# ANALYSIS AND PDE ON METRIC MEASURE SPACES: SOBOLEV FUNCTIONS AND VISCOSITY SOLUTIONS

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**ABSTRACT**

**ANALYSIS AND PDE ON METRIC MEASURE SPACES: SOBOLEV FUNCTIONS AND VISCOSITY SOLUTIONS**

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We study analysis and partial differential equations on metric measure spaces by investigating the properties of Sobolev functions or Sobolev mappings and studying the viscosity solutions to some partial differential equations.

This manuscript consists of two parts. The first part is focused on the theory of Sobolev spaces on metric measure spaces. We investigate the continuity of Sobolev functions in the critical case in some general metric spaces including compact connected one-dimensional spaces and fractals. We also construct a large class of pathological $n$-dimensional spheres in $\mathbb{R}^{n+1}$ by showing that for any Cantor set $C \subset \mathbb{R}^{n+1}$ there is a topological embedding $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ of the Sobolev class $W^{1,n}$ whose image contains the Cantor set $C$.

The second part is focused on the theory of viscosity solutions for nonlinear partial differential equations in metric spaces, including the Heisenberg group as an important special case. We study Hamilton-Jacobi equations on the Heisenberg group $\mathbb{H}$ and show uniqueness of viscosity solutions with exponential growth at infinity. Lipschitz and horizontal convexity preserving properties of these equations under appropriate assumptions are also investigated. In this part, we also study a recent game-theoretic approach to the viscosity solutions of various equations, including deterministic and stochastic games. Based on this interpretation, we give new proofs of convexity preserving properties of the mean curvature flow equations and normalized $p$-Laplace equations in the Euclidean space.

**Keywords:** Sobolev spaces, metric measure spaces, Heisenberg group, viscosity solution, convexity preserving, game-theoretic methods.

## TABLE OF CONTENTS


PREFACE . . . . . . . . . . ix

**1.0 INTRODUCTION** . . . . . . . . . . 1

**2.0 REVIEW OF METRIC MEASURE SPACES** . . . . . . . . . . 10

2.1 Basic definitions in metric measure spaces . . . . . . . . . . 10

2.2 Curves in metric measure spaces . . . . . . . . . . 14

2.3 Rectifiability of one-dimensional compact and connected spaces . . . . . . . 16

**3.0 SOBOLEV FUNCTIONS ON METRIC MEASURE SPACES** . . . . . 19

3.1 Sobolev functions in the Euclidean space . . . . . . . . . . 19

3.1.1 Definitions and basic properties . . . . . . . . . . 19

3.1.2 Equivalent characterizations . . . . . . . . . . 23

3.2 Sobolev functions on the metric measure spaces . . . . . . . . . . 24

3.2.1 Sobolev spaces $M^{1,p}$ . . . . . . . . . . 25

3.2.2 Sobolev spaces $P^{1,p}$ . . . . . . . . . . 26

3.2.3 Sobolev spaces $N^{1,p}$ . . . . . . . . . . 27

3.3 Spaces supporting the Poincaré inequality . . . . . . . . . . 29

**4.0 SOBOLEV FUNCTIONS IN THE CRITICAL CASE ON METRIC MEASURE SPACES** . . . . . . . . . . 32

4.1 On Compact, connected metric spaces with $\mathcal{H}^1(X) < \infty$ . . . . . . . . . . 33

4.1.1 Absolutely continuous functions on the metric spaces . . . . . . . . . . 33

4.1.2 Absolutely continuous characterization of Sobolev functions . . . . . . 34

4.1.3 Spaces supporting the Poincaré inequality . . . . . . . . . . 39

4.2 On $s$-Ahlfors regular metric spaces with $s \leq 1$ . . . . . . . . . . 41

5.0 SOBOLEV EMBEDDING OF A SPHERE CONTAINING AN ARBITRARY CANTOR SET IN THE IMAGE . . . 46
5.1 Introduction . . . 46
5.2 Cantor sets and trees . . . 48
5.2.1 Ternary Cantor set . . . 48
5.2.2 Cantor trees . . . 49
5.3 Sobolev tentacles . . . 53
5.4 Main result . . . 55
5.5 Uncountable inequivalent embeddings . . . 62
6.0 SEMILINEAR EVOLUTION EQUATIONS IN THE HEISENBERG GROUP . . . 65
6.1 The Heisenberg group $\mathbb{H}$ . . . 66
6.1.1 Definitions and basic properties . . . 66
6.1.2 Metrics on $\mathbb{H}$ . . . 68
6.1.3 Lipschitz continuity . . . 70
6.1.4 Horizontal convexity . . . 71
6.2 Review of viscosity solutions . . . 72
6.2.1 Viscosity solutions in $\mathbb{R}^n$ . . . 72
6.2.2 Viscosity solutions in $\mathbb{H}$ . . . 76
6.3 Uniqueness of unbounded solutions . . . 78
6.4 Lipschitz preserving properties . . . 85
6.4.1 Right invariant Lipschitz continuity preserving . . . 87
6.4.2 A special class of Hamilton-Jacobi equations . . . 89
6.5 Convexity preserving properties . . . 92
6.5.1 Right invariant h-convexity preserving . . . 93
6.5.2 Left invariant h-convexity preserving . . . 100
6.6 More examples . . . 101
7.0 A GAME-THEORETIC PROOF OF CONVEXITY PRESERVATION OF SOME NONLINEAR DIFFERENTIAL EQUATIONS . . . 103
7.1 Convexity preserving properties . . . 103

7.2 Game interpretations of viscosity solutions . . . . . . . . . . . . . . . . . . . . 105
7.2.1 Tug of War game and normalized parabolic $p$-Laplace equations . . . 105
7.2.2 The deterministic game and the mean curvature flow equations . . . . 107
7.2.3 A modified game . . . . . . . . . . . . . . . . . . . . . . . . . . . . 109
7.3 Game-theoretic proofs of convexity-preserving properties . . . . . . . . . . . 116
7.3.1 Convexity-preserving for the solutions to the normalized parabolic $p$-Laplace equations . . . . . . . . . . . . . . . . . . . . . . . . . . 116
7.3.2 Convexity-preserving for level sets of the mean curvature flow equations 118
7.3.3 Convexity-preserving for the solutions to the mean curvature flow equations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 121
7.3.4 Convexity-preserving with Neumann boundary condition of the mean curvature flow equations . . . . . . . . . . . . . . . . . . . . . . . . 125
7.3.4.1 Billiard games for the Neumann condition . . . . . . . . . . . 125
7.3.4.2 Convexity of level sets . . . . . . . . . . . . . . . . . . . . . 126
**BIBLIOGRAPHY** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 132

## PREFACE

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# 1.0 INTRODUCTION

Analysis on metric measure spaces is a new area in contemporary mathematics that has been developed since the nineties. Because of its general setting it plays a fundamental role in the unification of methods that previously have been developed separately for different areas of mathematics [4, 5, 55, 59]. Besides the pure mathematical importance of these problems, analysis on metric spaces has been widely applied to image reconstruction theory, optimal transport, control theory, robotics and mathematical biology.

In the first part of this manuscript, we study the Sobolev functions on general metric measure spaces. The theory of Sobolev spaces is one of the main tools in analysis on metric spaces. Let $(X, d, \mu)$ denote a metric space equipped with a doubling measure $\mu$. Given a Borel function $u : X \to \mathbb{R}$, we say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of $u$ if

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$$

for every rectifiable curve $\gamma : [a, b] \to X$. The *Newtonian-Sobolev space* $N^{1,p}(X)$ is defined as the collection of all $p$-integrable functions with $p$-integrable upper gradients [112]. The notion of upper gradient is an important generalization of the length of the gradient.

While the $N^{1,p}$ spaces can be defined on general metric measure spaces, without additional information about the structure of the underlying metric space the theory is not interesting. Indeed, if there are no rectifiable curves in the space then $N^{1,p} = L^p$. On the other hand the theory of $N^{1,p}$ spaces is very rich when $X$ supports the so called $p$-Poincaré inequality. Spaces supporting Poincaré inequalities introduced by Heinonen and Koskela [56], provide a good structure to study first-order analysis. Metric measure spaces supporting an abstract Poincaré inequality are highly connected. Spaces like fractals with limited connectedness do not belong to this class. However, in this case, one can define the *Hajłasz-Sobolev space* $M^{1,p}(X)$ as the collection of all $p$-integrable functions for which there exist nonnegative $g \in L^p(X)$ such that

$$|u(x) - u(y)| \leq d(x, y)\big(g(x) + g(y)\big), \tag{1.0.1}$$

for $x, y \in X \setminus E$ where $\mu(E) = 0$. We denote the collection of nonnegative Borel function $g$ as above by $D(u)$.

There are many other extensions of the classical theory of Sobolev spaces to the settings of metric measure spaces [4, 5, 24, 46, 49, 112]. Under suitable assumptions, some of or all these Sobolev spaces defined via different approaches coincide [59, 74]. We will review some basic definitions and theorems of Sobolev functions on metric measure spaces in Chapter 2 and Chapter 3.

In the Euclidean space, Sobolev functions $u \in W^{1,1}([a, b])$ are absolutely continuous. When $n \geq 2$, $u(x) = \log|\log|x||$ provides an example of a discontinuous Sobolev functions in $W^{1,n}(B^n(0, e^{-1}))$ in the $n$-dimensional space. We are interested in investigating the continuity of the Sobolev functions in the critical case, i.e., $p = n$, on metric measure spaces. From the above examples, we conjecture that the Sobolev functions in the critical case may lose the continuity if the Hausdorff dimension is greater than 1. In Chapter 4, we first generalize the characterization of the Sobolev functions by absolute continuity to some metric spaces $X$ that are compact, connected and have finite one-dimensional Hausdorff measure. Note that there are many extensions of the definitions of absolutely continuous functions to more general settings and much research has been done on the connections between these absolutely continuous functions and Sobolev functions. Malý [93] extended this definition to $n$-dimensional Euclidean space by replacing the pairwise disjoint subintervals $[a_i, b_i]$ with pairwise disjoint balls $B_i$ and $|u(b_i) - u(a_i)|$ with $(\operatorname{osc}(u(B_i)))^n$. Kinnunen and Tuominen [76, Theorem 4] proved that on a doubling space $X$ functions in the Hajłasz Sobolev space $M^{1,1}(X)$ [46] coincide with Hölder continuous functions outside a set of small Hausdorff content.

The definition of classical absolutely continuous functions can be modified in the following way in our settings. Recall that a simple curve is a curve without self intersections. In what follows $\mathcal{H}^s$ stands for the $s$-dimensional Hausdorff measure.

**Definition 1.0.1** ([121])**.** Let $(X, d, \mathcal{H}^1)$ be a compact, connected metric measure space with $\mathcal{H}^1(X) < \infty$. A function $u : X \to \mathbb{R}$ is said to be *absolutely continuous* (denoted $u \in AC(X)$) if for any $\epsilon > 0$, there is a positive number $\delta$ such that

$$\sum_i |u \circ \gamma_i(\ell_i) - u \circ \gamma_i(0)| < \epsilon,$$

for any finite collection of pairwise disjoint arc-length parametrized simple curves $\gamma_i \colon [0, \ell_i] \to X$ satisfying $\sum_i \ell_i < \delta$.

In the real line, absolutely continuous functions are differentiable almost everywhere and the fundamental theorem of calculus holds. For a compact, connected metric space $X$ with $\mathcal{H}^1(X) < \infty$, there exists a good parametrization (Theorem 2.3.3) that can decompose this space as a countable union of pairwise disjoint simple curves and a set with 1-dimensional Hausdorff measure zero [5, Theorem 4.4.8]. We have the following result.

**Theorem 1.0.2** ([121])**.** *Let $(X, d, \mathcal{H}^1)$ be a compact, connected metric measure space with $\mathcal{H}^1(X) < \infty$. If $u \in AC(X)$, then there is an upper gradient $g \in L^1(X)$ of $u$, such that, for any rectifiable curve $\gamma : [a, b] \to X$, we have*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g.$$

For an absolutely continuous function $u \in AC(X)$, if the upper gradient $g$ in Theorem 1.0.2 is $p$-integrable, we write $u \in AC^p(X)$. By definition, it follows that $AC^p(X) \cap L^p(X) \subset N^{1,p}(X)$. On the other hand, we can verify that all Sobolev functions in $N^{1,p}(X)$ belong to the class $AC(X)$ and the associated upper gradients $g$ are $p$-integrable. Thus, we have the following result.

**Theorem 1.0.3.** *Let $(X, d, \mathcal{H}^1)$ be a compact and connected metric measure space with $\mathcal{H}^1(X) < \infty$. Then $u \in N^{1,p}(X)$ if and only if $u \in AC^p(X)$ and $u \in L^p(X)$. In other words,*

$$N^{1,p}(X) = AC^p(X) \cap L^p(X).$$

We also prove that if a compact metric space is quasiconvex and 1-Ahlfors regular, then $X$ supports $p$-Poincaré inequality for $1 \le p < \infty$. Recall that a metric space $X$ is $s$-Ahlfors regular if there exists a constant $C_0 \ge 1$ such that $C_0^{-1} r^s \le \mu(B(r)) \le C_0 r^s$ for any ball $B(r) \subset X$ with $0 < r < \operatorname{diam}(X)$. A metric space is quasiconvex if there exists a constant $C$ such that any two points $x, y \in X$ can be joined by a rectifiable curve of length bounded by $Cd(x, y)$. It follows that absolutely continuous functions with $p$-integrable upper gradient can be identified with the Sobolev functions defined via different approaches.

**Theorem 1.0.4** ([121])**.** *Let $(X, d, \mathcal{H}^1)$ be a compact, quasiconvex and 1-Ahlfors regular metric measure space. Let $1 < p < \infty$. Then*

$$AC^p(X) \cap L^p(X) = N^{1,p}(X) = P^{1,p}(X) = M^{1,p}(X).$$

This result extends the characterization of Sobolev functions by absolute continuity known in the 1-dimensional intervals to more general 1-dimensional spaces. In the next theorem, we deal with some fractal type metric spaces. We prove the uniform continuity of Sobolev functions in $s$-Ahlfors regular spaces with $s \le 1$. Note that Ahlfors regular spaces include a large class of fractals generated by iterated function systems satisfying the open set condition.

**Theorem 1.0.5** ([122])**.** *Let $(X, d, \mathcal{H}^s)$ be an $s$-Ahlfors regular metric space and $0 < s \le 1$. If $u \in M^{1,s}(X, d, \mathcal{H}^s)$, then $u$ is uniformly continuous. Moreover, there exists a constant $C$, such that for any ball $B \subset X$,*

$$\operatorname{osc}_B |u| = \sup_{x,y \in B} |u(x) - u(y)| \le C \Big( \int_{2B} g^s d\mathcal{H}^s \Big)^{\frac{1}{s}},$$

*where $g \in D(u) \cap L^s(X)$.*

In Chapter 5, we generalized a construction of the famous Alexander horned sphere, which provides a counterexample to the Schoenflies Theorem in $\mathbb{R}^3$. Namely we managed to construct a Sobolev embedding from the $n$-dimensional sphere to the $(n+1)$-dimensional Euclidean space whose image contains an *arbitrary* Cantor set. Recall that a Cantor set is any compact, totally disconnected and perfect set. The main theorem can be stated as follows.

**Theorem 1.0.6** ([52]). *For any Cantor set $C \subset \mathbb{R}^{n+1}$, $n \geq 2$, there is an embedding $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that*

*(a) $f \in W^{1,n}(\mathbb{S}^n, \mathbb{R}^{n+1})$,*

*(b) $C \subset f(\mathbb{S}^n)$,*

*(c) $f^{-1}(C) \subset \mathbb{S}^n$ is a Cantor set of Hausdorff dimension zero,*

*(d) $f$ is a smooth diffeomorphism in $\mathbb{S}^n \setminus f^{-1}(C)$.*

Note that such a sphere $f(\mathbb{S}^n)$ can be very pathological due to topological complexity of wild Cantor sets in $\mathbb{R}^{n+1}$. Note also that the Cantor set $C$ may have positive $(n+1)$-dimensional Lebegue measure. The Sobolev regularity of such a homeomorphism has not been known previously. Moreover our construction generalizes not only the Alexander horned sphere, but also provides a new class of Sobolev homomorphisms without the Luzin property. We also prove in Theorem 5.5.1 that there are uncountably many such embeddings $f : \mathbb{S}^2 \to \mathbb{R}^3$ of class $W^{1,2}(\mathbb{S}^2, \mathbb{R}^3)$ which are not equivalent.

In the second part of this manuscript, we study viscosity solutions to some partial differential equations. Since 1980s, the theory of viscosity solution for nonlinear partial differential equations has been developed and applied to a wide range of fields. It provides existence and uniqueness of weak solutions to a very general class of fully nonlinear equations in the space of continuous functions [27, 80]. A recent trend is to extend this theory to metric measure spaces; see [3, 40] for some results of first order Hamilton-Jacobi equations on general metric spaces. We study viscosity solutions to a class of second order equations. We focus on the setting of sub-Riemannian manifolds [21], which not only play a very important role in general analysis on metric spaces but also has applications in many other fields including robotic control, neuroscience and digital image reconstruction. We aim to develop the viscosity solution theory for nonlinear parabolic equations in the Heisenberg group $\mathbb{H}$, which is known as the simplest example of sub-Riemmanian manifold.

In Chapter 6, we review some basic definitions and properties of Heisenberg group and the theory of viscosity solutions. Then we study viscosity solutions of the following semilinear parabolic equations

$$\begin{cases} u_t - \operatorname{tr}(A(\nabla_H^2 u)^*) + f(p, \nabla_H u) = 0 & \text{in } \mathbb{H} \times (0, \infty), \quad (1.0.2) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{H}, \quad (1.0.3) \end{cases}$$

in the Heisenberg group, where $A$ is a given $2 \times 2$ symmetric positive-semidefinite matrix and the function $f : \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following Lipschitz assumptions.

(A1) There exists $L_1 > 0$ such that

$$|f(p, w_1) - f(p, w_2)| \leq L_1 |w_1 - w_2| \tag{1.0.4}$$

for all $p \in \mathbb{H}$ and $w_1, w_2 \in \mathbb{R}^2$.

(A2) There exists $L_2(\rho) > 0$ depending on $\rho > 0$ such that

$$|f(p, w) - f(q, w)| \leq L_2(\rho) |p \cdot q^{-1}|_G \tag{1.0.5}$$

for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$ and all $w \in \mathbb{R}^2$.

Here $|\cdot|_G$ denotes the Korányi gauge in $\mathbb{H}$,

$$|p|_G = \left((x_p^2 + y_p^2)^2 + 16 z_p^2\right)^{\frac{1}{4}}$$

for all $p = (x_p, y_p, z_p) \in \mathbb{H}$. We first show uniqueness of viscosity solutions to the above equation with exponential growth at infinity.

**Theorem 1.0.7** (Uniqueness of solutions, [89])**.** *Assume that (A1) and (A2) hold. Let $u_0 \in C(\mathbb{H})$. Then there is at most one continuous viscosity solution $u$ of* (1.0.2)–(1.0.3) *satisfying the following exponential growth condition at infinity:*

*(G) For any $T > 0$, there exists $k > 0$ and $C_T > 0$ such that $|u(p, t)| \leq C_T e^{k\langle p \rangle}$ for all $(p, t) \in \mathbb{H} \times [0, T]$.*

Among many properties of the viscosity solutions, the Lipschitz and convexity preserving properties are known to be important for various linear and nonlinear parabolic equations arising in geometry, material sciences and image processing. Suppose that $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a solution of a certain parabolic equation with initial condition $u_0$. The property reads as follows: when $u_0$ is Lipschitz continuous (resp. convex), the unique solution $u(x, t)$ is Lipschitz (resp., convex) in $x$ as well for any $t \geq 0$. The convexity preserving property of viscosity solutions was proved to hold in a very general class of degenerate parabolic equations [39]. It is natural to ask whether the Lipschitz and convexity preserving properties also hold for nonlinear parabolic equations in the Heisenberg group. Note that the notion of convexity of functions in the Heisenberg group is known [30, 91]. More precisely, a function $u$ is said to be Lipschitz continuous in $\mathbb{H}$ if there exists $L > 0$ such that

$$|u(p) - u(q)| \leq L d_L(p, q)$$

for all $p, q \in \mathbb{H}$, and horizontally convex in $\mathbb{H}$ if

$$u(p \cdot h^{-1}) + u(p \cdot h) \geq 2u(p)$$

for any $p \in \mathbb{H}$ and any $h \in \mathbb{H}_0$, where

$$\mathbb{H}_0 = \{h \in \mathbb{H} : h = (h_1, h_2, 0) \text{ for } h_1, h_2 \in \mathbb{R}\}.$$

It turns out that in general such properties cannot be expected in the Heisenberg group. Some restrictions on the class of solutions proved to be necessary. In fact, we obtained the Lipschitz continuity and convexity preserving properties with respect to the right invariant metric $d_R(p, q) = |p \cdot q^{-1}|_G$, which is invariant only under right translations and therefore not equivalent to the usual gauge metric give by $d_L(p, q) = |p^{-1} \cdot q|_G$.

Let us also present our results in a simple case. A more general version of Lipschitz preserving is given in Theorem 6.4.2.

**Theorem 1.0.8** (Preservation of right invariant Lipschitz continuity, [89])**.** *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz. Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of*

$$u_t - \operatorname{tr}(A(\nabla_H^2 u)^*) + f(\nabla_H u) = 0 \quad \text{in } \mathbb{H} \times (0, \infty), \tag{1.0.6}$$

*with $u(\cdot, 0) = u_0(\cdot)$ satisfying the growth condition (G). If there exists $L > 0$ such that*

$$|u_0(p) - u_0(q)| \leq L d_R(p, q)$$

*for all $p, q \in \mathbb{H}$, then*

$$|u(p, t) - u(q, t)| \leq L d_R(p, q)$$

*for all $p, q \in \mathbb{H}$ and $t \geq 0$.*

For the case of first order Hamilton-Jacobi equations ($A = 0$), if in addition we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is in the form that $f(\xi) = m(|\xi|)$ with $m : \mathbb{R} \to \mathbb{R}$ locally uniformly continuous, then the Lipschitz preserving property of a bounded solution can be directly shown without the evenness assumption. We refer the reader to Theorem 6.4.4, which answers a question asked in [98]. A more general question on Lipschitz continuity of viscosity solutions was posed in [8], but it is not clear if our method here immediately applies to that general setting.

As for the case of $h$-convexity preserving property, we obtain the following:

**Theorem 1.0.9** (Preservation of right invariant $h$-convexity, [89])**.** *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz. Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of (1.0.6) with $u(\cdot, 0) = u_0(\cdot)$ satisfying the growth condition (G). Assume in addition that $f$ is concave in $\mathbb{R}^2$, i.e.,*

$$f(\xi) + f(\eta) \leq 2f\left(\frac{1}{2}(\xi + \eta)\right) \tag{1.0.7}$$

*for all $\xi, \eta \in \mathbb{R}^2$. If $u_0$ is right invariant $h$-convex in $\mathbb{H}$; that is,*

$$u_0(h^{-1} \cdot p) + u_0(h \cdot p) \geq 2u_0(p)$$

*for all $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$, then so is $u(\cdot, t)$ for all $t \geq 0$.*

Although the result above only concerns the right invariant definitions of Lipschitz continuity or convexity, we can obtain the left-invariant preserving properties under certain additional assumptions. For instance, the notions of left invariant and right invariant Lipschitz continuity or convexity are equivalent when functions are even or vertically even. This implies the Lipschitz or $h$-convexity preserving property of an even function or vertically even function.

On the other hand, there are also many affirmative examples for Lipschitz and convexity preserving in the Heisenberg group. In the last section of Chapter 6, we give a list of several such examples. It is certainly natural to ask whether the preserving of left invariant convexity holds without these symmetry assumptions. This remains as a future problem we aim to address.

In Chapter 7, we will study the game-theoretic interpretation of the viscosity solutions to some fully nonlinear partial differential equations [90]. The discrete game interpretations of various elliptic and parabolic PDEs ([78, 100, 101, 79, 97, 96], etc) have recently attracted great attention. The game related methods are also used as a new tool in different contexts. For example, the fattening phenomenon for mean curvature flow is rigorously proved via games without using parabolic theory by Liu [88]. Armstrong and Smart [7] proved the uniqueness for infinity harmonic functions using a method related to the tug-of-war games in [100]. A recent work [92] provides a new proof of Harnack's inequality for $p$-Laplacian by stochastic games. All of these results largely simplify the original PDE proofs and indicate a strong potential of applicability of the game-theoretic approach.

Due to the works [78, 96, 100, 101], one may find a family of discrete games, whose value functions $u^\varepsilon$ converge, as $\varepsilon \to 0$, to the unique solution $u$ of a class of quasilinear parabolic equations including level set mean curvature flow and normalized $p$-Laplace equations. We revisit the convexity preserving properties in the Euclidean space for these two classes of equations by respectively using the game-theoretic approximations first proposed in [96] and [78]. Our new proofs are based on investigating game strategies or iterated applications of Dynamic Programming Principles (DPP), which are very different from the standard proofs in the literature. We look to the convexity preserving property of $u^\varepsilon$ rather than that of $u$. For the $p$-Laplace equations ($2 \leq p \leq \infty$), the convexity of $u^\varepsilon$ follows directly from an iteration of the corresponding DPP. However, for the level set mean curvature flow equation, extra work is needed since the control set of the players in the game is not convex. We then introduce a modified game and manage to show the convexity by comparing the limits of value functions as $\varepsilon \to 0$. This game-theoretic method can also be applied to study convexity preserving of the level sets of mean curvature flow equations and the Neumann boundary problems.

## 2.0 REVIEW OF METRIC MEASURE SPACES

In this chapter, we give a review of some definitions and properties of metric measure spaces. The abstract metric measure spaces will play a crucial role in the first four chapters. The main references for this chapter are the books by Ambrosio and Tilli [5], Evans and Gariepy [33], Heinonen, Koskela, Shanmugalingam and Tyson [58]. Proofs for the theorems without specifications can be found in the above books.

## 2.1 BASIC DEFINITIONS IN METRIC MEASURE SPACES

Let $X$ denote a set, and $2^X$ the collection of subsets of $X$. Let $C$ denote a general constant whose value can change even in the same chain of estimates.

**Definition 2.1.1.** A mapping $\mu : 2^X \to [0, \infty]$ is called a measure on $X$ if

(1) $\mu(\emptyset) = 0$, and

(2) $\mu(A) \le \sum_{k=1}^{\infty} \mu(A_k)$ for any $A \subset \bigcup_{k=1}^{\infty} A_k$.

**Definition 2.1.2.** A set $A \subset X$ is $\mu$-measurable if for every set $B \subset X$,

$$\mu(B) = \mu(A \cap B) + \mu(B \setminus A).$$

Notice that in some textbooks, the mapping $\mu$ defined in Definition 2.1.1 is called "outer measure" while the term "measure" is defined by restricting $\mu$ to the collection of all $\mu$-measurable sets.

A collection of subsets $\mathcal{A} \subset 2^X$ is a $\sigma$-algebra if it satisfies $\emptyset, X \in \mathcal{A}$; $A \in \mathcal{A}$ implies that $X \setminus A \in \mathcal{A}$; $A_k \in \mathcal{A} (k = 1, \cdots)$ implies that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$. The collection of measurable sets forms a $\sigma$-algebra in $X$. The smallest $\sigma$-algebra in $X$ containing all open sets is called the Borel $\sigma$-algebra. The sets in the Borel $\sigma$-algebra are called Borel sets.

Let us recall some basic definitions.

(1) A measure $\mu$ on $X$ is regular if for every set $A \subset X$, there exists a $\mu$-measurable set $B$ such that $A \subset B$ and $\mu(A) = \mu(B)$.
(2) A measure $\mu$ on $X$ is called Borel if every Borel set is $\mu$-measurable.
(3) A measure $\mu$ on $X$ is Borel regular if $\mu$ is Borel and for every $A \subset X$, there exists a Borel set $B$ such that $A \subset B$ and $\mu(A) = \mu(B)$.
(4) A measure $\mu$ on $X$ is a Radon measure if $\mu$ is Borel regular and $\mu(K) < \infty$ for every compact set $K \subset X$.

**Definition 2.1.3.** A function $d : X \times X \to [0, \infty)$ is called a metric on $X$ if it satisfies

(1) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(2) $d(x, y) = 0$ if and only if $x = y$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then the pair $(X, d)$ is called a metric space. We denote open and closed balls in the metric space as

$$B(x, r) = \{y \in X : d(x, y) < r\} \quad \text{and} \quad \overline{B(x, r)} = \{y \in X : d(x, y) \leq r\}.$$

Sometimes $B(x, r)$ are abbreviated as $B$. We use $2B$ to denote a concentric ball of $B$ with twice the radius and $B^n$ denotes a ball in the Euclidean space $\mathbb{R}^n$.

A metric space $(X, d)$ equipped with a Borel measure $\mu$ is called a metric measure space and denoted as $(X, d, \mu)$. In what follows, we always assume that $0 < \mu(B) < \infty$, for every ball $B \subset X$. We say that $\mu$ is doubling if there exists a constant $C_d \geq 1$ such that for every ball $B \subset X$,

$$\mu(2B) \leq C_d \mu(B).$$

If $(X, d, \mu)$ is a doubling metric measure space, then there exists a constant $C > 0$ such that

$$\frac{\mu(B(x, r))}{\mu(B(x_0, r_0))} \geq C\Big(\frac{r}{r_0}\Big)^s$$

whenever $x \in B(x_0, r_0)$, $r \leq r_0$ and $s = \log C_d / \log 2$ is called the associated homogeneous dimension.

Let $\mu, \nu$ be Radon measures on a metric space $X$. Then $\nu$ is called absolutely continuous with respect to $\mu$, written $\nu << \mu$, provided $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \subset X$. The following theorem is often referred to as Radom-Nikodym theorem.

**Theorem 2.1.4.** *Let $\mu, \nu$ be Radon measures on a metric space $X$ with $\nu << \mu$. Then there is a $\mu$-measurable function $g : X \to [0, \infty)$ such that*

$$\nu(A) = \int_A g \; d\mu$$

*for all $\mu$-measurable sets $A \subset X$.*

Given a set $A$ in a metric space $(X, d)$, the diameter of this set is

$$\operatorname{diam}(A) = \sup\{d(x, y) | x, y \in A\}.$$

We next define a very important measure, called Hausdorff measure on a metric space.

**Definition 2.1.5.** (1) Let $A \subset X$, $0 \leq s < \infty$, $0 < \delta \leq \infty$. Define

$$\mathcal{H}^s_\delta(A) = \inf\Big\{ \sum_{j=1}^{\infty} \alpha(s) \big(\frac{\operatorname{diam} C_j}{2}\big)^s \mid A \subset \bigcup_{j=1}^{\infty} C_j, \operatorname{diam} C_j \leq \delta \Big\},$$

where

$$\alpha(s) = \frac{\pi^{s/2}}{\int_0^\infty e^{-x} x^{s/2} dx}.$$

(2) For $A$ and $s$ above, define

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

We call $\mathcal{H}^s$ $s$-dimensional Hausdorff measure on $X$.

It can be verified that $\mathcal{H}^s$ is a Borel regular measure and $n$-dimensional Hausdorff measure $\mathcal{H}^n$ coincides with the Lebesgue measure $\mathcal{L}^n$ on the Euclidean space $\mathbb{R}^n$.

**Definition 2.1.6.** The Hausdorff dimension of a set $A \subset X$ is defined to be

$$\mathcal{H}_{\dim}(A) = \inf\{0 \leq s < \infty \mid \mathcal{H}^s(A) = 0\}.$$

If $k > \mathcal{H}_{\dim}(A)$, then $\mathcal{H}^k(A) = 0$, and if $k < \mathcal{H}_{\dim}(A)$, then $\mathcal{H}^k(A) = \infty$. If $k = \mathcal{H}_{\dim}(A)$, nothing can be said about the value of $\mathcal{H}^k(A)$.

**Definition 2.1.7.** Let $(X, d, \mu)$ be a metric space equipped with a Borel measure $\mu$ and let $s > 0$. We say that a metric measure space $(X, d, \mu)$ is $s$-Ahlfors regular if there is a constant $C_A \geq 1$, such that

$$C_A^{-1} r^s \leq \mu(B(x,r)) \leq C_A r^s, \quad \text{for any } x \in X \text{ and } 0 < r < \operatorname{diam}(X).$$

If $X$ is $s$-Ahlfors regular with respect to a Borel regular measure $\mu$, then $X$ has Hausdorff dimension precisely $s$. Moreover, $\mu$ is comparable to the Hausdorff measure $\mathcal{H}^s$, that is, there exist constant $C \geq 1$ such that

$$C^{-1}\mathcal{H}^s(E) \leq \mu(E) \leq C\mathcal{H}^s(E),$$

for all Borel sets $E \subset X$. This implies that $X$ is $s$-Ahlfors regular with respect to $\mathcal{H}^s$ [55, Exercise 8.11.].

**Definition 2.1.8.** A metric space $(X, d)$ is uniformly perfect if there exists a constant $0 < c < 1$ such that for all $x \in X$ and $0 < r < \operatorname{diam}(X)$,

$$\overline{B(x,r)} \setminus B(x, cr) \neq \emptyset.$$

It is well known that $s$-Ahlfors regular metric spaces are uniformly perfect. Indeed, if $X$ is $s$-Ahlfors regular with constant $C_A$, then $\overline{B(x,r)} \setminus B(x, cr) \neq \emptyset$ for any $0 < c < C_A^{-2/s}$. Suppose to the contrary that $\overline{B(x,r)} \setminus B(x, cr) = \emptyset$. This would imply

$$C_A^{-1} r^s \leq \mathcal{H}^s(\overline{B(x,r)}) = \mathcal{H}^s(B(x,cr)) \leq C_A (cr)^s$$

so $C_A^{-2/s} \leq c$, which is a contradiction. This is observed in some earlier papers, for example [114].

Let $f : X \to \mathbb{R}$ be a function on a metric space. If there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L d(x, y)$$

for all $x, y \in X$, we say that $f$ is a Lipschitz function on $X$. The smallest constant $L$ with this property will be denoted by $\|f\|_{\text{Lip}}$. For $\alpha \in (0,1)$, if there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq Ld(x,y)^\alpha$$

for all $x, y \in X$, we say that $f$ is an $\alpha$-Hölder continuous function.

## 2.2 CURVES IN METRIC MEASURE SPACES

A curve in a metric space $(X, d)$ is a continuous map from an interval into $X$. We usually denote a curve as $\gamma : [a, b] \to X$, and the image of a curve $\gamma([a, b])$ as $\Gamma$.

We define the length of a curve $\gamma : [a, b] \to X$ as

$$\ell(\gamma) = \sup\{\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1})), a = t_0 < t_1 < \cdots < t_k = b\},$$

where the supremum is taken over all finite sequences of $\{t_i\} \subset [a, b]$ defined as above. If $\ell(\gamma) < \infty$, then we say the curve $\gamma$ is a rectifiable curve. Every rectifiable curve admits a parametrization by arc-length, that is, there exists a parametrization of $\Gamma$, denoted as $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ such that $\ell(\tilde{\gamma}|_{[0,t]}) = t$ for any $t \in [0, \ell(\gamma)]$. This arc-length parametrization $\tilde{\gamma}$ is a 1-Lipschitz mapping.

**Definition 2.2.1.** For a curve $\gamma : [a, b] \to X$ we define speed at a point $t \in (a, b)$ as the limit

$$|\gamma'|(t) = \lim_{h \to 0} \frac{d(\gamma(t + h), \gamma(t))}{|h|},$$

if the limit exists.

**Theorem 2.2.2.** *For every Lipschitz curve $\gamma : [a, b] \to X$ speed exists almost everywhere and*

$$\ell(\gamma) = \int_a^b |\gamma'|(t)\ dt.$$

If a rectifiable curve is parametrized by arc-length $\tilde{\gamma}$, then $|\tilde{\gamma}'(t)| = 1$ for almost every point $t \in [0, \ell(\tilde{\gamma})]$.

We say a metric space $X$ is quasiconvex if there is a constant $C \geq 1$ such that every two points $x, y \in X$ can be joined by a curve with length less than or equal to $Cd(x, y)$. A metric space is called proper if bounded and closed sets are compact. The following theorem guarantees the existence of geodesics in proper metric spaces.

**Theorem 2.2.3.** *Suppose a metric space $X$ is proper and path connected, that is, every two points $x, y$ can be joined by a rectifiable curve in $X$. Then there exists a shortest curve $\gamma : [a, b] \to X$ connecting $x$ and $y$, that is, $\gamma(a) = x$ and $\gamma(b) = y$. The shortest curve is injective.*

Next, we define the integral along a rectifiable curve.

**Definition 2.2.4.** Let $\gamma : [a, b] \to X$ be a rectifiable curve and $g : X \to [0, \infty]$ be a Borel measurable function. Then we define

$$\int_\gamma g = \int_0^{\ell(\gamma)} g(\tilde{\gamma}(t))\ dt,$$

where $\tilde{\gamma}$ is the arc-length parametrization of the given curve.

We also give the generalization of the Euclidean area formula to the case of Lipschitz maps $f$ from the Euclidean space $\mathbb{R}^n$ into a metric space $X$. The proof can be found in [77, Corollary 8].

**Theorem 2.2.5** (Area fomula)**.** *Let $f : \mathbb{R}^n \to X$ be Lipschitz. Then*

$$\int_{\mathbb{R}^n} \theta(x) J_n(mdf_x)\ dx = \int_X \sum_{x \in f^{-1}(y)} \theta(x)\ d\mathcal{H}^n(y)$$

*for any Borel function $\theta \colon \mathbb{R}^n \to [0, \infty]$ and*

$$\int_A \theta(f(x)) J_n(mdf_x)\ dx = \int_X \theta(y) \mathcal{H}^0(A \cap f^{-1}(y))\ d\mathcal{H}^n(y)$$

*for $A \in \mathcal{B}(\mathbb{R}^n)$ and any Borel function $\theta \colon X \to [0, \infty]$.*

Here $J_n(mdf_x)$ in the Jacobian of the metric derivative. We will however, be interested in a special case where the Lipschitz mapping into $X$ is just an injective arc-length parametrized rectifiable curve $\gamma : [0, \ell] \to X$. In this case, $J_1(mdf_x)$ in the above theorem is the metric derivative $|\gamma'|(t)$ and hence it equals 1. Let $\Gamma = \gamma([0, \ell])$ and $g : X \to [0, \infty]$ be a Borel function. Applying the area formula, we get

$$\int_0^\ell g(\gamma(s))\, ds = \int_\Gamma g(y)\, d\mathcal{H}^1(y).$$

If $N \subset X$ and $\mathcal{H}^1(N) = 0$, the area formula also implies that $\mathcal{H}^1(\gamma^{-1}(N)) = 0$.

## 2.3 RECTIFIABILITY OF ONE-DIMENSIONAL COMPACT AND CONNECTED SPACES

We list here several important results about the parametrization of compact and connected metric measure spaces with finite $\mathcal{H}^1$ measure.

The first result proved by Schul [110, Lemma 2.3] gives a Lipschitz parametrization of such spaces.

**Lemma 2.3.1.** *Let $K \subset X$ be a compact connected set of finite $\mathcal{H}^1$ measure. Then there is a Lipschitz function $\gamma : [0, 1] \to K$ such that $\gamma([0, 1]) = K$ and $\|\gamma\|_{\mathrm{Lip}} \leq 32\mathcal{H}^1(K)$. Moreover, if $K$ is 1-Ahlfors-regular, then*

$$\frac{R}{C} \leq \mathcal{H}^1(\gamma^{-1}(B(x, R))) \leq CR \quad \forall x \in K, 0 < R \leq \operatorname{diam}(K), \tag{2.3.1}$$

*where $C$ is a constant depending only on the 1-Ahlfors regularity constant of the set $K$.*

The proofs of the following two theorems can be found in [5, Theorem 4.4.7, Theorem 4.4.8].

**Theorem 2.3.2** (First rectifiability theorem)**.** *If $X$ is a compact and connected set and $\mathcal{H}^1(X) < \infty$, then every pair of points $x, y \in X$ can be connected by a injective rectifiable curve.*

We call curves without self-intersections simple curves, that is, there is an injective parametrization of such curves.

**Theorem 2.3.3** (Second rectifiability theorem)**.** *If $X$ is compact, connected, and $\mathcal{H}^1(X) < \infty$, then there exist countably many arc-length parametrized simple curves $\gamma_i\colon [0,\ell_i] \to X$ such that*

$$\mathcal{H}^1(X \setminus \bigcup_{i=1}^{\infty} \gamma_i([0,\ell_i])) = 0.$$

We briefly describe the construction of this parametrization. Since $X$ is compact, we can choose $x, y \in X$ such that

$$d(x,y) = \mathrm{diam}(X).$$

By Theorem 2.3.2, we can join $x, y$ by an arc-length parametrized simple curve $\gamma_0 : [0,\ell_0] \to X$ and we denote the range of this curve as $\Gamma_0$. Suppose that we have already constructed $\Gamma_0, \cdots, \Gamma_k$ with the following properties:

(1) $\Gamma_i \subset X$, $i = 0, \cdots, k$;

(3) Each curve $\gamma_i : [0,\ell_i] \to X$ with $\Gamma_i = \gamma_i([0,\ell_i])$ is a simple arc-length parametrized curve;

(2) Each intersection $\Gamma_i \cap \bigcup_{j<i} \Gamma_j$ consists of a single point, for $i = 1, \cdots, k$.

Let

$$d_k = \sup_{x \in X} d\big(x, \bigcup_{i=0}^{k} \Gamma_i\big).$$

If $d_k = 0$, then $X = \bigcup_{i=0}^{k} \Gamma_i$ and we are done. If $d_k > 0$ for all $k$, by compactness we can choose $x_k \in X$ and $y_k \in \cup_{i=0}^{k} \Gamma_i$ such that $d(x_k, y_k) = d_k$. Connect $x_k$ and $y_k$ with an arc-length parametrized simple curve $\gamma_{k+1}$ such that $\gamma_{k+1}(0) = x_k$ and $\gamma_{k+1}(\ell_{k+1}) = y_{k+1}$. Let

$$\tilde{t} = \inf\{t \in [0,\ell_{k+1}] |\ \gamma_{k+1}(t) \in \bigcup_{i=0}^{k} \Gamma_i\}$$

and define $\Gamma_{k+1} = \gamma_{k+1}([0,\tilde{t}])$. We get that $\Gamma_{k+1} \subset X$ is an arc-length parametrized simple curve and the intersection of $\Gamma_{k+1}$ and $\bigcup_{i=0}^{k} \Gamma_i$ consists of one single point. We can continue this construction. Since $\bigcup_{i=0}^{\infty} \Gamma_i$ may not be closed, we may have

$$X \setminus \bigcup_{i=0}^{\infty} \Gamma_i \neq \emptyset.$$

We omit the arguments to estimate the above set $X \setminus \bigcup_{i=0}^{\infty} \Gamma_i$ has 1-dimensional Hausdorff measure zero.

It is easy to see from this construction that these simple curves intersect with each other at most at one point. If two curves intersect with each other, then the intersection point must be the endpoint of one of these curves.

## 3.0 SOBOLEV FUNCTIONS ON METRIC MEASURE SPACES

The theory of Sobolev functions has been widely applied in different areas of mathematics including calculus of variations, partial differential equations, and so on. In this chapter, we will first give a brief review of the classical theory of Sobolev functions. We refer to the book by Evans and Gariepy [33] and notes by Hajłasz [48] for the definitions and proofs of the theorems.

There are various extensions of the classical theory of Sobolev functions to general metric measure spaces and their connections with variational problems, geometric function theory and many related fields have been studied since the nineties. These results constitute a significant part of analysis on metric measure spaces. In this chapter, we list several important definitions of Sobolev functions on metric measure spaces and also give the definition of spaces supporting Poincaré inequalities in the end. Doubling metric measure spaces that support Poincaré inequalities provide a setting in which different definitions of Sobolev spaces are equivalent. The main source for these materials are the survey paper by Hajłasz [47] and the book by Heinonen, Koskela, Shanmugalingam and Tyson [58].

### 3.1 SOBOLEV FUNCTIONS IN THE EUCLIDEAN SPACE

#### 3.1.1 Definitions and basic properties

In this section $\Omega$ will denote an open subset of $\mathbb{R}^n$, $C_c^\infty(\Omega)$ be the collection of all smooth functions with compact support in $\Omega$ and $C^\infty(\Omega)$ be the collection of all smooth functions in $\Omega$.

**Definition 3.1.1.** Let $u \in L^1_{\text{loc}}(\Omega)$ and $1 \le i \le n$. We say $g_i \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of $u$ with respect to $x_i$ if

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi$$

for all $\varphi \in C_c^\infty(\Omega)$.

It is easy to verify that given a function $u \in L^1_{\text{loc}}(\Omega)$, if the weak partial derivative exists, then it is uniquely defined $\mathcal{L}^n$ almost everywhere. We write

$$\frac{\partial u}{\partial x_i} = g_i$$

and

$$\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$$

**Definition 3.1.2.** For $1 \le p < \infty$, the function $u$ belong to the Sobolev space $W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and the weak partial derivatives $\partial u / \partial x_i$ exist and belong to $L^p(\Omega)$ for $1 \le i \le n$.

If $u \in W^{1,p}(\Omega)$, define

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)},$$

and we can verify that the above definition is a norm and Sobolev spaces equipped with this norm is a Banach space.

**Theorem 3.1.3** (Meyers-Serrin)**.** *The smooth functions $C^\infty(\Omega)$ are dense in $W^{1,p}(\Omega)$.*

According to this theorem, the Sobolev functions can also be defined as the completion of all smooth functions with respect to the Sobolev norm. Moreover, if $\Omega = \mathbb{R}^n$, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$.

The Sobolev embedding theorems are among the most important results in the theory of Sobolev spaces. We list some results below.

**Theorem 3.1.4** (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $1 \le p < n$, we define*

$$p^* = \frac{np}{n-p},$$

*and call $p^*$ the Sobolev conjugate of $p$. Then for all $u \in W^{1,p}(\mathbb{R}^n)$, there exists a constant $C(p,n)$, such that*

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

**Theorem 3.1.5.** *For each $n < p < \infty$, there exists a constant $C(p,n)$ such that*

$$|u(x) - u(y)| \le C|x-y|^{1-\frac{n}{p}}\|\nabla u\|_{L^p(B)}$$

*for all $x, y \in B$ and $u \in W^{1,p}(B)$.*

As a corollary, if $u \in W^{1,p}(\Omega)$ and $n < p < \infty$ then $u$ is locally Hölder continuous with exponent $1 - n/p$. However, when $1 \le p < n$, Sobolev functions need not be continuous. For example, $u(x) = 1/|x|^{1/2} \in W^{1,1}(B^2(0,1))$ and it is discontinuous in the origin.

The above examples and theorems show that in $\mathbb{R}^n$, the properties of Sobolev functions $W^{1,p}$ are very different when $p > n$ and $p < n$. We often call Sobolev functions in $W^{1,p}$ with $p = n$ the Sobolev functions in the critical case. The embedding result in the critical case is as follows.

**Theorem 3.1.6** (Trudinger)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exist constants $C_1, C_2$ depending on $\Omega$ only such that*

$$\fint_\Omega \exp\big(\frac{|u - u_\Omega|}{C_1\|\nabla u\|_{L^n(\Omega)}}\big)^{\frac{n}{n-1}} \le C_2.$$

*for any $u \in W^{1,n}(\Omega)$.*

Here and in what follows we will use notation

$$u_\Omega = \fint_\Omega u\ d\mu = \frac{1}{\mu(\Omega)} \int_\Omega u\ d\mu.$$

When $p = n$ and $n \geq 2$, $u(x) = \log|\log|x||$ provides an example of a discontinuous function in $W^{1,n}(B^n(0,1))$. When $n = 1$, functions in $W^{1,1}([a,b])$ are absolutely continuous. A function $u : [a,b] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{k} |u(b_i) - u(a_i)| < \varepsilon,$$

for any finite collection of pairwise disjoint subintervals $\bigcup_{i=1}^{k}[a_i, b_i] \subset [a,b]$ satisfying

$$\sum_{i=1}^{k} |b_i - a_i| < \delta.$$

We denote the collection of absolutely continuous functions on $[a,b]$ as $AC([a,b])$. An absolutely continuous function $u$ is differentiable almost everywhere and we denote this pointwise derivative as $u'$. In fact, $u \in AC([a,b])$ if and only if there exist $g \in L^1([a,b])$ such that

$$u(x) = u(a) + \int_a^x g(t)\ dt.$$

Moreover, $u'(t) = g(t)$ almost everywhere. In other words absolutely continuous functions are exactly the functions for which the fundamental theorem of calculus is true.

We denote the collection of all absolutely continuous functions with $u' \in L^p([a,b])$ as $AC^p([a,b])$. Upon choosing a representative, we have $AC^p([a,b]) = W^{1,p}([a,b])$.

The following result is the classical Poincaré inequality.

**Theorem 3.1.7.** *If $u \in W^{1,p}(B(r))$ and $1 \leq p < \infty$, then*

$$\Big(\fint_B |u - u_B|^p\ dx\Big)^{1/p} \leq C(n,p)r\Big(\fint_B |\nabla u|^p\ dx\Big)^{1/p}.$$

### 3.1.2 Equivalent characterizations

In this section, we give several equivalent definitions of Sobolev spaces in $\mathbb{R}^n$. In the previous section, we have seen the equivalence of Sobolev functions in $W^{1,p}([a,b])$ with absolutely continuous functions in $AC^p([a,b])$. This characterization has a higher dimensional analog.

If $\Omega \subset \mathbb{R}$ is open, we say that $u \in AC(\Omega)$ if $u$ is absolutely continuous on every compact interval in $\Omega$.

**Definition 3.1.8.** Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $u$ is absolutely continuou on lines, $u \in ACL(\Omega)$, if the function $u$ is Borel measurable and for almost every line $\ell$ parallel to one of the coordinate axes, $u|_\ell \in AC(\Omega \cap \ell)$.

Since absolutely continuous functions in the real line are differentiable almost everywhere, $u \in ACL(\Omega)$ has partial derivative almost everywhere and hence $\nabla u$ is defined almost everywhere. We say that $u \in ACL^p(\Omega)$ if $u \in L^p(\Omega) \cap ACL(\Omega)$ and $|\nabla u| \in L^p(\Omega)$.

**Theorem 3.1.9.** *For $1 \leq p < \infty$ and any open set $\Omega \subset \mathbb{R}^n$,*

$$W^{1,p}(\Omega) = ACL^p(\Omega).$$

*Moreover, the pointwise partial derivatives of an $ACL^p(\Omega)$ function equal the weak partial derivatives.*

Let $\mathcal{M}$ be the Hardy-Littlewood maximal function defined as

$$\mathcal{M}u(x) = \sup_{r>0} \fint_{B(x,r)} |u(y)|\, dy$$

and

$$\mathcal{M}_R u(x) = \sup_{0<r<R} \fint_{B(x,r)} |u(y)|\, dy$$

for $x \in \mathbb{R}^n$.

**Theorem 3.1.10.** *For $u \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, the following conditions are equivalent*

*(1)* $u \in W^{1,p}(\mathbb{R}^n)$,

*(2) There exists* $0 \le g \in L^p(\mathbb{R}^n)$ *and* $\sigma \ge 1$ *such that*

$$\fint_B |u - u_B| \ dx \le r\big(\fint_{\sigma B} g^p \ dx\big)^{1/p}$$

*on every ball* $B$ *of any radius* $r$*.*

*(3) There exists* $0 \le g \in L^p(\mathbb{R}^n)$ *and* $\sigma \ge 1$ *such that*

$$|u(x) - u(y)| \le |x - y|\big(g(x) + +g(y)\big) \quad \text{almost everywhere.}$$

The second condition follows from the Poincaré inequality 3.2.1 and Hölder inequality. It is first proved by Koskela and MacManus [82]. The third condition for $u \in W^{1,p}(\mathbb{R}^n)$ stems from the following inequality

$$|u(x) - u(y)| \le |x - y|\big((\mathcal{M}|\nabla u|^p(x))^{1/p} + (\mathcal{M}|\nabla u|^p(y))^{1/p}\big) \quad a.e..$$

This characterization is first proved by Hajłasz [46].

## 3.2 SOBOLEV FUNCTIONS ON THE METRIC MEASURE SPACES

Note that the conditions (2) and (3) in Theorem 3.1.10 do not involve derivatives and only involve metric and measure in $\mathbb{R}^n$. Each of this condition can be used to define a version of Sobolev spaces in general metric measure spaces and will be discussed in Sections 3.2.1 and 3.2.2. However, in Theorem 3.1.9, the notion of almost all lines parallel to coordinate axes and the notion of gradient do not apply to the general metric measure spaces. Instead, we will introduce the modulus of the path family and the notion of upper gradient as replacements. The Sobolev space based on this approach is called Newton Sobolev space and will be discussed in Section 3.2.3.

### 3.2.1 Sobolev spaces $M^{1,p}$

**Definition 3.2.1.** Let $(X, d, \mu)$ be a metric space equipped with a Borel measure $\mu$. For $0 < p < \infty$ we define the Hajłasz Sobolev space $M^{1,p}(X, d, \mu)$ to be the set of all functions $u \in L^p(X)$ for which there exists a nonnegative Borel function $g \geq 0$ such that

$$|u(x) - u(y)| \leq d(x,y)\big(g(x) + g(y)\big) \quad \text{almost everywhere.}$$

Denote by $D(u)$ the class of all nonnegative Borel functions $g$ that satisfy the above inequality. Thus $u \in M^{1,p}(X, d, \mu)$ if and only if $u \in L^p(X)$ and $D(u) \cap L^p \neq \emptyset$. The space $M^{1,p}(X, d, \mu)$ is linear and we define

$$||u||_{M^{1,p}} = ||u||_{L^p} + \inf_{g \in D(u)} ||g||_{L^p}.$$

When $p \geq 1$, $|| \cdot ||_{M^{1,p}}$ is a norm and $M^{1,p}(X)$ is a Banach space [46]. For studies and applications of Hajłasz Sobolev spaces with $0 < p < 1$, see for example [47, 68, 83, 84, 85].

According to Theorem 3.1.10, in the Euclidean space and $1 < p < \infty$,

$$M^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n) = W^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n).$$

The equivalence of $M^{1,p}$ and $W^{1,p}$ also holds on smooth domains $\Omega$ in $\mathbb{R}^n$. However, in general, $M^{1,1} \subsetneq W^{1,1}$. For example, function $u(x) = -x/(|x| \log |x|) \in W^{1,1}(I)$ but it does not belong to $M^{1,1}(I)$, where $I = (-1/4, 1/4)$.

An analog of the Sobolev embedding theorems holds for functions in $M^{1,p}(X, d, \mu)$, where $X$ is a doubling space with constant $C_d$ and the associated homogeneous dimension $s = \log C_d / \log 2$ plays the role of dimension in the Euclidean space. The following embedding theorem is proved by Hajłasz [46].

**Theorem 3.2.2.** *Let $X$ be a doubling space and fix a ball $B \subset X$ of radius $r$, $\sigma > 1$. Assume that $u \in M^{1,p}(\sigma B, d, \mu)$ and $g \in D(u) \cap L^p(X)$, where $0 < p < \infty$. There exist constants $C, C_1$ and $C_2$ depending on $C_d$, $p$ and $\sigma$ only such that*

*(1) If $0 < p < s$, then $u \in L^{p^*}(B)$, $p^* = sp/(s-p)$, and*

$$\inf_{c \in \mathbb{R}} \Big( \fint_B |u - c|^{p^*} \, d\mu \Big)^{1/p^*} \leq Cr \Big( \fint_{\sigma B} g^p \, d\mu \Big)^{1/p}.$$

*(2) If $p = s$, then*

$$\fint_B \exp\left(C_1 \frac{\mu(\sigma B)^{1/s}}{r} \frac{|u-u_B|}{\|g\|^s_L(\sigma B)}\right) d\mu \le C_2.$$

*(3) If $p > s$, then $u$ is Hölder continuous on $B$ and*

$$|u(x)-u(y)| \le C r^{s/p} d(x,y)^{1-s/p} \Big( \fint_{\sigma B} g^p \, d\mu \Big)^{1/p} \quad \text{for} \quad x, y \in B.$$

### 3.2.2 Sobolev spaces $P^{1,p}$

**Definition 3.2.3.** Fix $\sigma \ge 1$ and $0 < p < \infty$. We say that a pair $(u, g)$, $u \in L^1_{\text{loc}}(X)$, $0 \le g \in L^p_{\text{loc}}(X)$ satisfies the $p$-Poincaré inequality if the following inequality holds:

$$\fint_B |u - u_B| d\mu \le r \left( \fint_{\sigma B} g^p d\mu \right)^{\frac{1}{p}} \tag{3.2.1}$$

on every ball $B$ of radius $r$ and $\sigma B \subset X$.

The class of $u \in L^p(X)$ for which there exists $0 \le g \in L^p(X)$ so that the pair $(u, g)$ satisfies the $p$-Poincaré inequality will be denoted by $P^{1,p}_\sigma(X, d, \mu)$ and

$$P^{1,p}(X, d, \mu) = \bigcup_{\sigma \ge 1} P^{1,p}_\sigma(X, d, \mu).$$

By Theorem 3.1.10, the Sobolev space $P^{1,p}$ defined above is equivalent with the classical Sobolev space in the Euclidean setting.

**Theorem 3.2.4.** *Let $(X, d, \mu)$ be a doubling metric measure space and $s = \log_2 C_d$ is the associated homogeneous dimension. If $p \ge s/(s+1)$, then*

$$M^{1,p}(X) \subset P^{1,p}(X).$$

The above theorem is proved by Hajłasz [47]. We also give the general Sobolev embedding theorem for $P^{1,p}(X)$. This result is due to Hajłasz and Koskela [49].

**Theorem 3.2.5.** *Let $(X, d, \mu)$ be a doubling metric measure space and $s = \log_2 C_d$ be the associated homogeneous dimension. Assume the pair $u \in L^p(X)$ and $0 \le g \in L^p(X)$ satisfies the p-Poincaré inequality (3.2.1) with $0 < p < \infty$ and $\sigma > 1$.*

*(1) If $0 < p < s$, then for every $0 < h < p^* = sp/(s-p)$*

$$\inf_{c\in\mathbb{R}} \Big( ⨍_B |u-c|^h d\mu \Big)^{1/h} \le Cr \Big( ⨍_{6\sigma B} g^p \, d\mu \Big)^{1/p}.$$

*If in addition $g \in L^q$, $p < q < s$, then*

$$\inf_{c\in\mathbb{R}} \Big( ⨍_B |u-c|^{q^*} d\mu \Big)^{1/q^*} \le Cr \Big( ⨍_{6\sigma B} g^q \, d\mu \Big)^{1/q},$$

*where $q^* = sq/(s-q)$ and $B$ is any ball of radius $r$.*

*(2) If $p = s$, then*

$$\int_B \exp\Big( \frac{C_1 \mu(6\sigma B)^{1/s} |u - u_B|}{r \|g\|_{L^s(6\sigma B)}} \Big) \, d\mu \le C_2.$$

*(3) If $p > s$, then $u$ is locally Hölder continuous and*

$$|u(x) - u(y)| \le C r^{s/p} d(x,y)^{1-s/p} \Big( ⨍_{6\sigma B_0 g^p} \, d\mu \Big)^{1/p}$$

*for $x, y \in B$, where $B$ is an arbitrary ball of radius $r_0$.*

*The constants in the theorem depend on $p, q, h, C_d$ and $\sigma$.*

### 3.2.3 Sobolev spaces $N^{1,p}$

**Definition 3.2.6.** Let $\Gamma$ be a collection of non constant rectifiable curves, and let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \ge 1 \text{ for every } \gamma \in \Gamma.$$

For each $1 \le p < \infty$ we define

$$\mathrm{Mod_p}(\Gamma) = \inf_{\rho \in \mathrm{F}(\Gamma)} \int_{\mathrm{X}} \rho^{\mathrm{p}} \mathrm{d}\mu.$$

The number $\mathrm{Mod_p}(\Gamma)$ is called the $p$-modulus of the family $\Gamma$.

We can verify that $\mathrm{Mod_p}$ is a measure on the family of all nonconstant rectifiable curves in $X$. If some property holds for all nonconstant rectifiable curves except a subcollection $\Gamma$ with $\mathrm{Mod_p}(\Gamma) = 0$, then we say that the property holds for $p$-a.e. curve. By the following theorem, we can see that this notion of $p$-a.e. curve is a natural generalization of almost every line parallel to a given coordinate axe.

**Theorem 3.2.7.** *Points in* $Q^n = [0,1]^n = [0,1] \times Q^{n-1}$ *will be denoted by* $x = (x_1, x')$. *Let* $1 \le p < \infty$. *For a Borel subset* $E \subset Q^{n-1}$, *consider the family of straight segments passing through* $E$ *and parallel to* $x_1$, *i.e.,*

$$\Gamma_E = \{\gamma_{x'} : [0,1] \to Q^n : \gamma_{x'} = (t, x'), x' \in E\}.$$

*Then* $\mathrm{Mod_p}(\Gamma_\mathrm{E}) = 0$ *if and only if* $\mathcal{L}^n(E) = 0$.

**Definition 3.2.8.** Let $u : X \to \mathbb{R}$ be a Borel function. We say that a Borel function $g : X \to [0, \infty]$ is an upper gradient of $u$ if

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g$$

for every rectifiable curve $\gamma : [a, b] \to X$. We say that $g$ is a $p$-weak upper gradient of u if the above inequality holds for $p$-a.e. curve.

If $g$ is a $p$-weak upper gradient of $u$ which is finite a.e., then for every $\varepsilon > 0$, there is an upper gradient $g_\varepsilon$ of $u$ such that

$$g_\varepsilon \ge g \quad \text{everywhere,} \quad \text{and} \quad \|g_\varepsilon - g\|_{L^p} < \varepsilon.$$

This shows that a $p$-weak upper gradient of $u$ can be nicely approximated by an upper gradient.

Moreover, the notion of upper gradient is a natural generalization of the length of the gradient. If $u \in C^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$, then we can verify that $|\nabla u|$ is an upper gradient of $u$. Furthermore, $|\nabla u|$ is the least upper gradient in the sense that if $g \in L^1_{\mathrm{loc}}(\Omega)$ is another upper gradient of $u$, then $g \ge |\nabla u|$ a.e.. In fact, we have a stronger result.

**Theorem 3.2.9.** *Any function* $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$ *has a representative for which* $|\nabla u|$ *is a p-weak upper gradient. On the other hand, if* $g \in L^1_{\mathrm{loc}}(\Omega)$ *is a p-weak upper gradient of* $u$, *then* $g \ge |\nabla u|$ *a.e..*

Let $1 \le p < \infty$. $\widetilde{N}^{1,p}(X, d, \mu)$ is the class of all $L^p$-integrable Borel functions on $X$ for which there exists a $p$-integrable $p$-weak upper gradient. With each $u \in \widetilde{N}^{1,p}(X, d, \mu)$ we associate a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X,d,\mu)} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all $p$-weak upper gradients $g$ of $u$.

**Definition 3.2.10.** We define an equivalence relation in $\widetilde{N}^{1,p}(X, d, \mu)$ by $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X,d,\mu)} = 0.$$

Then the space $N^{1,p}(X, d, \mu)$ is defined as the quotient space $\widetilde{N}^{1,p}(X, d, \mu)/\sim$ and it is equipped with the norm

$$\|u\|_{N^{1,p}} = \|u\|_{\widetilde{N}^{1,p}}.$$

This space $N^{1,p}(X, d, \mu)$ with the above norm is a Banach space for $1 \le p < \infty$ and it is a natural generalization of the classical Sobolev space $W^{1,p}$ to the setting of metric spaces.

**Theorem 3.2.11.** *If $\Omega \subset \mathbb{R}^n$ is open and $1 \le p < \infty$, then*

$$N^{1,p}(\Omega, |\cdot|, \mathcal{L}^n) = W^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$$

*as sets and the norms are equal.*

## 3.3 SPACES SUPPORTING THE POINCARÉ INEQUALITY

The notion of an abstract Poincaré inequality on metric measure spaces was introduced by Heinonen and Koskela [56]. Metric measure spaces that are doubling and support an abstract Poincaré inequality provide a good structure to study the first-order analysis.

**Definition 3.3.1.** Let $p \geq 1$. A metric measure space $(X, d, \mu)$ is said to support a $p$-Poincaré inequality if there exists constants $C, \lambda \geq 1$ such that for all measurable functions, the following holds:

$$\fint_B |u - u_B| d\mu \leq C \mathrm{diam} B \left( \fint_{\lambda B} g^p d\mu \right)^{\frac{1}{p}},$$

for every pair of functions $u : X \to \mathbb{R}$ and $g \to [0, \infty]$ where $u$ is measurable, and $g$ is an upper gradient for $u$.

For examples of spaces supporting Poincaré inequalities, we give a partial list below: Euclidean spaces, compact Riemannian manifolds, complete Riemannian manifolds with nonnegative Ricci curvature [19, 108], Carnot groups and general sub-Riemannian manifolds equipped with Carnot-Caratheodory metrics [49, 54, 57], boundaries of hyperbolic buildings [18], metric spaces with quantitative topology [111] and the Laakso spaces [86]. If $X$ is complete, doubling and supports a $p$-Poincaré inequality for $p \geq 1$, then $X$ is quasiconvex [24, Theorem 17.1.][49, Proposition 4.4]. In this manuscript, we will prove that each compact, quasiconvex and 1-Ahlfors regular space also supports $p$-Poincaré inequality for $1 \leq p < \infty$.

As mentioned before, a doubling metric measure spaces that supports Poincaré inequalities provides a very good setting such that different definitions of Sobolev spaces are equivalent. The proof of the following theorem can be found in [47, Theorem 11.3]

**Theorem 3.3.2.** *Let $(X, d, \mu)$ be a complete metric measure space with $\mu$ Borel and doubling. If $1 < p < \infty$ and the space supports the $q$-Poincaré inequality for some $1 \leq q < p$, then*

$$M^{1,p}(X, d, \mu) = N^{1,p}(X, d, \mu) = P^{1,p}(X, d, \mu).$$

By Hölder's inequality, we know that any metric measure space that supports a $(1,p)$-Poincaré inequality also supports a $(1,q)$-Poincaré inequality for $1 \leq p \leq q$. Keith and Zhong [74, Theorem 1.0.1] proved that the parameter $p > 1$ in the space supporting Poincaré inequality is also open ended on the left in a doubling metric space. We list their result as below.

**Theorem 3.3.3.** *Let $p > 1$ and $(X, d, \mu)$ be a complete metric measure space with $\mu$ Borel and doubling, that admits a $(1,p)$-Poincaré inequality. Then there exists $\varepsilon > 0$ such that $(X, d, \mu)$ admits a $(1,q)$-Poincaré inequality for every $q > p - \varepsilon$, quantitatively.*

Combining the above two theorems, we have the following corollary.

**Corollary 3.3.4.** *Let $(X, d, \mu)$ be a complete metric measure space with $\mu$ Borel and doubling. If $1 < p < \infty$ and the space supports the p-Poincaré inequality, then*

$$M^{1,p}(X, d, \mu) = N^{1,p}(X, d, \mu) = P^{1,p}(X, d, \mu).$$

## 4.0 SOBOLEV FUNCTIONS IN THE CRITICAL CASE ON METRIC MEASURE SPACES

Let $\Omega \subset \mathbb{R}^n$ be a domain. We define the critical exponent of the Sobolev space $W^{1,p}(\Omega)$ to be $p = n$. This is because that when $p > n$, $W^{1,p}$ functions are Hölder continuous while for $p < n$ they need not be continuous at any point. This notion can be extended to the settings of metric measure spaces. If $(X, d, \mu)$ is a doubling metric measure space then we define the critical exponent for the Sobolev space $M^{1,p}(X)$, $P^{1,p}(X)$ or $N^{1,p}(X)$ to be equal to the homogeneous dimension $p = s$. This definition is suggested by the corresponding Sobolev embedding Theorems 3.2.2 and 3.2.5.

From the examples in the Euclidean spaces, we conjecture that the Sobolev functions in the critical case may be discontinuous if the homogeneous dimension satisfies $s > 1$. Thus we turn to investigate the continuity of Sobolev functions on metric measure spaces with dimensions less than or equal to 1. One natural generalization is to compact, connected 1-dimensional metric spaces. We give the definition of the absolutely continuous functions on such spaces and prove the equivalence of Newton Sobolev functions in $N^{1,1}(X)$ and absolutely continuous functions. Moreover, by proving that the spaces support Poincaré inequality, we get the equivalence of these absolutely continuous functions with Sobolev functions defined via different approaches.

Another direction is to look at metric spaces with dimension $s \leq 1$. We focus on the case of $s$-Ahlfors regular metric spaces which include a large class of fractals and we prove the uniform continuity of Hajłasz Sobolev functions with explicit estimates.

## 4.1 ON COMPACT, CONNECTED METRIC SPACES WITH $\mathcal{H}^1(X) < \infty$

### 4.1.1 Absolutely continuous functions on the metric spaces

We define absolutely continuous functions on spaces of finite 1-dimensional measure as follows.

**Definition 4.1.1.** Let $(X, d, \mathcal{H}^1)$ be a compact, connected metric space with $\mathcal{H}^1(X) < \infty$. A function $u : X \to \mathbb{R}$ is absolutely continuous if for any $\epsilon > 0$, there is a positive number $\delta$ such that

$$\sum_i |u \circ \gamma_i(\ell_i) - u \circ \gamma_i(0)| < \epsilon,$$

for any collection of pairwise disjoint arc-length parametrized simple curves $\gamma_i \colon [0, \ell_i] \to X$ with total length $\sum_i \ell_i < \delta$.

Let $V(f, [a, b])$ denote the total variation of a mapping $f : [a, b] \to X$ on $[a, b]$. The total variation of a mapping on a interval is defined as

$$V(f, [a, b]) := \sup \left\{ \sum_{i=1}^{k} d(f(t_i), f(t_{i-1})), a = t_0 < t_1 < \cdots < t_k = b \right\},$$

where the supremum is taken over all finite sequences of $\{t_i\} \subset [a, b]$ defined as above. We can replace $|u \circ \gamma_i(\ell_i) - u \circ \gamma_i(0)|$ by $V(u \circ \gamma_i, [0, \ell_i])$ in the above definition, and get an equivalent definition.

We denote the above class of absolutely continuous functions on $X$ by $AC(X)$. Let $X = [a, b]$, it is easy to verify this definition is consistent with the classical definition.

We prove that the absolutely continuous functions defined above are uniformly continuous on a quasiconvex metric space.

**Proposition 4.1.2.** *Let $(X, d, \mu)$ be a compact, quasiconvex metric measure space with $\mathcal{H}^1(X) < \infty$. If a function $u : X \to \mathbb{R}$ is absolutely continuous, then it is uniformly continuous.*

*Proof.* Given an arbitrary positive number $\epsilon > 0$, it suffices to find $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ whenever $d(x, y) < \delta$ for $x, y \in X$.

Since $X$ is compact and path-connected, two arbitrary points $x, y \in X$ can be joined by a shortest curve. We denote the arc-length parametrization of this shortest curve by $\gamma_0 : [0, \ell_0] \to X$ with $\gamma_0(0) = x$ and $\gamma_0(\ell_0) = y$. Thus,

$$|u(x) - u(y)| = |u(\gamma_0(0)) - u(\gamma_0(\ell_0))| = |u \circ \gamma_0(0) - u \circ \gamma_0(\ell_0)|.$$

The fact that the space is quasiconvex means that there is a curve connecting $x, y$ with length $\ell \leq Cd(x, y)$. Thus, the length of the shortest curve $\ell_0 \leq \ell \leq Cd(x, y)$.

Since $u \in AC(X)$, for an arbitrary positive number $\epsilon > 0$, there is $\delta_0 > 0$, such that if the length of a simple curve $\ell_0 < \delta_0$, then

$$|u \circ \gamma_0(\ell_0) - u \circ \gamma_0(0)| < \epsilon.$$

In particular, let $\delta < C^{-1}\delta_0$, then $\ell_0 \leq Cd(x, y) < \delta_0$. It implies that

$$|u(x) - u(y)| < \epsilon.$$

□

### 4.1.2 Absolutely continuous characterization of Sobolev functions

In this section, we will show that the absolutely continuous functions we define are actually the same as Newton Sobolev functions.

**Theorem 4.1.3.** *Let $(X, d, \mathcal{H}^1)$ be a compact, connected metric measure space with $\mathcal{H}^1(X) < \infty$. If $u \in AC(X)$, then there is an upper gradient $g \in L^1(X)$ for $u$, that is, for any rectifiable curve $\gamma : [a, b] \to X$, we have*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g.$$

*Proof.* By Theorem 2.3.3, we know that there is a countable collection of simple curves in $X$ and a set $N_1$ with $\mathcal{H}^1(N_1) = 0$ such that

$$X = \bigcup_{i=1}^{\infty} \gamma_i((0, \ell_i)) \cup N_1.$$

Let $\Gamma_i = \gamma_i([0, \ell_i])$, then the intersection of $\Gamma_i$ are $\Gamma_j$ is empty for $i \neq j$. Indeed, the simple curves from Theorem 2.3.3 meet at the endpoints only as we excluded the endpoints here. We denote $u \circ \gamma_i$ by $u_i$. If $u \in AC(X)$, then $u_i \in AC((0, \ell_i))$. Thus, $u_i'(t)$ exists almost everywhere for $t \in (0, \ell_i)$. Moreover, we have

$$u_i(\ell_i) - u_i(0) = \int_0^{\ell_i} u_i'(s)\ ds,$$

and

$$\bigvee_0^{\ell_i} u_i = \int_0^{\ell_i} |u_i'(s)|\ ds.$$

We denote the collection of points $x \in X$ such that $u_i'(\gamma_i^{-1}(x))$ does not exist as $N_2$, and $N_0 = N_1 \cup N_2$. It is clear that $\mathcal{H}^1(N_0) = 0$. Then we define a function $g : X \to \mathbb{R}$ as follows,

$$g(x) = \begin{cases} |u_i'(\gamma_i^{-1}(x))| & \text{if } x \in X \setminus N_0 \\ \infty & \text{if } x \in N_0. \end{cases} \tag{4.1.1}$$

This function $g$ is integrable on $X$. In fact, by definition and area formula, we have

$$\begin{aligned} \int_X g\ d\mathcal{H}^1 &= \int_{X \setminus N_0} g\ d\mathcal{H}^1 \\ &= \sum_i \int_{\Gamma_i} g\ d\mathcal{H}^1 \\ &= \sum_i \int_0^{\ell_i} g(\gamma_i(s))\ ds \\ &= \sum_i \int_0^{\ell_i} |u_i'(s)|\ ds \\ &= \sum_i \bigvee_0^{\ell_i} u_i. \end{aligned}$$

Since $\gamma_i$ is injective and $\mathcal{H}^1(X) < \infty$, it implies that

$$\sum_i \ell_{\gamma_i} = \sum_i \mathcal{H}^1(\Gamma_i) \leq \mathcal{H}^1(X) < \infty.$$

Thus, for any $\epsilon > 0$, there exists a natural number $n_0$ such that $\sum_{i=n_0+1}^{\infty} \ell_{\gamma_i} < \epsilon$. By Definition 4.1.1, it implies that $\sum_{i=n_0+1}^{\infty} \bigvee_0^{\ell_i} u_i$ can be sufficiently small.

$$\begin{aligned}\int_X g \; d\mathcal{H}^1 &= \sum_i \bigvee_0^{\ell_i} u_i \\ &= \sum_{i=1}^{n_0} \bigvee_0^{\ell_i} u_i + \sum_{i=n_0+1}^{\infty} \bigvee_0^{\ell_i} u_i \;\; < \infty.\end{aligned}$$

We next prove that $g$ is an upper gradient for the function $u \in AC(X)$, that is, for any rectifiable curve $\gamma : [a, b] \to X$, we have

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g.$$

Let $\gamma : [a, b] \to X$ be a rectifiable curve and $\Gamma = \gamma([a, b])$. Since $\Gamma$ is compact, connected and $\mathcal{H}^1(\Gamma) < \infty$, there exists a shortest curve joining $\gamma(a)$ and $\gamma(b)$ in $\Gamma$. We denote the arc-length parametrization of this injective curve by $\gamma_0 : [0, \ell] \to \Gamma$ with

$$\gamma_0(0) = \gamma(a) \quad \text{and} \quad \gamma_0(\ell) = \gamma(b). \tag{4.1.2}$$

Let $\Gamma_0 = \gamma_0([0, \ell])$. Applying the area formula we obtain that

$$\begin{aligned}\int_{\gamma_0} g &= \int_0^\ell g(\gamma_0(s)) \; ds \\ &= \int_{\Gamma_0} g(x) \; d\mathcal{H}^1 \\ &\leq \int_\Gamma g(x) \; d\mathcal{H}^1 \\ &\leq \int_\gamma g.\end{aligned} \tag{4.1.3}$$

Since $u \in AC(X)$, it follows that $u_0 = u \circ \gamma_0 \in AC([0, \ell])$ and $u_0'$ exists almost everywhere. Let $I = \{i \in \mathbb{N} : \mathcal{H}^1(\Gamma_0 \cap \Gamma_i) \neq 0\}$. Then $\Gamma_0$ is the union of $\bigcup_{i \in I}(\Gamma_0 \cap \Gamma_i)$ and a null set.

Let $i \in I$ and $x \in \Gamma_i \cap \Gamma_0$ such that $u_0'(\gamma_0^{-1}(x))$, $u_i'(\gamma_i^{-1}(x))$ both exist and $t = \gamma_0^{-1}(x)$ is a density point of $\gamma_0^{-1}(\Gamma_i \cap \Gamma_0)$. By definition,

$$\begin{aligned} |u_0'(t)| &= \lim_{h \to 0} \frac{|u_0(t+h) - u_0(t)|}{|h|} \\ &= \lim_{h \to 0} \frac{|u(\gamma_0(t+h)) - u(\gamma_0(t))|}{|h|}. \end{aligned}$$

Since $t$ is a density point in the measurable set $\gamma_0^{-1}(\Gamma_i \cap \Gamma_0)$, when $h$ is sufficiently small, there exists $t + h \in \gamma_0^{-1}(\Gamma_i \cap \Gamma_0)$. We denote $\gamma_i(s) = \gamma_0(t)$ and $\gamma_i(s') = \gamma_0(t+h)$. By construction, the curve $\gamma_i$ is the shortest curve joining $\gamma_i(s)$ and $\gamma_i(s')$. Since $\gamma_0$ and $\gamma_i$ are both parametrized by arc-length, it follows that

$$|s - s'| = \ell_{\gamma_i(s,s')} \le \ell_{\gamma_0(t,t+h)} = |h|.$$

It follows that

$$\begin{aligned} |u_0'(t)| &= \lim_{h \to 0} \frac{|u(\gamma_0(t+h)) - u(\gamma_0(t))|}{|h|} \\ &\le \lim_{|s-s'| \to 0} \frac{|u(\gamma_i(s')) - u(\gamma_i(s))|}{|s - s'|} \\ &= |u_i'(s)|. \end{aligned}$$

By area formula, for the injective arc-length parametrized curves $\gamma_i$ and $\gamma_0$, the $\mathcal{H}^1$ measure of the preimage of a null set is zero. Thus, it implies that

$$|u_0'(t)| \le |u_i'(s)| = |u_i'(\gamma_i^{-1}(\gamma_0(t)))| = g(\gamma_0(t))$$

holds almost everywhere for $t \in [0, \ell]$. It follows that

$$\begin{aligned} |u_0(\ell) - u_0(0)| &\le \int_0^\ell |u_0'(t)|\ dt \\ &\le \int_0^\ell g(\gamma_0(t))\ dt \\ &= \int_{\gamma_0} g. \end{aligned} \tag{4.1.4}$$

Combining inequalities (4.1.2), (4.1.3), (4.1.4), we get

$$
\begin{aligned}
|u(\gamma(a)) - u(\gamma(b))| &= |u(\gamma_0(0)) - u(\gamma_0(\ell))| \\
&= |u_0(0) - u_0(\ell)| \\
&\leq \int_{\gamma_0} g \\
&\leq \int_{\gamma} g.
\end{aligned}
\tag{4.1.5}
$$

Thus, we verify that $g \in L^1(X)$ is an upper gradient of $u$. $\square$

If $u \in AC(X) \cap L^1(X)$, then Theorem 4.1.3 implies that $u \in N^{1,1}(X)$. If we further assume that $u \in AC^p(X)$, that is, the upper gradient $g$ defined in (4.1.1) belongs to $L^p(X)$. It follows that $u \in N^{1,p}(X)$ and $AC^p(X) \subset N^{1,p}(X)$. On the other hand, we need the following lemma [59, Proposition 6.3.3].

**Lemma 4.1.4.** *Let $u\colon X \to \mathbb{R}$ be a function and $\gamma : [0, \ell] \to X$ be an arc-length parametrized rectifiable curve in $X$. Assume that $\rho : X \to [0, \infty]$ is a Borel function such that $\rho$ is integrable on $\gamma$ and the pair $(u, \rho)$ satisfies the upper gradient inequality on $\gamma$ and each of its compact subcurves. Then $u \circ \gamma$ is absolutely continuous and the inequality*

$$|(u \circ \gamma)'(t)| \leq (\rho \circ \gamma)(t),$$

*holds for almost every $t \in [0, \ell]$*

If $u \in N^{1,p}(X)$, then it has a $p$-integrable upper gradient $\rho$ and $(u, \rho)$ satisfies the upper gradient inequality for any arc-length parametrized simple rectifiable curves in $X$, that is,

$$|u(\gamma(\ell)) - u(\gamma(0))| \leq \int_0^\ell \rho(\gamma(s))ds.$$

The absolute continuity of integral implies immediately that $u \in AC(X)$.

Moreover, if $u \in N^{1,p}(X)$, the upper gradient $g$ defined in (4.1.1) is $p$-integrable. Let $u \in N^{1,p}$ and $\gamma_i$ be the simple curves as in the proof of Theorem 4.1.3. Lemma 4.1.4 implies that

$$|(u \circ \gamma_i)'(t)| \leq (\rho \circ \gamma_i)(t).$$

For almost everywhere $x \in X$, the above inequality implies that

$$g(x) = |(u \circ \gamma_i)'(\gamma_i^{-1}(x))| \leq \rho(x).$$

The upper gradient $g$ defined in (4.1.1) is bounded by a $p$-integrable function $\rho$ almost everywhere in $X$. It implies that $g \in L^p(X)$ and $u \in AC^p(X)$. Combining the above facts, we get the following characterization.

**Theorem 4.1.5.** *Let $(X, d, \mathcal{H}^1)$ be a compact and connected metric measure space with $\mathcal{H}^1(X) < \infty$ and let $1 \leq p < \infty$. Then $u \in N^{1,p}(X)$ if and only if $u \in AC^p(X) \cap L^p(X)$. In other words*

$$N^{1,p}(X) = AC^p(X) \cap L^p(X).$$

### 4.1.3 Spaces supporting the Poincaré inequality

In this section, we will show that a compact, quasiconvex and 1-Ahlfors regular metric measure space supports Poincaré inequality. Thus, it implies the equivalence of absolutely continuous functions with Sobolev functions defined via different approaches.

**Theorem 4.1.6.** *Let $(X, d, \mathcal{H}^1)$ be a compact, quasiconvex and 1-Ahlfors regular metric measure space. Then it supports p-Poincaré inequality for $1 \leq p < \infty$.*

*Proof.* It suffices to show that $X$ supports the 1-Poincaré inequality. Let $B(O, r) \subset X$ be an arbitrary ball in $X$ and $x, y \in B$. There exists a rectifiable curve that joins $x$ and $y$. Since $X$ is compact and quasiconvex, there exists a shortest curve connecting $x$ and $y$ [47, Theorem 3.9]. We denote the arc-length parametrization of this shortest curve as $\gamma : [0, \ell] \to X$ such that $\gamma(0) = x$ and $\gamma(\ell) = y$.

Let $u$ be a Borel function and $g$ be an upper gradient of $u$. Then

$$|u(x) - u(y)| \leq \int_0^\ell g(\gamma(s))\, ds. \tag{4.1.6}$$

Denote $\Gamma = \gamma([0, \ell])$ and $\lambda = 3C$, where C is the quasiconvex constant of $X$.

For an arbitrary point $z \in \Gamma$, let $\ell_{\gamma(x,z)}$ denotes the length of the shortest curve joining $x$ and $z$. Then $\ell_{\gamma(x,z)} \leq \ell$. Since $X$ is quasiconvex, there is a curve $\tilde{\gamma}(x,y)$ connecting $x$ and $y$ such that $\ell_{\tilde{\gamma}(x,y)} \leq Cd(x,y)$. It implies that

$$\begin{aligned} d(z,O) &\leq d(z,x) + d(x,O) \\ &\leq \ell_{\gamma(x,z)} + r \\ &\leq \ell + r \\ &\leq \ell_{\tilde{\gamma}(x,y)} + r \\ &\leq Cd(x,y) + r \\ &\leq \lambda r. \end{aligned}$$

Thus, every point $z \in \Gamma$ belongs to the ball $\lambda B = B(O, \lambda r)$. It implies that

$$\begin{aligned} \int_0^\ell g(\gamma(s))\ ds &= \int_\Gamma g(y)\ d\mathcal{H}^1(y) \\ &\leq \int_{\lambda B} g(y)\ d\mathcal{H}^1(y). \end{aligned} \tag{4.1.7}$$

Finally, combining (4.1.6), (4.1.7), we get

$$\begin{aligned} |u(x) - u_B| &\leq \fint_B |u(x) - u(y)|\ dz \\ &\leq \fint_B \int_0^\ell g(\gamma(s))\ dsdz \\ &\leq \fint_B \int_{\lambda B} g(y)\ d\mathcal{H}^1(y)dz \\ &= \mathcal{H}^1(\lambda B) \fint_{\lambda B} g(y)\ d\mathcal{H}^1(y). \end{aligned} \tag{4.1.8}$$

Since $X$ is 1-Ahlfors regular, it follows that

$$\fint_B |u(x) - u_B|\ d\mathcal{H}^1 \leq C\mathrm{diam}(B) \fint_{\lambda B} g(z)\ d\mathcal{H}^1(z).$$

$\square$

*Remark* 4.1.7. In fact, instead of assuming quasiconvexity, in the proof of this theorem, it suffice to assume a weaker condition that the space $X$ is C-LLC1, that is, there exists a constant $C \geq 1$ so that for each $x \in X$ and $r > 0$, any pair of points in $B(x, r)$ can be joined by a curve in $B(x, Cr)$. With this condition, we still have that the shortest curve connecting any two points $x$ and $y$ is contained in $B(x, Cd(x, y))$ and the argument in the above theorem works.

When a complete doubling metric measure $X$ supports the $p$-Poincaré inequality with $1 < p < \infty$, the Newton Sobolev spaces $N^{1,p}(X)$, the Poincaré Sobolev spaces $P^{1,p}(X)$ and Hajłasz Sobolev spaces $M^{1,p}(X)$ are equivalent [47, Theorem 11.3]. Combined with Theorem 4.1.5, we obtain the following corollary.

**Corollary 4.1.8.** *Let $(X, d, \mathcal{H}^1)$ be a compact, quasiconvex, 1-Ahlfors regular metric measure space. Then*

$$AC^p(X) \cap L^p(X) = N^{1,p}(X) = P^{1,p}(X) = M^{1,p}(X),$$

*for $1 < p < \infty$ and the first two equalities also hold for $p = 1$.*

## 4.2 ON $S$-AHLFORS REGULAR METRIC SPACES WITH $S \leq 1$

The theory of Newton Sobolev spaces and Poincaré Sobolev spaces apply very well to the spaces with sufficiently many nonconstant rectifiable curves. However, when comes to the spaces with limited connectivity properties, the theory of Hajłasz Sobolev spaces is rich without any assumption on connectivity of the space. Hajłasz Sobolev spaces on fractals have been investigated in [61, 102, 106, 120].

In this section, we will investigate the Hajłasz Sobolev functions in the critical case $M^{1,s}$ on $s$-Ahlfors regular spaces with $0 < s \leq 1$. An important class of Ahlfors regular spaces is provided by fractals. Many fractals are generated by iterated function systems. A contraction $S$ on $D \subset \mathbb{R}^n$ is a Lipschitz mapping with Lipschitz constant $0 < c < 1$, i.e.

$$|S(x) - S(y)| \leq c|x - y|,$$

for all $x, y \in D$. A finite family of contractions $\{S_1, \cdots, S_m\}$, with $m \geq 2$, is called an iterated function system or IFS. We say that $S_i$ satisfy the open set condition if there exists a non-empty bounded open set $V$ such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V$$

with the union disjoint. Hutchinson [64] proved that fractals generated by iterated function systems satisfying the open set condition are Ahlfors regular. For example, the ternary Cantor set $C$ equipped with the Euclidean distance is $s$-Ahlfors regular with $s = \frac{\log 2}{\log 3}$ [35, Example 2.7].

The main theorem of this section states as follows.

**Theorem 4.2.1.** *Let $(X, d, \mathcal{H}^s)$ be an $s$-Ahlfors regular metric space and $0 < s \leq 1$. If $u \in M^{1,s}(X, d, \mathcal{H}^s)$, then $u$ is uniformly continuous. Moreover, there exists a constant $C > 0$, such that for any ball $B \subset X$,*

$$\sup_{x,y \in B} |u(x) - u(y)| \leq C\Big(\int_{2B} g^s d\mathcal{H}^s\Big)^{\frac{1}{s}}, \tag{4.2.1}$$

*where $g \in D(u) \cap L^s(X)$.*

The restriction that $s \leq 1$ is necessary in our statement, since we apply the reverse Minkowski inequality in the proof. We conjecture that there always exist discontinuous Sobolev functions in $M^{1,s}(X)$ when $X$ is an $s$-Ahlfors regular space with $s > 1$.

We need the following lemma.

**Lemma 4.2.2.** *Let $X$ be a uniformly perfect space. Then there is a constant $0 < C_0 < 1$ such that for any ball $B(x, r)$ with $0 < r < \operatorname{diam}(X)$, there is a sequence of balls $\{B_i\}_{i=1}^{\infty} = \{B(x_i, C_0^i r)\}_{i=1}^{\infty} \subset B(x, r)$ satisfying*

*(1) $B_i \cap B_j = \emptyset$, if $i \neq j$,*

*(2) $B_i \subset B(x, C_0^{i-1} r) \setminus B(x, C_0^i r)$.*

*Proof.* Since $X$ is uniformly perfect, there exists a constant $0 < c < 1$ such that for all $x \in X$ and $0 < r < \mathrm{diam}(X)$,

$$\overline{B(x,r)} \setminus B(x, cr) \neq \emptyset.$$

Fix a number $C_1 = c/2$. Clearly, $0 < C_1 < 1$.

Fix a ball $B(x,r) \subset X$ with $0 < r < \mathrm{diam}(X)$ and let $y \in \overline{B(x,r)} \setminus B(x, cr)$. Then

$$B(y, C_1 r) \subset B(x, C_1^{-1} r) \setminus B(x, C_1 r).$$

We can construct a sequence of balls $B_1, B_2 \ldots \subset B(x,r)$ in the following way. Let $\hat{r}_i = C_1^{2i-1} r$ and $\hat{B}_i = B(x, \hat{r}_i)$. By the previous argument, for each $\hat{B}_i$, there exists $x_i$ such that

$$B(x_i, C_1 \hat{r}_i) \subset B(x, C_1^{-1} \hat{r}_i) \setminus B(x, C_1 \hat{r}_i).$$

Let $r_i = C_1 \hat{r}_i = C_1^{2i} r$, so

$$B(x_i, r_i) \subset B(x, r_{i-1}) \setminus B(x, r_i).$$

Let $C_0 = C_1^2$. Clearly, $0 < C_0 < 1$. Let $B_i = B(x_i, r_i) = B(x_i, C_0^i r)$. Since these balls are contained in disjoint annuli, $B_i \cap B_j = \emptyset$ for $i \neq j$.

From the construction, the balls in this sequence satisfy

$$B_i \subset B(x, C_0^{i-1} r) \setminus B(x, C_0^i r).$$

This completes the proof. □

We now can complete the proof of the main result, following some ideas from [45].

*Proof of Theorem 4.2.1.* By definition, for $u \in M^{1,s}(X, d, \mathcal{H}^s)$, there exists $g \in L^s(X)$ such that

$$|u(x) - u(y)| \leq d(x,y)\big(g(x) + g(y)\big),$$

when $x, y \in X \setminus E$ and $\mathcal{H}^s(E) = 0$.

Let $E_0 = \{x \in X \setminus E : g(x) < \infty\}$. Clearly, $\mathcal{H}^s(X \setminus E_0) = 0$. Fix a ball $B(z,r) \subset X$. We will prove that

$$\sup_{x,y \in B(z,r) \cap E_0} |u(x) - u(y)| \leq C \Big( \int_{B(z,2r)} g^s \, d\mathcal{H}^s \Big)^{\frac{1}{s}}.$$

Let $x, y \in B(z, r) \cap E_0$. According to Lemma 4.2.2, there exists a sequence of disjoint balls $\{B_i\}_{i=1}^{\infty} = \{B(x_i, C_0^i r)\}_{i=1}^{\infty} \subset B(x, r)$ such that

$$B_i = B(x_i, C_0^i r) \subset B(x, C_0^{i-1} r) \setminus B(x, C_0^i r).$$

Let $r_i = C_0^i r$, then

$$B_i = B(x_i, r_i) \subset B(x, C_0^{-1} r_i) \setminus B(x, r_i).$$

Clearly, for each $i \in \mathbb{N}$, we can find $z_i \in B_i \cap E_0$ such that

$$\begin{aligned} g(z_i) &\le \Big(\frac{1}{\mathcal{H}^s(B_i)}\Big)^{1/s} \Big(\int_{B_i} g^s d\mathcal{H}^s\Big)^{1/s} \\ &\le \frac{C}{r_i} \Big(\int_{B_i} g^s d\mathcal{H}^s\Big)^{1/s}. \end{aligned}$$

Notice that $z_i \in B_i \subset B(x, C_0^{-1} r_i)$ implies that $d(x, z_i) \le C_0^{-1} r_i$. It follows that

$$\begin{aligned} |u(x) - u(z_i)| &\le d(x, z_i)\Big(g(x) + g(z_i)\Big) \\ &\le C r_i g(x) + C\Big(\int_{B_i} g^s d\mathcal{H}^s\Big)^{1/s}. \end{aligned}$$

Hence, $\lim_{i\to\infty} u(z_i) = u(x)$. Thus,

$$\begin{aligned} |u(z_1) - u(x)| &\le \sum_{i=1}^{\infty} |u(z_i) - u(z_{i+1})| \\ &= \sum_{i=0}^{\infty} d(z_i, z_{i+1})\Big(g(z_i) + g(z_{i+1})\Big). \end{aligned} \tag{4.2.2}$$

Since $z_i \in B_i$, $z_{i+1} \in B_{i+1}$ and $B_i, B_{i+1} \subset B(x, C_0^{-1} r_i)$, we have that

$$d(z_i, z_{i+1}) \le C r_i.$$

Moreover, $r_{i+1} = C_0 r_i$ and so the inequality (4.2.2) implies that

$$\begin{aligned} |u(z_1) - u(x)| &\le \sum_{i=0}^{\infty} C r_i \Big(\frac{1}{r_i}\big(\int_{B_i} g^s d\mathcal{H}^s\big)^{1/s} + \frac{1}{r_{i+1}}\big(\int_{B_{i+1}} g^s d\mathcal{H}^s\big)^{1/s}\Big) \\ &\le C \sum_{i=0}^{\infty} \Big(\int_{B_i} g^s d\mathcal{H}^s\Big)^{1/s} \\ &\le C\Big(\sum_{i=0}^{\infty} \int_{B_i} g^s d\mathcal{H}^s\Big)^{1/s} \\ &\le C\Big(\int_{B(x,r)} g^s d\mathcal{H}^s\Big)^{1/s}. \end{aligned} \tag{4.2.3}$$

The second to last inequality follows from the reverse Minkowski inequality and the last inequality follows from the fact that the balls in the sequence $\{B_i\}_{i=1}^{\infty} \subset B(x,r)$ are pairwise disjoint.

Notice that $z_1 \in B_1 \subset B(x,r) \subset B(z,2r)$ and

$$g(z_1) \leq \frac{C}{r_1}\Big(\int_{B_1} g^s d\mathcal{H}^s\Big)^{1/s} \leq \frac{C}{r}\Big(\int_{B(z,2r)} g^s d\mathcal{H}^s\Big)^{1/s}.$$

Similarly, we can find $w_1 \in B(y,r) \subset B(z,2r)$ such that

$$|u(w_1) - u(y)| \leq C\Big(\int_{B(y,r)} g^s d\mathcal{H}^s\Big)^{1/s}, \tag{4.2.4}$$

and

$$g(w_1) \leq \frac{C}{r}\Big(\int_{B(z,2r)} g^s d\mathcal{H}^s\Big)^{1/s}.$$

Since $z_1, w_1 \in B(z,2r)$, we also have

$$\begin{aligned} |u(z_1) - u(w_1)| &\leq d(z_1,w_1)(g(z_1) + g(w_1)) \\ &\leq C\big(\int_{B(z,2r)} g^s d\mathcal{H}^s\big)^{1/s}. \end{aligned} \tag{4.2.5}$$

Applying triangle inequality and combining the inequalities (4.2.3), (4.2.4), (4.2.5), we have

$$\sup_{x,y \in B(z,r) \cap E_0} |u(x) - u(y)| \leq C\Big(\int_{B(z,2r)} g^s d\mathcal{H}^s\Big)^{\frac{1}{s}}.$$

This and the absolute continuity of the integral imply uniform continuity of $u|_{X \setminus E_0}$. Hence $u$ uniquely extends to a uniformly continuous function on $X$ and the inequality (4.2.1) follows. □

*Remark* 4.2.3. A closer inspection of the proof of the main theorem suggests that the estimate (4.2.1) also holds under the weaker conditions that the metric space $(X, d, \mathcal{H}^s)$ is uniformly perfect and satisfies the condition that there exists a constant $C > 0$ such that for all $x \in X$ and $0 < r < \operatorname{diam}(X)$,

$$\mathcal{H}^s(B(x,r)) \geq Cr^s.$$

However, to get the uniform continuity of the function by this inequality (4.2.1), the condition that $\mathcal{H}^s(B(x,r)) \leq Cr^s$ seems necessary.

## 5.0 SOBOLEV EMBEDDING OF A SPHERE CONTAINING AN ARBITRARY CANTOR SET IN THE IMAGE

Let $\mathbb{S}^n$ denotes the standard unit sphere in $\mathbb{R}^{n+1}$. In 1924 J. W. Alexander [1], constructed a homeomorphism $f : \mathbb{S}^2 \to f(\mathbb{S}^2) \subset \mathbb{R}^3$ so that the unbounded component of $\mathbb{R}^3 \setminus f(\mathbb{S}^2)$ is not simply connected. In particular it is not homeomorphic to the complement of the standard ball in $\mathbb{R}^3$. This famous construction known as the *Alexander horned sphere* can be easily generalized to higher dimensions. The aim of this chapter is to show that a large class of pathological topological $n$-dimensional spheres, including the Alexander horned sphere, can be realized as the images of Sobolev $W^{1,n}$ homeomorphisms, each of which being a smooth diffeomorphism outside of a Cantor set in $\mathbb{S}^n$ of Hausdorff dimension zero.

### 5.1 INTRODUCTION

Recall also that the *Sobolev space* $W^{1,p}$ consists of functions in $L^p$ whose distributional gradient is in $L^p$. By $f \in W^{1,p}(\mathbb{S}^n, \mathbb{R}^{n+1})$ we will mean that the components of the mapping $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ are in $W^{1,p}$.

By an *embedding* we will mean a homeomorphism onto the image i.e., $f : X \to Y$ is an embedding if $f : X \to f(X)$ is a homeomorphism. In the literature such an embedding is often called a topological embedding. The main result of this chapter reads as follows.

**Theorem 5.1.1.** *For any Cantor set $C \subset \mathbb{R}^{n+1}$, $n \geq 2$, there is an embedding $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that*

*(a)* $f \in W^{1,n}(\mathbb{S}^n, \mathbb{R}^{n+1})$,

*(b)* $C \subset f(\mathbb{S}^n)$,

*(c)* $f^{-1}(C) \subset \mathbb{S}^n$ *is a Cantor set of Hausdorff dimension zero,*

*(d)* $f$ *is a smooth diffeomorphism in* $\mathbb{S}^n \setminus f^{-1}(C)$.

Our construction resembles that of the Alexander horned sphere and it will be clear that it can be used to construct a version of the Alexander horned sphere $f : \mathbb{S}^n \to f(\mathbb{S}^n) \subset \mathbb{R}^{n+1}$ so that $f \in W^{1,n}$ and $f$ is a smooth diffeomorphism outside a Cantor set of Hausdorff dimension zero.

A similar technique to the one used in the proof of Theorem 5.1.1 has also been employed in a variety of different settings [15, 14, 22, 44, 51, 50, 66, 67, 71, 94, 104, 103, 116, 118, 117].

According to Theorem 5.1.1 we can construct a topological sphere in $\mathbb{R}^3$ that is $W^{1,2}$ homeomorphic to $\mathbb{S}^2$ and that contains Antoine's necklace. *Antoine's necklace* is a Cantor set in $\mathbb{R}^3$ whose complement is not simply connected. Hence the unbounded component of $\mathbb{R}^3 \setminus f(\mathbb{S}^2)$ is also not simply connected. This gives a different example with the same topological consequences as those of the Alexander horned sphere. In fact, using results of Sher [113] we will show that there are uncountably many "essentially different" examples. For a precise statement see Theorem 5.5.1.

One cannot in general demand the function constructed in the theorem to be in $W^{1,p}$, $p > n$. Indeed, if $f \in W^{1,p}(\mathbb{S}^n, \mathbb{R}^{n+1})$, then the image $f(\mathbb{S}^n)$ has finite $n$-dimensional Hausdorff measure, but a Cantor set in $\mathbb{R}^{n+1}$ may have positive $(n+1)$-dimensional measure and in that case it cannot be contained in the image of $f$. The fact that the image $f(\mathbb{S}^n)$ has finite $n$-dimensional measure follows from the area formula and the integrability of the Jacobian of $f$. The fact that the area formula is satisfied for mappings $f \in W^{1,p}(\mathbb{S}^n, \mathbb{R}^{n+1})$ with $p > n$ is well known and follows from the following observations. The area formula is true for Lipschitz mappings [33, Theorem 3.3.2]. The domain $\mathbb{S}^n$ is the union of countably many sets on which $f$ is Lipschitz continuous [33, Section 6.6.3] plus a set of measure zero. On Lipschitz pieces the area formula is satisfied. Since the mapping $f$ maps sets of measure zero to sets of $n$-dimensional Hausdorff measure zero [60, Theorem 4.2], the area formula is in fact true for $f$. The proof presented in [60, Theorem 4.2] is in the case of mappings into $\mathbb{R}^n$, but the same proof works in the case of mappings into $\mathbb{R}^{n+1}$.

It is well known, [60, Theorem 4.9], that any homeomorphism $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ of class $W^{1,n}$, where $\Omega \subset \mathbb{R}^n$ is open, has the *Lusin property*, i.e. it maps sets of measure zero to sets of measure zero. Reshetnyak [105] observed that this is no longer true for embeddings $f \in W^{1,n}(\mathbb{S}^n, \mathbb{R}^m)$, when $m > n \geq 2$. In his example he considered $n = 2$ and $m = 3$, see also [28, Example 5.1]. Later Väisälä [115] generalized it to any $n \geq 2$ and $m > n$. In the constructions of Reshetnyak and Väisälä a set of measure zero is mapped to a set of positive $n$-dimensional Hausdorff measure. Theorem 5.1.1 also provides an example of this type. Indeed, if a Cantor set $C \subset \mathbb{R}^{n+1}$ has positive $(n+1)$-dimensional measure, then the embedding $f \in W^{1,n}(\mathbb{S}^n, \mathbb{R}^{n+1})$ from Theorem 5.1.1 maps the set $f^{-1}(C)$ of Hausdorff dimension zero onto the set $C \subset f(\mathbb{S}^n)$ of positive $(n+1)$-dimensional measure. Actually, in a context of the Lebesgue area a similar example has already been constructed by Besicovitch [15, 14], but Besicovitch did not consider the Sobolev regularity of the mapping. The construction of Besicovitch is very different from that of Reshetnyak and Väisäla and it is more related to ours. While Besicovitch's construction deals with a particular Cantor set, we deal with *any* Cantor set and we prove that the resulting mapping $f$ belongs to the Sobolev space $W^{1,n}$.

## 5.2 CANTOR SETS AND TREES

The following classical result [119, Theorem 30.3] provides a characterization of spaces that are homeomorphic to the ternary Cantor set: *A metric space is homeomorphic to the ternary Cantor set if and only if it is compact, totally disconnected and has no isolated points.* Recall that the space is *totally disconnected* if the only non-empty connected subsets are one-point sets. In what follows by a *Cantor set* we will mean any subset of Euclidean space that is homeomorphic to the ternary Cantor set.

### 5.2.1 Ternary Cantor set

The ternary Cantor set will be denoted by $\mathfrak{C}$. It is constructed by removing the middle third of the unit interval $[0, 1]$, and then successively deleting the middle third of each resulting subinterval. Denote by $\mathfrak{I}^k$ all binary numbers $i_1 \dots i_k$ such that $i_j \in \{0,1\}$ for $j = 1, 2, \dots, k$, and by $\mathfrak{I}^\infty$ all binary infinite sequences $i_1 i_2 \dots$ Clearly, the ternary Cantor set $\mathfrak{C}$ can be written as

$$\mathfrak{C} = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k \in \mathfrak{I}^k} I_{i_1 \dots i_k},$$

where $I_{i_1 \dots i_k}$ is one of the $2^k$ closed intervals in the $k$-th level of the construction of the Cantor set $\mathfrak{C}$; the binary number $i_1 \dots i_k$ denotes the position of this interval: If $i_k = 0$, it is the left subinterval of $I_{i_1 \dots i_{k-1}}$, otherwise it is the right subinterval.

We also have that for any $k \in \mathbb{N}$

$$\mathfrak{C} = \bigcup_{i_1 \dots i_k \in \mathfrak{I}^k} \mathfrak{C}_{i_1 \dots i_k}$$

where $\mathfrak{C}_{i_1 \dots i_k} = \mathfrak{C} \cap I_{i_1 \dots i_k}$.

Note that points in the Cantor set $\mathfrak{C}$ can be uniquely encoded by infinite binary sequences. Indeed, if $\mathfrak{i} = i_1 i_2 \dots \in \mathfrak{I}^\infty$, then

$$\{\mathfrak{c}_\mathfrak{i}\} = \bigcap_{k=1}^{\infty} I_{i_1 \dots i_k} = \bigcap_{k=1}^{\infty} \mathfrak{C}_{i_1 \dots i_k}$$

consists of a single point $\mathfrak{c}_\mathfrak{i} \in \mathfrak{C}$ and

$$\mathfrak{C} = \bigcup_{\mathfrak{i} \in \mathfrak{I}^\infty} \{\mathfrak{c}_\mathfrak{i}\}.$$

### 5.2.2 Cantor trees

Let $C \subset \mathbb{R}^{n+1}$ be a Cantor set and let $f : \mathfrak{C} \to C$ be a homeomorphism. We will write $C_{i_1 \dots i_k} = f(\mathfrak{C}_{i_1 \dots i_k})$ and $c_\mathfrak{i} = f(\mathfrak{c}_\mathfrak{i})$ for $\mathfrak{i} \in \mathfrak{I}^\infty$. Since the mapping $f$ is uniformly continuous,

$$\max_{i_1 \dots i_k \in \mathfrak{I}^k} (\operatorname{diam} C_{i_1 \dots i_k}) \to 0 \quad \text{as } k \to \infty. \tag{5.2.1}$$

For each $k$ and each $i_1 \dots i_k \in \mathfrak{I}^k$ we select a point $A_{i_1 \dots i_k}$ such that

- The point $A_{i_1 \dots i_k}$ does not belong to $C$,
- The distance of the point $A_{i_1 \dots i_k}$ to $C_{i_1 \dots i_k}$ is less than $2^{-k}$,
- $A_{i_1 \dots i_k} \neq A_{j_1 \dots j_\ell}$ if $i_1 \dots i_k \neq j_1 \dots j_\ell$.

It is easy to see that if $\mathfrak{i} = i_1 i_2 \ldots \in \mathfrak{I}^\infty$, then

$$A_{i_1 \ldots i_k} \to c_{\mathfrak{i}} = f(\mathfrak{c}_{\mathfrak{i}}) \quad \text{as } k \to \infty.$$

Indeed, $c_{\mathfrak{i}} \in C_{i_1 \ldots i_k}$ so

$$|A_{i_1 \ldots i_k} - c_{\mathfrak{i}}| < 2^{-k} + \operatorname{diam} C_{i_1 \ldots i_k} \to 0 \quad \text{as } k \to \infty.$$

Now we are ready to build a *Cantor tree* by adding branches $J_{i_1 \ldots i_k i_{k+1}}$ connecting $A_{i_1 \ldots i_k}$ to $A_{i_1 \ldots i_k i_{k+1}}$. The precise construction goes as follows.

By translating the coordinate system we may assume that the distance between the origin in $\mathbb{R}^{n+1}$ and the Cantor set $C$ is greater than 100 (that is way too much, but there is nothing wrong with being generous).

Let $J_0$ and $J_1$ be smooth Jordan arcs (smoothly embedded arcs without self-intersections) of unit speed (i.e., parametrized by arc-length) connecting the origin 0 to the points $A_0$ and $A_1$ respectively. We also assume that

- The curve $J_0$ does not intersect with the curve $J_1$ (except for the common endpoint 0).
- The curves $J_0$ and $J_1$ avoid the Cantor set $C$.
- The curves $J_0$ and $J_1$ meet the unit ball $\mathbb{B}^n(0,1) \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ lying in the hyperplane of the first $n$ coordinates only at the origin and both curves exit $\mathbb{B}^n(0,1)$ on the same side of $\mathbb{B}^n(0,1)$ in $\mathbb{R}^{n+1}$.

The curves $J_0$ and $J_1$ will be called *branches of order* 1.

A simple topological observation is needed here. While the topological structure of a Cantor set $C$ inside $\mathbb{R}^{n+1}$ may be very complicated (think of Antoine's necklace), no Cantor set can separate open sets in $\mathbb{R}^{n+1}$. Indeed, by [63, Corollary 2 of Theorem IV 3] compact sets separating open sets in $\mathbb{R}^{n+1}$ must have topological dimension at least $n$ but the topological dimension of a Cantor set is 0, [63, Example II 3]. Hence we can connect points in the complement of a Cantor set by smooth Jordan arcs that avoid the Cantor set. Moreover we can construct such an arc in a way that it is arbitrarily close to the line segment connecting the endpoints.

Recall that the *$\varepsilon$-neighborhood* of a set $A$ is the set of all points whose distance to the set $A$ is less than $\varepsilon$.

Suppose that we have already constructed all branches $J_{i_1\ldots i_k}$ of order $k \geq 1$. The construction of the branches of order $k+1$ goes as follows. $\{J_{i_1\ldots i_{k+1}}\}$ is a family of $2^{k+1}$ curves such that

- $J_{i_1\ldots i_{k+1}}$ is a smooth Jordan arc parametrized by arc-length that connects $A_{i_1\ldots i_k}$ (an endpoint of the branch $J_{i_1\ldots i_k}$) to $A_{i_1\ldots i_k i_{k+1}}$.
- The curves $J_{i_1\ldots i_{k+1}}$ do not intersect with the Cantor set $C$, they do not intersect with each other (except for the common endpoints) and they do not intersect with previously constructed branches of orders less than or equal to $k$ (except for the common endpoints).
- The image of the curve $J_{i_1\ldots i_{k+1}}$ is contained in the $2^{-k}$-neighborhood of the line segment $\overline{A_{i_1\ldots i_k} A_{i_1\ldots i_{k+1}}}$.
- The angle between the branch $J_{i_1\ldots i_k}$ and each of the emerging branches $J_{i_1\ldots i_k 0}$ and $J_{i_1\ldots i_k 1}$ at the point $A_{i_1\ldots i_k}$ where the curves meet is larger than $\pi/2$.

The reason why we require the last condition about the angles is far from being clear at the moment, but it will be clarified in the end of this section.

In what follows, depending on the situation, $J_{i_1\ldots i_k}$ will denote either the curve (a map from an interval to $\mathbb{R}^{n+1}$) or its image (a subset of $\mathbb{R}^{n+1}$), but it will always be clear from the context what interpretation we use.

A *Cantor tree* is the closure of the union of all branches

$$T = \overline{\bigcup_{k=1}^{\infty} \bigcup_{i_1\ldots i_k \in \mathfrak{I}^k} J_{i_1\ldots i_k}}\,.$$

We also define $T_k$ to be the tree with branches of orders less than or equal to $k$ removed. Formally

$$T_k = T \setminus B_k \quad \text{where} \quad B_k = \bigcup_{s=1}^{k} \bigcup_{i_1\ldots i_s \in \mathfrak{I}^s} J_{i_1\ldots i_s}.$$

Note that the set $T_k$ is not closed – it does not contain the endpoints $A_{i_1\ldots i_k}$.

The branches $J_{i_1\ldots i_{k+1}}$ are very close to the sets $C_{i_1\ldots i_k}$ in the following sense.

**Lemma 5.2.1.** *A branch $J_{i_1\dots i_{k+1}}$ is contained in the $2^{-k+2} + \operatorname{diam} C_{i_1\dots i_k}$ neighborhood of $C_{i_1\dots i_k}$.*

*Proof.* A branch $J_{i_1\dots i_{k+1}}$ connects the points $A_{i_1\dots i_k}$ and $A_{i_1\dots i_{k+1}}$. The distance of $A_{i_1\dots i_k}$ and $A_{i_1\dots i_{k+1}}$ to the set $C_{i_1\dots i_k}$ is less than $2^{-k}$ (because $C_{i_1\dots i_{k+1}} \subset C_{i_1\dots i_k}$). Hence

$$|A_{i_1\dots i_k} - A_{i_1\dots i_{k+1}}| < 2 \cdot 2^{-k} + \operatorname{diam} C_{i_1\dots i_k}$$

so by the triangle inequality the line segment $\overline{A_{i_1\dots i_k} A_{i_1\dots i_{k+1}}}$ is contained in the $3 \cdot 2^{-k} + \operatorname{diam} C_{i_1\dots i_k}$ neighborhood of $C_{i_1\dots i_k}$. Since $J_{i_1\dots i_{k+1}}$ is contained in the $2^{-k}$ neighborhood of the line segment, the lemma follows. □

**Corollary 5.2.2.** *$T_k$ is contained in the*

$$\varepsilon_k := 2^{-k+2} + \max_{i_1\dots i_k \in \mathfrak{I}^k} (\operatorname{diam} C_{i_1\dots i_k})$$

*neighborhood of the Cantor set $C$ and $\varepsilon_k \to 0$ as $k \to \infty$.*

*Proof.* Indeed, the branches of $T_k$ are of the form $J_{i_1\dots i_{s+1}}$, $s \geq k$. Each such branch is contained in the $2^{-s+2} + \operatorname{diam} C_{i_1\dots i_s}$ neighborhood of $C_{i_1\dots i_s} \subset C$. Since $s \geq k$ and $C_{i_1\dots i_s} \subset C_{i_1\dots i_k}$ we have

$$2^{-s+2} + \operatorname{diam} C_{i_1\dots i_s} \leq 2^{-k+2} + \operatorname{diam} C_{i_1\dots i_k}$$

so $T_k$ is contained in the $\varepsilon_k$ neighborhood of $C$. The fact that $\varepsilon_k \to 0$ follows from (5.2.1). The proof is complete. □

Since the sets $B_k$ are compact and their complements $T_k = T \setminus B_k$ are in close proximity of $C$ by Corollary 5.2.2, it easily follows that

$$T = C \cup \bigcup_{k=1}^{\infty} \bigcup_{i_1\dots i_l \in \mathfrak{I}^k} J_{i_i\dots i_k}.$$

The idea of the proof of Theorem 5.1.1 is to build a surface that looks very similar to the tree $T$ with one dimensional branches $J_{i_1\dots i_k}$ of the tree $T$ replaced by smooth thin surfaces built around the curves $J_{i_1\dots i_k}$; such surfaces will be called *tentacles*. The parametric surface $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ will be constructed as a limit of smooth surfaces $f_k : \mathbb{S}^n \to \mathbb{R}^{n+1}$. This sequence will be defined by induction. In the step $k$ we replace all branches $J_{i_1\dots i_k}$ by smooth surfaces – tentacles. Such surfaces will be very close to the branches $J_{i_1\ldots i_k}$ and they will pass through the endpoints $A_{i_1\ldots i_k}$. The only place where the set $T_k$ gets close to the branch $J_{i_1\ldots i_k}$ is the endpoint $A_{i_1\ldots i_k}$ where the two branches $J_{i_1\ldots i_k 0}$ and $J_{i_1\ldots i_k 1}$ of the set $T_k$ emerge. The surface around $J_{i_1\ldots i_k}$, and passing through the point $A_{i_1\ldots i_k}$ will be orthogonal to the curve $J_{i_1\ldots i_k}$ at the point $A_{i_1\ldots i_k}$. Since the branches $J_{i_1\ldots i_k 0}$ and $J_{i_1\ldots i_k 1}$ emerging from that point form angles larger than $\pi/2$ with $J_{i_1\ldots i_k}$ the surface will not intersect the branches $J_{i_1\ldots i_k 0}$ and $J_{i_1\ldots i_k 1}$. By making the surfaces around $J_{i_1\ldots i_k}$ thin enough we can make them disjoint from the set $T_k$ (note that $A_{i_1\ldots i_k}$ does not belong to $T_k$).

## 5.3 SOBOLEV TENTACLES

It is well known and easy to prove that $\eta(x) = \log|\log|x|| \in W^{1,n}(\mathbb{B}^n(0, e^{-1}))$. Define the truncation of $\eta$ between levels $s$ and $t$, $0 < s < t < \infty$ by

$$\eta_s^t(x) = \begin{cases} t - s & \text{if } \eta(x) \geq t, \\ \eta(x) - s & \text{if } s \leq \eta(x) \leq t \\ 0 & \text{if } \eta(x) \leq s. \end{cases}$$

Fix an arbitrary $\tau > 0$. For every $\delta > 0$ there is a sufficiently large $s$ such that $\widetilde{\eta}_{\delta,\tau} := \eta_s^{s+\tau}$ is a Lipschitz function on $\mathbb{R}^n$ with the properties:

$$\operatorname{supp} \widetilde{\eta}_{\delta,\tau} \subset \mathbb{B}^n(0, \delta/2),$$

$$0 \leq \widetilde{\eta}_{\delta,\tau} \leq \tau \text{ and } \widetilde{\eta}_{\delta,\tau} = \tau \text{ in a neighborhood } \mathbb{B}(0, \delta') \text{ of } 0,$$

$$\int_{\mathbb{R}^n} |\nabla \widetilde{\eta}_{\delta,\tau}|^n < \delta^n.$$

The function $\widetilde{\eta}_{\delta,\tau}$ is not smooth because it is defined as a truncation, however, mollifying $\widetilde{\eta}_{\delta,\tau}$ gives a smooth function, denoted by $\eta_{\delta,\tau}$, with the same properties as those of $\widetilde{\eta}_{\delta,\tau}$ listed above. In particular

$$\int_{\mathbb{R}^n} |\nabla \eta_{\delta,\tau}|^n < \delta^n. \tag{5.3.1}$$

The graph of $\eta_{\delta,\tau}$ restricted to the ball $\overline{\mathbb{B}}^n(0,\delta)$ is contained in the cylinder

$$\{(x_1,\ldots,x_{n+1})|\, x_1^2+\ldots+x_n^2\le\delta^2,\ 0\le x_{n+1}\le\tau\}, \tag{5.3.2}$$

and it forms a slim "tower" around the $x_{n+1}$-axis. The function $\eta_{\delta,\tau}$ equals zero in the annulus $\mathbb{B}^n(0,\delta)\setminus\mathbb{B}^n(0,\delta/2)$ and equals $\tau$ in the ball $\mathbb{B}^n(0,\delta')$.

Consider now a smooth Jordan arc $\gamma:[-1,\tau+1]\to\mathbb{R}^{n+1}$ parametrized by arc-length. We want to construct a smooth mapping $\gamma_\delta:\mathbb{B}^n(0,\delta)\to\mathbb{R}^{n+1}$ whose image will be a smooth, thin, tentacle-shaped surface around the curve $\gamma|_{[0,\tau]}$. To do this we will apply a diffeomorphism $\Phi$ mapping the cylinder (5.3.2) onto a neighborhood of the image of the curve $\gamma$. The tentacle-like surface will be the image of the graph of $\eta_{\delta,\tau}$ under the diffeomorphism $\Phi$.

The construction of a diffeomorphism $\Phi$ follows a standard procedure. Let

$$v_1,\ldots,v_n:[-1,\tau+1]\to T\mathbb{R}^{n+1}$$

be a smooth orthonormal basis in the orthogonal complement of the tangent space to the curve $\gamma$, i.e., for every $t\in[-1,\tau+1]$, $\langle v_1(t),\ldots,v_n(t),\gamma'(t)\rangle$ is a positively oriented orthonormal basis of $T_{\gamma(t)}\mathbb{R}^{n+1}$. Now we define

$$\Phi(x_1,\ldots,x_{n+1})=\gamma(x_{n+1})+\sum_{i=1}^n x_iv_i(x_{n+1})\quad\text{for } x\in\mathbb{R}^{n+1}\text{ with } -1\le x_{n+1}\le\tau+1.$$

Clearly, $\Phi$ is smooth and its Jacobian equals 1 along the $x_{n+1}$ axis, $-1<x_{n+1}<\tau+1$. Hence $\Phi$ is a diffeomorphism in a neighborhood of any point on the $x_{n+1}$-axis, $-1<x_{n+1}<\tau+1$. Using compactness of the image of the curve $\gamma$ it easily follows that there is a $\delta_0>0$ such that for all $0<\delta<\delta_0$, $\Phi$ is a diffeomorphism in an open neighborhood of the cylinder (5.3.2). Now we define

$$\gamma_\delta:\overline{\mathbb{B}}^n(0,\delta)\to\mathbb{R}^{n+1},\quad \gamma_\delta(x_1,\ldots,x_n)=\Phi(x_1,\ldots,x_n,\eta_{\delta,\tau}(x_1,\ldots,x_n)).$$

Since

$$\frac{\partial\gamma_\delta}{\partial x_i}=\frac{\partial\Phi}{\partial x_i}+\frac{\partial\Phi}{\partial x_{n+1}}\frac{\partial\eta_{\delta,\tau}}{\partial x_i}$$

it follows that

$$|D\gamma_\delta| \leq \sqrt{n}\|D\Phi\|_\infty(1+|\nabla\eta_{\delta,\tau}|),$$

where $\|D\Phi\|_\infty$ is the supremum of the Hilbert-Schmidt norms $|D\Phi|$ over the cylinder (5.3.2). Hence using (5.3.1), for every $\varepsilon > 0$ we can find $\delta > 0$ so small that

$$\int_{\mathbb{B}^n(0,\delta)} |D\gamma_\delta|^n \leq C(n)\|D\Phi\|_\infty^n \delta^n < \varepsilon. \tag{5.3.3}$$

Observe that $\delta$ depends on $\gamma$ (because $\|D\Phi\|_\infty$ depends on $\gamma$).

The tentacle $\gamma_\delta$ maps the annulus $\overline{\mathbb{B}}^n(0,\delta)\setminus\mathbb{B}^n(0,\delta/2)$ onto the isometric annulus in the hyperplane orthogonal to $\gamma$ at $\gamma(0)$. Indeed, for $x \in \overline{\mathbb{B}}^n(0,\delta)\setminus\mathbb{B}^n(0,\delta/2)$, $\eta_{\delta,\tau}(x) = 0$ and hence

$$\gamma_\delta(x) = \Phi(x_1,\ldots,x_n,0) = \gamma(0) + \sum_{i=1}^n x_i v_i(0)$$

is an affine isometry. For a similar reason $\gamma_\delta$ maps the ball $\overline{\mathbb{B}}^n(0,\delta')$ onto the isometric ball in the hyperplane orthogonal to $\gamma$ at $\gamma(\tau)$. Finally for $x \in \overline{\mathbb{B}}^n(0,\delta/2)\setminus\mathbb{B}^n(0,\delta')$, $\gamma_\delta$ creates a smooth thin surface around the curve $\gamma$ that connects the annulus at $\gamma(0)$ with the ball at $\gamma(\tau)$.

Composition with a translation allows us to define a tentacle $\gamma_\delta : \overline{\mathbb{B}}^n(p,\delta) \to \mathbb{R}^{n+1}$ centered at any point $p \in \mathbb{R}^n$.

## 5.4 MAIN RESULT

We will construct the Sobolev embedded surface $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ as a limit of smooth embedded surfaces $f_k : \mathbb{S}^n \to \mathbb{R}^{n+1}$.

By replacing $\mathbb{S}^n$ with a diffeomorphic submanifold (still denoted by $\mathbb{S}^n$) we may assume that it contains the unit ball

$$\mathbb{B}^n = \mathbb{B}^n(0,1) \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}.$$

lying in the hyperplane of the first $n$ coordinates.

Since the distance of the Cantor set $C$ to the origin is larger than 100, the only parts of the Cantor tree $T$ that are close to $\mathbb{B}^n$ are the branches $J_0$ and $J_1$ that connect the origin to $A_0$ and $A_1$. Since the branches meet $\mathbb{B}^n$ only at the origin and leave $\mathbb{B}^n$ on the same side of $\mathbb{B}^n$, we can assume that $\mathbb{S}^n$ meets $T$ only at the origin.

Now we will describe the construction of $f_1$. We want to grow two tentacles from $\mathbb{S}^n$ near the branches $J_0$ and $J_1$ all the way to points $A_0$ and $A_1$, but we want to make sure that the tentacles do not touch the set $T_1$.

To do this we choose two distinct points $p_0, p_1 \in \mathbb{B}^n$ close to the origin and modify the curves $J_0$ and $J_1$ only near the origin, so that the modified Jordan arcs $\gamma^0$ and $\gamma^1$ emerge from the points $p_0$ and $p_1$ instead of the origin, and they are orthogonal to $\mathbb{B}^n$ at the points $p_0$ and $p_1$. The curves $\gamma^0$ and $\gamma^1$ quickly meet with $J_0$ and $J_1$ and from the points where they meet they coincide with $J_0$ and $J_1$, so all non-intersection properties of the curves are preserved. Since the curves are modified only at their beginnings, outside that place they are identical with $J_0$ and $J_1$.

Next, we find $\delta_1 > 0$ so small that the balls $\overline{\mathbb{B}}^n(p_0, \delta_1)$ and $\overline{\mathbb{B}}^n(p_1, \delta_1)$ are disjoint and contained in $\mathbb{B}^n$ and that there are disjoint tentacles

$$\gamma^i_{\delta_1} : \overline{\mathbb{B}}^n(p_i, \delta_1) \to \mathbb{R}^{n+1} \quad \text{for } i = 0, 1$$

along the curves $\gamma^0$ and $\gamma^1$ such that

$$\int_{\mathbb{B}(p_i,\delta_1)} |D\gamma^i_{\delta_1}|^n < 4^{-n} \quad \text{for } i = 0, 1.$$

Observe that $\gamma^i_{\delta_1}(p_i) = A_i$ for $i = 0, 1$.

Note that the images of small balls $\mathbb{B}^n(p_0, \delta_1')$ and $\mathbb{B}^n(p_1, \delta_1')$ are isometric balls, orthogonal to the curves $\gamma^0$ and $\gamma^1$ (and hence to the curves $J_0$ and $J_1$) at the endpoints $A_0$ and $A_1$. Since the branches $J_{00}$, $J_{01}$ form angles larger than $\pi/2$ with the curve $\gamma^0$ at $A_0$ and the branches $J_{10}$, $J_{11}$ form angles larger than $\pi/2$ with the curve $\gamma^1$ at $A_1$ we may guarantee, by making the tentacles sufficiently thin, that they are disjoint from the set $T_1$ (observe that the endpoints $A_0$ and $A_1$ *do not* belong to $T_1$).

Also each annulus $\overline{\mathbb{B}}^n(p_i, \delta_1) \setminus \mathbb{B}^n(p_i, \delta_1/2)$ for $i = 0, 1$ is mapped isometrically by $\gamma^i_{\delta_1}$ onto an annulus centered at $\gamma^i(0) = p_i$ in the hyperplane orthogonal to $\gamma^i$ at $\gamma^i(0) = p_i$.

Since the curve $\gamma^i$ is orthogonal to $\mathbb{B}^n$ at $\gamma^i(0) = p_i$, the annulus $\overline{\mathbb{B}}^n(p_i, \delta_1) \setminus \mathbb{B}^n(p_i, \delta_1/2)$ is mapped isometrically onto itself. By choosing an appropriate orthonormal frame $v_1^i, \ldots v_n^i$ in the definition of $\gamma_{\delta_1}^i$ we may assume that $\gamma_{\delta_1}^i$ is the identity in the annulus. This guarantees that the mapping

$$f_1(x) = \begin{cases} \gamma_{\delta_1}^0(x) & \text{if} \quad x \in \overline{\mathbb{B}}^n(p_0, \delta_1), \\ \gamma_{\delta_1}^1(x) & \text{if} \quad x \in \overline{\mathbb{B}}^n(p_1, \delta_1), \\ x & \text{if} \quad x \in \mathbb{S}^n \setminus (\mathbb{B}^n(p_0, \delta_1) \cup \mathbb{B}^n(p_1, \delta_1)) \end{cases}$$

is continuous and hence smooth. The construction guarantees also that $f_1$ is a smooth embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ with the image that is disjoint from $T_1$.

The mapping $f_1$ maps the small balls $\overline{\mathbb{B}}^n(p_0, \delta_1')$ and $\overline{\mathbb{B}}^n(p_1, \delta_1')$ onto isometric balls centered at $A_0$ and $A_1$ respectively with $f_1(p_i) = A_i$ for $i = 0, 1$. Now the mapping $f_2$ will be obtained from $f_1$ by adding four more tentacles: from the ball $f_1(\mathbb{B}^n(p_0, \delta_1'))$ centered at $A_0$ there will be two tentacles connecting this ball to the points $A_{00}$ and $A_{01}$ and from the ball at $f_1(\mathbb{B}^n(p_1, \delta_1'))$ centered at $A_1$ there will be two tentacles connecting this ball to the points $A_{10}$ and $A_{11}$. More precisely the inductive step is described as follows.

Suppose that we have already constructed a mapping $f_k$, $k \geq 1$ such that

- $f_k$ is a smooth embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ whose image is disjoint from $T_k$.
- There are $2^k$ disjoint balls $\overline{\mathbb{B}}^n(p_{i_1 \ldots i_k}, \delta_k) \subset \mathbb{B}^n$ and $2^k$ tentacles

  $$\gamma_{\delta_k}^{i_1 \ldots i_k} : \overline{\mathbb{B}}^n(p_{i_1 \ldots i_k}, \delta_k) \to \mathbb{R}^{n+1}$$

  for $i_1 \ldots i_k \in \mathfrak{I}^k$ such that
  - ◇ $\gamma_{\delta_k}^{i_1 \ldots i_k}(p_{i_1 \ldots i_k}) = A_{i_1 \ldots i_k}$.
  - ◇ The image of $\gamma_{\delta_k}^{i_1 \ldots i_k}$ is in the $2^{-k}$ neighborhood of the curve $J_{i_1 \ldots i_k}$.
  - ◇ We have

    $$\int_{\mathbb{B}^n(p_{i_1 \ldots i_k}, \delta_k)} |D\gamma_{\delta_k}^{i_1 \ldots i_k}|^n \, dx < 4^{-nk}. \tag{5.4.1}$$
- The mapping $f_k$ satisfies

  $$f_k = \begin{cases} \gamma_{\delta_k}^{i_1 \ldots i_k} & \text{in} \quad \overline{\mathbb{B}}^n(p_{i_1 \ldots i_k}, \delta_k) \text{ for } i_1 \ldots i_k \in \mathfrak{I}^k, \\ f_{k-1} & \text{in} \quad \mathbb{S}^n \setminus \bigcup_{i_1 \ldots i_k \in \mathfrak{I}^k} \mathbb{B}^n(p_{i_1 \ldots i_k}, \delta_k). \end{cases}$$

Observe that $f_k(p_{i_1\dots i_k}) = A_{i_1\dots i_k}$ and that for some small $0 < \delta'_k < \delta_k$, $f_k$ maps balls $\overline{\mathbb{B}}^n(p_{i_1\dots i_k}, \delta'_k)$ onto isometric balls centered at $A_{i_1\dots i_k}$.

Now we will describe the construction of the mapping $f_{k+1}$.

For each $i_1 \dots i_k \in \mathfrak{I}^k$ we choose two points

$$p_{i_1\dots i_k 0}, p_{i_1\dots i_k 1} \in \mathbb{B}^n(p_{i_1\dots i_k}, \delta'_k)$$

and modify the curves $J_{i_1\dots i_k 0}$ and $J_{i_1\dots i_k 1}$ to $\gamma^{i_1\dots i_k 0}$ and $\gamma^{i_1\dots i_k 1}$ in a pretty similar way as we did for the curves $\gamma^0$ and $\gamma^1$: the new curves $\gamma^{i_1\dots i_k 0}$ and $\gamma^{i_1\dots i_k 1}$ emerge from the points $f_k(p_{i_1\dots i_k 0})$ and $f_k(p_{i_1\dots i_k 1})$, they are orthogonal to the ball $f_k(\mathbb{B}^n(p_{i_1\dots i_k}, \delta'_k))$ at these points and then they quickly meet and coincide with $J_{i_1\dots i_k 0}$ and $J_{i_1\dots i_k 1}$.

We find $\delta_{k+1} > 0$ so small that

- The balls

  $$\overline{\mathbb{B}}^n(p_{i_1\dots i_k 0}, \delta_{k+1}), \overline{\mathbb{B}}^n(p_{i_1\dots i_k 1}, \delta_{k+1}) \subset \mathbb{B}^n(p_{i_1\dots i_k}, \delta'_k)$$

  are disjoint.
- There are tentacles

  $$\gamma^{i_1\dots i_k i}_{\delta_{k+1}} : \overline{\mathbb{B}}^n(p_{i_1\dots i_k i}, \delta_{k+1}) \to \mathbb{R}^{n+1} \quad \text{for } i = 0, 1$$

  such that

  ◇ $\gamma^{i_1\dots i_k i}_{\delta_{k+1}}(p_{i_1\dots i_k i}) = A_{i_1\dots i_k i}$.

  ◇ The image of $\gamma^{i_1\dots i_k i}_{\delta_{k+1}}$ is in the $2^{-(k+1)}$ neighborhood of the curve $J_{i_1\dots i_k i}$.

  ◇ The tentacles do not intersect and they avoid the set $T_{k+1}$

  ◇ We have

  $$\int_{\mathbb{B}^n(p_{i_1\dots i_k i}, \delta_{k+1})} |D\gamma^{i_1\dots i_k i}_{\delta_{k+1}}|^n \, dx < 4^{-n(k+1)}.$$

The condition about the distance of the tentacle to the curve $J_{i_1\dots i_k i}$ can be easily guaranteed, because the curve $\gamma^{i_1\dots i_k i}$ can be arbitrarily close to $J_{i_1\dots i_k i}$ and the tentacle can be arbitrarily thin.

Because the curves $\gamma^{i_1\dots i_k i}$, $i = 0, 1$, are orthogonal to the balls $f_k(\mathbb{B}^n(p_{i_1\dots i_k}, \delta'_k))$ at the points $\gamma^{i_1\dots i_k i}(0) = f_k(p_{i_1\dots i_k i})$, by choosing appropriate orthonormal frames in the definition of $\gamma^{i_1\dots i_k i}_{\delta_{k+1}}$ we may guarantee one more condition

- $\gamma^{i_1\ldots i_k i}_{\delta_{k+1}} = f_k$ in $\overline{\mathbb{B}}^n(p_{i_1\ldots i_k i}, \delta_{k+1}) \setminus \mathbb{B}^n(p_{i_1\ldots i_k i}, \delta_{k+1}/2)$.

We are using here the fact that both $\gamma^{i_1\ldots i_k i}_{\delta_{k+1}}$ and $f_k$ are isometries in that annulus. Now we define

$$f_{k+1} = \begin{cases} \gamma^{i_1\ldots i_k i_{k+1}}_{\delta_k} & \text{in} \quad \overline{\mathbb{B}}^n(p_{i_1\ldots i_{k+1}}, \delta_{k+1}) \text{ for } i_1\ldots i_{k+1} \in \mathfrak{I}^{k+1}, \\ f_k & \text{in} \quad \mathbb{S}^n \setminus \bigcup_{i_1\ldots i_{k+1}\in\mathfrak{I}^{k+1}} \mathbb{B}^n(p_{i_1\ldots i_{k+1}}, \delta_{k+1}). \end{cases}$$

As before $f_{k+1}$ is a smooth embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ whose image is disjoint from $T_{k+1}$.

Let

$$W_k = \bigcup_{i_1\ldots i_k \in \mathfrak{I}^k} \overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k).$$

Clearly, $W_k$ is a decreasing sequence of compact sets and

$$E := \bigcap_{k=1}^{\infty} W_k$$

is a Cantor set $E \subset \mathbb{B}^n \subset \mathbb{S}^n$. By making the sequence $\delta_k$ converge to zero sufficiently fast we may guarantee that the Hausdorff dimension of $E$ equals zero. Similarly as in the case of the ternary Cantor set, points in the set $E$ can be encoded by infinite binary sequences For $\mathfrak{i} = i_1 i_1 \ldots \in \mathfrak{I}^\infty$ we define

$$\{e_{\mathfrak{i}}\} = \bigcap_{k=1}^{\infty} \overline{\mathbb{B}}^n(p_{i_1\ldots i_k,\delta_k}) \quad \text{so} \quad E = \bigcup_{\mathfrak{i}\in\mathfrak{I}^\infty} \{e_{\mathfrak{i}}\}.$$

Now we define

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{S}^n \setminus W_1, \\ f_k(x) & \text{if} \quad x \in W_k \setminus W_{k+1},\ k = 1, 2, \ldots \\ c_{\mathfrak{i}} & \text{if} \quad x = e_{\mathfrak{i}} \in E,\ \mathfrak{i} \in \mathfrak{I}^\infty. \end{cases}$$

Recall that $c_{\mathfrak{i}} = f(\mathfrak{c}_{\mathfrak{i}})$ is a point of the Cantor set $C$.

**Lemma 5.4.1.** *$f = f_k$ when restricted to $\mathbb{S}^n \setminus W_{k+1}$.*

*Proof.* The lemma can be easily proved by induction. Let $f_0 = \mathrm{id}$. By the definition of $f$, $f = f_0$ in $\mathbb{S}^n \setminus W_1$. Suppose now that $f = f_k$ in $\mathbb{S}^n \setminus W_{k+1}$. According to the construction of $f_{k+1}$, $f_{k+1} = f_k$ in $\mathbb{S}^n \setminus W_{k+1}$, but the definition of $f$ yields $f = f_{k+1}$ in $W_{k+1} \setminus W_{k+2}$ so $f = f_{k+1}$ in

$$(\mathbb{S}^n \setminus W_{k+1}) \cup (W_{k+1} \setminus W_{k+2}) = \mathbb{S}^n \setminus W_{k+2}.$$

This proves the lemma. $\square$

Since each of the mappings $f_k$ is a smooth embedding whose image does not intersect with the Cantor set $C$ it follows from the lemma that $f$ restricted to the open set $\mathbb{S}^n \setminus E$ is a smooth embedding and $f(\mathbb{S}^n \setminus E) \cap C = \emptyset$. In the remaining Cantor set $E$, the mapping $f$ is defined as a bijection that maps $E$ onto $C$. Therefore the mapping $f$ is one-to-one in $\mathbb{S}^n$ and $C \subset f(\mathbb{S}^n)$.

It remains to prove that $f$ is continuous and that $f \in W^{1,n}$.

First we will prove that $f \in W^{1,n}$. The mapping $f$ is bounded and hence its components are in $L^n$. Since the mapping $f$ is smooth outside the Cantor set $E$ of Hausdorff dimension zero, according to the characterization of the Sobolev space by absolute continuity on lines, [33, Section 4.9.2], it suffices to show that the classical derivative of $f$ defined outside of $E$ (and hence a.e. in $\mathbb{S}^n$) belongs to $L^n(\mathbb{S}^n)$. We have

$$\begin{aligned}\int_{\mathbb{S}^n} |Df|^n &= \int_{\mathbb{S}^n \setminus E} |Df|^n = \int_{\mathbb{S}^n \setminus W_1} |Df|^n + \sum_{k=1}^{\infty} \int_{W_k \setminus W_{k+1}} |Df_k|^n \\ &\leq \int_{\mathbb{S}^n \setminus W_1} |Df|^n + \sum_{k=1}^{\infty} \int_{W_k} |Df_k|^n.\end{aligned}$$

Since $f(x) = x$ in $\mathbb{S}^n \setminus W_1$, we do not have to worry about the first term on the right hand side and it remains to estimate the infinite sum.

Note that $f_k = \gamma_{\delta_k}^{i_1 \ldots i_k}$ in $\overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k)$ so (5.4.1) yields

$$\int_{\overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k)} |Df_k|^n = \int_{\overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k)} |D\gamma_{\delta_k}^{i_1\ldots i_k}|^n < 4^{-nk}.$$

Hence

$$\int_{W_k} |Df_k|^n = \sum_{i_1\ldots i_k \in \mathfrak{I}^k} \int_{\overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k)} |Df_k|^n < 2^k \cdot 4^{-nk} < 2^{-nk}$$

so

$$\sum_{k=1}^{\infty} \int_{W_k} |Df_k|^n < \sum_{k=1}^{\infty} 2^{-nk} < \infty.$$

This completes the proof that $f \in W^{1,n}$.

*Remark* 5.4.2. Replacing the estimate in (5.4.1) by $\varepsilon 4^{-nk}$ one can easily modify the construction so that the mapping $f$ will have an arbitrarily small Sobolev norm $W^{1,n}$.

It remains to prove that $f$ is continuous. We will need

**Lemma 5.4.3.** *$f(\overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k))$ is contained in the*

$$r_k := 2^{-k+4} + \operatorname{diam} C_{i_1\ldots i_{k-1}}$$

*neighborhood of $C_{i_1\ldots i_{k-1}}$.*

*Proof.* If $x \in \overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k) \cap E$, then $x = e_{\mathfrak{i}}$ for some $\mathfrak{i} \in \mathfrak{I}^\infty$ with the first $k$ binary digits equal $i_1, \ldots, i_k$ i.e., $\mathfrak{i} = i_1 \ldots i_k \ldots$ Hence

$$f(x) = f(e_{\mathfrak{i}}) = c_{\mathfrak{i}} = c_{i_1\ldots i_k\ldots} \in C_{i_1\ldots i_{k-1}}.$$

If $x \in \overline{\mathbb{B}}^n(p_{i_1\ldots i_k}, \delta_k) \setminus E$, then there is $s \geq k$ such that

$$x \in \overline{\mathbb{B}}^n(p_{i_1\ldots i_k i_{k+1}\ldots i_s}, \delta_s) \setminus W_{s+1} \subset W_s \setminus W_{s+1}$$

for some binary numbers $i_{k+1}, \ldots, i_s$.

Since $f = f_s$ in $W_s \setminus W_{s+1}$ and $f_s = \gamma_{\delta_s}^{i_1\ldots i_s}$ in $\overline{\mathbb{B}}^n(p_{i_1\ldots i_s}, \delta_s)$ we conclude that $f(x) = \gamma_{\delta_s}^{i_1\ldots i_s}(x)$.

It remains to show that the image of $\gamma_{\delta_s}^{i_1\ldots i_s}$ is contained in the $r_k$ neighborhood of $C_{i_1\ldots i_{k-1}}$.

By Lemma 5.2.1, $J_{i_1\ldots i_s}$ is in the $2^{-s+3} + \operatorname{diam} C_{i_1\ldots i_{s-1}}$ neighborhood of $C_{i_1\ldots i_{s-1}}$. Also the image of $\gamma_{\delta_s}^{i_1\ldots i_s}$ is contained in the $2^{-s}$ neighborhood of $J_{i_1\ldots i_s}$ so the image of $\gamma_{\delta_s}^{i_1\ldots i_s}$ is contained in the

$$2^{-s+4} + \operatorname{diam} C_{i_1\ldots i_{s-1}}$$

neighborhood of $C_{i_1\ldots i_{s-1}}$. Since $C_{i_1\ldots i_{s-1}} \subset C_{i_1\ldots i_{k-1}}$ and

$$2^{-s+4} + \operatorname{diam} C_{i_1\ldots i_{s-1}} \leq 2^{-k+4} + \operatorname{diam} C_{i_1\ldots i_{k-1}}$$

the lemma follows. ∎

Now we are ready to complete the proof of continuity of $f$. Clearly, $f$ is continuous on $\mathbb{S}^n \setminus E$ so it remains to prove its continuity on the Cantor set $E$. Let $e_{\mathfrak{i}} \in E$, $\mathfrak{i} = i_1 i_2 \ldots \in \mathfrak{I}^\infty$. Since $f(e_{\mathfrak{i}}) = c_{\mathfrak{i}}$ we need to show that for any $\varepsilon > 0$ there is $\delta > 0$ such that if $|x - e_{\mathfrak{i}}| < \delta$, then $|f(x) - c_{\mathfrak{i}}| < \varepsilon$.

Let $\varepsilon > 0$ be given. Let $k$ be so large that

$$2^{-k+4} + 2 \operatorname{diam} C_{i_1 \ldots i_{k-1}} < \varepsilon$$

and let $\delta > 0$ be so small that

$$\mathbb{B}^n(e_{\mathfrak{i}}, \delta) \subset \mathbb{B}^n(p_{i_1 \ldots i_k}, \delta_k).$$

If $|x - e_{\mathfrak{i}}| < \delta$, then $x \in \overline{\mathbb{B}}^n(p_{i_1 \ldots i_k}, \delta_k)$ so by Lemma 5.4.3, $f(x)$ belongs to the $r_k$ neighborhood of the set $C_{i_1 \ldots i_{k-1}}$. Since $c_{\mathfrak{i}} \in C_{i_1 \ldots i_{k-1}}$, the distance $|f(x) - c_{\mathfrak{i}}|$ is less than

$$r_k + \operatorname{diam} C_{i_1 \ldots i_{k-1}} = 2^{-k+4} + 2 \operatorname{diam} C_{i_1 \ldots i_{k-1}} < \varepsilon.$$

The proof is complete. $\square$

## 5.5 UNCOUNTABLE INEQUIVALENT EMBEDDINGS

We say that two embeddings $f, g : \mathbb{S}^n \to \mathbb{R}^{n+1}$, are *equivalent* if there is a homeomorphism $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $h(f(\mathbb{S}^n)) = g(\mathbb{S}^n)$. In this section, we will prove that there are uncountably many inequivalent embeddings as in Theorem 5.1.1 in $\mathbb{R}^3$.

**Theorem 5.5.1.** *There are uncountably many embeddings $f : \mathbb{S}^2 \to \mathbb{R}^3$ of class $W^{1,2}(\mathbb{S}^2, \mathbb{R}^3)$ which are not equivalent.*

The theorem can be generalized to higher dimensions, but we consider the case $n = 2$ only because our proof is based on a result of Sher [113] about Cantor sets in $\mathbb{R}^3$. Generalization of Theorem 5.5.1 to $n \geq 3$ would require a generalization of Sher's result to higher dimensions. Since this would be a work of purely technical nature with predicted answer, we do not find it particularly interesting.

Antoine's necklace $A$ is a Cantor set that is constructed iteratively as follows: Inside a solid torus $A_0$ in $\mathbb{R}^3$ (iteration 0) we construct a chain $A_1$ (iteration 1) of linked solid tori so that the chain cannot be contracted to a point inside the torus $A_0$. $A_1$ is a subset of $\mathbb{R}^3$, the union of the linked tori. Iteration $n+1$ is obtained from the iteration $n$ by constructing a chain of tori inside each of the tori of $A_n$ (i.e. inside each of the connected components of $A_n$). Again $A_{n+1}$ is a subset of $\mathbb{R}^3$ – the union of all tori in this step of construction. We also assume that the maximum of the diameters of tori in iteration $n$ converges to zero as $n$ approaches to infinity. Antoine's necklace is the intersection $A = \bigcap_{n=0}^{\infty} A_n$. For more details we refer to [113].

Antoine's necklace has the following properties:

- $\mathbb{R}^3 \setminus A$ is not simply connected.
- For any $x \in A$ and any $r > 0$, $A \cap \mathbb{B}^3(x, r)$ contains Antoine's necklace.

The first property is well known [99, Chapter 18] while the second one is quite obvious: $\mathbb{B}^3(x, r)$ contains one of the tori $T$ of one of the iterations (actually infinitely many of such tori) and $T \cap A$ is also Antoine's necklace because of the iterative nature of the procedure.

We also need the following observation.

**Lemma 5.5.2.** *Let $A$ be Antoine's necklace and let $M$ be a smooth 2-dimensional surface in $\mathbb{R}^3$. Then $A \cap M$ is contained in the closure of $A \setminus M$, $A \cap M \subset \overline{A \setminus M}$.*

*Proof.* Suppose to the contrary that for some $x \in A \cap M$, and some $r > 0$ we have $\mathbb{B}^3(x, r) \cap (A \setminus M) = \emptyset$. By taking $r > 0$ sufficiently small we can assume that $\mathbb{B}^3(x, r) \cap M$ is diffeomorphic to a disc. More precisely, there is a diffeomorphism $\Phi$ of $\mathbb{R}^3$ which maps $\mathbb{B}^3(x, r) \cap M$ onto the ball $\mathbb{B}^2(0, 2)$ in the $xy$-coordinate plane. Note that by the second property listed above $\mathbb{B}^3(x, r) \cap A$ contains Antoine's necklace denoted by $\tilde{A}$. Since $\tilde{A} \cap (A \setminus M) = \emptyset$ we have $\tilde{A} \subset \mathbb{B}^3(x, r) \cap M$ so $\Phi(\tilde{A}) \subset \mathbb{B}^2(0, 2)$ and $(\mathbb{B}^2(0, 2) \times \mathbb{R}) \setminus \Phi(\tilde{A})$ is not simply connected. By [99, Theorem 13.7 p.93] there is a homeomorphism $h$ of the ball $\mathbb{B}^2(0,2)$ onto itself in such that $\Phi(\tilde{A})$ is mapped onto the standard ternary Cantor set $\mathfrak{C}$ on the $x$-axis. This homeomorphism can be trivially extended to a homeomorphism of $\mathbb{B}^2(0,2)\times\mathbb{R}$ by letting $H(x,y,z)=(h(x,y),z)$. Clearly the complement of $H(\Phi(\tilde{A}))$ in $\mathbb{B}^2(0,2)\times\mathbb{R}$ is not simply connected. On the other hand since $H(\Phi(\tilde{A}))=\mathfrak{C}$, the complement of this set is simply connected in $\mathbb{B}^2(0,2)\times\mathbb{R}$ which is a contradiction. $\square$

The key argument in our proof is the following result of Sher [113, Corollary 1] that we state as a lemma.

**Lemma 5.5.3.** *There is an uncountable family of Antoine's necklaces* $\{A_i\}_{i\in I}$ *such that for any* $i,j\in I$, $i\neq j$ *there is no homeomorphism* $h:\mathbb{R}^3\to\mathbb{R}^3$ *with the property that* $h(A_i)=A_j$.

For each of the sets $A_i$, let $f_i:\mathbb{S}^2\to\mathbb{R}^3$ be an embedding as in Theorem 5.1.1 with the property that $A_i\subset f_i(\mathbb{S}^2)$ and $f_i(\mathbb{S}^2)\setminus A_i$ is a smooth surface (but not closed). It remains to prove that for $i\neq j$ there is no homeomorphism $h:\mathbb{R}^3\to\mathbb{R}^3$ such that $h(f_i(\mathbb{S}^2))=f_j(\mathbb{S}^2)$. Suppose to the contrary that such a homeomorphism $h$ exists. We will show that $h(A_i)=A_j$ which is a contradiction with Lemma 5.5.3.

Clearly,

$$h(A_i)=(h(A_i)\cap A_j)\cup(h(A_i)\cap(f_j(\mathbb{S}^2)\setminus A_j)).$$

Since $f_j(\mathbb{S}^2)\setminus A_j$ is a smooth surface, Lemma 5.5.2 yields

$$h(A_i)\cap(f_j(\mathbb{S}^2)\setminus A_j)\subset\overline{h(A_i)\setminus(f_j(\mathbb{S}^2)\setminus A_j)}=\overline{h(A_i)\cap A_j}\subset A_j$$

so $h(A_i)\cap(f_j(\mathbb{S}^2)\setminus A_j)=\emptyset$ and hence $h(A_i)\subset A_j$. Applying the same argument to $h^{-1}$ we obtain that $h^{-1}(A_j)\subset A_i$ so $A_j\subset h(A_i)$ and hence $h(A_i)=A_j$. The proof is complete. $\square$

# 6.0 SEMILINEAR EVOLUTION EQUATIONS IN THE HEISENBERG GROUP

Since 1980s, the theory of viscosity solutions for nonlinear partial differential equations has been developed and applied to a wide range of fields. It provides existence and uniqueness of weak solutions to a very general class of fully nonlinear equations in the space of continuous functions [27, 80]. We refer to the book by Koike [80] and the notes by Manfredi [95] for a brief review of the classical theory of viscosity solutions.

A recent trend is to extend the viscosity solution theory to metric measure spaces. Sub-Riemannian spaces play a central role in analysis on metric measure spaces and apply to a wide range of fields including robotic control and image reconstruction. The Heisenberg group $\mathbb{H}$, which is known as the simplest example of sub-Riemmanian manifolds provides a good setting for the generalization of the viscosity solution theory. We will give a brief introduction of Heisenberg group in the first section.

This chapter is concerned with the uniqueness and the Lipschitz and convexity preserving properties for viscous Hamilton-Jacobi equations on the Heisenberg group $\mathbb{H}$:

$$\begin{cases} u_t - \operatorname{tr}(A(\nabla_H^2 u)^*) + f(p, \nabla_H u) = 0 & \text{in } \mathbb{H} \times (0, \infty), \qquad (6.0.1) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{H}, \qquad (6.0.2) \end{cases}$$

where $A$ is a given $2\times 2$ symmetric positive-semidefinite matrix and the function $f : \mathbb{H}\times\mathbb{R}^2 \to \mathbb{R}$ satisfies certain assumptions to be made explicit later. Here $\nabla_H u$, $(\nabla_H^2 u)^*$ are respectively the horizontal gradient and the horizontal symmetrized Hessian of the unknown function $u$ in space, and $u_0$ is a given locally uniformly continuous function in $\mathbb{H}$.

Many of our results in this work also hold for more general fully nonlinear degenerate parabolic equations of the type

$$u_t + F\left(p, \nabla_H u, (\nabla_H^2 u)^*\right) = 0 \quad \text{in } \mathbb{H} \times (0, \infty) \tag{6.0.3}$$

under proper regularity assumptions on $F$. We however focus on (6.0.1) for simplicity of exposition.

We first show uniqueness of viscosity solutions to the equations with exponential growth at infinity and then turn to investigate the Lipschitz and convexity preserving of the unique viscosity solution. It turns out that in general, such properties cannot be expected to hold in the Heisenberg group. Some restrictions on the class of solutions prove to be necessary. Thus, we pose several appropriate conditions and prove that the Lipschitz continuity and horizontal convexity property of the solutions is preserved under these assumptions. On the other hand, there are also many affirmative examples of Lipschitz and convexity preserving in the Heisenberg group.

## 6.1 THE HEISENBERG GROUP $\mathbb{H}$

### 6.1.1 Definitions and basic properties

The Heisenberg group $\mathbb{H}$ is the space $\mathbb{R}^3$ endowed with the non-commutative group multiplication

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p)\right),$$

for all $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ in $\mathbb{H}$. Note that the group inverse of $p = (x_q, y_q, z_q)$ is $p^{-1} = (-x_q, -y_q, -z_q)$. The Korányi gauge is given by

$$|p|_G = ((p_1^2 + p_2^2)^2 + 16 p_3^2)^{1/4},$$

and the left-invariant Korányi or gauge metric is

$$d_L(p, q) = |p^{-1} \cdot q|_G.$$

The Lie Algebra of $\mathbb{H}$ is generated by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z};$$
$$X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z};$$
$$X_3 = \frac{\partial}{\partial z}.$$

One may easily verify the commuting relation $X_3 = [X_1, X_2] = X_1X_2 - X_2X_1$.

The horizontal gradient of $u$ is given by

$$\nabla_H u = (X_1u, X_2u)$$

and the symmetrized second horizontal Hessian $(\nabla_H^2 u)^* \in S^{2\times 2}$ is given by

$$(\nabla_H^2 u)^* := \begin{pmatrix} X_1^2u & (X_1X_2u + X_2X_1u)/2 \\ (X_1X_2u + X_2X_1u)/2 & X_2^2u \end{pmatrix}.$$

Here $S^{n\times n}$ denotes the set of all $n \times n$ symmetric matrices.

We denote by $\mathbb{H}_0$ the set of horizontal vectors of the form $(h_1, h_2, 0)$, or equivalently left-invariant vector fields of the form $h_1X_1 + h_2X_2$. The horizontal subspace at a point $p$ is the two dimensional space spanned by $X_1(p)$ and $X_2(p)$ and it can be identified with the left translation of $\mathbb{H}_0$ by $p$, that is,

$$p \cdot \mathbb{H}_0 = \text{Linear} - \text{span}\{X_1(p), X_2(p)\}.$$

A piecewise smooth curve $s \mapsto \gamma(s) \in \mathbb{H}$ is called horizontal if its tangent vector $\gamma'(s)$ is in the linear span of $\{X_1(\gamma(s)), X_2(\gamma(s))\}$ for every $s$ such that $\gamma'(s)$ exists; in other words, there exist $a(s), b(s) \in \mathbb{R}$ satisfying

$$\gamma'(s) = a(s)X_1(\gamma(s)) + b(s)X_2(\gamma(s))$$

whenever $\gamma'(s)$ exists. We denote

$$\|\gamma'(s)\| = \left(a^2(s) + b^2(s)\right)^{\frac{1}{2}}.$$

Given $p, q \in \mathbb{H}$, denote

$$\Gamma(p,q) = \{\text{horizontal curves } \gamma(s)\ (s \in [0,1])\colon\ \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

Chow's theorem states that $\Gamma(p,q) \neq \emptyset$; see, for example, [13]. The Carnot-Carathéodory metric is then defined to be

$$d_{CC}(p,q) = \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \|\gamma'(s)\|\, ds.$$

We next give the Taylor expansion of a smooth function $u : \mathbb{H} \to \mathbb{R}$. By the fact that $|p|_G^2 \approx x^2 + y^2 + |z|$, we obtain the horizontal Taylor expansion at point $p_0 = (x_0, y_0, z_0)$:

$$u(p) = u(p_0) + \langle \nabla u(p_0), p_0^{-1} \cdot p\rangle + \frac{1}{2}\langle (\nabla_H^2 u(p_0))^* h, h\rangle + o(|p_0^{-1} \cdot p|^2), \tag{6.1.1}$$

where $\nabla u(p_0)$ is the full gradient of $u$ at $p_0$ and the vector $h$ is a horizontal projection of the vector $p_0^{-1} \cdot p$.

### 6.1.2 Metrics on $\mathbb{H}$

Besides the left-invariant Korányi metric $d_L$ and Carnot-Carathéodory metric $d_{CC}$, the function $d_R(p,q) = |p \cdot q^{-1}|_G$ for any $p, q \in \mathbb{H}$ defines another metric on $\mathbb{H}$, which is right invariant; in fact, $d_R(p,q) = d_L(p^{-1}, q^{-1})$ for any $p, q \in \mathbb{H}$.

It is known that $d_L$ is bi-Lipschitz equivalent to the Carnot-Carathéodory metric $d_{CC}$ [21, 95]. The metrics $d_L$ and $d_R$ are not bi-Lipschitz equivalent, which is indicated in the example below.

*Example* 6.1.1. One may choose

$$p = (1 - \varepsilon, 1 + \varepsilon, \varepsilon), \quad q = (1, 1, 0)$$

with $\varepsilon > 0$ small, then by direct calculation, we have $d_L(p,q)^4 = |p^{-1} \cdot q|_G^4 = 4\varepsilon^4$ and $d_R(p,q)^4 = |p \cdot q^{-1}|_G^4 = 4\varepsilon^4 + 64\varepsilon^2$, which indicates that one cannot expect the existence of a constant $C > 0$ such that $d_R(p,q) \leq C d_L(p,q)$ for all $p, q \in \mathbb{H}$. A variant of this example shows that the reverse inequality also fails in general.

Although the metrics above are not bi-Lipschitz equivalent, it turns out that one is locally Hölder continuous in relative to the other.

**Proposition 6.1.2.** *For any $\rho > 0$, there exists $C_\rho > 0$ such that*

$$d_L(p,q) \leq C_\rho d_R(p,q)^{\frac{1}{2}} \tag{6.1.2}$$

*and*

$$d_R(p,q) \leq C_\rho d_L(p,q)^{\frac{1}{2}} \tag{6.1.3}$$

*for any $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$.*

*Proof.* We only show (6.1.2). The proof of (6.1.3) is similar. Set $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$. It is then clear that we only need to show that there exists some $C > 0$ depending only on $\rho$ such that

$$\begin{aligned}&\left|z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p\right| \\ &\qquad \leq C\left((|x_p - x_q|^2 + |y_p - y_q|^2)^2 + 16\left(z_p - z_q - \frac{1}{2}x_p y_q + \frac{1}{2}x_q y_p\right)^2\right)^{\frac{1}{4}}\end{aligned}$$

for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$.

Let $\delta = ((|x_p - x_q|^2 + |y_p - y_q|^2)^2 + 16|z_p - z_q - \frac{1}{2}x_p y_q + \frac{1}{2}x_q y_p|^2)^{\frac{1}{4}} \leq 1$. Then it is clear that

$$\begin{aligned}\left|z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p\right| &\leq \frac{\delta^2}{4} + |x_p y_q - x_q y_p| \\ &= \frac{\delta^2}{4} + |(x_p - x_q)y_q - x_q(y_p - y_q)|\,.\end{aligned}$$

It follows that

$$\left|z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p\right| \leq \frac{\delta^2}{4} + (|x_p - x_q|^2 + |y_p - y_q|^2)^{\frac{1}{2}}(x_q^2 + y_q^2)^{\frac{1}{2}}.$$

Noticing that $x_q^2 + y_q^2 \leq \rho^2$, we then have

$$\left|z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p\right| \leq \frac{\delta^2}{4} + \rho(|x_p - x_q|^2 + |y_p - y_q|^2)^{\frac{1}{2}} \leq \frac{\delta^2}{4} + \rho\delta = \left(\frac{\delta}{4} + \rho\right)\delta.$$

We conclude the proof by choosing $C = 1/4 + \rho$. $\square$

### 6.1.3 Lipschitz continuity

We discuss Lipschitz continuity with respect to $d_L$ and $d_R$.

It is easily seen that the function $f_0 : \mathbb{H} \to \mathbb{R}$ given by $f_0(p) = |p|_G$ is a Lipschitz function with respect to $d_L$ and $d_R$, due to the triangle inequality. But there exist functions that are Lipschitz with respect to one of the metrics but not with respect to the other. An example, following Example 6.1.1, is as below.

*Example* 6.1.3. Fix $q = (1, 1, 0) \in \mathbb{H}$ as in Example 6.1.1. Let $f_q : \mathbb{H} \to \mathbb{R}$ defined by $f_q(p) = d_R(p, q)$ for every $p \in \mathbb{H}$, which satisfies

$$|f_q(p) - f_q(p')| = |d_R(p, q) - d_R(p', q)| \leq d_R(p, p')$$

for all $p, p' \in \mathbb{H}$. But there is no constant $L > 0$ such that

$$f_q(p) - f_q(p') \leq L d_L(p, p')$$

for all $p, p' \in \mathbb{H}$, for otherwise we may take $p = (1 - \varepsilon, 1 + \varepsilon, \varepsilon)$ and $p' = q$, and get

$$d_R(p, p') \leq L d_L(p, p'),$$

which is not true when $\varepsilon > 0$ small, as explained in Example 6.1.1. However, by Proposition 6.1.2, the function $f_q$ is still locally 1/2-Hölder continuous with respect to $d_L$.

On the other hand, not all functions that are (locally) Lipschitz with respect to $d_L$ or $d_R$ are (locally) Lipschitz with respect to the Euclidean metric. The simplest example is the function $f(p) = |p|_G$ for $p \in \mathbb{H}$.

We conclude this section by showing the equivalence of Lipschitz continuity with respect to both metrics for functions with symmetry. We include in our discussions two different types of evenness.

**Definition 6.1.4** (Even functions)**.** We say a function $f : \mathbb{H} \to \mathbb{R}$ is even (or symmetric about the origin) if $f(p) = f(p^{-1})$ for all $p \in \mathbb{H}$. We say $f$ is vertically even (or symmetric about the horizontal coordinate plane) if $f(p) = f(\overline{p})$ for all $p \in \mathbb{H}$, where

$$\overline{p} = (x, y, -z) \quad \text{for any } p = (x, y, z) \in \mathbb{H}. \tag{6.1.4}$$

Since $|p \cdot q^{-1}|_G = |\overline{p}^{-1} \cdot \overline{q}|_G = |(p^{-1})^{-1} \cdot q^{-1}|_G$ for any $p, q \in \mathbb{H}$, the following result is obvious.

**Proposition 6.1.5** (Equivalence of Lipschitz continuities)**.** *Let $f : \mathbb{H} \to \mathbb{R}$ be a function that is either even or vertically even in $\mathbb{H}$. Then $f$ is Lipschitz continuous with respect to $d_L$ if and only if $f$ is Lipschitz continuous with respect to $d_R$.*

### 6.1.4 Horizontal convexity

For any point $p \in \mathbb{H}$, we denote the line segment joining $p \cdot h^{-1}$ and $p \cdot h$ by $[p \cdot h^{-1}, p \cdot h]$, which is always a line segment contained in the horizontal subspace $p \cdot \mathbb{H}_0$.

**Definition 6.1.6** ([91, Definition 4.1])**.** Let $\Omega$ be an open set in $\mathbb{H}$ and $u : \Omega \to \mathbb{R}$ be an upper semicontinuous function. The function $u$ is said to be horizontally convex or h-convex in $\Omega$, if for every $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$ such that $[p \cdot h^{-1}, p \cdot h] \subset \Omega$, we have

$$u(p \cdot h^{-1}) + u(p \cdot h) \geq 2u(p). \tag{6.1.5}$$

One may also define convexity of a function through its second derivatives in the viscosity sense.

**Definition 6.1.7.** Let $\Omega$ be an open set in $\mathbb{H}$ and $u : \Omega \to \mathbb{R}$ be an upper semicontinuous function. The function $u$ is said to be v-convex in $\Omega$ if

$$(\nabla_H^2 u)^*(p) \geq 0 \quad \text{for all } p \in \mathbb{H} \tag{6.1.6}$$

in the viscosity sense.

It is clear that $u \in C^2(\Omega)$ is v-convex if it satisfies (6.1.6) everywhere in $\Omega$. It is known that the h-convexity and v-convexity are equivalent [91]. The following example shows that h-convexity is very different from convexity in the Euclidean sense.

*Example* 6.1.8. Let

$$f(x, y, z) = x^2y^2 + 2z^2 \tag{6.1.7}$$

for all $(x, y, z) \in \mathbb{H}$. It is not difficult to verify that $f$ is h-convex. Indeed, for any $p = (x, y, z) \in \mathbb{H}$ and $h = (h_1, h_2, 0) \in \mathbb{H}_0$, we have

$$\begin{aligned} & f(p \cdot h) + f(p \cdot h^{-1}) \\ &= 2x^2y^2 + 4z^2 + 3x^2h_2^2 + 3y^2h_1^2 + 2h_1^2h_2^2 + 6xyh_1h_2 \\ &\geq 2f(p) + 3(xh_2 + yh_1)^2 + 2h_1^2h_2^2 \geq 2f(p). \end{aligned}$$

The function $f$ is an example of (globally) h-convex functions in $\mathbb{H}$ that is not convex in $\mathbb{R}^3$.

## 6.2 REVIEW OF VISCOSITY SOLUTIONS

### 6.2.1 Viscosity solutions in $\mathbb{R}^n$

In this section, we review the viscosity solution theory for the general second-order fully nonlinear equations arising in various fields:

$$F(x, u(x), \nabla u(x), \nabla^2 u(x)) = 0, \tag{6.2.1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are independent variables, $u : \mathbb{R}^n \to \mathbb{R}$ is the unknown function, and $\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ and $\nabla^2 u$ respectively denote the gradient and Hessian of $u$ with respect to $x$. Here $F$ is a continuous function defined on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$, where $S^{n\times n}$ stands for the set of all symmetric $n \times n$ matrices.

We are particularly interested in the class of equations with the following assumptions:

(1) (Ellipticity) $F(x, r, p, X)$ is decreasing in $X$, i.e.,

$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ for all } X, Y \in S^{n\times n} \text{ such that } X \geq Y.$$

(2) (Monotonicity) $F(x, r, p, X)$ is increasing in $r$.

Regarding the notation $X \geq Y$, we recall that there is a natural partial ordering in $S^{n\times n}$; indeed, $X \geq Y$ means that $X - Y$ is positive semi-definite. We call $F$ is proper if the above two conditions are satisfied.

The viscosity approach is particularly useful when $F$ is not in the divergence form, i.e., the structure of $F$ does not allow integration. We provide the definition of viscosity solutions of (6.2.1) in a bounded domain $\Omega \subset \mathbb{R}^n$.

**Definition 6.2.1.** We say $u \in C(\Omega)$ is a *viscosity subsolution* (resp., *supersolution*) of (6.2.1) in $\Omega$ if for any test function $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ such that $u - \phi$ attains a maximum (resp., minimum) over $\Omega$ at $x_0$, we have

$$F(x_0, u(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leq 0 \quad (\text{resp., } F(x_0, u(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geq 0.)$$

We call $u$ is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

*Remark* 6.2.2. It is possible to give equivalent definitions by replacing the maximum /minimum in the definition above with "strict (unique) maximum/minimum" or "local maximum/minimum".

*Remark* 6.2.3. We usually take $C^1$ test functions if $F$ is a first order operator. In fact, one may choose to use smooth ($C^\infty$) functions as tests in Definition 6.2.1 and get another equivalent definition [38, Proposition 2.2.3].

*Remark* 6.2.4. It is also possible to define subsolutions for the space of all *upper semicontinuous functions* on $\overline{\Omega}$, denoted by $USC(\overline{\Omega})$. Similarly, the *lower semicontinuous function* space ($LSC(\overline{\Omega})$) can be used to define supersolutions.

We may think of the test for subsolutions in Definition 6.2.1 in a more geometric way by considering that $\phi$ touches $u$ from above at $x_0$. If $u$ is a viscosity subsolution or supersolution, we essentially require the derivatives of the test function $\phi$ to fulfill the corresponding inequality. The next result indicates that viscosity solutions are weak solutions which agree with classical solutions if their regularity is adequate.

**Proposition 6.2.5.** *Assume that $F$ is proper. Then $u$ is a classical solution of (6.2.1) in $\Omega$ if and only if $u$ is a viscosity solution and $u \in C^2(\Omega)$.*

There are various approaches to the existence theorem of viscosity solutions to a certain equation. We list several well-known methods.

1. Vanishing viscosity method: We may add an artificial viscosity in the equation to get a smoother solution and pass to the limit as the viscosity approaches 0.
2. Optimal control/differential game: We set up an optimal control or game problem, whose value function represents or approximates the viscosity solution.
3. Perron's method: The supremum of all subsolutions or the infimum of all supersolutions with boundary data constraint gives a viscosity solution.

The vanishing viscosity method is one of the classical methods to find solutions to fully nonlinear equations. One may refer to the early work [26] for more details. Perron's method is also classical but was first adopted in the theory of viscosity solutions by Ishii [65].

In general, proving the uniqueness is usually harder than proving the existence and stability in the study of the viscosity solutions. Thus, the comparison principle is the main issue of the viscosity solutions. In order to state the comparison principle, we introduce the following equivalent definitions of viscosity solutions.

**Definition 6.2.6.** A pair $(\eta, X)$, where $\eta \in \mathbb{R}^n$ and $X$ is an $n \times n$ symmetric matrix, belongs to the second order superjet (resp., subjet) of an upper semicontinuous (resp., lower semicontinuous) function $u$ at point $x_0$ if

$$u(x_0 + h) \leq u(x_0) + \langle \eta, h \rangle + \frac{1}{2}\langle X \cdot h, h \rangle + o(|h|^2)$$

$$(resp., u(x_0 + h) \geq u(x_0) + \langle \eta, h \rangle + \frac{1}{2}\langle X \cdot h, h \rangle + o(|h|^2))$$

as $h \to 0$. The collection of all of these pairs, is denoted by $J^{2,+}u(x_0)$ (resp., $J^{2,-}u(x_0)$).

We also introduce a sort of closure of jets $\overline{J}^{2,\pm}u(x_0)$ as the collection of $(\eta, X) \in \mathbb{R}^n \times S^n$ such that there exists a sequence $x_k \in \Omega$ and a sequence $(\eta_k, X_k) \in J^{2,\pm}u(x_k)$ satisfying

$$(x_k, u(x_k), \eta_k, X_k) \to (x_0, u(x_0), \eta, X)$$

as $k \to \infty$.

**Proposition 6.2.7.** *For $u : \Omega \to \mathbb{R}$, the following conditions are equivalent:*

*(1) $u$ is a viscosity subsolution (resp., supersolution)of* (6.2.1).

*(2) For $x \in \Omega$ and $(\eta, X) \in J^{2,+}u(x)$ (resp., $J^{2,-}u(x)$), we have $F(x, u(x), \eta, X) \leq 0$ (resp., $\geq 0$).*

*(3) For $x \in \Omega$ and $(\eta, X) \in \overline{J}^{2,+}u(x)$ (resp., $\overline{J}^{2,-}u(x)$), we have $F(x, u(x), \eta, X) \leq 0$ (resp., $\geq 0$).*

We next state the maximum principle for semicontinuous functions, which is essential for the proof of uniqueness of the viscosity solutions.

**Theorem 6.2.8** (Crandall-Ishii-Jensen-Lions)**.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let the function $u$ be upper semicontinuous and $v$ be lower semicontinuous in $\Omega$. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y \to x} u(y) \leq \liminf_{y \to x} v(y)$$

*where both sides are not $\infty$ or $-\infty$ simultaneously. If $u - v$ has an interior local maximum*

$$\sup_{\Omega}(u - v) > 0$$

*then we have: For $\tau > 0$, we can find points $p_\tau, q_\tau \in \Omega$ such that*

*(1)*

$$\lim_{\tau \to \infty} \tau\phi(p_\tau - q_\tau) = 0,$$

*where $\phi(p) = |p|^2$,*

*(2) there exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ and*

$$\sup_{\Omega}(u - v) = u(\hat{p}) - v(\hat{p}) > 0,$$

*(3) there exist $n \times n$ symmetric matrices $X_\tau, Y_\tau$ and vectors $\eta_\tau$ so that*

$$(\eta_\tau, X_\tau) \in \overline{J}^{2,+}(u, p_\tau),$$

*and*

$$(\eta_\tau, Y_\tau) \in \overline{J}^{2,+}(u, q_\tau),$$

*(4) $X_\tau \leq Y_\tau$.*

As a corollary, we give the comparison principle for the viscosity solutions of (6.2.1).

**Corollary 6.2.9.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $F$ in (6.2.1) be proper and Lipschitz. Let the function $u$ be a subsolution of (6.2.1) and $v$ be a supersolution of (6.2.1) in $\Omega$. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y \to x} u(y) \leq \liminf_{y \to x} v(y)$$

*where both sides are not $\infty$ or $-\infty$ simultaneously. Then we have $u(x) \leq v(x)$ for $x \in \Omega$.*

### 6.2.2 Viscosity solutions in $\mathbb{H}$

We also define the viscosity solutions in $\mathbb{H}$ via semijets.

**Definition 6.2.10.** A pair $(\eta, X)$, where $\eta \in \mathbb{R}^3$ and $X$ is an $2 \times 2$ symmetric matrix, belongs to the second order superjet (resp., subjet) of an upper semicontinuous (resp., lower semicontinuous) function $u$ at point $p_0$ if

$$u(p) \leq u(p_0) + \langle \eta, p_0^{-1} \cdot p \rangle + \frac{1}{2}\langle X \cdot h, h \rangle + o(|p_0^{-1} \cdot p|^2)$$

$$(resp., u(p_0 \cdot h) \geq u(p_0) + \langle \eta, h \rangle + \frac{1}{2}\langle X \cdot h, h \rangle + o(|p_0^{-1} \cdot p|^2)$$

as $|p_0^{-1} \cdot p| \to 0$, where $h$ is the horizontal projection of $p_0^{-1} \cdot p$. The collection of all of these pairs, is denoted by $J^{2,+}u(p_0)$ (resp., $J^{2,-}u(p_0)$).

As in the case of Euclidean spaces, we can also define the closure of jets $\overline{J}^{2,\pm}u(p_0)$ in a similar way. Consider a continuous function $F(p, u, \eta, X) : \mathbb{H} \times \mathbb{R} \times \mathbb{R}^3 \times S(\mathbb{R}^2) \to \mathbb{R}$. We will assume that $F$ is proper, that is, $F$ is increasing in $u$ and decreasing in $X$.

**Definition 6.2.11.** A lower semicontinuous (resp., upper semicontinuous) function $u$ is a viscosity supersolution of the equation

$$F(p, u(p), \nabla u(p), (\nabla^2 u(p))^*) = 0 \tag{6.2.2}$$

if whenever $(\eta, X) \in J^{2,-}u(p_0)$ (resp., $(\eta, X) \in J^{2,+}u(p_0)$) we have

$$F(p_0, u(p_0), \eta, X) \geq 0 \quad \text{(resp.,} \quad F(p_0, u(p_0), \eta, X) \leq 0).$$

Equivalently, if $\phi$ touches $u$ from below (resp., above) at $p_0$, is $C^2$ in $X_1, X_2$ and $C^1$ in $X_3$, then we must have

$$F(p_0, u(p_0), \nabla\phi(p_0), (\nabla^2\phi(p_0))^*) \geq 0 \quad \text{(resp.,} \quad F(p_0, u(p_0), \nabla\phi(p_0), (\nabla^2\phi(p_0))^*) \leq 0).$$

By the continuity of $F$, $J^{2,\pm}u(p_0)$ in the above definition can be replaced by $\overline{J}^{2,\pm}u(p_0)$. A viscosity solution is defined as being both a viscosity subsolution and a viscosity supersolution.

The following maximum principle is proved by Bieske [16].

**Theorem 6.2.12.** *Let $\Omega$ be a bounded domain in $\mathbb{H}$. Let the function $u$ be upper semicontinuous and $v$ be lower semicontinuous in $\Omega$. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y\to x} u(y) \leq \liminf_{y\to x} v(y)$$

*where both sides are not $\infty$ or $-\infty$ simultaneously. If $u-v$ has an interior local maximum*

$$\sup_{\Omega}(u-v) > 0$$

*then we have: For $\tau > 0$, we can find points $p_\tau, q_\tau \in \Omega$ such that*

*(1)*

$$\lim_{\tau\to\infty} \tau\phi(p_\tau \cdot q_\tau^{-1}) = 0,$$

*where $\phi(p) = x^4 + y^4 + z^2$,*

*(2) there exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ and*

$$\sup_{\Omega}(u-v) = u(\hat{p}) - v(\hat{p}) > 0,$$

*(3) there exist $2\times 2$ symmetric matrices $X_\tau, Y_\tau$ and vectors $\eta_\tau \in \mathbb{R}^3$ so that*

$$(\eta_\tau, X_\tau) \in \overline{J}^{2,+}(u, p_\tau),$$

*and*

$$(\eta_\tau, Y_\tau) \in \overline{J}^{2,+}(u, q_\tau),$$

*(4) $X_\tau \leq Y_\tau + o(1)$ as $\tau \to \infty$. This condition means if $\xi \in \mathbb{R}^2$, we have*

$$\langle X_\tau\xi, \xi\rangle - \langle Y_\tau\xi, \xi\rangle \leq a(\tau)|\xi|^2,$$

*where $a(\tau) \to 0$ as $\tau \to \infty$.*

Similarly, we give the comparison principle for the viscosity solutions of (6.2.2) in $\mathbb{H}$.

**Corollary 6.2.13.** *Let $\Omega$ be a bounded domain in $\mathbb{H}$ and $F$ in (6.2.2) be proper and Lipschitz. Let the function $u$ be a subsolution of (6.2.2) and $v$ be a supersolution of (6.2.2) in $\Omega$. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y\to x} u(y) \leq \liminf_{y\to x} v(y)$$

*where both sides are not $\infty$ or $-\infty$ simultaneously. Then we have $u(x) \leq v(x)$ for $x \in \Omega$.*

## 6.3 UNIQUENESS OF UNBOUNDED SOLUTIONS

Uniqueness of viscosity solutions of various nonlinear equations in the Heisenberg group is studied in [16, 17, 95, 98] etc. But most of these results are either for a bounded domain or for bounded solutions. It is less understood when the domain and solution are both unbounded in the Heisenberg group. To the best of our knowledge, the only known result on uniqueness for time-dependent equations in this case is due to Haller Martin [53], where a comparison principle is established for a class of nonlinear parabolic equations including the horizontal Gauss curvature flow of graphs in the Carnot group. The comparison principle in [53] is for solutions with polynomial growth at infinity while ours is for exponential growth, but our assumptions on the structure of the equations are stronger.

In this section, motivated by a Euclidean argument in [11], we present a proof of a comparison principle for (6.0.1) with exponential growth at space infinity. Our result and proof are different from those of [53].

We need the following Lipschitz continuity of $f$.

(A1) There exists $L_1 > 0$ such that

$$|f(p, w_1) - f(p, w_2)| \leq L_1 |w_1 - w_2| \tag{6.3.1}$$

for all $p \in \mathbb{H}$ and $w_1, w_2 \in \mathbb{R}^2$.

(A2) There exists $L_2(\rho) > 0$ depending on $\rho > 0$ such that

$$|f(p, w) - f(q, w)| \leq L_2(\rho) |p \cdot q^{-1}|_G \tag{6.3.2}$$

for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$ and all $w \in \mathbb{R}^2$.

Here $|\cdot|_G$ denotes the Korányi gauge in $\mathbb{H}$, i.e.,

$$|p|_G = \left((x_p^2 + y_p^2)^2 + 16z_p^2\right)^{\frac{1}{4}}$$

for all $p = (x_p, y_p, z_p) \in \mathbb{H}$. Note that (A2) is not the usual local Lipschitz continuity in $\mathbb{H}$, since the distance between $p, q \in \mathbb{H}$ defined by $d_R(p,q) = |p \cdot q^{-1}|_G$ is invariant only under right translations and therefore not equivalent to the usual gauge metric give by $d_L(p,q) = |p^{-1} \cdot q|_G$ or the Carnot-Carathéodory metric.

Our comparison principle is as below.

**Theorem 6.3.1** (Comparison principle for unbounded solutions)**.** *Assume that the Lipschitz conditions (A1) and (A2) hold. Let $u$ and $v$ be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of (6.0.1). Assume that for any fixed $T > 0$, there exist $k > 0$ and $C_T > 0$ depending on $T$ such that*

$$u(p,t) - v(p,t) \leq C_T e^{k\langle p\rangle} \tag{6.3.3}$$

*for all $(p,t) \in \mathbb{H} \times [0,T]$, where*

$$\langle p\rangle = (1 + x^4 + y^4 + 16z^2)^{\frac{1}{4}} \quad \text{for all } p = (x,y,z) \in \mathbb{H}. \tag{6.3.4}$$

*If $u(p,0) \leq v(p,0)$ for all $p \in \mathbb{H}$, then $u \leq v$ in $\mathbb{H} \times [0,\infty)$.*

*Proof of Theorem 6.3.1.* We aim to show that $u \leq v$ in $\mathbb{H} \times [0,T)$ for any fixed $T > 0$. By the growth assumption, there exist $k > 0$ and $C_T > 0$ satisfying condition (6.3.3). Take an arbitrary constant $\beta > \min\{k, 1\}$ and then $\alpha > 0$ to be determined later. Set

$$g(p,t) = e^{\alpha t + \beta\langle p\rangle} \tag{6.3.5}$$

for $(p,t) \in \mathbb{H} \times [0,\infty)$. Recall that $\langle p\rangle$ is a function of $p \in \mathbb{H}$ given in (6.3.4). If $p = (x,y,z)$, we have by direct calculations

$$\nabla_H \langle p\rangle = \left(\frac{x^3 - 4yz}{(1 + x^4 + y^4 + 16z^2)^{\frac{3}{4}}}, \frac{y^3 + 4xz}{(1 + x^4 + y^4 + 16z^2)^{\frac{3}{4}}}\right), \tag{6.3.6}$$

which implies that there exists $\mu > 0$ such that

$$|\nabla_H g(p,t)| \leq \beta\mu g(p,t) \tag{6.3.7}$$

for all $(p,t) \in \mathbb{H} \times [0,\infty)$.

We assume by contradiction that $u(p,t) - v(p,t)$ takes a positive value at some $(p,t) \in \mathbb{H} \times (0,\infty)$. Then there exists $\sigma \in (0,1)$ such that

$$u(p,t) - v(p,t) - 2\sigma g(p,t) - \frac{\sigma}{T-t}$$

attains a positive maximum at $(\hat{p},\hat{t}) \in \mathbb{H} \times [0,T)$. For all $\varepsilon > 0$ small, consider the function

$$\Phi(p,q,t,s) = u(p,t) - v(q,s) - \sigma\Psi_\varepsilon(p,q,t,s) - \frac{(t-s)^2}{\varepsilon} - \frac{\sigma}{T-t}$$

with

$$\Psi_\varepsilon(p,q,t,s) = \varphi_\varepsilon(p,q) + K(p,q,t,s),$$

$$\varphi_\varepsilon(p,q) = \frac{1}{\varepsilon}d_R(p,q)^4 = \frac{|p\cdot q^{-1}|^4}{\varepsilon}, \quad K(p,q,t,s) = g(p,t) + g(q,s).$$

Then $\Phi$ attains a positive maximum at some $(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{H}^2 \times [0,T)^2$. In particular,

$$\Phi(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \geq \Phi(\hat{p},\hat{p},\hat{t},\hat{t}),$$

which implies that

$$\begin{aligned}\frac{|p_\varepsilon\cdot q_\varepsilon^{-1}|^4}{\varepsilon} + \frac{(t_\varepsilon - s_\varepsilon)^2}{\varepsilon} \leq{}& u(p_\varepsilon,t_\varepsilon) - v(q_\varepsilon,s_\varepsilon) - \sigma g(p_\varepsilon,t_\varepsilon) - \sigma g(q_\varepsilon,s_\varepsilon) - \frac{\sigma}{T-t_\varepsilon}\\ &- \left(u(\hat{p},\hat{t}) - v(\hat{p},\hat{t}) - 2\sigma g(\hat{p},\hat{t}) - \frac{\sigma}{T-\hat{t}}\right).\end{aligned} \tag{6.3.8}$$

Since, due to (6.3.3), the terms $u(p_\varepsilon,t_\varepsilon) - v(q_\varepsilon,t_\varepsilon) - \sigma g(p_\varepsilon,t_\varepsilon) - \sigma g(q_\varepsilon,s_\varepsilon)$ are bounded from above uniformly in $\varepsilon$, we have

$$d_R(p_\varepsilon,q_\varepsilon) \to 0 \text{ and } t_\varepsilon - s_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.3.9}$$

We notice that $p_\varepsilon, q_\varepsilon$ are bounded, since otherwise the right hand side of (6.3.8) will tend to $-\infty$. Therefore, by taking a subsequence, still indexed by $\varepsilon$, we have $p_\varepsilon, q_\varepsilon \to \overline{p} \in \mathbb{H}$ and $t_\varepsilon, s_\varepsilon \to \overline{t} \in [0, T)$. It follows that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} u(p_\varepsilon, t_\varepsilon) - v(q_\varepsilon, s_\varepsilon) - \sigma g(p_\varepsilon, t_\varepsilon) - \sigma g(q_\varepsilon, s_\varepsilon) - \frac{\sigma}{T - t_\varepsilon}\\ &\leq u(\overline{p}, \overline{t}) - v(\overline{p}, \overline{t}) - 2\sigma g(\overline{p}, \overline{t}) - \frac{\sigma}{T - \overline{t}},\end{aligned}$$

which yields

$$\varphi_\varepsilon(p_\varepsilon, q_\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Also, it is easily seen that $\overline{t} > 0$ and therefore $t_\varepsilon, s_\varepsilon > 0$ thanks to the condition that $u(\cdot, 0) \leq v(\cdot, 0)$ in $\mathbb{H}$.

In order to apply the Crandall-Ishii lemma (cf. [27]) in our current case, let us recall the definition of semijets adapted to the Heisenberg group: for any $(p, t) \in \mathbb{H} \times (0, \infty)$ and any locally bounded upper semicontinuous function $u$ in $\mathbb{H} \times (0, \infty)$,

$$\begin{aligned}P_H^{2,+}u(p,t) = \Big\{(\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^{2\times 2} : u(q, s) \leq& u(p, t) + \tau(s - t)\\ &+\langle \zeta, p^{-1}\cdot q\rangle + \frac{1}{2}\langle Xh, h\rangle + o(|p^{-1}\cdot q|_G^2)\Big\},\end{aligned}$$

where $h$ denotes the horizontal projection of $p^{-1} \cdot q$. Similarly, we may define

$$\begin{aligned}P_H^{2,-}u(p,t) = \Big\{(\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^{2\times 2} : u(q, s) \geq& u(p, t) + \tau(s - t)\\ &+\langle \zeta, p^{-1}\cdot q\rangle + \frac{1}{2}\langle Xh, h\rangle + o(|p^{-1}\cdot q|_G^2)\Big\}\end{aligned}$$

for any locally bounded lower semicontinuous function $u$. Also, the closure $\overline{P}_H^{2,+}$ is the set of triples $(\tau, \zeta, X) \in \mathbb{R} \times \mathbb{R}^3 \times S^{2\times 2}$ that satisfy the following: there exist $(p_j, t_j) \in \mathbb{H} \times [0, \infty)$ and $(\tau_j, \zeta_j, X_j) \in P_H^{2,+}(p_j, t_j)$ such that

$$(p_j, t_j, u(p_j, t_j), \tau_j, \zeta_j, X_j) \to (p, t, u(p, t), \tau, \zeta, X) \qquad \text{as } j \to \infty.$$

The closure set $\overline{P}_H^{2,-}$ of $P_H^{2,-}$ can be similarly defined. We refer to [17] for more details.

We now apply the adaptation of the Crandall-Ishii lemma to the Heisenberg group [95, 17] and get for any $\lambda \in (0,1)$

$$(a_1, \zeta_1, X) \in \overline{P}_H^{2,+} u(p_\varepsilon, t_\varepsilon) \text{ and } (a_2, \zeta_2, Y) \in \overline{P}_H^{2,-} v(q_\varepsilon, s_\varepsilon)$$

such that

$$a_1 - a_2 = \alpha\sigma K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) + \frac{\sigma}{(T-t_\varepsilon)^2}, \tag{6.3.10}$$

$$\langle Xw, w\rangle - \langle Yw, w\rangle \leq \langle(\sigma M + \lambda\sigma^2 M^2) w_{p_\varepsilon} \oplus w_{q_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon}\rangle, \tag{6.3.11}$$

and the horizontal projections of $\zeta_1, \zeta_2 \in \mathbb{R}^3$ can be written respectively as $\xi + \eta_1$ and $\xi + \eta_2$ (in $\mathbb{R}^2$) with

$$\xi = \nabla_H^p \varphi_\varepsilon(p_\varepsilon, q_\varepsilon) = -\nabla_H^q \varphi_\varepsilon(p_\varepsilon, q_\varepsilon),$$

$$\eta_1 = \beta\sigma\nabla_H g(p_\varepsilon, t_\varepsilon), \quad \eta_2 = -\beta\sigma\nabla_H g(q_\varepsilon, s_\varepsilon).$$

Here $w = (w_1, w_2) \in \mathbb{R}^2$ is arbitrary, $M = (\nabla^2\Psi_\varepsilon)^*(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon)$ is a $6\times 6$ symmetric matrix, and

$$w_{p_\varepsilon} = \left(w_1, w_2, \frac{1}{2}w_2 x_{p_\varepsilon} - \frac{1}{2}w_1 y_{p_\varepsilon}\right) \tag{6.3.12}$$

and

$$w_{q_\varepsilon} = \left(w_1, w_2, \frac{1}{2}w_2 x_{p_\varepsilon} - \frac{1}{2}w_1 y_{q_\varepsilon}\right) \tag{6.3.13}$$

with $p_\varepsilon = (x_{p_\varepsilon}, y_{p_\varepsilon}, z_{p_\varepsilon})$ and $q_\varepsilon = (x_{q_\varepsilon}, y_{q_\varepsilon}, z_{q_\varepsilon})$.

It is easily seen that $M = M_1 + M_2$, where

$$M_1 = \nabla^2\varphi_\varepsilon(p_\varepsilon, q_\varepsilon)$$

and

$$M_2 = \nabla^2 K(p_\varepsilon, q_\varepsilon) = \begin{pmatrix} \nabla^2 g(p_\varepsilon, t_\varepsilon) & 0 \\ 0 & \nabla^2 g(q_\varepsilon, s_\varepsilon) \end{pmatrix}.$$

However, the algebraic complexity is quite more challenging in the non-commutative case. Following the calculation in the comparison arguments in [17] (and also [16, 95]), with the help of a computer algebra system, we get that there exists $C > 0$ such that

$$\langle(M_1 + \lambda M_1^2) w_{p_\varepsilon} \oplus w_{q_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon}\rangle \leq \frac{C}{\varepsilon}|w|^2 (z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon} y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon} y_{p_\varepsilon})^2 \tag{6.3.14}$$

for any $\lambda > 0$ small. We next follow the strategy in the Euclidean case from [11]. With the help of a computer algebra system, we simplify the left hand side of the following inequalities and obtain a constant $C_\beta > 0$ depending only on $\beta$, such that

$$\langle M_2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon})\rangle \leq \frac{1}{\varepsilon}|w|^2 C_\beta K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon), \tag{6.3.15}$$

$$\begin{aligned}&\langle M_1 M_2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon})\rangle \\ &\leq \frac{1}{\varepsilon}|w|^2 C_\beta K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon)\left|z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon}\right|\end{aligned} \tag{6.3.16}$$

and

$$\langle M_2^2(w_{p_\varepsilon} \oplus w_{q_\varepsilon}), (w_{p_\varepsilon} \oplus w_{q_\varepsilon})\rangle \leq |w|^2 C_\beta K^2(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon). \tag{6.3.17}$$

We remark that the existence of $C_\beta$ here is essentially due to the boundedness of $\nabla_H\langle p\rangle$ and $\nabla_H^2\langle p\rangle$ in $\mathbb{H}$.

By (6.3.11) and (6.3.14), we may take $\lambda > 0$ sufficiently small, depending on the size of $\varepsilon$, $\overline{p}, \overline{t}$, and $\beta$, such that

$$\begin{aligned}&\langle Xw, w\rangle - \langle Yw, w\rangle \\ &\leq \frac{C\sigma}{\varepsilon}|w|^2(z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon})^2 + 2\sigma|w|^2 C_\beta K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon).\end{aligned} \tag{6.3.18}$$

We next apply the definition of viscosity sub- and supersolutions and get

$$a_1 - \mathrm{tr}(AX) + f(p_\varepsilon, \xi + \eta_1) \leq 0 \tag{6.3.19}$$

and

$$a_2 - \mathrm{tr}(AY) + f(q_\varepsilon, \xi + \eta_2) \geq 0. \tag{6.3.20}$$

By subtracting (6.3.20) from (6.3.19), we have

$$a_1 - a_2 \leq \mathrm{tr}(AX) - \mathrm{tr}(AY) + f(q_\varepsilon, \xi + \eta_2) - f(p_\varepsilon, \xi + \eta_1),$$

which yields, by (6.3.18) and (A1),

$$\begin{aligned}a_1 - a_2 \leq{}& \frac{C\sigma}{\varepsilon}\|A\|(z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon})^2 + L_2(\rho)|p_\varepsilon \cdot q_\varepsilon^{-1}| \\ &+ (2\sigma C_\beta\|A\| + 2\beta\mu\sigma L_1)K(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon)\end{aligned} \tag{6.3.21}$$

with $\rho = |\overline{p}| + 1$ for $\varepsilon > 0$ sufficiently small.

Since we have (6.3.9), we now can take $\varepsilon > 0$ small to get

$$\frac{C}{\varepsilon}(z_{p_\varepsilon} - z_{q_\varepsilon} - \frac{1}{2}x_{p_\varepsilon}y_{q_\varepsilon} + \frac{1}{2}x_{q_\varepsilon}y_{p_\varepsilon})^2 + L_2(\rho)|p_\varepsilon \cdot q_\varepsilon^{-1}| \le \frac{\sigma}{T^2}.$$

Taking $\lambda > 0$ accordingly small and

$$\alpha > 2C_\beta\|A\| + \beta\mu L_f,$$

we reach a contradiction to (6.3.10).

□

An immediate consequence is certainly the uniqueness of solutions with at most exponential growth at space infinity.

**Corollary 6.3.2** (Uniqueness of solutions). *Assume that (A1) and (A2) hold. Let $u_0 \in C(\mathbb{H})$. Then there is at most one continuous viscosity solution $u$ of* (6.0.1)–(6.0.2) *satisfying the following exponential growth condition at infinity:*

*(G) For any $T > 0$, there exists $k > 0$ and $C_T > 0$ such that $|u(p,t)| \le C_T e^{k\langle p\rangle}$ for all $(p,t) \in \mathbb{H} \times [0,T]$.*

.

The existence of viscosity solutions of (6.0.1)–(6.0.2) is not the main topic of this chapter, but we remark that it is possible to adapt Perron's method [27] to our current case in the Heisenberg group, under various extra assumptions on the function $f$. For example, one may further assume on (6.0.1) that

(A3) $|f(p,\xi)| \le C_f(1 + |\xi|)$ for some $C_f > 0$ and all $p \in \mathbb{H}, \xi \in \mathbb{R}^2$.

In this case, it is not difficult to verify by computation that $\overline{u} = Cg(p,t) + C_f t$ and $\underline{u} = -Cg(p,t) - C_f t$ are respectively a supersolution and a subsolution of (6.0.1) for any $C > 0$ and $\beta > 0$ when $\alpha > 0$ is sufficiently large. Indeed, we have

$$\overline{u}_t = C\alpha g + C_f,$$

$$|\operatorname{tr}(A(\nabla_H^2 \overline{u})^*)| \le C\|A\|\beta^2\mu^2 g,$$

and

$$|f(p, \nabla_H \overline{u})| \leq CC_f \beta \mu g + C_f,$$

where $\mu$ is the same constant as in the proof of Theorem 6.3.1. Therefore, by (A3), we get

$$\overline{u}_t - \operatorname{tr}(A(\nabla_H^2 \overline{u})^*) + f(p, \nabla_H \overline{u}) \geq 0$$

when $\alpha > \|A\|\beta^2\mu^2 + C_f \beta\mu$. The verification for $\underline{u}$ is similar.

If there exist $C > 0$ and $k > 0$ such that

$$-Ce^{k\langle p \rangle} \leq u_0(p) \leq Ce^{k\langle p \rangle} \quad \text{for all } p \in \mathbb{H}, \tag{6.3.22}$$

then classical arguments [27] show that the supremum over all subsolutions bounded by $\underline{u}$ and $\overline{u}$ is in fact a unique continuous solution. We state the result below without more details in its proof.

**Corollary 6.3.3.** *Assume the Lipschitz conditions (A1), (A2) and the growth condition (A3). Let $u_0 \in C(\mathbb{H})$ satisfy (6.3.22) for some $C > 0$ and $k > 0$. Then there exists a unique continuous solution $u$ of (6.0.1)–(6.0.2) satisfying the exponential growth condition (G).*

## 6.4 LIPSCHITZ PRESERVING PROPERTIES

In the Euclidean space, Lipschitz continuity preserving of the solutions to some partial differential equations is a very important property: when the initial value $u_0$ is Lipschitz continuous , the unique solution $u(x,t)$ is Lipschitz continuous in $x$ as well for any $t \geq 0$.

In what follows, assuming appropriate growth conditions for the initial value $u_0$ and its derivatives, we sketch a proof of these properties for the unique smooth solution of the classical heat equation:

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{6.4.1}$$

with $u(\cdot, 0) = u_0(\cdot)$ in $\mathbb{R}^n$, where $\Delta u$ denotes the usual (Euclidean) Laplacian operator acted on $u$.

By differentiating the equation with respect to the space variables, one may easily see that each of the components of $\nabla u$ satisfies the heat equation (6.4.1), which, by the maximum principle, implies that $\nabla u(\cdot, t)$ is bounded for any $t \geq 0$ if $\nabla u_0$ is bounded in $\mathbb{R}^n$.

We intend to extend these preserving properties to nonlinear equations in the Heisenberg group $\mathbb{H}$. Notions and properties of Lipschitz continuity and convexity in the Heisenberg group are available in the literature [30, 91, 70]. Recall that a function $u$ is said to be Lipschitz continuous in $\mathbb{H}$ if there exists $L > 0$ such that

$$|u(p) - u(q)| \leq L d_L(p, q)$$

for all $p, q \in \mathbb{H}$. It is clear that Lipschitz continuity is left invariant.

As observed above, besides necessary applications of a comparison principle, the key in the straightforward proofs for the Euclidean case lies at differentiating the equation and interchanging derivatives. This is however not applicable directly in the Heisenberg group, since the mixed second derivatives in the Heisenberg group are not commutative in general. In fact, our counterexamples show that preserving of Lipschitz continuity fail even for very simple linear equations; see Examples 6.4.1.

On the other hand, there are many examples on Lipschitz preserving in the Heisenberg group. We have seen from the previous sections that there are some sufficient conditions to guarantee the equivalence between Lipschitz continuity of a function with respect to both metrics $d_L$ and $d_R$; see Definition 6.1.4, Proposition 6.1.5 and Proposition 6.5.6.

We thus can obtain the Lipschitz continuity properties by first investigating them with respect to the right invariant metric $d_R$ and then using the additional assumptions to get the Lipschitz continuity with respect to $d_L$.

In this section, we strengthen the assumption (A2) on $f$; we assume

(A2') the function $f(p, \xi)$ is globally Lipschitz continuous in $p$ with respect to the metric $d_R$, i.e., there exists $L_0 > 0$ such that

$$|f(p, \xi) - f(q, \xi)| \leq L_0 |p \cdot q^{-1}|_G \tag{6.4.2}$$

for all $p, q \in \mathbb{H}$ and $\xi \in \mathbb{R}^2$.

### 6.4.1 Right invariant Lipschitz continuity preserving

We first discuss the Lipschitz continuity based on the standard gauge metric $d_L$ (or equivalently, the Carnot-Carathéodory metric). It turns out that even the simplest first order linear equation will not preserve such Lipschitz continuity.

*Example* 6.4.1. Fix $h_0 = (1, 1) \in \mathbb{R}^2$. Let us consider the equation

$$u_t - \langle h_0, \nabla_H u \rangle = 0 \quad \text{in } \mathbb{H}$$

with $u(p, 0) = u_0(p) = |p|_G$ for $p \in \mathbb{H}$. By direct verification and Corollary 6.3.2, the unique solution is

$$u(p, t) = |p \cdot ht|_G = d_R(p, h^{-1}t),$$

where $h = (1, 1, 0) \in \mathbb{H}_0$. However, it is not Lipschitz continuous with respect to $d_L$. Indeed, similar to Example 6.1.3, one may choose $p_1 = (-t - \varepsilon, -t + \varepsilon, -\varepsilon t)$ and $p_2 = tv^{-1} = (-t, -t, 0)$, which gives

$$u(p_1, t) - u(p_2, t) = |p_1 \cdot p_2^{-1}|_G = (4\varepsilon^4 + 64\varepsilon^2 t^2)^{\frac{1}{4}}$$

but

$$d_L(p_1, p_2) = |p_1^{-1} \cdot p_2|_G = \sqrt{2}\varepsilon.$$

The example above directs us to first consider the Lipschitz continuity with respect to $d_R$. The following result is an immediate consequence of Theorem 6.3.1.

**Theorem 6.4.2** (Preserving of right invariant Lipschitz continuity)**.** *Assume that $f : \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the assumptions (A1), (A2') and (A3). Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of* (6.0.1)–(6.0.2) *satisfying the growth condition (G). If there exists $L > 0$ such that*

$$|u_0(p) - u_0(q)| \leq L d_R(p, q) \tag{6.4.3}$$

*for all $p, q \in \mathbb{H}$, then*

$$|u(p, t) - u(q, t)| \leq (L + L_0 t) d_R(p, q) \tag{6.4.4}$$

*for all $p, q \in \mathbb{H}$ and $t \geq 0$. In particular, there exists $C_\rho > 0$ depending on $\rho > 0$ and $t \geq 0$ such that*

$$|u(p,t) - u(q,t)| \leq C_\rho d_L(p,q)^{\frac{1}{2}} \tag{6.4.5}$$

*for all $p, q \in \mathbb{H}$ with $|p|, |q| \leq \rho$. Moreover, when $f$ does not depend on the space variable $p$, (6.4.4) holds with $L_0 = 0$.*

*Proof.* By symmetry, we only need to prove that

$$u(p,t) - u(h^{-1} \cdot p, t) \leq (L + L_0 t)|h|_G \tag{6.4.6}$$

for all $p, h \in \mathbb{H}$ and $t \geq 0$. It suffices to show that

$$v(p,t) = u(h^{-1} \cdot p, t) + (L + L_0 t)|h|_G$$

is a supersolution of (6.0.1)–(6.0.2) for any $h \in \mathbb{H}$. To this end, we recall the left invariance of horizontal derivatives in the Heisenberg group, which implies that $v$ is a supersolution of

$$v_t - \operatorname{tr}(A\nabla_H^2 v) + f(h^{-1} \cdot p, \nabla_H v) = L_0|h|_G \quad \text{in } \mathbb{H} \times (0, \infty).$$

Since

$$|f(h^{-1} \cdot p, \nabla_H v) - f(p, \nabla_H v)| \leq L_0|h|_G$$

due to (6.4.2), we easily see that $v$ is a supersolution of (6.0.1). Also, by (6.4.3), we have $u(p, 0) \leq v(p, 0)$ for all $p \in \mathbb{H}$. We conclude the proof of (6.4.6) by applying Theorem 6.3.1. The Hölder continuity (6.4.5) follows from Proposition 6.1.2. □

In view of Proposition 6.1.5, we may use the theorem above to show the preserving of Lipschitz continuity in the standard gauge metric under the assumption of evenness or vertical evenness.

**Corollary 6.4.3** (Lipschitz preserving of even solutions). *Assume that $f : \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the conditions (A1), (A2') and (A3). Let $u \in C(\mathbb{H} \times [0, \infty))$ be the unique solution of* (6.0.1)–(6.0.2) *satisfying the growth condition (G). Assume also that $u(\cdot, t)$ is an even or vertically even function. If there exists $L > 0$ such that*

$$|u_0(p) - u_0(q)| \leq L d_L(p, q) \tag{6.4.7}$$

*for all $p, q \in \mathbb{H}$, then*

$$|u(p, t) - u(q, t)| \leq (L + L_0 t) d_L(p, q) \tag{6.4.8}$$

*for all $p, q \in \mathbb{H}$ and $t \geq 0$. In particular, when $f$ does not depend on the space variable $p$, then* (6.4.8) *holds with $L_0 = 0$.*

### 6.4.2 A special class of Hamilton-Jacobi equations

For the case of first order Hamilton-Jacobi equations ($A = 0$), if in addition we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is in the form that $f(\xi) = m(|\xi|)$ with $m : \mathbb{R} \to \mathbb{R}$ locally uniformly continuous, then the Lipschitz preserving property of a bounded solution can be directly shown without the evenness assumption. More precisely, we study equations in the form of

$$u_t + m(|\nabla_H u|) = 0 \quad \text{ in } \mathbb{H} \times (0, \infty), \tag{6.4.9}$$

where $m : \mathbb{R} \to \mathbb{R}$ is a locally uniformly continuous function, with initial condition $u(\cdot, 0) = u_0(\cdot)$ bounded Lipschitz continuous with respect to $d_L$ in $\mathbb{H}$. Since the assumption on $m$ is quite weak, our uniqueness and existence results for unbounded solutions in Section 6.3 do not apply.

For solutions bounded in space, see [98] for a uniqueness theorem and a Hopf-Lax formula when the Hamiltonian $\xi \mapsto m(|\xi|)$ is assumed to be convex and coercive. For instance, when $m(|\xi|) = |\xi|^2/2$, the unique solution of (6.4.9) can be expressed as

$$u(p, t) = \inf_{q \in \mathbb{H}} \left\{ \frac{t}{2} d_{CC}^2 \left( 0, \left( \frac{q^{-1} \cdot p}{t} \right) \right) + u_0(q) \right\}. \tag{6.4.10}$$

The Lipschitz preserving property (with respect to $d_L$ or $d_{CC}$) was left as an open question in [98]; see also [8] for a related open question but for more general Hamiltonians. In contrast to the Euclidean case, it is not obvious how to prove the Lipschitz continuity by using the Hopf-Lax formula (6.4.10). We here give an answer to this question using a PDE approach.

**Theorem 6.4.4** (Lipschitz preserving for special Hamilton-Jacobi equations)**.** *Suppose that $m : \mathbb{R} \to \mathbb{R}$ is locally uniformly continuous. Let $u$ be the unique viscosity solution of* (6.4.9) *with $u(\cdot, 0) = u_0(\cdot)$ bounded in $\mathbb{H}$. If $u_0$ is Lipchitz with respect to $d_L$ in $\mathbb{H}$, i.e., there exists $L > 0$ such that* (6.4.7) *holds for any $p, q \in \mathbb{H}$, then for all $t \geq 0$*

$$|u(p,t) - u(q,t)| \leq L d_L(p,q)$$

*for all $p, q \in \mathbb{H}$.*

*Proof.* Under the assumptions above, it is known [98] that for any fixed $T > 0$, there is a unique bounded continuous viscosity solution in $\mathbb{H} \times [0, T)$. We only need to show that

$$u(p,t) - u(q,t) \leq L d_L(p,q)$$

for all $p, q \in \mathbb{H}$ and $t \in [0, T)$. The other part can be shown by a symmetric argument.

By Young's inequality applied to (6.4.7), we obtain

$$u_0(p) - u_0(q) \leq \frac{L d_L(p,q)^4}{4\delta^4} + \frac{3L\delta^{\frac{4}{3}}}{4} \tag{6.4.11}$$

for all $\delta > 0$ and $p, q \in \mathbb{H}$. It then suffices to show that

$$u(p,t) - u(q,t) \leq \frac{L d_L(p,q)^4}{4\delta^4} + \frac{3L\delta^{\frac{4}{3}}}{4} \tag{6.4.12}$$

for all $\delta > 0$ and $p, q \in \mathbb{H}$. To this end, we fix $\delta > 0$ and prove below that

$$u_L(p,t) = \inf_{q \in \mathbb{H}} \left\{ u(q,t) + C d_L(p,q)^4 \right\}$$

with $C = L/4\delta^4$ is a supersolution of (6.4.9). Suppose there exist a bounded open set $\mathcal{O} \subset \mathbb{H} \times (0, T)$, $\phi \in C^2(\mathcal{O})$ and $(\hat{p}, \hat{t}) \in \mathcal{O}$ such that

$$(u_L - \phi)(\hat{p}, \hat{t}) < (u_L - \phi)(p, t)$$

for all $(p,t) \in \mathcal{O}$. We may also assume that $\phi(p,t) \to -\infty$ when $(p,t) \to \partial\mathcal{O}$. Then for any $\varepsilon > 0$ sufficiently small,

$$\Phi_\varepsilon(p,q,t,s) = u(q,t) + Cd_L(p,q)^4 - \phi(p,s) + \frac{(t-s)^2}{\varepsilon}$$

attains a minimum at $(p_\varepsilon, q_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{H} \times \mathbb{H} \times [0,\infty) \times [0,\infty)$. A standard argument yields $p_\varepsilon, q_\varepsilon \to \hat{p}$ and $t_\varepsilon, s_\varepsilon \to \hat{t}$ as $\varepsilon \to 0$, which, in particular, implies that $t_\varepsilon, s_\varepsilon \neq 0$. The minimum also implies that

$$\nabla_H \phi_1(p_\varepsilon) = \nabla_H \phi(p_\varepsilon, s_\varepsilon) \text{ and } \phi_t(p_\varepsilon, s_\varepsilon) = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon}, \tag{6.4.13}$$

where $\phi_1(p) = Cd_L(p, q_\varepsilon)^4$.

We next apply the definition of supersolutions and get

$$a + m(|\nabla_H \phi_2(q_\varepsilon)|) \geq 0, \tag{6.4.14}$$

where

$$a = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon} \quad \text{and } \phi_2(q) = -Cd_L(p_\varepsilon, q)^4.$$

By (6.4.13), in order to prove that $u_L$ is a supersolution, we only need to substitute $\nabla_H \phi_2(q_\varepsilon)$ in (6.4.14) with $\nabla_H \phi_1(p_\varepsilon)$. By direct calculation, we have

$$\nabla_H^p d_L(p,q)^4 = 4\Big(\delta_1(\delta_1^2 + \delta_2^2) - 4\delta_2\delta_3,\ \delta_2(\delta_1^2 + \delta_2^2) + 4\delta_1\delta_3\Big)$$

and

$$\nabla_H^q d_L(p,q)^4 = 4\Big(-\delta_1(\delta_1^2 + \delta_2^2) - 4\delta_2\delta_3,\ -\delta_2(\delta_1^2 + \delta_2^2) + 4\delta_1\delta_3\Big)$$

with $p = (x_p, y_p, z_p)$, $q = (x_q, y_q, z_q)$ and

$$\delta_1 = x_p - x_q,\ \delta_2 = y_p - y_q,\ \delta_3 = z_p - z_q + \frac{1}{2}x_p y_q - \frac{1}{2}x_q y_p,$$

This reveals that $\nabla_H^p d_L(p,q)^4 \neq -\nabla_H^q d_L(p,q)^4$ and therefore $\nabla_H \phi_1(p_\varepsilon) \neq \nabla_H \phi_2(q_\varepsilon)$ in general; see [36] for more details on this aspect. However, their norms stay the same, i.e., $|\nabla_H^p d_L(p,q)^4| = |\nabla_H^q d_L(p,q)^4|$, which turns out to be a key ingredient in this proof. In fact, we have

$$|\nabla_H^p d_L(p,q)^4| = |\nabla_H^q d_L(p,q)^4| = 4d_L(p,q)^2(\delta_1^2 + \delta_2^2)^{\frac{1}{2}},$$

which implies that $|\nabla_H \phi_1(p_\varepsilon)| = |\nabla_H \phi_2(q_\varepsilon)|$ and their boundedness uniformly in $\varepsilon$. Hence, due to (6.4.13), the equation (6.4.14) is now rewritten as

$$\phi_t(p_\varepsilon, s_\varepsilon) + m(|\nabla_H \phi(p_\varepsilon, s_\varepsilon)|) \geq 0.$$

By sending $\varepsilon \to 0$ and using the continuity of $m$, we conclude the verification that $u_L$ is a supersolution. It follows that $v_L = u_L + 3L\delta^{\frac{4}{3}}/4$ is also a supersolution of (6.4.9). Thanks to (6.4.11), we have $u(p, 0) \leq v_L(p, 0)$, which implies (6.4.12) by Theorem 6.3.1. □

## 6.5 CONVEXITY PRESERVING PROPERTIES

It is well known that the convexity preserving property holds for a large class of fully nonlinear equations in the Euclidean space [39]. The definition reads as follows: when the initial value $u_0$ is convex, the unique solution $u(x, t)$ is convex in $x$ as well for any $t \geq 0$. Concerning convexity in the Heisenberg group, the notion of h-convexity (and equivalently v-convexity) turns out to be a natural extension of the Euclidean version. However, we cannot expect such convexity to be preserved in general. In fact, h-convexity is not preserved even for the first order linear equation.

*Example* 6.5.1 (Linear first order equations). We again consider the linear equation

$$u_t - \langle h_0, \nabla_H u \rangle = 0 \quad \text{in } \mathbb{H} \tag{6.5.1}$$

with $h_0 = (1, 1)$ and $u(x, y, z, 0) = f(x, y, z)$ with $f$ defined as in (6.1.7) for all $(x, y, z) \in \mathbb{H}$. Let $h = (1, 1, 0) \in \mathbb{H}_0$. As verified in Example 6.1.8, $u(\cdot, 0)$ is h-convex in $\mathbb{H}$. However, the unique solution

$$u(p, t) = f(p \cdot ht) = (x + t)^2 (y + t)^2 + 2\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 \tag{6.5.2}$$

is not h-convex for any $t > 0$. In fact, the symmetrized Hessian is given by

$$(\nabla_H^2 u)^*(p, t) = \begin{pmatrix} 2(y+t)^2 + (y-t)^2 & 4(x+t)(y+t) - (x-t)(y-t) \\ 4(x+t)(y+t) - (x-t)(y-t) & 2(x+t)^2 + (x-t)^2 \end{pmatrix}.$$

It is therefore easily seen that

$$(\nabla_H^2 u)^*(t,t,0,t) = \begin{pmatrix} 8t^2 & 16t^2 \\ 16t^2 & 8t^2 \end{pmatrix},$$

which shows that $u(\cdot,t)$ is not h-convex around the point $p=(t,t,0)\in\mathbb{H}$ for any $t>0$.

The loss of convexity preserving is due to the non commutativity of the Heisenberg group product. Although the h-convexity of a function is preserved under left translations, it is not necessarily preserved under right translations, as indicated in Example 6.5.1. We therefore consider right invariant h-convexity first and get left invariant h-convexity preserving in some special cases.

Finally, our study of the convexity preserving property in the Heisenberg group is also inspired by recent works on horizontal mean curvature flow in sub-Riemannian manifolds [20, 36]. The mean curvature flow in $\mathbb{R}^n$ is known to preserve convexity [62], but it is not clear if such a property also holds in $\mathbb{H}$ in general. Our analysis about convexity is only for the simpler equation (6.0.1). However, an explicit solution of the mean curvature flow in $\mathbb{H}$ that does preserve convexity can be found in Example 6.6.3; see also [36].

### 6.5.1 Right invariant h-convexity preserving

**Definition 6.5.2** (Right invariant h-convexity)**.** Let $\Omega$ be an open set in $\mathbb{H}$ and $u:\Omega\to\mathbb{R}$ be an upper semicontinuous function. The function $u$ is said to be right invariant horizontally convex or right h-convex in $\Omega$, if for every $p\in\mathbb{H}$ and $h\in\mathbb{H}_0$ such that $[h^{-1}\cdot p, h\cdot p]\subset\Omega$, we have

$$u(h^{-1}\cdot p)+u(h\cdot p)\geq 2u(p). \tag{6.5.3}$$

Following the proof of Theorem 6.3.1, we can show a convexity maximum principle and get the theorem below.

**Theorem 6.5.3** (Preservation of right invariant h-convexity)**.** *Suppose that the assumptions (A1), (A2) and (A3) hold. Let $u\in C(\mathbb{H}\times[0,\infty))$ be the unique viscosity solution of (6.0.1)–(6.0.2) satisfying the growth condition (G). Assume in addition that $f$ is right invariant* *concave in* $\mathbb{H} \times \mathbb{R}^2$*, i.e.,*

$$f(h^{-1} \cdot p, \xi) + f(h \cdot p, \eta) \leq 2f\left(p, \frac{1}{2}(\xi + \eta)\right) \tag{6.5.4}$$

*for all* $p \in \mathbb{H}$*,* $h \in \mathbb{H}_0$ *and* $\xi, \eta \in \mathbb{R}^2$*. If* $u_0$ *is right invariant h-convex in* $\mathbb{H}$*, then so is* $u(\cdot, t)$ *for all* $t \geq 0$*.*

*Proof of Theorem 6.5.3.* By definition, we aim to show that

$$u(h^{-1} \cdot p, t) + u(h \cdot p, t) \geq 2u(p, t)$$

for any $p \in \mathbb{H}, h \in \mathbb{H}_0, t \geq 0$. We assume by contradiction that there exist $(p_0, h_0, t_0) \in \mathbb{H} \times \mathbb{H}_0 \times [0, \infty)$ such that

$$u(h_0^{-1} \cdot p_0, t_0) + u(h_0 \cdot p_0, t_0) < 2u(p_0, t_0).$$

Then there exists a positive maximizer $(\hat{p}, \hat{h}, \hat{t}) \in \mathbb{H} \times \mathbb{H}_0 \times [0, T)$ of

$$2u(p, t) - u(h^{-1} \cdot p, t) - u(h \cdot p, t) - 3\sigma g(p, t) - \frac{\sigma}{m - |h|_G^4} - \frac{\sigma}{T - t}$$

with some constants $m > |\hat{h}|_G^4$, $T > \hat{t}$ and $\sigma > 0$ small. Here $g(p, t) = e^{\alpha t + \beta \langle p \rangle}$ with $\alpha > 0$ to be determined later and any fixed $\beta > k$. We next consider

$$\begin{aligned}\Phi(p, q, r, h, t, s, \tau) = 2u(r, \tau) - u(h^{-1} \cdot p, t) - u(h \cdot q, s) - \sigma \Psi_\varepsilon(p, q, r, t, s, \tau) \\ -\psi_\varepsilon(t, s, \tau) - \frac{\sigma}{m - |h|_G^4} - \frac{\sigma}{T - \tau},\end{aligned}$$

where

$$\psi_\varepsilon(t, s, \tau) = \frac{(t-s)^2}{\varepsilon} + \frac{(t-\tau)^2}{\varepsilon} + \frac{(s-\tau)^2}{\varepsilon},$$

$$\Psi_\varepsilon(p, q, r, t, s, \tau) = \phi_\varepsilon(p, q, r) + K(p, q, r, t, s, \tau)$$

with

$$\phi_\varepsilon(p, q, r) = \frac{|p \cdot r^{-1}|^4}{\varepsilon} + \frac{|q \cdot r^{-1}|^4}{\varepsilon}$$

and

$$K(p, q, r, t, s, \tau) = g(r, \tau) + g(p, t) + g(q, s).$$

It follows that $\Phi$ has a maximizer $(p_\varepsilon, q_\varepsilon, r_\varepsilon, h_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)$. As before, we denote

$$p_\varepsilon = (x_{p_\varepsilon}, y_{p_\varepsilon}, z_{p_\varepsilon}),\ q_\varepsilon = (x_{q_\varepsilon}, y_{q_\varepsilon}, z_{q_\varepsilon}),\ r_\varepsilon = (x_{r_\varepsilon}, y_{r_\varepsilon}, z_{r_\varepsilon}).$$

Due to the penalization at space infinity, we have $\phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon)$ bounded from above uniformly in $\varepsilon$. By a standard argument, we can show that there exists $\overline{p} \in \mathbb{H}, \overline{h} \in \mathbb{H}_0$ and $\overline{t} \in [0, T)$ such that, up to a subsequence,

$$p_\varepsilon, q_\varepsilon, r_\varepsilon \to \overline{p}, \quad h_\varepsilon \to \overline{h}, \quad t_\varepsilon, s_\varepsilon, \tau_\varepsilon \to \overline{t} \tag{6.5.5}$$

and

$$\phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon) \to 0 \tag{6.5.6}$$

as $\varepsilon \to 0$.

Since $u_0$ is right invariant h-convex, we have $\overline{t} > 0$ and therefore $t_\varepsilon, s_\varepsilon, \tau_\varepsilon > 0$ when $\varepsilon$ is sufficiently small.

Denote $u_-(p,t) = u(h_\varepsilon^{-1} \cdot p, t)$ and $u_+(p,t) = u(h_\varepsilon \cdot p, t)$. We now apply the Crandall-Ishii lemma in the Heisenberg group and get, for any $\lambda \in (0,1)$,

$$(a_1, \zeta_1, X_1) \in \overline{J}^{2,-} u_-(p_\varepsilon, t_\varepsilon),\ (a_2, \zeta_2, X_2) \in \overline{J}^{2,-} u_+(q_\varepsilon, s_\varepsilon),$$
$$(a_3, \zeta_3, X_3) \in \overline{J}^{2,+} u(r_\varepsilon, \tau_\varepsilon)$$

such that

$$2a_3 - a_1 - a_2 = \frac{\sigma}{(T - \tau_\varepsilon)^2} + \sigma\alpha K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon), \tag{6.5.7}$$

the horizontal projections of $\zeta_i$ can be expressed as $\xi_i + \eta_i$ $(i = 1, 2, 3)$ with

$$-\xi_1 = \sigma \nabla_H^p \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon),\ -\xi_2 = \sigma \nabla_H^q \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon),\ 2\xi_3 = \sigma \nabla_H^r \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon),$$

$$-\eta_1 = \sigma \nabla_H g(p_\varepsilon, t_\varepsilon),\ -\eta_2 = \sigma \nabla_H g(q_\varepsilon, s_\varepsilon),\ 2\eta_3 = \sigma \nabla_H g(r_\varepsilon, \tau_\varepsilon),$$

and

$$\langle (2X_3 - X_1 - X_2)w, w \rangle \le \langle (\sigma M + \lambda \sigma^2 M^2) w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle, \tag{6.5.8}$$

for all $w \in \mathbb{R}^2$, where $M = \nabla^2 \Psi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon, h_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)$ is a $9 \times 9$ symmetric matrix, $w_{p_\varepsilon}$, $w_{q_\varepsilon}$ are respectively taken as in (6.3.12) and (6.3.13), and

$$w_{r_\varepsilon} = \left(w_1, w_2, \frac{1}{2} w_2 x_{r_\varepsilon} - \frac{1}{2} w_1 y_{r_\varepsilon}\right).$$

By calculation, it is easily seen that

$$2\xi_3 = \xi_1 + \xi_2. \tag{6.5.9}$$

We next set

$$M_1 = \nabla^2 \phi_\varepsilon(p_\varepsilon, q_\varepsilon, r_\varepsilon)$$

and

$$M_2 = \nabla^2 K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)$$

Then $M = M_1 + M_2$. To investigate the right hand side of (6.5.8), we first give estimates the terms involving $M_1$, which is a variant of (6.3.14) for three space variables. Note that

$$M_1 = M_1' + M_1'',$$

where

$$M_1' = \frac{1}{\varepsilon} \nabla^2 |p \cdot r^{-1}|^4, \quad M_1'' = \frac{1}{\varepsilon} \nabla^2 |q \cdot r^{-1}|^4.$$

By direct calculations, we get

$$\langle M_1' w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle = 0,$$
$$\langle M_1'' w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle = 0$$

and

$$\langle M_1'^2 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle = \frac{512}{\varepsilon} m_1^2 |w|^2,$$
$$\langle M_1''^2 w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle = \frac{512}{\varepsilon} m_2^2 |w|^2,$$
$$\langle M_1' M_1'' w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon}, w_{p_\varepsilon} \oplus w_{q_\varepsilon} \oplus w_{r_\varepsilon} \rangle = \frac{256}{\varepsilon} m_1 m_2 |w|^2,$$

where

$$m_1 = z_{p_\varepsilon} - z_{r_\varepsilon} + \frac{1}{2} y_{p_\varepsilon} x_{r_\varepsilon} - \frac{1}{2} x_{p_\varepsilon} y_{r_\varepsilon}, \quad m_2 = z_{q_\varepsilon} - z_{r_\varepsilon} + \frac{1}{2} y_{q_\varepsilon} x_{r_\varepsilon} - \frac{1}{2} x_{q_\varepsilon} y_{r_\varepsilon}.$$

It follows that

$$\langle (M_1+\lambda M_1^2) w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon}, w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon}\rangle \leq \frac{512\lambda}{\varepsilon}(m_1^2+m_2^2+m_1m_2)|w|^2, \quad (6.5.10)$$

which implies that

$$\langle (M_1+\lambda M_1^2) w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon}, w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon}\rangle \leq \frac{C}{\varepsilon}|w|^2(|p_\varepsilon\cdot r_\varepsilon^{-1}|^4+|q_\varepsilon\cdot r_\varepsilon^{-1}|^4) \quad (6.5.11)$$

for some $C>0$ independent of $\varepsilon$ and $\lambda$. On the other hand, with the help of computer algebra system, we obtain estimates similar to (6.3.15), (6.3.16) and (6.3.17). In fact, we get a constant $C_\beta$ such that, when $\lambda>0$ is small enough (depending on $\varepsilon$),

$$\langle M_2(w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon}), (w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon})\rangle \leq C_\beta |w|^2 K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon) \quad (6.5.12)$$

$$\begin{aligned}&\langle \lambda(M_1M_2+M_2M_1+M_2^2)(w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon}), (w_{p_\varepsilon}\oplus w_{q_\varepsilon}\oplus w_{r_\varepsilon})\rangle\\ &\quad\leq 2C_\beta|w|^2K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon)\end{aligned} \quad (6.5.13)$$

for some constant $\mu>0$ independent of $\varepsilon, \beta$ and $\sigma$ and satisfying (6.3.7). As remarked in the proof of Theorem 6.3.1, we obtain the constant $C_\beta$ thanks to the boundedness of $\nabla_H\langle p\rangle$ and $\nabla_H^2\langle p\rangle$ in $\mathbb{H}$.

Combining (6.5.8), (6.5.11) and (6.5.13), we have

$$\begin{aligned}\langle (2X_3-X_1-X_2)w, w\rangle \leq{}& \frac{C\sigma}{\varepsilon}\|w\|^2(|p_\varepsilon\cdot r_\varepsilon^{-1}|^4+|q_\varepsilon\cdot r_\varepsilon^{-1}|^4)\\ &+2\sigma\|w\|^2C_\beta K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon).\end{aligned} \quad (6.5.14)$$

when $\lambda>0$ and $\sigma>0$ are sufficiently small.

Since the horizontal derivatives are left translation invariant, the functions $u_-$ and $u_+$ are respectively solutions of

$$(u_-)_t-\operatorname{tr}(A\nabla_H^2u_-)+f(h_\varepsilon^{-1}\cdot p, \nabla_H u_-)=0 \quad \text{in } \mathbb{H}\times(0,\infty)$$

and

$$(u_+)_t-\operatorname{tr}(A\nabla_H^2u_+)+f(h_\varepsilon\cdot p, \nabla_H u_+)=0 \quad \text{in } \mathbb{H}\times(0,\infty).$$

Applying the definition of viscosity subsolutions and supersolution, we have

$$a_1 - \operatorname{tr}(AX_1) + f(h_\varepsilon^{-1} \cdot p_\varepsilon, \xi_1 + \eta_1) \geq 0, \tag{6.5.15}$$

$$a_2 - \operatorname{tr}(AX_2) + f(h_\varepsilon \cdot q_\varepsilon, \xi_2 + \eta_2) \geq 0, \tag{6.5.16}$$

$$a_3 - \operatorname{tr}(AX_3) + f(r_\varepsilon, \xi_3 + \eta_3) \leq 0. \tag{6.5.17}$$

Subtracting (6.5.15) and (6.5.16) from twice (6.5.17), we get

$$2a_3 - a_1 - a_2 \leq \operatorname{tr} A(2X_3 - X_1 - X_2) + E, \tag{6.5.18}$$

where

$$E = f(h_\varepsilon^{-1} \cdot p_\varepsilon, \xi_1 + \eta_1) + f(h_\varepsilon \cdot q_\varepsilon, \xi_2 + \eta_2) - 2f(r_\varepsilon, \xi_3 + \eta_3)$$

It follows from the concavity assumption (6.5.4), the relation (6.5.9) and (A1)-(A2) that

$$\begin{aligned}
E \leq\ & f(h_\varepsilon^{-1} \cdot p_\varepsilon, \xi_1 + \eta_1) - f(h_\varepsilon^{-1} \cdot r_\varepsilon, \xi_1 + \eta_1) + f(h_\varepsilon \cdot q_\varepsilon, \xi_2 + \eta_2) - f(h_\varepsilon \cdot r_\varepsilon, \xi_2 + \eta_2) \\
& + f(h_\varepsilon^{-1} \cdot r_\varepsilon, \xi_1 + \eta_1) + f(h_\varepsilon \cdot r_\varepsilon, \xi_2 + \eta_2) - 2f(r_\varepsilon, \frac{1}{2}(\xi_1 + \xi_2 + \eta_1 + \eta_2)) \\
& + 2f(r_\varepsilon, \frac{1}{2}(\xi_1 + \xi_2 + \eta_1 + \eta_2)) - 2f(r_\varepsilon, \xi_3 + \eta_3) \\
\leq\ & L_R(|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot p_\varepsilon^{-1} \cdot h| + |h_\varepsilon^{-1} \cdot r_\varepsilon \cdot q_\varepsilon^{-1} \cdot h_\varepsilon|) + L_f|\eta_1 + \eta_2 - 2\eta_3|
\end{aligned} \tag{6.5.19}$$

with $R = (|\overline{p}| + 1)$ and $\varepsilon > 0$ small. Also, by (6.3.7), we have

$$|\eta_1 + \eta_2 - 2\eta_3| \leq 2(|\eta_1| + |\eta_2| + |\eta_3|) = 2\sigma\beta\mu K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon).$$

In view of (6.5.14), (6.5.18) and (6.5.19), we then obtain

$$\begin{aligned}
& 2a_3 - a_1 - a_2 \\
\leq & \frac{C\sigma}{\varepsilon}(|p_\varepsilon \cdot r_\varepsilon^{-1}|^4 + |q_\varepsilon \cdot r_\varepsilon^{-1}|^4) + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot p_\varepsilon^{-1} \cdot h| + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot q_\varepsilon^{-1} \cdot h_\varepsilon| \\
& + 2\sigma(C_\beta\|A\| + L_f\beta\mu)K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon).
\end{aligned} \tag{6.5.20}$$

In view of (6.5.5) and (6.5.6), we can take $\varepsilon > 0$ small such that

$$\frac{C\sigma}{\varepsilon}(|p_\varepsilon \cdot r_\varepsilon^{-1}|^4 + |q_\varepsilon \cdot r_\varepsilon^{-1}|^4) + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot p_\varepsilon^{-1} \cdot h| + L_R|h_\varepsilon^{-1} \cdot r_\varepsilon \cdot q_\varepsilon^{-1} \cdot h_\varepsilon| < \frac{\sigma}{T^2},$$

which, by (6.5.20), implies

$$2a_3 - a_1 - a_2 \leq \frac{\sigma}{T^2} + 2\sigma(C_\beta\|A\| + L_f\beta\mu)K(p_\varepsilon, q_\varepsilon, r_\varepsilon, t_\varepsilon, s_\varepsilon, \tau_\varepsilon).$$

It clearly contradicts (6.5.7) when $\alpha$ is chosen to satisfy

$$\alpha > 2\|A\|C_\beta + 2L_f\beta\mu.$$

□

One may further generalize this result for (6.0.3) by assuming that $F$ is concave in all arguments.

*Remark* 6.5.4. The concavity assumption (6.5.4) on the operator $f$ is stronger than the assumptions of the convexity results in the Euclidean space as shown in [39, 69]. In particular, the concavity of $\xi \mapsto f(p, \xi)$ is not needed in the Euclidean case. We here need this assumption, since there are no expressions of h-convexity in $\mathbb{H}$ corresponding to the following one for the Euclidean convexity

$$u(\xi) + u(\eta) \geq 2u\left(\frac{\xi + \eta}{2}\right)$$

for all $\xi, \eta \in \mathbb{R}^n$. It is not clear to us whether the assumption (6.5.4) can be weakened.

*Example* 6.5.5. Let us revisit Example 6.5.1. Since the equation (6.5.1) and the solution (6.5.2) satisfy all of the assumptions in Theorem 6.5.3, the right invariant h-convexity of the solution is preserved, though the h-convexity is not. Indeed, if $u(p, t)$ is given by (6.5.2), then by direct calculation we obtain, for all $p = (x, y, z)$, $h = (h_1, h_2, 0)$ and $t \geq 0$,

$$\begin{aligned}
&u(h \cdot p, t) + u(h^{-1} \cdot p, t) \\
&= (x + h_1 + t)^2(y + h_2 + t)^2 + 2\left(z + \frac{1}{2}h_1 y - \frac{1}{2}h_2 x + \frac{1}{2}(x + h_1)t - \frac{1}{2}(y + h_2)t\right)^2 \\
&= 2(x + t)^2(y + t)^2 + 4\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 + (h_1(y + t) - h_2(x + t))^2 + 2h_1^2(y + t)^2 \\
&\quad + 2h_2^2(x + t)^2 + 8(x + t)(y + t)h_1h_2 + 2h_1^2h_2^2 \\
&= 2(x + t)^2(y + t)^2 + 4\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 + 3(h_1(y + t) + h_2(y + t))^2 + 2h_1^2h_2^2 \\
&\geq 2(x + t)^2(y + t)^2 + 4\left(z + \frac{1}{2}xt - \frac{1}{2}yt\right)^2 = 2u(p, t).
\end{aligned}$$

### 6.5.2 Left invariant h-convexity preserving

We next discuss some special cases, where h-convexity and right invariant h-convexity are equivalent.

**Proposition 6.5.6** (Evenness). *Let $u$ be an even or vertically even function on $\mathbb{H}$ . Then $u$ is h-convex in $\mathbb{H}$ if and only if $u$ is right invariant h-convex in $\mathbb{H}$.*

*Proof.* By definition, $u$ is h-convex if $u$ satisfies (6.1.5) for any $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$. Since $u$ is even, it is easily seen that (6.1.5) holds if and only if

$$u(h \cdot \overline{p}) + u(h^{-1} \cdot \overline{p}) \geq u(\overline{p}),$$

where $\overline{p}$ is given as in (6.1.4), or

$$u(h \cdot p^{-1}) + u(h^{-1} \cdot p^{-1}) \geq u(p^{-1})$$

for all $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$, which is equivalent to saying

$$u(h \cdot p) + u(h^{-1} \cdot p) \geq u(p) \text{ for all } p \in \mathbb{H} \text{ and } h \in \mathbb{H}_0.$$

□

Another sufficient condition for equivalence between the h-convexity and the left h-convexity of a function $u$ on $\mathbb{H}$ is that $u$ has a separate structure; namely,

$$u(x, y, z) = f(x, y) + g(z) \tag{6.5.21}$$

for any $(x, y, z) \in \mathbb{H}$.

**Proposition 6.5.7** (Separability). *Let $u$ be a function on $\mathbb{H}$ with a separate structure as in (6.5.21). Then $u$ is h-convex in $\mathbb{H}$ if and only if $u$ is right invariant h-convex in $\mathbb{H}$.*

*Proof.* Suppose $u$ can be written as in (6.5.21). Setting $p=(x,y,z)$ and $h=(h_1,h_2)$, we then have

$$\begin{aligned}
u(p\cdot h) &= f(x+h_1,y+h_2)+g(z+\frac{1}{2}xh_2-\frac{1}{2}yh_1);\\
u(p\cdot h^{-1}) &= f(x-h_1,y-h_2)+g(z-\frac{1}{2}xh_2+\frac{1}{2}yh_1);\\
u(h\cdot p) &= f(x+h_1,y+h_2)+g(z+\frac{1}{2}yh_1-\frac{1}{2}xh_2);\\
u(h^{-1}\cdot p) &= f(x-h_1,y-h_2)+g(z-\frac{1}{2}yh_2+\frac{1}{2}xh_2).
\end{aligned}$$

It is easily seen that in this case

$$u(p\cdot h^{-1})+u(p\cdot h)=u(h^{-1}\cdot p)+u(h\cdot p),$$

which immediately yields the equivalence of (6.1.5) and (6.5.3) in $\mathbb{H}$. $\square$

The following result on preserving of the h-convexity itself is an immediate consequence of Theorem 6.5.3, Propositions 6.5.6 and 6.5.7.

**Corollary 6.5.8** (H-convexity preserving under evenness or separability)**.** *Assume that $f$ satisfies (A1)–(A3) and the concavity condition (6.5.4) for all $p\in\mathbb{H}$, $h\in\mathbb{H}_0$ and $\xi,\eta\in\mathbb{R}^2$. Let $u\in C(\mathbb{H}\times[0,\infty))$ be the unique viscosity solution of (6.0.1)–(6.0.2) satisfying the growth condition (G). Assume in addition that for any $t\geq 0$, $u(\cdot,t)$ either is an even or vertically even function or has a separable structure as in (6.5.21). If $u_0$ is h-convex in $\mathbb{H}$, then so is $u(\cdot,t)$ in $\mathbb{H}$ for all $t\geq 0$.*

## 6.6 MORE EXAMPLES

In this section, we provide more examples, where the h-convexity is preserved.

*Example* 6.6.1. Let $u_0(x,y,z)=(x^2+y^2)^2-8z^2$. It is not difficult to see that $u_0$ is an h-convex function in $\mathbb{H}$. Consider the heat equation

$$u_t-\Delta_H u=0\quad \text{in } \mathbb{H}\times(0,\infty) \tag{6.6.1}$$

with $u(\cdot, 0) = u_0$ in $\mathbb{H}$, where $\Delta_H$ denotes the horizontal Laplacian operator in the Heisenberg group, i.e., $\Delta_H u = \operatorname{tr}(\nabla_H^2 u)^*$. The unique solution of (6.6.1) in this case is

$$u(x, y, z, t) = (x^2 + y^2)^2 - 8z^2 + 12(x^2 + y^2)t + 24t^2 \tag{6.6.2}$$

for all $(x, y, z) \in \mathbb{H}$ and $t \geq 0$ and it actually preserves the h-convexity of the initial value $u_0$.

*Example* 6.6.2. The solution as in (6.6.2) looks special, since it can be written as the sum of a function of $x, y, t$ and a function of $z$. A more complicated solution of the heat equation (6.6.1) is

$$u(x, y, z, t) = (x^2+y^2)z^2 + \frac{1}{24}(x^2+y^2)^3 + (4z^2+2(x^2+y^2)^2)t + 17(x^2+y^2)t^2 + \frac{68}{3}t^3 \tag{6.6.3}$$

which contains mixed terms of $x, y$ and $z$. By direct calculation, one can also show that $u(\cdot, t)$ satisfies (6.1.6) in $\mathbb{H}$ in the classical sense for everywhere $t \geq 0$.

*Example* 6.6.3. We recall another example in [36] for the level-set mean curvature flow equation in $\mathbb{H}$. The equation is of the form

$$u_t - |\nabla_H u| \operatorname{div}_H \left( \frac{\nabla_H u}{|\nabla_H u|} \right) = 0 \quad \text{in } \mathbb{H} \times (0, \infty), \tag{6.6.4}$$

where $\operatorname{div}_H$ stands for the horizontal divergence operator in the Heisenberg group. An explicit solution is

$$u(x, y, z, t) = (x^2 + y^2)^2 + 16z^2 + 12(x^2 + y^2)t + 12t^2.$$

This is also an example of h-convexity preserving but unfortunately is not covered by our current results.

## 7.0 A GAME-THEORETIC PROOF OF CONVEXITY PRESERVATION OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

Among many properties of the viscosity solutions, the convexity preserving property is known to be important for various linear and nonlinear parabolic equations arising in geometry, material sciences and image processing. This property of viscosity solutions was proved to hold in a very general class of degenerate parabolic equations using the so called *convexity maximum principle* in the Euclidean spaces [39]. We have also seen some discussions of this property on the Heisenberg group in the previous chapter.

The discrete game interpretations of various elliptic and parabolic PDEs ([78, 100, 101, 79, 97, 96], etc) have recently attracted great attention. The game related methods are also used as a new tool in different contexts. We will introduce the game interpretation of the level set mean curvature flow equation and normalized $p$-Laplace equations.

Based on game-theoretic interpretations, we give a simpler proof of the convexity preserving of the level sets and solutions to the mean curvature flow equation in $\mathbb{R}^2$. Our method also applies to normalized parabolic $p$-Laplace equations. Our new proofs are based on investigating game strategies or iterated applications of Dynamic Programming Principles (DPP), which is very different from the standard proofs in the literature. We also use this method to study convexity preserving for the Neumann boundary problems.

### 7.1 CONVEXITY PRESERVING PROPERTIES

The mean curvature flow describes the motion of a surface in $\mathbb{R}^n$ governed by the law that the normal velocity is equal to the mean curvature. The convexity preserving property for mean curvature flow is a well-known result by Huisken [62], saying that an evolving surface by mean curvature stays convex if the initial surface is convex; see also [37] for the two dimensional case in detail. This property was later formulated in terms of level set method in [34] and [39].

$$\text{(MCF)}\quad \begin{cases} u_t - |\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \quad (7.1.1)\\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \quad (7.1.2)\end{cases}$$

where $u_0$ is assumed to be a Lipschitz continuous function in $\mathbb{R}^n$.

In [34] and [39], those authors proved the following.

**Theorem 7.1.1.** *Suppose $u_0$ is a Lipschitz continuous function on $\mathbb{R}^n$. Let $u$ be the unique viscosity solution of (MCF). If the set $\{x \in \mathbb{R}^n : u_0(x) \geq 0\}$ is convex, then the set $\{x \in \mathbb{R}^n : u(x,t) \geq 0\}$ is also convex for any $t \geq 0$.*

The proof in [34] is based on a regularization of the mean curvature operator for the corresponding stationary problem in a convex domain $\Omega \subset \mathbb{R}^n$, that is,

$$\text{(SP)}\quad \begin{cases} -\Delta_1^G u = -|\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 1 & \text{in } \Omega, \quad (7.1.3)\\ u = 0 & \text{on } \partial\Omega, \quad (7.1.4)\end{cases}$$

and an application of more classical convexity maximum principle by Korevaar [81] and Kennington [75] for $C^2$ solutions of uniformly elliptic equations. In two dimensions, Barron, Goebel and Jensen [12] recently gave a PDE proof for the convexity of level sets for any strict subsolution or strict supersolution of $-\Delta_1^G u = 0$.

The convexity (concavity) preserving property of viscosity solutions was later studied for a general class of degenerate parabolic equations in [39], including the mean curvature flow equation and normalized $p$-Laplace equations.

$$\text{(PL)}\quad \begin{cases} u_t - \operatorname{tr}\left(\left(I + (p-2)\dfrac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right)\nabla^2 u\right) = f(x) & \text{in } \mathbb{R}^n \times (0,\infty), \quad (7.1.5)\\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \quad (7.1.6)\end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given continuous function.

We list the theorem for the convexity preserving of the viscosity solutions to (MCF).

**Theorem 7.1.2.** *Suppose $u_0$ is a concave and Lipschitz continuous function on $\mathbb{R}^n$. Let $u$ be the unique viscosity solution of (MCF). Then $u(x,t)$ is also concave in $x$ for any $t \geq 0$.*

The classical argument relies on a generalized version of convexity comparison principle; we refer to [69] for related results. More precisely, it is shown, by adding more variables in the usual comparison theorem, that the continuous solution $u$ satisfies

$$u(x,t) + u(y,t) - 2u(z,t) \leq |x + y - 2z|$$

for any $x, y, z \in \mathbb{R}^n$ and $t \geq 0$ if the same holds initially:

$$u_0(x) + u_0(y) - 2u_0(z) \leq |x + y - 2z|.$$

The concavity preserving property is then a special case when $z = (x+y)/2$. We remark that the convexity of solutions to various PDEs has also been extensively studied in [72, 73, 107, 31, 2] etc.

## 7.2 GAME INTERPRETATIONS OF VISCOSITY SOLUTIONS

### 7.2.1 Tug of War game and normalized parabolic $p$-Laplace equations

Let us first discuss a game interpretation of parabolic normalized $\infty$-Laplace equation.

$$\begin{cases} u_t - \operatorname{tr}\left(\dfrac{\nabla u \otimes \nabla u}{|\nabla u|^2} \nabla^2 u\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

Fix $\varepsilon > 0$ be the stepsize of the game. The game starts from $x \in \mathbb{R}^n$. Repeat $N(= [2t/\varepsilon^2])$ times the following.

(1) Toss a fair coin;

(2) The winner of the first step choose a vector $v$ in the ball $B(0, \varepsilon)$ (abbreviated as $B_\varepsilon$);

(3) The game position moves from $x$ to $x + v$.

Fix any strategies $S_I$ of Player I and $S_{II}$ of Player II. Let $y_N$ denote the final stage of this game under $S_I$ and $S_{II}$. Player I (II) minimizes (maximizes) the value of $u_0$ at $y_N$.

Denote by $u^\varepsilon(x,t)$ the value function:

$$u^\varepsilon(x,t) = \inf_{S_I} \sup_{S_{II}} \mathbb{E}(u_0(y_N)),$$

where $\mathbb{E}$ stands for the expectation. This game described above is often referred to as tug-of-war game. Dynamic Programming Principle (DPP) of this game is

$$u^\varepsilon(x,t) = \frac{1}{2} \sup_{v\in B_\varepsilon} u^\varepsilon\left(x+v, t-\frac{\varepsilon^2}{2}\right) + \frac{1}{2} \inf_{v\in B_\varepsilon} u^\varepsilon\left(x+v, t-\frac{\varepsilon^2}{2}\right).$$

Following the argument in [96], one can prove $u^\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.

Then we discuss the game interpretation of normalized $p$-Laplace equations (PL) with $2 \le p \le \infty$. Fix $\varepsilon > 0$ be the stepsize of the game. The game starts from $x \in \mathbb{R}^n$. Let

$$\sigma_p = \begin{cases} \frac{1}{2(p+n)} & \text{if } 2 \le p < \infty, \\ \frac{1}{2} & \text{if } p = \infty. \end{cases}$$

Repeat $N (= [t/\sigma_p \varepsilon^2])$ times the following.

(1) Toss a biased coin with probabilities $\alpha$ to get heads and $\beta$ to get tails, $\alpha + \beta = 1$;

(2) If they get heads, they play a tug-of-war game;

(3) If they get tails, the game state moves randomly in the ball $B(x_n, \varepsilon)$.

Fix any strategies $S_I$ of Player I and $S_{II}$ of Player II. Let $y_N$ denote the final stage of this game under $S_I$ and $S_{II}$. Player I (II) minimizes (maximizes) the value of $u_0$ at $y_N$.

Denote by $u^\varepsilon(x,t)$ the value function:

$$u^\varepsilon(x,t) = \inf_{S_I} \sup_{S_{II}} \mathbb{E}_{S_I, S_{II}}(u_0(y_N)).$$

This game described above is often referred to as tug-of-war game with noise.

The equation is related to the DPP below:

$$\begin{aligned} u^\varepsilon(x,t) = \frac{\alpha}{2} \sup_{v\in B_\varepsilon} u^\varepsilon(x+v, t-\sigma_p\varepsilon^2) + \frac{\alpha}{2} \inf_{v\in B_\varepsilon} u^\varepsilon(x+v, t-\sigma_p\varepsilon^2) \\ + \beta \fint_{v\in B_\varepsilon} u^\varepsilon(x+v, t-\sigma_p\varepsilon^2)\, dv + \varepsilon^2 f(x). \end{aligned} \tag{7.2.1}$$

$$u^\varepsilon(x,t) = u_0(x) \text{ for all } x \in \mathbb{R}^n \text{ and } t < \varepsilon^2. \tag{7.2.2}$$

Here $\alpha \geq 0, \beta \geq 0$ satisfy $\alpha + \beta = 1$ and are determined by the choice of $p$. In fact, it is known from [96, 97] that

$$\alpha = \frac{p-2}{p+n} \text{ and } \beta = \frac{2+n}{p+n}.$$

The presence of the constant $\sigma_p$ in (7.2.1) is due to the different forms of DPP for our convenience later; in contrast to [96], our formula does not use averages with respect to $t$. One may show that $u^\varepsilon$ converges, as $\varepsilon \to 0$, to the unique solution $u$ of (PL) locally uniformly in $\mathbb{R}^n \times [0, \infty)$ by following [96].

### 7.2.2 The deterministic game and the mean curvature flow equations

Let us first review the game proposed in [78] for motion by curvature. A marker, representing the *game position* or *game state*, is initialized at $x \in \mathbb{R}^2$ from time 0. The maturity time given is denoted by $t$. Let the step size for space be $\varepsilon > 0$. Time $\varepsilon^2$ is consumed for each step. The total game steps $N$ can be regarded as $[t/\varepsilon^2]$.

Two players, Player I and Player II, participate in the game: Player I intends to maximize at the final state an *objective function*, which in our case is $u_0$, while Player II is to minimize it. At each turn,

(1) Player I chooses in $\mathbb{R}^2$ a unit vector $v$;

(2) Player II has the right to keep or reverse the choice of Player I, which determines a sign $b = \pm 1$;

(3) The marker is moved from the present state $x$ to $x + \sqrt{2}\varepsilon b v$.

To give a mathematical description, we denote

$$\mathbb{S}^1 = \{v \in \mathbb{R}^2 : |v| = 1\}.$$

Then the inductive *state equation* writes as

$$\begin{cases} z_{k+1} = z_k + \sqrt{2}\varepsilon b_k v_k, \quad k = 0, 1, \ldots, N-1; \\ z_0 = x, \end{cases}$$

where $v_k \in \mathbb{S}^1$ and $b_k = \pm 1$.

Hereafter, for any $x \in \mathbb{R}^n$ and $s \in [0, \infty)$, $z(s; x) = z_m$ stands for the game state at the step $m = [s/\varepsilon^2]$ starting from $x$ under the competing strategies so that our games look like continuous ones. The *value function* is defined as

$$u^\varepsilon(x,t) := \max_{v_1 \in \mathbb{S}^1} \min_{b_1 = \pm 1} \ldots \max_{v_N \in \mathbb{S}^1} \min_{b_N = \pm 1} u_0(z(t;x)). \tag{7.2.3}$$

It is also known [78, Appendix B] that $u^\varepsilon$ preserves Lipschitz continuity of $u_0$ in space. The *dynamic programming principle* is as below:

$$u^\varepsilon(x,t) = \max_{v \in \mathbb{S}^1} \min_{b = \pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon b v, t - \varepsilon^2), \tag{7.2.4}$$

It is employed to show the following theorem. We assume that

(A) $u_0$ is a bounded and Lipschitz continuous function in $\mathbb{R}^2$ satisfying that $u_0 \geq a$ and $u_0 - a$ has compact support $K_a$ for some constant $a \in \mathbb{R}$.

**Theorem 7.2.1** (Theorem 1.2 in [78])**.** *Assume that $u_0$ satisfies (A). Let $u^\varepsilon$ be the value function defined as in (7.2.3). Then $u^\varepsilon$ converges, as $\varepsilon \to 0$, to the unique viscosity solution of (MCF) uniformly on compact subsets of $\mathbb{R}^2 \times (0, \infty)$.*

We recall the definition of viscosity solutions for (MCF) below.

**Definition 7.2.2** ([38])**.** A locally bounded upper (resp., lower) semicontinuous function $u$ is called a subsolution (resp., supersolution) of (7.1.1) if for any $(x_0, t_0) \in \mathbb{R}^2$ and $\phi \in C^2(\mathbb{R}^2 \times [0, \infty))$ such that $u - \phi$ attains a (strict) maximum (resp., minimum) at $(x_0, t_0)$, we have

$$\phi_t - |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) \leq 0 \quad \text{at } (x_0, t_0), \tag{7.2.5}$$

$$\left(\text{resp.,} \quad \phi_t - |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) \geq 0 \quad \text{at } (x_0, t_0)\right)$$

when $\nabla \phi(x_0, t_0) \neq 0$ and

$$\phi_t(x_0, t_0) \leq 0, \quad (\text{resp.,} \quad \phi_t(x_0, t_0) \geq 0)$$

when $\nabla \phi(x_0, t_0) = 0$ and $\nabla^2 \phi(x_0, t_0) = O$.

A locally bounded continuous function $u$ is called a solution if it is both a subsolution and a supersolution. A subsolution (resp., supersolution, solution) $u$ of (7.1.1) is said to be a subsolution (resp., supersolution, solution) of (MCF) if it further satisfies $u(x,0) \leq u_0(x)$ (resp., $u(x,0) \geq u_0(x)$, $u(x,0) = u_0(x)$) for all $x \in \mathbb{R}^2$.

### 7.2.3 A modified game

Let us slightly change the rules of the game described above: we keep the basic setting of the game and the objectives of both Players, but at each round, we now ask that

(1) Player I chooses $v, v' \in \mathbb{S}^1$;

(2) Player II determines $b = \pm 1$ and $b' = \pm 1$;

(3) The marker is moved from the present state $x$ to $x + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v')$.

Under the new rules above, we may define the value function

$$w^\varepsilon(x,t) := \max_{v_1, v_1' \in \mathbb{S}^1} \min_{b_1, b_1' = \pm 1} \ldots \max_{v_N, v_N' \in \mathbb{S}^1} \min_{b_N, b_N' = \pm 1} u_0(\tilde{z}(t;x)), \tag{7.2.6}$$

where $\tilde{z}(t;x) = \tilde{z}_N$ is the solution of the following state equation:

$$\begin{cases} \tilde{z}_{k+1} = \tilde{z}_k + \dfrac{\sqrt{2}\varepsilon}{2}(b_k v_k + b_k' v_k'), \quad k = 0, 1, \ldots, N-1; \\ z_0 = x, \end{cases}$$

It is clear that the new dynamic programming principle is

$$w^\varepsilon(x,t) = \max_{v, v' \in \mathbb{S}^1} \min_{b, b' = \pm 1} w^\varepsilon\left(x + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v'), t - \varepsilon^2\right), \tag{7.2.7}$$

for any $t \geq \varepsilon^2$. The Lipschitz preserving property of $w^\varepsilon$ can be shown by following [78, Appendix B].

**Proposition 7.2.3** (Lipschitz continuity preserving property). *Let $w^\varepsilon$ be the value function associated to the modified game described above. If there exists $L > 0$ such that*

$$|u_0(x) - u_0(y)| \leq L|x - y| \text{ for any } x, y \in \mathbb{R}^2,$$

*then $w^\varepsilon(x,t)$ satisfies*

$$|w^\varepsilon(x,t) - w^\varepsilon(y,t)| \leq L|x - y| \text{ for any } x, y \in \mathbb{R}^2,\ t \geq 0 \text{ and } \varepsilon > 0. \tag{7.2.8}$$

*Proof.* We prove this result by induction. By (7.2.7), we have

$$w^\varepsilon(x,\varepsilon^2) = \max_{v,v'\in\mathbb{S}^1}\min_{b,b'=\pm 1} u_0\left(x + \frac{\sqrt{2}\varepsilon}{2}(bv+b'v')\right)$$
$$w^\varepsilon(y,\varepsilon^2) = \max_{v,v'\in\mathbb{S}^1}\min_{b,b'=\pm 1} u_0\left(y + \frac{\sqrt{2}\varepsilon}{2}(bv+b'v')\right).$$

Let $v_0, v_0'$ be the maximizer in the first relation and $b_0, b_0'$ be the minimizer (with respect to the choice of $v = v_0, v' = v_0'$) in the second. We have

$$w^\varepsilon(x,\varepsilon^2) - w^\varepsilon(y,\varepsilon^2) \le u_0\left(x + \frac{\sqrt{2}\varepsilon}{2}(b_0v_0+b_0'v_0')\right) - u_0\left(y + \frac{\sqrt{2}\varepsilon}{2}(b_0v_0+b_0'v_0')\right),$$

which, in view of the Lipschitz continuity of $u_0$, implies that

$$w^\varepsilon(x,\varepsilon^2) - w^\varepsilon(y,\varepsilon^2) \le L|x-y| \text{ for all } x,y\in\mathbb{R}^2 \text{ and } \varepsilon>0.$$

By applying this argument repeatedly and (7.2.7), we are led to

$$w^\varepsilon(x,t) - w^\varepsilon(y,t) \le L|x-y| \text{ for all } x,y\in\mathbb{R}^2 \text{ and } \varepsilon>0.$$

Then (7.2.8) follows immediately by interchanging the roles of $x$ and $y$. $\square$

We next compare this modified game with the original game. In fact, we can show that the relaxed upper limit

$$\overline{w}(x,t) = \limsup_{\varepsilon\to 0}{}^* w^\varepsilon(x,t) = \lim_{\varepsilon\to 0}\sup\{w^\delta(y,s) : |y-x|+|t-s|<\varepsilon, \delta<\varepsilon\}$$

is a subsolution of (MCF).

**Theorem 7.2.4** (Subsolution)**.** *Assume that $u_0$ satisfies (A). Let $w^\varepsilon$ be the value functions defined as in (7.2.6). Then $\overline{w}$ as defined above is a subsolution of* (7.1.1) *with $\overline{w}(\cdot,0)\le u_0$ in $\mathbb{R}^2$.*

We present a detailed proof of Theorem 7.2.4, since the game setting is quite different from the original games in [78]. The following elementary result, whose proof can be found in [41], is needed in our argument.

**Lemma 7.2.5** (Lemma 4.1 in [41])**.** *Suppose $p$ is a unit vector in $\mathbb{R}^2$ and $X$ is a real symmetric $2 \times 2$ matrix. Then there exists a constant $M > 0$ that depends only on the norm of $X$, such that for any unit vector $\xi \in \mathbb{R}^2$,*

$$|\langle Xp^{\perp}, p^{\perp}\rangle - \langle X\xi, \xi\rangle| \leq M|\langle \xi, p\rangle|, \tag{7.2.9}$$

*where $p^{\perp}$ denotes a unit orthonormal vector of $p$.*

*Remark* 7.2.6. By homogeneity, when $p$ is not necessarily a unit vector, the relation (7.2.9) still holds for all $\xi \in \mathbb{R}^2$ provided that $p^{\perp}$ is orthogonal to $p$ and $|p^{\perp}| = |p| = |\xi|$.

*Proof of Theorem 7.2.4.* We first show that $\overline{w}$ is a subsolution of (7.1.1). Let us assume that there exist $(x_0, t_0) \in Q = \mathbb{R}^2 \times (0, \infty)$ and $\phi \in C^2(Q)$ such that

$$\overline{w}(x,t) - \phi(x,t) < \overline{w}(x_0, t_0) - \phi(x_0, t_0) \text{ for all } (x,t) \in Q.$$

Then by definition of $\overline{w}$, there exists $(x_\varepsilon, t_\varepsilon) \in Q$ such that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$, $w^\varepsilon(x_\varepsilon, t_\varepsilon) \to \overline{w}(x_0, t_0)$, and

$$(w^\varepsilon - \phi)(x_\varepsilon, t_\varepsilon) \geq (w^\varepsilon - \phi)(x,t) - \varepsilon^3$$

for any $(x,t)$ in a neighborhood of $(x_0, t_0)$ with size independent of $\varepsilon$. By (7.2.7), we then have

$$\phi(x_\varepsilon, t_\varepsilon) \leq \max_{v,v' \in \mathbb{S}^1} \min_{b,b'=\pm 1} \phi\left(x_\varepsilon + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v'), t_\varepsilon - \varepsilon^2\right) + \varepsilon^3.$$

Applying Taylor expansion, we are led to

$$\begin{aligned} \varepsilon^2 \phi_t \leq \max_{v,v' \in \mathbb{S}^1} \min_{b,b'=\pm 1} & \left\{ \frac{\sqrt{2}\varepsilon}{2} \langle \nabla\phi, bv + b'v'\rangle \right. \\ & \left. + \frac{\varepsilon^2}{4}\langle \nabla^2\phi(bv + b'v'), (bv + b'v')\rangle \right\} + o(\varepsilon^2) \text{ at } (x_\varepsilon, t_\varepsilon). \end{aligned} \tag{7.2.10}$$

We denote by $I$ the right hand side evaluated at $(x_\varepsilon, t_\varepsilon)$.

One may first compare $\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v\rangle$ and $\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v'\rangle$ to determine $b$ or $b'$. For example, if

$$|\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v\rangle| \leq |\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v'\rangle|, \tag{7.2.11}$$

then we may choose $b'_0$ such that

$$-|\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v'\rangle| = \langle \nabla\phi(x_\varepsilon, t_\varepsilon), b'_0 v'\rangle$$

and thus $\langle \nabla\phi, bv + b'_0 v'\rangle \le 0$ for either $b = \pm 1$ by (7.2.11). Therefore we have at $(x_\varepsilon, t_\varepsilon)$

$$I \le \max_{v,v'} \min_{b} \left\{ -\frac{\sqrt{2}\varepsilon}{2} |\langle \nabla\phi, bv + b'_0 v'\rangle| + \frac{\varepsilon^2}{4} \langle \nabla^2\phi(bv + b'_0 v'), (bv + b'_0 v')\rangle \right\} + o(\varepsilon^2). \quad (7.2.12)$$

In what follows, we always assume (7.2.11) and keep the choice $b' = b'_0$.

**Case A.** Assume $\nabla\phi(x_0, t_0) \neq 0$, which in turn implies that $\nabla\phi(x_\varepsilon, t_\varepsilon) \neq 0$ for all $\varepsilon > 0$ small. We apply Lemma 7.2.5 and Remark 7.2.6 with

$$p = |bv + b'_0 v'| \frac{\nabla\phi(x_\varepsilon, t_\varepsilon)}{|\nabla\phi(x_\varepsilon, t_\varepsilon)|}, \quad \xi = bv + b'_0 v', \quad X = \nabla^2\phi(x_\varepsilon, t_\varepsilon),$$

and get

$$\begin{aligned} &\left| \langle \nabla^2\phi(bv + b'_0 v'), (bv + b'_0 v')\rangle - |bv + b'_0 v'|^2 \left\langle \nabla^2\phi \frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|} \right\rangle \right| \\ &\le \frac{M|bv + b'_0 v'|}{|\nabla\phi|} |\langle \nabla\phi, bv + b'_0 v'\rangle| \end{aligned}$$

for any $v, v' \in \mathbb{S}^1$. Hence, we have from (7.2.12)

$$I \le \varepsilon^2 \Delta_1^G \phi + \varepsilon\Phi + o(\varepsilon^2) \ \text{ at } (x_\varepsilon, t_\varepsilon), \quad (7.2.13)$$

where we denote

$$\Delta_1^G \phi = \left\langle \nabla^2\phi \frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|} \right\rangle = |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right),$$

and

$$\begin{aligned} \Phi(x,t) = \max_{v,v'} \min_{b} \Bigg\{ &-\frac{\sqrt{2}}{2} |\langle \nabla\phi(x,t), bv + b'_0 v'\rangle| + \varepsilon \left( \frac{|bv + b'_0 v'|^2}{4} - 1 \right) \Delta_1^G \phi(x,t) \\ &+ \frac{\varepsilon M}{4|\nabla\phi(x,t)|} |bv + b'_0 v'| |\langle \nabla\phi(x,t), bv + b'_0 v'\rangle| \Bigg\}. \end{aligned} \quad (7.2.14)$$

Here we denote for any function $\phi(x,t) \in C^1(\mathbb{R}^2)$ with $x = (x_1, x_2)$,

$$\nabla^\perp\phi = (-\partial\phi/\partial x_2, \partial\phi/\partial x_1).$$

We next estimate $\Phi(x_\varepsilon, t_\varepsilon)$ for $\varepsilon > 0$ sufficiently small. Indeed, we discuss two different cases. If there exists a subsequence $\varepsilon_k$ such that the maximum attains at $v$ and $v'$ (depending on $\varepsilon_k$) and as $k \to \infty$,

$$\text{either } |\langle \nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}), v\rangle| \to c \text{ or } |\langle \nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}), v'\rangle| \to c$$

for some $c > 0$, then it is clear that there exists $b = \pm 1$ also depending on $\varepsilon_k$ satisfying

$$|\langle \nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}), bv + b_0'v'\rangle| > c,$$

since

$$\begin{aligned}\max_b |\langle \nabla\phi(x,t), bv + b_0'v'\rangle| &\geq \frac{1}{2}(|\langle \nabla\phi(x,t), v + b_0'v'\rangle| + |\langle \nabla\phi(x,t), -v + b_0'v'\rangle|) \\ &= \max\{|\langle \nabla\phi(x,t), v\rangle|, |\langle \nabla\phi(x,t), b_0'v'\rangle|\}.\end{aligned}$$

In view of (7.2.14), this yields immediately that

$$\limsup_{\varepsilon_k \to 0} \Phi(x_{\varepsilon_k}, t_{\varepsilon_k}) \leq -\frac{\sqrt{2}c}{2}.$$

The remaining case is that, for any $\varepsilon > 0$ small, the maximum of $\Phi$ is attained at $v$ and $v'$ such that

$$|\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v\rangle| \to 0 \text{ and } |\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v'\rangle| \to 0 \text{ as } \varepsilon \to 0.$$

We denote $\eta_\varepsilon = \nabla\phi(x_\varepsilon, t_\varepsilon)/|\nabla\phi(x_\varepsilon, t_\varepsilon)|$ and $\eta_\varepsilon^\perp = \nabla^\perp\phi(x_\varepsilon, t_\varepsilon)/|\nabla\phi(x_\varepsilon, t_\varepsilon)|$. Let us take a modulus of continuity $\omega$ such that

$$\max\{|\langle \eta_\varepsilon, v\rangle|, |\langle \eta_\varepsilon, v'\rangle|\} \leq \omega(\varepsilon).$$

Without loss of generality, we may assume

$$\left|b_0'v' - \eta_\varepsilon^\perp\right| \leq \omega(\varepsilon).$$

(Otherwise, we have $|b_0'v' + \eta_\varepsilon^\perp| \leq \omega(\varepsilon)$ and apply the similar argument below.) Then we may pick $b$ such that

$$\left|bv - \eta_\varepsilon^\perp\right| \leq \omega(\varepsilon).$$

It follows that under this choice,

$$\left|\frac{1}{4}|bv + b_0'v'|^2 - 1\right| = \frac{1}{4}|bv - b_0'v'|^2 \le \frac{1}{2}\left(|bv - \eta_\varepsilon^\perp|^2 + |b_0'v' - \eta_\varepsilon^\perp|^2\right) \le \omega^2(\varepsilon).$$

It follows from (7.2.14) that

$$\Phi(x_\varepsilon, t_\varepsilon) \le C\varepsilon\omega^2(\varepsilon) + C\varepsilon\omega(\varepsilon)$$

for some $C > 0$ independent of $\varepsilon$. Hence, in either case, by (7.2.13), we have

$$\varepsilon^2\phi_t \le \varepsilon^2\Delta_1^G\phi + o(\varepsilon^2) \quad \text{at } (x_\varepsilon, t_\varepsilon). \tag{7.2.15}$$

Dividing the inequality by $\varepsilon^2$ and sending $\varepsilon \to 0$, we obtain (7.2.5).

**Case B.** It remains to show the viscosity inequality

$$\phi_t(x_0, t_0) \le 0 \tag{7.2.16}$$

under the conditions that $\nabla\phi(x_0, t_0) = 0$ and $\nabla^2\phi(x_0, t_0) = O$.

<u>Case B-1.</u> Suppose that there exists a subsequence $\nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}) \ne 0$ for $k$ arbitrarily large. Then we may repeat the argument as in Case A and reach (7.2.15) again. We then obtain (7.2.16) immediately, since in the present case $\nabla^2\phi(x_\varepsilon, t_\varepsilon) \to \nabla^2\phi(x_0, t_0) = O$ as $\varepsilon \to 0$.

<u>Case B-2.</u> If $\nabla\phi(x_\varepsilon, t_\varepsilon) = 0$ for all $\varepsilon > 0$, then it follows immediately from (7.2.10) that

$$\varepsilon^2\phi_t(x_\varepsilon, t_\varepsilon) \le C\varepsilon^2\|\nabla^2\phi(x_\varepsilon, t_\varepsilon)\| + o(\varepsilon^2),$$

for some $C > 0$ independent of $\varepsilon$, which yields (7.2.16).

2. We finally show that $\overline{w}(x, 0) \le u_0(x)$. To this end, we construct a game barrier at every fixed $x_0 \in \mathbb{R}^2$. It is easily seen that for any $\delta > 0$, there exits $K_\delta > 0$ such that

$$u_0(x) \le w_0(x) = u_0(x_0) + \delta + K_\delta|x - x_0|^2 \text{ for all } x \in \mathbb{R}^2.$$

We now play the games with objective function $w_0$ starting from any $x$ in a neighborhood of $x_0$. In the first step, we have

$$\left|x - x_0 + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v')\right|^2 = |x - x_0|^2 + \sqrt{2}\varepsilon\langle x - x_0, bv + b'v'\rangle + \frac{\varepsilon^2}{2}|bv + b'v'|^2.$$

Note that for any $v, v'$, Player II can pick $b'$ (or $b$) so that

$$\langle x - x_0, bv + b'v' \rangle \leq 0$$

for any $b$ (or any $b'$). This can be done by fixing $b'$ such that $\langle b'v', x - x_0 \rangle = -|\langle v', x - x_0 \rangle|$ if $|\langle v', x - x_0 \rangle| \geq |\langle v, x - x_0 \rangle|$. Such a strategy yields

$$\left| x - x_0 + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v') \right|^2 \leq 2\varepsilon^2 + |x - x_0|^2$$

for any $v, v'$. Repeating this strategy of Player II regardless of choices by Player I, we get

$$w^\varepsilon(x, t) \leq w_0(x) + 2K_\delta t.$$

This means $\overline{w}(x_0, 0) \leq w_0(x_0) \leq u_0(x_0) + \delta$. We conclude the proof by letting $\delta \to 0$. $\square$

The following comparison result is an immediate consequence of Theorem 7.2.4 and the proof is due to [25, 34].

**Corollary 7.2.7** (Comparison theorem)**.** *Suppose that $u$ is the viscosity solution of (MCF) and $\overline{w}$ is the upper relaxed limit of the game value $w^\varepsilon$ associated to the modified game. Then $\overline{w} \leq u$ in $\mathbb{R}^2 \times (0, \infty)$.*

## 7.3 GAME-THEORETIC PROOFS OF CONVEXITY-PRESERVING PROPERTIES

### 7.3.1 Convexity-preserving for the solutions to the normalized parabolic $p$-Laplace equations

Recall the DPP related to the normalized parabolic $p$-Laplace equations is:

$$\begin{aligned} u^\varepsilon(x,t) = \frac{\alpha}{2} \sup_{v\in B_\varepsilon} u^\varepsilon(x+v, t-\sigma_p\varepsilon^2) + \frac{\alpha}{2} \inf_{v\in B_\varepsilon} u^\varepsilon(x+v, t-\sigma_p\varepsilon^2) \\ + \beta \fint_{v\in B_\varepsilon} u^\varepsilon(x+v, t-\sigma_p\varepsilon^2)\, dv + \varepsilon^2 f(x), \end{aligned} \tag{7.3.1}$$

$$u^\varepsilon(x,t) = u_0(x) \text{ for all } x \in \mathbb{R}^n \text{ and } t < \varepsilon^2. \tag{7.3.2}$$

We next use the game to prove the following.

**Theorem 7.3.1.** *Assume that $u_0$ and $f$ are both convex and Lipschitz continuous in $\mathbb{R}^n$. Suppose that the function $u^\varepsilon$ satisfying (7.3.1)–(7.3.2) converges to the solution $u$ of (PL). Then the solution $u(x,t)$ is convex in $x$ for any $t \geq 0$; namely,*

$$\frac{1}{2}u(x-h,t) + \frac{1}{2}u(x+h,t) \geq u(x,t) \tag{7.3.3}$$

*for any $x, h \in \mathbb{R}^n$ and $t \geq 0$.*

*Proof.* It suffices to show (7.3.3) with $u$ replaced by $u^\varepsilon$. Let us take a look at the first game step:

$$\begin{aligned} u^\varepsilon(x\pm h, \sigma_p\varepsilon^2) = &\frac{\alpha}{2} \sup_{v\in B_\varepsilon} u_0(x\pm h+v) + \frac{\alpha}{2} \inf_{v\in B_\varepsilon} u_0(x\pm h+v) \\ &+ \beta \fint_{v\in B_\varepsilon} u_0(x\pm h+v)\, dv + \varepsilon^2 f(x\pm h), \end{aligned} \tag{7.3.4}$$

We assume that the infimum appearing in each equalities in (7.3.4) above is attained respectively at $v_1, v_2 \in B_\varepsilon$ and also assume that $\sup_{v \in B_\varepsilon} u_0(x+v) = u_0(x+\hat{v})$ for some $\hat{v} \in B_\varepsilon$. (These can be rigorously justified by taking errors into account.) Hence, we have

$$
\begin{aligned}
u^\varepsilon(x+h, \sigma_p \varepsilon^2) \geq\ & \frac{\alpha}{2} u_0(x+h+\hat{v}) + \frac{\alpha}{2} u_0(x+h+v_1) \\
& \qquad + \beta \fint_{v \in B_\varepsilon} u_0(x+h+v)\, dv + \varepsilon^2 f(x+h); \\
u^\varepsilon(x-h, \sigma_p \varepsilon^2) \geq\ & \frac{\alpha}{2} u_0(x-h+\hat{v}) + \frac{\alpha}{2} u_0(x-h+v_2) \\
& \qquad + \beta \fint_{v \in B_\varepsilon} u_0(x-h+v)\, dv + \varepsilon^2 f(x-h).
\end{aligned}
$$

By convexity of $u_0$ and $f$, we get

$$
\begin{aligned}
& u^\varepsilon(x+h, \sigma_p \varepsilon^2) + u^\varepsilon(x-h, \sigma_p \varepsilon^2) \\
& \geq \alpha u_0(x+\hat{v}) + \alpha u_0\left(x + \frac{v_1+v_2}{2}\right) + 2\beta \fint_{v \in B_\varepsilon} u_0(x+v)\, dv + 2\varepsilon^2 f(x) \\
& \geq \alpha \sup_{v \in B_\varepsilon} u_0(x+v) + \alpha \inf_{v \in B_\varepsilon} u_0(x+v) + 2\beta \fint_{v \in B_\varepsilon} u_0(x+v)\, dv + 2\varepsilon^2 f(x).
\end{aligned} \tag{7.3.5}
$$

Since, by definition, the right hand side is just $2u^\varepsilon(x, \sigma_p \varepsilon^2)$, we have

$$
u^\varepsilon(x+h, \sigma_p \varepsilon^2) + u^\varepsilon(x-h, \sigma_p \varepsilon^2) \geq 2u^\varepsilon(x, \sigma_p \varepsilon^2).
$$

We repeat the argument and end up with

$$
u^\varepsilon(x+h, t) + u^\varepsilon(x-h, t) \geq 2u^\varepsilon(x, t)
$$

for any $t \geq 0$. Sending $\varepsilon \to 0$, we get the convexity desired. $\square$

### 7.3.2 Convexity-preserving for level sets of the mean curvature flow equations

For future use, let $D_c^0$ and $E_c^0$ respectively denote the open and closed $c$-superlevel set of $u_0$ for any $c \geq a$; that is,

$$D_c^0 = \{x \in \mathbb{R}^2 : u_0(x) > c\}; \quad E_c^0 = \{x \in \mathbb{R}^2 : u_0(x) \geq c\}.$$

Then we have $D_c^0 \subset D_d^0$ and $E_c^0 \subset E_d^0$ for any $c \geq d$. Moreover, we have $D_c^0 = \bigcup_{d>c} E_d^0$.

**Lemma 7.3.2** (Monotonicity)**.** *Suppose that $u_0$ satisfies (A). Assume that the superlevel set $E_c^0$ for each $c \geq a$ is convex. Let $u^\varepsilon$ be the value function associated to the game. Then*

$$u^\varepsilon(x,t) \leq u^\varepsilon(x,s) \text{ for all } x \in \mathbb{R}^2,\ t \geq s \geq 0 \text{ and } \varepsilon > 0. \tag{7.3.6}$$

*In particular, $u(x,t) \leq u(x,s)$ for all $x \in \mathbb{R}^2$, $t \geq s \geq 0$.*

*Proof.* Let us fix $\varepsilon > 0$. We begin our proof by claiming that for every $x \in \mathbb{R}^2$ and any $v \in \mathbb{S}^1$, there exists $b = \pm 1$ such that

$$u_0\left(x + \sqrt{2}\varepsilon b v\right) \leq u_0(x). \tag{7.3.7}$$

Indeed, assume that for some $v \in \mathbb{S}^1$ and some $\lambda > u_0(x)$,

$$u_0\left(x - \sqrt{2}\varepsilon v\right),\ u_0\left(x + \sqrt{2}\varepsilon v\right) \geq \lambda.$$

By convexity of the superlevel set $E_\lambda^0$, $x = \frac{1}{2}(x - \sqrt{2}\varepsilon v) + \frac{1}{2}(x + \sqrt{2}\varepsilon v) \in E_\lambda^0$ which contradicts the assumption that $u_0(x) < \lambda$.

Now let us incorporate the claim in our games. Notice that for any game position $z(s;x)$ from an arbitrary $x \in \mathbb{R}^2$ after time $s > 0$, depending on the choices $v_1, b_1, v_2, b_2, \ldots, v_N, b_N$ of both players ($N = [s/\varepsilon^2]$), Player II may use the strategy described above to ensure

$$\max_{v \in \mathbb{S}^1} \min_{b = \pm 1} u_0\left(z(s;x) + \sqrt{2}\varepsilon b v\right) \leq u_0(z(s;x)) \text{ for any } v \in \mathbb{S}^1.$$

By applying this strategy from step $N$ to $N' = [t/\varepsilon^2]$, we obtain

$$\max_{v_{N+1}} \min_{b_{N+1}} \ldots \max_{v_{N'}} \min_{b_{N'}} u_0\left(z(s;x) + \sqrt{2}\varepsilon \sum_{i=N+1}^{N'} b_i v_i\right) \leq u_0(z(s;x)).$$

Taking the extrema on both sides of the inequality above over $b_N$, $v_N$, $b_{N-1}$, $v_{N-1}$, $\ldots$, $b_1$ and $v_1$ in order, we get (7.3.6) by definition. It follows immediately that $u(x,t) \leq u(x,s)$ for all $t \geq s$, by passing to the limit as $\varepsilon \to 0$ in (7.3.6) with application of Theorem 7.2.1.

$\square$

**Theorem 7.3.3** (Convexity of level sets)**.** *Suppose that $u_0$ satisfies (A). Assume that each superlevel set $E^0_c$ ($c \geq a$) of $u_0$ is convex. Let $u^\varepsilon$ be the value function associated to the game. Then every superlevel set of $u^\varepsilon(\cdot,t)$ is almost convex for any $t \geq 0$ and $\varepsilon > 0$ in the sense that there exists a modulus $\omega$ of continuity depending on $u_0$ such that for any $x, y \in \mathbb{R}^2$ and $t \geq 0$,*

$$u^\varepsilon\left(\frac{x+y}{2}, t\right) \geq c - \omega(\varepsilon) \tag{7.3.8}$$

*provided that $u^\varepsilon(x,t) \geq c$ and $u^\varepsilon(y,t) \geq c$. In particular, all superlevel sets of the solution $u(\cdot,t)$ of (MCF) are convex for every $t \geq 0$.*

*Proof of Theorem 7.3.3.* We assume $x \neq y$, since otherwise the statements are trivial. Since $u^\varepsilon(x,t) \geq c$ and $u^\varepsilon(y,t) \geq c$, we have $u^\varepsilon(x,s) \geq c$ and $u^\varepsilon(y,s) \geq c$ for $s \leq t$ as well in virtue of the monotonicity shown in Lemma 7.3.2. (In particular, $u_0(x) \geq c$ and $u_0(y) \geq c$.)

Then for any $s \leq t$, there must exist maximizing strategies $S^I_{x,s}$ and $S^I_{y,s}$ of Player I such that regardless of the choices of Player II, we have $u_0(z(s;x)) \geq c$ and $u_0(z(s;y)) \geq c$ if $S^I_{x,s}$ and $S^I_{y,s}$ are applied respectively in the games starting from $x$ and $y$.

We next consider the game started from $(x+y)/2$. In this case, Player I has the following possible of move: he keeps choosing $v = (x-y)/|x-y|$ until the game position enters $B_{\sqrt{2}\varepsilon}(x)$ or $B_{\sqrt{2}\varepsilon}(y)$. Here $B_r(\xi)$ denotes the open ball centered at $\xi \in \mathbb{R}^n$ with radius $r$. Without loss of generality, suppose that Player II chooses to let $z(\tau;(x+y)/2) \in B_{\sqrt{2}\varepsilon}(x)$ after time $\tau(\leq t)$. Then Player I may use $S^I_{x,s}$ with $s = t - \tau$ to bring the game position to $\xi \in \mathbb{R}^n$, which depends on the response of Player II to $S^I_{x,s}$. However, the same strategy of Player II may send $x$ to $z(s;x)$, which is in the $\sqrt{2}\varepsilon$-neighborhood of $\xi$. Since $u_0(z(s;x)) \geq c$, we get the following estimate:

$$u_0(\xi) \geq u_0(z(s;x)) - \omega_0(\sqrt{2}\varepsilon) \geq c - \omega_0(\sqrt{2}\varepsilon).$$

The remaining case is that Player II may choose to let the game position wander away from the neighborhoods of $x$ and $y$. But in this case the final position $\eta$ must still stay on the line segment between $x$ and $y$ and therefore

$$u_0(\eta) \geq c,$$

due to the facts that $u_0(x) \geq c$ and $u_0(y) \geq c$ and the assumption that superlevel sets of $u_0$ are convex.

Since each of game outcomes is just for one possible strategy of Player I and his optimal strategy should be even better, we end up with (7.3.8) with $\omega(x)$ set to be $\omega_0(\sqrt{2}x)$.

Now in order to prove the statement concerning $u$, we only need to take the limit using Theorem 7.2.1. More precisely, for any $\delta > 0$, there exists $\varepsilon > 0$ such that

$$u^\varepsilon(x,t) \geq c - \delta \text{ and } u^\varepsilon(y,t) \geq c - \delta.$$

Our argument above yields

$$u^\varepsilon\left(\frac{x+y}{2}, t\right) \geq c - \delta - \omega(\varepsilon).$$

We conclude the proof by letting $\delta \to 0$. □

*Remark* 7.3.4. In the theorem above, we do not assume the concavity of $u_0$ itself but the convexity of its level sets. Our convexity result is therefore only for the convexity of level sets as well. Also, to study convexity of a particular level, it is not restrictive to assume that all level sets of $u_0$ are convex curves. Note that changing the other level sets of $u_0$ will not affect the evolution of the particular level set in question [34, 25, 38].

*Remark* 7.3.5. An alternative method, without assuming the convexity of all level sets but strict convexity of the initial level set, is to investigate the convexity of the level sets of the solution to (SP) with $\Omega$ (strictly) convex. The stationary problem has a similar game-theoretic approximation [78] but the value function $u^\varepsilon(x)$ in this case is defined to be the optimized first exit time of the domain starting from $x \in \Omega$. It is not difficult to see that the argument in the proof of Theorem 7.3.3 applies exactly to this case as well. A new PDE proof for this problem was recently given in [12].

### 7.3.3 Convexity-preserving for the solutions to the mean curvature flow equations

We next show the convexity preserving property for the solution itself. Due to an opposite choice of orientation, we here discuss instead the equivalent concavity preserving property, which, under the assumption (A), means that the solution $u(x,t)$ of (MCF) is concave with respect to $x$ in its open superlevel set

$$D_a^t := \{x \in \mathbb{R}^n : u(x,t) > a\}$$

if $u_0$ is concave in $K_a$.

**Theorem 7.3.6** (Concavity preserving of the solution)**.** *Suppose that $u_0$ satisfies (A). Assume that $u_0$ is concave in $K_a$. Let $u^\varepsilon$ and $w^\varepsilon$ be respectively the value functions of the original game and the modified game. Then $u^\varepsilon(x,t)$ satisfies*

$$u^\varepsilon(x+h,t) + u^\varepsilon(x-h,t) \leq 2w^\varepsilon(x,t), \tag{7.3.9}$$

*for any $x, h \in \mathbb{R}^2$, $t > 0$ and $\varepsilon > 0$ if*

$$u^\varepsilon(x+h,t) > a \text{ and } u^\varepsilon(x-h,t) > a. \tag{7.3.10}$$

*Moreover, the solution $u(x,t)$ of (MCF) is concave with respect to $x$ in $D_a^t$ for any $t \geq 0$.*

To show this result, we need the following elementary result related to our games.

**Lemma 7.3.7.** *Let $\Omega$ be an open set in $\mathbb{R}^2$. Suppose that $w, W \in C(\mathbb{R}^2)$ satisfy*

*(1) $w > a$ in $\Omega$ for some $a \in \mathbb{R}$ and $w = a$ on $\mathbb{R}^2 \setminus \Omega$;*

*(2) for all $x, h \in \mathbb{R}^2$ such that $x \pm h \in \Omega$,*

$$w(x+h) + w(x-h) \leq W(x). \tag{7.3.11}$$

*Then for any constant $\lambda > 0$,*

$$\begin{aligned}\max_{v\in\mathbb{S}^1}\min_{b=\pm 1}\ w(x+h+\lambda bv)+\max_{v\in\mathbb{S}^1}\min_{b=\pm 1}\ w(x-h+\lambda bv)\\ \leq \max_{v^1,v^2\in\mathbb{S}^1}\min_{b^1,b^2=\pm 1} W\left(x+\frac{\lambda}{2}(b^1v^1+b^2v^2)\right)\end{aligned} \tag{7.3.12}$$

*provided that*

$$\max_{v\in\mathbb{S}^1}\min_{b=\pm 1}\ w(x+h+\lambda bv)>a \ \textit{ and } \ \max_{v\in\mathbb{S}^1}\min_{b=\pm 1}\ w(x-h+\lambda bv)>a. \tag{7.3.13}$$

*Proof.* We take $v_+$ and $v_-$ such that

$$\max_{v}\min_{b} w(x\pm h+\lambda bv)=\min_{b} w(x\pm h+\lambda bv_\pm)(>a), \tag{7.3.14}$$

Let us next pick $b_\pm$ such that

$$W\left(x+\frac{\lambda}{2}(b_+v_++b_-v_-)\right)=\min_{b^1}\min_{b^2} W\left(x+\frac{\lambda}{2}(b^1v_++b^2v_+)\right),$$

which implies that

$$W\left(x+\frac{\lambda}{2}(b_+v_++b_-v_-)\right)\leq \max_{v^1,v^2}\min_{b^1,b^2} W\left(x+\frac{\lambda}{2}(b^1v^1+b^2v^2)\right). \tag{7.3.15}$$

On the other hand, by (7.3.14), we have

$$\max_{v\in\mathbb{S}^1}\min_{b=\pm 1} w(x\pm h+\lambda bv)\leq w(x\pm h+\lambda b_\pm v_\pm). \tag{7.3.16}$$

Due to (7.3.16) and (7.3.13) we have $x\pm h+\lambda b_\pm v_\pm\in\Omega$, since $w(x\pm h+\lambda b_\pm v_\pm)>a$. We finally obtain (7.3.12) by combining (7.3.11), (7.3.15) and (7.3.16). $\square$

*Proof of Theorem 7.3.6.* Fix $\varepsilon > 0$ and set $\hat{D}_a^t := \{x \in \mathbb{R}^n : u^\varepsilon(x,t) > a\}$. We first apply Lemma 7.3.7 with $\lambda = \sqrt{2}\varepsilon$, $\overline{\Omega} = K_a$ and

$$w(x) = u_0(x) \text{ and } W(x) = 2u_0(x).$$

Since it is clear that (7.3.11) holds due to the concavity of $u_0$ in $K_a$, we get

$$\begin{aligned}&\max_{v\in\mathbb{S}^1}\min_{b=\pm1} w(x+h+\sqrt{2}\varepsilon bv) + \max_{v\in\mathbb{S}^1}\min_{b=\pm1} w(x-h+\sqrt{2}\varepsilon bv)\\ &\leq \max_{v^1,v^2\in\mathbb{S}^1}\min_{b^1,b^2=\pm1} W\left(x+\frac{\sqrt{2}\varepsilon}{2}(b^1v^1+b^2v^2)\right)\end{aligned}$$

if both terms on the left hand side are greater than $a$. This amounts to saying that

$$u^\varepsilon(x+h,\varepsilon^2) + u^\varepsilon(x-h,\varepsilon^2) \leq 2w^\varepsilon(x,\varepsilon^2) \tag{7.3.17}$$

for all $x, h$ provided that $u^\varepsilon(x\pm h,\varepsilon^2) > a$. Noticing that $u^\varepsilon(x,\varepsilon^2)$ and $w^\varepsilon(x,\varepsilon^2)$ are Lipschitz continuous in $x$, due to Proposition 7.2.3 and [78, Appendix B], we can continue using Lemma 7.3.7 with $\lambda = \sqrt{2}\varepsilon$, $\Omega = \hat{D}_a^{\varepsilon^2}$ and

$$w(x) = u^\varepsilon(x,\varepsilon^2) \text{ and } W(x) = 2w^\varepsilon(x,\varepsilon^2).$$

Analogously, we obtain

$$u^\varepsilon(x+h,2\varepsilon^2) + u^\varepsilon(x-h,2\varepsilon^2) \leq 2w^\varepsilon(x,2\varepsilon^2)$$

if $u^\varepsilon(x\pm h,2\varepsilon^2) > a$, thanks to the dynamic programming principle:

$$\begin{aligned}u^\varepsilon(x+h,2\varepsilon^2) &= \max_{v\in\mathbb{S}^1}\min_{b=\pm1} u^\varepsilon(x+h+\sqrt{2}\varepsilon bv,\varepsilon^2);\\ u^\varepsilon(x-h,2\varepsilon^2) &= \max_{v\in\mathbb{S}^1}\min_{b=\pm1} u^\varepsilon(x-h+\sqrt{2}\varepsilon bv,\varepsilon^2);\\ w^\varepsilon(x,2\varepsilon^2) &= \max_{v^1,v^2\in\mathbb{S}^1}\min_{b^1,b^2=\pm1} w^\varepsilon\left(x+\frac{\sqrt{2}\varepsilon}{2}(b^1v^1+b^2v^2),\varepsilon^2\right).\end{aligned}$$

We keep iterating the arguments above and eventually get

$$u^\varepsilon(x+h,N\varepsilon^2) + u^\varepsilon(x-h,N\varepsilon^2) \leq 2w^\varepsilon(x,N\varepsilon^2), \tag{7.3.18}$$

for any $x, h \in \mathbb{R}^n$ satisfying (7.3.10). This is exactly the desired inequality (7.3.9). Since $\varepsilon > 0$ is arbitrary in (7.3.18), by passing to the limits as $\varepsilon \to 0$ and applying Theorem 7.2.1, we get

$$u(x+h,t) + u(x-h,t) \leq 2\overline{w}(x,t)$$

provided that $x \pm h \in D_a^t$. The concavity preserving property for the solution $u$ follows immediately from the comparison that $\overline{w} \leq u$ in Corollary 7.2.7. □

The concavity or convexity preserving property does not precisely hold on the discrete level in general. In other words, one cannot expect in general that

$$u^\varepsilon(x-h,t) + u^\varepsilon(x+h,t) \leq 2u^\varepsilon(x,t)$$

for all $x, h \in \mathbb{R}^2$, $t > 0$ and $\varepsilon > 0$. We give an example to show this.

*Example* 7.3.8. Let $u_0$ be Lipschitz and concave in $\mathbb{R}^2$. Suppose that the level set $\{x \in \mathbb{R}^2 : u_0(x) = 0\}$ consists of the positive axes. We assume that $u_0 > 0$ in the first quadrant. Let $x_0 = \left(\sqrt{2}\varepsilon/2, \sqrt{2}\varepsilon/2\right)$ and $h = (\sqrt{2}\varepsilon/2, -\sqrt{2}\varepsilon/2)$.

When $\varepsilon > 0$ is taken small, it is easily seen that $u^\varepsilon(x_0 \pm h, \varepsilon^2) = 0$, since the maximizing choices of $v$ for Player I to start the game at $x_0 + h$ or $x_0 - h$ can be the ones along the axes. On the other hand, at the point $x_0$, the outcome after one step is always negative no matter what choice Player I makes, i.e., $u^\varepsilon(x_0, \varepsilon^2) < 0$. We therefore have

$$u^\varepsilon(x_0+h,\varepsilon^2) + u^\varepsilon(x_0-h,\varepsilon^2) \geq 2u^\varepsilon(x_0,\varepsilon^2),$$

although $u_0$ is concave. One may easily modify this example to have $u_0$ also satisfy (A).

### 7.3.4 Convexity-preserving with Neumann boundary condition of the mean curvature flow equations

Our game-theoretic approach to convexity can be extended to the Neumann problems as well. In this section, we assume that $\Omega$ is a smooth bounded convex domain in $\mathbb{R}^2$ and $\nu(x)$ denote the unit outward normal to $\partial\Omega$. We consider the Neumann boundary problem for the level set curvature flow equation in $\Omega$:

$$\text{(NP)} \quad \begin{cases} u_t - |\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \Omega \times (0, \infty), \quad (7.3.19) \\ \nabla u(x,t) \cdot \nu(x) = 0 & \text{for } x \in \partial\Omega \text{ and } t > 0, \quad (7.3.20) \\ u(x,0) = u_0(x) & \text{for all } x \in \overline{\Omega}. \quad (7.3.21) \end{cases}$$

Hereafter we assume an analogue of (A):

(A1) $u_0$ is a bounded Lipschitz continuous function in $\overline{\Omega}$ satisfying that $u_0 \geq a$ and $u_0 - a$ has compact support $K_a \subset \overline{\Omega}$ for some constant $a \in \mathbb{R}$.

We refer [43, 109] for existence and uniqueness for viscosity solutions of this problem.

**7.3.4.1 Billiard games for the Neumann condition** We first recall a billiard game interpretation for this problem in [42]. For any $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbb{S}^1$, let $S^t(x,v) \in \overline{\Omega}$ denote the position of billiard motion starting from $x$ with initial direction $v$ and with distance of travel equal to $t$; see [42, Definition 2.1].

In view of [42, Lemma 2.3], we have

$$S^t(x,v) = x + tv - \alpha^t(x,v), \tag{7.3.22}$$

where $\alpha^t(x,v) \in \mathbb{R}^2$ is called a boundary adjustor, satisfying $|\alpha^t(x,v)| \leq 2t$. More precisely, we may write

$$\alpha^t(x,v) = \sum_{k=0}^{\infty} c_k \nu(y_k), \tag{7.3.23}$$

with $c_k \in \mathbb{R}$, $y_k \in \partial\Omega \cap B_t(x)$.

We apply this billiard dynamics to our game setting by slightly changing the game rules introduced in Section 7.2.2. We restrict the starting point $x$ to be in $\overline{\Omega}$ and substitute the rule (3) with the following:

(3)' The marker is moved from the present state $x$ to $S^{\sqrt{2}\varepsilon}(x, bv)$.

We may define the value function $u^\varepsilon$ to be in the same form of (7.2.3) by establishing a new state equation:

$$\begin{cases} z_{k+1} = S^{\sqrt{2}\varepsilon}(z_k, b_k v_k), \quad k = 0, 1, \ldots, N-1; \\ z_0 = x, \end{cases} \tag{7.3.24}$$

where $v_k \in \mathbb{S}^1$ and $b_k = \pm 1$.

**Theorem 7.3.9.** *[42, Theorem 1.1] Assume that $\Omega$ is a smooth bounded convex domain in $\mathbb{R}^2$. Let $u^\varepsilon$ be the value function associated to the game above. Then $u^\varepsilon$ converges, as $\varepsilon \to 0$, to the unique viscosity solution of (NP) uniformly on compact subsets of $\overline{\Omega} \times [0, \infty)$.*

*Remark* 7.3.10. There is more than one way to generate the boundary condition of Neumann type in games. One may simply adopt direct constraints of game trajectories in the domain to establish tug-of-war game interpretations of Neumann boundary problem for the infinity Laplacian in [23, 6]. Their method relies on the special structure of the infinity Laplacian and is not applicable in our present case. A different but more relevant approach is presented in [29] for the Neumann problems for a general class of elliptic and parabolic equations without assuming the boundedness and convexity of the domain, where the author uses boundary projection instead of our billiard motion. The game convergence result in [29] does not include (NP), but it is possible to adapt the argument to our case.

**7.3.4.2 Convexity of level sets** As in the preceding sections, we take for any $t \geq 0$

$$D_c^t = \{x \in \overline{\Omega} : u(x,t) > c\}, E_c^t = \{x \in \overline{\Omega} : u(x,t) \geq c\} \text{ and } \Gamma_c^t = E_c^t \setminus D_c^t.$$

We aim to show that for any fixed $c \geq a$, $E_c^0$ being convex implies that $E_c^t$ is convex for any $t \geq 0$. We assume

(A2) $E_c^0$ is convex for any $c \geq a$.

It is clear that under the assumption (A2), for any $x \in \overline{\Omega}$ with $u_0(x) = c \geq a$, there exists a supporting line $L_x$ of $E_c^0$ passing through $x$; that is, $L_x$ divides $\mathbb{R}^2$ into two half planes, only one of which has nonempty intersection with $D_c^0$. We denote by $\xi(x)$ the outward unit normal to the half plane containing $D_c^0$.

We will also use a compatibility condition as given below:

**Definition 7.3.11** (Compatibility condition)**.** For any $u_0$ satisfying (A2), we say $u_0$ is (weakly) compatible with the Neumann boundary condition (7.3.20) if for any $\tau > 0$ small and $x_0 \in \partial\Omega$ there exists a supporting line $L_{x_0}$ with normal $\xi(x_0)$ satisfying

$$u_0(S^\tau(x_0, v)) \leq u_0(x_0) \tag{7.3.25}$$

for all $v \in \mathbb{S}^1$ with $\langle v, \xi(x_0)\rangle \geq 0$.

**Theorem 7.3.12** (Convexity preserving for the Neumann problem)**.** *Assume that $\Omega$ is a smooth bounded convex domain in $\mathbb{R}^2$. Let $u$ be the unique viscosity solution of (NP) with $u_0$ satisfying (A1) and (A2). Assume that $u_0$ also satisfies the compatibility condition as in Definition 7.3.11. Then the superlevel sets $E_c^t$ of $u(\cdot, t)$ for any $t \geq 0$ are convex for any $t \geq 0$.*

In order to prove this theorem, we again need a monotonicity result similar to Lemma 7.3.2.

**Lemma 7.3.13** (Monotonicity for the Neumann problem)**.** *Suppose that $u_0$ satisfies (A1) and (A2). Assume that $u_0$ also satisfies the compatibility condition. Let $u^\varepsilon$ be the value function associated to the billiard game described above. Then*

$$u^\varepsilon(x,t) \leq u^\varepsilon(x,s) \text{ for all } x \in \mathbb{R}^n,\ t \geq s \geq 0 \text{ and } \varepsilon > 0. \tag{7.3.26}$$

*In particular, the solution $u$ is monotone in time, i.e., $u(x,t) \leq u(x,s)$ for all $x \in \mathbb{R}^n$, $t \geq s \geq 0$.*

*Proof.* Let us fix $\varepsilon > 0$. We claim this time that for every $x \in \mathbb{R}^n$,

$$\sup_{v\in\mathbb{S}^1} \inf_{b=\pm 1} u_0\left(S^{\sqrt{2}\varepsilon}(x, bv)\right) \leq u_0(x). \tag{7.3.27}$$

Indeed, for any $v \in \mathbb{S}^1$, one may pick $\hat{b} = \pm 1$ such that $\langle \xi(x), \hat{b}v\rangle \geq 0$. If the billiard motion does not touch $\partial\Omega$ in the initial direction $\hat{b}v$, then by convexity of the superlevel set with level $u_0(x)$, we get

$$u_0\left(S^{\sqrt{2}\varepsilon}(x, \hat{b}v)\right) \leq u_0(x).$$

In the case that the billiard trajectory hits $\partial\Omega$, we have

$$S^{\sqrt{2}\varepsilon}(x, \hat{b}v) = S^{\sqrt{2}\varepsilon-\tau}(y, \hat{b}v), \tag{7.3.28}$$

where $y = x + \tau\hat{b}v \in \partial\Omega$ denotes the first hitting position with $\tau \geq 0$. For the same reason as above, $y$ must belong to the lower level of $u_0$, i.e., $u_0(y) \leq u_0(x)$. We next apply the compatibility condition to get

$$u_0(S^{\sqrt{2}\varepsilon-\tau}(y, \hat{b}v)) \leq u_0(y),$$

which by (7.3.28) yields

$$u_0(S^{\sqrt{2}\varepsilon}(x, \hat{b}v)) \leq u_0(x).$$

The rest of the proof, similar to that in the proof of Lemma 7.3.2 consists in the iteration of (7.3.27) and passing to the limit as $\varepsilon \to 0$ with application of Theorem 7.3.9. We omit it and refer the reader to the proof of Lemma 7.3.2. □

The proof of Theorem 7.3.12 is essentially the same as that of Theorem 7.3.3. However, in the proof of Theorem 7.3.3 we used the Lipschitz continuous dependence of game strategies on the initial positions and directions, which is not clear in the current case for the billiard motion. We thus provide a slightly different proof by choosing a subsequence of game values. This proof also works for Theorem 7.3.3.

*Proof of Theorem 7.3.12.* We take $x, y \in \overline{\Omega}$ with $x \neq y$ and $u(x,t), u(y,t) \geq c$ for some $t \geq 0$ and $c \geq a$. We aim to show that $u(\frac{x+y}{2}, t) \geq c$. We set $\varepsilon_m = \frac{|x-y|}{2\sqrt{2}m}$ for any positive integer $m$. In view of Theorem 7.3.9, for any $\delta > 0$ small, there exists $m$ sufficiently large, such that

$$u^{\varepsilon_m}(x,t), u^{\varepsilon_m}(y,t) \geq c - \delta.$$

We then have $u^\varepsilon(x,s) \geq c - \delta$ and $u^\varepsilon(y,s) \geq c - \delta$ for $s \leq t$ as well due to Lemma 7.3.13.

Then for any $s \leq t$, there must exist maximizing strategies $S^I_{x,s}$ and $S^I_{y,s}$ of Player I such that regardless of the choices of Player II, we have $u_0(z(s;x)) \geq c-2\delta$ and $u_0(z(s;y)) \geq c-2\delta$ if $S^I_{x,s}$ and $S^I_{y,s}$ are applied respectively in the games starting from $x$ and $y$.

We next consider the game started from $(x+y)/2$. If Player I keeps choosing $v = (x-y)/|x-y|$ until the game position reaches $x$ or $y$. Without loss of generality, suppose that Player II chooses to let $z(\tau; (x+y)/2) = x$ after time $\tau(\leq t)$. Then Player I may use $S^I_{x,s}$ with $s = t - \tau$ to bring the game position to $\xi \in \mathbb{R}^n$, which depends on the response of Player II to $S^I_{x,s}$. This yields

$$u_0(z(t; (x+y)/2)) = u_0(z(s; x)) \geq c - 2\delta.$$

Player II may choose to let the game position wander away from the neighborhoods of $x$ and $y$. In this case the final position $\eta$ must still stay on the line segment between $x$ and $y$ and therefore

$$u_0(\eta) \geq c - 2\delta,$$

due to the assumption that superlevel sets of $u_0$ are convex.

Since the above game estimate is for a fixed strategy of Player I, we get

$$u^{\varepsilon_m}\left(\frac{x+y}{2}, t\right) \geq c - 2\delta.$$

Thanks to Theorem 7.3.9, we conclude the proof by passing to the limit as $m \to \infty$ and then $\delta \to 0$. □

*Remark* 7.3.14. As mentioned in Remark 7.3.10, it is possible to have a game approximation result without the boundedness and convexity assumptions on $\Omega$, following [29]. Therefore one may also expect that the convexity preserving property still holds without assuming the boundedness and convexity of $\Omega$. In fact, without convexity of $\Omega$, we can use the same argument to prove that $E^t_c$ preserves convexity relative to $\Omega$ under the assumption (A1), (A2) and the compatibility condition. To be more precise, we have

$$u\left(\frac{x+y}{2}, t\right) \geq c$$

whenever $x, y \in \overline{\Omega}$ and $t \geq 0$ satisfy $u(x,t), u(y,t) \geq c$ and $kx + (1-k)y \in \overline{\Omega}$ for all $k \in [0,1]$.

We finally make some remarks on the compatibility condition in Definition 7.3.11. Let us discuss a smooth special case. Suppose $u_0$ is of class $C^2$ and concave in $\overline{\Omega}$. Then $u_0$ is compatible with the Neumann boundary condition if there is $\sigma > 0$ such that

$$\langle \nabla u_0(x_0), \nu(y)\rangle \geq 0 \text{ and } \nabla^2 u_0(x_0) \leq -\sigma I. \tag{7.3.29}$$

for any $x_0 \in \partial\Omega$ and any $y \in B_\sigma(x_0) \cap \partial\Omega$.

Indeed, in this case one may choose $\xi(x_0) = -\nabla u_0(x_0)/|\nabla u_0(x_0)|$. To simplify notation, we write $\xi_0$ instead of $\xi(x_0)$. For any $v \in \mathbb{S}^1$ with $\langle v, \xi_0\rangle \geq 0$, we write

$$v = \tau_1 \xi_0 + \tau_2 \xi_0^\perp,$$

where $\xi_0^\perp$ is the unit vector orthogonal to $\xi_0$ and $\tau_1 \geq 0, \tau_2 \in \mathbb{R}$ satisfy $\tau_1^2 + \tau_2^2 = \tau^2$. We apply Taylor expansion to get

$$\begin{aligned} u_0(S^\tau(x_0, v)) &= u_0(x_0) + \tau_1 \langle \nabla u_0(x_0), \xi_0\rangle - \langle \nabla u_0(x_0), \alpha^\tau(x_0, v)\rangle \\ &\quad + \frac{1}{2}\langle \nabla^2 u_0(x_0)(\tau v - \alpha^\tau(x_0, v)), (\tau v - \alpha^\tau(x_0, v))\rangle + o(|\tau v - \alpha^\tau(x_0, v)|^2), \end{aligned}$$

which, due to (7.3.29) and (7.3.23), implies (7.3.25) for $\tau$ sufficiently small.

It is worth pointing out that the curvature flow $\Gamma_t$ fails to preserve convexity in general if the compatibility condition as in Definition 7.3.11 is not satisfied.

*Example* 7.3.15. Consider the special case when $\Omega = (-1, 1) \times \mathbb{R}$ and $\Gamma_t$ can be expressed as the graph of a function $y = v(x, t)$ for $(x, t) \in [-1, 1] \times [0, \infty)$, we deduce

$$\begin{cases} v_t - \dfrac{v_{xx}}{1 + v_x^2} = 0 & \text{in } (-1, 1) \times (0, \infty), \qquad (7.3.30) \\ v_x(-1, t) = v_x(1, t) = 0 & \text{for all } t > 0, \qquad (7.3.31) \\ v(x, 0) = v_0(x) & \text{for all } x \in [-1, 1]. \qquad (7.3.32) \end{cases}$$

It is known [87, 9, 10] that there still exists a unique viscosity solution of this problem for any Lipschitz continuous function even when $v_0$ does not fulfill the compatibility condition. Suppose that $v_0$ is an even Lipschitz function on $[-1, 1]$ with

$$(v_0)_x(1) = -(v_0)_x(-1) > 0.$$

Then for any $t \geq 0$, the unique solution $v(x, t)$ must also be even in $x$.

On the other hand, one may extend $v_0$ to a Lipschitz function on $\mathbb{R}$ in a periodic manner and then solve the corresponding Cauchy problem in $\mathbb{R} \times [0, \infty)$. It is clear that the solution is still space periodic for any time. Moreover, it is shown by Ecker and Huisken [32] that the solution $\tilde{v}(x, t)$ is smooth for any $t > 0$, which implies that $\tilde{v}_x(\pm 1, t) = 0$ and $\tilde{v}$ cannot be convex around $x = \pm 1$. This means that the restriction of $\tilde{v}$ on $[-1, 1] \times [0, \infty)$ is the unique solution of (7.3.30)–(7.3.32). The failure of being convex near $x = \pm 1$ remains with such a restriction.

For the same reason as explained above, one cannot in general expect the viscosity solution of (NP) itself to be convexity preserving without assuming a more restrictive compatibility condition.

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