TUKEY QUOTIENTS, PRE-IDEALS, AND NEIGHBORHOOD FILTERS WITH CALIBRE (OMEGA 1, OMEGA)

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TUKEY QUOTIENTS, PRE-IDEALS, AND NEIGHBORHOOD FILTERS WITH CALIBRE (OMEGA 1, OMEGA)

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This work seeks to extract topological information from the order-properties of certain pre-ideals and pre-filters associated with topological spaces. In particular, we investigate the neighborhood filter of a subset of a space, the pre-ideal of all compact subsets of a space, and the ideal of all locally finite subcollections of an open cover of a space. The class of directed sets with calibre (ω_1, ω) (i.e. those whose uncountable subsets each contain an infinite subset with an upper bound) play a crucial role throughout our results. For example, we prove two optimal generalizations of Schneider's classic theorem that a compact space with a G_δ diagonal is metrizable. The first of these can be stated as: if X is (countably) compact and the neighborhood filter of the diagonal in X^2 has calibre (ω_1, ω) with respect to reverse set inclusion, then X is metrizable. Tukey quotients are used extensively and provide a unifying language for expressing many of the concepts studied here.

Keywords: directed sets, calibres, Tukey quotients, compact covers, P-paracompactness, metrizability, productivity, Lindelöf Σ-spaces, neighborhood filters, strong Pytkeev property, function spaces.

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1.0 INTRODUCTION

Given a set S, a *pre-ideal* on S is a collection $\mathcal{A}$ of subsets of S that is directed with respect to set inclusion $\subseteq$; that is, any pair of members of $\mathcal{A}$ are contained within a third member of $\mathcal{A}$. A *pre-filter* on S is a collection $\mathcal{A}$ of subsets of S that is directed with respect to reverse set inclusion $\supseteq$; that is, any pair of members of $\mathcal{A}$ contain a third member of $\mathcal{A}$ in their intersection. A pre-ideal (respectively, pre-filter) $\mathcal{A}$ on S is called an *ideal* (respectively, *filter*) if $\mathcal{A}$ contains every subset (respectively, superset) of each element of $\mathcal{A}$, or in other words, if $\mathcal{A}$ is 'downwards-closed' with respect to its relevant ordering. The broad aim of this work is to investigate the relationship between (i) the order properties of several pre-ideals and pre-filters which naturally occur in topology and (ii) the topological properties of the spaces from which those pre-ideals and pre-filters are derived.

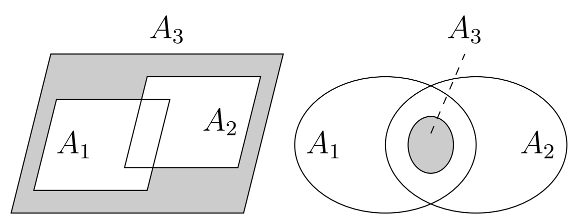

Figure 1: A pre-ideal (left) vs. a pre-filter (right)

This objective is by no means new. Indeed, many well-studied topological properties are intimately linked to the order structure of pre-ideals and pre-filters. For example, a fundamental property generalizing metrizability is the notion of *first countability*, which requires that each point in a space has a countable neighborhood base. Recall that a subset

C of a directed set P is called *cofinal* in P if for every p in P, there is a c in C such that $p \leq c$, and the *cofinality* of a directed set is the minimal cardinality of its cofinal subsets. Thus, a neighborhood base for a point x in a space X is precisely a cofinal (with respect to $\supseteq$) subset of the filter $\mathcal{N}_x^X$ of all neighborhoods of x, and so X is first countable if and only if $\mathcal{N}_x^X$ has countable cofinality for each x in X. Similarly, a space X is called *hemicompact* if the pre-ideal $\mathcal{K}(X)$ of all compact subsets of X has countable cofinality (with respect to $\subseteq$), and this generalizes compactness.

More to the point, much of Chapters 3 and 4 below is inspired by Schneider's classic metrization theorem. Here the *diagonal* of X is the subset $\Delta = \{(x, x) : x \in X\}$ of X^2 and $\mathcal{N}_\Delta^{X^2}$ is the filter of all neighborhoods of Δ in X^2.

Theorem 1 (Schneider, 1945)**.** *For a compact space X, the following are equivalent:*

(i) X is metrizable,

(ii) $\mathcal{K}(X^2 \setminus \Delta)$ has countable cofinality with respect to $\subseteq$,

(iii) $\mathcal{N}_\Delta^{X^2}$ has countable cofinality with respect to $\supseteq$, and

(iv) Δ is a G_δ subset of X^2.

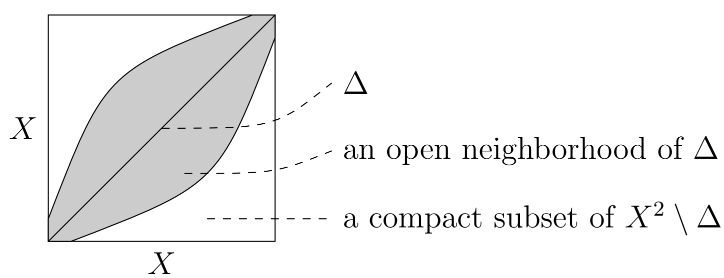

Figure 2: The diagonal of a compact space

The equivalence of (i) and (iv) in Theorem 1 is the usual statement of Schneider's theorem, but compactness makes (iv) equivalent to the (generally stronger) condition that Δ has a countable neighborhood base, which is precisely statement (iii). Also, the complement of any open neighborhood of Δ is a compact subset of $X^2 \setminus \Delta$, so (ii) and (iii) are easily seen to be equivalent. Thus, Schneider's theorem establishes a connection between the topological properties of a compact space X, the order properties of the filter $\mathcal{N}_\Delta^{X^2}$, and the order properties of the pre-ideal $\mathcal{K}(X^2 \setminus \Delta)$.

This version of Schneider's theorem illustrates a theme that will continue throughout this work, namely that the order properties of interest will all be related to the 'cofinal structure' of the pre-ideals, pre-filters, and other directed sets studied here. To compare the cofinal complexity of two directed sets, we will use *Tukey quotients* (see Section 2.2), which were introduced by Tukey [48] for studying the convergence of what are today call nets. In most contexts throughout this work, a Tukey quotient map from a directed set P to a directed set Q will be an order-preserving map whose image is cofinal in Q, and the existence of such a map is denoted by the relation '$P \geq_T Q$', which indicates that Q is no more cofinally complex than P. Tukey quotients provide us a convenient notation for expressing many of the order concepts relevant to our investigations. For example, a directed set P has countable cofinality if and only if $\omega \geq_T P$, where ω is the first infinite ordinal (considered as the set of all finite ordinals) with its usual ordering.

Let Y be any space. By a *cofinal compact cover* of Y, we mean a cofinal subset of the pre-ideal $\mathcal{K}(Y)$. Thus, $\mathcal{K}(Y)$ has countable cofinality (in other words, Y is hemicompact) if and only if Y has a countable cofinal compact cover $\mathcal{A}$. We can enumerate $\mathcal{A} = \{A_n : n < \omega\}$, and by replacing A_n with $A_0 \cup \cdots \cup A_n$, we may assume $A_n \subseteq A_{n+1}$ for each $n < \omega$. In this way, $\mathcal{A}$ is not just countable, but also ordered like the directed set ω. If P is a directed set, then we say a family $\mathcal{S}$ of subsets of a set S is P*-ordered* if $\mathcal{S} = \{S_p : p \in P\}$ where $S_{p_1} \subseteq S_{p_2}$ whenever $p_1 \leq p_2$ in P. Using Tukey quotients, the existence of a P-ordered cofinal compact cover for a space Y can be expressed by the relation '$P \geq_T \mathcal{K}(Y)$'. Thus, Schneider's theorem may be rephrased as:

Theorem 2 (Schneider, version 2)**.** *Let X be a compact space. If $X^2 \setminus \Delta$ has an ω-ordered cofinal compact cover, in other words $\omega \geq_T \mathcal{K}(X^2 \setminus \Delta)$, then X is metrizable.*

It is natural to ask: which directed sets can be used in place of ω in Theorem 2? Cascales and Orihuela [6] showed that the directed set ω^ω consisting of all functions from ω to ω works here when it is ordered pointwise (that is, $\alpha \leq \beta$ in ω^ω if and only if $\alpha(n) \leq \beta(n)$ for each $n < \omega$). Note that this result generalizes Schneider's theorem because any ω-ordered family is also ω^ω-ordered. Indeed if $\mathcal{A} = \{A_n : n < \omega\}$ is ω-ordered, then after defining $A_\alpha = A_{\alpha(0)}$ for each α in ω^ω, we have $\mathcal{A} = \{A_\alpha : \alpha \in \omega^\omega\}$ and $\mathcal{A}$ is ω^ω-ordered. This argument boils down to the fact that the projection from ω^ω onto the first factor of ω is a Tukey quotient map, so that $\omega^\omega \geq_T \omega$. Thus if $\omega \geq_T \mathcal{K}(X^2 \setminus \Delta)$, then transitivity of the Tukey quotient relation shows that $\omega^\omega \geq_T \mathcal{K}(X^2 \setminus \Delta)$ also.

Cascales, Orihuela, and Tkachuk [7] later showed that for any separable metrizable space M, the pre-ideal $\mathcal{K}(M)$ can also be used in place of ω in Theorem 2:

Theorem 3 (Cascales et al, [7])**.** *Let X be a compact space. If $\mathcal{K}(M) \geq_T \mathcal{K}(X^2 \setminus \Delta)$ for some separable metrizable M, then X is metrizable.*

This roughly says that if X is compact and the compact subsets of $X^2 \setminus \Delta$ are ordered like the compact subsets of a separable metrizable space, then X itself must be a separable metrizable space. Note that this generalizes the previous result by Cascales and Orihuela. Indeed, Lemma 50 below shows that $\mathcal{K}(\omega^\omega) \geq_T \omega^\omega$, so if $\omega^\omega \geq_T \mathcal{K}(X^2 \setminus \Delta)$, then transitivity gives $\mathcal{K}(M) \geq_T \mathcal{K}(X^2 \setminus \Delta)$ for the separable metrizable space $M = \omega^\omega$.

In Section 2.3, we discuss the *calibres* of a directed set P as a means of measuring the cofinal complexity of P, and in Lemma 55, we show that for every separable metrizable space M, the pre-ideal $\mathcal{K}(M)$ has calibre (ω_1, ω), which means that each uncountable subset of $\mathcal{K}(M)$ contains an infinite subset with an upper bound in $\mathcal{K}(M)$. In Theorem 63 of Chapter 3, we generalize Theorem 3 by proving that if P is any directed set with calibre (ω_1, ω), then P can be used in place of ω in Theorem 2. Moreover, we show in Theorem 67 that these are the only directed sets that can replace ω. Theorem 63 can also be stated as: if X is compact and $\mathcal{K}(X^2 \setminus \Delta)$ itself has calibre (ω_1, ω), then X is metrizable, so this really is a link between the internal cofinal structure of the pre-ideal $\mathcal{K}(X^2 \setminus \Delta)$ and the topological properties of X.

Although Tukey quotients provide a convenient notation for expressing the existence of P-ordered cofinal compact covers, they lack the flexibility to address general (non-cofinal) P-ordered compact covers, which are also considered in Chapter 3. Let Y be any space, and note that we can identify Y with the subset of $\mathcal{K}(Y)$ consisting of all the singletons. If $\mathcal{A}$ is a compact cover of Y, then even if $\mathcal{A}$ is not cofinal in $\mathcal{K}(Y)$, it is still 'cofinal for Y' in the sense that each singleton subset of Y is contained in a member of $\mathcal{A}$. In Section 2.2, we therefore consider a relative version of Tukey quotients which is defined for pairs (P', P) where P is a directed set and P' is a subset of P. This allows us to express the statement 'Y has a P-ordered compact cover' via the relation '$P \geq_T (Y, \mathcal{K}(Y))$'. For example, a space Y is called *σ-compact* if it has an ω-ordered compact cover, which we can indicate by '$\omega \geq_T (Y, \mathcal{K}(Y))$', and this should be compared to the fact that Y is hemicompact precisely when $\omega \geq_T \mathcal{K}(Y)$.

In Chapter 4, we investigate a different sort of P-ordered cover. If $\mathcal{C}$ is a family of subsets of a space X, then the family $LF(\mathcal{C})$ of all locally finite subsets of $\mathcal{C}$ is an ideal on $\mathcal{C}$, and we can identify $\mathcal{C}$ with the subset of $LF(\mathcal{C})$ consisting of all singletons. Given a directed set P, we say $\mathcal{C}$ is *P-locally finite* if $\mathcal{C}$ has a P-ordered cover of locally finite subcollections, that is, if $\mathcal{C} = \bigcup\{\mathcal{C}_p : p \in P\}$ where each $\mathcal{C}_p$ is locally finite and $\mathcal{C}_{p_1} \subseteq \mathcal{C}_{p_2}$ whenever $p_1 \leq p_2$ in P, or equivalently, if $P \geq_T (\mathcal{C}, LF(\mathcal{C}))$. We then define a space X to be *P-paracompact* if each open cover of X has a P-locally finite open refinement, and we call a space *P-metrizable* if it has a $(P \times \omega)$-locally finite base. We study these and closely related properties throughout Chapter 4, with particular attention given to the case where $P = \mathcal{K}(M)$ for some separable metrizable M. In Section 4.3.1, we even give a complete characterization of those spaces which are $\mathcal{K}(M)$-metrizable for some separable metrizable M.

If X is a compact space whose diagonal is a G_δ subset of X^2, then $X^2 \setminus \Delta$ is σ-compact and therefore paracompact. Hence, the following result by Gruenhage is yet another generalization of Schneider's Theorem.

Theorem 4 (Gruenhage, [28])**.** *If X is compact and $X^2 \setminus \Delta$ is paracompact, then X is metrizable.*

In Section 4.2, we generalize Theorem 4 by showing that paracompactness can be replaced with P-paracompactness for any P with calibre (ω_1, ω). We then show that this generalization is optimal in the sense that we cannot weaken the hypothesis to P-paracompactness for P in some larger class of directed sets.

Thus, we have two optimal generalizations of Schneider's theorem which demonstrate the importance of the class of directed sets with calibre (ω_1, ω), so we find it worthwhile to study the properties of this class. Because Todorčević's Example 13 from [45] shows that calibre (ω_1, ω) is generally not a productive property, in Chapter 5 we investigate to what extent calibre (ω_1, ω) is preserved by various types of products. In particular, we show that there is a large class $\mathcal{Q}$ of directed sets with calibre (ω_1, ω) such that every Σ-product of members of $\mathcal{Q}$ has calibre (ω_1, ω), and this class includes every $\mathcal{K}(M)$ where M is separable and metrizable. We also study directed sets whose countable powers have calibre (ω_1, ω) as well as those directed sets whose product with any other calibre (ω_1, ω) directed set has calibre (ω_1, ω).

The focus in Chapter 6 shifts to neighborhood filters. Gabriyelyan, Kąkol, and Leiderman showed in [22] that the order structure of neighborhood filters is related to the strong Pytkeev property. Here, a space Y is said to have the *strong Pytkeev property* if for each point y in Y, there is a countable family $\mathcal{D}_y$ of subsets of Y such that, whenever U is a neighborhood of y and A is a subset of Y with $y \in \overline{A} \setminus A$, there is a D in $\mathcal{D}_y$ such that D is contained in U and $D \cap A$ is infinite. The family $\mathcal{D}_y$ is a sort of 'network' for the neighborhood filter $\mathcal{N}_y^Y$ since each neighborhood of y contains a member of $\mathcal{D}_y$. So the strong Pytkeev property for a space Y asserts that for each point y in Y, there is a countable 'network' for $\mathcal{N}_y^Y$ with 'nice' properties. In Sections 6.1 and 6.2, we consider several variations of the strong Pytkeev property formed by altering the meaning of 'nice'. For example, relaxing the condition '$D \cap A$ is infinite' in the definition of the strong Pytkeev property to '$D \cap A$ is nonempty' produces a property called (cn).

Gabriyelyan et al. proved the following result, where $C_k(X)$ denotes the space of continuous real-valued functions on a space X with the compact-open topology, and $\mathbf{0}$ denotes the zero function on X.

Theorem 5 (Gabriyelyan et al., [22]). *Let X be a space, and suppose $\omega^\omega \geq_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$. Then the following are equivalent:*

(i) $C_k(X)$ has the strong Pytkeev property, and

(ii) X is Lindelöf.

In Theorem 179 of Section 6.4, we completely characterize when $C_k(X)$ has the strong Pytkeev property by proving that this occurs if and only if X is Lindelöf cofinally Σ (a property which is discussed in Section 6.3). Moreover, this result shows that several variations of the strong Pytkeev property (including (cn)) are equivalent to the strong Pytkeev property for spaces of the form $C_k(X)$. We also prove that this is equivalent to $C_k(X)$ being countably tight and having $P \geq_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$ for some directed set P in the class $\mathcal{Q}$ mentioned in the above discussion of Chapter 5.

2.0 PRELIMINARIES

This chapter is devoted to introducing many of the order-theoretic concepts and related topological notions which will be used extensively in the following chapters. Throughout this work, topological spaces are usually assumed to be Tychonoff, or at least T_3, unless stated otherwise. The *weight* of a space X, denoted $w(X)$, is the minimal cardinality of a base for X, and the cardinality of the reals $\mathbb{R}$ (i.e. the *cardinality of the continuum*) is denoted $\mathfrak{c} = 2^\omega$. We will identify a cardinal κ with the first ordinal of cardinality κ, and so we will usually use the first infinite ordinal ω and the first uncountable ordinal ω_1 in place of $\aleph_0$ and $\aleph_1$, respectively. We will also identify an ordinal with the set of all smaller ordinals, so for example, we have $\omega = \{0, 1, 2, \ldots, n, n+1, \ldots\}$.

2.1 DIRECTED SETS, IDEALS, AND FILTERS

A *partially ordered set* is a set P with a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. If P and Q are partially ordered sets, then a map $f : P \to Q$ is *order-preserving* if $f(p_1) \leq f(p_2)$ in Q whenever $p_1 \leq p_2$ in P. The map f is an *order-isomorphism* (and P and Q are *order-isomorphic*) if it is bijective and both f and its inverse are order-preserving.

For any p in a partially ordered set P, the *down set* of p is $\downarrow p = \{p' \in P : p' \leq p\}$. We say a subset P' of P is *bounded above* if P' is contained in $\downarrow q$ for some q in P, and q is called an *upper bound* of P'. A *directed set* is a partially ordered set P such that every finite subset of P is bounded above. Many of the results discussed below apply to partially ordered sets, but for consistency and simplicity, we restrict our attention to directed sets.

A directed set P is called *Dedekind complete* if each subset of P that is bounded above has a least upper bound. Examples of Dedekind complete directed sets include any ordinal κ (considered as the set of all ordinals less than κ) as well as the *power set* $\mathbb{P}(X)$ of a set X, that is, the family of all subsets of X, which is directed with respect to both set inclusion $\subseteq$ and reverse set inclusion $\supseteq$. In fact, most of the directed sets of interest in this work will be subsets of power sets that are directed with respect to either set inclusion or reverse set inclusion.

If X is a set, then a subset of $\mathbb{P}(X)$ which is directed by set inclusion is called a *pre-ideal* on X. An *ideal* on X is a pre-ideal $\mathcal{I}$ such that whenever S is in $\mathcal{I}$, each subset of S is also in $\mathcal{I}$. Similarly, a *pre-filter* on X is a subset of $\mathbb{P}(X)$ which is directed by reverse set inclusion, and a *filter* on X is a pre-filter $\mathcal{F}$ such that whenever S is in $\mathcal{F}$, each subset of X containing S is in $\mathcal{F}$. Every pre-ideal $\mathcal{S}$ on X generates an ideal $\downarrow\!\mathcal{S} = \{A \subseteq X : \exists B \in \mathcal{S} \text{ such that } A \subseteq B\}$, and every pre-filter $\mathcal{P}$ on X generates a filter $\uparrow\!\mathcal{P} = \{A \subseteq X : \exists B \in \mathcal{P} \text{ such that } A \supseteq B\}$.

For example, the set $\mathcal{K}(X)$ of all compact subsets of a topological space X is a pre-ideal on X which generates the ideal of all subsets of X with compact closure. Likewise, the set $\mathcal{N}_x$ of all neighborhoods of a point x in a space X is a filter on X, called the *neighborhood filter* of x, which is generated by the pre-filter of all open neighborhoods of x. Note that a neighborhood filter is *proper* – it does not contain the empty set – although our definition of filter does not require this property.

For any set X, let $c_X : \mathbb{P}(X) \to \mathbb{P}(X)$ be the *complementation map*, $c_X(A) = X \setminus A$, and for any family $\mathcal{A}$ of subsets of X, write $c_X(\mathcal{A})$ for the family $\{c_X(A) : A \in \mathcal{A}\}$. Clearly we have:

Lemma 6. *Let $\mathcal{I}$ and $\mathcal{F}$ be a (pre-)ideal and (pre-)filter on X, respectively. Then:*

(1) $c_X(\mathcal{I})$ is a (pre-)filter on X which is order-isomorphic to $\mathcal{I}$ via c_X, and

(2) $c_X(\mathcal{F})$ is a (pre-)ideal on X which is order-isomorphic to $\mathcal{F}$ via c_X.

Thus, c_X induces a natural duality between ideals and filters on X and between pre-ideals and pre-filters on X.

Lemma 7. *Let X be a set.*

(1) If $\mathcal{S}$ is a pre-ideal on X that is closed under arbitrary intersections, then $\mathcal{S}$ is Dedekind complete (with respect to $\subseteq$).

(2) If $\mathcal{P}$ is a pre-filter on X that is closed under arbitrary unions, then $\mathcal{P}$ is Dedekind complete (with respect to $\supseteq$).

In particular, ideals and filters are always Dedekind complete.

Proof. Let $\mathcal{A}$ be a subset of $\mathcal{S}$ with an upper bound in $\mathcal{S}$. Let $\mathcal{B}$ be the set of all upper bounds of $\mathcal{A}$ in $\mathcal{S}$. Then $\bigcap \mathcal{B}$ is the least upper bound of $\mathcal{A}$ in $\mathcal{S}$, which proves the first claim. The second claim is dual to (and follows from) the first; of course we must keep in mind that a 'least upper bound' with respect to $\supseteq$ is actually a greatest lower bound with respect to $\subseteq$. $\square$

Generally, the only filters we will be interested in are neighborhood filters, but in addition to $\mathcal{K}(X)$, there are several other Dedekind complete (pre-)ideals that will find use throughout this work. We list them below, where S denotes a set, X denotes a topological space, and $\mathcal{C}$ denotes a family of subsets of X:

$$[S]^{<\omega} = \{F \subseteq S : F \text{ is finite}\} \qquad CL(X) = \{C \subseteq X : C \text{ is closed in } X\}$$

$$[S]^{\leq\omega} = \{C \subseteq S : C \text{ is countable}\} \qquad LF(\mathcal{C}) = \{\mathcal{L} \subseteq \mathcal{C} : \mathcal{L} \text{ is locally finite in } X\}$$

$$\mathcal{K}(X) = \{K \subseteq X : K \text{ is compact}\} \qquad PF(\mathcal{C}) = \{\mathcal{L} \subseteq \mathcal{C} : \mathcal{L} \text{ is point finite in } X\}$$

2.2 TUKEY QUOTIENTS AND DIRECTED SET PAIRS

A subset C of a directed set P is called *cofinal* in P if for any p in P, there is a c in C such that $p \leq c$. The *cofinality* of a directed set P, denoted $cof(P)$, is the minimal cardinality of a cofinal subset of P. The properties of a directed set P that we are interested in are all related to its 'cofinal structure' or 'end behavior'. For example, if $P = \mathcal{N}_x$ is the neighborhood filter of a point x in a space X, then x is a point of first countability if and only if $\mathcal{N}_x$ has countable cofinality. Two directed sets do not need to be order-isomorphic to have the same cofinal structure. Instead, we use Tukey quotients to compare the cofinal complexity of directed sets.

Let P and Q be directed sets. A map $\phi : P \to Q$ is called a *Tukey quotient map* if every cofinal subset of P is mapped by ϕ to a cofinal subset of Q. If such a map exists, we say Q is a *Tukey quotient* of P and write $P \geq_T Q$. If $P \geq_T Q$ and $Q \geq_T P$, then we will say P and Q are *Tukey equivalent* and write $P =_T Q$. Tukey equivalence is an equivalence relation on the class of directed sets, and the *Tukey order*, $\geq_T$, is a partial order on the equivalence classes.

Tukey quotients provide us a convenient notation for expressing many topological properties. For example, if x is a point in a space X, then x is isolated if and only if $1 \geq_T \mathcal{N}_x$, and x is a point of first countability if and only if $\omega \geq_T \mathcal{N}_x$. Here, 1 denotes the singleton $\{0\}$ and ω is the first infinite ordinal. Similarly, X is compact if and only if $1 \geq_T \mathcal{K}(X)$, and X is hemicompact (that is, X has a countable family of compact sets that contain every other compact subset of X) if and only if $\omega \geq_T \mathcal{K}(X)$.

However, we are concerned not only with the order properties of directed sets, but also with the relative properties of subsets of those directed sets. The definition of Tukey quotients given above lacks the flexibility to address these concerns, so we will need an extended notion of Tukey quotients. If P is a directed set and P' is a subset of P, then we call (P', P) a *directed set pair*. Note that P' itself is a partially ordered set but not necessarily a directed set. We say a set C is *cofinal for* (P', P) if C is a subset of P such that for each p' in P', there is a c in C with $p' \leq c$.

A directed set pair (Q', Q) is called a *(relative) Tukey quotient* of the pair (P', P), written $(P', P) \geq_T (Q', Q)$, if there is a map $\phi : P \to Q$ which takes cofinal sets for (P', P) to cofinal sets for (Q', Q). Such a ϕ is called a *(relative) Tukey quotient map.* If $P' = P$, then we can abbreviate the directed set pair (P', P) to just P. In this way, the relation $(P, P) \geq_T (Q, Q)$ reduces to $P \geq_T Q$, and this coincides with the notion of Tukey quotients of directed sets given above. If $(P', P) \geq_T (Q', Q)$ and $(Q', Q) \geq_T (P', P)$, then we say the pairs are *Tukey equivalent* and write $(P', P) =_T (Q', Q)$. Tukey equivalence is an equivalence relation on the class of directed set pairs, and '$\geq_T$' is a partial order on the equivalence classes; in particular, Tukey quotients are transitive.

Occasionally, the following alternate description of the Tukey quotient relation will be useful. The non-relative version is well-known, and the proof of this version is given in [27].

Lemma 8. *$(P', P) \geq_T (Q', Q)$ if and only if there is a map $\psi : Q' \to P'$ such that for any subset U of Q' that is unbounded in Q, $\psi(U)$ is unbounded in P.*

Additionally, the following lemma says that for Dedekind complete directed sets, we may always assume our Tukey quotient maps have a nice form. The proof is also given in [27], and again, the non-relative version is well-known.

Lemma 9. *Let (P', P) and (Q', Q) be directed set pairs. If $\phi : P \to Q$ is an order-preserving map such that $\phi(P')$ is cofinal for Q' in Q, then ϕ witnesses that $(P', P) \geq_T (Q', Q)$. Conversely, if Q is Dedekind complete and $(P', P) \geq_T (Q', Q)$, then there is an order-preserving Tukey quotient map $\phi : P \to Q$ witnessing $(P', P) \geq_T (Q', Q)$.*

Our directed set pairs will usually have the following form. Let Y be a subset of a set X, and let $\mathcal{S}$ be a pre-ideal on X such that $\mathcal{S}$ contains every finite subset of Y (or equivalently, $\mathcal{S}$ contains every singleton subset of Y). Then we can identify Y with the subset $[Y]^1 = \{\{y\} : y \in Y\}$ of $\mathcal{S}$, so that $(Y, \mathcal{S})$ is a directed set pair. Recall from the introduction that if P is a directed set, then a family $\{S_p : p \in P\}$ of subsets of a set X is called *P-ordered* if $S_{p_1} \subseteq S_{p_2}$ whenever $p_1 \leq p_2$.

Lemma 10. *Let P be a directed set, and let $\mathcal{S}$ be a Dedekind complete pre-ideal of subsets of a set X such that $\mathcal{S}$ contains every finite subset of a subset Y of X. Then the following are equivalent:*

(1) Y has a P-ordered cover consisting of sets in $\mathcal{S}$.

(2) $P \geq_T (Y, \mathcal{S})$.

Proof. Suppose $\{S_p : p \in P\}$ is a P-ordered cover of Y such that each S_p is in $\mathcal{S}$. Then the map $\phi : P \to \mathcal{S}$ given by $\phi(p) = S_p$ is order-preserving and its image covers Y. Notice that a subset of $\mathcal{S}$ covers Y if and only if it is cofinal for $(X, \mathcal{S})$. By Lemma 9, ϕ is therefore a Tukey quotient map witnessing (2). Since $\mathcal{S}$ is Dedekind complete, then Lemma 9 also works in reverse to give the proof that (2) implies (1). □

The previous lemma implies, for example, that a space X is σ-compact if and only if $\omega \geq_T (X, \mathcal{K}(X))$. Indeed, any countable compact cover $\{K_n : n < \omega\}$ of X generates an

ω-ordered compact cover $\{K'_n = \bigcup_{i \leq n} K_i : n < \omega\}$. This should be compared with our comment above that X is hemicompact if and only if $\omega \geq_T \mathcal{K}(X)$.

Lemma 11. *If C is a cofinal subset of a directed set P, then (C is directed and) $C =_T P$.*

Proof. Let $\phi : C \to P$ be the inclusion map. Then ϕ is order-preserving and its image is cofinal in P, so $C \geq_T P$ by Lemma 9. Now let S be a subset of C and suppose $\phi(S)$ is bounded in P, say by p. Since C is cofinal, there is a c in C such that $p \leq c$, and so $S = \phi(S)$ is bounded in C. Hence, ϕ also witnesses $P \geq_T C$ by Lemma 8. □

We immediately obtain:

Corollary 12. *Let X be a set.*

(1) If $\mathcal{S}$ is a pre-ideal on X, then $\mathcal{S} =_T {\downarrow}\mathcal{S}$.

(2) If $\mathcal{P}$ is a pre-filter on X, then $\mathcal{P} =_T {\uparrow}\mathcal{P}$.

The previous corollary says that any pre-ideal (or pre-filter) has the same cofinal structure as the ideal (or filter) that it generates. In principle, therefore, we should have no need of pre-ideals (or pre-filters) since we could replace them with the generated ideals (or filters). While it is true that the pre-filters of interest here will actually be filters (specifically, neighborhood filters), it will often still be useful for us to consider pre-ideals, such as $\mathcal{K}(X)$, rather than the ideals they generate. Indeed, the pre-ideal may have some other nice structure in addition to its order structure. For example, $\mathcal{K}(X)$ has a natural topological structure via the Vietoris topology (which is metrizable if X is metrizable).

Corollary 13. *Let $\mathcal{S}$ and $\mathcal{P}$ be a pre-ideal and pre-filter, respectively, on a set X. If the ideal generated by $\mathcal{S}$ is dual (via complementation) to the filter generated by $\mathcal{P}$, then $\mathcal{S} =_T \mathcal{P}$.*

Proof. By Lemma 6, a dual ideal and filter are order-isomorphic, and therefore Tukey equivalent. The result then follows from Corollary 12 and the transitivity of '$=_T$'. □

The next result says that every Tukey equivalence class contains a neighborhood filter and a pre-ideal of compact subsets. So directed sets of the form $\mathcal{N}_x^X$ and $\mathcal{K}(X)$ are, in some sense, universal.

Theorem 14. *For any directed set P, we have:*

(1) $P =_T \mathcal{K}(X_P)$ *for some locally compact Hausdorff space* X_P*.*

(2) $P =_T \mathcal{N}_{x_P}^{K_P}$ *for some compact Hausdorff space* K_P *and point* x_P *in* K_p*.*

(3) $P =_T \mathcal{N}_e^{G_P}$ *for some topological group* G_P *with identity* e*.*

(4) $P \times \omega =_T \mathcal{N}_0^{L_P}$ *for some locally convex topological vector space* L_P*.*

Proof. Let $D(P)$ be P with the discrete topology. For each p in P, let K_p be the closure of $\downarrow p$ in the Stone-Cěch compactification $\beta D(P)$. Note that K_p is compact and open. Let $X_P = \bigcup\{K_p : p \in P\}$ considered as a subspace of $\beta D(P)$. Then X_P is locally compact and Hausdorff. The map $\phi : P \to \mathcal{K}(X_P)$ given by $\phi(p) = K_p$ is clearly order-preserving, and since $K_p \cap D(P) = \downarrow p$, then in fact ϕ is an order-isomorphism between P and $\phi(P)$. To see that the image of ϕ is cofinal in $\mathcal{K}(X_P)$, note that if L is any compact subset of X_P, then L is covered by a finite subset $\mathcal{F}$ of $\phi(P)$ (since each K_p is open), and since P is directed, then $\mathcal{F}$ has an upper bound K_q in $\phi(P)$, which gives $L \subseteq K_q$. By Lemma 11, we now have $P =_T \mathcal{K}(X_P)$, which proves (1).

Let $K_P = X_P \cup \{x_P\}$ be the one-point compactification of X_P. Then $\mathcal{K}(X_P)$ and $\mathcal{N}_{x_P}^{K_P}$ are Tukey equivalent by Lemma 13, which proves (2).

Let $\mathbb{Z}_2$ be the discrete two-point group under addition modulo 2, and define $G_P = C_k(X_P, \mathbb{Z}_2)$, the group of all continuous maps of X_P into $\mathbb{Z}_2$, with the compact-open topology. For each compact subset K of X_P, let $B_K = \{g \in G_P : g(K) \subseteq \{0\}\}$. Then $\mathcal{B} = \{B_K : K \in \mathcal{K}(X_P)\}$ is a neighborhood base for the zero function $e = \mathbf{0}$ in G_P. Since X_P is zero dimensional, then the map $K \mapsto B_K$ is an order isomorphism between $\mathcal{K}(X_P)$ and $\mathcal{B}$. Hence, $\mathcal{K}(X_P)$ is Tukey equivalent to $\mathcal{B}$, which is Tukey equivalent to $\mathcal{N}_e^{G_P}$ by Lemma 12, which proves (3).

Finally, let $L_p = C_k(X_p)$ be the space of continuous real-valued functions on X_p with the compact open topology (see Section 2.9). Then Lemma 62 below shows that $\mathcal{N}_0^{L_p} =_T \mathcal{K}(X_p) \times \omega =_T P \times \omega$, and so (4) is proven. $\square$

2.3 CALIBRES

In the previous section, we introduced Tukey quotients as a tool for comparing the cofinal complexity of directed sets (or directed set pairs). In this section, we introduce a family of properties which allow us to measure the cofinal complexity of a directed set (pair). Because directed sets have a favored direction, we will say only that a subset of a directed set P is 'bounded' when we really mean 'bounded above in P'.

Consider any cardinals $\kappa \geq \lambda \geq \mu \geq \nu$. We say a directed set P has *calibre* (κ, λ) if every κ-sized subset of P contains a λ-sized bounded subset. We say P has *calibre* (κ, λ, μ) if every κ-sized subset S_1 of P contains a λ-sized subset S_2 such that every μ-sized subset of S_2 is bounded. Finally, we say P has *calibre* $(\kappa, \lambda, \mu, \nu)$ if every κ-sized subset S_1 of P contains a λ-sized subset S_2 such that every μ-sized subset of S_2 contains a ν-sized bounded subset.

More generally, we say that a directed set pair (P', P) has *calibre* (κ, λ) (or P' has *relative calibre* (κ, λ) *in* P) if each κ-sized subset of P' contains a λ-sized subset that is bounded in P. The other calibres can be similarly relativized. Of course when $P' = P$, these relative calibres reduce to the calibres defined in the previous paragraph.

Lemma 15. *Consider any cardinals $\kappa \geq \lambda \geq \mu \geq \nu$. The following implications hold for the calibres of any directed set pair.*

(1) calibre $(\kappa, \lambda) \implies$ calibre $(\kappa, \lambda, \mu) \implies$ calibre $(\kappa, \lambda, \mu, \nu)$

(2) calibre $(\kappa, \lambda) \implies$ calibre (κ, μ)

(3) calibre $(\kappa, \lambda, \mu) \implies$ calibre (κ, λ, ν)

(4) calibre $(\kappa, \lambda, \mu) \implies$ calibre (κ, μ)

(5) calibre $(\kappa, \lambda) \iff$ calibre $(\kappa, \lambda, \lambda) \iff$ calibre $(\kappa, \lambda, \lambda, \lambda) \iff$ calibre $(\kappa, \kappa, \kappa, \lambda)$

Proof. Statements (1)–(4) are immediate from the definitions. To show the first three calibres in (5) are equivalent, it therefore suffices to apply (1) and verify that the definitions immediately show calibre $(\kappa, \lambda, \lambda, \lambda)$ implies calibre (κ, λ). It is then also straightforward to check that the definitions give the equivalence of calibres (κ, λ) and $(\kappa, \kappa, \kappa, \lambda)$. $\square$

A cardinal is called *regular* if it is equal to its own cofinality.

Lemma 16. *Let $f : S \to T$ be any function between sets, and let κ be a regular cardinal. Then for any κ-sized subset S' of S, there is a κ-sized subset S'' of S' such that f is either one-to-one or constant on S''.*

Proof. Let S' be a κ-sized subset of S. We have two cases to consider. If $|f(S')| = \kappa$, then we can easily pick a subset S'' of S' consisting of one member of $f^{-1}(t)$ for each t in $f(S')$, and so f is one-to-one on S''.

Now suppose $|f(S')| < \kappa$ instead, and note that $S' = \bigcup\{S' \cap f^{-1}(t) : t \in f(S')\}$. Since S' has size κ and κ is regular, we see that there must be some t_0 in $f(S')$ such that $|S' \cap f^{-1}(t_0)| = \kappa$. Thus, f is constant on $S'' = S' \cap f^{-1}(t_0)$. □

The key connection between calibres and Tukey quotients is given by the following lemma, which says that (relative) Tukey quotients preserve (relative) calibres.

Lemma 17. *Suppose $(P', P) \geq_T (Q', Q)$ and $\kappa \geq \lambda \geq \mu \geq \nu$. If κ is a regular cardinal and (P', P) has any of the calibres (κ, λ), (κ, λ, μ), or $(\kappa, \lambda, \mu, \nu)$, then (Q', Q) has the same calibre.*

Proof. By Lemma 8, there is a map $\psi : Q' \to P'$ such that $\psi(U)$ is unbounded in P whenever U is unbounded in Q. For statement 1, we prove the calibre (κ, λ, μ) case, and the others are very similar. Let S_1 be a κ-sized subset of Q'. By Lemma 16, we know there is a κ-sized subset of S_1' of S_1 such that ψ is one-to-one or constant on S_1'. Suppose ψ is constant on S_1', then we can arbitrarily fix a λ-sized subset S_2 of S_1', and then any μ-sized subset S_3 of S_2 will have $\psi(S_3) = \{p\}$, which is bounded in P, and so S_3 is bounded in Q. If ψ is one-to-one on S_1' instead, then since (P', P) has calibre (κ, λ, μ), we can find a λ-sized subset T of $\psi(S_1')$ such that any μ-sized subset of T is bounded in P. Then let $S_2 = S_1' \cap \psi^{-1}(T)$, which has size λ. Now for any μ-sized subset S_3 of T, we know $|\psi(S_3)| = \lambda$ also, so $\psi(S_3)$ is bounded in P, and therefore S_3 is bounded in Q. □

In fact, the proof of Lemma 17 really shows:

Lemma 18. *Let (P', P) and (Q', Q) be directed set pairs, let κ be a regular cardinal, and suppose there is a map $\psi : Q' \to P'$ such that for any μ-sized subset E of Q' which is* *unbounded in Q, the image $\psi(E)$ is unbounded in P. If (P', P) has calibre (κ, λ, μ), then (Q', Q) also has calibre (κ, λ, μ).*

Calibres are preserved by 'small enough' unions:

Lemma 19. *Let κ be an infinite cardinal, let $\gamma < cof(\kappa)$, let P be a directed set, and suppose (P'_α, P_α) is a directed set pair such that P_α is contained in P for each $\alpha < \gamma$. If each (P'_α, P_α) has calibre $(\kappa, \ldots)$, then so does $(\bigcup_{\alpha<\gamma} P'_\alpha, P)$.*

Proof. Let S be a subset of $\bigcup_\alpha P'_\alpha$ with size κ. Since $\gamma < cof(\kappa)$, then there must be a $\beta < \gamma$ such that $S' = S \cap P'_\beta$ has size κ. The result now follows from the fact that P'_β has relative calibre $(\kappa, \ldots)$ in P_β (and so also in P). □

Lemma 19 immediately gives:

Corollary 20. *Let $\lambda \leq \omega_1$. If (P'_n, P_n) has calibre (ω_1, λ) and P_n is a subset of a directed set P for each $n < \omega$, then $(\bigcup_n P'_n, P)$ has calibre (ω_1, λ).*

Although calibres are generally not productive (see Example 13 in Chapter 5), we do have the following nice relationship, which is one of the motivations for defining calibre $(\kappa, \lambda, \mu, \nu)$.

Lemma 21. *Let $\kappa \geq \lambda \geq \mu \geq \nu$ be cardinals such that κ and μ are regular. If (P', P) has calibre (κ, λ) and (Q', Q) has calibre (μ, ν), then $(P' \times Q', P \times Q)$ has calibre $(\kappa, \lambda, \mu, \nu)$.*

Proof. Fix a subset S_1 of $P' \times Q'$ with size κ. By Lemma 16, there is a κ-sized subset S'_1 of S_1 such that the projection $\pi_1 : P \times Q \to P$ is one-to-one or constant on S'_1. In either case, we can find a λ-sized subset S_2 of S'_1 such that $\pi_1(S_2) \subseteq P'$ is bounded in P.

Now fix an arbitrary subset S_3 of S_2 with size μ. Then Lemma 16 provides a μ-sized subset S'_3 of S_3 such that the projection $\pi_2 : P \times Q \to Q$ is one-to-one or constant on S'_3. In either case, we can find a ν-sized subset S_4 of S'_3 such that $\pi_2(S_4) \subseteq Q'$ is bounded in Q. Since S_4 is contained in S_2, then S_4 is bounded in $P \times Q$. □

Throughout this work, we will be dealing primarily with the cardinals ω and ω_1, both of which are regular, so the previous two lemmas apply. In fact, we are particularly interested in calibre (ω_1, ω). A very important class of directed sets with calibre (ω_1, ω) are those of the form $\mathcal{K}(M)$ for any separable metrizable M (see Section 2.8). Other examples include any directed set with countable cofinality (since they are Tukey quotients of ω), as well as those in the next lemma.

Lemma 22. *Let S be any set.*

(1) $[S]^{\leq\omega}$ has calibre ω (and so also calibre (ω_1, ω)).

(2) $[S]^{<\omega}$ has calibre (ω_1, ω) if and only if S is countable.

Proof. To prove (1), it suffices to notice that if $\mathcal{A}$ is a countable subset of $[S]^{\leq\omega}$, then $\bigcup \mathcal{A}$ is in $[S]^{\leq\omega}$ and is an upper bound for $\mathcal{A}$. Now we prove (2). If S is countable, then $[S]^{<\omega}$ is also countable, and so it is vacuously calibre (ω_1, ω). On the other hand, if S is uncountable, then no infinite subset of $\{\{s\} : s \in S\}$ has an upper bound in $[S]^{<\omega}$, and so it does not have calibre (ω_1, ω). $\square$

In the next result, $\mathfrak{b}$ is the minimal size of an unbounded subset of the directed set $(\omega^\omega, <^*)$, where for f and g in ω^ω, we say $f <^* g$ if and only if $f(n) < g(n)$ for all but finitely many n. See [27] for a proof.

Lemma 23. *$\mathcal{K}(\omega^\omega)$ has calibre (ω_1, ω_1) if and only if $\omega_1 < \mathfrak{b}$.*

Lemma 24. *Suppose (P', P) and (Q', Q) are directed set pairs such that (P', P) does not have calibre (ω_1, ω) and $|Q'| \leq \omega_1$. Then $(P', P) \geq_T (Q', Q)$.*

Proof. Since (P', P) does not have calibre (ω_1, ω), then there is an uncountable subset S of P' such that every infinite subset of S is unbounded in P. Fix a one-to-one map $\psi : Q' \to S \subseteq P'$. If U is an unbounded subset of Q', then U must be infinite since Q is directed. Hence, $\psi(U)$ is an infinite subset of S and so is unbounded in P. Then Lemma 8 implies that $P \geq_T Q$. $\square$

Corollary 25. *(P', P) does not have calibre (ω_1, ω) if and only if $(P', P) \geq_T [\omega_1]^{<\omega}$.*

Proof. Since $[\omega_1]^{<\omega}$ has cardinality ω_1, then Lemma 24 immediately gives one direction. The other direction follows from Lemmas 17 and 22. $\square$

In Chapter 5, we will see that a product of two directed sets with calibre (ω_1, ω) need not have calibre (ω_1, ω), but the next lemma says that if we strengthen the calibre of one of the factors, then the product will have calibre (ω_1, ω).

Lemma 26. *If (P', P) has calibre (ω, ω) or (ω_1, ω_1), and if (Q', Q) has calibre (ω_1, ω), then $(P' \times Q', P \times Q)$ has calibre (ω_1, ω).*

Proof. By Lemma 15, we know calibre (ω_1, ω) is equivalent to the calibres $(\omega_1, \omega_1, \omega_1, \omega)$ and $(\omega_1, \omega, \omega, \omega)$. Thus, it suffices to apply Lemma 21 in the case where $\kappa = \lambda = \mu = \omega_1$ and $\nu = \omega$, and in the case where $\kappa = \omega_1$ and $\lambda = \mu = \nu = \omega$. $\square$

2.4 TYPES OF DIRECTED SETS

Let Q be a directed set and recall that a family $\{S_q : q \in Q\}$ of subsets of a set S is called Q-ordered if $S_q \subseteq S_{q'}$ whenever $q \leq q'$ in Q. If R is a property of subsets of a directed set, then we say a directed set pair (P', P) is of *type* $\langle Q, R \rangle$ if P' is covered by a Q-ordered family $\{P_q : q \in Q\}$ of subsets of P such that each P_q has property R. A directed set P is of type $\langle Q, R \rangle$ if (P, P) is of type $\langle Q, R \rangle$. For any class $\mathcal{Q}$ of directed sets, let $\langle \mathcal{Q}, R \rangle$ denote the class of all directed sets with type $\langle Q, R \rangle$ for some Q in $\mathcal{Q}$. Likewise, if R' is a property of directed sets, then $\langle R', R \rangle$ denotes the class of all directed sets with type $\langle Q, R \rangle$ for some directed set Q with property R'. We first mention some trivial types that a directed set (pair) may have:

Lemma 27. *Let R be a property shared by all directed sets that have a maximal element. If there is an order-preserving map witnessing $Q \geq_T (P', P)$, then (P', P) has type $\langle Q, R \rangle$.*

In particular, P has type $\langle P$, calibre $(\kappa, \lambda, \mu)\rangle$ for any cardinals $\kappa \geq \lambda \geq \mu$.

Proof. Fix an order-preserving map $\phi : Q \to P$ whose image is cofinal for (P', P). The family $\{\downarrow\phi(q) : q \in Q\}$ is Q-ordered and covers P'. Also, each $\downarrow\phi(q)$ has a maximal element and therefore has property R. $\square$

Lemma 28. *Let κ be a regular cardinal, and suppose $\kappa \geq \lambda \geq \mu \geq \nu$. If Q has calibre (κ, λ) and (P', P) has type $\langle Q$, relative calibre $(\mu, \nu)\rangle$, then (P', P) has calibre $(\kappa, \lambda, \mu, \nu)$.*

Proof. Let $\{P_q : q \in Q\}$ be a Q-ordered family of subsets of P covering P', where Q has calibre (κ, λ) and each P_q has relative calibre (μ, ν) in P. Then we can choose a function $f : P' \to Q$ such that $p \in P_{f(p)}$ for each p in P'. Let S_1 be a subset of P' with size κ. By Lemma 16, there is a κ-sized subset S_1' of S_1 such that f is one-to-one or constant on S'. Since Q has calibre (κ, λ), then in either case, we can find a λ-sized subset S_2 of S_1' such that $f(S_2)$ is bounded, say by q, in Q. Then S_2 is contained in P_q. The proof is completed by applying the fact that P_q has relative calibre (μ, ν) in P. □

Let $R(P)$ denote the family of all subsets of P with a property R, and say R is *hereditary* if it is preserved by passing to subsets.

Lemma 29. *Let (P', P) be a directed set pair, and suppose R is a property of subsets of P.*

(1) If R is preserved by finite unions, then $R(P)$ is a pre-ideal.

(2) If $R(P)$ is a pre-ideal, then $R(P)$ is an ideal if and only if R is hereditary.

(3) If $R(P)$ is a pre-ideal and R is preserved by arbitrary intersections, then $R(P)$ is Dedekind complete (with respect to $\subseteq$).

(4) If $R(P)$ is a pre-ideal and each singleton subset of P' has property R, then $(P', R(P))$ is a directed set pair if we identify P' with the subset $[P']^1$ of $R(P)$ (ignoring the partial order P' inherits from P).

Proof. Lemma 7 gives statement (3), and the other three statements are obvious. □

Lemma 30. *Let (P', P) be a directed set pair, let Q be a directed set, and let R be a property of subsets of P.*

(1) (P', P) has type $\langle Q, R\rangle$ if and only if there is an order-preserving map $\phi : Q \to R(P)$ whose image covers P'.

(2) If $(P', R(P))$ is a directed set pair as in Lemma 29, and if (P', P) has type $\langle Q, R\rangle$, then $Q \geq_T (P', R(P))$.

Proof. Statement (1) is immediate from the definition of type $\langle Q, R\rangle$, and then statement (2) follows from 1 and Lemma 9 since a subset of $R(P)$ is cofinal for $(P', R(P))$ if and only if it covers P'. $\square$

The next lemma shows, in particular, that when $R(P)$ is Dedekind complete, the implication in statement (2) of Lemma 30 can be reversed.

Lemma 31. *Let (P', P) be a directed set pair, and let R be a property of subsets of P such that $R(P)$ is Dedekind complete pre-ideal and $(P', R(P))$ is a directed set pair as in Lemma 29. Let Q and Q' be any directed sets.*

(1) (P', P) has type $\langle Q, R\rangle$ if and only if $Q \geq_T (P', R(P))$.

(2) If $Q \geq_T Q'$, then type $\langle Q', R\rangle$ implies type $\langle Q, R\rangle$.

(3) If Q' is Dedekind complete, then 2 holds even if $R(P)$ is not Dedekind complete.

Proof. Statement (1) follows from Lemma 10, and then statement (2) follows from (1) and transitivity of Tukey quotients. To prove (3), let $\phi_1 : Q \to Q'$ be an order-preserving map witnessing $Q \geq_T Q'$, which exists by Lemma 9. If (P', P) has type $\langle Q', R\rangle$, then by Lemma 30, there is an order-preserving map $\phi_2 : Q' \to R(P)$ whose image covers P'. The map $\phi_2 \circ \phi_1$ then witnesses (via Lemma 30) that (P', P) also has type $\langle Q, R\rangle$. $\square$

2.5 COUNTABLE DIRECTEDNESS AND DETERMINEDNESS

We call a directed set pair (P', P) *countably directed*, or say P' is *relatively countably directed in P*, if each countable subset of P' is bounded (above) in P. Of course, we say P is *countably directed* if it is relatively countably directed in itself.

Lemma 32. *Let (P', P) be a directed set pair such that P' is directed (in itself). Then (P', P) is countably directed if and only if (P', P) has calibre (ω, ω).*

In particular, a directed set P is countably directed if and only if it has calibre (ω, ω).

Proof. The 'only if' part follows from Lemma 15, so now we assume (P', P) is not countably directed and show that (P', P) does not have calibre (ω, ω). Suppose $\{s_n : n < \omega\}$ is a countably infinite subset of P' with no upper bound in P. Since P' is directed, we can inductively choose points p_n in P' such that $p_n \geq s_n$ and $p_{n+1} \geq p_n$ for each $n < \omega$. Then no infinite subset of $\{p_n : n < \omega\}$ can have an upper bound in P, so P' does not have relative calibre (ω, ω) in P. $\square$

Lemma 33. *If Q has calibre (ω_1, ω) and (P', P) has type $\langle Q$, countably directed$\rangle$, then (P', P) has calibre (ω_1, ω).*

Proof. Since countable directedness implies relative calibre (ω, ω), then this follows from Lemma 28 and Lemma 15. $\square$

Unfortunately, the family of all countably directed (in themselves) subsets of a directed set P is very often not Dedekind complete. For example, suppose P has two incomparable elements p and p', and suppose there are two distinct upper bounds q and q' for the set $\{p, p'\}$. Then $\{p, p', q\}$ and $\{p, p', q'\}$ are distinct minimal upper bounds for the subfamily $\{\{p\}, \{p'\}\}$ of $R(P)$, so $\{\{p\}, \{p'\}\}$ has no least upper bound in $R(P)$. However, the family of all relatively countably directed subsets of P is Dedekind complete for any directed set P by Lemma 29.

Lemma 34. *If P is Dedekind complete and P' is a subset of P, then the following are equivalent:*

(i) (P', P) is of type $\langle Q$, relatively countably directed$\rangle$,

(ii) (P', P) is of type $\langle Q$, countably directed$\rangle$,

(iii) $D \geq_T (P', P)$ for some directed set D of type $\langle Q$, countably directed$\rangle$.

In particular, if D is of type $\langle Q$, countably directed$\rangle$, P is Dedekind complete, and $D \geq_T P$, then P is also of type $\langle Q$, countably directed$\rangle$.

Proof. Of course (ii) implies (i), and we now show (i) implies (ii). Suppose $\{P_q : q \in Q\}$ is a Q-ordered cover of P' by subsets of P that are relatively countably directed in P. For each countable subset C of P_q, let $\sup(C)$ denote the least upper bound of C in P. Observe that $\hat{P}_q = \{\sup(C) : C \in [P_q]^{\leq\omega}\}$, which contains P_q, is countably directed (in itself), and the family $\{\hat{P}_q : q \in Q\}$ is a Q-ordered cover of P'.

Next, we will show (ii) implies (iii), so assume (P', P) has type $\langle Q, \text{countably directed}\rangle$, as witnessed by some Q-ordered family $\mathcal{P}$ of subsets of P that are countably directed (in themselves) and cover P'. Then $D = \bigcup \mathcal{P}$ is a directed set of type $\langle Q, \text{countably directed}\rangle$, and the inclusion map $\phi : D \to P$ witnesses that $D \geq_T (P', P)$.

Finally, we prove (iii) implies (ii). Assume D is of type $\langle Q, \text{countably directed}\rangle$, as witnessed by $D = \bigcup\{D_q : q \in Q\}$, and suppose $D \geq_T (P', P)$. By Lemma 9, there is an order-preserving map $\phi : D \to P$ whose image is cofinal for P'. For each q in Q, define $P_q = \bigcup\{\downarrow p : p \in \phi(D_q)\}$. Since ϕ is order-preserving, then $\mathcal{P} = \{P_q : q \in Q\}$ is a Q-ordered family, and each P_q is countably directed since the corresponding $\phi(D_q)$ is countably directed. Additionally, $\mathcal{P}$ covers P' since the image of ϕ is cofinal for P'. □

Corollary 35. *Suppose (P', P) has type $\langle Q'$, countably directed$\rangle$ and $Q \geq_T Q'$. If P or Q' is Dedekind complete, then (P', P) also has type $\langle Q$, countably directed$\rangle$.*

Proof. If P is Dedekind complete, then the result follows from part (2) of Lemma 31 and the equivalence of (i) and (ii) in Lemma 34. If instead Q' is Dedekind complete, then it follows from part (3) of Lemma 31. □

We call a directed set pair (P', P) is *countably determined*, or say P' is *relatively countably determined in* P, if whenever S is a subset of P' whose countable subsets are all bounded in P, then S is bounded in P. Equivalently P' is relatively countably determined in P if every relatively unbounded subset of P' contains a countable relatively unbounded subset. Of course, we say P is *countably determined* if it is relatively countably determined in itself.

Lemma 36. *Let P be a countably determined and Dedekind complete directed set with a subset P', and let Q be any directed set. Then the following are equivalent:*
(i) (P', P) has type $\langle Q$, countably directed$\rangle$,
(ii) $Q \geq_T (P', P)$.

Proof. Lemmas 9 and 27 show that (ii) implies (i), so now we prove the converse. By (i), there is a Q-ordered family $\{P_q : q \in Q\}$ of subsets of P, where each P_q is countably directed. As P is countably determined, each P_q is bounded in P. Then the map $\phi : Q \to P$, where $\phi(q)$ is the least upper bound of P_q in P, witnesses (ii). □

Lemma 37. *Let $\mathcal{S}$ be a pre-ideal on a set X that contains every singleton from $Y = \bigcup \mathcal{S}$. Then we can identify Y with the subset $[Y]^1$ of $\mathcal{S}$, and the following are equivalent:*

(i) $\mathcal{S}$ is countably determined;

(ii) $(Y, \mathcal{S})$ is countably determined; and

(iii) for any subset A of Y that is not contained in any member of $\mathcal{S}$, there is a countable subset A' of A that is also not contained in any member of $\mathcal{S}$.

Proof. The equivalence of (ii) and (iii) is immediate from the definition of countably determined after noticing that, for any subset Z of Y, the subset $[Z]^1$ of $\mathcal{S}$ is unbounded in $\mathcal{S}$ if and only if Z is not contained in any member of $\mathcal{S}$.

Statement (i) certainly implies (ii), so it suffices to prove that (iii) implies (i). Let $\mathcal{R}$ be an unbounded subset of $\mathcal{S}$. Then there is no member of $\mathcal{S}$ containing $A = \bigcup \mathcal{R} \subseteq Y$, so by (iii), there is a countable subset A' of A that is also not contained in any member of $\mathcal{S}$. For each a in A', pick a member R_a of $\mathcal{R}$ that contains a. Then the countable subset $\mathcal{R}' = \{R_a : a \in A'\}$ of $\mathcal{R}$ has no upper bound in $\mathcal{S}$. □

Duality between filters and ideals yields the following convenient description of when a filter is countably determined.

Lemma 38. *Let $\mathcal{F}$ be a filter of subsets of a set Y. Then the directed set $(\mathcal{F}, \supseteq)$ is countably determined if and only if every subset A of Y that meets each member of $\mathcal{F}$ contains a countable subset $A' \subseteq A$ that also meets each member of $\mathcal{F}$.*

2.6 TOPOLOGICAL DIRECTED SETS AND Σ-PRODUCTS

A *topological directed set* is a directed set with a topology. Our primary example of a topological directed set is the pre-ideal $\mathcal{K}(X)$ of all compact subsets of a space X, but we will look closer at this example in Section 2.7. A topological directed set P is said to be *CSB* (convergent sequences bounded) if every convergent sequence in P is bounded (above), *CSBS* (convergent sequences [have] bounded subsequences) if every convergent sequence in P has a bounded (above) subsequence, and *KSB* (compact sets bounded) if every compact subset of P is bounded (above). Clearly KSB implies CSB which implies CSBS. We also say P is *DK* (down-sets compact) if $\downarrow p$ is compact for each p in P.

Occasionally, the following strengthening of CSB will be useful. We say P is *ECSB* (everywhere convergent sequences are bounded) if there is a base $\mathcal{B}$ for P such that whenever B is in $\mathcal{B}$ and $(p_n)_n$ is a sequence on B converging to p in B, then the sequence has an upper bound in B. Note that if P is ECSB and has weight $w(P) \leq \kappa$, then we can find such a base $\mathcal{B}$ of size $|\mathcal{B}| \leq \kappa$. Clearly ECSB implies CSB.

Lemma 39. *Let P be a topological directed set with CSBS, and let P' be a subset of P. If every uncountable subset of P' contains an infinite sequence that converges in P, then (P', P) has calibre (ω_1, ω).*

Proof. Let E be an uncountable subset of P'. Then there is an infinite sequence $(e_n)_n$ in E that converges in P. By passing to a subsequence, we may assume $e_n \neq e_m$ whenever $n \neq m$. Since P is CSBS, there is a subsequence $(e_{n_k})_k$ of $(e_n)_n$ which is bounded in P. Hence, $\{e_{n_k} : k < \omega\}$ is an infinite, bounded subset of E. $\square$

Let X be any space. A subset A of X is called *sequentially closed* if A contains the limit of every convergent sequence on A. Certainly every closed subset of X is sequentially closed, but if the converse holds, then X is called a *sequential* space. Thus, sequential spaces are those spaces whose topology is determined by sequences. Note that every Fréchet-Urysohn space is sequential, and in particular, every first countable space is sequential. Here, X is called *Fréchet-Urysohn* if whenever a point x in X is in the closure of a subset A of X, there is a sequence on A converging to x.

Lemma 40. *Let Q be a topological directed set, and let Q' be a subset of Q.*

(1) If Q' is sequential (in particular, first countable) and has countable extent, and if Q is CSBS, then (Q', Q) has calibre (ω_1, ω).

(2) If Q' is locally compact and has countable extent, and if Q is KSB, then (Q', Q) has calibre (ω_1, ω).

Proof. First we prove (1). Let S be an uncountable subset of Q'. By countable extent, we know S is not closed and discrete in Q', so there is a q in Q' such that q is in the closure of

$S \setminus \{q\}$. Since $S \setminus \{q\}$ is not closed, then it is not sequentially closed, so there is a sequence $(q_n)_n$ on $S \setminus \{q\}$ that converges to some q' not in $S \setminus \{q\}$. The sequence is therefore infinite, and so we may apply Lemma 39.

For the proof of (2), let S and q be as in the proof of (1). Since Q' is locally compact, there is a compact neighborhood C of q in Q'. Then C contains an infinite subset S_0 of S, and by KSB, C (and therefore S_0) has an upper bound in Q. □

Lemma 41. *Let Q be a DK topological directed set, and let Q' be a closed subset of Q. If (Q', Q) has calibre (ω_1, ω), then Q' has countable extent.*

Proof. Suppose, to get a contradiction, that S is an uncountable closed discrete subset of Q'. Then S contains an infinite subset S_1 with an upper bound q in Q. Now, $\downarrow q$ is a compact subset of Q, but S_1 is an infinite, closed, discrete subset of $\downarrow q$, which is a contradiction. □

If $\{P_\alpha : \alpha < \kappa\}$ is a family of directed sets, then the product $\prod_\alpha P_\alpha$ is naturally a directed set via the *product order* defined by $(p_\alpha)_\alpha \leq (q_\alpha)_\alpha$ if and only if $p_\alpha \leq q_\alpha$ for every $\alpha < \kappa$. Of course, if each P_α is a topological directed set, then we also equip the product with the Tychonoff product topology.

Lemma 42. *Let P_α be a (nonempty) topological directed set for each $\alpha < \kappa$, and let R be one of the properties CSB, KSB, DK, or Dedekind completeness. Then we have:*
(1) $\prod_\alpha P_\alpha$ has R if and only if each P_α has R.
(2) If $\kappa = \omega$, then (1) is also true when R is CSBS.

Proof. Statement (1) is obvious for Dedekind completeness from the definition of the product order, and the 'only if' portion of (1) is easy to prove for each remaining property R, so we only prove the 'if' portion. Let $P = \prod_\alpha P_\alpha$. For CSB, it suffices to notice that a sequence in P converges if and only if it converges pointwise, and it is bounded if and only if it is bounded pointwise. The statement for KSB follows from the fact that if K is a compact subset of P, then each projection $\pi_\alpha(K)$ is a compact subset of P_α. Likewise, the statement for DK follows from the fact that for any $p = (p_\alpha)_\alpha$ in P, we have $\downarrow p = \prod_\alpha (\downarrow p_\alpha)$.

Now we prove (2). Again, it is clear that if the product is CSBS, then so is each P_α, so we prove the converse. Let $(\mathbf{p}_n)_n$ be a sequence in $\prod_{i<\omega} P_i$, and write $\mathbf{p}_n = (p_i^n)_i$ for each

$n < \omega$. Then for each i, the sequence $(p_i^n)_n$ converges in P_i, and since each P_i is CSBS, then we can inductively choose a decreasing sequence $(S_i)_i$ of infinite subsets of ω such that $\{p_i^n : n \in S_i\}$ has an upper bound p_i' in P_i for each $i < \omega$. As each S_i is infinite, we can choose a strictly increasing sequence $(n_i)_i$ such that n_i is in S_i for each $i < \omega$. Then p_i' is an upper bound for $\{p_i^{n_j} : j \geq i\}$, and since P_i is directed, we can find an upper bound p_i for all of $\{p_i^{n_j} : j < \omega\}$. Then $\mathbf{p} = (p_i)_i$ is an upper bound for the subsequence $(\mathbf{p}_{n_i})_i$ of $(\mathbf{p}_n)_n$. $\square$

We will investigate the productivity properties of calibre (ω_1, ω) in detail in Section 5, but we note here that calibre (ω_1, ω) is generally not preserved by countable (or even finite) products. However, we do have the following result. Recall that a space is called *second countable* (abbreviated 2^o) if it has a countable base.

Theorem 43. *The class of second countable directed sets with CSBS is closed under countable products, and each member has calibre (ω_1, ω).*

Proof. Closure under countable products follows from Lemma 42 and the fact that countable products preserve second countability. That each member has calibre (ω_1, ω) follows from Lemma 40 since second countability implies both first countability and countable extent. $\square$

If $\{P_\alpha : \alpha < \kappa\}$ is a family of sets and if b_α is a point in P_α for each α, then the Σ*-product* $\Sigma_\alpha P_\alpha$ with *base point* $\mathbf{b} = (b_\alpha)_\alpha$ is the subset of $\prod_\alpha P_\alpha$ consisting of all $\mathbf{p} = (p_\alpha)_\alpha$ whose *support* $supp_{\mathbf{b}}(\mathbf{p}) = \{\alpha < \kappa : p_\alpha \neq b_\alpha\}$ is countable. When each P_α is (Tukey equivalent to) P, then we write ΣP^κ for $\Sigma_\alpha P_\alpha$. If each P_α is a topological directed set, then the Σ-product inherits the product order and topology from $\prod_\alpha P_\alpha$.

Note that for any directed set P, we can add an isolated point 0 to the 'bottom' of P without affecting the cofinal structure of P, the relevant topological properties of P (such as second countability, countable extent, or sequentiality), or any of the properties CSBS, CSB, KSB, or DK. Thus, when forming Σ-products of topological directed sets, we will assume each factor P_α has a minimum 0_α and base the Σ-product at the point $\mathbf{0} = (0_\alpha)_\alpha$. In this case, we write $supp(\mathbf{p})$ for $supp_{\mathbf{0}}(\mathbf{p})$.

Lemma 44. *Let P_α be a topological directed set with minimum 0_α for each $\alpha < \kappa$, and let C be a countable subset of $\Sigma_\alpha P_\alpha$. If C is bounded in $\prod_\alpha P_\alpha$, then C is also bounded in $\Sigma_\alpha P_\alpha$.*

Proof. Let $(p_\alpha)_\alpha$ be an upper bound for C in $\prod_\alpha P_\alpha$, and define $A = \bigcup\{supp(\mathbf{c}) : \mathbf{c} \in C\}$. Now let $q_\alpha = p_\alpha$ for each α in A and $q_\alpha = 0_\alpha$ otherwise. Since A is countable, then $(q_\alpha)_\alpha$ is an upper bound for C in $\Sigma_\alpha P_\alpha$. □

Lemma 45. *Let P_α be a topological directed set with minimum 0_α for each $\alpha < \kappa$, and let R be one of the properties CSBS, CSB, or DK. Then $\Sigma_\alpha P_\alpha$ has R if and only if every P_α has R.*

Proof. If $\Sigma_\alpha P_\alpha$ has R, then it is clear that each P_β has R after we identify P_β with the subset $\{\mathbf{p} \in \prod_\alpha P_\alpha : supp(\mathbf{p}) \subseteq \{\beta\}\}$ of $\Sigma_\alpha P_\alpha$. Conversely, if R is CSB or DK and each P_α has R, then we know $\prod_\alpha P_\alpha$ has R by Lemma 42. Now if R is CSB, then Lemma 44 passes R on to $\Sigma_\alpha P_\alpha$, while if R is DK, then it suffices to notice that for any $\mathbf{p}$ in the Σ-product, the down-set of $\mathbf{p}$ in $\Sigma_\alpha P_\alpha$ is the same as the down-set of $\mathbf{p}$ in $\prod_\alpha P_\alpha$.

It remains to show that if each P_α is CSBS, then so is the Σ-product. Let S be a sequence in $\Sigma_\alpha P_\alpha$, and let $A = \bigcup\{supp(\mathbf{p}) : \mathbf{p} \in S\}$, which is a countable subset of κ. By Lemma 42, we know that $P_A = \pi_A(\Sigma_\alpha P_\alpha) = \prod\{P_\alpha : \alpha \in A\}$ is CSBS, so there is a point $(q_\alpha)_{\alpha \in A}$ in P_A which is an upper bound for $\pi_A(S')$, where S' is some subsequence of S. By defining $q_\alpha = 0_\alpha$ for each α not in A, we obtain an upper bound $(q_\alpha)_{\alpha<\kappa}$ for S' in $\prod_\alpha P_\alpha$. □

Notice that we cannot take R in the previous lemma to be the property KSB . Indeed, if $P_\alpha = \{0, 1\}$ for each $\alpha < \omega_1$, then $K = \{\mathbf{p} \in \Sigma_\alpha P_\alpha : |\,supp(\mathbf{p})| \leq 1\}$ is compact but not bounded in $\Sigma_\alpha P_\alpha$.

2.7 $\mathcal{K}(X)$ AND $(X, \mathcal{K}(X))$

If X is any space, then the pre-ideal $\mathcal{K}(X)$ is naturally equipped with the *Vietoris topology*, which is generated by basic open sets of the form

$$\langle U_1, \ldots, U_n\rangle = \left\{K \in \mathcal{K}(X) : K \subseteq \bigcup_{i=1}^{n} U_i \text{ and } K \cap U_i \neq \emptyset \ \forall i \leq n\right\},$$

where $0 < n < \omega$ and each U_i is open in X.

Lemma 46. *For any Hausdorff space X, the pre-ideal $\mathcal{K}(X)$ is KSB, ECSB (hence CSB), DK, and Dedekind complete.*

Proof. Two well-known properties of the Vietoris topology on $\mathcal{K}(X)$ are that if X is compact, then $\mathcal{K}(X)$ is also compact, and if $\mathcal{A}$ is a compact subset of $\mathcal{K}(X)$, then $\bigcup \mathcal{A}$ is a compact subset of X. The first property implies $\mathcal{K}(X)$ is DK (since the down set of any K in $\mathcal{K}(X)$ is $\mathcal{K}(K)$), while the second property implies $\mathcal{K}(X)$ is KSB (since $\bigcup \mathcal{A}$ is an upper bound for $\mathcal{A}$ in $\mathcal{K}(X)$). Moreover, note that if $\mathcal{A}$ is a subset of some basic open set in $\mathcal{K}(X)$, then $\bigcup \mathcal{A}$ is an element of that basic open set. By taking $\mathcal{A}$ to be a convergent sequence, it follows that $\mathcal{K}(X)$ is ECSB. Finally, Dedekind completeness follows from Lemma 7. $\square$

Corollary 47. *Every directed set is Tukey equivalent to a Dedekind complete topological directed set with KSB and DK.*

Proof. Combine Lemma 14 and Lemma 46. $\square$

Lemma 48. *Let X be a space.*
(1) If $\mathcal{K}(X)$ has calibre (ω_1, ω), then $\mathcal{K}(X)$ has countable extent.
(2) If $\mathcal{K}(X)$ is sequential or X is locally compact, then the converse of (1) holds.

Proof. First, note that if X is locally compact, then so is $\mathcal{K}(X)$. Thus, both claims follow immediately from Lemmas 46, 41, and 40. $\square$

Recall that we can identify a space X with the subset $[X]^1$ of $\mathcal{K}(X)$, so that $(X, \mathcal{K}(X))$ is a directed set pair. Thus, we may speak of the relative calibres of X in $\mathcal{K}(X)$ (in other words, the calibres of $(X, \mathcal{K}(X))$). We will usually omit 'in $\mathcal{K}(X)$' and simply refer to the 'relative calibres of X'. Note that X has relative calibre (ω_1, ω) (in $\mathcal{K}(X)$) if and only if each uncountable subset of X contains an infinite subset with compact closure in X.

Lemma 49. *Let X be any space.*
(1) If X has relative calibre (ω_1, ω) in $\mathcal{K}(X)$, then X has countable extent.
(2) If X is sequential or locally compact, then the converse of (1) holds.

Proof. Since X is a closed subspace of $\mathcal{K}(X)$, then this follows immediately from Lemmas 46, 41, and 40. $\square$

If P is a topological directed set, then we can view P as a directed set itself or as a subspace of the topological directed set $\mathcal{K}(P)$ (while ignoring the original order on P). The following lemma says that for nice enough P, there is essentially no difference between these two views.

Lemma 50. *Let P be a Dedekind complete topological directed set with DK and KSB. Then $P =_T (P, \mathcal{K}(P)) =_T \mathcal{K}(P)$. In particular, this applies to $P = \omega$, $P = \omega^\omega$, and $P = \mathcal{K}(X)$ for any space X.*

Proof. We will show $\mathcal{K}(P) \geq_T (P, \mathcal{K}(P)) \geq_T P \geq_T \mathcal{K}(P)$. First, note that the identity map on $\mathcal{K}(P)$ is order-preserving and its image covers P, so the first Tukey quotient is established. The second Tukey quotient is witnessed by the map $\phi : \mathcal{K}(P) \to P$ where $\phi(K)$ is the least upper bound of K in P (which exists since P is KSB and Dedekind complete). Indeed, ϕ is clearly order-preserving, and the image of $[P]^1 = \{\{p\} : p \in P\}$ is the entirety of P. The final Tukey quotient is witnessed by the map $\psi : P \to \mathcal{K}(P)$ given by $\psi(p) = \downarrow p$, which is well-defined since P is DK. Indeed, ψ is clearly order-preserving, and its image is cofinal in $\mathcal{K}(P)$ since P is KSB.

For the final claim, it is clear that ω is Dedekind complete, DK, and KSB, and thus ω^ω is also by Lemma 42. Additionally, Lemma 46 tells us that $\mathcal{K}(X)$ also satisfies these properties. $\square$

Lemma 51. *Suppose Y is an F_σ subset of a space X and $\lambda \leq \omega_1$. If X has relative calibre (ω_1, λ) in $\mathcal{K}(X)$, then Y has the same relative calibre in $\mathcal{K}(Y)$.*

Proof. Say $Y = \bigcup_n C_n$ where each C_n is closed in X. Fix n and let S be an ω_1-sized subset of C_n. Since X has relative calibre (ω_1, λ), then there is a λ-sized subset S' of S with compact closure in X. But since $\overline{S}^{C_n} = \overline{S}^X$, then we see that C_n has relative calibre (ω_1, λ) in $\mathcal{K}(C_n)$. Now the result follows from Lemma 20. $\square$

Lemma 52. *For any spaces X_α, we have $\prod_\alpha \mathcal{K}(X_\alpha) =_T \mathcal{K}(\prod_\alpha X_\alpha)$.*

Proof. Consider the Tukey quotient maps $(K_\alpha)_\alpha \mapsto \prod_\alpha K_\alpha$, where each K_α is in $\mathcal{K}(X_\alpha)$, and $K \mapsto (\pi_\alpha(K))_\alpha$, where K is in $\mathcal{K}(\prod_\alpha X_\alpha)$ and $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ is the projection. $\square$

A space is called *totally imperfect* if every compact subset is countable. Bernstein sets (subsets of $\mathbb{R}$ that intersect each uncountable closed subset of $\mathbb{R}$ but contain none of them) are examples of totally imperfect subsets of the reals which have size $\mathfrak{c}$.

Lemma 53. *If B is a totally imperfect subset of $\mathbb{R}$ with size $\mathfrak{c}$ and S is any set with size $|S| \leq \mathfrak{c}$, then $\mathcal{K}(B) \geq_T [S]^{\leq\omega}$.*

Proof. Since $|S| \leq \mathfrak{c}$ we also have $|[S]^{\leq\omega}| \leq \mathfrak{c}$, so fix a surjection $f : B \to [S]^{\leq\omega}$. Define $\phi : \mathcal{K}(B) \to [S]^{\leq\omega}$ by $\phi(K) = \bigcup\{f(x) : x \in K\}$. This is a well-defined map into $[S]^{\leq\omega}$ since each compact subset of B is countable. Clearly ϕ is order-preserving and surjective, and hence a Tukey quotient map by Lemma 9. □

Lemma 54. *If X is first countable and totally imperfect, then $\mathcal{K}(X)$ is also first countable.*

Proof. Let $K \in \mathcal{K}(X)$. Then we can enumerate $K = \{x_i : i < \omega\}$ since X is totally imperfect. For each $i < \omega$, fix a countable neighborhood base $(B_{i,n})_n$ for x_i. For any $\mathbf{n} = (n_0, \dots, n_k)$ where each $n_i < \omega$, define $T_{\mathbf{n}} = \langle B_{1,n_1}, \dots, B_{k,n_k}\rangle$. Now let $U = \langle U_1, \dots, U_\ell\rangle$ be any basic neighborhood of K, so K is contained in $\bigcup_{j=1}^{\ell} U_j$ and intersects each U_j.

For each $i < \omega$, pick $n_i < \omega$ such that $x_i \in B_{i,n_i} \subseteq \bigcap\{U_j : x_i \in U_j\}$. Since K is compact, there is a $0 < k < \omega$ such that $\{B_{i,n_i} : i = 1, \dots, k\}$ covers K. Since K intersects each U_j, then by choosing k large enough, we can also ensure that for each $j \in \{1, \dots, \ell\}$, there is some $i \in \{1, \dots, k\}$ such that $B_{i,n_i} \subseteq U_j$. For $\mathbf{n} = (n_1, \dots, n_k)$, it follows that $K \in T_{\mathbf{n}} \subseteq U$. Hence, K has a countable neighborhood base: $\{T_{\mathbf{n}} : \mathbf{n} \in \bigcup_m \omega^m,\ K \in T_{\mathbf{n}}\}$. □

2.8 $\mathcal{K}(M)$ FOR SEPARABLE METRIZABLE M

Our interest in directed sets of the form $\mathcal{K}(M)$ where M is separable metrizable stems from their use in the work of Cascales, Orihuela, and Tkachuk [7], and the following fact provides one of our primary motivations for studying the property calibre (ω_1, ω).

Lemma 55. *Let M be a separable metrizable space, and let (P', P) be a directed set pair such that P is Dedekind complete. Then we have:*

(1) $\mathcal{K}(M)$ is second countable (2^o), KSB, ECSB (so also CSB and CSBS), DK, and Dedekind complete.

(2) $\mathcal{K}(M)$ has calibre (ω_1, ω).

(3) If $\mathcal{K}(M) \geq_T (P', P)$, then (P', P) has type $\langle 2^o$+ECSB, countably directed$\rangle$.

Proof. If M is separable and metrizable, then $\mathcal{K}(M)$ is also separable and metrizable via the Hausdorff metric. Hence, $\mathcal{K}(M)$ is second countable, and the rest of (1) comes from Lemma 46. Applying Theorem 43 then gives (2). Lemma 27 shows that $\mathcal{K}(M)$ has type $\langle 2^o\text{+ECSB}, \text{countably directed}\rangle$, so then Lemma 34 gives (3). □

Lemma 55 shows that each $\mathcal{K}(M)$ is a member of the class of second countable directed sets with CSB. We saw in Theorem 43 that this class is closed under countable products, and the same is essentially true for directed sets of the form $\mathcal{K}(M)$ as well. Indeed, Lemma 52 gives:

Lemma 56. *If M_n is separable and metrizable for each $n < \omega$, then $\prod_n \mathcal{K}(M_n) =_T \mathcal{K}(M')$, where M' is the separable metrizable space $M' = \prod_n M_n$.*

Let $\mathcal{K}(\mathcal{M})$ denote the class of all Tukey equivalence classes of directed sets of the form $\mathcal{K}(M)$ for some separable metrizable M. Then $\mathcal{K}(\mathcal{M})$ is actually a set since each M is homeomorphic to a subspace of the Hilbert cube. Applying Lemmas 56, 55, and 17, we obtain:

Corollary 57. *$\mathcal{K}(\mathcal{M})$ is closed under countable products and each member has calibre (ω_1, ω).*

In particular, if M_n is separable metrizable for each n, then $\prod_n \mathcal{K}(M_n)$ has calibre (ω_1, ω). In fact, we can generalize this to Σ-products:

Theorem 58. *Let $\{M_\alpha : \alpha < \kappa\}$ be a family of separable metrizable spaces. Then $\Sigma_\alpha \mathcal{K}(M_\alpha)$ (with base point $\mathbf{0} = (\emptyset)_\alpha$) has calibre (ω_1, ω).*

Proof. Since each $\mathcal{K}(M_\alpha)$ is second countable (hence cosmic) and CSB (hence CSBS), then this is a consequence of Theorem 136 in Chapter 5 below. □

Note that $\mathcal{K}(\mathcal{M})$ is a directed set under Tukey quotients since for any two separable metrizable spaces M_1 and M_2, $\mathcal{K}(M_1)$ and $\mathcal{K}(M_2)$ are each Tukey quotients of $\mathcal{K}(M_1 \oplus M_2)$. The next two results from Gartside and Mamatelashvili [27] tell us much more about the order structure of $\mathcal{K}(\mathcal{M})$. The first result says when subsets of $\mathcal{K}(\mathcal{M})$ have upper bounds; it implies that $\mathcal{K}(\mathcal{M})$ is countably directed. The second asserts the existence of an antichain of maximal possible size in $\mathcal{K}(\mathcal{M})$.

Theorem 59 (Gartside and Mamatelashvili, [27])**.** *Let $\{M_\alpha : \alpha < \kappa\}$ be a family of separable metrizable spaces.*

(1) If $\kappa \leq \mathfrak{c}$, then there is a separable metrizable M such that $\mathcal{K}(M) \geq_T \mathcal{K}(M_\alpha)$ for all α.

(2) If $\kappa > \mathfrak{c}$ and the M_α's are all distinct subsets of a given separable metrizable space (or pairwise non-homeomorphic), then for any separable metrizable space M, there is an α such that $\mathcal{K}(M) \not\geq_T (M_\alpha, \mathcal{K}(M_\alpha))$.

Theorem 60 (Gartside and Mamatelashvili, [27])**.** *There is a $2^{\mathfrak{c}}$-sized family $\mathcal{A}$ of separable metrizable spaces such that if A and A' are distinct elements of $\mathcal{A}$ then $\mathcal{K}(A) \not\geq_T (A', \mathcal{K}(A'))$ and $\mathcal{K}(A') \not\geq_T (A, \mathcal{K}(A))$.*

If M is compact, then the cofinal structure of the pre-ideal $\mathcal{K}(M)$ is not very interesting. Indeed, in this case $\mathcal{K}(M) =_T 1$. However, if M is not compact, then the following lemma says that $\mathcal{K}(M)$ is at least as cofinally complex as ω.

Lemma 61. *A metrizable space M is not compact if any only if $\mathcal{K}(M) \geq_T \omega$. Hence, $\mathcal{K}(M) \times \omega =_T \mathcal{K}(M)$ for any compact metrizable space M.*

Proof. If M is compact, then $\mathcal{K}(M) =_T 1 \not\geq_T \omega$ since ω has no maximum. On the other hand, if M is not compact, then there is a sequence $(x_n)_{n<\omega}$ in M which has no convergent subsequence. Note that any compact subset of M can contain at most finitely many members of this sequence, so we may define a map $\phi : \mathcal{K}(M) \to \omega$ by $\phi(K) = \min\{n < \omega : x_n \notin K\}$. Then ϕ is order-preserving and onto, so it witnesses $\mathcal{K}(M) \geq_T \omega$.

For the second claim, note that $\mathcal{K}(M) \times \omega \geq_T \mathcal{K}(M)$ via the natural projection map, and combining the map ϕ from the first claim with the identity map on $\mathcal{K}(M)$ gives a map witnessing $\mathcal{K}(M) \geq_T \mathcal{K}(M) \times \omega$. □

2.9 FUNCTION SPACES

Write $C(X)$ for the set of all real-valued continuous functions on a space X. For any f in $C(X)$, subset S of X, and $\epsilon > 0$, define $B(f, S, \epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon,\ \forall x \in S\}$. If $\mathcal{S}$ is a pre-ideal of compact sets covering X. Then the collection $\{B(f, S, \epsilon) : f \in C(X),\ S \in \mathcal{S},\ \epsilon > 0\}$ is the basis for a locally convex topological vector space topology on $C(X)$. We denote $C(X)$ with this topology by $C_{\mathcal{S}}(X)$.

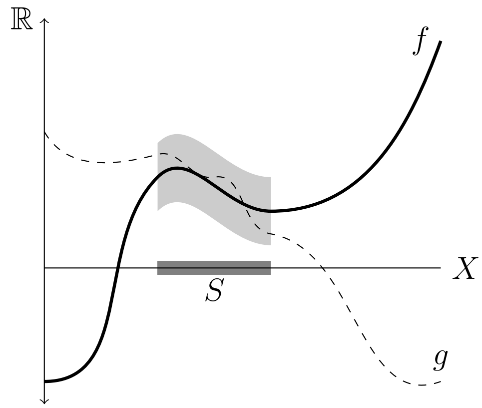

Figure 3: A basic neighborhood in $C_{\mathcal{S}}(X)$

Two special cases are of particular interest. If $\mathcal{S} = \mathcal{K}(X)$, then $C_{\mathcal{S}}(X)$ is $C(X)$ with the compact-open topology, denoted $C_k(X)$, while if $\mathcal{S} = [X]^{<\omega}$, then $C_{\mathcal{S}}(X)$ is $C(X)$ with the topology of pointwise convergence, denoted $C_p(X)$.

Lemma 62. *Let X be a space, and let $\mathcal{S}$ be a pre-ideal of compact subsets covering X. Then $\mathcal{N}_{\mathbf{0}}^{C_{\mathcal{S}}(X)} =_T \mathcal{S} \times \omega$.*

Proof. Let $\mathcal{B}_{\mathbf{0}} = \{B(\mathbf{0}, K, 1/n) : K \in \mathcal{S} \text{ and } 0 < n < \omega\}$. Then $\mathcal{B}_{\mathbf{0}}$ is cofinal in $\mathcal{N}_{\mathbf{0}}^{C_{\mathcal{S}}(X)}$ (that is, $\mathcal{B}_{\mathbf{0}}$ is a neighborhood base for $\mathbf{0}$). Thus, $\mathcal{N}_{\mathbf{0}}^{C_{\mathcal{S}}(X)} =_T \mathcal{B}_{\mathbf{0}}$ by Lemma 11. Now note that $B(\mathbf{0}, K, 1/n) = B(\mathbf{0}, L, 1/m)$ if and only if $K = L$ and $n = m$. Thus, the maps $(K, n) \mapsto B(\mathbf{0}, K, 1/n+1)$ and $B(\mathbf{0}, K, 1/n+1) \mapsto (K, n)$ are well-defined Tukey quotients showing that $\mathcal{B}_{\mathbf{0}} =_T \mathcal{S} \times \omega$. $\square$

3.0 P-ORDERED COMPACT COVERS

Recall that Theorem 3 by Cascales, Orihuela, and Tkachuk [7] says that if X is compact and $X^2 \setminus \Delta$ has a $\mathcal{K}(M)$-ordered cofinal compact cover for some separable metrizable M (i.e. $\mathcal{K}(M) \geq_T \mathcal{K}(X^2 \setminus \Delta)$), then X must be metrizable, and this result generalizes Schneider's Theorem 1. In Theorem 63 of Section 3.1, we generalize Theorem 3 by replacing $\mathcal{K}(M)$ with any directed set P having calibre (ω_1, ω). We further show in Theorem 67 that this generalization is optimal in the sense that the class of directed sets with calibre (ω_1, ω) cannot be replaced by a larger class of directed sets. Theorem 63 can also be stated as: if $\mathcal{K}(X^2 \setminus \Delta)$ has calibre (ω_1, ω) and X is compact, then X is metrizable, and we show in Section 3.2 that the hypothesis cannot be weakened to '$X^2 \setminus \Delta$ has relative calibre (ω_1, ω) in $\mathcal{K}(X^2 \setminus \Delta)$'.

In Section 3.3, rather than considering only the particular subset $X^2 \setminus \Delta$ of a square X^2, we consider what happens when for each subspace A of a space X, $\mathcal{K}(A)$ or $(A, \mathcal{K}(A))$ has calibre (ω_1, ω) or is a Tukey quotient of a directed set of the form $\mathcal{K}(M)$. This section seeks to answer several questions posed in [7].

3.1 A GENERALIZATION OF SCHNEIDER'S THEOREM

We now prove the following generalization of Theorem 3, which in turn generalizes Schneider's Theorem.

Theorem 63. *The following are equivalent for any countably compact space X:*

(i) X is compact and $P \geq_T \mathcal{K}(X^2 \setminus \Delta)$ for some directed set P with calibre (ω_1, ω),

(ii) X is compact and $\mathcal{K}(X^2 \setminus \Delta)$ has calibre (ω_1, ω),

(iii) $P \geq_T \mathcal{N}_\Delta^{X^2}$ *for some directed set* P *with calibre* (ω_1, ω)*,*

(iv) $\mathcal{N}_\Delta^{X^2}$ *has calibre* (ω_1, ω)*, and*

(v) X *is metrizable.*

Proof. Lemma 17 shows that (i) implies (ii), and of course (ii) implies (i) since we can take $P = \mathcal{K}(X^2 \setminus \Delta)$. So (i) and (ii) are equivalent, and similarly, (iii) and (iv) are equivalent. Also, (v) implies that X is actually compact, and then Schneider's theorem implies (iii) with $P = \omega$ (since the diagonal in a compact square is G_δ if and only if the diagonal has a countable neighborhood base). Hence, it suffices to show that (iv) implies (ii) and that (ii) implies (v).

Claim (A): Statement (iv) implies statement (ii).

It suffices to show X is Lindelöf. Indeed, then X is countably compact and Lindelöf, which gives compactness. By Lemma 13, $\mathcal{K}(X^2 \setminus \Delta)$ and $\mathcal{N}_\Delta^{X^2}$ are Tukey equivalent for compact X, so Lemma 17 shows that (iv) implies (ii). Thus, claim (A) follows from:

Claim (B): Statement (iv) implies X is hereditarily Lindelöf.

Assume not. Then X contains an uncountable right-separated sequence $\{y_\alpha : \alpha < \omega_1\}$; that is, each y_α has a neighborhood U_α such that $y_\beta \notin U_\alpha$ for all $\beta > \alpha$. For each $\alpha < \omega_1$, let $V_\alpha = X \setminus \{y_\alpha\}$ and $N_\alpha = U_\alpha^2 \cup V_\alpha^2$, which is a neighborhood of the diagonal. By (iv), we know there is an infinite $A \subseteq \omega_1$ and a neighborhood N of the diagonal such that $N_\alpha \supseteq N$ for all $\alpha \in A$. Without loss of generality, we may assume $N = \bigcup_{W \in \mathcal{W}} W^2$ for some open cover $\mathcal{W}$ of X.

We will show that each member of $\mathcal{W}$ contains at most one point of the infinite set $\{y_\alpha : \alpha \in A\}$. Indeed, if $y_\alpha, y_\beta \in W$ for some $\alpha, \beta \in A$, then we see that $(y_\alpha, y_\beta) \in W^2 \subseteq N \subseteq N_\alpha = U_\alpha^2 \cup V_\alpha^2$. Now since $y_\alpha \notin V_\alpha$, then we must have $y_\beta \in U_\alpha$, which gives $\beta \leq \alpha$. Similarly, we have $(y_\alpha, y_\beta) \in U_\beta^2 \cup V_\beta^2$, which implies that $\alpha \leq \beta$, and so $\alpha = \beta$. Hence, $\mathcal{W}$ witnesses that $\{y_\alpha : \alpha \in A\}$ is an infinite closed discrete subset of X, which contradicts that X is countably compact. Thus, claim (B) is proven.

Claim (C): Statement (ii) implies statement (v).

Let $\{(U_1^\alpha, U_2^\alpha) : \alpha < \kappa\}$ be a one-to-one enumeration of the pairs of open sets in X that have disjoint closures. Since X is compact, then it is normal, so for each α, we can find open sets V_1^α and V_2^α with disjoint closures such that $\overline{U_\ell^\alpha} \subseteq V_\ell^\alpha$ for $\ell = 1, 2$. We will say that any α and β less than κ are *comparable* if $U_\ell^\alpha \subseteq V_\ell^\beta$ and $U_\ell^\beta \subseteq V_\ell^\alpha$ for $\ell = 1, 2$. By an easy Zorn's lemma argument, there is a maximal incomparable subset A of κ. For any subsets E and F of X, let $R(E, F) = (E \times F) \cup (F \times E) \subseteq X^2$. Then for any α in A, define $K_\alpha = (\overline{V_1^\alpha} \times \overline{V_2^\alpha}) \cup R(\overline{U_1^\alpha}, C_1^\alpha) \cup R(\overline{U_2^\alpha}, C_2^\alpha)$ where $C_\ell^\alpha = X \setminus V_\ell^\alpha$ for $\ell = 1, 2$.

To establish (v), it suffices to show $X^2 \setminus \Delta$ is σ-compact since then the diagonal is G_δ, so X is metrizable by Schneider's theorem. Thus, claim (C) follows from:

Claim (D): $\mathcal{K} = \{K_\alpha : \alpha \in A\}$ is a countable subset of $\mathcal{K}(X^2 \setminus \Delta)$ covering $X^2 \setminus \Delta$.

Certainly each K_α is compact since it is closed in X^2, and K_α is disjoint from Δ since $\overline{V_1^\alpha} \cap \overline{V_2^\alpha} = \emptyset = \overline{U_\ell^\alpha} \cap C_\ell^\alpha$. To see that $\mathcal{K}$ covers $X^2 \setminus \Delta$, note that for any (x_1, x_2) in $X^2 \setminus \Delta$, we can find an $\alpha < \kappa$ such that $x_\ell \in U_\ell^\alpha$ for $\ell = 1, 2$. By the maximality of A, there is a β in A that is comparable with α. In particular, $U_\ell^\alpha \subseteq V_\ell^\beta$, and so $(x_1, x_2) \in \overline{V_1^\beta} \times \overline{V_2^\beta} \subseteq K_\beta \in \mathcal{K}$.

Now suppose, to get a contradiction, that $\mathcal{K}$ is uncountable. Then by (ii), $\mathcal{K}$ contains an infinite bounded subset $\mathcal{K}'$. Thus, $\bigcup \mathcal{K}'$ is contained in a compact set disjoint from the diagonal, or equivalently, the closure of $\bigcup \mathcal{K}'$ is disjoint from the diagonal. So to achieve our desired contradiction, it suffices to find a point on the diagonal which is in the closure of $\bigcup \mathcal{K}'$.

Write $\mathcal{K}' = \{K_{\alpha_i} : i < \omega\}$ where each α_i is in A and $\alpha_i \neq \alpha_j$ whenever $i \neq j$. Since the elements of A are incomparable, then for any $i < j$, there exists a point $x_{i,j} \in X$ witnessing one of the following four conditions.

$$(1)\ \ U_1^{\alpha_i} \not\subseteq V_1^{\alpha_j} \quad (2)\ \ U_2^{\alpha_i} \not\subseteq V_2^{\alpha_j} \quad (3)\ \ U_1^{\alpha_j} \not\subseteq V_1^{\alpha_i} \quad (4)\ \ U_2^{\alpha_j} \not\subseteq V_2^{\alpha_i}$$

Applying Ramsey's theorem, there is an infinite subset M of ω such that one of these four conditions is witnessed by all $x_{i,j}$ with $i, j \in M$ and $i < j$.

Since X is compact, then $\mathcal{K}(X^2 \setminus \Delta) =_T \mathcal{N}_\Delta^{X^2}$, so (iv) holds, and claim (B) gives that X is hereditarily Lindelöf. The singletons in X are therefore G_δ, and so by compactness of X, we see that X is first countable. Now, since X is compact and first countable, then every infinite subset of X has an accumulation point (and so a proper limit point), and whenever a point is in the closure of a set, it is the limit of a sequence on that set.

Hence, we may inductively construct a decreasing sequence $(S_i)_i$ of infinite subsets of M

such that $(x_{i,j})_{j \in S_i}$ converges to some point $x_{i,\infty} \in X$ for each $i < \omega$. Next, we inductively choose a strictly increasing sequence $(j_m)_m$ such that $j_m \in S_m$ for each $m < \omega$, and we let $J = \{j_m : m < \omega\}$. Then there is an infinite subset $J' \subseteq J$ such that $(x_{j,\infty})_{j \in J'}$ converges to some limit point x_∞.

Fix any open neighborhood W of x_∞. Pick an $i \in J'$ such that $x_{i,\infty} \in W$. Note that for each $m \geq i$, we have $j_m \in S_m \subseteq S_i$, and so $J' \cap S_i$ is infinite. As $(x_{j,\infty})_{j \in J'}$ converges to $x_\infty \in W$ and $(x_{i,j})_{j \in S_i}$ converges to $x_{i,\infty} \in W$, then we can find a $j \in J' \cap S_i$ with $j > i$ such that $x_{j,\infty} \in W$ and $x_{i,j} \in W$. There must then be a $k \in S_j$ such that $k > j$ and $x_{j,k} \in W$.

So we found $i < j < k$ in M such that $x_{i,j}$ and $x_{j,k}$ are both in W, and as noted above, one of the four conditions (1)–(4) is witnessed by both $x_{i,j}$ and $x_{j,k}$. If it is condition (1) or (2), then for some $\ell \in \{1, 2\}$, we have $x_{i,j} \in U_\ell^{\alpha_i} \setminus V_\ell^{\alpha_j} \subseteq C_\ell^{\alpha_j}$ and $x_{j,k} \in U_\ell^{\alpha_j} \setminus V_\ell^{\alpha_k} \subseteq U_\ell^{\alpha_j}$. If condition (3) or (4) is witnessed instead, then for some $\ell \in \{1, 2\}$, we have $x_{i,j} \in U_\ell^{\alpha_j} \setminus V_\ell^{\alpha_i} \subseteq U_\ell^{\alpha_j}$ and $x_{j,k} \in U_\ell^{\alpha_k} \setminus V_\ell^{\alpha_j} \subseteq C_\ell^{\alpha_j}$. In any case, we have $(x_{i,j}, x_{j,k}) \in R(\overline{U_\ell^{\alpha_j}}, C_\ell^{\alpha_j}) \subseteq K_{\alpha_j}$. Therefore $(x_{i,j}, x_{j,k})$ is in $K_{\alpha_j} \cap W^2$. Hence, every basic open neighborhood W^2 of (x_∞, x_∞) meets some member of $\mathcal{K}'$, and so (x_∞, x_∞) is in the closure of $\bigcup \mathcal{K}'$. This gives the desired contradiction to complete the proof of claim (D). $\square$

The next result and its corollary are an application of Theorem 63 in the spirit in which Cascales and Orihuela originally proved their theorem (see [6]) that for compact X, metrizability follows from the condition $\omega^\omega \geq_T \mathcal{K}(X^2 \setminus \Delta)$.

Theorem 64. *Let G be a topological group with identity e. If $\mathcal{N}_e^G$ has calibre (ω_1, ω), then every compact subset of G is metrizable.*

Proof. Let K be a compact subset of G. We write the group operation multiplicatively and denote Cartesian products by '×'. Let $\Delta = \Delta(K)$ be the diagonal in $K \times K$, and consider the map $\phi : \mathcal{N}_e^G \to \mathcal{N}_\Delta^{K \times K}$ given by $\phi(N) = (K \times K) \cap \bigcup\{gN \times gN : g \in K\}$.

Note that ϕ is order-preserving, and we now check that its image is cofinal. Let U be any neighborhood of Δ in $K \times K$. Then for each g in K, we can find a neighborhood B_g of g in G such that $(K \times K) \cap (B_g \times B_g) \subseteq U$. Choose a neighborhood N_g of e such that $N_g N_g$ is contained in $g^{-1} B_g$. By compactness of K, there is a finite subset F of K such that $\{f N_f : f \in F\}$ covers K. Let $N = \bigcap\{N_f : f \in F\}$, which is a neighborhood of e. Since each g in K is in fN_f for some f in F, then $gN \subseteq fN_fN_f \subseteq B_f$. Hence, $\phi(N) \subseteq (K \times K) \cap \bigcup\{B_f \times B_f : f \in F\} \subseteq U$, so the image of ϕ is cofinal.

Thus, ϕ witnesses $\mathcal{N}_e^G \geq_T \mathcal{N}_\Delta^{K\times K}$, so if $\mathcal{N}_e^G$ has calibre (ω_1, ω), then $\mathcal{N}_\Delta^{K\times K}$ does also. Hence, Theorem 63 implies that K is indeed metrizable. □

Recall that $C_k(X)$ denotes the space of continuous real-valued functions on X with the compact-open topology.

Corollary 65. *If $\mathcal{K}(X)$ has calibre (ω_1, ω), then every compact subset of $C_k(X)$ is metrizable.*

Proof. By Lemma 62, we know $\mathcal{K}(X) \times \omega =_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$. Note that ω has calibre (ω_1, ω_1) vacuously, so if $\mathcal{K}(X)$ has calibre (ω_1, ω), then Lemma 26 and Lemma 17 show that $\mathcal{N}_{\mathbf{0}}^{C_k(X)}$ also has calibre (ω_1, ω), and we can apply the previous result. □

As a simple consequence, we observe that for any cardinal κ, every compact subset of $C_k(\kappa)$ is metrizable. The previous results of this type in [6, 7] are restricted to cardinals with cofinality no more than the continuum, so this gives a small example of the value of our calibre result over Tukey quotients from $\mathcal{K}(M)$, where M is separable metrizable.

Our final goal in this section is to prove Theorem 67 below, which says that Theorem 63 is optimal. We start by proving the following lemma:

Lemma 66. *If X is a compact space with weight ω_1, then $[\omega_1]^{<\omega} \geq_T \mathcal{K}(X^2 \setminus \Delta)$.*

Proof. Let $\mathcal{B}$ be a base for X with size ω_1. Let $\mathcal{S} = \{\overline{B_1} \times \overline{B_2} : B_1, B_2 \in \mathcal{B} \text{ and } \overline{B_1} \cap \overline{B_2} = \emptyset\}$. Then $\mathcal{S}$ is a subset of $\mathcal{K}(X^2 \setminus \Delta)$ with size ω_1. Indeed, it is clear that $|\mathcal{S}| \leq \omega_1$, and if $\mathcal{S}$ was countable, then X would have a G_δ diagonal and so be metrizable, which contradicts that it has uncountable weight. Now define $\phi : [\mathcal{S}]^{<\omega} \to \mathcal{K}(X^2 \setminus \Delta)$ by $\phi(\mathcal{F}) = \bigcup \mathcal{F}$ for any finite $\mathcal{F} \subseteq \mathcal{S}$. Then ϕ is order-preserving, and since any compact subset of $X^2 \setminus \Delta$ can be covered by finitely many members of $\mathcal{S}$, then the image of ϕ is cofinal in $\mathcal{K}(X^2 \setminus \Delta)$. □

Theorem 67. *The following are equivalent for a directed set P:*

(i) P has calibre (ω_1, ω), and

(ii) For any compact X such that $P \geq_T \mathcal{K}(X^2 \setminus \Delta)$, X must be metrizable.

Proof. Theorem 63 shows that (i) implies (ii), and we now prove that the negation of (i) implies the negation of (ii). Suppose P does not have calibre (ω_1, ω). By Corollary 25, we have $P \geq_T [\omega_1]^{<\omega}$. Let $X = A(\omega_1)$, the one-point compactification of the discrete space of size ω_1. Then X is compact and has weight ω_1, so Lemma 66 gives that $[\omega_1]^{<\omega} \geq_T \mathcal{K}(X^2 \setminus \Delta)$. Hence, $P \geq_T \mathcal{K}(X^2 \setminus \Delta)$ by transitivity, but X is not metrizable. □

Theorem 67 says that we cannot enlarge the class of directed sets P used in the first equivalent condition of Theorem 63.

3.2 RELATIVE CALIBRES OF $X^2 \setminus \Delta$

Recall that the relative calibres of $X^2 \setminus \Delta$ refer to the calibres of the directed set pair $(X^2 \setminus \Delta, \mathcal{K}(X^2 \setminus \Delta))$ where we have identified $X^2 \setminus \Delta$ with the singletons in $\mathcal{K}(X^2 \setminus \Delta)$.

3.2.1 SMALL DIAGONALS

A space X is said to have a *small diagonal* if and only if every uncountable S contained in $X^2 \setminus \Delta$ contains an uncountable subset S_1 whose closure misses the diagonal. If X is compact then $\overline{S_1}$ is a compact subset of $X^2 \setminus \Delta$. The next lemma is then immediate.

Lemma 68. *Let X be compact. Then $X^2 \setminus \Delta$ has relative calibre (ω_1, ω_1) if and only if X has a small diagonal.*

Under various set theoretic hypotheses including PFA, it is known [12] that a compact space with a small diagonal is metrizable. Combining with the previous lemma gives:

Corollary 69 (Consistently)**.** *If X is compact and $X^2 \setminus \Delta$ has relative calibre (ω_1, ω_1), then X is metrizable.*

We can now recover a result from [7], albeit with a stronger set theoretic assumption, PFA rather than MA $+\neg$CH.

Theorem 70 (PFA)**.** *If X is compact and $\mathcal{K}(M) \geq_T (X^2 \setminus \Delta, \mathcal{K}(X^2 \setminus \Delta))$, where $\mathcal{K}(M)$ has calibre (ω_1, ω_1), then X is metrizable. In particular, this holds when $M = \omega^\omega$.*

This is because under PFA, $\omega_1 < \mathfrak{b}$, so $\mathcal{K}(\omega^\omega)$ has calibre (ω_1, ω_1) (see Lemma 23). Note that in [27] it is shown that consistently (in particular, under PFA) there are separable metrizable spaces M which are *not* Polish but such that $\mathcal{K}(M)$ has calibre (ω_1, ω_1).

3.2.2 A COUNTEREXAMPLE

In light of Theorem 63 and Corollary 69, it is natural to hope that if X is compact and $X^2 \setminus \Delta$ has relative calibre (ω_1, ω), then X must be metrizable, but we will see in Theorem 73 below that this is not the case. To build the desired counterexample, we generalize the construction of the Alexandrov duplicate of the unit interval, as follows. If τ is any topology on the closed unit interval $I = [0, 1]$, then $AD(\tau)$ will have the underlying set $I \times \{0, 1\}$. We define the topology on $AD(\tau)$ by giving points in $I \times \{1\}$ basic open neighborhoods of the form $U \times \{1\}$ for U in τ, while points in $I \times \{0\}$ are given basic open neighborhoods of the form $(V \times \{0, 1\}) \setminus (K \times \{1\})$, where V is open in the usual topology on I and K is a τ-compact subset of I. Notice that if τ is the discrete topology, then $AD(\tau)$ is the usual Alexandrov duplicate of the unit interval.

Lemma 71. *If τ is a first countable, locally countable, locally compact topology on I refining the usual topology, then $AD(\tau)$ is compact Hausdorff, first countable, and not metrizable.*

Proof. Since I is Hausdorff, then any two distinct points (x, i) and (y, j) in $AD(\tau)$ can easily be separated by open sets if $x \neq y$. That $(x, 0)$ and $(x, 1)$ can also be separated by open sets follows from the fact that τ is locally compact. Indeed, pick any τ-open neighborhood U of x that has compact closure in (I, τ). Then $U \times \{1\}$ and $AD(\tau) \setminus (\overline{U}^\tau \times \{1\})$ are disjoint open neighborhoods containing $(x, 1)$ and $(x, 0)$, respectively.

Let $\mathcal{U}$ be an open cover of $AD(\tau)$. For each $x \in I$, find $U_x \in \mathcal{U}$, V_x open in I, and a τ-compact subset K_x of I such that $(x, 0) \in B_x = (V_x \times \{0, 1\}) \setminus (K_x \times \{1\}) \subseteq U_x$. Then there is a finite $F \subseteq I$ such that $\{V_x : x \in F\}$ covers I, and so $\{B_x : x \in F\}$ covers all of $AD(\tau)$ except for a subset of the compact set $(\bigcup\{K_x : x \in F\}) \times \{1\}$. Thus, $AD(\tau)$ is compact.

Since τ is locally countable and $I \times \{1\}$ is an uncountable open subset of $AD(\tau)$ homeomorphic to (I, τ), then $AD(\tau)$ has no countable base. Thus, $AD(\tau)$ cannot be metrizable (by compactness).

To prove $AD(\tau)$ is first countable, we just need to find a countable neighborhood base for each point in $I \times \{0\}$. Fix $x \in I$ and a τ-open neighborhood U of x such that $K = \overline{U}^{\tau}$ is τ-compact. Also, fix a countable neighborhood base $\{V_n : n < \omega\}$ of x in I. Let $B_n = (V_n \times \{0,1\}) \setminus (K \times \{1\})$ for each n. Suppose $B = (V \times \{0,1\}) \setminus (L \times \{1\})$ is any other basic neighborhood of $(x,0)$ in $AD(\tau)$. Then $L \setminus U$ is τ-compact, and therefore compact in the usual topology on I, so it is also closed in I. Then there is an n such that $V_n \subseteq V \cap (I \setminus (L \setminus U)) = V \cap ((I \setminus L) \cup U) \subseteq (V \setminus L) \cup K$. Hence, $V_n \setminus K$ is contained in $V \setminus L$, and it follows that B_n is contained in B. □

Lemma 72. *Let $A = (I,\tau)$ denote I with some topology τ that refines the usual topology on I, and let $X = AD(\tau)$. If both A and A^2 have relative calibre (ω_1,ω) (in $\mathcal{K}(A)$ and $\mathcal{K}(A^2)$, respectively), then $X^2 \setminus \Delta$ has relative calibre (ω_1,ω) in $\mathcal{K}(X^2 \setminus \Delta)$.*

Proof. Notice that $X^2 \setminus \Delta$ is the union of four subspaces which are naturally homeomorphic to $I^2 \setminus \Delta$, $I \times A$, $A \times I$, and $A^2 \setminus \Delta$. By Lemma 20, it is sufficient to show that these four spaces all have relative calibre (ω_1,ω) in $\mathcal{K}(I^2 \setminus \Delta)$, $\mathcal{K}(I \times A)$, $\mathcal{K}(A \times I)$, and $\mathcal{K}(A^2 \setminus \Delta)$, respectively.

We know both A and A^2 have relative calibre (ω_1,ω), and of course I and I^2 have every relative calibre since they are compact. So $I \times A$ and $A \times I$ have relative calibre (ω_1,ω) by Lemma 26. Also, Δ is a G_δ subset of I^2, and since τ refines the usual topology on I, then Δ is also a G_δ subset of A^2. Thus, both $I^2 \setminus \Delta$ and $A^2 \setminus \Delta$ have relative calibre (ω_1,ω) by Lemma 51. □

Theorem 73. *There is a first countable, compact space X which is not metrizable even though $X^2 \setminus \Delta$ has relative calibre (ω_1,ω) in $\mathcal{K}(X^2 \setminus \Delta)$.*

Proof. By Lemma 71 and Lemma 72, it suffices to prove the existence of a first countable, locally countable, locally compact topology τ on I refining the usual topology such that both (I,τ) and its square have relative calibre (ω_1,ω). Indeed, then $X = AD(\tau)$ is as desired. Hence, this result follows from Proposition 74 below. □

3.2.3 A TOPOLOGY ON THE CLOSED UNIT INTERVAL

Our goal in this section is to prove:

Proposition 74. *There is a first countable, locally countable, locally compact topology τ on the closed unit interval I refining the usual topology such that both (I, τ) and its square have relative calibre (ω_1, ω).*

To prove this result, our first step will be to show (in Lemma 76) that the relative calibre portion of Proposition 74 follows from the property (λ_1^2) described below. For any set Z, a subset Y of $Z \times Z$ is called *small* if there is a countable subset $C \subseteq Z$ such that $Y \subseteq (C \times Z) \cup (Z \times C)$. In other words, Y is contained in the union of a countable family of 'horizontal' and 'vertical' lines. Y is called *big* if it is not small. Consider the following two properties, where τ denotes any topology on the closed unit interval I.

$$\text{For any } F \subseteq I^2 \text{ such that } \overline{F}^{I\times I} \text{ is big, } \overline{F}^{\tau\times\tau} \text{ is uncountable.} \tag{λ_1^2}$$

$$\text{For any } F \subseteq I \text{ such that } \overline{F}^{I} \text{ is uncountable, } \overline{F}^{\tau} \text{ is uncountable.} \tag{λ_1}$$

Lemma 75. *If τ is a topology on I satisfying (λ_1^2) above, then τ also satisfies (λ_1).*

Proof. For any $F \subseteq I$, consider $\Delta(F) = \{(x, x) : x \in F\}$. If $\overline{F}^{I}$ is uncountable, then $\overline{\Delta(F)}^{I\times I}$ is an uncountable subset of $\Delta(I)$, and so $\overline{\Delta(F)}^{I\times I}$ is big since any horizontal or vertical line meets $\Delta(I)$ in one point. Then by (λ_1^2), $\overline{\Delta(F)}^{\tau\times\tau}$ is uncountable, and it follows that $\overline{F}^{\tau}$ is also uncountable. $\square$

Lemma 76. *Let $A = (I, \tau)$, where τ is some first countable topology on I refining the usual topology and satisfying (λ_1^2). Then both A and A^2 have relative calibre (ω_1, ω).*

Proof. We will first verify that A has relative calibre (ω_1, ω) (in $\mathcal{K}(A)$). Fix an uncountable subset S of A. As I is hereditarily separable, we can find a countable subset $C \subseteq S$ such that $S \subseteq \overline{C}^{I}$. Then $\overline{C}^{I}$ is uncountable, and since τ also satisfies (λ_1) by Lemma 75, then $\overline{C}^{A}$ is uncountable. Since A is first countable, we can therefore find an infinite sequence $S_1 \subseteq C$ converging in A to a point x in $\overline{C}^{A} \setminus C$. The A-closure of S_1 is therefore $S_1 \cup \{x\}$, which is compact.

Now we will check that A^2 has relative calibre (ω_1, ω) in $\mathcal{K}(A^2)$. Suppose $S \subseteq A^2$ is uncountable. If S is small, then we may assume, without loss of generality, that S is contained in a horizontal or vertical line. Hence, we are done since this line is just a homeomorphic copy of A, which has relative calibre (ω_1, ω). So we will instead assume that S is big. Choose a countable subset $C \subseteq S$ such that $S \subseteq \overline{C}^{I\times I}$. Then $\overline{C}^{I\times I}$ is also big, so (λ_1^2) implies that $\overline{C}^{A\times A}$ is uncountable. Since A^2 is first countable, then as in the proof for A, we can find an infinite sequence S_1 in C converging in A^2 to a point outside of C, so that S_1 has compact A^2-closure. □

The next lemma is proven by van Douwen in [11].

Lemma 77. *For any big closed subset Y of $\mathbb{R}^2$ and for any subset S of $\mathbb{R}$ with $|S| < \mathfrak{c}$, there is a point (y_1, y_2) in Y such that $\{y_1, y_2\} \cap S = \emptyset$.*

We are now ready to prove Proposition 74. Our construction below is nearly the same as that of the space Λ by van Douwen in [11]. In fact, van Douwen introduces the condition (λ_1) and a clearly stronger condition (λ_ω), and he proves that his space Λ satisfies a condition (λ_ω^2), which is an 'upgrade' of (λ_ω) to dimension 2. Unfortunately, (λ_ω^2) is a weak upgrade and does not naturally imply (λ_1^2). This is why we consider the property (λ_1^2) given above, which is the natural dimension 2 'upgrade' of the condition (λ_1). We therefore cannot simply adopt van Douwen's Λ, but instead opt to prove directly the existence of τ satisfying (λ_1^2). In doing so, we note that there is a gap in van Douwen's construction and explain how to correct it.

Proof of Proposition 74. Consider the family $\mathcal{C}$ of all countable subsets of I^2 whose closures in I^2 are big. Note that $|\mathcal{C}| \leq \mathfrak{c}$, so we can enumerate $\mathcal{C} = \{C_\gamma : \gamma < \mathfrak{c}\}$ such that each member of $\mathcal{C}$ is listed $\mathfrak{c}$ times. Also, we may enumerate $I = \{x_\alpha : \alpha < \mathfrak{c}\}$ such that $x_\alpha \neq x_\beta$ when $\alpha \neq \beta$. Then define $X_\alpha = \{x_\beta : \beta < \alpha\}$ for each $\alpha < \mathfrak{c}$.

Let $\pi_1, \pi_2 : I^2 \to I$ be the natural projections. We will next construct injections $\psi_1, \psi_2 : \mathfrak{c} \to \mathfrak{c} \setminus \omega$ satisfying the following conditions for each $\gamma < \mathfrak{c}$:

$$\pi_1[C_\gamma] \cup \pi_2[C_\gamma] \subseteq X_{\psi_1(\gamma)} \cap X_{\psi_2(\gamma)}, \tag{3.1}$$

$$(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \in \overline{C_\gamma}^{I\times I}, \text{ and} \tag{3.2}$$

$$\psi_i(\gamma) \neq \psi_j(\delta) \text{ for any } i, j \text{ and } \delta \neq \gamma. \tag{3.3}$$

Fix $\gamma < \mathfrak{c}$ and suppose we have already defined $\psi_1(\delta)$ and $\psi_2(\delta)$ satisfying the above conditions for every $\delta < \gamma$. Since $D_\gamma = \pi_1[C_\gamma] \cup \pi_2[C_\gamma]$ is countable, then we may write $D_\gamma = \{x_{\beta_n} : n < \omega\}$. Since $\mathfrak{c}$ has uncountable cofinality, then $(\beta_n)_n$ is bounded above by some $\alpha < \mathfrak{c}$, and so D_γ is contained in X_α. Let $\alpha_\gamma = \min\{\alpha < \mathfrak{c} : D_\gamma \subseteq X_\alpha\}$. So $D_\gamma \subseteq X_\alpha$ for all $\alpha \geq \alpha_\gamma$. Note that $\alpha_\gamma \geq \omega$ since D_γ is infinite.

Consider $\Psi_\gamma = \{\psi_i(\delta) : \delta < \gamma, i = 1, 2\}$ and $S_\gamma = \{x_\beta : \beta \in \alpha_\gamma \cup \Psi_\gamma\}$. Since $|S_\gamma| < \mathfrak{c}$, then Lemma 77 shows that there is a point (y_1, y_2) in $\overline{C_\gamma}^{I\times I}$ such that $y_1, y_2 \notin S_\gamma$. Now for $i = 1, 2$, define $\psi_i(\gamma)$ to be the unique ordinal less than $\mathfrak{c}$ such that $y_i = x_{\psi_i(\gamma)}$. Then (3.2) is satisfied immediately. Also, since y_i is not in S_γ, then we have $\psi_i(\gamma) \geq \alpha_\gamma$, which implies that $D_\gamma \subseteq X_{\psi_i(\gamma)}$ for $i = 1, 2$. So (3.1) is satisfied. And of course, $\psi_i(\gamma) \geq \alpha_\gamma \geq \omega$ shows that each $\psi_i(\gamma)$ really is in $\mathfrak{c} \setminus \omega$, as desired. We also see that $\psi_i(\gamma)$ is not in Ψ_γ since y_i is not in S_γ, and so (3.3) is satisfied as well. Thus, by transfinite induction, our construction of the injections ψ_1 and ψ_2 is complete.

Choose a countable base $\{B_i : i < \omega\}$ for I with $B_1 = I$, and then define $E_j(x) = \bigcap\{B_i : i \leq j \text{ and } x \in B_i\}$ for each $x \in I$ and $j < \omega$. Then $\{E_j(x) : j < \omega\}$ is a neighborhood base for x in I such that:

$$\text{if } y \in E_j(x) \text{ and } i \geq j, \text{ then } E_i(y) \subseteq E_j(x). \tag{3.4}$$

Let $\Psi \subseteq \mathfrak{c} \setminus \omega$ be the union of the images of ψ_1 and ψ_2. By (3.2) and (3.3), there are well-defined sequences $s_\alpha = (s_\alpha^i)_{i<\omega}$ for each $\alpha \in \Psi$ satisfying:

$$s_\alpha^i \in E_i(x_\alpha) \text{ for each } \alpha \in \Psi \text{ and } i < \omega, \tag{3.5}$$

$$\text{if } \psi_1(\gamma) \neq \psi_2(\gamma), \text{ then } (s^i_{\psi_1(\gamma)}, s^i_{\psi_2(\gamma)}) \in C_\gamma \text{ for each } i < \omega, \text{ and} \tag{3.6}$$

$$\text{if } \alpha = \psi_1(\gamma) = \psi_2(\gamma)^{1}, \text{ then } (s_\alpha^{2i-1}, s_\alpha^{2i}) \in C_\gamma \text{ for each } i < \omega. \tag{3.6$'$}$$

Conditions (3.1), (3.6) and (3.6′) imply that the sequence s_α lies entirely in X_α for each $\alpha \in \Psi$, so the following construction by transfinite recursion makes sense. For each $\alpha \in \mathfrak{c} \setminus \Psi$, define $L_j(x_\alpha) = \{x_\alpha\}$ for all $j < \omega$. Now for any $\alpha \in \Psi$ (so $\alpha \geq \omega$), we may assume $L_i(x)$ has been defined for all $x \in X_\alpha$ and $i < \omega$, and so we define $L_j(x_\alpha) = \{x_\alpha\} \cup \bigcup_{i \geq j} L_i(s_\alpha^i)$ for each $j < \omega$. The next facts easily follow by transfinite induction on α and by (3.4):

$$\text{each } L_j(x) \text{ is countable,} \tag{3.7}$$

$$L_j(x) \subseteq E_j(x), \text{ and} \tag{3.8}$$

$$\text{if } y \in L_j(x), \text{ then } L_i(y) \subseteq L_j(x) \text{ for some } i < \omega. \tag{3.9}$$

Since $L_{j+1}(x) \subseteq L_j(x)$ for all $x \in I$ and $j < \omega$, then (3.9) implies that $\{L_j(x) : x \in I,\ j < \omega\}$ is a base generating a new topology on I. This new topology τ refines the usual topology on I because of (3.8), and since $\{L_j(x) : j < \omega\}$ is a neighborhood base at x, then τ is first countable. Also, τ is locally countable by (3.7). Additionally, it is easy to check that each $L_j(x_\alpha)$ is compact by transfinite induction, so τ is locally compact.

Note that the sequence s_α converges (with respect to τ) to x_α for each $\alpha \in \Psi$, and so by (3.6) and (3.6′), we have $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \in \overline{C_\gamma}^{\tau \times \tau}$ for all $\gamma \in \mathfrak{c}$. Since each member of $\mathcal{C}$ appears $\mathfrak{c}$ times in the enumeration $\{C_\gamma : \gamma < \mathfrak{c}\}$, and since $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \neq (x_{\psi_1(\delta)}, x_{\psi_2(\delta)})$ for $\gamma \neq \delta$ (as the ψ_i are injections), then $\overline{C}^{\tau \times \tau}$ has cardinality $\mathfrak{c}$ for each $C \in \mathcal{C}$. This implies (λ_1^2), and thus by Lemma 76, τ has all the desired properties. □

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[1]In [11], it was asserted that we could always make $\psi_1(\gamma) < \psi_2(\gamma)$, in which case (3.6′) would be unnecessary. However, there is a γ such that $C_\gamma \subseteq \mathbb{Q}^2 \setminus \Delta$ and $\overline{C_\gamma}^{I \times I} = C_\gamma \cup \Delta$. The rationals in [11] are each of the form x_α for some $\alpha < \omega$, and since $\psi_1, \psi_2 : \mathfrak{c} \to \mathfrak{c} \setminus \omega$, then $x_{\psi_i(\gamma)} \notin \mathbb{Q}$ so $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \notin C_\gamma$. Hence, (3.2) implies $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \in \Delta$, so we are forced to have $\psi_1(\gamma) = \psi_2(\gamma)$. As $C_\gamma \cap \Delta = \emptyset$, we cannot have $(s^i_{\psi_1(\gamma)}, s^i_{\psi_2(\gamma)}) \in C_\gamma$, which shows why a modified condition like (3.6′) is necessary.

3.3 HEREDITARY COMPACT COVER PROPERTIES

Previously in this chapter, we have considered the pre-ideal $\mathcal{K}(A)$ where A is the particular subset $A = X^2 \setminus \Delta$ of a square X^2. Cascales, Orihuela, and Tkachuk posed a number of questions in [7] asking what happens instead when for every subspace A of a space X, $\mathcal{K}(A)$ or $(A, \mathcal{K}(A))$ is a Tukey quotient of a directed set of the form $\mathcal{K}(M)$. In this section, we study this hereditary situation and answer some of the questions posed by Cascales et al.

Recall that a space X is called *Lindelöf* Σ if X has a compact cover $\mathcal{C}$ and a countable network $\mathcal{N}$ modulo $\mathcal{C}$. Here we say $\mathcal{N}$ is a *network for* X *modulo* $\mathcal{C}$ if for any C in $\mathcal{C}$ and any open set U containing C, there is an $N \in \mathcal{N}$ such that $C \subseteq N \subseteq U$. A space X is called *ω-bounded* if every countable subset of X has compact closure in X. Of course compact spaces are ω-bounded, so the next result by Cascales et al. shows that if $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$ for some separable metrizable M, then X is 'almost' Lindelöf Σ.

Proposition 78 (Cascales et al., [7])**.** *If X is a space and $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$, where M is a separable metrizable space, then there is a cover $\mathcal{C}$ of X and a countable collection $\mathcal{N}$ of subsets of X such that:*

(1) every element of $\mathcal{C}$ is ω-bounded, and

(2) $\mathcal{N}$ is a network for X modulo $\mathcal{C}$.

A space X is called an *$\aleph_0$-space* if it has a countable network modulo all of $\mathcal{K}(X)$. Recall also that X is called *cosmic* if it has a countable network (modulo the singletons). Also recall that a space satisfies the *countable chain condition* (ccc) if every family of disjoint open sets is countable. The next result answers questions 4.14-4.16 in [7].

Theorem 79. *Let X be a space.*

(1) If for every subspace A of X, there is some separable metrizable space M_A such that $\mathcal{K}(M_A) \geq_T \mathcal{K}(A)$, then X is an $\aleph_0$-space.

(2) If for every subspace A of a space X, there is some separable metrizable space M_A such that $\mathcal{K}(M_A) \geq_T (A, \mathcal{K}(A))$, then X is cosmic.

Proof. We can deduce the first claim from the second as follows. Let X be as in the first claim. Since $\mathcal{K}(A) \geq_T (A, \mathcal{K}(A))$, then from the second claim, we certainly know X is cosmic. So X has a coarser second countable topology, and hence $\mathcal{K}(X)$ has a coarser second countable topology. Now since $\mathcal{K}(M_X) \geq_T \mathcal{K}(X) =_T (\mathcal{K}(X), \mathcal{K}(\mathcal{K}(X))$ (see Lemma 50), then from Proposition 78, we know $\mathcal{K}(X)$ has a cover $\mathcal{C}$ of ω-bounded sets and a countable network $\mathcal{N}$ modulo $\mathcal{C}$. As $\mathcal{K}(X)$ has a coarser second countable topology, all the elements of $\mathcal{C}$ must be compact. Thus $\mathcal{K}(X)$ is Lindelöf Σ and has a coarser second countable topology, so it is cosmic. However, $\mathcal{K}(X)$ is cosmic if and only if X is an $\aleph_0$-space.

Now let X be as in the second claim. Since each $\mathcal{K}(M_A)$ has calibre (ω_1, ω), then by Lemma 41, every subspace of X has countable extent. Then X is hereditarily ccc and so also hereditarily separable. Fix an arbitrary subset A of X. Then Proposition 78 gives a collection $\mathcal{C}$ of ω-bounded subsets of A and a countable network $\mathcal{N}$ for A modulo $\mathcal{C}$. By hereditary separability, for each C in $\mathcal{C}$, we can find a countable dense subset D of C, and so $C = \overline{D}$ is compact by ω-boundedness of C. Hence, A is Lindelöf Σ and X is hereditarily Lindelöf Σ. It follows that X is cosmic [31]. $\square$

Instead of arbitrary separable metrizable spaces controlling the compact subsets of each subspace of X, as in the preceding theorem, we can restrict the M_A to be ω^ω, the irrationals. In the relative case, we have the following characterization of such spaces, which will be proved as part of Theorem 84 below.

Proposition 80. *Let X be a space. Then X is countable if and only if $\mathcal{K}(\omega^\omega) \geq_T (A, \mathcal{K}(A))$ for each subspace A of X.*

For metrizable spaces, we can also characterize the non-relative case as follows. Recall that separable completely metrizable spaces are called *Polish*, while a space X is *scattered* if each nonempty subspace Y of X contains a point which is isolated in Y.

Proposition 81. *Let X be a metrizable space. Then the following are equivalent:*

(i) for every subspace A of X, we have $\mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(A)$,

(ii) X is countable and Polish, and

(iii) X is countable and scattered.

The proof of Proposition 81 is deferred until after Theorem 84.

Call a space X *hereditarily relative calibre* (κ, λ, μ) if each subspace A of X has relative calibre (κ, λ, μ) in $\mathcal{K}(A)$ (and recall that calibre (κ, λ) is the same as calibre $(\kappa, \lambda, \lambda)$). Note that X is hereditarily relative calibre (ω_1, ω) (respectively, (ω_1, ω_1)) if and only if for each uncountable $S \subseteq X$, there is an infinite (respectively, uncountable) $S_1 \subseteq S$ such that $\overline{S_1}^S = \overline{S_1} \cap S$ is compact.

Observe that if X is a space such that 'for every subspace A of X, there is some separable metric space M_A such that $\mathcal{K}(M_A) \geq_T (A, \mathcal{K}(A))$', then it is also true that 'X is hereditarily relative calibre (ω_1, ω)'. Similarly, observe that consistently (precisely when $\omega_1 < \mathfrak{b}$ – see Lemma 23), if X is a space such that 'for every subspace A of X, we have $\mathcal{K}(\omega^\omega) \geq_T (A, \mathcal{K}(A))$', then it is also true that 'X is hereditarily relative calibre (ω_1, ω_1)'.

Call a space X *hereditarily calibre* (κ, λ, μ) if, for each subspace A of X, the partial order $\mathcal{K}(A)$ has calibre (κ, λ, μ). Observe that if X is a space such that 'for every subspace A of X, there is some separable metric space M_A such that $\mathcal{K}(M_A) \geq_T \mathcal{K}(A)$', then it is also true that 'X is hereditarily calibre (ω_1, ω)'. Similarly, observe that consistently (precisely when $\omega_1 < \mathfrak{b}$ – see Lemma 23), if X is a space such that 'for every subspace A of X, we have $\mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(A)$', then it is also true that 'X is hereditarily calibre (ω_1, ω_1)'.

Note also that 'hereditarily calibre (κ, λ, μ)' implies 'hereditarily relative calibre (κ, λ, μ)'. A further, and stronger, condition is that $\mathcal{K}(X)$ is hereditarily relative calibre (κ, λ, μ).

Lemma 82. *Let X be a space. If $\mathcal{K}(X)$ is hereditarily relative calibre (κ, λ, μ), then X is hereditarily calibre (κ, λ, μ).*

Proof. Take any subspace A of X. Since $\mathcal{K}(A)$ is a subspace of $\mathcal{K}(X)$, then $\mathcal{K}(A)$ has relative calibre (κ, λ, μ) in $\mathcal{K}(\mathcal{K}(A))$. So by Lemma 50, $\mathcal{K}(A)$ has calibre (κ, λ, μ). □

We now compare and contrast the situation when for every subspace A of a space X, there is a separable metrizable M_A such that $\mathcal{K}(M_A) \geq_T (A, \mathcal{K}(A))$, versus all subspaces having a (relative) calibre. In the weakest case, there is a clear difference between the two scenarios. The second part of Theorem 79 says that if each subspace A of X has a separable metric space M_A such that $\mathcal{K}(M_A) \geq_T (A, \mathcal{K}(A))$, then X is cosmic. Weakening the hypothesis on X to being hereditarily relative calibre (ω_1, ω) does not suffice to deduce cosmicity of X. For example, the Sorgenfrey line is not cosmic but is hereditarily ccc (so each subspace has countable extent) and first countable, and so we can apply Lemma 40 to $(Q', Q) = (A, \mathcal{K}(A))$

for any subspace A to see that the Sorgenfrey line is hereditarily relative calibre (ω_1, ω). Nor, consistently at least, is it sufficient to strengthen 'X hereditarily relative calibre (ω_1, ω)' to '$\mathcal{K}(X)$ hereditarily relative calibre (ω_1, ω)'.

Example 1 ($\mathfrak{b} = \omega_1$)**.** *There is an uncountable subspace X of the Sorgenfrey line such that $\mathcal{K}(X)$ is first countable and hereditarily ccc (see [17]).*

Hence $\mathcal{K}(X)$ is hereditarily relative calibre (ω_1, ω) (and so X is hereditarily calibre (ω_1, ω) by Lemma 82), but X is not cosmic.

In [17] it is shown that under the Open Coloring Axiom (OCA), if $\mathcal{K}(X)$ is first countable and hereditarily ccc, then X is cosmic. However, the argument given in that paper does not obviously show that OCA implies that 'if $\mathcal{K}(X)$ is hereditarily relative calibre (ω_1, ω), then X is cosmic'.

Moving from the relative calibre (ω_1, ω) case to relative calibre (ω_1, ω_1), however, we get equivalence between calibres and Tukey reduction, and equivalence to X being countable. We will use the following result. Recall that a space is called *analytic* if it is separable metrizable and a continuous image of a Polish space.

Theorem 83 (Christensen, [9])**.** *If M is a separable metrizable space, then*
(1) $\mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(M)$ if and only if M is Polish, and
(2) $\mathcal{K}(\omega^\omega) \geq_T (M, \mathcal{K}(M))$ if and only if M is analytic.

Theorem 84. *Let X be a space. Then the following are equivalent:*
(i) X is hereditarily relative calibre (ω_1, ω_1),
(ii) for every subspace A of X, we have $\mathcal{K}(\omega^\omega) \geq_T (A, \mathcal{K}(A))$,
(iii) for every subspace A of X, we have $\omega \geq_T (A, \mathcal{K}(A))$, and
(iv) X is countable.

Proof. We start by showing that (iv) implies (iii). Suppose X is countable and A is a subspace of X. Enumerate $A = \{a_n : n < \omega\}$. Define $\phi : \omega \to \mathcal{K}(A)$ by $\phi(n) = \{a_i : i \leq n\}$. Then ϕ is order-preserving and its image is a compact cover of A, which gives (iii). Of course (iii) implies (ii) since Lemma 50 shows that $\mathcal{K}(\omega^\omega) =_T \omega^\omega$ and projecting onto the fist coordinate gives $\omega^\omega \geq_T \omega$.

Next, we will prove that (ii) implies (iv). So assume, for a contradiction, that X satisfies (ii) but is uncountable. Then by Theorem 79, we know X is cosmic. Hence it has a coarser separable metrizable topology τ. Since any subset of X which is compact in the original topology is also compact in τ, we see that $X_\tau = (X, \tau)$ also satisfies (ii). In particular, $\mathcal{K}(\omega^\omega) \geq_T (X_\tau, \mathcal{K}(X_\tau))$, so X_τ is analytic by Christensen's Theorem 83. Hence X_τ contains a non-analytic subspace A (because an uncountable analytic space must contain a Cantor set, which contains non-analytic subspaces). But then we cannot have $\mathcal{K}(\omega^\omega) \geq_T (A, \mathcal{K}(A))$.

It is vacuously true that (iv) implies (i), so we complete the proof by showing that the negation of (iv) implies the negation of (i). Suppose X is an uncountable space. We have to show it contains a subspace A which is not relative calibre (ω_1, ω_1). Note that it suffices to find a subspace A of X satisfying,

$$A \text{ is uncountable, and all compact subsets of } A \text{ are countable,} \tag{3.10}$$

because then no uncountable subset of A can have compact closure in A. If X itself satisfies (3.10), then we are, of course, done. If not, then X contains an uncountable compact subspace, and so, without loss of generality, we can assume X is compact.

If X contains a right-separated subspace A of size ω_1, then every compact subset of A is contained in an initial (countable) interval, so A satisfies (3.10), and so we are done. If not then X is hereditarily Lindelöf. Since X is also compact, we see that X is first countable. Hence, X has size the continuum, $\mathfrak{c}$, and weight no more than $\mathfrak{c}$. Applying the fact that X is hereditarily Lindelöf again, we see that X contains no more than $\mathfrak{c}$ open subsets. So the collection $\mathcal{K}$ of all uncountable compact subsets of X has $|\mathcal{K}| \leq \mathfrak{c}$. Observe that each member of $\mathcal{K}$ has cardinality exactly $\mathfrak{c}$.

Next, we will follow the construction of Bernstein's set to form an uncountable subspace $A \subseteq X$ that does not contain any element of $\mathcal{K}$. Enumerate $\mathcal{K} = \{K_\alpha : \alpha < \mathfrak{c}\}$, possibly with repetitions. Using transfinite induction, we will construct uncountable sequences $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$ such that each $x_\alpha, y_\alpha \in K_\alpha$. We will also ensure that $x_\alpha \neq x_\beta$ and $y_\alpha \neq y_\beta$ whenever $\alpha \neq \beta$, and that $x_\alpha \neq y_\beta$ for any $\alpha, \beta < \mathfrak{c}$. Indeed, if $\beta < \mathfrak{c}$ and if we have already constructed x_α and y_α for each $\alpha < \beta$, then we can find distinct points $x_\beta, y_\beta \in K_\beta \setminus (\{x_\alpha : \alpha < \beta\} \cup \{y_\alpha : \alpha < \beta\})$ since $|K_\beta| = \mathfrak{c}$ and $\beta < \mathfrak{c}$. Now let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$.

Then A is uncountable and does not contain any K_α since $y_\alpha \notin A$. Thus A satisfies (3.10), and the proof is complete. $\square$

The Cantor-Bendixson process. For any space X, let $I(X)$ denote the subset of all points that are isolated in X and define $X' = X \setminus I(X)$, which is called the *derived set* or *Cantor-Bendixson derivative* of X. We can inductively define $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$ for each ordinal α, and $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ for each limit ordinal λ. By considering the cardinality of X, we can see that there must be an ordinal β such that $X^{(\alpha)} = X^{(\beta)}$ for all $\alpha \geq \beta$. The smallest ordinal β with this property is called the *Cantor-Bendixson rank* of X, while the subspace $X^{(\beta)}$ is called the *perfect kernel* of X since it is the largest *perfect* (i.e. closed with no isolated points) subset of X. As mentioned above, X is called scattered if each of its nonempty subspaces has an isolated point, and this is equivalent to its perfect kernel being empty. If X is scattered, then its Cantor-Bendixson rank is also known as the *scattered height* of X.

We are now ready to prove Proposition 81.

Proof of Proposition 81. That condition (ii) follows from (i) is immediate from Theorem 84 and Christensen's Theorem 83. On the other hand, if X is countable and Polish, then every subspace A of X is a G_δ subspace, hence also Polish, so $\mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(A)$ by Christensen's theorem. Thus, (i) and (ii) are equivalent.

Now we prove (ii) and (iii) are equivalent, so assume X is countable. Using the notation from the Cantor-Bendixson process above, note that each $X^{(\alpha)}$ is closed in X. Let β be the Cantor-Bendixson rank of X, so $C = X^{(\beta)}$ is the perfect kernel of X. Assume X is not scattered, so C is nonempty. Since C has no isolated points, then the complement of each singleton in C is open and dense in C, but the intersection of this countable family of open dense sets is empty. Then the Baire category theorem shows that C is not Polish, so neither is X as C is closed in X. Thus, (ii) implies (iii).

Now assume X is scattered, so C is empty, and let $Y_\alpha = X \setminus X^{(\alpha)}$, which is open in X for each α. We will show Y_α is Polish, by induction. Certainly $Y_0 = \emptyset$ is vacuously Polish, so now assume Y_α is Polish for some $\alpha < \beta$. Note that $Y_{\alpha+1} = Y_\alpha \cup I_\alpha$ where I_α is the set of isolated points in $X^{(\alpha)}$. Since I_α is (countable and) discrete, then it is Polish, and so $Y_{\alpha+1}$ is also Polish. Indeed, we can embed X in a Polish space Z (such as the Hilbert cube), and then a subspace of X is Polish if and only if it is G_δ in Z, and the union of two G_δ sets is G_δ. Now let $\lambda \leq \beta$ be any limit ordinal, and assume we already know Y_α is Polish for each $\alpha < \lambda$. Since X is countable, then so is λ, so we can find a sequence $(\alpha_n)_n$ of ordinals less than λ which converge to λ. Thus, $Y_\lambda = X \setminus X^{(\lambda)} = X \setminus \bigcap_{\alpha<\lambda} X^{(\alpha)} = \bigcup_{\alpha<\lambda} Y_\alpha = \bigcup_n Y_{\alpha_n}$. Now, since each Y_{α_n} is Polish and open in Y_λ, and since Y is metrizable, then Y_λ must also be Polish. Indeed, Y_λ is a continuous open image of the disjoint union $\bigoplus_n Y_{\alpha_n}$, which is Polish, so we can apply Exercise 5.5.8(d) in [13], and the induction is complete. In particular, $Y_\beta = X$ is Polish, so (iii) implies (ii). □

Theorem 85. *Let X be a metrizable space. Then the following are equivalent:*

(i) X is hereditarily calibre (ω_1, ω_1), and

(ii) Either X is homeomorphic to the disjoint sum of a countable (possibly empty) disjoint sum of convergent sequences and a countable (possibly empty) discrete space, or $\omega_1 < \mathfrak{b}$ and X is countable and scattered.

Proof. If X is homeomorphic to the disjoint sum of a countable (possibly empty) disjoint sum of convergent sequences and a countable (possibly empty) discrete space, then every subspace A of X is locally compact and countable, and so it is easily seen that $\mathcal{K}(A)$ has calibre (ω_1, ω_1).

Now suppose $\omega_1 < \mathfrak{b}$ and X is countable and scattered. Take any subspace A of X. By the preceding theorem, $\mathcal{K}(\omega^\omega) \geq_T \mathcal{K}(A)$. It follows from $\omega_1 < \mathfrak{b}$ that $\mathcal{K}(\omega^\omega)$ has calibre (ω_1, ω_1). Hence, $\mathcal{K}(A)$ also has calibre (ω_1, ω_1), and so (ii) implies (i).

For the converse, suppose that for every subspace A of X, the partial order $\mathcal{K}(A)$ has calibre (ω_1, ω_1). By Theorem 84, X is countable. Since $\mathcal{K}(\mathbb{Q})$ does not have calibre (ω_1, ω_1), the rationals $\mathbb{Q}$ do not embed in X. It follows that X is scattered. If X has scattered height 0, then it is discrete. If X has scattered height 1, then it is homeomorphic to the disjoint sum of a countable (non-empty) disjoint sum of convergent sequences and a countable (possibly empty) discrete space.

The remaining case is when X has scattered height at least 2. We have to show that $\omega_1 < \mathfrak{b}$. From the scattered height restriction on X, it follows that X contains a subspace A' which is homeomorphic to a convergent sequence of convergent sequences. Removing the limit points of the convergent sequences, but not the point that the sequence of convergent sequences converges to, we obtain a subspace A of X which is homeomorphic to the metric fan, F. By hypothesis, $\mathcal{K}(F)$ has calibre (ω_1, ω_1). As shown in [27], $\mathcal{K}(F)$ and $\mathcal{K}(\omega^\omega)$ are Tukey equivalent, and so share the same calibres. Specifically, $\mathcal{K}(F)$ has calibre (ω_1, ω_1) if and only if $\mathcal{K}(\omega^\omega)$ has calibre (ω_1, ω_1), and this in turn holds if and only if $\omega_1 < \mathfrak{b}$. □

Define $\mathcal{K}^{(1)}(X) = \mathcal{K}(X)$, and inductively, $\mathcal{K}^{(n+1)}(X) = \mathcal{K}(\mathcal{K}^{(n)}(X))$.

Theorem 86. *For any space X, the following are equivalent:*

(i) $\mathcal{K}^{(n)}(X)$ is hereditarily calibre (ω_1, ω_1) for every natural number n,

(ii) $\mathcal{K}(X)$ is hereditarily relative calibre (ω_1, ω_1),

(iii) $\mathcal{K}(X)$ is countable,

(iv) $\mathcal{K}(X)$ is countable and all compact subspaces of $\mathcal{K}(X)$ are finite, and

(v) X is countable and all compact subspaces of X are finite.

Proof. As noted previously, 'hereditarily calibre (ω_1, ω_1)' implies 'hereditarily relative calibre (ω_1, ω_1)', so (ii) follows from (i). And by Theorem 84, (ii) is equivalent to (iii).

If (iii) holds, then X is also countable, and we will show that all compact subspaces of X must be finite as follows. Suppose X has an infinite compact subspace K. Since K is countably infinite and compact, it contains an infinite convergent sequence S. But then S has uncountably many compact subspaces, which contradicts that $\mathcal{K}(X)$ is countable. Hence, (iii) implies (v).

If X is countable, then it has only countably many finite subsets, so (v) implies (iii). In fact, (v) implies (iv). To see this, assume (v) holds and K is a compact subset of $\mathcal{K}(X)$. Then $\hat{K} = \bigcup K$ is a compact subset of X, and so is finite. Hence, K is also finite since it is a subset of the power set of $\hat{K}$.

It remains to show that (iv) implies (i). Assume then that $\mathcal{K}(X)$ is countable and all its compact subspaces are finite. First note that (iv) trivially implies (iii), and hence (iv) and (v) are equivalent. Thus for every natural number n we have that $\mathcal{K}^{(n)}(X)$ is countable and all its compact subspaces are finite. So for a fixed n, every subspace of $\mathcal{K}^{(n)}(X)$ is hemicompact, and thus has calibre (ω_1, ω_1). $\square$

We now apply our results above to the case when an order condition is imposed on the compact subsets of each subspace of the square of a compact space. Claim (4) below answers questions 4.17-4.20 from [7].

Theorem 87. *Let X be compact.*

(1) If X^2 is hereditarily calibre (ω_1, ω), then X is metrizable.

(2) It is consistent and independent that 'X^2 is hereditarily relative calibre (ω_1, ω) implies X is metrizable'.

(3) If X (and a fortiori *X^2) is hereditarily relative calibre (ω_1, ω_1), then X is countable and metrizable.*

(4) If for every subspace A of X (and a fortiori *X^2) there is a separable metrizable space M_A such that $\mathcal{K}(M_A) \geq_T (A, \mathcal{K}(A))$, then X is metrizable.*

Proof. If X^2 is hereditarily calibre (ω_1, ω), then in particular $\mathcal{K}(A)$ has calibre (ω_1, ω) when $A = X^2 \setminus \Delta$, so Claim (1) follows from Theorem 63. Claim (3) follows immediately from Theorem 84, while Theorem 79 gives Claim (4) since a compact space is cosmic if and only if it is second countable and therefore separable and metrizable since it is T_3.

Now suppose X^2 is hereditarily relative calibre (ω_1, ω). Then every subspace of X^2 has countable extent (Lemma 49), so X^2 is hereditarily ccc. Under PFA [46] it follows that X^2 is hereditarily Lindelöf, and X is metrizable. However under the continuum hypothesis, Gruenhage [29] has constructed a compact, first countable, non-metrizable space X, whose square is hereditarily ccc (hence hereditarily has countable extent). Combining the first countability and hereditary countable extent of X^2 with Lemma 49, we deduce that X^2 is hereditarily relative calibre (ω_1, ω). $\square$

4.0 P-PARACOMPACTNESS AND P-METRIZABILITY

Let $\mathcal{C}$ be family of subsets of a space X, and let P be a directed set. As defined in the introduction, we say $\mathcal{C}$ is *P-locally finite* (respectively, *P-point finite*) if $\mathcal{C} = \bigcup\{\mathcal{C}_p : p \in P\}$ where each $\mathcal{C}_p$ is locally finite (respectively, point finite) in X and $\mathcal{C}_p \subseteq \mathcal{C}_{p'}$ whenever $p \leq p'$. In other words, $\mathcal{C}$ is P-locally finite if it has a P-ordered cover consisting of locally finite subcollections (and similarly for P-point finite). Note that 1-locally finite and ω-locally finite are equivalent to locally finite and σ-locally finite, respectively.

We say X is *P-paracompact* (respectively, *P-metacompact*) if every open cover of X has a P-locally finite (respectively, P-point finite) open refinement. Also, X is *P-metrizable* if it has a $(P \times \omega)$-locally finite base. Note that 1-paracompactness and 1-metrizability are equivalent to paracompactness and metrizability, respectively. In Section 4.1, we establish some basic lemmas about these properties and other closely related properties. The first main result of this chapter comes in Section 4.2, where we generalize Gruenhage's Theorem 4 by replacing paracompactness with P-paracompactness for P with calibre (ω_1, ω). We then show that this generalization is optimal in the sense that we cannot weaken the hypothesis to P-paracompactness for P in some larger class of directed sets.

In Section 4.3, we take a closer look at the situation where the directed set P has the form $\mathcal{K}(M)$ for some separable metrizable M, since these directed sets have more structure than a general P with calibre (ω_1, ω). In Section 4.3.1 in particular, this additional structure allows us to characterize $\mathcal{K}(M)$-metrizability, and partially characterize $\mathcal{K}(M)$-paracompactness, in terms of properties not referring to any separable metrizable space. Finally, in Section 4.4, we give constructions of some P-metrizable and P-paracompact spaces that provide useful counterexamples in Section 4.5.

Let κ be a cardinal, let $\mathcal{C}$ be a collection of subsets of a space, and let $\mathbf{P}$ be some property.

Throughout this chapter, we will say that $\mathcal{C}$ is κ-**P** if $\mathcal{C} = \bigcup\{\mathcal{C}_\alpha : \alpha < \kappa\}$ where each $\mathcal{C}_\alpha$ has **P**. Following tradition, we say 'σ-**P**' instead of 'ω-**P**'.

4.1 BASIC RESULTS AND RELATED PROPERTIES

Let $\mathcal{C}$ be a collection of subsets of a space X. Since the family $LF(\mathcal{C})$ of all locally finite subcollections of $\mathcal{C}$ is an ideal on $\mathcal{C}$ containing all finite subsets of $\mathcal{C}$, then we may identify $\mathcal{C}$ with the subset $[\mathcal{C}]^1$ of $LF(\mathcal{C})$, so that $(\mathcal{C}, LF(\mathcal{C}))$ is a directed set pair. We can now restate the definition of P-locally finite in terms of Tukey quotients, as follows.

Lemma 88. *Let $\mathcal{C}$ be family of subsets of a space X. Let P be a directed set. Then the following are equivalent:*
(i) $\mathcal{C}$ is P-locally finite,
(ii) there is an order-preserving map $\phi : P \to LF(\mathcal{C})$ whose image is cofinal for $(\mathcal{C}, LF(\mathcal{C}))$,
(iii) $P \geq_T (\mathcal{C}, LF(\mathcal{C}))$, and
(iv) $P \geq_T ([\mathcal{C}]^{<\omega}, LF(\mathcal{C}))$.
Hence if $\mathcal{C}$ is P-locally finite and $Q \geq_T P$ for some directed set Q, then $\mathcal{C}$ is Q-locally finite.

Proof. The equivalence of (i) and (ii) is immediate once the definitions are unpacked (in particular, '$\phi(P)$ is cofinal for $\mathcal{C}$ in $LF(\mathcal{C})$' means '$\bigcup \phi(P) = \mathcal{C}$'). Since $LF(\mathcal{C})$ is Dedekind complete, then (ii) and (iii) are equivalent by Lemma 9.

Since $[\mathcal{C}]^1$ is contained in $[\mathcal{C}]^{<\omega}$, then a Tukey quotient map witnessing (iv) will also witness (iii), so (iv) implies (iii). Now it suffices to show that (ii) implies (iv), so we show that any map ϕ as in (ii) will witness (iv) also. To this end, take any $\{C_1, \ldots, C_n\}$ in $[\mathcal{C}]^{<\omega}$, and pick $p_1, \ldots, p_n$ in P such that $C_i \in \phi(p_i)$ for $i = 1, \ldots, n$. Because P is directed, there is an upper bound, p_0, of $p_1, \ldots, p_n$, and since ϕ is order-preserving, then $\phi(p_0) \supseteq \{C_1, \ldots, C_n\}$, as required.

The final statement in the lemma follows from the equivalence of (i) and (iii) combined with transitivity of Tukey quotients. ☐

Lemma 88 allows us to simplify proofs by replacing a given directed set by anything equivalent or larger in the Tukey order. For example, a collection is $(P \times \omega)$-locally finite if and only if it is $(P \times [\omega]^{<\omega})$-locally finite. Lemma 88 (and the obvious analog for point finiteness) immediately gives:

Corollary 89. *If X is P-paracompact (respectively, P-metrizable, P-metacompact) and $Q \geq_T P$, then X is also Q-paracompact (respectively, Q-metrizable, Q-metacompact).*

Instead of requiring the same directed set P to be used for every open cover of a space X, we can generalize the notions of P-paracompactness and P-metacompactness as follows. If $\mathcal{P}$ is a class of directed sets, then we say X is *$\mathcal{P}$-paracompact* (respectively $\mathcal{P}$-metacompact) if every open cover $\mathcal{U}$ of X has a P-locally finite (respectively P-point finite) open refinement $\mathcal{V}$ for some P in $\mathcal{P}$ (so P depends on $\mathcal{U}$).

Also, we say a space X is *κ-paracompact* if each open cover of X has a κ-locally finite open refinement. Recall that a collection of subsets of X is *κ-locally finite* if it is a union of κ-many locally finite subcollections.

Lemma 90. *Let $\mathcal{C}$ be a collection of subsets of a space X, let P be a directed set, and let $\mathcal{P}$ be a class of directed sets whose members have cofinality at most κ, for some cardinal κ.*

(1) If $\mathcal{C}$ is P-locally finite, then it is $cof(P)$-locally finite.

(2) If X is $\mathcal{P}$-paracompact, then it is κ-paracompact.

(3) If X is P-paracompact, then it is $cof(P)$-paracompact.

Proof. Suppose that $\mathcal{C}$ is P-locally finite, say $\mathcal{C} = \bigcup\{\mathcal{C}_p : p \in P\}$ is P-ordered and each $\mathcal{C}_p$ is locally finite. Let Q be a cofinal subset of P of cardinality $cof(P)$. Then $\mathcal{C} = \bigcup\{\mathcal{C}_q : q \in Q\}$, so $\mathcal{C}$ is indeed the union of $cof(P)$-many locally finite subcollections. This establishes (1). Statement (2) then immediately follows, and statement (3) is just a special case of (2). $\square$

Recall that X is called a *Moore* space if it is T_3 and has a *development*, that is, a countable collection $\{\mathcal{G}_n : n < \omega\}$ of open covers such that: for any x in X and open U containing x, there is an $n < \omega$ such that $St(x, \mathcal{G}_n) = \{G \in \mathcal{G}_n : x \in G\}$ is contained in U. Of course a P-metrizable space is always P-paracompact, and the next lemma says, in particular, that for Moore spaces the two properties are equivalent.

Lemma 91. *Suppose $\mathcal{P}$ is a class of directed sets that is countably directed with respect to the Tukey order, $\geq_T$. If X is a $\mathcal{P}$-paracompact Moore space, then X is P-metrizable for some P in $\mathcal{P}$. In particular:*

(1) if X is a $\mathcal{K}(\mathcal{M})$-paracompact Moore space, then X is $\mathcal{K}(M)$-metrizable for some separable metrizable M, and

(2) if X is a P-paracompact Moore space for some directed set P, then X is P-metrizable.

Proof. Let $\{\mathcal{G}_n : n < \omega\}$ be a development for X, and for each n, find a P_n-locally finite open refinement $\mathcal{U}_n$ of $\mathcal{G}_n$ for some P_n in $\mathcal{P}$. Since $\mathcal{P}$ is countably directed with respect to $\geq_T$, then there is a P in $\mathcal{P}$ such that $P \geq_T P_n$ for every n. Thus, each $\mathcal{U}_n$ is P-locally finite: $\mathcal{U}_n = \bigcup\{\mathcal{U}_{n,p} : p \in P\}$. Define $\mathcal{B}_{n,p} = \bigcup\{\mathcal{U}_{i,p} : i \leq n\}$. Then $\mathcal{B} = \bigcup\{\mathcal{B}_{n,p} : n < \omega,\ p \in P\}$ is a $(P \times \omega)$-locally finite base for X, which completes the main claim.

Now note that (1) follows from Theorem 59 which shows that $\mathcal{K}(\mathcal{M})$ is countably directed, with respect to $\geq_T$. Also, (2) follows by taking $\mathcal{P} = \{P\}$. $\square$

A collection $\mathcal{C}$ of subsets of a space is *relatively locally finite* if it is locally finite in its union, that is, for each point x in $\bigcup\mathcal{C}$, there is an open neighborhood of x meeting only finitely many elements of $\mathcal{C}$. A space is called *κ-relatively paracompact* if every open cover has an open refinement which is κ-relatively locally finite (i.e. is a union of κ-many locally finite subcollections).

A space is *screenable* if every open cover has a σ-disjoint open refinement. Clearly a space with a σ-disjoint base is screenable. Observe that a pairwise disjoint collection of open sets is relatively locally finite, while a relatively locally finite family is point-finite. Hence:

Lemma 92. *Every σ-disjoint family of open sets is σ-relatively locally finite, and every σ-relatively locally finite family of open sets is σ-point finite.*

Once again, we are primarily interested in directed sets with calibre (ω_1, ω). Notice that if the directed set P in Lemma 88 has calibre (ω_1, ω), then by Lemma 17, the collection $\mathcal{C}$ has relative calibre (ω_1, ω) in $LF(\mathcal{C})$. Thus, we make the following definition. A family $\mathcal{C}$ of subsets of a space X is called (ω_1, ω)-*locally finite* (respectively, (ω_1, ω)-*point finite*) if $\mathcal{C}$ has relative calibre (ω_1, ω) in $LF(\mathcal{C})$ (respectively, in $PF(\mathcal{C})$). Note that this is the same as requiring that every uncountable subcollection of $\mathcal{C}$ must contain an infinite subcollection which is locally finite (respectively, point finite). We then call a space (ω_1, ω)-*paracompact* (respectively, (ω_1, ω)-*metacompact*) if every open cover has an (ω_1, ω)-locally finite (respectively, (ω_1, ω)-point finite) open refinement. We also say a space is (ω_1, ω)-*metrizable* if it has an (ω_1, ω)-locally finite base.

Lemma 93. *Let P be a directed set with calibre (ω_1, ω), let $\mathcal{P}$ be a class of directed sets with calibre (ω_1, ω), and let $\mathcal{C}$ be a family of subsets of a space X.*

(1) If $\mathcal{C}$ is P-locally finite (respectively, P-point finite), then it is (ω_1, ω)-locally finite (respectively, (ω_1, ω)-point finite).

(2) Hence if X is P-paracompact (respectively, P-metrizable or P-metacompact), then it is (ω_1, ω)-paracompact (respectively, (ω_1, ω)-metrizable or (ω_1, ω)-metacompact).

(3) If X is $\mathcal{P}$-paracompact (respectively, $\mathcal{P}$-metacompact), then it is (ω_1, ω)-paracompact (respectively, (ω_1, ω)-metacompact).

Proof. We essentially proved (1) at the start of this section. Indeed, if $\mathcal{C}$ is P-locally finite, then $P \geq_T (\mathcal{C}, LF(\mathcal{C}))$ by Lemma 88. Since P has calibre (ω_1, ω), then Lemma 17 implies that $\mathcal{C}$ has relative calibre (ω_1, ω) in $LF(\mathcal{C})$, which means $\mathcal{C}$ is (ω_1, ω)-locally finite. We can similarly prove that P-point finite collections are (ω_1, ω)-point finite, which completes (1). Statements (2) and (3) immediately follow from (1). $\square$

Clearly, (ω_1, ω)-metrizability always implies (ω_1, ω)-paracompactness, and similarly to Lemma 91, we see that these properties are actually equivalent for Moore spaces:

Lemma 94. *If X is an (ω_1, ω)-paracompact Moore space, then X is (ω_1, ω)-metrizable.*

Proof. Let $\{\mathcal{G}_n : n < \omega\}$ be a development of X, and let $\mathcal{U}_n$ be an (ω_1, ω)-locally finite open refinement of $\mathcal{G}_n$. Then $\mathcal{U} = \bigcup_n \mathcal{U}_n$ is a base for X, and it is (ω_1, ω)-locally finite by Lemma 20 (take $P = LF(\mathcal{U})$, $P_n = LF(\mathcal{U}_n)$, and $P_n' = \mathcal{U}_n = [\mathcal{U}_n]^1$). $\square$

Lemma 95. *An (ω_1, ω)-point finite collection of subsets of a space is point countable.*

Proof. Let $\mathcal{C}$ be a collection of subsets of a space X, and suppose $\mathcal{C}$ is not point countable. So there is a point x in X such that $\mathcal{C}_x = \{C \in \mathcal{C} : x \in C\}$ is uncountable. Then every infinite subfamily of $\mathcal{C}_x$ contains x in its intersection, and so is not point-finite. Thus, $\mathcal{C}$ is not (ω_1, ω)-point finite. $\square$

Lemma 96. *Every space with an (ω_1, ω)-point finite base (in particular, every (ω_1, ω)-metrizable space) is first countable.*

Proof. From Lemma 95, we see that a given (ω_1, ω)-point finite base must be point-countable, and so every point has a countable local base. $\square$

A space is called *metaLindelöf* if every open cover has a point countable open refinement. By Lemma 95, we know that every (ω_1, ω)-metacompact space (and so also every (ω_1, ω)-paracompact space) is metaLindelöf, and every (ω_1, ω)-metrizable space has a point countable base. From [37], we know that a countably compact space is (i) compact if it is metaLindelöf and (ii) metrizable if it has a point countable base. The same is not true for pseudocompact spaces. Indeed, there are pseudocompact spaces with a point countable base (hence metaLindelöf) which are not compact (and so not metrizable) [43]. However, a pseudocompact space is (i) compact if σ-metacompact and (ii) metrizable if it has a σ-point finite base [49]. Similarly, we can prove:

Lemma 97. *Let X be a pseudocompact space.*

(1) If X is (ω_1, ω)-paracompact, then X is compact.

(2) If X is (ω_1, ω)-metrizable, then X is metrizable.

Proof. Recall that X is pseudocompact if and only if each locally finite family of open subsets of X is finite. Thus, any open cover $\mathcal{V}$ of X with relative calibre (ω_1, ω) in $LF(\mathcal{V})$ must be countable. Both claims are now immediate. $\square$

In the presence of separability, (ω_1, ω)-paracompactness and (ω_1, ω)-metrizability also reduce to simpler properties. In fact, we have:

Lemma 98. *Let X be a space with a dense σ-compact subset.*

(1) If X is (ω_1, ω)-paracompact, then X is Lindelöf.

(2) If X is (ω_1, ω)-metrizable, then X is (separable) metrizable.

Proof. Note first that if K is a compact subset of X and $\mathcal{U}$ is a locally finite family of subsets of X, then there is an open V containing K which meets only finitely many elements of $\mathcal{U}$. So if $\mathcal{U}$ is (ω_1, ω)-locally finite, then every compact subset of X meets only countably many members of $\mathcal{U}$. It easily follows that if X has a dense σ-compact subset, then an (ω_1, ω)-locally finite open refinement of a given open cover, or an (ω_1, ω)-locally finite base, must be countable. □

4.2 ANOTHER GENERALIZATION OF SCHNEIDER'S THEOREM

Our main goal of this section is to prove Theorem 102 below, which generalizes Gruenhage's Theorem 4. We begin with a few lemmas, the first of which is extracted from the proof of Theorem 4. Here, $A(\kappa)$ denotes the one-point compactification of $D(\kappa)$, the discrete space of size κ. We call $A(\kappa)$ the *supersequence* of size κ.

Lemma 99 (Gruenhage, [28]). *Let X be compact and not metrizable. If $X^2 \setminus \Delta$ has a partition $\mathcal{S}$ whose members are open in $X^2 \setminus \Delta$ and Lindelöf, then X contains a subspace homeomorphic to the supersequence $A(\kappa)$ for some uncountable κ.*

Proof. Enumerate $\mathcal{S} = \{S_\alpha : \alpha < \gamma\}$. Since each S_α is Lindelöf and open in $X^2 \setminus \Delta$, we can write $S_\alpha = \bigcup_n (U_{\alpha,n} \times V_{\alpha,n})$, where $U_{\alpha,n}$ and $V_{\alpha,n}$ are disjoint open sets in X for each $n < \omega$. Define $\mathcal{W} = \{U_{\alpha,n} : \alpha < \gamma,\ n < \omega\} \cup \{V_{\alpha,n} : \alpha < \gamma,\ n < \omega\}$. Then $\mathcal{W}$ is a T_2-separating open cover of X. Since any compact space with a point countable T_1-separating open cover is metrizable [37], and by hypothesis X is not metrizable, $\mathcal{W}$ cannot be point countable. Hence, there is a point $x \in X$ contained in uncountably many members of $\mathcal{W}$.

Without loss of generality, there is an uncountable subset $A \subseteq \gamma$ and an $m < \omega$ such that $x \in \bigcap_{\alpha \in A} U_{\alpha,m}$. Because $U_{\alpha,m} \times V_{\alpha,m} \subseteq S_\alpha$, then $\{U_{\alpha,m} \times V_{\alpha,m} : \alpha \in A\}$ is a discrete collection in $X^2 \setminus \Delta$. It follows that $\{V_{\alpha,m} : \alpha \in A\}$ is a discrete collection in $X \setminus \{x\}$. Thus, if we choose a point $y_\alpha \in V_{\alpha,m}$ for each $\alpha \in A$, then $Y = \{y_\alpha : \alpha \in A\}$ is an uncountable closed discrete subspace of $X \setminus \{x\}$. As X is compact, $\overline{Y}^X = \{x\} \cup Y$ is the one-point compactification of Y, so $\overline{Y}^X$ is a copy of $A(\kappa)$ in X, where $\kappa = |Y|$. □

Lemma 100. *If $\kappa > \omega$, then $A(\kappa)^2 \setminus \Delta$ is not (ω_1, ω)-paracompact.*

Proof. Let $Y = A(\kappa)^2 \setminus \Delta$, and write $A(\kappa) = D(\kappa) \cup \{\infty\}$. Consider the following open cover of Y:

$$\mathcal{U} = \{(\{x\} \times A(\kappa)) \cap Y : x \in D(\kappa)\} \cup \{Y \setminus (A(\kappa) \times \{\infty\})\}.$$

Let $\mathcal{V}$ be any open refinement of $\mathcal{U}$. We will show that $\mathcal{V}$ does not have relative calibre (ω_1, ω) in $LF(\mathcal{V})$.

For each x in $D(\kappa)$, choose $V_x \in \mathcal{V}$ such that $(x, \infty) \in V_x$. As $\mathcal{V}$ refines $\mathcal{U}$, we have $V_x \subseteq \{x\} \times A(\kappa)$. Then there is a finite set $F_x \subseteq D(\kappa)$ such that $V_x = \{x\} \times (A(\kappa) \setminus F_x)$. Suppose $\mathcal{V}$ does have relative calibre (ω_1, ω) in $LF(\mathcal{V})$. Then there is a countably infinite subset $C \subseteq D(\kappa)$ such that $\{V_x : x \in C\}$ is locally finite. Let $E = \bigcup\{F_x : x \in C\}$, which is countable, and choose any $y \in D(\kappa) \setminus E$. Then $(x, y) \in V_x$ for each $x \in C$, so each neighborhood of (∞, y) intersects all but finitely many members of the infinite locally finite family $\{V_x : x \in C\}$, which is a contradiction. $\square$

Lemma 101. *Let X be a space and x and point in X. If $X \setminus \{x\}$ is (ω_1, ω)-paracompact or P-paracompact, then X has the same property.*

Proof. We prove the (ω_1, ω)-paracompact statement; the proof for P-paracompact is similar. Let $\mathcal{U}$ be any open cover of X. Then there is an (ω_1, ω)-locally finite open refinement $\mathcal{V}'$ of the open cover $\mathcal{U}' = \{U \setminus \{x\} : U \in \mathcal{U}\}$ of $X \setminus \{x\}$. Now choose any $U_x \in \mathcal{U}$ containing x. Then $\mathcal{V} = \mathcal{V}' \cup \{U_x\}$ is also (ω_1, ω)-locally finite and is an open refinement of $\mathcal{U}$. $\square$

Now we generalize Gruenhage's Theorem 4 to P-paracompactness.

Theorem 102. *The following are equivalent for a pseudocompact space X:*

(i) $X^2 \setminus \Delta$ is P-paracompact for some directed set P with calibre (ω_1, ω),

(ii) $X^2 \setminus \Delta$ is $\mathcal{P}$-paracompact, where $\mathcal{P}$ is a class of directed sets with calibre (ω_1, ω),

(iii) $X^2 \setminus \Delta$ is (ω_1, ω)-paracompact, and

(iv) X is metrizable.

Proof. We know (i) implies (ii) trivially, and Lemma 93 shows that (ii) implies (iii). We also know (iv) implies (i) since every metrizable space is paracompact, i.e. 1-paracompact, so we just need to show that (iii) implies (iv). First, pick any point $x \in X$ and note that, since $X \setminus \{x\}$ is homeomorphic to a closed subspace of $X^2 \setminus \Delta$, then $X \setminus \{x\}$ is (ω_1, ω)-paracompact, and so X is compact by Lemmas 101 and 97.

Now, $X^2 \setminus \Delta$ is locally compact and (ω_1, ω)-paracompact, so we can find an open cover $\mathcal{U}$ of $X^2 \setminus \Delta$ which is (ω_1, ω)-locally finite and such that $\overline{U}^{X^2 \setminus \Delta}$ is compact for each $U \in \mathcal{U}$. For each $n < \omega$, define a relation $\sim_n$ on $\mathcal{U}$ by $U \sim_n V$ if and only if there are $U_0, U_1, \ldots, U_n \in \mathcal{U}$ such that $U_0 = U$, $U_n = V$, and $U_i \cap U_{i-1} \neq \emptyset$ for all $0 < i < n$. Then define an equivalence relation $\sim$ on $\mathcal{U}$ by $U \sim V$ if and only if $U \sim_n V$ for some $n < \omega$.

Now let $\{[U_\alpha] : \alpha < \kappa\}$ be a one-to-one enumeration of the $\sim$-equivalence classes, and let $S_\alpha = \bigcup [U_\alpha]$. Then $\{S_\alpha : \alpha < \kappa\}$ is a partition of $X^2 \setminus \Delta$ consisting of open sets. Each S_α is thus also closed in $X^2 \setminus \Delta$, and it follows that $S_\alpha = \bigcup \{\overline{U}^{X^2 \setminus \Delta} : U \in [U_\alpha]\}$. Hence, we can show that each S_α is σ-compact by verifying that $[U_\alpha]$ is countable.

For each $U \in \mathcal{U}$ and $n < \omega$, let $[U]_n = \{V \in \mathcal{U} : U \sim_n V\}$. Clearly $[U]_0 = \{U\}$ is countable for each U. Now suppose that $[U]_1$ is uncountable for some U in $\mathcal{U}$. Since $\mathcal{U}$ has relative calibre (ω_1, ω) in $LF(\mathcal{U})$, then there is an infinite subset $\mathcal{V} \subseteq [U]_1$ which is locally finite. But since $\overline{U}^{X^2 \setminus \Delta}$ is compact and $\mathcal{V}$ is locally finite, then there should be only finitely many members of $\mathcal{V}$ intersecting U, which is a contradiction since every member of $[U]_1$ intersects U. Thus, each $[U]_1$ is countable, and since $[U]_{n+1} = \bigcup \{[V]_1 : V \in [U]_n\}$, then by induction, we see that each $[U]_n$ is countable. Hence, $[U] = \bigcup \{[U]_n : n < \omega\}$ is also countable.

Suppose X is not metrizable. Since each S_α is σ-compact, then by Lemma 99, we can find a subspace Y of X and an uncountable cardinal λ such that Y is homeomorphic to $A(\lambda)$. Since Y is compact, then $Y^2 \setminus \Delta$ is a closed subspace of $X^2 \setminus \Delta$, so $Y^2 \setminus \Delta$ is also (ω_1, ω)-paracompact. But this is a contradiction, according to Lemma 100. $\square$

Our final task in this section is to prove Theorem 104 below, which says that, in Theorem 102, we cannot replace the class of all directed sets with calibre (ω_1, ω) with a larger class of directed sets. Thus, Theorem 102 is optimal, in the same sense as Theorem 63.

Lemma 103. *Let P be a directed set which does not have calibre (ω_1, ω). Let X be any space with weight at most ω_1. Then X has a P-ordered base $\mathcal{B} = \bigcup \{\mathcal{B}_p : p \in P\}$ where each $\mathcal{B}_p$ is* *finite, and hence X is P-metrizable.*

Proof. Fix a base $\mathcal{B}$ for X of size at most ω_1. Since P does not have calibre (ω_1, ω), there is a subset S of P with size ω_1 such that no infinite subset of S has an upper bound. Fix a surjection $f : S \to \mathcal{B}$. For each p in P, define $\mathcal{B}_p = \{f(s) : s \in S,\ s \leq p\}$. Clearly if $p_1 \leq p_2$, then $\mathcal{B}_{p_1} \subseteq \mathcal{B}_{p_2}$, and since no infinite subset of S has an upper bound, then each $\mathcal{B}_p$ is finite. Lastly, $\mathcal{B} = \bigcup\{\mathcal{B}_p : p \in P\}$ since f is a surjection and $S \subseteq P$. □

Theorem 104. *The following are equivalent for any directed set P:*

(i) P has calibre (ω_1, ω), and

(ii) For any compact X such that $X^2 \setminus \Delta$ is P-paracompact, X must be metrizable.

Proof. Theorem 102 shows that (i) implies (ii), and we now prove that the negation of (i) implies the negation of (ii). Suppose P does not have calibre (ω_1, ω). Take any compact space X which has weight precisely ω_1 (for example, $X = A(\omega_1)$). By Lemma 103, $X^2 \setminus \Delta$ has a P-(locally) finite base, and so is P-paracompact, but X is not metrizable since it has uncountable weight. □

4.3 $\mathcal{K}(M)$-PARACOMPACTNESS AND $\mathcal{K}(M)$-METRIZABILITY

For any separable metrizable M, the pre-ideal $\mathcal{K}(M)$ is a second countable topological directed set with CSB, so it has more structure than general directed sets with calibre (ω_1, ω). Therefore, we devote this section to investigating P-paracompactness and P-metrizability in the case where $P = \mathcal{K}(M)$[1]. The results of this section are joint with Ziqin Feng.

In Section 4.3.1, we completely characterize $\mathcal{K}(M)$-metrizability, and partially characterize $\mathcal{K}(M)$-paracompactness, in terms of properties not referring to any separable metrizable space. In Section 4.3.2, we investigate the relationship of these $\mathcal{K}(M)$-ordered properties with normality, countable paracompactness, and the countable chain condition. Finally, in Section 4.3.3, we give a method for constructing $\mathcal{K}(M)$-metrizable spaces for every separable metrizable space M.

[1]In fact, many of the results of this section apply not only to directed sets of the form $\mathcal{K}(M)$ for separable

4.3.1 CHARACTERIZATIONS

Here we aim to give characterizations of when a space is $\mathcal{K}(M)$-paracompact or $\mathcal{K}(M)$-metrizable, for some separable metrizable M in terms of properties not referring to a separable metrizable space. This is completely successful for $\mathcal{K}(M)$-metrizability (see Theorem 108) but only partially so for $\mathcal{K}(M)$-paracompactness (see Theorem 109). Our first goal will be to give an alternate description of $\mathcal{K}(M)$-local finiteness in first countable spaces (see Proposition 106).

Lemma 105. *Suppose that X is first countable and $\mathcal{V} = \{\mathcal{V}_K : K \in \mathcal{K}(M)\}$ is a $\mathcal{K}(M)$-locally finite family of subsets of X for some separable metrizable space M. Then for any x in X and K in $\mathcal{K}(M)$, there is an open neighborhood T of K in $\mathcal{K}(X)$ such that $\mathcal{V}_T = \bigcup\{\mathcal{V}_L : L \in T\}$ is locally finite at x.*

Proof. Suppose, instead, that we can find x and K such that for any neighborhood T of K, the set $\mathcal{V}_T$ is not locally finite at x. Then let $\{B_m : m < \omega\}$ be a decreasing local base at x and $\{T_n : n < \omega\}$ be a decreasing local base at K. So for each $m, n < \omega$, the set $\{V \in \mathcal{V}_{T_n} : B_m \cap V \neq \emptyset\}$ is infinite. We can then inductively find, for any $m \leq n < \omega$, a set V_n^m in $\mathcal{V}_{T_n} \setminus \{V_m^m, \ldots, V_{n-1}^m\}$ such that $B_m \cap V_n^m \neq \emptyset$. By definition of $\mathcal{V}_{T_n}$, there is a $K_n^m \in T_n$ such that $V_n^m \in \mathcal{V}_{K_n^m}$.

Since $\{T_n : n < \omega\}$ is a decreasing local base at K, the set $\mathcal{K} = \{K_n^m : m \leq n < \omega\} \cup \{K\}$ is a compact subset of $\mathcal{K}(M)$. Hence, $\hat{K} = \bigcup \mathcal{K}$ is a compact subset of M such that $K_n^m \subseteq \hat{K}$ whenever $m \leq n$, and so we have $\{V_n^m : m \leq n < \omega\} \subseteq \mathcal{V}_{\hat{K}}$. But since B_m intersects V_n^m whenever $n \geq m$, then $\mathcal{V}_{\hat{K}}$ is not locally finite at x, which is a contradiction. $\square$

By analogy with the property 'weakly σ-point finite' used in [24], we call a family $\mathcal{C}$ of subsets of a space X *weakly σ-locally finite* if we can write $\mathcal{C} = \bigcup_n \mathcal{C}_n$ in such a way that:

$$\forall x \in X, \ \bigcup\{\mathcal{C}_n : \mathcal{C}_n \text{ is locally finite at } x\} = \mathcal{C}. \tag{4.1}$$

A space is *weakly σ-paracompact* if every open cover has a weakly σ-locally finite open refinement.

metrizable M, but to any second countable topological directed set which is CSB.

Proposition 106. *Let $\mathcal{V}$ be a collection of subsets of a first countable space X. Then the following are equivalent:*

(i) $\mathcal{V}$ is weakly σ-locally finite, and

(ii) $\mathcal{V}$ is $\mathcal{K}(M)$-locally finite for some separable metrizable M.

Moreover, (i) implies (ii) even if X is not first countable.

Proof. Assume (i) and write $\mathcal{V} = \bigcup_n \mathcal{V}_n$ where (4.1) is satisfied for each $x \in X$. Now, for each $V \in \mathcal{V}$, define $\sigma_V \in \{0,1\}^\omega$ by $\sigma_V(n) = 1$ if $V \in \mathcal{V}_n$ and $\sigma_V(n) = 0$ if $V \notin \mathcal{V}_n$. Then let M be the subspace $\{\sigma_V : V \in \mathcal{V}\}$ of the Cantor space $\{0,1\}^\omega$. For any compact subset K of M, let $\mathcal{V}_K = \{V \in \mathcal{V} : \sigma_V \in K\}$. Then $\mathcal{V} = \bigcup\{\mathcal{V}_K : K \in \mathcal{K}(M)\}$ is $\mathcal{K}(M)$-ordered.

Now we check that each $\mathcal{V}_K$ is locally finite in X. Fix K in $\mathcal{K}(M)$ and define $U_n = \{\sigma \in M : \sigma(n) = 1\} = \{\sigma_V : V \in \mathcal{V}_n\}$, which is open in M for each n. We claim that for any x in X, $\{U_n : \mathcal{V}_n$ is locally finite at $x\}$ covers K. Indeed, if σ_V is in K, then (4.1) implies that there is an m such that V is in $\mathcal{V}_m$ (so σ_V is in U_m) and $\mathcal{V}_m$ is locally finite at x. Thus, for each x in X, there is a finite subset F_x of ω such that $\{U_n : n \in F_x\}$ covers K and $\mathcal{V}_n$ is locally finite at x for each $n \in F_x$. Tracing the definitions shows that $\mathcal{V}_K$ is contained in $\bigcup\{\mathcal{V}_n : n \in F_x\}$ for each x, and is therefore locally finite at each x. Thus, (i) implies (ii).

For the converse, assume $\mathcal{V} = \bigcup\{\mathcal{V}_K : K \in \mathcal{K}(M)\}$ is $\mathcal{K}(M)$-locally finite. Fix a countable base $\mathcal{B}$ for $\mathcal{K}(M)$ and define $\mathcal{V}_B = \bigcup\{\mathcal{V}_K : K \in B\}$ for each $B \in \mathcal{B}$. Then for each $x \in X$, Lemma 105 guarantees that $\bigcup\{\mathcal{V}_B : \mathcal{V}_B$ is locally finite at $x\} = \mathcal{V}$. Hence, $\mathcal{V}$ is weakly σ-locally finite since $\mathcal{B}$ is countable. □

Lemma 107. *Let $\mathcal{V}$ be a weakly σ-locally finite family of subsets of a space X, and write $\mathcal{V} = \bigcup_n \mathcal{V}_n$ satisfying (4.1) of the definition. Define $X_n = \{x \in X : \mathcal{V}_n$ is locally finite at $x\}$ and $\mathcal{W}_n = \{V \cap X_n : V \in \mathcal{V}_n\}$ for each $n < \omega$, and let $\mathcal{W} = \bigcup_n \mathcal{W}_n$.*

Then each $\mathcal{W}_n$ is relatively locally finite, so $\mathcal{W}$ is σ-relatively locally finite. If $\mathcal{V}$ is an open cover, then $\mathcal{W}$ is an open refinement of $\mathcal{V}$, and if $\mathcal{V}$ is a base for X, then so is $\mathcal{W}$.

Proof. Since $\bigcup \mathcal{W}_n$ is contained in X_n, and $\mathcal{V}_n$ is locally finite on X_n, then it follows that $\mathcal{W}_n$ is locally finite in its union $\bigcup \mathcal{W}_n$, which proves the first claim. Note that each X_n is open and $\mathcal{W}$ refines $\mathcal{V}$, so to prove the final two claims, it suffices to check that whenever $x \in V \in \mathcal{V}$, then there is a $W \in \mathcal{W}$ such that $x \in W \subseteq V$. Indeed, if $x \in V \in \mathcal{V}$, then by property (4.1), there is an n such that $V \in \mathcal{V}_n$ and $\mathcal{V}_n$ is locally finite at x. Thus, $W = V \cap X_n$ is in $\mathcal{W}$ and $x \in W \subseteq V$. □

Theorem 108. *Let X be a space.*

(1) If X is $\mathcal{K}(M)$-metrizable for some separable metrizable M, then X is P-metrizable where P has calibre (ω_1, ω).

(2) If X is P-metrizable where P has calibre (ω_1, ω), then X is (ω_1, ω)-metrizable.

(3) X is $\mathcal{K}(M)$-metrizable for some separable metrizable M if and only if X has a weakly σ-locally finite base.

(4) If X is $\mathcal{K}(M)$-metrizable for some separable metrizable M, then X has a σ-relatively locally finite base.

(5) If X has a σ-disjoint base, then it has a σ-relatively locally finite base. If X has a σ-relatively locally finite base, then X has a σ-point finite base.

Proof. First of all, claims (1) and (2) follow immediately from Lemmas 55 and 93, respectively.

Proposition 106 shows that any weakly σ-locally finite base is $\mathcal{K}(M)$-locally finite for some separable metrizable space M, which gives one direction of (3). Conversely, if X is $\mathcal{K}(M)$-metrizable for some separable metrizable space M, then X is first countable by (1), (2), and Lemma 96. So by Proposition 106, we can see that X has a weakly σ-locally finite base, which completes the proof of (3).

Statement (4) follows from (3) and Lemma 107, and finally, Lemma 92 shows that (5) is true. □

We summarize these results in Figure 4. Arrows indicate implications, while examples next to an arrow demonstrate that the converse fails. The referenced examples appear in Section 4.5.

There is a clear logical difference between saying that a space is 'metrizable' and saying that it is 'paracompact'. Metrizability asserts the existence of a certain object (σ-locally finite base), while paracompactness says that for every object of one type (open cover) there is a certain object of another type (locally finite open refinement). This logical difference means that there is a unique '$\mathcal{K}(M)$-variant' of metrizability ($\mathcal{K}(M)$-metrizable, for some M) but

$$
\begin{tikzcd}
\exists M : \mathcal{K}(M)\text{-metrizable} \arrow[r, phantom, "\leftrightarrows"] \arrow[d, "\text{Ex. 8, 7}"'] & \text{weakly } \sigma\text{-locally finite base} \arrow[d, "\text{Ex. 4}"'] & \sigma\text{-disjoint base} \arrow[dl, "\text{Q. 1}"] \\
\exists P \text{ cal. } (\omega_1,\omega) : P\text{-metrizable} \arrow[d, "\text{Ex. 10}"'] & \sigma\text{-relatively locally finite base} \arrow[dr, "\text{Ex. 5, 6}"] & \\
(\omega_1,\omega)\text{-metrizable} & & \sigma\text{-point finite base}
\end{tikzcd}
$$

Figure 4: P-metrizability and related properties (summary of Theorem 108)

two '$\mathcal{K}(M)$-variants' of paracompactness, depending on whether the M used to organize an open refinement is chosen in advance ($\mathcal{K}(M)$-paracompact) or with the refinement ($\mathcal{K}(\mathcal{M})$-paracompact). This results in a more complex range of implications and examples.

Theorem 109. *Let X be a space.*

(1) If X is $\mathcal{K}(M)$-paracompact for some separable metrizable M, then X is also $\mathcal{K}(\mathcal{M})$-paracompact.

(2) If X is $\mathcal{K}(\mathcal{M})$-paracompact, then X is P-paracompact for some directed set P with calibre (ω_1, ω).

(3) If X is P-paracompact for some directed set P with calibre (ω_1, ω), then X is (ω_1, ω)-paracompact.

(4) If X is weakly σ-paracompact, then it is $\mathcal{K}(\mathcal{M})$-paracompact and σ-relatively paracompact.

(5) If X is first countable and $\mathcal{K}(\mathcal{M})$-paracompact, then X is weakly σ-paracompact (and hence, σ-relatively paracompact).

(6) If X is screenable, then it is σ-relatively paracompact. If X is σ-relatively paracompact, then X is σ-metacompact.

Proof. Claim (1) is immediate from the definitions. Claim (3) follows from Lemma 93.

For (2), suppose X is $\mathcal{K}(\mathcal{M})$-paracompact. Let $P = \Sigma\{\mathcal{K}(M) : M \subseteq I^\omega\}$. Then P has calibre (ω_1, ω) by Theorem 58. Take any open cover $\mathcal{U}$ of X. By hypothesis there is a separable metrizable M such that $\mathcal{U}$ has a $\mathcal{K}(M)$-locally finite open refinement $\mathcal{V}$. Without loss of generality, we can suppose M is a subspace of the Hilbert cube I^ω. Then, the projection from P onto $\mathcal{K}(M)$ witnesses that $P \geq_T \mathcal{K}(M)$, so $\mathcal{V}$ is P-locally finite.

For (4), suppose $\mathcal{W}$ is a weakly σ-locally finite open cover of X. By Proposition 106, $\mathcal{W}$ is also $\mathcal{K}(M)$-locally finite for some separable metrizable M. By Lemma 107, $\mathcal{W}$ has a σ-relatively locally finite open refinement $\mathcal{V}$.

Similarly, (5) follows immediately from Proposition 106 and Lemma 107, and finally, Lemma 92 shows that (6) is true. $\square$

Again, we summarize these results in Figure 5. Arrows indicate implications, while examples on their own next to an arrow demonstrate that the converse fails. Arrows with '+'-properties indicate that an additional assumption was used, and the adjacent example demonstrates its necessity. The referenced examples appear in Section 4.5.

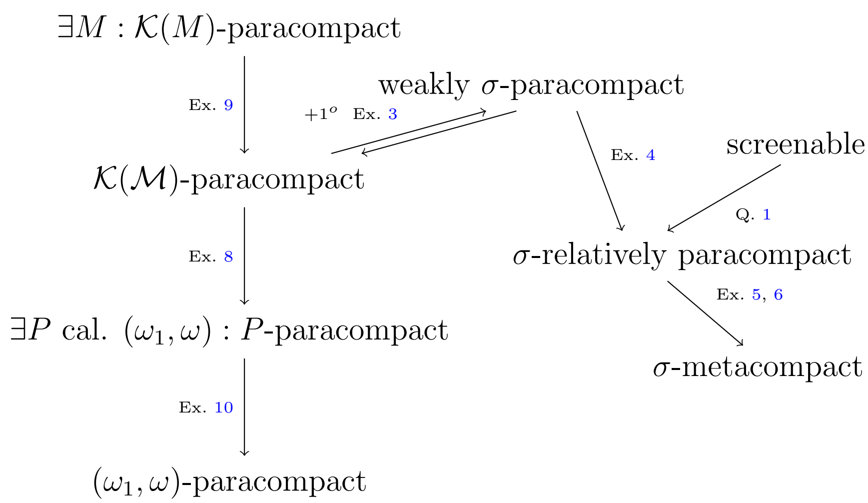

Figure 5: P-paracompactness and related properties (summary of Theorem 109)

4.3.2 WITH NORMAL, COUNTABLY PARACOMPACT, AND CCC

Recall that a space X is *countably paracompact* if and only if every increasing countable open cover $\{U_n : n < \omega\}$ of X is *shrinkable*, that is, it has an open refinement $\{V_n : n < \omega\}$

covering X such that $\overline{V_n} \subseteq U_n$; and X is normal and countably paracompact if and only if every countable (not necessarily increasing) open cover of X is shrinkable.

By Lemma 92, every screenable space is σ-relatively paracompact. Question 1 at the end of Section 4.5 asks if these two properties are distinct. If they are, then the next result generalizes Nagami's theorem [38] that a space is paracompact if and only if it is screenable, normal, and countably paracompact.

Theorem 110. *A space X is paracompact if and only if it is σ-relatively paracompact, normal, and countably paracompact.*

Proof. Of course paracompactness implies the other properties, so we only prove the 'if' portion. Let $\mathcal{U}$ be any open cover of X. Since X is σ-relatively paracompact, there is an open refinement $\mathcal{W}$ of $\mathcal{U}$ covering X which has the form $\mathcal{W} = \bigcup\{\mathcal{W}_n : n < \omega\}$ where each $\mathcal{W}_n$ is locally finite in $X_n = \bigcup \mathcal{W}_n$.

Since X is normal and countably paracompact, then we may shrink the open cover $\{X_n : n < \omega\}$ to get an open cover $\mathcal{Y} = \{Y_n : n < \omega\}$ such that $\overline{Y_n} \subseteq X_n$ for each n. Let $\mathcal{T}_n = \{W \cap Y_n : W \in \mathcal{W}_n\}$. Then $\mathcal{T}_n$ is locally finite in X and covers Y_n since $\mathcal{W}_n$ covers X_n. Now since $\mathcal{Y}$ covers X, then $\bigcup_n \mathcal{T}_n$ covers X, and it is also a σ-locally finite open refinement of $\mathcal{U}$. Hence, X is paracompact. □

Corollary 111. *Let X be a first countable space. Then X is paracompact if and only if it is $\mathcal{K}(\mathcal{M})$-paracompact, normal, and countably paracompact.*

Proof. Of course paracompactness implies the other properties. For the other direction, it suffices to recall that every first countable $\mathcal{K}(\mathcal{M})$-paracompact space is σ-relatively paracompact (Theorem 109), so we may apply Theorem 110. □

Since $\mathcal{K}(M)$-metrizable spaces are first countable (Lemma 96), we deduce:

Theorem 112. *Every $\mathcal{K}(M)$-metrizable space which is normal and countably paracompact is paracompact.*

Then it follows from Lemma 91 that:

Theorem 113. *Every $\mathcal{K}(\mathcal{M})$-paracompact, normal Moore space is metrizable.*

We may ask if normality can be dropped in Theorem 113 (and replaced with countable paracompactness), but there is a consistent counterexample, which we give next. A space Y is called a Δ-*space* if whenever we write Y as an increasing union of subsets, $Y = \bigcup_n S_n$ where $S_n \subseteq S_{n+1}$ for all n, there is a countable closed cover $\{C_n : n < \omega\}$ of Y such that $C_n \subseteq S_n$ for every n. A space Y is called a Q-*space* if every subset of Y is G_δ. Note that every Q-space is a Δ-space. A subset A of $\mathbb{R}$ is a called a Q-*set* if it is an uncountable Q-space, and A is called a Δ-*set* if it is an uncountable Δ-space.

Example 2 (Consistently)**.** *There is a $\mathcal{K}(M)$-metrizable, countably paracompact Moore space which is not normal.*

Proof. Knight [33] has shown it is consistent that there is a Δ-set A which is not a Q-set. Fix a subset A_1 of A which is not a G_δ, and let $A_2 = A \setminus A_1$. Let X denote the disjoint split X-machine $D(A; A_1, A_2)$ (see Section 4.4.3 below). Then X is a Moore space, $\mathcal{K}(M)$-metrizable for $M = A_1 \times A_2$ (see Lemmas 128 and 130), and countably paracompact (Lemma 131).

As A_2 is not an F_σ subset of X, then Lemma 130 also tells us that X is not metrizable and so not normal (by Theorem 113). $\square$

For any directed set P with calibre (ω_1, ω), we know (Lemma 98) that 'separable plus P-paracompact implies Lindelöf' and 'separable plus P-metrizable implies metrizable'. It is natural to ask when 'separable' can be relaxed to 'ccc' (*countable chain condition*: every pairwise disjoint family of open sets is countable).

The next lemma is well-known.

Lemma 114. *Every locally finite open cover $\mathcal{W}$ of a ccc space Y contains a countable subcollection whose closures cover Y.*

Now we can give a positive answer to our question in the case when P is a $\mathcal{K}(M)$.

Theorem 115. *Let X be a ccc space.*
(1) If X is first countable and $\mathcal{K}(\mathcal{M})$-paracompact, then X is Lindelöf.
(2) If X is $\mathcal{K}(M)$-metrizable, then X is metrizable.

Proof. We prove (1) first. So suppose X is first countable and $\mathcal{K}(\mathcal{M})$-paracompact. Then X is σ-relatively paracompact. Take any open cover $\mathcal{U}$. It has an open refinement $\mathcal{V} = \bigcup_n \mathcal{V}_n$

where each $\mathcal{V}_n$ is relatively locally finite, and (using regularity) we can additionally assume that the closure of each V in $\mathcal{V}$ is contained in some member of $\mathcal{U}$.

Fix n. Apply the preceding lemma to the ccc space $Y_n = \bigcup \mathcal{V}_n$ and the locally finite cover $\mathcal{V}_n$ to get a countable subcollection of $\mathcal{V}_n$ whose closures cover Y_n. Recalling that the closure of each V in $\mathcal{V}$ is contained in some member of $\mathcal{U}$, we obtain a countable subcollection of $\mathcal{U}$ covering $\bigcup \mathcal{V}_n$. Taking the union over all n of these countable subcollections yields a countable subcover of $\mathcal{U}$.

Now we establish (2). Suppose X is $\mathcal{K}(M)$-metrizable. Then every subspace is $\mathcal{K}(M)$-paracompact. In particular, every open subspace is $\mathcal{K}(M)$-paracompact and ccc, and hence Lindelöf. Thus X is hereditarily Lindelöf, so hereditarily ccc, and hence (see [25] for example) any point-finite family of open sets in X is countable. But as X is $\mathcal{K}(M)$-metrizable, it has a σ-point finite base, which we now see must be countable. Thus X is indeed (separable and) metrizable. $\square$

4.3.3 DIVERSITY OF $\mathcal{K}(M)$-METRIZABLE SPACES

In Theorem 118 below, we show that there exist $\mathcal{K}(M)$-metrizable spaces for every separable metrizable M. Further, there is a maximal 'antichain' of separable metrizable spaces with corresponding topological spaces which are $\mathcal{K}(M)$-metrizable for one, and only one, member M of the antichain. Our $\mathcal{K}(M)$-metrizable spaces in Theorem 118 will be formed from a construction which generalizes that of the Michael Line, as follows.

Let Y be a metrizable space and A any subset. Then define $M(Y, A)$ to be the space whose underlying set is Y, and whose topology refines that of Y by declaring the points in A to be isolated. When $Y = \mathbb{R}$ and A is the set of irrationals, this gives the usual Michael line. Recall that $CL(Y)$ denotes the pre-ideal of closed subsets of Y (with respect to the original topology on Y), and $\mathbb{P}(A)$ denotes the power set of A. Hence $CL(Y) \cap \mathbb{P}(A)$ is the pre-ideal of subsets of A that are closed in Y. Since this pre-ideal contains each finite subset of A, we may identify A with the subset $[A]^1$ of $CL(Y) \cap \mathbb{P}(A)$.

Lemma 116. *Let A be a subspace of a metrizable space Y, and let P be a directed set. Then $M(Y, A)$ is P-metrizable if and only if $P \times \omega \geq_T (A, CL(Y) \cap \mathbb{P}(A))$.*

Proof. Let $Q = P \times \omega$. Suppose, first, that $M(Y, A)$ is P-metrizable. Then $M(Y, A)$ has a Q-ordered base $\mathcal{B} = \bigcup\{\mathcal{B}_q : q \in Q\}$ where every $\mathcal{B}_q$ is locally finite. For each q, let $\mathcal{B}_q^A = \mathcal{B}_q \cap \{\{a\} : a \in A\}$, and $B_q = \bigcup \mathcal{B}_q^A$. Since all points of A are isolated, the $\mathcal{B}_q^A$ form a Q-ordered clopen cover of A by families locally finite in $M(Y, A)$. Hence the B_q form a Q-ordered cover of A by sets closed in $M(Y, A)$. By definition of the topology on $M(Y, A)$, the closure in Y (with its original topology) of a B_q, call it C_q, is contained in A. Hence, the family $\{C_q : q \in Q\}$ witness that $Q \geq_T (A, CL(Y) \cap \mathbb{P}(A))$.

Now suppose $\{C_q : q \in Q\}$ is a $(P \times \omega)$-ordered cover of A by subsets of A which are closed in Y. Let $\mathcal{B}' = \bigcup_n \mathcal{B}'_n$ be a base for Y (with its original, metrizable topology) such that $\mathcal{B}'_n \subseteq \mathcal{B}'_m$ when $n \leq m$ and each $\mathcal{B}'_n$ is locally finite. Define $\mathcal{B} = \bigcup\{\mathcal{B}_{q,n} : q \in Q, n < \omega\}$ where $\mathcal{B}_{q,n} = \mathcal{B}'_n \cup \{\{a\} : a \in C_q\}$. Then each $\mathcal{B}_{q,n}$ is locally finite in $M(Y, A)$ since C_q is closed in Y, and so $\mathcal{B}$ is a $(Q \times \omega)$-locally finite base for $M(Y, A)$. Hence, $M(Y, A)$ is P-metrizable since $Q \times \omega = P \times \omega \times \omega =_T P \times \omega$. $\square$

Lemma 117. *Let A be a subspace of a compact metrizable space Y. Then $M(Y, A)$ is $\mathcal{K}(A)$-metrizable, and if $M(Y, A)$ is P-metrizable for some directed set P, then $P \times \omega \geq_T (A, \mathcal{K}(A))$.*

Proof. Since Y is compact, then $CL(Y) \cap \mathbb{P}(A) = \mathcal{K}(A)$, and note that $\mathcal{K}(A) \times \omega \geq_T \mathcal{K}(A) \geq_T (A, \mathcal{K}(A))$. Both claims now follow immediately from Lemma 116. $\square$

Theorem 118. *For each separable metrizable space A, there is a hereditarily paracompact $\mathcal{K}(A)$-metrizable space M_A such that: if A' is any non-compact separable metrizable space and M_A is $\mathcal{K}(A')$-metrizable, then $\mathcal{K}(A') \geq_T (A, \mathcal{K}(A))$.*

Hence there is a $2^{\mathfrak{c}}$-sized family $\mathcal{A}$ of separable metrizable spaces such that:

(1) If A is in $\mathcal{A}$ then M_A is $\mathcal{K}(A)$-metrizable, but

(2) If A' is another member of $\mathcal{A}$, then M_A is not $\mathcal{K}(A')$-metrizable.

Proof. Fix a separable metrizable space A. Without loss of generality, we suppose A is a subspace of I^ω, the Hilbert cube. Let $M_A = M(I^\omega, A)$ (see Section 4.3.3). By Lemma 116, M_A is $\mathcal{K}(A)$-metrizable. Let A' be any non-compact separable metrizable space and suppose M_A is $\mathcal{K}(A')$-metrizable. By Lemma 117, we know $\mathcal{K}(A') \times \omega \geq_T (A, \mathcal{K}(A))$, and since A' is not compact, then $\mathcal{K}(A') \times \omega =_T \mathcal{K}(A')$ by Lemma 61, so we have $\mathcal{K}(A') \geq_T (A, \mathcal{K}(A))$.

Now take $\mathcal{A}$ to be the $2^{\mathfrak{c}}$-sized 'antichain' of Theorem 60. No member of $\mathcal{A}$ is compact. Statements (1) and (2) now immediately follow. □

4.4 USEFUL CONSTRUCTIONS

In this section, we describe constructions for generating topological spaces with certain properties that will be useful in the examples of Section 4.5.

4.4.1 THE X-MACHINE

For any space Y, let $X(Y)$ have the underlying set $(Y^2 \setminus \Delta) \cup Y$. Isolate all points of $Y^2 \setminus \Delta$. A basic open neighborhood of a point y in $Y \subseteq X(Y)$ is $\{y\} \cup (U \times \{y\}) \cup (\{y\} \times U)$ for any open neighborhood U of y in Y. Note that $X(\mathbb{R})$ is (homeomorphic) to $\mathbb{R} \times \mathbb{R}$ with points away from the x-axis isolated, and points on the x-axis have neighborhoods in the shape of an 'X' — and the closed upper half plane of this latter space is Heath's V-space. In other words, $X(\mathbb{R})$ is a symmetric version of Heath's V-space.

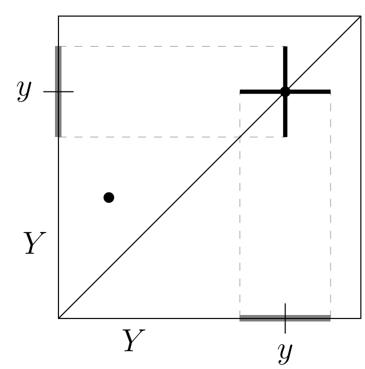

Figure 6: The X-space produced from a space Y

Lemma 119. *Let Y be any Hausdorff space. Then we have:*

(1) $X(Y)$ is zero-dimensional and Hausdorff and hence Tychonoff.

(2) $X(Y)$ is a Moore space if and only if Y is first countable.

(3) If Y is first countable and $X(Y)$ is P-paracompact for some directed set P, then $X(Y)$ is P-metrizable.

(4) If Y is first countable and $X(Y)$ is (ω_1, ω)-paracompact, then $X(Y)$ is (ω_1, ω)-metrizable.

(5) If Y is first countable, then $X(Y)$ has a σ-point finite base.

Proof. For (1), it is straightforward to check that the base for $X(Y)$ provided above consists of clopen sets, so $X(Y)$ is zero-dimensional, and it is easy to see $X(Y)$ is Hausdorff. Statement (3) follows from (2) and Lemma 91, while (4) follows from (2) and Lemma 94, so now we prove (2).

If Y is first countable, then fix a countable neighborhood base $\{U_n(y) : n < \omega\}$ at each y in Y. Then define $\mathcal{G}_n = \{\{y\} \cup (U_n(y) \times \{y\}) \cup (\{y\} \times U_n(y)) : y \in Y\}$ for each $n < \omega$ and $\mathcal{G}_\omega = \{\{x\} : x \in X(Y) \setminus Y\}$. Then $\{\mathcal{G}_n : n \leq \omega\}$ is a development for $X(Y)$. Conversely, if $X(Y)$ is a Moore space, then it is first countable, and by definition of the basic neighborhoods of each $y \in Y \subseteq X(Y)$, we see that Y is first countable at y. So we have proven (2).

Note that when Y is first countable, then each family $\mathcal{G}_n$ above is also point finite in $X(Y)$. Indeed, each point of Y is in precisely one member of $\mathcal{G}_n$ and each point in $Y^2 \setminus \Delta$ is in at most two members of $\mathcal{G}_n$. Let $\mathcal{V}_n$ be the set of singletons of points in $Y^2 \setminus \Delta$ not covered by $\mathcal{G}_n$, and let $\mathcal{G}'_n = \mathcal{G}_n \cup \mathcal{V}_n$. Then each $\mathcal{G}'_n$ is point finite, and their union is a base for $X(Y)$, so (5) is proven. $\square$

Let A be a subspace of a space Y. Then A is *relatively countably compact* if every subset of A which is closed discrete in Y is finite. We say Y is *RCCC* if every relatively countably compact subset of Y is countable. We note that a metrizable space Y is RCCC if and only if every compact subset of Y is countable (in other words, Y is totally imperfect).

Lemma 120. *Let Y be any space. If Y is RCCC, then $X(Y)$ is (ω_1, ω)-paracompact. Hence, if Y is RCCC and first countable, then $X(Y)$ is (ω_1, ω)-metrizable. If Y is metrizable and $X(Y)$ is (ω_1, ω)-paracompact, then Y is RCCC.*

Proof. Any open cover of $X(Y)$ has an open refinement of the form $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ contains one basic open neighborhood U_y for each point $y \in Y \subseteq X(Y)$, and $\mathcal{U}_2$ consists of the singletons for each point in $X(Y)$ not already covered by $\mathcal{U}_1$. Notice that $\mathcal{U}_2$ is locally discrete. Now assume Y is RCCC. Then to show $X(Y)$ is (ω_1, ω)-paracompact, it suffices to show that $\mathcal{U}_1$ is (ω_1, ω)-locally finite.

Suppose $\mathcal{V}$ is an uncountable subset of $\mathcal{U}_1$, so $\mathcal{V} = \{U_y : y \in A\}$ for some uncountable $A \subseteq Y$. Since Y is RCCC, then A is not relatively countably compact in Y, so there is an infinite subset S of A that is closed and discrete in Y. It is then easy to check that $\mathcal{W} = \{U_y : y \in S\}$ is an infinite locally finite subset of $\mathcal{V}$. Thus, the first claim has been proven, and the second claim follows from Lemma 119.

To prove the final claim, fix a metric generating the topology on Y, and for any $y \in Y$ and $n < \omega$, let $B_n(y)$ denote the open ball of radius $1/n+1$ centered at y. Assuming $X(Y)$ is (ω_1, ω)-paracompact, then for each $y \in Y$, we can find an $n_y < \omega$ such that $\{U_y = \{y\} \cup (B_{n_y}(y) \times \{y\}) \cup (\{y\} \times B_{n_y}(y)) : y \in Y\}$ is (ω_1, ω)-locally finite in $X(Y)$. Let A be any uncountable subset of Y. By counting, there is an uncountable subset A_1 of A and an $m < \omega$ such that $n_y = m$ for all $y \in A_1$. Then there is an infinite subset S of A_1 such that $\{U_y : y \in S\}$ is locally finite.

We claim S is closed and discrete in Y, which shows that A is not relatively countably compact in Y, so that Y is RCCC. To that end, suppose some $z \in Y$ is in the closure of $S \setminus \{z\}$. So any basic neighborhood $B_n(z)$ of z in Y contains an element y_n of $S \setminus \{z\}$. Hence, any basic neighborhood $\{z\} \cup (B_n(z) \times \{z\}) \cup (\{z\} \times B_n(z))$ of z in $X(Y)$ intersects U_{y_k} for each $k \geq \max\{n, m\}$, contradicting the fact that $\{U_y : y \in S\}$ is locally finite. □

Lemma 121. *Let Y be a space such that $w(Y) < |Y|$. Then $X(Y)$ is not $w(Y)$-relatively paracompact.*

Proof. Fix a base $\mathcal{B}$ for Y with cardinality $\kappa = w(Y)$, and suppose $X(Y)$ is κ-relatively paracompact. Then there is a collection $\mathcal{U} = \{U_y : y \in Y\} = \bigcup\{\mathcal{U}_\alpha : \alpha < \kappa\}$ where each U_y is a basic neighborhood of y in $X(Y)$ and each $\mathcal{U}_\alpha$ is locally finite in its union.

Since $|Y| > \kappa$, there is a $\beta < \kappa$ such that $Y_\beta = Y \cap (\bigcup \mathcal{U}_\beta)$ has size greater than κ. Note that $\{U_y : y \in Y_\beta\} = \mathcal{U}_\beta$ is locally finite on $Y_\beta \subseteq X(Y)$, so by shrinking the elements of $\mathcal{U}_\beta$, we can obtain a collection $\mathcal{V} = \{V_y : y \in Y_\beta\}$ where each V_y is a basic $X(Y)$-neighborhood of y that intersects only finitely many other members of $\mathcal{V}$. In fact, any two distinct members of $\mathcal{V}$ will intersect in at most two points, so without loss of generality, $\mathcal{V}$ is actually pairwise disjoint.

For each $y \in Y_\beta$, write $V_y = \{y\} \cup (B_y \times \{y\}) \cup (\{y\} \times B_y)$ for some $B_y \in \mathcal{B}$. Then there is a $B \in \mathcal{B}$ and a subset S of Y_β such that $|S| > \kappa$ and $B_y = B$ for every $y \in S$. Note that S is a subset of B. Now pick any two distinct points y_1 and y_2 in S. Then the point (y_1, y_2) is in the intersection of V_{y_1} and V_{y_2}, contradicting that $\mathcal{V}$ is pairwise disjoint. $\square$

For any space Y denote by Y_ω the space with underlying set Y and topology obtained by adding all co-countable subsets of Y to the original topology on Y. Note that if the original topology on Y is Hausdorff, then so is Y_ω.

Lemma 122. *Let Y be a space.*

(1) $X(Y_\omega)$ is P-paracompact where P is the directed set $[Y]^{\leq\omega}$.

(2) If $w(Y) \cdot \omega_1 < |Y|$, then $X(Y_\omega)$ is not $w(Y)$-relatively paracompact.

Proof. To prove (1), it suffices to show that any open cover for $X(Y_\omega)$ of the form $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ as in the proof of Lemma 120 is P-locally finite. Write $\mathcal{U}_1 = \{U_y : y \in Y_\omega\}$ where each U_y is a basic open neighborhood of y in $X(Y_\omega)$. Then for any countable subset C of Y, we have that C is closed and discrete in Y_ω, so the family $\mathcal{U}_C = \{U_y : y \in C\} \cup \mathcal{U}_2$ is locally finite in $X(Y_\omega)$. Thus, $\mathcal{U} = \bigcup\{\mathcal{U}_C : C \in P = [Y]^{\leq\omega}\}$ is P-locally finite.

For the proof of (2), assume $X(Y_\omega)$ is κ-relatively paracompact, where $\kappa = w(Y)$, and fix a base $\mathcal{B}$ for Y with size κ. Then $\{B \setminus C : B \in \mathcal{B},\ C \subseteq Y,\ |C| \leq \omega\}$ is a base for Y_ω. By only slightly modifying the proof of Lemma 121, we can find a $B \in \mathcal{B}$, a subset S of B with $|S| > \kappa \cdot \omega_1$, and countable sets $C_y \subseteq Y \setminus \{y\}$ for each $y \in S$ such that the collection $\mathcal{V} = \{V_y = \{y\} \cup ((B \setminus C_y) \times \{y\}) \cup (\{y\} \times (B \setminus C_y)) : y \in S\}$ is pairwise disjoint.

Choose an arbitrary subset A of S with size ω_1, and let $A' = A \cup (\bigcup\{C_y : y \in A\})$, which also has size ω_1. Then there is a y_1 in $S \setminus A'$ and a y_2 in $A \setminus C_{y_1}$. Hence, we have $y_1 \in B \setminus C_{y_2}$ and $y_2 \in B \setminus C_{y_1}$, which means V_{y_1} intersects V_{y_2}, which is a contradiction. $\square$

4.4.2 THE SPLIT X-MACHINE

For any point y in a space Y, write y^+ for $(y, +)$ and y^- for $(y, -)$. For any subset S of Y, let $S^+ = \{s^+ : s \in S\}$ and $S^- = \{s^- : s \in S\}$. Let $S(Y)$ have the underlying set $(Y \times Y \setminus \Delta) \cup Y^+ \cup Y^-$. Isolate all points of $Y^2 \setminus \Delta$. A basic open neighborhood of a point y^+ in $S(Y)$ is $\{y^+\} \cup (U \times \{y\})$ for any open neighborhood U of y in Y. A basic open neighborhood of a point y^- in $S(Y)$ is $\{y^-\} \cup (\{y\} \times U)$ for any open neighborhood U of y in Y. Note that $S(\mathbb{R})$ is a symmetric version of Heath's split V-space.

Lemma 123. *Let Y be any Hausdorff space. Then we have:*

(1) $S(Y)$ is zero-dimensional and Hausdorff and hence Tychonoff.

(2) $S(Y)$ is a Moore space if and only if Y is first countable.

(3) If Y is first countable and $S(Y)$ is P-paracompact for some directed set P, then $S(Y)$ is P-metrizable.

(4) If Y is first countable and $S(Y)$ is (ω_1, ω)-paracompact, then $S(Y)$ is (ω_1, ω)-metrizable.

Proof. A slightly modified version of the proof for Lemma 119 works here. □

Lemma 124. *For any space Y, the split X-space, $S(Y)$, is screenable and therefore σ-relatively paracompact. The space $S(Y)$ has a σ-disjoint base if and only if it has a σ-relatively locally finite base if and only if Y is first countable.*

Proof. Every open cover for $S(Y)$ has an open refinement of the form $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_0$ contains the singletons for every point in $Y^2 \setminus \Delta$, $\mathcal{U}_1$ contains one basic neighborhood of each point in Y^+, and $\mathcal{U}_2$ contains one basic neighborhood for each point in Y^-. Since each $\mathcal{U}_i$ is pairwise disjoint, then $S(Y)$ is screenable.

Lemma 92 gives that every σ-disjoint base is σ-relatively locally finite. The same lemma also shows that every σ-relatively locally finite base for $S(Y)$ is σ-point finite, and therefore point countable, which implies first countability for Y. Finally, if Y is first countable, say $\{B_n(y) : n < \omega\}$ is a countable base at y in Y, then $S(Y)$ has a σ-disjoint base $\mathcal{V} \cup \bigcup_n \mathcal{W}_n$, where $\mathcal{V}$ contains the singletons for each point in $Y^2 \setminus \Delta$ and $\mathcal{W}_n = \{\{y^+\} \cup (B_n(y) \times \{y\}) : y \in Y\} \cup \{\{y^-\} \cup (\{y\} \times B_n(y)) : y \in Y\}$. So the three conditions in the second statement are equivalent. □

Lemma 125. *Let Y be any space. If Y is RCCC, then $S(Y)$ is (ω_1,ω)-paracompact. Hence, if Y is RCCC and first countable, then $S(Y)$ is (ω_1,ω)-metrizable. If Y is metrizable and $S(Y)$ is (ω_1,ω)-paracompact, then Y is RCCC.*

Proof. The proof for Lemma 120 can be easily modified to work here. $\square$

For a subset Y of $\mathbb{R}$, define $H(Y) = \{(y,y') : y < y',\ y,y' \in Y\} \cup Y^+ \cup Y^-$ with the subspace topology from $S(Y)$. For y in Y and $n < \omega$, define $V_n(y,+) = \{y^+\} \cup (y - \frac{1}{n+1}, y) \times \{y\}$ and $V_n(y,-) = \{y^-\} \cup \{y\} \times (y, y + \frac{1}{n+1})$. These are basic neighborhoods of y^+ and y^-, respectively. As alluded to above, Heath's split V-space is (homeomorphic to) $H = H(\mathbb{R})$. This family of subspaces has some specific properties we identify.

Lemma 126. *For any subspace Y of $\mathbb{R}$, the space $H(Y)$ is (ω_1,ω)-metrizable.*

Proof. Let $\mathcal{B} = \{\{(y,y')\} : y < y',\ y,y' \in Y\} \cup \{V_n(y,+) : y \in Y,\ n < \omega\} \cup \{V_n(y,-) : y \in Y,\ n < \omega\}$. This is a basis for $H(Y)$. We show it is (ω_1,ω)-locally finite.

Let $\mathcal{B}_1$ be any uncountable subset of $\mathcal{B}$. There must be an $n < \omega$ and an uncountable subset $\mathcal{B}_2$ of $\mathcal{B}_1$ as in one of the following three cases.

Case 1: Each element of $\mathcal{B}_2$ is a singleton of the form $\{(y,y')\}$, where $y + \frac{1}{n+1} < y'$. Then $\mathcal{B}_2$ is clearly locally finite in $H(Y)$.

Case 2: $\mathcal{B}_2 = \{V_n(y,+) : y \in Y'\}$ for some uncountable $Y' \subseteq Y$. Since $\mathbb{R}$ with the 'left' Sorgenfrey topology (in other words, with base $\{(a,b] : a < b\}$) has countable extent, then Y' contains a strictly increasing sequence $(y_k)_k$ that converges in $\mathbb{R}$. It is then straightforward to check that $\mathcal{B}_3 = \{V_n(y_k,+) : k < \omega\}$ is locally finite in $H(Y)$.

Case 3: $\mathcal{B}_2 = \{V_n(y,-) : y \in Y'\}$ for some uncountable $Y' \subseteq Y$. A similar argument (using the 'right' Sorgenfrey topology and extracting a strictly decreasing convergent sequence) as for case 2 works here.

In any case, $\mathcal{B}_1$ contains an infinite locally finite subset, so the proof is complete. $\square$

Lemma 127. *For any subspace Y of $\mathbb{R}$ that is not RCCC, the space $H(Y)$ is not P-paracompact for any P with calibre (ω_1,ω).*

Proof. Fix a subspace Y of $\mathbb{R}$, an uncountable relatively countably compact subset A of Y, and a directed set P with calibre (ω_1,ω). To get a contradiction, suppose $\mathcal{B}$ is a P-locally finite base for $H(Y)$. Then according to Lemma 88, we have $P \geq_T ([\mathcal{B}]^{<\omega}, LF(\mathcal{B}))$, and so by Lemma 17, $[\mathcal{B}]^{<\omega}$ has relative calibre (ω_1, ω) in $LF(\mathcal{B})$.

For each $y \in A$, there are $W_y^+, W_y^- \in \mathcal{B}$ and an $n_y < \omega$ such that $V_{n_y}(y,+) \subseteq W_y^+ \subseteq V_0(y,+)$ and $V_{n_y}(y,-) \subseteq W_y^- \subseteq V_0(y,-)$. We can find an uncountable subset A_1 of A and an $n < \omega$ such that $n_y = n$ for all $y \in A_1$. Then the uncountable subset $\{\{W_y^+, W_y^-\} : y \in A_1\}$ of $[\mathcal{B}]^{<\omega}$ must contain an infinite subset with an upper bound in $LF(\mathcal{B})$. Thus, there exists an infinite $A_2 \subseteq A_1$ such that $\mathcal{W} = \bigcup\{\{W_y^+, W_y^-\} : y \in A_2\}$ is locally finite.

If A_2 contains an increasing sequence that converges to some point a in A, then $\mathcal{W}$ fails to be locally finite at a^+, since $V_n(y,-) \subseteq W_y^+$ for each $y \in A_2$. Thus, A_2 does not contain any increasing sequence that converges in Y, and similarly, we can show A_2 does not contain any decreasing sequence that converges in Y. But that contradicts the fact that A is relatively countably compact in Y. □

4.4.3 THE DISJOINT SETS SPLIT X-MACHINE

Here we highlight a useful subspace of the previous split X-space. Let Y be a space, and A_1, A_2 be a partition of Y. Let $D(Y; A_1, A_2)$ be the subspace $(Y^2 \setminus \Delta) \cup A_1^+ \cup A_2^-$ of $S(Y)$.

Lemma 128. *Let Y be any Hausdorff space with a partition A_1, A_2. Then we have:*

(1) $D(Y; A_1, A_2)$ is zero-dimensional and Hausdorff and hence Tychonoff.

(2) $D(Y; A_1, A_2)$ is a Moore space if and only if Y is first countable.

(3) If Y is first countable and $D(Y; A_1, A_2)$ is P-paracompact for some directed set P, then $D(Y; A_1, A_2)$ is P-metrizable.

(4) If Y is first countable and $D(Y; A_1, A_2)$ is (ω_1, ω)-paracompact, then $D(Y; A_1, A_2)$ is (ω_1, ω)-metrizable.

Proof. Once again, we can slightly modify the proof of Lemma 119 to work here. □

Lemma 129. *For any space Y and partition A_1, A_2, the space $D(Y; A_1, A_2)$ is screenable and hence σ-relatively paracompact.*

The space $D(Y; A_1, A_2)$ has a σ-disjoint base if and only if it has a σ-relatively locally finite base if and only if Y is first countable.

Proof. The proof of Lemma 124 works here with only slight modifications. ☐

Lemma 130. *Let Y be a space with partition A_1, A_2. Let P be a directed set. If $P \geq_T (A_i, CL(Y) \cap \mathbb{P}(A_i))$ for $i = 1, 2$, then $D(Y; A_1, A_2)$ is P-paracompact. If Y is metrizable and $D(Y; A_1, A_2)$ is P-paracompact, then $P \times \omega \geq_T (A_i, CL(Y) \cap \mathbb{P}(A_i))$ for $i = 1, 2$.*

Proof. Suppose $P \geq_T (A_i, CL(Y) \cap \mathbb{P}(A_i))$ for $i = 1, 2$. Then for each $i = 1, 2$, there is an order-preserving map $\phi_i : P \to CL(Y) \cap \mathbb{P}(A_i)$ whose image covers A_i. Each open cover of $D(Y; A_1, A_2)$ has an open refinement of the form $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ contains one basic neighborhood U_x of each x in A_1^+, $\mathcal{U}_2$ contains one basic neighborhood U_x of each x in A_2^-, and $\mathcal{U}_0$ contains the singletons for each point not covered by $\mathcal{U}_1 \cup \mathcal{U}_2$.

For each $p \in P$, let $\mathcal{U}_p = \{U_x : x \in \phi_1(p) \cup \phi_2(p)\} \cup \mathcal{U}_0$. Note that $\mathcal{U}_1$ is trivially locally finite in $D(Y; A_1, A_2) \setminus A_2^-$, and since $\phi_1(p) \subseteq A_1$ is closed in Y, then $\{U_x : x \in \phi_1(p)\}$ is locally finite in all of $D(Y; A_1, A_2)$. Similarly, the rest of $\mathcal{U}_p$ is locally finite also. Thus, $\bigcup\{\mathcal{U}_p : p \in P\} = \mathcal{U}$ is P-locally finite, which proves the first claim.

For the second claim, fix a metric for Y. Then for any $y \in Y$ and $n < \omega$, let $B_n(y)$ denote the open ball of radius $\frac{1}{n+1}$ centered at y, and define $U_n(y) = \{y^+\} \cup (B_n(y) \times \{y\})$ when $y \in A_1$ and $U_n(y) = \{y^-\} \cup (\{y\} \times B_n(y))$ when $y \in A_2$. Since $D(Y; A_1, A_2)$ is P-paracompact, then for any $y \in Y$, there is an $n_y < \omega$ such that $\mathcal{U} = \{U_{n_y}(y) : y \in Y\}$ is P-locally finite. Write $\mathcal{U} = \bigcup\{\mathcal{U}_p : p \in P\}$ where each $\mathcal{U}_p$ is locally finite and $\mathcal{U}_p \subseteq \mathcal{U}_{p'}$ whenever $p \leq p'$ in P.

Fix $p \in P$ and $n < \omega$ and define $A_{i,p,n} = \{y \in A_i : U_{n_y}(y) \in \mathcal{U}_p,\ n_y \leq n\}$. We will check that $\overline{A_{i,p,n}}^Y$ is contained in A_i. Let a be in $Y \setminus A_i = A_{3-i}$. Then a has a basic neighborhood $U_k(a)$ with $k \geq n$ that intersects $U_{n_y}(y)$ for only finitely many y in $A_{i,p,n}$, and since those basic neighborhoods intersect in only one point, then we can actually assume $U_k(a)$ does not intersect any $U_{n_y}(y)$ for $y \in A_{i,p,n}$. We claim that $B_k(a)$ does not intersect $A_{i,p,n}$, which shows that y is not in $\overline{A_{i,p,n}}^Y$. Indeed, if there were some y in $A_{i,p,n} \cap B_k(a)$, then a would be in $B_k(y) \subseteq B_{n_y}(y)$, so either (a, y) or (y, a) would be in $U_{n_y}(y) \cap U_k(a)$, which is a contradiction.

Then it is straightforward to check that $\phi_i : P \times \omega \to CL(Y) \cap \mathbb{P}(A_i)$ defined by $\phi_i(p, n) = \overline{A_{i,p,n}}^Y$ is order-preserving and its image covers A_i, so $P \times \omega \geq_T (A_i, CL(Y) \cap \mathbb{P}(A_i))$. ☐

Recall that a space Y is called a Δ-space if whenever we write Y as an increasing union of subsets, $Y = \bigcup_n S_n$ where $S_n \subseteq S_{n+1}$ for all n, there is a countable closed cover $\{C_n : n < \omega\}$ of Y such that $C_n \subseteq S_n$ for every n.

Lemma 131. *If Y is a Δ-space, then $D(Y; A_1, A_2)$ is countably paracompact.*

Proof. Let Y be a Δ-space. Recall that a space X is countably paracompact if and only if for every countable increasing open cover $\{U_n : n < \omega\}$ of X, there is an open cover $\{V_n : n < \omega\}$ of X such that $\overline{V_n} \subseteq U_n$ for each $n < \omega$. So, let $\{U_n : n < \omega\}$ be an increasing open cover of $D(Y; A_1, A_2)$.

For each $n < \omega$ and $i = 1, 2$, let $S_n^i \subseteq A_i$ such that $(S_n^1)^+ = U_n \cap A_1^+$ and $(S_n^2)^- = U_n \cap A_2^-$, and then define $S_n = S_n^1 \cup S_n^2$. Then $\{S_n : n < \omega\}$ forms an increasing cover of Y, so there is a closed cover $\{C_n : n < \omega\}$ of Y such that $C_n \subseteq S_n$ for each $n < \omega$. Let $C_n^i = C_n \cap A_i$ for each $n < \omega$ and $i = 1, 2$, so $C_n^i \subseteq S_n^i$ and $\bigcup_n C_n^i = A_i$.

For each y in C_n^1, pick a basic open set $B_{y,n}$ such that $y^+ \in B_{y,n} \subseteq U_n$, and for each y in C_n^2, pick a basic open $B_{y,n}$ such that $y^- \in B_{y,n} \subseteq U_n$. Let $W_n^i = \bigcup\{B_{y,n} : y \in C_n^i\}$ for $i = 1, 2$. Then W_n^i is open and contained in U_n.

Notice also that if $z \in \overline{W_n^1} \setminus W_n^1$, then $z = a^-$ for some a in $A_2 \cap \overline{C_n^1} \subseteq A_2 \cap C_n = C_n^2$. Thus, $\overline{W_n^1}$ is contained in $W_n^1 \cup (C_n^2)^- \subseteq U_n$. Similarly, we also have $\overline{W_n^2} \subseteq U_n$. Let $W_n = W_n^1 \cup W_n^2$. Then W_n is open, its closure is contained in U_n, and $\bigcup_n W_n$ contains $A_1^+ \cup A_2^-$.

Let $T = D(Y, A_1, A_2) \setminus \bigcup_n W_n$, which is a clopen set of isolated points. Then $T_n = T \cap U_n$ is also clopen for each n. Let $V_n = W_n \cup T_n$. Then $\{V_n : n < \omega\}$ is an open cover of $D(Y; A_1, A_2)$ such that $\overline{V_n} \subseteq U_n$. Thus, $D(Y; A_1, A_2)$ is countably paracompact. $\square$

4.5 COUNTEREXAMPLES

Here we give examples showing that the results of Section 4.3.1 do not hold if the given additional hypotheses are dropped, nor if $\mathcal{K}(M)$ is weakened to P with calibre (ω_1, ω). We also give examples distinguishing all the relevant properties ($\mathcal{K}(M)$-metrizable, P-metrizable for P with calibre (ω_1, ω), (ω_1, ω)-metrizable, etc).

Example 3 ($\neg$CH)**.** *There is a space X that is $\mathcal{K}(M)$-paracompact for some separable metrizable M (hence X is $\mathcal{K}(\mathcal{M})$-paracompact) but such that X is not σ-relatively paracompact (hence X is not weakly σ-paracompact).*

Proof. Let I_ω be $I = [0,1]$ with its topology refined by declaring each of its countable sets to be closed, and consider the space $X(I_\omega)$ (see Section 4.4.1). Note that I has a countable base, so if we assume $\neg$CH, then $w(I) \cdot \omega_1 < \mathfrak{c}$. By Lemma 122, we have that $X(I_\omega)$ is $[I]^{\leq\omega}$-paracompact but not σ-relatively paracompact. We also know from Lemma 53 that $\mathcal{K}(M) \geq_T [\mathfrak{c}]^{\leq\omega} = [I]^{\leq\omega}$, where M is a Bernstein set, and so $X(I_\omega)$ is $\mathcal{K}(M)$-paracompact. $\square$

Curiously, under CH the same example is paracompact.

Example 4. *There is a Moore space which has a σ-disjoint base (and hence has a σ-relatively locally finite base) which is not (ω_1, ω)-paracompact and so not $\mathcal{K}(\mathcal{M})$-paracompact.*

Proof. Let $X = S(I)$ (see Section 4.4.2). This space is well-known to be a Moore space with a σ-disjoint base, and so, as observed above, it has a σ-relatively locally finite base. The space X is not (ω_1, ω)-paracompact (see Lemma 125), and so not $\mathcal{K}(M)$-paracompact for any separable metrizable M $\square$

Example 5. *There is a Moore space with a σ-point finite base (hence, it is σ-metacompact) which is not σ-relatively paracompact.*

Proof. Take $X = X(\mathbb{R})$ (see Section 4.4.1). Since $w(\mathbb{R}) = \omega < |\mathbb{R}|$, we have that $X(\mathbb{R})$ is not σ-relatively paracompact by Lemma 121. By Lemma 119, we have that $X(\mathbb{R})$ is a Moore space with a σ-point finite base. $\square$

A space X is called *perfectly normal* if it is normal and each closed subset of X is G_δ. Equivalently, X is perfectly normal if and only if each open subset U of X can be written as $U = \bigcup\{V_n : n < \omega\} = \bigcup\{\overline{V_n} : n < \omega\}$ where each V_n is open. Recall the definition of Q-set given before Example 2. It is consistent and independent from ZFC that there exists a Q-set (for example, see [18]).

Example 6 ($\exists$ a Q-set)**.** *There is a Moore space which is perfectly normal, has a σ-point finite base, but is not σ-relatively paracompact.*

Proof. Let $Q \subseteq \mathbb{R}$ be a Q-set. By the same reasoning as in Example 5, $X(Q)$ is a Moore space with a σ-point finite base but not σ-relatively paracompact. It is well-known that the Heath V-space on a Q-set is perfectly normal, and the same idea works here, but we include the proof anyway. Let U be an open subset of $X(Q)$, so we can write $U = V \cup \bigcup\{B_y : y \in Y\}$ where V is some subset of $Q^2 \setminus \Delta$ and B_y is a basic (X-shaped) neighborhood of y for each y in some subset Y of Q. Since Q is a Q-set, then Y is an F_σ subset of Q, so write $Y = \bigcup_n C_n$ where each C_n is closed. Then $W_n = \bigcup\{B_y : y \in C_n\}$ is a clopen subset of $X(Q)$, as is the set V_n of all points in V that have Euclidean distance at least $1/n$ from the diagonal. Hence, each $U_n = V_n \cup W_n$ is clopen, so $U = \bigcup_n U_n = \bigcup_n \overline{U_n}$, which means $X(Q)$ is perfectly normal. $\square$

Recall that $CL(Y)$ denotes the pre-ideal of closed subsets of a space Y and $\mathbb{P}(A)$ denotes the power set of a set A.

Lemma 132. *Let $\{A_\alpha : \alpha < \kappa\}$ be a family of separable metrizable spaces and define $P = \Sigma_\alpha \mathcal{K}(A_\alpha)$. Then there is a metrizable and locally separable space Y with a subspace A homeomorphic to $\bigoplus\{A_\alpha : \alpha < \kappa\}$ such that:*

(1) $P \geq_T (A, CL(Y) \cap \mathbb{P}(A))$, and

(2) if Q is a directed set such that $Q \geq_T (A, CL(Y) \cap \mathbb{P}(A))$, then $Q \geq_T (A_\alpha, \mathcal{K}(A_\alpha))$ for each $\alpha < \kappa$.

Proof. We may assume each A_α is a subspace of Y_α, a copy of the Hilbert cube I^ω. Let $Y = \bigoplus\{Y_\alpha : \alpha < \kappa\}$. Define $A = \bigoplus\{A_\alpha : \alpha < \kappa\}$. Then the map $\phi : P \to CL(Y) \cap \mathbb{P}(A)$ given by $\phi((K_\alpha)_\alpha) = \bigoplus_\alpha K_\alpha$ is order-preserving, and its image covers A, which proves (1).

Now suppose $Q \geq_T (A, CL(Y) \cap \mathbb{P}(A))$ is witnessed by $\phi : Q \to CL(Y) \cap \mathbb{P}(A)$. Note that if C is a subset of A that is closed in Y, then $C \cap Y_\alpha$ is a subset of A_α that is closed in Y_α, and since Y_α is compact, then so is $C \cap Y_\alpha$. Thus, we have a map $\phi_\alpha : Q \to \mathcal{K}(A_\alpha)$ given by $\phi_\alpha(q) = \phi(q) \cap Y_\alpha$ that witnesses (2). $\square$

Example 7. *There is a hereditarily paracompact, first countable space which is P-metrizable for a directed set P with calibre (ω_1, ω), but is not $\mathcal{K}(M)$-metrizable for any separable metrizable space M.*

Proof. Take a family $\{A_\alpha \subseteq I^\omega : \alpha < \mathfrak{c}^+\}$ of distinct subsets of the Hilbert cube, I^ω, and let $P = \Sigma\{\mathcal{K}(A_\alpha) : \alpha < \mathfrak{c}^+\}$, which has calibre (ω_1, ω) by Theorem 58. Then Lemma 132 provides a metrizable space Y with a subspace A such that $P \times \omega \geq_T P \geq_T (A, CL(Y) \cap \mathbb{P}(A))$.

Let $X = M(Y, A)$. Then X is first countable and hereditarily paracompact, and X is P-metrizable by Lemma 116.

Suppose X is $\mathcal{K}(M)$-metrizable for some separable metrizable space M, and let $M' = M \times \omega$. Then Lemma 116 implies that $\mathcal{K}(M') =_T \mathcal{K}(M) \times \omega \geq_T (A, CL(Y) \cap \mathbb{P}(A))$. Thus, $\mathcal{K}(M') \geq_T (A_\alpha, \mathcal{K}(A_\alpha))$ for each $\alpha < \mathfrak{c}^+$ by part (2) of Lemma 132, but that contradicts (2) in Theorem 59. $\square$

Lemma 133. *Let Y be a metrizable and locally separable space. Let A be a subspace of Y.*

(1) There is a directed set P with calibre (ω_1, ω) such that $P \geq_T (A, CL(Y) \cap \mathbb{P}(A))$. Moreover, $P = \Sigma_\alpha \mathcal{K}(M_\alpha)$ where each M_α is separable metrizable.

(2) If the weight of Y is $\leq \mathfrak{c}$, then we can take $P = \mathcal{K}(M)$ in (1) where M is some separable metrizable space.

Proof. Using local separability, regularity and paracompactness of Y, we can find a closed locally finite cover $\mathcal{C} = \{C_\alpha : \alpha < w(Y)\}$ of Y of separable sets. Define $A_\alpha = C_\alpha \cap A$.

For (1), define $P = \Sigma_\alpha \mathcal{K}(A_\alpha)$, which has calibre (ω_1, ω) by Theorem 58. Any collection $\{K_\alpha : \alpha < w(Y)\}$ with $K_\alpha \in \mathcal{K}(A_\alpha)$ is locally finite in Y since $\mathcal{C}$ is locally finite. Therefore, $\bigcup_\alpha K_\alpha$ is closed in Y and is also a subset of A. The map $\Sigma\mathcal{K}(A_\alpha) \to CL(Y) \cap \mathbb{P}(A)$ given by $(K_\alpha)_\alpha \mapsto \bigcup_\alpha K_\alpha$ is then order-preserving, and its image covers A.

For (2), we use Theorem 59 to find a separable metrizable space M and Tukey quotient maps $\phi_\alpha : \mathcal{K}(M) \to \mathcal{K}(A_\alpha)$ for each $\alpha < w(Y) \leq \mathfrak{c}$. Then define $\phi : \mathcal{K}(M) \to CL(Y) \cap \mathbb{P}(A)$ by $\phi(K) = \bigcup_\alpha \phi_\alpha(K)$, which witnesses $\mathcal{K}(M) \geq_T (A, CL(Y) \cap \mathbb{P}(A))$. $\square$

Example 8. *There is a Moore space with a σ-disjoint base which is P-metrizable for a directed set P with calibre (ω_1, ω), but is not $\mathcal{K}(\mathcal{M})$-paracompact.*

Proof. Let Y, A, and P be as in the proof of Example 7, and define $B = Y \setminus A$. By Lemma 133, there is a $Q = \Sigma\{M_\alpha : \alpha < \kappa\}$, where each M_α is separable metrizable, such that $Q \geq_T (B, CL(Y) \cap \mathbb{P}(B))$. Then $P' = P \times Q$ is also a Σ-product of $\mathcal{K}(M)$'s and so has calibre (ω_1, ω) by Theorem 58.

Arguing as in Example 7, but with Lemma 130 replacing Lemma 116, we see that $X = D(Y; A, B)$ is P'-paracompact but not $\mathcal{K}(M)$-paracompact for any separable metrizable space M. Since Y is first countable, then X is P'-metrizable by Lemma 128. Since $D(Y; A, B)$ is a Moore space, it follows from Lemma 91 that it cannot be $\mathcal{K}(\mathcal{M})$-paracompact.

$\square$

Example 9. *There is a space which is $\mathcal{K}(\mathcal{M})$-paracompact but not $\mathcal{K}(M)$-paracompact for any separable metrizable M.*

Proof. Let $\{A_\alpha : \alpha < \mathfrak{c}^+\}$ be a family of distinct subsets of the Hilbert cube I^ω. For each $\alpha < \mathfrak{c}^+$, set $Y_\alpha = I^\omega$, $B_\alpha = Y_\alpha \setminus A_\alpha$, and $X_\alpha = D(Y_\alpha; A_\alpha, B_\alpha)$. Then define $X = \bigoplus_\alpha X_\alpha$, and let X^* be X with one additional point, $*$, where basic neighborhoods of $*$ have the form $U_C = \{*\} \cup \bigoplus \{X_\alpha : \alpha \in \mathfrak{c}^+ \setminus C\}$ for any countable subset C of $\mathfrak{c}^+$.

Fix a separable metrizable space M. We check that X^* is not $\mathcal{K}(M)$-paracompact. By Theorem 59, we have $\mathcal{K}(M \times \omega) \not\geq_T (A_\alpha, \mathcal{K}(A_\alpha))$ for some $\alpha < \mathfrak{c}^+$, and if X^* were $\mathcal{K}(M)$-paracompact, then the closed subspace X_α would also be $\mathcal{K}(M)$-paracompact. But since Y_α is metrizable, then by Lemma 130, we would have $\mathcal{K}(M \times \omega) =_T \mathcal{K}(M) \times \omega \geq_T (A_\alpha, CL(Y) \cap \mathbb{P}(A))$. However, $CL(Y) \cap \mathbb{P}(A) = \mathcal{K}(A_\alpha)$ since Y_α is compact, which gives a contradiction.

Now we show X^* is $\mathcal{K}(\mathcal{M})$-paracompact. Let $\mathcal{U}$ be any open cover of X^*, and pick a U_* in $\mathcal{U}$ containing $*$. Then there is a countable subset C of $\mathfrak{c}^+$ such that U_* contains X_α for each $\alpha \in \mathfrak{c}^+ \setminus C$. By Lemma 130, each X_α is $\mathcal{K}(M_\alpha)$-paracompact where $M_\alpha = A_\alpha \times B_\alpha$. By Theorem 59, there is a separable metrizable M such that $M \geq_T M_\alpha$ for each α in C. Thus, for each α in C, X_α is $\mathcal{K}(M)$-paracompact, so we can find a $\mathcal{K}(M)$-locally finite open refinement $\mathcal{V}_\alpha = \bigcup \{\mathcal{V}_{\alpha,K} : K \in \mathcal{K}(M)\}$ of $\mathcal{U}_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$. For each $K \in \mathcal{K}(M)$, define $\mathcal{V}_K = \{U_*\} \cup \bigcup \{\mathcal{V}_{\alpha,K} : \alpha \in C\}$. Then $\mathcal{V} = \bigcup \{\mathcal{V}_K : K \in \mathcal{K}(M)\}$ is a $\mathcal{K}(M)$-locally finite open refinement of $\mathcal{U}$ since each X_α is open in X^*. $\square$

Example 10. *There is a Moore space with a σ-disjoint base which is (ω_1, ω)-metrizable but not P-paracompact for any directed set P with calibre (ω_1, ω).*

Proof. Consider Heath's original split V space, $H = H(\mathbb{R})$. This is a Moore space with a σ-disjoint base. By Lemma 126, H is (ω_1, ω)-metrizable. Lemma 127 implies H is not P-paracompact for any directed set P with calibre (ω_1, ω). □

We know that 'first countable plus $\mathcal{K}(\mathcal{M})$-paracompact' implies 'σ-relatively paracompact' and '$\mathcal{K}(M)$-metrizable' implies 'σ-relatively locally finite base'. The last pair of examples show that '$\mathcal{K}(\mathcal{M})$-paracompact' cannot be replaced above by '(ω_1, ω)-paracompact' or 'P-paracompact, where P has calibre (ω_1, ω)'; nor can '$\mathcal{K}(M)$-metrizable' be similarly weakened.

Example 11. *There is a Moore space (hence, first countable) which is (ω_1, ω)-paracompact but not σ-relatively paracompact, and so it is not $\mathcal{K}(\mathcal{M})$-paracompact and not weakly σ-paracompact.*

Proof. Take $X = X(B)$, where B is a Bernstein set. We can see that B is RCCC since every compact subset of B is countable. By Lemma 120, $X(B)$ is (ω_1, ω)-paracompact. Since $|B| > \omega_0$ and $w(B) = \omega_0$, we see that $X(B)$ is not σ-relatively paracompact by Lemma 121. □

Example 12. *There is a P-paracompact space, where P has calibre (ω_1, ω), which is not σ-relatively paracompact, and also not $\mathcal{K}(\mathcal{M})$-paracompact.*

Proof. Let $Y = I^{\mathfrak{c}}$, and consider $X = X(Y_\omega)$. By Lemma 122, $X = X(Y_\omega)$ is $[Y]^{\leq\omega}$-paracompact. Since the directed set $[Y]^{\leq\omega}$ has calibre (ω_1, ω), we have that $X(Y_\omega)$ is (ω_1, ω)-paracompact.

Notice that Y has weight $w(Y) = \mathfrak{c}$. Therefore $w(Y) \cdot \omega_1 < 2^{\mathfrak{c}}$, so Lemma 122 also shows that $X(Y_\omega)$ is not $\mathfrak{c}$-relatively paracompact, and hence, it is not σ-relatively paracompact. Again, $X(Y_\omega)$ is not $\mathfrak{c}$-relatively paracompact, and so not $\mathfrak{c}$-paracompact. Since for every M in $\mathcal{M}$ we have $cof\, \mathcal{K}(M) \leq \mathfrak{c}$, then from Lemma 90 we see that $X(Y_\omega)$ is not $\mathcal{K}(\mathcal{M})$-paracompact. □

Answers to the next set of questions would help complete the diagrams in Section 4.3.1.

Question 1. *Is there a σ-relatively paracompact space which is not screenable? Is there a $\mathcal{K}(M)$-metrizable space without a σ-disjoint base? One which is a Moore space?*

5.0 PRODUCTIVITY OF CALIBRE (ω_1, ω)

In Sections 3.1 and 4.2, the class of directed sets with calibre (ω_1, ω) played a crucial role in our generalizations of Schneider's theorem, so it seems worthwhile to study this class in its own right. In this chapter, we study the preservation of calibre (ω_1, ω) under products. Generally, it is not a productive property, as the following example by Todorčević [45] demonstrates. Here, a subset S of ω_1 is called *stationary* if it intersects every subset of ω_1 that is closed (in the order topology) and unbounded (i.e. uncountable), and S is called *co-stationary* if $\omega_1 \setminus S$ is stationary.

Example 13 (Todorčević [45])**.** *If S is a stationary and co-stationary subset of ω_1, then $\mathcal{K}(S)$ and $\mathcal{K}(\omega_1 \setminus S)$ each have calibre (ω_1, ω) but $\mathcal{K}(S) \times \mathcal{K}(\omega_1 \setminus S)$ does not have calibre (ω_1, ω).*

However, some products do preserve calibre (ω_1, ω). For example, if P has calibre (ω, ω) or calibre (ω_1, ω_1) (each of which implies calibre (ω_1, ω)) and if Q has calibre (ω_1, ω), then the product of P and Q does have calibre (ω_1, ω) (see Lemma 26). Also, every finite power of a calibre (ω_1, ω) directed set has calibre (ω_1, ω) since if P is directed, then $P \geq_T P^n$ via the map $p \mapsto (p, \ldots, p)$. Further, Lemma 57 shows that any countable product of directed sets in the class $\mathcal{K}(\mathcal{M})$ has calibre (ω_1, ω).

5.1 UNCOUNTABLE PRODUCTS AND Σ-PRODUCTS

The following lemma shows that an uncountable product of directed sets with calibre (ω_1, ω) may fail to have calibre (ω_1, ω), even when each factor is very simple.

Lemma 134. *The directed set ω^{ω_1} does not have calibre (ω_1, ω).*

Proof. For each $\alpha < \omega_1$ fix an injection $g_\alpha : \alpha \to \omega$, and then define $f_\alpha : \omega_1 \to \omega$ by $f_\alpha(\beta) = g_\beta(\alpha)$ if $\beta > \alpha$ and $f_\alpha(\beta) = 0$ otherwise. Then $\{f_\alpha : \alpha < \omega_1\}$ is uncountable but every infinite subset is unbounded. To see this, take any countably infinite subset S of ω_1, and let $\beta = \sup S + 1$. Then for each α in S, we have $f_\alpha(\beta) = g_\beta(\alpha)$. Since S is infinite and g_β is injective, we see that the family $\{f_\alpha : \alpha \in S\}$ is unbounded on β. □

In particular, since $\omega =_T \mathcal{K}(\omega)$, then an uncountable product of members of the class $\mathcal{K}(\mathcal{M})$ need not have calibre (ω_1, ω). In fact, we now show that uncountable products have calibre (ω_1, ω) only in trivial circumstances.

Theorem 135. *If κ is an uncountable cardinal and P_α is a directed set for each $\alpha < \kappa$, then $\prod_\alpha P_\alpha$ has calibre (ω_1, ω) if and only if all countable subproducts have calibre (ω_1, ω) and all but countably many P_α are countably directed.*

Proof. If $\prod_\alpha P_\alpha$ has calibre (ω_1, ω), then certainly all countable subproducts have calibre (ω_1, ω) since projections are Tukey quotients. And note that a P_α is not countably directed if and only if $P_\alpha \geq_T \omega$. So if uncountably many P_α's are not countably directed, then $\prod_\alpha P_\alpha \geq_T \omega^{\omega_1}$, and so Lemma 134 shows $\prod_\alpha P_\alpha$ does not have calibre (ω_1, ω).

For the converse, let $C = \{\alpha : P_\alpha \text{ is not countably directed}\}$. By hypothesis C is countable and $\prod\{P_\alpha : \alpha \in C\}$ has calibre (ω_1, ω). Since P_α is countably directed for each $\alpha \notin C$, then $\prod\{P_\alpha : \alpha \notin C\}$ is countably directed, or equivalently has calibre (ω, ω) (see Lemma 32). Hence, $\prod_\alpha P_\alpha = \prod\{P_\alpha : \alpha \in C\} \times \prod\{P_\alpha : \alpha \notin C\}$ has calibre (ω_1, ω) by Lemma 26. □

Thus, uncountable products are generally 'too large' to have calibre (ω_1, ω), but we will see that this is not the case for Σ-products. As mentioned in Section 2.6, we can ensure that any topological directed set has a minimal element without altering any of its relevant properties, so when we form a Σ-product $\Sigma_\alpha P_\alpha$ of directed sets P_α for $\alpha < \kappa$, we will assume that each P_α has a minimal element 0_α and base the Σ-product at the point $(0_\alpha)_\alpha$.

By Theorem 43, we know any countable product of second countable directed sets with CSBS has calibre (ω_1, ω), and the next result generalizes this fact. Recall that a space X is called *cosmic* if it has a countable network $\mathcal{N}$; that is, for any x in X and open neighborhood U of x, there is an N in $\mathcal{N}$ such that $x \in N \subseteq U$.

Theorem 136. *Let $\{P_\alpha : \alpha < \kappa\}$ be a family of cosmic topological directed sets with CSBS. Then $\Sigma_\alpha P_\alpha$ has calibre (ω_1, ω).*

Proof. First, we simplify the factors. Suppose P is a topological directed set which is cosmic and has CSBS. Take a countable network $\mathcal{N}$ for P. We may assume each member of $\mathcal{N}$ is closed since P is T_3. Now refine the topology on P by declaring the elements of $\mathcal{N}$ to be open also. Then with this new topology, P is zero-dimensional, separable metrizable, and still has CSBS. Thus, we may assume that each P_α is a subspace of the Cantor set $\mathfrak{C} = \{0,1\}^\omega$, and so $\Sigma_\alpha P_\alpha$ is, topologically, a subspace of $\mathfrak{C}^\kappa$.

Fix a base $\mathcal{B} = \bigcup_n \mathcal{B}_n$ for $\mathfrak{C}$ such that each $\mathcal{B}_n$ is a finite clopen partition of $\mathfrak{C}$ and $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$. Let $\pi_\alpha : \mathfrak{C}^\kappa \to \mathfrak{C}$ be the natural projection for each $\alpha < \kappa$. Now, for any finite subset F of κ, and for any $n < \omega$, define $\mathcal{C}(F, n) = \{\Sigma_\alpha P_\alpha \cap \bigcap_{\alpha \in F} \pi_\alpha^{-1} U_\alpha : U_\alpha \in \mathcal{B}_n \text{ for each } \alpha \in F\}$. Note that each $\mathcal{C}(F, n)$ is a finite clopen partition of $\Sigma_\alpha P_\alpha$.

By Lemma 45, we know $\Sigma_\alpha P_\alpha$ is CSBS, so by Lemma 39, it suffices to show that any uncountable subset of $\Sigma_\alpha P_\alpha$ contains an infinite convergent sequence. So let E be an uncountable subset of the Σ-product. For each subset G of $\Sigma_\alpha P_\alpha$, if G meets E then we select a point $e(G)$ in $G \cap E$. Also, for each $e = (e_\alpha)_\alpha$ in E, fix a surjection $f_e : \omega \to supp(e) = \{\alpha < \kappa : e_\alpha \neq 0_\alpha\}$.

Let E_0 be the singleton $\{e(\Sigma P_\alpha)\}$. Now, inductively define $A_n = \{f_e(i) : e \in E_n,\ 1 \leq i \leq n\}$, $\mathcal{C}_n = \mathcal{C}(A_n, n)$, and $E_{n+1} = E_n \cup \{e(C \setminus E_n) : C \in \mathcal{C}_n \text{ and } (C \setminus E_n) \cap E \neq \emptyset\}$. Then the E_n's are finite subsets of E, the A_n's are finite subsets of κ, and each $\mathcal{C}_n$ is a finite clopen partition of ΣP_α. We set $E' = \bigcup_n E_n$, $A = \bigcup_n A_n$, and $\mathcal{C} = \bigcup \mathcal{C}_n$. Note that A is the union of $\{supp(e) : e \in E'\}$. Observe also that the projections, $\{\pi_A(C) : C \in \mathcal{C}\}$, of $\mathcal{C}$ into $P_A = \prod\{P_\alpha : \alpha \in A\}$ form a base for P_A.

Since E' is countable and E is uncountable, we can pick e in $E \setminus E'$. For each n, there is a (unique) C_n in $\mathcal{C}_n$ such that e is in C_n. Thus, $(C_n \setminus E_n) \cap E$ is nonempty for each n, so by the inductive construction above, the point $e_n = e(C_n \setminus E_n)$ is in $E_{n+1} \setminus E_n$.

Write $e = (e_\alpha)_\alpha$, and let $x = (x_\alpha)_\alpha$ be the point in $\Sigma_\alpha P_\alpha$ given by $x_\alpha = e_\alpha$ for each α in A

and $x_\alpha = 0_\alpha$ otherwise. The sets $\pi_A(C_n)$ form a decreasing local base at $\pi_A(x) = \pi_A(e)$ in P_A, so the $\pi_A(e_n)$'s converge in P_A to $\pi_A(x)$. In fact, since $x_\alpha = 0_\alpha$ for each $\alpha \notin A \supseteq \bigcup_n supp(e_n)$, then $(e_n)_n$ converges to x in $\Sigma_\alpha P_\alpha$. $\square$

Theorem 137. *Let $\mathcal{Q}$ be a class of directed sets. The following are equivalent:*

(i) For any family $\{P_\alpha : \alpha < \kappa\}$ of directed sets in the class $\mathcal{Q}$, $\Sigma_\alpha P_\alpha$ has calibre (ω_1, ω).

(ii) For any family $\{P_\alpha : \alpha < \kappa\}$ of directed sets in the class $\langle \mathcal{Q}$, countably directed$\rangle$, $\Sigma_\alpha P_\alpha$ has calibre (ω_1, ω).

Proof. By Lemma 27, a directed set Q always has type $\langle Q$, countably directed$\rangle$, so $\mathcal{Q}$ is contained in $\langle \mathcal{Q}$, countably directed$\rangle$, and (ii) immediately implies (i).

Assume (i) and let $\{P_\alpha : \alpha < \kappa\}$ be a family of directed sets in the class $\langle \mathcal{Q}$, countably directed$\rangle$ such that each P_α has a minimal element 0_α. So for every $\alpha < \kappa$, we have a directed set Q_α in $\mathcal{Q}$ such that $P_\alpha = \bigcup\{P_{\alpha,q} : q \in Q_\alpha\}$, where each $P_{\alpha,q}$ is countably directed and $P_{\alpha,q_1} \subseteq P_{\alpha,q_2}$ whenever $q_1 \leq q_2$ in Q_α. We may assume that Q_α contains a minimal element $0'_\alpha$ and that $P_{\alpha,0'_\alpha} = \{0_\alpha\}$.

Fix a map $\psi_\alpha : P_\alpha \to Q_\alpha$ for each $\alpha < \kappa$ such that $p \in P_{\alpha,\psi_\alpha(p)}$ for each p in P_α. We can choose ψ_α so that $\psi_\alpha(0_\alpha) = 0'_\alpha$. Then let $\psi = \prod_\alpha \psi_\alpha : \prod_\alpha P_\alpha \to \prod_\alpha Q_\alpha$, which restricts to a map from $\Sigma_\alpha P_\alpha$ into $\Sigma_\alpha Q_\alpha$. By (i), we know that $\Sigma_\alpha Q_\alpha$ has calibre (ω_1, ω), so according to Lemma 18, it suffices to show that for any unbounded countable subset E of $\Sigma_\alpha P_\alpha$, its image $\psi(E)$ is unbounded in $\Sigma_\alpha Q_\alpha$.

Let E be a countable subset of $\Sigma_\alpha P_\alpha$, and suppose $\psi(E)$ is bounded (above) by some $q = (q_\alpha)_\alpha$ in $\Sigma_\alpha Q_\alpha$. Then for every $e = (e_\alpha)_\alpha$ in E and any $\alpha < \kappa$, we have $\psi_\alpha(e_\alpha) \leq q_\alpha$ and so $e_\alpha \in P_{\alpha,\psi_\alpha(e_\alpha)} \subseteq P_{\alpha,q_\alpha}$. As P_{α,q_α} is countably directed, there is a p_α in P_{α,q_α} such that $e_\alpha \leq p_\alpha$ for each $e \in E$. Thus, $p = (p_\alpha)_\alpha$ is an upper bound for E in $\prod_\alpha P_\alpha$. Moreover, since $P_{\alpha,0'_\alpha} = \{0_\alpha\}$ for every α, then $supp(p) \subseteq supp(q)$, and so p is in $\Sigma_\alpha P_\alpha$. $\square$

Combining Theorem 136 and Theorem 137 gives:

Corollary 138. *If $\{P_\alpha : \alpha < \kappa\}$ is a family of topological directed sets in the class $\langle$cosmic+CSBS, countably directed$\rangle$, then $\Sigma_\alpha P_\alpha$ has calibre (ω_1, ω).*

5.2 FINITE AND COUNTABLE PRODUCTS

5.2.1 DEFINITIONS AND QUESTIONS

As mentioned in the introduction to this chapter, any finite power of a directed set with calibre (ω_1, ω) has calibre (ω_1, ω). We say a directed set P is *powerfully calibre* (ω_1, ω) if P^ω is calibre (ω_1, ω). By Lemma 57, we know that $\mathcal{K}(M)$ is powerfully calibre (ω_1, ω) for every separable metrizable M. Also, we say a directed set P is *productively calibre* (ω_1, ω) if $P \times Q$ is calibre (ω_1, ω) for every directed set Q with calibre (ω_1, ω). Todorčević's Example 13 shows that a directed set with calibre (ω_1, ω) need not be productively calibre (ω_1, ω), but by Lemma 26, we do know that if P has calibre (ω, ω) or calibre (ω_1, ω_1), then P is productively calibre (ω_1, ω). We will see in Corollary 140 below that calibre (ω, ω) and calibre (ω_1, ω_1) each also imply powerfully calibre (ω_1, ω).

Question 2. *Is there* any *directed set P which is productively calibre (ω_1, ω), but neither calibre (ω_1, ω_1) nor calibre (ω, ω)?*

In Section 5.3.2 below, we give an example of a directed set with calibre (ω_1, ω) that is neither productively calibre (ω_1, ω) nor powerfully calibre (ω_1, ω). It is inspired by Todorčević's Example 13. The next question remains open:

Question 3. *Is there a directed set P which is productively calibre (ω_1, ω) but not powerfully calibre (ω_1, ω)?*

On the other hand, in Section 5.3.3, we do give an example of a directed set (of the form $\mathcal{K}(X)$) which is powerfully calibre (ω_1, ω) but not productively calibre (ω_1, ω). However, we would still like to answer:

Question 4. *(a) Is $\mathcal{K}(\omega^\omega)$ productively calibre (ω_1, ω)? The answer is 'Yes' when $\mathfrak{b} > \omega_1$, in which case $\mathcal{K}(\omega^\omega)$ has calibre (ω_1, ω_1) (see Lemma 23). What about when $\mathfrak{b} = \omega_1$?*

(b) Is every $\mathcal{K}(M)$, where M is separable metrizable, productively calibre (ω_1, ω)? (We do know each $\mathcal{K}(M)$ is powerfully calibre (ω_1, ω).)

(c) If S is a stationary, co-stationary subset of ω_1, then is $\mathcal{K}(S)$ powerfully calibre (ω_1, ω)?

We would like a non-productive $\mathcal{K}(M)$ example and are hoping that the following is true:

$\mathcal{K}(\omega^\omega)$ is productive iff $\mathcal{K}(\omega^\omega)$ is calibre (ω_1, ω_1) iff $\mathfrak{b} > \omega_1$. It remains to show: if $\mathfrak{b} = \omega_1$ then there is an X such that $\mathcal{K}(X) \times \mathcal{K}(\omega^\omega)$ is not calibre (ω_1, ω). And the natural X is $M(\mathbb{Q} \cup U, U)$ where is U is an unbounded subset of ω^ω of size ω_1. (It would follow that under $\mathfrak{b} = \omega_1$ no non-locally compact separable metrizable space has productive $\mathcal{K}(M)$ – since $\mathcal{K}(M) \geq_T \mathcal{K}(\omega^\omega)$.)

5.2.2 POSITIVE RESULTS

Lemma 139. *If κ is an infinite regular cardinal and (P'_n, P_n) has calibre (κ, κ, ω) for each $n < \omega$, then $(\prod_n P'_n, \prod_n P_n)$ has calibre (κ, ω).*

Proof. Let S be a κ-sized subset of $\prod_n P'_n$. By Lemma 16, there is a κ-sized subset S'_0 of S such that the projection $\pi_0 : \prod_n P_n \to P_0$ is one-to-one or constant on S'_0. Since (P'_0, P_0) has calibre (κ, κ, ω), then we can find a κ-sized subset S_0 of S'_0 such that for any countable subset C of S_0, $\pi_0(C)$ is bounded in P_0. Continuing inductively, we obtain a decreasing sequence $(S_m)_{m<\omega}$ of κ-sized subsets of S such that for any $m < \omega$ and any countable subset C of S_m, the projection $\pi_m : \prod_n P_n \to P_m$ is bounded on C.

Now choose distinct $\mathbf{x}_m$ in S_m for each $m < \omega$. Since $C_n = \{\mathbf{x}_m : m \geq n\}$ is a countable subset of S_n, then we can find an upper bound x_n^∞ for $\pi_n(C_n)$ in P_n, and since P_n is directed, then we can also find an upper bound z_n for $\{x_n^\infty\} \cup \{\pi_n(\mathbf{x}_m) : m < n\}$ in P_n. Thus, $\mathbf{z} = (z_n)_n$ is an upper bound for $\{\mathbf{x}_m : m < \omega\}$ in $\prod_n P_n$. □

Corollary 140. *Calibre (ω_1, ω_1) and calibre (ω, ω) each imply powerfully calibre (ω_1, ω).*

Proof. If P has calibre (ω_1, ω_1), then it also has calibre $(\omega_1, \omega_1, \omega)$, and so P^ω has calibre (ω_1, ω) by Lemma 139. On the other hand, if P has calibre (ω, ω), then Lemma 139 shows that P^ω has calibre (ω, ω), which implies calibre (ω_1, ω). □

Lemma 141. *Let κ be an infinite regular cardinal, and let P_n and Q_n be directed sets for each $n < \omega$.*

(1) If P_n has type $\langle Q_n$, (relative) calibre $(\kappa, \kappa, \omega)\rangle$ for each $n < \omega$, then $\prod_n P_n$ has type $\langle \prod_n Q_n$, (relative) calibre $(\kappa, \omega)\rangle$.

(2) If P_n has type $\langle Q_n$, countably directed$\rangle$ for each $n < \omega$, then $\prod_n P_n$ has type $\langle \prod_n Q_n$, countably directed$\rangle$.

Proof. For (1), we prove the relative calibre case. The proof of the calibre case can be obtained by omitting the parts in square brackets. Say $P_n = \bigcup\{P_{n,q} : q \in Q_n\}$ where $P_{n,q} \subseteq P_{n,q'}$ whenever $q \le q'$ in Q_n and each $P_{n,q}$ has [relative] calibre (κ, κ, ω) [in P_n]. For each $\mathbf{q} = (q_n)_{n<\omega}$ in $\prod_n Q_n$, define $R_{\mathbf{q}} = \prod_n P_{n,q_n}$. Then $R_{\mathbf{q}}$ has [relative] calibre (κ, ω) [in $\prod_n P_n$] by Lemma 139, and $\{R_{\mathbf{q}} : \mathbf{q} \in \prod_n Q_n\}$ is a $(\prod_n Q_n)$-ordered cover of $\prod_n P_n$.

For (2), a similar proof works, but note that if each $P_{n,q}$ is countably directed (in itself), then so is each $R_{\mathbf{q}}$. □

Lemma 142. *If Q is productively calibre (ω_1, ω) and P has type $\langle Q$, relative calibre $(\omega, \omega)\rangle$, then P is also productively calibre (ω_1, ω).*

Proof. Let S be any directed set with calibre (ω_1, ω). By Lemma 27, S has type $\langle S$, relative calibre $(\omega, \omega)\rangle$, and so by Lemma 141, $P \times S$ has type $\langle Q \times S$, relative calibre $(\omega, \omega)\rangle$. Since $Q \times S$ has calibre (ω_1, ω), then by Lemma 28, $P \times S$ does also. □

Corollary 143. *If $P = \bigcup_{n<\omega} P_n$ where each P_n has relative calibre (ω, ω) in P, then P is productively calibre (ω_1, ω).*

Proof. We may assume $P_n \subseteq P_{n+1}$ for each n, and so P is of type $\langle Q$, relative calibre $(\omega, \omega)\rangle$, where $Q = \omega$ is productively calibre (ω_1, ω). Apply the previous Lemma. □

Corollary 144. *Let $\mathcal{Q}$ be a class of directed sets such that whenever Q_n is in $\mathcal{Q}$ for each $n < \omega$, there is a Q in $\mathcal{Q}$ with $Q \ge_T \prod_n Q_n$. Let $\mathcal{D}$ denote the class of all Dedekind complete directed sets.*

(1) The class $\mathcal{D} \cap \langle \mathcal{Q}$, countably directed$\rangle$ is closed under countable products.

(2) If $\mathcal{Q} \subseteq \mathcal{D}$, then $\langle \mathcal{Q}$, countably directed$\rangle$ is closed under countable products.

Proof. We prove (1) first. Suppose that for each $n < \omega$, P_n is Dedekind complete and has type $\langle Q_n$, countably directed$\rangle$ where Q_n is in $\mathcal{Q}$. By Lemma 141, $P = \prod_n P_n$ has type $\langle Q'$, countably directed$\rangle$ where $Q' = \prod_n Q_n$, and by assumption, there is a Q in $\mathcal{Q}$ such that

$Q \geq_T Q'$. Since each P_n is Dedekind complete, then so is P, and thus Lemma 35 implies that P has type $\langle Q, \text{countably directed}\rangle$.

For (2), we use the same notation as for (1). If each Q_n is Dedekind complete, then so is Q'. Thus, even if P is not Dedekind complete, we can use Lemma 35 to conclude that P has type $\langle Q, \text{countably directed}\rangle$. $\square$

Corollary 145. *Suppose P_n has a type $\langle Q_n$, relative calibre $(\omega, \omega)\rangle$ for each $n < \omega$. If $\prod_n Q_n$ has calibre (ω_1, ω), then so does $\prod_n P_n$. In particular:*

(1) If $\kappa = \omega$ or $\kappa = \omega_1$, then any countable product of members of the class $\langle$calibre κ, relative calibre $(\omega, \omega)\rangle$ has calibre (ω_1, ω).

(2) Each member of the class $\langle$powerfully calibre (ω_1, ω), relative calibre $(\omega, \omega)\rangle$ is powerfully calibre (ω_1, ω).

Proof. The main claim follows from Lemma 141 (with $\kappa = \omega$) and Lemma 28 (with $\kappa = \omega_1$ and $\lambda = \mu = \omega$).

Statement (1) then follows from Corollary 140. Statement (2) is the special case of the main claim where the P_n's are all the same, and the Q_n's are also all the same. $\square$

Corollary 146. *If $P = \bigcup_{n<\omega} P_n$ where each P_n has relative calibre (ω, ω) in P, then P is powerfully calibre (ω_1, ω).*

Proof. We may assume $P_n \subseteq P_{n+1}$ for each $n < \omega$. Then P has type $\langle Q, \text{relative calibre } (\omega, \omega)\rangle$, where $Q = \omega$ is powerfully calibre (ω_1, ω), so we may apply (2) in the previous corollary. $\square$

Theorem 147. *Each member of the class $\langle 2^o$+CSB, countably directed$\rangle$ has calibre (ω_1, ω). In particular, each member of the class $\langle \mathcal{K}(\mathcal{M})$, countably directed$\rangle$ has calibre (ω_1, ω). Additionally, the following classes are closed under countable products:*

(1) the class of all Dedekind complete directed sets with type $\langle 2^o$+CSB, countably directed$\rangle$

(2) $\langle 2^o$+CSB+Dedekind Complete, countably directed$\rangle$

(3) $\langle \mathcal{K}(\mathcal{M})$, countably directed$\rangle$.

Proof. By Theorem 43, every second countable and CSB directed set has calibre (ω_1, ω). Since countably directed implies (relative) calibre (ω, ω), then Lemma 28 shows that the members of the class $\langle 2^o{+}\text{CSB}, \text{countably directed}\rangle$ have calibre (ω_1, ω). Closure under countable products follows from Corollary 144, Theorem 43, and Lemma 56. $\square$

5.3 EXAMPLES

5.3.1 PRODUCTIVE AND POWERFUL

Example 14. *Let $\mathcal{I} = \bigcup_{c<\omega} \mathcal{I}_c$, where $\mathcal{I}_c = \{A \subseteq \omega : |A \cap 2^n| \leq n^c,\ \forall n \geq 2\}$. Then $\mathcal{I}$ is an ideal on ω (called the ideal of polynomial growth) and can naturally be viewed as a subspace of the Cantor set, $\mathbb{P}(\omega) = \{0,1\}^\omega$. We have:*

(1) each $\mathcal{I}_c$ has relative calibre (ω, ω) in $\mathcal{I}$,

(2) $\mathcal{I}$ is productively and powerfully calibre (ω_1, ω), and

(3) $\mathcal{I}$ is second countable but not CSBS.

Proof. This is Example 1 from [35], where it is shown that: for all infinite $\mathcal{S} \subseteq \mathcal{I}_c$, there is an infinite subset $\mathcal{S}'$ such that $\bigcup \mathcal{S}' \in \mathcal{I}_{c+1}$. This immediately gives (1). Statement (2) follows from Corollaries 146 and 143, respectively.

Of course $\mathcal{I}$ is second countable as a subspace of the Cantor set, and we now show $\mathcal{I}$ is not CSBS to complete (3). Consider $A_n = [2^n, 2^{n+1}) \subseteq \omega$ for each $n < \omega$. Then each A_n is in $\mathcal{I}$ since it is finite, and the sequence $(A_n)_{n<\omega}$ converges to $\emptyset$. Let $(A_{n_k})_{k<\omega}$ be any subsequence. Fix $c < \omega$ and find a $j < \omega$ such that $2^m > 2m^c$ for all $m \geq j$. If $A \subseteq \omega$ contains every A_{n_k}, then we have:

$$|A \cap 2^{n_j+1}| \geq |A_{n_j}| = 2^{n_j} = \frac{2^{n_j+1}}{2} > \frac{2(n_j+1)^c}{2} = (n_j+1)^c,$$

and so A is not in $\mathcal{I}_c$. Thus $(A_{n_k})_{k<\omega}$ has no upper bound in $\mathcal{I}$, and $\mathcal{I}$ is not CSBS. $\square$

5.3.2 NOT PRODUCTIVE AND NOT POWERFUL

Lemma 148. *Each stationary subset of ω_1 has countable extent.*

Proof. Let S be a stationary subset of ω_1 and let E be a closed discrete subset of S. For each s in S, there is an $\alpha_s < s$ such that $(\alpha_s, s] \cap E \subseteq \{s\}$. By the pressing down lemma, there is a stationary subset S' of S and an $\alpha < \omega_1$ such that $\alpha_s = \alpha$ for each s in S'. It follows that α is an upper bound for E, so E is countable. $\square$

Lemma 149. *If M is separable metrizable, D is a dense subset of M, and K is a compact subset of M disjoint from D, then there is a discrete subset D' of D whose closure in M is precisely $D' \cup K$.*

Proof. Fix a compatible metric ρ for X. Let A be a countable dense subset of K. Enumerate $A = \{x_n\}_n$ so that each element is repeated infinitely many times. For each n, pick d_n in D such that $\rho(d_n, x_n) < 1/n$. Let $D' = \{d_n\}_n$. Clearly K is contained in the closure of D', but note also that if $(d_{n_k})_k$ converges to x in M, then so does $(x_{n_k})_k$, and so all limit points of D' are in K. It follows that the closure of D' is contained in $D' \cup K$, and since D' is disjoint from K, then it also follows that D' is discrete. $\square$

Example 15. *There is a directed set P which is calibre (ω_1, ω) but is neither powerfully calibre (ω_1, ω) nor productively calibre (ω_1, ω).*

Proof. Let I be the isolated points in ω_1. Write ω_1 as a disjoint union of stationary (and co-stationary) sets, $\omega_1 = \bigcup_n S'_n$. Let $S_n = S'_n \cup I$, and let $X_n = M(\omega_1, S_n)$, that is, ω_1 with its topology refined by isolating the points in S_n. We consider $P = \mathcal{K}(X)$, where $X = \bigoplus_n X_n$.

$\mathcal{K}(S_1)$ is calibre (ω_1, ω) since S_1 is stationary (and co-stationary), but $\mathcal{K}(X) \times \mathcal{K}(S_1) =_T \mathcal{K}(X \times S_1)$ is not calibre (ω_1, ω) according to Lemma 48 since $\mathcal{K}(X \times S_1)$ does not have countable extent. Indeed, $\{(\alpha, \alpha) : \alpha \in S_1\}$ is an uncountable closed discrete subset of $X_1 \times S_1$, which is a closed subset of $X \times S_1$, which in turn is a closed subset of $\mathcal{K}(X \times S_1)$. Similarly $\mathcal{K}(X)^\omega =_T \mathcal{K}(X^\omega)$ does not have calibre (ω_1, ω) because $\{(\alpha, \alpha, \ldots) : \alpha \in \omega_1\}$ is an uncountable closed discrete subset of $\prod_n X_n$, which is a closed subset of X^ω, which in turn is a closed subset of $\mathcal{K}(X^\omega)$.

It remains to show that $\mathcal{K}(X)$ has calibre (ω_1, ω). Since $\mathcal{K}(X) = \bigcup_n \mathcal{K}(X_1 \oplus \cdots \oplus X_n)$, we see that $\mathcal{K}(X)$ has calibre (ω_1, ω) provided each $\mathcal{K}(X_1 \oplus \cdots \oplus X_n) =_T \mathcal{K}(X_1) \times \cdots \times \mathcal{K}(X_n)$ has calibre (ω_1, ω). This follows from the following three claims:

Claim (A): For any stationary (and co-stationary) subset S of ω_1 containing I we have $\mathcal{K}(Y) =_T CD(S)$, where $Y = M(\omega_1, S)$.

Define $\phi_1 : CD(S) \to \mathcal{K}(Y)$ by $\phi_1(E) = \overline{E}^Y$ and $\phi_2 : \mathcal{K}(Y) \to CD(S)$ by $\phi_2(K) = K \cap S$. Then ϕ_2 is well-defined since, for any compact subset K of Y, the topologies K inherits from Y and ω_1 coincide. To see that ϕ_1 is well-defined, first notice that for any closed subset E of S, the subspaces $\overline{E}^Y$ and $\overline{E}^{\omega_1}$ are equal as sets, but if E is also discrete, then in fact, their topologies coincide as well. By Lemma 148, each closed discrete subset E of S is countable, so $\overline{E}^{\omega_1}$ is compact and ϕ_1 is well-defined.

Clearly ϕ_1 and ϕ_2 are order-preserving, so it remains to show that their images are cofinal in $\mathcal{K}(Y)$ and $CD(S)$, respectively. Since $\phi_2(\phi_1(E)) = E$ for every E in $CD(S)$, we see that, in fact, ϕ_2 is onto.

Now we check that the image of ϕ_1 is cofinal. Let K be a compact subset of Y, and let $E_0 = \phi_2(K) \in CD(S)$. Then $K \cap S = E_0$ is contained in $\phi_1(E_0)$. We aim to find a closed discrete subset E_1 of S such that $\phi_1(E_1)$ contains $K_1 = K \setminus S$. Then $E = E_0 \cup E_1$ will be a closed discrete subset of S such that $\phi_1(E)$ contains K.

Note that K_1 is compact as a subspace of Y and so also compact as a subspace of ω_1. Hence, K_1 is contained in some interval $[0, \alpha]$. Now, $[0, \alpha]$ is countable and first countable, so it is separable and metrizable. Since S contains I, then $S \cap [0, \alpha]$ is dense in $[0, \alpha]$, and since K_1 is disjoint from S, then Lemma 149 provides a discrete subset E_1 of $S \cap [0, \alpha]$ whose closure in $[0, \alpha]$ is $E_1 \cup K_1$. Since K_1 is disjoint from S, then E_1 is closed in S, and $\overline{E_1}^Y = \overline{E_1}^{[0,\alpha]}$. Hence, K_1 is contained in $\phi_1(E_1)$, as desired.

Claim (B): For any subsets $S_1, \ldots, S_n$ of ω_1 we have $CD(S_1) \times \cdots \times CD(S_n) =_T CD(S_1 \oplus \cdots \oplus S_n)$.

The map $\phi : CD(S_1) \times \cdots \times CD(S_n) \to CD(S_1 \oplus \cdots \oplus S_n)$ given by $\phi(A_1, \ldots, A_n) = A_1 \oplus \cdots \oplus A_n$ is a well-defined order-isomorphism, which gives claim (B).

Claim (C): Let S be a stationary subset of ω_1, and let A be an uncountable subset of ω_1. If $\{E_\alpha : \alpha \in A\}$ is a family of closed discrete subsets of S such that $B_\gamma = \{\alpha \in A : E_\alpha \subseteq [0, \gamma]\}$

is countable for each $\gamma < \omega_1$, then there is a $\gamma < \omega_1$ and an uncountable $A' \subseteq A \setminus B_\gamma$ such that $\sup E_\alpha < \min(E_\beta \setminus [0, \gamma])$ whenever $\alpha < \beta$ in A'.

Let S, A, E_α, and B_γ be as in the statement of claim (C). Fix a bijection $f : \omega_1 \to A$ such that $\alpha < \beta$ if and only if $f(\alpha) < f(\beta)$. Each α in S has a neighborhood that intersects $E_{f(\alpha)}$ in at most one point, and since S is stationary, then by the pressing down lemma, we can find a stationary subset S' of S and a $\gamma < \omega_1$ such that $(\gamma, \alpha] \cap E_{f(\alpha)} \subseteq \{\alpha\}$ for each α in S'. For any α in the uncountable set $S' \setminus f^{-1}[B_\gamma]$, we therefore have $\min(E_{f(\alpha)} \setminus [0, \gamma]) \geq \alpha$. Note that, by Lemma 148, each E_α is countable. We can then inductively construct an uncountable subset S'' of $S' \setminus f^{-1}[B_\gamma]$ such that $\beta > \sup_{\alpha \in S'' \cap \beta} \sup E_{f(\alpha)}$ for each β in S''. Let $A' = f[S''] \subseteq A \setminus B_\gamma$. If $f(\alpha) < f(\beta)$ in A', then we have $\sup E_{f(\alpha)} < \beta \leq \min(E_{f(\beta)} \setminus [0, \gamma])$, which completes the proof of claim (C).

Claim (D): If $S_1, \ldots, S_n$ are subsets of ω_1 whose union is co-stationary then $CD(S_1 \oplus \cdots \oplus S_n)$ has calibre (ω_1, ω).

Let $S_1, \ldots, S_n$ be stationary subsets of ω_1 such that $\bigcup_i S_i$ is co-stationary. Let $\{E_\alpha : \alpha < \omega_1\}$ be an uncountable family of closed discrete subsets of $S_1 \oplus \cdots \oplus S_n$. Write $E_\alpha = E_\alpha^1 \oplus \cdots \oplus E_\alpha^n$ where E_α^i is a closed discrete subset of S_i. We may assume that for every $i \leq n$ and every $\gamma < \omega_1$, there are only countably many E_α^i contained in $[0, \gamma]$ (for example, by adding one point from $S_i \cap [\alpha, \omega_1)$ to E_α^i).

By applying claim (C) repeatedly, we can then find uncountable subsets $A_1' \supseteq A_2' \supseteq \cdots \supseteq A_n'$ of ω_1 and $\gamma_1, \ldots, \gamma_n < \omega_1$ such that for each i we have $E_\alpha^i \not\subseteq [0, \gamma_i]$ for all α in A_i' and $\sup E_\alpha^i < \min(E_\beta^i \setminus [0, \gamma_i])$ whenever $\alpha < \beta$ in A_i'. Let $\gamma = \max\{\gamma_1, \ldots, \gamma_n\}$. We can find an uncountable subset A_0 of A_n' such that $E_\alpha^i \not\subseteq [0, \gamma]$ for any α in A_0 and any i, and such that whenever $\alpha < \beta$ in A_0, we have $u_\alpha < \ell_\beta$; here $u_\alpha = \max_i \sup E_\alpha^i$ and $\ell_\alpha = \min_i \min(E_\alpha^i \setminus [0, \gamma])$ for any α in A_0.

Enumerate $S_i \cap [0, \gamma] = \{s_m^i : m < \omega\}$ for each $i \leq n$. We construct a decreasing sequence $(A_m)_{m<\omega}$ of uncountable subsets of A_0 and open neighborhoods U_m^i of s_m^i in S_i as follows. Assume we have already defined A_{m-1} for some $0 < m < \omega$. Since E_α^i is closed discrete in S_i and s_m^i has a countable neighborhood base in S_i for each $i \leq n$, we can find an uncountable subset A_m of A_{m-1} and open neighborhoods U_m^i of s_m^i in S_i such that $U_m^i \cap E_\alpha^i \subseteq \{s_m^i\}$ for every α in A_m.

For each $m < \omega$, let $C_m = \{u_\alpha : \alpha \in A_m\}$. Then $C = \bigcap_m \overline{C_m}^{\omega_1}$ is closed and unbounded in ω_1, and so is the subset C' of all limit points in C. As $\bigcup_i S_i$ is co-stationary, we can find a u_∞ in C' that is not in any S_i. Hence, we can choose an increasing sequence $(\alpha_m)_m$ such that α_m is in A_m and $(u_{\alpha_m})_m$ converges to u_∞. Since $\sup E^i_{\alpha_m} \leq u_{\alpha_m} < \ell_{\alpha_{m+1}} \leq \min(E^i_{\alpha_{m+1}} \setminus [0, \gamma])$ for each m, and since u_∞ is not in S_i, then we see that $\bigcup_m (E^i_{\alpha_m} \setminus [0, \gamma])$ is closed discrete in S_i for each i. On the other hand, $\bigcup_m (E^i_{\alpha_m} \cap [0, \gamma])$ is also closed discrete in S_i because for any s^i_k in $S_i \cap [0, \gamma]$, we have $U^i_k \cap E^i_{\alpha_m} \subseteq \{s^i_k\}$ for every $m \geq k$ (since $\alpha_m \in A_m \subseteq A_k$). Thus, $\bigcup_m E_{\alpha_m}$ is closed discrete in $S_1 \oplus \cdots \oplus S_n$, which completes the proof of claim (D). □

5.3.3 POWERFUL BUT NOT PRODUCTIVE AND Σ-PRODUCTS

We now show that there is a directed set which is powerfully calibre (ω_1, ω) but not productively calibre (ω_1, ω). We simultaneously show that the restriction in Theorem 136 to cosmic directed sets with CSBS is not arbitrary.

If ΣP_α has calibre (ω_1, ω) then every projection has the same calibre. In particular, every countable subproduct of the P_α's has calibre (ω_1, ω). It is natural to hope that this necessary condition for a Σ-product to have calibre (ω_1, ω) is also sufficient. We show that this is not the case, even when all the directed sets, P_α, are equal.

Example 16. *There is a directed set P such that:*

(1) the Σ-product, $P \times \Sigma\omega^{\omega_1}$ does not have calibre (ω_1, ω), but every countable subproduct does have calibre (ω_1, ω); and

(2) ΣP^{ω_1} does not have calibre (ω_1, ω), but P^ω has calibre (ω_1, ω).

Hence both P and $\Sigma\omega^{\omega_1}$ are powerfully calibre (ω_1, ω) but not productively calibre (ω_1, ω).

Proof. We prove the final claim first, so assume there is a directed set P satisfying (1) and (2). From (2) P is powerfully calibre (ω_1, ω). Since $\Sigma\omega^{\omega_1}$ is clearly Tukey equivalent to its countable power, from Theorem 136 we see that $\Sigma\omega^{\omega_1}$ is also powerfully calibre (ω_1, ω). Now (1) says that neither P nor $\Sigma\omega^{\omega_1}$ are productively calibre (ω_1, ω).

So we need to show that there is a directed set P with properties (1) and (2). Let X be any subset of the reals which is totally imperfect and has size ω_1 (say an ω_1-sized subset of a Bernstein set — see the end of Section 2.7). Index $X = \{x_\alpha : \alpha < \omega_1\}$. For each α let

$U_\alpha = \{x_\beta : \beta \geq \alpha\}$ and $C_\alpha = X \setminus U_\alpha = \{x_\beta : \beta < \alpha\}$. Refine the standard topology on X (inherited from $\mathbb{R}$) by adding the sets U_α, for α in ω_1, as open sets. We write X for the set X with this topology. It is a Hausdorff space. Let $P = \mathcal{K}(X)$. Then P is a Hausdorff topological directed set with CSB and DK. We verify two claims:

Claim (A): $\mathcal{K}(X) \times \Sigma\omega^{\omega_1}$ does not have calibre (ω_1, ω)

Claim (B): $\mathcal{K}(X)^\omega$ is calibre (ω_1, ω).

Since each compact subset of X is also a compact subset of $\mathbb{R}$, then note that $\mathcal{K}(X) \geq_T \mathcal{K}(X, \tau_\mathbb{R})$ where $\tau_\mathbb{R}$ is the topology X initially inherited from $\mathbb{R}$, and since every $\mathbb{R}$-compact subset of X is countable, then $(X, \tau_\mathbb{R})$ is not compact, and so Lemma 61 shows that $\mathcal{K}(X) \geq_T \omega$. Thus, we have $\Sigma\mathcal{K}(X)^{\omega_1} \geq_T \mathcal{K}(X) \times \Sigma\omega^{\omega_1}$ and $\mathcal{K}(X)^\omega \geq_T \mathcal{K}(X) \times \omega^\omega$. Hence claims (1) and (2) follow from claims (A) and (B).

Proof of (A): By Lemmas 45 and 41, we know it suffices to show $P \times \Sigma\omega^{\omega_1}$ contains an uncountable closed discrete set.

For each α, fix a decreasing local base, $(B_{\alpha,n})_n$, at x_α such that $B_{\alpha,1} \subseteq U_\alpha$. Define $\mathbf{y}_\alpha = (y_{\alpha,\beta})_\beta$ in $\Sigma\omega^{\omega_1}$ by $y_{\alpha,\beta} = \min\{n : x_\alpha \notin B_{\beta,n}\}$ if $\beta < \alpha$ and 0 for $\beta \geq \alpha$. Then let $E = \{(x_\alpha, \mathbf{y}_\alpha) : \alpha < \omega_1\} \subseteq P \times \Sigma\omega^{\omega_1}$. Obviously E is uncountable. We show E is closed discrete in $P \times \Sigma\omega^{\omega_1}$

Since X is closed in $P = \mathcal{K}(X)$, it suffices to show E is closed and discrete in $X \times \Sigma\omega^{\omega_1}$. Fix $(x_\beta, \mathbf{z})$ in $X \times \Sigma\omega^{\omega_1}$. We need to find an open neighborhood of $(x_\beta, \mathbf{z})$ that contains at most one member of E. Consider $V = V_{\beta,\mathbf{z}} = B_{\beta,z_\beta} \times \pi_\beta^{-1}\{z_\beta\}$, which contains $(x_\beta, \mathbf{z})$.

Suppose $(x_\alpha, \mathbf{y}_\alpha)$ is in $E \cap V$. Then x_α is in $B_{\beta,z_\beta} \subseteq U_\beta$, so $\beta \leq \alpha$. If $\beta < \alpha$, then $y_{\alpha,\beta} = \min\{n : x_\alpha \notin B_{\beta,n}\} > z_\beta$, which contradicts that $\mathbf{y}_\alpha$ is in $\pi_\beta^{-1}\{z_\beta\}$. Thus, $\alpha = \beta$, so V contains at most one member of E, namely $(x_\beta, \mathbf{y}_\beta)$, which proves claim (A) since X is T_1.

Proof of (B): We will show that $P = \mathcal{K}(X)^\omega$ is first countable and hereditarily ccc (all discrete subsets are countable). Then P has countable extent, is first countable and CSB, and so has calibre (ω_1, ω) by Lemma 40.

That $\mathcal{K}(X)$ (and hence $\mathcal{K}(X)^\omega$) is first countable follows from Lemma 54. So we focus on showing $\mathcal{K}(X)^\omega$ contains no uncountable discrete subsets. It is a standard fact (and straightforward to check) that the countable power Y^ω of a space Y is herditarily ccc if and only if every finite power if Y is hereditarily ccc. Thus, we show each finite power $\mathcal{K}(X)^n$ is hereditarily ccc.

Recall that basic neighborhoods in $\mathcal{K}(X)$ (with the Vietoris topology) have the form $\langle U_1, \ldots, U_n\rangle = \{K \in \mathcal{K}(X) : K \subseteq \bigcup_i U_i,\ K \cap U_i \neq \emptyset\ \forall i = 1, \ldots, n\}$ for some $0 < n < \omega$ and U_i open in X; further, the open sets $U_1, \ldots, U_n$ can be assumed to come from any specified base for X.

Suppose $\mathcal{K}(X)^n$ is not hereditarily ccc. Then there is an uncountable discrete subset $\mathcal{S} = \{\mathbf{K}_\lambda = (K_1^\lambda, \ldots, K_n^\lambda) : \lambda < \omega_1\}$ of $\mathcal{K}(X)^n$, enumerated in a one-to-one fashion. So for each $\lambda < \omega_1$, there is a basic open $\mathbf{V}_\lambda = \prod_{i=1}^n \langle V_{i,1}^\lambda, \ldots, V_{i,m_\lambda}^\lambda\rangle$ such that $\mathbf{V}_\lambda \cap \mathcal{S} = \{\mathbf{K}_\lambda\}$.

Let $\mathcal{B}$ be a countable base for the usual topology on $X = \{x_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$, and define $C(\alpha) = \{x_\beta : \beta < \alpha\} = X \setminus U_\alpha$ for each $\alpha < \omega_1$. Then we can choose each $V_{i,j}^\lambda$ to have the form $V_{i,j}^\lambda = B_{i,j}^\lambda \setminus C(\alpha_{i,j}^\lambda)$ for some $B_{i,j}^\lambda \in \mathcal{B}$ and $\alpha_{i,j}^\lambda < \omega_1$. By replacing $\mathcal{S}$ with an appropriate uncountable subset, we may assume there is some $0 < m < \omega$ and $B_{i,j} \in \mathcal{B}$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ such that $m_\lambda = m$ and $B_{i,j}^\lambda = B_{i,j}$ for each $\lambda < \omega_1$. Then $\mathbf{V}_\lambda$ is uniquely determined by the corresponding $n \times m$ matrix $(\alpha_{i,j}^\lambda)_{i,j}$.

Consider the countable subset $\{\mathbf{K}_\ell : \ell < \omega\}$ of $\mathcal{S}$. If there are $k < \ell < \omega$ such that $\alpha_{i,j}^k \leq \alpha_{i,j}^\ell$ for each i, j, then $V_{i,j}^\ell = B_{i,j} \setminus C(\alpha_{i,j}^\ell) \subseteq B_{i,j} \setminus C(\alpha_{i,j}^k) = V_{i,j}^k$, which implies that $\mathbf{V}_\ell \subseteq \mathbf{V}_k$. But that is a contradiction since $\mathbf{K}_\ell$ is in $\mathbf{V}_\ell \setminus \mathbf{V}_k$. Hence, for any $k < \ell < \omega$, we have $\alpha_{i,j}^k > \alpha_{i,j}^\ell$ for some $i = i(k, \ell) \in \{1, \ldots, n\}$ and $j = j(k, \ell) \in \{1, \ldots, m\}$.

By Ramsey's Theorem, there is an infinite subset A of ω, an $i \in \{1, \ldots, n\}$, and a $j \in \{1, \ldots, m\}$ such that for any $k < \ell$ in A, we have $i(k, \ell) = i$ and $j(k, \ell) = j$. But then $\{\alpha_{i,j}^\ell : \ell \in A\}$ forms an infinite decreasing sequence in ω_1, which contradicts that ω_1 is well-ordered. Therefore, such $\mathcal{S}$ cannot exist. $\square$

Corollary 150. *There is a directed set P which is powerfully calibre (ω_1, ω) but not productively calibre (ω_1, ω).*

Proof. Let P be the directed set in Example 16. By (2) in that example, we know P is powerfully calibre (ω_1, ω), and by (1), we know that P is not productively calibre (ω_1, ω) since $\Sigma\omega^{\omega_1}$ has calibre (ω_1, ω) according to Theorem 136. $\square$

6.0 NETWORK PROPERTIES AND NEIGHBORHOOD FILTERS

Recall that $C_k(X)$ denotes the space of continuous real-valued functions on a space X with the compact-open topology. The main inspiration for this chapter comes from a result of Gabriyelyan et al. [22] which says that for spaces of the form $C_k(X)$ such that $\omega^\omega \geq_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$, the strong Pytkeev property is equivalent to countable tightness, and these are equivalent to X being Lindelöf. Here, a space Y is said to have the *strong Pytkeev property* at a point y in Y if there is a countable family $\mathcal{D}_y$ of subsets of Y such that, whenever U is a neighborhood of y and A is a subset of Y with $y \in \overline{A} \setminus A$, there is a D in $\mathcal{D}_y$ such that D is contained in U and $D \cap A$ is infinite. A space then has the *strong Pytkeev property* if it has the strong Pytkeev property at every point.

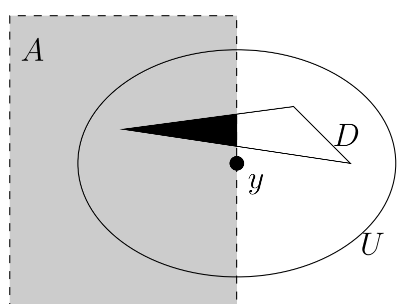

Figure 7: The strong Pytkeev property

The family $\mathcal{D}_y$ in the definition of the strong Pytkeev property is a sort of 'network' for the neighborhood filter $\mathcal{N}_y^Y$ since each neighborhood of y contains a member of $\mathcal{D}_y$ (and we could also assume each member of $\mathcal{D}_y$ contains y). So the strong Pytkeev property for a space Y asserts that for each point y in Y, there is a countable 'network' for $\mathcal{N}_y^Y$ with 'nice' properties. We investigate several variations of the strong Pytkeev property formed by altering the meaning of 'nice', some of which were introduced in [20]. For example, relaxing the condition '$D \cap A$ is infinite' in the definition of the strong Pytkeev property to '$D \cap A$ is nonempty', yields a property called (cn). In fact, we define these countable network properties (each receiving a designation of the form '(c·)') for any pre-filter in Section 6.2, and we describe their dual versions (receiving designations of the form '(c·)′') for pre-ideals in Section 6.1. Throughout this chapter, we investigate the order properties of neighborhood filters and their relationship with the (c·) and (c·)′ network properties.

Returning to the inspiration for this chapter, in Theorem 179 of Section 6.4, we give a complete characterization of when the space $C_k(X)$ has the strong Pytkeev property by proving that this occurs if and only if X is Lindelöf cofinally Σ (a property which is discussed in Section 6.3). We also prove that this is equivalent to $C_k(X)$ being countably tight and having $P \geq_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$ for some directed set P in the class $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$ (see Section 2.4). In fact, this result shows that several of the (c·) network properties (including (cn)) are equivalent to the strong Pytkeev property for spaces of the form $C_k(X)$. In Theorem 181, we analogously characterize precisely when $C_p(X)$ has the property (cn).

The class $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$ also plays important roles elsewhere in this chapter. For example, Gabriyelyan et al. proved in [21] that if a Fréchet-Urysohn topological group G has $\omega^\omega \geq_T \mathcal{N}_e^G$, where e is the identity in G, then G is metrizable. We generalize this result in Corollary 187 by showing that if P is in the class $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$, then any Fréchet-Urysohn topological group G with $P \geq_T \mathcal{N}_e^G$ is metrizable.

6.1 THE (C·)′ PROPERTIES FOR PRE-IDEALS

Fix a pre-ideal $\mathcal{S}$ on a set X. We say a family $\mathcal{N}$ of subsets of X is a *(pre-ideal) network* for $\mathcal{S}$ if each member of $\mathcal{S}$ is contained in some member of $\mathcal{N}$, and if $\bigcup \mathcal{N} = \bigcup \mathcal{S}$. Below, we define a family of properties (c·)′ that a countable network $\mathcal{N}$ for $\mathcal{S}$ may satisfy. We will also say that $\mathcal{S}$ itself satisfies (c·)′ if there is a countable network $\mathcal{N}$ satisfying (c·)′ for $\mathcal{S}$. Recall that pre-ideals and pre-filters on X are in duality (see Lemma 6). Consequently, there is a family of dual properties (c·) for pre-filters, which we describe in the next section.

(cp$_\omega$)$'$ If S is in $\mathcal{S}$, and if A is a countable subset of $\bigcup\mathcal{S}$ such that $A \setminus N$ is finite for each $N \in \mathcal{N}$ containing S, then there is a $T \in \mathcal{S}$ such that $A \subseteq T$.

(cp)$'$ If S is in $\mathcal{S}$, and if A is a subset of $\bigcup\mathcal{S}$ such that $A \setminus N$ is finite for each $N \in \mathcal{N}$ containing S, then there is a $T \in \mathcal{S}$ such that $A \subseteq T$.

(cs)$'$ If S is in $\mathcal{S}$, and if A is a countably infinite subset of X such that $A \cap N$ is infinite for each $N \in \mathcal{N}$ containing S, then there is a $T \in \mathcal{S}$ such that $A \cap T$ is infinite.

(cs*)$'$ If S is in $\mathcal{S}$, and if A is a countably infinite subset of X such that $A \setminus N$ is finite for each $N \in \mathcal{N}$ containing S, then there is a $T \in \mathcal{S}$ such that $A \cap T$ is infinite.

(cn)$'$ For any $S \in \mathcal{S}$, there is a $T \in \mathcal{S}$ such that $S \subseteq \bigcap\{N \in \mathcal{N} : S \subseteq N\} \subseteq T$.

(cn*)$'$ If S is in $\mathcal{S}$, and if A is a subset of $\bigcup\mathcal{S}$ such that $A \subseteq N$ for each $N \in \mathcal{N}$ containing S, then there is a $T \in \mathcal{S}$ such that $A \subseteq T$.

If X is a topological space, then we define the following additional properties (c·)$'$ which a countable pre-ideal network $\mathcal{N}$ may satisfy for a pre-ideal $\mathcal{S}$ on X. Once again, we say $\mathcal{S}$ itself has the property (c·)$'$ if there is a countable network $\mathcal{N}$ satisfying (c·)$'$ for $\mathcal{S}$.

(cck)$'$ For any $S \in \mathcal{S}$, there is a $T \in \mathcal{S}$ containing S such that for any closed, countably compact $C \subseteq X \setminus T$, there is an $N \in \mathcal{N}$ such that $S \subseteq N \subseteq X \setminus C$.

(ck)$'$ For any $S \in \mathcal{S}$, there is a $T \in \mathcal{S}$ containing S such that for any compact $K \subseteq X \setminus T$, there is an $N \in \mathcal{N}$ such that $S \subseteq N \subseteq X \setminus K$.

(ck$^+$)$'$ For any $S \in \mathcal{S}$, there is a $T \in \mathcal{S}$ containing S such that for any compact $K \subseteq X \setminus T$, there is an $N \in \mathcal{N}$ such that $T \subseteq N \subseteq X \setminus K$.

(cc)$'$ For any $S \in \mathcal{S}$, there is a $T \in \mathcal{S}$ containing S such that for any closed $C \subseteq X \setminus T$, there is an $N \in \mathcal{N}$ such that $S \subseteq N \subseteq X \setminus C$.

(cc$^+$)$'$ For any $S \in \mathcal{S}$, there is a $T \in \mathcal{S}$ containing S such that for any closed $C \subseteq X \setminus T$, there is an $N \in \mathcal{N}$ such that $T \subseteq N \subseteq X \setminus C$.

In most of our cases of interest, the pre-ideal $\mathcal{S}$ will cover X, so that the requirement in (cp)$'$ and some of the other (c·)$'$ properties that A be a subset of $\bigcup\mathcal{S}$ is redundant, as is the requirement that a pre-ideal network $\mathcal{N}$ for $\mathcal{S}$ satisfy $\bigcup\mathcal{N} = \bigcup\mathcal{S}$.

The following lemma says that, in the above properties, there is no difference whether we consider a pre-ideal $\mathcal{S}$ or any other pre-ideal between $\mathcal{S}$ and the ideal $\downarrow\!\mathcal{S}$ generated by $\mathcal{S}$. In principle, then, we could simply use ideals and eliminate all use of pre-ideals, but in practice, pre-ideals like $\mathcal{K}(X)$ are very natural and may have additional useful structure (like a topology) that the generated ideal may not.

Lemma 151. *Let $\mathcal{S}$ and $\mathcal{S}'$ be pre-ideals on X such that $\mathcal{S} \subseteq \mathcal{S}' \subseteq \downarrow\!\mathcal{S}$. Let $\mathcal{N}$ be a countable family of subsets of X, and let (c$\cdot$)$'$ be any of the above properties. Then $\mathcal{N}$ is a network satisfying (c$\cdot$)$'$ for $\mathcal{S}$ if and only if $\mathcal{N}$ is a network satisfying (c$\cdot$)$'$ for $\mathcal{S}'$.*

Proof. We prove the lemma only for the property (cp)$'$ since the others are similar. First, note that every member of $\mathcal{S}$ is contained in some member of $\mathcal{N}$ if and only if every member of $\mathcal{S}'$ is contained in some member of $\mathcal{N}$. Also, $\bigcup \mathcal{S} = \bigcup \mathcal{S}'$. Thus, $\mathcal{N}$ is a network for $\mathcal{S}$ if and only if it is a network for $\mathcal{S}'$. Now assume $\mathcal{N}$ satisfies (cp)$'$ for $\mathcal{S}'$. Let S be any element of $\mathcal{S}$ and let A any subset of X such that $A \setminus N$ is finite for each N in $\mathcal{N}$ containing S. We know there is a T'_A in $\mathcal{S}'$ such that A is contained in T'_A. Since $\mathcal{S}'$ is contained in $\downarrow\!\mathcal{S}$, there must be a T_A in $\mathcal{S}$ containing T'_A, and so $A \subseteq T'_A \subseteq T_A$.

Next, assume $\mathcal{N}$ satisfies (cp)$'$ for $\mathcal{S}$. Let S' be any element of $\mathcal{S}'$, and let A be a subset of X such that $A \setminus N$ is finite for each N in $\mathcal{N}$ containing S'. Then we also know $A \setminus N$ is finite for each N in $\mathcal{N}$ containing S, and so A is contained in some member of $\mathcal{S}$, which is thus also a member of $\mathcal{S}'$. □

Our first goal in this section is to establish the basic relationships between these network properties. We start by showing that (cn)$'$ and (cn*)$'$ are equivalent, as are (cs)$'$ and (cs*)$'$.

Lemma 152. *Let $\mathcal{S}$ be a pre-ideal on a set X, and let $\mathcal{N}$ be a countable pre-ideal network for $\mathcal{S}$. Then $\mathcal{N}$ is (cn)$'$ for $\mathcal{S}$ if and only if $\mathcal{N}$ is (cn*)$'$ for $\mathcal{S}$.*

Proof. Fix S in $\mathcal{S}$ and assume $\mathcal{N}$ satisfies (cn*)$'$ for $\mathcal{S}$. Note that $A = \bigcap\{N \in \mathcal{N} : S \subseteq N\}$ is contained in $\bigcup \mathcal{N}$, which equals $\bigcup \mathcal{S}$ since $\mathcal{N}$ is a network for $\mathcal{S}$, and of course A is contained in each member of $\mathcal{N}$ containing S. By (cn*)$'$, we therefore have a $T \in \mathcal{S}$ such that $A \subseteq T$, which gives (cn)$'$. Conversely, if we assume (cn)$'$, then any A as in (cn*)$'$ will be contained in $\bigcap\{N \in \mathcal{N} : S \subseteq N\}$, which is contained in some member of $\mathcal{S}$, by (cn)$'$. □

Lemma 153. *Let $\mathcal{S}$ be a pre-ideal on a set X, and let $\mathcal{N}$ be a countable network for $\mathcal{S}$.*

(1) If $\mathcal{N}$ is (cs*)$'$ for $\mathcal{S}$, then the family $\mathcal{N}'$ of all finite intersections of elements of $\mathcal{N}$ is (cs)$'$ for $\mathcal{S}$.

(2) If $\mathcal{N}$ is (cs)$'$ for $\mathcal{S}$, then it is (cs*)$'$ for $\mathcal{S}$.

Proof. Statement (2) is immediate from the definitions of (cs)$'$ and (cs*)$'$, so we check statement (1). Let S be in $\mathcal{S}$, and let A' be a subset of X such that $A' \cap N'$ is infinite for each N' in $\mathcal{N}'$ containing S. Enumerate $\{N \in \mathcal{N} : S \subseteq N\} = \{N_k : k < \omega\}$ and let $N'_k = \bigcap\{N_i : i < k\} \in \mathcal{N}'$. Since each $A' \cap N'_k$ is infinite, we can inductively find distinct points x_k in $A' \cap N'_k$ for each $k < \omega$. Then each N_k contains all but finitely many elements of $A = \{x_k : k < \omega\}$, so by assumption, there is a T in $\mathcal{S}$ such that $A \cap T$ is infinite. Thus, $A' \cap T$ is also infinite, as desired. □

Theorem 154. *Let $\mathcal{S}$ be a pre-ideal on a set X.*

(1) If $\mathcal{S}$ is (cn*)$'$ (equivalently, (cn)$'$) and contains every singleton of $\bigcup\mathcal{S}$, then $\mathcal{S}$ is countably determined (with respect to $\subseteq$).

Now let $\mathcal{N}$ be a countable pre-ideal network for $\mathcal{S}$.

(2) If $\mathcal{N}$ is (cp)$'$ for $\mathcal{S}$, then it is (cn*)$'$ (and hence (cn)$'$).

(3) If $\mathcal{S}$ contains every singleton of $\bigcup\mathcal{S}$, then $\mathcal{N}$ is (cp)$'$ for $\mathcal{S}$ if and only if $\mathcal{N}$ is (cp$_\omega$)$'$ and $\mathcal{S}$ is countably determined.

(4) If $\mathcal{N}$ is (cp$_\omega$)$'$ for $\mathcal{S}$, then it is also (cs*)$'$ (and hence is (cs)$'$ if $\mathcal{N}$ is closed under finite intersections).

If X is a topological space, then we also have:

(5) If $\mathcal{N}$ is (cck)$'$ for $\mathcal{S}$, then it is also (ck)$'$.

(6) If $\mathcal{N}$ is (ck$^+$)$'$ for $\mathcal{S}$, then it is also (ck)$'$.

(7) If $\mathcal{N}$ is (cc$^+$)$'$ for $\mathcal{S}$, then it is also (cc)$'$.

(8) If $\mathcal{N}$ is (cc)$'$ for $\mathcal{S}$, then it is also (cck)$'$.

(9) If $\mathcal{N}$ is (cc$^+$)$'$ for $\mathcal{S}$, then it is also (ck$^+$)$'$.

(10) If $\mathcal{N}$ is (ck)$'$ for $\mathcal{S}$, then it is also (cn)$'$.

(11) Suppose every member of $\mathcal{S}$ is contained in a closed member of $\mathcal{S}$, and $\mathcal{S}$ contains every singleton of $\bigcup\mathcal{S}$. Assume also that $\mathcal{N}$ is closed under finite intersections. If $\mathcal{N}$ is (cp)$'$

for $\mathcal{S}$, then it is also (cck)$'$.

Proof. We start by proving (1). The 'equivalently' part follows from Lemma 152. Fix a countable network $\mathcal{N}$ satisfying (cn*)$'$ for $\mathcal{S}$. Let A be a subset of $\bigcup\mathcal{S}$ that is not contained in any member of $\mathcal{S}$. Then for each S in $\mathcal{S}$, (cn*)$'$ shows that there must be an N_S in $\mathcal{N}$ containing S but not containing A. Let $\mathcal{N}' = \{N_S : S \in \mathcal{S}\}$ which is countable, being a subset of $\mathcal{N}$. So for each N in $\mathcal{N}'$, there is a point a_N in $A \setminus N$, and since N_S contains S, then in particular, a_{N_S} is not in S. Thus, $A' = \{a_N : N \in \mathcal{N}'\}$ is a countable subset of A not contained in any member of $\mathcal{S}$, so $\mathcal{S}$ is countably determined by Lemma 37.

Statements (2) is immediate from the definitions, with the 'hence' part following from Lemma 152. For (3), it is evident that (cp)$'$ implies (cp$_\omega$)$'$, and according to (1) and (2), (cp)$'$ also implies $\mathcal{S}$ is countably determined. The converse in (3) follows from Lemma 37.

Next we prove (4), so assume $\mathcal{N}$ is (cp$_\omega$)$'$ for $\mathcal{S}$. Let S be in $\mathcal{S}$ and suppose A is a countably infinite subset of X such that $A \setminus N$ is finite for each $N \in \mathcal{N}$ containing S. Recall that $\bigcup\mathcal{N} = \bigcup\mathcal{S}$ since $\mathcal{N}$ is a network for $\mathcal{S}$. Thus, $A \setminus \bigcup\mathcal{S}$ is finite, so $A_0 = A \cap (\bigcup\mathcal{S})$ is countably infinite and $A_0 \setminus N$ is finite for each $N \in \mathcal{N}$ containing S. By (cp$_\omega$)$'$, there must then be a $T \in \mathcal{S}$ containing A_0. Thus, $A \cap T = A_0$ is infinite, which gives (cs*)$'$. The 'hence' part follows from Lemma 153.

Statements (5)–(9) are all immediate from the definitions. To prove (10), let S be in $\mathcal{S}$, and take T as in (ck)$'$. Then for each $x \notin T$, there is an N_x in $\mathcal{N}$ such that $S \subseteq N_x \subseteq X \setminus \{x\}$. Thus, $\bigcap\{N \in \mathcal{N} : S \subseteq N\} \subseteq \bigcap\{N_x : x \notin T\} \subseteq T$.

Finally, we establish (11). Fix an arbitrary S in $\mathcal{S}$. Since $\mathcal{N}$ is closed under finite intersections, then there is a decreasing family $\mathcal{N}_S = \{N_k : n < \omega\}$ in $\{N \in \mathcal{N} : S \subseteq N\}$ such that for any N in $\mathcal{N}$ containing S, there is a k with $N_k \subseteq N$. It then follows from (cp)$'$ that whenever A is a subset of X such that $A \setminus N_k$ is finite for each k, there is a member of $\mathcal{S}$ containing A, and so by our additional assumption on $\mathcal{S}$, there is a closed member of $\mathcal{S}$ containing A. Also, $\mathcal{S}$ is countably determined by (3), and $\bigcap\mathcal{N} \subseteq \bigcup\mathcal{N} = \bigcup\mathcal{S}$ since $\mathcal{N}$ is a pre-filter network for $\mathcal{S}$. Hence, we can apply Theorem 155 below to find a T in $\mathcal{S}$ such that $T \supseteq \bigcap\mathcal{N}_S \supseteq S$ and such that for every closed, countably compact subset C of $X \setminus T$, there is a member of $\mathcal{N}_S$ disjoint from C. This shows that $\mathcal{N}$ witnesses (cck)$'$. $\square$

Theorem 155. *Let X be a space with a pre-ideal $\mathcal{S}$ that contains every singleton of $\bigcup \mathcal{S}$ and is countably determined. Let $\mathcal{N} = \{N_n : n < \omega\}$ be a decreasing sequence of subsets of X satisfying: whenever A is a subset of X such that $A \setminus N_n$ is finite for each n, there is a closed member of $\mathcal{S}$ containing A. Further, assume $\bigcap \mathcal{N}$ is contained in $\bigcup \mathcal{S}$.*

Then there is a T in $\mathcal{S}$ containing $\bigcap \mathcal{N}$ such that, whenever C is a subset of $X \setminus T$ satisfying

$$C \cap S \text{ is closed and countably compact for each closed } S \in \mathcal{S}, \tag{6.1}$$

there is a member of $\mathcal{N}$ which is disjoint from C.

Proof. For a contradiction, suppose that for every member T of $\mathcal{S}$ containing $L = \bigcap \mathcal{N}$, there is a subset C_T of $X \setminus T$ satisfying (6.1) that meets each member of $\mathcal{N}$. Then we can choose a point $x_{T,n}$ in $C_T \cap N_n$ for each n.

For each T in $\mathcal{S}$ containing L, let $A_T = \{x_{T,n} : n < \omega\}$. Since $x_{T,n} \in N_m$ for all $n \geq m$, we see that $A_T \setminus N_m$ is finite for every m. So by hypothesis, there is a closed S_T in $\mathcal{S}$ such that $A_T \subseteq S_T$. Thus, A_T is contained in $C_T \cap S_T$, which is closed and countably compact by (6.1), so $\overline{A_T}$ is a closed, countably compact subset of C_T. Hence, A_T has a limit point x_T in $C_T \cap S_T$.

Let $A = \{x_T : T \in \mathcal{S} \text{ and } T \supseteq L\} \subseteq \bigcup \mathcal{S}$. As x_T is not in T, then A is not contained in any member of $\mathcal{S}$ that contains L. Thus, $A \cup L$ is a subset of $\bigcup \mathcal{S}$ that is not contained in any member of $\mathcal{S}$. Since $\mathcal{S}$ is countably determined and contains each singleton of $\bigcup \mathcal{S}$, then Lemma 37 provides a countable subset A_0 of A and a countable subset L_0 of L such that no member of $\mathcal{S}$ contains $A_0 \cup L_0$.

Write $A_0 = \{x_{T_n} : n < \omega\}$ and let $A' = \{x_{T_n,m} : n \leq m < \omega\}$. Then we have $(A' \cup L) \setminus N_k = A' \setminus N_k \subseteq \{x_{T_n,m} : n \leq m < k\}$ is finite for each k. So by hypothesis, there is a closed S in $\mathcal{S}$ containing $A' \cup L$. But then $S \supseteq \overline{A'} \cup L \supseteq A_0 \cup L_0$, which is a contradiction. $\square$

Note that, in particular, (6.1) in Theorem 155 is satisfied whenever C itself is (closed, countably) compact. Alternatively, if every closed member of $\mathcal{S}$ is (closed, countably) compact, then (6.1) is satisfied for every closed set C.

In the remainder of this section, we deduce network properties from the order properties of a pre-ideal. More precisely, we show that if a pre-ideal is sufficiently simple, in the sense that it has countable cofinality or is of type $\langle 2^o{+}\text{CSB}, \text{countably directed}\rangle$, then it has a countable network satisfying some of the (c$\cdot$)$'$ properties from above. This works best when the pre-ideal (or pre-filter) is countably determined.

Lemma 156. *Let $\mathcal{S}$ be a pre-ideal on a set X. If $\omega \geq_T \mathcal{S}$, then $\mathcal{S}$ is (cc$^+$)$'$.*

Proof. Fix an increasing cofinal sequence $(S_n)_n$ for $\mathcal{S}$, and let $\mathcal{N} = \{S_n : n < \omega\}$. Now, given an S in $\mathcal{S}$, we can find n such that $S_n \supseteq S$, and then choosing $T = N = S_n$ fulfills (cc$^+$)$'$. □

Theorem 157. *Let $\mathcal{S}$ be a pre-ideal on a set X, and suppose $\mathcal{S} = \bigcup\{\mathcal{S}_q : q \in Q\}$ witnesses that $\mathcal{S}$ is of type $\langle Q$, countably directed$\rangle$, where Q is second countable and CSB. Fix a countable base $\mathcal{B}$ for Q, and for each B in $\mathcal{B}$, define $\mathcal{S}_B = \bigcup\{\mathcal{S}_q : q \in B\}$ and $N_B = \bigcup \mathcal{S}_B$.*

Now fix some q in Q, and select a decreasing local base $(B_n)_n \subseteq \mathcal{B}$ for q. Define $C_q = \bigcap\{N_B : q \in B \in \mathcal{B}\} = \bigcap\{N_{B_n} : n < \omega\}$. Then the following statements hold.

(a) If A_n is a finite subset of N_{B_n} for each n, then there is an S in $\mathcal{S}$ such that $\bigcup_n A_n \subseteq S$.

(b) If A_n is a finite subset of N_{B_n} for each n, and if $\mathcal{B}$ witnesses that Q is ECSB, then for every m, there is an S_m in $\mathcal{S}_{B_m}$ such that $\bigcup_{n\geq m} A_n \subseteq S_m$.

(c) If A is a countable subset of X such that $A \setminus N_{B_n}$ is finite for all n, then there is an S in $\mathcal{S}$ such that $A \subseteq S$.

(d) If $\mathcal{S}$ contains every singleton of $\bigcup \mathcal{S}$ and is countably determined, then statement (c) holds even when A is uncountable.

(e) If $\mathcal{S}$ contains every singleton of $\bigcup \mathcal{S}$ and is countably determined, then C_q is contained in some S from $\mathcal{S}$.

(f) $\bigcup \mathcal{S}_q \subseteq C_q$.

Proof. For (a), if $A = \bigcup_n A_n$ is finite, then since $\mathcal{S}$ is directed, there is nothing to do. So assume A is infinite. As the N_{B_n}'s are decreasing we can assume each A_n is non-empty, say $A_n = \{a_1^n, \ldots, a_{r_n}^n\}$. Since A_n is contained in N_{B_n}, then for $i = 1, \ldots, r_n$, there are elements q_i^n in B_n and $S_i^n \in \mathcal{S}_{q_i^n}$ such that $a_i^n \in S_i^n$. Now $\{q_i^n : n < \omega \text{ and } 1 \leq i \leq r_n\}$ is a sequence in

Q converging to q, so by CSB, it has an upper bound $\hat{q}$. Then $\{S_i^n : n < \omega \text{ and } 1 \le i \le r_n\}$ is a subset of $\mathcal{S}_{\hat{q}}$, and the latter set is countably directed, so the former set is bounded in $\mathcal{S}_{\hat{q}}$, say by $\hat{S}$. It follows that $\bigcup_n A_n \subseteq \bigcup_{n,i} S_i^n \subseteq \hat{S} \in \mathcal{S}$, as required for (b).

The proof of (b) is just a slight modification of the proof of (a). In place of $\hat{q}$, ECSB provides an upper bound $\hat{q}_m$ for $\{q_i^n : n \ge m \text{ and } 1 \le i \le r_n\}$ in B_m for each m, and in place of $\hat{S}$, we obtain $\hat{S}_m \in \mathcal{S}_{\hat{q}_m} \subseteq \mathcal{S}_{B_m}$ which is an upper bound for $\{S_i^n : n \ge m \text{ and } 1 \le i \le r_n\}$.

Let A be as in (c), and enumerate $A \cap C_q = \{a_n : n < \omega\}$. Now define $A' = A \setminus N_{B_0}$ and $A_n = (A \cap N_{B_n} \setminus N_{B_{n+1}}) \cup \{a_n\}$, which are finite. We may apply (a) to find S_1 in $\mathcal{S}$ such that $\bigcup_{n<\omega} A_n \subseteq S_1$. Since $\mathcal{S}$ covers X and is directed, then we can find an S in $\mathcal{S}$ containing S_1 and A', so that A is contained in S.

Claim (d) follows from (c) and Lemma 37, and claim (e) follows from (d) since $C_q \setminus N_{B_n} = \emptyset$ for each n. Finally, (f) is immediate from the definitions. $\square$

Corollary 158. *Let X be a set with a pre-ideal $\mathcal{S}$ that contains every singleton from $\bigcup \mathcal{S}$. Suppose also that $\mathcal{S}$ is in the class $\langle 2^o$+CSB, countably directed$\rangle$, so it has a Q-ordered cover $\{\mathcal{S}_q : q \in Q\}$ of countably directed subsets of $\mathcal{S}$ for some second countable Q with CSB. Using the notation from Theorem 157, let $\mathcal{N} = \{N_B : B \in \mathcal{B}\}$. Then:*

(1) $\mathcal{N}$ is a (countable) pre-filter network for $\mathcal{S}$ satisfying (cp$_\omega$)$'$, and

(2) if $\mathcal{S}$ is countably determined, then $\mathcal{N}$ is (cp)$'$ for $\mathcal{S}$.

Proof. For any S in $\mathcal{S}$, pick q in Q such that S is in $\mathcal{S}_q$, and then note that for any B in $\mathcal{B}$ such that $q \in B$, we have $S \subseteq \bigcup \mathcal{S}_q \subseteq \bigcup \mathcal{S}_B = N_B$. Thus, each member of $\mathcal{S}$ is contained in some member of $\mathcal{N}$, and of course $\bigcup \mathcal{N} = \bigcup \mathcal{S}$, so $\mathcal{N}$ is a network for $\mathcal{S}$. If A is a countable subset of X such that $A \setminus N$ is finite for each N in $\mathcal{N}$ containing S, then $A \setminus N_B$ is finite for each B in $\mathcal{B}$ containing q, so Theorem 157 (c) shows that some member of $\mathcal{S}$ contains A. Thus, (1) has been proven, and (2) similarly follows from Theorem 157 (d). $\square$

6.2 THE (C·) PROPERTIES FOR (NEIGHBORHOOD) FILTERS

Fix a pre-filter $\mathcal{F}$ on a set X. We say a family $\mathcal{D}$ of subsets of X is a *(pre-filter) network* for $\mathcal{F}$ if each member of $\mathcal{F}$ contains some member of $\mathcal{D}$, and if $\bigcap \mathcal{D} = \bigcap \mathcal{F}$. Below, we define a family of properties (c·) that a countable network $\mathcal{D}$ for $\mathcal{F}$ may satisfy. We will also say that $\mathcal{F}$ itself satisfies (c·) if there is a countable network $\mathcal{D}$ satisfying (c·) for $\mathcal{F}$. These (c·) properties are naturally dual to the (c·)′ properties for pre-ideals described in the previous section, as we will see in Lemma 159 below.

(cp$_\omega$) If F is in $\mathcal{F}$, and if A is a countable subset of $X \setminus \bigcap \mathcal{F}$ intersecting each member of $\mathcal{F}$, then there is a $D \in \mathcal{D}$ contained in F such that $A \cap D$ is infinite.

(cp) If F is in $\mathcal{F}$, and if A is a subset of $X \setminus \bigcap \mathcal{F}$ intersecting each member of $\mathcal{F}$, then there is a $D \in \mathcal{D}$ contained in F such that $A \cap D$ is infinite.

(cs) If F is in $\mathcal{F}$, and if A is a countably infinite subset of X such that $A \setminus G$ is finite for each $G \in \mathcal{F}$, then there is a $D \in \mathcal{D}$ contained in F such that $A \setminus D$ is finite.

(cs*) If F is in $\mathcal{F}$, and if A is a countably infinite subset of X such that $A \setminus G$ is finite for each $G \in \mathcal{F}$, then there is a $D \in \mathcal{D}$ contained in F such that $A \cap D$ is infinite.

(cn) For any $F \in \mathcal{F}$, there is a $G \in \mathcal{F}$ such that $G \subseteq \bigcup\{D \in \mathcal{D} : D \subseteq F\} \subseteq F$.

(cn*) If F is in $\mathcal{F}$, and if A is a subset of $X \setminus \bigcap \mathcal{F}$ intersecting each member of $\mathcal{F}$, then there is a $D \in \mathcal{D}$ contained in F such that $A \cap D$ is nonempty.

If X is a topological space, then we define the following additional properties (c·) which a countable pre-filter network $\mathcal{D}$ may satisfy for a pre-filter $\mathcal{F}$ on X. Once again, we say $\mathcal{F}$ itself has the property (c·) if there is a countable network $\mathcal{D}$ satisfying (c·) for $\mathcal{F}$.

(cck) For any $F \in \mathcal{F}$, there is a $G \in \mathcal{F}$ contained in F such that for any closed, countably compact $C \subseteq G$, there is a $D \in \mathcal{D}$ such that $C \subseteq D \subseteq F$.

(ck) For any $F \in \mathcal{F}$, there is a $G \in \mathcal{G}$ contained in F such that for any compact $K \subseteq G$, there is a $D \in \mathcal{D}$ such that $K \subseteq D \subseteq F$.

(ck$^+$) For any $F \in \mathcal{F}$, there is a $G \in \mathcal{G}$ contained in F such that for any compact $K \subseteq G$, there is a $D \in \mathcal{D}$ such that $K \subseteq D \subseteq G$.

(cc) For any $F \in \mathcal{F}$, there is a $G \in \mathcal{F}$ contained in F such that for any closed $C \subseteq G$, there is a $D \in \mathcal{D}$ such that $C \subseteq D \subseteq F$.

(cc$^+$) For any $F \in \mathcal{F}$, there is a $G \in \mathcal{F}$ contained in F such that for any closed $C \subseteq G$, there is a $D \in \mathcal{D}$ such that $C \subseteq D \subseteq G$.

Recall that c_X denotes the complementation map on a set X, and for any family $\mathcal{A}$ of subsets of X, $c_X(\mathcal{A})$ denotes the family $\{c_X(A) : A \in \mathcal{A}\}$ of subsets of X.

Lemma 159. *Let $\mathcal{F}$ be a pre-filter on a set (or space) X, let $\mathcal{D}$ be a family of subsets of X, let (c·) be any of the pre-filter network properties above, and let (c·)$'$ be the corresponding pre-ideal network property from Section 6.1. Then $\mathcal{D}$ is a countable pre-filter network for $\mathcal{F}$ satisfying (c·) if and only if $\mathcal{N} = c_X(\mathcal{D})$ is a countable pre-ideal network for $\mathcal{S} = c_X(\mathcal{F})$ satisfying (c·)$'$.*

Proof. First, note that $\mathcal{S}$ is a pre-ideal by Lemma 6. Now observe that each member of $\mathcal{F}$ contains a member of $\mathcal{D}$ if and only if each member of $\mathcal{S}$ is contained in a member of $\mathcal{N}$, while $\bigcap \mathcal{D} = \bigcap \mathcal{F}$ if and only if $\bigcup \mathcal{N} = \bigcup \mathcal{S}$. Thus, $\mathcal{D}$ is a countable pre-filter network for $\mathcal{F}$ if and only if $\mathcal{N}$ is a countable pre-ideal network for $\mathcal{S}$.

For most of the network properties, the statement for (c·) is obtained directly from the corresponding (c·)' statement by replacing S with $X \setminus F$, N with $X \setminus D$, and T with $X \setminus G$. We have also used DeMorgan's Laws and the fact that $E \subseteq H$ if and only if $c_X(H) \subseteq c_X(E)$ for any subsets E and H of X. In this way, the duality between (cn), (cck), (ck), (ck$^+$), (cc), (cc$^+$), and their (c·)$'$ analogs is clear.

In the case of (cp$_\omega$), (cp), (cs), (cs*), and (cn*), we have also replaced part of the corresponding (c·)$'$ statement with its contrapositive. For example, if (cp)$'$ is dualized in the obvious way, we obtain the statement: 'If F is in $\mathcal{F}$, and if A is a subset of $X \setminus \bigcap \mathcal{F}$ such that $A \cap D$ is finite for each $D \in \mathcal{D}$ contained in F, then there is a $G \in \mathcal{F}$ such that $A \cap G = \emptyset$.' Now, replacing 'if … $A \cap D$ is finite for each $D \in \mathcal{D}$ contained in F, then there is a $G \in \mathcal{F}$ such that $A \cap G = \emptyset$' with its contrapositive gives precisely the statement of (cp). □

Note that (cp) is equivalent to the strong Pytkeev property when $\mathcal{F}$ is the neighborhood filter of a point. For neighborhood filters of points, the (cn) and (ck) properties were introduced in [20] while studying the (cp) property, and the (cs) and (cs*) properties were introduced in [30] and [23], respectively.

The dual versions of Lemmas 152 and 153 follow automatically: a pre-filter network $\mathcal{D}$ for a pre-filter $\mathcal{F}$ on X satisfies (cn) if and only if $\mathcal{D}$ is (cn*) for $\mathcal{F}$; and if $\mathcal{D}$ is closed under finite unions, then it is (cs) for $\mathcal{F}$ if and only if it is (cs*) for $\mathcal{F}$. Hence $\mathcal{F}$ itself is (cn) if and only if it is (cn*), and $\mathcal{F}$ is (cs) if and only if it is (cs*).

We also have an obvious dual version of Theorem 154 for pre-filters. Here we replace every occurrence of $\mathcal{S}$ with $\mathcal{F}$, $\mathcal{N}$ with $\mathcal{D}$, (c$\cdot$)$'$ with (c$\cdot$), and 'contains every singleton of $\bigcup \mathcal{S}$' with 'contains the complement of every singleton in $X \setminus \bigcap \mathcal{F}$'. For statement (11), we need to additionally replace 'closed under finite intersections' with 'closed under finite unions' and replace 'every member of $\mathcal{S}$ is contained in a closed member of $\mathcal{S}$' with 'every member of $\mathcal{F}$ contains an open member of $\mathcal{F}$'

The next two results are dual to Theorem 157 and Corollary 158, respectively. They follow automatically from duality and the definitions.

Theorem 160. *Let $\mathcal{F}$ be a pre-filter of subsets of a set Y, and suppose $\mathcal{F} = \bigcup\{\mathcal{F}_q : q \in Q\}$ witnesses that the directed set $(\mathcal{F}, \supseteq)$ is of type $\langle Q, \text{countably directed}\rangle$, where Q is second countable and CSB. Let $\mathcal{B}$ be a countable base for Q, and for each B in $\mathcal{B}$, define $\mathcal{F}_B = \bigcup\{\mathcal{F}_q : q \in B\}$ and $D_B = \bigcap \mathcal{F}_B$.*

Fix a q in Q, and select a decreasing local base $(B_n)_n \subseteq \mathcal{B}$ for q. Define $W_q = \bigcup\{D_B : q \in B \in \mathcal{B}\}$, and note that $W_q = \bigcup\{D_{B_n} : n < \omega\}$. Then the following statements hold.

(a) If A_n is a finite subset of Y disjoint from D_{B_n} for each n, then there is an F in $\mathcal{F}$ which is disjoint from $\bigcup_n A_n$.

(b) If A_n is a finite subset of Y disjoint from D_{B_n} for each n, and if $\mathcal{B}$ witnesses that Q is ECSB, then for every m, there is an F_m in $\mathcal{F}_{B_m}$ which is disjoint from $\bigcup_{n \geq m} A_n$.

(c) If A is a countable subset of $Y \setminus (\bigcap \mathcal{P})$ such that $A \cap D_{B_n}$ is finite for all n, then there is a member of $\mathcal{F}$ that is disjoint from A.

(d) If $(\mathcal{F}, \supseteq)$ contains every complement of points in $Y \setminus \bigcap \mathcal{F}$ and is countably determined, then statement (c) holds even when A is uncountable.

(e) If $(\mathcal{F}, \supseteq)$ contains the complement of every point in $X \setminus \bigcap \mathcal{F}$ and is countably determined, then W_q is in $\mathcal{F}$.

(f) $W_q \subseteq \bigcap \mathcal{F}_q$.

Corollary 161. *Let $\mathcal{F}$ be a pre-filter of subsets of a set Y such that $(\mathcal{F}, \supseteq)$ is in the class $\langle Q$, countably directed$\rangle$, where Q is second countable and CSB. Let $\mathcal{B}$ be a countable base for Q, and for each B in $\mathcal{B}$, define $\mathcal{F}_B = \bigcup\{\mathcal{F}_q : q \in B\}$ and $D_B = \bigcap \mathcal{F}_B$. Let $\mathcal{D} = \{D_B : LB \in \mathcal{B}\}$. Then:*

(1) $\mathcal{D}$ is a countable pre-filter network for $\mathcal{F}$ satisfying (cp$_\omega$), and

(2) if $\mathcal{F}$ contains the complement of each singleton in $Y \setminus \bigcap \mathcal{F}$ and is countably determined, then $\mathcal{D}$ is (cp) for $\mathcal{F}$.

For variety we prove the next result for pre-filters and leave it to the reader to derive the dual pre-ideal version.

Lemma 162. *Let $\mathcal{F}$ be a pre-filter of subsets of a space Y such that the interior F° of F is in $\mathcal{F}$ whenever F is in $\mathcal{F}$. Suppose $\mathcal{F} = \bigcup\{\mathcal{F}_q : q \in Q\}$ witnesses that $(\mathcal{F}, \supseteq)$ is of type $\langle Q$, countably directed$\rangle$ for some directed set Q.*

(1) We can ensure that F° is in $\mathcal{F}_q$ whenever F is in $\mathcal{F}_q$.

Suppose $\mathcal{B}$ is a countable base for Q witnessing that Q is ECSB. Fix q in Q and let D_B, $(B_n)_n$, and W_q be as in Theorem 160. Then:

(2) Every compact subset of W_q is contained in some D_{B_n}.

(3) If $\mathcal{F}$ is countably determined, then the family $\mathcal{D} = \{D_B : B \in \mathcal{B}\}$ is a (countable) pre-filter network satisfying (ck$^+$) for $\mathcal{F}$.

Proof. If we replace each $\mathcal{F}_q$ with $\mathcal{F}_q \cup \{F^\circ : F \in \mathcal{F}_q\}$, then (1) holds and the $\mathcal{F}_q$'s still witness that $\mathcal{F}$ is of type $\langle Q, \text{countably directed}\rangle$.

We prove (2) next. Let K be a compact subset of W_q, and suppose, for a contradiction, that there is a point a_n in $K \setminus D_{B_n}$ for each n. Then pick $q_n \in B_n$ and $F_n \in \mathcal{F}_{q_n}$ such that $a_n \notin F_n$. Set $A_k = \{a_n : n \geq k\}$. Note that $K \supseteq \overline{A_1} \supseteq \overline{A_2} \supseteq \cdots$, so by compactness, $\bigcap_k \overline{A_k}$ is a non-empty subset of K and, hence, of W_q.

For each k, we have $(q_n)_{n \geq k} \to q$, so by ECSB, this sequence has an upper bound $\hat{q}_k$ in B_k. Thus, $\{F_n : n \geq k\}$ is contained in $\mathcal{F}_{\hat{q}_k}$, which is countably directed, so there is an $\hat{F}_k$ in $\mathcal{F}_{\hat{q}_k}$ such that $\hat{F}_k \subseteq F_n$ for all $n \geq k$. By (1), we may assume $\hat{F}_k$ is open. Note that $\hat{F}_k \cap A_k = \emptyset$, so $D_{B_k} \cap \overline{A_k} \subseteq \hat{F}_k \cap \overline{A_k} = \emptyset$. Hence, $W_q = \bigcup_n D_{B_n}$ is disjoint from $\bigcap_k \overline{A_k}$, which is a contradiction.

Finally, we prove (3). Note that for any F in $\mathcal{F}$, there are $q \in Q$ and $B \in \mathcal{B}$ such that $q \in B$ and $F \in \mathcal{F}_q \subseteq \mathcal{F}_B$, so that $D_B \subseteq F$, which means each member of $\mathcal{F}$ contains a member of $\mathcal{D}$. It is also clear that $\bigcap \mathcal{D} = \bigcap \mathcal{F}$, so we see that $\mathcal{D}$ is a countable pre-filter network for $\mathcal{F}$. To complete the proof of (3), it therefore suffices to check that for any F in $\mathcal{F}_q \subseteq \mathcal{F}$, the set $G = W_q$ is as required for (ck$^+$). Indeed, W_q is in $\mathcal{F}$ and $W_q \subseteq F$ by parts (e) and (f) of Theorem 160, and then statement (2) and the definition of W_q give the conclusion of (ck$^+$). □

The (pre-) filters of interest here have the following form. Let A be a subset of a space X, and let $\mathcal{N}_A$ denote the family of all neighborhoods of A – in other words, the subsets of X containing A in their interior. Then $\mathcal{N}_A$ is a filter on X called the *neighborhood filter of A*. Like any filter, $\mathcal{N}_A$ is directed by reverse set inclusion $\supseteq$. When $A = \{x\}$ is a singleton, we denote the neighborhood filter by $\mathcal{N}_x$. We extend some standard concepts for neighborhood filters of points to neighborhood filters of sets, as follows. We say A is *first countable* in X if $\omega \geq_T \mathcal{N}_A$ (i.e. $\mathcal{N}_A$ has a countable base), while a subset S of X *converges to* A if each neighborhood of A contains all but finitely many elements of S. We also say that A has *countable tightness* in X if for any subset S of X that meets every neighborhood of A, there is a countable subset of S which also meets every neighborhood of A.

For any of the pre-filter properties (c$\cdot$) introduced above, we say a space X has the property (c$\cdot$)$_A$ for some subset A of X if the filter $\mathcal{N}_A$ has the property (c$\cdot$). We abbreviate (c$\cdot$)$_{\{x\}}$ to (c$\cdot$)$_x$ and say simply that the space X has the property (c$\cdot$) if it has (c$\cdot$)$_x$ at each point x in X.

Now we see that a space is (cp) if and only if it is strongly Pytkeev, and the dual version of Theorem 154 shows that (cn) is a natural weakening of the strong Pytkeev property. Observe also that X is (cs)$_A$ if and only if there is a countable local network $\mathcal{D}$ at A such that for any sequence S converging to A and any neighborhood U of A, there is a D in $\mathcal{D}$ such that D is contained in U and contains all but finitely many elements of S.

Recall that a space X is called *homogeneous* if for for any two points x and y in X, there is a self-homeomorphism of X taking x to y. Thus, the neighborhood structure at each point in a homogeneous space is identical. The next lemma is then clear, and in particular, observe that to verify a topological group has a local network property, it suffices to establish it at the identity.

Lemma 163. *A homogeneous space X is (c$\cdot$) if and only if it is $(\mathrm{c}\cdot)_x$ at some point x in X.*

Lemma 38 applied to $\mathcal{F} = \mathcal{N}_A$ tells us when $\mathcal{N}_A$ is countably determined.

Lemma 164. *Let A be a subset of a space X. Then $\mathcal{N}_A$ is countably determined if and only if A has countable tightness in X.*

The dual version of Theorem 154 immediately gives the following basic relations.

Theorem 165. *Let A be a subset of a space X. Then:*

(1) $(cp)_A \implies (cck)_A$,

(2) $(cck)_A \implies (ck)_A$,

(3) $(ck^+)_A \implies (ck)_A$,

(4) $(ck)_A \implies (cn)_A$,

(5) $(cn)_A \implies \mathcal{N}_A$ is countably determined,

(6) $(cp)_A \iff (cp_\omega)_A$ and $\mathcal{N}_A$ is countably determined, and

(7) $(cp_\omega)_A \implies (cs)_A$.

However, the next set of relationships depend on properties characteristic of neighborhood filters. Note that the neighborhood filter of a subset A of a space X has a base of closed sets if A is compact, or if A is closed and X is normal. Also observe that for a point x in a space X, properties (1)–(7) above, and (1) and (2) below all hold for $\mathcal{N}_x$.

Theorem 166. *Let A be a subset of a space X. Then:*

(1) if A is closed and countably compact, then $(cck)_A \implies (cs)_A$,

(2) if $\mathcal{N}_A$ has a base of closed sets, then $(cc)_A$ if and only if $(cc^+)_A$ if and only if A is first countable in X, and

(3) if A has a closed and countably compact neighborhood and $\mathcal{N}_A$ has a base of closed sets, then A is first countable in X if and only if $(cck)_A$.

Proof. For (1), assume $\mathcal{D}$ is a local network at A witnessing $(\mathrm{cck})_A$. Let S be a countable subset of X such that each neighborhood of A contains all but finitely many points of S, and let U be an arbitrary neighborhood of A. Let V be the open neighborhood of A contained in U given by (cck).

Note that $C = A \cup (S \cap V)$ is a closed, countably compact subset of V, so by (cck) there is a D in $\mathcal{D}$ with $S \cap V \subseteq C \subseteq D \subseteq U$. Thus, $S \setminus D \subseteq S \setminus V$ is finite and $D \subseteq U$, so $\mathcal{D}$ witnesses $(\text{cs})_A$.

By Lemma 156 and Theorem 154 (and duality), for (2) it suffices to show that if A is compact, or A is closed and X normal, and $(\text{cc})_A$ holds then A is first countable in X. So fix countable $\mathcal{D}$ which is (cc) for $\mathcal{N}_A$. Let $\mathcal{B} = \{D \in \mathcal{D} : A \subseteq D^\circ\}$. Then $\mathcal{B}$ is a countable family of neighborhoods of A. We verify it is a base for $\mathcal{N}_A$. To see this take any open $U \supseteq A$. Let V be neighborhood of A contained in U given by (cc) for $\mathcal{D}$. Using the hypothesis on A, find an open T such that $A \subseteq T \subseteq \overline{T} \subseteq V$. Set $C = \overline{T}$, and apply (cc) to get D in $\mathcal{D}$ such that $A \subseteq T \subseteq C \subseteq D \subseteq U$. Then D is in $\mathcal{B}$ and contained in U, and we are done.

For (3) observe that the hypotheses imply A has a neighborhood base of closed countably compact sets. So $(\text{cck})_A$ easily gives $(\text{cc})_A$, and we can apply part (2). □

From our general results, we can deduce local network properties of neighborhood filters from their order structure.

Theorem 167. *Let X be a space and A a subset of X.*

(1) If $\mathcal{N}_A$ is in the class $\langle 2^o$+CSB, countably directed$\rangle$, then $(\text{cp}_\omega)_A$ holds.

Suppose, additionally, that A is countably tight in X. Then:

(1′) if $\mathcal{N}_A$ is in $\langle 2^o$+CSB, countably directed$\rangle$, then $(\text{cp})_A$ holds; and

(2′) if $\mathcal{N}_A$ is in $\langle 2^o$+ECSB, countably directed$\rangle$, then $(\text{ck}^+)_A$ holds.

In particular, for any point x of countable tightness in X, we have:

(1″) if $\mathcal{N}_x$ is in $\langle 2^o$+CSB, countably directed$\rangle$, then $(\text{cp})_x$ holds; and

(2″) if $\mathcal{N}_x$ is in $\langle 2^o$+ECSB, countably directed$\rangle$, then $(\text{ck}^+)_x$ holds.

Proof. For statements (1), (1′), and (2′), first recall that $\mathcal{N}_A$ is countably determined if and only if A is countably tight in X (Lemma 164). Now (1) and (1′) follow from Corollary 161, while statement (2′) follows from Lemma 162. Statements (1″) and (2″) are just the special case where $A = \{x\}$. □

6.3 $\mathcal{K}(X)$ AND LINDELÖF COFINALLY Σ-SPACES

Recall that if $\mathcal{N}$ and $\mathcal{A}$ are families of subsets of a space X, then $\mathcal{N}$ is called a *network for X modulo $\mathcal{A}$* if whenever U is an open subset of X containing some member A of $\mathcal{A}$, there is an N in $\mathcal{N}$ such that $A \subseteq N \subseteq U$. Then X is called *Lindelöf* Σ if it has a countable network modulo some compact cover of X. Similarly, we say X is *Lindelöf cofinally* Σ if X has a countable network modulo some cofinal (with respect to $\subseteq$) subset of $\mathcal{K}(X)$.

More generally, if $\mathcal{S}$ is a pre-ideal of compact subsets of a space X and if $\mathcal{S}'$ is a subset of $\mathcal{S}$, then we say X is *Lindelöf* $(\mathcal{S}', \mathcal{S})$*-cofinally* Σ if X has a countable network modulo a cofinal family for the directed set pair $(\mathcal{S}', \mathcal{S})$ (see Section 2.2). We will primarily focus on when $\mathcal{S}' = \mathcal{S}$, in which case we just say X is *Lindelöf* $\mathcal{S}$*-cofinally* Σ. Thus, X is Lindelöf cofinally Σ if and only if it is Lindelöf $\mathcal{K}(X)$-cofinally Σ, and since a cofinal family for $(X, \mathcal{K}(X))$ is precisely a compact cover of X, then X is Lindelöf Σ if and only if it is Lindelöf $(X, \mathcal{K}(X))$-cofinally Σ.

Lemma 168. *Let $\mathcal{S}$ be a pre-ideal of compact subsets of X such that $\mathcal{S}$ contains the closed subsets of each of its members. Let $\mathcal{S}'$ be a subset of $\mathcal{S}$. Then the following are equivalent:*

(i) X is Lindelöf $(\mathcal{S}', \mathcal{S})$-cofinally Σ, and

(ii) there is a countable family $\mathcal{N}$ of subsets of X, and for each K in $\mathcal{S}'$, there is a L_K in $\mathcal{S}$ containing K such that for every open set U containing L_K, there is an N in $\mathcal{N}$ with $K \subseteq N \subseteq U$.

Proof. Assume (i), so X has a countable network $\mathcal{N}$ modulo a family $\mathcal{L}$ of subsets of X which is cofinal for $(\mathcal{S}', \mathcal{S})$. Thus, $\mathcal{L}$ is a subset of $\mathcal{S}$, and for each K in $\mathcal{S}'$, there is an L_K in $\mathcal{L}$ such that $K \subseteq L_K$. If U is any open set containing L_K, then as $\mathcal{N}$ is a network modulo $\mathcal{L}$, there is an N in $\mathcal{N}$ such that $L_K \subseteq N \subseteq U$. But since K is contained in L_K, then we have shown that (i) implies (ii).

Now let $\mathcal{N}$ and the L_K's be as in (ii). We may assume each member of $\mathcal{N}$ is closed. Indeed, if K is in $\mathcal{S}'$ and U is an open set containing L_K, then by regularity, we can find an open set V such that $L_K \subseteq V \subseteq \overline{V} \subseteq U$. Then there is an N in $\mathcal{N}$ such that $K \subseteq N \subseteq V$, and so $K \subseteq \overline{N} \subseteq U$. Thus, we can replace $\mathcal{N}$ with $\{\overline{N} : N \in \mathcal{N}\}$. We may further assume

$\mathcal{N}$ is closed under finite intersections.

For each K in $\mathcal{S}$, let $A_K = \bigcap\{N \in \mathcal{N} : K \subseteq N\}$. Then we claim that $A_K \subseteq L_K$. Indeed, if $x \notin L_K$, then there is an $N \in \mathcal{N}$ such that $K \subseteq N \subseteq X \setminus \{x\}$, so $x \notin A_K$ also. Now, since each N in $\mathcal{N}$ is closed, then A_K is a closed subset of L_K and hence A_K is in $\mathcal{S}$. Since $K \subseteq A_K$ for each K in $\mathcal{S}'$, then the family $\mathcal{A} = \{A_K : K \in \mathcal{S}'\}$ is cofinal for $(\mathcal{S}', \mathcal{S})$, so to establish (i), it suffices to show that $\mathcal{N}$ is a network modulo $\mathcal{A}$.

Fix a K in $\mathcal{S}'$. Since $\mathcal{N}$ is countable and closed under finite intersections, then we can find a decreasing nested sequence $(N_i)_{i<\omega}$ of sets in $\mathcal{N}$ such that for any $N \in \mathcal{N}$ containing K, there is an $i < \omega$ with $K \subseteq N_i \subseteq N$. Fix an open set U containing A_K and assume, to get a contradiction, that $N_i \not\subseteq U$ for each $i < \omega$. Then we may find $x_i \in N_i \setminus U$. Since any open V containing L_K must contain some N_i, and since $(N_i)_i$ is decreasing, then $(x_i)_i$ converges to L_K, so $Q = L_K \cup \{x_i : i < \omega\}$ is compact.

Hence, there must be an x_∞ in Q such that for every neighborhood T of x_∞, the set $\{i < \omega : x_i \in T\}$ is infinite. Note that since each x_i is not in U, then neither is x_∞. So x_∞ is not in A_K, which means there must then be a $j < \omega$ such that $x_\infty \notin N_j$. However, $T = X \setminus N_j$ is now an open neighborhood of x_∞ such that $\{i < \omega : x_i \in T\} \subseteq \{0, \ldots, j-1\}$ is finite. This contradiction shows that there must be an N_i in $\mathcal{N}$ such that $K \subseteq N_i \subseteq U$, and so $A_K \subseteq N_i \subseteq U$, which completes the proof. □

Theorem 169. *Let $\mathcal{S}$ be a pre-ideal of compact subsets covering a space X. Then the following are equivalent:*

(i) X is Lindelöf $\mathcal{S}$-cofinally Σ;

(ii) $\mathcal{S}$ is $(cc^+)'$; and

(iii) $\mathcal{S}$ is $(cc)'$.

Suppose $\mathcal{S}$ satisfies the additional property: whenever L is in $\mathcal{S}$ and S is a sequence on X converging to L, $S \cup L$ is in $\mathcal{S}$. Then statements (i)-(iii) are also equivalent to:

(iv) $\mathcal{S}$ is $(cp)'$.

Moreover, (iv) implies (i)-(iii) even if $\mathcal{S}$ does not satisfy the additional property.

Proof. Applying Lemma 151, we can assume $\mathcal{S}$ is closed under taking closed subsets. Suppose (i) holds and $\mathcal{N}$ is a countable network for X modulo a cofinal subset $\mathcal{A}$ of $(\mathcal{S}, \subseteq)$. Note that $\mathcal{N}$ is also a pre-ideal network for $\mathcal{S}$. We verify $\mathcal{N}$ satisfies $(\mathrm{cc}^+)'$ for $\mathcal{S}$, and so (ii) holds. Take any K in $\mathcal{S}$. As $\mathcal{A}$ is cofinal in $\mathcal{S}$ there is an L in $\mathcal{A}$ such that $K \subseteq L$. Take any closed set C. Let $U = X \setminus C$. Since (i) holds, there is an N in $\mathcal{N}$ such that $L \subseteq N \subseteq U = X \setminus C$, as required for $\mathcal{N}$ to be $(\mathrm{cc}^+)'$.

Evidently (ii) implies (iii), so for the equivalence of (i)–(iii), it remains to prove that (iii) implies (i). But this follows from Lemma 168 since for $\mathcal{S}' = \mathcal{S}$, statement (iii) here is equivalent to statement (ii) there. Thus the equivalence of (i)–(iii) is proven.

Now we show that (iv) implies (iii). Suppose $\mathcal{N}$ satisfies $(\mathrm{cp})'$ for $\mathcal{S}$ and is closed under finite intersections. By Theorem 154, $\mathcal{N}$ also satisfies $(\mathrm{ck})'$, and we verify this can be improved to $(\mathrm{cc})'$. Take any S in $\mathcal{S}$. Fix a decreasing sequence $(N_i)_i$ from $\mathcal{N}$ with each N_i containing S and such that for every N from $\mathcal{N}$ if $N \supseteq S$ then for some i we have $S \subseteq N_i \subseteq N$. Let T be given by $(\mathrm{ck})'$, so $T \supseteq S$ and for every $X \setminus K$ containing T, where K is compact, there is an N in $\mathcal{N}$ such that $S \subseteq N \subseteq X \setminus K$. Take any $X \setminus C$ containing T, where C is closed. We are done if some N_i is contained in $X \setminus C$. If not then pick a_i in $N_i \setminus U$. Set $A = \{a_i : i < \omega\}$. Then A satisfies the condition to apply $(\mathrm{cp})'$, and we see that A is contained in some member of $\mathcal{S}$. In particular, $K = \overline{A}$ is compact, and disjoint from T. Then by $(\mathrm{ck})'$, and definition of the N_i's there is an N_i such that $N_i \subseteq X \setminus K \subseteq X \setminus \{a_i\}$ – contradicting $a_i \in N_i$.

Finally, we suppose $\mathcal{S}$ satisfies (ii) and the additional property given before statement (iv). Let $\mathcal{N}$ be a countable family of closed sets satisfying $(\mathrm{cc}^+)'$ for $\mathcal{S}$. We verify $\mathcal{N}$ also satisfies $(\mathrm{cp})'$, which gives (iv). Take any K in $\mathcal{S}$. Fix the L in $\mathcal{S}$ given by $(\mathrm{cc}^+)'$, so $L \supseteq K$ and for every open U containing L there is an N in $\mathcal{N}$ such that $L \subseteq N \subseteq U$. Take any subset A of X such that $A \setminus N$ is finite for each N in $\mathcal{N}$ containing K. Set $A_0 = A \cap L$ and $A_1 = A \setminus A_0$. We need to show A is contained in some member of $\mathcal{S}$. This is achieved if we show $A_1 \cup L$ is in $\mathcal{S}$. But the assumption on A and the $(\mathrm{cc}^+)'$ property implies that A_1 is a sequence converging to L. So the proof is completed by applying the additional hypothesis on $\mathcal{S}$. □

Example 17 below shows that we cannot weaken $(\mathrm{cc})'$ to $(\mathrm{ck}^+)'$ in the previous theorem. For general pre-ideals $\mathcal{S}$ of compact sets, we also cannot remove the additional hypothesis for (cp)$'$ to be added to the list of equivalences (see Example 18). Note that for $\mathcal{S} = \mathcal{K}(X)$ the additional hypothesis is satisfied.

Evidently, if X is Lindelöf cofinally Σ, then it is Lindelöf Σ and, in particular, Lindelöf. We can strengthen this to k-Lindelöf. A collection $\mathcal{U}$ of subsets of a space X is called a *k-cover* if every compact subset of X is contained in some member of $\mathcal{U}$. A space X is *k-Lindelöf* if every open k-cover contains a countable subcollection which is also a k-cover.

Lemma 170. *If X is Lindelöf cofinally Σ, then X is k-Lindelöf.*

Proof. Since X is Lindelöf cofinally Σ, there is a countable family $\mathcal{N}$ which is a network for X modulo some cofinal subset $\mathcal{A}$ of $\mathcal{K}(X)$. Let $\mathcal{U}$ be an open k-cover. For each K in $\mathcal{A}$, pick $U_K \in \mathcal{U}$ and then $N_K \in \mathcal{N}$ such that $K \subseteq N_K \subseteq U_K$. Since $\mathcal{N}$ is countable, we can enumerate $\{N_K : K \in \mathcal{A}\} = \{N_i : i < \omega\}$ and then pick U_i in $\mathcal{U}$ such that $N_i \subseteq U_i$. By the cofinality of $\mathcal{A}$ in $\mathcal{K}(X)$, every compact subset of X is contained in some U_i. Thus, the countable subcollection $\{U_i : i < \omega\}$ of $\mathcal{U}$ is a k-cover. $\square$

Theorem 171. *The following are equivalent for any space X:*

(i) $\mathcal{K}(X)$ is in the class $\langle 2^o + CSB$, countably directed$\rangle$ and is countably determined;

(ii) $\mathcal{K}(M) \geq_T \mathcal{K}(X)$ for some separable metrizable M, and $\mathcal{K}(X)$ is countably determined;

(iii) $\mathcal{K}(X)$ is (cp)$'$;

(iv) X is Lindelöf cofinally Σ (in other words, $\mathcal{K}(X)$ is (cc$^+$)$'$); and

(v) X is the compact-covering image of a perfect preimage of a separable metrizable space.

Proof. By Lemma 27 and Lemma 9, we know (ii) implies (i), and applying Corollary 158 to $\mathcal{S} = \mathcal{K}(X)$ shows that (i) implies (iii). From Theorem 169, we also know (iii) is equivalent to (iv). We now show (iv) and (v) are equivalent and (iii) implies (ii).

Suppose there are a space Z, a separable metrizable space M, a perfect map $f : Z \to M$, and compact-covering map $g : Z \to X$. Suppose $\mathcal{B}$ is a countable base of M that is closed under finite unions and intersections. Let $\mathcal{C} = \{g(f^{-1}(K)) : K \in \mathcal{K}(M)\}$ and $\mathcal{N} = \{g(f^{-1}(B)) : B \in \mathcal{B}\}$. Then $\mathcal{C}$ is cofinal in $\mathcal{K}(X)$ and $\mathcal{N}$ is a network modulo $\mathcal{C}$. Thus, (v) implies (iv).

Now suppose X is Lindelöf cofinally Σ, so there is a countable network $\mathcal{N}$ modulo some cofinal subset $\mathcal{C}$ of $\mathcal{K}(X)$. Let $D(\mathcal{N})$ be $\mathcal{N}$ with the discrete topology. For any $m \in D(\mathcal{N})^\omega$ define $C_m = \bigcap\{m(n) : n < \omega\}$. Define $M = \{m \in D(\mathcal{N})^\omega : C_m \in \mathcal{C}\}$. Then M is separable and metrizable. Consider the closed subspace $Z = \bigcup_{m \in M}\{m\} \times C_m$ of $M \times \beta X$. Since βX is compact, the projection map π_M of $M \times \beta X$ into M is a closed map, and so $f = \pi_M|Z$ is a perfect map. Furthermore, $g = \pi_{\beta X}|Z$ maps into X and is compact-covering since $g(\{m\} \times C_m) = C_m$ and the C_m's are cofinal in $\mathcal{K}(X)$. Thus, (iv) implies (v).

From Lemma 154, we know (iii) implies that $\mathcal{K}(X)$ is countably determined. Since we already showed (iii) is equivalent to (v), then there are a space Z, a separable metrizable space M, a perfect map $f : Z \to M$, and a compact-covering map $g : Z \to X$. Hence, the map $K \mapsto g(f^{-1}(K))$ witnesses $\mathcal{K}(M) \geq_T \mathcal{K}(X)$, so (iii) implies (ii). $\square$

Once again, we identify a space X with the subset $[X]^1$ of $\mathcal{K}(X)$. Recall that a space is called *ω-bounded* if every countable subset is relatively compact (has compact closure).

Lemma 172. *For any space X, the following are equivalent:*

(i) $\mathcal{K}(X)$ is countably determined,

(ii) $(X, \mathcal{K}(X))$ is countably determined, and

(iii) every ω-bounded subset of X is relatively compact.

Proof. The equivalence of (ii) and (iii) is immediate from the definitions since for any subset Y of X, the corresponding subset $[Y]^1$ of $\mathcal{K}(X)$ is bounded if and only if Y is relatively compact. The equivalence of (i) and (ii) is a special case of Lemma 37. $\square$

Every ω-bounded space is countably compact, so a sufficient condition on a space X to have $(X, \mathcal{K}(X))$ countably determined is that every countably compact subset is compact, which occurs, for example, if X is Lindelöf.

The next fact generalizes Proposition 78 by Cascales, Orihuela, and Tkachuk.

Proposition 173. *If $(X, \mathcal{K}(X))$ is in the class $\langle 2^o$+ECSB, countably directed$\rangle$, then X has a countable network $\mathcal{N}$ modulo a cover $\mathcal{C}$ consisting of ω-bounded subsets.*

Proof. Let $(X, \mathcal{K}(X))$ be of type $\langle Q, \text{countably directed}\rangle$, where Q is a directed set with a countable base $\mathcal{B}$ witnessing ECSB. Then there is a family $\{\mathcal{S}_q : q \in Q\}$ of countably directed subsets of $\mathcal{K}(X)$ covering $\{\{x\} : x \in X\}$ such that $\mathcal{S}_q \subseteq \mathcal{S}_{q'}$ whenever $q \leq q'$ in Q. Since the down set of a countably directed set is still countably directed, we may suppose each $\mathcal{S}_q$ is downwards closed. Since Q is directed, it follows that every finite subset of X is in some $\mathcal{S}_q$.

Let $\mathcal{S}_B$, N_B, and C_q be as in Theorem 157. We will show that $\mathcal{N} = \{N_B : B \in \mathcal{B}\}$ is a network modulo the cover $\mathcal{C} = \{C_q : q \in Q\}$ of X and that each C_q is ω-bounded. We fix q in Q and let $\{B_n : n < \omega\} \subseteq \mathcal{B}$ be a decreasing neighborhood base at q.

Let $A = \{a_n : n < \omega\}$ be any countable subset of C_q. Applying Theorem 157 to the sets $A_n = \{a_n\} \subseteq C_q \subseteq N_{B_n}$ shows that for each $m < \omega$, there is an S_m in $\mathcal{S}_{B_m}$ such that $\{a_n : n \geq m\} \subseteq S_m$. Since each S_m is compact, then the closure $\overline{A}$ of A in X is compact, and $\overline{A} \setminus A$ is contained in $\bigcap_m S_m \subseteq \bigcap_m N_{B_m} = C_q$. Since A is also contained in C_q, we conclude that C_q is ω-bounded.

Now, suppose U is an open set containing C_q. Of course N_{B_n} contains C_q for each n, so to get a contradiction, assume $N_{B_n} \not\subseteq U$ for each n and pick $x_n \in N_{B_n} \setminus U$. Note that $A = \{x_n : n < \omega\}$ is infinite because otherwise, there would be an $x \in A$ such that $x \in N_{B_n} \setminus U$ for infinitely many $n < \omega$, but since $(N_{B_n})_n$ is decreasing, then x would be in $\bigcap_n N_{B_n} = C_q \subseteq U$, which is a contradiction. Repeating the argument from the previous paragraph shows that A has compact closure $\overline{A}$ in X and that $\overline{A} \setminus A \subseteq C_q \subseteq U$. Since A is infinite, then $\overline{A} \setminus A$ is nonempty, which is a contradiction since A is disjoint from U and $\overline{A} \setminus A$ is contained in U. Thus, $\mathcal{N}$ is a network for X modulo $\mathcal{C}$. ☐

Theorem 174. *Let X be a space. The following are equivalent:*

(i) $(X, \mathcal{K}(X))$ *has type* $\langle 2^o$*+CSB, countably directed*$\rangle$ *and is countably determined;*

(ii) $\mathcal{K}(M) \geq_T (X, \mathcal{K}(X))$ *for some separable metrizable space M, and $\mathcal{K}(X)$ is countably determined;*

(iii) X is Lindelöf Σ;

(iv) X is the continuous image of a perfect pre-image of a separable metrizable space; and

(v) there is a countable family $\mathcal{N}$ of subsets of X and, for each point x of X, there is a compact set L_x containing x such that, for any open U containing L_x, there is an N in $\mathcal{N}$ with $x \in N \subseteq U$.

Proof. Assume X is a Lindelöf Σ-space. It is shown in [7] that X has a $\mathcal{K}(M)$-ordered compact cover for some separable metrizable M. Also, since Lindelöf Σ-spaces are Lindelöf, then Lemma 172 and the comment following it show that $\mathcal{K}(X)$ is countably determined. Hence, (iii) implies (ii). Next, Lemma 172 and Lemma 55 show that (ii) implies (i). Now assume (i). Then Proposition 173 says that X has a countable network $\mathcal{N}$ modulo a cover $\mathcal{C}$ consisting of ω-bounded sets, and since $(X, \mathcal{K}(X))$ is also countably determined, then Lemma 172 shows that each member of $\mathcal{C}$ has compact closure. As X is T_3, the family $\{\overline{N} : N \in \mathcal{N}\}$ is a countable network for X modulo the compact cover $\{\overline{C} : C \in \mathcal{C}\}$, so (i) implies (iii), and the equivalence of (i)–(iii) is established.

The equivalence of (iii) and (iv) is well-known, and in fact, a slight variation of the proof of the equivalence of (iv) and (v) in Theorem 171 works here. Also, (iii) immediately implies (v) because (iii) is just the strengthening of (v) obtained by changing '$x \in N \subseteq U$' to '$L_x \subseteq N \subseteq U$'. We finish by showing that (v) implies (iii), so assume (v), and let $\mathcal{S}$ be the set of all closed subsets of all finite unions of L_x's. Then $\mathcal{S}$ is a pre-ideal on X that contains every closed subset of each of its members, and $\mathcal{S}$ contains the family $[X]^1$ of singleton subsets of X, which we identify with X, as usual. Then Theorem 168 shows that (v) implies X is Lindelöf $(X, \mathcal{S})$-cofinally Σ. In particular, X is Lindelöf Σ since a cofinal family for $(X, \mathcal{S})$ is a compact cover of X. □

6.4 THE STRONG PYTKEEV PROPERTY IN FUNCTION SPACES

Recall from Section 2.9 that for any pre-ideal $\mathcal{S}$ of compact sets covering a space X, $C_{\mathcal{S}}(X)$ denotes the space of all continuous real-valued functions on X with the topology of uniform convergence on members of $\mathcal{S}$. We wish to determine when $C_{\mathcal{S}}(X)$ has various (c$\cdot$) properties in terms of properties of $\mathcal{S}$ and X. Since $C_{\mathcal{S}}(X)$ is a topological group, and hence homogeneous, then it has one of the properties (c$\cdot$) if and only if it satisfies $(\mathrm{c}\cdot)_{\mathbf{0}}$, where $\mathbf{0}$ is the zero function on X.

Lemma 175. *Let X be a space, and let $\mathcal{S}$ be a pre-ideal of compact subsets covering X. If $C_{\mathcal{S}}(X)$ is (cn), then $\mathcal{S}$ is (cc)$'$.*

Proof. Let $\mathcal{D}$ be a countable local network at $\mathbf{0}$ witnessing that $C_{\mathcal{S}}(X)$ satisfies $(\text{cn})_{\mathbf{0}}$. For each $D \in \mathcal{D}$, define $N_D = \bigcap\{g^{-1}(\mathbb{R} \setminus \{1\}) : g \in D\}$ and $\mathcal{N} = \{N_D : D \in \mathcal{D}\}$. We show $\mathcal{N}$ is (cc)$'$ for $\mathcal{S}$.

Fix a (nonempty) K in $\mathcal{S}$, and let $W = B(\mathbf{0}, K, 1/2)$. By $(\text{cn})_{\mathbf{0}}$, the set $T = \bigcup\{D \in \mathcal{D} : D \subseteq W\}$ is a neighborhood of $\mathbf{0}$, so there is a member L_K of $\mathcal{S}$ containing K and $\epsilon > 0$ such that $B(\mathbf{0}, L_K, \epsilon) \subseteq T$.

Now suppose $U \subseteq X$ is any open set containing L_K. Pick $f_U \in C_{\mathcal{S}}(X)$ such that $f_U(L_K) = \{0\}$ and $f_U(X \setminus U) \subseteq \{1\}$. Then f_U is in T, so we can find a $D \in \mathcal{D}$ such that $f_U \in D \subseteq W$. Since $f_U \in D$, then $N_D \subseteq f_U^{-1}(\mathbb{R} \setminus \{1\}) \subseteq U$, and since $D \subseteq B(\mathbf{0}, K, 1/2)$, then $K \subseteq N_D$ – as required for $\mathcal{N}$ to be (cc)$'$ for $\mathcal{S}$. $\square$

The next lemma says, in particular, that the converse of the previous lemma is true if we strengthen (cc)$'$ to (cc$^+$)$'$.

Lemma 176. *Let X be a space. Let $\mathcal{S}$ be a pre-ideal of compact subsets covering X.*
(1) If $\mathcal{S}$ is (cc$^+$)$'$, then $C_{\mathcal{S}}(X)$ is (cn).
(2) If $\mathcal{S}$ is (cp)$'$, then $C_{\mathcal{S}}(X)$ is (cp).

Proof. We first prove (1), so assume $\mathcal{S}$ is (cc$^+$)$'$. We show that $C_{\mathcal{S}}(X)$ has the property $(\text{cn})_{\mathbf{0}}$, which implies, by homogeneity, that $C_{\mathcal{S}}(X)$ is (cn). Let $\mathcal{N}$ be (cc$^+$)$'$ for $\mathcal{S}$. For each K in $\mathcal{S}$, fix the L_K in $\mathcal{S}$ given by property (cc$^+$)$'$. Define $D_{N,n} = B(\mathbf{0}, N, 1/n)$ for each N in $\mathcal{N}$ and $0 < n < \omega$. We show $\mathcal{D} = \{D_{N,n} : N \in \mathcal{N},\ 0 < n < \omega\}$ is (cn) for $\mathcal{N}_{\mathbf{0}}^{C_{\mathcal{S}}(X)}$.

Fix a neighborhood U of $\mathbf{0}$, and choose a K in $\mathcal{S}$ and an $0 < n < \omega$ such that $B(\mathbf{0}, K, 1/n) \subseteq U$. Then $V = B(\mathbf{0}, L_K, 1/n)$ is also contained in U. Consider an arbitrary f in V. Since $L_K \subseteq f^{-1}(-1/n, 1/n)$, then by (cc$^+$)$'$, we can find an N in $\mathcal{N}$ such that $L_K \subseteq N \subseteq f^{-1}(-1/n, 1/n)$. It follows that, for the set $D_{N,n}$ from $\mathcal{D}$, we have $f \in D_{N,n} \subseteq V \subseteq U$.

Now we prove (2), so we assume $\mathcal{S}$ is (cp)$'$ and show that $C_{\mathcal{S}}(X)$ has the property $(\text{cp})_{\mathbf{0}}$. Theorem 169 shows that $\mathcal{S}$ is (cc$^+$)$'$, so there is a family $\mathcal{N}$, closed under finite unions and intersections, witnessing both (cp)$'$ and (cc$^+$)$'$ for $\mathcal{S}$. Let L_K and $D_{N,n}$ be as in part (1). Then we already know $\mathcal{D}$ is (cn) for $\mathcal{N}_{\mathbf{0}}^{C_{\mathcal{S}}(X)}$, and now we verify that, in fact, $\mathcal{D}$ is (cp) for $\mathcal{N}_{\mathbf{0}}^{C_{\mathcal{S}}(X)}$.

Fix a neighborhood U of $\mathbf{0}$, and take any subset A of $C_{\mathcal{S}}(X)$ such that $\mathbf{0}$ is in $\overline{A} \setminus A$. We need to show that some member of $\mathcal{D}$ is contained in U and meets A in an infinite set. Choose K in $\mathcal{S}$ and $0 < n < \omega$ such that $B(\mathbf{0}, K, 1/n)$ is contained in U, and then let $V = B(\mathbf{0}, L_K, 1/n)$, which is also contained in U. Select a decreasing sequence $(N_i)_i$ from $\mathcal{N}$ such that if N is in $\mathcal{N}$ and $L_K \subseteq N$, then for some i we have $L_K \subseteq N_i \subseteq N$. From part (1) and the choice of the $(N_i)_i$, we have that V is the increasing union of $\{D_{N_i,n} : i < \omega\}$.

Thus, it suffices to show that $D_{N_i,n} \cap A$ is infinite for some i. If this is not the case, then $A_0 = D_{N_1,n} \cap A$ and $A_i = (D_{N_i,n} \setminus D_{N_{i-1},n}) \cap A$ for $i > 0$ are all finite. For any $i > 0$, we can therefore find a finite subset B_i of $N_{i-1} \setminus N_i$ such each f in A_i satisfies $|f(x)| \geq 1/n$ for some x in B_i. Let $B = \bigcup_i B_i$. Then for any N in $\mathcal{N}$ containing L_k, we can find an i such that $L_K \subseteq N_i \subseteq N$, so that $B \setminus N \subseteq B \setminus N_i \subseteq B_1 \cup \cdots \cup B_i$ and $B \setminus N$ is finite. Hence, by the (cp)$'$ property of $\mathcal{N}$, there is an element L' of $\mathcal{S}$ containing B. But now $B(\mathbf{0}, L_K \cup L', 1/n) \setminus A_0$, which is contained in V, is an open neighborhood of $\mathbf{0}$ in $C_{\mathcal{S}}(X)$ that is disjoint from A since $A \cap V = \bigcup_i A_i$. This contradicts that $\mathbf{0}$ is in $\overline{A}$. □

The next theorem is immediate from Lemmas 175 and 176, and Theorem 169.

Theorem 177. *Let $\mathcal{S}$ be pre-ideal of compact sets covering a space X. Then the following are equivalent:*

(i) $C_{\mathcal{S}}(X)$ is (cn),

(ii) $\mathcal{S}$ is (cc)$'$,

(iii) $\mathcal{S}$ is (cc$^+$)$'$, and

(iv) X is Lindelöf $\mathcal{S}$-cofinally Σ.

Suppose $\mathcal{S}$ additionally satisfies: $S \cup K$ is in $\mathcal{S}$ whenever K is in $\mathcal{S}$ and S is a sequence in X converging to K. Then the following are also equivalent to statements (i)–(iv) above:

(v) $C_{\mathcal{S}}(X)$ is strongly Pytkeev (i.e. is (cp))

(vi) $\mathcal{S}$ is (cp)$'$.

Now we look closer at the case were $\mathcal{S} = \mathcal{K}(X)$. Recall that, in this situation, $C_{\mathcal{S}}(X)$ is written as $C_k(X)$ and denotes the topological vector space of all continuous real-valued functions on X, endowed with the compact-open topology. From Lemma 62, we have the following description of the neighborhood filter of $\mathbf{0}$ in $C_k(X)$.

Lemma 178. *For any space X, we have $\mathcal{K}(X) \times \omega =_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$.*

By Theorem 166, we know that $C_k(X)$ is (cc$^+$) if and only if it is (cc), if and only if it is first countable, which occurs if and only if X is hemicompact (i.e. $\mathcal{K}(X)$ has countable cofinality). The next theorem deals with all the other (c$\cdot$) properties of $C_k(X)$, except (cs), for which see Example 21. It also firmly connects them to order properties of the neighborhood filter of $\mathbf{0}$ in $C_k(X)$ and also to $\mathcal{K}(X)$.

Theorem 179. *The following are equivalent for any space X:*

(i) $\mathcal{N}_{\mathbf{0}}^{C_k(X)}$ is countably determined and of type $\langle 2^o$+CSB, countably directed$\rangle$,

(ii) $C_k(X)$ is countably tight and $\mathcal{K}(M) \geq_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$ for some separable metrizable M,

(iii) $C_k(X)$ is strongly Pytkeev (in other words, (cp)),

(iv) $C_k(X)$ is (ck$^+$),

(v) $C_k(X)$ is (cn),

(vi) $\mathcal{K}(X)$ is (cc)$'$,

(vii) X is Lindelöf cofinally Σ (in other words, $\mathcal{K}(X)$ is (cc$^+$)$'$),

(viii) X is k-Lindelöf and $\mathcal{K}(M') \geq_T \mathcal{K}(X)$ for some separable metrizable space M', and

(ix) $\mathcal{K}(X)$ is countably determined and of type $\langle 2^o$+CSB, countably directed$\rangle$.

Proof. Lemma 164 and Lemma 55 show that (ii) implies (i). Then Lemma 164 and Theorem 167 show that (i) implies (iii). By Theorem 177, we already know (iii), (v), (vi), and (vii) are equivalent, and Theorem 171 shows that (vii) is also equivalent to (ix). Next, Lemma 170 and Theorem 171 show that (vii) implies (viii). If we assume (viii), then by Lemma 178, setting $M = M' \times \omega$, we see that $\mathcal{K}(M) = \mathcal{K}(M' \times \omega) \geq_T \mathcal{K}(X \times \omega) \geq_T \mathcal{N}_{\mathbf{0}}^{C_k(X)}$. Also, since X is k-Lindelöf, then $C_k(X)$ is countably tight (see [36]), so (ii) holds.

So far we have shown that (i)–(iii) and (v)–(ix) are all equivalent, so it remains to show (iv) is equivalent to the rest. By Lemma 55 and part (2$'$) of Theorem 167, we see that (ii) implies (iv), and of course (iv) implies (v) by Lemma 165. $\square$

Finally, we look at the case where $\mathcal{S} = [X]^{<\omega}$ is the ideal of all finite subsets of a space X. In this case, $C_{\mathcal{S}}(X)$ is denoted as $C_p(X)$, which is the space of all continuous real-valued functions on X with the topology of pointwise convergence. Regarding local networks for $C_p(X)$, Sakai proved the following result in [42].

Theorem 180 (Sakai)**.** *If $C_p(X)$ is (cs), then X is countable (and $C_p(X)$ is separable metrizable).*

Since (cp_ω) and (ck) both imply (cs), all our local network conditions on $C_p(X)$ – except for (cn) – therefore imply that X is countable. We will see shortly that for $C_k(X)$ the condition (cn) implies (cp) and (ck^+). However, $C_p(X)$ is different; indeed, every cosmic space is (cn), and $C_p(X)$ is cosmic if and only if X is cosmic. But more is true, and $C_p(X)$ can be (cn) for non-trivial reasons. We now characterize exactly when $C_p(X)$ is (cn). This result parallels Theorem 179.

Theorem 181. *For any space X, the following are equivalent:*

(i) $C_p(X)$ is (cn),

(ii) $[X]^{<\omega}$ is (cc)$'$,

(iii) $[X]^{<\omega}$ is (cc^+)$'$ (in other words, X is Lindelöf $[X]^{<\omega}$-cofinally Σ), and

(iv) X has a countable network modulo a cover consisting of finite sets.

Proof. The equivalence of (i) and (ii) follows from Lemmas 175 and 176. The equivalence of (ii) and (iii) follows from Theorem 169. Of course (iii) implies (iv) since any cofinal subset of $[X]^{<\omega}$ covers X, so it remains to show that (iv) implies (iii). Let $\mathcal{N}$ be a countable network for X modulo a cover $\mathcal{A}$ consisting of finite subsets of X. Let $\mathcal{A}'$ and $\mathcal{N}'$ denote the sets of all finite unions of elements of $\mathcal{A}$ and $\mathcal{N}$, respectively. Then $\mathcal{A}'$ is cofinal in $[X]^{<\omega}$, and $\mathcal{N}'$ is a countable network for X modulo $\mathcal{A}'$. □

In the terminology of [34], the spaces satisfying (iv) in the previous theorem are precisely those in the class $L\Sigma(<\omega)$. Proposition 2.2 of [34] gives the following alternate characterizations of these spaces.

Proposition 182 (Kubiś, Okunev, Szeptycki, [34])**.** *The following are equivalent for any space X:*

(i) X is $L\Sigma(<\omega)$,

(ii) there is a second countable space M and an upper semi-continuous map $f : M \to [X]^{<\omega}$ whose image covers X,

(iii) X is a continuous image of a space L such that there are a second countable space M and a perfect map $g : L \to M$ whose fibres are finite, and

(iv) X is a continuous image of a closed subspace F of $M \times K$ for some second countable space M and compact space K such that $F \cap \pi_M^{-1}(m)$ is finite for each m in M.

6.5 CONNECTING NETWORKS TO OTHER PROPERTIES

A space X is (α_4) at a point x if: whenever $\{(x_{m,n})_{n<\omega} : m < \omega\}$ is a family of sequences in X that all converge to x, then there are strictly increasing sequences $(m_k)_k$ and $(n_k)_k$ in ω such that $\lim_i x_{m_i,n_i} = x$.

Lemma 183. *Suppose A is a countably compact subset of a Fréchet-Urysohn space X. If X satisfies (cs)$_A$, and if X is (α_4) at each point in A, then A has a countable neighborhood base in X.*

Proof. Let $\mathcal{D}$ be a countable local network for A witnessing (cs)$_A$. Without loss of generality, $\mathcal{D}$ is closed under finite unions. We will show that for every neighborhood U of A, there is a D from $\mathcal{D}$ such that $A \subseteq D^o \subseteq U$.

Fix a neighborhood U of A. By (cs)$_A$, there is a neighborhood V of A contained in U such that every convergent sequence on V is contained in some member of $\mathcal{D}_U = \{D \in \mathcal{D} : D \subseteq U\}$. Since $\mathcal{D}$ is countable and closed under finite unions, we can find an increasing sequence $(D_n)_n$ in $\mathcal{D}_U$ such that every compact subset of V is contained in some D_n.

Suppose, for a contradiction, that D_n is not a neighborhood of A for each $n < \omega$. Then A meets the closure of $V \setminus D_n$ for each n. Since $(A \cap \overline{V \setminus D_n})_n$ is a decreasing sequence of nonempty closed subsets of the countably compact space A, then the intersection of this sequence is nonempty, so there is an x in A which is in the closure of each $V \setminus D_n$. Since X is Fréchet-Urysohn, we can find a sequence $(x_{n,m})_m$ in $V \setminus D_n$ converging to x for each $n < \omega$. Applying (α_4), there are strictly increasing sequences $(n_i)_i$ and $(m_i)_i$ in ω such that $(x_{n_i,m_i})_i$ converges to x.

Then $K = \{x_{n_i,m_i} : i < \omega\} \cup \{x\}$ is a convergent sequence on V, and so there must be a

$k < \omega$ such that $K \subseteq D_k$. On the other hand, x_{n_k,m_k} is not in D_{n_k} by definition, and since $n_k \geq k$, then $D_{n_k} \supseteq D_k$. Hence, x_{n_k,m_k} is in K but not in D_k, which is a contradiction. $\square$

Corollary 184. *If X is Fréchet-Urysohn, (α_4) and (cs)$_x$ at a some x in X, then x is a point of first countability.*

Theorem 63 shows that if X is countably compact and if the neighborhood filter $\mathcal{N}_\Delta$ of the diagonal Δ in X^2 has calibre (ω_1, ω), then X must be metrizable. We can now give another metrization result for countably compact spaces.

Proposition 185. *If X is countably compact, Fréchet-Urysohn, and (α_4), and if X^2 satisfies (cs)$_\Delta$, then X is metrizable*

Proof. Corollary 184 shows X is first countable, and so X^2 is also first countable, which means X^2 is both Fréchet-Urysohn and (α_4). Also, Δ is a countably compact subset of X^2 since it is closed. Thus, we can apply Lemma 183 to see that Δ has a countable neighborhood base. Finally, Chaber [8] showed that a countably compact space with a G_δ-diagonal is metrizable. $\square$

For topological groups, the (α_4) assumption in Corollary 184 is redundant. Indeed, Nyikos proved:

Lemma 186 (Nyikos [40])**.** *If G is a topological group that is Fréchet-Urysohn, then it is (α_4) at every point.*

Thus, we obtain:

Corollary 187. *Let G be a Fréchet-Urysohn group.*

(1) If G is (cs) (or, a fortiori*, (ck) or strongly Pytkeev), then it is metrizable.*

(2) If there is a directed set P in the class $\langle 2^o$+CSB, countably directed$\rangle$ such that $P \geq_T \mathcal{N}_e$, then G is metrizable.

Proof. For part (1), Corollary 184 and Lemma 186 show that G is first countable, but every first countable topological group is metrizable. For part (2), recall that Fréchet-Urysohn implies countably tight, so G is strongly Pytkeev by Theorem 167 and Lemma 34, and thus we may apply (1). $\square$

In Example 20, we will show that (cs) cannot be replaced with (cn) in the previous result. However, we still can relate (cn) to other properties, as the next few results show.

A *point-network* (respectively, *strong point-network*) for a space X is a collection $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where each $\mathcal{W}(x)$ is a collection of subsets of X containing x such that for any point x in an open set U, there is an open set V with $x \in V \subseteq U$ satisfying: for each y in V, there is a W in $\mathcal{W}(y)$ such that $x \in W \subseteq U$ (respectively, $x \in W \subseteq V$). We note that we can take U and V to be basic open sets. Point-networks are also known as 'condition (F)', and as the 'Collins-Roscoe structuring mechanism' after the authors who introduced them [10]. The term 'point network' was suggested by Gruenhage. A point-network is called *countable* if each collection $\mathcal{W}(x)$ is countable. Spaces with a countable point-network are sometimes called 'Collins-Roscoe' spaces.

Lemma 188. *If a topological group G is (cn), then G has a countable point-network.*

Hence, G, and each of its finite powers, is hereditarily metaLindelöf and (monotonically) monolithic.

Proof. Fix a countable family $\mathcal{D}$ satisfying $(\mathrm{cn})_e$, where e is the identity in G. We may suppose if D is in $\mathcal{D}$ then so is D^{-1}. Define $\mathcal{W} = \{\mathcal{W}(g) : g \in G\}$ by $\mathcal{W}(g) = \{gD : D \in \mathcal{D}\}$. Note that each $\mathcal{W}(g)$ is countable, and now we show this is a point-network. Fix an arbitrary g in G and a basic open neighborhood U of g, so $U = gU_e$ for some open neighborhood U_e of e. Pick a neighborhood T of e such that $TT^{-1} \subseteq U_e$. Since $\mathcal{D}$ satisfies $(\mathrm{cn})_e$, then we know $V_e = \bigcup\{D \in \mathcal{D} : D \subseteq T\}$ is a neighborhood of e, and so $V = gV_e$ is a neighborhood of g. Also, note that $V_e \subseteq T \subseteq U_e$, so V is contained in U.

Now consider any h in V. For some D in $\mathcal{D}$ such that $D \subseteq T$, we have $h \in gD$, and so $h = gd$ for some d in D. Note that $g = hd^{-1}$ and D^{-1} is in $\mathcal{D}$, so $W = hD^{-1}$ is in $\mathcal{W}(y)$. It remains to note that $g = hd^{-1} \in hD^{-1} = W = gdD^{-1} \subseteq gDD^{-1} \subseteq gTT^{-1} \subseteq U$. $\square$

Recall that a space is called *Baire* if any countable union of closed nowhere dense subsets has empty interior. A space if *locally Baire* if each point has a neighborhood base whose members are Baire.

Lemma 189. *Let A be a nonempty subset of a locally Baire space X such that $(\mathrm{cn})_A$ holds and A has neighborhood base of closed sets. Then A has countable π-character in X.*

Proof. Let $\mathcal{D}$ be a local network at A witnessing $(\mathrm{cn})_A$. Since A has a neighborhood base of closed sets, we may assume each member of $\mathcal{D}$ is closed. Consider any open U containing A. Then $T = \bigcup\{D \in \mathcal{D} : D \subseteq U\}$ is a neighborhood of A by $(\mathrm{cn})_A$. Fix an x in A. Then there is an open, Baire neighborhood V of x contained in T. Since $\{D \cap V : D \in \mathcal{D},\ D \subseteq U\}$ is a countable closed cover of the Baire space V, then some member of this cover has nonempty interior in V, and hence also in X. Thus, $\{D^\circ : D \in \mathcal{D}\}$ is a countable π-base at A. $\square$

Since any point in a (regular) space has a neighborhood base of closed sets, then we obtain:

Corollary 190. *If x is a point in a locally Baire space X such that $(cn)_x$ holds, then x has countable π-character*

Sakai gives the following definition in [42]. A space X has the property (#) at a point x in X if for each sequence $(U_n)_{n<\omega}$ of open subsets of X such that x is in $\bigcap_n \overline{U_n}$, there is a sequence $(F_n)_{n<\omega}$ of finite sets with $F_n \subseteq U_n$ such that x is in $\overline{\bigcup_n F_n}$.

In [42], it is shown that (#) combined with the strong Pytkeev property at x (that is, $(\mathrm{cp})_x$) guarantees first countability at x. We generalize Sakai's result as follows. Say X has the property $(\#)'$ at a subset A if for each decreasing sequence $(U_n)_{n<\omega}$ of open subsets of X with $A \cap \overline{U_n} \neq \emptyset$ for each n, there is a sequence $(F_n)_{n<\omega}$ of finite sets with $F_n \subseteq U_n$ such that $A \cap \overline{\bigcup_n F_n} \neq \emptyset$. If $A = \{x\}$ is a singleton, then we just say X has the property $(\#)'$ at x. If X satisfies (#) at x, then it clearly satisfies $(\#)'$ at x.

Lemma 191. *If A is a countably compact subset of X and X has the property $(\#)'$ at each point in A, then X has the property $(\#)'$ at A.*

Proof. Suppose $(U_n)_n$ is a decreasing family of open subsets of X such that A meets $\overline{U_n}$ for each n. Then $(A \cap \overline{U_n})_n$ is a decreasing sequence of nonempty closed subsets of the countably compact space A, so the intersection of this sequence is nonempty. Thus, there is an x in $A \cap \bigcap_n \overline{U_n}$. The property $(\#)'$ at x then provides a sequence $(F_n)_n$ of finite sets $F_n \subseteq U_n$ such that x is in $\overline{\bigcup_n F_n}$. Hence, A meets $\overline{\bigcup_n F_n}$. $\square$

Lemma 192. *Suppose X satisfies $(cp_\omega)_A$ and $(\#)'$ for a subset A of X which has a neighborhood base of closed sets. Then A has a countable neighborhood base in X.*

Proof. Let $\mathcal{D}$ be a countable local network at A witnessing $(\mathrm{cp}_\omega)_A$. Since A has a neighborhood base of closed sets, then we may assume each member of $\mathcal{D}$ is closed. We will show that $\{\bigcup \mathcal{D}' : \mathcal{D}' \subseteq \mathcal{D} \text{ finite}\} \cap \mathcal{N}_A$ is a (clearly countable) neighborhood base at A. Fix an open neighborhood U of A and enumerate $\{D \in \mathcal{D} : D \subseteq U\} = \{D_n : n < \omega\}$. It suffices to find an n such that $B_n = D_0 \cup \cdots \cup D_n$ is a neighborhood of A.

Suppose no such n exists. Then A meets the closure of the open set $U_n = U \setminus B_n$ for each $n < \omega$. Since $(U_n)_n$ is a decreasing sequence, then property $(\#)'$ provides finite sets $F_n \subseteq U_n$ such that A meets the closure of the countable set $C = \bigcup_n F_n$. Note also that C is disjoint from A since each D_n contains A. Hence, $(\mathrm{cp}_\omega)_A$ provides a D in $\mathcal{D}$ such that $D \subseteq U$ and $D \cap C$ is infinite. But then $D = D_n$ for some $n < \omega$, and since $D_n \cap F_m$ is empty for each $m \geq n$, we see that $D \cap C \subseteq F_0 \cup \cdots \cup F_{n-1}$ is actually finite, a contradiction. □

Since every point in a (regular) space has a neighborhood base of closed sets, then we recover Sakai's result from [42] mentioned above (with slightly weakened hypotheses).

Corollary 193. *If X satisfies $(\#)'$ and $(cp_\omega)_x$ at a point x in X, then x is a point of first countability.*

We cannot weaken the hypothesis $(\mathrm{cp}_\omega)_x$ to $(\mathrm{cn})_x$ in the previous result. Indeed, if $Y = \mathbb{R}$ with the discrete topology, then the one-point compactification $X = \alpha Y = Y \cup \{\infty\}$ (that is, the supersequence of size $\mathfrak{c}$) has both properties $(\#)$ and $(\mathrm{cn})_\infty$ at ∞, but X is clearly not first countable at ∞.

6.6 MORE ABOUT NEIGHBORHOOD FILTERS

6.6.1 IN METRIZABLE SPACES

Lemma 194. *Let A be a nowhere dense subset of a first countable space X. Let $\mathcal{U}$ be a family of open subsets of X, which is discrete for A (every point in A has an open neighborhood in X meeting at most one member of $\mathcal{U}$), and each member of which meets A. Then $\mathcal{N}_A^X \geq_T \omega^{\mathcal{U}}$.*

Proof. For each U in $\mathcal{U}$, pick $x_U \in U \cap A$. For each x in A, fix an open neighborhood T_x

of x such that T_x meets exactly one element of $\mathcal{U}$, say U_x. Since X is first countable (and Tychonoff), then for each U in $\mathcal{U}$, we can fix a decreasing local base $(B_n^U)_{n<\omega}$ at x_U such that $B_0^U = T_{x_U} \cap U$ and $B_n^U \supseteq \overline{B_{n+1}^U}$. As A is nowhere dense in X, we can pick z_n^U in $(B_n^U \setminus \overline{B_{n+1}^U}) \setminus A$.

Define $\phi : \mathcal{N}_A^X \to \omega^{\mathcal{U}}$ by $\phi(V) = (n_U)_U$, where n_U is the minimal $n < \omega$ such that $B_n^U \subseteq V$. Then ϕ is order-preserving, and we now show it is a surjection. Take any $(n_U)_U$ in $\omega^{\mathcal{U}}$. Let $V = V_0 \cup V_1$ where $V_0 = \bigcup\{B_{n_U}^U : U \in \mathcal{U}\}$ and $V_1 = X \setminus \{z_{n_U-1}^U : U \in \mathcal{U} \text{ and } n_U > 0\}$. If x is in $A \setminus V_0$, then either T_x or $T_x \setminus \{z_{n_{U_x}-1}^{U_x}\}$ is a neighborhood of x contained in V_1. Thus, V is a neighborhood of A. Moreover, since $z_{n_U-1}^U$ is not in V for each U such that $n_U > 0$, then for every U in $\mathcal{U}$, the smallest n such that B_n^U is contained in V is $n = n_U$. In other words, $\phi(V)_U = n_U$, so ϕ is surjective, as claimed. □

Corollary 195. *Let A be a closed nowhere dense subset of metrizable X. If $\mathcal{N}_A^X$ has calibre (ω_1, ω), then A is separable.*

Proof. Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a base for X, where each $\mathcal{B}_n$ is discrete. Set $\mathcal{B}_n^A = \{B \in \mathcal{B}_n : A \cap B \neq \emptyset\}$, and $\mathcal{B}^A = \bigcup_n \mathcal{B}_n^A$. Then $\mathcal{B}^A$ is a base for A in X. So A is second countable (and so separable) if we can show each $\mathcal{B}_n^A$ is countable. By Theorem 135, we know the directed set ω^κ is not calibre (ω_1, ω) when κ is uncountable since ω is not countably directed. It follows from Lemma 194 that every discrete family $\mathcal{U}$ of open sets in X each member of which meets A must be countable. In particular, each $\mathcal{B}_n^A$ is countable, as desired. □

Lemma 196. *If X is a metrizable space and A is a separable subset of X, then X can be embedded in a metrizable space $\tilde{X}$ such that the closure K of A in $\tilde{X}$ is compact and $\tilde{X} \setminus X = K \setminus A$.*

Proof. Let D be a countable dense subset of A. Recall that X embeds in $H(\kappa)^\omega$, where κ is the weight of X and $H(\kappa)$ is the metric hedgehog with κ many spines (see 4.4.9 in [13]). Viewing D as a subset of $H(\kappa)^\omega$, we see that for each $n < \omega$, the projection $\pi_n(D)$ is a countable subset of $H(\kappa)$, and so $\pi_n(D)$ is contained in countably many spines of the hedgehog. By rearranging the spines in each factor, we may assume there is one subspace Y

of $H(\kappa)$ consisting of countably many spines (so $Y \cong H(\omega)$) that contains every projection $\pi_n(D)$. Thus, $D \subseteq Y^\omega \subseteq H(\kappa)^\omega$.

Let I be the closed unit interval $[0,1]$ and consider the subspace $H = \{(x, \frac{x}{n}) : x \in I,\ n \in \mathbb{N}\}$ of $\mathbb{R}^2$. Then H is homeomorphic to $H(\omega)$ and so also to Y. Now let $F = H \cup (I \times \{0\})$, which is the closure of H in $\mathbb{R}^2$ and is compact. So F is H with one additional 'limit spine'. By gluing the origin in F to the center of the hedgehog $H(\kappa)$, we obtain a metrizable space $\tilde{H}(\kappa)$ which has a total of $\kappa + \omega = \kappa$ many spines, one of which is the limit of a countable sequence of spines. We can identify $H(\kappa)$ with the subspace of non-limit spines in $\tilde{H}(\kappa)$ in such a way that Y is identified with the countable sequence of spines which converge to the limit spine.

We now have $D \subseteq Y^\omega \subseteq H(\kappa)^\omega \subseteq \tilde{H}(\kappa)^\omega$, and the closure $\tilde{Y}$ of Y in $\tilde{H}(\kappa)$ is compact (being homeomorphic to F). Let K be the closure of D in $\tilde{H}(\kappa)^\omega$. Since D is dense in A, we see that K is also the closure of A in $\tilde{H}(\kappa)^\omega$. Moreover, K is contained in $\tilde{Y}^\omega$ and so is compact. Finally, $\tilde{X} = X \cup K \subseteq \tilde{H}(\kappa)^\omega$ is the desired metrizable space. □

In particular, Lemma 196 shows that every separable metrizable space M has a metrizable compactification γM. Let $\check{M} = \gamma M \setminus M$, which is called the *remainder* of M in the compactification γM. Then $\check{M}$ is also separable metrizable. Although different choices of the compactification γM will yield different remainders $\check{M}$, Lemma 197 below says that the Tukey class of $\mathcal{K}(\check{M})$ is independent of the choice of (metrizable) compactification.

Lemma 197. *If γX and δX are compactifications of a space X, then $\mathcal{K}(\gamma X \setminus X) =_T \mathcal{K}(\delta X \setminus X)$.*

Proof. It clearly suffices to prove this when $\delta X = \beta X$, the Stone-Cech compactification of X. By maximality of βX, there is a continuous $F : \beta X \to \gamma X$ which is the identity on X and carries $\beta X \setminus X$ onto $\gamma X \setminus X$. Then $\phi_1 : \mathcal{K}(\beta X \setminus X) \to \mathcal{K}(\gamma X \setminus X)$ defined by $\phi_1(K) = F(K)$ is easily checked to be a Tukey quotient. As is $\phi_2 : \mathcal{K}(\gamma X \setminus X) \to \mathcal{K}(\beta X \setminus X)$ defined by $\phi_2(L) = F^{-1}L$ □

Theorem 198. *Let X be a metrizable space, let A be a separable subset of X, and let $\check{A}$ be the remainder of A in any (separable) metrizable compactification of A.*

(1) If A is compact, then $\omega \geq_T \mathcal{N}_A^X$.

(2) If A is non-compact, then $\mathcal{K}(\check{A} \times \omega^\omega) \geq_T \mathcal{N}_A^X$.

(3) If A is nowhere dense in X and non-compact, then $\mathcal{N}_A^X =_T \mathcal{K}(\check{A} \times \omega^\omega)$.

Proof. If A is compact, then certainly it has a countable neighborhood base in X, which gives (1). So assume A is not compact. By Lemma 196, X embeds in a metrizable space $\tilde{X} = X \cup K$, where K is the closure of A in $\tilde{X}$ and K is compact. Let $M = K \setminus A$, which is non-empty since A is not compact. Then $\mathcal{K}(M) =_T \mathcal{K}(\check{A})$ by Lemma 197, and so $\mathcal{K}(\check{A} \times \omega^\omega) =_T \mathcal{K}(\check{A}) \times \mathcal{K}(\omega^\omega) =_T \mathcal{K}(M) \times \omega^\omega$.

Since $\mathcal{N}_A^{\tilde{X}} \geq_T \mathcal{N}_A^X$ (by tracing neighborhoods in $\tilde{X}$ down onto X), then to prove (2), it suffices to show $\mathcal{K}(M) \times \omega^\omega \geq_T \mathcal{N}_A^{\tilde{X}}$. Fix a compatible metric for $\tilde{X}$. For any compact subset L of $\tilde{X}$ and any $n < \omega$, let $B_n(L)$ be the open ball of radius $\frac{1}{n+1}$ around L and let $\overline{B}_n(L)$ be the corresponding closed ball. We may assume the metric is bounded by 3/4 so that $B_0(L) = \tilde{X}$ for any L. For each σ in ω^ω and each compact subset L of M, let $L_\sigma = L \cup \bigcup_n [\overline{B}_n(L) \setminus B_{\sigma(n)}(K)]$. Note that each L_σ is a closed subset of $\tilde{X}$ such that $L_\sigma \cap K = L$, and in particular, L_σ is disjoint from A. Now define $\phi_1 : \mathcal{K}(M) \times \omega^\omega \to \mathcal{N}_A^{\tilde{X}}$ by $\phi_1(L, \sigma) = \tilde{X} \setminus L_\sigma$.

Since ϕ_1 is clearly order-preserving, it remains to show that the image of ϕ_1 is cofinal in $\mathcal{N}_A^{\tilde{X}}$. Let U be any open subset of $\tilde{X}$ containing A, and let $L = K \setminus U$, which is a compact subset of M. We wish to find a σ such that $\phi_1(L, \sigma)$ is contained in U, or in other words, such that $\tilde{X} \setminus U$ is contained in L_σ. Let C_n be the closed set $\overline{B}_n(L) \setminus B_{n+1}(L)$ for each n, and note that these sets, together with L, form a cover of $\tilde{X}$. Thus, we have

$$\tilde{X} \setminus U = (\tilde{X} \setminus U) \cap \left(L \cup \bigcup_n C_n \right) = L \cup \bigcup_n (C_n \setminus U) = L \cup \bigcup_n [\overline{B}_n(L) \setminus V_n],$$

where $V_n = U \cup B_{n+1}(L)$ for each n. Note that each V_n is a neighborhood of K in $\tilde{X}$, so we can find σ such that $B_{\sigma(n)}(K)$ is contained in V_n for each n. It follows that $\tilde{X} \setminus U$ is contained in L_σ, as desired to complete the proof of (2).

Now we assume further that A is nowhere dense in X and show that $\mathcal{N}_A^X \geq_T \mathcal{K}(M) \times \omega^\omega$. Since A is not compact, there is a sequence $(a_n)_n$ on A which does not converge to any element of A. As X is metrizable, we can find an open set U_n around each a_n so that

$\mathcal{U} = \{U_n : n < \omega\}$ is discrete on A. Since A is nowhere dense, we can apply Lemma 194 to see that $\mathcal{N}_A^X \geq_T \omega^{\mathcal{U}} =_T \omega^\omega$. Thus it remains to show $\mathcal{N}_A^X \geq_T \mathcal{K}(M)$.

Define $\phi_2 : \mathcal{N}_A^X \to \mathcal{K}(M)$ by $\phi_2(U) = \overline{(X \setminus U)}^{\tilde{X}} \cap K$. First note that as U is a neighborhood of A in X, its complement in X has closure in $\tilde{X}$ disjoint from A. Hence, $\phi_2(U)$ really is a compact subset of $M = K \setminus A$. Clearly ϕ_2 also preserves the relevant orders (reverse inclusion on $\mathcal{N}_A^X$, inclusion on $\mathcal{K}(M)$), so it suffices to show the image of ϕ_2 is cofinal. In fact, we will show ϕ_2 is a surjection.

Fix any compact subset L of M and choose a countable dense subset D of L. Enumerate $D = \{x_n : n < \omega\}$ such that each element is listed infinitely many times. For each n, let B_n denote the open ball of radius $\frac{1}{n+1}$ around x_n in $\tilde{X}$. Since A is dense in K, then B_n meets A, and since A is nowhere dense in X, then B_n also meets $X \setminus \overline{A}$, say at y_n. Let $C = \{y_n : n < \omega\}$. Then by construction, the closure of C in $\tilde{X}$ is $C \cup L$, which is disjoint from A. So if we let $U = X \setminus C$, then U is in $\mathcal{N}_A^X$, and $\phi_2(U) = (C \cup L) \cap K = L$, which completes the proof of (3). □

In the remainder of this section, we investigate the neighborhood filter of the diagonal Δ of a metrizable space X. Recall the Cantor-Bendixson process described in Section 3.3, wherein $I(X)$ denotes the set of isolated points in X, and $X' = X \setminus I(X)$ is the derived set of X. For any subset S of X, let $\Delta(S) = \{(x, x) : x \in S\}$ be the diagonal of S in X^2. Write Δ for $\Delta(X)$.

Lemma 199. *Fix a space X. Then $\Delta(X')$ is a closed nowhere dense subset of X^2, and $\mathcal{N}_\Delta^{X^2} =_T \mathcal{N}_{\Delta(X')}^{X^2}$.*

Proof. Since X' is closed in X, then $\Delta(X')$ is closed in Δ and so also in X^2. Also, for any point x in X', we know x is not isolated in X, so every basic neighborhood of (x, x) in X^2 must intersect $X^2 \setminus \Delta$. Thus, $\Delta(X')$ is also nowhere dense in X^2.

Now for any neighborhood U of $\Delta(X')$ in X^2, let $\phi_1(U) = \Delta(I(X)) \cup U$, which is a neighborhood of $\Delta = \Delta(X)$ in X^2 since each point of $I(X)$ is isolated in X. Moreover, every member of $\mathcal{N}_\Delta^{X^2}$ contains such a neighborhood, so the order-preserving map $\phi_1 : \mathcal{N}_{\Delta(X')}^{X^2} \to \mathcal{N}_\Delta^{X^2}$ witnesses $\mathcal{N}_{\Delta(X')}^{X^2} \geq_T \mathcal{N}_\Delta^{X^2}$.

For each x in X, let $P_x = (X \times \{x\}) \cup (\{x\} \times X)$, and for any neighborhood V of $\Delta(X)$ in X^2, define $\phi_2(V) = V \setminus \{(x,x) : P_x \cap V = \{(x,x)\}\}$. Note that $\phi_2(V)$ is a neighborhood of $\Delta(X')$ in X^2 since each x in X' has a neighborhood W such that W^2 is contained in V, and since W is not a singleton, then W^2 is also contained in $\phi_2(V)$. Evidently $\phi_2 : \mathcal{N}_\Delta^{X^2} \to \mathcal{N}_{\Delta(X')}^{X^2}$ is order-preserving, and if U is in $\mathcal{N}_{\Delta(X')}^{X^2}$, then $\phi_2(\phi_1(U)) = U$. Hence, ϕ_2 shows $\mathcal{N}_\Delta^{X^2} \geq_T \mathcal{N}_{\Delta(X')}^{X^2}$. $\square$

A point x in a space X is called a *P-point* if every countable intersection of neighborhoods of x is a neighborhood of x. In other words, x is a P-point when $\mathcal{N}_x$ is countably directed.

Lemma 200. *Let X be a space, and let E be a closed discrete subspace of X. Then $\mathcal{N}_\Delta^{X^2} \geq_T \prod\{\mathcal{N}_x^X : x \in E\}$.*

Hence, if $\mathcal{N}_\Delta^{X^2}$ has calibre (ω_1, ω), and if A is a closed subset of X such that at most countably many points of A are P-points in X, then A has countable extent.

Proof. Define $\phi : \mathcal{N}_\Delta^{X^2} \to \prod\{\mathcal{N}_x^X : x \in E\}$ by $\phi(W) = (\pi_2[W \cap (\{x\} \times X)])_{x \in E}$, which is clearly order-preserving. To see that the image of ϕ is cofinal, fix any $(U_x)_x$ in $\prod\{\mathcal{N}_x^X : x \in E\}$, and for each x in E, fix a neighborhood V_x of x such that $V_x \cap E = \{x\}$. Then $W = (X \setminus E)^2 \cup \bigcup\{(U_x \cap V_x)^2 : x \in E\}$ is a neighborhood of Δ, and $\phi(W) = (U_x \cap V_x)_x$, which is 'above' $(U_x)_x$ in $\prod\{\mathcal{N}_x^X : x \in E\}$.

Now suppose $\mathcal{N}_\Delta^{X^2}$ has calibre (ω_1, ω), and assume A is a closed subset of X that does not have countable extent, so there is an uncountable closed discrete subset E of A. Since E is also closed in X, then from the first part, we know $\prod\{\mathcal{N}_x^X : x \in E\}$ must also have calibre (ω_1, ω). Then Theorem 135 shows that $\mathcal{N}_x^X$ is countably directed for all but countably many points x in E. It follows that uncountably many of the points in E are P-points in X. $\square$

Theorem 201. *Let X be metrizable. Then the following are equivalent:*

(i) X' is separable,

(ii) $\mathcal{N}_\Delta^{X^2} =_T \mathcal{K}(\check{X}' \times \omega^\omega)$,

(iii) $\mathcal{K}(M) \geq_T \mathcal{N}_\Delta^{X^2}$, for some separable metrizable M,

(iv) $P \geq_T \mathcal{N}_\Delta^{X^2}$, for some second countable P with CSB, and

(v) $\mathcal{N}_\Delta^{X^2}$ is calibre (ω_1, ω).

In particular, $\omega^\omega \geq_T \mathcal{N}_\Delta^{X^2}$ *if and only if* X' *is* σ*-compact.*

Proof. If X' is separable, then $\Delta' = \Delta(X')$ is a separable, closed and nowhere dense subset of X^2. By Theorem 198, $\mathcal{N}_{\Delta'}^{X^2} =_T \mathcal{K}(\check{\Delta'} \times \omega^\omega)$. Since X' and Δ' are homeomorphic, combining this with Lemma 199, we see $\mathcal{N}_\Delta^{X^2} =_T \mathcal{K}(\check{X'} \times \omega^\omega)$, so (i) implies (ii).

Taking $M = \check{X'} \times \omega^\omega$ shows that (ii) implies (iii), while Lemma 55 shows that (iii) implies (iv), and Lemma 40 shows that (iv) implies (v). Now assume (v). Since X is first countable, then none of the points in X' are P-points in X, so Lemma 200 shows that X' has countable extent and is therefore separable. Thus, the equivalence of (i)-(v) is established.

Finally, if X' is σ-compact, then X' is separable since it is metrizable, and likewise if $\omega^\omega \geq_T \mathcal{N}_\Delta^{X^2}$, then from the equivalence of (i)-(v), we know X' is separable. So in either case, the equivalence of (i)-(v) shows that $\mathcal{N}_\Delta^{X^2} =_T \mathcal{K}(\check{X'} \times \omega^\omega) =_T \mathcal{K}(\check{X'}) \times \omega^\omega$. Hence $\omega^\omega \geq_T \mathcal{N}_\Delta^{X^2}$ if and only if $\omega^\omega \geq_T \mathcal{K}(\check{X'})$, and by Christensen's Theorem 83, this occurs if and only if $\check{X'} = \gamma(X') \setminus X'$ is Polish (i.e. absolutely G_δ) or equivalently X' is σ-compact (i.e. absolutely F_σ). ∎

If X itself is separable, rather than just X', then the next result says we can replace $\mathcal{K}(\check{X'} \times \omega^\omega)$ with $\mathcal{K}(\check{X} \times \omega^\omega)$ in the implication '(i) $\implies$ (ii)' of the previous theorem.

Lemma 202. *Let* X *be separable metrizable but not compact. Then* $\mathcal{K}(\check{X'}) \times \omega^\omega \geq_T \mathcal{K}(\check{X}) \geq_T \mathcal{K}(\check{X'})$*. Hence,* $\mathcal{N}_\Delta^{X^2} =_T \mathcal{K}(\check{X} \times \omega^\omega)$*.*

Proof. Let γX be a separable metrizable compactification of X, so we can take $\check{X} = \gamma X \setminus X$ and $\check{X'} = \overline{X'}^{\gamma X} \setminus X'$. Note that $\check{X'} = \check{X} \cap \overline{X'}^{\gamma X}$, so $\check{X'}$ is a closed subset of $\check{X}$, and we immediately get that $\mathcal{K}(\check{X}) \geq_T \mathcal{K}(\check{X'})$.

Next we show $\mathcal{K}(\check{X'}) \times \omega^\omega \geq_T \mathcal{K}(\check{X})$. We know from Lemma 52 and Theorem 198 that $\mathcal{K}(\check{X'}) \times \omega^\omega =_T \mathcal{K}(\check{X'} \times \omega^\omega) \geq_T \mathcal{N}_{X'}^{\gamma X \setminus I(X)}$, and by Corollary 13, we see that $\mathcal{K}(\check{X}) =_T \mathcal{N}_X^{\gamma X}$, so it suffices to show $\mathcal{N}_X^{\gamma X} =_T \mathcal{N}_{X'}^{\gamma X \setminus I(X)}$. Define $\phi : \mathcal{N}_{X'}^{\gamma X \setminus I(X)} \to \mathcal{N}_X^{\gamma X}$ by $\phi(U) = U \cup I(X)$ which is well-defined since $I(X)$ is open in γX. Then ϕ is an order-isomorphism with inverse $\phi^{-1} : \mathcal{N}_X^{\gamma X} \to \mathcal{N}_{X'}^{\gamma X \setminus I(X)}$ defined by $\phi^{-1}(U) = U \setminus I(X)$. Thus, the first claim is proven.

Finally, note that $\omega^\omega = \omega^\omega \times \omega^\omega$, so the first claim gives $\mathcal{K}(\check{X'}) \times \omega^\omega =_T \mathcal{K}(\check{X'}) \times \omega^\omega \times \omega^\omega \geq_T \mathcal{K}(\check{X}) \times \omega^\omega \geq_T \mathcal{K}(\check{X'}) \times \omega^\omega$, so that $\mathcal{K}(\check{X}) \times \omega^\omega =_T \mathcal{K}(\check{X'}) \times \omega^\omega$. Thus, the second claim follows from Lemma 52 and Theorem 201. □

6.6.2 IN TOPOLOGICAL VECTOR SPACES

In [6], Cascales and Orihuela introduced the class $\mathfrak{G}$ of all locally convex topological vector spaces E such that the dual E^* has an ω^ω-ordered cover $\{A_\alpha : \alpha \in \omega^\omega\}$ (that is, $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ in ω^ω) such that the countable subsets of each A_α are equicontinuous. Since the equicontinuous families in E^* are related via polars to the neighborhoods of 0 in E, which are, in turn, related to the neighborhoods of the diagonal in E^2, then Cascales and Orihuela were able to use their Theorem 3 to prove two key results about the class $\mathfrak{G}$ in [6]: if E is in $\mathfrak{G}$, then (i) every compact subspace of E is metrizable and (ii) every weakly compact subspace of E is Talagrand compact. Here, a compact space X is called *Talagrand compact* if it is compact and $C_p(X)$ is K-analytic, and a space Y is called *K-analytic* if there is a space Z with compact subspaces $K_{n,m}$ for each $n, m < \omega$ such that Y is a continuous image of $\bigcap_n \bigcup_m K_{n,m}$.

We show that both of these results hold more generally for any locally convex space (lcs) satisfying the conclusion of Theorem 203 below. In fact (i) holds for a much wider class of lcs, and every weakly compact subspace of a lcs satisfying the conclusion of Theorem 203 is Gul'ko compact. Here, a compact space X is called *Gul'ko compact* if $C_p(X)$ is Lindelöf Σ. Let us say that a locally convex space E has *(neighborhood) type* $\langle Q, R\rangle$ if there is a directed set P of type $\langle Q, R\rangle$ such that $P \geq_T \mathcal{N}_0^E$.

If $E = (E, \tau)$ is a topological vector space, and if F is a subset of E^*, the set of all continuous linear functionals on (E, τ), then $\sigma(E, F)$ denotes the coarsest topology on E with respect to which the members of F remain continuous. Thus, $\sigma(E, E^*)$ is the weak topology on E, and $\sigma(E^*, E)$ is the weak-$*$ topology on the dual E^*. Note that these topologies depend on the original choice of topology τ for E.

Theorem 203. *Let $E = (E, \tau)$ be a lcs in the class $\mathfrak{G}$. Then E has a lcs topology τ^s such that $\sigma(E, E^*) \subseteq \tau^s \subseteq \tau$ and (E, τ^s) is a lcs with neighborhood type $\langle \omega^\omega$, countably directed$\rangle$.*

Proof. Let $\{A_\alpha : \alpha \in \omega^\omega\}$ be an ω^ω-ordered cover of E^* witnessing that $E = (E, \tau)$ is in the class $\mathfrak{G}$. Let τ^s be the topology induced by convergence on countable subsets of the

A_α's. Then (E,τ^s) is also a Hausdorff lcs, τ^s contains the weak topology $\sigma(E,E^*)$, and since countable subsets of each A_α are equicontinuous, we have $\tau^s \subseteq \tau$. Clearly $P = \bigcup\{[A_\alpha]^{\leq\omega} : \alpha \in \omega^\omega\}$ is of type $\langle\omega^\omega, \text{countably directed}\rangle$ and $P \geq_T \mathcal{N}_0^{(E,\tau^s)}$. $\square$

Since every directed set in the class $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$ has calibre (ω_1,ω) (see Lemma 33 and Lemma 43), we can deduce the following from Theorem 64.

Proposition 204. *If a lcs E has neighborhood type $\langle 2^o$+CSB, countably directed$\rangle$, then every compact subset of E is metrizable.*

Now we give our extension of Cascales and Orihuela's result that every weakly compact subset of a class $\mathfrak{G}$ space is Talagrand compact.

Proposition 205. *If a lcs E has a weaker topology τ^s containing $\sigma(E,E^*)$ such that (E,τ^s) has neighborhood type $\langle 2^o$+CSB, countably directed$\rangle$ (respectively, $\langle\omega^\omega$, countably directed$\rangle$), then every weakly compact subset of E is Gul'ko (respectively, Talagrand) compact.*

Proof. Suppose there is a type $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$ directed set P and a weaker locally convex topology τ^s containing $\sigma(E,E^*)$ such that $P \geq_T \mathcal{N}_0^{(E,\tau^s)}$. Since each member of E^* is continuous on (E,weak), and since the weak-$*$ topology is equivalent to the pointwise convergence topology, then $F = (E^*,\text{weak-}*)$ is a subspace of $C_p(E,\text{weak})$. Let K be a weakly compact subset of E, and let $\pi : C_p(E,\text{weak}) \to C_p(K)$ be the restriction map. Then $S = \pi(F)$ separates points in K.

For each τ^s-neighborhood V of 0 in E (which is also a neighborhood of 0 in the original topology on E), let $\phi(V)$ be the polar $\{l \in E^* : |l(x)| \leq 1 \ \forall x \in V\}$ of V, which is compact in F by the Banach-Alaoglu-Bourbaki theorem. Then ϕ is an order-preserving map from $\mathcal{N}_0^{(E,\tau^s)}$ to $\mathcal{K}(F)$ whose image covers F, so ϕ witnesses that $\mathcal{N}_0^{(E,\tau^s)} \geq_T (F,\mathcal{K}(F))$. The map π above also witnesses that $(F,\mathcal{K}(F)) \geq_T (S,\mathcal{K}(S))$, so by transitivity, we therefore have $P \geq_T (S,\mathcal{K}(S))$.

Baturov's theorem (see III.6.1 in [1]) shows that for any subspace Y of $C_p(K)$, Y is Lindelöf if and only if Y has countable extent. In particular, the countably compact subsets of $C_p(K)$ are Lindelöf and therefore compact. Hence, $(S,\mathcal{K}(S))$ is countably determined (see Lemma 172 and the comment that follows it) and has type $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$, so we can apply Theorem 174 to see that S is Lindelöf Σ. Thus K is Gul'ko compact since it is compact and S is a Lindelöf Σ subset of $C_p(K)$ separating the points in K (see IV.2.10 in [1]).

Now suppose there is a directed set P with type $\langle \omega^\omega, \text{countably directed}\rangle$ and a weaker locally convex topology τ^s containing $\sigma(E, E^*)$ such that $P \geq_T \mathcal{N}_0^{(E,\tau^s)}$. Let K be a weakly compact subset of E. Repeating the argument for the $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$ case, we find a Lindelöf Σ subset S of $C_p(K)$ which separates the points of K and has $P \geq_T (S, \mathcal{K}(S))$. Since $(S, \mathcal{K}(S))$ has type $\langle \omega^\omega, \text{countably directed}\rangle$ (by Lemma 34) and $\mathcal{K}(S)$ is countably determined (see Lemma 172 and the comment after it), then $\omega^\omega \geq_T (S, \mathcal{K}(S))$ by Lemma 36.

It is well-known (see [44], for example) that a Lindelöf space Y is K-analytic if and only if it has an ω^ω-ordered compact cover, i.e. $\omega^\omega \geq_T (Y, \mathcal{K}(Y))$. Thus, S is K-analytic, so $C_p(K)$ is K-analytic by IV.2.14 in [1], which means K is Talagrand compact. $\square$

Recall $C_p(X)$ is the topological vector space of all continuous real-valued functions on a space X, endowed with the topology of pointwise convergence. According to Lemma 62, $\mathcal{N}_{\mathbf{0}}^{C_p(X)} =_T [X]^{<\omega} \times \omega$. In particular, up to Tukey equivalence, $\mathcal{N}_{\mathbf{0}}^{C_p(X)}$ is determined entirely by the cardinality of the space X, and is not at all dependent on the topology. In fact we can prove a much broader result.

Let (L, τ) be a topological vector space. Then τ is said to be a *weak topology* for L if it is the weakest topology induced by L^*, the set of all continuous linear functionals on (L, τ). In other words, τ is a weak topology if it coincides with the weak topology $\sigma(L, L^*)$ generated by τ. It is known, see [26] for example, that (i) τ is a weak topology if and only if (ii) (L, τ) embeds as a dense linear subspace of $\mathbb{R}^H$, where H is a Hamel basis for L^*, and if and only if (iii) (L, τ) embeds as a linear subspace of some power of $\mathbb{R}$.

Of course, the weak topology on a Banach space is a weak topology. Also, if $\tau = \sigma(E^*, E)$ is the weak-$*$ topology on the dual E^* of a Banach space E, then (E^*, τ) is a linear subspace of $\mathbb{R}^E$, so the weak-$*$ topology is a weak topology by (iii). In both cases, if the weak topology is metrizable, then the Banach space (and its dual) is finite dimensional.

Similarly, note by (iii) that the topology τ_p of pointwise convergence on $C(X)$ is weak, and any Hamel basis H for $C(X)^*$ has cardinality $|X|$. Observe also that $L_p(X)$, the weak dual of $C_p(X)$ (in other words, $C_p(X)^*$ with the weak topology induced by all continuous linear functionals on $C_p(X)$) has a weak topology, and its dual is $C(X)$. If $C(X)$ has a countable Hamel basis $H = \{f_n : n \in \mathbb{N}\}$, then $C_p(X) = \bigcup_n \{\sum_{i=1}^n \lambda_i f_i : \lambda_i \in [-n, n]\}$ is σ-compact, and so by Velichko's theorem (see I.2.1 in [1]), X is finite.

Lemma 206. *Let A be a subset of a space X such that A has a neighborhood base of closed sets. Let D be a dense subset of X containing A. Then $\mathcal{N}_A^X =_T \mathcal{N}_A^D$.*

Proof. Clearly the map $U \mapsto U \cap D$ shows $\mathcal{N}_A^X \geq_T \mathcal{N}_A^D$. For the converse, consider the map $\mathcal{N}_A^D \to \mathcal{N}_A^X$ given by $U \mapsto (\overline{U})^\circ$ (where the interior and closure are taken in X). This map is well-defined, order-preserving, and has cofinal image because D is dense and A has a base of closed neighborhoods in X. □

Theorem 207. *Let (L, τ) be a topological vector space, and suppose τ is a weak topology. Then $\mathcal{N}_{\mathbf{0}}^{(L,\tau)} =_T [\kappa]^{<\omega}$, where $\kappa = |H| \cdot \aleph_0$ and H is a Hamel basis for L^*.*

Proof. Fix H a Hamel basis for L^*. Then (L, τ) embeds densely in $\mathbb{R}^H$. By Lemma 206, $\mathcal{N}_{\mathbf{0}}^L =_T \mathcal{N}_{\mathbf{0}}^{\mathbb{R}^H}$. By Lemma 62, we see that $\mathcal{N}_{\mathbf{0}}^{\mathbb{R}^H} =_T [H]^{<\omega} \times \omega =_T [\kappa]^{<\omega}$ (give H the discrete topology and use $\mathcal{S} = \mathcal{K}(H) = [H]^{<\omega}$ so that $C_{\mathcal{S}}(H) = \mathbb{R}^H$). □

Theorem 208. *Let (L, τ) be a topological vector space, and suppose τ is a weak topology. Then the following are equivalent:*

(i) $\omega^\omega \geq_T \mathcal{N}_0^L$,

(ii) $\mathcal{K}(M) \geq_T \mathcal{N}_{\mathbf{0}}^L$ *for some separable metric space M,*

(iii) $\mathcal{N}_{\mathbf{0}}^L$ *has calibre (ω_1, ω),*

(iv) L^* has countable algebraic dimension, and

(v) L is separable metrizable.

Proof. Certainly (i) implies (ii) since $\omega^\omega =_T \mathcal{K}(\omega^\omega)$. Also, since $\mathcal{K}(M)$ has calibre (ω_1, ω) for any separable metric space M, then (ii) implies (iii). If (iv) holds, then as τ is weak, (L, τ) embeds in a countable power of $\mathbb{R}$, so it is separable metrizable, which is (v). And if (v) holds, then L is first countable, and $\omega^\omega \geq_T \omega \geq_T \mathcal{N}_{\mathbf{0}}^L$, so (v) implies (i).

It remains to show (iii) implies (iv). Let H be a Hamel basis for L^*. Then $\mathcal{N}_0^L =_T [H \times \omega]^{<\omega}$, and $[H \times \omega]^{<\omega}$ has calibre (ω_1, ω) if and only if H is countable. □

Corollary 209. *For any space X, the neighborhood filter of $\mathbf{0}$ in $C_p(X)$ has calibre (ω_1, ω) if and only if X is countable, while the neighborhood filter of $\mathbf{0}$ in $L_p(X)$ has calibre (ω_1, ω) if and only if X is finite.*

Corollary 210. *Let B be a Banach space. Then $\mathbf{0}$ in B with the weak topology has neighborhood filter with calibre (ω_1, ω) if and only if B is finite dimensional. And the $\mathbf{0}$ in B^* with the weak-$*$ topology has neighborhood filter with calibre (ω_1, ω) if and only if B^* (equivalently, B) is finite dimensional.*

6.7 EXAMPLES

Example 17. *Let τ be any topology on $\mathbb{R}$ refining the usual topology. Then the pre-ideal $\mathcal{K}(\mathbb{R}, \tau)$ of all τ-compact subsets of $\mathbb{R}$ is (ck$^+$)$'$. However, for many choices of τ, such as the discrete topology and the Michael-line topology, $(\mathbb{R}, \tau)$ is not Lindelöf cofinally Σ (equivalently, $\mathcal{K}(\mathbb{R}, \tau)$ is not (cc)$'$).*

Proof. Let $\mathcal{N}$ be any countable network for the reals with the usual topology, which is closed under finite unions. Since any τ-compact subset K of $\mathbb{R}$ is also compact in the usual topology, then K is the intersection of all elements of $\mathcal{N}$ containing it. Hence if we let $L_K = K$, then it is clear $\mathcal{N}$ satisfies (ck$^+$)$'$ for $\mathcal{K}(\mathbb{R}, \tau)$. Now note that the Michael line and the discrete topology on $\mathbb{R}$ are not even Lindelöf. □

Example 18. *Let $X = \mathbb{R}$ be the reals with the usual topology. Let $\mathcal{S} = [X]^{<\omega}$, the ideal of all finite subsets of X. Then $\mathcal{S}$ is (cc$^+$)$'$ but not (cp)$'$.*

Proof. Setting $\mathcal{A} = \mathcal{S}$ and $\mathcal{N}$ to be a countable base for $\mathbb{R}$ closed under finite unions and intersections. Then $\mathcal{N}$ is a countable network for $\mathcal{S}$ modulo the cofinal family $\mathcal{A}$. So (cc$^+$)$'$ holds by Theorem 169.

Now suppose $\mathcal{N}$ is a countable family of subsets of X, which, without loss of generality, we assume is closed under finite unions and intersections. We show $\mathcal{N}$ is not (cp)$'$ for $\mathcal{S}$. And so $\mathcal{S}$ is not (cp)$'$.

Take any S in $\mathcal{S}$ and from $\mathcal{N}_S = \{N \in \mathcal{N} : S \subseteq N\}$ select a decreasing sequence $(N_n^S)_n$ such that every member of $\mathcal{N}_S$ is contained in some N_n^S. Since $\mathcal{N}$ is countable while $\mathcal{S}$ has no countable cofinal family, there must be some S in $\mathcal{S}$ such that $(N_n^S)_n$ does not stabilize (no m such that for all $n \geq m$ we have $N_n^S = N_m^S$). Then passing to a subsequence we can suppose there is an a_n in N_n^S but not N_{n+1}^S. Let $A = \{a_n : n < \omega\}$. By construction, we know that for every N in $\mathcal{N}$ containing S, all but finitely many points of A are contained in N. But A is infinite, and so not contained in any element of $\mathcal{S} = [X]^{<\omega}$. □

Example 19. *$C_p(X)$ is (cn) but not (cs) for each of the following spaces X: the closed unit interval I, the Double arrow space, and the Alexandrov duplicate of I.*

Proof. In each case X is uncountable, so $C_p(X)$ is not (cs) by Theorem 180. But $C_p(X)$ can be checked to be (cn) by applying Theorem 181 assisted by Theorem 182. □

The following example shows that (ck) cannot be weakened to (cn) in Corollary 187. Here $A(\kappa)$ denotes the one-point compactification of the discrete space of size κ.

Example 20. *For every uncountable $\kappa \leq \mathfrak{c}$, the locally convex space $C_p(A(\kappa))$ is Fréchet-Urysohn and (cn), but not metrizable.*

Proof. Certainly $C_p(A(\kappa))$ is not metrizable when κ is uncountable, but it is Fréchet-Urysohn since $A(\kappa)$ is a compact scattered space. Since $\kappa \leq \mathfrak{c}$, then $A(\kappa)$ is a continuous image of the Alexandrov duplicate of the unit interval, which in turn is the pre-image of $I = [0, 1]$ under a perfect map with finite fibres. So Theorem 182 shows that $A(\kappa)$ is in $L\Sigma(<\omega)$, and thus Theorem 181 shows that $C_p(A(\kappa))$ is (cn). □

Example 21. *For any cardinal κ with uncountable cofinality, $C_k(\kappa)$ is (cp_ω) and so (cs), but not (cn).*

Proof. By Theorem 179, we know $C_k(\kappa)$ is not (cn) since κ is not Lindelöf when it has uncountable cofinality. But $\mathcal{K}(\kappa) =_T cof(\kappa)$, which is countably directed since κ has uncountable cofinality, and hence $\mathcal{K}(\kappa)$ is of type $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$, as is $\mathcal{K}(\kappa \times \omega) =_T \mathcal{K}(\kappa) \times \omega$. Hence $\mathcal{N}_{\mathbf{0}}^{C_k(\kappa)}$ is $\langle 2^o\text{+CSB}, \text{countably directed}\rangle$ (Lemma 178). So $C_k(\kappa)$ is (cp_ω) (Theorem 167 (1)). □

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