RATIONAL STRUCTURES AND FRACTIONAL DIFFERENTIAL REFINEMENTS

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In the following thesis, we explore the notion of rational Fivebrane structures. This is done through a combination of obstruction theory and rational homotopy theory. We show that these structures can be classified to some degree by the underlying Spin bundle. From there we turn our focus to the differential setting. Using this relation to the Spin bundle, we apply the classical machinery of Cheeger and Simons to understand differential rational Fivebrane classes. Finally we use these classes to obtain information for differential trivializations in the integral case. In doing this we introduce the exact braid diagram.

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PREFACE

This work would not have been possible if not for the help and support of a great many people. I would first like to thank my thesis adviser, Hisham Sati, without whose direction I could not have made it this far. His guidance and support throughout my academic career has been immeasurably important to me.

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I dedicate this dissertation in loving memory of my uncle, Brian Wheeler.

1.0 INTRODUCTION

The overall goal of this thesis is to study the topological and differential geometric properties of higher structures related to the orthogonal group, $\mathrm{O}(n)$. In particular, we focus on String, Fivebrane, and Ninebrane structures. These groups arise from the Whitehead tower over $\mathrm{SO}(n)$ and the classifying spaces can be obtained similarly from the Whitehead tower over $B\mathrm{SO}(n)$. As we will be looking at these groups stably, we drop the n in general. Fivebrane and Ninebrane structures were introduced in [30] and [29], respectively. In [30], the name Fivebrane arose from the observation that these structures corresponded to the anomaly cancellation involved in the dual Green-Schwarz mechanism related to a geometric object in string theory known as the NS 5-brane. The differential properties of Fivebrane structures were studied in [31] using the theory of stacks. Part of our work focuses on putting these structures into the classical context of Cheeger-Simons differential characters [5].

An unfortunate side effect of the killing of homotopy groups in the Whitehead tower is that cohomologically these spaces become more complicated. To overcome this complication, we investigate how these structures behave when torsion is removed by placing ourselves in the setting of rational homotopy. We find that by doing this, we obtain some very pleasing results. For one, the Whitehead tower in rational homotopy theory gives us a sequence of rationalizations for our original tower. We discover that when classifying Fivebrane and Ninebrane structures rationally, in essence all of the necessary topological data is stored within the underlying principal Spin bundle. This is attractive as it allows us to use more familiar and classical tools to describe these new structures. In the case of Fivebrane structures, we define rational Fivebrane classes which live on the total space of a String bundle as well as what we call *rational Spin-Fivebrane classes* which live on the underlying Spin bundle. Using that the homomorphism $\rho : \mathrm{String} \to \mathrm{Spin}$ is an isomorphism on degree 7 rational cohomology and that ρ induces a bundle map μ between the String bundle, $\pi_{\text{String}} : P \to M$, and its underlying Spin bundle, $\pi_{\text{Spin}} : Q \to M$, we are able to prove the following theorem.

Theorem 1.0.1. *1. For every rational Spin-Fivebrane class $F \in H^7(Q; \mathbb{Q})$, $\mu^* F$ is a rational Fivebrane class;*

2. *Any rational Fivebrane structure $F \in H^7(P; \mathbb{Q})$ can be described by a class in $H^7(Q; \mathbb{Q})$;*

3. *Two classes $F, F' \in H^7(Q; \mathbb{Q})$ will give the same Fivebrane structure if $F - F' = S \cdot \pi^*_{\text{Spin}} \phi$ where $S \in H^3(Q; \mathbb{Q})$ is the String structure class and $\phi \in H^4(M; \mathbb{Q})$.*

A similar theorem then holds in the case of Ninebrane structures. From here we ask the question of how this theory looks in the context of differential cohomology. Using differential characters as a model for differential cohomology, we describe differential refinements for rational cohomology. Then by using the model for the String group given in [25], we find that the total space can be modeled similarly and in this context it makes sense to study the differential cohomology of a String bundle. We are able to apply this notion of rational differential refinements to rational Fivebrane structures to get *differential rational Fivebrane classes*. From this definition we prove the following,

Proposition 1.0.2. *Let $\pi_{\text{String}} : P \to X$ be a principal String-bundle where $\pi_{\text{Spin}} : (Q, \theta) \to X$ is the underlying principal Spin-bundle equipped with a connection θ. It follows that*

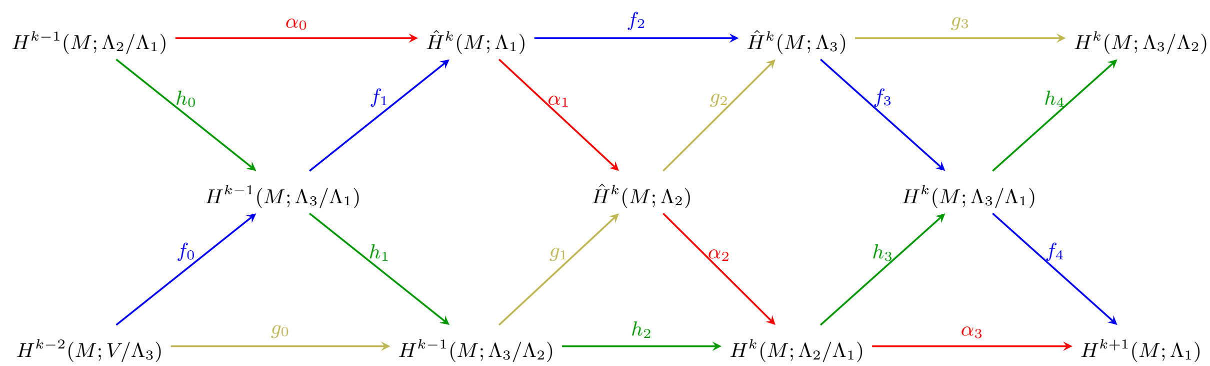
1. *Given any rational Fivebrane class $F \in H^7(P; \mathbb{Q})$, there exists a differential rational Fivebrane class $\widehat{F} \in \widehat{H}^7(P; \mathbb{Q})$ with $I(\widehat{F}) = F$, where I picks out the underlying characteristic class.*

2. *Any differential character $\hat{h} \in \widehat{H}^7(P; \mathbb{Q})$, which has curvature $CS_\theta(\frac{1}{6} p_2) - \pi^*_{\text{String}} \rho$ for some differential form $\rho \in \Omega^7(X)$ and has $I(\hat{h})$ a rational Fivebrane class, is a differential rational Fivebrane class.*

Furthermore, we find that the set of all differential rational Fivebrane classes is a torsor for $\widehat{H}^7(X; \mathbb{Q})$ where X is the base space. From here, we investigate whether the notion of differential trivialization given in **??** is appropriate for studying differential Fivebrane structures and to what degree. To pursue this, we look at a different notion for differential trivialization, which was introduced by Hopkins and Singer in [14] and studied further by

Redden in [27]. We show that, under certain conditions, the two definitions agree, specifically in the case of differential String classes and differential rational Fivebrane classes.

Having defined differential refinements for rational classes, we would like to know if and when these rational differential characters can be used to describe *integral* differential trivializations. To this end, we describe the notion of a *fractional differential refinement* and present some results in that direction. More importantly, we discuss an interesting line of research that arose when studying these objects. We introduce what we call an *exact braid diagram*. It is a combination of four long exact sequences into the following braid diagram



The outline for how the thesis is structured is as follows. In Chapter 2 we give the necessary background required for this thesis including relevant tools from obstruction theory, topological properties of the connected covers of $\mathrm{O}(n)$, and a construction of the Whitehead tower. Chapter 3 begins with a brief introduction to rational homotopy theory and then presents our work on rational structures.

In Chapter 4, we review the concept of differential cohomology and provide the relevant theory required in Chapters 5 and 6. We also discuss two different models for differential cohomology: that of the differential characters of Cheeger-Simons [5], and the Hopkins-Singer [14] model of differential cocyles. We also introduce the concept of differential trivializations as defined by Becker [2].

Then in Chapter 5, we introduce rational differential Fivebrane structures and a generalized version for bundles with highly connected fibers, and we provide some interesting results concerning them. We also give conditions for when Becker's definition for differential trivializations agrees with one given by Hopkins and Singer.

In Chapter 6, we discuss fractional differential refinements and we introduce our exact braid diagram. The majority of original material is found in Chapters 3, 5, and 6.

2.0 BACKGROUND

We will begin by recalling some material from [3],[7]. All maps considered here will be at least continuous, all spaces will be connected unless otherwise noted, and by connected we mean that any two points are connected by a path (i.e. path-connected). Let X and Y be topological spaces, and let $f, g : X \to Y$ be two maps. We say that f and g are **homotopic** if there exists a **homotopy** $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ and we write $f \simeq g$. X and Y are said to be **homotopy equivalent** if there exists a homotopy $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

Definition 2.0.1. The $\mathbf{n^{th}}$**-homotopy group of X**, denoted $\pi_n(X, x_0)$, is the group of homotopy classes of maps $\{f : I^n \to X \mid f(\partial I^n) = x_0\}$.

Given an abelian group G and a nonnegative integer n, there are a class of spaces called **Eilenberg-Maclane spaces** which we denote as $K(G, n)$ that satisfy the following property

$$\pi_i(K(G,n)) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n\,. \end{cases}$$

Definition 2.0.2. The **loop space** $\Omega(X, x_0)$ is defined as the space $\{\gamma : S^1 \to X \mid \gamma(s_0) = x_0\}$ equipped with the compact-open topology where s_0 is a choice of fixed point of S^1.

If we assume that X is connected, then this definition does not depend on the choice of basepoint of X and we will denote it simply as $\Omega(X)$.

Definition 2.0.3. A map $p : E \to B$ is said to have the **homotopy lifting property** with respect to a space X if, given a homotopy $G : X \times I \to B$ and a map $\tilde{g}_0 : X \to E$ satisfying $p \circ \tilde{g}_0 = g_0$, there exists a homotopy $\tilde{G} : X \times I \to E$ which fits into the following commuting diagram,

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{g_0}} & E \\ {\scriptstyle \iota_0}\downarrow & \overset{\tilde{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{G} & B\,. \end{array}$$

We define a **fibration** to be any map $p : E \to B$ having the homotopy lifting property with respect to all spaces X. (In particular, this is what is known as a Hurewicz fibration).

Theorem 2.0.4. *Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geqslant 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. The map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geqslant 1$. Hence if B is connected, then there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F, x_0) \xrightarrow{p^*} \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0.$$

2.0.1 Homotopy Fiber

Let $f : X \to Y$ be a continuous map. Choose a basepoint $y_0 \in Y$ and define the path space PY of Y by

$$PY = \{\gamma \in Y^I \mid \gamma(0) = y_0\},$$

where Y^I is the space of mappings $I \to Y$ with the compact-open topology. The space PY is contractible and there is a canonical map $ev_1 : PY \to Y$ which sends $\gamma \mapsto \gamma(1)$. The homotopy fiber of f is defined to be the pullback

$$\begin{array}{ccc} F(f) & \xrightarrow{p} & PY \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle ev_1} \\ X & \xrightarrow{f} & Y \end{array}$$

Explicitly, we have

$$F(f) = \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x_0) \text{ and } \gamma(1) = f(x)\}.$$

One thing to note here is that when f is just the inclusion of a point $f : * \hookrightarrow Y$, then the homotopy fiber of f is just the loop space $\Omega(Y)$.

We also have a more general construction of the path space. Consider the map $f : X \to Y$. Let $P(f) = \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x)\}$. Thus the path space PY is just the space $P(* \hookrightarrow Y)$.

Proposition 2.0.5. *The map $p : P(f) \to Y$, where $p(x, \gamma) = \gamma(1)$, is a fibration.*

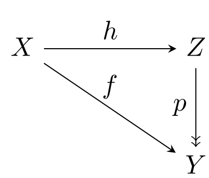
Proof. Let $g_t : A \to Y$ and $\tilde{g}_0 : X \to P(f)$ such that $p\tilde{g}_0 = g_0$ be given. In order to show that $p : P(f) \to Y$ is a fibration, we must show that $\tilde{g}_0$ extends to a homotopy $\tilde{g}_t : A \to P(f)$ lifting g_t. Now by the definition of $P(f)$, we can define a map $h : A \to X$ and a map $\gamma_a : A \times I \to Y$ such that $\tilde{g}_0(a) = (h(a), \gamma_a)$. As $\tilde{g}_0(a) \in P(f)$, then $\gamma_a(0) = f(h(a))$. Now let $\eta_a = \gamma_a \cdot g_{[0,t]}(a)$, where $g_{[0,t]}(a)$ denotes the path $\alpha(s) = g_{st}(a)$ which starts at $g_0(a)$ and ends at $g_t(a)$. Then η_a is well defined since $g_0(a) = p\tilde{g}_0(a) = \gamma_a(1)$, $\eta_a(0) = \gamma_a(0) = f(h(a))$, and $\eta_a(1) = g_t(x)$. Finally, define $\tilde{g}_t : A \to P(f)$ by $\tilde{g}_t(a) = (h(a), \eta_a)$. Then $p\tilde{g}_t = \eta_a(1) = g_t(a)$, and thus $\tilde{g}_t$ is a lift for g_t. □

The fact that this construction gives us a fibration is very useful. One immediate consequence is that by applying Theorem 2.0.4, we get:

Corollary 2.0.6. *Given a connected space X, then $\pi_n(\Omega(X)) \cong \pi_{n+1}(X)$.*

Another important result is that the space $P(f)$ is homotopy equivalent to X and that, because of this, any map $f : X \to Y$ factors through a fibration followed by a homotopy equivalence.

Proposition 2.0.7. *For any map $f : X \to Y$, there is a space Z, a homotopy equivalence $h : X \to Z$, and a fibration $p : Z \to Y$ such that $f = ph$.*



Proof. Let $Z = P(f)$. Then by Proposition 1, the map $p : P(f) \to Y$ is a fibration. To show that X and $P(f)$ are homotopy equivalent, notice that we may include X into $P(f)$ as $h : x \mapsto (x, c_{f(x)})$, where $c_{f(x)}$ is the constant path in Y at $f(x)$. Now by continuously truncating the paths in Y, we see that $P(f)$ deformation retracts onto X. Hence h is a homotopy equivalence and $f = ph$. □

This process of replacing maps with this factorization is called *homotopy fiber replacement* and is another important tool that we will use in constructing the Whitehead tower.

2.0.2 Fiber Bundles

Here we follow along the lines of [23],[39].

Definition 2.0.8. A **fiber bundle** is given by a map $p : E \to B$ such that for every $b \in B$ there is an open neighborhood $U_b \subset B$ with a diffeomorphism $\phi : p^{-1}(U_b) \to U_b \times \pi^{-1}(b)$. The space $\pi^{-1}(b)$ is called the fiber over b which we denote as F_b

One property of a fiber bundle $p : E \to B$ is that for any two elements $b, b' \in B$, the fibers F_b and $F_{b'}$ are diffeomorphic and in general we will denote the fiber of B as just F. A fiber bundle $p : E \to B$ can be described locally using a **collection of local trivializations**. This is a collection $\{U_i, \phi_i\}$ where $\{U_i\}$ is a covering of B and ϕ_i are diffeomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times F$. In this sense a fiber bundle is locally just a cartesian product. Given a covering of local trivializations $\{U_i, \phi_i\}$, we define the **transition functions** $t_{ij} = \phi_i \circ \phi_j^{-1} : U_{ij} \times F \to U_{ij} \times F$ where we are letting U_{ij} denote the intersection $U_i \cap U_j$. We can equivalently think of the transition functions as maps $t_{ij} : U_{ij} \to Aut(F)$.

Having defined what a fiber bundle is, it is important to have an idea of what a suitable morphism between two fiber bundles should be. This leads us to the following definition.

Definition 2.0.9. A **morphism between fiber bundles** $p : E \to B$ and $p' : E' \to B'$ is a pair of maps $F : E \to E'$ and $f : B \to B'$ such that $p' \circ F = f \circ p$. In other words F maps the fiber over $b \in B$ to the fiber over $f(b) \in B'$.

We say that a morphism of bundles is an isomorphism if the map F induces an isomorphism on each fiber.

Definition 2.0.10. Given a fiber bundle $p : E \to B$. If the fibers are homeomorphic to vector spaces and if the vector space operations vary continuously over B, then $p : E \to B$ is called a **vector bundle**.

In other words a vector bundle is a continuous family of vector spaces parametrized by the base B.

2.0.3 Principal G-Bundles

Let G be a topological group. A certain class of fiber bundles that we will be interested in are those where each fiber can be considered a torsor for G.

Definition 2.0.11. A **principal G-bundle** is given by a surjective map $\pi : P \to M$ along with a free and transitive right action of G on P:

$$r : P \times G \to P$$

such that

1. for $p \in P, g \in G$, then $\pi(pg) = \pi(p)$;

2. for any $x \in M$ there is an open neighborhood $U \ni x$ in M and a G-diffeomorphism $\phi : \pi^{-1}(U) \cong U \times G$ where G acts on $U \times G$ via $((u, h), g) \mapsto (u, hg)$.

An important quality of principal G-bundles is that they can be classified in a nice homotopic way. The total space of principal G-bundle is a free G-space, i.e. a topological space with a continuous free right action of G. There is a universal G-space in that every free G-space P admits a G-equivariant map $F : P \to EG$ which is unique up to G-homotopy. By quotienting out by the action of G, this map descends to a map $f : P/G \to EG/G$. The space EG/G is denoted by BG. The quotient map $\pi_{EG} : EG \to BG$ is a principal G-bundle, and we call this bundle the **universal G-bundle**. The space BG is called the the classifying space for G. The importance of this is that it relates principal G-bundles over a space X to maps between X and BG.

Theorem 2.0.12. *Given a topological group G and a space X, then there is a 1-1 correspondence between the set of homotopy classes of maps in $Maps(X, BG)$ and isomorphism* *classes of principal G-bundles.*

$$[M, BG] \longleftrightarrow \{G\text{-}principal\ bundles\ over\ M\}/\sim$$
$$[f] \mapsto f^*EG \to M$$

Now for an arbitrary vector bundle, one can construct a principal G-bundle out of this bundle where G is the automorphism group of F. For the more general setting, suppose that F is a vector space and that $\rho : G \to \mathrm{Aut}(F)$ is a representation of G. Then we define a G-structure for a vector bundle as follows.

Definition 2.0.13. A vector bundle $F \to E \to B$ is said to have a **G-structure** if there is a homomorphism $\rho : G \to \mathrm{Aut}(F)$ and if there is a trivialization such that the transition functions define maps $t_{ij} : U_{ij} \to Im(\rho)$ for every i, j.

Given a vector bundle $F \to E \to B$ with a set of local trivializations $\{U_i, \phi_i\}$ giving the bundle a G-structure, we can construct a principal G-bundle $P \to B$ with

$$P = \left(\bigcup_i U_i \times G \right) \Big/ \sim$$

where $(x, g) \sim (y, h)$ means $x = y \in U_{ij}$ and $g = t_{ij}(h)$. This is called the **associated principal G-bundle** to the vector bundle $F \to E \to B$. In fact, given a cover of B and functions $t_{ij} : U_{ij} \to G$ satisfying

- $t_{ii} = id_G$, and
- $t_{ij} \cdot t_{jk} = t_{ik}$ for every $x \in U_{ijk}$,

then one can construct a principal G-bundle over B in the same way as with the associated principal G-bundle.

Lemma 2.0.14. *Given a homomorphism between groups $\phi : H \to G$ and a principal H-bundle $p : E \to B$, then there is principal G-bundle $p' : E' \to B$ and a bundle morphism $\Phi : E \to E'$.*

Proof. Given a principal H-bundle and a set of local trivializations $\{U_i, \phi_i\}$, we can define transition functions $t_{ij} : U_{ij} \to H$ as $t_{ij} = \phi_j \circ \phi_i^{-1}$. These transition functions determine E up to isomorphism, i.e. there is a bundle isomorphism $\Psi : E \cong E'' := (\bigcup_i U_i \times G) / \sim$. Now using the homomorphism $\phi : H \to G$, we can define functions $\tilde{\tau}_{ij} : U_{ij} \to H$ as $\tilde{\tau}_{ij} = \phi \circ t_{ij}$. To see that these are transition functions, we have

- $\tilde{\tau}_{ii} = \phi \circ t_{ii} = \phi(id_G) = id_H$, and
- $\tilde{\tau}_{ij} \cdot_H \tilde{\tau}_{jk} = (\phi \circ t_{ij}) \cdot_H (\phi \circ t_{jk}) = \phi(t_{ij} \cdot_G t_{jk}) = \phi \circ t_{ik} = \tilde{\tau}_{ik}$.

Thus we can form a principal H-bundle $E' = (\bigcup_i U_i \times H)/ \sim$. Since the map $(id_{U_i}, \phi) : U_i \times G \to U_i \times H$ respects the equivalence relations, then it extends to a bundle morphism $\Phi' : E'' \to E'$. By setting $\Phi = \Phi' \circ \Psi$, we obtain a bundle morphism $\Phi : E \to E'$. □

2.1 OBSTRUCTION THEORY

In studying geometric structures on manifolds, a common question which arises is whether the structure group of a vector bundle has a lift to a different group. We will rephrase this statement more precisely later on. First however, it will be convenient to address what we mean by a lift and to give conditions for when a lift exists. The standard reference is [35], but we provide a slightly more modern point of view ([7] is another good reference).

Let $F \to E \xrightarrow{p} B$ be a fibration. As we mentioned earlier, given a map $f : X \to B$, we say that f lifts to E if there is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

Lemma 2.1.1. *A map $f : X \to B$ lifts to E if and only if the fibration $f^*E \to X$ given by the pullback of f admits a section.*

Proof. Suppose that f lifts to a map $\tilde{f} : X \to E$, and let $f^*E \to X$ be the fibration over X given by the pullback along f. Now by definition of the pullback $f^*E = \{(x, e) \in X \times E \mid f(x) = p(e)\}$. We define a section $s : X \to f^*E$ by $s(x) = (x, \tilde{f}(x))$.

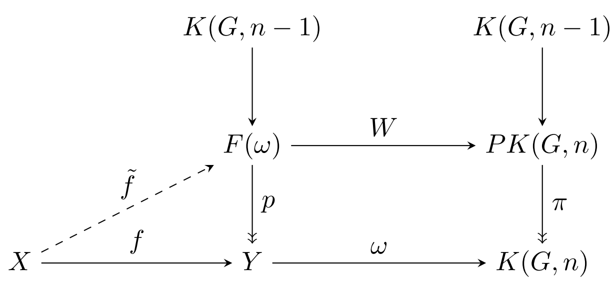
Conversely, suppose that $f^*E \to X$ admits a section $s : X \to f^*E$. Then as f^*E is a pullback, we have a map $F : f^*E \to E$ which covers f, i.e. $p \circ F(x, e) = f(x)$. Thus the map $F \circ s$ defines a lift of f. □

Lemma 2.1.2. *Let $F \to E \to X$ be a fiber bundle where X is a CW-complex and $s_i : X_i \to E$ is a section on the ith-skeleton of X. Then the obstruction to lifting this section from the ith-skeleton to the $(i+1)$st-skeleton is given by a cohomology class in $H^i(X; G)$ where $G \simeq \pi_i(F)$.*

Proof. For a cell complex, the X_{i+1} skeleton is formed by attaching disks D_{i+1} via attaching maps $e : \partial D_{i+1} \to X_i$. Given such a disk D_{i+1}, we can pull back the bundle $E \to X$ along the inclusion $D_{i+1} \hookrightarrow B$. As D_{i+1} is contractible, we have that $e^*E \cong F \times D_{i+1}$. By assumtpion, we have a section $s_i : X_i \to E$ which induces a section on the pullback $s : \partial D_{i+1} \to e^*E \cong F \times D_{i+1}$ and hence gives a map $s \in \pi_i(F)$. This in turns gives us an assignment of an element of $G \cong \pi_i(F)$ for each $i+1$-cell of X. Hence this describes a cochain $\omega(s_i) \in C^{i+1}(X; G)$. In fact, $\omega(s_i)$ is a cocycle. □

Remark 2.1.3. With some slight adjustments, this proof can generalize to all fibrations instead of just fiber bundles.

If we consider a particular cohomology class $\omega \in H^n(Y; G)$, then by Brown's representability, there is a corresponding homotopy class of maps $[\omega] \in [Y, K(G, n)]$. Let ω also denote a choice of representative of this class and consider the homotopy fiber $F(\omega)$.



Lemma 2.1.4. *Let $f : X \to Y$ and $\omega : Y \to K(G, n)$. Then the map f lifts from Y to the homotopy fiber $F(\omega)$ if and only if $f^*[\omega] = 0 \in H^n(X; G)$.*

Proof. First suppose that a lift $\tilde{f} : X \to F(\omega)$ of f exists. Let $W : F(\omega) \to PK(G, n)$ be the map covering ω. Now as $PK(G, n)$ is a contractible space, then there is a homotopy $h_t : PK(G, n) \times I \to PK(G, n)$ such that $h_1 = id$ and $h_0 = *$. Then $h_t \circ W \circ \tilde{f}$ is a homotopy $W \circ \tilde{f} \simeq *$. It follows that $\omega \circ f = \pi \circ W \circ \tilde{f} \simeq *$. Hence $f^*[\omega] = 0$.

Now assume that $f^*[\omega] = 0$. This implies that the composition $\omega \circ f$ is homotopic to the constant map. Let h_t denote this homotopy where $h_0 = *$ and $h_1 = \omega \circ f$. Since the constant map has a lift to $PK(G, n)$ and since π is a fibration, then the homotopy h_t lifts to a homotopy $\tilde{h}_t$ and it follows that $\tilde{h}_1$ is a lift of $\omega \circ f$. The proof follows by applying Lemma 2.1.1. □

2.2 TOPOLOGICAL TRANSGRESSION

There are several different definitions and constructions for transgression maps. As we are following [2] in our construction of differential trivializations, we will use the definition described there, and introduce it presently. In general, the transgression map can be defined for a continuous map $f : E \to X$. It gives a map

$$\tau : \text{Trans}(f) \to \frac{H^{*-1}(E_x; \Lambda)}{\iota_{E_x}^* H^{*-1}(X; \Lambda)}$$

where $\text{Trans}(f) := \{u \in H^*(X; \Lambda) \mid f^*u = 0\}$ is the set of transgressive elements of $H^*(X; \Lambda)$, and E_x is the fiber $f^{-1}(x)$ for some fixed point $x \in \text{im}(f)$. As there are several ways to define the transgression map and not all are necessarily equivalent, we take a moment to fix a working definition for our purposes.

We consider the long exact sequences of the mapping cone cohomology for the maps, $\iota_{E_x} : E_x \to E$ and $\iota_x : x \to X$ and have the following commutative diagram.

$$\begin{array}{ccccccc} & & & & H^*(X, \{x\}; \Lambda) & \xrightarrow{\iota_x^*} & H^*(X; \Lambda) \\ & & & & \downarrow f^* & & \downarrow f^* \\ H^{*-1}(E; \Lambda) & \xrightarrow{\iota_{E_x}^*} & H^{*-1}(E_x; \Lambda) & \xrightarrow{\delta_f} & H^*(E, E_x; \Lambda) & \longrightarrow & H^*(E; \Lambda) \end{array}$$

As $H^i(\{x\}; \Lambda) = 0$ for $i \geqslant 1$, then ι_x^* is an isomorphism and by exactness, $f^*(\iota_x^*)^{-1}u \in \text{Im}(\delta_f)$ for any $u \in \text{Trans}(f)$. Thus we can choose $y \in H^{*-1}(E_x; \Lambda)$ which is unique up to an element $im(\iota_{E_x}^*)$ and we set $\tau(u) = [y]$.

Lemma 2.2.1. *$Trans(f)$ is zero on the set of decomposable elements.*

Proof. Let $f : E \to X$ be a continuous map and let $u \in H^k(X;\Lambda) \cap \mathrm{Trans}(f)$ such that $f^*(u) = 0$. On the level of cochains, the transgression map gives a class $\tau(u) := [\iota_x^*\nu - \pi^*\alpha] \in H^{k-1}(E_x;\Lambda)$ where $\nu \in C^{k-1}(E;\Lambda)$ and $\alpha \in C^{k-1}(\{x\};\Lambda)$ are such that there exists a representative $\mu \in Z^k(X;\Lambda)$ of u with $\delta\alpha = \iota_x^*\mu$ and $\delta\nu = \pi^*\mu$.

Now consider another class $w \in H^l(X;\Lambda)$ and let $\omega \in Z^l(X;\Lambda)$ be a representative. We notice that

$$\delta(\alpha \cup \iota_x^*\omega) = (\delta\alpha) \cup \iota_x^*\omega = \iota_x^*(\mu \cup \omega) \text{ and } \delta(\nu \cup \pi^*\omega) = (\delta\nu) \cup \pi^*\omega = \pi^*(\mu \cup \omega).$$

Then it follows that the transgression of $u \cup w$ is given by

$$\tau(u \cup w) = [\iota_x^*(\nu \cup \pi^*\omega) - \pi^*(\alpha \cup \iota_x^*\omega)] = [(\iota_x^*\nu - \pi^*\alpha) \cup \pi^*\iota_x^*\omega] = \tau(u) \cup [\delta\iota_x^*\zeta] = 0.$$

In other words, the transgression, τ, is equal to zero on decomposable elements. □

In the case where our map is the projection for a universal G-bundle, the transgression map is well defined over all of $H^*(BG;\Lambda)$, as $H^*(EG;\Lambda) = 0$.

Corollary 2.2.2. *Let $\pi : EG \to BG$ be a universal G-bundle. Then the transgression map τ gives a morphism of degree -1*

$$\tau : H^*(BG;\Lambda) \to H^{*-1}(G;\Lambda)$$

such that the set of all decomposable elements in $H^*(BG;\Lambda)$ is contained in the kernel of τ.

There is a nice relation between the transgression and the differentials $d_n : E_n^{0,n-1} \to E_n^{n,0}$ of the Serre spectral sequence for a fibration.

Lemma 2.2.3. *The transgression map τ is a left inverse of d_n.*

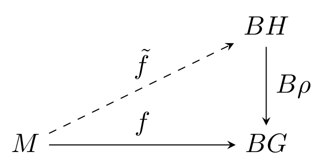
A proof of this can be found in [12]. The benefit of this lemma is that it will allow us to use knowledge about the differentials to compute the values of the transgression map on $H^*(BG;\Lambda)$.

2.3 TOPOLOGICAL TRIVIALIZATIONS

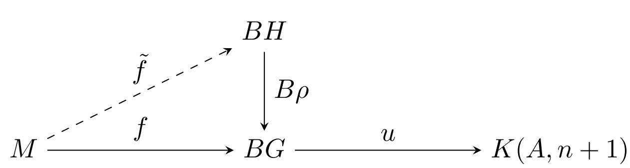
In what follows, we will assume that M is a paracompact manifold and that G is a Lie group with finitely many components. Recall that a principal G-bundle is a fiber bundle where each fiber is a torsor for G. One way of prescribing the information of principal G-bundle is through transition functions. Transition functions assign smooth G-valued functions to the intersections of open sets belonging to some predetermined covering of M. The basic idea is that choosing a trivialization for an open set amounts to a smooth assignment of values in G. Then on intersections we have two different local trivializations. The difference between these trivializations prescribes an element in G for each element in the intersection.

Given a homomorphism $\rho : H \to G$, one asks whether there is a choice of local trivializations which take values only in the image of ρ. If such a local trivialization exists, then $\pi : P \to M$ can be extended so that it sits inside of an H-bundle. We say that $\pi : P \to M$ lifts to a principal H-bundle.

In terms of universal bundles, the assignment of a classifying space to each Lie group is functorial, and thus for each homomorphism $\rho : H \to G$ we have a map $B\rho : BH \to BG$. Then the lifting of a G-bundle to an H-bundle is given by a choice of lift of the classifying map $f : M \to BG$ to a map $\tilde{f} : M \to BH$ along $B\rho : BH \to BG$.



For our purposes, we will be focusing on the higher groups which arise out of the Whitehead tower sitting over BO. We will clarify what these groups are in the following sections. With respect to obstruction theory, this amounts to studying lifts of bundles where the space BH is actually the homotopy fiber of a map $u : BG \to K(A, n+1)$, where A is some abelian group. Our lifting triangle now becomes



and via obstruction theory such a lift $\tilde{f}$ exists if and only if $u(f) \simeq *$. In fact, following Lemma 2.1.4, the map u corresponds to a cohomology class $[u] \in H^{n+1}(BG, A)$ and we let $u(P)$ denote the pullback of this class along the classifying map, $u(P) = f^*[u] \in H^n(M; A)$. Then a lift $\tilde{f}$ exists exists if and only if $u(P) = 0$. This leads us to make the following definition.

Definition 2.3.1. A u**-trivialization** is a lift $\tilde{f} : X \to BH$ of the classifying map f.

We'll denote the set of u-trivializations by $\mathcal{T}_0(u)$. We can place an equivalence relation on $\mathcal{T}_0(u)$ by considering them up to homotopies, h_t, which are lifts of f for each t. We let $T_0(u)$ denote the set of equivalence classes of $\mathcal{T}(u)$. In the situation we are studying, related to each trivialization is a cohomology class on the total space which only depends on the homotopy class of the lift. Further, the restriction of this class along the fiber is identified with the transgression of the obstructing class.

Proposition 2.3.2. *Let $\pi : P \to X$ be a principal G-bundle.*

1. *If $u(P)$=0, then the set of u-trivializations up to homotopy has a free and transitive action of $H^n(X; A)$.*
2. *Every u-trivialization $\tilde{f}$ determines a canonical cohomology class $\tilde{F} \in H^n(P; A)$ which depends only on the homotopy class of $\tilde{f}$.*
3. *For any $x \in X$, we have $\iota_x^*[\tilde{F}] = \tau(u)$.*
4. *$H^n(X; A)$ acts equivariantly on homotopy classes of lifts and $H^n(P; A)$ via π^*.*

In other words, we have a mapping

$$\Gamma : T_0(u) \to \{\tilde{F} \in H^n(P; A) | \iota_x^* \tilde{F} = \tau(u) \text{ for every } x \in X\}$$

and moreover this mapping is equivariant.

Proposition 2.3.3. *If G is $(n-1)$-connected then the mapping Γ is a bijection.*

Detailed proofs of these propositions can be found in [28]. We will set

$$T_1(u) := \{\tilde{F} \in H^n(P; A) | \iota_x^* \tilde{F} = \tau(u) \text{ for every } x \in X\}$$

and thus we have that $T_0(u) \cong T_1(u)$ when G is $(n-1)$-connected.

2.3.1 Trivializations as Cochains

A further notion for a u-trivialization arises by considering what happens at the level of cochains. Given that the obstruction to lifting the classifying map is a cohomology class $u \in H^{n+1}(X; A)$, one can choose a cochain representative of this class and consider all the n-cochains which trivialize this cocycle. In other words, we can choose a cochain representative $\mu \in Z^{n+1}(X; A)$ of u and define a trivialization of μ to be a choice of a cochain $\eta \in C^n(X; A)$ such that $\delta(\eta) = \mu$. In our case where we are studying the obstruction class for a principal G-bundle, we can choose a cochain $\nu \in Z^{n+1}(BG; A)$ and set $\mu := f^*\nu$. One of the reasons we are interested in this perspective is that we will have a similar notion of trivializations for differential cocycles arising from [14, 27]. Following the notation from [27], we define the category of trivializations of μ as

$$\mathcal{T}_2(\mu) = \pi_{\leqslant 1}\{C^{n-2}(X; A) \xrightarrow{\delta} C^{n-1}(X; A) \xrightarrow{\delta} \delta^{-1}(\mu)\}.$$

This denotes the category where the objects are cochains $\eta \in C^n(X; A)$ satisfying $\delta(\eta) = \mu$ and a morphism between two cohcains η, η' is a cochain $\tau \in C^{n-1}(X; A)$ such that $\delta(\tau) = \eta' - \eta$. Two such morphisms τ, τ' are then identified if there is a cochain $\zeta \in C^{n-2}(X; A)$ such that $\delta(\zeta) = \tau' - \tau$. We denote the set of equivalence classes as $T_2(\mu) := \pi_0(\mathcal{T}_2(\mu))$. The group $H^n(X; A)$ acts on $T_2(\mu)$ by sending $\eta \in T_2(\mu)$ to $\eta + \alpha$ where α is a cocyle representative of a class in $H^n(X; A)$. This is well defined as given two cocycles $\alpha, \alpha' \in Z^n(X; A)$ such that $[\alpha] = [\alpha']$, then $(\eta + \alpha) - (\eta + \alpha') = \delta\beta$ for some $\beta \in C^{n-1}(X; A)$ and thus $\eta + \alpha \sim \eta + \alpha'$ in $T_2(\mu)$. Moreover one can show that $T_2(\mu)$ is a torsor for $H^n(X; A)$.

The question now is whether there is a relation between the different notions of trivializations given by $T_0([\nu])$, $T_1([\nu])$, and $T_2(\mu)$.

To answer that question, we let $\eta \in T_2(\mu)$. Then $\eta \in C^n(X; A)$ and $\delta(\eta) = \mu$. It follows that $\pi^*\eta \in T_2(\pi^*\mu)$. Unfortunately this cochain is neither closed nor does it represent the generator $\tau([\nu]) \in H^n(F; A)$. However, we may use the commutativity of the following diagram to remedy this situation.

$$\begin{array}{ccc} P & \xrightarrow{F} & EG \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_{EG}} \\ M & \xrightarrow{f} & BG \end{array}$$

Since $H^k(EG; A) = 0$ for $k > 0$, we can make a choice of cochain $\omega \in C^n(EG; A)$ satisfying $\delta\omega = \pi_{EG}^*\mu$. Then $F^*\omega \in T_2(\pi^*\mu(P))$ and since $T_2(\pi^*\mu(P))$ is a torsor for $H^n(P; A)$, the difference $\pi^*\eta - F^*\omega$ is a cocycle representative for a class in $H^n(P; A)$. Denote this class by $S_\eta := [\pi^*\eta - F^*\omega]$. Furthermore, notice that by construction, given any $x \in BG$ the cochain $\iota_x^*\omega$ is a cocycle representative for $\tau([\nu]) \in H^n(G; A)$. Then for any $x \in X$, $\iota_x^* S_\eta = \tau([\nu])$. Hence $S_\eta \in T_0([\nu])$. Thus we have a map

$$\Pi : T_2(\mu) \to T_1([\nu]) \tag{2.1}$$

$$\eta \mapsto S_\eta. \tag{2.2}$$

This map is equivariant as given a cocycle α representing a class $[\alpha] \in H^n(X; A)$, we have $\Pi(\eta + \alpha) = [\pi^*(\eta + \alpha) - F^*\omega] = [\pi^*\eta - F^*\omega] + \pi^*[\alpha] = \Pi(\eta) + \pi^*[\alpha]$. It turns out that these two notions of u-trivializations are also equivalent when G is $(n-1)$-connected.

Lemma 2.3.4. *If $u(P) = 0$ and G is $(n-1)$-connected, then the map Π is an isomorphism.*

Proof. Suppose $\Pi(\eta) = 0$. Then there exists a cochain $\nu \in C^{n-1}(P; A)$ such that $\pi^*\eta - F^*\omega = \delta\nu$. This in turn would imply that $[\delta\iota_x^*\nu] = \tau(u)$ which contradicts the assumption that $\tau(u)$ is a generator for $H^n(G; A)$. Hence Π is injective.

Now as $u(P) = 0$, then $T_2(\mu)$ is nonempty and we may choose $v \in T_2(u)$. Let $S \in T_1(u)$. Using the assumption that G is $(n-1)$-connected, it follows from Lemma 2.3.3 that $T_1(u)$ is a torsor for $H^n(M; A)$, and thus we have a class $x \in H^n(M; A)$ such that $S - \Pi(v) = \pi^*x$. Let $\chi \in Z^n(M; A)$ be a cocycle representative for x, then $\Pi(v + x) = S$. □

2.4 THE WHITEHEAD TOWER

For an arbitrary connected space X, there is a well known sequence of spaces $\cdots \to W_3 \to W_2 \to W_1 \to X$ such that the homotopy classes of W_n are isomorphic to those of X for $k > n$ and otherwise they are zero. This is what is called the *Whitehead tower of* X, and we formalize this in what follows and provide the details for constructing each space W_n. The idea in the construction of the Whitehead tower will be to build upon the spaces used in forming the Postnikov tower which we introduce first. The original reference is [42], but the presentation is new (see also [10]).

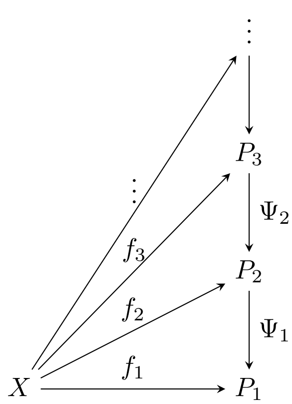
2.4.1 The Postnikov Tower

We provide the following theorem which defines what is a Postnikov tower and gives its existence for any connected space X.

Lemma 2.4.1. *Let* (X, x_0) *be a pointed space. Then there exists a relative CW complex* $\iota : X \to Y$, *constructed by adjoining* $(n+1)$*-cells only, such that* $\iota_* : X \to X'$ *is a bijection for* $k < n$ *and such that* $\pi_n(Y, y_0) = 0$.

Lemma 2.4.2. *A compact subspace of a CW complex is contained in a finite subcomplex.*

Theorem 2.4.3. *For any connected space X, there is a 'tower' of fibrations*



where each triangle commutes and the following properties are satisfied:

(1) $\pi_k(P_n) = 0$ *for* $k > n$;

(2) $\pi_k(X) \to \pi_k(P_n)$ *is an isomorphism for* $k \leqslant n$*;*

(3) the fiber F_n *of* Ψ_{n-1} *has the property that* $\pi_n(F_n)$ *is a* $K(\pi_n(X), n)$*-space.*

Proof. We start by constructing spaces Y_n such that

$$\pi_k(Y_n) \cong \begin{cases} \pi_k(X) & \text{if } k \leqslant n \\ 0 & \text{if } k > n. \end{cases}$$

The generators of $\pi_k(X)$ are homotopy classes of maps $[S^k, X]$. In order to construct Y_n, we glue $(n+2)$-cells along the generators of $\pi_{n+1}(X)$ and denote this new space as $Y_n^{(n+1)}$. Then by the previous Lemma 2.4.1, the inclusion map $\iota : X \to Y_n^{(n+1)}$ induces a map on the homotopy groups $\iota_* : \pi_k(X, x_o) \to \pi_k(Y_n^{(n+1)})$ such that ι_* is bijective for $k < n+1$, and $\pi_{n+1}(Y_n^{(n+1)}) = 0$. Then applying Lemma 2.4.1 again, we attach $(n+3)$-cells to $Y_n^{(n+1)}$ to obtain a new space $Y_n^{(n+2)}$ for which $\pi_k(X, x_0) \cong \pi_k(Y_n^{(n+2)})$ for $k < n+1$ and $\pi_{n+1}(Y_n^{(n+2)}) = \pi_{n+2}(Y_n^{(n+2)}) = 0$. Repeating this process indefininetly, we obtain a sequence of nested spaces

$$X \subset Y_n^{(n+1)} \subset Y_n^{(n+2)} \cdots .$$

Let $Y_n = \bigcup_{m>n} Y_n^{(m)}$ and endow it with the weak topology ($A \subset Y_n$ is open iff $A \cap Y_n^{(m)}$ is open in $Y_n^{(m)}$, for all $n > m$).

For any map $K \to Y_n$ where K is a compact set, by Lemma 2.4.2 it must factor through some $Y_n^{(m)}$. As a representative of $\pi_k(Y_n)$ is a map $f : S^k \to Y_n$, then by this observation, the image of f must land in $Y_n^{(m)}$ for some m, and thus be a representative of $\pi_k(Y_n^{(m)})$ as well. By construction of each $Y_n^{(m)}$, we have

$$\pi_k(Y_n) = \pi_k(\bigcup_{n<m} Y_n^{(m)}) \cong \begin{cases} \pi_k(X) & \text{if } k \leqslant n \\ 0 & \text{if } k > n. \end{cases}$$

as was our aim.

Next we notice that there are canonical inclusions $\Phi_n : Y_{n+1} \to Y_n$ giving the following commuting diagram

$$\begin{array}{ccc} & & Y_{n+1} \\ & \overset{\iota_{n+1}}{\nearrow} & \downarrow \Phi_n \\ X & \xrightarrow{\iota_n} & Y_n\,. \end{array}$$

Roughly speaking, this follows as Y_n has to adjoin more cells than Y_{n+1} in order to eliminate the n-th homotopy group of X. Thus X is approximated by smaller and smaller relative CW-complexes.

Let $P_1(X) = Y_1$ and let $f_1 : X \to P_1(X)$ just be the inclusion map $\iota_1 : X \to Y_1$. Applying Proposition 2.0.7 to the map $\Phi_1 : Y_2 \to Y_1$, we obtain a space $P(\Phi_1)$ fitting into a sequence of maps

$$\begin{array}{ccc} Y_2 & \xrightarrow[\simeq]{j_2} & P(\Phi_1) \\ \downarrow \Phi_1 & & \downarrow \psi_1 \\ Y_1 & \xrightarrow{=} & P_1(X) \end{array}$$

where j_2 is a homotopy equivalence, ψ_1 is a fibration, and $\Phi_1 = \psi_1 \circ j_2$. Let $P_2(X) := P(\Phi_1)$. In the same way, we can factor the map $j_2 \circ \Phi_2 : Y_3 \to P_2(X)$ to obtain a space which $P(j_2 \circ \Phi_2)$ which is homotopically equivalent to Y_3. Iterating this process gives us a sequence of spaces $\cdots P_3(X) \to P_2(X) \to P_1(X)$ where $P_n(X) \simeq Y_n$. Thus the spaces P_n satisfy parts (1) and (2) of our theorem.

$$\pi_k(P_n) \cong \pi_k(Y_n) \cong \begin{cases} \pi_k(X) & \text{if } k \leqslant n \\ 0 & \text{if } k > n. \end{cases}$$

In order to show that the fiber F_n of each ψ_{n-1} is a $K(\pi_n(X), n)$, we employ the long exact sequence arising from a fibration

$$\cdots \to \pi_{k+1}(P_n) \to \pi_{k+1}(P_{n-1}) \to \pi_k(F_n) \to \pi_k(P_n) \to \pi_k(P_{n-1}) \to \cdots$$

- When $k > n$, $\pi_{k+1}(P_{n-1}) = \pi_k(P_n) = 0$, and thus by exactness $\pi_k(F_n) = 0$.
- If $k = n$, then $\pi_{n+1}(P_{n-1}) = \pi_n(P_{n-1}) = 0$. Thus by exactness $\pi_n(F_n) \cong \pi_n(P_{n-1})$, and by construction, $\pi_n(P_{n-1}) \cong \pi_k(X)$.

- If $k < n$, then the long exact sequence becomes

$$\cdots \to \pi_{k+1}(X) \xrightarrow{(1)} \pi_{k+1}(X) \xrightarrow{\alpha} \pi_k(F_n) \xrightarrow{\beta} \pi_k(X) \xrightarrow{(2)} \pi_k(X) \to \cdots$$

  where (2) is an isomorphism, and (1) is an isomorphism for $k < n-1$ and a surjection for $k = n-1$. Then by exactness, Im(β)=Ker((2))=0 and Ker(α)=Im((1))=$\pi_{k+1}(X)$ as (1) is at least a surjection. Thus Ker(β)=Im(α)=0, and by the first isomorphism theorem from algebra, $0 = \text{Im}(\beta) \cong \pi_k(F_n)$.

Hence it follows that F_n is a K$(\pi_n(X), n)$. □

We now have all the ingredients required to build the Whitehead tower.

2.4.2 Construction of the Whitehead Tower

As the Postnikov tower gives us a sequence of nested spaces for X such that the k^{th} homotopy groups of X and P_n are isomorphic for $k \leqslant n$ and 0 for $k > n$, the Whitehead tower provides the opposite, which is to say that there is a sequence of spaces W_n such that $\pi_k(W_n) = 0$ for $k \leqslant n$ and $\pi_k(W_n) \cong \pi_k(X)$ for $k > n$. In order to construct the Whitehead tower, we first start with the Postnikov tower. Then define $\tilde{W_n}$ to be the space of paths in P_n from the basepoint to X. This should be thought of as the pullback of the evaluation map $\epsilon : P_n^I \to P_n \times P_n$ along the map $f : X \to P_n \times P_n$, where P_n^I is the space of smooth curves in Y, $\epsilon : \gamma \mapsto (\gamma(0), \gamma(1))$, and $f : x \mapsto (x_0, x)$.

Claim 2.4.4. *The map $f^*\epsilon : \tilde{W_n} \to X$ is a fibration.*

Proof. As the pullback of a fibration is a fibration, it suffices for us to show that the map $\epsilon : Y_n^I \to Y_n \times Y_n$ is a fibration. Suppose that we are a given a commutative square

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{f} & Y_n \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \epsilon} \\ A \times I & \xrightarrow{g} & Y_n \times Y_n \end{array}$$

where ι is the inclusion map, A is an arbitrary space, and f and g are arbitrary maps giving us a commuting square. Then $Y_n^I \xrightarrow{\epsilon} Y_n \times Y_n$ is a Hurewicz fibration if there is an $h : A \times I \times I \to X^I$ such that $g = \epsilon \circ h$ and $f = h \circ \iota$. Now notice that the map $f : A \times \{0\} \to X^I$ can be thought of as a map $f : A \times \{0\} \times I \to X$, and similarly we may think of the map g as a map from $A \times I \times \{0, 1\} \to X$. Then the commutative square can be thought of as a map $\Phi : (A \times \{0\} \times I) \cup (A \times I \times \{0, 1\}) \to X$ and the existence of h corresponds to an extension of Φ to $A \times I \times I$. Now as $(\{0\} \times I) \cup (I \times \{0, 1\})$ is just a retraction r of I^2, then the map $\bar{\Phi} = \Phi \circ (id, r)$ gives us an extension of Φ to $A \times I \times I$. Hence $Y_n^I \xrightarrow{\epsilon} Y_n \times Y_n$ is a fibration. □

Now as $X \subseteq \cdots \subseteq P_{n+1} \subseteq P_n \cdots \subseteq P_2 \subseteq P_1$, then any path in P_{n+1} is a path in P_n. It follows that this gives a sequence of inclusions $\cdots \xrightarrow{\subseteq} \tilde{W}_2 \xrightarrow{\subseteq} \tilde{W}_1 \xrightarrow{\subseteq} X$. Applying Proposition 2 for each inclusion map, we obtain the following diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\subseteq} & \tilde{W}_3 & \xrightarrow{\subseteq} & \tilde{W}_2 & \xrightarrow{\subseteq} & \tilde{W}_1 & \xrightarrow{\subseteq} & X \\
 & & \simeq\downarrow & & \simeq\downarrow & & =\downarrow & & =\downarrow \\
\cdots & \xrightarrow{p_4} & W_3 & \xrightarrow{p_3} & W_2 & \xrightarrow{p_2} & W_1 & \xrightarrow{p_1} & X
\end{array}$$

where the vertical maps are homotopy equivalences and the bottom maps p_n are fibrations. The bottom of the diagram turns out to satisfy the properties of the Whitehead tower.

Claim 2.4.5. *The spaces W_n satisfy the following:*

(1) $\pi_k(W_n) = 0$ for $k \leqslant n$;

(2) $\pi_k(W_n) \cong \pi_k(X)$ for $k > n$;

(3) The map $W_n \xrightarrow{p_n} W_{n-1}$ is a fibration whose fiber is a $K(\pi_n(X), n-1)$-space.

Proof. For each $n \in \mathbb{N}$ consider the fibrations $\tilde{W}_n \to X$ and $W_n \to W_{n-1}$. Since $\tilde{W}_n$ is the space of paths in P_n from the basepoint x_0 to X, then the fiber $p^{-1}(x_0)$ is just ΩP_n. Now the long exact sequence of fibrations for $\tilde{W}_n \to X$ is

$$\cdots \to \pi_k(\Omega P_n) \to \pi_k(\tilde{W}_n) \to \pi_k(X) \to \pi_{k-1}(\Omega P_n) \to \cdots ,$$

and using the fact that $\pi_k(\Omega P_n) = \pi_{k+1}(P_n)$, this becomes

$$\cdots \to \pi_{k+1}(P_n) \to \pi_k(\tilde{W}_n) \to \pi_k(X) \to \pi_k(P_n) \to \cdots .$$

Now by the properties of the Postnikov tower, the homotopy groups of P_n are isomorphic to $\pi_k(X)$ for $k \leqslant n$ and are 0 otherwise. If $k > n$, then $\pi_{k+1}(P_n) = \pi_k(P_n) = 0$ implies that $\pi_k(\tilde{W}_n) \cong \pi_k(X)$. If $k \leqslant n$, then $\pi_k(X) \to \pi_k(P_n)$ is an isomorphism and hence $\pi_k(\tilde{W}_n) = 0$. As W_n and $\tilde{W}_n$ are homotopy equivalent, then the same holds true for the homotopy classes of W_n. Thus it remains to show that the fiber of the fibration $W_n \xrightarrow{p_n} W_{n-1}$ is a $K(\pi_n(X), n-1)$-space. Let F_n denote the fiber of the fibration $W_n \xrightarrow{p_n} W_{n-1}$ and consider the corresponding long exact sequence

$$\cdots \to \pi_{k+1}(W_n) \to \pi_{k+1}(W_{n-1}) \to \pi_k(F_n) \to \pi_k(W_n) \to \pi_k(W_{n-1}) \to \cdots$$

- When $k > n$, the maps $\pi_{k+1}(W_n) \to \pi_{k+1}(W_{n-1})$ and $\pi_k(W_n) \to \pi_k(W_{n-1})$ are isomorphisms, and thus $\pi_k(F_n) = 0$.
- If $k = n$, then $\pi_{n+1}(W_n) \to \pi_{n+1}(W_{n-1})$ is an isomorphism and thus the sequence $\pi_{n+1}(W_n) \to \pi_{n+1}(W_{n-1}) \to \pi_n(F_n) \to 0$ shows that $\pi_n(F_n) = 0$.
- If $k = n-1$, then the long exact sequence becomes $0 \to \pi_n(W_{n-1}) \to \pi_{n-1}(F_n) \to 0 \to 0$ gives us that $\pi_{n-1}(F_n) \cong \pi_n(W_{n-1}) \cong \pi_n(X)$.
- Finally, if $k \leqslant n-2$, then the sequence is $0 \to 0 \to \pi_k(F_n) \to 0 \to 0$ and hence $\pi_k(F_n) = 0$.

It follows that F_n is a $K(\pi_n(X), n-1)$-space. □

Thus we have constructed a series of spaces that become increasingly connected. We summarize the properties of these spaces in the following theorem.

Theorem 2.4.6. *For each space X, there is a tower of fibrations $\cdots W_3 \xrightarrow{p_3} W_2 \xrightarrow{p_2} W_1 \xrightarrow{p_1} X$ such that:*

(1) The map $\pi_k(W_n) \to \pi_k(X)$ is an isomorphism for all $k > n$;

(2) $\pi_k(W_n) = 0$ for $k \leqslant n$;

(3) For each n, the fiber of $W_n \xrightarrow{p_n} W_{n-1}$ is a $K(\pi_n(X), n-1)$-space.

This is the Whitehead tower.

Remark 2.4.7. We introduce notation here and stress a discrepancy between this new notation and that just given in the construction of the Whitehead tower. Let $X\langle n\rangle$ denote the connected cover of X obtained in the Whitehead tower by killing the first $n-1$ homotopy groups of X. Thus comparing this notation with that in the previous theorem, we have that $X\langle n\rangle = W_{n-1}$.

2.5 SPECTRAL SEQUENCES

A spectral sequence may be thought of as a powerful machine for computing homology and cohomology. In what follows, we will make repeated use of this machine so it is important that we introduce it now. For a more comprehensive presentation of spectral sequences, we refer the reader to [20]. There are many types of Spectral Sequences, but for our needs we only need the Leray-Serre spectral sequence for fibrations. Let $F \to E \to B$ be a fibration. The Leray-Serre spectral sequence (we will sometimes say Serre spectral sequence as well) provides a way to calculate the cohomology of the total space E from the cohomologies of the fiber F and the base B.

The Serre spectral sequence roughly works as follows. It prescribes a filtration for the cohomology of the total space. More precisely, there is a sequence of submodules $0 \subset F_r^r \subset \cdots \subset F_0^r = H^r(E;A)$. Associated to this filtration is a sequence of chain complexes $\{E_j^{r,s}, d_j\}$ where $d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}$ and the groups $E_{j+1}^{r,s}$ are defined inductively as the cohomology $H(E_j^{r,s}, d_j)$. This sequence then converges to the cohomology $H^*(E)$ of the total space of the fibration. By convergence, we mean that for some $N > 0$, the groups $E_j^{r,s}$ stabilize i.e. $E_j^{r,s} \cong E_N^{r,s}$ for all $j > N$, and we denote the limits of these groups by $E_\infty^{r,s}$. Moreover we have that $E_\infty^{r,n-r} \cong F_r^n/F_{r+1}^n$. Thus using short exact sequences of the form

$$0 \to F_{r+1}^n \to F_r^n \to F_r^n/F_{r+1}^n \to 0,$$

the task of calculating the cohomology groups $H^*(E;A)$ reduces to an extension problem. If the coefficients A are taken to be a field, then these sequences split and the cohomology groups $H^*(E;A)$ can be read off as

$$H^p(E;A) \cong \bigoplus_{r+s=p} E_\infty^{r,s}.$$

One should note however that these splittings are not natural.

Theorem 2.5.1 (The Leray-Serre Cohomology Spectral Sequence)**.** *Let $p: E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there is a cohomology spectral sequence $E_j^{*,*}$ converging to $H^*(E;A)$, with $E_2^{*,*} = H^r(B;H^s(F;A))$, such that*

1. *the differential d_j has bidegree $(j, -j+1)$:*

$$d_j : E_j^{r,s} \to E_j^{r+j,s-j+1};$$

2. *for each j, $E_j^{*,*}$ is a bigraded ring with ring multiplication maps*

$$E_j^{p,q} \otimes E_j^{r,s} \to E_j^{p+r,q+s};$$

3. *the differential $d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}$ is an anti-derivation in the sense that it satisfies the product rule:*

$$d_j(ab) = d_j(a) \cdot b + (-1)^{u+v} a \cdot d_j(b)$$

where $a \in E_j^{u,v}$;

4. *the product in the ring E_{j+1} is induced by the product in the ring E_j, and the product in E_∞ is induced by the cup product in $H^*(E;A)$.*

Definition 2.5.2. A **morphism of spectral sequences** $\{(E_j, d_j)\} \to \{(E'_j, d'_j)\}$ is a of homomorphisms $\psi_j : E_j \to E'_j$ of bidegree $(0,0)$ such that the morphism on $H(E_j, d_j) = E_{j+1}$ induced by ϕ_j coincides with ϕ_{j+1}.

Proposition 2.5.3. *A morphism of fibrations $(F, f) : \{E \to B\} \to \{E' \to B'\}$ induces a morphism of the corresponding Leray-Serre spectral sequence.*

The Leray-Serre spectral sequence will be an important tool for us. It is one of the main tools we will employ in Section 3.2. The bundles studied in Section 3.2 have highly connected fibers which makes the computations for the cohomology groups of the total space much nicer for low degrees. Moreover these bundles will be principal G-bundles. The high connectivity of these groups means that they cannot be Lie groups, but the groups we will be studying will be connected covers of the Spin group. Because of this, Lemma 2.0.14 gives us a bundle morphism between the bundles and what we are calling there underlying Spin bundles. Once the existence of a bundle morphism between these bundles is established, Proposition 2.5.3 allows us to compare the cohomologies of these bundles.

2.6 HIGHER CONNECTED COVERS OF $\mathrm{O}(N)$

A classical result due to R. Bott which comes from what is known as Bott periodicity, is that the stable homotopy groups of $O(n)$ are isomorphic mod 8, i.e. $\pi_i(\mathrm{O}(n)) \cong \pi_j(\mathrm{O}(n))$ if $i - j = 8k$ and n is sufficiently large. To make this a little more precise, we define the stable group O of $\mathrm{O}(n)$ as the colimit

$$\mathrm{O}(1) \hookrightarrow \mathrm{O}(2) \hookrightarrow \cdots \hookrightarrow \mathrm{O}(n) \hookrightarrow \cdots \hookrightarrow \mathrm{O} = \bigcup_n \mathrm{O}(n).$$

Then Bott periodicity gives us

Theorem 2.6.1. *The homotopy groups $\pi(\mathrm{O})$ are isomorphic mod 8, i.e. $\pi_n(\mathrm{O}) \cong \pi_{n+8}(\mathrm{O})$.*

Furthermore, the first eight homotopy groups are calculated to be

$i\,(\mathrm{mod}\,8)$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

We define the subgroup $\mathrm{SO}(n)$, the special orthogonal group, to be the connected component of the identity element of O(n). There is an exact sequence

$$1 \to \mathrm{SO}(n) \to \mathrm{O}(n) \to \mathbb{Z}_2 \to 1.$$

Applying the classifying functor B to the projection $\mathrm{O}(n) \to \mathbb{Z}_2$ gives us a map $B\mathrm{O}(n) \to B\mathbb{Z}_2$.

Theorem 2.6.2. *The cohomology groups of* $\mathrm{SO}(n)$ *are*

$$H^*(\mathrm{SO}(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, x_3, \ldots, x_{2m-1}]/(x_i^{a_i})$$
$$H^*(\mathrm{SO}(n); \mathbb{R}) \cong \Lambda[x_{4i-1} \mid 0 < 2i < n] \otimes \Lambda[y_{n-1} \mid n \textit{ even}]$$

where a_i *is smallest number such that* $ia_i \geqslant n$, $m = \frac{n}{2}$

Details for showing this can be found in [21].

2.6.1 The Spin Group

We define the group $\mathrm{Spin}(n)$ to be the simply connected cover of $\mathrm{SO}(n)$ and as $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, it follows that $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a $\mathbb{Z}_2$-principal bundle. Corresponding to this bundle is a classifying map $\beta \in \mathrm{Maps}(\mathrm{SO}(n), B\mathbb{Z}_2)$. As $B\mathbb{Z}_2$ is a $K(\mathbb{Z}_2, 1)$, the homotopy class of β defines also cohomology class $[\beta] \in H^1(\mathrm{SO}(n); \mathbb{Z}_2)$. Now applying the long exact sequence of homotopy groups to the bundle $\mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n)$, we find that for $n \geqslant 3$

$$\pi_i(\mathrm{Spin}(n)) \cong \begin{cases} \pi_i(\mathrm{SO}(n)) & \text{if } i \geqslant 2 \\ 0 & \text{if } i < 2. \end{cases}$$

From this viewpoint, we can think of $\mathrm{Spin}(n)$ as being the first space in the Whitehead tower corresponding to $\mathrm{SO}(n)$, i.e. $\mathrm{Spin}(n) = \mathrm{SO}(n)\langle 3 \rangle$.

2.6.2 Whitehead Tower of O(n)

The fact that $\mathrm{Spin}(n)$ can be defined via the Whitehead tower is an important idea which is one of the main reasons for the direction of our research. As $\mathrm{Spin}(n)$ is a simply connected compact Lie group, then we have that $\mathrm{Spin}(n)$ is 2-connected which is the reason that we said $\mathrm{Spin}(n) = \mathrm{SO}(n)\langle 3 \rangle$ and not just $\mathrm{SO}(n)\langle 2 \rangle$.

Since $\pi_i(\mathrm{SO}(n)) \cong \pi_i(\mathrm{O}(n))$ for $i > 0$, we think of the Whitehead tower of $\mathrm{SO}(n)$ as extending to be a Whitehead tower over $\mathrm{O}(n)$. By Bott periodicity, we know that the next non-zero homotopy group is $\pi_7(\mathrm{O}(n)) = \mathbb{Z}$. By killing $\pi_3(\mathrm{Spin}(n))$ we get the space $\mathrm{O}(n)\langle 7 \rangle$ which in the literature is called the String group. By going farther up the Whitehead tower, we get higher and higher groups. While a priori, these spaces being defined by the Whitehead construction may not necessarily be groups, there are models in the literature which allow one to realize these spaces as groups (see [25, 29, 31, 33]). However as we are only concerning ourselves with properties of these groups up to homotopy, it will suffice for us to define these groups solely as spaces in the Whitehead tower (though we will still call them groups). The following are some of the groups which will appear in what follows [29, 30]:

k	3	7	8	9	11	14
$\mathrm{SO}(n)\langle k\rangle$	Spin(n)	String(n)	Fivebrane(n)	2Orient(n)	2Spin(n)	Ninebrane(n)

2.6.3 Cohomology of Spin

The cohomology of the stable Spin group is relatively easy to describe. The following result was shown in [38].

Theorem 2.6.3. *The cohomology of BSpin is given by*

$$H^*(BSpin;\mathbb{Z}) = \mathbb{Z}[Q_1, Q_2, \cdots] \oplus \hat{T},$$

where

1. $Q_i \in H^{4i}(BSpin;\mathbb{Z})$;
2. *if* $i \neq 2^r$, *then* $Q_i = \pi^* p_i$;
3. *if* $i = 2^r$, *then* $\pi^* p_{2j} = 2Q_{2j} + Q_j^2 - \pi^*\Phi_{2j}$ *and* $\pi^* p_1 = 2Q_1$, *and*
   $\rho_2(Q_j) = \pi^*(W_{4j} + \Psi_{4j}), \quad \rho_2(\Phi_j) = (\Psi_{2j})^2.$

Here $\pi : B\mathrm{Spin} \to B\mathrm{SO}$, $\Phi_i \in H^{4i}(BSO(n);\mathbb{Z})$, $\Psi_i \in H^i(BSO;\mathbb{Z}_2)$, *and* p_i *are the Pontrjagin classes, viewed as generators of the cohomology of* $B\mathrm{SO}$.

In contrast, the cohomology of the unstable Spin group, Spin(n), is much more involved.

Theorem 2.6.4. *The cohomology of* Spin(n) *is given by*

$$H^*(Spin(n);\mathbb{R}) \cong \Lambda[\{x_{4i-1} \mid 0 < 2i < n\}] \otimes [\{y_{n-1} \mid n \text{ even}\}]$$

$$H^*(Spin(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[u_i \mid 1 \leqslant i < n, i \neq 2^r] \cup \{u\}$$

Here $deg(u) = 2^s - 1$ *where* $2^{s-1} < n \leqslant 2^s$, *and* $Sq^j(u_i) = \binom{i}{j} u_{i+j}$, $Sq^j(u_i) = 0$ *for* $i < j$, *and* $u^2 = 0$. *The homomorphism* Sq^j *refers to the* j*th Steenrod square.*

A proof of this can be found in [21]. We use this theorem to calculate the integral (co)homology classes of Spin(n). We will only be considered in relatively low degrees (but higher than what is traditionally considered). We will also assume that n is sufficiently large enough so that the unstable classes don't play a role. Through repeated application of the Universal Coefficients Theorem (UCT), we obtain the following.

Lemma 2.6.5. *The first seven (co)homology classes of* $H^*(\mathrm{Spin}(n);\mathbb{Z})$ *for* $n > 9$ *are*

k	*1-2*	*3*	*4*	*5*	*6*	*7*
$H_k(\mathrm{Spin}(n);\mathbb{Z})$	*0*	$\mathbb{Z}$	*0*	$\mathbb{Z}_2$	*0*	$\mathbb{Z}$
$H^k(\mathrm{Spin}(n);\mathbb{Z})$	*0*	$\mathbb{Z}$	*0*	*0*	$\mathbb{Z}_2$	$\mathbb{Z}$

Proof. To calculate these groups, we use a combination of the Universal Coefficients Theorem. By the Hurewicz isomorphism, $H_3(\mathrm{Spin}(n);\mathbb{Z}) \cong \pi_3(\mathrm{Spin}(n)) \cong \mathbb{Z}$. Then the exact sequence in UCT gives

$$0 \to \mathrm{Ext}(H_2(\mathrm{Spin}(n);\mathbb{Z});\mathbb{Z}) \to H^3(\mathrm{Spin}(n);\mathbb{Z}) \to \mathrm{Hom}(H_3(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}) \to 0$$

$$0 \to 0 \to H^3(\mathrm{Spin}(n);\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0$$

and thus $H^3(\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}$.

For degree 4, the UCT exact sequence becomes

$$0 \to \mathrm{Ext}(H_3(\mathrm{Spin}(n);\mathbb{Z});\mathbb{Z}) \to H^4(\mathrm{Spin}(n);\mathbb{Z}) \to \mathrm{Hom}(H_4(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}) \to 0$$

$$0 \to \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) \to H^4(\mathrm{Spin}(n);\mathbb{Z}) \to \mathrm{Hom}(H_4(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}) \to 0$$

Furthermore, we know that $H^4(\mathrm{Spin}(n);\mathbb{R}) \cong H^4(\mathrm{Spin}(n);\mathbb{Z}_p) = 0$ for every prime p. There are corresponding UCT exact sequences

$$0 \to \mathrm{Ext}(\mathbb{Z};\mathbb{Z}_p) \to H^4(\mathrm{Spin}(n);\mathbb{Z}_p) \to \mathrm{Hom}(H_4(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}_p) \to 0$$

$$0 \to 0 \to 0 \to \mathrm{Hom}(H_4(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}_p) \to 0$$

and

$$0 \to \mathrm{Ext}(H_3(\mathrm{Spin}(n);\mathbb{Z});\mathbb{R}) \to H^4(\mathrm{Spin}(n);\mathbb{R}) \to \mathrm{Hom}(H_4(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{R}) \to 0$$

$$0 \to 0 \to 0 \to \mathrm{Hom}(H_4(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{R}) \to 0,$$

from which we can deduce that $H_4(\mathrm{Spin}(n);\mathbb{Z}) = 0$ and thus $H^4(\mathrm{Spin}(n);\mathbb{Z}) = 0$.

For degree 5, since we determined $H_4(\mathrm{Spin}(n);\mathbb{Z}) = 0$, then it follows that for any abelian group G, $H^5(\mathrm{Spin}(n);G) \cong \mathrm{Hom}(H_5(\mathrm{Spin}(n);G),\mathbb{Z})$. We know that $H^5(\mathrm{Spin}(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H^5(\mathrm{Spin}(n);\mathbb{Z}_p) = 0$ for p odd. Thus $H_5(\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}_2$ and hence $H^5(\mathrm{Spin}(n);\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) = 0$.

For degree 6, $\mathrm{Ext}(H_5(\mathrm{Spin}(n);\mathbb{Z});G) \cong \mathrm{Ext}(\mathbb{Z}_2;G) \cong G/2G$. We also know that $H^6(\mathrm{Spin}(n);\mathbb{Z}_2) = \mathbb{Z}_2[u_3^2] = \mathbb{Z}_2[u_6] \cong \mathbb{Z}_2$. Thus from the UCT exact sequence for $H^6(\mathrm{Spin}(n);\mathbb{Z}_2)$ it follows that $\mathrm{Hom}(H_6(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}_2) = 0$ which means that $H_6(\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}_p$ for some prime p. However, since $H^6(\mathrm{Spin}(n);\mathbb{Z}_p) = 0$ for any prime p, it follows that $H_6(\mathrm{Spin}(n);\mathbb{Z}) = 0$. Thus the UCT exact sequence becomes

$$0 \to \mathrm{Ext}(H_5(\mathrm{Spin}(n);\mathbb{Z});\mathbb{Z}) \to H^6(\mathrm{Spin}(n);\mathbb{Z}) \to \mathrm{Hom}(H_6(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}) \to 0$$

$$0 \to \mathbb{Z}_2 \to H^6(\mathrm{Spin}(n);\mathbb{Z}) \to 0 \to 0$$

and we have $H^6(\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}_2$.

Finally, for degree 7, we first note that $H^7(\mathrm{Spin}(n);\mathbb{Z}_p) \cong \mathrm{Hom}(H_7(\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}_p)$ follows from our previous result of $H_6(\mathrm{Spin}(n);\mathbb{Z}) = 0$. Now since $H^7(\mathrm{Spin}(n);\mathbb{Z}_p) \cong \mathbb{Z}_p$ for any p it follows that $H_7(\mathrm{Spin}(n);\mathbb{Z})\mathbb{Z}) \cong \mathbb{Z}$, and consequently $H^7(\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}$. □

2.7 O$\langle N \rangle$-STRUCTURES

Let $E \to B$ be a real vector bundle. Recall from Section 2.0.2 that a bundle is said to have a G-structure if there is a homomorphism $\rho : G \to \mathrm{Aut(F)} \cong GL(n, \mathbb{R})$ and a trivialization $\{U_i, \phi_i)\}$ such that the transition functions define maps $t_{ij} : U_{ij} \to \rho(G)$. A lifting of the structure group from G to a group H is given by a map $\tau : H \to GL(n, \mathbb{R})$ which factors through ρ, i.e. $\tau = \rho \circ \tau'$ for some homomorphism $\tau' : H \to G$, and a trivialization whose transition functions map to $\tau(H)$.

Another way to view a lifting of the structure group is as follows. Given a bundle $E \to B$ with a G-structure, we have the associated principal G-bundle $P \to M$. Then there is a map $f : M \to BG$ such that $P \cong f^*EG$. The map $\tau' : H \to G$ induces a map $B\tau' : BH \to BG$. Then the structure group lifts to H if the classifying map f lifts along $B\tau'$,

$$\begin{array}{ccc} & & BH \\ & \overset{\tilde{f}}{\nearrow} & \downarrow B\tau' \\ M & \xrightarrow{\quad f \quad} & BG\,. \end{array}$$

To summarize, we have the following definition.

Definition 2.7.1. An **H-structure** on a principal G-bundle $\pi : P \to M$ is a lift of the classifying map $f : M \to BG$ along the map $B\tau' : BH \to BG$.

Following along the lines of [28], suppose further that H is obtained as the homotopy fiber of a map $\omega : G \to K(A, n)$ for some abelian group A and where G is $(n-1)$-connected. Then applying the classifying functor B, it follows that BH is the homotopy fiber of the map $B\omega : BG \to K(\pi, n+1)$. We'll let $u \in H^{(}M; A)$ denote the cohomology class given by $f^*[B\omega]$. For another general approach involving the use of stacks, we direct the reader to [31].

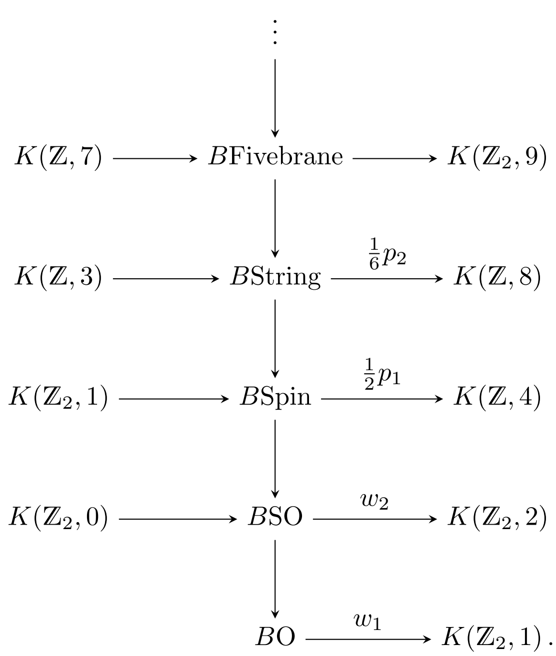
Proposition 2.7.2. *For a principal G-bundle $P \to M$ there is a 1-1 correspondence between homotopy classes of maps $\tilde{f} : M \to BH$ lifting the classifying map $f : M \to BG$ and cohomology classes $[\phi] \in H^n(P; \pi)$ such that $\iota_x^* \phi = \tau(u) \in H^n(G; \pi)$ where ι_x is the inclusion of the fiber into P.*

The proof of this proposition can be found in [28]. In the Whitehead tower for $B\mathrm{O}$, we have a sequence of spaces $B\mathrm{O}\langle n\rangle$ and fibrations $p_{n+1} : B\mathrm{O}\langle n+1\rangle \to B\mathrm{O}\langle n\rangle$ where the fibers are $K(\pi_{n-1}(\mathrm{O}), n-1)$. The obstructions for lifting along these fibrations then are given by a homotopy class of maps to a $K(\pi_{n-1}(\mathrm{O}), n)$. (In what follows, we will assume that we are studying the corresponding stable groups.)

Proposition 2.7.3. *The obstruction to lifting a map $f : X \to B\mathrm{O}\langle n\rangle$ along p_{n+1} : $B\mathrm{O}\langle n+1\rangle \to B\mathrm{O}\langle n\rangle$ is given by a cohomology class $o_n(f) \in H^n(X; \pi_n(\mathrm{O}))$.*

If we are given a manifold M with a String structure, we may want to ask whether this structure lifts to a Fivebrane structure. Let $\pi_{\mathrm{String}} : P \to M$ denote the associated principal String bundle. As shown in [30], the bundle lifts to a Fivebrane bundle when $\frac{1}{6}p_2(M) = 0$, where $p_2(M)$ is the second Pontrjagin class of M.

As we are mainly interested in these spaces up to homotopy, we define $B\mathrm{O}\langle n+1\rangle$ to be the homotopy fiber of a map representing the generator of $H^n(B\mathrm{O}; \pi_{n-1}(\mathrm{O})) \cong \pi_{n-1}(\mathrm{O})$. Thus, for example, the group String is classified by the space $B\mathrm{String} = F(\frac{1}{2}p_1)$ as $\frac{1}{2}p_1$ is a generator for $H^4(B\mathrm{Spin}; \mathbb{Z})$. The Whitehead tower of BO is as follows (see [29, 30] for more details)



2.7.1 Fivebrane **structures**

Definition 2.7.4. Let $\pi_{\text{String}} : P \to M$ be a String-principal bundle. We say that P admits a **Fivebrane structure** if the classifying map $f : M \to B\text{String}$ lifts to $B\text{Fivebrane}$.

We are mainly interested in the classes that characterize the structure, rather than the ones that provide the obstruction to having such a structure. The latter are considered extensively in [29, 30].

Definition 2.7.5. A **Fivebrane class** is a cohomology class $F \in H^7(P;\mathbb{Z})$ such that $\iota_x^* F = \tau(\frac{1}{2}p_1) \in H^7(\text{String};\mathbb{Z})$ for each inclusion of the fiber $\iota_x : \text{String} \to P$.

Looking at the Serre spectral sequence arising from the fibration $\text{String} \to P \to M$, we find that $E_\infty^{0,7} \cong \text{Ker}(d_8 : H^7(\text{String};\mathbb{Z}) \to H^8(M;\mathbb{Z}))$, $E_\infty^{7,0} \cong H^7(M;\mathbb{Z})$, and $E_\infty^{r,7-r} = 0$ for $1 < r < 7$. Since there is a filtration of $F_7^7 \subset \cdots F_0^7 = H^7(P;\mathbb{Z})$ where $E_\infty^{r,7-1} \cong F_r^7 / F_{r+1}^7$, then $H^7(P;Z)$ fits into the following exact sequence,

$$0 \to H^7(M;\mathbb{Z}) \to H^7(P;\mathbb{Z}) \to H^7(\text{String};\mathbb{Z}) \xrightarrow{d_8} H^8(M;\mathbb{Z}).$$

A choice of lift then corresponds to a cohomology class in $H^7(P;\mathbb{Z})$ which maps to the generator of $H^7(\text{String};\mathbb{Z})$. Since d_8 maps the generator of $H^7(\text{String};\mathbb{Z})$ to the obstruction for lifting the String structure on M to a Fivebrane structure, then if a lift indeed exists, this sequence becomes a short exact sequence,

$$0 \to H^7(M;\mathbb{Z}) \to H^7(P;\mathbb{Z}) \to H^7(\text{String};\mathbb{Z}) \to 0,$$

and the choice of lift determines a splitting for this short exact sequence. Thus we obtain the following Lemma.

Lemma 2.7.6. *$H^7(P;\mathbb{Z}) \cong H^7(M;\mathbb{Z}) \bigoplus Ker(d_8)$ where d_8 is the transgression mapping $H^7(\text{String};\mathbb{Z}) \xrightarrow{d_8} H^8(M;\mathbb{Z})$.*

In other words a Fivebrane structure is determined up to isomorphism by the cohomology class in $H^7(P;\mathbb{Z})$ which maps to the generator of $H^7(\text{String};\mathbb{Z})$. Furthermore, the short exact sequence tells us that Fivebrane structures are a torsor for $H^7(M;\mathbb{Z})$ as given two different Fivebrane structures that lift the String structure on M with corresponding classes $F, F' \in H^7(P;\mathbb{Z})$, then by exactness there is a class $\mu \in H^7(M;\mathbb{Z})$ such that $\pi^*_{\text{String}}(\mu) = F - F'$. Putting all of this together gives us the following.

Theorem 2.7.7. *1. A* String-*principal bundle admits a* Fivebrane-*structure if and only if P admits a* Fivebrane *class.*

2. The set of isomorphism classes of Fivebrane-*structures is a torsor for $H^7(M;\mathbb{Z})$.*

3.0 RATIONAL STRUCTURES

3.1 RATIONAL HOMOTOPY THEORY

As we are interested in understanding rational structures corresponding to principal bundles, it will be convenient and useful to place these ideas in the realm of rational homotopy theory. With this in mind, we provide a brief overview of the subject here. For more detailed resources on rational homotopy theory, the reader is directed to [8, 9, 13].

The idea behind rational homotopy theory is that most of the difficulties encountered when understanding the homotopic properties of an object arise from torsion. By tensoring coefficients with a field of characteristic zero, this torsion disappears. The resulting object is much nicer to deal with, but still has useful information that describes the original object.

3.1.1 Rationalizations of a Space

To begin we consider what it would mean to "rationalize" a topological space. First let us define a rational space to be any space for which all homotopy groups are products of $\mathbb{Q}$.

Definition 3.1.1. We say that **the rationalization of a space** X is a map $\ell : X \to X_{\mathbb{Q}}$ such that

1. The induced map $\ell_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_{\mathbb{Q}})$ is an isomorphism.
2. This map l is universal in that given any map $f : X \to Y$ where Y is a rational space, then f factors through ℓ, i.e. there is a map $g : X_{\mathbb{Q}} \to Y$ such that $f = g \circ \ell$. The choice of g is unique up to homotopy.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \Big\downarrow{\scriptstyle \ell} & \overset{g}{\nearrow} & \\ X_0 & & \end{array}$$

Now in general, not every topological space may admit a rationalization. Thus we will restrict ourselves to simply connected spaces. In fact it holds that all nilpotent spaces admit a rationalization. We define nilpotent spaces as follows.

Definition 3.1.2. A CW-complex X is **nilpotent** if each fibration $X_n \to X_{n-1}$ in its Postnikov decomposition is a principal fibration.

Recall that the Postnikov tower is a sequence of spaces

$$X \to \cdots X_n \to X_{n-1} \to \cdots \to X_2 \to X_1$$

such that the homotopy groups $\pi_i(X_n)$ are trivial for $i > n$ and isomorphic to $\pi_i(X)$ for $i \leqslant n$ with a map $f : X \to X_n$ inducing these isomorphisms on the level of homotopy. Moreover, the maps $X_n \to X_{n-1}$ are $K(\pi_n, n)$-fibrations. It follows from this theorem that for a nilpotent space we may construct each fibration as the pullback of a map representing the Postnikov invariant $k_{n+1} \in H^{n+1}(X_{n-1}, \pi_n(X))$.

$$\begin{array}{ccc} X_n & \xrightarrow{\qquad\qquad} & PK(\pi_n(X), n+1) \\ \Big\downarrow & & \Big\downarrow \\ X_{n-1} & \xrightarrow{\quad k_{n+1} \quad} & K(\pi_n(X), n+1) \end{array}$$

Theorem 3.1.3. *Let $\ell : X \to Y$ be a map between nilpotent spaces. Then the following are equivalent.*

1. *The map $\ell : X \to Y$ is the rationalization of X.*
2. *ℓ induces an isomorphism $\ell_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y)$.*
3. *ℓ induces an isomorphism $\ell_* : H_*(X) \otimes \mathbb{Q} \to H_*(Y)$.*

A very simple example of a rational space is the Eilenberg-MacLane space $K(\mathbb{Q}, n)$. With this theorem one can show that $K(\mathbb{Q}, n)$ is the rationalization of $K(\mathbb{Z}, n)$ and in general that $K(\pi \otimes \mathbb{Q}, n)$ is the rationalization for $K(\pi, n)$ where π is any abelian group. From here it becomes clear why we chose the category of nilpotent spaces. Given a simply connected space X, we can construct the rationalization of X by using its Postnikov tower. The idea is that at each stage of the Postnikov tower we can replace the space X_n with a rationalization of X_n. To do this, we start by setting $(X_2)_{\mathbb{Q}} = K(\pi_2(X), 2)$. Now supposing that we have rationalized each space up to $n-1$, we use a rationalization of the k-invariant to construct $(X_n)_{\mathbb{Q}}$.

$$\begin{array}{ccccc}
X_n & \longrightarrow & PK(\pi_n(X), n+1) & & \\
\downarrow & & \downarrow & & \\
X_{n-1} & \xrightarrow{k_{n+1}} & K(\pi_n(X), n+1) & & \\
& \searrow^{\ell_{n-1}} & & \searrow^{\ell'} & \\
& (X_{n-1})_0 & \overset{\tilde{k}_{n+1}}{\dashrightarrow} & & K(\pi_n(X) \otimes \mathbb{Q}, n+1)
\end{array}$$

The map $\tilde{k}_{n+1}$ is guaranteed to exist by the universal property for rationalizations as $\ell' \circ k_{n+1}$ is a map from X to a $\mathbb{Q}$ space. Thus $(X_n)_{\mathbb{Q}}$ is defined as the pullback along $\tilde{k}_{n+1}$ of the path fibration

$$\begin{array}{ccc}
(X_n)_0 & \longrightarrow & PK(\pi_n(X) \otimes \mathbb{Q}, n+1) \\
\downarrow & & \downarrow \\
(X_{n-1})_0 & \xrightarrow{\tilde{k}_{n+1}} & K(\pi_n(X) \otimes \mathbb{Q}, n+1)
\end{array}$$

By the universal property of pullbacks, there is a map $\ell_n : X_n \to (X_n)_0$ and by the commutativity it follows that it induces an isomorphism $(\ell_n)_* : \pi_*(X_*) \otimes \mathbb{Q} \to \pi_*((X_n)_0)$. Hence by Theorem 3.1.3, $\ell_n : X_n \to (X_n)_{\mathbb{Q}}$ is indeed the rationalization of X_n which completes the inductive step. Thus we have a sequence of spaces

$$\cdots \to (X_n)_0 \to \cdots \to (X_3)_{\mathbb{Q}} \to (X_2)_{\mathbb{Q}}$$

for which $f : X \to \varprojlim (X_n)_{\mathbb{Q}}$ is a rationalization for X. We summarize with the following theorem.

Theorem 3.1.4. *Every simply connected space admits a rationalization.*

As we are primarily concerned with studying the geometric structures arising from the Whitehead tower over $\mathrm{BO}(n)$, then we should also have an understanding of this tower rationally. Recall that in constructing the Whitehead tower, we built off of the Postnikov tower. Now as we noted that for simply connected spaces the fibrations in the Postnikov tower can be made to be principal fibrations, the same is true for the Whitehead tower. Then, as with the Postnikov tower, we may construct each step of the Whitehead tower via pullbacks along representatives of the obstruction classes

$$\begin{array}{ccc} W_n & \longrightarrow & PK(\pi_{n-1}(X), n) \\ \downarrow & & \downarrow \\ W_{n-1} & \longrightarrow & K(\pi_{n-1}(X)n). \end{array}$$

The rational Whitehead tower is then constructed by pullbacks along maps representing the rationalizations of these obstruction classes.

3.1.2 Minimal Models

Definition 3.1.5. A **commutative differential graded algebra (CDGA)** is a graded vector space $\mathcal{A}^*$ over $\mathbb{Q}$ equipped with

1. a differentiation $d : \mathcal{A}^* \to \mathcal{A}^{*+1}$ where $d^2 = 0$;
2. a multiplication $\mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}$ where

$$\alpha\beta = (-1)^{pq}\beta\alpha;$$

3. $d(\alpha\beta) = d(\alpha)\beta + (-1)^p \alpha d(\beta)$.

Definition 3.1.6. A CDGA $\mathcal{A}^*$ is **minimal** if

1. $\mathcal{A}^*$ is a graded-commutative algebra on generators of degrees $\geqslant 2$.
2. $d(\mathcal{A}^*) \subset \mathcal{A}^+ \wedge \mathcal{A}^+$ where $\mathcal{A}^* = \bigoplus_{k>0} \mathcal{A}^k$ and $\mathcal{A}^+$ denotes the set of decomposable elements.

Theorem 3.1.7. *Every simply connected CDGA has a minimal model. Moreover if $\rho : \mathcal{M} \to \mathcal{A}$ and $\rho' : \mathcal{M}' \to \mathcal{A}$ are two minimal models for $\mathcal{A}$, then there is an isomorphism $I : \mathcal{M} \to \mathcal{M}'$ and a homotopy H of ρ to $\rho' \circ I$. The isomorphism itself is determined by conditions up to homotopy.*

3.2 RATIONAL STRUCTURES

Here we are interested in studying the above structures when ignoring torsion. In the rational setting, the vanishing of an obstruction is equivalent to the vanishing of any multiple of that obstruction. For example, recall that for a Spin bundle to admit a String structure, the obstruction $\frac{1}{2}p_1$ must be zero. In the rational setting, this is equivalent to p_1 being zero.

3.2.1 The Rational Whitehead Tower

For a simply connected space X, we have the Whitehead tower sitting above it, and we have a rationalization, $\ell : X \to X_{\mathbb{Q}}$, of X. The rationalization is again simply connected and thus we can consider its Whitehead tower. We will call the Whitehead tower of $X_{\mathbb{Q}}$ the **rational Whitehead tower of** X. A nice property of the rational Whitehead tower is the following.

Proposition 3.2.1. *Let X be a simply connected space with Whitehead tower*

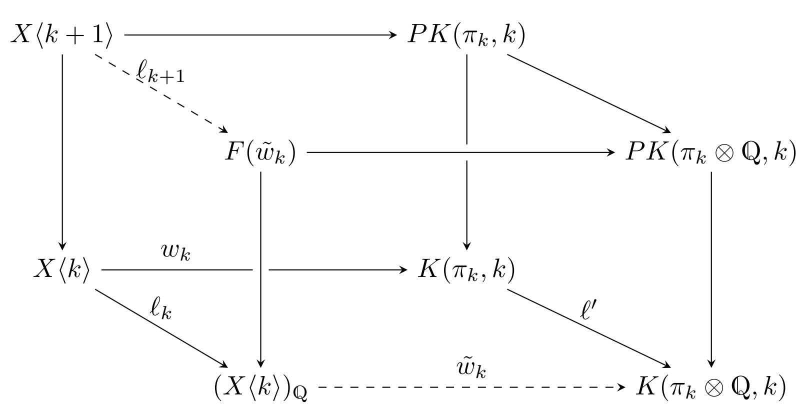
$$\cdots \to X\langle k+1\rangle \to X\langle k\rangle \to X\langle k-1\rangle \to \cdots \to X_3 \to X.$$

Let $\ell : X \to X_{\mathbb{Q}}$ be a rationalization of X and consider the rational Whitehead tower of X

$$\cdots \to X_{\mathbb{Q}}\langle k+1\rangle \to X_{\mathbb{Q}}\langle k\rangle \to X_{\mathbb{Q}}\langle k-1\rangle \to \cdots X_{\mathbb{Q}}\langle 3\rangle \to X_{\mathbb{Q}}.$$

Then there exist maps $\ell_k : X\langle k\rangle \to X_{\mathbb{Q}}\langle k\rangle$ such that each is a rationalization of $X\langle k\rangle$.

Proof. The proof follows along similar lines to Theorem 3.1.4 . For convenience, we will use π_k in place of $\pi_k(X)$. Suppose that we have such rationalizations for the first k spaces in the Whitehead tower. Then at the k-th stage, we have the following commutative diagram



where $F(\tilde{w}_k)$ denotes the homotopy fiber of $\tilde{w}_k$. As in the case with the Postnikov tower, the map $\tilde{w}_k$ exists by the universal property for rationalizations since $\ell' \circ w_k$ maps X to the rational space $K(\pi_k \otimes \mathbb{Q}, k)$. Furthermore, since w_k is a generator of $H^k(X; \pi_k)$, then $\tilde{w}_k$ is a generator of $H^k(X\langle k\rangle; \pi_k \otimes \mathbb{Q}) \cong H^k(X; \pi_k \otimes \mathbb{Q})$. Thus $F(\tilde{w}_k)$ fits into the next stage of the rational Whitehead tower. We also have the map $\ell_{k+1} : X\langle k+1\rangle \to F(\tilde{w}_k)$ coming from the universal property for pullbacks and this map induces an isomorphism

$$\ell_{k+1*} : \pi_*(X\langle k+1\rangle) \otimes \mathbb{Q} \to \pi_*(F(\tilde{w}_k).$$

This gives us our rationalization $\ell_{k+1} : X\langle k+1\rangle) \to X_{\mathbb{Q}}\langle k+1\rangle$ where we've set $X_{\mathbb{Q}}\langle k+1\rangle := F(\tilde{w}_k)$. Recall that the Whitehead tower is only unique up to a weak equivalence. What we have shown above is that by constructing the rational Whitehead tower by using the rationalizations of the obstructions from the Whitehead tower, we obtain the desired result. We still need to show that any Whitehead tower over $X_{\mathbb{Q}}$ satisfies the Proposition. Given two different Whitehead towers over $X_{\mathbb{Q}}$, there are weak equivalences $p_k : X\langle k\rangle \to X'\langle k\rangle$ between each connected covering. Then it follows that $p_{k*} : \pi_*(X_{\mathbb{Q}}\langle k\rangle) \to \pi_*(X'_{\mathbb{Q}}\langle k\rangle)$ is an isomorphism for each k and thus

$$(p_k \circ \ell)_* : \pi_*(X\langle k\rangle) \otimes \mathbb{Q} \to \pi_*(X'_{\mathbb{Q}}\langle k\rangle)$$

is also an isomorphism. Applying Theorem 3.1.3, we have that this is also a rationalization of X and hence any Whitehead tower above $X_{\mathbb{Q}}$ also satisfies this Proposition. □

3.2.2 Rational structures

The theory of $\mathrm{O}\langle n\rangle$-structures on a manifold can be translated to the setting of rational homotopy theory. Suppose, as we did at the beginning of the Section 2.7, that we have a homomorphism of groups $\tau : H \to G$ where H is obtained as the homotopy fiber of a map $\omega : G \to K(\pi, n)$ for some Abelian group π. Furthermore, let us assume that G is simply connected. Then we have the rationalization of G, $l_G : G \to G_{\mathbb{Q}}$ and the rationalization of H can be constructed as the homotopy fiber of the map $\omega_{\mathbb{Q}} : G_{\mathbb{Q}} \to K(\pi \otimes \mathbb{Q}, n)$. Consequently we now have a map $\tau_{\mathbb{Q}} : H_{\mathbb{Q}} \to G_{\mathbb{Q}}$, which is the rationalization of τ, and in the same fashion we can construct a map $B\tau_{\mathbb{Q}} : BH_{\mathbb{Q}} \to BG_{\mathbb{Q}}$.

Let $\pi : P \to M$ be a principal G-bundle and let $f : M \to BG$ be it's classifying map. Composing f with the rationalization of BG gives a map $f_{\mathbb{Q}} : M \to BG_{\mathbb{Q}}$. In Section 2.7, we defined an H-structure to be a lift of the classifying map along the map $B\tau : BH \to BG$. We can form an analogous definition in the rational setting.

Definition 3.2.2. A **rational H-structure** on a principal G-bundle, $\pi : P \to M$, is a lift of the rationalization of the classifying map, $f_{\mathbb{Q}}$, along the map $B\tau_{\mathbb{Q}}$.

We can apply this to the case of $\mathrm{O}\langle n\rangle$-structures. The obstruction theory involved also transfers over nicely. For example, as in Proposition 2.7.3, we have the following.

Proposition 3.2.3. *The obstruction to lifting a map $f_{\mathbb{Q}} : M \to B\mathrm{O}\langle n\rangle_{\mathbb{Q}}$ along $(p_{n+1})_{\mathbb{Q}} : B\mathrm{O}\langle n+1\rangle_{\mathbb{Q}} \to B\mathrm{O}\langle n\rangle_{\mathbb{Q}}$ is given by a cohomology class $o_n(f) \in H^n(X; \pi_n(\mathrm{O}) \otimes \mathbb{Q})$.*

We also can consider the rational Whitehead tower of BO.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\mathbb{Q},11) & \longrightarrow & B\mathrm{Ninebrane}_{\mathbb{Q}} & \longrightarrow & K(\mathbb{Q},16) \\
 & & \downarrow & & \\
K(\mathbb{Q},7) & \longrightarrow & B2\mathrm{Spin}_{\mathbb{Q}} & \xrightarrow{(\frac{1}{240}p_3)_{\mathbb{Q}}} & K(\mathbb{Q},12) \\
 & & \downarrow & & \\
K(\mathbb{Q},3) & \longrightarrow & B\mathrm{String}_{\mathbb{Q}} & \xrightarrow{(\frac{1}{6}p_2)_{\mathbb{Q}}} & K(\mathbb{Q},8) \\
 & & \downarrow & & \\
 & & B\mathrm{Spin}_{\mathbb{Q}} & \xrightarrow{(\frac{1}{2}p_1)_{\mathbb{Q}}} & K(\mathbb{Q},4)\,.
\end{array}
$$

Now since the first two homotopy groups are torsion, it follows that $B\mathrm{Spin}$ is the localization for both $B\mathrm{O}$ and $B\mathrm{SO}$ as well as the first step in this tower. As we showed in the previous section, the rational Whitehead tower gives us a sequence of rationalizations for the usual Whitehead tower, and at each level there exist maps $\ell_n : B\mathrm{O}\langle n\rangle \to B\mathrm{O}\langle n\rangle$. In what follows, we will focus on the cases when $n = 8, 12$.

3.2.3 Rational Fivebrane structures

We start by discussing the $n = 8$ case, i.e. when we have a principal String-bundle and wish to see whether it admits a rational Fivebrane structure. As String is 6-connected with $\pi_7(\text{String}) \cong \pi_7(O) \cong \mathbb{Z}$, it follows from the Hurewicz and Universal Coefficients Theorem that $H^7(\text{String}; \mathbb{Q}) \cong \mathbb{Q}$. Following the definition for a manifold M to have a Fivebrane structure, we make the following definitions.

Definition 3.2.4. Given a String-principal bundle $\pi_{\text{String}} : P \to M$, P is said to have a **rational Fivebrane-structure** if there is a lift of the rationalized classifying map $f : M \to B\text{String}_Q$ to the homotopy fiber $F((\frac{1}{6}p_2)_{\mathbb{Q}})$.

Now, we set $a_7 \in H^7(\text{String}; \mathbb{Q})$ to be the generator given by $a_7 := \tau((\frac{1}{6}p_2)_{\mathbb{Q}})$ and make the following definition.

Definition 3.2.5. A **rational Fivebrane structure class** is a cohomology class $F \in H^7(P; \mathbb{Q})$ such that $\iota_x^* F = a_7 \in H^7(\text{String}; \mathbb{Q})$ for each inclusion $\iota_x : \text{String} \to P$.

As with the integral case, it follows that these rational Fivebrane structure classes form a torsor for $H^7(M; \mathbb{Q})$. At this stage, our goal is to describe higher structures (beyond Spin) using Spin structures to the extent of which it is possible. We will do this for Fivebrane and Ninebrane structures.

Lemma 3.2.6. *The map* $\rho : \text{String} \to \text{Spin}$ *induces an isomorphism* $\rho^* : H^7(\text{Spin}; \mathbb{Q}) \xrightarrow{\cong} H^7(\text{String}; \mathbb{Q})$.

We've defined rational Fivebrane classes solely as any class in $H^7(P; \mathbb{Q})$ which restricts to a certain generator in $H^7(\text{String}; \mathbb{Q})$. We make two notes regarding this. The first is that the transgression map is invariant under rationalization. Thus if we have a generator $\frac{1}{6}p_2 \in H^8(B\text{String}; \mathbb{Z})$, then $(\frac{1}{6}p_2)_{\mathbb{Q}}$ is a generator for $H^8(B\text{String}; \mathbb{Q})$. More importantly, we have $\tau((\frac{1}{6}p_2)_{\mathbb{Q}}) = (\tau(\frac{1}{6}p_2))_{\mathbb{Q}}$. The consequence of this is the following.

Proposition 3.2.7. *The rationalization of any Fivebrane class is a rational Fivebrane class.*

Proof. Every Fivebrane class $F \in H^7(P; \mathbb{Z})$ satisfies $\iota_x^* F = \tau(\frac{1}{6}p_2)$. Then by naturality of rationalization and what we noted above, the rational class $F_{\mathbb{Q}}$ satisfies

$$\iota_x^* F_{\mathbb{Q}} = (\iota_x^* F)_{\mathbb{Q}} = (\tau(\tfrac{1}{6}p_2))_{\mathbb{Q}} = a_7.$$

Hence $F_{\mathbb{Q}}$ is a rational Fivebrane class. □

Thus for any ordinary Fivebrane class, there is a corresponding rational Fivebrane class. The second thing we note is that with the isomorphism from Lemma 3.2.6, we can define a generator of $H^7(\text{Spin};\mathbb{Q})$ as $(\rho^*)^{-1}(\tau(\frac{1}{6}p_2))$. For simplicity we'll denote this class as $\tilde{a}_7$. We will also set $a_3 := \tau((\frac{1}{2}p_1)_{\mathbb{Q}}) \in H^3(\text{Spin};\mathbb{Q})$. Then by considering the underlying Spin bundle for our String bundle, we can define classes here similar to how Fivebrane classes are defined cohomologically. Let $\pi_{\text{Spin}} : Q \to M$ denote the underlying Spin bundle.

Definition 3.2.8. A **rational Spin-Fivebrane class** is a cohomology class $F_{\mathbb{Q}}$ in $H^7(Q;\mathbb{Q})$ such that $\iota_x^* F_{\mathbb{Q}} = \tilde{a}_7 \in H^7(\text{Spin};\mathbb{Q})$ for each $x \in M$.

The question now becomes how these two definitions are related. It is not too difficult to show that every rational Spin-Fivebrane class gets mapped by μ^* to a rational Fivebrane class, however we can say more.

Theorem 3.2.9. *1. For every rational Spin-Fivebrane class $F \in H^7(Q;\mathbb{Q})$, $\mu^* F$ is a rational Fivebrane class.*

2. *For any rational* Fivebrane *structure $F \in H^7(P;\mathbb{Q})$ there is a Spin-Fivebrane class in $\tilde{\in} H^7(Q;\mathbb{Q})$ such that $\mu^* \tilde{F} = F$.*

3. *Two classes $F, F' \in H^7(Q;\mathbb{Q})$ will give the same Fivebrane structure if $F - F' = S \cdot \pi_{\text{Spin}}^* \phi$ where $S \in H^3(Q;\mathbb{Q})$ is the String structure class and $\phi \in H^4(M;\mathbb{Q})$.*

Proof. The main ingredient that will be used in the proof is the corresponding Serre spectral sequences for the bundles Q and P along with the spectral sequences for the universal bundles over the classifying spaces $B\text{Spin}$ and $B\text{String}$. Let f_{Spin} and f_{String} be the classifying map of Q and P respectively.

The second page of the spectral sequence for Q is as follows

$H^*(\text{Spin})$						
7	a_7					
$\vdots$	0	0	0	0	0	
3	a_3	$\cdots$	$a_3\omega_4$	$\cdots$	$\cdots$	
$\vdots$	0	0	0	0	0	
0	$\mathbb{Q}$	$\cdots$	ω_4	$\cdots$	ω_7	
	0	$\cdots$	4	$\cdots$	7	$H^*(M)$

As the spectral sequence converges to the cohomology of the total space and as are coefficents are $\mathbb{Q}$, it follows that we have a non-canonical splitting

$$H^7(Q;\mathbb{Q}) \cong E_\infty^{7,0} \oplus E_\infty^{4,3} \oplus E_\infty^{0,7}.$$

Thus we want to calculate each of these terms. On $E_\infty^{7,0}$, we have that the differentials d_r are all zero since $E_r^{p,q} = 0$ for $q < 0$. Thus the only differential of interest is $d_4 : E_4^{3,3} \to E_4^{7,0}$. Let us determine how the differential acts on generators of $E_4^{3,3}$. Using that $E_4^{3,3} \cong E_2^{3,3} \cong \mathbb{Q}[a_3] \otimes H^3(M)$, then a typical generator is of the form $a_3\omega_3$ where $\omega_3 \in H^3(M;\mathbb{Q})$. We also know that $d_4(a_3) = kp_1$ where p_1 is the first Pontrjagin class of M and k is a scalar, and since M admits a String structure, then $p_1 = 0$. Thus $d_4(a_3\omega_3) = kp_1\omega_3 + a_3 d_4(\omega) = 0$ for any generator of $E_4^{3,3}$ and thus $E_\infty^{7,0} = H^7(M;\mathbb{Q})$.

For $E_\infty^{0,7}$, the only relevant differentials are $d_5 : E_5^{0,7} \to E_5^{5,3}$ and $d_8 : E_8^{0,7} \to E_8^{8,0}$. Since d_r is zero for $r \leqslant 4$ and $6 \leqslant r < 8$, then it follows that $E_5^{0,7} \cong \mathbb{Q}[a_7]$. To see what d_5 and d_8 map $\mathbb{R}[a_7]$ to, we will use the spectral sequence for the universal bundle $Spin(n) \to E\text{Spin} \to B\text{Spin}$ along with naturality of the bundle map coming from the classifying map $f : M \to B\text{Spin}$. Let $F_r^{p,q}$ represent the spectral sequence for the universal bundle. Then the map $f : M \to B\text{Spin}$ induces maps $f^* : F_r^{p,q} \to E_r^{p,q}$ such that f^* is the identity when $p = 0$. Thus $d_5(a_7) = d_5(f^*a_7) = f^*d_5(a_7) = 0$ since $H^5(B\text{Spin};\mathbb{Q}) = 0$ which means $F_*^{5,3} = 0$. By the same reasoning, since d_8 maps a_7 to the generator of $H^8(B\text{Spin};\mathbb{Q})$ then for Q, $d_8(a_7) = kp_2$ where p_2 is the second Pontrjagin class and k is some scalar. Since Q has a Fivebrane structure, then $d_8(a_7) = 0$ and thus $E_\infty^{0,7} = E_2^{0,7} \cong \mathbb{R}[a_7]$. It follows that

$$H^7(Q;\mathbb{Q}) \cong \mathbb{Q}[a_7] \oplus E_\infty^{4,3} \oplus H^7(M;\mathbb{Q}).$$

Through a similar argument, we find that $H^7(P;\mathbb{Q}) \cong \mathbb{Q}[a_7] \oplus H^7(M;\mathbb{Q})$ where we now have $E_\infty^{4,3} = 0$ since $H^3(\text{String};\mathbb{Q}) = 0$. The bundle morphism $\mu : P \to Q$ induces a homomorphism $\mu^* : H^7(Q;\mathbb{Q}) \to H^7(P;\mathbb{Q})$ and thus a homomorphism between each page of the spectral sequences. It follows that μ^* is surjective, and that $\text{Ker}(\mu^*) = E_\infty^{4,3}$. To finish the proof, it only remains for us to show that $E_\infty^{4,3} \cong \mathbb{Q}[a_3] \otimes H^4(M;\mathbb{Q})$. Indeed, the only nontrivial differential is $d_4 : E_4^{4,3} \to E_4^{8,0}$ and since we've already shown that $d_4(a_3) = 0$ then it follows that d_4 is also trivial.

□

Thus by studying Fivebrane structures in the rational setting, we are able to capitalize on several of the nice properties that come with rational cohomology. The lack of torsion is what allows us to break up the cohomology of our total space into direct sums. The actual computations are much easier as the cohomology of Spin becomes much cleaner without torsion.

Theorem 3.2.9 demonstrates the degree to which the underlying Spin-bundle can be used to classify lifts of the String-bundles rationally. As we remarked before, by defining these classes via there restriction on each fiber, we have many nice parallels between the integral and rational cases as well as between those classes defined on the Spin-bundle and those on the String-bundle. The difference between the integral and rational case is torsion and the Bockstein sequence corresponding to the short exact sequence

$$1 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 1$$

prescribes to what degree that they differ. Thus two Fivebrane structures are identified rationally if their difference corresponds to a torsion class in $H^7(M;\mathbb{Z})$.

Theorem 3.2.9 gives us a similar understanding of what happens now when going from Spin-Fivebrane to Fivebrane classes rationally. Part 1 tells us that every rational Spin-Fivebrane class corresponds to a rational Fivebrane class. Part 2 of Theorem 3.2.9 then tells us when any two rational Spin-Fivebrane class corresponds to the same rational Fivebrane class. In some sense we find that rationally all the information for Fivebrane structures is encoded in the underlying Spin bundles. In fact, it follows that if $H^4(M;\mathbb{Q}) = 0$, then the set of Fivebrane classes and Spin-Fivebrane classes are bijective.

3.2.4 Rational Ninebrane structures

We now extend this result for higher connected groups of $B\mathrm{O}$. Following [29] let 2Spin and Ninebrane denote the groups $B\mathrm{O}\langle 12\rangle$ and $B\mathrm{O}\langle 16\rangle$ respectively (see the table at the end of Section 2.6.2). Notice that in our Whitehead tower, $B\mathrm{O}\langle k\rangle$ for $k = 9, 10$ is obtained by killing homotopy groups that are completely torsion. Hence rationally, $H^*(B\mathrm{O}\langle k\rangle;\mathbb{Q}) \cong H^*(2\mathrm{Spin};\mathbb{Q})$ for $k = 9, 10$. So to follow along the lines of rational Fivebrane structures, we may define rational Ninebrane structures, and so on, for all the n-connected covers of O which correspond to the killing of integral homotopy groups.

Definition 3.2.10. Given a 2Spin-principal bundle $\pi_{2\mathrm{Spin}} : T \to M$, T is said to have a **rational Ninebrane structure** if there is a lift of the rational classifying map $f : M \to B2\mathrm{Spin}_{\mathbb{Q}}$ to the homotopy fiber $F(\frac{1}{240}p_3)_{\mathbb{Q}}$.

Definition 3.2.11. A **rational Ninebrane class** is a cohomology class $N_{\mathbb{Q}} \in H^{11}(T;\mathbb{Q})$ such that $\iota_x^* N = a_{11} \in H^{11}(2\mathrm{Spin};\mathbb{Q})$ for each inclusion $\iota_x : 2\mathrm{Spin} \to T$.

Now, just as we did in the case of Fivebrane structures, we will relate these classes to ones on the underlying Spin bundle. In order to do this, as we compared degree 7 rational cohomology between Spin and String we need to compare the degree 11 rational cohomology of Spin and 2Spin.

Lemma 3.2.12. *The map* $\rho : 2\mathrm{Spin} \to \mathrm{Spin}$ *induces an isomorphism* $\rho^* : H^{11}(\mathrm{Spin};\mathbb{Q}) \xrightarrow{\cong} H^{11}(2\mathrm{Spin};\mathbb{Q})$.

Proof. The first thing to notice is that, as $\pi_n(\mathrm{O}) \cong \mathbb{Z}_2$ for $n = 0, 1 (\mathrm{mod}\ 8)$, then $\mathrm{O}\langle 9\rangle$ and

$2Spin$ each arise from killing a homotopy class with $\mathbb{Z}_2$ torsion. Thus by the Leray-Hirsch Theorem $H^*(\text{Fivebrane};\mathbb{Q}) \cong H^*(\mathrm{O}\langle 9\rangle;\mathbb{R}) \cong H^*(\text{2Spin};\mathbb{Q})$ since $H^*(K(\mathbb{Z}_2,n);\mathbb{Q}) = 0$. Now using the following fibrations

$$\text{Fivebrane} \to \text{String} \to K(\mathbb{Z},7)$$

$$K(\mathbb{Z},2) \to \text{String} \to \text{Spin},$$

we look at the corresponding Serre Spectral sequences. For the first fibration, as $H^k(\text{Fivebrane};\mathbb{Q}) = 0$ for $k \leqslant 10$, then it follows that $E^{p,q} = 0$ for $q \leqslant 10$. Hence

$$H^{11}(\text{String};\mathbb{Q}) \cong \bigoplus_{p+q=11} E_\infty^{p,q} = E_\infty^{0,11} = E_2^{0,11} \cong H^{11}(\text{Fivebrane}(n);\mathbb{Q}).$$

Finally, for the second fibration, the second page of our spectral sequence is as follows:

$H^*(K(\mathbb{Z},2))$								
	0	0	0	0	0	0	0	0
4	x^2	0	a_3x^2	0	a_7x^2	0	$a_3a_7x^2$	$a_{11}x^2$
3	0	0	0	0	0	0	0	0
2	x	0	a_3x	0	a_7x	0	a_3a_7x	$a_{11}x$
1	0	0	0	0	0	0	0	0
0	$\mathbb{Q}$	0	a_3	0	a_7	0	a_3a_7	a_{11}
	0	$\cdots$	3	$\cdots$	7	$\cdots$	10	11

$H^*(\text{Spin})$

Now $H^{11}(\text{String};\mathbb{Q}) \cong \bigoplus_{p+q=11} E_\infty^{p,q} = E_\infty^{11,0} \oplus E_\infty^{10,1} \oplus E_\infty^{7,4} \oplus E_\infty^{3,8} \oplus E_\infty^{0,11}$. Also note that as $H^3(\text{String};\mathbb{Q}) = 0$, then the differential $d_3 : E_3^{0,2} \to E_3^{0,3}$ is an isomorphism and thus $d_3(x^n) = nx^{n-1}a_3$. Consequently d_3 is an isomorphism between the zeroth and third columns, and hence it follows that $E_\infty^{0,q} = E^{3,q} = 0$ for every $q > 0$. Furthermore we have $d_3(a_7x^2) = 2a_3a_7x \neq 0$ and thus $E_\infty^{7,4} = 0$. Therefore it follows that $H^{11}(\text{String};\mathbb{Q}) \cong E_\infty^{11,0} \cong H^{11}(\text{Spin};\mathbb{Q})$. $\square$

Now paralleling the Fivebrane case, we can use this Lemma to relate rational Ninebrane classes to classes on the underlying Spin bundle.

Definition 3.2.13. A **rational Spin-Ninebrane class** is a cohomology class $N_{\mathbb{Q}}$ in $H^{11}(Q;\mathbb{Q})$ such that $\iota_x^* N_{\mathbb{Q}} = \tilde{a}_{11} \in H^{11}(\text{Spin};\mathbb{Q})$ for each $x \in M$.

Theorem 3.2.14. *1. For every rational Spin-Ninebrane class $N_{\mathbb{Q}} \in H^{11}(Q;\mathbb{Q})$, $\mu^* N_{\mathbb{Q}}$ is a rational Fivebrane class;*

2. *Any rational Ninebrane structure $N_{\mathbb{Q}} \in H^{11}(T;\mathbb{Q})$ can be described by a class in $H^{11}(Q;\mathbb{Q})$;*

3. *Two classes $N_{\mathbb{Q}}, N'_{\mathbb{Q}} \in H^{11}(Q;\mathbb{Q})$ will give the same rational Ninebrane structure if $N_{\mathbb{Q}} - N'_{\mathbb{Q}} = S \cdot \pi^*_{\text{Spin}}\phi + F \cdot \pi^*_{\text{Spin}}\psi$ where $S \in H^3(Q;\mathbb{Q})$ is the String structure class, $F \in H^7(Q;\mathbb{Q})$ is the Fivebrane structure class, $\phi \in H^8(M;\mathbb{Q})$, and $\psi \in H^4(M;\mathbb{Q})$.*

Proof. The proof follows along similar lines as the proof of Proposition 3.2.9. Given a 2Spin-bundle T over a manifold M, we have an induced Spin-bundle over M, by Lemma 2.0.14, induced by the fibration $\rho : 2\text{Spin} \to \text{Spin}$. By Lemma 3.2.12, we also know that this fibration induces an isomorphism on rational cohomology of degree 11. In keeping with our notation, we'll denote this induced Spin-bundle as Q. Now as before, we'll compare the Serre spectral sequences corresponding to the rational cohomology for both bundles. As $H^k(2\text{Spin};\mathbb{Q}) = 0$ for $0 < k < 11$, it follows easily that $H^{11}(T;\mathbb{Q}) = \mathbb{Q}[a_{11}] \oplus H^{11}(M;\mathbb{Q})$. Now for the bundle Q, the second page of the Serre spectral sequence is provided below.

We want to calculate the entries $E_\infty^{p,q}$ such that $p + q = 11$. It follows immediately that $E_\infty^{1,10} = E_\infty^{2,9} = E_\infty^{3,8} = E_\infty^{5,6} = E_\infty^{6,5} = E_\infty^{7,4} = E_\infty^{9,2} = E_\infty^{10,1} = 0$. Thus

$$H^{11}(T;\mathbb{Q}) \cong E_\infty^{11,0} \oplus E_\infty^{7,4} \oplus E_\infty^{3,8} \oplus E_\infty^{0,11}.$$

On inspection of the universal Spin-bundle, we find that $d_4(\tilde{a}_3) = b_4$, $d_8(\tilde{a}_7) = b_8$, and $d_{12}(\tilde{a}_{11}) = b_{12}$, where $b_i \in H^i(B\text{Spin};\mathbb{Q})$ and $\tilde{a}_i \in H^i(B\text{Spin};\mathbb{Q})$ are generators. We also find that for all other possible differentials, $d_r(a_i) = 0$. Using functoriality of the differential maps and using the classifying map of Q to compare with the universal Spin-bundle, it follows that $d_r(a_3) = 0$ for $r \neq 4$, $d_r(a_7) = 0$ for $r \neq 8$, and $d_r(a_{11}) = 0$ for $r \neq 12$. Then we may proceed along the same lines as in Proposition 3.2.9 and find that

$$\begin{aligned}
E_\infty^{11,0} &\cong \mathbb{Q}[a_{11}] \\
E_\infty^{7,4} &\cong \mathbb{Q}[a_7] \otimes H^4(M;\mathbb{Q}) \\
E_\infty^{3,8} &\cong \mathbb{Q}[a_3] \otimes H^8(M;\mathbb{Q}) \\
E_\infty^{0,11} &\cong H^{11}(M;\mathbb{Q})
\end{aligned}$$

and thus the theorem follows.

$H^*(\mathrm{Spin})$										
11	a_{11}									
10	$a_3 a_7$	0								
$\vdots$	0	0	0	0						
7	a_7	$\cdots$	$\cdots$	$a_7 H^3(M)$	$a_7 H^4(M)$					
$\vdots$	0	0	0	0	0	0				
3	a_3	$\cdots$	$\cdots$	$a_3 H^3(M)$	$a_3 H^4(M)$		$a_3 H^8(M)$			
$\vdots$	0	0	0	0	0	0	0	0		
0	$\mathbb{Q}$	0	$H^2(M)$	$H^3(M)$	$H^4(M)$		$H^8(M)$		$H^{11}(M)$	
	0	1	2	3	4	$\cdots$	8	$\cdots$	11	$H^*(M)$

□

This theorem shows again that rationally, for Ninebrane structures, most of the information is encoded in the underlying Spin bundle. While the kernel of the map which assigns rational Spin-Ninebrane classes to rational Ninebrane classes is larger, we still have a surjection. In fact, this process should extend further to higher geometric structures. The reason is that we are making use of the fact that rationally there is an isomorphism,

$$H^{12}(B\mathrm{Spin};\mathbb{Q})/(p_1,p_2) \cong H^{12}(B2\mathrm{Spin};\mathbb{Q}).$$

Through minimal models, it becomes clear that isomorphisms such as these continue to occur for higher connected covers of Spin. The difficulty in extending this definition then becomes more of a problem with determining the kernel of these maps.

4.0 DIFFERENTIAL COHOMOLOGY

4.1 DIFFERENTIAL CHARACTERS

In this section we introduce differential characters and give an overview of the constructions for Cheeger-Simons characters and Cheeger-Chern-Simons characters. We follow closely along the lines of Becker and Bär [1, 2]. The model for differential cohomology that is presented in [1] is a generalization to the model given by Cheeger and Simons in [5] which are known as differential characters. Becker and Bär's construction differs slightly in that they build these characters using what are called *geometric chains* as a model for homology. For our purposes, it will not be necessary to discuss the details of this model. However, the benefit of using this model is that it allows us to extend differential cohomology to the category of smooth spaces, and using the model for the String group given by Nikolaus, Sachse, and Wockel in [25], we can study its differential cohomology. This also allows us to study smooth principal String-bundles and to define the group $\widehat{H}^*(P;\Lambda)$ when $\pi : P \to M$ is a smooth principal String-bundle. We'll start by reviewing the definition of a smooth space and then proceed to introduce the version of differential characters given in [1].

4.1.1 Smooth Spaces

The model for differential cohomology in [1] is constructed using smooth homology and stratifold homology. This allows one to extend differential cohomology to a broader category of spaces. We recall these spaces here.

Definition 4.1.1. A **differential space** is a pair $(M, C^\infty(M))$ where M is a topological space and $C^\infty(M)$ is a subset of the set $C^0(M)$ of all continuous real-valued functions satisfying the following properties,

- (Initial topology): M has the weakest topology for which all the functions in $C^\infty(M)$ are continuous,
- (Locality): If $f \in C^0(M)$ and for every $x \in M$ there is an open neighborhood U of x and a function $g \in C^\infty(M)$ such that $f = g$ on U, then $f \in C^\infty(M)$,
- (Composition with smooth functions): If $f_1, \ldots, f_k \in C^\infty(M)$ and g is a smooth function defined on an open neighborhood of $f_1(M) \times \cdots \times f_k(M) \subset \mathbb{R}^k$, then $g \circ (f_1, \ldots, f_k) \in C^\infty(M)$.

Definition 4.1.2. We say that a differential space is a **smooth space** if all of the following hold:

- (*Continuous versus smooth singular (co-)homology*): The inclusion of the complex of smooth singular chains into that of continuous singular chains induces isomorphisms for the corresponding homology and cohomology theories,
- (*de Rham theorem*): Integration of differential forms induces an isomorphism from de Rham cohomology to smooth singular cohomology with real coefficients,
- (*Stratifold versus singular homology*): Pushing forward fundamental cycles induces an isomorphism from the bordism theory of oriented p-stratifolds to smooth singular homology theory with integral coefficients.

One important example is that smooth finite dimensional manifolds satisfy all of these properties and thus the concept of smooth spaces generalizes smooth manifolds. A more general definition for manifolds can be given by changing the types of spaces with which the manifold is modeled. A good reference for infinite dimensional manifolds is [18] where the authors define a smooth manifold as one which is modeled on what they call "convenient" locally convex vector space. One nice example of a convenient locally convex vector space is a Fréchet space. We briefly recall the definition here.

Definition 4.1.3. A **Fréchet space** is a complete locally convex Hausdorff metrizable vector space.

Now suppose V, W are Fréchet spaces and that $U \subset V$ is open. Then the function $f : U \subset V \to W$ is differentiable at $u \in U$ in the direction of $v \in V$ if the limit

$$Df(u; v) := \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}$$

exists. Furthermore we say that f is continuously differentiable on U if the limit exists for every $(u, v) \in U \times V$ and if the function $Df : U \times V \to W$ is continuous. We can ask whether Df is differentiable and continue this process. We say that f is smooth or C^∞ if all its derivatives exist and are continuous.

Definition 4.1.4. A **Fréchet manifold** is a Hausdorff space with a coordinate atlas where each chart takes values in some Fréchet space and all transition functions are smooth maps.

We have a few nice results for Fréchet manifolds which we'll use.

Theorem 4.1.5. *A Fréchet manifold is metrizable if and only if it is smoothly paracompact.*

Theorem 4.1.6. *A metrizable manifold has the homotopy type of a CW-complex. In particular, weak equivalences between metrizable manifolds are weak equivalences.*

Proofs of these theorems can be found in [26] as well as [25]. The major example that we'll be interested in is, as we mentioned above, the model given in [25] for the String group. By a smooth model for the String group, we mean the following.

Definition 4.1.7. Let G be a compact, simple and 1-connected Lie-group. A **smooth String group model** for G is a Lie group $\widehat{G}$ along with a smooth homomorphism

$$q : \widehat{G} \to G$$

such that q is a Serre fibration, $\pi_k(\widehat{G}) = 0$ for $k \leqslant 3$ and $\pi_i(q)$ is an isomorphism for $i > 3$.

It is important that we understand what is meant by smooth here. For [25] Lie group is modeled on arbitrary locally convex vector spaces. In fact they construct such a smooth String group model which they denote String_G and they prove the following.

Theorem 4.1.8 (Nikolaus, Sachse, Wöckel)**.** String_G *is a smooth string group model, it is metrizable, and there exists a Fréchet Lie group structure on* String_G*.*

Now our aim is to study the differential cohomological aspects of the String group and to use it in our study of higher geometric structures. In order to do this, we must show that the smooth String group model in [25] satisfies the definition for a smooth space in [1]. It is not too difficult to show that this object is a differentiable space. To show that this space is smooth, we'll need a few tools.

The first theorem we use comes from [17] and allows us to attack the question of when stratifold homology is isomorphic to the usual singular homology.

Theorem 4.1.9 (Kreck). *There exists a natural transformation*

$$\sigma : SH_k(X) \to H_k(X)$$

which is an isomorphism for all CW-complexes X and for all k. This natural transformation also commutes with the $\times$-product.

Combining this with Theorem 4.1.6 shows that the natural transformation is an isomorphism for String_G as well. We also have from [18], the following.

Theorem 4.1.10. *Let M be a smooth manifold which is smoothly paracompact. Then the de Rham cohomology of M coincides with the singular cohomology of M with coefficients in $\mathbb{R}$ via a canonical isomorphism induced by integration of p-forms over smooth singular simplices.*

Now as we noted above, smooth manifolds in [18] refer to those modeled on convenient vector spaces and that in particular Fréchet spaces fall under this designation. Thus it follows that the de Rham cohomology and the real singular cohomology on String_G are isomorphic. Now in the proof of this theorem, they actually use smooth singular cohomology with real coefficients to build the isomorphism between de Rham and singular cohomology. The argument they use also works for the integers and it follows that singular cohomology and smooth singular cohomology are isomorphic for String_G. The last thing to verify is that there is an isomorphism between smooth homology and singular homology. For this one only needs to show that smooth homology on Fréchet manifolds satisfies the Eilenberg-Steenrod axioms and that the homology of a point is just $\mathbb{Z}$. From this we get our desired result.

Proposition 4.1.11. *The model String_G satisfies definitions 4.1.1 and 4.1.2.*

Proof. The set of smooth function $C^\infty(\mathrm{String}_G; \mathbb{R})$ for the String_G model are defined using the Fréchet derivative. In other words, a function $f : \mathrm{String}_G \to \mathbb{R}$ is smooth if for each chart $\phi : U \to E_U$, the function $f \circ \phi^{-1} : \phi(E_U) \to \mathbb{R}$ is smooth. By construction then the set of smooth functions satisfies the properties of definition 4.1.1. We need to show that String_G also satisfies the properties of definition 4.1.2.

Using Theorem 4.1.5, String_G is a smoothly paracompact manifold. Then applying Theorem 4.1.10 gives us that the de Rham cohomology of String_G is isomorphic to real singular cohomology.

To see that continuous singular homology and smooth singular homology agree, we use that by Theorem 4.1.6 String_G has the homotopy type of a CW-complex. Now the set of smooth singular k-chains on String_G consists of smooth maps $\sigma : \Delta_k \to \mathrm{String}_G$ and we denote this homology as $H_*^{sm}(\mathrm{String}_G; \mathbb{Z})$. As $H_*^{sm}(\{*\}; \mathbb{Z})$ of a point is still just $\mathbb{Z}$, then this homology theory satisfies the dimension axiom. As an ordinary homology theory for a CW-complex is completely determined by the Eilenberg-Steenrod axioms and in fact $H_*^{sm}(\mathrm{String}_G; \mathbb{Z})$ is naturally isomorphic to $H_*(\mathrm{String}_G; \mathbb{Z})$.

In order to show that smooth singular cohomology and singular cohomology agree, we will use sheaf cohomology. Let C^q_∞ represent the sheaf associated to the presheaf given by

$$U \mapsto C^q_\infty(U, \mathbb{Z}),$$

of smooth singular q-cochains. Then the following sequence of sheaves

$$\mathbb{Z} \to C^0_\infty \to C^1_\infty \to C^2_\infty \to \cdots$$

is an acyclic resolution of the constant sheaf $\mathbb{Z}$. That the sequence is exact follows from the fact that given some open set U which is diffeomorphic to a radial neighborhood in the modeling Fréchet space, then U is smoothly contractible to a point. Then on U the complex of presheafs

$$\cdots \to C^{q-1}_\infty(U, \mathbb{Z}) \xrightarrow{\delta^*} C^q_\infty(U, \mathbb{Z}) \xrightarrow{\delta^*} C^{q+1}_\infty(U, \mathbb{Z}) \to \cdots \tag{4.1}$$

is exact since $H^q(U; \mathbb{Z}) \cong H^q(\{*\}; \mathbb{Z}) = 0$ for $q > 0$. From this we have that the same is true at the level of stalks and thus the complex C^q_∞ is exact for positive degrees. For degree 0, since String_G is path connected then we have that $\mathrm{Ker}(\delta^* : C^0_\infty \to C^1_\infty) = \mathbb{Z}$ as any two points can be connected by a smooth path. Recall that a sheaf $\mathcal{F}$ is flasque if, for each inclusion $V \subseteq U$ of open sets, the restriction morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Moreover it is a classic result from sheaf theory that flasque sheaves are acyclic. Now since every smooth function $\sigma : \Delta_q \to V$ is also a smooth function into U, it follows that the sheaf C_∞^q is flasque. Thus (4.1) is indeed an acyclic resolution of the constant sheaf $\mathbb{Z}$, and thus we can conclude that smooth singular cohomology is naturally isomorphic to singular cohomology.

Finally, using again the fact that String_G has the homotopy type of a CW-complex, it follows from Theorem 4.1.9 that the stratifold homology and singular homology of String_G are isomorphic. Hence we can conclude that String_G is a smooth space. □

4.1.2 Differential Characters

Here we give the definition for differential characters and then proceed to review some of the useful and important properties of these objects. One highlight is that here we will be studying the general case where characters take values in $\mathbb{R}/\Lambda$, and where Λ is a completely disconnected subgroup of $\mathbb{R}$. We will usully be considering the case when $\Lambda = \mathbb{Z}$ or $\mathbb{Q}$ but there are other coefficients which would be useful for certain purposes.

Definition 4.1.12. For a smooth space M, the **group of differential characters** of M is defined as

$$\widehat{H}^k(M;\Lambda) := \left\{ h \in \mathrm{Hom}(Z_{k-1}(M;\mathbb{Z})) \to \mathbb{R}/\Lambda \mid h(\partial c) \equiv_\Lambda \int_c \omega, \forall c \in C_k(M;\mathbb{Z}) \right\}.$$

In other words, they are morphisms from $(k-1)$-cycles to $\mathbb{R}/\Lambda$ such that on boundaries they agree modulo Λ to the integration of a form ω. The form ω is unique and we define the curvature of a differential character h to be $\mathrm{curv}(h) := \omega$. Moreover the form ω is closed with Λ-periods, $\omega \in \Omega^k_\Lambda(M)$.

In order to have a better understanding of what differential characters are, we consider several maps which relate these characters to real and integral cohomology classes. The first map defines the characteristic class corresponding to a character $h \in \widehat{H}^k(M;\Lambda)$,

$$I : \widehat{H}^k(M;\Lambda) \to H^k(M;\Lambda).$$

In order to define this map we first note that, as the set of $Z_{k-1}(M;\mathbb{Z})$ of cocycles is a free $\mathbb{Z}$-module, then the functor $\mathrm{Hom}(Z_{k-1}(M;\mathbb{Z}),-)$ is exact. Thus applying this functor to the short exact sequence of coefficients

$$1 \to \Lambda \xrightarrow{i} \mathbb{R} \xrightarrow{\mathrm{mod}\ \Lambda} \mathbb{R}/\Lambda \to 1$$

gives a short exact sequence

$$1 \to \mathrm{Hom}(Z_{k-1}(M;\mathbb{Z}),\Lambda) \to \mathrm{Hom}(Z_{k-1}(M;\mathbb{Z}),\mathbb{R}) \to \mathrm{Hom}(Z_{k-1}(M;\mathbb{Z}),\mathbb{R}/\Lambda) \to 1.$$

Consequently, a character h has a lift to a morphism $\tilde{h} \in \mathrm{Hom}(Z_{k-1}(M;\mathbb{Z}),\mathbb{R})$ such that

$$h(z) = \tilde{h}(z) \bmod \Lambda$$

for any $z \in Z_{k-1}(M;\mathbb{Z})$. We then define a map $\mu^{\tilde{h}} : C_k(M;\mathbb{Z}) \to \Lambda$ as

$$c \mapsto \int_c \mathrm{curv}(h) - \tilde{h}(\partial c).$$

By reducing modulo Λ, we have

$$\begin{aligned}\left(\int_c \mathrm{curv}(h) - \tilde{h}(\partial c)\right) \bmod \Lambda &= \left(\int_c \mathrm{curv}(h)\right) \bmod \Lambda - (\tilde{h}(\partial c)) \bmod \Lambda \\ &= h(\partial c) - h(\partial c) \\ &= 0.\end{aligned}$$

Hence $\mu^{\tilde{h}} \in \mathrm{Hom}(C_k(M;\mathbb{Z}),\Lambda)$. We also have

$$\delta\mu^{\tilde{h}}(c) = \mu^{\tilde{h}}(\partial c) = \int_{\partial c} \mathrm{curv}(h) - \tilde{h}(\partial^2 c) = 0$$

and thus $\mu^{\tilde{h}}$ describes a class $[\mu^{\tilde{h}}] \in H^k(M;\Lambda)$.

We define a map $a : \Omega^{k-1}(M) \to \hat{H}^k(M;\Lambda)$ which associates to each form $\eta \in \Omega^{k-1}(M)$ the homomorphism

$$a(\eta) : z \mapsto \left(\int_z \eta\right) \bmod \Lambda.$$

Evaluating $a(\eta)$ on boundaries, we find

$$a(\eta)(\partial c) = \left(\int_{\partial c} \eta\right) \bmod \Lambda = \left(\int_c d\eta\right) \bmod \Lambda.$$

Hence $\operatorname{curv}(a(\eta)) = d\eta$.

Next we notice that $\int_z \eta$ defines a lift for $a(\eta)$. Then composition of a with I is

$$I(a(\eta)) = \left[\int_c d\eta - \int_{\partial c} \eta\right] = 0.$$

Furthermore, if $\eta \in \Omega_\Lambda(M)$, then $\int_c \eta \in \Lambda$ and thus $a(\eta) = 0$. Thus a descends to a morphism

$$a : \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_\Lambda(M)} \to \widehat{H}^k(M;\Lambda).$$

In fact, the topogical trivialization map and the characteristic map fit into a short exact sequence. This is one of three short exact sequences for differential cohomology originally introduced in [5] that we will use heavily.

Proposition 4.1.13. *The following are exact sequences.*

$$0 \to H^{k-1}(M;\mathbb{R}/\Lambda) \xrightarrow{j} \widehat{H}^k(M;\Lambda) \xrightarrow{curv} \Omega^{k+1}_\Lambda(M) \to 0 \tag{4.2}$$

$$0 \to \Omega^{k-1}(M)/\Omega^{k-1}_\Lambda(M) \xrightarrow{a} \widehat{H}^k(M;\Lambda) \xrightarrow{I} H^k(M;\Lambda) \to 0 \tag{4.3}$$

$$0 \to H^{k-1}(M;\mathbb{R})/H^{k-1}(M;\mathbb{R})_\Lambda \to \widehat{H}^k(M;\Lambda) \to R^k(M;\Lambda) \to 0 \tag{4.4}$$

In the last sequence, the group $R^k(M;\Lambda)$ is given by

$$R^k(M;\Lambda) := \{(\omega, u) \in \Omega^k_\Lambda(M) \times H^k(M;\Lambda) \mid r(u) = [\omega]_{\mathrm{dR}}\}.$$

In other words, $R^k(M;\Lambda)$ is the pullback of the following diagram,

$$\begin{array}{ccc} & & \Omega^k_\Lambda(M) \\ & & \downarrow{\scriptstyle \int} \\ H^k(M;\Lambda) & \xrightarrow{h} & \operatorname{Hom}(H_k(M;\mathbb{Z}),\Lambda) \end{array}$$

where the map $\int$ sends a differential form to the homomorphism represented by integrating the form along a cocycle representative of a homology class. Since we are considering forms with Λ-periods, this map is well-defined. The map h comes from the universal coefficients theorem. Alternatively, there is an isomorphism

$$H^k(M;\mathbb{R})_\Lambda \cong \mathrm{Hom}(H_k(M;\mathbb{Z}),\Lambda)$$

where we've set $H^k(M;\mathbb{R})_\Lambda := \mathrm{Im}(H^k(M;\Lambda) \to H^k(M;\mathbb{R})$ and under this isomorphism we have an equivalent pullback diagram where the map $\int$ becomes the map dR which sends a form to its representative combined with the de Rham isomorphism, and the map h becomes the map including cohomology with Λ coefficients into real cohomology. A nice result which can be found in [1] is that these short exact sequences fit into the following diagram where each row and each column is exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^{k-1}(M;\mathbb{R})}{H^{k-1}(M;\mathbb{R})_\Lambda} & \longrightarrow & \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_\Lambda(M)} & \xrightarrow{d} & d\Omega^{k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow & & \\
0 & \longrightarrow & H^{k-1}(M;\mathbb{R}/\Lambda) & \xrightarrow{j} & \hat{H}^k(M;\Lambda) & \xrightarrow{\text{curv}} & \Omega^k_\Lambda(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle I} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{k-1}(M;\mathbb{Z}),\Lambda) & \longrightarrow & H^k(M;\Lambda) & \longrightarrow & \mathrm{Hom}(H_k(M;\mathbb{Z}),\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This diagram is related to the diamond diagram (see [4, 34]). As we will be interested in studying differential cohomology for different choices of Λ, it is important to understand how one might relate differential cohomology for two different choices of Λ. In [27], the following was proven when we have an inclusion $\Lambda_1 \subset \Lambda_2$.

Proposition 4.1.14. *Let $\Lambda_1 \subset \Lambda_2$ be two proper subgroups of $\mathbb{R}$. Then the inclusion $\Lambda_1 \overset{i}{\hookrightarrow} \Lambda_2 \to \mathbb{R}$ induces a long exact sequence*

$$\cdots \longrightarrow H^{k-2}(M;\mathbb{R}/\Lambda_2) \xrightarrow{\beta} H^{k-1}(M;\Lambda_2/\Lambda_1) \longrightarrow \widehat{H}^k(M;\Lambda_1) \xrightarrow{\hat{i}} \widehat{H}^k(M;\Lambda_2) \xrightarrow[I \bmod \Lambda_1]{} H^k(M;\Lambda_2/\Lambda_1) \xrightarrow[\beta]{} H^k(M;\Lambda_1) \longrightarrow \cdots$$

and a short exact sequence

$$0 \to \text{Ker } i_* \to \widehat{H}^k(M;\Lambda_1) \xrightarrow{\hat{i}} \widehat{H}^k(M;\Lambda_2) \to \frac{H^k(M;\Lambda_2)}{H^k(M;\Lambda_1)} \to 0,$$

where $i_* : H^{k-1}(M;\mathbb{R}/\Lambda_1) \to H^{k-1}(M;\mathbb{R}/\Lambda_2)$*.*

4.1.3 Relative Differential Characters

We now consider an extension of relative cohomology to differential characters. This will be needed in the next section in order to define the Cheeger-Chern-Simons characters. Again, we follow along the lines of [2]. To begin we recall the definition for the mapping cone complex for a map $\phi : P \to M$.

Definition 4.1.15. For a cochain complex A^* and a map $\phi : P \to M$, we define the **mapping cone complex** of ϕ, $(A^*(\phi), d_\phi)$ by

$$A^k(\phi) = A^k(M) \oplus A^{k-1}(P)$$

$$d_\phi = (d_M, \phi^* - d_P) : A^k(\phi) \to A^{k+1}(\phi)$$

where d_M and d_P refer to the differentials for the cochain complex $A^*(M)$ and $A^*(P)$, respectively. We let $H^*(A^*(\phi))$ denote the corresponding cohomology.

The mapping cone complex fits into a short exact sequence of complexes

$$0 \to A^*(P) \xrightarrow{i} A^{*+1}(\phi) \xrightarrow{p} A^{*+1}(M) \to 0$$

which induces a long exact sequence in cohomology

$$\cdots \to H^k(A^*(M)) \xrightarrow{\phi^*} H^k(A^*(P)) \xrightarrow{i} H^{k+1}(A^*(\phi)) \xrightarrow{p} H^{k+1}(A^*(M)) \to \cdots .$$

Using the complex $C^*(M;\Lambda)$ of smooth singular cochains allows us to define the mapping cohomology for singular cohomology with Λ coefficients. If ϕ is just the inclusion of a subspace $A \hookrightarrow M$, then the resulting cohomology is just the usual relative cohomology (this can be checked easily by applying the Five Lemma). As with singular cohomology, we have an equivalent definition for relative differential characters which refine the mapping cone cohomology.

Definition 4.1.16. The **group of relative differential characters** is defined as the set

$$\widehat{H}^k(\phi;\Lambda) = \left\{ h \in \mathrm{Hom}(Z_{k-1}(\phi), \mathbb{R}/\Lambda) \mid h(\partial_\phi(a,b)) \equiv_\Lambda \int_{(a,b)} (\omega,\nu),\ \forall (a,b) \in C_k(\phi,\mathbb{Z}) \right\}$$

where $\int_{(a,b)}(\omega,\nu) := \int_a \omega + \int_b \nu$.

Again the pair (ω,ν) is unique and has Λ-periods, and we define the form $\mathrm{curv}(h) := \omega \in \Omega^k(M)$ to be the curvature of a relative differential character h and the form $\mathrm{cov}(h) := \nu \in \Omega^{k-1}(P)$ to be its covariant derivative. As with differential characters, we can define a map which assigns a characteristic class $I(h) \in H^k(\phi;\Lambda)$, and we construct this map in a similar fashion by defining a cocycle

$$\mu^{\tilde{h}} := (\mathrm{curv},\ \mathrm{cov})(h) - \delta_\phi \tilde{h}$$

where $\tilde{h}$ is a real lift of h. Similarly we also can define the topological trivialization $a_\phi : \Omega^{k-1}(\phi) \to \hat{H}^k(\phi;\Lambda)$ where we define

$$a_\phi(\omega,\nu)(s,t) = \left(\int_{(s,t)} (\omega,\nu) \right) \bmod \Lambda.$$

As before, these maps fit into a pair of short exact sequences

$$0 \to \frac{\Omega^{k-1}(\phi)}{\Omega^{k-1}_\Lambda(\phi} \xrightarrow{a_\phi} \widehat{H}^k(\phi;\Lambda) \xrightarrow{I} H^k(\phi;\Lambda) \to 0$$

$$0 \to H^{k-1}(\phi;\mathbb{R}/\Lambda) \xrightarrow{j} \widehat{H}^k(\phi;\Lambda) \xrightarrow{(\mathrm{curv},\mathrm{cov})} \Omega^k_\Lambda(\phi) \to 0$$

and there is a long exact sequence

$$\cdots \longrightarrow H^{k-2}(P;\mathbb{R}/\Lambda) \longrightarrow H^{k-1}(\phi;\mathbb{R}/\Lambda) \longrightarrow H^{k-1}(M;\mathbb{R}/\Lambda) \xrightarrow{j \circ \phi^*}$$

$$\widehat{H}^k(P;\Lambda) \xrightarrow{\breve{\imath}_\phi} \widehat{H}^{k+1}(\phi;\Lambda) \xrightarrow{\breve{p}_\phi} \widehat{H}^{k+1}(M;\Lambda) \xrightarrow{\phi^* \circ c}$$

$$H^{k+1}(P;\Lambda) \longrightarrow H^{k+2}(\phi;\Lambda) \longrightarrow H^{k+2}(M;\Lambda) \longrightarrow \cdots$$

which extends to the left as the long exact sequence of the mapping cone with $\mathbb{R}/\Lambda$ coefficients and to the right as the long exact sequence with Λ coefficients. We can also define the group $R^k(\phi;\Lambda)$ as

$$R^k(\phi;\Lambda) := \{(\omega,\nu,u_\phi) \in \Omega^k_\Lambda(\phi) \times H^k(\phi;\Lambda) \mid r(u_\phi) = [(\omega,\nu)]_{\mathrm{dR}}\}$$

which can be thought of as a pullback just like before. Furthermore this group fits into the following short exact sequence

$$0 \to H^k(\phi;\mathbb{R})/H^k(\phi;\mathbb{R})_\Lambda \to \widehat{H}^k(\phi;\Lambda) \to R^k(\phi;\Lambda) \to 0.$$

Thus the relative characters behave in a similar fashion to the ordinary differential characters of Cheeger and Simons. At this point, we've introduced most of the important aspects of the differential character model for differential cohomology. For most of what follows, we will favor this model. However for some of the constructions later, we will also need the Hopkins-Singer model for differential cohomology [14].

4.2 DIFFERENTIAL COCYCLES

The group of differential characters is just one model for differential cohomology. Another model which will be of interest to us is that of differential cocycles which was introduced by Hopkins and Singer in [14]. In this paper they construct a complex $\check{C}(q)^*(M)$ which fits into the homotopy cartesian square

$$\begin{array}{ccc} \check{C}(q)^*(M) & \longrightarrow & \Omega^{*\geq q}(M) \\ \downarrow & & \downarrow \\ C^*(M;\Lambda) & \longrightarrow & C^*(M;\mathbb{R}). \end{array}$$

where

$$\check{C}(q)^n(M) = \begin{cases} C^n(M;\Lambda) \times C^{n-1}(M;\mathbb{R}) \times \Omega^n(M) & n \geqslant q \\ C^n(M;\Lambda) \times C^{n-1}(M;\mathbb{R}) & n < q \end{cases}$$

with differential defined by

$$d(c,h,\omega) = (\delta c, \omega - c - \delta h, d\omega)$$
$$d(c,h) = \begin{cases} (\delta c, -c - \delta h, 0) & (c,h) \in \check{C}(q)^{q-1} \\ (\delta c, -c - \delta h) & \text{otherwise.} \end{cases}$$

They go on to show that the cohomology of this complex is given by natural isomorphisms

$$\check{H}(q)^n(M) \cong \begin{cases} H^n(M;\Lambda) & n > q \\ H^{n-1}(M;\mathbb{R}/\Lambda) & n < q, \end{cases}$$

but most importantly they show that when $q = n$ that the cohomology fits into several nice short exact sequences. These are exactly the same sequences from Proposition 4.1.13 and thus $\check{H}(n)^n(M;\Lambda)$ fits into the large square of short exact sequences. It follows then from Simons and Sullivan's work in [34] that there is a natural isomorphism

$$\widehat{H}^n(M;\Lambda) \cong \check{H}(n)^n(M;\Lambda).$$

To see how this isomorphism is defined, take a cocycle representative for a class in $\check{H}(n)^n(M;\Lambda)$. This is a triple (c,h,ω) satisfying

$$\delta c = 0$$
$$d\omega = 0$$
$$\delta h = \omega - c.$$

Then we can define a homomorphism

$$\chi : Z_{n-1} \to \mathbb{R}/\Lambda$$

by

$$\chi(z) := h(z) \bmod \Lambda.$$

This gives a differential character with curvature ω, as for any chain $b \in C_k(X;\Lambda)$, we have

$$\left(\chi(\partial b) - \int_b \omega\right) \bmod \Lambda = \left(\delta h(b) - \int_b \omega\right) \bmod \Lambda = (c(b)) \bmod \Lambda = 0.$$

We find that the map

$$(c, h, \omega) \mapsto \chi$$

describes the isomorphism between the cohomology of differential cocycles and differential characters.

4.3 CHERN-WEIL THEORY

One of the major results of Cheeger and Simons in [5] is the construction of a canonical lift of the Chern-Weil homomorphism. We will use these characters a fair amount and thus we will need to have some understanding of how the Chern-Weil homomorphism is constructed. Continuing the discussion from Section 2.0.2, let us recall what is meant by a connection on a principal G-bundle. Get G be a Lie group and let $\mathfrak{g}$ denote its Lie algebra. Given a principal G-bundle $\pi : P \to M$ and $u \in P$, we define the subspace $V_uP \subset T_uP$ to be the set of vectors which are tangent to the fiber G_p over $p = \pi(u)$. If $A \in \mathfrak{g}$, then $u \cdot \exp(tA)$ defines a path that lies in G_p. Thus the vector $A^\#$ given by $A^\#(f(u \cdot \exp(tA)) \mid_t = 0$ defines an element in V_uP and we have a map

$$\# : \mathfrak{g} \to V_uP.$$

In fact this map is an isomorphism. A connection then prescribes a continuous splitting of the tangent space $T_uP = V_uP \oplus H_uP$ and we call H_uP the horizontal vector space of T_uP. For our purposes, we think of a connection instead as a Lie algebra valued 1-form.

Definition 4.3.1. A **connection** on $\pi : P \to M$ is a Lie-algebra 1-form $\theta \in \mathfrak{g} \otimes T^*P$ such that

1. $\theta(A^\#) = A$ for $A \in \mathfrak{g}$
2. $R_g^*\theta = \mathrm{Ad}_{g^{-1}}\omega$ for $g \in G$.

Definition 4.3.2. The **curvature** F_θ of a connection θ is given by the covariant derivative $D\theta$.

Let $I^k(G)$ denote the set of invariant polynomials of G, i.e.

$$I^k(G) = \{\lambda : \mathfrak{g}^{\otimes^n} \to \mathbb{R} \mid \lambda \text{ is symmetric, bilinear, and } Ad_g^*\lambda = \lambda\}$$

Now suppose $\pi : (P, \theta) \to M$ is a principal G-bundle equipped with a connection θ. Let $F_\theta \in \Omega^2(M; Ad(P))$ denote the curvature 2-form corresponding to θ. Then the Chern-Weil homomorphism,

$$CW_\theta : I^k(G) \to \Omega^{2k}(M),$$

is defined by $CW_\theta(\lambda) = \lambda(F_\theta^k)$. In fact this homomorphism gives a closed form for each invariant polynomial and the de Rham class corresponding to this form is independent of the connection. Thus using the de Rham isomorphism, we get a well defined homomorphism

$$CW_\theta : I^k(G) \to H^{2k}(M; \mathbb{R}).$$

The form $CW_\theta(\lambda)$ is known as the Chern-Weil form of λ. Now as a result of this map being well defined, it follows that the forms corresponding to an invariant polynomial for two different connections must differ by an exact form. In other words, given any two connections θ_0, θ_1 on $\pi : P \to M$, the Chern-Simons form $CS(\theta_0, \theta_1, \lambda) \in \Omega^{2k-1}(M)$ is the differential form satisying $dCS(\theta_0, \theta_1, \lambda) = CW_{\theta_0}(\lambda) - CW_{\theta_1}(\lambda)$. There is another notion of the Chern-Simons form which is a $(2k-1)$-form on the total space P. To construct this version, one first pulls back the bundle $\pi : P \to M$ along π to get a principal G-bundle $\pi^* P \to P$. This bundle is trivial as it admits a global section given by the map

$$\sigma : P \to \pi^* P, \quad p \mapsto (p, p).$$

Thus there is bundle isomorphism between $\pi^* P \cong G \times P$. The trivial bundle $G \times P$ has a canonical flat connection. Pulling this connection back along the isomorphism to $\pi^* P$ defines a flat connection on $\pi^* P$ which we will denote as θ_{taut}. Thus $\pi^*\theta$ and θ_{taut} are two connections on $\pi^* P$ and we define the Chern-Simons form on P by

$$CS_\theta(\lambda) := CS(\theta_{taut}, \pi^*\theta, \lambda).$$

Since θ_{taut} is flat, then $CW_{\theta_{taut}}(\lambda)$ is always zero and thus

$$dCS_\theta(\lambda) = CW_{\pi^*\theta}(\lambda) = \pi^* CW_\theta(\lambda) \in \Omega^{2k-1}(P).$$

4.4 CHEEGER-SIMONS AND CHEEGER-CHERN-SIMONS CHARACTERS

There is a universal connection Θ on the universal bundle $\pi_{EG} : EG \to BG$ such that given any principal G-bundle $\pi : (P, \theta) \to M$ with connection θ, then there is a classifying map $f : M \to BG$ such that $f^*\Theta = \theta$, and then for the connection Θ, the Chern-Weil homomorphism describes a map $I^k(G) \to H^{2k}(BG; \mathbb{R})$. We put these maps together to get the following diagram

$$\begin{array}{ccccc} I^k(G) & \xrightarrow{CW_\Theta} & H^{2k}(BG;\mathbb{R}) & \longleftarrow & H^{2k}(BG;\Lambda) \\ \downarrow{\scriptstyle CW_\theta} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \Omega^{2k}(X) & \xrightarrow{dR} & H^{2k}(X;\mathbb{R}) & \longleftarrow & H^{2k}(X;\Lambda)\,. \end{array}$$

Define the set $K^{2k}(G; \Lambda)$ to be the pullback of the top row and $R^{2k}(M; \mathbb{Z})$ to be the pullback of the bottom row.

$$K^{2k}(G; \Lambda) := \{(\lambda, u) \in I^k(G) \times H^{2k}(BG; \Lambda) \mid CW(\lambda) = [u]_{\mathbb{R}}\},$$
$$R^n(M; \Lambda) := \{(\omega, w) \in \Omega^n_0(M) \times H^n(M; \Lambda) \mid CW_\theta(\lambda) = [w]_{\mathbb{R}}\}.$$

As we mentioned before, in [5] Cheeger and Simons show that the map

$$CW \times f^* : K^{2k}(G; \Lambda) \to R^{2k}(M; \Lambda)$$

lifts to a map

$$\widehat{CW}_\theta : K^{2k}(G; \Lambda) \to \widehat{H}^{2k}(M; \Lambda),$$

i.e. we have the following lifting diagram,

$$\begin{array}{ccc} & & \hat{H}^{2k}(X;\Lambda) \\ & \overset{\widehat{CW}_\theta}{\nearrow} & \downarrow{\scriptstyle (\mathrm{curv},\mathrm{c})} \\ K^{2k}(G;\Lambda) & \xrightarrow[CW_\theta \times f^*]{} & R^{2k}(X;\Lambda)\,. \end{array}$$

To put this more precisely, we have the following theorem from [5].

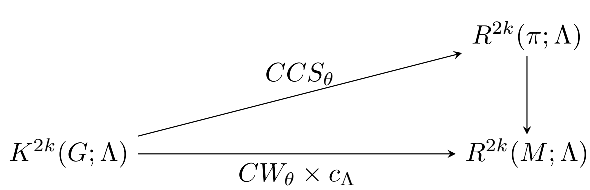
Theorem 4.4.1. *Let $(\omega, u) \in K^{2k}(G; \Lambda)$. For each principal G-bundle with connection, $\pi : (P, \theta) \to M$, there exists a unique element $\widehat{CW}_\theta(\lambda, u) \in \widehat{H}^{2k}(M; \Lambda)$ satisfying:*

1. *curv*$(\widehat{CW}_\theta(\lambda, u)) = CW_\theta(\lambda)$;
2. $I(\widehat{CW}_\theta(\lambda, u)) = u(P)$;
3. *if $\pi' : (P', \theta') \to M'$ is another principal G-bundle with connection and $\phi : (P, \theta) \to (P', \theta')$ is a connection preserving bundle morphism, then $\phi^*(\widehat{CW}'_\theta(\lambda, u) = \widehat{CW}_\theta(\lambda, u)$.*

We'll give here a sketch of how the proof in [5] goes. There is a universal connection on the universal bundle which classifies principal G-bundles with connection. Letting F be the curvature of this connection, and letting $(\lambda, u) \in K^{2k}(G; \Lambda)$, Chern-Weil theory says that the pair $(\lambda(F^k), u)$ represents an element in $R^{2k}(BG; \Lambda)$. We know also that $H^{odd}(BG; \mathbb{R}) = 0$ and thus from the proposition, the map $(\text{curv}, I) : \widehat{H}^{2k}(BG; \Lambda) \to R^{2k}(BG; \Lambda)$ is an isomorphism. Thus we define $\widehat{CW}_\theta(\lambda, u) = (\text{curv}, I)^{-1}(\lambda(F^k), u)$. Then for G-bundles with connection, these characters are defined via pullback. To fully prove the theorem, one needs to introduce the notion of classifying objects. The reason for this is that it is not clear whether $\widehat{H}^{2k}(BG; \Lambda)$ is well-defined and thus one introduces objects which approximate it. The technical parts deal with showing that this construction is independent of the choice of classifying objects. We will call elements in the image of $\widehat{CW}_\theta$ Cheeger-Simons characters.

In [2], Becker introduces the notion of the Cheeger-Chern-Simons character which lives in the relative differential cohomology of the map $\pi : P \to M$. To construct these characters, we have to construct two lifts. The first is a lift of the map $CW_\theta \times f^* : K^{2k}(BG; \Lambda) \to R^{2k}(M; \Lambda)$ to $R^{2k}(\pi; \Lambda)$ where we are letting c_Λ denote the map which represents pulling back along the classifying map. We recreate the proofs from [2] to ensure that these theorems extend to Λ coefficients in general.

Proposition 4.4.2. *Let $\pi : (P, \theta) \to M$ be a principal G-bundle with connection. The Chern-Weil map has a canonical natural lift CCS_θ such that the following diagram commutes.*



Proof. Consider the long exact sequence for the mapping cone complex of $\pi_{EG} : EG \to BG$.

$$\cdots \to H^{2k-1}(EG; \Lambda) \to H^{2k}(\pi_{EG}; \Lambda) \xrightarrow{\cong} H^{2k}(BG; \Lambda) \to H^{2k}(EG; \Lambda) \to \cdots$$

Since EG is contractible, it follows that we have an isomorphism $p : H^{2k}(\pi_{EG}; \Lambda) \xrightarrow{\cong} H^{2k}(BG; \Lambda)$. Given a universal characteristic class $u \in H^{2k}(BG : \Lambda)$, we let $\tilde{u} := p^{-1}(u) \in H^{2k}(\pi_{EG}; \Lambda)$. Define the lift $CCS_\theta : K^{2k}(BG; \Lambda) \to R^{2k}(\pi; \Lambda)$ by

$$CCS_\theta(\lambda, u) := (CW_\theta(\lambda), CS_\theta(\lambda), f^*\tilde{u}).$$

As each component is natural with respect to pullbacks, then so is the whole function, and with respect to the universal connection Θ on $\pi_{EG} : EG \to BG$, we have

$$\begin{aligned} CCS_\theta(\lambda, u) &= f^* CCS_\Theta(\lambda, u) \\ &= f^*(CW_\Theta(\lambda), CS_\Theta(\lambda), \tilde{u}). \end{aligned}$$

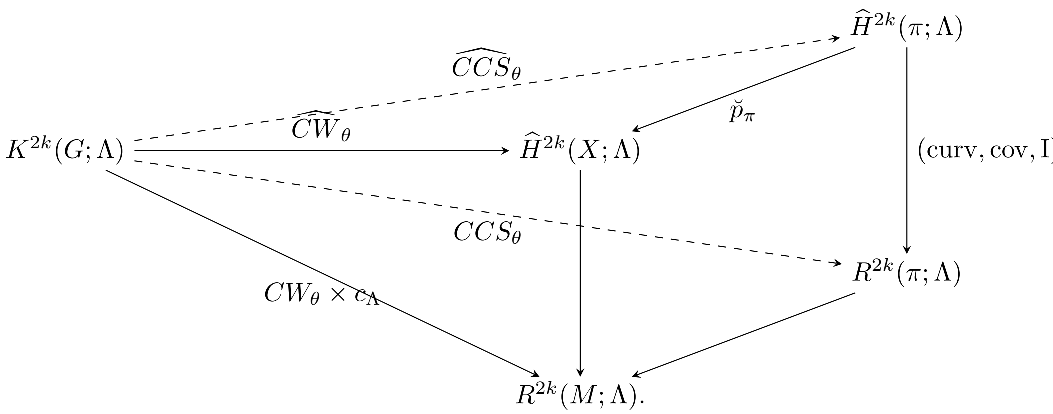
Furthermore, composing this map with the forgetful map $R^{2k}(\pi; \mathbb{Z}) \to R^{2k}(M; \mathbb{Z})$ gives the map $CW_\theta \times f^*$. It remains to show that this map is well defined. Thus we must show that $(CW_\theta(\lambda), CS_\theta(\lambda) \in \Omega^{2k}_\Lambda(\pi)$ and that $[CW_\theta(\lambda), CS_\theta(\lambda)]_{dR} = [f^*\tilde{u}]_\mathbb{R}$. To do this we'll prove it for $(CW_\Theta(\lambda), CS_\Theta(\lambda), \tilde{u})$ on the universal bundle. Since $dCS_\Theta(\lambda) = dCS(\pi^*_{EG}\Theta, \theta_{taut}, \lambda) = \pi^*_{EG} CW_\Theta(\lambda)$ and since $CW_\Theta(\lambda)$ is a closed form, it follows that $d_{\pi_{EG}}(CW_\Theta(\lambda), CS_\Theta(\lambda) = 0$. By definition of $(\lambda, u) \in K^{2k}(G; \Lambda)$, $CW_\Theta(\lambda)$ has Λ periods. □

From here, Becker shows that this map $CCS_\theta : K^{2k}(G; \Lambda) \to R^{2k}(\pi; \Lambda)$ in turn lifts to a map $\widehat{CCS}_\theta : K^{2k}(G; \Lambda) \to \widehat{H}^{2k}(\pi; \Lambda)$. To make this statement precise as well as to elaborate on what this implies, he provides us with the following proposition.

Proposition 4.4.3. *Let $(\lambda, u) \in K^{2k}(G; \Lambda)$. Then for any principal G-bundle with connection $\pi : (P, \theta) \to M$, there is a unique relative differential character $\widehat{CCS}_\theta(\lambda, u) \in \widehat{H}^{2k}(\pi; \Lambda)$ such that*

1. *$(curv, cov, I)(\widehat{CCS}_\theta(\lambda, u) = (CW(\lambda), CS_\theta(\lambda), \tilde{u}(P))$,*
2. *$\breve{p}_\pi(\widehat{CCS}_\theta(\lambda, u)) = \widehat{CW}_\theta(\lambda, \tilde{u})$,*
3. *For any smooth map $f : M' \to M$, $f^*\widehat{CCS}_\theta(\lambda, u) = \widehat{CCS}_{f^*\theta}(\lambda, u)$,*

and thus the following diagram commutes



The important point to note here is that the differential character $\widehat{CCS}_\theta(\Lambda, u)$ is mapped to $\widehat{CW}_\theta(\lambda, u)$. In defining differential u-trivializtions, this character will play a similar role as the transgression of the obstruction for a geometric structure does in the topological setting. Now while the relative cohomology of the bundle map can be identified with the cohomology of the fiber when classifying geometric structures, the same does not necessarily hold in the differential setting. However this is not as much of a concern as with differential trivializations we will be looking to classify topological trivializations of $\widehat{CW}_\theta(\lambda, u)$ and are not as concerned with classifying when $\widehat{CW}_\theta(\lambda, u) = 0$.

4.4.1 Differential Transgression

We also have a transgression in differential cohomology. Again, we follow Becker's perspective for the construction. As we used the long exact sequence for the mapping cone cohomology in constructing the transgression for ordinary cohomology, we use the differential cohomological analog to construct the differential version. Given a smooth map $f : E \to M$ and letting $E_x := f^{-1}(x)$ denote the fiber of f, consider the long exact sequences corresponding to the map f and $f_x : E_x \to \{x\}$, the restriction of f to the fiber over x. We obtain the following diagram

$$\begin{array}{ccccccc}
\widehat{H}^{n-1}(E;\Lambda) & \xrightarrow{\check{\iota}_f} & \widehat{H}^{n}(f;\Lambda) & \xrightarrow{\check{p}_f} & \widehat{H}^{n}(X;\Lambda) & \xrightarrow{c \circ f^*} & H^{n}(E;\Lambda) \\
\downarrow{\scriptstyle \iota_x^*} & & \downarrow{\scriptstyle \iota_{E_x}^*} & & \downarrow{\scriptstyle \iota_x^*} & & \\
\widehat{H}^{n-1}(E_x;\Lambda) & \xrightarrow{\check{\iota}_{f_x}} & \widehat{H}^{n}(f_x;\Lambda) & \longrightarrow & \widehat{H}^{n}(\{x\};\Lambda) & &
\end{array}$$

We define the set of transgressive differential characters as any element $h \in \widehat{H}^n(M;\Lambda)$ satisfying $I(f^*h) = 0$.

$$\widehat{\mathrm{Trans}}(f) := \left\{ h \in \widehat{H}^*(M;\Lambda) \mid I(f^*h) = 0 \right\}$$

Equivalently, we may think of these characters as those whose characteristic class is transgressive. The transgression is then defined as follows for $n > 2$. Given $h \in \widehat{\mathrm{Trans}}(f)$, by exactness there is a character $\tilde{h} \in \widehat{H}^n(f;\Lambda)$ such that $\breve{p}_f(\tilde{h}) = h$. Since $\widehat{H}^n(\{x\},\Lambda) = 0$ for $n > 1$, then by exactness along the bottom, there is a character $q \in \widehat{H}^{n-1}(E_x, A)$ such that $\breve{\imath}_{f_x}(q) = \iota_{E_x}^*\tilde{h}$. Again by exactness, the choice of $\tilde{h}$ is unique up to some character in the image of $\breve{\imath}_f$ and the choice of q corresponding to $\iota_{E_x}^*\tilde{h}$ is unique as $\widehat{H}^{n-1}(\{x\};\Lambda) = 0$. Hence this defines a map

$$\widehat{\tau}_f : \widehat{\mathrm{Trans}}(f) \cap \widehat{H}^n(M;\Lambda) \to \frac{\widehat{H}^{n-1}(E_x;\Lambda)}{\iota_{E_x}^*\widehat{H}^{n-1}(E;\Lambda)}.$$

The differential cohomology transgression satisfies some nice properties. It is natural with respect to pairs of maps. The set of transgressive elements $\widehat{\mathrm{Trans}}(f)$ is an ideal in $\widehat{H}^*(M;\Lambda)$. It covers the transgression on normal cohomology. Let's make these statements more precise. Proofs of the following propositions can be found in Becker.

Proposition 4.4.4. *The map $\widehat{\tau}$ is a natural functor. In other words, given smooth functions $\Phi : E' \to E$ and $\phi : M' \to M$ such that the following diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\Phi} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{\phi} & X \end{array}$$

commutes, then the transgressions along f and f' are related through,

$$\Phi^* \circ \widehat{\tau}_f = \widehat{\tau}_{f'} \circ \phi^*.$$

Now an analog to Lemma 2.2.1 is the following.

Proposition 4.4.5. *Let $\pi : E \to M$ be a fiber bundle and $h_1, h_2 \in \widehat{H}^*(M,\Lambda)$. If h_1 or h_2 is transgressive, then $h_1 * h_2$ is transgressive and $\widehat{\tau}_\pi(h_1 * h_2) = 0$.*

4.5 DIFFERENTIAL TRIVIALIZATIONS

4.5.1 Differential Character Version

We start by introducing the notion of a differential trivialization as defined in Becker's paper [2]. The way that these trivializations will be defined is analogous to the topological trivializations where they correspond to certain cohomology classes of the total space. The basic premise is that we want a suitable differential character representing the topological trivialization. As Becker mentions, the naïve way would be to simply define the differential trivialization of some universal class $u \in H^{2k}(M;\Lambda)$ as all differential characters $\hat{q} \in \widehat{H}^{2k-1}(P;\Lambda)$ such that $I(\hat{q})$ is a topological trivialization. However, this definition is seen as deficient in that the set of all such differential trivializations is huge and more importantly, for specific cases such as for geometric String structures, this definition is too broad. However Becker's differential trivializations do in fact correspond to geometric String structures.

Let G be a Lie group with finitely many components and let $\pi : (P, \theta) \to M$ be a principal G-bundle with connection. Let $f : M \to BG$ be the classifying map and let $u(P) \in H^{2k}(M;\Lambda)$ be a universal characteristic class.

Definition 4.5.1. A **differential u-trivialization** on (P, θ) is a differential character $\hat{q} \in \widehat{H}^{2k-1}(P;\Lambda)$ such that

$$-\breve{\iota}_\pi(\hat{q}) = \widehat{CCS}_\theta(\tfrac{1}{2}p_1) - a_\pi(\rho, 0)$$

for some $\rho \in \Omega^3(M)$.

To have a better understanding of how we should think of the form ρ, we have the following Lemma,

Lemma 4.5.2. *Given a differential u-trivialization $\hat{q}$, the differential form ρ is uniquely determined by $\hat{q}$ and satisfies $\breve{p}_{id}(a_{id}(\rho, 0)) = \widehat{CW}_\theta(\lambda, u)$. In particular, we have $d\rho = CW_\theta(\lambda)$ and thus $u(P) = 0$.*

Conversely, any $\rho \in \Omega^{n-1}(M)$ such that $a(\rho, 0) = \widehat{CW}_\theta(\lambda, u)$ uniquely determine a differential u-trivialization up to characters of the form $j(\pi^* w) \in \widehat{H}^{2k-1}(P;\Lambda)$ for some $w \in H^{2k-2}(M;\mathbb{R}/\Lambda)$.

Some explanation here is useful in understanding the reason for this definition. In the integral cohomology setting, given a characteristic class $u(P) \in H^{2k}(M;\mathbb{Z})$ corresponding to a principal G-bundle $\pi : P \to M$, recall that a u-trivialization class of $u(P)$ is given a cohomology class $q \in H^{2k-1}(P)$ such that $\iota_x^* q = \tau(u)$ where $\tau : H^*(BG;\Lambda) \to H^{*-1}(G;\mathbb{Z})$ is the transgression. Then the existence of a trivialization means that $u(P) = 0$.

Now suppose we have $\rho \in \Omega^{2k-1}(M)$ such that $d\rho = CW_\theta(\lambda)$. Immediately this implies that $[CW_\theta(\lambda)]_{\mathbb{R}} = 0$. Now if we look at the difference of $h = \widehat{CCS}_\theta(\lambda, u) - a_\pi(\rho, 0) \in \widehat{H}^{2k}(\pi, \Lambda)$, we find that

$$I(h) = I(\widehat{CCS}_\theta(\lambda, u)) = \tau(u),$$

$$(\text{curv,cov})(h) = (CW_\theta(\lambda) - CW_\theta(\lambda), CS_\theta(\lambda) - \pi^*\rho) = (0, CS_\theta(\lambda) - \pi^*\rho).$$

Then $\tilde{h} = \breve{p}_\pi(h) \in \widehat{H}^{2k}(M;\Lambda)$ satisfies $I(\tilde{h}) = u(P)$ and $\text{curv}(\tilde{h}) = 0$. Furthemore, since $\widehat{H}^{2k}(\pi;\Lambda)$ fits into a long exact sequence, then $\breve{p}_\pi \circ \breve{\iota}_\pi = 0$. Thus if we have a differential u-trivialization $\hat{q}$ as defined above, then $u(P) = 0$ and $q = I(\hat{q})$ is a trivialization of the integral class $u(P)$. Moreover, it is shown in [2] that if $-\breve{\iota}_\pi(\hat{q}) = \widehat{CCS}_\theta - a_\pi(\rho, 0)$ for any $\rho \in \Omega^{2k-1}(M)$, then ρ is uniquely determined by $\hat{q}$.

The following propositions help to further characterize this definition of differential u-trivializations.

Proposition 4.5.3. *Let $\pi : (P, \theta) \to M$ be a principal G-bundle with connection and let u be a universal characteristic class and λ a corresponding invariant polynomial.*

1. *If $u(P) = 0$, then there exists a differential u-trivialization.*
2. *If $\hat{q} \in \widehat{H}^{2k-1}(P;\Lambda)$ is a differential u-trivialization, then $I(\hat{q})$ is a u-trivialization class.*
3. *For $\hat{q}$ a differential u-trivialization with differential form ρ, curv$(\hat{q}) = CS_\theta(\lambda) - \pi^*\rho$, and for an arbitrary point $x \in M$,*

$$\widehat{H}^{2k-1}(P_x;\Lambda) \ni \iota_{P_x}^* \hat{q} = \widehat{\tau}(\widehat{CW}_\theta(\lambda, u)) \in \widehat{H}^{2k-1}(G;\Lambda).$$

Proposition 4.5.4. *Let $\pi : (P, \theta) \to M$ be a principal G-bundle with connection and let u be a universal characteristic class and λ a corresponding invariant polynomial. Then the differential cohomology group $\widehat{H}^{2k-1}(M;\Lambda)$ acts on the set of all differential u-trivializations* *by* $(\hat{q}, h) \mapsto \hat{q} + \pi^* h$*. Moreover, the set of differential* u*-trivializations is a torsor for* $\pi^* \widehat{H}^{2k-2}(M; \Lambda)$.

Proofs of Propositions 4.5.3 and 4.5.4 can be found in [2]. Proposition 4.5.3 tells us that differential u-trivializations exist whenever u-trivializations exist. In fact, if we assume that G is a $(2k-2)$-connected Lie group with finite components, it can be shown that every u-trivialization admits a differential u-trivialization as a refinement and that the set of differential u-trivializations is a torsor for $\widehat{H}^{2k-2}(M; \Lambda)$. However if our definition of a Lie group includes G being a finite dimensional manifold, then this condition only makes sense for $k < 3$. In the following chapter, we focus on placing this concept of a differential u-trivializations in the setting of rational Spin-Fivebrane and rational Fivebrane structures.

5.0 DIFFERENTIAL RATIONAL FIVEBRANE STRUCTURES

5.1 DIFFERENTIAL RATIONAL FIVEBRANE STRUCTURES

Taking the definition of differential u-trivializations that we have chosen, the goal now is to generalize this concept to describe differential characters on $O\langle n\rangle$-bundles. Our main example will be smooth String-bundles which admit rational Fivebrane structures and thus for our purposes we will set $\Lambda = \mathbb{Q}$. The main obstacle in extending the concept of differential u-trivializations is that for the higher connected covers of a Lie group G, they are no longer guaranteed to be finite dimensional. Thus the classical construction for the Chern-Weil homomorphism as well as the Cheeger-Simons characters are no longer applicable. We take somewhat of a naïve approach to extending these characters, but we end up obtaining some interesting results.

Let us begin by setting $\pi_{\text{String}} : P \to M$ to be a principal String-bundle where its underlying principal Spin-bundle, which we will denote $\pi_{\text{Spin}} : (Q, \theta) \to M$, is equipped with a connection θ. Recall from Section 3.2 that a rational Spin-Fivebrane class is an element $F \in H^7(Q; \mathbb{Q})$ such that $\iota_x^* F \in H^7(\text{Spin}; \mathbb{Q})$ corresponds to the transgression of $\frac{1}{6}p_2$ for every $x \in M$ and that the existence of such a class is equivalent to $(\frac{1}{6}p_2(P))_{\mathbb{Q}} \in H^8(M; \mathbb{Q})$ being trivial. Now while $\frac{1}{6}p_2$ is not an integral class in $H^8(B\text{Spin}; \mathbb{Z})$, it is a well defined rational class $\frac{1}{6}p_2 \in H^8(B\text{Spin}; \mathbb{Q})$ and thus we can define differential characters

$$\widehat{CW}_\theta(\tfrac{1}{6}p_2, \mathbb{Q}) \in \widehat{H}^8(M; \mathbb{Q})$$
$$\widehat{CCS}_\theta(\tfrac{1}{6}p_2, \mathbb{Q}) \in \widehat{H}^8(\pi_{\text{Spin}}; \mathbb{Q}).$$

As we are only considering the principal Spin-bundle at the moment, the definition and results from above can be applied directly to get the following definition.

Definition 5.1.1. A **differential rational Spin-Fivebrane class** is a differential character $\hat{q} \in \widehat{H}^7(Q;\mathbb{Q})$ such that

$$-\breve{\iota}_{\pi_{\mathrm{Spin}}}(\hat{q}) = \widehat{CCS}_\theta(\tfrac{1}{6}p_2,\mathbb{Q}) - a_{\pi_{\mathrm{Spin}}}(\rho,0)$$

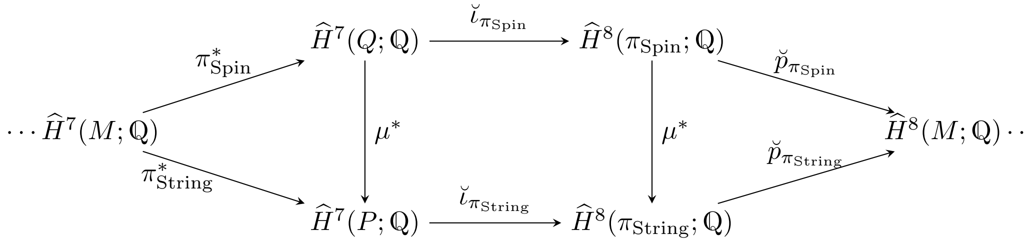
for some $\rho \in \Omega^{2k-1}(M)$.

In particular, a differential rational Spin-Fivebrane class is a differential $\frac{1}{6}p_2$-trivialization on the bundle $\pi_{\mathrm{Spin}} : (Q,\theta) \to M$ in the setting of rational cohomology. From Lemma 4.5.2 and Proposition 4.5.3 we have the following facts about these differential characters.

Corollary 5.1.2. *Let $\hat{q} \in \widehat{H}^7(P;\mathbb{Q})$ be a differential rational Spin-Fivebrane class. Then we have the following:*

1. *$\hat{q}$ uniquely determines the differential form ρ;*
2. *given $\rho \in \Omega^7(M)$ such that $a(\rho) = \widehat{CW}_\theta(\frac{1}{6}p_2,\mathbb{Q})$, ρ determines $\hat{q}$ up to a character of the form $\pi^* j(w)$ where $w \in H^6(M;\mathbb{R}/\mathbb{Q})$;*
3. *$\breve{p}_{id}(a_{id}(\rho,0)) = \widehat{CW}_\theta(\frac{1}{6}p_2,\mathbb{Q})$;*
4. *$d\rho = CW_\theta(\frac{1}{6}p_2,\mathbb{Q})$;*
5. *$\hat{q}$ exists iff $(\frac{1}{6}p_2)_{\mathbb{Q}} = 0$;*
6. *$I(\hat{q})$ is a rational Spin-Fivebrane class;*
7. *$curv(\hat{q}) = CS_\theta(\frac{1}{6}p_2,\mathbb{Q}) - \pi^*\rho$;*
8. *$\iota_x^*\hat{q} = \widehat{\tau}(\widehat{CW}_\theta(\frac{1}{6}p_2,\mathbb{Q}))$ for every $x \in M$; and*
9. *the set of all differential rational Spin-Fivebrane classes is a torsor for $\pi^*\widehat{H}^7(M;\mathbb{Q})$.*

Parts 1-4 follow from Lemma 4.5.2, parts 5-8 follow from Proposition 4.5.3 and part 9 is Proposition 4.5.4. Now considering that we have a principal String-bundle P with a bundle morphism $\mu : P \to Q$ to its underlying Spin-bundle. Then μ induces a morphism $\mu^* : \widehat{H}^*(Q;\mathbb{Q}) \to \widehat{H}^*(P;\mathbb{Q})$ as well as a morphism $\mu^* : \widehat{H}^*(\pi_{\mathrm{Spin}};\mathbb{Q}) \to \widehat{H}^*(\pi_{\mathrm{String}};\mathbb{Q})$. We can consider the character $\widehat{S} := \mu^*\widehat{CCS}_\theta(\frac{1}{6}p_2,\mathbb{Q}) \in \widehat{H}^8(\pi_{\mathrm{String}};\mathbb{Q})$ in the image of this map. As the map $\rho : \mathrm{String} \to \mathrm{Spin}$ is surjective, then so is the bundle map $\mu : P \to Q$. Thus the induced map on differential forms is injective, and as $\mathrm{curv}(\widehat{S}) = (\mu^* CW_\theta(\frac{1}{6}p_2), CS_\theta(\frac{1}{6}p_2))$ then $\widehat{S}$ is non-zero. Furthermore, as the bundle morphism $\mu : P \to Q$ induces a morphism between the long exact sequences for relative differential cohomology, then we have the following commutative diagram,



and it follows that

$$\breve{p}_{\pi_{\text{String}}}(\widehat{S}) = \breve{p}_{\pi_{\text{String}}}(\mu^*\widehat{CCS}_\theta(\tfrac{1}{6}p_2)) = \breve{p}_{\pi_{\text{String}}}(\widehat{CCS}_\theta(\tfrac{1}{6}p_2)) = \widehat{CW}_\theta(\tfrac{1}{6}p_2). \tag{5.1}$$

From this perspective we're able to extend the definition for differential trivializations to characters on the total space of our String bundle.

Definition 5.1.3. A **differential rational Fivebrane class** is a differential character $\widehat{F} \in \widehat{H}^7(P;\mathbb{Q})$ such that

$$-\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}) = \widehat{S} - a_{\pi_{\text{String}}}(\rho, 0)$$

for some $\rho \in \Omega^{2k-1}(M)$.

Using the previous commutative diagram, we can make the following statement.

Proposition 5.1.4. *For every differential rational Spin-Fivebrane class $\hat{q}$ with differential form ρ, $\mu^*\hat{q}$ is a differential rational Fivebrane class with differential form ρ.*

The set of differential rational Fivebrane classes share many similar properties to those of differential u-trivializations. In analogy to Lemma 4.5.2, we have the following.

Lemma 5.1.5. *Let $\pi_{\text{String}} : (P, \theta) \to M$ be a String bundle where θ is a connection on the underlying Spin bundle, and let $\hat{F}$ be a differential rational Fivebrane class with differential form ρ. Then the form $\rho \in \Omega^7(M)$ is uniquely determined by $\hat{F}$ and $\breve{p}_{id}(a_{id}(\rho, 0)) = \widehat{CW}_\theta(\frac{1}{6}p_2)$.*

Conversely, any $\rho \in \Omega^7(M)$ such that $a(\rho) = \widehat{CW}_\theta(\frac{1}{6}p_2)$ uniquely determines a differential rational Fivebrane class up to characters of the form $j(\pi^*_{\text{String}} w) \in \widehat{H}^7(P;\mathbb{Q})$ for some $w \in H^6(M;\mathbb{R}/\mathbb{Q})$.

Proof. While we ommitted the proof before, we provide one now to illustrate that it remains unaffected when considering characters on the principal String-bundle. Suppose we have two forms ρ, ρ' satisfying definition 5.1.3. Then $a_{\pi_{\text{String}}}(\rho - \rho') = 0$ and thus

$$(0,0) = (\text{curv}, \text{cov})(a_{\pi_{\text{String}}}(\rho - \rho', 0) = d_\pi(\rho - \rho', 0) = (d\rho - d\rho', \pi^*(\rho - \rho')).$$

Now as $\pi^*_{\text{String}} : \Omega^*(M) \to \Omega^*(P)$ is an injection and since $\pi^*_{\text{String}}(\rho - \rho') = 0$, it follows that $\rho = \rho'$. Hence the form ρ is unique.

By comparing the long exact sequences for the relative differential cohomology of the maps $\pi_{\text{String}} : P \to M$ and $id_M : M \to M$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
H^6(M;\mathbb{R}/\mathbb{Q}) & \xrightarrow{\pi^*_{\text{String}} \circ j} & \widehat{H}^7(P;\mathbb{Q}) & \xrightarrow{\breve{\imath}_{\pi_{\text{String}}}} & \widehat{H}^8(\pi_{\text{String}};\mathbb{Q}) & \xrightarrow{\breve{p}_{\pi_{\text{String}}}} & \widehat{H}^8(M;\mathbb{Q}) \\
\uparrow {\scriptstyle id^*_M} & & \uparrow {\scriptstyle \pi^*_{\text{String}}} & & \uparrow {\scriptstyle (id_M, \pi_{\text{String}})^*} & & \uparrow {\scriptstyle id^*_M} \\
H^6(M;\mathbb{R}/\mathbb{Q}) & \xrightarrow[j]{} & \widehat{H}^7(M;\mathbb{Q}) & \xrightarrow[\breve{\imath}_{id}]{} & \widehat{H}^8(id_X;\mathbb{Q}) & \xrightarrow[\breve{p}_{id}]{} & \widehat{H}^8(M;\mathbb{Q})
\end{array}$$

We have a character $a_{id}(\rho, 0) \in \widehat{H}^8(id_x)$ corresponding to ρ and by the commutativity of the diagram, we have

$$\breve{p}_{id}(a_{id}(\rho, 0)) = \breve{p}_{\pi_{\text{String}}}(a_\pi(\rho, 0)) = \breve{p}_{\pi_{\text{String}}}(\widehat{S} + \breve{\imath}(\hat{q})) = \widehat{CW}_\theta(\tfrac{1}{6}p_2).$$

Thus $a(\rho) = \breve{p}_{\pi_{\text{String}}}(a_{\pi_{\text{String}}}(\rho, 0)) = \widehat{CW}_\theta(\frac{1}{6}p_2)$.

For the converse, suppose that $a(\rho) = \widehat{CW}_\theta(\frac{1}{6}p_2)$. Then $\breve{p}_{\pi_{\text{String}}}(a_{\pi_{\text{String}}}(\rho, 0)) = \widehat{CW}_\theta(\frac{1}{6}p_2)$ and thus $\breve{p}_{\pi_{\text{String}}}(\widehat{S} - a_{\pi_{\text{String}}}(\rho, 0))) = 0$. Then by the long exact sequence

$$\cdots \to H^6(M;\mathbb{R}/\mathbb{Q}) \xrightarrow{\pi^*_{\text{String}} \circ j} \widehat{H}^7(P;\mathbb{Q}) \xrightarrow{\breve{\imath}_{\pi_{\text{String}}}} \widehat{H}^8(\pi_{\text{String}};\mathbb{Q}) \xrightarrow{\breve{p}_{\pi_{\text{String}}}} \widehat{H}^8(M;\mathbb{Q}) \to \cdots$$

there is a character $\hat{q} \in \widehat{H}^7(P;\mathbb{Q})$ which maps to $\widehat{S} - a_{\pi_{\text{String}}}(\rho, 0)$ and the difference between any two such characters lies in the image of $\pi^*_{\text{String}} \circ j$. □

A major benefit of studying characters on the principal String-bundle is that the fibers of this bundle are 6-connected. In [41], Waldorf shows that given a principal Spin-bundle $\pi : Q \to M$, the map $\pi^* : \widehat{H}^3(M;\mathbb{Z}) \to \widehat{H}^3(Q;\mathbb{Z})$ is injective. The proof however only really relies on the fact that we have a bundle whose fiber is 2-connected. In fact, this result can be generalized.

Lemma 5.1.6. *Let $\pi : P \to X$ be a fiber bundle whose fiber, F, is $(n-1)$-connected. Then the map $\pi^* : \widehat{H}^n(M;\Lambda) \to \widehat{H}^n(P;\Lambda)$ is injective.*

Proof. We start by considering the Serre spectral sequence corresponding to $F \to P \to M$. It follows that the first nontrivial differential is $d_n : E^{k,r} \to E^{k+n+1,r-n}$. From this we obtain an exact sequence (in the same manner as we did in Section 2.7.1)

$$0 \to H^n(M;\Lambda) \xrightarrow{\pi^*} H^n(P;\Lambda) \xrightarrow{\iota_x^*} H^n(F;\Lambda) \xrightarrow{d_n} H^{n+1}(M;\Lambda) \tag{5.2}$$

and for $k < n$, the map $\pi^* : H^k(M;\Lambda) \to H^k(P;\Lambda)$ is an isomorphism. In order to prove that $\pi^* : \widehat{H}^n(M;\Lambda) \to \widehat{H}^n(P;\Lambda)$ is injective, we show that if $\widehat{\eta} \in \widehat{H}^n(M;\Lambda)$ is nonzero, then $\pi^*\widehat{\eta}$ is nonzero.

Suppose $\eta = I(\widehat{\eta})$ is nonzero. Using the short exact sequence (5.2), the map $\pi^* : H^n(M;\Lambda) \to H^n(P;\Lambda)$ is injective, and thus $\pi^*\eta$ is nonzero. Then it follows from (4.3) that $\pi^*\widehat{\eta}$ must be nonzero.

Now suppose that $\eta = 0$. Then there is a form $\xi \in \Omega^{n-1}(M)$ such that $a(\xi) = \widehat{\eta}$. Since π is a surjective submersion, then $\pi^* : \Omega^*(M) \to \Omega^*(P)$ is an injection. Thus if ξ is not closed, then $\pi^*\xi$ is not closed and therefore $\pi^*(\widehat{\eta}) = \pi^*a(\xi) = a(\pi^*\xi)$ is nonzero. On the other hand, suppose that ξ is closed but does not have Λ periods. It follows from the fact that $\pi^* : H^{n-1}(M;\Lambda) \to H^{n-1}(P;\Lambda)$ is an isomorphism then that $\pi^*\xi$ must also not have Λ periods and thus again, $a(\pi^*\xi)$ is nonzero. From this it follows that the map is injective. □

Now by Corollary 5.1.2, we know that the set of differential rational Spin-Fivebrane classes is a torsor for $\pi^*_{\mathrm{Spin}}\widehat{H}^7(M;\mathbb{Q})$. This also holds true for differential rational Fivebrane classes, except now we have from the previous Lemma that $\pi^*_{\mathrm{String}} : \widehat{H}^7(M;\mathbb{Q}) \to \widehat{H}^7(P;\mathbb{Q})$ is injective.

Proposition 5.1.7. *Let $\pi_{\mathrm{String}} : P \to M$ be a principal String-bundle and let $\pi_{\mathrm{Spin}} : (Q,\theta) \to M$ be the underlying Spin bundle with connection. The set of all differential rational Fivebrane classes is a torsor for $\widehat{H}^7(M;\mathbb{Q})$.*

Proof. As in Lemma 5.1.5, the proof here is very similar to the one provided by Becker in [2], but again we are providing it here as we are dealing with characters on the overlying principal String-bundle. To show that differential rational Fivebrane classes are a torsor for $\widehat{H}^{n-1}(M;\Lambda)$, we'll need to first show what the action is and then that this action is free and transitive. The action of $\widehat{H}^{n-1}(M;\Lambda)$ is just the addition of elements in the image of π^*_{String}. To see that this is in fact an action on the set of differential rational Fivebrane classes, take a differential character $\hat{h} \in \widehat{H}^{n-1}(M;\Lambda)$ and let $\widehat{F}$ be a differential rational Fivebrane class. Then, using the fact that $\breve{\iota}_{id}(\hat{h}) = a_{id}(-\text{curv}(\hat{h}), 0)$, we have

$$\begin{aligned}
-\breve{\iota}_{\pi_{\text{String}}}(\widehat{F} + \pi^*_{\text{String}}\hat{h}) &= -\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}) - \breve{\iota}_{\pi_{\text{String}}}(\pi^*_{\text{String}}\hat{h}) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho, 0) - (id_X, \pi_{\text{String}})^* a_{id}(-\text{curv}(\hat{h}), 0) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho, 0) - a_{\pi_{\text{String}}}(-\text{curv}(\hat{h}), 0) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho - \text{curv}(\hat{h}), 0).
\end{aligned}$$

Hence $\widehat{F} + \pi^*_{\text{String}}\hat{h}$ is also a differential rational Fivebrane class with differential form $\rho - \text{curv}(\hat{h})$. To show that the action is free, it is equivalent to show that the map $\pi^*_{\text{String}} : \widehat{H}^{n-1}(M;\Lambda) \to \widehat{H}^{n-1}(P;\Lambda)$ and this follows exactly from Lemma 5.1.6. Now to see that the action is transitive, take two different differential rational Fivebrane class $\widehat{F}, \widehat{F}'$ with respective differential forms ρ, ρ'. Then it follows that

$$\begin{aligned}
-\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}' - \widehat{F}) &= -a_{\pi_{\text{String}}}(\rho', 0) - (-a_{\pi_{\text{String}}}(\rho, 0)) \\
&= a_{\pi_{\text{String}}}(\rho - \rho', 0) \\
&= \pi^* a_{id}(\rho' - \rho) \\
&= \pi^* \breve{\iota}_{id}(\hat{h}') \\
&= \breve{\iota}_{\pi_{\text{String}}}(\pi^*_{\text{String}}\hat{h}').
\end{aligned}$$

From this we have that $\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}' - \widehat{F} + \pi^*_{\text{String}}\hat{h}') = 0$ Thus by the long exact sequence for relative differential cohomology, there is a class $v \in H^{n-2}(M;\mathbb{R}/\Lambda)$ such that $\pi^*_{\text{String}} j(v) = \widehat{F}' - \widehat{F} + \pi^*_{\text{String}}\hat{h}'$. Thus if we let $\hat{h} = \hat{h}' - j(v)$, then $\widehat{F}' = \widehat{F} + \pi^*_{\text{String}}\hat{h}$. □

It's important to note that as $\pi_{\text{Spin}} : Q \to M$ is the underlying principal Spin-bundle, then we know that π_{String} factors as $\pi_{\text{String}} = \pi_{\text{Spin}} \circ \mu$. Therefore if π^*_{String} is injective, then so is π^*_{Spin}. In other words, we have that the set of differential rational Spin-Fivebrane classes is also a torsor for $\widehat{H}^7(M;\mathbb{Q})$ when the principal Spin-bundle admits a String structure. Even more, we have that in the presence of a String structure, Definition 5.1.1 and Definition 5.1.3 become equivalent.

Proposition 5.1.8. *For a principal String-bundle, the set of differential rational Fivebrane classes are in bijective correspondence to the set of differential rational Spin-Fivebrane classes.*

Proof. As noted above, π_{String} and π_{Spin} are both injective. Thus the set of differential rational Spin-Fivebrane classes and differential rational Fivebrane classes are both torsors for $\widehat{H}^7(M;\mathbb{Q})$. The action of $\widehat{H}^7(M;\mathbb{Q})$ is given by adding $(\hat{x}, \hat{q}) \mapsto \hat{q} + \pi^*_{\text{Spin}}\hat{x}$, for any $\hat{x} \in \widehat{H}^7(M;\mathbb{Q})$, when $\hat{q}$ is a differential rational Spin-Fivebrane class, and $(\hat{F}, \hat{x}) \mapsto \hat{F} + \pi^*_{\text{String}}\hat{x}$ when $\hat{F}$ is a differential rational Fivebrane class. By Proposition 5.1.4, μ^* maps differential rational Spin-Fivebrane classes to differential rational Fivebrane classes. Since μ^* is equivariant with respect to the action of $\widehat{H}^7(M;\mathbb{Q})$, it follows that μ^* maps these sets bijectively. □

From here we can easily prove the following Corollary.

Corollary 5.1.9. *Let $\pi_{\text{String}} : P \to M$ be a principal String-bundle where $\pi_{\text{Spin}} : (Q, \theta) \to M$ is the underlying principal Spin-bundle equipped with a connection. Then a differential rational Fivebrane class exists iff $(\frac{1}{6}p_2(P))_{\mathbb{Q}} = 0$. If $\widehat{F}$ is a differential rational Fivebrane class with differential form $\rho \in \Omega^7(M)$, then the following are true.*

1. *$I(\widehat{F})$ is a rational Fivebrane class; and*
2. *$curv(\widehat{F}) = \mu^* CS_\theta(\frac{1}{6}p_2) - \pi^*_{\text{String}}\rho$.*

Proof. If $(\frac{1}{6}p_2(P))_{\mathbb{Q}} = 0$, then there exists a rational Spin-Fivebrane class $\hat{q}$ and thus by Proposition 5.1.4, $\mu^*\hat{q}$ is a rational Fivebrane class. Conversely, if we have $\widehat{F}$ a differential rational Fivebrane class with differential form ρ, then $a(\rho) = \widehat{CW}_{\theta,\mathbb{Q}}(\frac{1}{6}p_2)$ and thus $(\frac{1}{6}p_2(P))_{\mathbb{Q}} = 0$. To show1, from Proposition 5.1.8 every differential rational Fivebrane class

$\widehat{F} \in \widehat{H}^7(P;\mathbb{Q})$ with differential form ρ, there is a corresponding differential rational Spin-Fivebrane class $\hat{q} \in \widehat{H}^7(Q;\mathbb{Q})$ with differential form ρ such that $\mu^*\hat{q} = \widehat{F}$. By Corollary 5.1.2, $I(\hat{q})$ is a rational Spin-Fivebrane class, and thus by Proposition 3.2.9, $I(\widehat{F}) = \mu^* I(\hat{q})$ is a rational Fivebrane class. Finally for 2, again we have some $\hat{q} \in \widehat{H}^7(Q;\mathbb{Q})$ such that $\mu^*\hat{q} = \widehat{F}$ and thus

$$\mathrm{curv}(\widehat{F}) = \mu^*\mathrm{curv}(\hat{q}) = \mu^*(CS_\theta(\tfrac{1}{6}p_2) - \pi^*_{\mathrm{Spin}}\rho) = \mu^* CS_\theta(\tfrac{1}{6}p_2) - \pi^*_{\mathrm{String}}\rho.$$

□

Now for the most part we've demonstrated how differential trivializations defined this way on the principal String-bundle have the same properties as those on the Spin-bundle, there are some extra properties that we gain from these trivializations. One of the properties is that every rational Fivebrane class has a differential refinement which is a differential rational Fivebrane class. The second benefit is that one can infer whether any character $\hat{h} \in \widehat{H}^7(P;\mathbb{Q})$ is a differential Fivebrane class just by looking at its characteristic class and curvature.

Proposition 5.1.10. *Let $\pi_{\mathrm{String}} : P \to M$ be a principal String-bundle where $\pi_{\mathrm{Spin}} : (Q,\theta) \to M$ is the underlying principal Spin-bundle equipped with a connection. It follows that:*

1. *given any rational Fivebrane class $F \in H^7(P;\mathbb{Q})$, then there exists a differential character $\widehat{F} \in \widehat{H}^7(P;\mathbb{Q})$ with $I(\widehat{F}) = F$; and*
2. *any differential character $\hat{h} \in \widehat{H}^7(P;\mathbb{Q})$, which has curvature $\mu^* CS_\theta(\frac{1}{6}p_2) - \pi^*_{\mathrm{String}}\rho$ for some differential form $\rho \in \Omega^7(M)$ and has $I(\hat{h})$ a rational Fivebrane class, is a differential u-trivialization.*

Proof. For the proof of part 1, let $F \in H^7(P;\mathbb{Q})$ be a rational Fivebrane class. By Corollary 5.1.9 we know that a differential rational Fivebrane class exists. So let $\widehat{F}'$ be an arbitrary differential rational Fivebrane class with differential form ρ'. Then $I(\widehat{F}')$ is a rational Fivebrane class and since $H^7(M;\mathbb{Q})$ is a torsor for the set of rational Fivebrane classes, there is an element $x \in H^7(M;\mathbb{Q})$ such that $F - I(\widehat{F}') = \pi^*_{\mathrm{String}} x$. Let $\hat{x} \in \widehat{H}^7(M;\mathbb{Q})$ be any differential refinement of x. Then set $\widehat{F} := \widehat{F}' + \pi^*_{\mathrm{String}}\hat{x}$ to get a differential character where

$I(\widehat{F}) = F$. Now it remains to show that this differential character is a differential rational Fivebrane class. Set $\rho := \rho' - \text{curv}(\hat{x})$ and use the fact that $\breve{\iota}_{id}(\hat{x}) = a_{id}(-\text{curv}(\hat{x}), 0)$. Then

$$\begin{aligned}
-\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}) &= -\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}') - \breve{\iota}_{\pi_{\text{String}}}(\pi^*_{\text{String}}\hat{x}) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho', 0) - (id_M, \pi_{\text{String}})^* \breve{\iota}_{id}(\hat{x}) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho', 0) - (id_M, \pi)^* a_{id}(-\text{curv}(\hat{x}), 0) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho' - \text{curv}(\hat{x}), 0) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho, 0).
\end{aligned}$$

Thus $\widehat{F}$ is a differential rational Fivebrane class with characteristic class $I(\widehat{F}) = F$.

For the proof of part 2, let $\widehat{F} \in \widehat{H}^7(P;\mathbb{Q})$ be a differential character with $\text{curv}(\widehat{F}) = \mu^* CS_\theta(\frac{1}{6}p_2) - \pi^*_{\text{String}}\rho$ and where $I(\widehat{F})$ is a rational Fivebrane class. By part 1, there exists a differential rational Fivebrane class $\widehat{F}'$ with differential form ρ' such that $I(\widehat{F}') = I(\widehat{F})$. Then $\text{curv}(\widehat{F}') = \mu^* CS_\theta(\frac{1}{6}p_2) - \pi^*_{\text{String}}\rho'$ and thus $\text{curv}(\widehat{F} - \widehat{F}') = \pi^*_{\text{String}}(\rho' - \rho)$. Using the fact that the pullback $\pi^*_{\text{String}} : \Omega^*(M) \to \Omega^*(P)$ is injective and that the pullback $\pi^*_{\text{String}} : H^k(M;\mathbb{Q}) \to H^k(P;\mathbb{Q})$ is also injective for $k = 7$ and an isomorphism for $k < 7$, we have that since $\pi^*_{\text{String}}(\rho' - \rho)$ is closed and since $[\pi^*_{\text{String}}\rho]_{dR} = [I(\widehat{F})]_{\mathbb{R}} = [I(\widehat{F}')]_{\mathbb{R}} = [\rho']_{dR}$ then $\pi^*_{\text{String}}[\rho' - \rho]_{dR} = 0$ and by the injectivity of π^*_{String} there is a form $\eta \in \Omega^6(X)$ such that $d\eta = \rho' - \rho$.

Setting $\widehat{F}'' = \widehat{F}' + a(\pi^*_{\text{String}}\eta)$ gives a differential character satisfying

$$\begin{aligned}
\text{curv}(\widehat{F}'') = \text{curv}(\widehat{F}') + d\pi^*_{\text{String}}\eta &= \text{curv}(\widehat{F}) \\
I(\widehat{F}'') = I(\widehat{F}') &= I(\widehat{F}).
\end{aligned}$$

Then there is a form $\gamma \in \Omega^6(P)$ satisfying $a(\gamma) = \widehat{F} - \widehat{F}''$. In fact we can choose γ so that $\gamma = \pi^*_{\text{String}}\nu$ for some $\nu \in \Omega^6(M)$. To see this, note that $d\gamma = \text{curv}(\widehat{F} - \widehat{F}'') = 0$ and thus from the top short exact sequence of the differential cohomology square, we know that there is class

$$[u] \in \frac{H^6(P;\mathbb{R})}{H^6(P;\mathbb{R})_{\mathbb{Q}}}$$

which gets mapped to γ. In Lemma 5.1.6, we used the fact that for a fiber bundle with $(n-1)$-connected fiber, the map $\pi^*_{\text{String}} : H^k(M;\Lambda) \to H^k(P;\Lambda)$ is an isomorphism for $k < n$. In our present case where String is 6-connected, then

$$\pi^*_{\text{String}} : \frac{H^6(M;\mathbb{R})}{H^6(M;\mathbb{R})_{\mathbb{Q}}} \to \frac{H^6(P;\mathbb{R})}{H^6(P;\mathbb{R})_{\mathbb{Q}}}$$

is an isomorphism. Thus we have a class $[v]$ such that $\pi^*_{\text{String}}[v] = [u]$ and corresponding to this class we have a closed differential form ν such that $\gamma = \pi^*_{\text{String}}\nu$. Thus

$$\widehat{F} = \widehat{F}'' + a_P(\pi^*_{\text{String}}\nu) = \widehat{F}' + a_P(\pi^*_{\text{String}}(\eta+\nu)),$$

and as $\widehat{F}'$ is a differential rational Fivebrane class with differential form ρ',

$$\begin{aligned}
-\breve{\iota}_{\pi}(\widehat{F}) &= -\breve{\iota}_{\pi_{\text{String}}}(\widehat{F}') - \breve{\iota}_{\pi_{\text{String}}}(a(\pi^*(\eta+\nu))) \\
&= (\widehat{S} - a_{\pi_{\text{String}}}(\rho', 0)) - (a_{\pi_{\text{String}}}(0, \pi^*_{\text{String}}(\eta+\nu))) \\
&= (\widehat{S} - a_{\pi_{\text{String}}}(\rho', 0)) + a_{\pi_{\text{String}}}(d\eta, 0) + (a_{\pi_{\text{String}}}(d_{\pi}(\eta+\nu))) \\
&= \widehat{S} - a_{\pi_{\text{String}}}(\rho', 0).
\end{aligned}$$

In the second to last line, we use the fact that $d_{\pi_{\text{String}}}(\eta+\nu, 0) = (d\eta + d\nu, \pi^*_{\text{String}}(\eta+\nu) - 0)$ and that $d\Omega^6(\pi_{\text{String}}) \subset \Omega^6_{\mathbb{Q}}(\pi_{\text{String}})$ (i.e. exact forms lie in the kernel of $a_{\pi_{\text{String}}}$). Hence we have that $\widehat{F}$ is indeed a differential rational Fivebrane class. □

Now as we showed in Section 4.1.1, the model for String given in [25] satisfies the conditions in [1] to be a smooth space. Hence if we assume that the fibers of our smooth principal String bundle are diffeomorphic to this model, then we can consider the differential cohomology of String when studying this bundle. The first thing we note about the differential cohomology of String is that as a space it is 6-connected, then there is a canonical class in $H^7(\text{String};\mathbb{Z})$.

Proposition 5.1.11. *Given a generator s_7 of $H^7(\text{String};\mathbb{Z})$, there is a canonical character $\widehat{s}_7$ in $\widehat{H}^7(\text{String};\mathbb{Z})$.*

Proof. For convenience, let's say that s_7 corresponds to the generator $a_7 \in H^7(\text{Spin};\mathbb{Z})$ in that $3 \cdot a_7 = s_7$ where a_7 is the transgression of a generator of $H^8(B\text{String};\mathbb{Z})$. From [6] we know that the Chern-Simons construction for an invariant polynomial $\lambda \in I^k(\text{Spin})$ gives a closed form $TP(\lambda) \in \Omega^{2k-1}(\text{Spin})$ and for a bundle with connection, $TP(\lambda)$ corresponds to the pullback of the Chern-Simons form $CS_\theta(\lambda)$ along the inclusion of the fiber after identifying the fiber with Spin. Now if we take the second Pontryagin class $p_2 \in I^8(\text{Spin})$, we have that the following cohomology classes are identified $[TP(p_2)]_{dR} = [2a_7]_{\mathbb{R}}$. Thus by pulling back this form along the smooth map $q : \text{String} \to \text{Spin}$, we get a form $\eta := \frac{1}{6}q^*TP(p_2)$ such that $[\eta]_{dR} = [s_7]_{\mathbb{R}}$. From Proposition 4.1.11, we know that the groups $\widehat{H}^*(\text{String};\mathbb{Z})$ are well defined. Using the short exact sequence

$$1 \to \frac{H^{k-1}(\text{String};\mathbb{R})}{H^{k-1}(\text{String};\mathbb{R})_\Lambda} \to \widehat{H}^k(\text{String};\Lambda) \to A^k(\text{String};\Lambda) \to 1$$

where

$$A^k(\text{String};\Lambda) = \big\{(u,\omega) \in H^k(\text{String};\Lambda) \times \Omega^k(\text{String}) \mid [u]_{\mathbb{R}} = [\omega]_{dR}\big\}$$

and using the fact that $H^i(\text{String};\mathbb{R}) = 0$ for $i < 7$, it follows that $\widehat{H}^7(\text{String};\Lambda) \cong A^7(\text{String};\Lambda)$. Since $(s_7,\eta) \in A^7(\text{String};\mathbb{Z})$, we use this isomorphism to get a unique differential character corresponding to this pair which we denote as $\widehat{s}_7$. □

Let us return to the setting of rational coefficients. Recall that we defined the character $\widehat{S} := \mu^*\widehat{CCS}_\theta(\frac{1}{6}p_2) \in \widehat{H}^7(\pi_{\text{String}};\mathbb{Q})$ and used this character to define differential rational Fivebrane classes. We show that for cohomology, the degree 8 relative cohomology of a String bundle is isomorphic to degree 7 cohomology of the fiber. Moreover we can show that this map lifts to a map in differential cohomology which identifies the differential character $\widehat{(s_7)}_{\mathbb{Q}} \in \widehat{H}^7(\text{String};\mathbb{Q})$ to $\widehat{S}$.

Proposition 5.1.12. *1. There is an isomorphism* $\beta : H^8(\pi_{\text{String}};\mathbb{Q}) \xrightarrow{\cong} H^7(\text{String};\mathbb{Q})$.

2. $I(\widehat{S}) = \beta^{-1}(s_7)_{\mathbb{Q}}$.

3. There is a morphism $\widehat{\beta} : \widehat{H}^8(\pi_{\text{String}};\mathbb{Q}) \to \widehat{H}^7(\text{String};\mathbb{Q})$ *which lifts* β*, and* $\widehat{\beta}(\widehat{S}) = \widehat{(s_7)}_{\mathbb{Q}}$.

Proof. For the first, we'll first need to construct the map $\beta : H^8(\pi_{\text{String}};\mathbb{Q}) \to H^7(\text{String};\mathbb{Q})$. Let's start by considering the following diagram where the top and bottom rows are exact sequences.

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^7(M;\mathbb{Q}) & \xrightarrow{\pi^*_{\text{String}}} & H^7(P;\mathbb{Q}) & \xrightarrow{\breve{\iota}_{\pi_{\text{String}}}} & H^8(\pi_{\text{String}};\mathbb{Q}) & \xrightarrow{\breve{p}_{\pi_{\text{String}}}} & H^8(M;\mathbb{Q}) & \xrightarrow{\pi^*_{\text{String}}} & H^8(P;\mathbb{Q}) \\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle =}\downarrow & & \\
0 & \longrightarrow & H^7(M;\mathbb{Q}) & \xrightarrow{\pi^*_{\text{String}}} & H^7(P;\mathbb{Q}) & \xrightarrow{\iota^*_x} & H^7(\text{String};\mathbb{Q}) & \xrightarrow{d_7} & H^8(M;\mathbb{Q}) & &
\end{array}
$$

The top line represents the long exact sequence of the mapping cone, and the bottom line is the Serre exact sequence. We define the map $\beta : H^8(\pi_{\text{String}};\mathbb{Q}) \to H^7(\text{String};\mathbb{Q})$ by starting at the level of cocycles. Let $(u, v) \in Z^8(\pi_{\text{String}};\mathbb{Q})$ be a representative of some class $[(u, v)]$. Then we set $\beta([(u, v)]) := [\iota^*_x v]$ corresponding to the $x \in X$ for the fiber we've chosen. Now first we'll show that this map is well defined. Since (u, v) is closed, it follows that $\delta(u) = 0$ and $\pi^* u = \delta(v)$. Then

$$\delta(\iota^*_x v) = \iota^*_x \delta(v) = \iota^*_x \pi^*_{\text{String}} u = 0.$$

Hence $\iota^*_x v$ represents a cohomology class in $H^7(\text{String};\mathbb{Q})$. Suppose we have $[(u, v)] = 0$. Then it follows that there is a cochain $(r, s) \in C^7(\pi_{\text{String}};\mathbb{Q})$ such that $(u, v) = (\delta(r), \pi^*_{\text{String}} r - \delta(s))$. Then

$$\beta([(u, v)]) = [\iota^*_x v] = [\iota^*_x(\pi^*_{\text{String}} r - \delta(s))] = [\delta(\iota^*_x s)] = 0.$$

Thus β is well defined. It remains to show that this map makes the diagram commute. The square to the left of β commutes by construction. For the square on the right, d_7 denotes the differential arising from the seventh page of the Serre spectral sequence. We must show that $d_7 \circ \beta = \breve{p}_{\pi_{\text{String}}}$. Recall that the transgression τ is a right inverse of d_7. Given a cohomology class $[(u, v)] \in H^8(\pi';\mathbb{Q})$, we have $\breve{p}_{\pi_{\text{String}}}([(u, v)]) = [u]$. Now as $\delta v = \pi^*_{\text{String}} u$, it follows that $[u]$ is transgressive. On the cochain level, τ maps u to a cocycle in $y \in C^7(P_x;\mathbb{Q})$ such that $y = \iota^* \tilde{v}$ where $\delta\tilde{v} = (\pi_{\text{String}})^* u$. Thus $v - \tilde{v}$ represents a cohomology class in $H^7(P;\mathbb{Q})$ and thus $d_7(\iota^*_x(v - \tilde{v})) = 0$. It follows that

$$d_7(\beta([u, v]) = d_7(\iota^*_x v) = d_7(\iota^*_x \tilde{v}) = [u] = \breve{p}_{\pi_{\text{String}}}([u, v]),$$

and thus the right square commutes as well.

Now that we've constructed β and shown that it makes our diagram commute. We'd like to show that this map is in fact an isomorphism. Injectivity of β follows from a general version of the Five Lemma. For surjectivity, we use some diagram chasing as the bottom line doesn't continue to the right. Let $y \in H^7(\text{String};\mathbb{Q})$. Then $\pi^*_{\text{String}} d_7(y) = 0$ and thus by exactness of the top line, there is an element $x \in H^8(\pi_{\text{String}};\mathbb{Q})$ such that $\breve{p}_{\pi_{\text{String}}}(x) = d_7(y)$. Then $y - \beta(x) \in \operatorname{Ker}(d_7)$ and thus again by exactness, there exists an element $w \in H^7(P;\mathbb{Q})$ such that $\iota_x^* w = y - \beta(x)$. Set $z := \breve{\iota}_{\pi_{\text{String}}}(w) + x \in H^8(\pi_{\text{String}};\mathbb{Q})$. Then by commutativity,

$$\beta(z) = \iota_x^* w + \beta(x) = y - \beta(x) + \beta(x) = y.$$

Hence β is an isomorphism and we can identify $H^7(\text{String};\mathbb{Q})$ and $H^8(\pi_{\text{String}};\mathbb{Q})$.

From equation (5.1) it follows that

$$d_7(\beta(I(\widehat{S}))) = \breve{p}_{\pi_{\text{String}}}(I(\widehat{S})) = \tfrac{1}{6}p_2 = d_7(s_7)_{\mathbb{Q}},$$

and since s_7 is the unique element which gets mapped by d_7 to $\frac{1}{6}p_2$, it follows that $\beta(I(\widehat{S})) = (s_7)_{\mathbb{Q}}$, proving 2.

For part 3, we use the fact that $\widehat{(s_7)}_{\mathbb{Q}}$ is the unique differential character in $\widehat{H}^7(\text{String};\mathbb{Q})$ satisfying $\operatorname{curv}(\widehat{(s_7)}_{\mathbb{Q}}) = \iota_x^* CS_\theta(\frac{1}{6}p_2)$ and $I(\widehat{(s_7)}_{\mathbb{Q}}) = (s_7)_{\mathbb{Q}}$. Thus it remains to show that $\operatorname{curv}(\widehat{S}) = \iota_x^* CS_\theta(\frac{1}{6}p_2)$. To do this, consider the fiber P_x of P at the point $x \in M$ as the pullback

$$\begin{array}{ccc} P_x & \xrightarrow{\iota_x} & P \\ {\scriptstyle (\pi_{\text{String}})_x}\downarrow & & \downarrow{\scriptstyle \pi_{\text{String}}} \\ \{x\} & \xrightarrow{\iota_x} & M. \end{array}$$

From this we have a relative map $\iota_x : (E_x, \{x\}) \to (E, M)$ which induces a map on relative differential cohomology $\iota_x^* : \widehat{H}^*(\pi_{\text{String}};\mathbb{Q}) \to \widehat{H}^*((\pi_{\text{String}})_x;\mathbb{Q})$. Now consider the following commutative diagram where the top and bottom lines are part of the long exact sequences for relative cohomology.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \widehat{H}^7(P;\mathbb{Q}) & \xrightarrow{\breve{\iota}_{\pi_{\text{String}}}} & \widehat{H}^8(\pi_{\text{String}};\mathbb{Q}) & \xrightarrow{\breve{p}_{\pi_{\text{String}}}} & \widehat{H}^8(M;\mathbb{Q}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \iota_x^*} & & \downarrow{\scriptstyle \iota_x^*} & & \downarrow{\scriptstyle \iota_x^*} & & \\ \cdots & \longrightarrow & \widehat{H}^7(P_x;\mathbb{Q}) & \xrightarrow{\breve{\iota}_{(\pi_{\text{String}})_x}} & \widehat{H}^8((\pi_{\text{String}})_x;\mathbb{Q}) & \xrightarrow{\breve{p}_{(\pi_{\text{String}})_x}} & \widehat{H}^8(\{x\};\mathbb{Q}) & \longrightarrow & \cdots \end{array}$$

On forms the map $\iota_x^* : \Omega^7(M)\times\Omega^6(P) \to \omega^7(\{x\})\times\Omega^6(P_x)$ and thus for $\hat{q} \in \widehat{H}^8(\pi_{\text{String}}; \mathbb{Q})$, we have that $(\text{curv},\text{cov})(\hat{q}) = (0, \iota_x^*\text{cov}(\hat{q}))$. Furthermore as $\widehat{H}^i(\{x\}; \mathbb{Q}) = 0$ for $i = 7, 8$ then the map $\breve{\iota}_{(\pi_{\text{String}})_x}$ is an isomorphism. Thus we have a map $\widehat{\beta} := (\breve{\iota}_{(\pi_{\text{String}})_x})^{-1}\iota_x^* : \widehat{H}^8(\pi_{\text{String}}; \mathbb{Q})$ and a quick check shows us that this map is a refinement of the map β. Finally, we find that the curvature of $\widehat{\beta}(\widehat{S})$ is

$$\text{curv}(\widehat{\beta}(\widehat{S})) = \text{curv}((\breve{\iota}_{(\pi_{\text{String}})_x})^{-1}\iota_x^*\widehat{S}) = \text{cov}(\iota_x^*\widehat{S}) := \iota_x^* CS_\theta(\tfrac{1}{6}p_2).$$

Hence it follows that $\widehat{\beta}(\widehat{S}) = \widehat{(s_7)}_{\mathbb{Q}}$. □

5.2 DIFFERENTIAL COCYCLE TRIVIALIZATIONS

In Hopkins and Singer [14], there is alternative notion of trivialization that they introduce there which involves their differential cocycle model for differential cohomology. In [27], Redden unpackages this version of differential trivializations further and proves a variety of nice properties for these objects. One of the nice things he finds about these differential trivializations is that they are a torsor for $\widehat{H}^{2k-1}(M; \Lambda)$. Now recall that in the topological setting that we also had two different notions of trivializations, one for which we related isomorphism classes of certain cohomology classes on the total space of a principal bundle and another based on cocycles and coboundaries. We have an analogous situation here where now the differential trivializations of Hopkins and Singer are the cocycle version and Becker's trivializations are the principal bundle cohomology class version. A natural question then arises whether the two definitions are related. This possibility seems more likely once we know that both definitions describe torsors for the group $\widehat{H}^{2k-1}(M; \Lambda)$, and in fact it turns out that they are indeed related. We briefly recall Redden's definintion for differential trivializations and then we construct a map which gives a correlation between these two notions.

Given a differential character $\hat{x} \in \widehat{H}^k(M; \Lambda)$, we can choose a differential cocycle $(c, h, \omega) \in \check{Z}(k)^k(M; \Lambda)$ representing $\hat{x}$. For such a cocycle, Redden defines the category of trivializations of $\hat{x}$ as

$$\mathcal{T}riv(\hat{x}) := \pi_{\leqslant 1}\left\{\check{C}(k-1)^{k-3} \xrightarrow{d} \check{C}(k-1)^{k-2}(M;\Lambda) \xrightarrow{d} d^{-1}(\hat{x})\right\}$$

where the objects are differential cochains which get mapped by d to (c,h,ω). The morphisms of this category are given by differential cochains in $\check{C}(k-1)^{k-2}(X;\Lambda)$ where $(r,s,t) \in \check{C}(k-1)^{k-2}(M;\Lambda)$ is a morphism from (c,h,ω) to (c',h',ω') if $d(r,s,t) = (c'-c, h'-h, \omega'-\omega)$. Now for any two objects (c,h,ω), $(c',h',\omega') \in \mathcal{T}riv(\hat{h})$ it follows that there difference is a differential cocycle. By taking equivalence classes of these objects where we identify any two trivializations which have a morphism connecting them, then two differential cocyles represent the same isomorphism class of differential trivializations if the difference between them represents $0 \in \widehat{H}^{k-1}(M;\Lambda)$. It follows that the set of isomorphism classes of differential trivializations is a torsor for $\widehat{H}^{k-1}(M;\Lambda)$. Let's denote these isomorphism classes as

$$\mathrm{Triv}(\hat{x}) := \pi_0\mathcal{T}riv(\hat{x}).$$

We want to compare this definition of a differential trivialization with Becker's definition of a differential trivialization. In particular, we will focus on the case of differential String classes. Becker defines a differential String class to be a differential $\frac{1}{2}p_1$-trivialization on a principal Spin-bundle with connection. As the group Spin is 2-connected, he shows that these differential String classes have following additional properties.

Proposition 5.2.1. *Let $\pi : (Q,\theta) \to M$ be a principal* Spin*-bundle with connection. Then $\hat{q} \in \widehat{H}^3(Q;\mathbb{Z})$ is a differential String class if and only if $I(\hat{q})$ is a String class and $curv(\hat{q}) = CS_\theta(\frac{1}{2}p_1) - \pi^*\rho$ for some $\rho \in \Omega^3(M)$. Furthermore, the set of differential String classes is a torsor for $\widehat{H}^3(M;\mathbb{Z})$.*

To compare this definition of differential String classes with $\mathrm{Triv}(\frac{1}{2}\widehat{p_1})$, we will construct a map $\widehat{\Pi} : \mathrm{Triv}(\frac{1}{2}\widehat{p_1}) \to \{\text{differential String classes}\}$. As the set $\mathrm{Triv}(\hat{x})$ requires a differential cocycle representative, then in order to compare these definitions, a choice of cocycle representative for $\widehat{CW}_\theta(\frac{1}{6}p_2)$ must be made. Thus the map $\widehat{\Pi}$ depends on this choice of representative. Nevertheless, regardless of this choice, we show that this map $\widehat{\Pi}$ is equivariant and that when the set of differential u-trivializations is a torsor for $\pi^*\widehat{H}^{2k-1}(M;\Lambda)$, (as in the case of differential String classes) then this map gives a 1-1 correspondence.

Proposition 5.2.2. *Let* $\pi_{\text{Spin}} : (Q, \theta) \to M$ *be a principal* Spin*-bundle with connection* Θ*. Suppose further that* $\frac{1}{2}p_1(Q) = 0$*. Then the set of differential* String *classes of* Q *is in 1-1 correspondence with the set* $\text{Triv}(\frac{1}{2}\check{p}_1)$ *where* $\frac{1}{2}\check{p}_1$ *is a differential cocycle representative of* $\widehat{CW}_\theta(\frac{1}{2}p_1)$*.*

Proof. Recall that $\widehat{CW}_\Theta(\frac{1}{2}p_1)$ is the unique canonical differential character in $\widehat{H}^{2k}(B\text{Spin}; \mathbb{Z})$ corresponding to $\frac{1}{2}p_1 \in K^{2k}(\text{Spin}; \mathbb{Z})$. Following the construction of the morphism $I : \widehat{H}^*(-; \mathbb{Z}) \to H^*(-; \Lambda)$, we may choose a lift $\widetilde{CW}_\Theta(\frac{1}{2}p_1)$ of $\widehat{CW}_\Theta(\frac{1}{2}p_1)$ and obtain a cocycle

$$\mu := \mu^{\widetilde{CW}_\Theta(\frac{1}{2}p_1)} = \int CW_\Theta(\tfrac{1}{2}p_1) - \delta\widetilde{CW}_\Theta(\lambda, u)$$

which represents $\frac{1}{2}p_1$ in $H^4(B\text{Spin}; \mathbb{Z})$. Then $(\mu, \widetilde{CW}_\Theta(\frac{1}{2}p_1), CW_\Theta(\frac{1}{2}p_1))$ is a differential cocycle representative for $\widehat{CW}_\theta(\frac{1}{2}p_1)$. Now under the universal bundle map $\pi_{E\text{Spin}} : E\text{Spin} \to B\text{Spin}$, we can take the pullback to get $\check{z} := (\pi^*_{E\text{Spin}}\mu, \pi^*_{E\text{Spin}}\widetilde{CW}_\Theta(\frac{1}{2}p_1), CW_\Theta(\frac{1}{2}p_1)) \in \check{Z}(3)^4(EG; \Lambda)$ representing $\pi^*_{EG}\widehat{CW}_\Theta(\frac{1}{2}p_1)$. Now as $\pi^*_{E\text{Spin}}\frac{1}{2}p_1 \in H^4(E\text{Spin}; \Lambda) = 0$, it follows from [27] that the set $\text{Triv}(\check{z})$ is nonempty. Here we make a choice of differential cochain (r, s, τ) trivializing $\check{z}$. Moreover we'll further require that the differential form $\tau \in \Omega^3(E\text{Spin})$ is the Chern-Simons form $\tau := CS_\Theta(\frac{1}{2}p_1)$. Let $f : M \to B\text{Spin}$ be a classifying map for the bundle $\pi_{\text{Spin}} : (Q, \theta) \to M$ such that $f^*\Theta = \theta$. Suppose that $\frac{1}{2}p_1(P) := f^*\frac{1}{2}p_1 = 0 \in H^4(M; \Lambda)$. Then $f^*\mu$ admits topological cocycle trivializations and $\widehat{CW}_\theta(\frac{1}{2}p_1) \in \widehat{H}^4(M; \mathbb{Z})$ admits differential cocycle trivializations. Set $\frac{1}{2}\check{p}_1 = (f^*\mu, f^*\widetilde{CW}_\Theta(\frac{1}{2}p_1), CW_\theta(\frac{1}{2}p_1))$, and consider the following commutative diagram describing the bundle morphism $F : P \to E\text{Spin}$,

$$\begin{array}{ccc} P & \xrightarrow{F} & E\text{Spin} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{E\text{Spin}}} \\ M & \xrightarrow{f} & B\text{Spin}\,. \end{array}$$

Given a differential cochain $(x, y, \omega) \in \text{Triv}(\frac{1}{2}\check{p}_1)$, consider the differential cochain given by

$$(F^*r - \pi^*x, F^*s - \pi^*y, CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega) \in \check{C}(3)^3(P; \mathbb{Z}).$$

We claim that this differential cochain is closed and more importantly, that the differential character corresponding to this differential cocycle is a differential u-trivialization in the sense of Becker. Notice that this cochain is constructed in a similar manner as in the topological case where in constructing the functor Π in 2.1, we had to make a choice of cocycle representative u for $[u]$ and a choice of trivialization $v \in C^{k-1}(E\mathrm{Spin};\Lambda)$ such that $\delta v = \pi^*_{E\mathrm{Spin}}$. We can assume that the representatives we chose were $u = \mu$ and $v = r$. To see that we actually defined a differential cocycle, we have

$$d(F^*r - \pi^*x, F^*s - \pi^*y, CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega) =$$
$$= \left(F^*\delta r - \pi^*\delta x, (CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega) - (F^*r - \pi^*x) - \delta(F^*s - \pi^*y), dCS_\theta(\tfrac{1}{2}p_1) - \pi^*d\omega\right).$$

We have by commutativity that

$$F^*\delta r - \pi^*\delta x = F^*\pi^*_{EG}\mu - \pi^*f^*\mu = 0$$

and we know that $dCS_\theta(\frac{1}{2}p_1) = \pi^*CW_\theta(\frac{1}{2}p_1) = \pi^*d\omega$. It remains to show show that the middle real valued cochain vanishes, and indeed it does as we have

$$\begin{aligned}
(CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega) - (F^*r - \pi^*x) - \delta(F^*s - \pi^*y) = \\
&= CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega - F^*r + \pi^*x - F^*(CS_\Theta(\tfrac{1}{2}p_1) - r - \pi^*_{EG}\widetilde{CW}_\Theta(\tfrac{1}{2}p_1)) \\
&\qquad + \pi^*(\omega - x - f^*\widetilde{CW}_\Theta(\tfrac{1}{2}p_1)) \\
&= (CS_\theta(\tfrac{1}{2}p_1) - F^*CS_\Theta(\tfrac{1}{2}p_1)) + (F^*\pi^*_{EG}\widetilde{CW}_\Theta(\tfrac{1}{2}p_1) - \pi^*f^*\widetilde{CW}_\Theta(\tfrac{1}{2}p_1)) \\
&= 0.
\end{aligned}$$

Let's denote this cocycle as $\check{w} := (F^*r - \pi^*x, F^*s - \pi^*y, CS_\theta(\frac{1}{2}p_1) - \pi^*\omega)$. There is a corresponding differential character in $\widehat{H}^{2k-1}(P;\Lambda)$ which we will denote by $[\check{w}]$. We now must show that this differential character is in fact a differential String class.

As was hinted at earlier, recall that in Section 2.3 we defined the set $T_2(f^*\alpha)$ for some $\alpha \in Z^{*+1}(BG;\mathbb{Z})$ to be the set of cochains in $C^*(M;\mathbb{Z})$ up to isomorphism which trivialize $f^*\alpha$. We also constructed a map $\Pi : T_2(\alpha) \to T_1([\alpha])$ where $T_1([\alpha])$ was the set of $[\alpha]$-trivializations, and we showed that when the fiber was $(n-1)$-connected, this map was an equivariant bijection. As $(x, y, \omega) \in \mathrm{Triv}(\check{x})$, then it follows that $x \in T_2(f^*\mu)$ and

$\Pi(x) = F^*r - \pi^*x \in T_1(\frac{1}{2}p_1)$. In other words, as $I([\check{w}]) = [F^*r - \pi^*x] = \Pi(x)$, then $I([\check{w}])$ is a String class. Additionally, we note that $\text{curv}(\hat{w}) = CS_\theta(\frac{1}{2}p_1) - \pi^*\omega$. Now by definition differential String classes are differential $\frac{1}{2}p_1$-trivializations. By Proposition 5.2.1, an equivalent definition for a differential String class is any character $\hat{q} \in \widehat{H}^3(P;\mathbb{Z})$ such that $I(\hat{q})$ is a $\frac{1}{2}p_1$-trivialization and $\text{curv}(\hat{q}) = CS_\theta(\frac{1}{2}p_1) - \pi^*\rho$ where $\rho \in \Omega^3(M)$. Hence it follows that $\check{w}$ is in fact a differential String class, and we have defined a map

$$\widehat{\Pi} : \text{Triv}(\tfrac{1}{2}\check{p}_1) \to \{\text{differential } \tfrac{1}{2}p_1\text{-trivializations}\},$$

$$(x, y, \omega) \mapsto [(F^*r - \pi^*x, F^*s - \pi^*y, CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega)].$$

To conclude our proof, it suffices to show that this map is equivariant under the action of $\widehat{H}^3(X;\mathbb{Z})$. Then as both sets are torsors for $\widehat{H}^3(X;\mathbb{Z})$, it will follow that this map is in fact a bijection. However in order to do this, we redefine the action of $\widehat{H}^3(X;\mathbb{Z})$ on $\text{Triv}(\frac{1}{2}\check{p}_1)$ as $([(\alpha,\beta,\gamma)], (x,y,\omega) \mapsto (x-\alpha, y-\beta, \omega-\gamma)$ where $(\alpha,\beta,\gamma) \in \check{Z}(2)^3(X;\mathbb{Z})$ and $(x,y,\omega) \in C(3)^3(X;\mathbb{Z})$. As the previous action defined by addition was free and transitive, then it remains true for this action. Thus given $\hat{a} \in \widehat{H}^3(X;\mathbb{Z})$ and $(x,y,\omega) \in \text{Triv}(\frac{1}{2}\check{p}_1)$, we must show that $\widehat{\Pi}((x,y,\omega)\cdot\hat{a}) = \widehat{\Pi}((x,u,\omega)\cdot\hat{a}$. Picking a differential cocycle representative $(\alpha,\beta,\gamma) \in \check{Z}(2)^3(X;\mathbb{Z})$ of $\hat{a}$, we find that

$$\begin{aligned}
\widehat{\Pi}((x,y,\omega)\cdot\hat{a}) &= \widehat{\Pi}((x-\alpha, y-\beta, \omega-\gamma)) \\
&= [(F^* - \pi^*(x-\alpha), F^*s - \pi^*(y-\beta), CS_\theta(\tfrac{1}{2}p_1) - \pi^*(\omega-\gamma)] \\
&= [(F^*r - \pi^*x + \pi^*\alpha, F^*s - \pi^*y + \pi^*\beta, CS_\theta(\tfrac{1}{2}p_1) - \pi^*\omega + \pi^*\gamma)] \\
&= \widehat{\Pi}((x,y,\omega)) - \pi^*\hat{a} \\
&= \widehat{\Pi}((x,y,\omega))\cdot\hat{a}.
\end{aligned}$$

Hence $\widehat{\Pi}$ is equivariant and the proof follows. □

Remark 5.2.3. One needs to be cautious in that no smooth structure was specified for the universal Spin-bundle and in the notion of a differential form on the base space and total space is ill-defined. However there is a classical result by Narasimhan and Ramanan [24] that for G a Lie group with finite components, there is a system of N-classifying bundles. Briefly, for each positive integer N and for every manifold M of dimension less than N, there is a principal G-bundle $\pi_N : EG_N \to BG_N$ with connection Θ_N such that the set of all G bundles with connection over M are classified by maps from M to BG_N. Moreover they construct such a system of N-classifying objects where the universal bundle for each N is finite dimensional. Using this one can show that the concept of a differential form is well defined on these spaces.

The main point of this result is that, in the particular cases of differential String structures, the definitions for differential trivializations provided by Redden in [27] and Becker in [2] agree. Now that we have shown that these definitions are related for the case of differential String classes, we will do the same for differential rational Fivebrane classes. Thus we would like to compare our definition 5.1.3 for differential rational Fivebrane classes to the set $\mathrm{Triv}((\frac{1}{6}\widehat{p_2})_{\mathbb{Q}})$.

Proposition 5.2.4. *Let $\pi_{\mathrm{String}} : P \to M$ be a principal* String*-bundle and let $\pi_{\mathrm{Spin}} : (Q, \theta) \to M$ be the underlying principal* Spin*-bundle equipped with a connection. Then there is a 1-1 correspondence between the set of differential rational* Fivebrane *classes and the set Triv$(\frac{1}{6}\check{p}_2)$ where $\frac{1}{6}\check{p}_2$ is some differential cocycle representative of $\widehat{CW}_\theta(\frac{1}{6}p_2, \mathbb{Q})$.*

Proof. The proof follows along the same lines as Proposition 5.2.2. Since rationally $\frac{1}{6}p_2 \in H^8(B\mathrm{Spin}; \mathbb{Q})$, we can choose a differential cocycle representative given by $\check{\nu} := (\nu, \widetilde{CW}_\Theta(\frac{1}{6}p_2), \widetilde{CW}_\Theta(\frac{1}{6}p_2)) \in \check{Z}(7)^8(B\mathrm{Spin}; \mathbb{Q})$. Given a bundle morphism $(F, f) : (Q, M) \to (E\mathrm{Spin}, B\mathrm{Spin})$, we can construct a map

$$\widehat{\Gamma} : \mathrm{Triv}(f^*\check{\nu}) \to \{\text{differential Spin-Fivebrane classes}\}$$

by setting $\widehat{\Gamma}(\check{x}) = [F^*\check{r} - \pi^*\check{x}]$ where $\check{r}$ is some differential cocycle trivialization of $\pi^*_{E\mathrm{Spin}}\check{\nu}$ with $\mathrm{curv}(\check{r}) = CS_\Theta(\frac{1}{6}p_2)$. As this map is equivariant with respect to the action and as Spin-Fivebrane classes and $\mathrm{Triv}(f^*\check{\nu})$ are both torsors for $\widehat{H}^7(M; \mathbb{Q})$. Thus we obtain a 1-1 correspondence between differential rational Spin-Fivebrane classes and the set $\mathrm{Triv}(f^*\check{\nu})$, and thus by applying Proposition 5.1.8 there is a 1-1 correspondence between differential rational Fivebrane classes and $\mathrm{Triv}(f^*\check{\nu})$. □

Thus by picking a differential cocycle representative for $(\widehat{\frac{1}{6}p_2})_Q \in \widehat{H}^8(B\mathrm{Spin};\mathbb{Q})$, the concept of a differential trivialization as described in [14, 27] coincides with the our concept of a differential rational Fivebrane class. In the String setting, these notions of differential trivializations agree and have been shown to classify isomorphism classes of geometric String structures in the sense of Waldorf [40, 41]. It poses an interesting problem of how to give a corresponding geometric description for differential rational Fivebrane structures and whether these would be classified by our differential rational Fivebrane classes.

6.0 FRACTIONAL DIFFERENTIAL CHARACTERS

The purpose of the following chapter is to study the idea of what we are calling fractional differential refinements. In essence we would like to determine, for each integer k and for each differential character $\hat{u} \in \widehat{H}^k(X;\Lambda)$, whether there exists a differential character $\hat{v} \in \widehat{H}^k(X;\Lambda)$ such that $k\hat{v} = \hat{u}$ and if so, to what degree can we say that a choice of such a character can be unique. The main drive for this problem was understanding the class $\frac{1}{6}p_2(P)$ for a principal String-bundle. The reason being that we have focused ourselves on understanding how far we can extend the classical Chern-Simons and Cheeger-Simons constructions. As we've pointed out previously, one important assumption in these theories is that the group G is a Lie group with finitely many components, and String is not a Lie group in the classical sense. To this end, we've studied the underlying principal Spin-bundle and noted that rationally the cohomologies are very similar. In focusing on what happens for integral cohomology, we come to a natural question of what it means to be a fractional cohomology class. In the case for ordinary cohomology, this is nicely summarized by a Bockstein exact sequence. For differential cohomology, the issue becomes a little muddier. What follows is our attempt to understand fractions on the level of differential cohomology.

6.1 A BRAID DIAGRAM

In attacking this question, it becomes clear that it is important to understand what happens when changing the coefficients for differential characters. Recall from Proposition 4.1.14, that given two completely disconnected subgroups Λ_1, Λ_2 of $\mathbb{R}$ such that $\Lambda_1 \subset \Lambda_2 \subset \mathbb{R}$, then there is a long exact sequence

$$\cdots \longrightarrow H^{k-2}(M;\mathbb{R}/\Lambda_2) \xrightarrow{\beta} H^{k-1}(M;\Lambda_2/\Lambda_1) \longrightarrow \widehat{H}^k(M;\Lambda_1) \xrightarrow{\hat{i}} \widehat{H}^k(M;\Lambda_2) \xrightarrow[I \bmod \Lambda_1]{} H^k(M;\Lambda_2/\Lambda_1) \xrightarrow[\beta]{} H^k(M;\Lambda_1) \longrightarrow \cdots$$

describing what happens when we refine our lattice. A very nice extension of this sequence occurs when we consider what happens when we have three subgroups. Suppose now that there are three completely disconnected nested subgroups $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3$ of $\mathbb{R}$. Then there is a short exact sequence of groups

$$1 \to \Lambda_2/\Lambda_1 \to \Lambda_3/\Lambda_1 \to \Lambda_3/\Lambda_2 \to 1$$

which leads to a long exact sequence in cohomology. Moreover this long exact sequence along with the long exact sequences in differential cohomology fit into a long exact braid diagram.

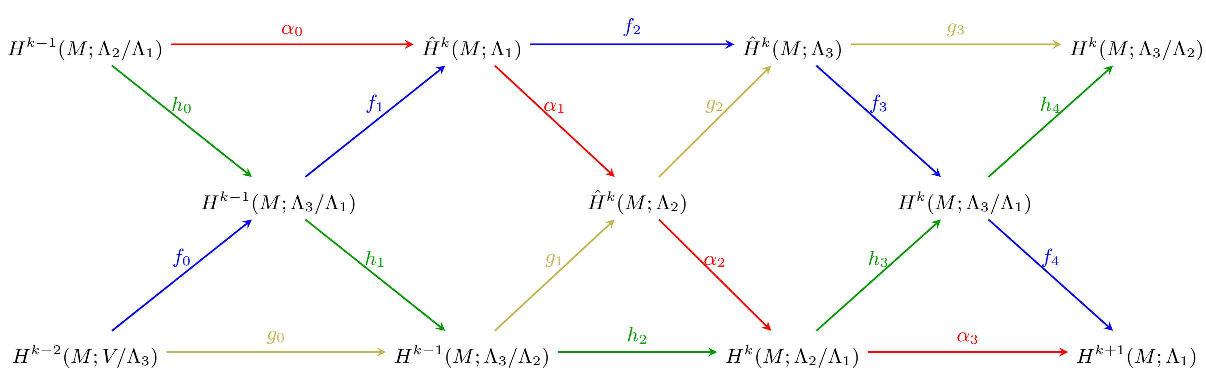
Proposition 6.1.1. *The long exact sequences in differential cohomology arising from*

$$f: \quad \Lambda_1 \hookrightarrow \Lambda_2 \to \mathbb{R}$$
$$g: \quad \Lambda_1 \hookrightarrow \Lambda_3 \to \mathbb{R}$$
$$h: \quad \Lambda_2 \hookrightarrow \Lambda_3 \to \mathbb{R}$$

and the Bockstein long exact sequence arising from

$$\alpha: \quad 1 \to \Lambda_2/\Lambda_1 \to \Lambda_3/\Lambda_1 \to \Lambda_3/\Lambda_2 \to 1 \tag{6.1}$$

fit into the following braided diagram



where all triangles and squares commute.

These continue on to the right and left as exact braids involving Bockstein sequences. To elaborate, we know that from the inclusion of lattices

$$\Lambda_1 \hookrightarrow \Lambda_2 \hookrightarrow \Lambda_3 \to \mathbb{R}$$

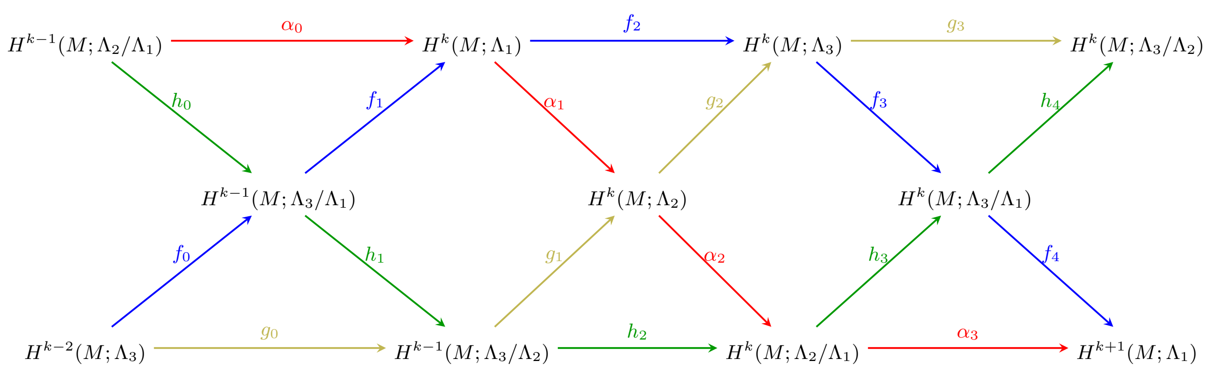
we can construct several short exact sequences of groups which take either of the following two forms

$$1 \to \Lambda_i \to \Lambda_j \to \Lambda_j/\Lambda_i \to 1 \tag{6.2}$$

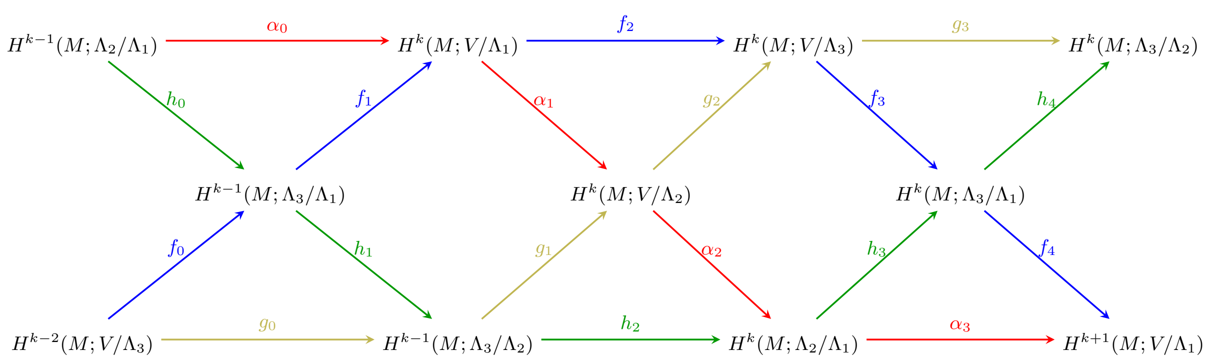
or

$$1 \to \Lambda_j/\Lambda_i \to \mathbb{R}/\Lambda_i \to \mathbb{R}/\Lambda_j \to 1 \tag{6.3}$$

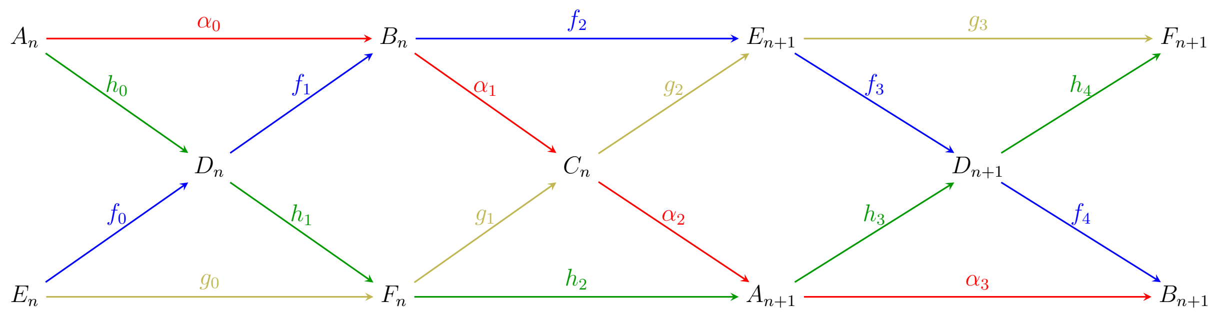
where $i < j$ along with the short exact sequence (6.1). If consider the long exact sequences arising out of short exact sequences of the form (6.1) and (6.2), then we find that these fit into the following exact braid diagram



and this is what our braid in Proposition 6.1.1 looks like as it continues to the right. Equivalently, by considering the long exact sequences arising from short exact sequences of the form (6.1) and (6.3), we obtain another exact braid diagram



which depicts how the braid in Proposition 6.1.1 continues to the left. Braid diagrams of these kind were studied in general in [11, 15]. Consider the following braid diagram



where the sequences (A, B, C), (E, F, C), (B, E, D), and (A, D, F) are all exact and where all triangles and squares commute. Then we have the following Propositions.

Proposition 6.1.2. *The following is an exact Mayer-Vietoris sequence.*

$$\cdots \to D_n \to C_n \to A_{n+1} \oplus E_{n+1} \to D_{n+1} \to \cdots \tag{6.4}$$

Proposition 6.1.3. *The top sequence*

$$X := \{\cdots \to A_n \to B_n \to E_{n+1} \to F_{n+1} \to \cdots\}$$

and the bottom sequence

$$Y := \{\cdots \to E_n \to F_n \to A_{n+1} \to B_{n+1} \to \cdots\}$$

are quasi-isomorphic chain complexes.

A proof of Proposition 6.1.2 can be found in both [11, 15]. For the proof of Proposition 6.1.3, it is straight forward to check that both of these sequences are in fact chain complexes. To show that they are isomorphic requires some diagram chasing. From these, we immediately obtain the following corollaries concerning our braid diagram in differential cohomology.

Corollary 6.1.4. *This gives rise to two Mayer-Vietoris exact sequences*

$$\cdots \longrightarrow H^{k-2}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\alpha_1 \circ f_1} H^{k-1}(M;\mathbb{Z}[\tfrac{1}{n}]) \xrightarrow{g_2 \oplus \alpha_2} \hat{H}^k(M;0) \oplus H^{k-1}(M;\mathbb{Z}[\tfrac{1}{n}]/\mathbb{Z})$$
$$\xrightarrow{f_3 - h_3} \hat{H}^k(M;\mathbb{Z}) \xrightarrow[g_4 \circ h_4]{} H^k(M;\mathbb{Z}[\tfrac{1}{n}]) \longrightarrow H^k(M;\mathbb{Z}[\tfrac{1}{n}]/\mathbb{Z}) \oplus H^{k+1}(M;0) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{k-2}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{h_1 \oplus f_1} H^{k-2}(M;\mathbb{R}/\mathbb{Z}[\tfrac{1}{n}]) \oplus H^{k-1}(M;\mathbb{Z}) \xrightarrow{g_1 - \alpha_1} H^{k-1}(M;\mathbb{Z}[\tfrac{1}{n}])$$
$$\xrightarrow{f_3 \circ g_2} \hat{H}^k(M;\mathbb{Z}) \xrightarrow[f_4 \oplus h_4]{} H^k(M;\mathbb{Z}) \oplus \hat{H}^K(M;\mathbb{Z}[\tfrac{1}{n}]) \longrightarrow H^k(M;\mathbb{Z}[\tfrac{1}{n}]) \longrightarrow \cdots$$

and

Corollary 6.1.5. *There is an isomorphism*

$$\frac{\ker\left\{\widehat{H}^k(X;\Lambda_1) \to \widehat{H}^k(X;\Lambda_3\right\}}{\operatorname{im}\left\{H^{k-1}(X;\Lambda_2) \to \widehat{H}^k(X;\Lambda_1)\right\}} \cong \frac{\ker\left\{H^{k-1}(X;\Lambda_3/\Lambda_2) \to H^k(X;\Lambda_2)\right\}}{\operatorname{im}\left\{H^{k-2}(X;\mathbb{R}/\Lambda_3) \to H^{k-1}(X;\Lambda_3/\Lambda_2)\right\}}.$$

Corollary 6.1.4 follows directly from Proposition 6.1.2. Corollary 6.1.5 follows because the left ratio comes from the cohomology of the top line and the right from the cohomology of the bottom line. This is an isomorphism by definition of quasi-isomorphism.

6.2 FRACTIONAL DIFFERENTIAL CHARACTERS

For this section, we take a direct approach at defining fractions of differential characters. As we mentioned above, for each $\hat{u} \in \widehat{H}^n(X;\Lambda)$, we are interested in characterizing the set of characters $\hat{v} \in \widehat{H}^n(X;\mathbb{Z})$ such that $k\hat{v} = \hat{u}$ for a specific k. To this end we make the following definition.

Definition 6.2.1. A **k-fractional differential refinement** of a differential character $\hat{u} \in \widehat{H}^n(X;\Lambda)$ is any character $\hat{v} \in \widehat{H}^n(X;\Lambda)$ satisfying $k\hat{v} = \hat{u}$.

In studying these characters, it will be helpful to understand how $\mathbb{R}/\mathbb{Z}$ encodes torsion of X. There is a useful homomorphism

$$\chi_k : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \qquad \chi_k(i) = \tfrac{i}{k} \tag{6.5}$$

The first question we would like address is whether or not such characters exist for a particular differential character. To answer this, we have the following proposition.

Proposition 6.2.2. *Given a differential character $\hat{u} \in \widehat{H}^n(X;\mathbb{Z})$, then the following are equivalent:*

1. *The set of k-fractional differential refinements of $\hat{u}$ is nonempty;*
2. $I(\hat{u}) \bmod k = 0$*;*
3. $\frac{1}{k}$*curv*$(\hat{u})$ *has integral periods.*

Proof. Suppose that we have a differential character $\hat{v}$ such that $k\hat{v} = \hat{u}$. Then $kI(\hat{v}) = I(\hat{u})$, and by the long exact Bockstein sequence

$$\cdots \to H^{n-1}(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta} H^n(X;\mathbb{Z}) \xrightarrow{\times k} H^n(X;\mathbb{Z}) \xrightarrow{\text{mod } k} H^n(X;\mathbb{Z}/k\mathbb{Z}) \to \cdots, \tag{6.6}$$

$I(\hat{u})$ mod $k = 0$.

Conversely, assume that $I(\hat{u})$ mod $k = 0$. Then by 6.6, there is a class $v \in H^n(X;\mathbb{Z})$ such that $kv = I(\hat{u})$. From this we can deduce that

$$k[v]_{\mathbb{R}} = [I(\hat{u})]_{\mathbb{R}} = [\text{curv}(\hat{u})]_{\text{dR}}$$

and thus

$$[v]_{\mathbb{R}} = [\tfrac{1}{k}\text{curv}(\hat{u})]_{\text{dR}}.$$

It follows then that $\frac{1}{k}\text{curv}(\hat{u}) \in \Omega^n_{\mathbb{Z}}(X)$. More importantly, we have that $(v, \frac{1}{k}\text{curv}(\hat{u})) \in R^n(X;\mathbb{Z})$. Using the exactness of 4.4 gives the existence of a character $\hat{v} \in \widehat{H}^n(X;\mathbb{Z})$ such that $I(\hat{v}) = v$ and $\text{curv}(\hat{v}) = \frac{1}{k}\text{curv}(\hat{u})$. Furthermore we have that

$$\text{curv}(k\hat{v}) = \text{curv}(\hat{u})$$
$$I(k\hat{v}) = I(\hat{u}).$$

Then using 4.4 again, there is a unique class $w \in \frac{H^{n-1}(X;\mathbb{R})}{H^{n-1}(X;\mathbb{Z})_{\mathbb{R}}}$ such that $h(w) = \hat{u} - k\hat{v}$. We can choose a class $z \in \frac{H^{n-1}(X;\mathbb{R})}{H^{n-1}(X;\mathbb{Z})_{\mathbb{R}}}$ satisfying $kz = w$. Finally, by setting $\hat{z} = \hat{v} + z$, we obtain a differential character satisfying

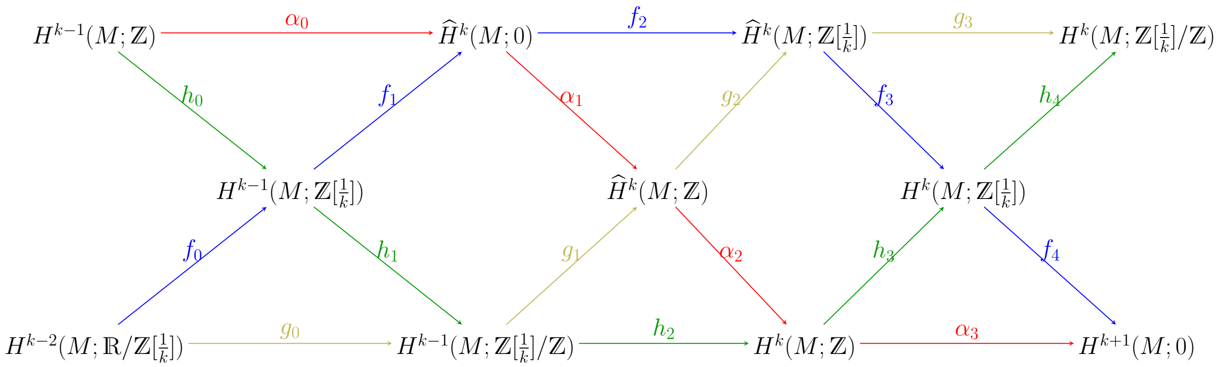
$$k\hat{z} = k(\hat{v} + z) = k\hat{v} + w = \hat{u}.$$

□

Proposition 6.2.3. *Let $\hat{v}$, $\hat{v}'$ be two k-fractional differential refinements of $\hat{u}$. Then the following are equivalent.*

1. $I(\hat{v}) = I(\hat{v}')$;
2. $\hat{v} - \hat{v}' = a(\omega)$, *where* ω *has* $\mathbb{Z}[\frac{1}{k}]$*-periods;*
3. $\hat{v} - \hat{v}' \in \mathrm{Im}\{H^{n-1}(X; \mathbb{Z}[\frac{1}{k}]) \to \widehat{H}^n(X; \mathbb{Z})\}$.

Proof. For this proof, we consider the exact braid corresponding to the inclusion $0 \subset \mathbb{Z} \subset \mathbb{Z}[\frac{1}{k}]$ where by $\mathbb{Z}[\frac{1}{k}]$ we mean the infinite cyclic subgroup of $\mathbb{R}$ additively generated by $\frac{1}{k}$ (i.e. $\{\ldots, \frac{-3}{k}, \frac{-2}{k}, \frac{-1}{k}, \frac{0}{k}\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \ldots\}$).



Assuming that $I(\hat{v}) = I(\hat{v}')$, then by exactness there is a differential form $\omega \in \frac{\Omega^{n-1}(X)}{d\Omega^{n-2}(X)}$ such that $\alpha_1(\omega) = \hat{v} - \hat{v}'$. Since $k(\hat{v} - \hat{v}') = 0$, then $\hat{v} - \hat{v}' = g_1(\mu)$ for some $\mu \in H^{n-1}(X; \mathbb{Z}\frac{1}{k}/\mathbb{Z})$. Then $f_2(\omega) = 0$ by exactness and it follows that ω has $\mathbb{Z}[\frac{1}{k}]$-periods. Conversely if we have a form $\omega \in \Omega^{n-1}(X)$ such that $a(\omega) = \hat{v} - \hat{v}'$ then $I(\hat{v}) = I(\hat{v}')$ by exactness. Thus we've shown 1 and 2 are equivalent.

Now if 1 holds, then as we previously noted, there is a class $\mu \in H^{n-1}(X; \mathbb{Z}[\frac{1}{k}])$ such that $g_1(\mu) = \hat{v} - \hat{v}'$. From this we have $\hat{v} - \hat{v}' \in \mathrm{Ker}(g_2)$ and thus

$$\hat{v} - \hat{v}' \in \mathrm{Ker}\left\{g_2 \oplus \alpha_2 : \widehat{H}^k(M; \mathbb{Z}) \to \widehat{H}^k(M; \mathbb{Z}[\tfrac{1}{k}]) \oplus H^k(M; \mathbb{Z})\right\}.$$

Then using the second Mayer-Vietoris sequence from Corollary 6.1.4, this is equivalent to

$$\hat{v} - \hat{v}' \in \mathrm{Im}\left\{g_1 \circ h_1\right\}$$

□

Thus under certain topological conditions, we can determine for a space X, integer k and differential character $\hat{u} \in \widehat{H}^n(X;\mathbb{Z})$, whether there is a unique differential $\hat{v} \in \widehat{H}^n(X;\mathbb{Z})$ such that $k\hat{v} = \hat{u}$. We should point out that this is not the same as a canonical refinement for a class v. However consider the example of a String bundle with connection on the underlying Spin bundle which admits a Fivebrane structure. It was shown in [31] through the use of ∞-stacks that there is a canonical differential refinement associated with this class. Then for the base space to have a unique 6-fractional differential character should imply that the canonical differential refinement of $\frac{1}{6}p_2$ is this unique character.

6.3 CONCLUDING REMARKS

By studying rational structures on a manifold we found that we could obtain much of the topological data describing these structures from a Spin bundle over this manifold. This becomes even more interesting when one realizes that we can define differential refinements for rational cohomology. Once we translated the theory of differential trivializations to the rational setting, it was natural for us to ask to what degree we could reverse this process and what type of information could we obtain about the original differential cohomology. By pursuing this notion of fractional differential refinements, we discovered this very exciting braid diagram. We conjecture that this type of diagram can be an very powerful calculational tool in differential cohomology. It also seems that this diagram will be very useful in understanding the role that torsion plays in differential cohomology.

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