MOTIVIC INTEGRATION, THE SATAKE TRANSFORM AND THE FUNDAMENTAL LEMMA.

by

Jorge E. Cely

B.S., Universidad de Los Andes, Bogotá, 2007

M.Sc., Universidad de Los Andes, Bogotá, 2009

Submitted to the Graduate Faculty of
the Kenneth P. Dietrich School of Arts and Sciences in partial
fulfillment
of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2016

UNIVERSITY OF PITTSBURGH

KENNETH P. DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Jorge E. Cely

It was defended on

July 8, 2016

and approved by

Professor Thomas C. Hales, Department of Mathematics, University of Pittsburgh

Professor Bogdan Ion, Department of Mathematics, University of Pittsburgh

Professor Kiumars Kaveh, Department of Mathematics, University of Pittsburgh

Professor Julia Gordon, Department of Mathematics, University of British Columbia

Dissertation Director: Professor Thomas C. Hales, Department of Mathematics, University of Pittsburgh

Copyright © by Jorge E. Cely
2016

MOTIVIC INTEGRATION, THE SATAKE TRANSFORM AND THE FUNDAMENTAL LEMMA.

Jorge E. Cely, PhD

University of Pittsburgh, 2016

The purpose of this work is to use motivic integration for the study of reductive groups over p-adic fields (towards applications of the fundamental lemma for groups). We study spherical Hecke algebras from a motivic point of view. We get a field independent description of the spherical Hecke algebra of a reductive group and its structure. We investigate the Satake isomorphism from the motivic point of view. We prove that some data of the Satake isomorphism is motivic.

TABLE OF CONTENTS

1.0 INTRODUCTION 1

2.0 MOTIVIC INTEGRATION 3

2.1 Model Theory 3

2.1.1 First-order logic 3

2.1.2 Languages and structures 4

2.1.3 Valued fields 8

2.2 Cluckers-Loeser's theory of motivic integration 9

2.2.1 Denef-Pas language 9

2.2.2 Definable subassignments 11

2.2.3 Grothendieck rings and semirings 13

2.2.4 Constructible motivic functions 14

2.2.5 The motivic measure 17

2.2.6 Integrals with parameters 22

2.2.7 Volume forms 24

2.3 Specialization of motivic functions 24

2.4 Transfer principles 26

2.4.1 Ax-Kochen-Eršhov principle 26

2.4.2 Transfer principles for motivic functions 27

3.0 THE FUNDAMENTAL LEMMA 29

3.1 Reductive groups over non-archimedean local fields 29

3.1.1 Unramified groups 31

3.1.2 Absolute and relative theory 33

3.1.3 Classification of unramified groups 40
3.1.4 The complex dual group and the L-group 41
3.2 The statement of the fundamental lemma 43
3.2.1 The spherical Hecke algebra 43
3.2.2 Orbital integrals 46
3.2.3 Endoscopic groups 50
3.2.4 The fundamental lemma - Lie algebra version 51
4.0 MOTIVIC SPHERICAL HECKE ALGEBRAS 52
4.1 Fixed choices 52
4.2 Motivic identities and motivic identities up to a null function 53
4.3 The split case 55
4.4 Definability of the Cartan decomposition 56
4.5 Motivic K-average property 67
4.5.1 K-average property 68
4.6 Motivic K-invariant functions 70
4.7 The structure theorem up to a null function 75
4.8 The structure theorem as a motivic identity 77
5.0 ON THE MOTIVIC NATURE OF THE SATAKE TRANSFORM . . 78
5.1 The Satake transform 78
5.1.1 Choice of bases 80
5.2 Motivic data on the Satake transform 80
5.3 A motivic Satake transform 82
5.4 Some representation theory 83
6.0 TOWARDS A MOTIVIC FUNDAMENTAL LEMMA 87
6.1 From G to H 87
Appendix 89
A.1 Multiplicative characters of p-adic fields 90
BIBLIOGRAPHY 92

1.0 INTRODUCTION

In late nineties, Hales started to use motivic integration to representation theory of p-adic groups.

Arithmetic motivic integration, as a generalization of p-adic integration, can be viewed as a universal theory of integration for local fields that is independent of p. In 2004, Cluckers and Loeser developed a theory of motivic integration based the model theory of certain valued fields. This is the framework of our work. Cluckers, Hales and Loeser used this later theory to prove that it is possible to transfer the Lie algebra variant of the fundamental lemma. A lot of work has been done by Gordon, Cluckers and Halupczok.

This work follows those lines and leaves some open questions on that area.

The purpose of this work is to use motivic integration for the study of reductive groups over p-adic fields (towards applications of the fundamental lemma for groups). In Chapter 2 we present the theory of motivic integration that is used here. In Chapter 3 we present all the background material on reductive p-adic groups and other ingredients in the fundamental lemma. The new contributions are in the following chapters. In Chapter 4 we study spherical Hecke algebras from a motivic point of view. We get a field independent description of the spherical Hecke algebra of a reductive group and its structure. Even though some of our results are modulo a null function, that is enough for applications of the transfer principle. In Chapter 5 we study the Satake isomorphism from the motivic point of view. We prove that some data of the Satake isomorphism is motivic. Using results from Chapter 4 we define a motivic version of the Satake transform (up to null functions). We believe that a motivic Satake can lead to a motivic version of the fundamental lemma for non-unit elements in the spherical Hecke algebra. Although we do not have a result like that, in Chapter 6 we discuss that possibility.

2.0 MOTIVIC INTEGRATION

2.1 MODEL THEORY

Model theory is the branch of mathematical logic that deals with the relations between mathematical structures and formal languages used to describe them.

Consider a mathematical structure and a formal language capable of expressing properties of the mathematical structure. Now if we assume that the formal language has a logic, the general question that arises is, what is the relation between the syntactic component (with the logic included) and the semantics of the structure? This is a broad question but it is at the heart of model theory. Firstly, we restrict to first-order model theory which uses first-order logic. That is enough for our purposes. Some references are [7], the first book in the subject; [30] and [42]. These are more recent references, and they include some applications. Historically the major developments in model theory (theory and applications) have occurred in first-order model theory. We include an short introduction and comments on the subject because the theory of motivic integration needed in this work uses the model theory of certain valued fields. We begin with the logic.

2.1.1 First-order logic

The idea is to use first-order logic to study mathematical objects. These are the logical symbols:

- Logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Quantifier symbols $\exists, \forall$

- Equality symbol = and parenthesis $($, $)$
- Variables $x_1, x_2, \ldots$

The description of a mathematical object, from the point of view of the logic, is given by the choice of a language.

2.1.2 Languages and structures

Given a mathematical object $\mathcal{M}$ the idea is to associate to $\mathcal{M}$ a first-order language $\mathcal{L}$ consisting of non-logical symbols that capture some structure of $\mathcal{M}$. The symbols in a first-order language are of three kinds:

- Predicate symbols. We attach to each predicate symbol a positive natural number that represents the arity of the predicate.
- Function symbols. We attach to each function symbol a positive natural number that represents the arity of the function.
- Constant symbols.

There is no restriction on the number of non-logical symbols. Clearly the language depends on the mathematical structure $\mathcal{M}$, but it is not unique. One has to choose it. This is a very important matter because it determines how much structure is controlled by the logic.

Definition 1. *Let $\mathcal{L}$ be a first-order language. An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following:*

i) A non-empty set M, called the universe of the structure or the underlying set (sometimes M is just denoted $\mathcal{M}$).

ii) For each function symbol $f \in \mathcal{L}$ with arity n_f, a function $f^{\mathcal{M}} : M^{n_f} \to M$.

iii) For each predicate symbol $R \in \mathcal{L}$ with arity n_R, a set $R^{\mathcal{M}} \subset M^{n_R}$.

iv) For each $c \in \mathcal{L}$, constant symbol, a member $c^{\mathcal{M}} \in M$.

Example 2. *The set $\mathbb{Z}$ of integer numbers.*

The following are some possibilities of languages for this object.

- *$\mathcal{L} = \emptyset$. The $\mathcal{L}$-structure of $\mathbb{Z}$ is the one of a countably infinite set.*

- $\mathcal{L}_{Pres} = \{+, \leq, 0, 1, \equiv_d,\ d = 2, 3, 4, \ldots\}$. *The function symbol* $+$ *(with arity 2) is interpreted as the addition in* $\mathbb{Z}$. *The relation symbol* $<$ *(with arity 2) is interpreted as the order in* $\mathbb{Z}$. *The constant symbols* $0, 1$ *are interpreted in* $\mathbb{Z}$ *as the natural numbers* 0, 1. *The symbol* $\equiv_n$ *is interpreted as the binary relation* $a \equiv b \bmod (n)$ *in* $\mathbb{Z}$. *The* $\mathcal{L}_{Pres}$*-structure of* $\mathbb{Z}$ *is called the Presburger arithmetic (arithmetic without multiplication). From the model theoretic perspective, this structure is well-understood. We use it in this work.*

- $\mathcal{L}_{rings} = \{+, \times, 0, 1\}$. *The function symbol* $\times$ *is interpreted as the usual multiplication in* $\mathbb{Z}$. *The* $\mathcal{L}_{rings}$*-structure of* $\mathbb{Z}$ *seems to be the natural place for the study of number theory. By Gödel's incompleteness theorems, this structure is extremely complicated from the point of view of logic and model theory.*

We give now some standard definitions to conclude with the definition of an $\mathcal{L}$-formula.

Definition 3. *Let* $\mathcal{L}$ *be a first-order language. The set of* $\mathcal{L}$*-terms is the smallest set* $\mathcal{T}$ *such that*

i) For each constant symbol $c \in \mathcal{L}$*, we have* $c \in \mathcal{T}$.

ii) Each variable symbol $x_i \in \mathcal{T}$.

iii) For each function symbol $f \in \mathcal{L}$ *with arity* n_f *and* $t_1, \ldots, t_{n_f} \in \mathcal{T}$*, we have* $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

Definition 4. *Let* $\mathcal{L}$ *be a first-order language. We say that* ϕ *is an atomic* $\mathcal{L}$*-formula if* ϕ *is either of the form*

i) $t_1 = t_2$*, where* t_1, t_2 *are* $\mathcal{L}$*-terms, or*

ii) $R(t_1, \ldots, t_{n_R})$*, where* R *is a predicate symbol with arity* n_R *and* $t_1, \ldots, t_{n_R}$ *are* $\mathcal{L}$*-terms.*

The set of $\mathcal{L}$*-formulas* $\mathcal{F}$ *is defined as the smallest set that contains all the atomic* $\mathcal{L}$*-formulas and such that*

i) If $\phi \in \mathcal{F}$*, then* $\neg\phi \in \mathcal{F}$.

ii) If $\phi, \psi \in \mathcal{F}$*, then* $(\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi) \in \mathcal{F}$.

iii) If $\phi \in \mathcal{F}$*, then* $\exists x\, \phi \in \mathcal{F}$ *and* $\forall x\, \phi \in \mathcal{F}$.

In the language of rings $\mathcal{L}_{rings} = \{0, 1, +, \times\}$ the following are examples of formulas

$$x + 1 = 0, \quad \exists x\,(x^3 + y = 2), \quad \text{and} \quad \forall x\,\forall y \neg(xy = yx).$$

The following strings of symbols are not $\mathcal{L}_{rings}$-formulas

$$x \to y, \quad x^y = 1, \quad \text{and} \quad ((\neg 1(\wedge x.$$

Notice that x^3 is an abbreviation of the $\mathcal{L}_{rings}$-term $x \times x \times x$, 2 is an abbreviation of $1+1$.

A free variable in an $\mathcal{L}$-formula is a variable that is not bounded by a quantifier. An $\mathcal{L}$-sentence is a an $\mathcal{L}$-formula without free variables.

In model theory one associates invariants of a logical nature to an $\mathcal{L}$-structure $\mathcal{M}$. For example $Th(\mathcal{M})$, the set of first-order $\mathcal{L}$-sentences which are true in $\mathcal{M}$, is an object called the $\mathcal{L}$-theory of $\mathcal{M}$. It is fundamental and vastly studied in model theory. But rather than look at sentences and theories, we can look at formulas with free variables in a given language. That leads to the concept of a definable set in that language. From here one gets the important concept of an $\mathcal{L}$-type which is essential in classification theory. Regarding the use of model theory in motivic integration we do not need to go further into classification theory. We need to look at the category of definable sets on certain valued fields. A definable set in a structure (the precise definition is given below) is the set of solutions of a formula with free variables in a prescribed language. This approach allows us to uniformly handle those definable sets in a class of structures where the defining formulas makes sense. When the purpose is to use and understand the definable sets in a given structure or a class of structures it is very desirable to have quantifier elimination (*i.e.* every formula is equivalent to a formula without quantifiers). This is an old and important topic in model theory and it plays an important role in the theory of motivic integration developed by Cluckers and Loeser.

Definition 5. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $X \subset M^n$ is a definable set if and only if there is an $\mathcal{L}$-formula $\phi(x_1, ..., x_n, y_1, ..., y_m)$ and $b_1, \ldots, b_m \in M^m$ such that*

$$X = \{a \in M^n : \mathcal{M} \models \phi(a, b_1, \ldots, b_m)\},$$

i.e., X consists of all the tuples $a \in M^n$ for which $\phi(a, b_1, \ldots, b_m)$ is true in $\mathcal{M}$. We say that X is definable with parameters in A or definable over A (or A-definable) if $b_1, \ldots, b_m \in A \subset M$. We say that a function $f : M^n \to M^m$ is definable if the graph of f in $M^{n \times m}$ is a definable set.

Example 6. *Let $(\mathbb{R}, +, -, \times, 0, 1)$ the $\mathcal{L}_{rings}$-structure of the real field.*

- *The formula $x^2 + y^2 = 1$ defines the unit circle in $\mathbb{R}^2$.*
- *The usual linear order $\leq$ of the reals is definable*

$$x \leq y \;\;\Leftrightarrow\;\; \exists z\,(y = x + z^2).$$

- *The function $x \mapsto x^3$ is definable because its graph is a definable subset of $\mathbb{R}^2$.*

Example 7. *Let $(\mathbb{Z}, +, -, \times, 0, 1)$ the $\mathcal{L}_{rings}$-structure of the ring of integers $\mathbb{Z}$. Lagrange's four square theorem (any natural number can be represented as the sum of four integer squares) implies that $\mathbb{N}$ is definable by the $\mathcal{L}_{ring}$-formula (y as the free variable)*

$$\exists x_1\, \exists x_2\, \exists x_3\, \exists x_4\; (x_1^2 + x_2^2 + x_3^2 + x_4^2 = y).$$

2.1.3 Valued fields

The motivic measure has a universal character with respect to the Haar measures on locally compact non-archimedean fields. It works as a measure that does not depend on any particular locally compact field. Local fields are the non-discrete locally compact fields; in the non-archimedean case they are finite extensions of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. These are valued fields, and the formulas in $\mathcal{L}_{rings}$ do not capture the extra structure (*e.g.* the valuation) on these fields. So a bigger language is need.

Definition 8. *A valued field is a field with a map* $\mathrm{ord} : F \to G \cup \{\infty\}$ *to an ordered abelian group* G *such that.*

- $\mathrm{ord}(x) = \infty$ *if and only if* $x = 0$*,*
- $\mathrm{ord}(xy) = \mathrm{ord}(x) + \mathrm{ord}(y)$ *for all* $x, y \in F$*,*
- $\mathrm{ord}(x + y) \geq \min\{\mathrm{ord}(x), \mathrm{ord}(y)\}$ *for all* $x, y \in F$*.*

The map ord *is called the valuation map of* F*.*

The following examples of valued fields are widely used in mathematics.

Example 9. *p-adic numbers.*
The field $\mathbb{Q}_p$*, its finite extensions, its algebraic closure* $\bar{\mathbb{Q}}_p$ *and its complete algebraic closure* $\mathbb{C}_p$*. All of them have the p-adic valuation.*

Example 10. *Formal Laurent series.*
For any field K*, the field* $K((t))$ *equipped with the* t*-adic valuation.*

Example 11. *Puiseux series.*
For any field K*, the field*

$$\bigcup_{n=1}^{\infty} K((t^{1/n})),$$

equipped with the t*-adic valuation.*

There are various ways (languages) in which valued fields may be regarded as first-order structures. Before we introduce in the next section the language used in Cluckers-Loeser's theory, we mention other possibilities. To begin, a valued field is first and foremost a field, so any reasonable language must start with $\mathcal{L}_{ring}$, the language of rings. Suppose F is a valued field with valuation map ord.

- Let $\mathcal{L}_V = \mathcal{L}_{ring} \cup \{V\}$, with V a unary relation symbol. We regard the valued field F as a $\mathcal{L}_V$-structure by interpreting V as the valuation ring. If F is a non-archimedean local field, $V^F = \mathcal{O}_F$, the ring of integers of F. The value group $\Gamma = F^\times/(V^F)^\times$ is present in F viewed as an $\mathcal{L}_V$-structure, we say that Γ is interpretable in $(F, +, \cdot, 0, 1, V)$.
- Let $\mathcal{L}_{div} = \mathcal{L}_{ring} \cup \{|\}$, with $|$ a binary relation symbol interpreted in F by

$$x \,|\, y \Leftrightarrow \mathrm{ord}(x) \leq \mathrm{ord}(y).$$

  In the language $\mathcal{L}_{div}$ the valuation ring is definable by the formula $1 \,|\, x$. These two languages, $\mathcal{L}_V$ and $\mathcal{L}_{div}$ describe the same structure because each structure is definable in the other.
- Macintyre's language, $\mathcal{L}_{Mac} = \mathcal{L}_{div} \cup \{P_n : n \geq 1\}$, each P_n is a unary predicate symbol, interpreted in F by

$$P_n^F = \{x \in F^\times \,|\, \exists y \in F\,(y^n = x)\},$$

  the set of n^{th} powers in F.

2.2 CLUCKERS-LOESER'S THEORY OF MOTIVIC INTEGRATION

2.2.1 Denef-Pas language

Before we introduce the Denef-Pas language, we need to describe a slight generalization of first-order languages. A three-sorted language has three sorts of variables; the description of functions symbols, relation symbols and constant symbols must specify the sort of the variables involved. Regarding the quantifiers, they range over the corresponding sort. For example, in the Denef-Pas language, an existential quantifier for a valued field variable ranges over the valued field sort. A structure for a three-sorted language has a universe with three disjoint sorts.

The Denef-Pas language is a three-sorted language designed for valued fields. The three sorts correspond respectively to the valued field (denoted by VF), the residue field (denoted by RF) and the value group (which is always $\mathbb{Z}$).

The Denef-Pas language, denoted by $\mathcal{L}_{DP}$, consist of the following symbols

- $\mathcal{L}_{\text{rings}}$ for the valued field sort VF,
- $\mathcal{L}_{\text{rings}}$ for the residue field sort RF,
- $\mathcal{L}_{\text{Pres}}$ for the valued group sort $\mathbb{Z}$,
- $\text{ord} : \text{VF}^{\times} \to \mathbb{Z}$ a function symbol for the valuation map,
- $\text{ac} : \text{VF} \to \text{RF}$ a function symbol for the angular component map.

The language of rings $\mathcal{L}_{\text{rings}} = \{+, \times, 0, 1\}$, already introduced, is the standard language used in model theory for the study of rings, fields and skew fields. This language consists of two function symbols $+, \times$ interpreted as the addition and the multiplication in the corresponding field; and two constant symbols $0, 1$ which are interpreted in the expected way, as the additive unit and the multiplicative unit in the field. Notice that a quantifier-free formula with n free variables in the language of rings defines a constructible set (a boolean combination of zero sets of polynomials in n variables).

The Presburger language $\mathcal{L}_{\text{Pres}} = \{+, \leq, 0, 1, \equiv_d,\ d = 2, 3, 4, \ldots\}$ is described above in example 2. The description of the definable sets and functions by formulas in $\mathcal{L}_{\text{Pres}}$ is relatively simple because of quantifer elimination proved by Presburger [48]. For instance, the definable subsets of $\mathbb{Z}$ in the Presburger language are finite unions of arithmetic progressions (in positive or negative direction) and points.

The function symbol $\text{ord} : \text{VF}^{\times} \to \mathbb{Z}$ is interpreted as the valuation map, so clearly by construction, this language only can be used for valued fields with value group $\mathbb{Z}$; for example it is suitable for non-archimedean local fields but not for algebraically closed valued fields.

The function symbol $\text{ac} : \text{VF} \to \text{RF}$ is interpreted as an angular component map, this requires us to fix a uniformizer (a generator of the unique non-zero prime ideal of the discrete valuation ring) in the valued field. We give more details in the next sections.

In summary, any formula in $\mathcal{L}_{DP}$ can be interpreted in any discretely valued field once a uniformizer is chosen. Hence, any discretely valued field is an $\mathcal{L}_{DP}$-structure.

2.2.2 Definable subassignments

The main references for this section are [20] and [10]. We quote some parts of these references. Let K be a field containing the ground field k. The field of formal Laurent series $K((t))$ is a valued field with

$$\mathrm{ord}(f) = N, \quad ac(f) = a_N$$

where $f = \sum_{i=N}^{\infty} a_i t^i$ and $N \neq 0$. In this case the uniformizer is t. Then $(K((t)), K, \mathbb{Z})$ is a structure in the Denef-Pas language.

Let $\mathbf{Field}_k$ be the category of fields containing the ground field k.

Definition 12. *We will denote by $h[m,n,r]$ the functor from the category $\mathbf{Field}_k$ to the category* **Sets** *by*

$$h[m,n,r](K) = K((t))^m \times K^n \times \mathbb{Z}^r,$$

where $K((t))$ is the field of formal Laurent series with coefficients in K.

Some examples: $h[1,0,0](K) = K((t))$, $h[0,0,r](K) = \mathbb{Z}^r$ and $h[0,0,0]$ is the functor that assigns to each field K a one-point set.

Definition 13. *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to$* **Sets** *be a functor. A subassignment of F is a collection of subsets $h(C) \subseteq F(C)$, for each object in $\mathcal{C}$.*

Note that there is no requirement about the morphisms so a subassignment has not to be a subfunctor. The subassignments play an important role, they are the manner in which we describe sets in a uniform way, but the idea is to do this in a definable way and that motivates the following definition.

Definition 14. *Let h be a subassignment of the functor $h[m,n,r]$, we say that h is a definable subassignment if there exists a formula ϕ in the Denef-Pas language with coefficients in $k((t))$*

and m free variables of the valued field sort, with coeffients in k and n free variables of the residue field sort, and r free variables of the value sort, such that for each $K \in \mathbf{Field}_k$, $h(K)$ is the definable subset of $K((t))^m \times K^n \times \mathbb{Z}^r$ given (or defined) by the formula ϕ.

The definable subassignments are just the connection (in a uniform way) between the formulas and their corresponding definable sets. The following definition leaves us in position to talk about a category of definable subassignments.

Definition 15. *Let h_1 and h_2 be two definable subassignments (probably of different functors). A morphism between h_1 and h_2 is a definable subassignment G such that $G(K)$ is the graph of a function from $h_1(K)$ to $h_2(K)$ (note that this is just a definable function in the Denef-Pas language between two definable sets), for each element in $\mathbf{Field}_k$.*

We will denote by $\mathbf{Def}_k$ the category of definable subassignments. Note that it is possible to define set-theoretic operations on subassignments in a natural way, for instance, $(h_1 \cap h_2)(K) := h_1(K) \cap h_2(K)$.

A point on a subassignment $S \in \mathbf{Def}_k$ is, by definition, a pair (s_0, K) where $K \in \mathbf{Field}_k$, and $s_0 \in S(K)$. We denote the collection of points of S by $|S|$.

The relative situation is defined as follows. Let S be an object in $\mathbf{Def}_k$. We define the category $\mathbf{Def}_S$ as follows: the objects are the definable subassignments equipped with a morphism to S, so for $Y \in \mathbf{Def}_k$, $[Y \to S] \in \mathbf{Def}_S$ and morphisms are commutative triangles

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[dl] \\ S & \end{tikzcd}$$

For $S \in \mathbf{Def}_k$, we define the following object in $\mathbf{Def}_S$,

$$S[m, n, r] := S \times K((t))^m \times K^n \times \mathbb{Z}^r$$

the morphism to S is given by the projection onto the first factor.

We construct another category. Given $S \in \mathbf{Def}_k$, we define the category of R-definable subassignments over S, denoted by $\mathbf{RDef}_S$, whose objects are definable subassignments of $S[0,n,0]$ for some $n \geq 0$, the morphism to S of this elements should be the projection on the first factor, and the morphisms between two elements in this new category are morphisms over S, similar to above. The idea is that the elements in this category correspond to definable elements in the residue sort attached somehow to S. For example, $\mathbf{RDef}_{Spec\,k}$ is basically the category of definable subsets in the residue sort. They are the definable subsets in the language of rings.

2.2.3 Grothendieck rings and semirings

Various theories of motivic integration (if not all) are associated with Grothendieck rings. As a first example we can consider the Grothendieck ring of algebraic varieties over a field k, denoted $K_0(\mathrm{Var}_k)$. It is defined as the free abelian group generated by the isomorphism classes of k-varieties modulo the set of relations of the form $[X - Y] = [X] - [Y]$, whenever Y is a closed subvariety of X. The product operation is defined using the product operation on varieties. If k has characteristic zero, Poonen proved that $K_0(\mathrm{Var}_k)$ is not a domain [47]. There are open questions about this ring.

Let $S \in \mathbf{Def}_k$. The Grothendieck semigroup of R-definable subassignments over S, denoted by $SK_0(\mathbf{RDef}_S)$ is defined as the quotient of the free abelian semigroup over symbols $[Y \to S]$ with $Y \to S \in \mathbf{RDef}_S$ by relations

$$[\emptyset \to S] = 0,$$

$$[Y \to S] = [Y' \to S]$$

if $Y \to S$ is isomorphic to $Y' \to S$ (over S) and

$$[(Y \cup Y') \to S] + [(Y \cap Y') \to S] = [Y \to S] + [Y' \to S]$$

for Y and Y' definable subassignments of some $S[0,n,0] \to S$. Similarly, one defines the Grothendieck group of R-definable subassignments over S, denoted by $K_0(\mathbf{RDef}_S)$, as the quotient of the free abelian group over symbols $[Y \to S]$ with $Y \to S \in \mathbf{RDef}_S$ by the same relations above. The cartesian fiber product over S induces a semiring, resp. ring, structure on $SK_0(\mathbf{RDef}_S)$, resp. $K_0(\mathbf{RDef}_S)$, by

$$[Y \to S][Y' \to S] = [Y \otimes_S Y' \to S].$$

The multiplicative unit in the ring and the semiring is $[S \to S]$. The ring $K_0(\mathbf{RDef}_S)$ is nothing but the ring obtained from $SK_0(\mathbf{RDef}_S)$ by inverting additively every element. In Lebesgue's theory of integration, the positive functions play an important role. In this framework the situation is similar, that is the reason why Grothendieck semirings have to be considered.

2.2.4 Constructible motivic functions

In this section we define the ring of constructible motivic functions over a fixed subassignment. We begin with the definition of the ring of values of these functions. Let $\mathbb{L}$ be a formal symbol (not a symbol in a first-order language). One considers the ring

$$A = \mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, \left(\frac{1}{1-\mathbb{L}^{-n}}\right) : n \geq 0\right].$$

Let q be a real number greater than 1. We have a ring homomorphism

$$v_q : A \to \mathbb{R}$$

defined by $\mathbb{L} \mapsto q$. We consider the semiring

$$A_+ = \{x \in A \,|\, v_q(x) \geq 0, \forall q > 1\}$$

for the functions taking "positive" values. In fact, there are two separate constructions: the semiring of constructible motivic functions over a definable subassignment S (associated to "positive" functions on S, the semiring A_+ and the Grothendieck semiring $SK_0(\mathbf{RDef}_S)$, and the ring of constructible motivic functions on S (associated to the ring A and the Grothendieck ring $K_0(\mathbf{RDef}_S)$).

Let $S \in \mathbf{Def}_k$ be a definable subassignment. The semiring and the ring of constructible motivic functions over S is built from functions of two types. The first resembles functions that appear naturally in the p-adic setting. This is not accidental, and it is explained in more detail in the section on specialization of constructible motivic functions.

We consider the subring $\mathcal{P}(S)$ of the ring of functions $|S| \to A$ generated by

- Constant functions in A.
- Definable functions $\alpha : S \to \mathbb{Z}$ in the Denef-Pas language.
- Functions of the form $\mathbb{L}^\alpha$, where $\alpha : S \to \mathbb{Z}$ is a definable function in the Denef-Pas language.

We define $\mathcal{P}_+(S)$ as the semiring of functions in $\mathcal{P}(S)$ taking values in A_+.

The functions of the second type are elements in the Grothendieck ring $K_0(\mathbf{RDef}_S)$, or in the Grothendieck semiring $SK_0(\mathbf{RDef}_S)$. So strictly speaking these elements are not functions over S. Nevertheless, if we think of specialization to p-adic integration (explained below in detail), we get a family of functions associated to S. Let K be a non-archimedean local field with residue field $\mathbb{F}_q$. Let $[Y \to S] \in \mathbf{RDef}_S$. This element gives the following integer-valued function on $S(K)$:

$$\begin{aligned} &S(K) \to \mathbb{Z}, \\ &\qquad x \mapsto \#\{y \in Y(\mathbb{F}_q^n) \,|\, y \mapsto x\}, \end{aligned}$$

where $Y \in S[0, n, 0]$. That is, the cardinality of the fiber of Y over x.

To put together the two kind of functions we proceed as follows. In $K_0(\mathbf{RDef}_S)$ and $SK_0(\mathbf{RDef}_S)$, the isomorphism class of the subassignment $x \neq 0$ of $S[0, 1, 0]$ is denoted by $\mathbb{L} - 1$. We denote by $\mathcal{P}^0(S)$ the subring of $\mathcal{P}(S)$ generated by

- the characteristic functions $\mathbf{1}_Y : S \to \mathbb{Z}$, where Y is a definable subassignment of S, and
- the constant function $\mathbb{L} - 1$.

The subsemiring $\mathcal{P}^0_+(S)$ contains the same functions as $\mathcal{P}^0(S)$, but it is viewed as a subsemiring of $\mathcal{P}_+(S)$. We have canonical morphisms $\mathcal{P}^0(S) \to K_0(\mathbf{RDef}_S)$ and $\mathcal{P}^0_+(S) \to SK_0(\mathbf{RDef}_S)$ given by $Y \mapsto [Y]$ and $(\mathbb{L}-1) \mapsto [x \neq 0]$. These morphisms allow us to define the following

$$C_+(S) = SK_0(\mathbf{RDef}_S) \otimes_{\mathcal{P}^0_+(S)} \mathcal{P}_+(S)$$

and

$$C(S) = K_0(\mathbf{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S).$$

We now introduce some functorialities. Suppose $f : S \to S'$ is a morphism in $\mathbf{Def}_k$. The fiber product induces a natural pullback $f^* : SK_0(\mathbf{RDef}_{S'}) \to SK_0(\mathbf{RDef}_S)$, namely, by sending $[Y \to S']$ to $[Y \times_{S'} S \to S]$. It is shown in [12] that the morphism f^* can be naturally extended to

$$f^* : C_+(S') \to C_+(S).$$

Similar constructions apply for $K_0(\mathbf{RDef}_S)$ and $C(S)$.

If $f : S \to S'$ is a morphism in $\mathbf{RDef}_{S'}$, composition with f induces a morphism $f_! : SK_0(\mathbf{RDef}_S) \to SK_0(\mathbf{RDef}_{S'})$. It is shown in [12] that $f_!$ can be naturally be extended to

$$f_! : C_+(S) \to C_+(S').$$

Similar constructions apply for $K_0(\mathbf{RDef}_S)$ and $C(S)$.

Some dimension theory for subassignments is needed. Suppose $S \in \mathbf{Def}_k$ is a subassignment in $h[m,n,r]$. Let $\mathrm{VF}(S)$ the image of S under the projection $h[m,n,r] \to h[m,0,0]$. Note that each Zariski closed set in $\mathbb{A}^m_{k((t))}$ defines a (definable) subassignment in $h[m,0,0]$. Define the Zariski closure $\overline{\mathrm{VF}(S)}$ of the subassignment $\mathrm{VF}(S)$ to be the intersection of all the Zariski closed sets (viewed as definable subassignments) containing $\mathrm{VF}(S)$. The dimension of S is defined as the dimension of $\overline{\mathrm{VF}(S)}$, viewed as an affine subvariety of $\mathbb{A}^m_{k((t))}$. The dimension of S is denoted by $\dim(S)$. The following is a result in [12].

Proposition 16 (c.f. [12], Thm. 3.3.1)**.** *Any two isomorphic objects in* $\mathbf{Def}_k$ *have the same dimension.*

For every non-negative integer d, the ideal of $C_+(S)$ generated by functions $\mathbf{1}_Z$, where Z is a definable subassignment of S with $\dim(Z) \leq d$, is denoted by $C_+^{\leq d}(S)$. We now set

$$\mathcal{C}_+(S) := \bigoplus_{d \geq 1} \mathcal{C}_+^d(S)$$

where

$$\mathcal{C}_+^d(S) := C_+^{\leq d}(S)/C_+^{\leq d-1}(S).$$

It is a graded abelian semigroup and also a $C_+(S)$-semimodule. We call the elements of $\mathcal{C}_+(S)$ positive constructible motivic functions (or just positive constructible functions) on S. If ϕ is a function lying in $C_+^{\leq d}(S)$ but not in $C_+^{\leq d-1}(S)$, its image in $\mathcal{C}_+^d(S)$ is denoted by $[\phi]$. In a similar way, one can define from $C(S)$ the ring $\mathcal{C}(S)$ of constructible motivic functions (or just constructible functions) over S.
The quotient of $C_+^{\leq d}(S)$ by $C_+^{\leq d-1}(S)$ is made because of the necessity to include functions defined almost everywhere (from the point of view of classical measure theory, this is standard). This is related to the problem of differentiation of functions with respect to the valued field variables. It can be proved that, if $f : S \to S'$ is an isomorphism in $\mathbf{Def}_k$, one may define a function called, the order of the jacobian of f, denoted by $\mathrm{ordjac} f$. This function is equal almost everywhere to a definable function, hence we may define $\mathbb{L}^{\mathrm{ordjac} f}$ in $\mathcal{C}_+^d(S)$ when S is of dimension d.

The constructible functions over a subassignment S are built from definable functions on S. The next step is the construction of the S-integrable functions via pushforward of morphisms.

2.2.5 The motivic measure

The following is the main theorem by Cluckers and Loeser in [12]. It constructs the functor of S-integrable functions, and this leads to the definition of the motivic measure, as we explain after the statement of the theorem (quoted from [11] and [12]).

Theorem 17 (c.f. [12], Thm. 10.1.1)**.** *Let k be a field of characteristic zero and let S be an object in $\mathbf{Def}_k$. There exists a unique functor*

$$\mathbf{Def}_S \to \mathbf{Abelian\ Semigroups}$$

$$Z \mapsto I_S\mathcal{C}_+(Z)$$

assigning to every morphism $f : Z \to Y$ in $\mathbf{Def}_S$ a morphism $f_! : I_S\mathcal{C}_+(Z) \to I_S\mathcal{C}_+(Y)$ and satisfying the following axioms:

1. ***Functoriality***
   (a) *For every composable morphisms f and g in $\mathbf{Def}_S$, $(f \circ g)_! = f_! \circ g_!$. In particular, $\mathrm{id}_! = \mathrm{id}$.*
   (b) *(Naturality) Let $\delta : S \to S'$ be a morphism in $\mathbf{Def}_k$ and denote by $\delta_+ : \mathbf{Def}_S \to \mathbf{Def}_{S'}$ the functor induced by composition with δ. For every Z in $\mathbf{Def}_S$, we have the inclusion $I_{S'}\mathcal{C}_+(\delta_+(Z)) \subset I_S\mathcal{C}_+(Z)$, and for $\phi \in I_{S'}\mathcal{C}_+(\delta_+(Z))$, $f_!(\phi)$ is the same function computed in I_S or in $I_{S'}$.*
   (c) *(Fubini) If $f : X \to Y$ is a morphism in $\mathbf{Def}_S$, a positive constructible function ϕ on X belongs to $I_S\mathcal{C}_+(X)$ if and only if ϕ belongs to $I_Y\mathcal{C}_+(X)$ and $f_!(\phi)$ belongs to $I_S\mathcal{C}_+(Y)$.*
2. ***Integrability***
   (a) *For every Z in $\mathbf{Def}_S$, $I_S\mathcal{C}_+(Z)$ is a graded subsemigroup of $\mathcal{C}_+(Z)$.*
   (b) *$I_S\mathcal{C}_+(S) = \mathcal{C}_+(S)$.*
3. ***Disjoint union***
   If Z is the disjoint union of two definable subassignments Z_1 and Z_2, then the isomorphism $\mathcal{C}_+(Z) \cong \mathcal{C}_+(Z_1) \oplus \mathcal{C}_+(Z_2)$ induces an isomorphism $I_S\mathcal{C}_+(Z) \cong I_S\mathcal{C}_+(Z_1) \oplus I_S\mathcal{C}_+(Z_2)$, under which $f_! = f_{|Z_1 !} \oplus f_{|Z_2 !}$.
4. ***Projection formula***
   For every α in $C_+(Y)$ and every β in $I_S\mathcal{C}_+(Z)$, $\alpha f_!(\beta) \in I_S\mathcal{C}_+(Y)$ if and only if $f^*(\alpha)\beta \in I_S\mathcal{C}_+(Z)$. If these conditions are verified, then $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.

5. ***Inclusions***

   If $i : Z \to Z'$ is the inclusion of definable subassignments of the same object of $\mathbf{Def}_S$, then $i_!$ is induced by extension by zero outside Z. For every $\phi \in C_+(Z)$, $[\phi] \in I_S\mathcal{C}_+(Z)$ if and only if $[i_!(\phi)] \in I_S\mathcal{C}_+(Z')$. If this is the case, then $i_!([\phi]) = [i_!(\phi)]$.

6. ***Integration along residue field variables***

   This axiom means that integrating with respect to variables in the residue field just amounts to taking the pushforward induced by composition at the level of Grothendieck semirings.

   Let Y be an object of $\mathbf{Def}_S$ and denote by π the projection $Y[0,n,0] \to Y$. Let $[\phi]$ be a function in $\mathcal{C}_+(Y[0,n,0])$. Then $[\phi] \in I_S\mathcal{C}_+(Z)$ if and only if, $[\pi_!(\phi)] \in I_S\mathcal{C}_+(Y)$. Furthermore, when this holds, $\pi_!([\phi]) = [\pi_!(\phi)]$.

7. ***Integration along $\mathbb{Z}$-variables***

   This axiom states that integration along $\mathbb{Z}$-variables corresponds to summing over the integers. Some preliminary constructions are needed.

   Let $\phi \in \mathcal{P}(S[0,0,r])$, so $\phi : |S| \times \mathbb{Z}^r \to A$. We say that ϕ is S-integrable if for every $q > 1$ and every x in $|S|$, the series $\sum_{i\in\mathbb{Z}^r} v_q(\phi(x,i))$ is summable. It can be proved that if ϕ is S-integrable there exists a unique function $\mu_S(\phi) \in \mathcal{P}(S)$ such that

$$\sum_{i\in\mathbb{Z}^r} v_q(\phi(x,i)) = v_q(\mu_S(\phi)(x)),$$

   for all $q > 1$ and all $x \in |S|$. We denote by $I_S\mathcal{P}_+(S[0,0,r])$ the set of S-integrable functions in $\mathcal{P}_+(S[0,0,r])$ and we set

$$I_SC_+(S[0,0,r]) = C_+(S) \otimes_{\mathcal{P}_+(S)} I_S\mathcal{P}_+(S[0,0,r]).$$

   Hence $I_S\mathcal{P}_+(S[0,0,r])$ is a sub $C_+(S)$-semimodule of $C_+(S[0,0,r])$ and μ_S may be extended by tensoring to

$$\mu_S : I_SC_+(S[0,0,r]) \to C_+(S).$$

   We now state the axiom:

   Let $Y \in \mathbf{Def}_S$ and let $\pi : Y[0,0,1] \to Y$ be the projection. Let $[\phi]$ be a function in $\mathcal{C}_+(Y[0,0,r])$. Then $[\phi] \in I_S\mathcal{C}_+(Y[0,0,1])$ if and only if there exists ϕ' in $C_+(Y[0,0,r])$

with $[\phi'] = [\phi]$ *which is* Y*-integrable in the previous sense and such that* $[\mu_Y(\phi')] \in I_S\mathcal{C}(Y)$*. Futhermore, when this holds,* $\pi_!([\phi]) = [\mu_Y(\phi')]$*.*

8. ***Volume of balls***

By analogy with the p-adic case, it is natural to require that the volume of the ball

$$\{z \in h[1,0,0] \,|\, \mathrm{ord}(z-c) = \alpha \textit{ and } \mathrm{ac}(z-c) = \xi\},$$

with $\alpha \in \mathbb{Z}$*,* $c \in k((t))$ *and* $\xi \in k^\times$*, should be* $\mathbb{L}^{-\alpha-1}$*. This axiom is the relative version of this statement.*

Let $Y \in \mathbf{Def}_S$ *and let* Z *be the definable subassignment of* $Y[1,0,0]$ *defined by*

$$\mathrm{ord}(z - c(y)) = \alpha(y) \ \wedge \ \mathrm{ac}(z - c(y)) = \xi(y),$$

with z *the coordinate (free variable) on the* $\mathbb{A}^1_{k((t))}$*-factor and* α*,* ξ *and* c *definable functions on* Y *with values respectively in* $\mathbb{Z}$*,* $h[0,1,0]\backslash\{0\}$*, and* $h[1,0,0]$*. We denote by* $f : Z \to Y$ *the morphism induced by the projection. Then* $[\mathbf{1}_Z] \in I_S\mathcal{C}_+(Z)$ *if and only if* $\mathbb{L}^{-\alpha-1}[\mathbf{1}_Y] \in I_S\mathcal{C}_+(Y)$*, and, if both are true, then*

$$f_!([\mathbf{1}_Z]) = \mathbb{L}^{-\alpha-1}[\mathbf{1}_Y].$$

9. ***Graphs***

This axiom gives an expression for the pushforward for graph projections. It is a special case of the change of variables theorem that can be proved in the theory.

Let $Y \in \mathbf{Def}_S$ *and let* $c : Y \to h[1,0,0]$ *be a definable morphism. Let* Z *be the definable subassignment of* $Y[1,0,0]$ *defined by* $z - c(y) = 0$ *with* z *the coordinate (free variable) on the* VF*-sort. We denote by* $f : Z \to Y$ *the projection. Then* $[\mathbf{1}_Z] \in I_S\mathcal{C}_+(Z)$ *if and only if* $\mathbb{L}^{(\mathrm{ordjac} f)\circ f^{-1}} \in I_S\mathcal{C}_+(Y)$*, and, if both are true, then*

$$f_!([\mathbf{1}_Z]) = \mathbb{L}^{(\mathrm{ordjac} f)\circ f^{-1}}.$$

Definition 18. *Let* $f : Z \to S$ *be a morphism in* $\mathbf{Def}_k$*. The elements of* $I_S\mathcal{C}_+(Z)$ *are called* S*-integrable positive functions over* Z*.*

As before $S \in \mathbf{Def}_k$. The idea is to extend the construction from $\mathcal{C}_+(S)$ to $\mathcal{C}(S)$. Let $Z \in \mathbf{Def}_S$. Recall $I_S\mathcal{C}_+(Z)$ is a subsemigroup of $\mathcal{C}_+(Z)$, and $\mathcal{C}_+(Z)$ can be mapped (naturally) into $\mathcal{C}(Z)$. The subgroup $I_S\mathcal{C}(Z)$ of $\mathcal{S}$ is defined as the subgroup generated by the image of $I_S\mathcal{C}_+(Z)$ in $\mathcal{C}(Z)$. Then, it can be proved that if $f : Z \to Y$ is a morphism in $\mathbf{Def}_S$, the morphism $f_! : I_S\mathcal{C}_+(Z) \to I_S\mathcal{C}_+(Y)$ has a natural extension

$$f_! : I_S\mathcal{C}(Z) \to I_S\mathcal{C}(Y).$$

As in definition 18, for a morphism $f : Z \to S$ in $\mathbf{Def}_k$, the elements of $I_S\mathcal{C}(Z)$ are called S-integrable functions over Z.

Let us explain the relation with motivic measures. The definable subassignments will play the role of measurable sets. When $S = h[0,0,0]$, the functor that assigns to each field $K \in \mathbf{Field}_K$ a one-point set (this is the final object in $\mathbf{Def}_k$). We omit the "S" in the notation. That is, one writes $I\mathcal{C}_+(Z)$ for $I_S\mathcal{C}_+(Z)$, we say integrable instead of S-integrable and so on. Note that,

$$I\mathcal{C}_+(h[0,0,0]) = \mathcal{C}_+(h[0,0,0]) = SK_0(\mathbf{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} A_+$$

and

$$I\mathcal{C}(h[0,0,0]) = \mathcal{C}(h[0,0,0]) = K_0(\mathbf{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} A.$$

Let Z be an object in $\mathbf{Def}_k$ and let $f : Z \to h[0,0,0]$ be a morphism in $\mathbf{Def}_k$ (note that such a morphism always exists and it is unique, it depends on Z). Then we have

$$\begin{aligned} f_! : I\mathcal{C}(Z) &\to K_0(\mathbf{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} A \\ \phi &\mapsto f_!(\phi). \end{aligned}$$

For $\phi \in I\mathcal{C}(Z)$, we define the motivic measure (or motivic integral) $\mu(\phi)$ by

$$\mu(\phi) = \int_Z \phi \, d\mu := f_!(\phi).$$

If $\phi \in I\mathcal{C}_+(Z)$, $f_!(\phi)$ is "positive", that is, $f_!(\phi) \in SK_0(\mathbf{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} A_+$.

Let Z be an object in $\mathbf{Def}_k$ of dimension d. Let ϕ be a function in $C_+(Z)$, or in $C(Z)$. We say that ϕ is integrable if its class $[\phi]_d$ in $\mathcal{C}_+(Z)$, resp. in $\mathcal{C}(Z)$, is integrable, and we have

$$\mu(\phi) = \int_Z \phi \, d\mu := f_!([\phi]_d).$$

For $Z \in \mathbf{Def}_k$, the motivic volume of Z is defined as

$$\int_Z \mathbf{1}_Z \, d\mu = f_!([\mathbf{1}_Z]),$$

provided $\mathbf{1}_Z$ is integrable. Observe that a motivic volume is an element in the semiring $SK_0(\mathbf{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} A_+$.

Suppose Z is a subassignment of some $h[m,n,0]$. We say that Z is bounded if there exists a positive integer s such that Z is contained in the subassignment B_s of $h[m,n,0]$ defined by

$$\bigwedge_{i=1}^{m} \mathrm{ord}(x_i) \geq -s,$$

clearly the variables x_i are free VF-variables.

The following result of Cluckers and Loeser [12] gives a sufficient condition for the integrability of characteristic functions of certain definable subassignments.

Proposition 19 (c.f. [12], Prop. 12.2.2)**.** *If Z is a bounded definable subassignment of $h[m,n,0]$, then $\mathbf{1}_Z$ is integrable. Hence the motivic volume of Z exists.*

2.2.6 Integrals with parameters

This theory of integration can be extended to integrals depending on parameters. Let us fix Λ in $\mathbf{Def}_k$ playing the role of parameter space. Let $S \in \mathbf{Def}_\Lambda$. The ideal $C_+^{\leq d}(S \to \Lambda)$ of $C_+(S)$ generated by functions $\mathbf{1}_Z$ with Z a definable subassignment of S such that all the fibers of $Z \to \Lambda$ are of dimension $\leq d$ (we give below a more detailed explanation of these fibers and their dimensions). We set

$$\mathcal{C}_+(S \to \Lambda) := \bigoplus_{d \geq 1} \mathcal{C}_+^d(S \to \Lambda)$$

where

$$\mathcal{C}_+^d(S\to\Lambda) := C_+^{\leq d}(S)/C_+^{\leq d-1}(S\to\Lambda).$$

It is a graded abelian semigroup and it has also the structure of a $C_+(S)$-semimodule. If ϕ belongs to $C_+^{\leq d}(S\to\Lambda)$ but not to $C_+^{\leq d-1}(S\to\Lambda)$ we write, as before, $[\phi]$ for its image in $\mathcal{C}_+^d(S\to\Lambda)$. The following is the relative version of Theorem 17.

Theorem 20 (c.f. [12], Thm. 14.1.1)**.** *Let Λ be in $\mathbf{Def}_k$. Let S be in $\mathbf{Def}_\Lambda$. There is a unique functor*

$$\mathbf{Def}_S \to \mathbf{Abelian\ Semigroups}$$

$$Z \mapsto I_S\mathcal{C}_+(Z\to\Lambda),$$

assigning to every morphism $f: Z\to Y$ in $\mathbf{Def}_k$ a morphism $f_{!\Lambda}: I_S\mathcal{C}_+(Z\to\Lambda)\to I_S\mathcal{C}_+(Y\to\Lambda)$ and satisfying the axioms similar to $A1-A9$ of Theorem 17 replacing $I_S\mathcal{C}_+(-)$ by $I_S\mathcal{C}_+(-\to\Lambda)$ and the following two changes. In $A1(b)$, δ should be a morphism in $\mathbf{Def}_\Lambda$. In $A9$, one should replace the function ordjac *by its relative version* ordjac_Λ.

We define now the relative motivic measure. Let $f: Z\to\Lambda$ be a morphism in $\mathbf{Def}_\Lambda$ (since $\Lambda\to\Lambda$ is the final object in $\mathbf{Def}_\Lambda$, this morphism is unique and it only depends on Z). We simplify a bit the notation. We write $I\mathcal{C}_+(Z\to\Lambda)$ for $I_\Lambda\mathcal{C}_+(Z\to\Lambda)$. The relative motivic measure μ_Λ is defined as $f_{!\Lambda}$, more specifically

$$\mu_\Lambda := f_{!\Lambda} : I\mathcal{C}_+(Z\to\Lambda)\to\mathcal{C}_+(\Lambda\to\Lambda) = C_+(\Lambda),$$

and for ϕ in $I\mathcal{C}_+(Z\to\Lambda)$,

$$\mu_\Lambda(\phi) = \int_Z \phi\, d\mu_\Lambda := f_{!\Lambda}(\phi).$$

The relative motivic volume of a definable subassignment Z in $\mathbf{Def}_\Lambda$ is by definition $\mu_\Lambda(\mathbf{1}_Z)$, provided $\mathbf{1}_Z$ is integrable. The following result shows that the relative motivic measure corresponds to integration along the fibers of Λ. Before we state this result, it is necessary to explain some terminology. Suppose $Z\in\mathbf{Def}_\Lambda$. Recall that a point λ in Λ is a tuple (x_λ, K) where K in $\mathbf{Field}_k$, and $x_\lambda\in\Lambda(K)$. We write $k(\lambda)=K$. We denote by $Z_\lambda\subset Z$ the fiber of λ under $Z\to\Lambda$. The fiber Z_λ is an object in $\mathbf{Def}_{k(\lambda)}$. Note that the base field is $k(\lambda)$ instead of k. Thus, the dimension of Z_λ, as a definable subassignment, exists. The only thing to bear in mind is that this dimension is taken with respect to $k(\lambda)$. There exists a natural restriction morphism $i_\lambda^* : \mathcal{C}_+(Z \to \Lambda) \to \mathcal{C}_+(Z_\lambda)$ that respects the grading.

Proposition 21 (c.f. [12], Coro. 14.2.2)**.** *Let Z be in $\mathbf{Def}_\Lambda$. Let $\phi \in \mathcal{C}_+(Z \to \Lambda)$. Then, ϕ belongs to $I_\lambda \mathcal{C}_+(Z \to \Lambda)$ if and only if for every point $\lambda \in \Lambda$, $i_\lambda^*(\phi) \in I\mathcal{C}_+(Z_\lambda)$. If these are true, then*

$$i_\lambda^*(\mu_\Lambda(\phi)) = \mu_\lambda(i_\lambda^*(\phi)),$$

for every point $\lambda \in \Lambda$. Here μ_λ denotes the motivic measure on $\mathbf{Def}_{k(\lambda)}$.

As in the absolute case, it is possible to extend the theory from the case of "positive" functions and get an analogue $\mathcal{C}(S \to \Lambda)$, motivic measures and so on.

2.2.7 Volume forms

Cluckers and Loeser extended the theory of motivic integration to integration of volume forms. They define a spaces of motivic volume forms $|\Omega(S)|$ on S, a definable subassignment. Associated with each motivic volume form α, there is a constructible motivic function $|\alpha|$. For details, see [12].

Theorem 22 (c.f. [12] Thm. 15.3.1)**.** *Let $f : S \to S'$ be a morphism of definable subassignments. Assume f is an isomorphism of definable subassignments. A volume form α in $|\Omega(S')|$ is integrable if and only if $f^*(\alpha)$ is integrable. When this holds we have*

$$\int_S f^*(\alpha) = \int_{S'} \alpha.$$

2.3 SPECIALIZATION OF MOTIVIC FUNCTIONS

Definition 23. *Let $M > 0$ be a positive integer number. We denote by $\mathscr{A}_M$ the set of non-archimedean local fields of characteristic zero such that the characteristic of the residue field is greater or equal than M. We denote by $\mathscr{B}_M$ the set of non-archimedean local fields of positive characteristic such that the characteristic of the residue field is greater or equal than M. Finally, we define $\mathscr{C}_M = \mathscr{A}_M \cup \mathscr{B}_M$.*

Let S be a definable subassignment. Let $F \in \mathscr{C}_M$ and let $\phi \in \mathcal{C}(S)$. Notice that if $S \subset h[m,n,r]$, then S determines a definable set $S_F \subset F^m \times k_F^n \times \mathbb{Z}^r$. The specialization map is a ring homomorphism $\phi \mapsto \phi_F$ taking values in the ring of $\mathbb{C}$-valued functions on S_F. For details, see [13].

The following is the specialization principle.

Theorem 24. *Let $f : S \to \Lambda$ be a morphism in $\mathbf{Def}_k$. Let ϕ be in $C(S)$ and suppose ϕ is S-integrable. Then there exists $M \geq 0$ such that for every $F \in \mathcal{A}_M \cup \mathcal{B}_M$ we have*

$$(\mu_\Lambda(\phi))_F = \mu_{\Lambda_K}(\phi_F).$$

That is, with integral notation,

$$\left(\int_S \phi \, d\mu_\Lambda \right)_F = \int_{S_F} \phi_F \, d\mu_{\Lambda_F}.$$

Let X be a definable subassignment. A motivic function f over X determines, via specialization, the collection $(f_F)_F$ of functions $f_F : X_F \to \mathbb{C}$ of the form

$$f_F(x) = \sum_{i=1}^{N} q_F^{\alpha_{iF}(x)} (\#(\mathrm{Y}_{iF})_x) \ \cdot \ \prod_{j=1}^{N_i'} \beta_{ijF}(x) \ \cdot \ \prod_{l=1}^{N_i''} \frac{1}{1 - q_F^{a_{il}}},$$

for $x \in X_F$, where

- $a_{il} \in \mathbb{Z} \setminus \{0\}$,
- $\alpha_i : X \to \mathbb{Z}$ and $\beta_{ij} : X \to \mathbb{Z}$ are definable functions,
- $Y_i \subset X \times \mathrm{RF}^{r_i}$ is a definable subassignment. For $x \in X_F$, $(Y_{iF})_x = \{y \in k_F^{r_i} \mid (x,y) \in Y_{iF}\}$ is a finite set.

2.4 TRANSFER PRINCIPLES

The transfer principle for motivic integrals is the tool that allow us to move the truth of the fundamental lemma for non-unit elements in the Hecke algebra, from positive to zero characteristic, and vice versa (we will explain this later). We start with a transfer principle for local fields and an application of it. This principle is known as The Ax-Kochen-Eršhov principle. In fact, the Ax-Kochen-Eršhov principle can be viewed as a particular case of the transfer principle for integrals proved by Cluckers and Loeser. The Ax-Kochen-Eršhov principle was obtained independently by Ax-Kochen [2] and Eršhov [18] in the sixties.

2.4.1 Ax-Kochen-Eršhov principle

Although the non-archimedian local fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are quite different fields, in principle because of the characteristic ($\operatorname{char} \mathbb{F}_p((t)) = p$ and $\operatorname{char}(\mathbb{Q}_p) = 0$), the Ax-Kochen-Eršhov principle establishes that asymptotically they have similar field structures.

Theorem 25 (Ax-Kochen-Eršhov principle. c.f. [2], Thm. 6 and [18])**.** *Let F_1 and F_2 be two Henselian valued fields in the language $\mathcal{L}_V$. The $\mathcal{L}_V$-structures of these fields are $(F_i, \mathcal{O}_{F_i})$ for $i = 1, 2$, and with $\mathcal{O}_{F_i}$ being the discrete valuation ring of F_i. Suppose that*

- *The residue fields of F_1 and F_2 satisfy the same sentences in the language of rings.*
- *The value groups of F_1 and F_2 satisfy the same sentences in the language of ordered groups $\mathcal{L}_{og} = \{+, 0, <\}$ (the function symbol $+$ is interpreted as the binary operation of the group, 0 as the identity element and $<$ as the linear relation, compatible with the group operation).*

If the residue characteristic is 0, then $(F_1, \mathcal{O}_{F_1})$ and $(F_2, \mathcal{O}_{F_2})$ satisfy the same sentences in the language $\mathcal{L}_V$.

This theorem says that, from the model-theoretic perspective, a Henselian valued field is determined by the residue field and the value group.

Corollary 26 (Ax-Kochen-Eršhov principle. c.f. [2], Thm. 6 and [18])**.** *Let ϕ be a sentence* *in the language $\mathcal{L}_V$. Then*

$$(\mathbb{Q}_p, \mathbb{Z}_p) \models \phi \iff (\mathbb{F}_p((t)), \mathbb{F}_p[[t]]) \models \phi$$

for all but finitely many primes p.

The following theorem is an asymptotic version of a conjecture of E. Artin on the existence of solutions of homogeneous polynomials in several variables over p-adic fields.

Theorem 27 (c.f. [2], Thm. 5)**.** *For every degree $d \geq 1$ there exists a lower bound n_d such that for $p \geq n_d$, every homogeneous polynomial over $\mathbb{Q}_p$ of degree d in n variables such that $n > d^2$ has a non-trivial zero in $\mathbb{Q}_p^n$.*

Regarding the proof, one has to prove that for a fix $d \geq 1$, the property that every homogeneous polynomial of degree d in n variables such that $n > d^2$ has a non-trivial zero, is equivalent to the property that every homogeneous polynomial of degree d in $d^2 + 1$ variables has a non-trivial. The latter can be expressed by a $\mathcal{L}_{rings}$-sentence ψ_d. Then it is proved that

$$(\mathbb{F}_p((t)), \mathbb{F}_p[[t]]) \models \psi_d,$$

and by Corollary 26 the result is also true in $(\mathbb{Q}_p, \mathbb{Z}_p)$. We sketched the main steps in the proof just because our work follows the same philosophy but in the more general and powerful context of constructible motivic functions. In the next section we present the transfer principle that we use in this work.

2.4.2 Transfer principles for motivic functions

The next two results are due to Cluckers and Loeser. The following is the abstract transfer principle for constructible motivic functions.

Theorem 28 (c.f. [13], Thm. 9.2.1)**.** *Let S be a definable subassignment, and let f be a constructible motivic function on S. Then there exists $M > 0$ such that for every $F_1, F_2 \in \mathcal{A}_M \cup \mathcal{B}_M$ with isomorphic residue fields,*

$$f_{F_1} = 0 \iff f_{F_2} = 0.$$

The following is the transfer principle for integrals with parameters.

Theorem 29 (c.f. [13], 9.2.4)**.** *Let* $S \to \Lambda$ *and* $S' \to \Lambda$ *be morphisms in* $\mathbf{Def}_\Lambda$*. Let* ϕ *and* ϕ' *be relatively integrable functions in* $C(S)$ *and* $C(S')$*, respectively. Then there exists* $M > 0$ *such that for every* F_1 *and* F_2 *in* $\mathcal{A}_M \cup \mathcal{B}_M$ *with isomorphic residue fields,*

$$\mu_{\Lambda_{F_1}}(\phi_{F_1}) = \mu_{\Lambda_{F_1}}(\phi'_{F_1}) \quad \Longleftrightarrow \quad \mu_{\Lambda_{F_2}}(\phi_{F_2}) = \mu_{\Lambda_{F_2}}(\phi'_{F_2}).$$

That is, with integral notation,

$$\int_{S_{F_1}} (\phi_{F_1})\, d\mu_{F_1} = \int_{S'_{F_1}} (\phi'_{F_1})\, d\mu_{F_1} \quad \Longleftrightarrow \quad \int_{S_{F_2}} (\phi_{F_2})\, d\mu_{F_2} = \int_{S'_{F_2}} (\phi'_{F_2})\, d\mu_{F_2}$$

3.0 THE FUNDAMENTAL LEMMA

3.1 REDUCTIVE GROUPS OVER NON-ARCHIMEDEAN LOCAL FIELDS

In this and the next sections we introduce material on reductive groups over p-adic fields needed for the fundamental lemma.

We start with a p-adic field F. Its algebraic closure is denoted by $\bar{F}$ and its maximal unramified extension in $\bar{F}$ by F^{un}. Associated to the local field F we have the following objects,

- $\mathcal{O}_F$ the ring of integers,
- $\mathcal{M}_F$ the unique maximal ideal of $\mathcal{O}_F$,
- k_F the residue field,
- $q = |k_F|$ the cardinality of the residue field (a power of a prime number p), and
- $\varpi_F \in \mathcal{O}_F$ a uniformizer (*i.e.* a generator of $\mathcal{M}_F$, so $\mathcal{M}_F = \varpi_F \mathcal{O}_F$). We will use ϖ for ϖ_F since the field F will be clear by the context.

Every element $x \in F^\times$ can be uniquely written as $x = u \cdot \varpi_F^n$ for $n \in \mathbb{Z}$ and $u \in \mathcal{O}_F^\times$. Then, $\mathrm{ord}_F(x) = n$ and $\mathrm{ord}_F(0) = \infty$. We set $|x|_F = q^{-\mathrm{ord}_F(x)}$ for $x \in F^\times$ and $|0|_F = 0$.

Our main references for the theory of reductic groups are Springer and Humphreys [54], [32], respectively. We also follow very close the expository paper on the fundamental lemma by Hales [26].

There is a general assumption on the linear algebraic groups considered in the fundamental lemma. They are connected (with respect to the Zariski topology) reductive linear algebraic groups defined over a p-adic field F. Here when we say that a group is defined over F, that statement has the algebro-geometric meaning (although related to the model theoretic meaning). Following Springer, a group G is definable over a p-adic field F if the polynomials defining G have coefficients in F. This leads to the notion of F-structure ([54], page 6) on G. When we say G is F-group, we mean G is a group definable over the field F in the algebro-geometric context.

Let F be a p-adic field. If G is reductive group defined over F, the set of F-points of G is denoted by $G(F)$. The following examples give the F-points of various connected reductive linear algebraic groups. Let $M(n, F)$ be the algebra of $n \times n$ matrices with coefficients in F.

Example 30 (*The general linear group*)**.** *The general linear group $GL(n)$ with coefficients in F is*

$$GL(n, F) = \{X \in M(n, F) \mid \det(X) \neq 0\}.$$

We can view this group as an affine algebraic variety in n^2+1 variables with defining equation $\det(X)y - 1 = 0$, so it is definable over the prime field of F.

Example 31 (*The special linear group*)**.** *The special linear group $SL(n)$ with coefficients in F is*

$$SL(n, F) = \{X \in GL(n, F) \mid \det(X) = 1\}.$$

This group is in fact semisimple.

Example 32 (*The special orthogonal group*)**.** *Let $J \in GL(n, F)$ be an invertible symmetric matrix. The special orthogonal group $SO(n, J)$ with respect to J with coefficients in F is*

$$SO(n, J, F) = \{X \in GL(n, F) \mid X^t JX = J \ \det(X) = 1\}.$$

This group is in fact semisimple.

Example 33 (*The symplectic group*)**.** *Let $n = 2k$. Let $J = GL(2k, F)$ be an invertible skew-symmetric matrix $J^t = -J$. The symplectic group $Sp(2k, J)$ with respect to J with coefficients in F is*

$$Sp(2k, J, F) = \{X \in GL(2k, F) \mid X^t JX = J\}.$$

Example 34 (*The unitary group*)**.** *Let E/F be a separable quadratic extension. We denote by $\bar{x}$ the Galois conjugate of $x \in E$ with respect to the non-trivial element in* $\mathrm{Gal}(E/F)$*. For $X \in M(n, E)$, we denote by $\bar{X}$ the matrix obtained by taking the Galois conjugate of each entry of X. Let $J \in GL(n, E)$ be such that $\bar{J}^t = J$ and* $\det(J) \neq 0$*. The unitary group $U(n, J)$ with respect to J and E/F is*

$$U(n, J, F) = \{X \in GL(n, E) \mid \bar{X}^t J X = J\}.$$

3.1.1 Unramified groups

From now on, F denotes a p-adic field.

Definition 35. *Let G be a connected reductive group defined over F. A Borel subgroup $B \subset G$ is an algebraic subgroup such that $B \times_F \bar{F} \subset G \times_F \bar{F}$ is a maximal solvable algebraic subgroup.*

Definition 36. *Let G be a connected reductive group defined over F. The group G is said to be an unramified reductive group if it satisfies the following two conditions:*

- *G splits over an unramified field extension. That is, there is an unramified extension F_1/F and a Cartan (maximal torus defined over F) in G that is F_1- isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. We can just say, G is F_1-split.*
- *G is quasi-split. This means G has a Borel subgroup that is defined over F. More explicitly, there is an F-subgroup $B \subset G$ such that $B \times_F \bar{F}$ is a Borel subgroup of $G \times_F \bar{F}$.*

We say that G is a split group if G splits over F. That is, if G has F-Cartan subgroup T that is isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ over F (we say T is a F-split Cartan subgroup of G). In terms of the F-points of G, $T(F) \cong F^\times \times \cdots \times F^\times$. Recall that, if G is a split connected reductive group defined over F, then G is quasi-split, and hence unramified.

Example 37.

- *The general linear group $GL(n)$ is unramified. It is split and the subgroup of upper triangular matrices in $GL(n)$ is a Borel subgroup defined over F.*
- *The special linear groups $SL(n)$ is unramified. It is split and the subgroup of upper triangualr matrices in $SL(n)$ is a Borel subgroup defined over F.*

- *If J has non-zero entries along the cross-diagonal and zeros elsewhere, that is, J has the form*
$$J = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$$
then $SO(n)$ and $Sp(n)$ are split (with respect to this J), and hence unramified. The subgroups of upper triangular matrices in the corresponding groups form Borel subgroups defined over F.
- *In the unitary group $U(n)$, if E/F is unramified and J has the same form as above, then $U(n)$ splits over the unramified extension E. The group is also quasi-split and the subgroup of upper triangular matrices in $U(n)$ is a Borel subgroup defined over F.*

The following is the definition of the hyperspecial subgroup. The standard reference is the paper by Tits [56].

Definition 38. *Let G be a connected reductive group defined over F. A subgroup K of $G(F)$ is called hyperspecial if there exists $\mathcal{G}$ such that the following conditions are satisfied.*

- *$\mathcal{G}$ is a smooth group scheme over $\mathcal{O}_F$,*
- *$G = \mathcal{G} \times_{\mathcal{O}_F} F$,*
- *$\mathcal{G} \times_{\mathcal{O}_F} k_F$ is connected reductive, and*
- *$K = \mathcal{G}(\mathcal{O}_F)$.*

Lemma 39. *Let G be an unramified group defined over F. Then G is the localization of a smooth affine group scheme $\mathcal{G}$ defined over the ring of integers $\mathcal{O}_F$ whose special fiber over the residue field of k_F is connected reductive. Moreover, a reductive algebraic group G defined over F is unramified if and only if $G(F)$ contains a hyperspecial maximal compact subgroup.*

See [56] for the proof. We give some examples.

Example 40. • *$GL(n, \mathcal{O}_F)$ is a hyperspecial maximal compact subgroup of $GL(n, F)$.*

- *$SL(n, \mathcal{O}_F)$ is a hyperspecial maximal compact subgroup of $SL(n, F)$.*
- *Next consider the case of G being equal to $SO(n)$, $Sp(n)$ or $U(n)$, and each cross-diagonal entry is a unit in the ring of integers $\mathcal{O}_F$. There is a group scheme $\mathcal{G}$ over $\mathcal{O}_F$ defined by the equations $X^tJX = J$ and $\det(X) = 1$ (special orthogonal), $X^tJX = J$ (symplectic),*

and $\bar{X}^t J X = J$ (unitary). The group $\mathcal{G}(\mathcal{O}_F)$ is a hyperspecial maximal compact subgroup of $G(F)$.

3.1.2 Absolute and relative theory

The excellent paper by Springer [53] provides more detail on the material discussed here.

Let G be a connected unramified reductive group defined over F. Roughly speaking the relative theory (or relative case) corresponds to the set of F-point in G *i.e.,* $G(F)$ and associated objects over F. And the absolute theory (or absolute case) corresponds to the situation over the algebraic closure (*e.g.* the F-group G viewed as a $\bar{F}$-group $G \times_F \bar{F}$ with the underlying set of $\bar{F}$-points $G(\bar{F})$). In the practice we do not need to go that far for the understanding of the absolute case, it is enough to go up to an unramified extension where G is split. All the relevant data will be the same. More formally, for any F-algebra R, an R-point of G is a morphism $\mathrm{Spec}(R) \to G$. The set of R-points forms a group, which we denote $G(R)$. In the relative case we take $R = F$, and in the absolute case $R = \bar{F}$ or $R = F_1$.

We begin with the description of the relative theory.

We fix B a Borel subgroup of G defined over F. We fix a maximal F-split torus $A \subset B$ in G (any two maximal F-split tori of G are conjugate over F). The relative group of characters of G is

$$X_F^* := \mathrm{Hom}(A, \mathbb{G}_m) = \{\chi : A \to \mathbb{G}_m \mid \chi \text{ is a } F\text{-homomorphism}\}.$$

The relative group of cocharacters of G is

$$X_{*F} := \mathrm{Hom}(\mathbb{G}_m, A) = \{\lambda : \mathbb{G}_m \to A \mid \lambda \text{ is a } F\text{-homomorphism}\}.$$

Since A is F-split, for some $d \geq 1$, $A \cong \mathbb{G}_m^d$, the isomorphism being over F. Thus, we can identify any character $\chi \in X_F^*$ with a tuple $(n_1, \dots, n_d) \in \mathbb{Z}^d$, where χ is defined as

$$(x_1, \dots, x_d) \mapsto x_1^{n_1} \cdots x_d^{n_d}, \tag{3.1}$$

and any tuple in $\mathbb{Z}^d$ corresponds to a unique character in X_F^* defined similar to 3.1. Similarly, any cocharacter $\lambda \in X_{*F}$ is identified with $(m_1, \ldots, m_d)$, where λ is defined as

$$x \mapsto (x^{n_1}, \ldots, x^{n_d}), \tag{3.2}$$

and, as before, any tuple in $\mathbb{Z}^d$ corresponds to a unique cocharacter in X_{*F} defined similar to 3.2. We say that G has relative rank d or F-rank$(G) = d$. Based on this discussion, we see that we can think of X_F^* and X_{*F} as definable objects in $h[0, 0, d]$ (so they are present in any Denef-Pas structure) via the following identification

$$X_F^* \cong \mathbb{Z}^d \qquad \text{and} \qquad X_{*F} \cong \mathbb{Z}^d.$$

Observe that this identifications depend only on the relative rank of G.

Composition defines a paring

$$\langle\ ,\ \rangle : X_F^* \times X_{*F} \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z},$$

defined by $\langle \chi, \lambda \rangle = n \in \mathbb{Z}$ if $(\chi \circ \lambda) : \mathbb{G}_m \to \mathbb{G}_m$ is given by $x \mapsto x^n$.

Suppose that $\chi \in X_F^*$ is identified with $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, and $\lambda \in X_{*F}$ is identified with $(m_1, \ldots, m_d) \in \mathbb{Z}^d$. Then,

$$\langle \chi, \lambda \rangle = n_1 m_1 + \cdots + n_d m_d. \tag{3.3}$$

This proves that the paring $\langle\ ,\ \rangle$, viewed as a function $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}$ is not definable in the Presburger language because it uses multiplication of integers (and of course not definable in the Denef-Pas language). Nevertheless, we have the following two results.

Lemma 41. *The pairing $\langle\ ,\ \rangle$ is a constructible function over $\mathbb{Z}^d \times \mathbb{Z}^d$.*

Proof. For $i \in \{1, \ldots, 2d\}$, let $\pi_i : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}$ be the projection onto the i-coordinate. These functions are clearly definable in the Denef-Pas language (in fact, the projections are always definable in the first-order logic), and hence they are constructible functions over $\mathbb{Z}^d \times \mathbb{Z}^d$. Suppose $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are tuples of d $\mathbb{Z}$-variables. Thus, according to 3.3

$$\langle x, y \rangle = \pi_1(x)\pi_{d+1}(y) + \cdots + \pi_d(x)\pi_{d+d}(y)$$

Therefore, the pairing is a sum of products of constructible functions on $\mathbb{Z}^d \times \mathbb{Z}^d$. The result follows from the fact that the constructible functions over a definable subassignment form a ring. $\square$

Lemma 42. *Let $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ be a fixed d-tuple of integer numbers. The function $\langle \cdot, m\rangle : \mathbb{Z}^d \to \mathbb{Z}$ is definable in the Denef-Pas language and hence it is a constructible function over $\mathbb{Z}^d$. The same holds if we fix the first component of the pairing.*

Proof. The function

$$\langle \cdot, m\rangle : \mathbb{Z}^d \to \mathbb{Z}$$

is definable in the Presburger language (and hence in the Denef-Pas language) by the formula

$$\underbrace{x_1 + \cdots + x_1}_{m_1 \text{ times}} + \cdots + \underbrace{x_d + \cdots + x_d}_{m_d \text{ times}} = x,$$

where $x, x_1, \dots, x_d$ are $\mathbb{Z}$-free variables. In the case in which m_i is negative, it is possible to express additive inverses in a definable way or we can just add a function symbol interpreted as the additive inverse function in $\mathbb{Z}$, there is no harm with that. In the case in which the first component is fixed, the proof is the same. $\square$

While the relative character and cocharacter groups are intrinsic to A, we now consider the relative root system of G with respect to A. Although this root system is attached to A, it determines a lot of structure on G. The relative root system of G with respect to A,

$$\Psi_F(G) = \langle X_F^*, X_{*,F}, \Phi_F, \Phi_F^\vee \rangle$$

consists of the following

- X_F^* is the group of F-characters of A.
- $X_{*,F}$ is the group of F-cocharacters of A.
- $\Phi_F \subset X_F^*$ is the set of relative roots.
- $\Phi_F^\vee \subset X_{*F}$ is the set of relative coroots.

See Springer [53, §3.], for details on the construction of the relative roots and coroots. The sets of relative roots and relative coroots are finite subsets of X_F^* and X_{*F}, respectively. There is a bijection $\alpha \mapsto \alpha^\vee$ of Φ_F onto $\Phi_F^\vee$. For each $\alpha \in \Phi_F$ we define an endomorphism $s_\alpha : X_F^* \to X_F^*$ and $s_{\alpha^\vee} : X_{*F} \to X_{*F}$, defined by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad s_{\alpha^\vee}(u) = u - \langle \alpha, u \rangle \alpha^\vee.$$

The relative Weyl group W_F is identified with the group of automorphisms of X_F^* generated by the s_α and with the group of automorphisms of X_{*F} generated by the $s_{\alpha^\vee}$. This group is finite and the action on X_F^* and on X_{*F} permutes relative roots and relative coroots. Let $N(A)$ and $Z(A)$ denote normalizer and centralizer of A in G. These are F-subgroups. The relative Weyl group of G is isomorphic to $N(A)/Z(A)$, and any coset of $N(A)/Z(A)$ can be represented by an element in $N(A)(F)$.

Intuitively, it is clear that the generators of the relative Weyl group are definable functions, given that they are linear functions (reflections), that the products in the defining formulas are "scalar" products, and that can be expressed by formulas in the Presburger language. In the following lemma we give the details.

Lemma 43. *The generators of the relative Weyl group s_α (or $s_{\alpha^\vee}$), viewed as automorphisms of $\mathbb{Z}^d$, are definable functions in the Presburger language (and hence definable in the Denef-Pas language).*

Proof. Let us consider the case of s_α. The case $s_{\alpha^\vee}$ is similar. Under the identifications $X_F^* \cong \mathbb{Z}^d$ and $X_F^* \cong \mathbb{Z}^d$, the relative root α corresponds to $(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, and $\alpha^\vee$ corresponds to $(\alpha'_1, \ldots, \alpha'_d) \in \mathbb{Z}^d$. The function s_α is now rewriten as $s_\alpha : \mathbb{Z}^d \to \mathbb{Z}^d$. Consider the $\mathbb{Z}$-variables $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. The function s_α in terms of the variables x and y is given by

$$y = x - \langle x, \alpha^\vee \rangle \alpha.$$

Firstly, by the Lemma 42, the pairing $\langle \ , \alpha^\vee \rangle$ is a definable function in $\mathcal{L}_{DP}$. The product between the pairing (a term in $\mathbb{Z}$) and α (a fixed tuple of integer numbers) is not a problem, because it corresponds to the “scalar” product between a $\mathbb{Z}$-term and a tuple of fixed integers. The $\mathcal{L}_{DP}$-formula defining the graph of s_α is the following

$$\bigwedge_{i=1}^{d} \big(y_i = x_i - \underbrace{\big(\langle x, \alpha^\vee\rangle + \cdots + \langle x, \alpha^\vee\rangle\big)}_{\alpha_i \text{ times}}\big).$$

□

Remark 44. *Lemma 43 stress the fact that the action of the relative Weyl group on characters and cocharacters is definable. Nevertheless, we observe that any automorphism of $\mathbb{Z}^d$ is definable in the Presburger language (without using parameters) because it must be linear, so it is given by a matrix with entries in $\mathbb{Z}$ and determinant one.*

Lemma 43 clearly implies the definability of the action of W_F on characters and cocharacters. More explicitly, suppose $W_F = \{w_1, \dots, w_l\}$. For each $i \in \{1, \dots l\}$, we denote by $w_i(\lambda)$ the action of w_i on $\lambda \in \mathbb{Z}^d$ (representing characters or cocharacters). The action of each element in W_F corresponds to a composition of finitely many actions of generators, and these are definable by Lemma 43. Clearly the composition of these actions is definable (it may be thought as the product of matrices with entries in $\mathbb{Z}$). Hence the action of each element in the Weyl group is definable. That is, for each $i \in \{1, \dots l\}$, $w_i(\lambda) : \mathbb{Z}^d \to \mathbb{Z}^d$ is given by a definable function.

The definability of the action of W_F in X_F^* implies the following theorem. As usual, X_F^* is identified with the definable set $\mathbb{Z}^d$. For $\lambda \in \mathbb{Z}^d$, $W_F(\lambda)$ represents the orbit of λ under the action of W_F. The corresponding statement for X_{*F} is true. The proof is literally the same since $X_F^* \cong \mathbb{Z}^d \cong X_{*F}$, as groups.

Theorem 45. *a) For each $\lambda \in \mathbb{Z}^d$, the orbit $W_F(\lambda) \subset \mathbb{Z}^d$ is definable in the Presburger language.*

b) The function $\#W_F : \mathbb{Z}^d \to \mathbb{Z}$ given by

$$\lambda \mapsto \#W_F(\lambda),$$

is given by a constructible motivic function.

Proof. Suppose $W_F = \{w_1, \ldots, w_l\}$. The formula defining $W_F(\lambda)$, with x a d-tuple of free $\mathbb{Z}$-variables is

$$\bigvee_{i=1}^{l} x = w_i(\lambda).$$

This proves part a). For part b) we observe that the function $\#W_F$ can be described by the following integral

$$\#W_F(\lambda) = \int_{\mathbb{Z}^d} \mathrm{char}(W_F(\lambda)).$$

The integral is with respect to the counting measure in $\mathbb{Z}^d$ (recall that the counting measure is the motivic measure on the $\mathbb{Z}$-sort). By part a), the function $\mathrm{char}(W_F(\lambda))$ is a constructible motivic function. Then the function $\#W_F$ is expressed as the integral of a constructible motivic function over a definable set, hence $\#W_F$ is a constructible motivic function. $\square$

We now describe the absolute case. Let T be the Cartan subgroup (recall, this is a maximal torus in G defined over F) containing A. Hence $T = Z_G(A)$. That is, T is a maximally split Cartan subgroup of G (*i.e.* the torus T contains a F-subtorus that is F-split which has largest possible dimension). From the general theory we know that, G is split if and only if $A = T$. Recall that G is F_1-split for some unramified extension F_1/F. Therefore T splits over F_1. The constructions are basically the same, but now over T. Recall that any two F_1-split Cartan subgroups of G are conjugate over F_1, so although all the following objects are somehow attached to T, they really determine structure on G, and the choice of a different Cartan will give the equivalent objects and same structure on G. Observe that $A \subset B$, implies $T \subset B$. The group of characters of G is

$$X^* := \mathrm{Hom}(T, \mathbb{G}_m) = \{\chi : T \to \mathbb{G}_m \mid \chi \text{ is a } F_1\text{-homomorphism}\}.$$

The group of cocharacters of G is

$$X_* := \mathrm{Hom}(\mathbb{G}_m, T) = \{\lambda : \mathbb{G}_m \to T \mid \lambda \text{ is a } F_1\text{-homomorphism}\}.$$

Since T is F_1-split, for some $e \geq d \geq 1$, $T \cong \mathbb{G}_m^e$, the isomorphism being over F_1. Thus, as in the relative case, we can identify X^* and X_* with $\mathbb{Z}^e$. The absolute rank of G, denoted as rank(G) is e, the rank of the Cartan subgroup T. The absolute root data of G with respect to T,

$$\Psi_F(G) = \langle X^*, X_*, \Phi, \Phi^\vee, \sigma \rangle,$$

consists of the following

- X^* is the group of F_1-characters of T.
- X_* is the group of F_1-cocharacters of T.
- $\Phi \subset X^*$ is the set of roots.
- $\Phi^\vee \subset X_*$ is the set of coroots.
- σ is an automorphism of finite order of X^* sending a set of simple roots (see below for a definition) in Φ to itself. The automorphism σ is obtained from the action on X^* induced from the Frobenius element of $\mathrm{Gal}(F_1/F)$ on the maximally split Cartan subgroup T of G. We explain this in more detail.

  Let $\tau \in \mathrm{Gal}(F_1/F)$ and let $\chi \in X^*$. Consider

$$^{\tau}\chi := (\tau \circ \chi \circ \tau^{-1}) : T \to \mathbb{G}_m,$$

the factor τ^{-1} acts componentwise on T (recall everything here is over F_1). It can be proved that

- $^{\tau}\chi \in X^*$, that is, $^{\tau}\chi$ is a group homomorphism defined over F_1;
- $^{\mathrm{id}}\chi = \chi$, and
- $^{\tau\delta}\chi = {}^{\tau}({}^{\delta}\chi)$.

Hence, we have an action of $\mathrm{Gal}(F_1/F)$ on X^*. Since F_1 is an unramified extension of F, $\mathrm{Gal}(F_1/F)$ is a cyclic group generated by the Frobenius element $\mathfrak{Frob}$ (It comes from the Frobenius element in the finite residue field.) The action of $\mathfrak{Frob}$ corresponds to the group automorphism $\sigma : X^* \to X^*$.

See Springer [53, §2.], for details on the construction of the roots and coroots. The sets of roots and coroots are finite subsets of X^* and X_*, respectively. There is a pairing between characters and cocharacters, defined in the same way. There is a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. For each $\alpha \in \Phi$ we define an endomorphism $s_\alpha : X^* \to X^*$ and $s_{\alpha^\vee} : X_* \to X_*$.

The absolute Weyl group W is defined as the finite group generated by s_α or $s_{\alpha^\vee}$. This finite group acts on X^* and on X_* and permutes roots and coroots. Lemma 41, 42, 43 and Theorem 45 are valid in the absolute case.

The choice of the Borel $B \supset T$ is equivalent to a choice of subsets $\Delta \subset \Phi$ and $\Delta^\vee \subset \Phi^\vee$ satisfying the following properties:

- The bijection $\Phi \longleftrightarrow \Phi^\vee$ restricts to a bijection $\Delta \longleftrightarrow \Delta^\vee$.
- There exists an element $v \in X^*$ with trivial stabilizer in W such that

$$\Delta^\vee = \{\alpha^\vee \in \Phi^\vee \mid \langle v, \alpha^\vee \rangle > 0\}.$$

The roots in Δ are called simple roots, and the coroots in $\Delta^\vee$ are called simple coroots.

The septuple

$$\Psi_0(G) = \langle X^*, X_*, \Phi, \Phi^\vee, \Delta, \Delta^\vee, \sigma \rangle$$

is called the absolute based root data of G.

3.1.3 Classification of unramified groups

The group G, being a connected unramified reductive group defined over G, is classified by its absolute root data

$$\Psi(G) = \left\langle X^*, X_*, \Phi, \Phi^\vee, \sigma \right\rangle.$$

That is, if G' is a connected unramified reductive group defined over F, then G is isomorphic to G' over F if and only if G and G' have isomorphic absolute root data. Let us recall that an automorphism of the root data $\Psi(G)$ consist of an automorphism of X^* that leaves invariant Φ, and an automorphism of X_* that leaves invariant $\Phi^\vee$. The pairing between characters and cocharacters must be respected.

If G is a split group, then $\sigma = 1$, and the first four elements $\langle X^*, X_*, \Phi, \Phi^\vee \rangle$ classify these groups. If G is a split group, then the absolute root data and the relative root data are the same.

3.1.4 The complex dual group and the L-group

Let G be a connected unramified reductive group defined over F. The absolute root data of G is denoted by $\Psi(G) = \left\langle X^*, X_*, \Phi, \Phi^\vee, \sigma \right\rangle$. The dual group $\hat{G}$ is defined as the connected unramified reductive group over $\mathbb{C}$ that corresponds to the root data

$$\Psi(G)^\vee = \langle X_*, X^*, \Phi^\vee, \Phi \rangle,$$

notice that this root data is obtained by the exchange of characters with cocharacters, and roots with coroots in the absolute root data of G. Hence, by definition, $\Psi(\hat{G}) \cong \Psi(G)^\vee$. The groups G and $\hat{G}$ have isomorphic Weyl groups.

The simplest case of this duality is for tori. If T is a torus defined over F then

$$\hat{T} = X^* \otimes \mathbb{C}^\times,$$

where X^* is the group of characters of T.

Example 46. *The following are pairs of Langlands dual groups:*

$$GL(n) \longleftrightarrow GL(n),$$
$$SL(n) \longleftrightarrow PGL(n),$$
$$SO(2n) \longleftrightarrow SO(2n),$$
$$SO(2n+1) \longleftrightarrow Sp(2n).$$

The dual group $\hat{G}$, being a group defined over $\mathbb{C}$, is a $\mathbb{C}$-split group. The dual group of the F_1-split Cartan $T \subset G$ is denoted by $\hat{T}$. The choice of the Borel B in G determines a Borel $\hat{B} \subset \hat{G}$ that contains $\hat{T}$. Equivalently, this corresponds to interchanging simple roots with simple coroots, considering the absolute based root data of G. Note that there is no relative case for the dual group. It is by definition defined over $\mathbb{C}$, and we omit the adjective "absolute" when we refer to the dual group. Let us consider the based root data of $\hat{G}$,

$$\Psi_0(\hat{G}) = \langle X_*, X^*, \Phi^\vee, \Phi, \Delta^\vee, \Delta \rangle.$$

There is a split exact sequence

$$1 \longrightarrow \mathrm{Int}(\hat{G}) \longrightarrow \mathrm{Aut}(\hat{G}) \longrightarrow \mathrm{Aut}(\Psi_0(\hat{G})) \longrightarrow 1,$$

where $\mathrm{Aut}(\hat{G})$ denotes the set of automorphisms of $\hat{G}$ over $\mathbb{C}$, and $\mathrm{Int}(\hat{G})$ denotes the set of inner automorphisms of $\hat{G}$ over $\mathbb{C}$. An automorphism of a based root data is an automorphism of a root data that also leaves invariant the set of simple roots and the set of simple coroots.

To get a splitting, we make a choice of root vectors $x_{\alpha^\vee}$, for each $\alpha^\vee \in \Delta^\vee$. This choice defines a splitting $(\hat{G}, \hat{B}, \hat{T}, \{x_{\alpha^\vee}\}_{\alpha^\vee \in \Delta^\vee})$ of $\hat{G}$ and gives a canonical isomorphism

$$\mathrm{Aut}(\Psi_0(\hat{G})) \xrightarrow{\sim} \mathrm{Aut}(\hat{G}, \hat{B}, \hat{T}, \{x_{\alpha^\vee}\}_{\alpha^\vee \in \Delta^\vee}) \subset \mathrm{Aut}(\hat{G}).$$

Since G is defined over F, there is an action of $\mathrm{Gal}(F_1/F)$ on $G(F_1)$, that is, an homomorphism

$$\sigma_G : \mathrm{Gal}(F_1/F) \to \mathrm{Aut}(G(F_1)) \subset \mathrm{Aut}(G).$$

Hence, by composition, we have a homomorphism

$$\mathrm{Gal}(F_1/F) \to \mathrm{Aut}(\Psi_0(G)). \tag{3.4}$$

The finite group $\mathrm{Gal}(F_1/F)$ acts on the root data of G, this action induces an action on $\hat{G}$. The version of the L-group that we use is

$${}^L G = \hat{G} \rtimes \mathrm{Gal}(F_1/F).$$

If G is a split group, then ${}^L G = \hat{G}$.

3.2 THE STATEMENT OF THE FUNDAMENTAL LEMMA

3.2.1 The spherical Hecke algebra

Let G be a connected unramified reductive group defined over F. Let K be a hyperspecial maximal compact subgroup of $G(F)$. Before defining the spherical Hecke algebra of G, which is the object of our interest, we say some words about the Hecke algebra $\mathcal{H}(G)$. This algebra consist of all compactly supported locally constant complex-valued functions on $G(F)$. This space is also known as the Schwartz-Bruhat space of G. In the case of $G(F)$, being totally disconnected locally compact space (which is the case of p-adic groups with the p-adic topology), we have

$$\mathcal{H}(G) = \mathrm{span}_{\mathbb{C}}\{\mathrm{char}(A) : A \in \mathcal{T}_G\},$$

where $\mathcal{T}_G$ is the set of all open and compact sets in $G(F)$ and $\mathrm{char}(A)$ is the characteristic function of A. The compact open subgroups of $G(F)$,

$$K_m = \{g \in K : g \equiv 1 \bmod \varpi^m\}$$

give a neighborhood basis at the identity element, and by translations we get a basis for any element in the group. For a fixed $m \geqslant 1$, the formula defining K in conjunction with the formula

$$\bigwedge_{i \neq j} \mathrm{ord}(g_{ij}) \geqslant m \wedge \bigwedge_{i=1}^{n} \mathrm{ord}(g_{ii} - 1) \geqslant m,$$

define the subgroup K_m, where $g = (g_{ij})$ is a $n \times n$ matrix. Then each element of the neighborhood basis of the identity is definable in the Denef-Pas language. Moreover this neighborhood basis is uniformly definable: just let m vary. Any element in $\mathcal{T}_G$ is a finite union of translations of subgroups K_m and since each of these is definable we have that all the elements in $\mathcal{T}_G$ are definable with parameters in $G(F)$. Observe that if K_m is definable without parameters in $G(F)$, then $\mathrm{char}(K_m)$ is a constructible motivic function.

We fix the left Haar measure on $G(F)$ such that K has volume 1. We define the product on $\mathcal{H}(G)$ by the convolution formula,

$$(f_1 * f_2)(g) = \int_{G(F)} f_1(x) f_2(x^{-1}g) dx.$$

With this convolution product $\mathcal{H}(G)$ is an associative algebra. Let $\mathcal{H}(G, K)$ be the subalgebra of $\mathcal{H}(G)$ of bi-invariant functions under K (*i.e.*, $f(kg) = f(gk') = f(g)$ for all $g \in G(F)$ and $k, k' \in K$), this algebra is called the spherical Hecke algebra of G.

We fix A a maximal split torus of G, and a Borel B containing A. Suppose G, B and A are defined over $\mathcal{O}_F$. By choosing a root basis $\Delta \subset \Phi$ of positive and indecomposible roots, we determine a positive Weyl chamber P^+ in X_{*F} defined by

$$P^+ = \{\lambda \in X_{*F} \mid \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Recall that X_{*F} is identified $\mathbb{Z}^r$.

Lemma 47. *Under the hypothesis in this section. The positive Weyl chamber P^+ is a definable subset of $\mathbb{Z}^r$.*

Proof. Suppose $\Delta = \{\alpha_1, \ldots, \alpha_k\}$. So each α_i is a tuple in $\mathbb{Z}$. Although they will be used as parameters in the defining formula, the point is that this choice is field independent (they are fixed choices, as defined in the next chapter). The $\mathcal{L}_{Pres}$-formula with free variables represented by λ (corresponding to a definable set in $\mathbb{Z}^r$) define P^+

$$\bigwedge_{i=1}^{k} \langle \alpha_i, \lambda \rangle \geq 0.$$

The paring is given by the dot product, and it can be expressed in $\mathcal{L}_{Pres}$ because $\alpha_1, \ldots, \alpha_k$ are fixed (they are used as parameters in the formula). □

For $\lambda \in X_{*F}$, we interpret ϖ^λ as follows, $\lambda : F^\times \to A(F)$ is a cocharacter (defined over F) of the F-torus A, $\varpi \in F^\times$ and $\varpi^\lambda := \lambda(\varpi)$. The following is a fundamental result for the understanding of the spherical Hecke algebra.

Theorem 48 (Cartan decomposition, [56]). *The group $G(F)$ is the disjoint union of double cosets $K\varpi^\lambda K$, where λ runs through the cocharacters indexed by P^+. That is*

$$G(F) = \bigsqcup_{\lambda \in P^+} K\varpi^\lambda K.$$

$\square$

Each function $f \in \mathcal{H}(G, K)$ is constant on double cosets $K\varpi^\lambda K$. Since it is also compactly supported, it is a finite linear combination of the characteristic functions $\mathrm{char}(K\varpi^\lambda K)$ of these double cosets. Thus

$$\mathcal{H}(G, H) = \mathrm{span}_{\mathbb{C}}\{\mathrm{char}(K\varpi^\lambda K) \mid \lambda \in X_{*F}\}.$$

For these generators the convolution product has the following form

$$\mathrm{char}(K\varpi^\lambda K) * \mathrm{char}(K\varpi^\mu K) = \sum_\nu n_{\lambda\mu}(\nu)\mathrm{char}(K\varpi^\nu K)$$

with $n_{\lambda\mu}(\nu) \in \mathbb{Z}$. In [22], it is proved that

$$n_{\lambda\mu}(\nu) = \#\{(i,j) \,|\, \varpi^\nu \in x_i y_j K\} \tag{3.5}$$

where $K\varpi^\lambda K = \bigsqcup x_i K$ and $K\varpi^\mu K = \bigsqcup y_j K$. We prove in the next chapter that these integers can be obtained by a motivic function, so the convolution product on generators of the spherical Hecke algebra is, in some sense, motivic.

3.2.2 Orbital integrals

As in the previous sections, G is a connected unramified reductive group defined over F with F_1-split Cartan subgroup T and absolute Weyl group W. We follow the survey article by Ngô [46] for our exposition in this section.

Definition 49. *Let $\gamma \in G(F)$. The centralizer of γ is a subgroup I_γ of G defined over F. The set of F-points of I_γ is*

$$I_\gamma(F) = \{g \in G(F)\,,\, g\gamma = \gamma g\}.$$

We say that γ is a strongly regular semisimple if I_γ is a torus.

If γ is a strongly regular semisimple element of $G(F)$, then the F-torus I_γ is not necessarily an F-split torus.

Definition 50. *Let γ be a strongly regular semisimple element of $G(F)$. Let dg be a Haar measure on $G(F)$ and let dt be a Haar measure on $I_\gamma(F)$. The orbital integral over the conjugacy class of γ (or the orbit of γ) is the distribution*

$$\mathcal{O}_\gamma(\cdot, dg/dt) : \mathcal{H}(G) \to \mathbb{C}$$

given by

$$\mathcal{O}_\gamma(f, dg/dt) = \int_{I_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\frac{dg}{dt}.$$

Notice that the orbital integral $\mathcal{O}_\gamma$ does not depend on γ but on its conjugacy class. It also depends on the choice of Haar measures dg and dt on $G(F)$ and $I_\gamma(F)$, respectively.

Theorem 51 (Chevalley restriction theorem)**.** *Let G be a connected unramified reductive group defined over F and let $T \subset G$ be an F_1-split Cartan defined over F, with absolute Weyl group W. The G-invariant polynomial functions on G are isomorphic to the W-invariant polynomial functions on T. More explicitly, the restriction of functions along the inclusion $T \subset G$, induces an isomorphism*

$$F_1[G]^G \xrightarrow{\sim} F_1[T]^W.$$

From this isomorphism one can deduce the F_1-morphisms of varieties

$$\chi_G : G \to T/W = \mathrm{Spec}(F_1[T]^W).$$

There is a similar result for the Lie algebra $\mathfrak{g}$ of G. In this context one gets an adjoint quotient $\mathfrak{g} \to \mathfrak{h}/W$, with Cartan subalgebra $\mathfrak{h}$.

The morphism χ_G is called the characteristic polynomial map of G. The following example explains the reason for this name.

Example 52. *Suppose $G = GL(n)$, the general linear group. Let us consider first the case of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$, with the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. By Chevalley restriction theorem, $F_1[\mathfrak{g}]^G \cong F_1[\mathfrak{h}]^W$, so this is the polynomial ring generated by the coefficients of the characteristic polynomial. Concretely, $\mathfrak{h}/W \cong \mathbb{A}^n$, and the adjoint quotient $\mathfrak{g} \to \mathbb{A}^n$, maps $x \in \mathfrak{g}$ to the tuple $(c_0(x), \ldots, c_{n-1}(x))$, where*

$$p_x(t) = t^n + c_{n-1}(x)t^{n-1} + \cdot + c_0(x)$$

is the characteristic polynomial of x. The polynomial c_i is homogeneous of degree $n-i$. The group case G is similar; χ_G maps $x \in G$ to the coefficients of its characteristic polinomial. In this case

$$T/W = \mathrm{Spec}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n} \cong \mathbb{A}^{n-1} \times \mathbb{A} \backslash \{0\},$$

where S_n is the n-symmetric group.

Let us recall that, $\gamma, \gamma' \in G(F)$ are conjugate over F if $g^{-1}\gamma g = \gamma'$ for some $g \in G(F)$.

Definition 53. *Let γ and γ' be strongly regular semisimple elements in $G(F)$. They are stably conjugate if $g^{-1}\gamma g = \gamma'$ for some $g \in G(\bar{F})$. The stable conjugacy class of $\gamma \in G(F)$ is the set of strongly regular semisimple elements of $G(F)$ which are stably conjugate with γ.*

For strongly regular semisimple elements $\gamma, \gamma' \in G(F)$, they are stably conjugate if and only if $\chi_G(\gamma) = \chi_G(\gamma')$ (*i.e.* they have the same characteristic polynomial). By the Chevalley restriction theorem, this is saying that each stable conjugacy class of regular semisimple elements corresponds to an element of T/W.

See the article by Kottwitz [34] for a complete discussion of the notion of stable conjugacy in reductive groups. For $GL(n)$, these two notions are the same. That is, two strongly regular semisimple elements in $GL(n,F)$ are conjugate over F if and only if they are stably conjugate over $\bar{F}$. More generally, two elements in $GL(n,F)$ are conjugate over F if and only if they are conjugate over $\bar{F}$ (*i.e.*, conjugate by an element in $G(\bar{F})$).

The next example lies outside the p-adic context. It is just to illustrate with a simple example.

Example 54. *In $SL(2,\mathbb{R})$, the following two elements*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are conjugate by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL(2,\mathbb{C}).$$

A matrix calculation shows that they cannot be conjugate by an element in $SL(2,\mathbb{R})$. The obstruction on the conjugation is measured by a Galois cohomology group.

This example is taken from [26].

Example 55. *Let $G = SL(2)$ and let $F = \mathbb{Q}_p$. Let us assume that $p \neq 2$ and that u is not a square in $\mathbb{Q}_p$. Let $\epsilon = \sqrt{u} \in \bar{\mathbb{Q}}_p$. A matrix calculation shows that*

$$\begin{pmatrix} 1+p & 1 \\ 2p+p^2 & 1+p \end{pmatrix}, \quad \begin{pmatrix} 1+p & u^{-1} \\ (2p+p^2)u & 1+p \end{pmatrix}$$

are stably conjugate by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \in SL(2,\bar{\mathbb{Q}}_P),$$

but they are not conjugate by a matrix of $SL(2,\mathbb{Q}_p)$.

Let γ be a strongly regular semisimple element in $G(F)$. There are possible various $G(F)$-conjugacy classes inside the stable conjugacy class of γ. The set of $G(F)$-conjugacy classes in the stable conjugacy class of γ can be identified with the subset A_γ of elements in $H^1(\mathrm{Gal}(F_1/F), I_\gamma)$ whose image in $H^1(\mathrm{Gal}(F_1/F), G)$ is trivial. When F is a local field (archimedean and non-archimedean), $H^1(\mathrm{Gal}(F_1/F), I_\gamma)$ is a finite abelian group. Let $\kappa : A_\gamma \to \mathbb{C}^\times$ be a character. The κ-orbital integral is the linear combination

$$\mathcal{O}^\kappa_\gamma(f) := \sum_{\gamma'} \kappa(\mathrm{cl}(\gamma'))\mathcal{O}_{\gamma'}(f).$$

The sum runs over a set of representatives γ' of $G(F)$-conjugacy classes within the stable conjugacy class of γ, $\mathrm{cl}(\gamma')$ is the class of γ' in A_γ, and $f \in \mathcal{H}(G)$. Since $\gamma, \gamma' \in G(F)$ are strongly regular semisimple, the tori $I_{\gamma'}$ for γ' in the stable conjugacy class of γ, are isomorphic in a canonical way. This allows us to transfer a Haar measure from I_γ to $I_{\gamma'}$. Thus, the Haar measures on the different centralizers $I_{\gamma'}$ (and hence, the Haar measures on the orbits) are choosen in a consistent way. If $\kappa = e$ is the trivial character, we obtain the stable orbital integral

$$\mathcal{SO}_\gamma(f) := \mathcal{O}^e_\gamma(f) = \sum_{\gamma'} \mathcal{O}_{\gamma'}(f).$$

Note that the stable orbital integral $\mathcal{SO}_\gamma$ depends only on $\chi_G(\gamma)$, its characteristic polynomial. Therefore, for $a \in T/W(F)$ we can define the stable orbital integral at a as

$$\mathcal{SO}_a(f) := \mathcal{SO}_\gamma(f),$$

for any $\gamma \in G(F)$ such that $\chi_G(\gamma) = a$.

For κ-orbital integrals the choice of the base point γ in the stable conjugacy class is not a trivial issue. For any γ' in the stable conjugacy class of γ, it is true that A_γ and $A_{\gamma'}$ are canonically isomorphic, so the character κ on A_γ induces a character κ' on $A_{\gamma'}$. Now, the two objects that we must compare are

$$\mathcal{O}^\kappa_\gamma \quad \mathcal{O}^{\kappa'}_{\gamma'}.$$

They are not equal, they differ by the scalar $\kappa(\mathrm{cl}(\gamma'))$ ($\mathrm{cl}(\gamma')$ is the $G(F)$-conjugacy class of γ' in A_γ). These factors, which come from a choice of a base point in the stable conjugacy class, are known as the transfer factors. They were introduced by Langlands and Shelstad in [39], and they play an important but technical role in the theory of endoscopy. We will see that they are the source of a problem for us. Basically, we cannot describe them using the Denef-Pas language.

Notice that because of the transfer factors, we cannot define, a priori, the κ-orbital integral $\mathcal{O}_a^\kappa$ for a characteristic polynomial, that is, an element $a \in T(F)/W$ as in the case of stable orbital integrals. We observe that in the case of Lie algebras, the Konstant section $\iota_{\mathfrak{g}} : \mathfrak{h}/W \to \mathfrak{g}$ of the characteristic polynomial map $\chi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{h}/W$ allows to have a canonical representative, and the definition

$$\mathcal{O}_a^\kappa := \mathcal{O}_{\iota_{\mathfrak{g}}(a)}^\kappa$$

for $a \in \mathfrak{h}/W$, simplifies the situation in the Lie algebra case [35].

3.2.3 Endoscopic groups

Roughly, one can say that endoscopy theory is a series of techniques developed by Langlands and Shelstad to understand conjugation in terms of stable conjugation. This is a simplified definition of a vast field in representation theory of algebraic groups and automorphic forms. Our exposition of endoscopic groups is based on the paper by Hales [26], and the paper by Ngô [46]. We also recommend the article by Labesse [37] for an introduction to endoscopy theory.

Definition 56. *Let G be a connected unramified reductive group defined over F. Let $\Psi(G) = \langle X^*, X_*, \Phi, \Phi^\vee, \sigma\rangle$ be the absolute root data of G. An unramified endoscopic group H of G is an unramified reductive group over F whose classifying data (i.e. the absolute root data of H) has the form*

$$\Psi(H) = \langle X^*, X_*, \Phi_H, \Phi_H^\vee, \sigma_H\rangle.$$

The first two entries on the absolute root data are the same for G as for H. The data for H is subject to the following constraints: there exists an element $s \in \hat{T} \cong \mathrm{Hom}(X_*, \mathbb{C}^\times)$ and a

Weyl group element $w \in W$, the absolute Weyl group of G, such that

- $\Phi_H^\vee = \{\alpha \in \Phi^\vee \mid s(\alpha) = 1\}$,
- $\sigma_H = w \circ \sigma$, *and*
- $\sigma_H(s) = s$.

This definition implies that G and H have the same absolute rank. Since there is a bijection between roots and coroots of G and $\Phi_H^\vee \subset \Phi^\vee$. The roots $\Phi_H \subset \Phi$ are completely determined. The absolute Weyl group of H, denoted by W_H, is a subgroup of W. The endoscopic group H is not a subgroup of G in general.

Example 57. *The unramified endoscopic groups of $SL(2)$ are $U_E(1)$ (E/F is an unramified quadratic extension), $\mathbb{G}_m$, and $SL(2)$ itself. See [26] for details of this computation.*

3.2.4 The fundamental lemma - Lie algebra version

The Lie algebra version of the fundamental lemma takes the following form

$$\mathcal{SO}_{a_H}(\mathbf{1}_{\mathfrak{h}(\mathcal{O}_F)}) = \Delta(a_H, a_G)\mathcal{O}^\kappa_{a_G}(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}),$$

It follows from Langlands-Shelstad descent that the fundamental lemma for the Lie algebras implies the original statement (for groups). It is the Lie algebra variant which Ngô proved (for fields of positive characteristic).

4.0 MOTIVIC SPHERICAL HECKE ALGEBRAS

4.1 FIXED CHOICES

We follow the convention of [19] about a fixed choice. Thus, by a fixed choice we mean a fixed set that is completely independent of the Denef-Pas language or any p-adic field. In particular, a fixed choice is a field independent object.

Let G be a split reductive algebraic group defined over $\mathbb{Q}$ with Cartan and Borel subgroups $T \subset B \subset G$, all defined over $\mathbb{Q}$. Let $(X^*, X_*, \Phi, \Phi^\vee)$ the root datum of G with respect to T. The complex dual group of G, denoted by $\widehat{G}$ is the complex reductive group that corresponds to the root datum $(X_*, X^*, \Phi^\vee, \Phi)$. The Langlands dual group (L-group) that we use in this case is just the the complex dual group, so ${}^L G = \widehat{G}$.

Let H be a split endoscopic group of G defined over $\mathbb{Q}$. Let T_H and B_H be Cartan and Borel subgroups of H defined over $\mathbb{Q}$. Similarly as we did for G, we consider the root datum of H, and by switching characters with cocharacters and roots with coroots we get the root datum of $\widehat{H}$, the complex dual group of H. We can take $\widehat{H} = C_{\widehat{G}}(s)^\circ$ for some semisimple element $s \in \widehat{T}$. We take ${}^L H = \widehat{H}$.

In endoscopy theory an embeding of L-groups $\xi : {}^L H \to {}^L G$ is fixed. In our case we can give a very explicit description of this embedding. Since ${}^L H = \widehat{H} = C_{\widehat{G}}(s)^\circ \subset \widehat{G} = {}^L G$, we take $\xi : \widehat{H} \to \widehat{G}$ as the inclusion map.

The Weyl groups W and W_H of G and H respectively, are also viewed as fixed choices.

We note that in the following development we will make further fixed choices.

Let F be a p-adic field of characteristic zero. By extension of scalars the groups G, B, T can be considered as F-groups. In [10], it is shown how reductive unramified groups can be realized as definable subassignments in the Denef-Pas language. As is observed in [10] and [8], given G a split group defined over $\mathbb{Q}$ we can fix an embedding

$$\rho : G \to GL(n)$$

that is a morphism between two algebraic varieties defined over $\mathbb{Q}$, hence it is definable in the Denef-Pas language (in fact it is definable in the language of rings without using extra parameters). Thus, it is clear that the set of F-points $G(F)$ of the group G is a definable subset of $GL(n, F)$. We fix this n throughout. The unramified case requires the definability of the Frobenius. The details can be found in [10].

Given a split group G, as a subgroup of $GL(n)$, any closed subgroup of G is definable in the Denef-Pas language. In particular, a fixed Borel subgroup B is definable.

4.2 MOTIVIC IDENTITIES AND MOTIVIC IDENTITIES UP TO A NULL FUNCTION

A motivic identity, as the name indicates, is an identity between two constructible functions on a common subassignment. We want to define motivic identities up to a null function. These are functions that are equal to zero in almost all the specializations to non-archimedean local fields. The following is the formal definition.

Definition 58. *Let S be a definable subassignment and let $f \in \mathcal{C}(S)$. We say that f is a null constructible motivic function (or a null function, for short) if there exists $M > 0$ such that*

$$f_F = 0 \quad \textit{for all} \quad F \in \mathscr{C}_M.$$

Definition 59. *Let S be a definable subassignment and let $f, g \in \mathcal{C}(S)$. We say that f and g are equal up to a null function, or $f = g$ is a motivic identity up to a null function if*

$$f = g + n,$$

where $n \in \mathcal{C}(S)$ is a null function. We may also write $f = g \mod(\text{null})$. It is clearly equivalent to say that the difference of f and g is a null function or that f and g specialize to the same function for almost all non-archimedean local fields, that is

$$f_F = g_F \quad \textit{for all} \quad F \in \mathscr{C}_M,$$

for some $M > 0$.

It should be clear that it is easier to prove a motivic identity up to a null function instead of a genuine motivic identity. Maybe at the risk of being redundant, we emphasize the difference. When we allow null functions, we have to verify the identity for almost all non-archimedean local fields. Constructible motivic functions specialize to actual functions on such fields, and the description of the functions is uniform in all non-archimedean fields. The following two steps provide a general strategy to prove that a relation arising from the theory of reductive groups over p-adic fields is a motivic identity up to a null function.

1) Prove that the relation can be described in terms of constructible motivic functions (*i.e.,* as a potential identity of motivic functions). Let's say for simplicity $f = 0$ is such a relation. Basically, one is proving the existence of a constructible motivic function that specializes to f. Hence we can promote f from the p-adic level to the motivic one. Notice that in particular we have a uniform but also field independent description of f.
2) Having part 1), what remains is to prove that $f_F = 0$ for all $F \in \mathscr{C}_M$, for some $M > 1$. In most of the cases such identity was already implicit (basically because we start with a p-adic relation $f = 0$) if not obvious by construction.

At the motivic level, we do not really have functions (in the classical meaning of the word). They are elements in the ring of constructible motivic functions. To prove a motivic identity implies that it is going to be true in any Denef-Pas structure where it makes sense to consider an interpretation. In principle, these structures include more than non-archimedean local fields.

We observe that the pullback of a null function is a null function and the pushforward of a null function is also a null function, over the corresponding subassignments.

Remark 60. *Although everything in the ring of constructible motivic functions is determined by definable data, the tools from model theory cannot be applied directly. Nevertheless, we believe that the relation between motivic identities and motivic indentities up to a null function might be related to the completeness or non-completeness of a first-order theory in the Denef-Pas language. See §2.7. in [12]. We point out an important issue in this analysis. The specialization map does not correspond to the usual model-theoretic interpretation of a first-order language inside a structure, in principle because a constructible motivic function is somehow beyond the first-order setting.*

4.3 THE SPLIT CASE

We work the split case. Let us assume from now on and until the end of this work (unless we say the contrary) that G is a F-split group. Therefore the relative data equals the absolute data. For instance, $X^*_F = X^*$, $X_{*F} = X_*$, $W_F = W$ and so on.

We start with the description of the character δ in the definition of the Satake transform. Let $\Phi^+ \subset \Phi$ be a set of positive roots (this set is a fixed choice). Consider

$$2\rho = \sum_{\alpha \in \Phi^+} \alpha \in X^*.$$

Clearly, ρ is a fixed choice as well.

According to Gross [22], p.6, for $\lambda \in X_*$,

$$\delta^{1/2}(\varpi^\lambda) = q^{-\langle \rho,\lambda\rangle} \in q^{(1/2)\mathbb{Z}}.$$

Therefore, since δ is unramified, $\delta^{1/2}$ takes values in the group $q^{(1/2)\mathbb{Z}}$. If $\rho \in X^*$ then $\delta^{1/2}$ takes values in the group $q^{\mathbb{Z}}$. As it is observed in [8, §B.3.1.], the theory can be expanded without any harm to include roots of q. In our situation we just need to consider the square root of q. This observation combined with Lemma 42 gives that the function on $\delta^{1/2} : T \to \mathbb{C}$ is given by a constructible motivic function on T. That is, there exists a constructible motivic function on T that specializes to $\delta^{1/2}$.

Remark 61. *The previous computation of δ is valid in the unramified case, where it is necessary to work with positive roots in X^*_F.*

4.4 DEFINABILITY OF THE CARTAN DECOMPOSITION

The spherical Hecke algebra $\mathcal{H}(G, K)$ is in principle a p-adic object. A first attempt to deal with this object at a motivic level is to work just with generators. In this case, the set of generators can be identified with $\mathbb{Z}^d$, which is a definable set. In many situations this is enough. In this section we explain how the definability of the Cartan decomposition allow us to define motivic objects that describe in a field independent way $\mathcal{H}(G, K)$ and its algebra structure. The definability of the Cartan decomposition was proved in [52, Lemma B.12]. We explain it in detail and then we define the notion of a motivic Hecke algebra.

Lemma 62. *The elements in the group of characters X^*_F and the group of cocharacters X_{*F} are definable functions in the Denef-Pas language.*

Proof. Let $\chi : T(F) \to F^\times$ be an element of X^*_F. By definition χ is a morphism of algebraic varieties defined over F, in the sense of algebraic geometry. Clearly $T(F)$ and $F^\times$, as the set of F-points of algebraic varieties defined over F, are definable in the Denef-Pas language (using just VF-variables) using parameters in F. Then the map χ being an algebraic morphism between two definable sets over F is a F-definable function in the Denef-Pas language. The proof for the cocharacters is similar. $\square$

Lemma 63. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. Let $\lambda \in X_*$. The double coset $K\varpi^\lambda K \subset G(F)$ is a definable set in the Denef-Pas language. Therefore the characteristic function of $K\varpi^\lambda K$ is a constructible motivic function.*

Proof. Let $\chi_1, \dots \chi_r \in X^*$ be generators of X^*. By the Lemma 62 these characters are definable in the Denef-Pas language. Let D_λ^G the set defined by following Denef-Pas formula (VF-free variables)

$$\varphi_\lambda(g) = (g \in G) \wedge \exists\, k_1, k_2 \in K \;\; \exists\, a \in T \left(g = k_1 a k_2 \;\wedge\; \bigwedge_{i=1}^{r} \operatorname{ord}(\chi_i(a)) = \langle \chi_i, \lambda \rangle \right).$$

Note that $\langle \chi_i, \lambda \rangle \in \mathbb{Z}$. We claim that $D_\lambda^G = K\varpi^\lambda K$. We prove the two inclusions. If $g \in K\varpi^\lambda K$, there are $k_1, k_2 \in K$ such that $k_1 g k_2 = \varpi^\lambda \in T$. For each i, $\operatorname{ord}(\chi_i(\varpi^\lambda)) = \operatorname{ord}(\varpi^{\langle \chi_i, \lambda \rangle}) = \langle \chi_i, \lambda \rangle$. We now prove the other inclusion. Suppose $k_1 a k_2 \in D_\lambda^G$ then for each $i \in \{1, \dots, d\}$

$$\chi_i(a) = u_i \varpi^{\langle \chi_i, \lambda \rangle} \;\text{ for some }\; u_i \in \mathcal{O}_F^\times.$$

This implies that $a\varpi^{-\lambda} \in K$, so $k_1 a k_2 = k_1(a\varpi^{-\lambda})\varpi^\lambda k_2 \in K\varpi^\lambda K$. $\square$

The lemma shows that each double coset in the Cartan decomposition is independent of the uniformizer of the field (for any two uniformizers ϖ and π, $K\varpi^\lambda K = D_\lambda = K\pi^\lambda K$. So we can describe each double coset without a choice of a uniformizer (*i.e.,* without the choice of a p-adic element), and moreover that description can be expressed in the Denef-Pas language. But as we see in the proof there is a choice of generators of X^*, the group of F-characters of T. This is not a problem since this choice is independent of the Denef-Pas language or the p-adic field. In other words, this choice of generators is a fixed choice. So we fix a set of generators $\chi_1, \dots \chi_d \in X^*$ throughout the rest of our discussion. The following proposition shows how the Cartan decomposition can be parameterized by a definable set.

Proposition 64 (Definability of the Cartan decomposition). *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. There exists a formula in the Denef-Pas language that describes a parametrization of the double cosets in the Cartan decomposition of G. More explicitly, there exists a definable set $\mathcal{D}^G \subset \mathbb{Z}^d \times G$ in the Denef-Pas language such that*

$$(\lambda, g) \in \mathcal{D}^G \iff g \in D^G_\lambda,$$

where $\lambda = (z_1, \ldots, z_d)$ is an d-tuple of $\mathbb{Z}$-variables and g is a tuple of VF-variables.

Proof. The set $\mathcal{D}^G$ is defined by the formula

$$\varphi(\lambda, g) = (g \in G) \wedge \exists\, k_1, k_2 \in K \ \ \exists\, a \in T \left(g = k_1 a k_2 \ \wedge \ \bigwedge_{i=1}^{d} \mathrm{ord}(\chi_i(a)) = \langle \chi_i, \lambda \rangle \right).$$

Where $\langle \chi_i, \lambda \rangle$ is a formula in the free variables $z_1, \ldots, z_d$ that describes the pairing between $x_i = (m_{i1}, \ldots, m_{id}) \in \mathbb{Z}^r$ and $\lambda = (z_1, \ldots, z_d)$, a tuple of d free variables of type $\mathbb{Z}$. Since each χ_i is fixed that pairing can be expressed in the Presburger language, see Lemma 42. The result follows from Lemma 63. □

In Proposition 64 we put together all the generators in a definable way. More explicitly, we have constructed $\mathcal{D}^G$, a definable subassignment of $h[n^2, 0, d]$, that encodes the Cartan decomposition. It is possible to encode this in a single constructible function on $\mathbb{Z}^d \times G$. That is, there exists a constructible motivic function $\mathrm{char}_{\mathcal{D}^G} \in \mathcal{C}(\mathbb{Z}^d \times G)$ defined by

$$\mathrm{char}_{\mathcal{D}^G}(\lambda, g) = \begin{cases} 0 & \text{if} \quad g \notin D^G_\lambda \\ 1 & \text{if} \quad g \in D^G_\lambda \end{cases}$$

Clearly, the function $\mathrm{char}_{\mathcal{D}^G}$ is a constructible motivic function by Proposition 64.

Remark 65. *In Proposition 64, we consider the Cartan decomposition that runs over X_* instead of P^+ (as stated in Theorem 48). The corresponding result with that refined version of the Cartan decomposition (running over P^+) is still valid since P^+ is a definable set, as proved in Lemma 47.*

Let us make some comments on the generators of $\mathcal{H}(G,K)$. The Cartan decomposition (Theorem 48) says that the set of functions $\mathrm{char}(K\varpi^\lambda K)$ with $\lambda \in P^+$ is a basis for $\mathcal{H}(G,K)$, as a vector space over $\mathbb{C}$. If we consider λ running over X_* the set of functions $\mathrm{char}(K\varpi^\lambda K)$ is a set of generators.

By Lemma 63, for each $\lambda \in X_* \cong \mathbb{Z}^d$,

$$\mathbf{1}_\lambda^G := \mathrm{char}(D_\lambda^G) : G \to \mathbb{Z},$$

is the description of the generator $\mathrm{char}(K\varpi^\lambda K)$ in a field independent way *i.e.*, $\mathbf{1}_\lambda^G \in \mathcal{C}(G)$.

P-adically, the Haar measure on a split group G is the measure attached to an invariant differential form α of top degree. An explicit invariant differential form is given in [19, §2.3.]. We define the integral of $f \in \mathcal{C}(G)$ with respect to the motivic Haar measure to be

$$\int_G f\,|\alpha|$$

where $|\alpha|$ is a constructible function. For applications to group theory of reductive groups, we will always use motivic Haar integrals. If G is reductive, a left invariant differential form of top degree is also right invariant of top degree.

Lemma 66. *Let G, K be a split reductive group defined over F, and a hyperspecial subgroup of G. If α is a differential form of top degree and if $f \in \mathcal{C}(G)$, then*

$$\int_G f(xg)\,dx = \int_G f(x)\,dx$$

as functions of g in $\mathcal{C}(G)$.

Proof. Consider the following maps

- $\mu : G \times G \to G$ given by $(x,g) \mapsto xg$,
- $\pi_2 : G \times G \to G$ given by $(x,g) \mapsto g$, and
- $\psi : G \times G \to G \times G$ given by $(x,g) \mapsto (x,xg)$

Clearly, ψ is an isomorphism and $\mu = \pi_2 \circ \psi$. The lemma follows from Theorem 22 applied to the isomorphism ψ. $\square$

Remark 67. *Lemma 66 allows us to use the standard change of variables formulas for motivic Haar measure.*

The following lemma is due to Hales.

Lemma 68. *Let G, K be a split reductive group defined over F, and a hyperspecial subgroup of G. Let α be the invariant differential of top degree from [19, §2.3.]. Then*

$$\mathrm{vol}(K) := \int_G \mathrm{char}(K)\,|\alpha|$$

is an invertible element in the ring $\mathcal{C}(\mathrm{pt})$.

Proof. Let K_1 be as in §3.2.1. Then by the invariance of the measure

$$\int_G \mathrm{char}(K)\,|\alpha| = [K : K_1] \int_G \mathrm{char}(K_1)\,|\alpha|.$$

By $[K : K_1]$ we mean the motivic class of the reduction of K to the residue field. So it is enough to show $[K : K_1]$ and $\int_G \mathrm{char}(K_1)\,|\alpha|$ are invertible. The class $[K, K_1]$ is a product of factors $\mathbb{L}^i$ and $(\mathbb{L}^i - 1)$, with i a positive integer by [5]. These are invertible in A, hence in $\mathcal{C}(\mathrm{pt})$. By the explicit formula for α in [19, §2.3.], the integral

$$\int_G \mathrm{char}(K_1)\,|\alpha| = \int_{K_1} |\alpha|$$

is a product of integrals of the form

$$\int_{\mathrm{ord}(x)\geq 1} dx = \frac{1}{\mathbb{L}},$$

which is invertible, and where dx gives the additive Haar measure. $\square$

Remark 69. *Because of the Lemma 68, we can normalize the motivic Haar measure to give K volume 1 or consider $\frac{1}{\mathrm{vol}(K)}$ as a motivic constant.*

We make two comments with respect to our notation.

1. The motivic integral
$$\int_G f(x)\,dx$$
means integral of $f \in \mathcal{C}(G)$ with respect to the motivic Haar measure. That is, $\vartheta_!(f|\alpha|)$, where $\vartheta : G \to \{\mathrm{pt}\}$.
2. The motivic integral
$$\int_G f(xg)\,dg$$
means $\pi_{1!}(\mu^*(f|\alpha|))$ where $\pi_1 : G \times G \to G$ is the projection on the first component (denoted by x), and $\mu : G \times G \to G$ is the multiplication map, that is, $(x, g) \mapsto xg$. Clearly $f \in \mathcal{C}(G)$.

Similarly, the integral
$$\int_K f(xk)\,dk$$
means $\pi_!(\mu^*(f|\alpha|))$ where $\pi : K \times G \to G$ is the projection onto G, and $\mu : G \times K \to G$ still represents the multiplication map but restricted to K in the second component.

Definition 70. *Let G be a split reductive group defined over F. Let $f_1, f_2 \in \mathcal{C}(G)$, the convolution product of f_1 and f_2 is defined as*
$$(f_1 * f_2)(g) = \int_G f_1(x) \cdot f_2(x^{-1}g)\,dx,$$
where dx denotes the motivic Haar measure on G, normalized such that K gets volume 1. More precisely, in terms of pullbacks and pushforwards, it is given by the formula 4.5 that appears below.

Observe that the convolution product of two constructible motivic functions is a constructible motivic function since it is the integral over a definable set of a constructible motivic function. Note that the map $(x, g) \mapsto x^{-1}g$ is a definable morphism.

Let $\mathcal{C}_{bd}(G)$ be the ring of bounded constructible motivic functions on G. By Proposition 19, all such functions are integrable.

The ring $\mathcal{C}_{bd}(G)$ is a (possibly non-associative) algebra under convolution. There is a morphism $\mathcal{C}(\mathrm{pt}) \to (\mathcal{C}(G), \cdot)$ of algebras given by

$$c \mapsto \vartheta^* c$$

where $\vartheta : G \to \{\mathrm{pt}\}$.

If $f \in \mathcal{C}_{bd}(G)$ and $c \in \mathcal{C}(\mathrm{pt})$, then

$$\vartheta^* c \cdot f \in \mathcal{C}_{bd}(G).$$

So $\mathcal{C}_{bd}(G)$ is a $\mathcal{C}(\mathrm{pt})$-module. This allows us the following definition.

Definition 71. *Let G, K be a split reductive group defined over F, and a hyperspecial subgroup of G. Let $\mathcal{H}_{\mathrm{pt}}(G,K)^{\mathrm{mot}}$ be the convolution subalgebra of $(\mathcal{C}_{bd}(G), *)$ generated by*

$$\vartheta^* c \cdot \mathbf{1}_\lambda^G$$

with $\lambda \in P^+$ and $c \in \mathcal{C}(\mathrm{pt})$. This algebra is called the $\mathcal{C}(\mathrm{pt})$-motivic spherical Hecke algebra of G with respect to K. The motivic spherical Hecke algebra of G with respect to K is defined as

$$\mathcal{H}(G,K)^{\mathrm{mot}} := \mathcal{H}_{\mathrm{pt}}(G,K)^{\mathrm{mot}} \otimes_{\mathbb{Z}} \mathbb{C}.$$

See §2.9 in [10] for details and comments of this tensoring with $\mathbb{C}$.

Lemma 72. *Let G, K be a split reductive group defined over F, and a hyperspecial subgroup of G. If $c, c_1, c_2 \in \mathcal{C}(\mathrm{pt})$ and $f, f_1, f_2 \in \mathcal{C}(G)$ then*

a)

$$\int_G \vartheta^* c \cdot f \, dx = c \int_G f \, dx.$$

b)

$$(\vartheta^* c_1 \cdot f_1) * (\vartheta^* c_2 \cdot f_2) = \int_G \vartheta^*(c_1 c_2) \cdot f_1(x) \cdot f_2(x^{-1}g) \, dx = \vartheta^*(c_1 c_2) \cdot (f_1 * f_2).$$

Proof. This is the projection formula on Theorem 17. $\square$

We stress the fact that $\mathcal{H}(G,K)^{\mathrm{mot}}$ is a field independent object. The following lemma states that the convolution product in $\mathcal{H}(G,K)^{\mathrm{mot}}$ specializes to the convolution product on the Hecke algebra of $\mathcal{H}(G,K)$. Thus, we see that the motivic spherical Hecke algebra is related with the p-adic spherical Hecke algebra in an expected and desirable way.

Lemma 73. *Let G, K be a split reductive group defined over F, and a hyperspecial subgroup of G. Let $\lambda,\mu \in P^+$. The constructible motivic function $\mathbf{1}^G_\lambda * \mathbf{1}^G_\mu$ specializes to*

$$\operatorname{char}(K\varpi^\lambda K) * \operatorname{char}(K\varpi^\mu K).$$

Proof. We have that

$$\left(\mathbf{1}^G_\lambda * \mathbf{1}^G_\mu\right)(g) = \int_G \mathbf{1}^G_\lambda(x)\cdot \mathbf{1}^G_\mu(x^{-1}g)\,dx.$$

By Lemma 63 and the defining formula of the convolution product, we get that $\mathbf{1}^G_\lambda * \mathbf{1}^G_\mu$ specializes to $\operatorname{char}(K\varpi^\lambda K) * \operatorname{char}(K\varpi^\mu K)$. The specialization result on generators implies the result for any two elements in $\mathcal{H}(G,K)^{\mathrm{mot}}$. $\square$

Some basic questions on the structure of $\mathcal{H}(G,K)^{\mathrm{mot}}$ arise. Can we prove motivically that this product is associative? Is there an identity element? Is it commutative? Of course the first attempt to answer each of these questions is to look at the proof in the p-adic setting and then try to put that proof in the motivic setting. The properties of $\mathcal{H}(G,K)^{\mathrm{mot}}$ can be thought as universal properties for spherical Hecke algebras over p-adic fields.

The next proposition shows the existence of an identity element in $\mathcal{H}(G,K)^{\mathrm{mot}}$.

Proposition 74. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. The element $\epsilon = \frac{1}{\mathrm{vol}(K)}\mathbf{1}_K$ is a constructible motivic function on G and it is a left and right identity for the elements in $\mathcal{H}(G,K)^{\mathrm{mot}}$.*

Proof. By Lemma 68 and the definability of K, it is clear that ϵ is a constructible motivic function on G. It is enough to prove the result for the generators of $\mathcal{H}(G,K)^{\mathrm{mot}}$. Let $\lambda \in P^+$.

Then

$$\begin{aligned}
\epsilon * \mathbf{1}_\lambda^G &= \int_G \epsilon(x) \cdot \mathbf{1}_\lambda^G(x^{-1}g)\, dx \\
&= \frac{1}{\mathrm{vol}(K)} \int_G \mathbf{1}_K(x) \cdot \mathbf{1}_\lambda^G(x^{-1}g)\, dx \\
&= \frac{1}{\mathrm{vol}(K)} \int_K \mathbf{1}_\lambda^G(x^{-1}g)\, dx \\
&= \frac{1}{\mathrm{vol}(K)} \int_{Kg} \mathbf{1}_\lambda^G(y)\, dy \\
&= \mathbf{1}_\lambda^G(g).
\end{aligned}$$

The change of variables used was $y = x^{-1}g$, which is justified by Lemma 66. The proof of $\mathbf{1}_\lambda^G * \epsilon = \mathbf{1}_\lambda^G$ is similar and it is omitted. □

Remark 75. *What is the representation theory of $\mathcal{H}(G, K)^{\mathrm{mot}}$? Given the relation between representations of $G(F)$ and representations of the corresponding Hecke algebra or some spherical Hecke algebra, the study of representations of $\mathcal{H}(G, K)^{\mathrm{mot}}$ might lead to a reasonable study of representations of p-adic groups in a field independent way.*

Remark 76. *It is natural to wonder about the existence of a motivic object corresponding to the Hecke algebra $\mathcal{H}(G)$. We do not develop this but we notice that any reasonable attempt should include the definability of a basis of open and compact subgroups of G. Our discussion in §3.2.1 might be a good starting point.*

The next result shows that the convolution product in the torus can be completely described by a motivic identity.

Lemma 77. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. For $\lambda, \mu \in X_*$, the following identity of constructible motivic functions is true*

$$\mathbf{1}_\lambda^T * \mathbf{1}_\mu^T = \mathbf{1}_{\lambda+\mu}^T. \tag{4.1}$$

Proof. Let d^*x the motivic measure on T such that the specialization of the volume of $K^\circ = K \cap T$ is 1. The following is the proof of the identity, notice that the proof is inside the motivic framework. The change of variables is justified by Lemma 66.

$$\begin{aligned}
\left(\mathbf{1}_{\lambda}^{T} * \mathbf{1}_{\mu}^{T}\right)(g) &= \int_{T} \mathbf{1}_{\lambda}^{T}(x) \cdot \mathbf{1}_{\mu}^{T}(x^{-1}g)\, d^{*}x \\
&= \int_{D_{\lambda}^{T}} \mathbf{1}_{\mu}^{T}(x^{-1}g)\, d^{*}x \\
&= \int_{(D_{\lambda}^{T})g^{-1}} \mathbf{1}_{\mu}^{T}(x^{-1})\, d^{*}x \\
&= \int_{g(D_{-\lambda}^{T})} \mathbf{1}_{\mu}^{T}(x)\, d^{*}x \\
&= \int_{g(D_{-\lambda}^{T}) \cap D_{\mu}^{T}} \mathbf{1}\, d^{*}x \\
&= \left(\mathbf{1}_{(\lambda+\mu)}^{T}\right)(g).
\end{aligned}$$

□

Lemma 78. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. The specialization map*

$$\mathcal{H}(T, K^{\circ})^{\mathrm{mot}} \to \mathcal{H}(T, K^{\circ}),$$

is a $\mathbb{C}$-algebra epimorphism.

Proof. The result follows from Lemma 77 and the specialization Lemma 73. It is clear that the specialization map is surjective. □

Although $\mathcal{H}(T, K^{\circ})$ can be canonically identified with $\mathbb{C}[X_*]$ (see §5.1), a field independent object, Lemma 78 shows that $\mathcal{H}(T, K^{\circ})$ and its algebraic structure come from motivic data. One might ask about the same kind of result when G is F-split and reductive. The situation is a bit more complicated but basically one can prove that the spherical Hecke algebra of G and its algebraic structure are determined by motivic data. What we will call the structure theorem (a generalization of Lemma 77) is the description of the convolution product as a linear combination of generators in the motivic spherical Hecke algebra. This is represented as an identity of constructible motivic functions on G, as in 4.1.

The following lemma combines a simple but useful idea of Hales on averages with the specialization map.

Lemma 79 (Hales' averaging lemma on specialization)**.** *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. Let $f \in \mathcal{C}(G)$ such that f specializes to a function in $\mathcal{H}(G, K)$. Let $\lambda \in X_*$.*

a) The motivic constant

$$\frac{1}{\mathrm{vol}(K^\circ)} \int_{D_\lambda^T} f(x)\, dx \tag{4.2}$$

specializes to $f_F(\varpi^\lambda)$, where F is a non-archimedean local field of sufficiently large residue characteristic, and ϖ is a uniformizer in F.

b) The constructible motivic function

$$\frac{1}{\mathrm{vol}(K^\circ)} \int_{D_\lambda^T} f(xy)\ dx \tag{4.3}$$

specializes to $f_F(\varpi^\lambda y)$, where F is a non-archimedean local field of sufficiently large residue characteristic, and ϖ is a uniformizer in F.

Proof. We begin with part *a*). Let F be a non-archimedean local field of sufficiently large residue characteristic. The specialization of the motivic constant 4.2 is

$$\frac{1}{\mathrm{vol}_F(K^\circ)} \int_{\varpi^\lambda K^\circ} f_F(x)\, d_F x.$$

The change of variable $x = \varpi^\lambda k$ gives

$$\begin{aligned}
\frac{1}{\mathrm{vol}_F(K^\circ)} \int_{\varpi^\lambda K^\circ} f_F(x)\, d_F x &= \frac{1}{\mathrm{vol}_F(K^\circ)} \int_{K^\circ} f_F(\varpi^\lambda k)\, d_F k \\
&= \frac{1}{\mathrm{vol}_F(K^\circ)} \int_{K^\circ} f_F(\varpi^\lambda)\, d_F k \\
&= \frac{1}{\mathrm{vol}_F(K^\circ)} \cdot f_F(\varpi^\lambda) \cdot \int_{K^\circ} \mathbf{1}_{G(F)}\, d_F k \\
&= \frac{1}{\mathrm{vol}_F(K^\circ)} \cdot f_F(\varpi^\lambda) \cdot \mathrm{vol}_F(K^\circ) \\
&= f_F(\varpi^\lambda).
\end{aligned}$$

Note that there is no hypothesis on the normalization of the motivic measure. The proof of b) is basically the same. Using the change of variable $x = k\varpi^\lambda$, the constructible motivic function 4.3 specializes to

$$\begin{aligned}
\frac{1}{\mathrm{vol}_F(K^\circ)} \int_{\varpi^\lambda K^\circ} f_F(xy)\, d_F x &= \frac{1}{\mathrm{vol}_F(K^\circ)} \int_{K^\circ} f_F(k\varpi^\lambda y)\, d_F k \\
&= \frac{1}{\mathrm{vol}_F(K^\circ)} \int_{K^\circ} f_F(\varpi^\lambda y)\, d_F k \\
&= \frac{1}{\mathrm{vol}_F(K^\circ)} \cdot f_F(\varpi^\lambda y) \cdot \int_{K^\circ} \mathbf{1}_{G(F)}\, d_F k \\
&= \frac{1}{\mathrm{vol}_F(K^\circ)} \cdot f_F(\varpi^\lambda y) \cdot \mathrm{vol}_F(K^\circ) \\
&= f_F(\varpi^\lambda y).
\end{aligned}$$

□

4.5 MOTIVIC K-AVERAGE PROPERTY

Definition 80. *Let X be a definable set and let H be a definable group. Suppose H acts on X in a definable way. This is, the action $\theta : H \times X \to X$ is a definable map. We say $f \in \mathcal{C}(X)$ is H-invariant if*

$$\theta^* f = \pi^* f,$$

where $\pi : H \times X \to X$ is the projection map.

Let G be a split reductive group and let K be a hyperspecial subgroup of G. We say that $f \in \mathcal{C}(G)$ is K-bi-invariant if it is invariant by the left and right actions of K on G.

Definition 81. *Let X be a definable set and let H be a definable group. Suppose H acts on X in a definable way. This is, the action $\theta : H \times X \to X$ is a definable map. Suppose H is bounded and with motivic Haar measure dh. The H average of $f \in \mathcal{C}(X)$ over H is*

$$\frac{1}{\mathrm{vol}(H)} \int_H f(hx)\, dh \in \mathcal{C}(X)$$

where $hx := \theta(h,x)$. More formally

$$\frac{1}{\mathrm{vol}(H)}\int_H f(hx)\,dh = \pi_!(\theta^* f) \in \mathcal{C}(X).$$

We say that $f \in \mathcal{C}(X)$ has the H-average property if

$$\pi_!(\theta^* f) = \pi_!(\pi^* f) = \mathrm{vol}(H)\cdot f.$$

Definition 82. *Let Y and Z be two definable sets. Let H be a definable group. Suppose H acts on both Y and Z in a definable way. We denote these actions by $\theta_Y : H \times Y \to Y$ and $\theta_Z : H \times Z \to Z$. We say that $\phi : Z \to Y$ is H-equivariant if the following diagram commutes*

4.5.1 K-average property

Definition 83. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. Let $f \in \mathcal{C}(G)$. We say that f has the right K-average property if*

$$f(g) = \frac{1}{\mathrm{vol}(K)}\int_K f(gk)\,dk. \tag{4.4}$$

Since $\mathrm{vol}(K) = \int_K \mathbf{1}$, *the defining equation 4.4 can be rewritten as*

$$\int_K f(g)\,dk = \int_K f(gk)\,dk.$$

Similarly, one can define the notion of left K-average for constructible motivic functions on G. We say that a function $f \in \mathcal{C}(G)$ has the K-average property if it has both.

Observe that the specialization of a function with the K-average property is a K-bi-invariant function.

The following lemma has a very concrete statement. Its proof is just an observation on the notation.

Lemma 84. *If f is a motivic K-bi-invariant function, then f has the K-average property.*

Proof. We prove the left K-average property for f, the right one is similar. We want to prove that

$$\int_K f(kx)\,dk = \int_K f(x)\,dk.$$

By just changing the notation, we have

$$\int_K f(kx)\,dk = \pi_!(\phi^* f) \quad \text{and} \quad \int_K f(x)\,dk = \pi_!(\pi^* f).$$

Since f is motivic K-bi-invariant, in particular we have $\phi^* f = \pi^* f$. The result is now obvious. □

Proposition 85. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. Suppose $f_1, f_2 \in \mathcal{C}(G)$ are bi-K-invariant. Then $f_1 \cdot f_2$ is bi-K-invariant.*

Proof. We prove that $f_1 \cdot f_2$ is right K-invariant. Left K-invariance is similar. Since f_2 is right K-invariant we have

$$f_2(g) = \frac{1}{\mathrm{vol}(K)} \int_K f_2(gk_1)\,dk_1.$$

Then,

$$f_1(gk_2) \cdot f_2(g) = \frac{1}{\mathrm{vol}(K)} \int_K f_1(gk_2) \cdot f_2(gk_1)\,dk_1.$$

Taking integral over K

$$\int_K f_1(gk_2) \cdot f_2(g)\,dk_2 = \frac{1}{\mathrm{vol}(K)} \int_K \int_K f_1(gk_2) \cdot f_2(gk_1)\,dk_1 dk_2.$$

Using the right K-invariance of f_1 the left side of the equation becomes $f_1(g) \cdot f_2(g) \cdot \mathrm{vol}(K)$. Then, using the change of variable $k_1 = k_2 k'$ we get

$$\begin{aligned}
f_1(g) \cdot f_2(g) &= \frac{1}{\mathrm{vol}(K)^2} \int_K \int_K f_1(gk_2) \cdot f_2(gk_1)\,dk_1 dk_2 \\
&= \frac{1}{\mathrm{vol}(K)^2} \int_K \int_K f_1(gk_2) \cdot f_2(gk_2k')\,dk' dk_2 \\
&= \frac{1}{\mathrm{vol}(K)^2} \int_K f_1(gk_2) \left[\int_K f_2(gk_2k')\,dk' \right] dk_2 \\
&= \frac{1}{\mathrm{vol}(K)^2} \int_K f_1(gk_2) \cdot f_2(gk_2) \cdot \mathrm{vol}(K)\,dk_2 \\
&= \frac{1}{\mathrm{vol}(K)} \int_K f_1(gk_2) \cdot f_2(gk_2)\,dk_2.
\end{aligned}$$

□

Proposition 86. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. Suppose $f_1, f_2 \in \mathcal{C}(G)$ are bi-K-invariant. Then $f_1 * f_2$ is bi-K-invariant.*

Proof. The following motivic computation gives the result corresponding to left K-invariance. We use the change of variable $y = kx$, the Fubini theorem for motivic integrals and the fact that f_1 is left K invariant.

$$\begin{aligned}
\int_K (f_1 * f_2)(kg)\, dk &= \int_K \int_G f_1(x) \cdot f_2(x^{-1}kg)\, dxdk \\
&= \int_K \int_G f_1(k^{-1}y) \cdot f_2(y^{-1}g)\, dydk \\
&= \int_G \int_K f_1(k^{-1}y) \cdot f_2(y^{-1}g)\, dkdy \\
&= \int_G f_2(y^{-1}g) \left[\int_K f_1(k^{-1}y)\, dk \right] dy \\
&= \mathrm{vol}(K) \int_G f_1(y) \cdot f_2(y^{-1}g)\, dy \\
&= \mathrm{vol}(K)(f_1 * f_2)(g).
\end{aligned}$$

Thus

$$(f_1 * f_2)(g) = \frac{1}{\mathrm{vol}(K)} \int_K (f_1 * f_2)(kg)\, dk.$$

□

4.6 MOTIVIC K-INVARIANT FUNCTIONS

Our definition of motivic K-invariance is in terms of morphisms. As usual, let G be a split reductive group defined over F and let K be a hyperspecial subgroup of G. Let $\phi : K \times G \to G$ be the morphism defined by $(k, g) \mapsto kg$. Similarly, let $\psi : K \times G \to G$ be the morphism defined by $(k, g) \mapsto gk$. Let $\pi : K \times G \mapsto G$ be the projection on the second component $(k, g) \mapsto g$.

Definition 87. *Let G, K be a split reductive group defined over F and a hyperspecial subgroup of G. Let $f \in \mathcal{C}(G)$ be a constructible motivic function on G. We say that f is K-left invariant if*

$$\phi^* f = \pi^* f.$$

Similarly, we say that f is K-right invariant if $\psi^* f = \pi^* f$. The function f is called K-bi-invariant if it is both, left and right K-invariant. We denote the space of K-bi invariant functions on G by $\mathcal{C}(G)^K$.

Notice that if this definition were for actual functions on groups, this would be a right definition for K-invariance because it would be the same as the standard one.

In the following lemmas we prove the basic properties of K-bi-invariant functions. The context is always the same. Let G, K be a split reductive group defined over F and a hyperspecial subgroup of G.

Recall that constructible motivic functions are not actual functions so although some results are not really complicated they are not totally straightforward.

Lemma 88. *For each $\lambda \in P^+$, the function $\mathbf{1}_\lambda^G$ is K-bi-invariant.*

Proof. We prove K-left invariance, K-right invariance is similar. The function $\mathbf{1}_\lambda^G$ is a Presburger function on G, that is, $\mathbf{1}_\lambda^G : G \to \mathbb{Z}$ is a Denef-Pas definable function on G. Thus

$$\phi^* \mathbf{1}_\lambda^G = \mathbf{1}_\lambda^G \circ \phi \quad \text{and} \quad \pi^* \mathbf{1}_\lambda^G = \mathbf{1}_\lambda^G \circ \pi.$$

Now, $\mathbf{1}_\lambda^G \circ \phi = \mathbf{1}_\lambda^G \circ \pi$ because for any $g \in G$ and any $k \in K$ (in any Denef-Pas structure)

$$kg \in D_\lambda^G \quad \text{if and only if} \quad g \in D_\lambda^G.$$

□

Lemma 89. *Let f_1 and f_2 be two K-bi-invariant functions. Then $f_1 \cdot f_2$ is a K-bi-invariant function.*

Proof. We prove K-left invariance, K-right invariance is similar. Notice that the result follows from

$$\phi^*(f_1 \cdot f_2) = \phi^* f_1 \cdot \phi^* f_2 \quad \text{and} \quad \pi^*(f_1 \cdot f_2) = \pi^* f_1 \cdot \pi^* f_2,$$

which are true since the pullback operation is a ring homomorphism. □

Observe that constant functions on G are trivially K-bi-invariant. By the previous two Lemmas 88 and 89 we have that motivic functions of the form

$$c \cdot \mathbf{1}_\lambda^G,$$

where c is a motivic constant, are K-invariant.

Is the convolution product of two K-bi-invariant functions a K-bi-invariant function? This is the question we now try to answer. Under a weak assumption, we can state a positive answer to this question. We have reasons to believe these assumptions can be removed. As the reader will notice, the following is a bit technical.

Consider the following morphisms

- $\pi_1 : G \times G \to G, \quad (x, g) \mapsto x,$
- $\pi_2 : G \times G \to G, \quad (x, g) \mapsto g,$
- $\gamma : G \times G \to G, \quad (x, g) \mapsto x^{-1}g,$
- $\delta : G \times G \times K \to K \times G, \quad (x, g, k) \mapsto (k, x^{-1}g),$
- $\dot{\pi}_1 : G \times G \times K \to G, \quad (x, g, k) \mapsto x,$
- $\alpha : G \times G \times K \to K \times G, \quad (x, g, k) \mapsto (1, x^{-1}g),$
- $\pi_{12} : G \times G \times K \to G \times G, \quad (x, g, k) \mapsto (x, g)$ and
- $\pi_{23} : G \times G \times K \to K \times G, \quad (x, g, k) \mapsto (k, g).$

Recall that we already defined:

- $\phi : K \times G \to G, \quad (k, g) \mapsto kg,$
- $\psi : K \times G \to G, \quad (k, g) \mapsto gk$ and
- $\pi : K \times G \to G, \quad (k, g) \mapsto g.$

Then, for $f_1, f_2 \in \mathcal{C}(G)$ we can write

$$f_1 * f_2 = \pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2). \tag{4.5}$$

Lemma 90. *Let f_1 and f_2 be two motivic K-bi-invariant functions. Then*

a) $\alpha^*(\psi^* f_2) = \delta^*(\psi^* f_2)$.

b) $\pi_{23_!}(\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2)) = \pi_{23_!}(\dot{\pi}_1^* f_1 \cdot \delta^*(\psi^* f_2))$.

Proof. We start with part a). Since f_2 is motivic K-invariant, we have $\psi^* f_2 = \pi^* f_2$. Thus,

$$\alpha^*(\psi^* f_2) = \alpha^*(\pi^* f_2) \quad \text{and} \quad \delta^*(\psi^* f_2) = \delta^*(\pi^* f_2).$$

Now, notice that $\pi \circ \alpha = \pi \circ \delta$, which implies $(\pi \circ \alpha)^* = (\pi \circ \delta)^*$. Therefore,

$$\alpha^*(\psi^* f_2) = \alpha^*(\pi^* f_2) = (\pi \circ \alpha)^* f_2 = (\pi \circ \delta)^* f_2 = \delta^*(\pi^* f_2) = \delta^*(\psi^* f_2).$$

Part b) follows from part a). Notice that we only use motivic K-bi-invariance for f_2. $\square$

We make two assumptions in the following lemma:

- If $f \in \mathcal{C}(G \times G \times K)^K$ for the action of K on the right of K. Then $\pi_{23!}(f) \in \mathcal{C}(K \times G)^K$ for the action of K on K in $K \times G$.
- If $f \in \mathcal{C}(K \times G)^K$, then $\pi^* \pi_!(f) = \mathrm{vol}(K) f$.

Lemma 91. *Assume the assumptions made above. Let f_1 and f_2 be two motivic K-bi-invariant functions. Then $f_1 * f_2$ is a motivic K-bi-invariant function.*

Proof. We prove that $f_1 * f_2$ is motivic K-right invariant. The other one is similar. We start with the following claim

$$\pi_{23_!}(\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2)) = \pi^*(\pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2)). \tag{4.6}$$

Consider the following commutative diagram

$$\begin{tikzcd} G \times G \times K \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{23}"'] & G \times G \arrow[d, "\pi_2"] \\ K \times G \arrow[r, "\pi"] & G \end{tikzcd}$$

It implies

$$\pi_! \circ \pi_{23_!} = \pi_{2_!} \circ \pi_{12_!}. \tag{4.7}$$

Consider $\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2) \in \mathcal{C}(G \times G \times K)$. If we apply $\pi_{12_!}$, calculations gives

$$\pi_{12_!}(\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2)) = \pi_1^* f_1 \cdot \gamma^* f_2 \quad \in \mathcal{C}(G \times G). \tag{4.8}$$

Applying $\pi_{2_!}$ to 4.8 we get

$$(\pi_{2_!} \circ \pi_{12_!})(\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2)) = \pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2) \quad \in \mathcal{C}(G). \tag{4.9}$$

By 4.7, this last equation 4.9 becomes

$$(\pi_! \circ \pi_{23_!})(\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2)) = \pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2) \quad \in \mathcal{C}(G). \tag{4.10}$$

Finally, we apply π^* to 4.10 and we get

$$(\pi^* \circ \pi_! \circ \pi_{23_!})(\dot{\pi}_1^* f_1 \cdot \alpha^*(\psi^* f_2)) = \pi^*(\pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2)) \quad \in \mathcal{C}(K \times G). \tag{4.11}$$

We now prove the following

$$\pi_{23_!}(\dot{\pi}_1^* f_1 \cdot \delta^*(\psi^* f_2)) = \psi^*(\pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2)). \tag{4.12}$$

By part b) of Lemma 90 and equations 4.6 and 4.12, we have

$$\psi^*(\pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2)) = \pi^*(\pi_{2_!}(\pi_1^* f_1 \cdot \gamma^* f_2)),$$

which with a different notation is

$$\psi^*(f_1 * f_2) = \pi^*(f_1 * f_2).$$

$\square$

4.7 THE STRUCTURE THEOREM UP TO A NULL FUNCTION

The convolution product in p-adic spherical Hecke algebras is determined by motivic data, that is the content of the next theorem.

Theorem 92. *Let G, K be a split reductive group defined over F, and a hyperspecial subgroup of G. The function $n : P^+ \times P^+ \times P^+ \to \mathbb{Z}$ given by*

$$(\lambda, \mu, \nu) \mapsto n_{\lambda\mu}(\nu),$$

defined by equation 3.5, is the specialization of a constructible motivic function on $P^+ \times P^+ \times P^+$. The result is still true if we consider n as a function with domain $\mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d$.

Proof. Given $\lambda, \mu, \nu \in P^+$, we know that

$$n_{\lambda\mu}(\nu) = (\mathrm{char}(K\varpi^\lambda K) * \mathrm{char}(K\varpi^\mu K))(\varpi^\nu).$$

By Lemma 73, the right hand side is the specialization of the constructible motivic function $\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G$ evaluated at ϖ^ν. We need to eliminate any p-adic reference, so here we use Lemma 79 on averages and specializations and we get that the motivic constant

$$\frac{1}{\mathrm{vol}(K^\circ)} \int_{D_\nu^T} \left(\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G\right)(x)\, dx$$

specializes to the p-adic value $\left(\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G\right)(\varpi^\nu) = n_{\lambda\mu}(\nu)$. Thus, as a function over $P^+ \times P^+ \times P^+$, that is, letting vary $\lambda, \mu, \nu \in P^+$ the constructible motivic function on $P^+ \times P^+ \times P^+$ given by

$$(\lambda, \mu, \nu) \mapsto \frac{1}{\mathrm{vol}(K^\circ)} \int_{D_\nu^T} \left(\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G\right)(x)\, dx$$

specializes to the p-adic function $n_{\lambda\mu}(\nu)$. Therefore, $n_{\lambda\mu}(\nu)$ is the specialization of a constructible motivic function on $P^+ \times P^+ \times P^+$. $\square$

We think this is a good moment to say some words about notation.

Remark 93. *From now on, by Theorem 92, we can talk about the constructible motivic function* $n_{\lambda\mu}(\nu)$ *and its motivic definition*

$$\begin{aligned} n_{\lambda\mu}(\nu) &= \frac{1}{\mathrm{vol}(K^\circ)} \int_{D_\nu^T} (\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G)(x)\, dx \\ &= \frac{1}{\mathrm{vol}(K^\circ)} \int_{D_\nu^T} \int_G \mathbf{1}_\lambda^G(y) \cdot \mathbf{1}_\mu^G(y^{-1}x)\, dydx. \end{aligned}$$

Notice that here there is a slight abuse of notation since $n_{\lambda\mu}(\nu)$ *was defined as a p-adic object. Strictly speaking we have on the left side a p-adic object and on the right a motivic one. Since we just proved that* $n_{\lambda\mu}(\nu)$ *is the specialization of a constructible motivic function, we promote* $n_{\lambda\mu}(\nu)$ *to the "level" of motivic object and we do not change its name, so we accept the equality above. This might not be the only abuse of notation of this kind. In fact this is the case every time we say that certain function, defined in principle by p-adic objects, is the specialization of a constructible motivic function.*
We make another comment on notation. We defined $\mathbf{1}_\lambda^G$ *as a constructible function on* G *and sometimes it might be written in that way even if the intension is to think of it as an actual characteristic function of the corresponding double coset (i.e., a p-adic object). There is no harm in this.*

Theorem 94 (Structure theorem up to a null function)**.** *Let* G *and* K *be a split reductive group defined over* F *and a hyperspecial subgroup of* G*. Let* $\lambda, \mu \in P^+$*. Then*

$$\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G = \sum_\nu n_{\lambda\mu}(\nu) \cdot \mathbf{1}_\nu^G \mod (\mathrm{null}).$$

The sum is finite.

Proof. Notice that all the terms in the desired identity are described my motivic functions. Here Theorem 92 is essential. Now, by construction we know that the identity holds for all p-adic fields. This completes the proof. □

4.8 THE STRUCTURE THEOREM AS A MOTIVIC IDENTITY

Although the structure theorem up to null functions (Theorem 94) is enough for applications regarding Cluckers-Loeser transfer principle, we believe it is true as a genuine motivic identity.

Conjecture 95. *Let G and K be a split reductive group defined over F and a hyperspecial subgroup of G. Let $\lambda, \mu \in P^+$. Then*

$$\mathbf{1}_\lambda^G * \mathbf{1}_\mu^G = \sum_\nu n_{\lambda\mu}(\nu) \cdot \mathbf{1}_\nu^G$$

is a motivic identity. The sum is finite.

5.0 ON THE MOTIVIC NATURE OF THE SATAKE TRANSFORM

5.1 THE SATAKE TRANSFORM

The Satake isomorphism plays an essential role in the group version of the fundamental lemma. It gives the way to connect elements in the spherical Hecke algebra of G with elements on the spherical Hecke algebra of an endoscopic group of G. The isomorphism, as the name suggest, is due to Satake [50], and it might be viewed as the p-adic analog of a well-known result of Harish-Chandra in the context of real Lie groups. Langlands' interpretation of the Satake isomorphism is important in the study of spherical representations of unramified reductive groups.

Let us recall the context. Let F be a p-adic field of characteristic zero. Let G be a connected unramified reductive group defined over F. From §5.1. until the end of this work G will be split. Let K be a hyperspecial maximal compact subgroup of G. Let A be a maximal F-split torus of G and let $T \supset A$ be a Cartan that splits over F_1.

The Satake transform

$$\mathcal{S} : \mathcal{H}(G, K) \to \mathcal{H}(T, K^\circ)$$

defined by

$$f \mapsto \left[t \mapsto \delta(t)^{1/2} \cdot \int_N f(tn)\, dn \right]$$

allows us to identify the spherical Hecke algebra of G with the spherical Hecke algebra of T ($K^\circ = T(F) \cap K$), the latter being much simpler and with an explicit description as we explain below. The function δ is an unramified character of T (see Definition 100). The Haar measures are normalized so that K, $T(F) \cap K$ and $N(F) \cap K$ all get volume 1. Satake proved in [50] that the image of $\mathcal{S}$ is $\mathcal{H}(T, K^\circ)^W$ and that

$$\mathcal{S} : \mathcal{H}(G, K) \to \mathcal{H}(T, K^\circ)^W$$

is an isomorphism of $\mathbb{C}$-algebras.

We now explain a well-known description of $\mathcal{H}(T, K^\circ)$. Consider the Cartan decomposition

$$T(F) = \coprod_{\lambda \in X_{*F}} \varpi^\lambda K^\circ.$$

In particular, this decomposition implies that $\{ \operatorname{char}(\varpi^\lambda K^\circ) \mid \lambda \in X_{*F}\}$ forms a basis for $\mathcal{H}(T, K^\circ)$, as a vector space over $\mathbb{C}$. Let $\mathbb{C}[X_{*F}]$ be the group algebra of X_{*F}. The correspondence

$$\operatorname{char}(\varpi^\lambda K^\circ) \mapsto \lambda \qquad \text{for all } \lambda \in X_{*F},$$

extends by linearity to $\mathcal{H}(T, K^\circ)$ and gives a bijection from $\mathcal{H}(T, K^\circ)$ to $\mathbb{C}[X_{*F}]$ that preserves the $\mathbb{C}$-linear structure. The following computation shows that this correspondence respects the multiplication. Let dx be the Haar measure on $T(F)$ such that K° has measure one.

$$\begin{aligned}
(\operatorname{char}(\varpi^\lambda K^\circ) * \operatorname{char}(\varpi^\mu K^\circ))(g) &= \int_T \operatorname{char}(\varpi^\lambda K^\circ)(x) \cdot \operatorname{char}(\varpi^\mu K^\circ)(x^{-1}g)\, dx \\
&= \int_{\varpi^\lambda K^\circ} \operatorname{char}(\varpi^\mu K^\circ)(x^{-1}g)\, dx \\
&= \int_{g^{-1}\varpi^\lambda K^\circ} \operatorname{char}(\varpi^\mu K^\circ)(x^{-1})\, dx \\
&= \int_{g\varpi^{-\lambda} K^\circ} \operatorname{char}(\varpi^\mu K^\circ)(x)\, dx \\
&= \int_{(g\varpi^{-\lambda} K^\circ)\cap(\varpi^\mu K^\circ)} \mathbf{1}\, dx \\
&= \operatorname{char}(\varpi^{\lambda+\mu} K^\circ)(g).
\end{aligned}$$

Therefore, as $\mathbb{C}$-algebras

$$\mathcal{H}(T, K^\circ) \cong \mathbb{C}[X_{*F}].$$

Observe that this isomorphism identifies the p-adic object $\mathcal{H}(T, K^\circ)$ with a complex (field independent) object $\mathbb{C}[X_{*F}]$.

5.1.1 Choice of bases

From now on we assume G is a F-split reductive group. Almost by definition, the spherical Hecke algebra $\mathcal{H}(G,K)$ is a $\mathbb{C}$-algebra generated by the characteristic functions of all the double cosets of the form $K\varpi^\lambda K$ for $\lambda \in X_{*F} = X_*$. The Cartan decomposition gives the basis $\{\mathbf{1}_\lambda^G : \lambda \in P^+\}$. The basis we use is the same but each vector (characteristic function) is rescaled as follows. For each $\lambda \in P^+$, we have

$$q^{-\langle \rho,\lambda\rangle}\mathbf{1}_\lambda^G.$$

On the W-invariant space of $\mathcal{H}(T,K^\circ)$ the basis is as follows. Given $\mu \in X_*$, let W_μ be the stabilizer of μ in W and let $W(\mu)$ be the orbit of μ in X_* under the action of W. The basis of $\mathcal{H}(T,K^\circ)^W$ is given by

$$m_\mu = \frac{1}{|W_\mu|}\sum_{w\in W} w(\mathbf{1}_\mu^T) = \frac{1}{|W_\mu|}\sum_{w\in W}\mathbf{1}_{w(\mu)}^T,$$

for $\mu \in X_*$.

5.2 MOTIVIC DATA ON THE SATAKE TRANSFORM

Proposition 96. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. For each $\lambda \in P^+$, there exists a constructible motivic function on T that specializes to $\mathcal{S}(\mathbf{1}_\lambda^G) : T \to \mathbb{C}$. Moreover, the family of constructible motivic functions on T specializing onto the elements in the image of the Satake transform is parametrized by the definable set P^+.*

Proof. The proof is simple, we just have to observe that all the ingredients in the definition of $\mathcal{S}(\mathbf{1}_\lambda^G)$ are motivic. This is almost straightforward. Consider the function $\alpha : T \to X_*$ defined as: $\alpha(t) \in X_* \cong \mathbb{Z}^r$ is the only element in the cocharacter group such that $t \in D_{\alpha(t)}^T$. The definability of the Cartan decomposition, Proposition 64, implies the definability of the function α, notice that T and X_* are definable sets. Then, the constructible motivic function

$$\mathbb{L}^{-\langle\rho,\alpha(t)\rangle}\int_N \mathbf{1}^G_\lambda(tn)\,dn,$$

specializes to the function $\mathcal{S}(\mathbf{1}^G_\lambda)$. The last part of the statement is obvious. $\square$

The remark 93 applies here, and from now on we can talk about $\mathcal{S}(\mathbf{1}^G_\lambda)$ as a motivic function on T.

We have the following p-adic identities.

$$\mathcal{S}\Big(q^{-\langle\rho,\lambda\rangle}\mathbf{1}^G_\lambda\Big) = q^{-\langle\rho,\lambda\rangle}\mathcal{S}\Big(\mathbf{1}^G_\lambda\Big) = q^{-\langle\rho,\lambda\rangle}\sum_{\mu\le\lambda} C^G_{\lambda\mu}\,m_\mu$$

$$\begin{aligned}
\mathcal{S}\big(\mathbf{1}^G_\lambda\big)(\varpi^\nu) &= \sum_{\mu\le\lambda} C^G_{\lambda\mu}\,m_\mu(\varpi^\nu) && (5.1)\\
&= C^G_{\lambda\nu}\,m_\nu(\varpi^\nu) && (5.2)\\
&= C^G_{\lambda\nu}\frac{1}{|W_\nu|}\sum_{w\in W} w(\mathbf{1}^T_\nu)(\varpi^\nu) && (5.3)\\
&= C^G_{\lambda\nu} && (5.4)
\end{aligned}$$

$$C^G_{\lambda\nu} = \mathcal{S}\Big(\mathbf{1}^G_\lambda\Big)(\varpi^\nu) = q^{\langle\rho,\nu\rangle}\int_N \mathbf{1}^G_\lambda(\varpi^\nu n)dn$$

$$\begin{aligned}
C^G_{\lambda\lambda} &= \mathcal{S}\Big(\mathbf{1}^G_\lambda\Big)(\varpi^\lambda) && (5.5)\\
&= q^{\langle\rho,\lambda\rangle}\int_N \mathbf{1}^G_\lambda(\varpi^\lambda n)dn && (5.6)\\
&= q^{\langle\rho,\lambda\rangle}\int_{N\cap(\varpi^{-\lambda}K\varpi^\lambda K)} dn && (5.7)\\
&= q^{\langle\rho,\lambda\rangle}\sum_i\int_{N\cap(\varpi^{-\lambda}x_iK)} dn && (5.8)\\
&= q^{\langle\rho,\lambda\rangle}\#\{i : \varpi^{-\lambda}t(x_i)\in T\cap K\} && (5.9)\\
&= q^{\langle\rho,\lambda\rangle} && (5.10)
\end{aligned}$$

Theorem 97. *Let G, T, K be a split reductive group defined over F, a Cartan subgroup of G and a hyperspecial subgroup of G. There exists a constructible motivic function on $P^+\times P^+$ that specializes to the function $P^+\times P^+\to\mathbb{C}$ given by $(\lambda,\mu)\mapsto C^G_{\lambda\mu}$.*

Proof. Let $\lambda, \mu \in P^+$. We use the Hales' average lemma on specializations 79 with the constructible motivic function $\mathcal{S}(\mathbf{1}^G_\lambda)$ which specializes to an element in $\mathcal{H}(T, K^\circ)$, by Proposition 96. Notice that this is an hypothesis on Lemma 79. Thus, the motivic constant

$$\frac{1}{\mathrm{vol}(K^\circ)} \int_{D^T_\mu} \mathcal{S}(\mathbf{1}^G_\lambda)(x)\, dx$$

specializes to the p-adic value $\mathcal{S}(\mathbf{1}^G_\lambda)(\varpi^\mu) = C^G_{\lambda\mu}$. We let vary $\lambda, \mu \in P^+$ and the result follows. □

5.3 A MOTIVIC SATAKE TRANSFORM

In the previous chapter we described the motivic spherical Hecke algebras. They are basically the field independent description of the p-adic spherical Hecke algebras. Having these objects, we define on them a Satake transform; we call it the motivic Satake transform. We prove that the motivic Satake transform is an isomorphism.

The framework is as in §5.1. In particular, G is split.

Lemma 98. *For each $\lambda \in P^+$, the constructible motivic function $\mathcal{S}\big(\mathbf{1}^G_\lambda\big) \in \mathcal{C}(T)$ is in $\mathcal{H}(T, K^\circ)^{\mathrm{mot}} \mod (\mathrm{null})$.*

Proof. It is true in specializations of fields in $\mathscr{C}_M$, for some $M > 0$. □

Definition 99. *The motivic Satake transform is defined as*

$$\mathfrak{S} : \mathcal{H}(G, K)^{\mathrm{mot}} \to \mathcal{H}(T, K^\circ)^{\mathrm{mot}} \mod (\mathrm{null}),$$

given by

$$\mathfrak{S}\big(c\mathbf{1}^G_\lambda\big) := c\mathcal{S}\big(\mathbf{1}^G_\lambda\big) \mod (\mathrm{null})$$

for each $\lambda \in P^+$ and c a motivic constant. By Theorem 94, using linearity with respect to motivic constants, we get the definition of the motivic Satake for every element in $\mathcal{H}(G, K)^{\mathrm{mot}}$. Notice that Lemma 98 guarantees that this definition makes sense.

5.4 SOME REPRESENTATION THEORY

This sections contains a brief discussion about basic representation theory of p-adic groups. It is included in this work in regard to future possible conections with our work.

Although many of the definitions and result that we present in this section are valid in more general contexts, we continue with our assumptions. Let G be a connected unramified reductive group defined over F, and let K be a hyperspecial maximal compact subgroup of $G(F)$. We consider $G(F)$ with the Haar measure that gives volume 1 to K. We fix a Cartan subgroup $T \subset G$ that splits over an unramified extension $F_1 \supset F$.

A representation (π, V) of G is a homomorphism

$$\pi : G(F) \to GL(V),$$

where V is a complex vector space, which is often infinite dimensional. The space V may have more structure *e.g.* a Hilbert space. The representation is called smooth if the stabilizer of every $v \in V$ is an open subgroup of $G(F)$. A smooth representation is called irreducible if it has no proper nontrivial invariant subspaces. The representation is called admissible if it is smooth, and if for every open compact subgroup $D \subset G(F)$, the space

$$V^D := \{v \in V \mid \pi(g)v = v \ \ \forall g \in D\},$$

of D-fixed vectors is finite dimensional.

Let (π, V) be a smooth representation of G. For $f \in \mathcal{H}(G)$, we define $\pi(f) \in End(V)$ by

$$\pi(f)v = \int_G f(g)\, \pi(g)v \ dg. \tag{5.11}$$

Observe that $g \mapsto f(g)\, \pi(g)v \in V$ only takes finitely many values since the stabilizer of v is open and f is locally constant and compactly supported. Therefore, the integral 5.11, for a given $v \in V$, is just a finite sum of vectors in V. In the literature, this type of integral is called a Bochner integral. Thus, we have described $\pi(\cdot) : \mathcal{H}(G) \to End(V)$. It can be proved that this map is a representation of $\mathcal{H}(G)$.

An irreducible admissible representation (π, V) of G is called K-spherical (or unramified) if it has a non-zero K-fixed vector. These representations are important in the theory of automorphic forms.

We recall the notion of unramified character of the Cartan subgroup T (our main reference is Cartier's article [6]). Recall X^* and X_* are the groups of F_1-characters and F_1-cocharacters of T, respectively. Let $X^*_{/F} \subset X^*$ be the group of F-characters and let $X_{*/F} \subset X_*$ be the group of F-cocharacters of T. There exists a standard homomorphism

$$H_T : T(F) \to X_{*/F}$$

given by

$$q^{\langle H_T(t), \chi \rangle} = |\chi(t)|_F, \qquad \forall\, \chi \in X^*_{/F}$$

Recall that X_{*F} (do not confuse with $X_{*/F}$) denotes the set of F-cocharacters of A. In our situation where T splits over an unramified extension $F_1 \supset F$, it can be proved that $H_T(T(F)) = X_{*F}$. This is explained in Borel's paper [3], Section 9.5.

Definition 100. *A character $\chi : T(F) \to \mathbb{C}^\times$ is unramified if $\chi|_{\mathrm{Ker}(H_T)} = 1$ i.e., factors through $H_T(T(F)) = X_{*F} \cong \mathbb{Z}^d$,*

$$\begin{tikzcd} T(F) \arrow[r, "\chi"] \arrow[d] & \mathbb{C}^\times \\ \mathbb{Z}^d \arrow[ur] & \end{tikzcd}$$

For a suitable choice of K, we can assume that $\mathrm{Ker}(H_T) = K^\circ = K \cap T(F)$. Thus a character χ of T is said to be unramified if it is trivial on the greatest compact subgroup K°. See Appendix A.1 for a discussion on multiplicative characters over p-adic fields.

We denote by X^{un} the set of unramified characters of T. As in the case of multiplicative characters, one may ask about the motivic nature of characters and unramified characters. The situation is very similar. Some unramified characters are given by constructible motivic functions.

The relative Weyl group W_F acts on X^{un} by

$$\chi^w(t) = \chi(w^{-1}tw),$$

for $\chi \in X^{un}$, $t \in T(F)$ and $w \in W_F$. If we fix a basis $m_1, \ldots, m_d$ of $T(F)/K^\circ \cong \mathbb{Z}^d$, where d is the relative rank of G, we get an isomorphism:

$$(\mathbb{C}^\times)^d \xrightarrow{\sim} X^{un}$$

where $z = (z_1, \ldots, z_d) \in (\mathbb{C}^\times)^d$ is mapped to χ_z given by $\chi_z(m_i) = z_i$, for $i = 1, \ldots, d$.

Let χ be an unramified character of T. Consider the Haar measure on $T(F)$ that assigns to K° volume 1. The Fourier transform

$$f \mapsto \int_{T(F)} f(t)\chi(t)\, dt$$

is an homomorphism from $\mathcal{H}(T, K^\circ)$ to $\mathbb{C}$. Now, define

$$\omega_\chi : \mathcal{H}(G, K) \to \mathbb{C}$$

by

$$\omega_\chi(f) = \int_{T(F)} \mathcal{S}f(t)\chi(t)\, dt.$$

As a consequence of the Satake isomorphism we have the next two results.

Proposition 101 (c.f. [6], Cor. 4.1)**.** *The algebra $\mathcal{H}(G, K)$ is commutative and finitely generated over $\mathbb{C}$.*

Proposition 102 (c.f. [6], Cor. 4.2)**.** *Any algebra homomorphism from $\mathcal{H}(G, K)$ into $\mathbb{C}$ is of the form ω_χ for some unramified character χ of T. Moreover, one has $\omega_\chi = \omega_{\chi'}$ if and only if there exists an element $w \in W_F$ such that $\chi' = \chi^w$.*

Let (π, V) be an irreducible K-spherical representation of G. Then we have a representation of $\mathcal{H}(G,K)$ on the one dimensional vector space $V^K = \langle v_0 \rangle$, for some non-zero vector $v_0 \in V$. See [6], section 4.4. We denote by $\lambda_\pi : \mathcal{H}(G,K) \to \mathbb{C}$ this representation and it is given by

$$\lambda_\pi(f)v_0 = \pi(f)v_0 = \int_{G(F)} f(g)\pi(g)v_0 \, dg$$

where this integral, as we observed before, is a finite sum. It can be proved (c.f. [6] §1.5) that

$$\lambda_\pi(f) = \mathrm{tr}(\pi(f)).$$

By 102, there exists an unramified character χ_π, unique up to conjugation by W_0 such that

$$\lambda_\pi = \omega_{\chi_\pi}.$$

We consider the function λ_π on generators of the spherical Hecke algebra $\mathcal{H}(G,K)$. Let $\lambda \in X_{*F}$.

$$\begin{aligned}
\lambda_\pi(\mathbf{1}_\lambda^G)v_0 &= \pi(\mathbf{1}_\lambda^G)v_0 \\
&= \int_{G(F)} \mathbf{1}_\lambda^G(g)\pi(g)v_0 \, dg \\
&= \int_{K\varpi^\lambda K} \pi(g)v_0 \, dg
\end{aligned}$$

6.0 TOWARDS A MOTIVIC FUNDAMENTAL LEMMA

In this chapter we discuss possible applications of our work to the fundamental lemma for groups. We want to show that all terms describing the fundamental lemma are given by constructible motivic functions. In the case of groups and all the elements in the spherical Hecke algebra, the Satake transform plays a crucial role.

6.1 FROM G TO H

As above, G is a split connected reductive group. By a little abuse of notation we denote the restriction of the inclusion ξ to $\widehat{T}_H$ by $\xi : \widehat{T}_H \to \widehat{G}$, so $\widehat{T}_H$ is a subtorus of $\widehat{G}$ of the same rank. By endoscopy theory, we know that $\operatorname{rank}(\widehat{T}) = \operatorname{rank}(\widehat{T}_H)$, so $\xi(\widehat{T}_H) = \widehat{T}_H = g\widehat{T}g^{-1}$ for some $g \in \widehat{G}$. As part of our fixed choices we can take $\widehat{T}_H = \widehat{T}$. Notice that $\widehat{G}$ is a complex group, so no p-adic objects are involved here. Hence this fixed choice is valid. Thus $\mathbb{C}[\widehat{T}_H] = \mathbb{C}[\widehat{T}]$. There exists a canonical (almost tautological) way to identify the group ring $\mathbb{C}[X_*]$ with the ring $\mathbb{C}[\widehat{T}]$ because X_* is the character group of $\widehat{T}$. We denote by $\eta : \mathbb{C}[X_*] \to \mathbb{C}[\widehat{T}]$ this isomorphism.

On the endoscopic side we have similar identifications. Let $\dot{X}^*$ be the character group of T_H and let $\dot{X}_*$ be the cocharacter group of T_H. We denote by $\mathcal{S}^H : \mathcal{H}(H, K_H) \to \mathcal{H}(T_H, K_H^\circ)^{W_H}$ the Satake isomorphism on the endoscopic side. Through a (fixed) choice of generators of $\dot{X}^*$ we identify $\mathcal{H}(T_H, K_H^\circ)$ with $\mathbb{C}[\dot{X}_*]$ in a field independent way. There exists a canonical isomorphism $\dot{\eta} : \mathbb{C}[\widehat{T}_H] \to \mathbb{C}[\dot{X}_*]$. Thus, putting together the isomorphisms that we have defined we obtain

$$\mathcal{H}(T, K^\circ) \cong \mathbb{C}[X_*] \xrightarrow{\eta} \mathbb{C}[\widehat{T}] = \mathbb{C}[\widehat{T}_H] \xrightarrow{\dot{\eta}} \mathbb{C}[\dot{X}_*] \cong \mathcal{H}(T_H, K_H^\circ).$$

We denote by $\mathsf{j} : \mathcal{H}(T, K^\circ) \to \mathcal{H}(T_H, K_H^\circ)$ the composition of these isomorphisms, it is clearly a $\mathbb{C}$-algebra isomorphism. Now, although the spherical Hecke algebras of T and T_H are p-adic objects, the identification with not p-adic objects in a field independent way allows us to think the map j as a fixed choice. To clarify, with some abuse of notation, if we write $\mathsf{j} : \mathbb{C}[X_*] \to \mathbb{C}[\dot{X}_*]$, it is clear that this map is a fixed choice.
Since W_H is a subgroup of W, $\mathcal{H}(T, K^\circ)^W \subset \mathcal{H}(T, K^\circ)^{W_H}$, hence

$$\mathsf{j}\Big(\mathcal{H}(T, K^\circ)^W\Big) \subset \mathcal{H}(T_H, K_H^\circ)^{W_H}.$$

We now define a map that is essential in the fundamental lemma

$$\mathfrak{b} : \mathcal{H}(G, K) \to \mathcal{H}(H, K_H)$$
$$f \mapsto ((\mathcal{S}^H)^{-1} \circ \mathsf{j} \circ \mathcal{S})\, f$$

This function is the conection between the spherical Hecke algebra of G and the spherical Hecke algebra of its endoscopy group H.

Theorem 103. *Let G, T, K be a split reductive group defined over $\mathbb{Q}$, a Cartan subgroup of G and a hyperspecial subgroup of G. Let F be a p-adic field of characteristic zero with uniformizer ϖ. There exists a definable set $\mathcal{Z} \subset \mathbb{Z}^r \times \mathbb{Z}^r$ in the Denef-Pas language such that*

$$(\lambda, \mu) \in \mathcal{Z} \iff C^G_{\lambda\mu} \neq 0.$$

Proof. Since the character δ is never zero, $C^G_{\lambda\mu} \neq 0$ if and only if the integral

$$\int_N \mathbf{1}^G_\lambda(\varpi^\mu n)dn = \int_N \int_{D^T_\mu} \mathbf{1}^G_\lambda(xn)dxdn \neq 0.$$

by the Lemma **??**. Consider the following first-order formula

$$\psi(\lambda, \mu) = \exists x \in N\ \exists y \in D^T_\mu\ \big(yx \in D^G_\lambda\big)$$

Since $\mathbf{1}^G_\lambda$ is K-bi-invariant, the double integral is non-zero if and only if $\psi(\lambda, \mu)$ holds.

□

Let $\lambda \in X_*$. We compute $\mathfrak{b}$ in steps. First we apply the Satake isomorphism

$$\mathcal{S}(\mathbf{1}_\lambda^G) = \sum_\mu C_{\lambda\mu}^G \mathbf{1}_\mu^T \in \mathcal{H}(T, K^\circ)^W$$

then the map j

$$\mathsf{j}(\mathcal{S}(\mathbf{1}_\lambda^G)) = \sum_\mu C_{\lambda\mu}^G \mathbf{1}_\mu^{T_H} \in \mathcal{H}(T_H, K_H^\circ)^{W_H}$$

and finally inverse Satake on the H

$$\mathfrak{b}(\mathbf{1}_\lambda^G) = \sum_\nu d_{\lambda\nu} \mathbf{1}_\nu^H \in \mathcal{H}(H, K_H).$$

Similarly, for $\nu \in X_*$

$$\mathcal{S}^H(\mathbf{1}_\nu^H) = \sum_\gamma C_{\nu\gamma}^H \mathbf{1}_\gamma^{T_H} \in \mathcal{H}(T_H, K_H^\circ).$$

Now,

$$\begin{aligned}
\mathcal{S}^H(\mathfrak{b}(\mathbf{1}_\lambda^G)) &= \sum_\nu d_{\lambda\nu} \mathcal{S}^H(\mathbf{1}_\nu^H) \\
&= \sum_\nu d_{\lambda\nu} \sum_\gamma C_{\nu\gamma}^H \mathbf{1}_\gamma^{T_H} \\
&= \sum_{\nu,\gamma} d_{\lambda\nu} C_{\nu\gamma}^H \mathbf{1}_\gamma^{T_H} \\
&= \sum_\mu C_{\lambda\mu}^G \mathbf{1}_\mu^{T_H} \in \mathcal{H}(T_H, K_H^\circ)^{W_H}
\end{aligned}$$

Hence, for each μ

$$C_{\lambda\mu}^G = \sum_\nu d_{\lambda\nu} C_{\nu\mu}^H.$$

A.1 MULTIPLICATIVE CHARACTERS OF P-ADIC FIELDS

We discuss multiplicative characters on p-adic fields and their relationship with motivic functions. In general, a multiplicative character is not the specialization of a motivic function on $\mathrm{VF}^{\times}$. This is the main problem with the transfer factors, the non-motivic nature of the multiplicative characters used to describe them. Nevertheless, some multiplicative characters can be described by motivic functions. Maybe some extensions of the theory will allow to treat an important class of multiplicative characters inside the motivic framework. We follow Sally's article [49].

We describe the characters of $\mathbb{Q}_p^{\times}$. In the case of a finite extension of $\mathbb{Q}_p^{\times}$, we can fix a uniformizer ϖ in the field and everything follows in the same manner but with respect to the ϖ-adic expansion.

Recall that any $x \in \mathbb{Q}_p^{\times}$ can be written uniquely in the form $x = p^{\mathrm{ord}(x)} \cdot u$, with $\mathrm{ord}(x) \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^{\times}$. This gives an isomorphism $\mathbb{Q}_p^{\times} \cong \mathbb{Z} \times \mathbb{Z}_p^{\times}$. Hence, a character $\chi : \mathbb{Q}_p^{\times} \to \mathbb{C}^{\times}$ may be written as

$$\chi(x) = \chi(p^{\mathrm{ord}(x)} \cdot u) = |x|_p^s \, \omega(u) = p^{-\mathrm{ord}(x)s} \, \omega(u),$$

where $s \in \mathbb{C}$ and ω is a character of $\mathbb{Z}_p^{\times}$. This character is unitary if and only if $s \in \mathbb{C}$ is pure imaginary. The character ω of $\mathbb{Z}_p^{\times}$ is always unitary. By continuity of ω, some group $\mathbb{Z}_p^{(n)} = 1 + p^n\mathbb{Z}_p$ for $n \geq 1$ or $\mathbb{Z}_p^{(0)} = \mathbb{Z}_p^{\times}$ for $n = 0$ is in its kernel. Note that $\mathbb{Z}_p^{(n)}$, for $n \geq 0$, are open compact subgroups of $\mathbb{Q}_p^{\times}$. It is easy to see that $\mathbb{Z}_p/\mathbb{Z}_p^{(n)} \cong (\mathbb{Z}/p^n\mathbb{Z})^{\times}$. Thus ω can be viewed as a character of the finite abelian group of $(\mathbb{Z}/p^n\mathbb{Z})^{\times}$. The conductor of χ is defined as the minimal natural number n such that $\mathbb{Z}_p^{(n)} \subset \mathrm{Ker}(\chi)$. Our previous discussion guarantees the existence of such a minimal number. If the conductor of χ is zero, we say that the character χ is unramified. For a given conductor $n \geq 1$, there are only finitely many possibilities for ω. To be precise, there are $p-2$ possibilities if $n = 1$ and $p^{n-2}(p-1)^2$ if $n > 1$.

In summary, a multiplicative character of $\mathbb{Q}_p^\times$ is determined by an additive character of $\mathbb{Z}$ and a character of the finite abelian group $(\mathbb{Z}/p^n\mathbb{Z})^\times$, for some $n \geq 0$. It is completely valid to ask about the motivic nature of the multiplicative characters. This corresponds to a description of the multiplicative characters in a form that is independent of p. The main obstacle for this is the character on the finite abelian group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We do not see how to describe such a finite group and its character using the Denef-Pas language. In the case of an unramified character χ there is no contribution of this finite group that depends on p. In fact, the character depends just on $\mathbb{Z}$ and it has the form $x \mapsto |x|_p^s$ for some $s \in \mathbb{C}$. In other words, an unramified character χ factors through $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \cong \mathbb{Z}$,

$$\begin{tikzcd} \mathbb{Q}_p^\times \arrow[r, "\chi"] \arrow[d] & \mathbb{C}^\times \\ \mathbb{Z} \arrow[ur] & \end{tikzcd}$$

Therefore each unramified character can be considered as a fixed choice. Nonetheless, in the case where $s \in \mathbb{Z}$ (fixed), the unramified character $x \mapsto |x|_p^s$ can be obtained by the specialization of the constructible motivic function $\mathrm{VF}^\times \to A$ given by

$$x \mapsto \mathbb{L}^{-\mathrm{ord}(x)s}.$$

Observe that the product $\mathrm{ord}(x)s$ can be represented by a formula in the Denef-Pas language as long as $s \in \mathbb{Z}$ is fixed. So the family of unramified characters parametrized by $s \in \mathbb{Z}$ is not definable in the Denef-Pas language.

BIBLIOGRAPHY

[1] J. D. Adler, L. Corwin and P. J. Sally Jr, *Discrete series characters of division algebras and GLn over a p-adic field*, in Contributions to Automorphic Forms, Geometry and Number Theory. The Johns Hopkins University Press, Baltimore MD, 2004

[2] J. Ax and S. Kochen, *Diophantine problems over local fields. I*, Amer. J. Math. **87** (1965), 605-630.

[3] A. Borel, *Automorphic L-functions*, in Automorphic forms, representations and L-functions, Corvallis, Symp. in Pure Math, AMS 33, part 2, Providence, RI, 1977.

[4] D. Bump, *Orbital integrals and the Satake Isomorphism*, available at http://sporadic.stanford.edu/bump/orbital.ps

[5] R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Vol. 20, Wiley New York, 1985.

[6] P. Cartier,*Representations of p-adic groups: a survey*, in Automorphic forms, representations and L-functions, Corvallis, Symp. in Pure Math, AMS 33, part 1, Providence, RI, 1977.

[7] C. C. Chang and H. J. Keisler, *Model Theory*, 3rd edition, Third ed., Studies in Logic and the Foundations of Math. 73, North-Holland, Amsterdam, 1990.

[8] R. Cluckers, J. Gordon and I. Halupczok, *Local integrability results in harmonic analysis on reductive groups in large positive characteristic*(2011), available at arXiv:1111.7057.

[9] R. Cluckers, J. Gordon and I. Halupczok, *Motivic functions, integrability, and uniform in p bounds for orbital integrals* , Electronic Math. Research Announcements in Math. Sciences. **21** (2014), 137-152.

[10] R. Cluckers, T. C. Hales and F. Loeser, *Transfer principle for the Fundamental Lemma*, in Stabilization of the Trace Formula, Shimura varieties, and arithmetic applications, I. International Press, 2011.

[11] R. Cluckers and F. Loeser, *Ax-Kochen-Ershov theorems for p-adic integrals and motivic integration*, in Geometric methods in algebra and number theory, Prog. Math. vol. 235, Birkhauster Boston, Boston, MA 2005.

[12] R. Cluckers and F. Loeser, *Constructible motivic functions and motivic integration*, Invent. Math.**173** (2008), no. 1, 23-121.

[13] R. Cluckers and F. Loeser, *Constructible exponential functions, motivic Fourier transform and transfer principles*, Annals of Mathematics.**171** (2010), no. 2, 1011-1065.

[14] C. Cunningham and T. Hales, *Good Orbital Integrals*, Represent. Theory **8** (2004), 414-457.

[15] J. F. Dat, *Lemme fondamental et endoscopie, une approche géométrique*, Bourbaki Seminar, (940), 2004.

[16] S. DeBacker, *The fundamental lemma: what is it and what do we know?*, Current developments in mathematics, 2005, Int. Press, Somerville, MA, 2007.

[17] J. Denef and F. Loeser, *Definable sets, motives and p-adic integrals*, J. Amer. Math. Soc. **14** (2001), no. 2, 429469.

[18] Y. Eršhov, *On elementary theories of local fields*, Algebra i Logika (Seminar) **42** (1965), no. 2, 5-30.

[19] J. Gordon and T. Hales, *Endoscopic transfer of orbital integrals in large residual characteristic* (2015), http://arxiv.org/pdf/1502.07368.pdf.

[20] J. Gordon and Y. Yaffe, *An overview of arithmetic motivic integration.*, in Ottawa Lectures on p-adic Groups, Fields Institute Monographs Series, AMS, Providence, RI, 2009.

[21] J. Gordon and D. Roe, *The canonical measure on a reductive p-adic group is motivic*, http://arxiv.org/pdf/1507.02619.pdf.

[22] B. Gross. *On the Satake isomorphism*, in Galois representations in arithmetic algebraic geometry. London Math. Soc. Lecture Note Series 254, Cambridge Univ. Press, Cambridge, 1998.

[23] T. Hales, *A simple definition of transfer factors for unramified groups*, Contemporary Mathematics. Vol. **145** (1993), 109-134.

[24] T. Hales, *On the fundamental lemma for standard endoscopy: Reduction to unit elements*, Can. J. Math. Vol.**47** (1995), no. 5, 974-994.

[25] T. Hales,*What is a motivic measure?*, Bull. Amer. Math. Soc. (N.S)**42** (2005), no. 2, 119-135.

[26] T. Hales, *A statement for the fundamental lemma*, in Harmonic Analysis, the Trace Formula, and Shimura Varieties. Clay Math. Proc. Vol. **4**, AMS, Providence, RI, 2005.

[27] T. Hales, *Orbital integrals are motivic*, Proc. Amer. Math. Soc., **133** (2005), 1515-1525.

[28] T. Hales, *The fundamental lemma and the Hitchin fibration [after Ngô Bao Châu]*, Bourbaki Seminar, (1035), 2011.

[29] T. Hales, *Lectures at MSRI. Introductory Workshop: Geometric Representation Theory, 2014*, available at https://www.msri.org/people/29066.

[30] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications 42, Cambridge University Press, Cambridge, 1993.

[31] R. Howe, *Harish-Chandra Homomorphisms for* $\mathfrak{p}$*-adic groups*, CBMS Regional Conference Series in Mathematics, Volume 59, AMS, Providence, RI, 1985.

[32] J. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics 21, Springer-Verlag, New York, 1975.

[33] J. Igusa, *An introduction to the theory of local zeta functions*. AMS Studies in Advanced Mathematics, 14, International Press, Cambridge, MA, 2000.

[34] R. E. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), no. 4, 785-806.

[35] R. E. Kottwitz, *Transfer factors for Lie algebras*, Represent. Theory **3** (1999), 127-138.

[36] R. Kottwitz, *Harmonic analysis on reductive p-adic groups and Lie algebras*, in Harmonic Analysis, the Trace Formula, and Shimura Varieties. Clay Math. Proc. Vol. **4**, AMS, Providence, RI, 2005.

[37] J. P. Labesse, *Introduction to Endoscopy*, available at http://www.math.utah.edu/ptrapa/src2006/labesse.pdf.

[38] R. P. Langlands, *Les débuts d'une formule des traces stables*, Publications de l'Université Paris VII, 13, 1983.

[39] R. P. Langlands and D. Shelstad, *On the definition of the transfer factors*, Math. Ann. **87** (1987), 219-271.

[40] I. G. Macdonald, *Spherical functions on a p-adic Chevalley group*, Bull. Amer. Math. Soc. **74** (1968), 520-525.

[41] I. G. Macdonald, *Spherical functions on a group of p-adic type, Ramanujan*, Inst. Publ., Madras, 1971.

[42] D. Marker, *Model theory. An introduction*. Graduate Texts in Mathematics, 217. Springer-Verlag, New York, 2002.

[43] F. Murnaghan, *Representations of reductive p-adic groups. Course Notes*, available at http://www.math.toronto.edu/murnaghan/courses/mat1197/notes.pdf.

[44] D. Nadler, *The geometric nature of the fundamental lemma*, Bulletin of the American Mathematical Society **49** (2012), 150.

[45] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. **111** (2010), no. 1, 1-169

[46] B. C. Ngô, *Survey on the fundamental lemma*, available at https://www.math.uchicago.edu/∼ngo/survey.pdf.

[47] B. Poonen, *The Grothendieck ring of varieties is not a domain*, Math. Res. Lett. 9 (2002), no. 4, 493 497. MR MR1928868 (2003g:14010).

[48] M. Presburger, *On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation*, Hist. Philos. Logic. **12** (1991), 225-233.

[49] P. Sally, *An introduction to p-adic fields, harmonic analysis and the representation theory of SL2*, Letters in Mathematical Physics **46** (1998), 1-47.

[50] I. Satake, *Theory of spherical functions on reductive algebraic groups over p-adic groups*, Inst. Hautes Etudes Sci. Publ.Math. **18** (1963), 1-69.

[51] F. Shahidi, *Eisenstein Series and Automorphic L-functions*, vol. 58 of American Mathematical Society Colloquium Publications. AMS, Providence, RI, 2010.

[52] S. W. Shin and N. Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L-functions*, with appendices by R. Kottwitz [A] and J.Gordon, R. Cluckers and I. Halupczok [B], preprint available at http://arxiv.org/abs/1208.1945.

[53] T. Springer, *Reductive groups*, in Automorphic forms, representations and L-functions, Corvallis, Symp. in Pure Math, AMS 33, part 1, Providence, RI, 1977.

[54] T. Springer, *Linear Algebraic Groups*, Progress in Mathematics, 9, Birkhäuser, Boston Inc, 1998.

[55] T. Tamagawa, *On the ζ-function of a division algebra*, Annals of Mathematics. **77** (1963), no. 2, 387-405.

[56] J. Tits, *Reductive groups over local fields*, in Automorphic forms, representations and L-functions, Corvallis, Symp. in Pure Math, AMS 33, part 1, Providence , RI, 1977.

[57] R. Vakil *The Rising Sea. Foundations of Algebraic Geometry*, available at http://math.stanford.edu/∼vakil/

[58] Z. Yun, *The fundamental lemma of Jacquet and Rallis*, with an appendix by J. Gordon, Duke Math. J., **156** (2011), 167-227.