numerical analysis of solutions of the CDS model, under the first case, there is no interest rate become zero. But when we loose the condition, one can notice that some interest rate become zero and then reflected. This phenomenon caused the bad performance of the CDS model, i.e. the solution of the PDE problem may diverge. Therefore, mathematical analysis on the behavior of the solution of the PDE problem under general condition is necessary and it extends the application of CIR model.

At the end, the major contributions of this thesis can be summarized as below:

- The methods in this study greatly extend the area of applicability of structure models in finance problems;
- From mathematical point of view, replacing the boundary condition $u \in L^\infty$ by $u_r \in L^\infty$ is a novel idea. We regard it as a significant contribution to the PDE theory;
- From financial point of view, the removal of the condition $\kappa > \frac{\sigma^2}{2}$ made the application of the CIR model robust. One goal of this thesis is to introduce this new theoretical development of the CIR model to the public.

Last thing to mention about is that one can modify some of treaties in the CDS contract. In our model, we only assume that once the buyer or the seller default, the contract terminates with no more further rights and obligations between two parties. Actually either party can auction sale the contract at the time of its default. This renders to various kinds of models to be studied in the future.

APPENDIX A

CODE OF CIR MODEL CALIBRATION

A.1 BASIC INTEREST RATE TENDENCY

```
TimeUSLibor1M = Range[2005, 2015, 10/(Length[USDLIBOR1MData] - 1)];

TimeUSDLibor3M = Range[2005, 2015, 10/(Length[USDLIBOR3MData] - 1)];

TimeUSDLibor3MLong =Range[1990, 2015, 25/(Length[USDLIBOR3MDataLong] - 1)];

TimeJPYTibor1Y = Range[2005, 2015, 10/(Length[JPYTIBOR1YData] - 1)];

USD1MData = N[USDLIBOR1MData/100];

USD3MData = N[USDLIBOR3MData/100];

USD3MDataLong = N[USDLIBOR3MDataLong/100];

JPY1YData = N[JPYTIBOR1YData/100];

ListPlot[Transpose[{TimeUSLibor1M, USD1MData}], PlotRange -> All,
 PlotStyle -> Red, Joined -> True,
 AxesLabel -> {"Time", "USD Libor 1M-10Y"}]

ListPlot[Transpose[{TimeUSDLibor3M, USD3MData}], PlotRange -> All,
 PlotStyle -> Green, Joined -> True,
 AxesLabel -> {"Time", "USD Libor 3M-10Y"}]

ListPlot[Transpose[{TimeUSDLibor3MLong, USD3MDataLong}],
 PlotRange -> All, PlotStyle -> Pink, Joined -> True,
 AxesLabel -> {"Time", "USD Libor 3M-25Y"}]

ListPlot[Transpose[{TimeJPYTibor1Y, JPY1YData}], PlotRange -> All,
 PlotStyle -> Blue, Joined -> True,
 AxesLabel -> {"Time", "JPY Tibor 1Y-10Y"}]

MatrixForm[{{Mean[USD1MData], Variance[USD1MData],
   StandardDeviation[USD1MData]},

{Mean[USD3MData], Variance[USD3MData],
   StandardDeviation[USD3MData]},

{Mean[USD3MDataLong], Variance[USD3MDataLong],
   StandardDeviation[USD3MDataLong]},

{Mean[JPY1YData], Variance[JPY1YData], StandardDeviation[JPY1YData]}}]
```

A.2 CALIBRATION OF CIR INTEREST RATE MODEL

```
T = Length[USD1MData] - 1;

Delta_t = 1/250;

USD1M = Table[Max[USD1MData[[i]], 0.0001], {i, 1, T + 1}];

X = Table[(USD1M[[i + 1]] - USD1M[[i]])/Sqrt[Delta_t* USD1M[[i]]], {i, 1, T}];

Y = Table[Sqrt[Delta_t/USD1M[[i]]], {i, 1, T}];

Z = Table[Sqrt[Delta_t* USD1M[[i]]], {i, 1, T}];

M = { { Y.Y,    -Y.Z}, {Y.Z, - Z.Z} };

f = {X.Y, X.Z};

{{Kappa, Beta} = Inverse[M] . f ,

Sigma = StandardDeviation[X - Kappa*Y + Beta* Z]

Mu = 2 Kappa/Sigma^2;

Print["Kappa=", Kappa]

Print["Beta=", Beta]

Print["Sigma=", Sigma]

Max[(X - Kappa*  Y + Beta* Z/Sigma)], Min[(X - Kappa*  Y + Beta* Z/Sigma)],

Mean[(X - Kappa*  Y + Beta* Z/Sigma)], StandardDeviation[(X - Kappa*  Y + Beta* Z/Sigma)]

QuantilePicture[x_] := Quantile[(X - Kappa*  Y + Beta* Z/Sigma), x];
```

APPENDIX B

CODE OF SIMULATING CDS MODEL

B.1 CODE OF MONTE-CARLO SIMULATION

```
clear;
tic;

global T n K dt s dr rbar sigma beta kappa a b B1 B2 p q r0 h

%%Some basic assumptions;

T=10;  % Time to expiry;
n=360; dt=T/n; % Number of discrete in T;
               % Length of discrete in T;
K=10000; % Number of random paths for statistics;
s=1500; dr=1/s;  % Number of discrete in Interest Rate;
                 % Length of discrete in r;
%% Parameters defined by ourselves;
rbar=0.03;
beta=0.15;
kappa=rbar*beta; % CIR Model: dr=(kappa-beta*r)dt+sigma*sqrt(r)*dw
sigma=0.01;

%% Parameters defined by actual calibration;
% rbar=kappa/beta;
% beta=0.265533;
% kappa=0.00507039; % CIR Model: dr=(kappa-beta*r)dt+sigma*sqrt(r)*dw
% sigma=0.08086;
%% Some basic assumptions;
a=0.5;           %  lambda=a*r+b
b=0.1;
B2=0.06; p=0.5;   % lambda_1= p*H(r-B_2)
B1=0.04; q=0.3;   % lambda_2= q*H(B_1-r)
r0=0.06; % Current interest rate
h=0.125; % premimum for unit (K=1) insurance

%% r(i,:) = Interest rate r_t of the i-th Status;
r=zeros(n+1,K);
r(1,:)=r0;

%% v0,v1,v2 are varibles for calculating CDS model;

% v0=\int_0^s (r_\theta+\lamda_{\theta}+\lamda_{1\,\theta}+\lamda_{2\,\theta})d\theta;
v0=zeros(n+1,K);

% v1 =\int_0^T \lambda_s e^{-v0} ds;
```

```
v1=zeros(n+1,K);

% v2 =\int_0^T  e^{-v0} ds;
v2=zeros(n+1,K);

%% Ksi are n*K random numbers which distributed N(0,1);
randn('state', 0);
Ksi=normrnd(0,1,n,K);

%% Calculation of r, v0,v1 and v2;
for i = 1 : n

    v0(i+1,:)=v0(i,:)+ ( (a+1)*r(i,:)+b ...
                          +p*(r(i,:)>B2)  ...
                          +q*(r(i,:)<B1)     )*dt;

    v1(i+1,:)=v1(i,:)+(a*r(i,:)+b).*exp(-v0(i,:))*dt;

    v2(i+1,:)=v2(i,:)+exp(-v0(i,:))*dt;

    r(i+1,:) =max(0, ...
                  r(i,:)+ (kappa - beta * r(i,:))*dt ...
                        + sigma * sqrt(r(i,:)).* Ksi(i,:) * sqrt(dt));

end



%% Statistics of Interest Rate r ;
t=(0:dt:T)';
MeanR=rbar+(r0-rbar)* exp(-beta*t);
Rt=mean(r,2);
stdRt=std(r')';

%% Value of u from the Monte-Carlo method
% MonteCarlov1=mean(v1,2); stdv1=std(v1')';

% MonteCarlov2=mean(v2,2); stdv2=std(v2')';

MonteCarlou=mean(v1-h*v2,2) ; % u here represents the numerical
                               % solution of CDS model with K=1;
stdu=std((v1-h*v2)')';

%% Another Numerical Method;
%  Using Finite Difference Method to calculate CDS model;
%  For details, please refer to function "RHS";

%  Relative tolerance=1e-5, absolute Tol=1e-5;
options = odeset('RelTol',1e-5,'AbsTol',ones(s-1,1)*1e-5);
[T1,Y] = ode45(@RHS,[0 T],zeros(s-1,1),options);

%% Plots of solutions and comparsion of two numerical methods;

%Mean interest rate and deviation rate included 95% confidence interval;
figure(1)
hold on
plot(t,100*Rt,'b', t, 100* MeanR,'r');
plot(t,100*(Rt+1.96/sqrt(K)*stdRt),':g');
plot(t,100*(Rt-1.96/sqrt(K)*stdRt),':g');
title(['expected interest rate and deviation rate from ',...
       num2str(K),' sample curves']);
xlabel('T (year)');
ylabel('r ( %)');
legend('Monte Carlo Curve','ODE Curve without noise','95% Interval')
hold off

% M-C Method: CDS model value, included 95% confidence interval;
% figure(2)
```

```
% hold on
% plot((1:n+1)*dt,Rt1,'-b');
% plot((1:n+1)*dt,Rt1+1.96/sqrt(K)*stdRt1,':g');
% plot((1:n+1)*dt,Rt1-1.96/sqrt(K)*stdRt1,':g');
% title([' M-C CDS value u(r,T) with r=', num2str(r0)]);
% xlabel('T (year)');
% ylabel('u(r,T) ($)');
% hold off

% Finite Difference Method: CDS model value with interest rate r=r0;
% figure(3)
% plot(T1,Y(:,r0/dr))
% title(['Finite Difference Method CDS value u(r,T) with r=', num2str(r0)]);
% xlabel('T (year)');
% ylabel('u(r,T) ($)');

% M-C vs Finite Difference Method for CDS model with r=r0;

figure(2)
hold on
plot(t,MonteCarlou,'-r')
plot(T1,Y(:,r0/dr),'-g')
title(['Two methods for CDS value u(r,T) with r=', num2str(r0)]);
xlabel('T (year)');
ylabel('u(r,T) ($)');
legend('M-C Method','Finite Difference Method')
hold off


% figure(3)
% hold on
%
% plot(t,r(:,1))
% plot(t,r(:,1000))
% plot(t,r(:,2000))
% plot(t,r(:,3050))
% plot(t,r(:,5750))
% hold off

timeused = toc
```

B.2 CODE OF FINITE DIFFERENCE METHOD

```
function dV=RHS(t,V)

global sigma beta kappa a b s dr h B2 p B1 q

dV=zeros(s-1,1);

for j = 2:(s-2)
    rj=j*dr;

        dV(j)=sigma^2./2*rj*(V(j+1)+V(j-1)-2*V(j))/(dr^2) ...
               +(kappa-beta*rj)*(V(j+1)-V(j-1))/(2*dr) ...
               -((a+1)*rj+b+p*(rj>B2)+q*(rj<B1))*V(j)+a*rj+b-h;

end

j=1;
rj=j*dr;

    dV(j)=sigma^2./2*rj*(V(j+1)+V(j)-2*V(j))/(dr^2) ...
           +(kappa-beta*rj)*(V(j+1)-V(j))/(2*dr) ...
           -((a+1)*rj+b+p*(rj>B2)+q*(rj<B1))*V(j)+a*rj+b-h;
```

```
j=s-1;
rj=j*dr;

    dV(j)=sigma^2./2*rj*(V(j)+V(j-1)-2*V(j))/(dr^2) ...
           +(kappa-beta*rj)*(V(j)-V(j-1))/(2*dr) ...
           -((a+1)*rj+b+p*(rj>B2)+q*(rj<B1))*V(j)+a*rj+b-h;
```

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