CONTRIBUTIONS TO THE THEORY OF SENSITIVITY AND STABILITY ANALYSIS OF MULTI-CRITERIA DECISION MODELS, WITH APPLICATIONS TO MEDICAL DECISION MAKING

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Patients are faced with multiple alternatives when selecting the preferred method for colorectal cancer screening, and there are multiple criteria to be considered in the decision process. We model patients' choices using a multi-criteria decision model, and propose a new approach for characterizing the idiosyncratic preference regions for individuals and for groups of similar patients.

We propose an extension of the sensitivity and stability analyses for Analytic Network Models developed by May et al. (2013). We study ANP models to understand how preference regions are created, and how boundaries can be characterized, as the number of criteria increases. For the two-criteria and three-criteria sensitivity and stability analyses, piecewise linear functions and triangular mesh generation, respectively, are used to approximate the boundaries between two adjacent preference regions. We use optimization methods to find the best approximations for the core stability and solution stability regions for cases where two and three criteria are perturbed simultaneously, and there exist an arbitrary number of alternatives. We define sensitivity and stability measures that can be implemented in practice, and that can be considered as a starting point in any medical decision making process.

We apply our newly developed methodology to randomly chosen patients, and show how insights derived from the sensitivity and stability of patients' preferences might be used within the medical decision making process. Individualized stability analysis is informative, but the generalization to groups of similar patients may be even more important for healthcare providers. Our comparisons reveal that a patient's age may be an effective discriminating factor that should be taken into consideration when extending the individualized sensitivity and stability analysis to groups of patients with similar characteristics.

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1.0 INTRODUCTION

"Sometimes you just need some reassurance about

your Choice before you take any decision"

Auliq Ice

1.1. BACKGROUND

How do people make decisions? How influenced are they by changes that might appear in their preferences? Those are questions that need to be answered. Our judgments are based on current knowledge, experience and expertise. What happens if additional information is available? How will the additional information influence our preferences, and, ultimately, our decision making process? Can this change in preferences be quantified mathematically and predicted? Could a mathematical measure of the change guide the decision making process towards better outcomes?

An important component of the decision making process is a determination of the sensitivity and the stability of the decision maker's initially most preferred alternative. Should one implement the most preferred alternative, or, instead, select the next most preferred alternative if it is less sensitive to changes in the environment, and, hence, may be more stable? Sensitivity and stability analysis provide insights for answering that question. Determining how sensitive and stable our preferences and decisions are, and how they are influenced by changes in the available information, may provide information regarding the most appropriate solution to be chosen.

Because the decision making environment we consider in this dissertation is complex and involves multiple criteria, we need a methodology that represents that complexity, and helps to measure influences among the components of the system used to make decisions. One such methodology is the Analytic Hierarchy Process/Analytical Network Process (Saaty, 1980; Saaty, 1986; Saaty and Vargas, 2013). May et al. (2013) proposed an innovative approach for studying the sensitivity and stability analysis of Analytic Network Process (ANP) models. In that paper, the authors (1) generated a systematic non-random sample of the perturbation space; (2) approximated, with hyperplanes, the boundaries of the preference regions in the perturbation space in which the different alternatives are ranked first, and (3) found stability regions within the preference regions, using the Euclidean center method (Arbel and Vargas, 2007). The Euclidean center method generates spheres to measure stability. A basic assumption that simplifies the implementation of the Euclidean center method is that the region in which the maximum volume spheres are inscribed must be convex. In general, preference regions generated by perturbations of ANP models are simply connected, but may be non-convex.

Medical decision making is characterized by multiple criteria, subjective information, and sometimes incomplete information, so it is a highly suitable area for the application of our work. In medical decision making, preferences may change rapidly, and are influenced by multiple factors. Within the context of medical decision making, we chose to apply the methodology developed in this dissertation to the problem of selecting the most appropriate colorectal cancer screening option. That problem is particularly challenging, because patient preferences are related to the level of medical knowledge the patient has, and to the information he was exposed to before making a medical decision. We analyzed how sensitive and how stable individual patients' preferences are, as additional information about the criteria, with respect to which the screening alternatives are compared, is given to the patients. Even though individualized analyses might be more beneficial, healthcare providers are also interested in learning about generalized analyses of the sensitivity and stability of preferences that can be applied directly to patients categorized as having an average risk of being diagnosed with colorectal cancer. Preliminary results of the generalization of the analysis to groups of patients with similar preferences, using age as discriminating factor, are presented.

1.2. OBJECTIVES OF THIS DISSERTATION

There are several *objectives* of the research presented in this dissertation:

1. To extend the sensitivity and stability methodology of May et al. (2013) to two and three-dimensional spaces;
2. To develop measures of: (a) the core stability of the most preferred alternative, (b) the solution stability, and (c) criteria sensitivity, and,
3. To apply the methodology developed to a medical decision making problem.

The primary *contributions* of this research are:

1. To extend the procedure in May et al. (2013) by estimating solution stability using ellipsoids instead of spheres, because ellipsoids cover more volume of the preference region than spheres do;
2. To approximate the boundaries of the preference regions using piecewise linear functions instead of hyperplanes, for the two-dimensional case, and by triangular

mesh generation, for the three-dimensional case; and to measure the classification error induced by the piecewise linear boundary approximations;

3. To develop an algorithm to approximate the maximal-volume ellipsoid that can be inscribed in a non-convex region within a higher dimensional space ($\mathrm{n} \geq 2$);
4. To develop a measure of the core stability of the most preferred alternative, as given by the maximal spheroid inscribed in the perturbation region, centered at 0^n, and of the direction of change in stability derived from the vector of perturbations that determines the fastest change in preferences;
5. To develop a measure of the solution stability, by comparing the relative areas of each preference region within the perturbation space to the relative areas of the corresponding maximal-volume inscribed ellipsoids;
6. To determine the alternative that is the most stable, among a set of alternatives, with respect to given pairs or triplets of criteria;
7. To apply the methodology to support the selection of the most appropriate option among colorectal cancer screening alternatives. We apply the methodology to a single patient, and compare the results for the situations when pairs and triplets of criteria are changed simultaneously. We study the feasibility of generalizing the methodology to groups of patients with similar characteristics, considering age as the discriminating factor.

A measure of the area (volume in *n*-dimensional space, $\mathrm{n} \geq 2$) of the preference regions is an estimate of the stability of the solutions. Thus, if a solution/alternative is the most preferred, should one implement that alternative, even if its stability is low, or should one find the most stable alternative? In addition, we could determine under what conditions the most preferred alternative becomes the most stable one. Chapter 4 provides such an exposition, in the context of a medical decision making problem, and how the healthcare provider can use the results provided by the sensitivity and stability analysis to guide or influence the patient in the medical decision making process.

1.3. OVERVIEW OF CHAPTERS

The remainder of the dissertation is organized as follows. In Chapter 2 we present a review of the literature on approximating nonlinear boundaries using piecewise linear functions, on approximating the nonlinear boundaries in higher dimensional space using mesh generation, on the problem of inscribing the maximal ellipsoid within a convex set, and regarding how patients make decisions about colorectal cancer screening using AHP/ANP models.

In Chapter 3, we extend May et al.'s stability region methodology, propose an algorithm to approximate the nonlinear boundaries of two-dimensional regions using piecewise linear functions and triangular mesh generation for the three-dimensional case, develop a nonlinear programming model for inscribing the maximum ellipsoid in a higher dimensional region ($n \geq 2$) that might be non-convex, and propose measures of sensitivity and stability that can be used as practical guidelines.

In Chapter 4 we apply the methodology to a medical decision making problem about choosing the most appropriate colorectal cancer screening option for a patient categorized as having an average risk of being diagnosed with colorectal cancer. Individualized sensitivity and stability analysis are performed for individual patients for the two and three-criteria cases, and the medical implications, based on the analysis, are presented. The analyses for two different patients are compared, to illustrate the similarities and the differences in their preferences. Using age as a discriminating factor, we present preliminary results of the generalization of the sensitivity and stability analysis to groups of patients with similar characteristics.

In Chapter 5 we summarize the conclusions of our research, and present future research directions that address the limitations of the current approach.

2.0 LITERATURE REVIEW

2.1. INTRODUCTION

This dissertation is concerned with: (1) the development of sensitivity and stability analysis for multi-criteria decision models, such as the Analytic Hierarchy Process (AHP) and the Analytic Network Process (ANP); (2) the approximation of the boundaries of regions in two and three-dimensional space; (3) inscribing maximal area/volume ellipsoids in non-convex sets; and (4) the development of stability measures that can be applied to difficult decision processes, such as medical decision making, specifically as related to colorectal cancer screening.

This literature review identifies two types of sensitivity analyses previously developed for the AHP models: (1) the sensitivity analysis of judgements, which analyzes the effect of judgments' randomness on the stability of the rank order of alternatives; (2) the sensitivity analysis of priorities, based on changes in the vectors of the supermatrix, and how such changes impact the ranking of alternatives. May et al. (2013) includes a thorough review of the AHP/ANP sensitivity literature; the other topics are discussed in this chapter.

2.2. PIECEWISE LINEAR APPROXIMATION

In this dissertation, instead of approximating each boundary of the preference regions with a single hyperplane, as in May et al. (2013), we use piecewise linear hyperplanes. In the two-dimensional case, those hyperplanes are line segments. A *piecewise linear hyperplane* is a collection of linear subspaces with dimensionality $n-1$, where n is the dimension of the current space. The primary reason for using a set of hyperplanes, instead of a single hyperplane, to model each boundary, is because the preference regions are, in general, not convex. Note that the literature on piecewise linear approximations is usually based on an assumption that the functions to be approximated are known; that is not true for the boundaries that we address in this dissertation. See Lin et al. (2013) for a review of piecewise linearization methods. The AHP/ANP solution process (Saaty, 1980) does not provide the user with functions that define the boundaries of the preference regions (May et al., 2013). The preference regions are initially estimated pointwise, by perturbing the data and repeating the solution process. Because the region labeling of each point in space is determined by a numerical process, which is subject to numerical error, the boundaries of the preference regions are not well defined. Our problem consists of separating sets of data, the preference regions, with hyperplanes. The equivalent of the problem, in the two-dimensional case, is to find piecewise linear approximations of the boundaries between the preference regions, within the perturbation space.

Williams (1978) developed an algorithm for approximating planar curves by the smallest possible number of linear segments. The solution obtained does not necessarily provide the minimum number of segments for a given error tolerance parameter. The algorithm also cannot deal with a reverse in the direction of the curve. Dunham (1986) presents an algorithm that finds the optimal linear approximation to a planar curve, by generating the minimum number of segments needed to describe the shape. The method requires, as inputs, the end points of the curve. Sato (1992) proposed an optimal piecewise linear approximation of a planar curve, using dynamic programming. The algorithm determines a point choice function, which establishes a relationship between the original points that describe the curve, and the points selected to be used for the piecewise linear approximation to the curve. The primary goal of the dynamic programming model is to determine the optimal point choice function that generates the maximal arc length of the curve.

Bennett and Mangasarian (1992) formulated a linear program that discriminates between *k* disjoint sets, by generating a *k*-piecewise linear surface, if one exists, or an approximate separation for the case when the sets are not separable. The function obtained is convex, and is the maximum of *k* linear functions. Bredensteiner and Bennett (1999) proposed two methods to discriminate between multiple classes, using a piecewise linear separator. The first method uses a robust linear programming (RLP) model to construct a discriminant function that separates a class from the other *k-1* classes. The *k*-RLP model is a generalization of the RLP developed by Bennett and Mangasarian (1992), and it is limited to piecewise linear discriminants. The second method constructs a piecewise linear support vector machine (SVM), using a single quadratic program. The model can be generalized to the piecewise nonlinear case.

May et al. (2013) used the approach developed in Bennett and Mangasarian (1992) to construct hyperplanes that separate the preference regions; each pair of preference regions was separated by a single hyperplane. As a result, the regions defined by the separating hyperplanes are convex. May et al. used the Euclidean center method to inscribe spheres in the separated regions. Given the goal of using a geometric shape to capture as much as possible of the area of a plane region, because the area of the plane region is a measure of the stability of the associated solution, ellipsoids are superior to spheres. That is because, in general, they may include more from the area/volume of the preference regions in which they are inscribed. We next discuss the literature on the inscription of maximal area ellipsoids in plane regions.

2.3. MESH GENERATION IN HIGHER DIMENSIONAL SPACE

Mesh generation is a methodology that generates a mesh of polygons, or of polyhedrons, to approximate the geometric surfaces characterized by both convex and non-convex regions (Thompson et al., 1985). A *polygonal mesh* is a collection of vertices, edges and faces that define the space that was approximated (Edelsbrunner, 2001). The different types of mesh generation are differentiated by the shape used to approximate the faces of the polygon (e.g. triangles, quadrilaterals or convex polygons).

When the number of criteria used in the network increases, so does the dimensionality of the space of the preference regions. In the three-dimensional case, the area/volume of the preference regions can be approximated by pyramids, whose top vertices are at the center of the region, and whose triangular bases define the boundary approximation. To keep the boundary approximation piecewise linear, so that we can use linear constraints in our nonlinear programming model, we approximate the separating boundaries using Delaunay's triangulation. Thus, given a set of points on the boundary, Delaunay's method ensures that the circumcircle associated with each triplet contains no other point from the set of points considered, in its interior, and maximizes the minimum angle of all the angles described by the triangles formed (Delaunay, 1934). There are multiple algorithms based on the Delaunay principle. We use the one described by de Berg et al. (2008), which provides an unstructured grid connecting triplets of the boundary points (Mavriplis, 1996). Based on that matrix, we construct the set of inequalities that describe the separating hyperplanes in the three-dimensional space.

2.4. MAXIMAL INSCRIBED ELLIPSOID

The problem of inscribing a maximal ellipsoid within a given convex set is not new. John (1948) proved that each convex body in *n*-dimensional Euclidean space R^n contains a unique ellipsoid of maximal volume. Based on John's theorem, Ball (1992) obtained the necessary and sufficient conditions for the maximal inscribed ellipsoid within a convex body to be the closed Euclidean unit ball B_2^n. Maximal inscribed ellipsoids are used repeatedly as part of the ellipsoid method for convex optimization (Tarasov et al. 1988), so there has been considerable interest in efficient methods for approximating them in polytopes. Khachiyan and Todd (1993) proposed a polynomial-time algorithm for inscribing the maximal ellipsoid in a polytope, by modeling it as the problem of inscribing the maximal paraboloid in a polyhedral cone. Anstreicher (2002) includes an optimization model that obtains an ellipsoid whose volume is at least a factor $e^{-\varepsilon}$ of the maximum possible in O(m3.5 ln(mR/ε)) operations, and shows that the computational complexity may be reduced even more if the analytic center of the polyhedron is computed before approximating the maximum volume inscribed ellipsoid.

The current literature on the inscription of the maximal ellipsoid within a given set does not cover the non-convex case. When faced with non-convex sets, researchers use convexification methods in order to apply the algorithms already developed for the convex case (Bajaj and Dey, 1990; Lien and Amato, 2006, 2007).

2.5. MEDICAL DECISION MAKING USING AHP/ANP MODELS

Analytic Hierarchy Process (AHP) models have been proven to be a “friendly” technique to ascertain patients’ preferences, and to help them make medical decisions. An initial attempt to categorize the AHP applications in healthcare, and to identify the methodological impact, was made by Liberatore et al. (2008). The authors reviewed 50 articles, classified them into seven categories based on their purpose, and concluded that AHP could be a promising decision support tool for medical decision making. Schmidt et al. (2015) provide a comprehensive review of the last ten years of applications of the AHP models in healthcare. Their research purpose was to identify how accurately the method has been implemented in studies, and to assess the quality of the results obtained. The authors concluded that there is an increased interest in applying the AHP models to various healthcare problems, but there exist inconsistencies in how the studies were conducted. They propose the development of standard guidelines that will help unify the research framework in healthcare and medical decision making, by enabling a consistent application of the AHP methodology.

Dolan (1994) conducted a study to determine if patients have the capability and the willingness to use the AHP methodology to make clinical decisions. A group of 20 volunteers were interviewed, and asked to use an AHP-based model to choose among five colorectal cancer screening options. His results showed that 90% of the patients were capable of performing the pairwise comparisons necessary, and suggested that AHP models could be a practical tool in assisting in the medical decision making process.

Recently, Dolan and his team conducted an extended pilot study, created to help patients make the best medical decision regarding colorectal cancer screening. He designed an AHP-based model. His model includes four main criteria, one of which has three sub-criteria, to ascertain patients' preferences among ten screening alternatives. In his initial results, Dolan et al. (2002) concluded that the AHP model improved the patients' decision making process, but he could not find any statistical evidence of the effect of the model on the implementation of the decision obtained. In their later studies, Dolan et al. (2013) emphasized even more the usefulness of the AHP-based model in identifying the preferences of individual patients and the necessity of including patients' preferences in the medical decision making process. They aggregated the individual results of 484 patients to determine the average weights over the four main criteria in the model. The purpose was to generalize the individual preferences to those of the group and, finally, to the entire population of patients faced with the medical decision of choosing among colorectal cancer screening options. Dolan et al. (2014) also refined their initial model by combining multiple multi-criteria decision analysis methods in order to determine the best approach to assess, as accurately as possible, the patients' preferences regarding colorectal cancer screening options.

Analytic Network Process (ANP) models may be challenging to apply to medical decision making problems because of the extensive input data they require, so there are very few articles which use the methodology in studies. Saaty and Vargas (1998) proposed a model that combined the ANP framework with statistics to support the medical diagnosis process, by incorporating both statistical data and expert judgments. Carter et al. (1999) compared the results provided by three multi-criteria decision methods – AHP, ANP, and a Markov process, when used to determine the optimal treatment strategy in early-stage breast cancer. Wang et al. (2014) studied using the AHP and ANP methodologies for choosing the most appropriate intervention for type-2 diabetes.

3.0 METHODOLOGY

3.1. INTRODUCTION

Chapter 3 presents the methodological contributions made, in this dissertation, to the theory of sensitivity and stability analysis of multi-criteria decision models. We focus on the Analytical Hierarchy Process (AHP)/Analytical Network Process (ANP) models, when two or three criteria are perturbed simultaneously. The methodology has been developed for the most general case of an Analytical Network Process model – the situation in which there is feedback between the criteria and alternatives clusters and within the clusters. Analyzing the most general case ensures the applicability of the technique to simpler models, such as the Analytical Hierarchy Process (AHP). We begin by extending the study of stability from spheres to ellipsoids, because ellipsoids capture more volume within the perturbations space than do spheres. We approximate the separating boundaries using piecewise linear functions for the two-dimensional case, and use mesh generation via triangles for the three-dimensional case. We propose nonlinear programing models to inscribe the maximal ellipsoid within a non-convex set defined in two and three-dimensional space. Finally, we define *measures* of sensitivity and stability to enable the practical application of the methodology.

3.2. STABILITY ELLIPSOIDS IN CONVEX SETS

In May et al. (2013), the *core stability* of the solution to an AHP/ANP model was defined as the region of the perturbation space in which the initial solution remains most preferred. To study core stability, not all of the boundaries of the perturbation space are required. What is required is (1) the solution to the original problem, and (2) the boundaries that separate the preference region for the most preferred alternative from the regions associated with the other alternatives, if those regions share a boundary with the region associated with the most preferred alternative.

Assume that the k^{th} alternative is the most preferred alternative in the original study. In order to determine the region over which the k^{th} alternative remains ranked first (core stability), we first identify all the hyperplanes that separate the k^{th} alternative from the other alternatives. To do this (May et al. 2013) proceeded as follows.

The sets to be separated are created by selecting the points in the perturbation space and finding out which alternative has the largest limiting priority, as given by the supermatrix. Let $\mathbf{X}(i)$ be the subset of $\mathbf{X}$ for which the i^{th} alternative is ranked first. Let $B(i, j)$ be the boundary separating $\mathbf{X}(i)$ and $\mathbf{X}(j)$. The boundary need not be linear but we use linear boundaries because doing so reduces the amount of computation necessary and simplifies the interpretation of the results.

The stability problem can be modeled as a set of binary classification problems. A binary classification problem consists of discriminating between two given point sets A and B, with m_1 and m_2 points, respectively, in the n-dimensional real space $\mathbf{R}^n$, by using as few of the n dimensions of the space as possible. Following Mangasarian (1997), geometrically our approach constructs a plane in $\mathbf{R}^n$ defined by

$$P = \{x \mid x \in \mathbf{R}^n; x^T w = \gamma\}$$

with normal $w \in \mathbf{R}^n$ and distance $\frac{|\gamma|}{\|w\|_2}$ to the origin ($\|w\|_2$ is the Euclidean norm of the vector w), while suppressing as many of the components of w as possible. In addition, the set $\mathbb{A}$ must lie, to the extent possible, in the open half space $\{x \mid x \in \mathbf{R}^n; x^T w > \gamma\}$ and the set $\mathbb{B}$ in the open half space $\{x \mid x \in \mathbf{R}^n; x^T w < \gamma\}$. Let A and B denote the matrices representing the sets $\mathbb{A}$ and $\mathbb{B}$, respectively, where $A \in \mathbf{R}^{m_1 \times n}$ and $B \in \mathbf{R}^{m_2 \times n}$. The problem is to find variables w and γ such that $Aw > \gamma$ and $Bw < \gamma$. Because strict inequalities are not possible in an LP formulation, the variables are rescaled by the positive constant $\min_{i=1,\ldots,m_1,\, j=1,\ldots,m_2} \{A_i w - \gamma, -B_j w - \gamma\}$ where A_i and B_j are the i^th^ and j^th^ rows of the corresponding matrices. To keep the notation simple, the variables are denoted the same as before the rescaling took place. Bennett and Mangasarian (1992) proposed the following robust, i.e., $w = 0$ is excluded, linear programming formulation:

Find w, γ, y and z to

$$\text{Minimize } \frac{e^T y}{m_1} + \frac{e^T z}{m_2}$$

Subject to

$$\begin{aligned} -Aw + e\gamma + e &\le y \\ Bw - e\gamma + e &\le z \\ y \ge 0, z &\ge 0 \end{aligned}$$

where $e = (1, \ldots, 1)^T$.

In our problem, instead of separating just two sets, we need to separate n sets. Then the robust linear programing formulation is given by:

Find vectors $w_i, i=1,...,n$, $y_{ij}, i, j=1,...,n, i \neq j$, and γ to

Minimize $$\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{e^T \bar{y}_{ij}}{m_i} + \frac{e^T \bar{y}_{ji}}{m_j}$$

Subject to

$$\begin{aligned}
&-\mathbf{A}_i(w_i - w_j) + e(\gamma_i - \gamma_j) + e \leq y_{ij} \\
&\mathbf{A}_j(w_j - w_i) + e(\gamma_j - \gamma_i) + e \leq y_{ji} \\
&y_{ij} \geq 0, i, j = 1,...,n, i \neq j
\end{aligned}$$

where $m_i, i=1,...,n$ are the number of points in the sets.

The result is a set of hyperplanes that separate the sets in pairs. For two sets *i* and *j*, the equation of the separating hyperplane is given by $x^T(w_i - w_j) = \gamma_i - \gamma_j$. Thus, the boundary separating alternatives i and k is given by

$$B(i,k): x^T(w_i - w_k) = \gamma_i - \gamma_k,\ i = 1,2,...,n, i \neq k\colon, \tag{3.1}$$

Using the hyperplanes generated by the relation (3.1), we calculate the sphere of stability for the k^{th} alternative. The spheres of solution stability for the other alternatives, the ones dominated by the k^{th} alternative, may be used to identify the regions over which those dominated alternatives would become dominant and remain dominant.

In perturbation space, the *sphere of core stability* for the k^{th} alternative is centered at $\mathbf{0}$ = (0,…,0), the point at which no element in the supermatrix is perturbed away from its original value. The radius of that sphere is the distance from the origin to the nearest boundary. The distance from any point $\mathbf{x}$ to a hyperplane described by (3.1) is given by

$$d_{ij}(\mathbf{x}) = \frac{\left|\gamma_i - \gamma_j - \mathbf{x}^T(w_i - w_j)\right|}{\left\|w_i - w_j\right\|_2} \tag{3.2}$$

The region in perturbation space over which the k^{th} alternative is the most preferred alternative, $\boldsymbol{X}(k)$, is defined by

$$\mathbf{X}(k)=\left\{\mathbf{x}=(x_1,...,x_n)\in\mathbf{R}^n \mid -1\le x_i\le 1, \mathbf{x}^T(w_i-w_k)\le\gamma_i-\gamma_k, i=1,...,n, i\ne k\right\}$$

Thus, the minimum distance from $\mathbf{0}$ = (0, …, 0) to a boundary, beyond which the k^{th} alternative ceases to be the initially most preferred alternative, is the greatest radius of a hypersphere of perturbations, centered at $\mathbf{0}$ = (0, …, 0), within which the k^{th} alternative remains the most preferred alternative. The equation of the hypersphere is given by:

$$\mathbf{x}^T\mathbf{x}=\left(\min_{1\le i\le n, i\ne k}\left\{\frac{|\gamma_i-\gamma_k|}{\|w_i-w_k\|_2},1\right\}\right)^2 . \qquad (3.3)$$

Hence, as long as a perturbation $\mathbf{x}=(x_1,x_2,...,x_n)^T$ satisfies

$$\mathbf{x}^T\mathbf{x}\le\left(\min_{1\le i\le n, i\ne k}\left\{\frac{|\gamma_i-\gamma_k|}{\|w_i-w_k\|_2},1\right\}\right)^2$$

the k$^{\text{th}}$ alternative remains the most preferred alternative.

The *spheres of solution stability* for the dominated alternatives, that is, the regions in the perturbation space over which each of them become dominant and remain dominant, may be constructed using similar reasoning. For each alternative *i*, the center and the radius of the sphere of stability need to be determined. Because, by assumption, all of the boundaries of the partitions in perturbation space are linear, the problem of determining the spheres of stability for the dominated alternatives is known as the *Euclidean center problem* of a linear programming model (Dantzig and Thapa 1997). The spheres of stability are the largest spheres that can be inscribed in the regions $(h), h=1,2,\ldots,n, h\ne k$.

Hyperspheres may not be the largest volume stability regions that can be inscribed in the preference regions. In general, ellipsoids should capture more volume than spheres. To show this, consider the following convex polyhedron $P = \{x \mid a_i^T x \le b_i, i = 1,...,m\}$. An ellipsoid is defined by $E(d,B) = \{x \in \mathbb{R}^n \mid (x-d)^T B^{-2} (x-d) \le 1\}$, where d is the center of the ellipsoid and $B \in \mathbb{R}^{nxn}$ is a diagonal matrix with positive entries given by the radii of the ellipsoid, i.e., $\sum_{i=1}^{m} \left(\frac{x_i - d_i}{r_i}\right)^2 \le 1$. The maximal volume inscribed ellipsoid problem consists of finding the matrix B and the vector d, such that $E(d,B)$ has maximum volume and it is contained in the polyhedron P. Following the formulation by Boyd and Vandenberghe (2004), B and d may be found by solving the problem:

$$\begin{aligned} &Max\{\ln(\text{Det } B)\} \\ &s.t., \\ &\|Ba_i\|_2 + a_i^T d \le b_i,\ i = 1,2,...,m \end{aligned} \tag{3.4}$$

where $\|Ba_i\|_2 = \left[\lambda_{\max}(a_i^T B^T B a_i)\right]^{1/2}$ is the spectral norm of Ba_i, i.e., the principal eigenvalue of the matrix $(Ba)^T(Ba)$, and $\text{Det}\, B$ is the determinant of the matrix B (the volume of the ellipsoid).

Example: The example used in May et al. (2013) has the supermatrix given in Table 1.

Table 1. The supermatrix from May et al. (2013)

Goal		**Goal**				
		C1	**C2**	**A1**	**A2**	**A3**
	0	0	0	0	0	0
C1	0.2	0	0	0.07	0.35	0.63
C2	0.8	0	0	0.63	0.35	0.07
A1	0	0.5	0.1	0	0.18	0.12
A2	0	0.4	0.4	0.18	0	0.18
A3	0	0.1	0.5	0.12	0.12	0

The preference regions generated by perturbing criteria **C1** and **C2** have the corner points given in Table 2.

Table 2. Cartesian coordinates of the preference regions

	Cartesian coordinates (x,y)
Set A1	(1,0); (1,-1); (0.212,-1); (0.962,0)
Set A2	(1,0.05); (1,1); (0.25,1); (-1,0.375); (-1,-1); (0.212,-1)
Set A3	(0,1); (0,0.875); (-1,0.375); (-1,1)

Using the implementation of convex optimization in MATLAB by Grant and Boyd (2008), and Grant and Boyd (2013) we obtain the ellipsoids shown in Figure 1.

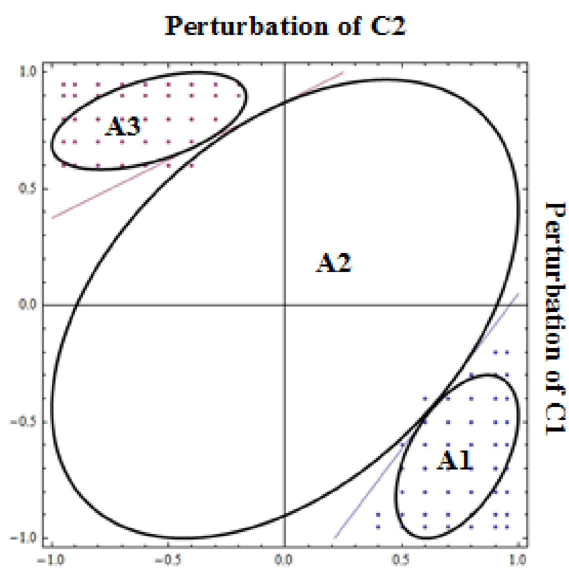


Figure 1. Core and solution stability ellipsoids within the perturbation space

Table 3 provides a comparison of the ellipsoids in Figure 1 with the spheres that were obtained in May et al. (2013). Table 3 shows that the ellipsoids capture about 50% more of the volume of the preference regions than the spheres.

Table 3. Volume of the maximum ellipsoids versus maximum spheres

Alternative	Ellipsoid center	Ellipsoid semi-axes	Volume of the inscribed ellipsoid	Volume of the inscribed sphere
Set A1	(0.7373; -0.6498)	a = 0.2063 b = 0.3860	0.2502	0.2164
Set A2	(0,-0.0152)	a = 0.7464 b = 1.1884	2.7869	1.8625
Set A3	(-0.5833,0.7917)	a = 0.1739 b = 0.432	0.2360	0.1789
Total			3.2731 (81.82%)	2.2578 (56.44%)

Given that the inscribed geometric shapes are intended to model, as closely as possible, the regions in which they are inscribed, the more volume captured, the better the model. As the example shows, ellipsoids are superior to spheres for measuring stability. The difficulty in applying the mathematical programming formulation (3.4) to the AHP/ANP problem, though, is that the preference regions are not always convex. Using a single hyperplane to separate adjacent pairs of regions, as was done in May et el. (2013), creates convex regions. Such a simple separation approach is accurate only when the true boundary is linear, and, as is shown in the next section, the true boundaries between adjacent preference regions are generally nonlinear. Thus, a more precise way of proceeding is to approximate the nonlinear boundaries by piecewise linear functions, which remain tractable because of their linearity, and reduce the approximation error as compared with the use of hyperplanes. As the number of dimensions within the space increases (e.g. $n = 3$), mesh generation methods, using triangulation, are needed to approximate the separating planes between the preference regions.

3.3. PIECEWISE LINEAR BOUNDARY APPROXIMATIONS - TWO-DIMENSIONAL CASE

Consider the network depicted in Figure 2. The supermatrix associated with that network is given by:

$$W = \begin{pmatrix} 0 & 0 & 0 \\ W_{21} & W_{22} & W_{23} \\ 0 & W_{32} & W_{33} \end{pmatrix}$$

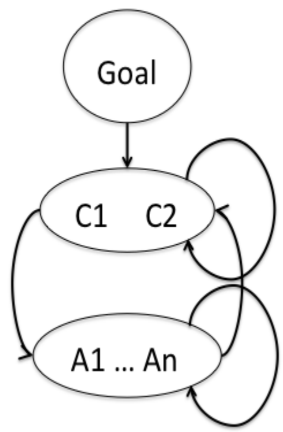


Figure 2. A network with two criteria and *n* alternatives

Because the matrix W is column stochastic, perturbations of it need to be made in such a way as to preserve both stochasticity and the proportionality of the unperturbed entries in a column. Perturbations are created by increasing or decreasing all of the entries of a row by the same percentage. Because all the entries in a supermatrix must be between zero and one, perturbations that increase a value are based on a percentage of the distance between the entry's original value and the distance to the upper bound of one. Perturbations that decrease a value are based on a percentage of the distance between the entry's original value and the distance to the lower bound of zero. That is, given an entry w_{ij}, a perturbation $-1 \le \delta_{ij} \le 1$ results in a new entry w'_{ij} given by

$$w'_{ij} = \begin{cases} w_{ij} + \delta_{ij} w_{ij} & if \;\; \delta_{ij} \le 0 \\ w_{ij} + \delta_{ij}(1 - w_{ij}) & if \;\; \delta_{ij} > 0 \end{cases} \quad (3.5)$$

The sets to be separated are created by selecting the points in the perturbation space, and then finding out which alternative has the greatest limiting priority, as given by the supermatrix. Let $\mathbf{X}(i)$ be the subset of the space of perturbations **X** for which the i^{th} alternative is ranked first.

Theorem 1. *The preference regions* $\{\mathbf{X}(i),\ i = 1, \ldots, n\}$ *are simply connected.*

<u>Proof</u>:

The space of perturbations **X** is the product space $[-1,1]^m$, where *m* is the number of elements perturbed. By definition, **X** is compact. In addition, it is simply connected, because each perturbation induces a supermatrix **W**. By Perron–Froebenius theory (Keener 1993), because **W** is a stochastic matrix, there always exists a real positive eigenvalue equal to 1 and a corresponding real nonnegative eigenvector *w*. Wilkinson (1965) showed that small continuous perturbations of the entries of **W** induce small continuous perturbations of its eigenvectors, and, specifically, of *w*. Thus, the space of perturbations is mapped, via a continuous function, into the space of priorities represented by the principal eigenvector of the supermatrix **W**. In addition, the limiting priorities obtained from the supermatrix **W** add to unity. Hence, the space of perturbations is mapped, via a continuous function, into the hyperplane $e^T w = 1$. Because the image of a simply connected space via a continuous function is also simply connected, the space of priorities resulting from the space of perturbations is also simply connected. In addition, it is also compact.

The space of perturbations $\mathbf{X}$ may be written as the union of subspaces $\{\mathbf{X}(i), i = 1, \ldots, n\}$. Thus, $\mathbf{X} = \bigcup_{i=1}^{n} \mathbf{X}(i)$, but not all the preference regions have non-null intersections. Because the space $\mathbf{X}$ is compact and a subset of $\mathbb{R}^n$, it is closed and bounded. The subspaces $\mathbf{X}(i)$ must also be closed and bounded, because, otherwise, there would be holes between them. If the $\mathbf{X}(i)$ were open sets instead of closed ones, then there would be holes in $\mathbf{X}$, and $\mathbf{X}$ would not be simply connected. Of course, the $\mathbf{X}(i)$'s could be open sets if two adjacent regions have a non-null intersection, i.e., they overlap. However, an overlap would imply that perturbations could yield non-unique priorities, i.e., there could be perturbations that would yield multiple dominating alternatives. That is not possible, unless the perturbations are exactly on the boundaries of the dominating alternatives, and, in such a case, the priorities would be the same as for the dominating alternatives. Thus, there are no holes in $\mathbf{X}$, and because it is compact and simply connected, all the $\mathbf{X}(i)$'s must also be compact and simply connected. Q.E.D.

Corollary: *The preference regions* $\{\mathbf{X}(i),\ i = 1, \ldots, n\}$ *are closed, in the topological sense.*

Proof: Follows from Theorem 1. Q.E.D.

Let $B(i, j)$ be the boundary separating $\mathbf{X}(i)$ and $\mathbf{X}(j)$. Let $\mathbf{w}(i)$ be the priority values of the i^{th} alternative corresponding to the perturbations in $\mathbf{X}(i)$. Because the regions are simply connected, the boundary between the regions $\mathbf{X}(i)$ and $\mathbf{X}(j)$ consists of perturbations points for which $\mathbf{w}(i) = \mathbf{w}(j)$. In addition, it is possible that a preference region could have multiple neighbors. Thus, if the preference regions $\mathbf{X}(i)$, $\mathbf{X}(j)$ and $\mathbf{X}(k)$ are neighbors, then the boundary $B(i,j,k)$ will consist of all the perturbations for which $\mathbf{w}(i) = \mathbf{w}(j) = \mathbf{w}(k)$.

Let $\Delta(i,j) \subseteq \mathbf{X}(i) \cap \mathbf{X}(j)$ be the set of perturbations that form the boundary between $\mathbf{X}(i)$ and $\mathbf{X}(j)$. The boundary between any pair of preference regions $\mathbf{X}(i)$ and $\mathbf{X}(j)$ is given by a function $f_{ij}(\delta) = 0$, $\delta \in \Delta(i,j)$. In the two-dimensional case, the boundary consists of points (δ_i, δ_j). A difficulty is that the equality of the priorities, i.e., $\mathbf{w}(i) = \mathbf{w}(j)$, is impossible in practice, due to the limited precision of numerical computation. Thus, the best one can do is to find the perturbations for which the difference between $\mathbf{w}(i)$ and $\mathbf{w}(j)$ is sufficiently small, that is, $|\mathbf{w}(i) - \mathbf{w}(j)| < \varepsilon$.

3.3.1. An algorithm to generate the piecewise linear approximations

To approximate $\Delta(i,j) \subseteq \mathbf{X}(i) \cap \mathbf{X}(j)$ with a piecewise linear function of order *m* (i.e., *m* segments), we proceed as follows:

(1) Select (*m*+*1*) points in $\Delta(i,j)$, $\{x_k = (\delta_{ik}, \delta_{jk}),\ k = 0,1,\dots,m\}$.

(2) Order the points selected $\{x_k = (\delta_{ik}, \delta_{jk}),\ k = 0,1,\dots,m\}$ according to the proximity of adjacent points, as measured by the Euclidean distance, i.e., find a point $x_{k+1} = (\delta_{i,k+1}, \delta_{j,k+1})$ that is the closest point to $x_k = (\delta_{ik}, \delta_{jk})$.

(3) Minimize the number of segments required, by determining if several points lie in a straight line:

a. Select the first three adjacent points and compute the area of the triangle formed by them. If they are in a straight line, then select the first and third point.

b. Select another adjacent point, and repeat step 3(a) until no more points are aligned for the segment being constructed.

(4) Construct the linear segments:

a. If the point $x_k = (\delta_{ik}, \delta_{jk})$ is the closest point to the point $x_h = (\delta_{ih}, \delta_{jh})$, then join them with a line:

$$a_j(h,k)\delta_i + b_i(h,k)\delta_j = c_{ij}(h,k)$$

$$a_j(h,k) = \delta_{j,k} - \delta_{j,h},\ b_i(h,k) = \delta_{i,h} - \delta_{i,k},\ c_{ij}(h,k) = \delta_{j,k}\delta_{i,k} - 2\delta_{j,k}\delta_{i,h} + \delta_{j,h}\delta_{i,h}.$$

b. Select the last point of the segment constructed in (4.a), and proceed as in (3) to build the next segment.

(5) Stop when the last point is reached.

The boundary $\Delta(i,j) \subseteq \mathrm{X}(i) \cap \mathrm{X}(j)$ is then described by a set of linear equations

$$B(i,j):\ \{a_j(h,k)\delta_i + b_i(h,k)\delta_j = c_{ij}(h,k), \delta_{ih} \le \delta_i \le \delta_{i,k}, h \ge 0, k \le m\} \qquad (3.6)$$

<u>Example</u>: Consider the network structure depicted in Figure 2 and the randomly generated supermatrix given in Table 4.

Table 4. A randomly generated supermatrix for a network with the structure of Figure 2

Goal		Goal							
		C1	**C2**	**A1**	**A2**	**A3**	**A4**	**A5**	**A6**
	0	0	0	0	0	0	0	0	0
C1	0.4831	0	0.0254	0.1359	0.1711	0.1616	0.1636	0.0821	0.2676
C2	0.5169	0.0984	0	0.2016	0.1523	0.1936	0.1782	0.3915	0.0126
A1	0	0.1236	0.0386	0	0.1279	0.0203	0.2929	0.3562	0.0101
A2	0	0.1297	0.1920	0.1643	0	0.1372	0.1059	0.0423	0.2323
A3	0	0.1528	0.1990	0.1896	0.0974	0	0.0052	0.0317	0.3322
A4	0	0.1522	0.2486	0.0543	0.1519	0.1695	0	0.0444	0.1109
A5	0	0.1864	0.1727	0.1485	0.1210	0.0572	0.1351	0	0.0343
A6	0	0.1569	0.1237	0.1060	0.1785	0.2606	0.1190	0.0517	0

The limiting priorities resulting from the supermatrix in Table 4 are given in Table 5:

Table 5. Priorities of the six alternatives from the supermatrix of Table 4

Alternative	Priority
A1	0.1588
A2	0.1751
A3	0.1762
A4	0.1667
A5	0.1497
A6	0.1732

Next, we perturb all the entries of rows 2 and 3 of the Table 4 supermatrix, using formula (3.5), with a perturbation increment $\alpha = 0.002$. The number of perturbation levels is $m = 999$ for each of the two criteria, so, for the given value of the increment, the matrix of perturbations has 998,001 rows. The perturbation regions for which each of the six alternatives dominate within the space $[-1,1]^2$ are portrayed in Figure 3.

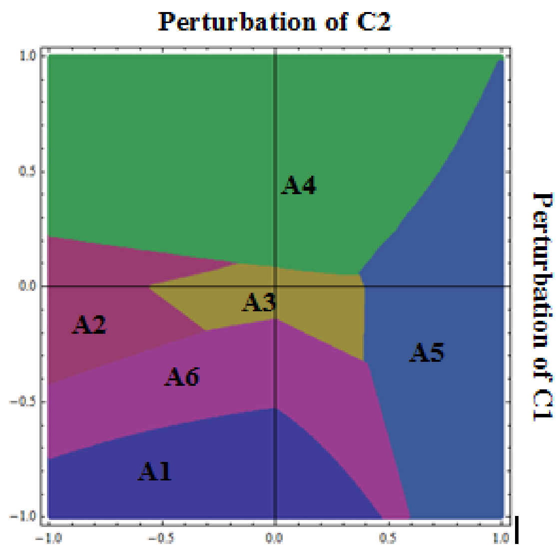


Figure 3. Dominance regions within the perturbation space

The perturbation space $[-1,1]^2$ is partitioned into six regions. Thus, we need to determine the piecewise linear boundaries between the following pairs of alternatives: A1-A6; A2-A3; A2-A4; A2-A6; A3-A4; A3-A5; A3-A6; A4-A5 and A5-A6. To obtain those boundaries, we only need the points classified as *boundary points*. Figure 4 shows the boundary points between the six regions obtained from the sample of 998,001 perturbations, using a cut-off value of $\varepsilon = 10^{-5}$, i.e., we consider that a point belongs to the boundary between two regions *i* and *j* if the priorities of the alternatives A_i and A_j satisfy $|\mathbf{w}(i) - \mathbf{w}(j)| < \varepsilon = 10^{-5}$.

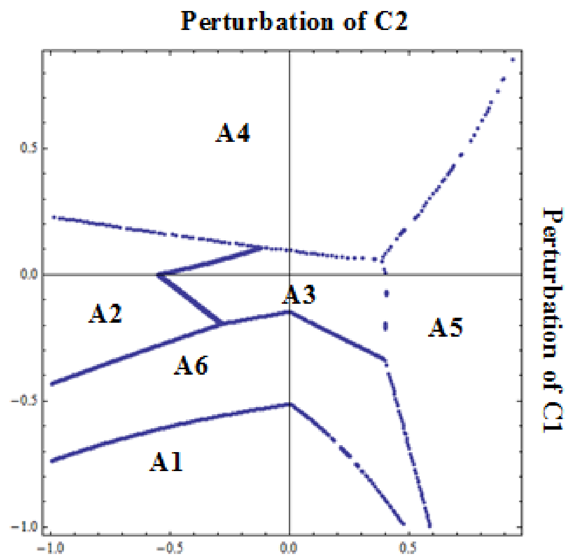


Figure 4. Boundary points between pairs of alternatives for $\varepsilon = 10^{-5}$

In addition to the boundary points, we need to find points that may be on the boundary of more than two regions. To find them, we consider all $\sum_{k=2}^{6} C_k^6 = 57$ possible combinations of three regions. We identified the following three-region intersection points: regions A3, A4 and A5, point (0.400; -0.128); regions A2, A3 and A4, point (-0.121; 0.108); regions A2, A3 and A6, point (-0.285; -0.195) and regions A3, A5 and A6, point (0.400; -0.342).

Next, in order to simplify the problem, rather than using all the points on the boundaries between two regions, we select boundary points using a grid of $\beta = 0.1$ over the perturbations of the first criterion (C1). The points obtained in such a fashion are used to construct the piecewise linear approximations. The ordering process begins from an intersection point, if one exists. Using the ordered points in pairs, we determine the piecewise linear approximation for the given boundaries following the algorithm described in section 3.3.1. The systems of inequalities describing each of the six regions for the example are given in *Appendix A*.

Figure 5 shows the approximated piecewise linear boundaries within the perturbation space.

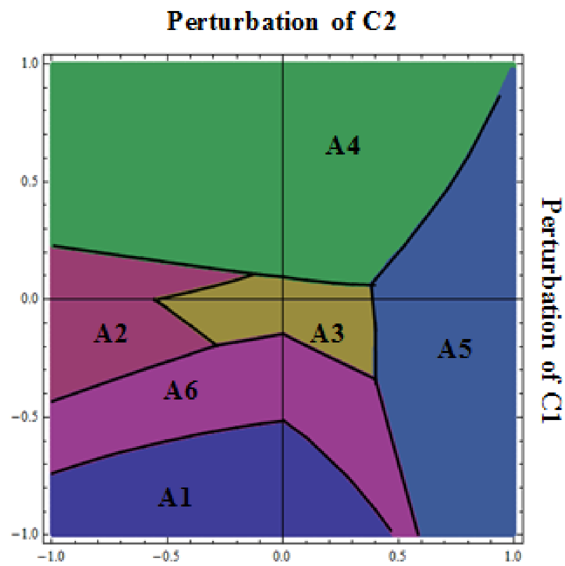


Figure 5. Approximated piecewise linear boundaries within the perturbation space

3.3.2. Goodness of fit

To determine how well the piecewise linear algorithm approximates the nonlinear boundaries between pairs of regions, we calculate the percentage of points misclassified by the system of linear inequalities describing each region. That is, we know from the 988,001 points generated in the $[-1,1]^2$, the actual region to which each point belongs. We compare those actual regions with the regions to which the points are assigned by the piecewise linear boundaries. The numbers of correctly classified points, and the number of misclassified points, are given in Table 6. Note that the number of misclassified points is less than 1% for each of the six regions. Thus, it appears that the piecewise linear algorithm provides a good approximation to the nonlinear boundaries between the preference regions.

Table 6. Goodness of fit – piecewise linear algorithm

Alternative	Total points (within the region)	Correct classified points (counts and %)	Misclassified points (counts and %)
A1	130,112	129,759 (99.729%)	353 (0.271%)
A2	80,539	80,401 (99.829%)	138 (0.171%)
A3	58,673	58,210 (99.211%)	463 (0.789%)
A4	379,268	378,967 (99.921%)	301 (0.079%)
A5	206,933	206,598 (99.838%)	335 (0.162%)
A6	142,476	142,372 (99.927%)	104 (0.073%)
TOTAL	**998,001**	**996,307 (99.83%)**	**1,694 (0.169%)**

3.4. STABILITY ELLIPSOIDS FOR NON-CONVEX SETS – TWO-DIMENSIONAL CASE

The problems of core and solution stability within the preference regions are of high importance when analyzing how sensitive the model results are to changes in their inputs. In Section 3.2, we showed that ellipsoids are superior to spheres when constructing the stability regions, because they capture more volume. The formulation by Boyd and Vandenberghe (2004) is applicable only to convex sets, but for the example in Figure 3, only three of the preference regions (A1, A4 and A5) are convex.

In this section we propose a nonlinear programming model for approximating the maximal volume ellipsoid that can be inscribed in a two-dimensional non-convex set. Given the set of boundary points $\Delta(i,j)$ and the system of inequalities describing the boundaries between each pair of regions, i.e. $B(i,j)$, we formulated the following model:

$$
\begin{cases}
max\ \pi\|a\|_2\|b\|_2 \\
s.t. \\
a^T b = 0 \\
a^T a - b^T b \geq 0 \\
\min_{1\leq i\leq m}\left\|p^i - f^1\right\| - \left\|p^i - f^2\right\| \geq 2\|a\| \\
\min_{1\leq j\leq n}\left(r_j x_A + s_j y_A \leq t_j\right) \\
\min_{1\leq j\leq n}\left(r_j x_{A'} + s_j y_{A'} \leq t_j\right) \\
\min_{1\leq j\leq n}\left(r_j x_B + s_j y_B \leq t_j\right) \\
\min_{1\leq j\leq n}\left(r_j x_{B'} + s_j y_{B'} \leq t_j\right) \\
min\left\{p_1^i\right\} \leq c_1, x_A, x_{A'}, x_B, x_{B'} \leq max\left\{p_1^i\right\} \\
min\left\{p_2^i\right\} \leq c_2, y_A, y_{A'}, y_B, y_{B'} \leq max\left\{p_2^i\right\} \\
-1 \leq a_1, a_2, b_1, b_2 \leq 1 \\
i = 1,2,\dots,m \\
j = 1,2,\dots,n
\end{cases}
\tag{3.7}
$$

where (c_1, c_2) are the coordinates of the ellipsoid center C; a_1, a_2 are the offsets for the semi-major axis a; b_1, b_2 are the offsets for the semi-minor axis b; $\|a\|_2$, $\|b\|_2$ are the Euclidean norms of the two semi-axes; $p^1, p^2, \dots, p^m$ are the boundary points describing the preference region, with $p^1 = (p_1^1, p_2^1)$; $f^1\left(x_{f1}, y_{f1}\right), f^2\left(x_{f2}, y_{f2}\right)$ the focal points of the ellipsoid; $A(x_A, y_A)$, $A'(x_{A'}, y_{A'})$, $B(x_B, y_B)$, $B'(x_{B'}, y_{B'})$ are the coordinates of the extreme points of the semi-major axis a and semi-minor axis b, respectively; and $r_j x + s_j y \leq t_j,\ \ u_j \leq x \leq v_j$ are the inequalities that define the boundaries.

<u>**Example**</u>: Using the example described previously, we determined the maximal volume ellipsoids that can be inscribed in the six preference regions identified. Three of the regions approximated in Figure 5, A1, A4 and A5, are convex. For those regions, ellipsoids may be determined using the formulation in (3.4). The other three approximated preference regions, A2, A3 and A6, are non-convex, for which we use the mathematical program (3.7). A representation of the stability ellipsoids is depicted in Figure 6.

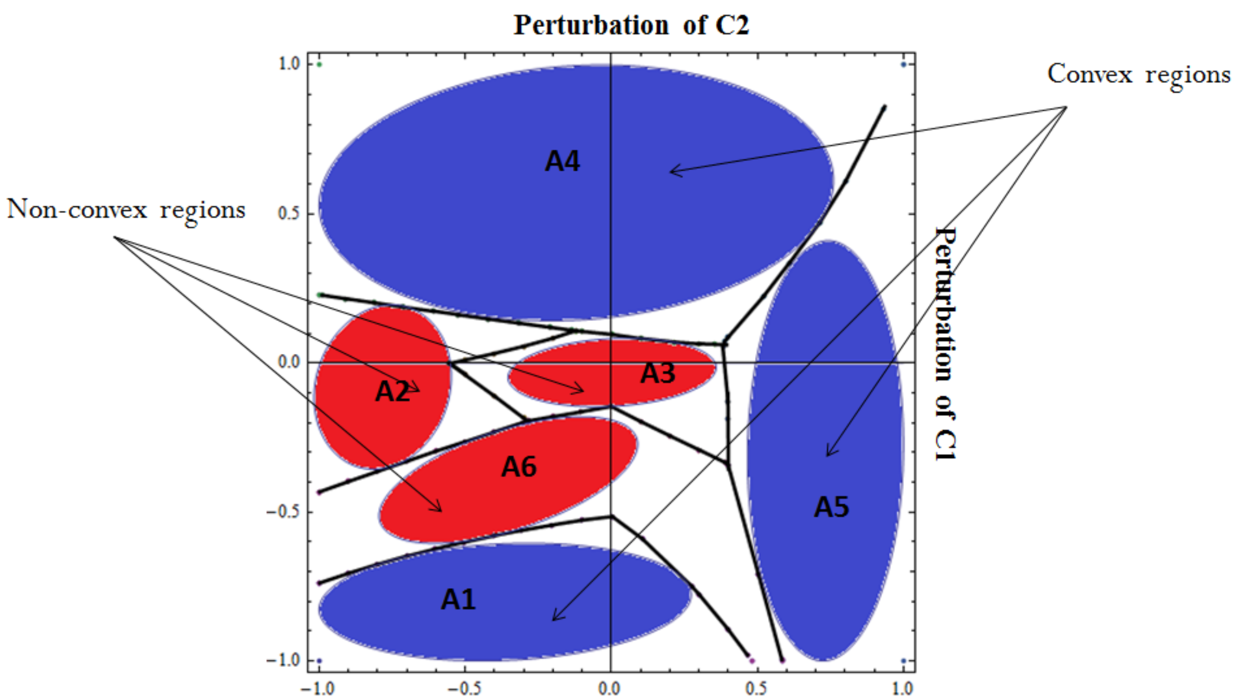


Figure 6. Solution stability ellipsoids within the perturbation space

3.5. TRIANGULAR MESH GENERATION FOR THREE-DIMENSIONAL BOUNDARY APPROXIMATIONS

As the number of criteria within the network increases, analyzing the sensitivity and stability of the solution obtained becomes more complex. In order to retain the notion of piecewise linear boundaries in spaces of dimension greater than two, we use mesh generation, particularly triangulation, to approximate the boundaries between the preference regions in a higher dimensional space ($\boldsymbol{n > 2}$). Mesh generation uses piecewise polygons or convex polyhedrons to approximate the geometric surfaces characterized by non-convex regions (Thompson et al., 1985; Edelsbrunner, 2001). To differentiate among the preference regions in a three-dimensional space, we approximate the separating boundaries using Delaunay's triangulation method, which ensures that the circumcircle associated with each triplet of points in the boundary contains no other boundary point in its interior (Delaunay, 1934; de Berg et al. 2008). The method gives us an unstructured grid, connecting triplets of the boundary points, which can be used to construct the set of inequalities describing the separating hyperplanes (Mavriplis, 1996).

Consider the network represented in Figure 7. The supermatrix associated with the network in Figure 7 is column stochastic, and the simultaneous perturbations with respect to all three criteria are made using relation (3.8)

$$w'_{ijk} = \begin{cases} w_{ijk} + \delta_{ijk} w_{ijk} & if \;\; \delta_{ijk} \le 0 \\ w_{ijk} + \delta_{ijk}(1 - w_{ijk}) & if \;\; \delta_{ijk} > 0 \end{cases} \tag{3.8}$$

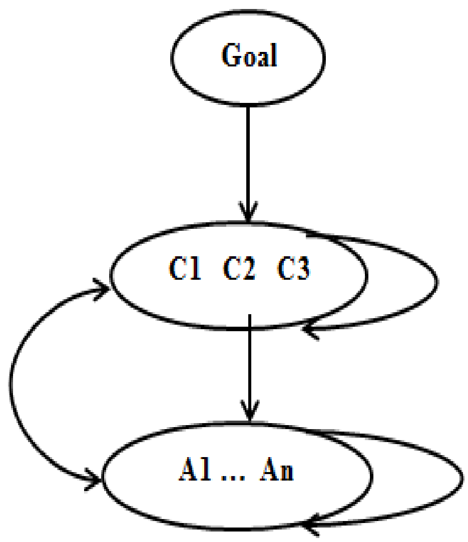


Figure 7. A network with three criteria and *n* alternatives

Similarly, as with the two-dimensional case, the sets to be separated are created by selecting only the boundary points in the three-dimensional perturbation space, and determining which alternative dominates by having the largest limiting priority.

3.5.1. An algorithm to generate the piecewise triangular approximation

Let $B(i,j)$ be the boundary between regions $\boldsymbol{X}(i)$ and $\boldsymbol{X}(j)$, and $\Delta(i,j) \subseteq \boldsymbol{X}(i) \cap \boldsymbol{X}(j)$ the set of perturbations for which $\boldsymbol{w}(i) = \boldsymbol{w}(j)$. For the three-dimensional case, the boundary $B(i,j)$ is given by the function $f_{ij}(\delta) = 0$, $\delta \in \Delta(i,j)$, with $\delta = (\delta_i, \delta_j, \delta_k)$. To approximate $\Delta(i,j) \subseteq \boldsymbol{X}(i) \cap \boldsymbol{X}(j)$ with a piecewise triangular mesh, we proceed as follows:

1. Select ($m+1$) points in $\Delta(i,j)$, with $\{x_h = (\delta_{ih}, \delta_{jh}, \delta_{kh}), h = 0,1,\dots,m\}$.
2. Determine the triplets of boundary points from $\Delta(i,j)$ which can form a triangle within the three-dimensional perturbation space, using the Delaunay triangulation algorithm as developed by de Berg et al. (2008)[1].
3. The output matrix M gives the number of possible combinations of vertices for the triangular mesh, based on the order of the points in the original input for Step (2):

   $$M = \begin{pmatrix} point\ no.\ x_{i1} & point\ no.\ x_{j2} & point\ no.\ x_{k3} \\ & \dots & \\ & \dots & \end{pmatrix},$$

   where $i = 1,\dots,m+1; j = 1,\dots,m; k = 1,\dots,m-1$

   and $M' = \begin{pmatrix} (\delta_{i1}, \delta_{j1}, \delta_{k1}) & (\delta_{i2}, \delta_{j2}, \delta_{k2}) & (\delta_{i3}, \delta_{j3}, \delta_{k3}) \\ & \dots & \\ & \dots & \end{pmatrix}$ is the matrix containing the corresponding Cartesian coordinates for each triplet.
4. Construct the separating hyperplane generated by any set of three vertices from M. We use the first point in M' for illustrative purposes. Let $x_1 = (\delta_{i1}, \delta_{j1}, \delta_{k1})$, $x_2 = (\delta_{i2}, \delta_{j2}, \delta_{k2})$ and $x_3 = (\delta_{i3}, \delta_{j3}, \delta_{k3})$,

[1] de Berg, M., Cheong, O., van Kreveld, M., Overmars, M., 2008. Computatioal geometry. Algorithm and applications. 3ed, Springer, pp. 199-208.

a. Determine the direction vectors $\overrightarrow{x_1x_2} = (\delta_{i2} - \delta_{i1})\hat{\imath} + (\delta_{j2} - \delta_{j1})\hat{\jmath} + (\delta_{k2} - \delta_{k1})\hat{k}$ and $\overrightarrow{x_1x_3} = (\delta_{i3} - \delta_{i1})\hat{\imath} + (\delta_{j3} - \delta_{j1})\hat{\jmath} + (\delta_{k3} - \delta_{k1})\hat{k}$.

b. Calculate the normal vector

$$\overrightarrow{x_1x_2} \times \overrightarrow{x_1x_3} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \delta_{i2} - \delta_{i1} & \delta_{j2} - \delta_{j1} & \delta_{k2} - \delta_{k1} \\ \delta_{i3} - \delta_{i1} & \delta_{j3} - \delta_{j1} & \delta_{k3} - \delta_{k1} \end{vmatrix}$$

c. The equation of the hyperplane is given by $ax + by + cz + d = 0$, where

$a = \begin{vmatrix} \delta_{j2} - \delta_{j1} & \delta_{k2} - \delta_{k1} \\ \delta_{j3} - \delta_{j1} & \delta_{k3} - \delta_{k1} \end{vmatrix}$, $b = -\begin{vmatrix} \delta_{k2} - \delta_{k1} & \delta_{i2} - \delta_{i1} \\ \delta_{k3} - \delta_{k1} & \delta_{i3} - \delta_{i1} \end{vmatrix}$,

$c = \begin{vmatrix} \delta_{i2} - \delta_{i1} & \delta_{j2} - \delta_{j1} \\ \delta_{i3} - \delta_{i1} & \delta_{j3} - \delta_{j1} \end{vmatrix}$, and $d = -(a\delta_{i1} + b\delta_{j1} + c\delta_{k1})$.

Set the conditions for any given point $x_h = (\delta_{ih}, \delta_{jh}, \delta_{kh})$ to be in the interior of the triangle $\Delta x_1x_2x_3$, using the barycentric coordinates approach developed by Shirley et al. (2009): $\alpha = \frac{|\overrightarrow{x_hx_2} \times \overrightarrow{x_hx_3}|}{|\overrightarrow{x_1x_2} \times \overrightarrow{x_1x_3}|}$, $\beta = \frac{|\overrightarrow{x_hx_3} \times \overrightarrow{x_hx_1}|}{|\overrightarrow{x_1x_2} \times \overrightarrow{x_1x_3}|}$, $\gamma = 1 - \alpha - \beta$ and $0 \leq \alpha, \beta, \gamma \leq 1$, where:

$$|\overrightarrow{x_1x_2} \times \overrightarrow{x_1x_3}| = det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \delta_{i2} - \delta_{i1} & \delta_{j2} - \delta_{j1} & \delta_{k2} - \delta_{k1} \\ \delta_{i3} - \delta_{i1} & \delta_{j3} - \delta_{j1} & \delta_{k3} - \delta_{k1} \end{vmatrix}$$

$$|\overrightarrow{x_hx_2} \times \overrightarrow{x_hx_3}| = det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \delta_{i2} - \delta_{ih} & \delta_{j2} - \delta_{jh} & \delta_{k2} - \delta_{kh} \\ \delta_{i3} - \delta_{ih} & \delta_{j3} - \delta_{jh} & \delta_{k3} - \delta_{kh} \end{vmatrix}$$

and $$|\overrightarrow{x_hx_3} \times \overrightarrow{x_hx_1}| = det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \delta_{i3} - \delta_{ih} & \delta_{j3} - \delta_{jh} & \delta_{k3} - \delta_{kh} \\ \delta_{i1} - \delta_{ih} & \delta_{j1} - \delta_{jh} & \delta_{k1} - \delta_{kh} \end{vmatrix}.$$

5. Stop when the last triplet from matrix *M* is reached.

The boundary $\Delta(i,j) \subseteq \boldsymbol{X}(i) \cap \boldsymbol{X}(j)$ is then described by a set of *n* planar equations, where *n* is the total number of triangles generated in Step (3), of the form:

$$B(i,j):\{a_k x + b_k y + c_k z + d_k = 0, k = 1, \dots, n\} \quad (3.9)$$

Example: Consider the network represented in Figure 7 and the associated randomly generated supermatrix from Table 7.

Table 7. A randomly generated supermatrix for a network with the structure of Figure 7

Goal		Goal					
		C1	**C2**	**C3**	**A1**	**A2**	**A3**
	0	0	0	0	0	0	0
C1	0.3	0	0	0	0.07	0.13	0.5
C2	0.4	0	0	0	0.5	0.07	0.13
C3	0.3	0	0	0	0.13	0.5	0.07
A1	0	0.5	0.1	0.4	0	0.18	0.12
A2	0	0.4	0.4	0.5	0.18	0	0.18
A3	0	0.1	0.5	0.1	0.12	0.12	0

The limiting priorities resulting from the supermatrix given in Table 7 are 0.3348 for alternative *A1*, 0.4125 for alternative *A2*, and 0.2516 for alternative *A3*. We perturb simultaneously all the entries of rows 2, 3 and 4 of the supermatrix in Table 7, using formula (3.8), with a perturbation increment $\alpha = 0.02$. The level of perturbation α was chosen so as to result in fewer than one million perturbed matrices. The number of perturbation levels is $m = 99$ for each of the three criteria perturbed, so the matrix of perturbations has 970,300 rows. Different perspectives of the perturbation regions for each of the three alternatives dominating within the perturbation space $[-1; 1]^3$ are represented in Figure 8 and Figure 9.

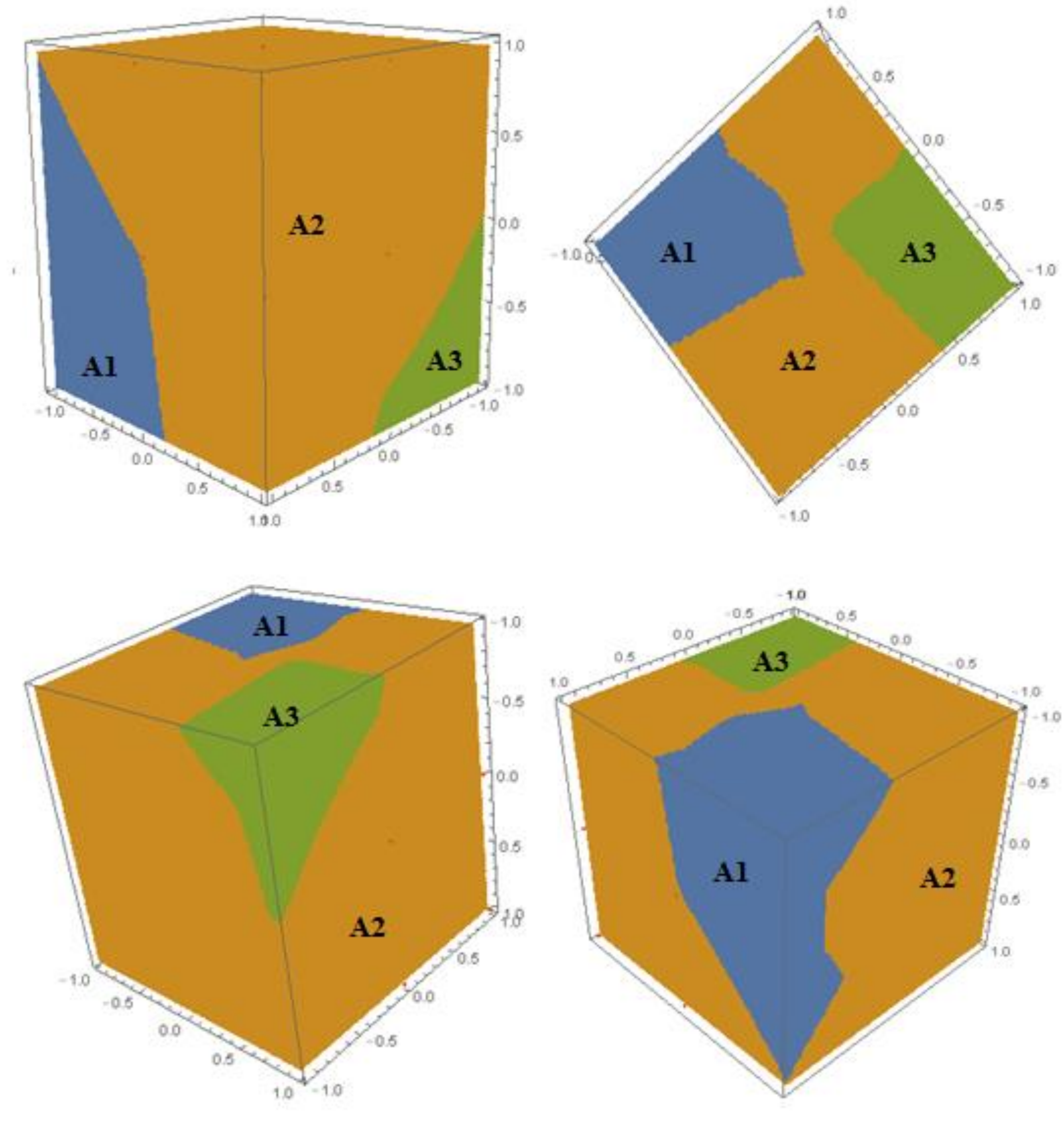


Figure 8. Preference regions within the three-dimensional perturbation space

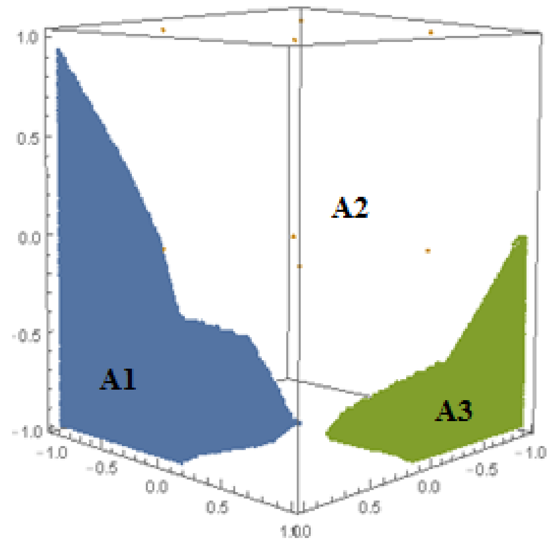


Figure 9. Preference regions within the three-dimensional perturbation space for the dominating alternatives

The three-dimensional perturbation space $[-1; 1]^3$ is partitioned into three regions, thus we need to determine the piecewise triangular meshes that approximate the nonlinear boundaries between regions A1-A2 and A2-A3, using only the boundary points between the pairs of regions. Figure 10 shows the boundary points used as input to the algorithm described in Section 3.5.1. The points were obtained from the initial sample of 970,300 perturbations, using, as the selection criterion, $|\boldsymbol{w}(i) - \boldsymbol{w}(j)| < \varepsilon = 10^{-4}$. To obtain "enough" points for generating good approximations of the triangular meshes we used a more lenient threshold for the three-dimensional case.

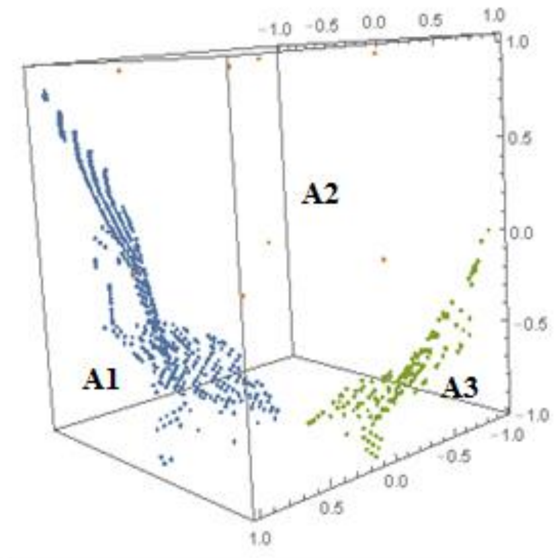


Figure 10. Boundary points between pairs of alternatives for $\boldsymbol{\varepsilon = 10^{-4}}$

Using the MATLAB functions implemented to generate the Delaunay triangulation, we determined the boundaries between the preference regions. Figure 11 shows multiple perspectives of the approximated piecewise triangular boundaries, within the three-dimensional perturbation space, between regions A1 and A2, and between A2 and A3, respectively. The combinations of boundary points generating the triangles within the piecewise triangular boundaries and the associated planar equations defining the regions over which each alternative dominates, are detailed in *Appendix B.*

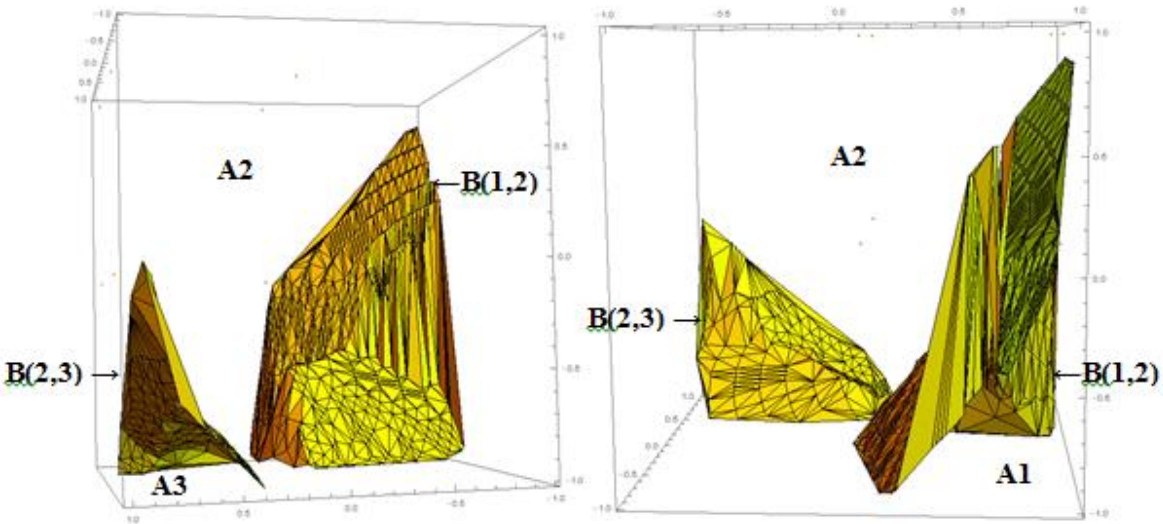


Figure 11. Approximated piecewise triangular boundaries within the three-dimensional perturbation space

3.6. STABILITY ELLIPSOIDS FOR NON-CONVEX SETS – THREE-DIMENSIONAL CASE

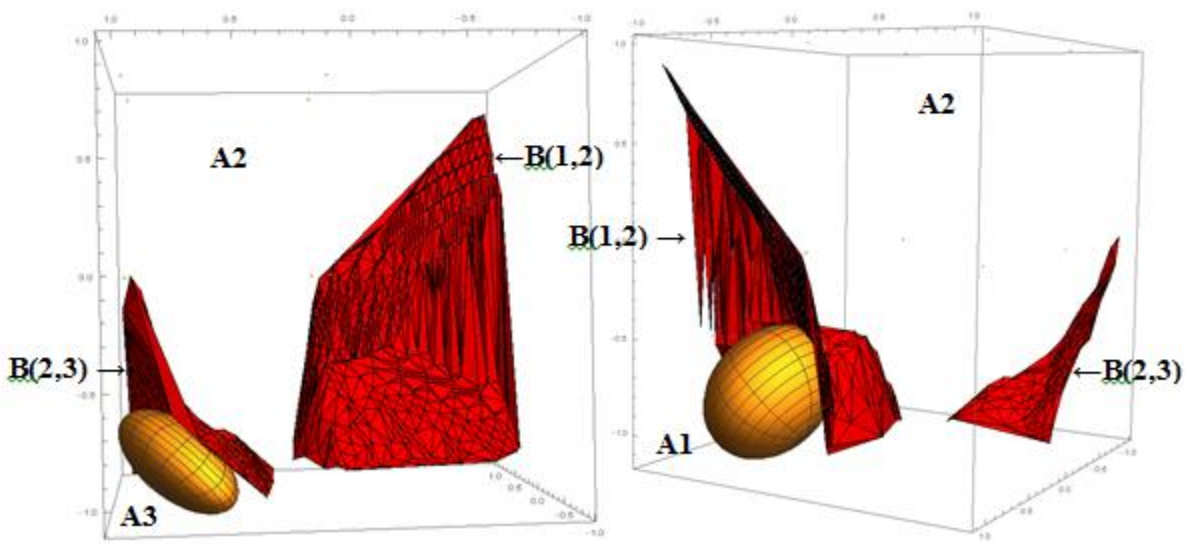
We need to define the maximal inscribed ellipsoid within the three-dimensional preference regions to determine how stable the initial solution is, and to describe the stability region of the dominated alternatives, as the complexity of the network increases. The complexity of the analysis increases as the number of the simultaneously perturbed criteria increases. In addition, the preference regions are non-convex. So, we propose an extension of the nonlinear programming model from section 3.4 to the three-dimensional case.

Given the set of boundary points $\Delta(i,j)$ separating two regions, and the matrix M' containing the corresponding Cartesian coordinates for each triplet describing the piecewise triangular boundary $B(i,j)$, we can formulate the following nonlinear programming model to approximate the maximal inscribed ellipsoid within a three-dimensional non-convex set:

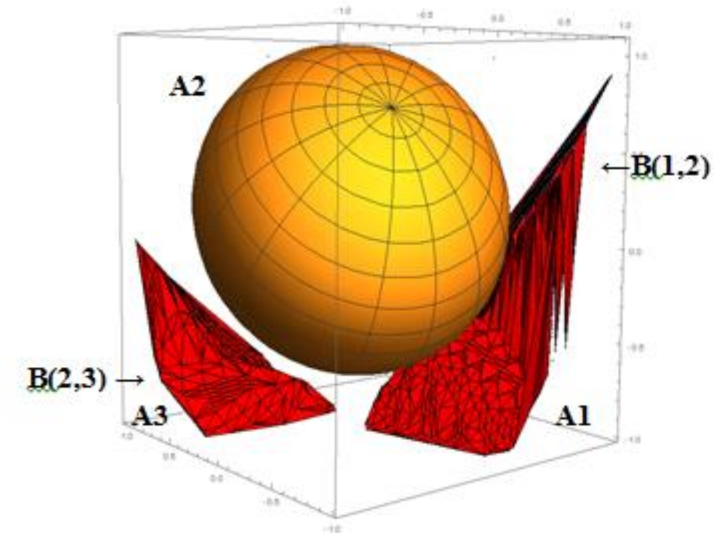
$$\begin{cases} max\ \frac{4}{3}\pi\|a\|_2\|b\|_2\|c\|_2 \\ s.t. \\ a^T b = 0 \\ a^T c = 0 \\ b^T c = 0 \\ a^T a - b^T b \geq 0 \\ b^T b - c^T c \geq 0 \\ \frac{\left(p_1^i - ce_1\right)^2}{\|a\|_2} + \frac{\left(p_2^i - ce_2\right)^2}{\|b\|_2} + \frac{\left(p_3^i - ce_3\right)^2}{\|c\|_2} \geq 1 \\ \bar{\alpha}^{1T} = (\bar{A}^T - \bar{O}^T){M'}^{-1} \geq 0 \\ \bar{\alpha}^{2T} = \left(\overline{A'}^T - \bar{O}^T\right){M'}^{-1} \geq 0 \\ \bar{\alpha}^{3T} = (\bar{B}^T - \bar{O}^T){M'}^{-1} \geq 0 \\ \bar{\alpha}^{4T} = \left(\overline{B'}^T - \bar{O}^T\right){M'}^{-1} \geq 0 \\ \bar{\alpha}^{5T} = (\bar{C}^T - \bar{O}^T){M'}^{-1} \geq 0 \\ \bar{\alpha}^{6T} = \left(\overline{C'}^T - \bar{O}^T\right){M'}^{-1} \geq 0 \\ \sum_{i=1}^{3} \alpha_i^k \leq 1 \\ -1 \leq ce_1, ce_2, ce_3 \leq 1 \\ -1 \leq a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \leq 1 \\ i = 1,2, \dots, m \\ j = 1,2, \dots, n \end{cases} \tag{3.10}$$

where (ce_1, ce_2, ce_3) are the coordinates of the ellipsoid center *C*; a_1, a_2, a_3 are the offsets for the semi-axis *a*; b_1, b_2, b_3 are the offsets for the semi-axis *b*; c_1, c_2, c_3 are the offsets for the semi-axis *c*, with $a \geq b \geq c$; $\|a\|_2$, $\|b\|_2$, $\|c\|_2$ are the Euclidean norms of the three semi-axes; $p^1, p^2, \dots, p^m$ are the boundary points describing the preference region, with $p^1 = (p_1^1, p_2^1, p_3^1)$; $A(x_A, y_A, z_A)$, $A'(x_{A'}, y_{A'}, z_{A'})$, $B(x_B, y_B, z_B)$, $B'(x_{B'}, y_{B'}, z_{B'})$, $C(x_C, y_C, z_C)$, $C'(x_{C'}, y_{C'}, z_{C'})$ are the coordinates of the extreme points of the semi-axes *a*, *b* and *c*, respectively; M' is the matrix with the Cartesian coordinates for each triplet describing the piecewise triangular boundary $B(i,j)$; $\bar{\alpha}^k$ is the vector of coefficients associated with the linear combination determined by each triplet from matrix M'; and $\bar{O}^T$ is an arbitrary point in the three-dimensional perturbation space, used to construct pyramids associated with each of the triangles approximating the boundary $B(i,j)$ to ensure that the ellipsoids were contained in the region described by the boundary.

Example: For the example in Section 3.5, we approximate the maximal inscribed ellipsoids within the non-convex preference regions, using the nonlinear optimization model (3.10). Figure 12 depicts the maximal stability ellipsoids for each of the three preference regions within the three-dimensional perturbation space.



a) Preference region A3 b) Preference region A1



c) Preference region A2

Figure 12. Solution stability ellipsoids within the three-dimensional perturbation space

3.7. MEASURES OF SENSITIVITY AND STABILITY

Interpreting the sensitivity and stability measures begins with the most preferred alternative in the initial study. The study of *core stability* tells us how the most preferred alternative remains most preferred after perturbations have taken place. We measure core stability using the largest sphere centered at the origin of the perturbation space, i.e., (0,0,0).

Definition 1: The ***Core stability*** of the most preferred alternative A_i is given by the maximum sphere that can be inscribed in the perturbation region, centered at 0^n. The dimensionality of the perturbation space $[-1;1]^n$ is given by the number of criteria perturbed simultaneously

Definition 2: The ***direction of change in stability*** of A_i is a vector of perturbations $(\delta_1, .., \delta_n)$, where *n* represents the number of criteria perturbed, for which the most preferred alternative A_i

is replaced by another alternative A_j. The distance from the origin is given by $\left[\sum_{i=1}^{n} {\delta_i}^2\right]^{1/2} = r$, where *r* is the radius of the core stability sphere.

Definition 3: The ***perturbation stability*** over a set of criteria $\left(C_i \ldots C_j\right)$ is given by the distance from the center of the core solution stability sphere for the most preferred alternative A_i to the closest boundary $B(i, j)$, i.e., the radius of the core stability sphere.

Combining the perturbation stability with the direction of change in stability, we obtain insight about the maximum change in the weights of the criteria that keeps the most preferred alternative invariant, i.e., still the most preferred. Alternatively, it could be viewed as the minimum change in the weights of the criteria that will make another alternative most preferred. Mathematically, the minimum change is given by *r*, the radius of the core solution stability sphere. The direction of change will determine which alternative becomes the most preferred one.

The preference regions in perturbation space are simply connected, closed, and not-necessarily-convex sets. Because we approximate the boundaries of the preference regions by piecewise linear functions, their areas, in the two-dimensional case, or volumes, in the three-dimensional case, can be obtained analytically, by adding the areas/volumes of all the triangles/pyramids that can be formed by points on the boundary of the regions. The relative area/volume of the preference regions could be used as a measure of solution stability. The relative volume of the stability ellipsoids generated is an approximation to the relative areas of the preference regions. Further, we determine the *solution stability* for all the alternatives dominating in the perturbation space, by using the largest ellipsoid that can be inscribed in the preference region.

Definition 4: The ***solution stability*** of an alternative A_i is given by the maximum volume of the inscribed ellipsoid in the associated preference region.

For the most preferred alternative A_i, the relation between the maximum core stability sphere and the maximum solution stability ellipsoid is a measure of the *level of stability* of a decision. The level of stability measure should be particularly useful in difficult decisions, e.g., medical decisions involving the selection of alternative treatments.

Definition 5: The ***level of solution stability*** of the most preferred alternative A_i is determined by the ratio of the volume of the core stability sphere to the volume of the stability ellipsoid of the most preferred alternative.

The current sensitivity and stability analysis is developed at the level of perturbing simultaneously pairs and triplets of criteria. Given this restrictive setting, an analysis of the stability evolution on sets of criteria is necessary to understand how perturbations in the model inputs change the most preferred solution or make the solution invariant.

Definition 6: An alternative A_i is ***pairwise more stable*** than another alternative A_j *with respect to a set of criteria* if the stability ellipsoid of A_i is larger than that of A_j.

Definition 7: An alternative is ***more stable*** than all other alternatives *with respect to a set of criteria* if the alternative is pairwise more stable than all other alternatives.

Definition 8: An alternative is ***the most stable alternative among a set of alternatives*** if it is more stable than all other alternatives *with respect to a set of criteria.*

The theoretical guidelines described in this section can be a starting point for any sensitivity and stability analysis performed on practical multi-criteria decision making models.

4.0 AN ILLUSTRATION OF SENSITIVITY AND STABILITY ANALYSIS IN MEDICAL DECISION MAKING

4.1. INTRODUCTION

Medical decision making is an important research area for both patients and providers. How patients make medical decisions, given their limited knowledge, and how providers can learn from the decisions patients make, and generalize their findings to groups of patients with similar characteristics, are two questions that still need to be answered. In this chapter, we apply the methodology developed in Chapter 3 to the colorectal cancer screening problem, showing how the decision process evolves from simultaneous changes in two criteria to changes in three criteria, and from a single patient to groups of patients.

4.2. SINGLE PATIENT STABILITY ANALYSIS FOR COLORECTAL CANCER SCREENING – PAIRS OF CRITERIA

Colorectal cancer is one of the leading causes of mortality among cancer patients in the United States. It is also one of the most preventable forms of cancer. The current colorectal cancer screening guidelines recommend multiple screening options for patients who are classified as having an average risk for colorectal cancer (U.S. Preventive Service Task Force

2008). Currently, there are ten screening options from which patients and healthcare providers may choose (Table 8). The alternatives available are either invasive or non-invasive procedures, or a combination of the two types.

Table 8. Screening options for colorectal cancer

Screening options		
Annual Fecal Occult Blood Test with sensitivity 20%	**A1**	Non-invasive
Annual Fecal Occult Blood Test with sensitivity 40%	**A2**	Non-invasive
Flexible Sigmoidoscopy every 5 years	**A3**	Invasive
Fecal DNA test every 5 years	**A4**	Non-invasive
Annual immunochemical fecal occult blood test	**A5**	Non-invasive
Annual Fecal Occult Blood Test and flexible Sigmoidoscopy every 5 years	**A6**	Both
CT colonography	**A7**	Non-invasive
Double contrast barium enema	**A8**	Invasive
Annual Immunochemical Fecal Occult Blood Test and flexible Sigmoidoscopy every 5 years	**A9**	Both
Colonoscopy every 10 years	**A10**	Invasive

Dolan et al. (2013) proposed an Analytic Hierarchy Process (AHP)–based model (see Figure 13) to ascertain patients' preferences among the ten alternatives for colorectal cancer screening. Participants used the standard AHP pairwise comparison method to compare the screening options, and to judge the relative priorities of the criteria involved in the decision. Data were collected from primary healthcare practices in Rochester, NY, Birmingham, AL, and Indianapolis, IN. From the total sample of 484 patients reported in the paper, we used 395 patients for the purposes of our analysis. The 395 patients considered had complete demographic information and judgments elicitation. Because selection of screening alternatives should be a decision made between the patient and the healthcare provider, we illustrate the applicability of our methodology by analyzing first the sensitivity and the stability of the initial selection of a single patient. The results of the analysis provide information to the healthcare provider about the patient's preferences over the set of screening alternatives and how fast those preferences could change, if the patient is presented with additional information about the criteria used to discriminate among the alternatives.

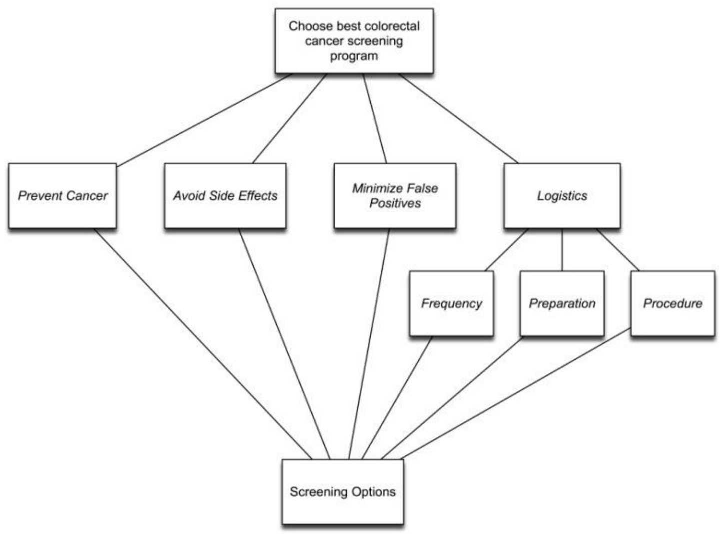


Figure 13. The AHP model for ascertaining patient preferences[2]

Consider a 70 year old male patient who requires screening. He is asked to state his preferences regarding the ten colorectal cancer screening options, based on the model presented in Figure 13. His judgments, with respect to the six criteria (Table 9) and ten alternatives, are summarized in the supermatrix presented in Table 10.

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[2] Dolan, J.G., Boohaker, E., Allison, J., Imperiale, T.F. (2013). Can streamlined multi-criteria decision analysis be used to implement shared decision making for colorectal cancer screening? Medical Decision Making, 34(6).

Table 9. The six criteria considered in the AHP-based model

	Criteria
C1	Prevent Cancer
C2	Avoid Side Effects
C3	Minimize False Positives
C4	Procedure Frequency
C5	Procedure Preparation
C6	Procedure Complexity

Table 10. The supermatrix for the 70 year-old male patient

Goal		**Goal**															
		C1	**C2**	**C3**	**C4**	**C5**	**C6**	**A1**	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**	**A9**	**A10**
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C1	**0.6890**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C2	**0.2060**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C3	**0.0770**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C4	**0.0015**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C5	**0.0220**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C6	**0.0054**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A1	0	0	0.0249	0.0171	0.0292	0.0224	0.1310	1	0	0	0	0	0	0	0	0	0
A2	0	0.0005	0.0015	0.0171	0.0794	0.0535	0.0064	0	1	0	0	0	0	0	0	0	0
A3	0	0.0013	0.6675	0.3910	0.6696	0.0930	0.0167	0	0	1	0	0	0	0	0	0	0
A4	0	0.0042	0.0001	0	0.2216	0.2236	0.0449	0	0	0	1	0	0	0	0	0	0
A5	0	0.0058	0.0104	0.0036	0	0.6072	0.6049	0	0	0	0	1	0	0	0	0	0
A6	0	0.0254	0.0005	0.0002	0	0	0.1405	0	0	0	0	0	1	0	0	0	0
A7	0	0.0254	0.2150	0.0669	0	0	0.0129	0	0	0	0	0	0	1	0	0	0
A8	0	0.0744	0.0763	0.1113	0	0	0.0422	0	0	0	0	0	0	0	1	0	0
A9	0	0.2029	0.0032	0.0012	0	0	0	0	0	0	0	0	0	0	0	1	0
A10	0	0.6597	0	0.3910	0	0	0	0	0	0	0	0	0	0	0	0	1

The limiting priorities corresponding to the supermatrix from Table 10, and the ranking of the ten screening options, are given in Table 11.

Table 11. Priorities and ranking of the ten screening alternatives

Ranking	Alternative		Priority
1	**Colonoscopy every 10 years**	**A10**	**0.4842**
2	Flexible Sigmoidoscopy every 5 years	**A3**	0.1715
3	Annual Immunochemical Fecal Occult Blood Test and flexible Sigmoidoscopy every 5 years	**A9**	0.1404
4	Double contrast barium enema	**A8**	0.0757
5	CT colonography	**A7**	0.0669
6	Annual immunochemical fecal occult blood test	**A5**	0.0230
7	Annual Fecal Occult Blood Test and flexible Sigmoidoscopy every 5 years	**A6**	0.0184
8	Fecal DNA test every 5 years	**A4**	0.0084
9	Annual Fecal Occult Blood Test with sensitivity 20%	**A1**	0.0077
10	Annual Fecal Occult Blood Test with sensitivity 40%	**A2**	0.0033

According to the priorities in Table 11, the alternative most preferred by the patient, given his current preferences, is Colonoscopy every 10 years (A10). The two most important criteria for the patient are Prevent Cancer (C1) and Avoid Side Effects (C2). They capture about 89% of the criteria weights. We would like to know:

1. *Is the most preferred alternative also the most stable one?*
2. *Which criteria are most sensitive to input changes?*

To answer these questions, we considered all pairs of criteria - 15 possibilities, based on the six criteria associated with the model presented in Figure 13. We then perturbed simultaneously both criteria in the pair, using a 0.002 perturbation increment. The perturbation threshold was chosen so as to provide “enough” sampling data for the sensitivity and stability analysis. Using these data we calculated: (a) the volume of each preference region present in the perturbation space; (b) the volume of the core stability sphere around the origin; and (c) the volumes of the maximal solution ellipsoids. In addition, we identified the sets of perturbations that would lead to a change in preference away from A10. The results for the 70 years old male patient are displayed in Table 12. A graphical representation of the preference regions in the perturbation space, the core stability sphere, and the solution stability ellipsoids can be seen in *Appendix C.*

Table 12. Results of the new sensitivity and stability analysis for the patient

Criteria pair	Preferred alternatives (region volume and % total perturbation space)			Stability sphere radius*	Stability sphere (volume and % preference region)	Direction of fastest change (perturbation values)		Direction of fastest change (new criteria weights)**		New preferred alternative	Stability ellipsoid (volume and % preference region)			Ratio core stability sphere/solution stability ellipsoid
	A3	**A5**	**A10**		**A10**	**pi**	**pj**	**Ci**	**Cj**		**A3**	**A5**	**A10**	**A10**
					Original criteria weights (C1-C2)* **			**0.6890**	**0.2060**					
C1-C2	1.2105 (30.26%)	-	2.7894 (69.74%)	0.4511	0.6393 (22.92%)	-0.304	0.333	0.4541	0.4454	**A3**	0.5602 (46.28%)	-	2.0931 (75.04%)	30.54%
					Original criteria weights (C1-C3)			**0.6890**	**0.0770**					
C1-C3	0.9840 (24.60%)	-	3.0160 (75.40%)	0.3607	0.4087 (13.55%)	-0.360	-0.024	0.5871	0.1001	**A3**	0.7103 (72.18%)	-	2.0576 (68.22%)	19.86%
					Original criteria weights (C1-C4)			**0.6890**	**0.0015**					
C1-C4	1.5750 (39.38%)	-	2.4250 (60.63%)	0.3939	0.4874 (20.10%)	-0.380	0.104	0.5067	0.1250	**A3**	0.9424 (60.27%)	-	1.9191 (79.14%)	25.40%
					Original criteria weights (C1-C5)			**0.6890**	**0.0220**					
C1-C5	0.8210 (20.53%)	0.6112 (15.28%)	2.5676 (64.19%)	0.4173	0.5471 (21.31%)	-0.416	0.034	0.5382	0.0739	**A3**	0.6529 (79.52%)	0.3730 (61.03%)	1.9915 (77.56%)	27.47%
					Original criteria weights (C1-C6)			**0.6890**	**0.0054**					
C1-C6	0.7849 (19.62%)	0.6224 (15.56%)	2.5926 (64.82%)	0.4100	0.5281 (20.37%)	-0.410	0.006	0.5611	0.0158	**A3**	0.5822 (74.18%)	0.3882 (62.37%)	2.0158 (77.75%)	26.20%
					Original criteria weights (C2-C3)			**0.2060**	**0.0770**					
C2-C3	1.8190 (45.48%)	-	2.1809 (54.52%)	0.2649	0.2205 (10.11%)	0.232	0.128	0.2994	0.1497	**A3**	1.0851 (59.65%)	-	1.5763 (72.28%)	13.99%
					Original criteria weights (C2-C4)			**0.2060**	**0.0015**					
C2-C4	2.2841 (57.10%)	-	1.7158 (42.90%)	0.1980	0.1232 (7.18%)	0.122	0.156	0.2416	0.1255	**A3**	1.0681 (46.76%)	-	1.3079 (76.23%)	9.42%
					Original criteria weights (C2-C5)			**0.2060**	**0.0220**					
C2-C5	1.3428 (33.57%)	0.8249 (20.62%)	1.8322 (45.81%)	0.2800	0.2463 (13.44%)	0.316	-0.114	0.3657	0.0156	**A3**	1.0044 (74.80%)	0.5782 (70.09%)	1.4703 (80.25%)	16.75%
					Original criteria weights (C2-C6)			**0.2060**	**0.0054**					
C2-C6	1.2791 (31.98%)	0.8841 (22.10%)	1.8367 (45.92%)	0.2733	0.2347 (12.78%)	0.236	0.138	0.2968	0.1076	**A3**	0.9532 (74.52%)	0.6208 (70.22%)	1.4627 (79.64%)	16.04%
					Original criteria weights (C3-C4)			**0.0770**	**0.0015**					
C3-C4	1.5221 (38.05%)	-	2.4778 (61.95%)	0.2960	0.2753 (11.11%)	0.090	0.282	0.1172	0.2073	**A3**	1.0461 (68.73%)	-	2.0007 (81.00%)	13.71%
					Original criteria weights (C3-C5)			**0.0770**	**0.0220**					
C3-C5	0.2047 (5.12%)	1.0773 (26.93%)	2.7179 (67.95%)	0.4267	0.5720 (21.05%)	0.048	0.424	0.0831	0.2991	**A5**	0.1239 (60.53%)	0.7499 (69.61%)	2.1493 (79.08%)	26.61%
					Original criteria weights (C3-C6)			**0.0770**	**0.0054**					
C3-C6	0.1237 (3.09%)	1.1064 (27.66%)	2.7698 (69.25%)	0.4287	0.5774 (20.85%)	0.026	0.428	0.0692	0.2953	**A5**	0.0838 (67.74%)	0.8799 (79.53%)	2.2345 (80.67%)	25.84%
					Original criteria weights (C4-C5)			**0.0015**	**0.0220**					
C4-C5	1.2672 (31.68%)	0.8883 (22.21%)	1.8443 (46.11%)	0.2918	0.2675 (14.50%)	0.266	0.120	0.1830	0.0955	**A3**	0.9011 (71.11%)	0.6851 (77.12%)	1.4716 (79.79%)	18.18%
					Original criteria weights (C4-C6)			**0.0015**	**0.0054**					
C4-C6	1.2142 (30.36%)	0.9347 (23.37%)	1.8509 (46.27%)	0.3010	0.2846 (15.38%)	0.286	0.094	0.1966	0.0678	**A3**	0.9048 (74.52%)	0.7296 (78.06%)	1.4698 (79.41%)	19.37%
					Original criteria weights (C5-C6)* **			**0.0220**	**0.0054**					
C5-C6	-	2.0421 (51.05%)	1.9578 (48.95%)	0.3012	0.2850 (14.56%)	0.213	0.213	0.1577	0.1488	**A5**	-	0.8733 (42.76%)	1.6054 (82.00%)	17.75%
Min	**0.1237**	**0.6112**	**1.7158**	**0.1980**	**0.1232**						**0.0838**	**0.3730**	**1.3079**	
Min (%)	**3.09%**	**15.28%**	**42.90%**		**7.18%**						**46.28%**	**42.76%**	**68.22%**	**9.42%**
Max	**2.2841**	**2.0421**	**3.0160**	**0.4511**	**0.6393**						**1.0851**	**0.8799**	**2.2345**	
Max (%)	**57.10%**	**51.05%**	**75.40%**		**22.92%**						**79.52%**	**79.53%**	**82.00%**	**30.54%**

*Calculated from the origin (0,0) where the most preferred alternative is A10
**The new criteria weights after normalization
***When two or more vectors of perturbations are at tie, the average was reported

Consider the first row of Table 12. We first selected the pair of criteria that are perturbed simultaneously – Prevent Cancer (C1) and Avoid Side Effects (C2). We then identified the resulting preference regions for all alternatives that are most preferred across all perturbations of the two selected criteria. For perturbations of C1 and C2, only Flexible Sigmoidoscopy every 5 years (A3) and Colonoscopy every 10 years (A10) would be selected. We approximated the areas of the two associated regions as a percentage of the total perturbation space. The total perturbation space has an area of four because the perturbation space is defined as $[-1; 1]^2$. For the pair of criteria Prevent Cancer (C1) and Avoid Side Effects (C2), the screening alternative Flexible Sigmoidoscopy every 5 years (A3) covers approximately 30% of the perturbation space, while the other dominating alternative, Colonoscopy every 10 years (A10), dominates over a region that is twice as large.

Next, we estimated the volume of the core stability sphere, and its relative volume with respect to the preference region of the initial most preferred alternative – in our case, Colonoscopy every 10 years (A10).

To determine the direction of the shortest-length vector that results in a switch in the most preferred alternative, i.e., from A10 to A3, we identified the associated perturbation pair (-0.304, 0.333) and calculated the new adjusted criteria weights (0.4541, 0.4454). We also determined what would be the "new" most preferred alternative, A3, based on the minimum change necessary in the criteria weights, as given by the perturbation pair.

We were also interested in approximating the preference regions using a regular shape, by inscribing the maximal ellipsoid within those regions. For the first pair of criteria, only two screening alternatives dominate. We calculated the volume of the ellipsoids, and their relative measures, with respect to the associated preference regions. Note that the stability ellipsoid for the screening alternative Colonoscopy every 10 years (A10) captures approximately 75% of its preference region, while the stability ellipsoid for Flexible Sigmoidoscopy every 5 years (A3) covers only 46% of its region. A reason for the reduced coverage for A3 is that certain preference regions are non-convex, which makes it more difficult to capture as much of their area using a regular shape. The maximal inscribed ellipsoid is the best approximation of the interior of a convex hull that we have available at this point. We do not have a metric that assesses the goodness-of-fit of the ellipsoid approximation, but future research might analyze how the sensitivity and stability analysis is affected by the quality of the ellipsoid approximation.

One last measure, displayed in Table 12, is the ratio between the stability sphere and the stability ellipsoid for the most preferred alternative - Colonoscopy every 10 years (A10). This ratio will be used to characterize the level of solution stability.

Overall, the results for the 70 years old patient, summarized in Table 12, indicate that only three of the ten possible screening alternatives could ever dominate - Flexible Sigmoidoscopy every 5 years (A3), Annual Immunochemical Fecal Occult Blood Test (A5) and Colonoscopy every 10 years (A10). For half of the pairs of criteria considered, only the screening alternatives Flexible Sigmoidoscopy every 5 years (A3) and Colonoscopy every 10 years (A10) appear in the perturbation space. For the other half, which includes either Procedure Preparation (C5) or Procedure Complexity (C6) in the pair of criteria, a third screening alternative appears – the Annual Immunochemical Fecal Occult Blood Test (A5).

The volume of the stability sphere varies between a minimum of 0.1232 and a maximum of 0.6393, providing information about the region in which the initially most preferred alternative remains the most preferred one, despite the perturbations applied to the criteria. For all pairs considered, only one vector of perturbations determines the fastest switch in preference.

If the minimum perturbation determining a change in preferences is applied, the patient most frequently switches from the initially most preferred screening alternative, Colonoscopy every 10 years (A10), to Flexible Sigmoidoscopy every 5 years (A3). Exceptions are three pairs of criteria: Minimize False Positive (C3) - Procedure Preparation (C5); Minimize False Positive (C3) - Procedure Complexity (C6); and Procedure Preparation (C5) - Procedure Complexity (C6), for which the increase in the importance of the criteria causes the patient to switch to a non-invasive screening procedure, Annual Immunochemical Fecal Occult Blood Test (A5). The change in preferences may be accomplished by different combinations of perturbations applied to the criteria weights, namely, (1) one criterion is increased while the second one is decreased; (2) both criteria are decreased; or (3) both criteria are increased simultaneously.

For the preference regions identified for each pair of criteria, we constructed the stability ellipsoids as the maximal regular shape that can be inscribed in the preference region space. The maximal ellipsoid captures between 68% and 82% of the region associated with the screening alternative Colonoscopy every 10 years (A10). The maximal ellipsoid approximation captures less of the region for the other two alternatives – between 46% and 79% for Flexible Sigmoidoscopy every 5 years (A3) and between 42% and 79% for Annual Immunochemical Fecal Occult Blood Test (A5). The decrease in the proportion captured is most likely due to the non-convexity of those preference regions. Note that the stability sphere may capture as little as 9% of the ellipsoid and as much as 30%. This means that the stability sphere captures a similar percentage of the preference region, so that the maximal ellipsoid may be used to approximate the preference region when sampling the perturbation space becomes difficult. We can conclude that the maximal stability ellipsoid may be considered a good approximation of the preference region when increasing the dimensionality of the problem. Smaller values of the ratio would have led to an opposite conclusion.

The analysis of the sensitivity and stability of the initial solution begins with the identification of the ***core stability***, meaning the largest sphere that can be inscribed around the origin of the perturbation space in which the initial solution lies. Using the results from Table 12, the initially most preferred screening option, Colonoscopy every 10 years (A10), is *more stable* with respect to the pair of criteria Prevent Cancer (C1) and Avoid Side Effects (C2), where the core stability sphere has the greatest volume, and it is *less stable* with respect to the pair of criteria Avoid Side Effects (C2) and Procedure Frequency (C4), where the core stability sphere has the smallest volume. For all other pairs of criteria considered in the analysis, the area of the core stability sphere is between these two extreme values. Comparing the volume of the core stability sphere in the two extreme situations with the volume of the associated preference region, the ratio of the two measures indicates that the core stability measure captures more of the preference region for the first pair of criteria, Prevent Cancer (C1) - Avoid Side Effects (C2), and less of the preference region for the second pair of criteria, Avoid Side Effects (C2) - Procedure Frequency (C4).

Plots of the core stability spheres and of the solution stability ellipsoids for the pairs of criteria Prevent Cancer (C1) - Avoid Side Effects (C2) and Avoid Side Effects (C2) - Procedure Frequency (C4) are depicted in Figure 14.

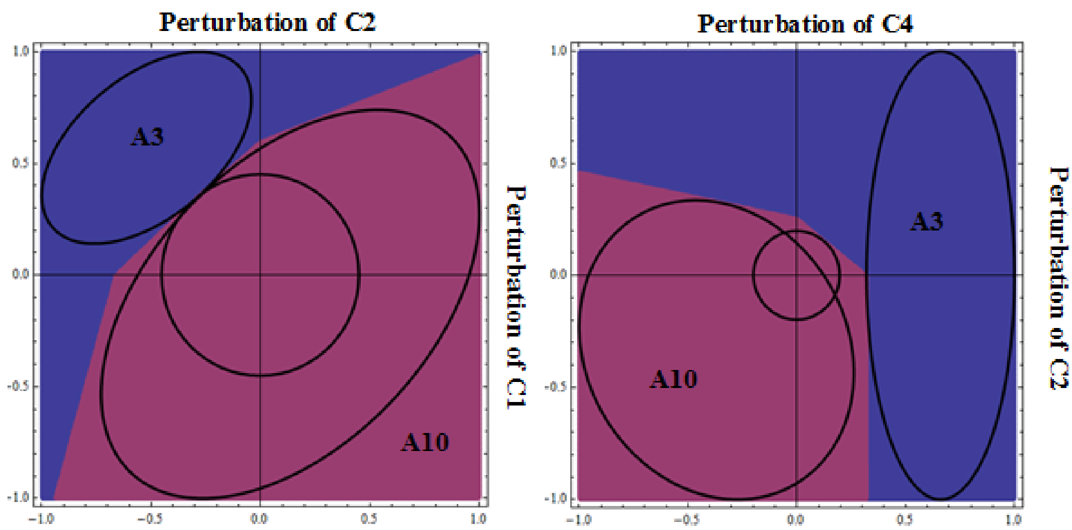


Figure 14. The perturbation space for the pairs of criteria $C_1 \times C_2$ and $C_2 \times C_4$

The core stability spheres provide information about the *direction of change in stability* and about the *perturbation stability* over a given pair of criteria, determined by the magnitude of the perturbation necessary to determine a change in preference. The set of perturbations (δ_1, δ_2) over the pair of criteria (C_1, C_2), for which the most preferred alternative – in this case, Colonoscopy every 10 years (A10), is replaced by another alternative A_j, is the vector indicating the direction of change in stability. Given a vector of perturbations, we can calculate the distance from the origin to the closest separating boundary $B(i, j)$, and its length will be equal to the length of the radius r of the core stability sphere. Identification of the set (δ_1, δ_2) enables the determination of the minimum change to be made in the weights of the two criteria, necessary to make a different alternative most preferred.

For the pair of criteria Prevent Cancer (C1) - Avoid Side Effects (C2), the vector of minimum perturbations, if made, change the most preferred alternative from Colonoscopy every 10 years (A10) to Flexible Sigmoidoscopy every 5 years (A3). Analyzing the direction of the change in stability, the change occurs when the importance of the criterion Prevent Cancer (C1) decreases and the importance of the criterion Avoid Side Effects (C2) increases. Initially, the difference between those two criteria weights was significant. As their importance become almost equal, the patient changes his preference to Flexible Sigmoidoscopy every 5 years (A3).

For the pair of criteria Avoid Side Effects (C2) - Procedure Frequency (C4), the vector of minimum perturbations has the same effect as the vector for the pair of criteria Prevent Cancer (C1) - Avoid Side Effects (C2). The most preferred screening alternative changes from Colonoscopy every 10 years (A10) to Flexible Sigmoidoscopy every 5 years (A3). But, for C2-C4, the change happens only when the importance of both criteria increase significantly, as compared with their initial values. Comparing the magnitude of the minimum change necessary for the two extreme situations analyzed, the change in preference for the pair of criteria Avoid Side Effects (C2) - Procedure Frequency (C4) occurs more quickly than does the change for C1-C2, because the values of the perturbations determining the minimum change are smaller for C2-C4.

A different change in the most preferred alternative happens when the pair of criteria contains either Procedure Preparation (C5) or Procedure Complexity (C6). Consider the pair of criteria Minimize False Positive (C3) and Procedure Preparation (C5). The vector of perturbations tells us that, as the importance of both criteria increases, with the criterion Procedure Preparation (C5) experiencing a greater increase in its weight, the patient switches his preference from Colonoscopy every 10 years (A10) to Annual Immunochemical Fecal Occult Blood Test (A5). A similar change occurs for the pairs of criteria Minimize False Positive (C3) - Procedure Complexity (C6), and Procedure Preparation (C5) - Procedure Complexity (C6).

The perturbation stability analysis of all pairs of criteria indicates that the only other two screening alternatives that could become most preferred, under changes in the criteria weights, are Flexible Sigmoidoscopy every 5 years (A3) and Annual Immunochemical Fecal Occult Blood Test (A5). How quickly the change in the most preferred alternative occurs is a function of the direction of change in stability, given by the vector of perturbations associated with each of the pairs of criteria. For nine pairs of criteria, the change in the most preferred alternative happens when both criteria weights are increased. For one set of criteria, decreasing the weight of both criteria brings about the switch. For five pairs of criteria, one of the criteria needs to be increased, and the second one is needs to be decreased.

For all the alternatives present in the perturbation space, we construct the maximal inscribed ellipsoid within the preference region to define the *solution stability*. In our example, alternatives Colonoscopy every 10 years (A10), Flexible Sigmoidoscopy every 5 years (A3), and Annual Immunochemical Fecal Occult Blood Test (A5) always dominate, within the perturbation space determined by all pairs of criteria. For all pairs of criteria, the volume of the maximal ellipsoid, as a measure of the solution stability, varies between 46% and 82% of the preference region. Knowing that the preference regions for real-life applications are not necessarily convex, we consider, at this point, the maximal ellipsoids to be a good approximation of how stable an alternative can be with respect to its preference region.

The screening alternative Colonoscopy every 10 years (A10) is characterized by two levels of stability: core stability, for being the alternative initially most preferred by the patient; and solution stability, defined by the maximal ellipsoid inscribed within the preference region. The *level of solution stability*, calculated as the ratio of the core stability sphere to the maximal inscribed ellipsoid, provides information about how much of the preference space, approximated by the maximum ellipsoid, is covered by the core stability sphere. As the dimensionality of the perturbation space increases, sampling a sufficient number of points to adequately approximate the preference regions may become a challenge. The maximum ellipsoids provide good approximations of the preference regions, and the comparison of the ellipse that includes the origin with the core stability sphere provides a measure of the stability of the initially most preferred alternative.

Insight can also be derived from a consideration of the evolution of the most preferred alternative as the criteria weights are varied. With respect to all sets of criteria, the screening alternative Colonoscopy every 10 years (A10) is *pairwise more stable*, because the associated solution stability ellipsoids have a greater volume than do the other alternatives that may be most preferred. That observation implies that Colonoscopy every ten years (A10) *is the most stable alternative among all sets of alternatives* with respect to all pairs of criteria, because it is more stable than the other alternatives that may be most preferred in this case, namely, Flexible Sigmoidoscopy every 5 years (A3) and Annual Immunochemical Fecal Occult Blood Test (A5).

Based on the measures of sensitivity and stability we defined, the screening alternative Colonoscopy every 10 years (A10) is not only the most preferred one, but also the *most stable* of the entire set of alternatives. Thus, it should be used for screening the patient. We should take into consideration that small changes in the criteria weights, as the patient obtains more knowledge, could drastically change the most preferred alternative. The results obtained should be treated as a starting point for understanding the patient's preferences regarding his optimal screening alternative for colorectal cancer.

4.2.1. Medical implications of the stability analysis

The findings in this section, about how sensitive and stable are the preferences of the 70 year old patient regarding the set of ten colorectal cancer screening alternative, are of interest from a methodological point of view. But what might be the medical implications of our results? How can our findings help the healthcare provider better understand the underlying structure of the patient's preferences, and how to incorporate that structure into the medical decision making process?

The first medical implication of the analysis is that only three of the original colorectal cancer screening alternatives appear to be appropriate for this patient: two invasive procedures, Colonoscopy every 10 years (A10) and Flexible Sigmoidoscopy every 5 years (A3); and one non-invasive procedure, Annual Immunochemical Fecal Occult Blood Test (A5). The early identification of that subset of the alternatives could help the healthcare provider during the discussions with the patient, and could help ensure that the screening procedure chosen is one that best meets the patient preferences and priorities. The medical decision making process will incorporate, in this situation, both the patient's preferences, as revealed by the analysis, and the healthcare provider's expertise, based on the medical literature and on the medical history of the patient. The analysis of a larger set of similar patients, which would identify the subset of options that are candidates for domination, might help guideline panels and other policy makers eliminate potential options that are less compatible with the majority of patients' preferences.

Another medical implication of the sensitivity and stability analysis that could make it clinically useful is the quick identification of the key pairs of criteria that could determine a switch in patient preferences. For the patient analyzed, Colonoscopy every 10 years (A10) was the initially most preferred screening option. If the healthcare provider considers, based on the patient’s medical history, that this screening alternative might not be the most appropriate one, he could consider attempting to influence the patient’s decision, by taking into consideration the sensitivity and stability analysis results. For example, suppose the healthcare provider believes that Flexible Sigmoidoscopy every 5 years (A3) is a medically superior alternative for the patient. Our analysis tells the healthcare provider that he should emphasize to the patient the importance of the possible side effects that might result, at the patient’s age, from having a Colonoscopy, as compared with having a Flexible Sigmoidoscopy, and how important it is for the patient to increase the frequency of the screening procedure. That strategy for the healthcare provider is provided by the analysis of the core stability of the initially most preferred screening alternative, and by the direction of fastest change in stability of the pair of criteria Avoid Side Effects (C2) - Procedure Frequency (C4). If, on the other hand, the healthcare provider would like the patient to follow up with a non-invasive colorectal cancer screening option – the Annual Immunochemical Fecal Occult Blood Test, he should emphasize the extensive preparation necessary to undergo a Colonoscopy procedure, and how complex the procedure is at the patient’s current age. That strategy for the healthcare provider results from the analysis made with the pair of criteria Procedure Preparation (C5) - Procedure Complexity (C6).

In conclusion, the sensitivity and stability analysis of the patient’s preferences regarding the colorectal cancer screening options might be more useful for the healthcare provider than for the patient. Understanding the patient’s preferences could help the healthcare provider to guide the patient through the medical decision making process to an outcome that is both acceptable to the patient and medically appropriate.

4.3. COMPARING THE STABILITY ANALYSES OF TWO PATIENTS

Custom tailoring a sensitivity and stability analysis for an individual patient might be appropriate if there are particular considerations involved with that patient. For routine cases, patients classified as having an average risk for colorectal cancer, it may be sufficient to partition the space of patients based on their preferences, and to produce a generic analysis that would be a good approximation across each partition of the space. A generic analysis could be created by generalizing the sensitivity and stability analysis for individual patients in each partition, if we could first identify the characteristics that are relevant to determine how similar the patients' preferences are. In this section, we compare the results of the sensitivity and stability analyses for two patients, to determine those differentiating characteristics. The extension of the individual sensitivity and stability analysis to groups of patients with similar characteristics is discussed in Section 4.5.

Consider the patient analyzed in Section 4.2, a 70-year-old male from the Midwest (Patient A), and a different patient, an 80-year-old female from the Southeast (Patient B). The two patients were chosen randomly from the data set. For both patients, we used the model depicted in Figure 13 to ascertain their preferences among the ten colorectal cancer screening alternatives. The importance given to each of the six criteria by the two patients is shown in Table 13.

Table 13. Importance of the criteria for the two patients

	Patient A		Patient B	
Ranking	**Criteria**	**Criteria weights**	**Criteria**	**Criteria weights**
1	Prevent Cancer (C1)	0.6890	Prevent Cancer (C1)	0.5810
2	Avoid Side Effects (C2)	0.2060	Minimize False Positives (C3)	0.2480
3	Minimize False Positives (C3)	0.0770	Avoid Side Effects (C2)	0.1130
4	Procedure Preparation (C5)	0.0220	Procedure Complexity (C6)	0.0379
5	Procedure Complexity (C6)	0.0054	Procedure Frequency (C4)	0.0145
6	Procedure Frequency (C4)	0.0015	Procedure Preparation (C5)	0.0056

Patient B's judgments, with respect to the six criteria (Table 13) and to the ten alternatives, are summarized in the supermatrix presented in Table 14.

Table 14. The supermatrix for the 80 year-old female patient (Patient B)

Goal		Goal															
		C1	**C2**	**C3**	**C4**	**C5**	**C6**	**A1**	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**	**A9**	**A10**
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C1	0.5810	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C2	0.1130	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C3	0.2480	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C4	0.0145	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C5	0.0056	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C6	0.0379	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A1	0	0.0007	0.2071	0.0010	0.0437	0.4650	0.1817	1	0	0	0	0	0	0	0	0	0
A2	0	0.0015	0.1748	0.0010	0.0824	0.3205	0.1578	0	1	0	0	0	0	0	0	0	0
A3	0	0.0136	0.5023	0.3409	0.6492	0.1072	0.2323	0	0	1	0	0	0	0	0	0	0
A4	0	0.0042	0.0078	0.0002	0.2247	0.0743	0.1935	0	0	0	1	0	0	0	0	0	0
A5	0	0.0042	0.0620	0.0266	0.0000	0.0330	0.0921	0	0	0	0	1	0	0	0	0	0
A6	0	0.0156	0.0152	0.0026	0.0000	0.0000	0.0921	0	0	0	0	0	1	0	0	0	0
A7	0	0.2264	0.0022	0.1409	0.0000	0.0000	0.0073	0	0	0	0	0	0	1	0	0	0
A8	0	0.1106	0.0225	0.1409	0.0000	0.0000	0.0432	0	0	0	0	0	0	0	1	0	0
A9	0	0.0408	0.0053	0.0051	0.0000	0.0000	0.0000	0	0	0	0	0	0	0	0	1	0
A10	0	0.5823	0.0007	0.3409	0.0000	0.0000	0.0000	0	0	0	0	0	0	0	0	0	1

The limiting priorities corresponding to the supermatrix from Table 14, and the ranking of the ten screening options, are given in Table 15.

Table 15. Priorities and ranking of the ten screening alternatives for Patient B

Ranking	Alternative		Priority
1	**Colonoscopy every 10 years**	**A10**	**0.4229**
2	Flexible Sigmoidoscopy every 5 years	**A3**	0.1681
3	CT colonography	**A7**	0.1670
4	Double contrast barium enema	**A8**	0.1034
5	Annual Fecal Occult Blood Test with sensitivity 20%	**A1**	0.0342
6	Annual Fecal Occult Blood Test with sensitivity 40%	**A2**	0.0299
7	Annual Immunochemical Fecal Occult Blood Test and flexible Sigmoidoscopy every 5 years	**A9**	0.0256
8	Annual immunochemical fecal occult blood test	**A5**	0.0197
9	Annual Fecal Occult Blood Test and flexible Sigmoidoscopy every 5 years	**A6**	0.0149
10	Fecal DNA test every 5 years	**A4**	0.0144

Our goal, in this section, is to determine how similar are the preferences of the two patients considered, using the results of their individual sensitivity and stability analyses, and what factors might be used to differentiate between the two patients. We first performed the individual sensitivity and stability analysis for Patient B, following the approach used in Section 4.2. The numerical results of the analysis are shown in Table 16.

Table 16. Results of the new sensitivity and stability analysis for Patient B

Criteria pair	Preferred alternatives (region volume and % total perturbation space)			Stability sphere radius*	Stability sphere (volume and % preference region)	Direction of fastest change (perturbation values)		Direction of fastest change (new criteria weights)**		New preferred alternative	Stability ellipsoid (volume and % preference region)			Ratio core stability sphere/solution stability ellipsoid
	A1	A3	A10		A10	pi	pj	Ci	Cj		A1	A3	A10	A10
					Original criteria weights (C1-C2)***			0.5810	0.1130					
C1-C2	-	1.1502 (28.76%)	2.8497 (71.24%)	0.4765	0.7133 (25.03%)	-0.337	0.337	0.3492	0.3734	A3	-	0.4855 (42.21%)	2.1574 (75.71%)	33.06%
					Original criteria weights (C1-C3)			0.5810	0.2480					
C1-C3	-	0.9411 (23.53%)	3.0588 (76.47%)	0.4268	0.5723 (18.71%)	-0.420	-0.076	0.4572	0.3109	A3	-	0.6635 (70.50%)	1.9603 (64.09%)	29.19%
					Original criteria weights (C1-C4)			0.5810	0.0145					
C1-C4	-	1.5476 (38.69%)	2.4523 (61.31%)	0.3979	0.4974 (20.28%)	-0.266	0.296	0.3750	0.2693	A3	-	0.5384 (34.79%)	1.9354 (78.92%)	25.70%
					Original criteria weights (C1-C5)			0.5810	0.0056					
C1-C5	0.5816 (14.54%)	0.5723 (14.31%)	2.8460 (71.15%)	0.5860	1.0788 (37.91%)	-0.586	0.006	0.3614	0.0173	A1	0.3623 (62.29%)	0.4491 (78.47%)	2.0735 (72.86%)	52.03%
					Original criteria weights (C1-C6)			0.5810	0.0379					
C1-C6	-	0.9066 (22.67%)	3.0933 (77.33%)	0.5849	1.0748 (34.74%)	-0.580	0.076	0.3315	0.1509	A3	-	0.6377 (70.34%)	2.2555 (72.92%)	47.65%
					Original criteria weights (C2-C3)			0.1130	0.2480					
C2-C3	-	1.6898 (42.25%)	2.3101 (57.75%)	0.2682	0.2260 (9.78%)	0.252	0.092	0.2603	0.2454	A3	-	0.8984 (53.17%)	1.8192 (78.75%)	12.42%
					Original criteria weights (C2-C4)			0.1130	0.0145					
C2-C4	-	2.3139 (57.85%)	1.6860 (42.15%)	0.2016	0.1277 (7.57%)	0.120	0.162	0.1733	0.1376	A3	-	1.0241 (44.26%)	1.3332 (79.07%)	9.58%
					Original criteria weights (C2-C5)			0.1130	0.0056					
C2-C5	1.0256 (25.64%)	1.0930 (27.33%)	1.8813 (47.03%)	0.2874	0.2595 (13.79%)	0.244	0.152	0.2409	0.1146	A3	0.7319 (71.36%)	0.7829 (71.63%)	1.4972 (79.58%)	17.33%
					Original criteria weights (C2-C6)			0.1130	0.0379					
C2-C6	-	2.0670 (51.68%)	1.9329 (48.32%)	0.2612	0.2143 (11.09%)	0.208	0.158	0.2226	0.1421	A3	-	1.0115 (48.94%)	1.5478 (80.08%)	13.85%
					Original criteria weights (C3-C4)			0.2480	0.0145					
C3-C4	-	1.5992 (39.98%)	2.4007 (60.02%)	0.2479	0.1931 (8.04%)	0.076	0.236	0.2366	0.1916	A3	-	1.0272 (64.23%)	1.8013 (75.03%)	10.72%
					Original criteria weights (C3-C5)			0.2480	0.0056					
C3-C5	0.9207 (23.02%)	0.3538 (8.85%)	2.7254 (68.14%)	0.4451	0.6224 (22.84%)	0.032	0.444	0.1856	0.3051	A1	0.6888 (74.81%)	0.2130 (60.20%)	1.9832 (72.77%)	31.38%
					Original criteria weights (C3-C6)			0.2480	0.0379					
C3-C6	-	1.3108 (32.77%)	2.6891 (67.23%)	0.3971	0.4954 (18.42%)	0.176	0.356	0.2579	0.2579	A3	-	0.6723 (51.29%)	2.1215 (78.89%)	23.35%
					Original criteria weights (C4-C5)			0.0145	0.0056					
C4-C5	0.7728 (19.32%)	1.4062 (35.16%)	1.8209 (45.52%)	0.2600	0.2124 (11.66%)	0.244	0.090	0.1917	0.0715	A3	0.6005 (77.70%)	0.9958 (70.81%)	1.4482 (79.53%)	14.66%
					Original criteria weights (C4-C6)			0.0145	0.0379					
C4-C6	-	2.1197 (52.99%)	1.8802 (47.01%)	0.2424	0.1846 (9.82%)	0.214	0.114	0.1707	0.1118	A3	-	1.1058 (52.17%)	1.4703 (78.20%)	12.55%
					Original criteria weights (C5-C6)			0.0056	0.0379					
C5-C6	1.3252 (33.13%)	0.5602 (14.01%)	2.1145 (52.86%)	0.3711	0.4326 (20.46%)	0.320	0.188	0.2160	0.1460	A1	0.8356 (63.05%)	0.4070 (72.65%)	1.5302 (72.37%)	28.27%
Min	**0.5816**	**0.3538**	**1.6860**	**0.2016**	**0.1277**						**0.3623**	**0.2130**	**1.3332**	
Min (%)	**14.54%**	**8.85%**	**42.15%**		**8.04%**						**62.29%**	**34.79%**	**64.09%**	**9.58%**
Max	**1.3252**	**2.3139**	**3.0933**	**0.5860**	**1.0788**						**0.8356**	**1.1058**	**2.2555**	
Max (%)	**33.13%**	**57.85%**	**77.33%**		**37.91%**						**77.70%**	**78.47%**	**80.08%**	**52.03%**

*Calculated from the origin (0,0) where the most preferred alternative is A10
**The new criteria weights after normalization
***When two or more vectors of perturbations are at tie, the average was reported

The sensitivity and stability analysis described in Table 16 for Patient B was performed by simultaneously perturbing pairs of criteria. For each of the 15 pairs of criteria considered in the analysis, we identified first the dominating preference regions within the perturbation space. For Patient B, only three of the ten screening alternatives ever dominate in the perturbation space: the non-invasive screening procedure, Annual Fecal Occult Blood Test with sensitivity 20% (A1), and the invasive procedures, Flexible Sigmoidoscopy every 5 years (A3), and

Colonoscopy every 10 years (A10). For each of the preference regions, we approximated its area as a fraction of the total perturbation space and as a proportion of the maximal volume possible. Note that the non-invasive screening alternative Annual Fecal Occult Blood Test with sensitivity 20% (A1) dominates in the perturbations space only when the importance of the criterion Procedure Preparation (C5) is increased. Even though C5 has the smallest weight in the initial patient ranking (Table 13), as its importance increases, the patient could switch from the initially most preferred invasive screening alternative, Colonoscopy every 10 years (A10), to a non-invasive screening alternative, Annual Fecal Occult Blood Test with sensitivity 20% (A1).

From a medical perspective, these results tell the healthcare provider that: (1) only three out of the ten colorectal cancer screening options are viable for this patient; and (2) it might be more beneficial for the patient to follow-up with a less invasive or even a non-invasive colorectal cancer screening procedure. If, based on Patient B's medical history, the non-invasive screening alternative might be the most appropriate option, the healthcare provider might consider emphasizing the increased level of preparation necessary to undertake a Colonoscopy (A10) compared with the Annual Fecal Occult Blood Test with sensitivity 20% (A1) in order to guide the patient's preferences.

For the majority of the pairs of criteria, the initially most preferred screening alternative, Colonoscopy every 10 years (A10), captures most of the perturbation space – from a minimum of 42% of the space, up to a maximum of 77%, followed, in size, by Flexible Sigmoidoscopy every 5 years (A3) – between 8% and 57% of the perturbation space. Changes occur when the third screening alternative dominates in the perturbation space, as a result of the increase in the weight of the criterion Procedure Preparation (C5). The increase in C5 increases the volume of the preference region associated with the screening alternative Annual Fecal Occult Blood Test with sensitivity 20% (A1) above the volume of the alternative Flexible Sigmoidoscopy every 5 years (A3), indicating that Patient B will most likely change her preference from A3 to A1.

Clinically, this information tells the healthcare provider that, if he or the patient considers that Procedure Preparation (C5) is an important criterion to be taken into consideration when making the medical decision regarding the most appropriate colorectal cancer screening option, the patient will more likely follow up with the Annual Fecal Occult Blood Test with sensitivity 20% (A1) rather than with Flexible Sigmoidoscopy every 5 years (A3).

Next, we were interested in estimating the volume of the stability sphere, and in calculating its relative volume with respect to that of the preference region. For Patient B, the initially most preferred screening alternative was Colonoscopy every 10 years (A10). The calculated volume of the stability sphere varies between a minimum of 0.1277 and a maximum of 1.0788, providing information about the area in which the initially most preferred screening alternative remains the most preferred one, when perturbations are applied to each pair of criteria.

Consider the stability sphere of smallest radius. Denote its radius by r, and by C_i and C_j the pair of criteria defining the perturbation space that includes the smallest sphere. Then there exists a vector v, in the C_ixC_j space, such that the point $(0,0) + \left(\frac{r}{\|v\|} + \varepsilon\right) \cdot v$ lies in a region dominated by an alternative other than A10. The direction of the vector v is the direction of the fastest preference change. For patient B, the pair of criteria determining the smallest stability sphere, and the fastest switch in preferences, is Avoid Side Effects (C2) and Procedure Frequency (C4). If we perturb the criteria weights for C2-C4 by the values determining the minimum change – increasing the importance of both criteria, Patient B switches her preference from the initially most preferred screening alternative, Colonoscopy every 10 years (A10), to Flexible Sigmoidoscopy every 5 years (A3).

At the other extreme, the slowest change in preferences, and the largest stability sphere around the original most preferred alternative, A10, occurs in the space determined by the pair of criteria Prevent Cancer (C1) and Procedure Preparation (C5). The pair of perturbations applied to the criteria weights of C1 and of C5, by decreasing the importance of the first criterion and increasing the importance of the second criterion, results in a change in Patient B's preferences from Colonoscopy every 10 years (A10), to Annual Fecal Occult Blood Test with sensitivity 20% (A1).

Across all pairwise-perturbation combinations considered, Patient B's preferences change when one of the following pattern appears (1) both criteria are increased (e.g. C2-C4), (2) one criterion is decreased and the second one is increased (e.g. C1-C2) or (3) both criteria are decreased simultaneously (e.g. C1-C3).

From a medical point of view, analysis of the stability sphere volume and of the direction of fastest change provides the healthcare provider with the following insights. One, if A10 is not the most appropriate screening option for Patient B, the care provider might consider discussing with the patient the importance of the criteria C2 and C4, because that pair determines the fastest change in preferences. Two, if Colonoscopy every 10 years (A10) is the best screening option for the patient, the care provider might stress the importance of the pair of criteria Prevent Cancer (C1) and Procedure Preparation (C5). Three, if the non-invasive screening alternative Annual Fecal Occult Blood Test with sensitivity 20% (A1) might be the most appropriate option, the healthcare provider might focus on the increased level of preparation necessary to undertake A10.

We also approximated the preference regions, using the maximal stability ellipsoids that can be inscribed in those regions. For each pair of criteria, we calculated the volume of the maximal ellipsoid, and its relative measures with respect to the associated preference regions. The number of ellipsoids that need to be inscribed in each perturbation space generated by a pair of criteria C_ixC_j is given by the number of distinct preference regions that occur for that pair of criteria. For eleven pairs of criteria, two screening alternatives always dominate - Flexible Sigmoidoscopy every 5 years (A3) and Colonoscopy every 10 years (A10). For the other four pairs of criteria, a third screening option appears – Annual Fecal Occult Blood Test with sensitivity 20% (A1). Note that the stability ellipsoids capture more of the preference regions for alternatives A10 – between 64% and 80%, and for A3 – between 62% and 77%, as compared with the area captured by the stability ellipsoids for alternative A1 – with a range of values between 34% and 78%.

One reason for the difference in captured ratio is that the preference regions for screening alternative A1 are more non-convex than are the preference regions associated with the other two screening alternatives, A3 and A10, decreasing the goodness-of-fit of the (convex) ellipsoid to the preference region (Figure 16). Due to the model's complexity, the preference regions for A1, A3, and A10, within the perturbation space generated by a pair of criteria C_ixC_j, are non-convex. We approximated the maximal inscribed ellipsoid using the nonlinear optimization model described in Chapter 3. We do not currently have a metric to assess the level of goodness-of-fit of our approximation, as compared to the maximal inscribed ellipsoid within a preference region which was first convexified. Future research will analyze how the sensitivity and stability analysis could be affected by the quality of the ellipsoid approximation. Overall, the stability ellipsoids are considered a good approximation of the preference regions as we increase the number of criteria perturbed simultaneously.

From a clinical perspective, these results tell the healthcare provider that, for Patient B, the invasive colorectal cancer screening options, Colonoscopy every 10 years (A10) and Flexible Sigmoidoscopy every 5 years (A3), dominate more of the perturbation space as additional information is presented to the patient with respect to pairs of criteria. Based on the analysis of the 15 pairs of criteria, there are only five situations in which Patient B prefers the non-invasive screening option Annual Fecal Occult Blood Test with sensitivity 20% (A1). Those situations occur only when Patient B gives more importance to the criterion Procedure Preparation (C5). Taking into consideration Patient B's age (she is 80 years old), criterion C5 might be of greater importance than initially considered. That, and her medical history, might influence a decision as to which is the best colorectal cancer screening option for her.

The last measure of interest is the ratio between the stability sphere and the stability ellipsoid for the most preferred alternative, Colonoscopy every 10 years (A10). The ratio is used to characterize the level of solution stability. For Patient B, the ratio is between 9% and 52%, meaning that the stability of the most preferred initial screening alternative is highly imprecise, and depends on the particular pair of criteria considered.

As mentioned at the beginning of this section, our goal is to isolate the factors that could help us identify how similar or how different are the patients' preferences regarding the colorectal cancer screening options available. To identify these characteristics, we compare the sensitivity and stability analyses of the two patients considered, Patient A and Patient B, focusing on the elements that might be the basis for generalization. Figures 15 and 16 graphically show which alternatives dominate within the perturbation space, as perturbations are applied to each of the 15 pairs of criteria. A pairwise stability matrix of preferences was defined for each of the two patients, for Patient A in Figure 15 and for Patient B in Figure 16, to display the preference regions, core stability spheres, and solution stability ellipsoids.

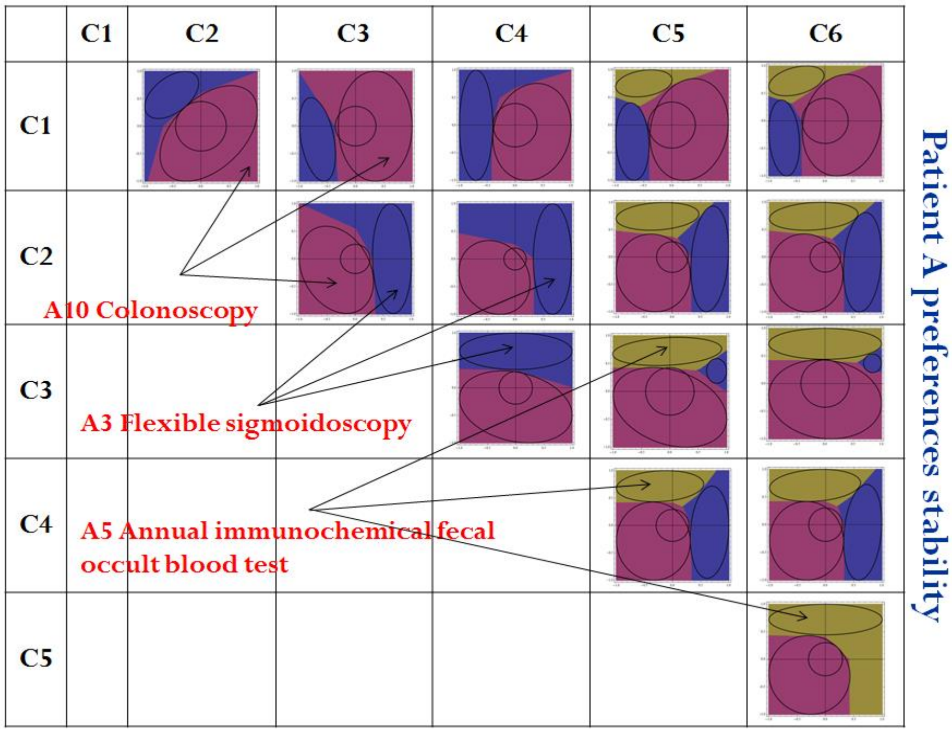


Figure 15. Patient A pairwise stability matrix of preferences

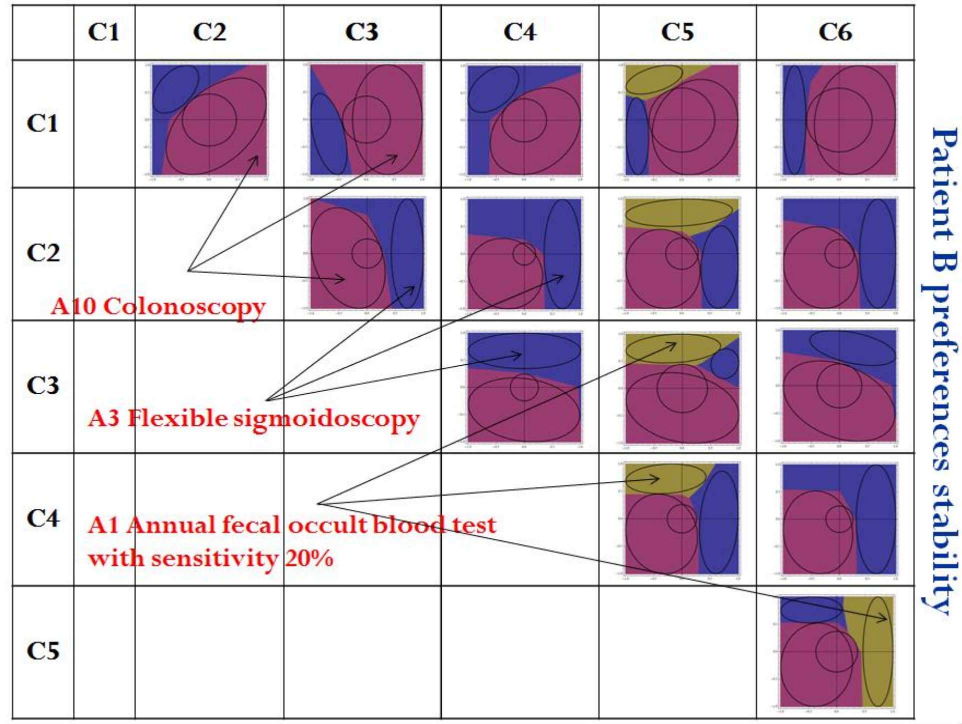


Figure 16. Patient B pairwise stability matrix of preferences

In order to compare the sensitivity and stability analyses of the two patients, we would like to know:

1. *How stable are the preferences of the patients?*
2. *What are the differences and similarities between the two patients?*

Analyzing the pairwise stability matrices of preferences (Figure 15 and Figure 16), for both patients, only three of the ten screening alternatives ever dominate in any of the perturbation space: Colonoscopy every 10 years (A10), Flexible Sigmoidoscopy every 5 years (A3), and Annual Immunochemical Fecal Occult Blood Test (A5) for Patient A; and Colonoscopy every 10 years (A10), Flexible Sigmoidoscopy every 5 years (A3), and Annual Fecal Occult Blood Test with sensitivity 20% (A1) for Patient B. The set of viable alternatives, for either patient, are a combination of invasive and non-invasive procedures. While the two invasive procedures are the same for both patients, Colonoscopy every 10 years (A10) and Flexible Sigmoidoscopy every 5 years (A5), the preferences for a non-invasive procedure are different. Patient A prefers an Annual Immunochemical Fecal Occult Blood Test (A5), while Patient B prefers an Annual Fecal Occult Blood Test with sensitivity 20% (A1).

These findings suggest to the healthcare provider that the two patients have both similarities and differences in preferences. One, only three colorectal cancer screening options are viable for each patient, a fact that could better guide the medical decision making process regarding the best colorectal cancer screening option to be chosen. Two, the preferred invasive screening procedures are the same, A3 and A10, but the non-invasive one is different – A5 for Patient A and A1 for Patient B. One of the factors that could determine the similarity in preferences is test accuracy - Colonoscopy every 10 years (A10) and Flexible Sigmoidoscopy every 5 years (A5) have the highest accuracy rates, while age could determine the difference in preferences. Medically, the Annual Immunochemical Fecal Occult Blood Test is preferred for the younger patients.

The study of sensitivity and stability of preferences begins with ***core stability***. Core stability is defined as the maximum sphere that can be inscribed around the origin. It provides the region where the initially most preferred screening alternative remains the most preferred one, despite the perturbations applied to the criteria. For both patients, alternative A10 is the initially most preferred screening option. For Patient A, A10 is *most stable* with respect to the pair of criteria Prevent Cancer (C1) and Avoid Side Effects (C2), the combination of criteria that determines the largest core stability sphere, while for Patient B, the most preferred alternative A10 is *most stable* with respect to the pair of criteria Prevent Cancer (C1) and Procedure Preparation (C5). The greatest stability of the initially most preferred screening alternative A10 is achieved under different conditions for the two patients. Even though both patients consider the criterion Prevent Cancer (C1) to be very important, the second criterion from the pair determining the highest stability is different. For Patient A it is C2, and for Patient B it is C5. Age can be considered the differentiating factor in this situation, too. Patient A is a younger patient, 70 years old, so he is more concerned with avoiding the possible side effects. Patient B is 80 years old, and she is more concerned with the complexity of the preparation for the procedure. At the other extreme, the most preferred screening alternative Colonoscopy every 10 years (A10) is *least stable* with respect to the pair of criteria Avoid Side Effects (C2) and Procedure Frequency (C4) for both patients, because that combination of criteria generates the smallest core stability sphere. The implication is that, for both patients, the initially most preferred screening alternative A10 is more sensitive with respect to the pair of criteria C2 and C4. That pair will determine the fastest change in patients' preferences.

If, from a medical perspective, the healthcare provider would like to recommend Colonoscopy every 10 years (A10) as the screening option to be chosen by the patients, he should highlight to Patient A how minimal are the possible side effects at his age, while to Patient B he should emphasize that the procedure preparation is not as complex as in the past. Alternatively, if the care provider considers that Colonoscopy every 10 years (A10) is not the most appropriate screening procedures to be used, he should emphasize, to both patients, the importance of avoiding side effects at their age, and that a more frequent colorectal cancer procedure might be preferable at their age. As patients consider C2 and C4 to be more important, they are predicted to switch their preferences toward the colorectal cancer procedure Flexible Sigmoidoscopy every 5 years (A3).

The differences in core stability are determined by the variations in the criteria weights for the two patients (Table 13). Patient A gives greater weight to the importance of the criteria Prevent Cancer (C1) and Side Effects (C2), while Patient B considers the criteria Prevent Cancer (C1) and Minimize False Positives (C3) to be more important. We also compared the ratio of the volume of the core stability sphere to the volume of the associated preference region. For both patients, the *more stable* the initially preferred screening alternative, the greater the volume captured by the core stability sphere. Conversely, the *less stable* the initial preferred screening alternative, the smaller the volume of the core stability sphere is, as compared to the volume of the associated preference region.

The core stability sphere provides information about *the direction of change in stability*, and about the *perturbation stability*, given by the magnitude of the minimum perturbation necessary to determine the change. Consider the situation in which the initially most preferred screening option for Patient A is also the *most stable* one for the pair of criteria Prevent Cancer (C1) and Avoid Side Effects (C2). The vector of minimum perturbations associated with this pair of criteria determines a change from A10 to A3. The direction of change in stability indicates that the change in preferences occurs when a decrease in the importance of C1 is combined with an increase in the importance of C2. The perturbation stability indicates that the magnitude of the perturbations needs to be applied in such a way as to make the two criteria almost equal in their importance. Initially, criterion C1 had the greatest importance. For Patient B, the initially most preferred screening alternative A10 is *most stable* with respect to the pair of criteria Prevent Cancer (C1) and Procedure Preparation (C5). Following the perturbation vector associated with C1 and C5 is associated with a change in preference from A10 to Annual Fecal Occult Blood Test with sensitivity 20% (A1). The direction of change in stability shows that the switch in preferences happens if the importance of C1 decreases, while the importance of C5 increases. The perturbation stability indicates that the magnitude of perturbations necessary to be applied to the two criteria weights need to be significant for C1. Its importance needs to decrease by almost a half, accompanied with a slight increase in C5.

From a clinical perspective these results means that if it is to Patient A's benefit to follow-up with a less invasive and more frequent colorectal cancer procedure, such as Flexible Sigmoidoscopy every 5 years (A3), even though A10 has the greatest stability, the care provider should talk with the patient about the importance of avoiding any possible side effects at his age, while assuring him that the accuracy of the test in detecting and preventing cancer is as high as the accuracy for Colonoscopy. For Patient B, if the healthcare provider places increased emphasis on the extensive procedure preparation, combined with the assurance that, at the patient's current age, an annual non-invasive colorectal cancer screening option might be more beneficial to prevent cancer, the patient is predicted to switch his preferences towards the screening alternative Annual Fecal Occult Blood Test with sensitivity 20% (A1).

For both patients, the initially most preferred screening option is *least stable* with respect to the pair of criteria Side Effects (C2) and Procedure Frequency (C4). For both patients, the unique vector of minimum perturbations is associated with a change in preferred alternative from A10 to Flexible Sigmoidoscopy every 5 years (A3). The direction of change in stability is the same for both patients. A simultaneous increase in both of the two criteria weights of approximately equal magnitude leads to the switch in preferences.

A different change in patients' preferences occurs when the pair of criteria analyzed contains, for Patient A, either Procedure Preparation (C5) or Procedure Complexity (C6), and for Patient B only Procedure Preparation (C5). In such a situation, Patient A is inclined to choose a non-invasive screening procedure, switching from Colonoscopy every 10 years (A10) to Annual Immunochemical Fecal Occult Blood Test (A5), and the direction of change in stability is a function of the pair of criteria considered. For Patient A there are only three combinations of criteria that determine the switch to the non-invasive screening option: (1) Minimize False Positives (C3) - Procedure Preparation (C5); (2) Minimize False Positives (C3) - Procedure Complexity (C6); and (3) Procedure Preparation (C5) or Procedure Complexity (C6). When both Procedure Preparation (C5) and Procedure Complexity (C6) are paired with Minimize False Positives (C3), the perturbation stability indicates that the weight of criterion C3 requires a small positive perturbation, while either C5 or C6 needs to be increased considerably. When the two criteria C5 and C6 are paired together, then the change in preferences occurs when both criteria are increased by almost the same amount.

Patient B switches from A10 to the non-invasive screening option Annual Fecal Occult Blood Test with sensitivity 20% (A1) when the criterion Procedure Preparation (C5) is perturbed along with one of the following three criteria (1) Prevent Cancer (C1); (2) Minimize False Positive (C3); and (3) Procedure Complexity (C6). The direction of change in stability is a function of the criterion with which C5 is paired. When paired with C1, the criterion with the greatest weight for Patient B, perturbation stability indicates that the change in preferences occurs when C1 is significantly decreased while C5 is just slightly increased. When paired with either C3 or C6, the switch to A1 occurs when criterion C5 is increased considerably more than either C3 or C6.

In summary, the core stability analysis implies that even though the two patients have the same initially most preferred alternative, Colonoscopy every 10 years (A10), the stability of that solution differs for the two patients. A primary reason for the stability differences is the differences in weights assigned to the six criteria by the two patients. Medically, these findings are more important for the healthcare provider. Using the results of how stable is the initially most preferred alternative, the healthcare provider can decide which pair of criteria to emphasize when discussing the possible screening options with the patient. Based on the patient's age and medical history (which was not accessible to us during this research), the care provider can "guide" the patient towards the most appropriate colorectal cancer screening option, while taking into consideration the patient's preference.

For both patients, only three out of the ten screening alternatives ever dominate in the perturbation space. For Patient A, the viable screening options are: Colonoscopy every 10 years (A10), Flexible Sigmoidoscopy every 5 years (A3), and Annual Immunochemical Fecal Occult Blood Test (A5). For Patient B, the viable screening options are: A10, A3, and Annual Fecal Occult Blood Test with sensitivity 20% (A1). The ***solution stability*** for the preference regions within the perturbation space over which each alternative dominates is approximated by the volume of the maximal ellipsoid that can be inscribed within the regions. To obtain a good sampling of the perturbation space requires intensive numerical computation. As a result, the maximal ellipsoid is considered, at this point of our research, to be the best approximation of the preference region when sampling the space becomes computationally difficult. For Patient A, for all pairs of criteria, the volume of the maximal ellipsoid varies between 42% and 82% of the preference region. Similarly, for Patient B, those volumes vary between 34% and 80% of the preference region. These percentages tell us that the entire preference region cannot be covered by a single convex, regular shape, due to non-convex shapes of the preference regions. For Patient A, the preferences regarding the screening options are *more solution stable* than are the preferences for Patient B because the maximal ellipsoid for Patient A covers more of his preference region than does the corresponding region for Patient B. Future research may identify a more appropriate way of approximating the preference region. Currently, for any new pair of criteria C_ixC_j, if it satisfies any of the equations of the maximal ellipsoids, we can identify the preference region associated. As each pair of criteria is perturbed, the region described by the solution stability ellipsoid differs between the two patients. So it might be of interest to know the range of values that will satisfy each of the maximal ellipsoid associated with a dominating preference region.

From a medical perspective, the maximal ellipsoids are useful in identifying what will be the most preferred colorectal cancer screening option as patient preferences change, and as the importance of the criteria weights is adjusted. Having available the range of values over which each preference is solution stable could guide the healthcare provider when discussing with the patient the set of colorectal cancer screening options viable for him based on his preferences.

For both patients, the initially most preferred screening alternative, A10, is characterized by both core stability and solution stability. Comparing the ***level of solution stability,*** calculated as the ratio between the area of the core stability sphere and area of the solution stability ellipsoid, for Patient A, the ratio is between 9% and 30%, while for Patient B it is between 9% and 52%. This means that the preferences of Patient B with respect to the initially most preferred screening alternative have *a higher level of solution stability* for certain pairs of criteria, as compared with the preferences of Patient A. The maximal level of solution stability is greater for Patient B.

Clinically, the greater level of solution stability may tell the care provider that it might be easier to influence Patient A’s preferences, and to recommend either a less invasive screening option or a non-invasive one, than it would be to influence the preferences of Patient B. Age might be the differentiating factor in this situation. As patients age, they may be increasingly reluctant to change or to adjust their preferences, leading to longer discussions between the Patient and the healthcare provider.

We are also interested in how the patients' preferences for the most preferred screening alternative, Colonoscopy every 10 years (A10), change or remain unchanged as the criterion weights change. For both patients, the initially most preferred alternative is ***pairwise most stable*** with respect to all pairs of criteria, because the stability ellipsoids for A10 have greater volumes than do the stability ellipsoids of other dominating alternatives. This means that A10 *is the most stable alternative among all sets of alternatives*, because it is the most stable screening alternative with respect to the other screening options for both patients.

In conclusion, for both patients, Colonoscopy every 10 years (A10) is not only the most preferred one. It is also the *most stable* of the entire set of alternatives. But small changes in a patient's judgments could drastically change the preferred screening option. The switch in preference can be either in the direction of a less invasive screening option, e.g. Flexible Sigmoidoscopy every 5 years (A3), or towards a non-invasive screening alternative, A5 for Patient A and A1 for Patient B.

In this section, we presented the sensitivity and stability analysis for two patients faced with the same medical decision: to choose the best colorectal cancer screening procedure, based on their current knowledge and preferences. The individual analysis of preferences is time consuming, and it may not be feasible in all circumstances. Thus, our goal was to identify the factors that could help us cluster the patients. Our results revealed that: (1) the *age* of the patient appears to be the most important differentiating factor when patients decide over the non-invasive procedure: (2) the criteria that determine the core stability of the initially most preferred screening alternative are also influenced by patient's *age*. Patient A is younger (70 years old) so he is more worried about the possible side effects associated with the procedure, while Patient B (80 years old) is more concerned with the complexity of the preparations for the screening procedure; (3) when deciding the frequency of the screening procedure recommended, the healthcare provider takes into consideration the patients *age* – a more frequent procedure might be recommended if the patient is older; (4) if the healthcare provided would like to guide the patient towards a screening option different from the initially most preferred one, he should take into consideration the patient's *age* . Older patients might change their preferences more slowly than do younger ones. Therefore, we are going to use the ***age*** of the patients as the discriminating factor when clustering the patients, and generalizing the individual sensitivity and stability analysis to groups of patients. Preliminary results of our approach regarding generalization will be presented in the last section of this chapter.

4.4. SENSITIVITY AND STABILITY ANALYSIS WHEN MORE THAN TWO CRITERIA ARE PERTURBED

The majority of the AHP/ANP models are designed with more than two criteria, with respect to which the set of alternatives are evaluated. For example, the AHP-based model proposed by Dolan et al. (2013), to determine patients' preferences regarding colorectal cancer screening options, has six criteria (Figure 13). Our analysis of the sensitivity and stability of preferences began by looking at the changes that happen when pairs of criteria are perturbed simultaneously. But how sensitive and stable is the most preferred screening option when *three criteria* are perturbed at once? This section discusses the changes that occur in our analysis, from both the methodological and application perspective, when we perform a three-criteria sensitivity and stability analysis. For illustrative purposes, we consider the 70-year-old patient analyzed in Section 4.2.

As the number of criteria changed simultaneously increases, the methodology for determining how sensitive and stable the initially most preferred alternative is, needs to be adjusted to accommodate the characteristics of three-dimensional space. Table 17 presents the methodological differences between the two-dimensional and the three-dimensional sensitivity and stability analyses.

The first change that occurs, as the number of criteria perturbed simultaneously increases from two to three, is the number of combinations that need to be generated. The number of triplets C_ixC_jxC_k analyzed is greater than the number of pairs C_ixC_j, and is a function of the number of criteria considered in the model. This change is also associated with the increase of the perturbation space dimensionality. Each triplet is analyzed over the cube $[-1,1]^3$. After the number of triplets needed to be analyzed is determined, it is necessary to sample the perturbation space associated with each triplet in such a way as to obtain enough points for the sensitivity and stability analysis, while also keeping the data set at a manageable number of points. Given those considerations, the mesh size for the three-dimensional case, β_2, is going to be greater than the mesh size used in the two-dimensional case, β_1. We set $\beta_1 = 0.002$ for the two-criteria sensitivity analysis. For the three-criteria analysis, the mesh size was $\beta_2 = 0.02$.

Table 17. Methodological differences between the two and three-dimensional cases

	Two-dimensional case	**Three-dimensional case**
Number of criteria simultaneously perturbed	2 criteria	3 criteria
Number of combinations analyzed (*n* – the number of criteria in the model)	C_n^2 pairs	C_n^3 triplets
Perturbation space dimensionality and type	$[-1,1]^2$ - square	$[-1,1]^3$ - cube
Perturbation level (for sampling the space)	β_1	β_2
Dominating alternatives within the perturbation space	A_i, A_j, A_k	A_i, A_j, A_k
Characterization of preference regions	Convex and non-convex	Non-convex
Characterization of the boundaries between the preference regions	Piecewise linear approximation	Piecewise triangular approximation
Core stability	Sphere	Hypersphere
Direction of the fastest change in core stability	$v = (p_1, p_2)$	$v = (p_1, p_2, p_3)$
Solution stability	Ellipse	Ellipsoid

The next step in the sensitivity and stability analysis is to determine the set of alternatives that appear in the perturbation space associated with each triplet of criteria. The preference region where each alternative dominates is calculated based on the largest limiting priority. Due to the complexity of the models, the majority of the preference regions within a three-dimensional perturbation space will be non-convex, compared with the two-dimensional case, where both convex and non-convex preference regions were obtained. Using our empirical results, the following statement appears to hold.

Conjecture 1: *Every alternative that dominates in a non-empty region of the $[-1,1]^2$ space of perturbations also dominates in at least one non-empty region of the $[-1,1]^3$ space of perturbations.*

We do not yet have a mathematical proof of Conjecture 1, nor a counterexample to it. Future research will determine the correlation between the dominance of an alternative over a non-empty region and the dimensionality of the perturbation space. What seems more intuitive is that if an alternative dominates in a region of the $[-1,1]^3$ perturbation space that it should also dominate in all the 2-dimensional subspaces generated from the 3-dimensional region.

After identifying the regions within the perturbation space dominated by different alternatives, we need to approximate the separating boundaries between those regions. For the three-dimensional case, the boundaries are calculated using a piecewise triangular approximation – the extension of the piecewise linear boundaries from the two-dimensional space. The algorithm calculating the boundaries is described in Section 3.5.1. Determining the separating boundaries becomes more complex in the three-dimensional case because of the large number of triangles that need to be generated. Given the extensive numerical computations necessary, the goodness-of-fit of the boundaries approximation via a triangular mesh will typically be less than the goodness-of-fit in the two-dimensional case.

The analysis of the *core stability* of the initially most preferred alternative is obtained by generating the largest ellipse, for the two-dimensional case, and the largest ellipsoid, for the three-dimensional case, that can be inscribed around the origin of the space $[-1,1]^m$, where *m* is the number of criteria perturbed simultaneously. As the dimensionality of the perturbation space used for the analysis of the sensitivity and stability of preferences increases, the volume of the core stability hypersphere decreases. Based on our empirical results obtained so far, the following conjecture can be stated.

Conjecture 2: *The volume of the core stability hypersphere decreases as the perturbation space dimensionality increases.*

Based on the methodology we developed, a smaller volume of the core stability hypersphere is associated with a lower stability of the initially most preferred alternative. This means that the following statement can be considered.

Conjecture 3: *As the number of criteria perturbed simultaneously increases, the initially most preferred alternative is less stable with respect to a given set of criteria (pair or triplet).*

How fast the preferences change from the initially most preferred alternative to another alternative is given by the direction of fastest change vector. The length of that vector represents the minimum perturbation necessary to be made in order to determine a switch in preferences. Its direction indicates the way in which the criteria should be changed in order to bring about the change. The number of elements of the vector is equal to the number of criteria perturbed simultaneously. For the three-criteria analysis, the vector has the form $v = (p_1, p_2, p_3)$. We have not yet identified if there is any preservation of the direction of the fastest change as the number of criteria perturbed simultaneously is increased. Future research will address how the direction of fastest change in preferences is affected by an increase in the number of criteria changed at once.

The *solution stability* analysis of all of the preference regions associated with an alternative and present in the perturbation space is obtained by generating the maximal inscribed ellipse/ellipsoid within a given preference region. Based on the preference region type, convex or non-convex, different optimization models are used to calculate the largest ellipsoid that can be inscribed within each region. For the convex case, an already established model exists (see Section 3.2). For the non-convex case, we proposed a new non-linear programming model, which was extended to the three-dimensional case also (see Section 3.4 and Section 3.6).

In the first part of this section, we presented the methodological differences that result when the number of criteria changed simultaneously is increased to three. The second part focuses on the changes that occur in the sensitivity and stability analysis from an application perspective. To illustrate our findings, we consider the 70-year- old male patient analyzed in Section 4.2. His preferences were assessed using the AHP-based model depicted in Figure 13. Based on his judgements, the importance of the six criteria considered in the model was presented in Table 9, and the supermatrix associated was synthetized in Table 10.

As the number of criteria perturbed simultaneously is increased to ***three***, the number of combinations necessary to be analyzed increases to 20. In the two-dimensional case, only 15 pairs were analyzed. For each triplet, the three-dimensional perturbation space was generated using an incremental perturbation level of $\beta_2 = 0.02$, that yielded 970,200 number of points utilized to be used in the analysis. The results of the three-criteria sensitivity and stability analysis for the patient are reported in Table 18. For each of the 20 triplets, we identified the preference regions associated with a given perturbation space, the core stability hypersphere, the direction of the fastest change in preference, and the new preferred alternative when the switch happens. The maximal inscribed ellipsoids within the preference regions are not presented in this section due to the increased computational resources needed to generate all, and are left as a subject of future research. The practical differences between the two-criteria and the three-criteria analyses for the patient analyzed are presented in Table 19.

Table 18. Results of the three-criteria sensitivity and stability analysis for the patient

Criteria triplet	Preferred alternatives (region volume and % total perturbation space)			Stability sphere radius*	Stability sphere (volume and % preference region)	Direction of fastest change (perturbation values)*			Direction of fastest change (new criteria weights)**			New preferred alternative
	A3	A5	A10		A10	pi	pj	pk	Ci	Cj	Ck	
					Original criteria weights (C1-C2-C3)				**0.6890**	**0.2060**	**0.0770**	
C1-C2-C3	2.0770 (25.96%)	0.7423 (9.28%)	5.1808 (64.76%)	0.4535	0.3907 (7.54%)	-0.32	0.32	0	0.4528	0.4446	0.0745	**A3**
					Original criteria weights (C1-C2-C4)				**0.6890**	**0.2060**	**0.0015**	
C1-C2-C4	3.9193 (48.99%)	-	4.0807 (51.01%)	0.3262	0.1454 (3.56%)	-0.16	0.18	0.22	0.4618	0.2784	0.1765	**A3**
					Original criteria weights (C1-C2-C5)				**0.6890**	**0.2060**	**0.0220**	
C1-C2-C5	2.1935 (27.42%)	0.9156 (11.45%)	4.8909 (61.14%)	0.4477	0.3758 (7.68%)	-0.28	0.34	0.08	0.4290	0.4116	0.0867	**A3**
					Original criteria weights (C1-C2-C6)				**0.6890**	**0.2060**	**0.0054**	
C1-C2-C6	2.0558 (25.70%)	1.0114 (12.64%)	4.9328 (61.66%)	0.4525	0.3882 (7.87%)	-0.32	0.32	0	0.4529	0.4447	0.0053	**A3**
					Original criteria weights (C1-C3-C4)				**0.6890**	**0.0770**	**0.0015**	
C1-C3-C4	2.8541 (35.68%)	-	5.1459 (64.32%)	0.3606	0.1963 (3.82%)	-0.36	0	0.02	0.5705	0.0996	0.0278	**A3**
					Original criteria weights (C1-C3-C5)				**0.6890**	**0.0770**	**0.0220**	
C1-C3-C5	1.3548 (16.93%)	0.8625 (10.78%)	5.7828 (72.28%)	0.3600	0.1954 (3.38%)	-0.36	0	0	0.5856	0.1023	0.0292	**A3**
					Original criteria weights (C1-C3-C6)				**0.6890**	**0.0770**	**0.0054**	
C1-C3-C6	1.2824 (16.03%)	0.8788 (10.99%)	5.8388 (72.98%)	0.3600	0.1954 (3.35%)	-0.36	0	0	0.5856	0.1023	0.0072	**A3**
					Original criteria weights (C1-C4-C5)				**0.6890**	**0.0015**	**0.0220**	
C1-C4-C5	2.6263 (32.83%)	0.9515 (11.89%)	4.4222 (55.28%)	0.3904	0.2492 (5.64%)	-0.38	0.08	-0.04	0.5221	0.0995	0.0258	**A3**
					Original criteria weights (C1-C4-C6)				**0.6890**	**0.0015**	**0.0054**	
C1-C4-C6	2.4618 (30.77%)	1.0145 (12.68%)	4.5237 (56.55%)	0.3929	0.2541 (5.62%)	-0.38	0.1	0	0.5091	0.1208	0.0065	**A3**
					Original criteria weights (C1-C5-C6)				**0.6890**	**0.0220**	**0.0054**	
C1-C5-C6	1.0670 (13.34%)	2.6812 (33.51%)	4.2518 (53.15%)	0.4000	0.2681 (6.31%)	-0.40	0	0	0.5699	0.0303	0.0075	**A3**
					Original criteria weights (C2-C3-C4)				**0.2060**	**0.0770**	**0.0015**	
C2-C3-C4	5.2577 (65.72%)	-	2.7423 (34.28%)	0.1800	0.0244 (0.89%)	0.12	0.06	0.12	0.2370	0.1041	0.0954	**A3**
					Original criteria weights (C2-C3-C5)				**0.2060**	**0.0770**	**0.0220**	
C2-C3-C5	3.7755 (47.19%)	1.3034 (16.29%)	2.9211 (36.51%)	0.2236	0.0468 (1.60%)	0.16	0.10	0.12	0.2490	0.1266	0.1042	**A3**
					Original criteria weights (C2-C3-C6)				**0.2060**	**0.0770**	**0.0054**	
C2-C3-C6	3.5964 (44.95%)	1.4522 (18.15%)	2.9514 (36.89%)	0.2332	0.0531 (1.80%)	0.16	0.12	0.12	0.2452	0.1382	0.0919	**A3**
					Original criteria weights (C2-C4-C5)				**0.2060**	**0.0015**	**0.0220**	
C2-C4-C5	4.5192 (56.49%)	1.1298 (14.12%)	2.3510 (29.39%)	0.1822	0.0253 (1.08%)	0.10	0.14	0.06	0.2232	0.1105	0.0631	**A3**
					Original criteria weights (C2-C4-C6)				**0.2060**	**0.0015**	**0.0054**	
C2-C4-C6	4.3829 (54.79%)	1.2604 (15.76%)	2.3566 (29.46%)	0.1876	0.0277 (1.17%)	0.12	0.12	0.08	0.2325	0.0937	0.0656	**A3**
					Original criteria weights (C2-C5-C6)				**0.2060**	**0.0220**	**0.0054**	
C2-C5-C6	2.2083 (27.60%)	3.3190 (41.49%)	2.4727 (30.91%)	0.2349	0.0543 (2.20%)	0.16	0.14	0.14	0.2441	0.1165	0.0769	**A3**
					Original criteria weights (C3-C4-C5)				**0.0770**	**0.0015**	**0.0220**	
C3-C4-C5	3.1414 (39.27%)	1.5235 (19.04%)	3.3351 (41.69%)	0.2720	0.0843 (2.53%)	0.08	0.24	0.10	0.1068	0.1708	0.0848	**A3**
					Original criteria weights (C3-C4-C6)				**0.0770**	**0.0015**	**0.0054**	
C3-C4-C6	2.9670 (37.09%)	1.6424 (20.53%)	3.3905 (42.38%)	0.2835	0.0955 (2.82%)	0.08	0.26	0.08	0.1067	0.1847	0.0601	**A3**
					Original criteria weights (C3-C5-C6)				**0.0770**	**0.0220**	**0.0054**	
C3-C5-C6	0.3929 (4.91%)	3.9741 (49.68%)	3.6330 (45.41%)	0.3000	0.1131 (3.11%)	0.04	0.21	0.21	0.0784	0.1565	0.1475	**A5**
					Original criteria weights (C4-C5-C6)				**0.0015**	**0.0220**	**0.0054**	
C4-C5-C6	2.0280 (25.35%)	3.4687 (43.36%)	2.5032 (31.29%)	0.2786	0.0905 (3.62%)	0.24	0.10	0.10	0.1677	0.0833	0.0730	**A3**
Min	**0.3929**	**0.7423**	**2.3510**	**0.1800**	**0.0244**							
Min (%)	**4.91%**	**9.28%**	**29.39%**		**0.89%**							
Max	**5.2577**	**3.9741**	**5.8388**	**0.4535**	**0.3907**							
Max (%)	**65.72%**	**49.68%**	**72.98%**		**7.87%**							

*Calculated from the origin (0,0) where the most preferred alternative is A10
**The new criteria weights after normalization
***When two or more vectors of perturbations are at tie, the average was reported

Table 19. Two-criteria versus three-criteria sensitivity and stability analysis

	Two-criteria case	Three-criteria case
Number of criteria perturbed simultaneously	2 criteria	3 criteria
Number of combinations analyzed	15 pairs	20 triplets
Perturbation level used to generate the perturbation space	$\beta_1 = 0.002$	$\beta_2 = 0.02$
Number of points within each perturbation space	998,001	970,299
Initially most preferred alternative	**A10 -** Colonoscopy every 10 years	**A10 -** Colonoscopy every 10 years
Dominating alternatives within the perturbation space	**A3 -** Flexible sigmoidoscopy every 5 years **A5 -** Annual immunochemical fecal occult blood test **A10 -** Colonoscopy every 10 years	**A3 -** Flexible sigmoidoscopy every 5 years **A5 -** Annual immunochemical fecal occult blood test **A10 -** Colonoscopy every 10 years
Criteria determining the presence of the non-invasive screening option Annual immunochemical fecal occult blood test (A5)	**C5** – Procedure Preparation **C6 -** Procedure Complexity	**C3 –** Minimize False Positive **C5** – Procedure Preparation **C6 -** Procedure Complexity
Smallest core stability sphere/spheroid	**C2-C4** Avoid Side Effects and Procedure Frequency	**C2-C3-C4** Avoid Side Effect, Minimize False Positive and Procedure Frequency
Direction of the fastest change in stability and vector's magnitude	$v = (0.122, 0.156)$ $\lVert v \rVert = 0.198$	$v = (0.12, 0.06, 0.12)$ $\lVert v \rVert = 0.180$
New preferred alternative based on the fastest change	**A3 -** Flexible sigmoidoscopy every 5 years	**A3 -** Flexible sigmoidoscopy every 5 years
Largest core stability sphere/spheroid	**C1-C2** Prevent Cancer and Avoid Side Effects	**C1-C2-C3** Prevent Cancer, Avoid Side Effects and Minimize False Positive
Direction of the slowest change in stability and vector's magnitude	$v = (-0.304, 0.333)$ $\lVert v \rVert = 0.450$	$v = (-0.32, 0.32, 0)$ $\lVert v \rVert = 0.452$
New preferred alternative based on the slowest change	**A3 -** Flexible sigmoidoscopy every 5 years	**A3 -** Flexible sigmoidoscopy every 5 years

As hypothesized in Conjecture 1, the colorectal cancer screening alternatives that dominate in the three-criteria analysis are the same ones that dominate in the perturbation space for the two-criteria analysis. The alternatives are: Colonoscopy every 10 years (A10), Flexible Sigmoidoscopy every 5 years (A3), and Annual Immunochemical Fecal Occult Blood Test (A5). But not all the screening options are present in all the 20 combinations analyzed. The difference that appears in the three-dimensional case it is that the non-invasive screening procedure Annual Immunochemical Fecal Occult Blood Test (A5) is preferred by the patient whenever one of the criteria Minimize False Positive (C3), Procedure Preparation (C5), and Procedure Complexity (C6), or a combination of those criteria, is present in the triplet. By comparison, in the two-dimensional case, only criteria C5 and C6 determined the presence of screening alternative A5 within the perturbation space. Another difference for three-criteria sensitivity analysis it is that the presence of alternative A5 decreases the volume of another screening option, the Flexible Sigmoidoscopy every 5 years (A3).

From a medical perspective, the results above indicate that if the healthcare provider would like the patient to follow-up with the non-invasive colorectal cancer screening option Annual Immunochemical Fecal Occult Blood Test (A5), he should add, into the discussion, that the percentage of false positive tests for A5 is equivalent to that obtained from the initially most preferred screening option, A10.

Analyzing the *core stability* of the initially most preferred screening alternative Colonoscopy every 10 years (A10), consistent with Conjecture 2, the core stability spheroid in the three-criteria case has a smaller volume than does the core stability sphere in the two-criteria case. This means that, as stated in Conjecture 3, as the number of criteria perturbed simultaneously is increased from two to three, the initially most preferred screening alternative is *less stable* with respect to the triplets, because it covers less of the preference region. The fastest change in preferences, which is associated with the smallest core stability hypersphere, is generated by the triplet of criteria Avoid Side Effects (C2), Minimize False Positive (C3), and Procedure Frequency (C4). The direction of the fastest change vector indicates that the switch in preferences from A10 to Flexible Sigmoidoscopy every 5 years (A3) occurs when all the three criteria are increased. In the two-criteria case, two of the criteria in the triplet, C2 and C4, are the same ones generating the smallest spheroid in the two-dimensional case. Adding criterion C3 determines a faster change in patient's preferences. The magnitude of the direction of change vector in the three-criteria case (0.180) is smaller than the magnitude of the vector for the two-criteria case (0.198).

If the healthcare provider would like to bring out a change in the patient's preferences to the less invasive screening procedure Flexible Sigmoidoscopy every 5 years (A3), he can use these findings to inject into the discussion with the patient, information about the reduced side effects associated with A3 (C2); the necessity of a more frequent procedure at the patient's age and based on his medical history (C4) – screening alternative A10 was performed every 10 years, and about the similar percentage of false positive tests obtained with either of the two procedures (C3) – A10 and A3. If initially, based solely on the two-criteria results, the care provider talked only about criteria C2 and C4, adding the medical information about C3 will bring out a faster switch in preferences. So, from a medical perspective, if the goal of the healthcare provider is to suggest to the patient that he/she should follow-up with another screening alternative, other than the initially most preferred one, talking about the importance of three criteria will result in a much faster change in preferences.

In contrast, the largest stability hypersphere is associated with the triplet of criteria Prevent Cancer (C1), Avoid Side Effects (C2), and Minimize False Positive (C3), similar with the two-criteria analysis, in which the pair C1 and C2 generated the largest core stability sphere. With respect to C1, C2, and C3, the initially most preferred screening alternative, A10, will remain the most preferred one over the widest range of perturbations. In this situation, the change in preferences will happen more slowly. and only when the first criterion C1 is decreased, the second criterion C2 is increased, and the third criterion C3 is also increased, as a result of which the patient will switch to the less invasive screening procedure, Flexible Sigmoidoscopy every 5 years (A3). The direction of the slowest change in preferences is given by the minimum perturbation vector $v = (-0.32, 0.32, 0)$.

From a medical decision making perspective, this tells the healthcare provider that if he would like the patient to follow-up with the invasive screening procedure, Colonoscopy every 10 years (A10), he should explain to the patient (1) the increased benefits of a colonoscopy in preventing cancer, compared with other procedures, (2) the relatively decreased probability of side effects given the patient's age, and (3) how reliable the procedure is when minimizing false positive results.

The screening alternative selected as the result of a switch generated in the direction of the vector of fastest change appears to differ in the three-criteria case from the one selected in the two-criteria case. In 19 of the triplets considered for analysis, the change in preferences is from the initially most preferred screening alternative A10 to the less invasive screening option A3. Only once does the fastest change in preference indicate a change towards the non-invasive screening option A5. In the two-criteria case the switch from A10 to A5 happened three times.

Based on all the triplets analyzed, the initially most preferred screening option, Colonoscopy every 10 years (A10), is the *most stable* alternative with respect to all triplets of criteria, but its stability is less than it was in the two-criteria analysis, as hypothesized in Conjecture 3. This result might influence the way the healthcare provider might use the information provided by the sensitivity and stability analysis to guide the patient during the medical decision making process.

In this section we presented the changes that result when the number of criteria perturbed simultaneously is increased, and the medical implications associated with such a change. Which of the two approaches should be used in practice? Is the two-criteria sensitivity and stability analysis sufficient to guide both the healthcare provider and the patient though the medical decision making process? Or are the insights from the three-criteria analysis necessary to provide the additional information to lead to a best outcome? Future research will look into this issue, and try to identify what will be the "optimal" number of criteria that should be perturbed simultaneously, from the practical perspective. Also future research will investigate the behavior of the direction of fastest change as the number of criteria changed at once is increased.

4.5. GENERALIZATION OF SENSITIVITY AND STABILITY ANALYSIS TO GROUPS OF PATIENTS WITH SIMILAR CHARACTERISTICS

Individualized sensitivity and stability analysis provides insights for understanding a single patient's preferences regarding the available colorectal cancer screening options. The drawback of individual analysis is the time required to perform it, a time requirement that may make individual-level analysis infeasible within the time allocated to a medical appointment. In-depth, detailed analysis might be useful for patients who are predicted to have a high risk of being diagnosed with colorectal cancer, but a different approach might be more practical in order to inform healthcare providers about how preferences change for average risk patients.

In this section, we present preliminary results about generalizing our methodology for individual sensitivity and stability analysis to groups of patients with similar characteristics, because we believe that group-level analyses would be useful for discussions with average risk patients. Based on our findings from Section 4.3, where we compared two patients' sensitivity and stability analysis, we consider ***age*** as the discriminating factor when clustering our patients.

We divided the 395 patients from our initial dataset into seven age groups between 50 and 80 years-old. Patient participation in the study that generated our data set was voluntary, and the majority of patients who agreed to participate were between 50 and 65 years old. Within each age group we determined the similarity of patients' preferences. We used the Hilbert metric to calculate the distances between the set of alternatives' priorities for a given patient with the priorities of each of the other patients from that age group. The Hilbert metric was chosen as a measure of similarity because it is the only metric defined in the eigenvectors/eigenvalues space (Genest and Shang, 1996). Let $w^1 = (w_1^1, \dots, w_{10}^1)$ and $w^2 = (w_1^2, \dots, w_{10}^2)$ be the limiting priorities of the ten screening alternatives for two patients. The distance between the two sets of priorities, as given by the Hilbert metric, is defined as:

$$d_1(w^1, w^2) = log \frac{max_k\left(\frac{w_k^1}{w_k^2}\right)}{min_k\left(\frac{w_k^1}{w_k^2}\right)} \tag{4.1}$$

where $k = 1, \dots, 10$.

The distances between each pair of patients' preferences among the ten colorectal cancer screening alternatives were calculated using the relation (4.1), and summarized in the Hilbert metric matrix for each age group. For example, in the 50-year-old age group, there are 68 patients, so the Hilbert metric matrix associated with this age group contains 2,278 distinct entries. An excerpt of that matrix is presented in Table 20.

Table 20. Excerpt of the Hilbert metric matrix associated with the age group 50

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12
A1	0.0000	1.9910	2.5623	5.1882	2.6299	2.6747	2.6921	2.7375	1.6164	1.5608	1.7953	2.9320
A2	1.9910	0.0000	3.7836	3.4597	2.6688	1.1215	2.5008	1.2559	0.8797	1.4166	0.7169	2.2916
A3	2.5623	3.7836	0.0000	5.5868	3.7931	3.6109	3.5515	3.9368	3.3884	3.9244	3.4081	3.9035
A4	5.1882	3.4597	5.5868	0.0000	2.5583	3.4885	2.7179	2.4882	3.6747	3.8435	3.5927	2.8002
A5	2.6299	2.6688	3.7931	2.5583	0.0000	2.3896	1.2142	2.1321	2.0230	2.3937	2.3366	1.3085
A6	2.6747	1.1215	3.6109	3.4885	2.3896	0.0000	1.9060	1.7977	1.1100	2.1002	0.9647	3.2632
A7	2.6921	2.5008	3.5515	2.7179	1.2142	1.9060	0.0000	1.8283	1.9101	2.2429	1.8769	2.3193
A8	2.7375	1.2559	3.9368	2.4882	2.1321	1.7977	1.8283	0.0000	1.6923	1.5065	1.6620	1.4655
A9	1.6164	0.8797	3.3884	3.6747	2.0230	1.1100	1.9101	1.6923	0.0000	1.4179	0.4997	2.3910
A10	1.5608	1.4166	3.9244	3.8435	2.3937	2.1002	2.2429	1.5065	1.4179	0.0000	1.5666	1.6512
A11	1.7953	0.7169	3.4081	3.5927	2.3366	0.9647	1.8769	1.6620	0.4997	1.5666	0.0000	2.4735

Within each age bracket, we used the SPSS two-step clustering (auto-clustering) routine (Zhang et al., 1996; Chiu et al., 2001) to group the patients, using, as input, the Hilbert metric matrix which measures the similarity of the patients' preferences. The optimal number of clusters is determined automatically, based on the AIC criterion. The following graph shows the steps we followed in our analysis for generalization (Figure 17).

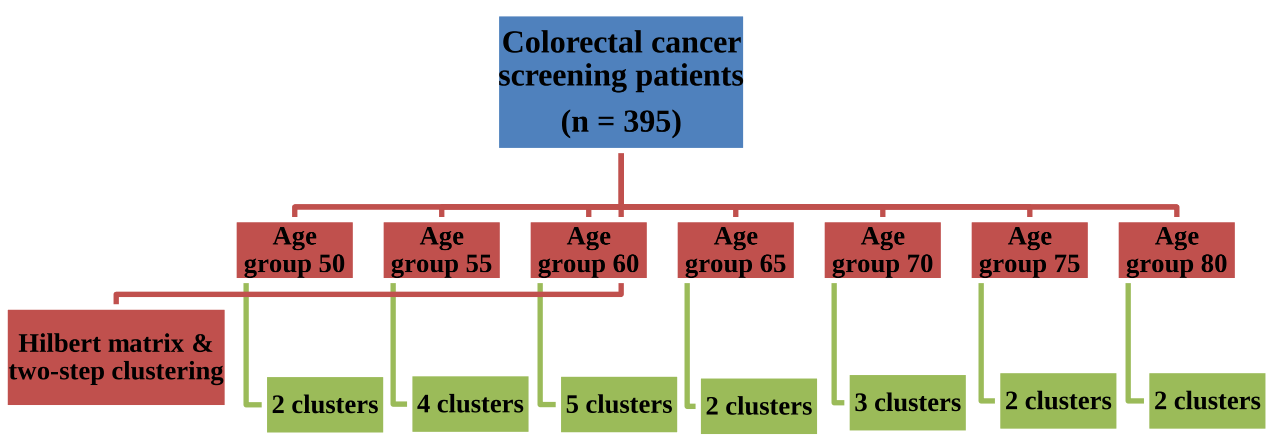


Figure 17. Generalization process to identify clusters of patients with similar characteristics

For each age group and cluster type we determined: the number of patients, the most preferred alternatives and the associated priorities range, and the range of values for the six criteria included in the model. The results are presented in Table 21.

The majority of the 395 patients in the study were between 50 and 65 years of age. Analyzing the number of clusters into which the patients were grouped, note that patients in the 50, 65, 75, and 80 year-old groups have preferences that are less volatile than those in the 55, 60, and 70 year old groups, because they are subdivided into fewer clusters. Because we only have a fixed data set to work with, we do not know if that difference is due to a structural change in decision behavior, or to an idiosyncrasy of the available data. Surveying a large set of patients might resolve that uncertainty.

Considering the combinations of the most preferred initial screening alternative associated with each cluster, we can conjecture that, based on the age group, only certain screening options are viable. For example, the 80-year-old group of patients appears to prefer one of the following three screening alternatives: Flexible Sigmoidoscopy every 5 years (A3), Colonoscopy every 10 years (A10), and Annual Fecal Occult Blood Test with sensitivity 20% (A1). Additionally, for each screening alternative that is preferred by the patients, we identified the range of values associated with the priorities. That additional information may help to predict the preference structure of a new patient.

Table 21. Primary characteristics of the clusters based on age group

Patients age group	Number of patients	Cluster number	Number of patients in cluster	Most preferred alternative (priorities range)	Criteria range C1	C2	C3	C4	C5	C6
50	68	***Cluster 1***	47	**A10** [0.217;0.450]	[0.534;0.724]	[0.027;0.295]	[0.025;0.282]	[0.002;0.084]	[0.002;0.059]	[0.002;0.204]
		Cluster 2	21	**A1** (0.130) **A3** [0.177;0.490] **A4** (0.269) **A8** [0.134;0.159] **A10** [0.191;0.385]	[0.099;0.634]	[0.044;0.567]	[0.042;0.697]	[0.003;0.371]	[0.005;0.143]	[0.003;0.206]
55	73	***Cluster 1***	8	**A1** [0.166;0.206] **A2** (0.279) **A3** [0.132;0.265]	[0.048;0.602]	[0.040;0.270]	[0.040;0.585]	[0.040;0.196]	[0.009;0.196]	[0.024;0.387]
		Cluster 2	24	**A3** [0.138;0.395] **A8** [0.142;0.159] **A10** [0.157;0.402]	[0.251;0.720]	[0.027;0.602]	[0.026;0.268]	[0.002;0.193]	[0.004;0.210]	[0.005;0.218]
		Cluster 3	19	**A10** [0.257;0.504]	[0.230;0.727]	[0.020;0.305]	[0.074;0.670]	[0.003;0.028]	[0.002;0.015]	[0.001;0.057]
		Cluster 4	22	**A10** [0.199;0.333]	[0.486;0.703]	[0.044;0.319]	[0.046;0.280]	[0.002;0.064]	[0.003;0.213]	[0.003;0.134]
60	77	***Cluster 1***	12	**A3** [0.190;0.415] **A10** [0.238;0.445]	[0.098;0.674]	[0.025;0.576]	[0.039;0.604]	[0.002;0.108]	[0.004;0.176]	[0.004;0.389]
		Cluster 2	1	**A3** (0.545)	0.038	0.211	0.038	0.555	0.128	0.030
		Cluster 3	18	**A3** [0.213;0.250] **A10** [0.170;0.356]	[0.091;0.750]	[0.052;0.273]	[0.032;0.663]	[0.006;0.223]	[0.004;0.104]	[0.004;0.251]
		Cluster 4	23	**A3** [0.197;0.224] **A10** [0.205;0.312]	[0.320;0.640]	[0.044;0.283]	[0.044;0.547]	[0.003;0.052]	[0.003;0.066]	[0.003;0.107]
		Cluster 5	23	**A3** (0.316) **A10** [0.260;0.406]	[0.438;0.714]	[0.052;0.429]	[0.031;0.285]	[0.003;0.037]	[0.004;0.029]	[0.004;0.087]
65	70	***Cluster 1***	25	**A2** (0.172) **A3** [0.119;0.348] **A8** (0.394) **A10** [0.147;0.421]	[0.090;0.710]	[0.025;0.590]	[0.020;0.260]	[0.006;0.359]	[0.004;0.223]	[0.005;0.526]
		Cluster 2	45	**A3** (0.215) **A8** (0.172) **A10** [0.208;0.465]	[0.312;0.701]	[0.050;0.307]	[0.039;0.514]	[0.002;0.177]	[0.002;0.111]	[0.002;0.216]
70	45	***Cluster 1***	7	**A2** (0.278) **A3** (0.304;0.362) **A10** (0.273;0.485)	[0.120;0.729]	[0.075;0.529]	[0.025;0.077]	[0.002;0.100]	[0.022;0.183]	[0.005;0.518]
		Cluster 2	17	**A2** (0.178) **A3** [0.169;0.228] **A7** (0.162) **A10** [0.154;0.291]	[0.150;0.750]	[0.040;0.302]	[0.030;0.300]	[0.006;0.255]	[0.012;0.209]	[0.018;0.334]
		Cluster 3	21	**A3** [0.244;0.247] **A10** [0.197;0.362]	[0.460;0.700]	[0.095;0.304]	[0.050;0.282]	[0.003;0.067]	[0.002;0.038]	[0.005;0.084]

Table 21. Primary characteristics of the clusters based on age group (cont.)

75	35	***Cluster 1***	20	**A3** (0.209) **A7** [0.178;0.288] **A10** [0.213;0.306]	[0.534;0.675]	[0.067;0.272]	[0.050;0.281]	[0.003;0.059]	[0.004;0.071]	[0.003;0.144]
		Cluster 2	15	**A3** [0.125;0.454] **A7** [0.181;0.236] **A10** [0.226;0.282]	[0.107;0.691]	[0.031;0.621]	[0.042;0.302]	[0.005;0.155]	[0.012;0.105]	[0.006;0.244]
80	27	***Cluster 1***	17	**A3** (0.273) **A10** [0.206;0.447]	[0.487;0.664]	[0.043;0.312]	[0.030;0.266]	[0.003;0.107]	[0.002;0.055]	[0.007;0.156]
		Cluster 2	10	**A1** [0.181;0.399] **A3** [0.126;0.271]	[0.070;0.573]	[0.020;0.400]	[0.046;0.670]	[0.009;0.409]	[0.018;0.427]	[0.029;0.480]

From a medical perspective, the clustering analysis tells the healthcare provider, based on the patient's age group, the expected degree of fluctuation in the patient's preferences. Also, as shown in Table 21, the set of screening alternatives viable for each age group is less than the ten colorectal cancer screening options currently available. So, identifying the appropriate alternatives for each age group could simplify the medical decision making process, by focusing only on the options that are likely to be more important for the patient. Having additional numerical information about how important the initially most preferred screening alternative is for the patient could help the care provider when discussing other alternatives with the patient.

Our current approach for the sensitivity and stability analyses for multi-criteria decision making models is based on the assumption that we perturb simultaneously pairs or triplets of criteria. That is the reason why we also identified, for each age group and cluster, the range of values for the six criteria used to compare the ten colorectal cancer screening options. The combined information with respect to the most preferred screening option, the associated interval of priorities, and the range of values for the six criteria could provide guidance towards the identification of the type of any new patient within a certain age group.

From a clinical perspective, our future results will help the healthcare provider to construct guidelines that will enable a better understanding of patients' preferences regarding the colorectal cancer screening options.

5.0 CONCLUSIONS AND FURTHER RESEARCH

In this dissertation, we extended the sensitivity and stability analysis for multi-criteria decision models, such as the Analytical Hierarchy Process/Analytical Network Process, previously developed by May et al. (2013). We applied our new methodology to a real medical decision making problem, and showed how the results of the methodology could be used in actual clinical situations.

After studying multiple simple ANP models, we found that the boundaries between the preference regions are typically nonlinear, so that using a single hyperplane to separate adjacent preference regions, as was done in May et al., may introduce unacceptably large errors in defining the separating boundaries between regions. We developed an algorithm to approximate the nonlinear boundaries using piecewise linear functions for the two-criteria sensitivity and stability analysis. A measure of the goodness-of-fit of the proposed algorithm measures the increased precision of the piecewise-linear approximation. In order to accommodate three-criteria perturbations, we used an algorithm that generates a triangular mesh to approximate the nonlinear boundaries between the preference regions.

Stability of the solution is an important concept for AHP/ANP models. Stability can be defined for the initially most preferred alternative and also for the other alternatives that appear in the perturbation space as criteria are changed simultaneously. The *core stability*, which is directly related to the initially preferred alternative, was measured by the largest sphere, for the two-criteria case, or by the largest hypersphere or 3-sphere, for the three-criteria case, that can be inscribed around the origin of the perturbation space. To measure the *solution stability*, we proposed the adoption of ellipsoids to describe it, rather than spheres, because they capture more of the perturbation space associated with a given alternative. To obtain the maximal volume ellipsoids that can be inscribed within a convex set we used a convex optimization program. Due to the nonlinearity of the boundaries, some of the preference regions may not be convex. Currently, in the literature, we are not aware of the existence of an optimization model to solve the problem of inscribing a maximal volume ellipsoid in a non-convex region. We proposed two nonlinear programming models to approximate it in a dimensional space greater than or equal to two.

A set of sensitivity and stability measures were defined to enable the application of the methodology to practical problems. The theoretical guidelines constitute a starting point for the application of our analysis to multi-criteria decision making problems. The practical insights that result from our methodology are specific to each real-life problem.

To illustrate our approach, we applied the methodology developed to actual data from a medical decision making problem involving the task of determining the best screening option for a patient having an average risk for colorectal cancer. Single patient sensitivity and stability analyses were developed for situations in which pairs and triplets of criteria were perturbed simultaneously. Important theoretical and medical insights were identified, to validate the importance of sensitivity and stability analysis for multi-criteria decision making models when applied to real-life situations. Our findings revealed that the results of the analysis of a patient's preferences may be useful to the healthcare provider. Knowing how a patient's preferences change based on additional knowledge might help the care provider to guide the patient towards a specific medical decision that might not be the patient's initially most preferred screening option. The comparison between two randomly chosen patients revealed that the generalization of the sensitivity and stability analysis to groups of patients with similar characteristics might be approached by using *age* as a discriminatory factor. Preliminary results of the generalization to groups of patients, and the medical implications, are presented in Section 4.5.

Currently, we are working on performing the two-criteria individualized sensitivity and stability analyses for all 395 patients within our dataset. The information provided by the analysis of preferences will be added to the current results, in order to enable the prediction of preference behavior of a new patient, based on our generalization.

Our current analysis is limited to the study of the sensitivity and stability of preferences as two and three criteria are changed simultaneously. Further work of the theory developed in this dissertation might include:

(1) Extending the methodology to the *n*-dimensional case – *n* criteria and *m* alternatives within a given network;

(2) Determining a more efficient way of generating the perturbation space using sparse grid stochastic collocation method for the *n*-dimensional case;

(3) Deriving practical guidelines from the stability measures that could be used in the medical decision making process or other decision processes involving multi-criteria decision making models;

(4) Studying the numerical accuracy of the approximation of the solution stability using ellipsoids, and assessing if the sensitivity and stability analysis is affected by the quality of the ellipsoid-based approximation to the region;

(5) Determining if the set of alternatives that dominate a region in the perturbation space is a function of the dimensionality of the perturbation space;

(6) Studying how the direction of fastest change in preferences is affected by the dimension of the perturbation space;

(7) Identifying the "optimal" number of criteria that should be perturbed simultaneously, from a practical perspective;

(8) Addressing how to preserve the direction of fastest change as the number of criteria perturbed increases;

(9) Exploring how stability results change with changes in patient preferences over a neighborhood of similar matrices;

(10) Showing how insights derived for a particular individual may be applied, perhaps with minor adjustments, to patients with sufficiently similar preferences;

(11) Extending the single patient model to a team-based architecture, in which the doctors' expertise is combined with patient preferences, resulting in a *shared decision making model*. The result of a shared decision making model is an individualized patient choice that may be closely tailored to a patient's expressed desires, presumably resulting in improved patient satisfaction and superior clinical outcomes.

APPENDIX A. THE SYSTEMS OF INEQUALITIES THAT DEFINE THE REGIONS WHERE THE ALTERNATIVES DOMINATE

Region A1

$$\begin{cases} 0.327x - y \geq 0.412, & x \in [-0.998;-0.9] \\ 0.3x - y \geq 0.436, & x \in (-0.9;-0.8] \\ 0.28x - y \geq 0.452, & x \in (-0.8;-0.7] \\ 0.245x - y \geq 0.477, & x \in (-0.7;-0.602] \\ 0.229x - y \geq 0.486, & x \in (-0.602;-0.506] \\ 0.208x - y \geq 0.497, & x \in (-0.506;-0.4] \\ 0.191x - y \geq 0.503, & x \in (-0.4;-0.306] \\ 0.173x - y \geq 0.509, & x \in (-0.306;-0.202] \\ 0.157x - y \geq 0.512, & x \in (-0.202;-0.1] \\ 0.115x - y \geq 0.516, & x \in (-0.1;0.004] \\ -0.725x - y \geq 0.513, & x \in (0.004;0.106] \\ -0.941x - y \geq 0.49, & x \in (0.106;0.276] \\ -1.083x - y \geq 0.451, & x \in (0.276;0.3] \\ -1.18x - y \geq 0.422, & x \in (0.3;0.4] \\ -1.303x - y \geq 0.373, & x \in (0.4;0.466] \end{cases}$$

Region A2

$$\begin{cases} -0.154x - y \geq -0.089, & x \in (-0.134;-0.1 \\ -0.132x - y \geq -0.092, & x \in (-0.316;-0.1 \\ -0.135x - y \geq -0.091, & x \in (-0.524;-0.3 \\ -0.14x - y \geq -0.089, & x \in (-0.61;-0.52 \\ -0.137x - y \geq -0.09, & x \in (-0.712;-0.6 \\ -0.143x - y \geq -0.086, & x \in (-0.908;-0.7 \\ -0.146x - y \geq -0.083, & x \in [-0.998;-0.9 \\ 0.333x - y \leq 0.1, & x \in (-0.288;-0.285] \\ 0.286x - y \leq 0.114, & x \in (-0.302;-0.28 \\ 0.32x - y \leq 0.103, & x \in (-0.402;-0.302 \\ 0.327x - y \leq 0.101, & x \in (-0.5;-0.402] \\ 0.32x - y \leq 0.104, & x \in (-0.6;-0.5] \\ 0.34x - y \leq 0.092, & x \in (-0.9;-0.6] \\ 0.347x - y \leq 0.086, & x \in [-0.998,-0.9] \\ 0.313x - y \leq -0.146, & x \in (-0.2;-0.12] \\ 0.27x - y \leq -0.137, & x \in (-0.3;-0.2] \\ 0.24x - y \leq -0.128, & x \in (-0.4;-0.3] \\ 0.22 - y \leq -0.12, & x \in [-0.556;-0.4] \\ -0.661x - y \geq 0.367, & x \in [-0.556;-0.5 \\ -0.712x - y \geq 0.396, & x \in (-0.5;-0.4] \\ -0.74x - y \geq 0.407, & x \in (-0.4;-0.3] \\ -0.733x - y \geq 0.405, & x \in (-0.3;-0.285 \end{cases}$$

Region A3

$$\begin{cases} -0.095x - y \geq -0.096, & x \in [-0.121;-0.1] \\ -0.098x - y \geq -0.096, & x \in (-0.1;0.002] \\ -0.14x - y \geq -0.096, & x \in (0.002;0.102] \\ -0.105x - y \geq -0.093, & x \in (0.102;0.216] \\ -0.07x - y \geq -0.085, & x \in (0.216;0.302] \\ -0.038x - y \geq -0.076, & x \in (0.302;0.395] \\ -169.712x - y \geq -67.698, & x \in [0.395;0.401] \\ -10.182x - y \geq -3.942, & x \in [0.382;0.4] \\ -0.8x - y \leq 0.022, & x \in (0.39;0.4] \\ -0.477x - y \leq 0.148, & x \in (0.302;0.39] \\ -0.471x - y \leq 0.15, & x \in (0.2;0.302] \\ -0.48x - y \leq 0.148, & x \in (0.1;0.2] \\ -0.5x - y \leq 0.146, & x \in (0;0.1] \\ 0.173x - y \leq 0.146, & x \in (-0.104;0] \\ 0.167x - y \leq 0.147, & x \in (-0.2;-0.104] \\ 0.171x - y \leq 0.146, & x \in (-0.282;-0.2] \\ 0.333x - y \leq 0.1, & x \in [-0.285;-0.282] \\ 0.313x - y \geq -0.146, & x \in (-0.2;-0.12] \\ 0.27x - y \geq -0.137, & x \in (-0.3;-0.2] \\ 0.24x - y \geq -0.128, & x \in (-0.4;-0.3] \\ 0.22 - y \geq -0.12, & x \in [-0.556;-0.4] \\ -0.661x - y \leq 0.367, & x \in [-0.556;-0.5] \\ -0.712x - y \leq 0.396, & x \in (-0.5;-0.4] \\ -0.74x - y \leq 0.407, & x \in (-0.4;-0.3] \\ -0.733x - y \leq 0.405, & x \in (-0.3;-0.285] \end{cases}$$

Region A4

$$\begin{cases} -0.154x - y \leq -0.089, & x \in (-0.134;-0.1 \\ -0.132x - y \leq -0.092, & x \in (-0.316;-0.1 \\ -0.135x - y \leq -0.091, & x \in (-0.524;-0.3 \\ -0.14x - y \leq -0.089, & x \in (-0.61;-0.52 \\ -0.137x - y \leq -0.09, & x \in (-0.712;-0.6 \\ -0.143x - y \leq -0.086, & x \in (-0.908;-0.7 \\ -0.146x - y \leq -0.083, & x \in [-0.998;-0.9 \\ -0.095x - y \leq -0.096, & x \in [-0.121;-0. \\ -0.098x - y \leq -0.096, & x \in (-0.1;0.002 \\ -0.14x - y \leq -0.096, & x \in (0.002;0.102 \\ -0.105x - y \leq -0.093, & x \in (0.102;0.21 \\ -0.07x - y \leq -0.085, & x \in (0.216;0.302 \\ -0.038x - y \leq -0.076, & x \in (0.302;0.39 \\ 1.167x - y \leq 0.381, & x \in [0.396;0.4] \\ 1.131x - y \leq 0.366, & x \in (0.4;0.522] \\ 1.273x - y \leq 0.44, & x \in (0.522;0.61] \\ 1.308x - y \leq 0.462, & x \in (0.61;0.714] \\ 1.568x - y \leq 0.648, & x \in (0.714;0.802] \\ 1.875x - y \leq 0.894, & x \in (0.802;0.998] \end{cases}$$

Region A5

$$\begin{cases} 1.167x - y \geq 0.381, & x \in [0.396; 0.4] \\ 1.131x - y \geq 0.366, & x \in (0.4; 0.522] \\ 1.273x - y \geq 0.44, & x \in (0.522; 0.61] \\ 1.308x - y \geq 0.462, & x \in (0.61; 0.714] \\ 1.568x - y \geq 0.648, & x \in (0.714; 0.802] \\ 1.875x - y \geq 0.894, & x \in (0.802; 0.998] \\ -169.712x - y \leq -67.698, & x \in [0.395; 0.401] \\ -10.182x - y \leq -3.942, & x \in [0.382; 0.4] \\ -3.538x - y \leq -1.073, & x \in [0.4; 0.504] \\ -3.575x - y \leq -1.092, & x \in (0.504; 0.584] \end{cases}$$

Region A6

$$\begin{cases} 0.327x - y \leq 0.412, & x \in [-0.998; -0.9] \\ 0.3x - y \leq 0.436, & x \in (-0.9; -0.8] \\ 0.28x - y \leq 0.452, & x \in (-0.8; -0.7] \\ 0.245x - y \leq 0.477, & x \in (-0.7; -0.602] \\ 0.229x - y \leq 0.486, & x \in (-0.602; -0.506] \\ 0.208x - y \leq 0.497, & x \in (-0.506; -0.4] \\ 0.191x - y \leq 0.503, & x \in (-0.4; -0.306] \\ 0.173x - y \leq 0.509, & x \in (-0.306; -0.202] \\ 0.157x - y \leq 0.512, & x \in (-0.202; -0.1] \\ 0.115x - y \leq 0.516, & x \in (-0.1; 0.004] \\ -0.725x - y \leq 0.513, & x \in (0.004; 0.106] \\ -0.941x - y \leq 0.49, & x \in (0.106; 0.276] \\ -1.083x - y \leq 0.451, & x \in (0.276; 0.3] \\ -1.18x - y \leq 0.422, & x \in (0.3; 0.4] \\ -1.303x - y \leq 0.373, & x \in (0.4; 0.466] \\ 0.333x - y \geq 0.1, & x \in (-0.288; -0.285] \\ 0.286x - y \geq 0.114, & x \in (-0.302; -0.288] \\ 0.32x - y \geq 0.103, & x \in (-0.402; -0.302] \\ 0.327x - y \geq 0.101, & x \in (-0.5; -0.402] \\ 0.32x - y \geq 0.104, & x \in (-0.6; -0.5] \\ 0.34x - y \geq 0.092, & x \in (-0.9; -0.6] \\ 0.347x - y \geq 0.086, & x \in [-0.998, -0.9] \\ -0.8x - y \geq 0.022, & x \in (0.39; 0.4] \\ -0.477x - y \geq 0.148, & x \in (0.302; 0.39] \\ -0.471x - y \geq 0.15, & x \in (0.2; 0.302] \\ -0.48x - y \geq 0.148, & x \in (0.1; 0.2] \\ -0.5x - y \geq 0.146, & x \in (0; 0.1] \\ 0.173x - y \geq 0.146, & x \in (-0.104; 0] \\ 0.167x - y \geq 0.147, & x \in (-0.2; -0.104] \\ 0.171x - y \geq 0.146, & x \in (-0.282; -0.2] \\ 0.333x - y \geq 0.1, & x \in [-0.285; -0.282] \\ -3.538x - y \geq -1.073, & x \in [0.4; 0.504] \\ -3.575x - y \geq -1.092, & x \in (0.504; 0.584] \end{cases}$$

APPENDIX B. THE OUTPUT MATRIX M AND THE ASSOCIATED PLANAR EQUATIONS THAT DEFINE THE REGIONS WHERE THE ALTERNATIVES DOMINATE

Region A1

$$M_{12} = \begin{bmatrix} \boldsymbol{V1} & \boldsymbol{V2} & \boldsymbol{V3} \\ 208 & 282 & 263 \\ 171 & 157 & 147 \\ & \cdots & \\ 468 & 482 & 428 \\ 483 & 493 & 482 \end{bmatrix} \rightarrow \begin{cases} 0.0016x + 0.0024y + 0.0032z + 0.00104 \leq 0 \\ -0.066x + 0.0288y + 0.0056z + 0.0547 \leq 0 \\ \ldots \\ -0.0036x + 0.0032y + 0.0028z + 0.00399 \leq 0 \\ -0.0004x + 0.0004y + 0.0004z + 0.00041 \leq 0 \end{cases}$$

Total number of planar equations describing region A1 is 922.

Region A2

$$M_{12} = \begin{bmatrix} \boldsymbol{V1} & \boldsymbol{V2} & \boldsymbol{V3} \\ 208 & 282 & 263 \\ 171 & 157 & 147 \\ & \cdots & \\ 468 & 482 & 428 \\ 483 & 493 & 482 \end{bmatrix} \rightarrow \begin{cases} 0.0016x + 0.0024y + 0.0032z + 0.00104 \geq 0 \\ -0.066x + 0.0288y + 0.0056z + 0.0547 \geq 0 \\ \ldots \\ -0.0036x + 0.0032y + 0.0028z + 0.00399 \geq 0 \\ -0.0004x + 0.0004y + 0.0004z + 0.00041 \geq 0 \end{cases}$$

$$M_{23} = \begin{bmatrix} \boldsymbol{V1} & \boldsymbol{V2} & \boldsymbol{V3} \\ 17 & 16 & 36 \\ 13 & 17 & 28 \\ & \cdots & \\ 130 & 107 & 134 \\ 146 & 130 & 142 \end{bmatrix} \rightarrow \begin{cases} -0.0004x - 0.0028y + 0.0028z + 0.00346 \geq 0 \\ -0.0012x - 0.0088y + 0.0084z + 0.01055 \geq 0 \\ \ldots \\ -0.0004x - 0.0032y + 0.0028z + 0.00363 \geq 0 \\ 0.0008x - 0.002y + 0.0032z + 0.0037 \geq 0 \end{cases}$$

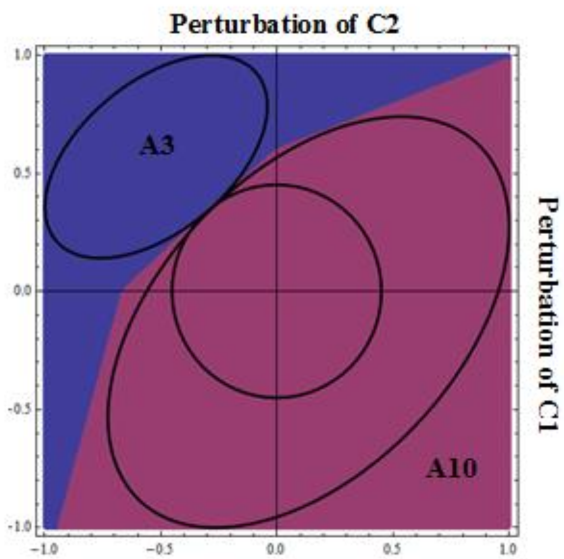
Total number of planar equations describing region A2 is 922+291 = 1213.

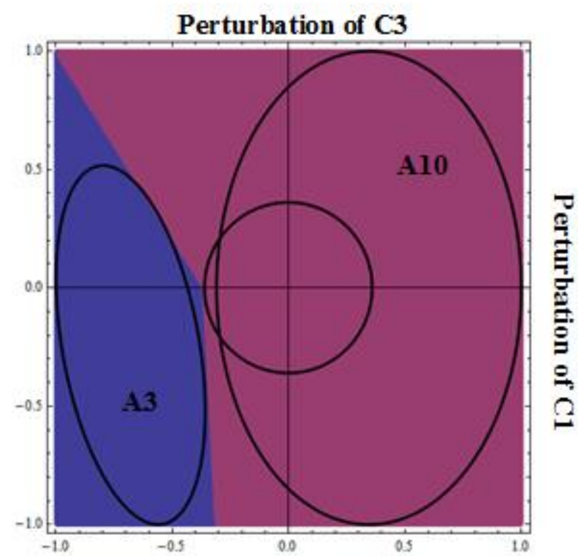
Region A3

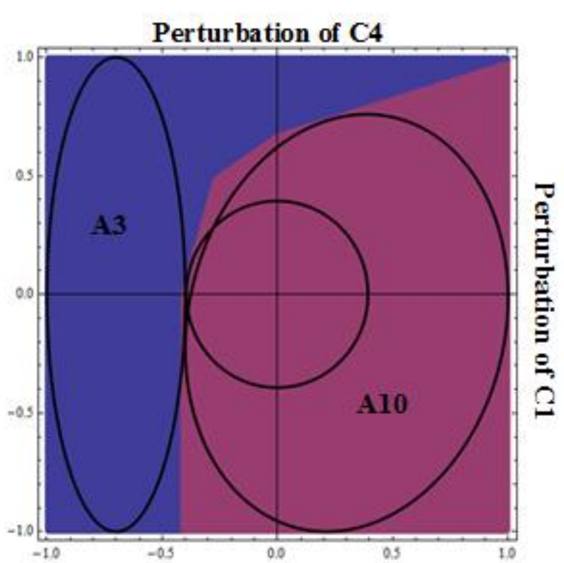
$$M_{23} = \begin{bmatrix} \boldsymbol{V1} & \boldsymbol{V2} & \boldsymbol{V3} \\ 17 & 16 & 36 \\ 13 & 17 & 28 \\ & \cdots & \\ 130 & 107 & 134 \\ 146 & 130 & 142 \end{bmatrix} \rightarrow \begin{cases} -0.0004x - 0.0028y + 0.0028z + 0.00346 \leq 0 \\ -0.0012x - 0.0088y + 0.0084z + 0.01055 \leq 0 \\ \ldots \\ -0.0004x - 0.0032y + 0.0028z + 0.00363 \leq 0 \\ 0.0008x - 0.002y + 0.0032z + 0.0037 \leq 0 \end{cases}$$

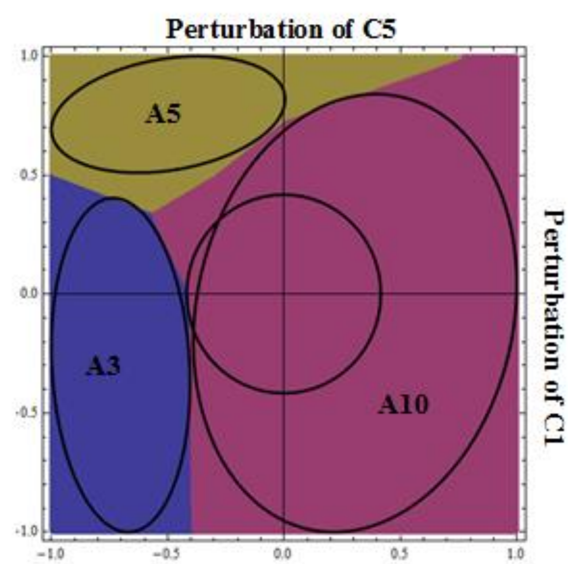
Total number of planar equations describing region A3 is 291.

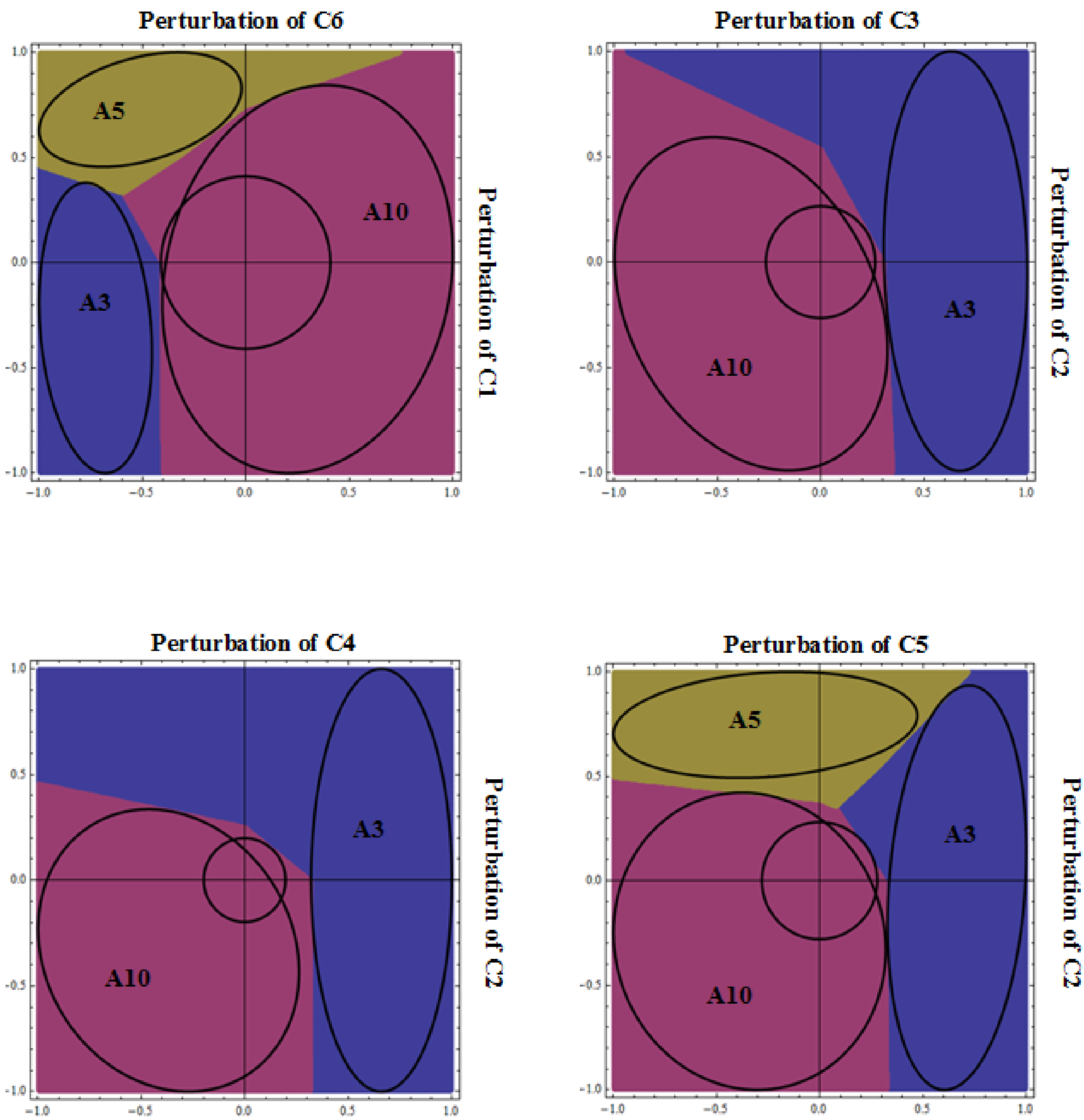
APPENDIX C. CORE (SPHERE) AND SOLUTION STABILITY (ELLIPSOIDS) REGIONS WITHIN THE PERTURBATION SPACE FOR ALL TWO-CRITERIA PAIRS

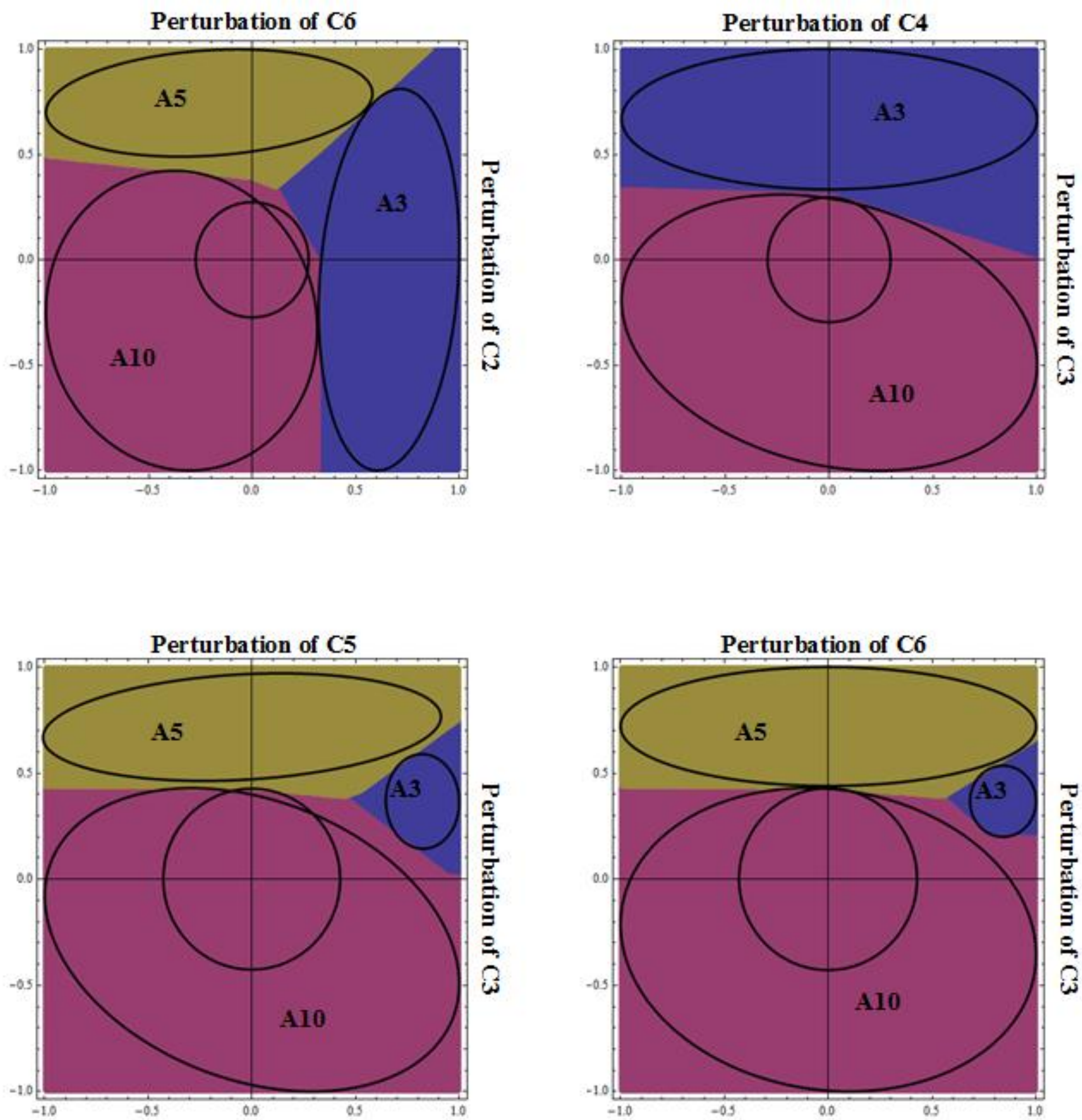


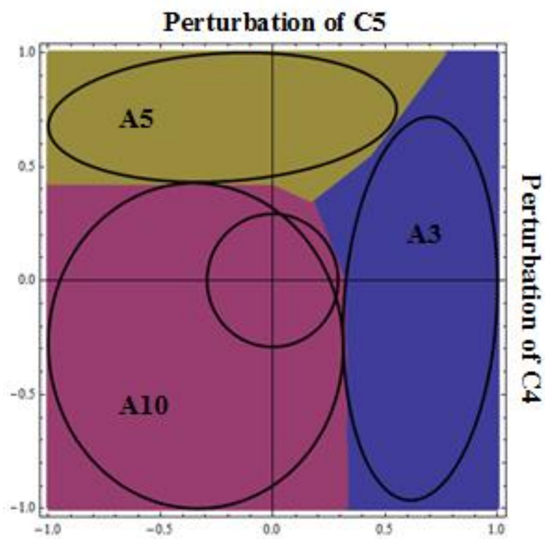


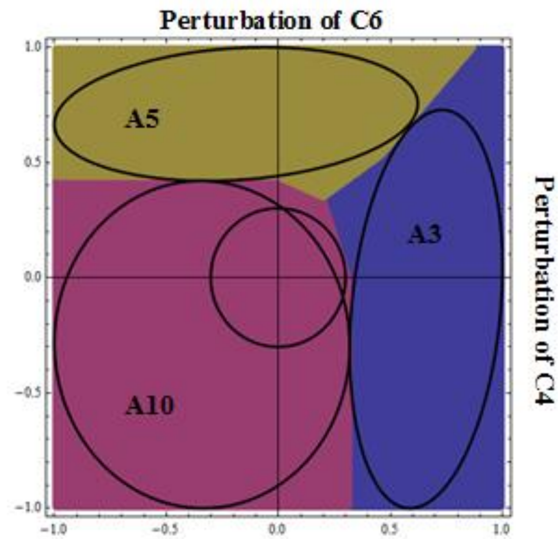


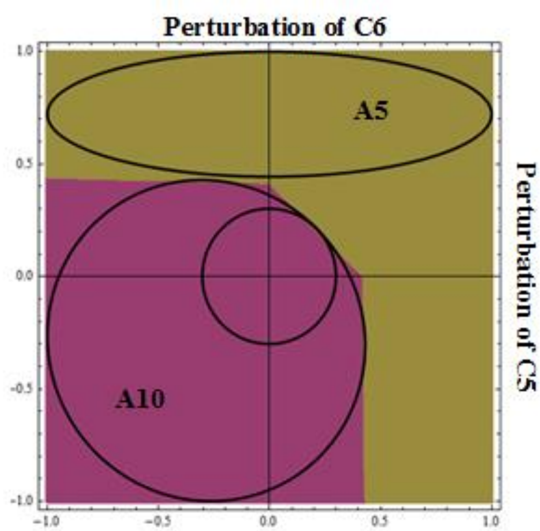



Perturbation of C6
A5
A10
A3
Perturbation of C1
Perturbation of C3
A10
A3
Perturbation of C2
Perturbation of C4
A3
A10
Perturbation of C2
Perturbation of C5
A5
A3
A10
Perturbation of C2


Perturbation of C6
A5
A3
A10
Perturbation of C2
Perturbation of C4
A3
A10
Perturbation of C3
Perturbation of C5
A5
A3
A10
Perturbation of C3
Perturbation of C6
A5
A3
A10
Perturbation of C3


Perturbation of C5
A5
A3
A10
Perturbation of C4
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0


Perturbation of C6
A5
A3
A10
Perturbation of C4
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0


Perturbation of C6
A5
A10
Perturbation of C5
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0

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