ESSAYS ON INFORMATION ECONOMICS

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This dissertation comprises three essays on information economics. I study the role of information in various decision making environments.

In the first chapter, I propose an alternative way to study the value of information in a game. A decision problem is similar to another if the optimal decision rule for the latter, when applied to the former, is better than making a decision without any information in the former. In a game, if the induced decision problem by a change in the strategies of other players is similar to the problem originally faced by the player, the player benefits more from her own information after the change. Using the concept of similarity, I study the value of information in various games, even when a closed form solution is unavailable.

The second chapter studies a persuasion game between a decision maker (DM) and an expert. Prior to the communication stage, the expert exerts costly effort to obtain decisive information about the state of nature. The expert may feign ignorance but cannot misreport. We show that monitoring of information acquisition hampers the expert's incentives to acquire information. Contrary to everyday experiences, monitoring is always suboptimal if the expert's bias is large, yet sometimes optimal if the expert's bias is small.

The third chapter studies a model in which partisan voting is rationalized by Knightian decision theory under uncertainty (Bewley, 2002). When uncertainty is large, some voters become hard-core supporters of their current party due to status quo bias. I characterize equilibria of the model that are robust to electorate size. With costly information acquisition, partisan behaviors arise naturally from status quo biases in large elections. In the selected informative voting equilibrium, swing voters rationally mix between two alternatives: either

they acquire information and vote informatively or they do not acquire information and vote to cancel the partisans' votes.

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PREFACE

Six years of Ph.D. studies is a long journey and I am grateful that I was accompanied by great teachers, peers, and friends throughout this endeavor. Given the depth of my indebtedness, a few words of gratitude can scarcely do them justice, but I will give it a try.

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1.0 SIMILARITY AND VALUE OF INFORMATION

1.1 INTRODUCTION

Good information enables good decisions. Blackwell's order formalizes this intuition, and ranks different information structures for a decision maker. However, the value of information in a game depends not only on how one plays the game according to his information, but also on how the others play the game. In this paper, we study the value of information in games. Our approach is to view the best-responding problem in games as an individual decision problem. In an individual decision problem, the value of information is how much the decision maker can gain by making an optimal decision based on the information. We introduce a concept called similarity, and show that the concept of similarity is closely related to the value of information in individual decision problems. We then apply the notion of similarity to games. We study the complementarity and substitutability of information in quadratic games with information structures that are not studied in the literature. Moreover, we investigate the value of information in a global game and a multi-expert persuasion game.

Most of the studies that explore the value of information in games focus on tractable but specific models, e.g., a normal-quadratic setting. This approach provides interesting and powerful insights in many economic environments, but the need to solve the model and to arrive at an explicit solution severely limits our ability to prove general results and casts doubt on the applicability of our insights to general environments. In this paper, we view the best-responding problem in games as an individual decision problem, and therefore the comparison of the value of information in different equilibria as a comparison of different payoff functions for a particular information structure.

In a game, we can define an interim payoff function based on the strategies of other players. When other players change their strategies, a new interim payoff function can be defined. Therefore, a change in the value of one's information induced by the changes in the other players' strategies can be viewed as a change in the value of own information induced by the changes in the interim payoff function. For a particular information structure, we can determine how the changes in the strategies of other players change the value of information in that game, if we have an ordering on the interim payoff functions.

In order to find such an ordering, we develop a partial order on payoff functions for a fixed information structure in an individual decision-making environment. Consider two decision problems with different payoff functions but the same information structure. In one decision problem, if the use of the optimal decision rule of another problem is better than knowing nothing, we say that the former is similar to the latter. The implication of a similar relation between these two decision problems is that the value of information in a grand problem that combines the two is certainly higher than that in the former (Proposition 1.2). Moreover, when the default actions are the same in these two decision problems, if one of these two problems is not similar to the grand problem, then the value of information in the grand problem is lower than that in the other (Proposition 1.3).

Using these two results, we explore games with payoff externality and apply the concept of similarity to study a change in value of information caused by the change of the other players' strategies. We define an *original* decision problem based on the interim payoff function before the change in the strategies of other players, and a *new* decision problem after the change. We then construct a *marginal* decision problem based on the difference between the interim payoff functions after and before the change of the strategies of other players. If the marginal decision problem is similar to the original problem, then the change in strategies of other players increases the value of information for the player, according to Proposition 1.2. If the marginal decision problem is not similar to the new problem and the default actions are the same in the original problem and the marginal problem, then the change in strategies of other players reduces the value of information for the player, according to Proposition 1.3.

We apply such a construction to quadratic games, a global game, and a multi-expert persuasion game. We find conditions under which a player's benefit from having information about the state of the world is higher in one equilibrium than in another. The condition is the similarity of the problem induced by the change in other players' strategies from one equilibrium to another equilibrium to the problem induced by one of the two equilibria. In quadratic games, we find conditions on information structures under which we are able to draw an unambiguous comparison. We first establish results for affine information structure, which is assumed in most of the papers that study quadratic games. We find that complementarity in actions translates into complementarity in information, while the substitutability in actions translates into substitutability in information when the complementarity between action and state is strong enough (Proposition 1.4). Then we extend our results to monotone information structure (Proposition 1.5), with which a closed form solution is in general not available and therefore does not receive much attention in the existing literature. In a global game, we discuss how to compare the value of information in two different equilibria. We show that investors find their information less valuable when the original market condition is good (bad) and becomes even better (worse) (Proposition 1.6). In a multi-expert persuasion game, we study how the presence of another strategic expert changes the existing expert's value of information. We find that information is substitute when experts have opposite extreme biases (Proposition 1.7).

1.1.1 Related Literature

A sizable literature studies the value of information in individual decision problems. The literature focuses on the ordering of information structures for a family of payoff functions (i.e., Blackwell, 1951, 1953; Lehman, 1988; Athey and Levin, 2000; Persico, 2000). The present paper differs from this strand of literature in the objects the ordering is on. In this paper, we studies how to rank two payoff functions for the same information structure in terms of how much the decision maker benefit from having the corresponding information structure.

The majority of the literature on the value of information in game-theoretic setting focuses on the linear-quadratic model. Vives (1984) studies duopolistic competition with exogenous information. In equilibrium, the expected profit increases with the precision of the competitor's information when the goods are complements. In a beauty contest game, Morris and Shin (2002) note that public information may reduce social welfare because of excessive coordination motives. The provision of public information causes investors to over-react to public information while neglecting socially valuable private information. However, additional public information is always beneficial to welfare in models of technological spillover (Angeletos and Pavan, 2004) and monopolistic competition (Hellwig, 2005). By using a general quadratic payoff function and normally distributed public and private signals, Angeletos and Pavan (2007) consider a class of Bayesian games with a continuum of players. They find mixed results on the value of information. Hellwig and Veldkamp (2009) find that players want to coordinate their information in models with complementarity. Jiménez-Martínez (2014) and Ui and Yoshizawa (2015) provide general treatments with mixed findings. In this paper, we first consider a linear-quadratic model (Proposition 1.4) and then generalize the results to monotone information structures, which are not studies in the literature (Proposition 1.5). Aside from quadratic games, Szkup and Trevino (2014) also consider a global game and establish that complementarity in actions does not necessarily lead to complementarity in information acquisition. We look into a similar game and study the value of information when a full characterization of the equilibrium is not available.

In a general setting, Van Zandt and Vives (2007) note that action complementarities work in the same direction as state complementarities, thus players want to coordinate in information. Gendron-Saulnier and Gordon (2015) work with general payoff functions, but with a more restrictive information structure. In this paper, we also do not focus on solving the game per se. We concentrate on the change in a player's payoff resulting from the change of the other players' strategies, which can be caused by a change in information, payoff functions or the game environment itself, and discuss how it would affect the value of information in equilibrium.

The rest of the paper is organized as follows. Section 1.2 propose the framework. Section 1.3 introduces the concept of similarity, and discusses the relationship between similarity and value of information. Section 1.4 contains application of similarity in various games. Section 1.5 concludes. Some of the proofs are relegated to the Appendix A.

1.2 THE DECISION PROBLEM

The stochastic environment consists of an unknown state of the world Θ, with realization $\theta \in \Theta$, and a signal X with realization $x \in X$.[1] Given a prior $H \in \triangle(\Theta)$, the distribution of the signal induces a joint distribution over states and signals, $F : \Theta \times X \to [0, 1]$. We call F an *information structure*. Let $F_X(.|\theta)$ be the signal distribution conditional on $\Theta = \theta$ and $F_X(.)$ be the marginal distribution of signal, i.e., $F_X(.) = E_\Theta[F_X(.|\theta)]$. Let $F_\Theta(.|x)$ be the conditional distribution of Θ given $X = x$. The posterior is consistent with the prior, i.e., for all $\theta \in \Theta$, $E_X[F_\Theta(\theta|x)] = H(\theta)$.

After observing the signal realization, a decision maker takes an action $a \in A \subset \mathbb{R}$. His payoff function is $u : A \times \Theta \to \mathbb{R}$. A payoff function u and an information structure F constitute a *decision problem* $\langle F, u\rangle$. The (ex ante) value of the decision problem $\langle F, u\rangle$ is

$$V(F, u) = E_X\left[\sup_{a\in A}\int_\Theta u(a, \theta)\, dF_\Theta(\theta|x)\right].$$

We assume throughout this paper that all the distributions and payoff functions involved are regular enough so that the expected payoff is always well-defined. A *decision rule* $\sigma : X \to \triangle(A)$ assigns a distribution over actions to every signal realization. We call σ^* an optimal decision rule for the decision problem $\langle F, u\rangle$, if $\sigma^*(x)$ is optimal for every $x \in X$, i.e.,

$$\sigma^*(x) \in \arg\max_{s\in\triangle(A)}\int_\Theta u(s, \theta)\, dF_\Theta(\theta|x).$$

We denote the set of optimal decision rules for decision problem $\langle F, u\rangle$ by $\sum^*_{F,u}$ and a generic element of the set $\sum^*_{F,u}$ by $\sigma^*_{F,u}$. We assume throughout this paper that given an information structure F, an optimal decision rule exists. This can be ensured by, for example, assuming that the action space A is compact and u is continuous in a. However, we do not impose such restrictions formally, as an optimal decision rule often exists in applications even if these assumptions are violated.

[1] By abuse of notations, we use Θ and X to denote both the random variables and the sets of realizations.

A null information structure F_ϕ, or interchangeably ϕ, is an information structure that satisfies $(F_\phi)_\Theta(\theta|x) = (F_\phi)_\Theta(\theta|x') = H(\theta)$ for all $x, x' \in X$. The (ex ante) value of the decision problem $\langle \phi, u \rangle$ is

$$\begin{aligned} V(\phi, u) &= E_X\left[\max_{a\in A} \int_\Theta u(a,\theta)\, d(F_\phi)_\Theta(\theta|x)\right] \\ &= \max_{a\in A} \int_\Theta u(a,\theta)\, dH(\theta). \end{aligned}$$

We call a^* an optimal default action, or simply default action, for the decision problem $\langle F, u \rangle$, if a^* is optimal given the prior H, i.e.,

$$a^* \in \arg\max_{a\in A} \int_\Theta u(a,\theta)\, dH(\theta).$$

Note that a^* only depends on the prior, so a^* is also an optimal decision rule for the decision problem $\langle \phi, u \rangle$ and the set of default actions for the decision problem $\langle F, u \rangle$ is the same as that for the decision problem $\langle F', u \rangle$. Therefore, we denote the set of optimal default actions for the decision problem $\langle F, u \rangle$ by $\sum^*_{\phi,u}$ and a generic element of the set $\sum^*_{\phi,u}$ by $a^*_{\phi,u}$.

The value of a decision rule σ in the decision problem $\langle F, u \rangle$ is

$$V(\sigma, F, u) = E_\Theta\left[\int_X u(\sigma(x),\theta)\, dF_X(x|\theta)\right].$$

In the decision problem $\langle F, u \rangle$, when an optimal decision rule $\sigma^*_{F,u}$ is used, we will achieve $V(F,u)$, i.e., $V\left(\sigma^*_{F,u}, F, u\right) = V(F,u)$; when an optimal default action $a^*_{\phi,u}$ is used, we will achieve $V(\phi,u)$, i.e., $V\left(a^*_{\phi,u}, F, u\right) = V(\phi,u)$.

The *value of information structure* F in the decision problem $\langle F, u \rangle$ is defined as

$$V(F,u) - V(\phi,u).$$

The value of information is the payoff difference between an optimal decision rule based on the information structure F and a default action based on prior.

1.3 SIMILARITY

In this section, we introduce the concept of similarity and discuss its properties. Consider a decision maker adopting a decision rule. The decision maker finds a decision rule favorable when it is better than his default action. If the decision rule that is favorable is an optimal decision rule for another decision problem, then we say that his own decision problem is similar to the other one. Formally,

Definition 1.1 (Similarity)**.** *$\langle F, v\rangle$ is similar to $\langle F, u\rangle$ if and only if there exists $\sigma^*_{F,u} \in \sum^*_{F,u}$ such that*

$$V\left(\sigma^*_{F,u}, F, v\right) \geq V\left(a^*_{\phi,v}, F, v\right).$$

We say that $\langle F, v\rangle$ is *strongly similar* to $\langle F, u\rangle$, when the inequality is strict.[2] A decision problem $\langle F, v\rangle$ is similar to another decision problem $\langle F, u\rangle$ if the optimal decision rule of $\langle F, u\rangle$, $\sigma^*_{F,u}$, gives a higher payoff in $\langle F, v\rangle$ than the optimal prior-based decision rule for $\langle F, v\rangle$, $a^*_{\phi,v}$. We denote a similar relation by $\overset{S}{\Rightarrow}$, and a strongly similar relation by $\overset{S}{\rightarrow}$. Conversely, a decision problem $\langle F, v\rangle$ is *not similar* to another decision problem $\langle F, u\rangle$ if and only if for all $\sigma^*_{F,u} \in \sum^*_{F,u}$, $V\left(\sigma^*_{F,u}, F, v\right) < V\left(a^*_{\phi,v}, F, v\right)$. We denote this by $\langle F, v\rangle \overset{S}{\not\Rightarrow} \langle F, u\rangle$. $\overset{S}{\not\rightarrow}$ is defined analogously.

The notion of similarity is implicitly invoked in many situations in our daily life. When the optimal decision is difficult to find, we often utilize pre-existing decision rules designed for similar situations. For instance, we use manuals, guide books, recipes on a daily basis. No manual is written for a particular user, no guide book perfectly matches the case we are handling, and no recipe exactly knows what food our baby boy is fond of. We use them because figuring out the best action takes too much effort. We would rather rely on suboptimal but ready-to-use pre-existing knowledge, as long as we find that we are facing a similar situation. Consider the tax preparation software as an example. Going through tax documents is not a unpleasant experience to some people. Using a tax preparation software that specifies a typical taxpayer similar to ourselves is still a valuable option, because we

[2]Similarity is reflexive for any information structure F, but strong similarity is not necessarily reflexive. Consider a constant payoff function u, we have, $V\left(\sigma^*_{F,u}, F, u\right) = V\left(a^*_{\phi,u}, F, u\right)$.

benefit from following the procedure pre-programmed in the software compared to figuring it out by our own.

The use of pre-existing rules and the development of fixed decision rules could be the result of optimization cost, legal restriction, consistency or other regarding concerns. The most relevant reason here is that positive optimization cost implies bounded rationality. If we take optimization costs seriously, there is a circularity problem that we would never reach "a optimization problem which fully incorporates the cost of its own solution" (Conlisk, 1988). At some stage of the decision making, the decision maker would have to use some decision making procedure that is not a standard optimization. The procedure that involves using a fixed decision rule for a class of problems is certainly a reasonable candidate. But how would a decision maker decide which rule to use? For instance, in Mohlin (2014), the decision maker groups observations to make predictions based on a variance-bias trade-off. In this situation, if it is costly to figure out to which group the new observation belongs, how does decision maker pick which prediction rule at the first stage? Our notion of similarity suggests a particular way to make decision in such a situation.

Given two payoff functions u and v, we define $w = u + v$ as

$$w(a,\theta) = (u+v)(a,\theta) = u(a,\theta) + v(a,\theta) \text{ for all } a \in A \text{ and } \theta \in \Theta.$$

We have,

Proposition 1.1 (Additivity). *Suppose* $\left|\sum^{*}_{F,u}\right| = 1$, *if* $\langle F, v\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$ *and* $\langle F, v'\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$, *then* $\langle F, av + bv'\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$ *for all* $a, b \geq 0$.

Proof. A decision rule σ generates a joint distribution over actions and states. By definition, $(u+v)(a,\theta) = u(a,\theta) + v(a,\theta)$. Therefore, given a decision rule σ, $V(\sigma, F, av + bv') = aV(\sigma, F, u) + bV(\sigma, F, v')$. $V\left(\sigma^{*}_{F,u}, F, av + bv'\right) = aV\left(\sigma^{*}_{F,u}, F, v\right) + bV\left(\sigma^{*}_{F,u}, F, v'\right)$. If $\langle F, v\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$ and $\langle F, v'\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$, $V\left(\sigma^{*}_{F,u}, F, v\right) \geq V(\phi, v)$ and $V\left(\sigma^{*}_{F,u}, F, v'\right) \geq V(\phi, v')$. By the optimality of $a^{*}_{\phi,v}$ and $a^{*}_{\phi,v'}$, $aV(\phi, v) + bV(\phi, v') \geq V(\phi, av + bv')$. □

When the optimal default action in $\langle F, u\rangle$ is unique, similarity is additive, because the value of a fixed decision rule in $\langle F, v + v'\rangle$ is the sum of the value of the decision rule in $\langle F, v\rangle$ and $\langle F, v'\rangle$.

The rest of this section consists of a series of examples, which show that the similarity relation does not have nice properties in a general environment. Specifically, similarity depends on the information structure. Change in the common information structure changes the similarity between two decision problems. Example 1.1 illustrates such dependence.

Example 1.1 ($\langle F, v\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$, and $\langle G, v\rangle \overset{S}{\not\Rightarrow} \langle G, u\rangle$)**.** *Let* $\Theta = \{\theta_1, \theta_2\}$ *and* $A = \{a_1, a_2, a_3\}$. *Consider two payoff functions,* u *and* v*, as shown in the tables below, i.e.,* $u(a_1, \theta_1) = 1$. ε *is a small positive number.*

u

	θ_1	θ_2
a_1	1	$-1-\varepsilon$
a_2	0	0
a_3	$-1-\varepsilon$	1

v

	θ_1	θ_2
a_1	1	-4
a_2	0	0
a_3	-4	1

The prior probability of $\Theta = \theta_1$ *is* $\frac{1}{2}$*, so the optimal default action in both problems, regardless the information structure, is* a_2*, i.e.,* $a^*_{\phi,u} = a^*_{\phi,v} = a_2$. $V(\phi, u) = V(\phi, v) = 0$.

Consider the following two information structures, F *and* G. $X = \{x_1, x_2\}$. *The elements in the tables represent the joint distribution over the signals and the states, i.e.,* $\Pr(X = x_1, \Theta = \theta_1) = 0.45$.

F

	θ_1	θ_2
x_1	0.45	0.05
x_2	0.05	0.45

G

	θ_1	θ_2
x_1	0.3	0.2
x_2	0.2	0.3

According to F*,* $\Pr(\theta_1|x_1) = 0.9$ *and* $\Pr(\theta_1|x_2) = 0.1$. $\sigma^*_{F,u}(x_1) = a_1$ *and* $\sigma^*_{F,u}(x_2) = a_3$. $V\left(\sigma^*_{F,u}, F, v\right) = 0.9 \cdot 1 - 0.1 \cdot 4 = 0.5 > V(\phi, v)$. *Therefore,* $\langle F, v\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$.

According to G*,* $\Pr(\theta_1|x_1) = 0.6$ *and* $\Pr(\theta_1|x_2) = 0.4$. $\sigma^*_{G,u}(x_1) = a_1$ *and* $\sigma^*_{G,u}(x_2) = a_3$. $V\left(\sigma^*_{G,u}, G, v\right) = 0.6 \cdot 1 - 0.4 \cdot 4 = -1 < V(\phi, v)$. *Therefore,* $\langle G, v\rangle \overset{S}{\not\Rightarrow} \langle G, u\rangle$.

The similarity relation is clearly not complete, and it is also not symmetric or transitive in general. When $\langle F, v\rangle$ is similar to $\langle F, u\rangle$, $\sigma^*_{F,u}$ is preferred to $a^*_{\phi,v}$ in $\langle F, v\rangle$. It says nothing about the comparison of $\sigma^*_{F,v}$ and $a^*_{\phi,v}$ in $\langle F, u\rangle$. The following example demonstrates the subtlety of similarity as it relies on the comparison of two suboptimal decision rules.

Example 1.2 ($\langle F,v\rangle \overset{S}{\Rightarrow} \langle F,u\rangle$, and $\langle F,u\rangle \overset{S}{\nRightarrow} \langle F,v\rangle$). *Let $\Theta=\{\theta_1,\theta_2\}$, and $A=\{a_1,a_2,a_3\}$. The payoff functions, u and v, are shown as follows,*

	u			v	
	θ_1	θ_2		θ_1	θ_2
a_1	1	0	a_1	1	0
a_2	-10	-10	a_2	0	1
a_3	0	1	a_3	0	0

The prior probability of $\Theta=\theta_1$ is $p>\frac{1}{2}$, so the optimal default action in both problems, regardless of the information structure, is a_1, i.e., $a^*_{\phi,u}=a^*_{\phi,v}=a_1$. $V(\phi,u)=V(\phi,v)=p$.

$X=\{x_1,x_2\}$. Signals are perfect, i.e., according to F, $\Pr(\theta_1|x_1)=\Pr(\theta_2|x_2)=1$. In $\langle F,u\rangle$, $\sigma^*_{F,u}(x_1)=a_1$, and $\sigma^*_{F,u}(x_2)=a_3$. $V\left(\sigma^*_{F,u},F,v\right)=p\cdot 1+(1-p)\cdot 0=p=V(\phi,v)$. Thus, $\langle F,v\rangle \overset{S}{\Rightarrow} \langle F,u\rangle$. In $\langle F,v\rangle$, $\sigma^*_{F,v}(x_1)=a_1$, and $\sigma^*_{F,v}(x_2)=a_2$. $V\left(\sigma^*_{F,v},F,u\right)=p\cdot 1+(1-p)\cdot(-10)<V(\phi,u)$. Thus, $\langle F,u\rangle \overset{S}{\nRightarrow} \langle F,v\rangle$.

The following example shows that a similar relation may not be transitive. When $\langle F,v\rangle$ is similar to $\langle F,u\rangle$, $\sigma^*_{F,u}$ is preferred to $\sigma^*_{\phi,v}$ in $\langle F,v\rangle$. No restriction is imposed on $\sigma^*_{F,v}$. Therefore, transitivity is not guaranteed.

Example 1.3 (A similarity relation that is not transitive). *Let θ follow the uniform distribution on $[-1,1]$. Consider a family of payoff functions, indexed by $k\in Z_+$, $u_k(a,\theta)=-(a-\theta-kb)^2$. Under the null information ϕ, $a^*_{\phi,u_k}=kb$. Signals are perfect, i.e., conditional on x, the posterior distribution of θ is a degenerated distribution at $\theta=x$. Therefore, $\sigma^*_{F,u_k}=x+kb$. For b small enough, $\langle F,u_k\rangle \overset{S}{\Rightarrow} \langle F,u_{k-1}\rangle$ for all $k\geq 1$. For any $k\geq 1$, conditional on θ,*

$$u_k\left(\sigma^*_{F,u_0}(\theta),\theta\right)=-(\theta-\theta-kb)^2=-(kb)^2,$$

and

$$u_k\left(a^*_{\phi,u_k}(\theta),\theta\right)=-(kb-\theta-kb)^2=-\theta^2.$$

For k large enough, $u_k\left(\sigma^*_{F,u_0}(\theta),\theta\right)<u_k\left(a^*_{\phi,u_k}(\theta),\theta\right)$ for any state realization θ. Thus, $\langle F,u_k\rangle \overset{S}{\nRightarrow} \langle F,u_0\rangle$ for k large enough.

1.3.1 Similarity and value of information

In this section, we investigate how the similarity relation relates to the value of information in decision problems.

Even if $\langle F, v\rangle$ is similar to $\langle F, u\rangle$, we may not know how to compare $V(F,u)$ and $V(F,v)$. However, we will now show that the concept of similarity can be used to compare $\langle F, u\rangle$ and another decision problem, using $\langle F, v\rangle$ as a bridge between the two.

Proposition 1.2. *If $\langle F, v\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$, then the value of information F is higher in $\langle F, u+v\rangle$ than in $\langle F, u\rangle$, i.e.,*

$$V(F, u+v) - V(\phi, u+v) \geq V(F,u) - V(\phi, u). \tag{1.1}$$

Proof. Take $\sigma^*_{F,u} \in \sum^*_{F,u}$ such that $V\left(\sigma^*_{F,u}, F, v\right) \geq V(\phi, v)$. We have

$$\begin{aligned}
& V(F, u+v) - V(\phi, u+v) \\
\geq\ & V\left(\sigma^*_{F,u}, F, u+v\right) - V(\phi, u+v) \\
=\ & \left[V\left(\sigma^*_{F,u}, F, u\right) - V\left(a^*_{\phi,u+v}, \phi, u\right)\right] + \left[V\left(\sigma^*_{F,u}, F, v\right) - V\left(a^*_{\phi,u+v}, \phi, v\right)\right] \\
\geq\ & \left[V\left(\sigma^*_{F,u}, F, u\right) - V(\phi, u)\right] + \left[V\left(\sigma^*_{F,u}, F, v\right) - V(\phi, v)\right] \\
\geq\ & V\left(\sigma^*_{F,u}, F, u\right) - V(\phi, u),
\end{aligned}$$

where the first inequality follows from the suboptimality of $\sigma^*_{F,u}$ in $\langle F, u+v\rangle$, the second last inequality follows the suboptimality of $a^*_{\phi,u+v}$ in both $\langle \phi, u\rangle$ and $\langle \phi, v\rangle$, and the last inequality follows from the assumption that $\langle F, v\rangle \overset{S}{\Rightarrow} \langle F, u\rangle$. □

We call $\langle F, u\rangle$, $\langle F, v\rangle$, and $\langle F, u+v\rangle$ the original problem, the marginal problem, and the new problem, respectively. The value of information in the new problem is higher than that in the original problem if the marginal problem is similar to the original problem. The intuitive explanation is indicated in the proof. If the marginal problem $\langle F, v\rangle$ is similar to the original problem $\langle F, u\rangle$, the original optimal decision rule $\sigma^*_{F,u}$ has an advantage in the marginal problem $\langle F, v\rangle$ over the default action. This implies that the optimal decision rule for the new problem that combines $\langle F, u\rangle$ and $\langle F, v\rangle$ must provide an even higher payoff.

The opposite direction of this statement is not true, as illustrated by the following example.

Example 1.4. *Let* $\Theta = \{\theta_1, \theta_2\}$*, and* $A = \{a_1, a_2\}$*. The payoff function* u *is given by* $u(a_1, \theta_1) = u(a_2, \theta_2) = 1$ *and* $u(a_1, \theta_2) = u(a_2, \theta_1) = 0$*. The payoff function* $(u + v)$ *is given by* $(u+v)(a_1, \theta_1) = (u+v)(a_2, \theta_2) = 0$ *and* $(u+v)(a_1, \theta_2) = (u+v)(a_2, \theta_1) = 100$*. The prior probability of* $\Theta = \theta_1$ *is* $\frac{1}{2}$*, so* $V(\phi, u) = \frac{1}{2}$ *and* $V(\phi, u+v) = 50$*. Signals are perfect, i.e., according to* F*,* $\Pr(\theta_1|x_1) = \Pr(\theta_2|x_2) = 1$*. Therefore,* $V(F, u) = 1$ *and* $V(F, u+v) = 100$*.* $V(F, u+v) - V(\phi, u+v) \geq V(F, u) - V(\phi, u)$*. Since the payoff function* v *is given by* $v(a_1, \theta_1) = v(a_2, \theta_2) = -1$*, and* $v(a_1, \theta_2) = v(a_2, \theta_1) = 100$*,* $V\left(\sigma^*_{F,u}, F, v\right) = -1 < V(\phi, v) = 49\frac{1}{2}$*. Thus,* $\langle F, v\rangle \overset{S}{\not\twoheadrightarrow} \langle F, u\rangle$*.*

While $V(F, u+v) - V(\phi, u+v) \geq V(F, u) - V(\phi, u)$ does not imply the similarity of $\langle F, v\rangle$ to $\langle F, u\rangle$, it implies the similarity of $\langle F, v\rangle$ to $\langle F, u\rangle$ under an extra assumption, shown in the following proposition.

Proposition 1.3. *Suppose* $\Sigma^*_{\phi,u} \cap \Sigma^*_{\phi,v} \neq \phi$ *, if* $\langle F, v\rangle \overset{S}{\twoheadrightarrow} \langle F, u\rangle$*, then the value of information* F *is lower in* $\langle F, u+v\rangle$ *than in* $\langle F, u\rangle$*, i.e.,*

$$V(F, u+v) - V(\phi, u+v) \leq V(F, u) - V(\phi, u). \tag{1.2}$$

Proof. Since $V(\phi, u+v) \leq V(\phi, u) + V(\phi, v)$, $a \in \Sigma^*_{\phi,v} \cap \Sigma^*_{\phi,u}$ implies that $a \in \Sigma^*_{\phi,u+v}$. Similar to the proof for Proposition 1.2, we have

$$\begin{aligned} & V(F, u) - V(\phi, u) \\ \geq\ & V\left(\sigma^*_{F,u+v}, F, u\right) - V(\phi, u) \\ =\ & \left[V\left(\sigma^*_{F,u+v}, F, u+v\right) - V(\phi, u+v)\right] - \left[V\left(\sigma^*_{F,u+v}, F, v\right) - V(\phi, v)\right] \\ \geq\ & V(F, u+v) - V(\phi, u+v), \end{aligned}$$

where the equality follows the assumption that $\Sigma^*_{\phi,u} \cap \Sigma^*_{\phi,v} \neq \phi$, and last inequality follows from the assumption that $\langle F, v\rangle \overset{S}{\twoheadrightarrow} \langle F, u\rangle$. □

The assumption of a common default action in two decision problems can sometimes be restrictive. However, in certain cases with adjustment costs, this assumption appears to be rather natural. For example, a trader only changes his portfolio when he obtains access to some new information, a company only enters a market if it detects a new business opportunity, and we stick to our usual dinning place unless we hear good comments about a newly opened restaurant. This is also a reasonable assumption when u and v are close and actions are discrete.

The concept of similarity is quite intuitive, and it provides an intuitive way to rank payoff functions for a given information structure, according to the value of information in the corresponding decision problems. It is a partial order, as is Blackwell's order. Blackwell's order starts from an information structure that is more valuable to all payoff functions to one that is less valuable by a garbling of the former. An ordering based on similarity relations starts from a payoff function that values one information structure less to one that values more.

In Section 1.4, we are going to repeatedly use the relationship between a similarity relation and the value of information. We construct a marginal problem, compare it with either the original problem or the new problem, and then apply Proposition 1.2 or 1.3 to get conclusions on changes in the value of information.

1.4 APPLICATIONS

In this section, we apply the notion of similarity to games. We find conditions under which a player's benefit from having information about the state of the world is higher in one equilibrium than in another. The condition is the similarity of the problem induced by the change in other players' strategies from one equilibrium to another equilibrium to the problem induced by one of the two equilibria.

As in an individual decision problem, we measure the value of information by how much a player benefits from having that information. In measuring the value of information in games, we abstract away from the issue of *interactive knowledge*. In games, changing the information structure also changes the mutual knowledge of the information structure. The value of information measured without abstracting away from such a change in the mutual knowledge consists of two components: One is the instrumental value of information, which is how much a player can benefit from having the information fixing the strategies of the other players. The other is the strategic value of information, which is how much a player can benefit form a change in the *equilibrium strategies* of other players under the assumption that any change in the information structure is also common knowledge. Throughout this section, we assume that the strategies of the other players are *fixed at a particular equilibrium under a particular information structure* and thus abstract away from the strategic value of information. The value of information for a particular player is thus the payoff difference between best-responding with and without information. It is important to note that although we focus only on the instrumental value, the strategic aspects of the game in consideration still matter. This is because the strategies of the other players must be equilibrium strategies. Conceptually, the instrumental value of information measures the incentives for a player to acquire information *covertly*.

To apply Propositions 1.2 and 1.3, we compare the equilibrium strategy in one equilibrium to the optimal default action in the induced marginal problem for the player with the fixed information structure. There are several difficulties in performing this comparison. The first is to characterize the equilibrium strategies for all players. We need the equilibrium strategies in constructing the induced marginal problem, as well as in the comparison we conduct in the induced marginal problem. To cope with this difficulty, we focus on games that have monotone equilibria. The second difficulty is to characterize the optimal default action in the marginal problem. Given two monotone equilibria, we in general have no idea what the optimal default action is in the induced marginal problem. For the class of games we consider in this section, fortunately, we have ways to get around this difficulty. In the quadratic games considered in Section 1.4.3, any action is optimal in the induced marginal problem. In the global game considered in Section 1.4.4, the investor only chooses either to invest or not, and this simplifies the choice of optimal default action in the induced marginal problem. In the persuasion game considered in Section 1.4.5, there is only one action the player can take when there is no information. The third and main difficulty is the comparison of two endogenous objects that do not necessarily have closed form expressions in the induced marginal problem: one is the optimal default action for the marginal problem and the other is an equilibrium strategy in one equilibrium. In the quadratic game, we find conditions for information structures under which we are able to have an unambiguous comparison. In the global game, the investor always prefers the other investors to invest more, and the investor only cares about the aggregate level of investment in the economy. Therefore, the equilibrium investment strategy distribution is reduced to a single parameter. In the persuasion game, the presence of an extra expert with opposite extreme bias has two effects that have the same direction, which leads to an unambiguous comparison. In the corresponding sections, we will discuss in detail how these features allow us to compare two endogenous objects and draw conclusions on the change in the value of information in the corresponding games.

1.4.1 Games and the induced marginal problem

Consider a game with N players and let I denote the set of players. Player $i \in I$ receives a signal $x_i \in X_i \subseteq \mathbb{R}$. Denote $X = \times_{i \in I} X_i$. An information structure F for the game is a joint distribution over states and signals, $F : \Theta \times X \to [0,1]$. Player i chooses an action $a_i \in A_i \subseteq \mathbb{R}$. Denote $A = \times_{i \in I} A_i$. Player i's payoff function is $u_i : A \times \Theta \to \mathbb{R}$. A strategy of player i is a mapping from the received signals to distributions over actions, $\sigma_i : X_i \to \triangle (A_i)$. Given the information structure F, let $F_X (.|\theta)$ denote the signal distribution conditional on $\Theta = \theta$, and $F_{X_i} (.|\theta)$ denote player i's signal distribution conditional on $\Theta = \theta$. The players have a common prior, thus $E_{X_i} [F_\Theta (\theta|x_i)] = E_{X_j} [F_\Theta (\theta|x_j)] = H(\theta)$ for all $i, j \in I$.

In order to consider the value of information in games, we need to incorporate the other players' strategies. Fixing the strategies of the other players, player i faces an individual decision problem. We call the situation in which the other players use $\sigma_{-i} (x_{-i})$ and $\sigma'_{-i} (x_{-i})$ the original problem and the new problem, respectively, and define the original problem $\langle F, \widetilde{u}_i \rangle$ and the new problem $\langle F_i, \widetilde{w}_i \rangle$ correspondingly. We define $X_{-i} \times \Theta$ as the new state space. Given any $\sigma_{-i} (x_{-i})$, we can define the corresponding decision problem $\langle F_i, \widetilde{u}_i \rangle$ and discuss the value of information for player i in $\langle F_i, \widetilde{u}_i \rangle$, which is equivalent to the value of information given the other players are using $\sigma_{-i} (x_{-i})$. Similar for $\sigma'_{-i} (x_{-i})$ and $\langle F_i, \widetilde{w}_i \rangle$.

Given $\sigma_{-i} (x_{-i})$, player i's payoff function is

$$\widetilde{u}_i (a_i, x_{-i}, \theta) = \int_{A_{-i}} u_i (a_i, a_{-i}, \theta) \, d\sigma_{-i} (a_{-i}|x_{-i}) \, ,$$

in the original problem $\langle F_i, \widetilde{u}_i \rangle$. With a standard abuse of notation, we write $\widetilde{u}_i(a_i, x_{-i}, \theta) = u_i(a_i, \sigma_{-i}(x_{-i}), \theta)$. Given $\sigma'_{-i}(x_{-i})$, player i's payoff function is

$$\widetilde{w}_i(a_i, x_{-i}, \theta) = u_i\left(a_i, \sigma'_{-i}(x_{-i}), \theta\right),$$

in the new problem $\langle F_i, \widetilde{w}_i \rangle$. Then, in the marginal problem induced, player i has information structure F_i and the following payoff function,

$$\widetilde{v}_i(a_i, x_{-i}, \theta) = \widetilde{w}_i(a_i, x_{-i}, \theta) - \widetilde{u}_i(a_i, x_{-i}, \theta).$$

The construction of the marginal problem depends on $\sigma_{-i}(x_{-i})$ and $\sigma'_{-i}(x_{-i})$, and $\widetilde{v}_i(a_i, x_{-i}, \theta)$ is the change in payoffs due to a change in other players' strategy, from $\sigma_{-i}(x_{-i})$ to $\sigma'_{-i}(x_{-i})$. In the next section, we will discuss how to compare the value of information in two different equilibria based on the induced marginal problem defined above.

1.4.2 Value of information in equilibrium

In a game, the value of information depends on the equilibrium strategies. Given an information structure F, $\sigma^*(x)$ is an equilibrium strategy profile if and only if

$$\sigma_i^*(x_i) \in \arg\max_{s_i \in \triangle(A_i)} \int_{X_{-i}\times\Theta} u_i\left(s_i, \sigma^*_{-i}(x_{-i}), \theta\right) dF_{X_{-i}\times\Theta}(x_{-i}, \theta | x_i) \tag{1.3}$$

for all $i \in I$. In order to highlight the dependence of the equilibrium strategy profile $\sigma^*(x)$ on the information structure F, we denote it by $\sigma^*(x; F)$. Denote player i's payoff by playing $\sigma_i^*(x_i; F)$ when the other players are playing the equilibrium strategy $\sigma^*_{-i}(x_{-i}; F)$ by $V(F_i; F)$, i.e.,

$$V(F_i; F) = E_{X\times\Theta}\left[u_i(\sigma^*(x; F), \theta)\right],$$

and it is also player i's payoff in the equilibrium corresponding to $\sigma^*(x; F)$.

Given the other players' equilibrium strategies $\sigma^*_{-i}(x_{-i}; F)$, a_i^* is player i's (optimal) default action if and only if

$$a_i^* \in \arg\max_{a_i \in A_i} \int_{X_{-i}\times\Theta} u_i\left(a_i, \sigma^*_{-i}(x_{-i}; F), \theta\right) dF(x_{-i}, \theta).$$

Analogously, we denote it by $a_i^*(F)$. Note that the optimal default action a_i^* depends on the information F_i not because player i has information F_i, but *the other players believe that player i has information F_i*. Denote player i's payoff by playing $a_i^*(F)$ when the other players are playing the equilibrium strategy $\sigma_{-i}^*(x_{-i};F)$ by $V(\phi;F)$, i.e.,

$$V(\phi;F) = E_{X_{-i}\times\Theta}\left[u_i\left(a_i^*(F), \sigma_{-i}^*(x_{-i};F), \theta\right)\right],$$

and it is the payoff player i can get by unilaterally and covertly deviating from F_i to ϕ in the equilibrium $\sigma^*(x;F)$.

Given an information structure F, we define the value of information F_i to player i in equilibrium by

$$V(F_i;F) - V(\phi;F). \tag{1.4}$$

The difference between $V(F_i;F)$ and $V(\phi;F)$ measures how much player i benefits from information F_i when holding the other's belief of player i's information constant at F_i. As mentioned at the beginning of this section, the value of information defined by (1.4) does not measure the strategic value of information. Therefore, (1.4) is non-negative. The value of information as defined by (1.4) can be interpreted as a measure of the incentives of player i to deviate from acquiring the equilibrium level of information F_i in a game with *covert information acquisition*. The higher is the value, the more likely that the equilibrium with information F_i can be sustained given a fixed cost of information acquisition.[3] With (1.4) at hand, we can now define complementarity and substitutability in information in games. For each player i, consider two information levels $\{\phi, \overline{F}_i\}$, we define,

[3]This notion of value of information is not the only notion that one could think about in this situation. Readers might also consider the value of information in the following sense,

$$E_{X\times\Theta}\left[u_i\left(\sigma^*(x;F), \theta\right)\right] - E_{X\times\Theta}\left[u_i\left(\sigma^*(x;\phi,F_{-i}), \theta\right)\right]. \tag{1.5}$$

The main difference between these two notions is that, when player i switches to the default action, player i's opponents continue to use the strategies $\sigma_{-i}^*(x_{-i};F)$ in (1.4), while player i's opponents switch to $\sigma_{-i}^*(x_{-i};\phi,F_{-i})$ in (1.5). Conceptually, when the value of information is defined by (1.5), the value of information is determined not only by information per se but also from the change in the mutual knowledge situation. That is, the other players always know when the information of player i changes, and they also know that all the other players know that, etc. For a detailed discussion of these issues, see Bassan, Scarsini and Zamir (1997). However, when there is a continuum of players, the equilibrium strategies of the other players do not depend on the information acquisition of a particular player. In that case, the two notions (1.4) and (1.5) are equivalent. This is true, for instance, in the global game considered in Section *4.4*.

Definition 1.2. *Given an information structure $F = \left(\overline{F}_1, ..., \overline{F}_N\right)$, we say that player i has complementarity (substitutability) in information with player j if and only if for all equilibria $\sigma^*\left(x; \overline{F}_j, F_{-j}\right)$ and $\sigma^*\left(x; \phi, F_{-j}\right)$,*

$$V\left(\overline{F}_i; \overline{F}_j, F_{-j}\right) - V\left(\phi; \overline{F}_j, F_{-j}\right) \geq (\leq)\, V\left(\overline{F}_i; \phi, F_{-j}\right) - V\left(\phi; \phi, F_{-j}\right). \tag{1.6}$$

In words, information is complementary if the value of information increases with the other players' information level. To incorporate the possibility of multiple equilibria, our definition of complementarity/substitutability requires the inequality (1.6) to be satisfied for all combinations of equilibria under the information profiles $\left(\overline{F}_j, F_{-j}\right)$ and $\left(\phi, F_{-j}\right)$. An equally legitimate definition requires the inequality (1.6) to be satisfied for a pair of equilibria. In all our applications, however, the equilibria involved are unique.[4] The distinction of the two definitions thus is immaterial.

In Section 1.4.3, we consider a quadratic game, and discuss how a change in the other players' information affects the player's value of information in the equilibrium. We first draw conclusions on the value of information in quadratic games with affine information structure under which the equilibrium strategy is linear. We then extend the results to quadratic game with monotone information structure under which a closed form equilibrium strategy is in general unavailable. In Section 1.4.4, we consider a global game, and discuss how to compare the value of information in two different equilibria. We show that investors find their information less valuable when the original market condition is good (bad) and becomes even better (worse). In Section 1.4.5, we consider a persuasion game, and discuss how the presence of another strategic expert changes the existing expert's value of information. For the existing expert, adding another strategic expert with an opposite bias always reduces his value of information. Some of the proofs are relegated to the Appendix A.

[4] Strictly speaking, the persuasion game we consider in Section *4.5* has more than one equilibrium, as the experts are free to send different messages when he is indifferent. Clearly, this multiplicity has no bearing on our calculations.

1.4.3 Quadratic game

The quadratic payoff function has been employed extensively to study complementarity /substitutability of information in the literature. Consider the *generalized* quadratic payoff function, under which player i's payoff from the action profile $a = (a_1, ..., a_N)$ when the state of the world is θ is given by

$$u_i(a, \theta) = -a_i^2 + 2\alpha a_i \sum_{j \neq i} a_j + 2\beta_i \theta a_i + f_i(a_{-i}, \theta), \tag{1.7}$$

where $\alpha, \beta_i \in \mathbb{R}$ are constants and $f_i : \mathbb{R}^N \to \mathbb{R}$ is a measurable function. In words, player i's payoff can be decomposed into two parts, a quadratic part that is quadratic in player i's own action a_i and a functional part that is independent of a_i. In the quadratic part, α is the coefficient of the effect of the interaction between player i's action a_i and the other players' aggregate action $\sum_{j \neq i} a_j$, and β_i is the coefficient of the effect of the interaction between player i's action a_i and the state of the world θ.[5] As a result, player i's best response depends on both α and β_i, but not on the function f_i. We assume that $(N-1)|\alpha| < 1$ and $\beta_i \geq 0$.[6]

Given a generalized quadratic payoff function, the single parameter α characterizes the interaction between player i and the aggregate action. If $\alpha \geq (\leq) 0$, there is strategic complementarity (substitutability) in player i's and the aggregate actions. How would the interaction of actions affects the interactions of information acquisitions? In Vives (1984), complementarity (substitutability) in information depends on complementarity (substitutability) in

[5] If $N = 2$, and $f_i(a_{-i}, \theta) = 0$, we obtain the model of oligopolistic competition of Vives (1984). In Vives (1984), firm i's profit $\pi_i = (\alpha - \beta q_i - \gamma q_j) q_i$, where $\alpha - \beta q_i - \gamma q_j$ is the market price of good i, α is the state of the world, q_i is the firm's quantity supplied, and q_j is the opponent's quantity supplied. Market price of good i is decreasing in own quantity supplied, i.e., $\beta \geq 0$.

Suppose N is large and the impact of other players' actions on player i is only through the aggregate action, i.e., $f_i(a_{-i}, \theta) = 0$. Our quadratic game becomes a version of the investment game of Angeletos and Pavan (2004). In Angeletos and Pavan (2004), there is a continuum of players. $u_i = [(1-\alpha)\theta + \alpha K] k_i - \frac{1}{2} k_i^2$, where θ is the state of the world, K is the total investment of the economy, i.e., $K = \int_0^1 k_i di$, and k_i is player i's investment level. α is a parameter that measures how much player i's investment benefits from the total investment.

Next, suppose, for some $r \in [0, 1]$, $\alpha = \frac{r}{N-1}$, $\beta = (1-r)$ and $f_i(a_{-i}, \theta) = -\left(\frac{r}{N-1} \sum_{j \neq i} a_j + (1-r)\theta\right)^2$, we get a two-player version of the beauty contest as in Hellwig and Veldkamp (2009), $u_i = -[a_i - (1-r)\theta - r a_j]^2$. In Hellwig and Veldkamp (2009), $u_i = -\frac{1}{(1-r)^2}(p_i - p^*)^2$, where p_i is player i's position, and $p^* = (1-r)s + r\overline{p}$, s is the state of the world and $\overline{p}$ is the average position of all players. r is a paremeter that measures player i's incentive to match with the average position $\overline{p}$.

[6] The assumption that $(N-1)|\alpha| < 1$ implies that the feedback effect from an increase in own action a_i on player i's best response through the aggregate action $\sum_{j \neq i} a_j$ is less than 1-to-1.

actions. In Morris and Shin (2002) and Hellwig and Veldkamp (2009), coordination motives in actions translate into coordination motives in information acquisitions in a beauty contest. In Angeletos and Pavan (2004), complementarity in investment translates into complementarity in information.[7]

In this section, we explore the marginal problem resulting from a change in the other players' strategies and apply the concept of similarity to investigate the complementarity/substitutability in information. For simplicity, we only consider a zero-one problem, i.e., player i has either no information ϕ or a fixed amount of information $\overline{F}_i$. Denote player i's information level by $F_i \in \left\{\phi, \overline{F}_i\right\}$. To simplify the notation, we only consider the case with two players, i.e., $N = 2$, the result can be easily extended to the multiple-player case.[8] Let $\sigma^*(x; F)$ be an equilibrium strategy profile defined in (1.3). We construct a decision problem corresponding to the equilibrium under information structure $\left(\overline{F}_i, \phi\right)$, and call it the original problem. The value of information in the original problem is $V\left(\overline{F}_i; \overline{F}_i, \phi\right) - V\left(\phi; \overline{F}_i, \phi\right)$. Similarly, we construct a decision problem corresponding to the equilibrium under information structure $\left(\overline{F}_i, \overline{F}_j\right)$, and call it the new problem. The value of information in the new problem is $V\left(\overline{F}_i; \overline{F}_i, \overline{F}_j\right) - V\left(\phi; \overline{F}_i, \overline{F}_j\right)$. Thus, determining the complementarity of information is equivalent to comparing the value of information in the original problem and the new problem. When the induced marginal problem is similar to the original problem, the value of information is higher in the new problem than in the original problem, according to Proposition 1.2. When the induced marginal problem is not strongly similar to the new problem, the value of information is higher in the original problem than in the new problem, according to Proposition 1.3.

Before constructing the marginal problem, we state a simple observation that will be useful in the construction and the proofs. Let $\sigma^*(x; F)$ be an equilibrium strategy profile defined in (1.3), then we have

Lemma 1.1. *Given any information structure F and any equilibrium strategy profile of the*

[7]In Angeletos and Pavan (2004), the authors refer the complementarity in a quadratic game as moderate complementarity, while they refer the complementarity in a global game as strong complementarity.

[8]Notice that the multiplier α for the interaction between a_i and a_j is same for all $i, j \in I$.

quadratic game $\sigma^*(x;F)$*, for* $i\in\{1,2\}$*,*

$$E\left(\sigma_i^*\left(x_i;F\right)\right)=\overline{a}_i,$$

where $(\overline{a}_1,\overline{a}_2)$ *is the equilibrium actions pair under the information structure* (ϕ,ϕ)*.*

By Lemma 1.1, player j must take the action $\overline{a}_j$ in equilibrium under information structure $\left(\overline{F}_i,\phi\right)$. Therefore, in the original problem, which corresponds to the equilibrium under information structure $\left(\overline{F}_i,\phi\right)$, player i has information $\overline{F}_i$, and the payoff function is

$$\widetilde{u}_i\left(a_i,x_j,\theta\right)=-a_i^2+2\alpha a_i\overline{a}_j+2\beta_i\theta a_i+f_i\left(\overline{a}_j,\theta\right). \tag{1.8}$$

Let $\overline{F}=\left(\overline{F}_1,\overline{F}_2\right)$. In the new problem, which corresponds to the equilibrium under information structure $\overline{F}$, player i has information $\overline{F}_i$, and the payoff function is

$$\widetilde{w}_i\left(a_i,x_j,\theta\right)=-a_i^2+2\alpha a_i\sigma_j^*\left(x_j;\overline{F}\right)+2\beta_i\theta a_i+f_i\left(\sigma_j^*\left(x_j;\overline{F}\right),\theta\right). \tag{1.9}$$

Therefore, in the induced marginal problem, player i has information $\overline{F}_i$, and the payoff function is

$$\widetilde{v}_i\left(a_i,x_j,\theta\right)=\underbrace{2\alpha a_i\left(\sigma_j^*\left(x_j;\overline{F}\right)-\overline{a}_j\right)}_{\text{interaction between } a_i \text{ and } \left(\sigma_j^*\left(x_j;\overline{F}\right)-\overline{a}_j\right)}+\underbrace{f_i\left(\sigma_j^*\left(x_j;\overline{F}\right),\theta\right)-f_i\left(\overline{a}_j,\theta\right)}_{\text{other terms independent of } a_i}. \tag{1.10}$$

In the induced marginal problem $\left\langle\overline{F}_i,\widetilde{v}_i\right\rangle$, the payoff function $\widetilde{v}_i$ captures the impact of a change in player j's strategy on player i's payoff. Intuitively, $\widetilde{v}_i$ is the incremental payoff change with respect to a change in player j's strategy due to a change in player j's information, i.e., from $\overline{a}_j$ to $\sigma_j^*\left(x_j;\overline{F}\right)$. In the marginal problem $\left\langle\overline{F}_i,\widetilde{v}_i\right\rangle$, we have,

Lemma 1.2. $\overline{a}_i$ *is an optimal default action in the marginal problem* $\left\langle\overline{F}_i,\widetilde{v}_i\right\rangle$*.*

To prove Lemma 1.2, take unconditional expectation of (1.10) and apply Lemma 1.1 to conclude that the expected payoff in the marginal problem $E\left(\widetilde{v}_i\left(a_i,x_j,\theta\right)\right)$ is independent of a_i. Thus, we can simply take $\overline{a}_i$ to be the optimal default action in the marginal problem. We have,

Lemma 1.3. *Given the original problem* $\langle \overline{F}_i, \widetilde{u}_i \rangle$*, the marginal problem* $\langle \overline{F}_i, \widetilde{v}_i \rangle$*, and the new problem* $\langle \overline{F}_i, \widetilde{w}_i \rangle$*,*

1) the marginal problem is similar to the original problem if and only if

$$\alpha Cov(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right), \sigma_j^*\left(x_j; \overline{F}\right)) \geq 0,$$

and

2) the marginal problem is similar to the new problem if and only if

$$\alpha Cov(\sigma_i^*\left(x_i; \overline{F}\right), \sigma_j^*\left(x_j; \overline{F}\right)) \geq 0.$$

In the marginal problem $\langle \overline{F}_i, \widetilde{v}_i \rangle$, for every realization of (x_i, x_j, θ), the payoff difference between the strategy σ_i and the optimal default action $\overline{a}_i$ is

$$\underbrace{2\alpha\left(\sigma_i\left(x_i\right) - \overline{a}_i\right)\left(\sigma_j^*\left(x_j; \overline{F}\right) - \overline{a}_j\right)}_{\text{interaction between } \left(\sigma_i(x_i) - \overline{a}_i\right) \text{ and } \left(\sigma_j^*\left(x_j; \overline{F}\right) - \overline{a}_j\right)}, \tag{1.11}$$

which depends only on the interaction between players i and j. Intuitively, (1.11) measures the incremental payoff due to some strategy σ_i with respect to the default action $\overline{a}_i$ given a change in the strategy of player j due to a change in player j's information. The *ex ante* payoff difference between the equilibrium strategy $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ in the original problem $\langle \overline{F}_i, \widetilde{u}_i \rangle$ and the optimal default action $\overline{a}_i$ in the marginal problem $\langle \overline{F}_i, \widetilde{v}_i \rangle$ can be written as

$$\begin{aligned} & V\left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right), \overline{F}_i, \widetilde{v}_i\right) - V\left(\phi, \widetilde{v}_i\right) \\ = \; & 2\alpha E\left[\left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right) - \overline{a}_i\right)\left(\sigma_j^*\left(x_j; \overline{F}\right) - \overline{a}_j\right)\right] \\ = \; & 2\alpha Cov\left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right), \sigma_j^*\left(x_j; \overline{F}\right)\right), \end{aligned} \tag{1.12}$$

where the first equality follows from Lemma 1.2 and the second equality follows from Lemma 1.1. Similarly, we have

$$V\left(\sigma_i^*\left(x_i; \overline{F}\right), \overline{F}_i, \widetilde{v}_i\right) - V\left(\phi, \widetilde{v}_i\right) = 2\alpha Cov\left(\sigma_i^*\left(x_i; \overline{F}\right), \sigma_j^*\left(x_j; \overline{F}\right)\right). \tag{1.13}$$

In Section 1.4.3.1, we investigate the sign of (1.12) and (1.13) given an affine information structure, under which the equilibrium strategies are linear. The most studied Normal environment is a special case of affine information structures. In Section 1.4.3.2, we extend the results to monotone information structures, under which a closed form solution is in general unavailable.

1.4.3.1 Affine Information Structure

The first class of information structures we consider is the affine information structure. This class of information structures includes the most familiar Normal environment, which the majority of papers in the literature assume. As pointed out by Vives (1988), the class of affine information structures includes many cases other than the Normal environment. The signals could distribute according to Binomial, Negative Binomial, Poisson, Gamma or Exponential distributions when natural conjugate priors are assigned.

Definition 1.3 (Affine information structure)**.** *An information structure is affine if and only if for all* $i, j \in I$*,*

A1 $E(\theta|x_i) = A_i x_i + B_i$*, where* $0 \leq A_i < 1$ *and* $B_i \in \mathbb{R}$*;*

A2 $E(x_j|x_i) = E(x_j) + E(\theta|x_i) - E(\theta)$*.*

Consider the Normal environment as an example. Let θ distribute according to a Normal distribution with mean μ and finite variance σ^2. Player i receives a signal x_i such that $x_i = \theta + \varepsilon_i$, where ε_i is a noise term independent of both θ and ε_j and distributed according to a Normal distribution with mean 0 and variance υ. Under these assumptions, $E(\theta|x_i) = \frac{\sigma^2}{\sigma^2+\upsilon} x_i + \frac{\upsilon}{\sigma^2+\upsilon}\mu$. $(A1)$ and $(A2)$ are satisfied. Under such information structures, we establish the existence of a unique linear equilibrium in the following lemma.

Lemma 1.4. *In a quadratic game with affine information structure, the equilibrium is unique and linear. Moreover, if the players have complementarity in actions, then the equilibrium is also increasing.*

Given Lemma 1.4, the signs of (1.12) and (1.13) depend on how the signals x_i and x_j are correlated. Given $(A1)$ and $(A2)$, we have

Proposition 1.4. *In a quadratic game with affine information structure,*

1. *when the players have complementarity in actions, i.e.,* $\alpha \geq 0$*, the players have complementarity in information;*
2. *when the players have substitutability in actions, i.e.,* $\alpha \leq 0$*,*
   a. *if the equilibrium strategy* $\sigma_j^*\left(x_j; \overline{F}\right)$ *is decreasing, then player j's information is complementary to player i's information, and*

b. if the equilibrium $\sigma^*\left(x;\overline{F}\right)$ *is increasing, then the players have substitutability in information.*

When the players have complementarity in actions, there exists a unique increasing linear equilibrium under both information structure $\left(\overline{F}_i,\phi\right)$ and $\overline{F}$. Therefore, (1.12) is positive if and only if the signal x_i and x_j are positively correlated. When (1.12) is positive, the marginal problem $\left(\overline{F}_i,\widetilde{v}_i\right)$ is similar to the original problem $\left(\overline{F}_i,\widetilde{u}_i\right)$. Therefore, the value of information is higher in the equilibrium under information structure $\overline{F}$ than in the equilibrium under information structure $\left(\overline{F}_i,\phi\right)$, by Proposition 1.2. The generalization of part (*1*) of Proposition 1.4 to more than 2 players is straightforward. When $N>2$ and $\alpha\geq 0$, there still exists a unique increasing equilibrium that is linear. By the same argument, complementarity in actions translates into complementarity in information. This result is related to the complementarity in information in Vives (1984), Morris and Shin (2002), Angeletos and Pavan (2004), and Hellwig and Veldkamp (2009).

When the players have substitutability in actions, there is no guarantee that an increasing equilibrium exists under information structure $\overline{F}$. However, it is easy to see that the strategy $\sigma^*\left(x;\overline{F}_i,\phi\right)$ is increasing. Suppose further that the equilibrium strategy $\sigma_j^*\left(x_j;\overline{F}\right)$ is decreasing. In this case, the linearity of the strategies and the fact that the signals x_i and x_j are positive correlated allow us to conclude that (1.12) is positive. Again, the marginal problem $\left(\overline{F}_i,\widetilde{v}_i\right)$ is similar to the original problem $\left(\overline{F}_i,\widetilde{u}_i\right)$. Therefore, the value of information is higher in the equilibrium under information structure $\overline{F}$ than in the equilibrium under information structure $\left(\overline{F}_i,\phi\right)$. If there are more than 2 players, the generalization of part (*2.a*) of Proposition 1.4 requires that $\sigma_j^*\left(x_j;\overline{F}\right)$ is decreasing for all $j\neq i$. Moreover, when the players have substitutability in actions but the β's are large enough relative to α, an equilibrium under information structure $\overline{F}$ is increasing. By an argument similar to that for part (*2.a*), (1.13) is positive; therefore, the marginal problem $\left\langle\overline{F}_i,\widetilde{v}_i\right\rangle$ is not strongly similar to the new problem $\left\langle\overline{F}_i,\widetilde{w}_i\right\rangle$. By Proposition 1.3, the value of information is lower in the equilibrium under information structure $\overline{F}$ than in the equilibrium under information structure $\left(\overline{F}_i,\phi\right)$, as in Vives (1984).

1.4.3.2 Monotone Information structure

The second class of information structures we consider is the monotone information structure. Under a monotone information structure, we do not have a closed form solution in general. For this reason, to the best of my knowledge, no paper has studied a quadratic game with a monotone information structure that is not affine. However, we are still able to extend our previous results to monotone information structure.

Definition 1.4 (Monotone information structure)**.** *An information structure is monotone if and only if*

M1 *X_i and Θ are compact and convex subsets of $\mathbb{R}$.*

M2 *For all $i \in I$, $E(\theta|x_i)$ is continuous and increasing in x_i.*

M3 *For all $i, j \in I$, $i \neq j$, $F(x_j|x_i)$ is continuous in x_i and x_j and decreasing in x_i. i.e. The conditional distribution of x_j given x_i can be ordered by first order stochastic dominance.*

Given such an information structure, there exists an unique equilibrium. Moreover, if the players have complementarity in actions, the equilibrium is also increasing, as established in Lemma 1.5. Intuitively, $(M1)$ ensures that the players' best responses are bounded. $(M2)$ implies that given a higher signal x_i the expected state of the world is higher and $(M3)$ implies that player j's expected action is also higher as long as player j's strategy is increasing. Complementarity then ensures that player i would also take a higher action given a higher signal.

Lemma 1.5. *In a quadratic game with monotone information structure, the equilibrium is unique and continuous. Moreover, if the players have complementarity in actions, the equilibrium is also increasing.*

The proof of Lemma 1.5 is simple. Given that our assumption that $|\alpha| < 1$, the best response mapping is a contraction. The contraction mapping theorem then implies that the game has a unique equilibrium.[9] When $\alpha \geq 0$, $(M2)$ and $(M3)$ ensure that the best response mapping also preserves monotonicity, implying an increasing fixed point.

[9]The same approach is used in Mason and Valentinyi (2010) to establish the existence and uniqueness of monotone pure strategy equilibrium in Bayesian games under a different set of assumptions.

Given that $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ are increasing in the signals, $(M3)$ ensures that the covariance between the two strategies must be positive. Just like in the case of the affine information structure, when the players have substitutability in actions, we need additional assumptions on the equilibrium strategies to show the complementarity/substitutability in information. Thus, we have,

Proposition 1.5. *In a quadratic game with monotone information structure,*

1. *when the players have complementarity in actions, i.e., $\alpha \geq 0$, the players have complementarity in information;*
2. *when the players have substitutability in actions, i.e., $\alpha \leq 0$,*
   a. *if the equilibrium strategy $\sigma_j^*\left(x_j; \overline{F}\right)$ is decreasing, then player j's information is complementary to player i's information, and*
   b. *if the equilibrium $\sigma^*\left(x; \overline{F}\right)$ is increasing, then the players have substitutability in information.*

Intuitively, if a player's action is more likely to move together with the state of the world when the player acquires more information, then complementarity in actions leads to complementarity in information, and substitutability in actions to substitutability in information. When the players have complementarity in action, the unique equilibrium is increasing under both information structure $\left(\overline{F}_i, \phi\right)$ and $\overline{F}$. Considering the corresponding equilibrium strategy profiles $\sigma^*\left(x; \overline{F}_i, \phi\right)$ and $\sigma^*\left(x; \overline{F}\right)$, (1.12) is positive by $(M3)$. Therefore, the marginal problem $\left(\overline{F}_i, \widetilde{v}_i\right)$ is similar to the original problem $\left(\overline{F}_i, \widetilde{u}_i\right)$, and the value of information is higher in the equilibrium under information structure $\overline{F}$ than in the equilibrium under information structure $\left(\overline{F}_i, \phi\right)$, by Proposition 1.2.[10] The result extends to $N > 2$ immediately. With the same set of extra assumptions on $\sigma^*\left(x; \overline{F}\right)$ we made in Proposition 1.4, we can also conclude the complementarity/substitutability in information when the players have substitutability in actions. In the proof, the assumption of $(M3)$ enables us to determine the signs

[10]However, notice that the complementarity concluded here is only relative to the null information. In our environment, extra information always leads to a strategy that is more responsive. When the baseline information level is not zero, there is no guarantee that the new strategy is more responsive to the signal then the old strategy. For details, see Jiménez-Martínez (2012) and Ui and Yoshizawa (2015).

of (1.12) and (1.13) as long as the equilibrium strategy is monotone, without relying on a closed form solution.

The monotone information structure goes beyond the normal-quadratic setting. Adhering to the quadratic payoff function greatly simplifies the induced marginal problem. By Lemma 1.3, we only need to evaluate the sign of the covariance between two equilibrium strategies to apply either Proposition 1.2 or Proposition 1.3. Extending our results with monotone information structure beyond the quadratic payoff function, however, will be challenging, as Lemma 1.3 no long holds.

1.4.4 A Global game

In this section, we apply the concept of similarity to obtain sufficient conditions for a decrease in the value of information in a global game with heterogeneous agents. While our sufficient conditions depend on an endogenous object, the conditions give us some insights on how the value of information changes in the global game.

Consider a global game with a continuum of investors indexed by $i \in [0,1]$. Investors choose simultaneously whether to invest (I) or not (N). The economic fundamental is characterized by $\theta \in \Theta \subseteq \mathbb{R}$. For each individual i, investment costs $T_i \geq 0$, where T_i is a differentiable function of the index i with bounded derivative. The return to a successful investment is 1 and an investment is successful if and only if the proportion of investors who choose to invest is high enough, i.e., $p > 1 - \theta$. The payoff to no investment is always 0. To summarize, the payoff function for investor i is given by

$$u_i\left(I, p, \theta\right) = \begin{cases} 1 - T_i & \text{if } p > 1 - \theta, \\ -T_i & \text{o.w.,} \end{cases} \quad \text{and } u\left(N, p, \theta\right) = 0.$$

The payoff function u_i exhibits complementarity in investor i's own action and the other investors' average action p since

$$u_i\left(I, p', \theta\right) - u_i\left(N, p', \theta\right) \geq u_i\left(I, p, \theta\right) - u_i\left(N, p, \theta\right) \text{ if } p' \geq p.$$

We consider how the value of investor i's private information changes with the other investors' strategies. We impose the following assumptions on the information structure.

Definition 1.5 (Monotone information structure in global games)**.** *An information structure F of the global game is monotone if and only if it satisfies the followings:*

MG1.1 *θ and x_i are continuous random variables,*

MG1.2 *The support set Θ is convex, and the support set X_i is convex and compact,*

MG2 *Conditional on θ, x_i is i.i.d.,*

MG3 *$F(x_i|\theta') \leq F(x_i|\theta)$ if $\theta' > \theta$,*

MG4 *There exists $k > 0$ such that for all $x_i \in X_i$ and $\bar{\theta} \in [0,1]$, $\frac{\partial \Pr\left(\theta > \bar{\theta}|x_i\right)}{\partial x_i} \geq k$.*

($MG2$) assumes that the investors are homogeneous in private information. ($MG3$) and ($MG4$) impose monotonicity on the information structure, which, together with the complementarity of own action with both the state of the world and the average action, guarantees the existence of a monotone equilibrium in cut-off strategies. Notice that ($MG4$) also implies that $[0,1] \subseteq \Theta$. We will focus on equilibrium in cut-off strategies. Investor i's strategy σ_i is a cut-off strategy if and only if there exists $\tau_i \in X_i$ such that

$$\sigma_i = \begin{cases} I & \text{if } x_i > \tau_i, \\ N & \text{if } x_i < \tau_i. \end{cases}$$

With a slight abuse of notation, we also denote a cut-off strategy σ_i by its cut-off τ_i. Given that each investor is using a cut-off strategy, ($MG3$) implies that each investor is more likely to invest when the state is high. Thus, we have,

Lemma 1.6. *Given that each investor i uses a cut-off strategy τ_i and the cutoff τ_i is continuous in i, then the average action p is an increasing function of θ.*

Given the cut-off strategies of the investors, the average action p is a deterministic function of the state of the world θ. Since, by Lemma 1.6, $p(\theta)$ is an increasing function that is bounded between 0 and 1, there exists a unique cut-off $\overline{\theta} \in [0,1]$ such that $p\left(\overline{\theta}\right) + \overline{\theta} = 1$. As $\Pr(p > 1 - \theta|x_i = \tau_i) = 1 - F\left(\overline{\theta}|x_i = \tau_i\right)$, the cut-off $\overline{\theta}$ then implies a unique best response in cut-off strategy for each investor. An application of Glicksberg's fixed point theorem shows that there exists an equilibrium in cut-off strategies.

Lemma 1.7. *In a global game with a monotone information structure, there exists an equilibrium in cut-off strategies.*

Holding investor i's cost fixed, consider two games with different cost functions, T'_{-i} and T''_{-i}. In the equilibria in cut-off strategies considered, investor j's optimal cut-off strategies are τ'_j and τ''_j, and the corresponding cut-offs for the state are $\overline{\theta}'$ and $\overline{\theta}''$, respectively.

Investor i's payoff function is

$$\widetilde{u}_i\left(I,\theta\right) = 1_{\left\{\theta > \overline{\theta}'\right\}} - T_i, \text{ and } \widetilde{u}_i\left(N,\theta\right) = 0$$

in the original problem. Investor i's payoff function is

$$\widetilde{w}_i\left(I,\theta\right) = 1_{\left\{\theta > \overline{\theta}''\right\}} - T_i, \text{ and } \widetilde{w}_i\left(N,\theta\right) = 0$$

in the new problem. Therefore, investor i's payoff function is

$$\widetilde{v}_i\left(I,\theta\right) = 1_{\left\{\theta > \overline{\theta}''\right\}} - 1_{\left\{\theta > \overline{\theta}'\right\}}, \text{ and } \widetilde{v}_i\left(N,\theta\right) = 0$$

in the marginal problem. Given the payoff function in the marginal problem, we have,

Lemma 1.8. *(No) Investment is an optimal default action in the marginal problem if and only if* $\overline{\theta}'' \leq (\geq)\, \overline{\theta}'$.

If $\overline{\theta}'' > \overline{\theta}'$, $\widetilde{v}_i\left(I,\theta\right) = -1$ for $\theta \in (\overline{\theta}', \overline{\theta}'']$, and $\widetilde{v}_i\left(I,\theta\right) = 0$ otherwise. Thus, no investment strictly dominates any other cut-off strategies in the marginal problem, and no investment is the optimal default action in the marginal problem. Similarly, investment strictly dominates any other cut-off strategies in the marginal problem if $\overline{\theta}'' < \overline{\theta}'$, and investment is the optimal default action in the marginal problem. Therefore, unless $\overline{\theta}'' = \overline{\theta}'$, the marginal problem is similar to neither the original problem nor the new problem. Consider the two cases in which the default actions in the original and marginal problems are the same and apply Proposition 1.3, we immediately conclude that,

Proposition 1.6. *Investor i's value of information (strictly) decreases if,*

1) the cut-off $\overline{\theta}$ *in the new equilibrium is (strictly) higher than in the original equilibrium, and no investment is an optimal default action in the original equilibrium, or*

2) the cut-off $\overline{\theta}$ *in the new equilibrium is (strictly) lower than in the original equilibrium, and investment is an optimal default action in the original equilibrium.*

Applying the concept of similarity to the marginal problem, we obtain two sufficient conditions for one direction of change in the value of information. Although the cut-off $\overline{\theta}$ is an endogenous object, the conditions provide us some idea of how the value of information changes. Intuitively, when the market sentiment is "bad" enough so that not investing is optimal in the absence of information, a further decrease in the aggregate investment renders information even less useful, as the investor may as well withdraw from the market. Similar intuition applies to the opposite case when the market sentiment is "good". Notice that when both investment and no investment are optimal default actions in the original problem, Proposition 1.6 implies that investor i's value of information must decrease, regardless of how the strategies of the other investors change.

1.4.5 Multi-expert persuasion game

In this section, we apply the concept of similarity to deduce the substitutability of information in a multi-expert persuasion game studied by Bhattacharya and Mukherjee (2013). Some assumptions in Bhattacharya and Mukherjee (2013) that are unimportant for our purpose are relaxed. The model has a single decision-maker (DM) and two experts. The players have a commonly known prior belief on the state of the world $\theta \in [0,1]$ that is distributed according to a probability density function f that is continuous, bounded above, and has full support. The DM's payoff $u^{DM}(y,\theta)$ depends on the state of the world θ and the action $y \in [0,1]$ she takes. The function u^{DM} is twice differentiable. Moreover, given $\theta \in [0,1]$, the function $u^{DM}(.,\theta)$ is strictly concave and is maximized at $y = \theta$. The payoff of expert $i \in \{1,2\}$ is given by the function $u_i(y,\theta)$. The two experts have opposite and extreme biases. That is, given $\theta \in [0,1]$, $u_1(.,\theta)$ is strictly increasing and $u_2(.,\theta)$ is strictly decreasing. If expert i acquires information, expert i receives signal $x_i = \theta$ with probability $p_i \in (0,1)$ and $x_i = \varphi$ with probability $1 - p_i$, independent of the signal of expert j. With the null information structure, expert i always receives the null signal $x_i = \varphi$.

The sequence of events is as follows. First, each player receives a signal privately and then simultaneously sends a message to the DM. The message m_i that expert i can send is restricted to $\{\varphi, x_i\}$. In other words, he can only choose to reveal or conceal his signal. In particular, expert i can only send $m_i = \varphi$ if $x_i = \varphi$. Next, the DM chooses y based on the messages she receives and the game ends.

Bhattacharya and Mukherjee (2013) show that the equilibrium of this game is characterized by a null action $\mathbf{y}^*$, which the DM takes when $m_1 = m_2 = \varphi$. Moreover, expert 1 (2) reveals his signal if $x_1 > \mathbf{y}^*$ ($x_2 < \mathbf{y}^*$) and hides his signal if $x_1 < \mathbf{y}^*$ ($x_2 > \mathbf{y}^*$). When the received signal is equal to $\mathbf{y}^*$, the expert is indifferent between sending $\mathbf{y}^*$ or φ and either message can be sent in equilibrium. When a nonempty message, i.e., $m_i \neq \varphi$, is received, the DM chooses the optimal action $y = m_i$.

We are interested in how the presence of expert 2 changes the value of information to expert 1. In answering this question, we consider two equilibria: in one equilibrium, expert 2 has no information; and in another equilibrium, expert 2 has information. We call the former the original equilibrium and the latter the new equilibrium. Let $\mathbf{y}_n^*$ ($\mathbf{y}_o^*$) be the null action in the equilibrium in which expert 1 has information and expert 2 has information (no information). We have,

Lemma 1.9. *The equilibrium null action shifts towards expert* 1*'s preferred action after expert* 2 *have acquired information, but it does not reach the upper bound. i.e.,* $\mathbf{y}_o^* < \mathbf{y}_n^* < 1$.

We construct the original problem and the new problem corresponding to the original equilibrium and the new equilibrium, respectively. Then we define the induced marginal problem accordingly. Given null information, expert 1 can only send $m_1 = \varphi$; therefore, expert 1's default action is identical in all three problems. The payoff difference between expert 1's optimal decision rule for the new problem, the cut-off strategy $\mathbf{y}_n^*$, and the default action $m_1 = \varphi$ in the marginal problem is

$$-p_1 \int_{\mathbf{y}_n^*}^{1} \left\{ u_1\left(\mathbf{y}_n^*, \theta\right) - u_1\left(\mathbf{y}_o^*, \theta\right) \right\} f\left(\theta\right) d\theta. \tag{1.14}$$

By Lemma 1.9, (1.14) is always negative. Thus, the marginal problem is not similar to the new problem.[11] Applying Proposition 1.3, we conclude that

Proposition 1.7. *In the persuasion game of Bhattacharya and Mukherjee (2013), experts with opposite and extreme biases have substitutability in information.*

[11]Similarly, we can also show that the marginal problem is not similar to the original problem.

This result can be understood intuitively. When an expert with opposite and extreme bias acquires information and reports strategically, the DM increases her null action from $\mathbf{y}_o^*$ to $\mathbf{y}_n^*$. This change has two effects on the value of expert 1's information. First, whenever expert 1 reveals his information in the new equilibrium, the DM's null action is lifted from $\mathbf{y}_n^*$ to θ instead of from $\mathbf{y}_o^*$. Since $\mathbf{y}_o^* < \mathbf{y}_n^*$, the gain from revealing information is less in the new equilibrium. The decrease in the expected gain from revealing information is captured by (1.14), which we obtain from the marginal problem. Second, Expert 1 has less opportunity to use his information in the new equilibrium. Since he only uses his information when the state is higher than the null action, $\mathbf{y}_o^* < \mathbf{y}_n^*$ implies that information is less likely to be useful to expert 1 in the new problem. Notice that the second effect does not appear in our calculation because, by applying Proposition 1.3, we have bypassed it by noticing that it must be negative.[12]

One might be interested in knowing if the answer to our question changes when the experts have the *same* bias. In this case, applying the concept of similarity yields no prediction. Although the negative forces identified in the previous paragraph seem to turn positive when the DM's null action decreases, one must add to it the effect of expert 2's disclosure. Since the experts have the same ordinal preference, whenever expert 1 would like to reveal information, so does expert 2. Thus, expert 1's disclosure may be unnecessary and duplicate expert 2's. This effect reduces the value of information.[13] Without specifying the functions involved any further, one cannot compare the positive and negative effects and decide if there is complementarity in information.

[12]The negative effect arises from the suboptimality of the optimal strategy for the new problem in the original problem.

[13]The negative effect of *duplicated disclosure* on the value of information is best illustrated by the presence of a non-strategic expert 2, i.e., expert 2 sends $m_2 = \varphi$ only when $x_2 = \varphi$. In this case, a change from an equilibrium in which expert 2 has no information to an equilibrium in which expert 2 has information decreases the value of information to expert 1. This is because with some probability, the DM learns the true state from expert 2 and expert 1's disclosure does not make a difference. In Appendix A, we demonstrate this by showing that the marginal problem is not similar to the new problem.

1.5 CONCLUSION

We introduce the concept of similarity in this paper. This concept speaks to a lot of decision making situations. Given a fixed information structure, the ranking of value of information is closely related to the idea of similarity. We apply the concept of similarity to quadratic game, persuasion game and global game. In all three games, we study the value of information in equilibrium without relying on a closed form solution. The same approach may be used to explore a wider range of games.

In the future, we hope to explore the concept of similarity in individual decision problems. In our daily lives, we often make decisions based on our instinct rather than rational calculations. Our instinct in turn builds on years of experience with problems that are similar but not identical to the present problem. Also, firms develop payment structures, corporate cultures, and ranking systems. None of these are tailored incentive schemes, but they work reasonably well for employees that do not have too much idiosyncrasy. Therefore, studying how similarity may play a role in designing decision rules for a family of decision problems may advance our understanding of decision making under bounded rationality. Finally, we show in this paper that similarity does not have nice properties in general, in the future, we hope to find useful conditions under which the similarity relation is symmetric and transitive.

2.0 WHEN MONITORING HURTS: ENDOGENOUS INFORMATION ACQUISITION IN A GAME OF PERSUASION

2.1 INTRODUCTION

Conflict of interest is an important issue for many regulatory procedures and market practices. For example, in the aviation industry, dangerous mechanical flaws have been found to be known to the manufacturers but were kept from the public before they caused major accidents. In the crash of Turkish Airlines Flight 981, technical fault with the cargo door was recognized by the McDonnel Douglas Company even during the design process of DC-10. But the company was financially strained and eager to put the new model into the market. 346 people were killed in the incident. A lot of questions have been raised subsequently on engineering ethics and on the National Transportation Safety Board's (NTSB) regulatory policy.

It is widely recognized that when experts have their own interest in the decision, they may influence the decision maker by withholding information. In the DC-10 example, reporting the technical fault would definitely sabotage the sale of the new model. Yet, one dimension of the problem that is often overlooked is that the incentives for expert to collect decision-relevant data also depend on the decision maker's policy. If the expert does not expect to influence the decision, resource would not be put to such activities. We study the interactions between the expert and the decision maker (DM) in a formal model and ask the following question: Is it beneficial for a DM to monitor the expert's data collection activities?

We show that monitoring may hurt the DM. More specifically, we consider a model with a DM, who needs to make a state-contingent decision, and an expert, who has the capacity to acquire information about the state of nature. The DM always wants to match the action with the state, but the expert is biased and prefers a higher final action. The DM could learn from the expert about the state by communication, but cannot commit to her actions. Before communication begins, the expert puts in effort to conduct investigations. Higher effort results in a higher probability of obtaining a piece of hard evidence about the exact realization of the state of nature. In the communication stage, the expert chooses whether to present or conceal the hard evidence he has obtained. As information conveyed by the hard evidence is assumed to be verifiable by the DM, the DM will always choose the action that is most appropriate whenever hard evidence is presented. When no evidence is presented, however, the DM needs to make inference on the lack of evidence. As a result, the expert only reveals information when signals are favorable. We distinguish two games with respect to the monitoring of effort. In the *overt game*, the expert's effort is publicly monitored by the DM. In the *covert game,* the expert's effort is private.

When the expert's bias is extreme, the expert always acquire more information in the covert game than in the overt game. The logic of this result can be understood intuitively as follows. Suppose in the overt game the DM observes a high effort and no evidence is presented afterwards. The DM rationally concludes there is a high probability that the expert is withholding unfavorable evidence, and thus the state is likely to be relatively low. Therefore, the final action taken by the DM upon observing a high effort would also be low in the lack of hard evidence. This consideration is foreseen by the expert in the information acquisition stage. Thus, he would lower his effort accordingly to avoid the unfavorable action in case of concealing. On the other hand, such strategic consideration is absent in the covert game. As a result, the expert exerts more effort without the DM's monitoring. This argument shows that when the effort level is observable to the DM, an increase in effort has two opposite effects on the expert's gain, namely, the information precision effect and the default action effect. The information precision effect is always positive, as the expert is free to use the evidence once he has obtained it. This creates an option value. The default action effect, however, arises from the DM's adjustment of expectation and is always negative. In the covert game, the expert faces no default action effect when choosing the effort level, because the default action depends on expected effort level only. This argument carries on to the situation with a less extreme expert.

The DM's expected payoff is strictly increasing in the expert's effort level when the expert's gain from the DM's decision is strictly monotone. With this assumption, we are able to show that the DM strictly prefers to play the covert game in a very general setting. When this assumption does not hold, i.e., an expert with a small upward bias, the DM does not necessarily always prefer higher effort. To discuss the case of a less extreme expert, we employ a "uniform-quadratic" specification of our model. We show that this result does not apply to settings in which the preferences of the expert and the DM are partially aligned. Monitoring *could* be beneficial if the DM and expert's interests are partially aligned. However, contrary to everyday experiences, monitoring is beneficial only when the expert has a small enough bias, but never so if the expert has a large bias.

2.1.1 Related Literature

Our paper belongs to the literature of "persuasion games" introduced by Milgrom and Roberts (1986). This class of games differs from another important class of games, the "cheap talk" games introduced by Crawford and Sobel (1982), by the assumption of verifiable information transmission. Although our game is not a cheap talk game, parallel development in the cheap talk literature is also worth noting.

The related literature can be roughly grouped into three categories. The first category, to which this paper belongs, consists of single-expert models with endogenous information acquisition before communication begins. The second category consists of multiple-expert models with exogenous "expertise". The third category consists of multiple-expert models with endogenous information acquisition.

In the first category, the paper closest to ours is Henry (2009). He considers a persuasion game setting in which the unraveling result of Milgrom and Roberts (1986) holds. As a result, the DM's payoff is always monotone in the expert's effort, so monitoring always hurts regardless of the expert's bias. In contrast, bias matters in our setting. Another related paper in the first category is Che and Kartik (2009). They consider a covert persuasion game similar to ours and show that a greater difference between the prior beliefs of the expert and the DM provides more incentives for the expert to exert effort but also causes him to disclose less of the information he acquires. The choice of effort level is always private in their paper. Argenziano, Squintani and Severinov (2015) and Pei (2015) consider both overt and covert games in cheap talk settings. Pei (2015), in particular, shows that monitoring can hurt, but the logic of his result is quite different from ours.

In the second category, the paper closest to ours is Bhattacharya and Mukherjee (2013). In a multiple-expert persuasion game, they show that from the DM's standpoint, higher "quality" of the expert (i.e., a higher probability of acquiring information) does not necessarily improve the DM's ex ante welfare. Moreover, the DM always prefers expert with "extreme" preference. Our paper further endogenizes the "quality" parameter in their model. Many papers in the cheap talk literature are devoted to the study of competition between experts. Examples include Battaglini (2002), Krishna and Morgan (2001), and many others.

In the last category, Kartik, Xu Lee and Suen (2015) show in a multiple-expert covert persuasion game that competition reduces the experts' effort and it is possible that the DM gets worse off by hearing from one more expert.

In our model, the DM (principal) is unable to commit to an action. Contract based on the expert's (agent) report or effort cannot be made ex ante. This assumption is key to our results. With commitment, additional signal of the agent's action can never be detrimental to the principal. The principal can simply credibly ignore the information. In the "classical" moral hazard principal-agent problem of Holmström (1979), additional information about agent's action can never hurt the principal. It is strictly beneficial if and only if the principal does not already observe a signal that is a sufficient statistic for the additional information. In a cheap talk setting, Szalay (2005) incorporates both commitment power and costly information acquisition.

The rest of the paper is organized as follows. Section 2.2 introduces the model. Section 2.3 provides the characterizations of the equilibria. Section 2.4 presents the main results. Section 2.5 extends the analysis to the "uniform-quadratic" case. Section 2.6 concludes. Appendix B contains all proofs omitted in the text.

2.2 MODEL

We study a persuasion game between a decision-maker (DM) and an expert. The DM needs to choose an action $y \in Y$ that is most appropriate given the underlying state of the nature $\theta \in \Theta$. We assume that Θ is a compact subset of Y and Y is a compact and convex subset of the real line $\mathbb{R}$ itself.[1] The payoff of the DM from taking action $y \in Y$ in state $\theta \in \Theta$ is given by the function $u^{DM}(y, \theta)$. The function $u^{DM} : Y \times Y \to \mathbb{R}$ is twice differentiable.[2] Moreover, given $\theta \in \Theta$, the function $u^{DM}(\cdot, \theta)$ is strictly concave and is maximized at $y = \theta$. As $\Theta \subseteq Y$, the DM would always take the action $y = \theta$ if θ is known to her. The expert's gain from the DM's action is given by a twice differentiable function $u^E : Y \times Y \to \mathbb{R}$. We call $u^E(y, \theta)$ the *gain function* of the expert. Notice that in our model the expert's gain could be negative even though its name may suggest otherwise. We make the following assumption on u^E throughout

Assumption **MON:** Given $\theta \in \Theta$, the function $u^E(\cdot, \theta)$ is strictly increasing.

That is, the expert always prefers a higher action. This leads to a conflict of interest between the DM and the expert. This assumption on the expert's gain function are admittedly restrictive. Nevertheless, it is also realistic and seems to match many real-world examples of persuasion. For example, the DM could be the congress or any decision-making body in a government, the expert could be an interest group from an industry and the action y could be taken as the amount of subsidy on that industry. This assumption will be relaxed in Section 2.5. Note also that the standard concavity assumption of utility function is not imposed to the expert here.

The DM and the expert have a commonly known prior belief on the state of nature that is given by a probability distribution function $F : Y \to \mathbb{R}$. The support of probability distribution function $F(\cdot)$ is equal to Θ. The distribution admits a probability density function f that is continuous and bounded above.

[1] The assumption that the sets Θ and Y are bounded is not important for our results. Allowing these sets to be unbounded brings technical complications but no additional economic insights. For example, extra assumptions must be made to make sure all the integrals involved in our calculations exist.

[2] The extension of the domain of u^{DM} from $Y \times \Theta$ to $Y \times Y$ allows us to simplify some notations. It is of no consequence otherwise.

The expert can potentially obtain hard evidence about the realized state θ by engaging in costly investigation. If his investigation is successful, he knows exactly the realized state of nature θ and is able to demonstrate the truth by presenting the verifiable hard evidence he obtained. In the beginning of the game, the expert chooses the probability that his investigation is successful, $p \in [0, \overline{p}]$, where $0 < \overline{p} < 1$, at a cost $c(p)$. We will refer to p as the probability of success or the *effort level* interchangeably. We say a cost function $c(\cdot)$ is *regular* if it is twice differentiable, strictly increasing, and convex and it satisfies the Inada conditions $c'(0) = 0$ and $c'(p) \to \infty$ as $p \to \overline{p}$. Except in Example 2.2 of Section 2.5, we assume *regular* cost function throughout this paper. The parameter $\overline{p}$ is a upper bound on the probability of success that the expert can achieve and thus represents a limit of the investigation technology.

After the probability of success p is chosen, the expert privately observes the outcome of the investigation. At this stage, the expert can either be "informed" (the investigation is successful) or "uninformed" (the investigation is not successful) about the true state of nature. Next, the expert sends a report m to the DM. If the expert is informed, he has the choice of disclosing the state by presenting the verifiable evidence he obtained (i.e., $m = \theta$) or conceal it (i.e., $m = \phi$). On the other hand, an uninformed expert has no choice but to send an empty report (i.e., $m = \phi$). After receiving the report, the DM chooses the action y and payoffs are realized.

Following Argenziano, Squintani and Severinov (2014), we call the game in which the choice of the effort level p is publicly observed the *overt game* and the game in which the DM does not observe the expert's choice of p the *covert game*. The focus of this paper is to compare these two games.

A pure strategy of the expert in both games is given by a pair $(\tilde{p}, m(p, \theta))$, where $\tilde{p} \in [0, \overline{p}]$ is the expert's choice of effort level and $m(p, \theta) \in \{\theta, \phi\}$ specifies the expert's report if he has exerted effort $p \in [0, \overline{p}]$ and he is informed that the realized state is $\theta \in \Theta$. Notice that $m(p, \theta)$ should be well-defined given all $p \in [0, \overline{p}]$ and $\theta \in \Theta$. In the overt game, a pure strategy of the DM is a function $y\,(p, m)$, which specifies the action y of the DM if choice of effort p is observed and report m is received. In the covert game, a pure strategy $y\,(m)$ of the DM is a function of the report m only, as the choice of effort p is not observed by the

DM.

The solution concept we used is perfect Bayesian equilibrium (PBE) in pure strategies. We will simply refer to it as equilibrium at times. Let $\mu_O(\theta|p,m)$ be the posterior belief of the DM upon receiving the expert's report m and observing the choice of effort p in the overt game. Let $\mu_C(p,\theta|m)$ be the posterior belief of the DM upon receiving the expert's report m in the covert game. We have the following definitions.

Definition 2.1. *A strategy profile* $\left(\left(p_O^*, m_O^*(p,\theta)\right), y_O^*(p,m)\right)$ *along with a belief* $\mu_O^*(\theta|p,m)$ *constitutes a PBE of the overt game if the following holds:*

1. *The expert choose* p_O^* *to maximize expected payoff given* $m_O^*(p,\theta)$ *and* $y_O^*(p,m)$.
2. *For all* $p \in [0,\bar{p}]$, $\theta \in \Theta$, $m_O^*(p,\theta) = \theta$ *if and only if*
$$u^E\left(y_O^*(p,\theta),\theta\right) > u^E\left(y_O^*(p,\phi),\theta\right).$$
3. *For all* $p \in [0,\bar{p}]$, $m \in \Theta \cup \{\phi\}$, *the DM's action* $y_O^*(p,m)$ *satisfies*
$$y_O^*(p,m) = \arg\max_{y\in Y} \int_\Theta u^{DM}(y,\theta)\, d\mu_O^*(\theta|p,m).$$
4. *The posterior belief of the DM* $\mu_O^*(\theta|p,\phi)$ *is obtained by using Bayes rule given the prior belief* $F(\theta)$, *the choice of effort* p *observed, the actual report of the expert* m *and the reporting strategy* $m_O^*(p,\theta)$ *whenever possible. Also, if the expert makes an out-of-equilibrium report that reveals the state* θ, *the off-equilibrium belief is not allowed to put weight on any* $\theta' \in \Theta$ *other than* θ. *That is, for all* $\overline{\theta} \in \Theta$, $\theta \in \Theta$,
$$\int_{\left\{\theta'\in\Theta:\theta'\leq\overline{\theta}\right\}} d\mu_O^*(\theta'|p,\theta) = \mathbf{1}_{\left\{\overline{\theta}\geq\theta\right\}}\left(\overline{\theta}\right),$$
where $\mathbf{1}_{\left\{\overline{\theta}\geq\theta\right\}}$ *is indicator function for the event* $\left\{\overline{\theta}\in\Theta|\overline{\theta}\geq\theta\right\}$.

Definition 2.2. *A strategy profile* $\left(\left(p_C^*, m_C^*(p,\theta)\right), y_C^*(m)\right)$ *along with a belief* $\mu_C^*(p,\theta|m)$ *constitutes a PBE of the covert game if the following holds:*

1. *The expert choose* p_C^* *to maximize expected payoff given* $m_C^*(p,\theta)$ *and* $y_C^*(m)$.
2. *For all* $p \in [0,\bar{p}]$, $\theta \in \Theta$, $m_C^*(p,\theta) = \theta$ *if and only if*
$$u^E\left(y_C^*(\theta),\theta\right) > u^E\left(y_C^*(\phi),\theta\right).$$

3. *For all $m \in \Theta \cup \{\phi\}$, the DM's action $y_C^*(m)$ satisfies*

$$y_C^*(m) = \arg\max_{y \in Y} \int_{[0,\overline{p}]\times\Theta} u^{DM}(y,\theta)\, d\mu_C^*(p,\theta|m).$$

4. *The posterior belief of the DM $\mu_C^*(p,\theta|m)$ is obtained by using Bayes rule given the prior belief $F(\theta)$, the actual report of the expert m and the reporting strategy $m_C^*(p,\theta)$ whenever possible. Moreover, the belief is consistent with the equilibrium choice of effort level p_C^*. That is, for all $p \in [0, p_C^*)$, for all $m \in \Theta \cup \{\phi\}$,*

$$\int_{[0,p]\times\Theta} d\mu_C^*(p',\theta|m) = 0.$$

Moreover, for all $p \in [p_C^*, \overline{p}]$, for all $m \in \Theta \cup \{\phi\}$, $\int_{[0,p]\times\Theta} d\mu_C^*(p',\theta|m)$ is independent of p. Also, if the expert makes an out-of-equilibrium report that reveals the state θ, the off-equilibrium belief is not allowed to put weight on any $\theta' \in \Theta$ other than θ. That is, for all $\overline{\theta} \in \Theta$, $\theta \in \Theta$,

$$\int_{[0,\overline{p}]\times\left\{\theta'\in\Theta:\theta'\leq\overline{\theta}\right\}} d\mu_C^*(p',\theta'|\theta) = \mathbf{1}_{\left\{\overline{\theta}\geq\theta\right\}}\left(\overline{\theta}\right),$$

where $\mathbf{1}_{\left\{\overline{\theta}\geq\theta\right\}}$ is indicator function for the event $\left\{\overline{\theta}\in\Theta|\overline{\theta}\geq\theta\right\}$.

Two remarks are in order. First, we have assumed in condition 2 of Definitions 2.1 and 2.2 that when the expert is indifferent between sending a true report and concealing the evidence, the expert always conceals the evidence. As the probability that the expert is indifferent is always zero, this restriction is without loss of generality. Second, following Bhattacharya and Mukherjee (2013), we impose an off-the-equilibrium path belief restriction in Definitions 2.1 and 2.2 that is not a part of the canonical definition of the PBE. The extra assumptions are needed because our modelling assumptions require that whenever the hard evidence is presented, the DM is able to verify the true state, even if the hard evidence is not supposed to be presented in equilibrium.

To begin our analysis of the overt and covert games, we introduce an auxiliary game $\Gamma(p)$ for each $p \in [0,\overline{p}]$. An auxiliary game $\Gamma(p)$ is a proper subgame of the overt game, which starts after the effort level is publicly chosen to be p. Our equilibrium notion for the overt game requires any equilibrium of the overt game to generate a PBE in each of the auxiliary game $\Gamma(p)$. Notice that an auxiliary game $\Gamma(p)$ is not a subgame of the covert game. However, it will be important for us to analyze the covert game as we shall see in the next section.

Definition 2.3. *Given $p \in [0, \overline{p}]$, a strategy profile $(m^*(\theta), y^*(m))$ along with a belief $\mu^*(\theta|m)$ constitutes a PBE of the auxiliary game $\Gamma(p)$ if the following holds:*

1. *For all $\theta \in \Theta$, $m(\theta) = \theta$ if and only if*

$$u^E(y^*(\theta), \theta) > u^E(y^*(\phi), \theta).$$

2. *For all $m \in \Theta \cup \{\phi\}$, the DM's action $y^*(m)$ satisfies*

$$y^*(m) = \arg\max_{y \in Y} \int_{\Theta} u^{DM}(y, \theta)\, d\mu^*(\theta|m).$$

3. *The posterior belief of the DM $\mu^*(\theta|m)$ is obtained by using Bayes rule given the prior belief $F(\theta)$, the actual report of the expert m and the reporting strategy $m^*(\theta)$ whenever possible. Also, if the expert makes an out-of-equilibrium report that reveals the state θ, the off-equilibrium belief is only allowed to put weight on θ. That is, for all $\overline{\theta} \in \Theta$, $\theta \in \Theta$,*

$$\int_{\left\{\theta' \in \Theta : \theta' \leq \overline{\theta}\right\}} d\mu^*(\theta'|\theta) = \mathbf{1}_{\left\{\overline{\theta} \geq \theta\right\}}\left(\overline{\theta}\right),$$

where $\mathbf{1}_{\left\{\overline{\theta} \geq \theta\right\}}$ is indicator function for the event $\left\{\overline{\theta} \in \Theta | \overline{\theta} \geq \theta\right\}$.

2.3 EQUILIBRIUM CHARACTERIZATIONS

Without loss of generality, we further assume $Y = [0,1]$ from now on to simplify the presentation. To begin with, we characterize equilibrium of the auxiliary game $\Gamma(p)$. Our auxiliary game $\Gamma(p)$ is similar to the single-expert version of Bhattacharya and Mukherjee (2013). It can be characterized accordingly.

Proposition 2.1. *Given $p \in [0, \overline{p}]$, the unique equilibrium of the auxiliary game $\Gamma(p)$ is characterized by a default action $\mathbf{y}^*(p) \in (0,1)$ such that the DM takes the default action $\mathbf{y}^*(p)$ whenever the expert fails to report the state and the corresponding action if the expert reports the state. That is,*

$$y^*(m) = \begin{cases} \theta & \text{if } m = \theta \\ \mathbf{y}^*(p) & \text{if } m = \phi \end{cases},$$

where $\mathbf{y}^*(p)$ is pinned down by

$$p \int_0^{\mathbf{y}^*(p)} u_1^{DM}(\mathbf{y}^*(p), \theta) f(\theta) \, d\theta + (1-p) \int_0^1 u_1^{DM}(\mathbf{y}^*(p), \theta) f(\theta) \, d\theta = 0. \tag{2.1}$$

Moreover, the expert reports the state when $\theta > \mathbf{y}^*(p)$ and remains silent when $\theta \leq \mathbf{y}^*(p)$. That is,

$$m^*(\theta) = \begin{cases} \theta & \text{if } \theta > \mathbf{y}^*(p) \\ \phi & \text{if } \theta \leq \mathbf{y}^*(p) \end{cases}.$$

Proof. See Appendix B. □

To understand the characterization, we work backward from the last stage of the game. If the state is revealed, the DM trivially takes the action that exactly matches the state. If the expert fails to reveal the state, the DM takes a default action $\mathbf{y}^*(p)$ that maximizes her expected payoff taking into account the possibility that the expert fails to obtain any information as well as the possibility that the expert does know the true state but withholds the information under the equilibrium reporting strategy m^*. Next, as the gain of the expert is strictly increasing in the DM's action, the expert reports the state when $\theta > \mathbf{y}^*(p)$ and remains silent when $\theta \leq \mathbf{y}^*(p)$. The first order condition of the DM's optimization problem together with the reporting strategy m^* give us (2.1). Proposition 2.1 implies that $y^*(m)$ is unique given p and $m^*(\theta)$ is unique given $\mathbf{y}^*(p)$. To stress the dependence, we also write $y^*(m)$ and $m^*(\theta)$ as $y^*(m|p)$ and $m^*(\theta|\mathbf{y}^*(p))$, respectively.

The importance of Proposition 2.1 is that the default action $\mathbf{y}^*(p)$ fully characterizes the equilibrium of the auxiliary game $\Gamma(p)$. Using the function $\mathbf{y}^*(p)$, we write the ex ante equilibrium payoff of the DM and the ex ante equilibrium gain of the expert of the auxiliary game $\Gamma(p)$ as functions of p and $\mathbf{y}^*(p)$ only. Define

$$\begin{aligned} U^{DM}(p, \mathbf{y}^*(p)) \equiv \; & p\int_{\mathbf{y}^*(p)}^{1} u^{DM}(\theta,\theta) f(\theta)\, d\theta + p\int_{0}^{\mathbf{y}^*(p)} u^{DM}(\mathbf{y}^*(p),\theta) f(\theta)\, d\theta \\ & + (1-p)\int_{0}^{1} u^{DM}(\mathbf{y}^*(p),\theta) f(\theta)\, d\theta. \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} U^{E}(p, \mathbf{y}^*(p)) \equiv \; & p\int_{\mathbf{y}^*(p)}^{1} u^{E}(\theta,\theta) f(\theta)\, d\theta + p\int_{0}^{\mathbf{y}^*(p)} u^{E}(\mathbf{y}^*(p),\theta) f(\theta)\, d\theta \\ & + (1-p)\int_{0}^{1} u^{E}(\mathbf{y}^*(p),\theta) f(\theta)\, d\theta \end{aligned} \tag{2.3}$$

We have the following result.

Proposition 2.2. *There always exists an equilibrium of the overt game. A strategy profile* $\left(\left(p_O^*, m_O^*(p,\theta)\right), y_O^*(p,m)\right)$ *of the overt game is an equilibrium strategy profile if and only if*

$$p_O^* = \arg\max_{p\in[0,\overline{p}]} U^E(p, y^*(p)) - c(p), \tag{2.4}$$

$$m_O^*(p,\theta) = m^*(\theta|\mathbf{y}^*(p))$$

and

$$y_O^*(p,m) = y^*(m|p).$$

Proof. See Appendix B. □

To proceed, we take a look at the first order condition for (2.4), if p is endogenous and observable to the DM, the expert's optimal choice of p must satisfy

$$\frac{dU^E(p,\mathbf{y}^*(p))}{dp}\begin{cases}=c'(p) & \text{if } p>0\\ \leq c'(p) & \text{if } p=0\end{cases}. \tag{2.5}$$

Note that (2.5) is only necessary for (2.4). Moreover, the effect of a publicly observed change in probability p on the expert's gain can be decomposed in the following way,

$$\frac{dU^E(p,\mathbf{y}^*(p))}{dp}=\underbrace{U_1^E(p,\mathbf{y}^*(p))}_{\text{information precision effect}}+\underbrace{U_2^E(p,\mathbf{y}^*(p))\,\mathbf{y}^{*\prime}(p)}_{\text{default action effect}}, \tag{2.6}$$

where the partial derivative $U_1^E(p,\mathbf{y}^*(p))$ represents the effect of a small increase in the probability p on the expected gain of the expert, *holding the strategic interactions between the DM and the expert unchanged*. The term $U_2^E(p,\mathbf{y}^*(p))$ represents the effect of a small increase in the default action $\mathbf{y}^*(p)$ on the expected gain of the expert, *holding the probability p constant.* The derivative $y^{*\prime}(p)$ measures how the default action $\mathbf{y}^*(p)$ responds to a publicly observed change in the probability p. We will show later that the information precision effect is always positive and the default action effect is always negative.

Next, suppose that p is endogenous but unobservable to the DM, the DM's default action is fixed as a particular $\mathbf{y}\in(0,1)$, not responding to p. Because the optimal reporting strategy of the expert depends on the DM's default action only, it is also fixed. Therefore, fixing the DM's default action also keeps the strategic interactions unchanged. The relevant first order condition becomes

$$U_1^E(p,\mathbf{y}^*(p))\begin{cases}=c'(p) & \text{if } p>0\\ \leq c'(p) & \text{if } p=0\end{cases}. \tag{2.7}$$

This leads to our equilibrium characterization of the covert game.

Proposition 2.3. *There always exists an equilibrium of the covert game. A strategy profile $((p_C^*, m_C^*(p,\theta)), y_C^*(m))$ of the covert game is an equilibrium strategy profile if and only if*

$$U_1^E(p_C^*,\mathbf{y}^*(p_C^*))\begin{cases}=c'(p_C^*) & \text{if } p_C^*>0\\ \leq c'(p_C^*) & \text{if } p_C^*=0\end{cases},$$

$$m_C^*(p,\theta)=m^*(\theta|\mathbf{y}^*(p_C^*)),$$

and

$$y_C^*(m) = y^*\left(m|p_C^*\right).$$

Proof. See Appendix B. □

Unlike the case of the overt game, checking the relevant second order condition shows that the first order condition (2.7) can be used to fully characterize the equilibrium effort level p_C^*. In Propositions 2.1-2.3, we have characterized the equilibrium strategy profiles without referencing to the equilibrium beliefs directly. We will, therefore, simply refer to an equilibrium strategy profile as equilibrium hereafter.

2.4 MAIN RESULTS

To compare the equilibria of the overt and covert games, we first investigate the dependence of the DM's ex ante equilibrium payoff $U^{DM}\left(p, \mathbf{y}^*\left(p\right)\right)$ on p. This would allow us to rank equilibria of the overt and covert games according to the DM's ex ante welfare.

Proposition 2.4. *The DM's ex ante welfare in the equilibrium of the auxiliary game* $\Gamma\left(p\right)$ *always increases with the probability* p*. That is, for each* $p' \in [0, \overline{p}]$*,*

$$\frac{dU^{DM}\left(p, \mathbf{y}^*\left(p\right)\right)}{dp}|_{p=p'} > 0.$$

Proof. See Appendix B. □

As we have restricted ourselves to equilibria in pure strategies, the equilibrium effort level must be a constant in both games. Proposition 2.4 shows that the DM always prefers an equilibrium with higher effort level. Therefore, to compare the DM's welfare, we only need to compare the effort levels. Next, we turn to study the properties of $\frac{dU^E(p,\mathbf{y}^*(p))}{dp}$ and $U_1^E\left(p, \mathbf{y}^*\left(p\right)\right)$. This would allow us to study the equilibrium effort levels through (2.5) and (2.7). We prove in Lemma B.1 in the Appendix B that for all $p \in [0, \overline{p}]$,

$$\int_{\mathbf{y}^*(p)}^{1} f\left(\theta\right) d\theta > 0. \tag{2.8}$$

That is, the probability that the true state is higher than the equilibrium default action $\mathbf{y}^*(p)$ is always positive. Next, partial-differentiating (2.3) with respect to p leads to

$$U_1^E(p, \mathbf{y}^*(p)) = \int_{\mathbf{y}^*(p)}^{1} \left\{u^E(\theta, \theta) - u^E(\mathbf{y}^*(p), \theta)\right\} f(\theta)\, d\theta > 0. \tag{2.9}$$

As $u^E(\cdot, \theta)$ is strictly increasing, by (2.8), the expression in (2.9) is strictly positive. Intuitively, investment in concealable information is an option investment. The expert only reveals the state when the state is favorable (i.e., $\theta > \mathbf{y}^*(p)$) and gains $u^E(\theta) - u^E(\mathbf{y}^*(p))$ by doing so. Thus, the value of information cannot be negative to the expert in the covert game. It follows from (2.7) and (2.9) that all equilibria of the covert game involve positive equilibrium effort. Next, by Assumption **MON**, we have, for all p, $p' \in [0, \overline{p}]$,

$$U_2^E(p, \mathbf{y}^*(p')) = p \int_0^{\mathbf{y}^*(p')} u_1^E(\mathbf{y}^*(p'), \theta) f(\theta)\, d\theta + (1-p) \int_0^1 u_1^E(\mathbf{y}^*(p'), \theta) f(\theta)\, d\theta > 0. \tag{2.10}$$

Note that the inequality is true even for off-equilibrium $\mathbf{y}^*$. It will become be clear later from the proof of Theorem 2.1 that this property allows us to compare *all* equilibrium of the covert game with *all* equilibrium of the overt game, without which only set ordering is possible. Finally, an application of the implicit function theorem to (2.1) leads to

$$\mathbf{y}^{*\prime}(p) = \frac{\displaystyle\int_{\mathbf{y}^*(p)}^{1} u_1^{DM}(\mathbf{y}^*(p), \theta) f(\theta)\, d\theta}{\displaystyle p \int_0^{\mathbf{y}^*(p)} u_{11}^{DM}(\mathbf{y}^*(p), \theta) f(\theta)\, d\theta + (1-p) \int_0^1 u_{11}^{DM}(\mathbf{y}^*(p), \theta) f(\theta)\, d\theta} < 0. \tag{2.11}$$

Because $u_1^{DM}(\mathbf{y}^*(p), \theta) > 0$ for all $\theta > \mathbf{y}^*(p)$ and $u_{11}^{DM}(\mathbf{y}^*(p), \theta) < 0$ for all $\theta \in [0, 1]$, the expression in (2.11) is strictly less than zero by (2.8). Intuitively, an increase in the probability p makes it more likely that the expert is withholding information given that the expert fails to reveal the state. It is thus optimal for the DM to adjust her default action downwards, for the expert only withholds information in lower states. (2.6), (2.10), and (2.11) together imply that

$$\frac{dU^E(p, \mathbf{y}^*(p))}{dp} < U_1^E(p, \mathbf{y}^*(p)), \tag{2.12}$$

which suggests that the expert has less incentives to acquire information when the effort choice is observed. However, there is a subtlety here. In an equilibrium of the covert game, the default action is independent of p, so the marginal gain from the expert's viewpoint at a particular p is

$$U_1^E\left(p, \mathbf{y}^*\left(p_C^*\right)\right) = \int_{\mathbf{y}^*\left(p_C^*\right)}^1 \left\{u^E\left(\theta,\theta\right) - u^E\left(\mathbf{y}^*\left(p_C^*\right),\theta\right)\right\} f\left(\theta\right) d\theta$$

which depends on the equilibrium p_C^* (but not on p). Therefore, a low equilibrium effort level may result from an expectation of high default action, which is in turn due to a skeptical belief of effort level. Therefore, to compare the two games in a meaningful way, we must look beyond (2.12) and make an equilibrium analysis. The self-fulfilling property of the covert games also leads to multiplicity of equilibria. To see this, total-differentiating (2.9), we have

$$\frac{dU_1^E\left(p, \mathbf{y}^*\left(p\right)\right)}{dp} = -\left(\int_{\mathbf{y}^*\left(p\right)}^1 u_1^E\left(\mathbf{y}^*\left(p\right),\theta\right) f\left(\theta\right) d\theta\right) \mathbf{y}^{*\prime}\left(p\right) > 0. \tag{2.13}$$

The expression in (2.13) is strictly positive by Assumption **MON** and (2.11). Unlike usual models of marginal analysis where uniqueness is obtained by virtue of decreasing marginal benefit and increasing marginal cost, (2.13) implies that both sides of the equilibrium condition (2.7) are increasing in p. Therefore, as in Che and Kartik (2009), multiplicity cannot be ruled out without further restrictions on the cost function $c\left(\cdot\right)$. However, Theorem 2.1, the main result of this paper, implies that equilibrium multiplicity has no bearing on our main message, as our comparison of the two games will not depend on how the equilibria are picked.

Theorem 2.1. *The effort level of any equilibrium of the covert game is strictly larger than the effort level of any equilibrium of the overt game. Moreover, the DM strictly prefers any equilibrium of the covert game over any equilibrium of the overt game.*

Proof. It follows directly from (2.7) and (2.9) that all equilibria of the covert game involve positive equilibrium effort. Next, suppose by way of contradiction that there exists an equilibrium effort level p_O^* of the overt game and an equilibrium effort level p_C^* of the covert game such that

$$p_C^* \leq p_O^*.$$

There are two cases.

1. Suppose $p_C^* = p_O^*$, then, as $p_C^* > 0$, (2.5) and (2.7) imply

$$U_1^E\left(p_C^*, \mathbf{y}^*\left(p_C^*\right)\right) = c'\left(p_C^*\right) = c'\left(p_O^*\right) = \frac{dU^E\left(p, \mathbf{y}^*\left(p\right)\right)}{dp}|_{p=p_O^*},$$

which contradicts (2.12).

2. Next, suppose $p_C^* < p_O^*$, then by (2.11), we have $\mathbf{y}^*\left(p_C^*\right) > \mathbf{y}^*\left(p_O^*\right)$. By the definitions of p_C^* and p_O^*, we must have

$$U^E\left(p_C^*, \mathbf{y}^*\left(p_C^*\right)\right) - c\left(p_C^*\right) \geq U^E\left(p_O^*, \mathbf{y}^*\left(p_C^*\right)\right) - c\left(p_O^*\right),$$

and

$$U^E\left(p_O^*, \mathbf{y}^*\left(p_O^*\right)\right) - c\left(p_O^*\right) \geq U^E\left(p_C^*, \mathbf{y}^*\left(p_C^*\right)\right) - c\left(p_C^*\right),$$

which imply

$$U^E\left(p_O^*, \mathbf{y}^*\left(p_O^*\right)\right) \geq U^E\left(p_O^*, \mathbf{y}^*\left(p_C^*\right)\right).$$

By (2.10), we must have $\mathbf{y}^*\left(p_C^*\right) \leq \mathbf{y}^*\left(p_O^*\right)$. In both cases, we have reached a contradiction. The last statement follows directly from Proposition 2.4.

□

We close this section with an example that serves to illustrate the economic force behind our results. In this example, we assume that the DM's payoff function is quadratic[3] and the expert's gain function is concave in y and independent of θ. In this case, the default action effect, $U_2^E\left(p, \mathbf{y}^*\left(p\right)\right)\mathbf{y}^{*\prime}\left(p\right)$, overwhelms the information precision effect, $U_1^E\left(p, \mathbf{y}^*\left(p\right)\right)$, so that the value of information to the expert is never positive in the overt game. As a result, the only equilibrium of the overt game involves zero effort.

Example 2.1. *Suppose the DM's payoff function is quadratic and the expert's gain function is concave in y and independent of θ, then, the unique equilibrium of the overt game involves zero effort level. However, all equilibrium of the covert game involves a positive effort level.*

Proof. See Appendix B. □

[3]Notice that given our assumptions on the DM's payoff function $u^{DM}\left(y, \theta\right)$, it is without loss of generality to assume the quadratic function take the form $-\left(y-\theta\right)^2$.

The intuition behind this example is the following. When the DM's payoff function is quadratic, the equilibrium action $y^*(m)$ equals the conditional expected value of θ. As a result, the unconditional expected value of the DM's action must also equal to the mean of the distribution F. Because the expert is risk-averse and is unable to influence the mean value of the DM's action, the expert optimally chooses to acquire no information in the overt game and the DM responds by taking action equal to the mean of the distribution F. In the covert game, however, the expert is always tempted to exert a positive level of effort. This is because, by (2.8), there is always a positive probability that the true state is higher than the default action and the expert would be able to gain by being able to demonstrate that.

It can be shown that if the expert's gain function is strictly convex in y instead, equilibrium of the overt game in Example 2.1 would also involve *positive* effort level. This is because the expert is "risk-loving" and is willing to take addition risk by investing in information. However, as we have noted in Section 2.2, concavity of the function $u^E(\cdot,\theta)$ has no bearing on our main results. Theorem 2.1 still holds and any equilibrium effort level of the covert game must be strictly larger than any equilibrium effort level of the overt game.

2.5 UNIFORM-QUADRATIC CASE

Up to this point, we have assumed that the expert's gain function is strictly increasing in the DM's action. This assumption, though realistic, is also restrictive. It is not difficult to think of situations in which the interests of involved parties are partially aligned. That is, matching the state of nature with the appropriate action is to some degree also the objective of the expert. In this section, we extend our analysis to the "uniform-quadratic" specification that is widely used in the literature of strategic communication. The specific functional forms used in this section allow us to solve the games explicitly and demonstrate that monitoring may be beneficial if the expert's bias is small.

We assume throughout this section that $\Theta = Y = [0,1]$, $f(\theta) = 1$ and $u^{DM}(y,\theta) = -(y-\theta)^2$. Moreover, the expert's gain function is given by $u^E(y,\theta) = -(y-\theta-b)^2$, where $b > 0$. Notice that $u^E(y,\theta)$ is strictly increasing in y for all $y < \theta + b$ and strictly decreasing for all $y > \theta + b$. Therefore, Assumption **MON** is satisfied only when $b > 1$. With these specifications, the conclusion of Theorem 2.1 applies more generally to the case when $b > \frac{1}{8}$. To show that, we first characterize the equilibrium of auxiliary game of our "uniform-quadratic" model.

Proposition 2.5. *Consider the "uniform-quadratic" specification, given $p \in [0, \overline{p}]$, the unique equilibrium of the auxiliary game $\Gamma(p)$ is characterized by a default action $\mathbf{y}^*(p) \in (0, \frac{1}{2}]$ such that the DM takes the default action $\mathbf{y}^*(p)$ whenever the expert fails to report the state and the corresponding action if the expert reports the state. That is,*

$$y^*(m) = \begin{cases} \theta & \text{if } m = \theta \\ \mathbf{y}^*(p) & \text{if } m = \phi \end{cases},$$

where $\mathbf{y}^*(p)$ is given by

$$\mathbf{y}^*(p) = \begin{cases} \frac{1}{2} - 2b^2\left(\frac{p}{1-p}\right) & \text{if } p \leq \check{p} \\ \frac{\sqrt{1-p}}{1+\sqrt{1-p}} & \text{if } p > \check{p} \end{cases}. \tag{2.14}$$

Moreover, the reporting strategy m^* is given by

$$m^*(\theta) = \begin{cases} \theta & \text{if } \theta \notin [\max\{\mathbf{y}^*(p) - 2b, 0\}, \mathbf{y}^*(p)] \\ \phi & \text{if } \theta \in [\max\{\mathbf{y}^*(p) - 2b, 0\}, \mathbf{y}^*(p)] \end{cases} \tag{2.15}$$

where $\check{p} \equiv \frac{1-4b}{4b^2-4b+1}$.

Proof. See Appendix B. ☐

Notice that $\mathbf{y}^*(p)$ is strictly decreasing. Therefore, if $b < \frac{1}{4}$, by (2.15), there exists $p' < \check{p}$ such that the equilibrium reporting strategy m^* of the game $\Gamma(p')$ is no longer a cutoff strategy. That is, under the optimal strategy, the expert reports the state if $\theta < \mathbf{y}^*(p') - 2b$ or $\theta > \mathbf{y}^*(p')$. On the other hand, if $b \geq \frac{1}{4}$, the expert always employs a cutoff reporting strategy in equilibrium. Although $\mathbf{y}^*(p)$ is still strictly decreasing in the new specification, it is not differentiable at $\check{p}$. This, however, can be handled easily because the left hand limit of the function $\mathbf{y}^{*\prime}(p)$ can be used instead. Following the strategy we used in proving Theorem 2.1, we can show that the first part of Theorem 2.1 still applies with the "uniform-quadratic" specification.

Theorem 2.2. *Consider the "uniform-quadratic" specification. The effort level of any equilibrium of the covert game is strictly larger than the effort level of any equilibrium of the overt game.*

Proof. See Appendix B. □

However, the second part of Theorem 2.1, which states that the DM's ex ante welfare is higher in the covert game, does not carry over. Given the equilibrium strategies specified in Proposition 2.5, we can compute the DM's ex ante payoff function $U^{DM}(\cdot, \mathbf{y}^*(\cdot))$ explicitly, from which we can show:

Proposition 2.6. *Consider the "uniform-quadratic" specification. If $b > \frac{1}{8}$, the DM's ex ante payoff function $U^{DM}(\cdot, \mathbf{y}^*(\cdot))$ is strictly increasing. If $0 < b < \frac{1}{8}$, there exists $\hat{p} \in [0, \check{p})$, such that the DM's ex ante payoff function $U^{DM}(\cdot, \mathbf{y}^*(\cdot))$ is strictly increasing on $[0, \hat{p}) \cup (\check{p}, 1]$ and strictly decreasing on $(\hat{p}, \check{p})$.*

Proof. See Appendix B. □

Figure 1 illustrates the typical shape of the function $U^{DM}(p, \mathbf{y}^*(p))$ when $b < \frac{1}{8}$.

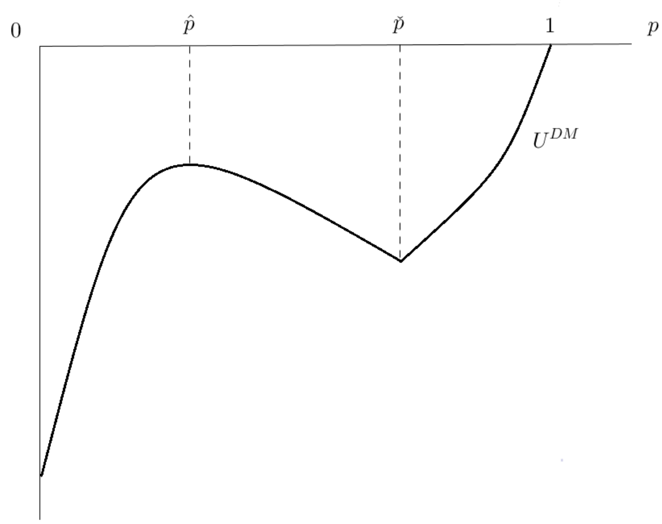

Figure 1: The DM's ex ante payoff function U^{DM} $(0 < b < \frac{1}{8})$

Why would the DM's ex ante payoff decrease with effort? As shown in Proposition 2.1, when Assumption **MON** is satisfied, the expert always uses a cutoff strategy. A reporting strategy with higher p and lower $\mathbf{y}^*$ is more informative than one with lower p and higher $\mathbf{y}^*$ in the sense of Blackwell. In general, Blackwell ordering on the pair (p, y^*) is a partial ordering. However, when Assumption **MON** is satisfied, (2.11) suggests higher p implies lower y^* in equilibrium. We can order the informativeness of equilibrium reporting strategy by the effort level p alone, so the order is complete for all equilibrium pairs (p, y^*). The expert also discloses more when he exerts higher effort. Therefore, the DM's ex ante payoff function cannot decrease with the effort level p (Proposition 2.4). With the "uniform-quadratic" specification, the equilibrium reporting strategy is not necessarily a cutoff strategy. As a result, the "informativeness" of the equilibrium reporting strategy cannot be ordered by p alone and the DM's ex ante payoff does not necessarily increase with effort. More information acquired does not necessarily translate into more information disclosed.

Corollary 2.1 is a simple consequence of Proposition 2.6.

Corollary 2.1. *Consider the "uniform-quadratic" specification. If either* $b \geq \frac{1}{8}$ *or* $\overline{p} \leq \hat{p}$*, the DM strictly prefers any equilibrium of the covert game over any equilibrium of the overt game.*

Corollary 2.1 states that the DM strictly prefers to not monitor the expert if either the bias of the expert is large enough ($b \geq \frac{1}{8}$) or the investigation technology is limited ($\overline{p} \leq \hat{p}$). Example 2.2 further shows that there are cases in which the DM prefers to monitor the expert's effort even though the equilibrium effort decreases under monitoring.

Example 2.2. *Consider the "uniform-quadratic" specification, suppose* $b = \frac{1}{12}$, $\overline{p} = \frac{24}{25}$, *and* $c(p) = 0$, *then*

$$\hat{p} = 1 - \frac{\sqrt{17}}{85} \in \arg\max_{p \in [0, \overline{p}]} U^{DM}\left(p, \mathbf{y}^*(p)\right),$$

and the unique equilibrium effort of the covert game p_C^* *and the unique equilibrium effort of the overt game* p_O^* *are given by*

$$p_C^* = \overline{p} = \frac{24}{25} \text{ and } p_O^* = 1 - \frac{\sqrt{17}}{85}.$$

Moreover, the DM strictly prefers the overt game equilibrium over the covert game equilibrium.

Proof. See Appendix B. ☐

Notice that the cost function in this example violates the assumptions we made in Section 2.2 and is thus not regular. Nevertheless, the main implications of the example will still hold if we replace the zero cost function by a regular cost function that is sufficiently close to the zero function in the L^p space, where $1 \leq p < \infty$. We will still have, $p_C^* > p_O^*$ but $U^{DM}\left(p_C^*, \mathbf{y}^*\left(p_C^*\right)\right) < U^{DM}\left(p_O^*, \mathbf{y}^*\left(p_O^*\right)\right)$.

In Example 2.2, the DM is strictly worse off in the covert game. It is easy to show that with the "uniform-quadratic" specification, the DM's ex ante payoff and the expert's ex ante gain are related by

$$U^E\left(p, \mathbf{y}^*\left(p\right)\right) = U^{DM}\left(p, \mathbf{y}^*\left(p\right)\right) - b^2. \tag{2.16}$$

As the expert's ex ante payoff is expected gain less cost, the expert is also strictly worse off in the covert game in the example. In general, however, equilibrium of the covert game and equilibrium of the overt game cannot be Pareto-ranked.

It is clear from Proposition 2.6 and Example 2.2 that we can always find a regular cost function such that the DM strictly prefers the overt game equilibrium over the covert game equilibrium whenever $0 < b < \frac{1}{8}$.

Corollary 2.2. *Consider the "uniform-quadratic" specification. If $0 < b < \frac{1}{8}$, there exists a technological limit $\overline{p}$ and a regular cost function $c : [0, \overline{p}] \to \mathbb{R}$ such that the DM strictly prefers the overt game equilibrium over the covert game equilibrium.*

2.6 CONCLUSION

In this paper, we study a persuasion game between a decision maker and an expert. We show that monitoring of information acquisition hampers the expert's incentives to acquire information. If the expert's gain from the DM's decision is strictly monotone, the ex ante welfare of the DM in any equilibrium of the covert game is strictly higher than any equilibrium of the overt game. We also show that this result continues to hold in a "uniform-quadratic" framework if the bias of the expert is large. If the expert's bias is small, however, the DM may still prefer to monitor even though it decreases effort.

3.0 PARTISAN VOTING AND UNCERTAINTY

3.1 INTRODUCTION

In everyday politics, partisans are considered to be hard-core supporters who do not change their position no matter what happens. Indeed, partisan voting is an important phenomenon. According to Huffpost Politics, the latest poll estimate of party identification in the U.S. is: independent 31.1%, Democrat 35.6%, and Republican 28.0%. Thus, more than 60% of the U.S. population consider themselves partisans. The population share of independent voters has also dropped from the peak estimate of 39.6% in 2011 to the current estimate of 31.1%, the lowest value in the past 8 years.[1] Moreover, voters are likely to consider themselves independent even though they are not. Burden and Klofstad (2005) identify more partisan voters by asking a set of questions related to party identification than asking about party identity directly.

Is partisan voting rational? Party supporters may appear to be stubborn and irresponsive to persuasion. In some voting models, partisans are assumed to stick to some parties (i.e., Feddersen and Pesendorfer, 1996; Palfrey and Rosenthal, 1983; Myatt, 2007). Under such assumption, partisan voters are not rational, since they do not take useful information into account. In other models, there is no fundamental difference between swing voters and party supporters in terms of rationality (i.e., Feddersen and Pesendorfer, 1999; Aragones and Palfrey, 2002; Gul and Pesendorfer, 2009; Krishna and Morgan, 2011). Some voters vote according to their information, and others do not. Swing voters and partisan voters are classified by their responsiveness to information and a voter's responsiveness to information

[1] On May 5, 2016. The latest poll estimate can be found at: http://elections.huffingtonpost.com/pollster/party-identification

depends on his preference intensity.

In this paper, we consider an alternative rationalization for partisan voting, and discuss its implications. Facing uncertainty, a Knightian decision maker's behavior is affected by his status quo. Voters who have a particular party as their status quo behave differently from those who do not. The status quo bias is more powerful under larger uncertainty. When the status quo bias is strong enough, partisan voters become hard-core supporters who are loyal to their own party regardless of any useful information. When the status quo bias is not strong enough, partisan voters may overcome the status quo bias and vote against their own party.

The model in this paper is built on Myerson's large Poisson game in a common value setting similar to Krishna and Morgan (2012). In such games, the size of the electorate is random. Myerson (1998 & 2000) shows the equivalence of qualitative predictions between Poisson voting model and standard voting model with a fixed electorate. In our model, voters are assumed to be Knightian decision makers with multiple priors. With multiple priors, two alternatives may be incomparable. If a voter treats a party as his status quo, he will stick to his own party when he is not able to compare two alternatives. Thus, in our model, partisans are voters with a particular status quo and swing voters are voters without a status quo.

We characterize four types of voting equilibrium: 1) fully informative voting equilibrium, 2) uninformative voting equilibrium, 3) partisan voting equilibrium, and 4) partial partisan voting equilibrium. Unlike expected utility models, the Knightian model can support equilibria that are robust to electorate size with a set of parameters. We study such equilibria in large elections which we call *limit voting equilibria*. When costly information acquisition is introduced into a large election, there exists an equilibrium in which the swing voters acquire some information with positive probability when the number of swing voters is more than the difference between the numbers of partisans of the two parties. In the selected informative voting equilibrium, swing voters rationally mix between two alternatives: either they acquire information and vote informatively or they do not acquire information and vote to balance the partisans' votes.

In Section 3.2, we briefly introduce Knightian decision theory and Bewley's inertia assumption, and argue that party identity is a natural candidate for status quo. In Section 3.3, the model is presented. In Section 3.4, we introduce the notion of limit voting equilibrium and study it in our model. Section 3.5 discusses a selection criterion for limit voting equilibria based on the idea of justifiable preferences. Section 3.6 presents an extension with costly information acquisition. Section 3.7 concludes. Most proofs are relegated to the Appendix C.

3.2 INCOMPLETE PREFERENCES AND STATUS QUO

Under uncertainty, completeness is not necessarily a reasonable axiom for individual decision problems. Bewley (1987, 1989 & 2002) develops Knightian decision theory, which relaxes the axiom of completeness.

Under the completeness axiom, individual decision maker is able to rank any pair of alternatives. If preference is not complete, some alternatives are incomparable. Bewley (2002) axiomatizes a model allowing for incompleteness with subjective probabilities.

Consider a finite state space N, the set of all probability distributions over N, $\triangle(N) := \{\pi \in R^N : \pi_i \geq 0\ \forall i = 1, ...N, \sum_{i=1}^{N} \pi_i = 1\}$, and two random monetary payoffs, $x, y \in X^N$, where $X \subset R$ is finite. Bewley characterizes incomplete preference relations represented by a unique nonempty, closed, convex set of probability distribution Π and a continuous, strictly increasing, concave function $u : X \to R$, unique up to positive affine transformation, such that

$$x \succ y \quad \textit{if and only if} \quad \sum_{i=1}^{N} \pi_i u(x_i) > \sum_{i=1}^{N} \pi_i u(y_i) \quad \textit{for all } \pi \in \Pi. \tag{3.1}$$

If the set of probabilities Π is a singleton, (3.1) is equivalent to an expected utility representation, so the ordering is complete. If Π is not a singleton, comparisons between two alternatives are done "one probability distribution at a time." A strict preference is obtained only when one alternative is "strictly preferred" to the other unanimously according to any $\pi \in \Pi$.

In some situations, a Knightian decision maker cannot make up his mind. Bewley's inertia assumption helps to settle some choice problems among incomparable alternatives. If there is a *status quo*, a Knightian decision maker always choose the status quo as long as no other alternative is strictly preferred to it according to every probability distribution. For instance, consider two alternatives x and y, x is preferred to y for some $\pi \in \Pi$, and y is preferred to x for some other $\pi' \in \Pi$. Knightian decision rule concludes that x and y are incomparable. A decision maker without any status quo will choose either x or y, or randomize. A decision maker with x (y) as status quo will always choose x (y). When x and y are comparable to each other, these three types of decision makers will make the same choice.

3.2.1 Party Identity as Status Quo

Campbell, Converse, Miller and Stokes (1960) in their classic *The American Voter* wrote:

> *Only in the exceptional case does the sense of individual attachment to party reflects a formal membership or an active connection with a party apparatus. Nor does it simply denote a voting record, although the influence of party allegiance on electoral behavior is strong, generally this tie is a psychological identification, which can persist without a consistent record of party support. Most Americans have this sense of attachment with one party or the other. And for the individual who does, the strength and direction of party identification are facts of central importance in accounting for attitude and behavior.*
>
> *In characterizing the relation of individual to party as a psychological identification we invoke a concept that has played an important if somewhat varied role in psychological theories of the relation of individual or individual to group. We use the concept here to characterize the individual's affective orientation to an important group-object in his environment.*

One difficulty in applying Bewley's inertia assumption is "identifying a plausible candidate for the role of status quo" (Lopomo, Rigotti and Shannon, 2014). In the case of partisan voting, we find party identity, an "affective orientation" as Campbell, Converse, Miller and Stokes (1960) put it, a natural candidate for the status quo. For all possible priors, party supporters compare two parties. They stick to their own parties as long as it is preferred for some priors. Therefore, to motivate a party supporter to vote against his own party, the incentives must be strong enough. Swing voters can be considered as voters without a party identity. As long as two parties are incomparable in the Knightian sense, a swing voter can cast a vote in any manner and still be rational. If complete preference ordering is assumed, such behaviors can never occur.

3.3 THE MODEL

Two party candidates, A and B, compete in an election decided by majority voting. In the event of a tie, the winning candidate is chosen by a fair coin toss. There are two states of nature, α and β. Voters have a compact set of prior probabilities $\left[\underline{p}, \overline{p}\right]$, where $\overline{p} \in \left(\frac{1}{2}, 1\right)$ and $\underline{p} \in (0, \overline{p})$.[2] Each $p \in \left[\underline{p}, \overline{p}\right]$ is a prior probability that the true state of nature is α. Candidate A is the better choice in state α, and candidate B is the better choice in state β. In state α, the payoff of any voters is 1 if candidate A is elected and -1 if B is elected. In state β, things reverse.

The size of the electorate is a random variable that follows the Poisson distribution with mean n. The probability that there are m voters is $e^{-n}\frac{n^m}{m!}$. After the electorate size is drawn, voters' party identities are determined randomly. There are three types of voters: one type of partisan voters, labeled A or partisans of A, takes party candidate A as their status quo choice; another type of partisan voters, labeled B or partisans of B, takes party candidate B as their status quo; swing voters, labeled S, have no party candidate as their status quo choice. A voter's type is A and B with probability λ_A and λ_B, respectively, independent of the state. Otherwise, he is a swing voter, with probability λ_S. Therefore, $\lambda_A + \lambda_B + \lambda_S = 1$. For each type i voter, $\lambda_i > 0$. Therefore, the sizes of partisans of A, partisans of B and swing voters follow the Poisson distributions with mean n_A, n_B, and n_S, respectively, given by

$$n_A = \lambda_A n, \;\; n_B = \lambda_B n, \;\; n_S = \lambda_S n.$$

Before casting a vote, every voter receives a private signal regarding the true state of nature. Conditional on the true state, signals are independent. The signal takes one of two

[2]Since our model is symmetric, the assumption that $\overline{p} > \frac{1}{2}$ is without loss of generality. If $\overline{p} \leq \frac{1}{2}$, then $1 - \underline{p} > 1 - \overline{p} \geq \frac{1}{2}$, the results of our analysis apply if we simply switch the roles of two party candidates, A and B.

values, a or b. The probability of receiving each signal is

$$\frac{1}{2} < P[a \mid \alpha] = P[b \mid \beta] = q < 1. \tag{3.2}$$

That is, we assume that signal is informative but inconclusive. We will relax this assumption and allow $q = \frac{1}{2}$ when we allow costly information acquisition in Section 3.6. The posterior beliefs of the states after receiving the signals for each $p \in [\underline{p}, \overline{p}]$ are

$$\begin{aligned} q_p(\alpha \mid a) &= \frac{pq}{pq + (1-p)(1-q)} > p, \text{ and} \\ q_p(\beta \mid b) &= \frac{(1-p)q}{(1-p)q + p(1-q)} > 1-p. \end{aligned}$$

3.3.1 Pivotal Voting

An elementary event is a singleton set consisting of a pair of vote totals (k, l), where k is the number of votes for party candidate A and l the votes for party candidate B. An event is an union of elementary events. An event is pivotal if a single vote can affect the final outcome of the election. There are two types of elementary events where one vote can have an effect on the final outcome: 1) there is a tie, or 2) party candidate A has one vote less or more than party candidate B. Let $T = \{(k, k) : k \geq 0\}$ denote the event that there is a tie, and let $T_{-1} = \{(k-1, k) : k \geq 1\}$ denote the event that A has one vote less than B, and let $T_{+1} = \{(k, k-1) : k \geq 1\}$ denote the event that A has one vote more than B. The event piv_A (pivotal if vote for A) is defined by $piv_A := T \cup T_{-1}$. The event piv_B is defined similarly. Let ϕ_A and ϕ_B denote the expected number of votes for A and B in state α, respectively. Abstention is not allowed, so $\phi_A + \phi_B = n$. τ_A and τ_B are defined similarly for the corresponding expected votes in state β.

Let $((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B))$ be a voting profile, where γ_s^i is the probability of voting for party candidate A for a type i voter with signal s.

Suppose the expected size of the electorate is n, consider the event that there are k votes in favor of party candidate A and l votes in favor of party candidate B. The probability of such event in state α is

$$\Pr[\{(k, l)\} \mid \alpha] = e^{-n} \frac{\phi_A^k}{k!} \frac{\phi_B^l}{l!}.$$

The probability of a tie in state α is

$$\Pr[T \mid \alpha] = e^{-n} \sum_{k=0}^{\infty} \frac{\phi_A^k}{k!} \frac{\phi_B^k}{k!},$$

while the probability that A has one vote less than B in state α is

$$\Pr[T_{-1} \mid \alpha] = e^{-n} \sum_{k=1}^{\infty} \frac{\phi_A^{k-1}}{(k-1)!} \frac{\phi_B^k}{k!},$$

and the probability that B has one vote less than A in state α is

$$\Pr[T_{+1} \mid \alpha] = e^{-n} \sum_{k=1}^{\infty} \frac{\phi_A^k}{k!} \frac{\phi_B^{k-1}}{(k-1)!}.$$

The corresponding probabilities in state β are obtained by substituting ϕ for τ. In state θ,

$$\begin{aligned} \Pr[piv_A \mid \theta] &= \Pr[T \mid \theta] + \Pr[T_{-1} \mid \theta], \\ \Pr[piv_B \mid \theta] &= \Pr[T \mid \theta] + \Pr[T_{+1} \mid \theta]. \end{aligned}$$

The probability of pivotal voting could be approximated using modified Bessel functions:[3]

$$\Pr[T \mid \alpha] \approx e^{-n} I_0(2\sqrt{\phi_A \phi_B}) = \frac{e^{-\left(\sqrt{\phi_A}-\sqrt{\phi_B}\right)^2}}{\sqrt{2\pi \cdot 2\sqrt{\phi_A \phi_B}}} = e^{-n} \frac{e^{2\sqrt{\phi_A \phi_B}}}{\sqrt{2\pi \cdot 2\sqrt{\phi_A \phi_B}}}; \tag{3.3}$$

when n is large,

$$\Pr[T_{\pm m} \mid \alpha] \approx \left(\frac{\phi_A}{\phi_B}\right)^{\pm \frac{m}{2}} \Pr[T \mid \alpha]. \tag{3.4}$$

The approximation is useful when we study the large population properties. Since

$$\Pr[piv_A \mid \alpha] \approx \Pr[T \mid \alpha] \left[1 + \left(\frac{\phi_A}{\phi_B}\right)^{-\frac{1}{2}}\right],$$

so $\Pr[piv_A \mid \alpha]$ is approximately the product of $\Pr[T \mid \alpha]$ and a function independent of population size.

[3] For details, see Krishna and Morgan (2012).

3.3.2 Voting Under Knightian Uncertainty

With multiple priors, voters are Knightian. If one party is strictly preferred to the other, all voters vote for the dominant party. If two parties are incomparable, partisans stick to their own parties, and the voting behavior of swing voters is not determined. Let $u(i)$ denote the payoff for voting for candidate i. Following Bewley (2002), the strict preference relation is characterized by

$$A \succ B \;\Leftrightarrow\; \forall p \in [\underline{p}, \overline{p}],\; E_p[u(A)] > E_p[u(B)],$$

where $[\underline{p}, \overline{p}]$ is the set of priors. Notice that the function u is an endogenous object that depends on the equilibrium voting profile. If A is strictly preferred to B, we say that A dominates B.

Definition 3.1 (Dominance)**.** *Given a signal s, party candidate i dominates party candidate j if and only if*

$$E_p[u(i)|s] > E_p[u(j)|s],\; \forall p \in [\underline{p}, \overline{p}].$$

Next, we define maximal and optimal voting choices in terms of dominance in an environment with uncertainty. Then, a voting equilibrium under uncertainty is defined in terms of maximal and optimal voting choices.

Definition 3.2 (Maximal and Optimal Choices)**.** *Given a signal s, party candidate i is an optimal choice if and only if party candidate i dominates party candidate $j \neq i$. Party candidate i is a maximal choice if and only if party candidate i is not dominated by party candidate $j \neq i$.*

Given a signal s, A is optimal if $E_p[u(A)|s]$ is strictly larger than $E_p[u(B)|s]$ for all $p \in [\underline{p}, \overline{p}]$; A is maximal if $E_p[u(A)|s]$ is at least as large as $E_p[u(B)|s]$ for some $p \in [\underline{p}, \overline{p}]$. Clearly, an optimal choice is maximal. The converse, however, may not hold.

Definition 3.3 (Voting Equilibrium under Uncertainty)**.** *A voting profile $((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B))$ is a voting equilibrium under uncertainty if and only if*

i) partisan voters vote for their own parties exclusively if it is a maximal choice, and

ii) if there is an optimal choice, all voters vote for it exclusively.

	Partisan of A	*Swing voter*	*Partisan of B*
$\forall p \in [\underline{p}, \overline{p}],\ E_p[u(A)\|s] > E_p[u(B)\|s]$	A	A	A
$\exists p \in [\underline{p}, \overline{p}],\ E_p[u(A)\|s] \geq E_p[u(B)\|s],\ and$ $\exists p \in [\underline{p}, \overline{p}],\ E_p[u(A)\|s] \leq E_p[u(B)\|s]$	A	*not determined*	B
$\forall p \in [\underline{p}, \overline{p}],\ E_p[u(A)\|s] < E_p[u(B)\|s]$	B	B	B

Table 1: Voting behaviors of partisans of A, partisans of B and swing voters

In equilibrium, partisans vote against their own parties only when the opponent is strictly preferred for every $p \in [\underline{p}, \overline{p}]$. Otherwise, they always vote for their own party candidate. By definition, they never mix. This is a restriction on the partisan voter's behavior imposed by the existence of a status quo. When there is an optimal choice, the model has a clear prediction on the swing voters' voting behavior. However, when there is no optimal choice, no prediction is made on how swing voters vote.

3.3.3 Equilibrium Characterization

In a voting equilibrium under uncertainty, party candidate i is a maximal voting choice given signal s if and only if

$$\exists p \in [\underline{p}, \overline{p}], \text{ s.t.} \qquad E_p \left[u\left(i\right) \ \mid s \right] \geq E_p \left[u\left(j\right) \ \mid s \right], \ i \neq j$$

and party candidate i is an optimal voting choice given signal s if and only if

$$\forall p \in [\underline{p}, \overline{p}], \text{ s.t.} \qquad E_p \left[u\left(i\right) \ \mid s \right] > E_p \left[u\left(j\right) \ \mid s \right], \ i \neq j.$$

Table 1 summarizes the voting behaviors of partisans and swing voters that are consistent with our definition of voting equilibrium under uncertainty (Definition 3.3).

In a voting equilibrium under uncertainty, a rational voter, no matter a partisan or not, compares the expected utility of voting for two candidates for every $p \in [\underline{p}, \overline{p}]$. Given signal s, the difference between the expected utility of voting for party candidate A and B for a particular $p \in [\underline{p}, \overline{p}]$ is

$$\begin{aligned} & E_p\left[u\left(A\right) \mid s\right] - E_p\left[u\left(B\right) \mid s\right] \\ = \quad & q_p(\alpha \mid s)\left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) - q_p(\beta \mid s)\left(\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]\right) \end{aligned} \tag{3.5}$$

where $\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]$ is the increase in expected utility by voting for party candidate A instead of B when the true state is α, while $\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]$ is the decrease in expected utility when true the state is β. We have,

Lemma 3.1. *All voting equilibria under certainty satisfy the following three properties:*

i) no mixing in partisans' strategies: $\forall i \in \{A, B\}, \forall s \in \{a, b\}, \gamma_s^i \in \{0, 1\}$;

ii) monotonicity across voters' strategies: $\forall s \in \{a, b\}, \gamma_s^A \geq \gamma_s^S \geq \gamma$;

iii) monotonicity of partisans' strategies: $\forall i \in \{A, B\}, \gamma_a^i \geq \gamma_b^i$.

Properties $i)$ and $ii)$ simply state the observations reported in Table 1. Property $iii)$ follows from the fact that the signals are informative, i.e., $q_p(\alpha \mid a) > q_p(\alpha \mid b)$. By (3.5), this implies that party candidate A (B) is more likely to be an optimal voting choice given signal a (b). Thus, if a partisan of B (A) votes for party candidate A (B) after receiving signal b (a), he must vote for A after receiving signal a (b).

Our first proposition follows immediately from Lemma 3.1. It states that there are only four possible types of voting equilibrium, as illustrated in Table 2.

Proposition 3.1. *All voting equilibria under uncertainty are one of the following four types:*

1) fully informative voting equilibrium, i.e., $\forall i \in \{A, S, B\}, (\gamma_a^i, \gamma_b^i) = (1, 0)$;

2) uninformative voting equilibrium, i.e., $\forall i \in \{A, S, B\}, (\gamma_a^i, \gamma_b^i) = (1, 1)$ *or* $\forall i \in \{A, S, B\}, (\gamma_a^i, \gamma_b^i) = (0, 0)$;

3) full partisan voting equilibrium, i.e., $\left(\left(\gamma_a^A, \gamma_b^A\right), \left(\gamma_a^B, \gamma_b^B\right)\right) = ((1, 1), (0, 0))$;

4) partial partisan voting equilibrium,

i.e., $((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)) = \left((1, 1), \left(1, \gamma_b^S\right), (1, 0)\right)$ *or*

$((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)) = \left((1, 0), \left(\gamma_a^S, 0\right), (0, 0)\right)$.

2.1: Fully informative voting

	A	S	B
a	A	A	A
b	B	B	B

2.2: Uninformative voting

	A	S	B
a	A	A	A
b	A	A	A

or

	A	S	B
a	B	B	B
b	B	B	B

2.3: Full partisan voting

	A	S	B
a	A	γ_a^S	B
b	A	γ_b^S	B

2.4: Partial partisan voting

	A	S	B
a	A	A	A
b	A	γ_b^S	B

or

	A	S	B
a	A	γ_a^S	B
b	B	B	B

Table 2: Four types of voting equilibria

To prove Proposition 3.1, we simply list all the possible combinations of maximal/optimal choices given signals and apply Lemma 3.1.

In Table 2, the rows correspond to the signals received, and the columns correspond to the voters' party identities. For instance, the entry in the first row and first column of Table 2.1 can be read as "in a fully informative voting equilibrium, given signal a, partisans of A vote for party candidate A."

In a fully informative voting equilibrium, votes represent the realized signals. In an uninformative voting equilibrium, neither information nor preference is revealed by the votes. Besides these two extreme cases, in a partisan voting equilibrium, both preference and information find their way to express themselves. In a full partisan voting equilibrium, partisans vote along their loyalty, while swing voters are responsive to information. In a partial partisan voting equilibrium, partisans of one party stick to their status quo, while swing voters and partisans of the other party respond to their signals.

To further characterize the equilibria, denote $\frac{q_p(\beta \mid a)}{q_p(\alpha \mid a)}$ and $\frac{q_p(\beta \mid b)}{q_p(\alpha \mid b)}$ by Q_p^a, and Q_p^b, respectively. We have

$$Q_p^a = \frac{(1-p)(1-q)}{pq} \text{ and } Q_p^b = \frac{(1-p)q}{p(1-q)}.$$

Q_p^s is the ratio of the posterior probabilities of the two states given signal s and prior p, or signal ratio in short. We also define the ratio of the pivotal probabilities in the two states, Ω, or pivotal ratio in short, by

$$\Omega = \frac{\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]}{\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]}.$$

By comparing these two ratios, Ω and Q_p^s, voters can decide their votes based on the information derived from the scenario of being pivotal for the final outcome of the election and the information derived from their private signals. If it is more likely to be pivotal in one state than the other, it is wise to vote for the corresponding party candidate, since not much damage can be done even if the choice is incorrect.

Notice that Q_p^a and Q_p^b are decreasing in p. As a result, party candidate A is maximal (optimal) given signal s if and only if $\Omega \geq Q_{\overline{p}}^s$ ($\Omega > Q_{\underline{p}}^s$). Similarly, party candidate B is maximal (optimal) given signal s if and only if $\Omega \leq Q_{\underline{p}}^s$ ($\Omega < Q_{\overline{p}}^s$). This observation greatly simplifies equilibrium characterization: we only need to check the inequalities for the boundary beliefs, $\underline{p}$ and $\overline{p}$, instead of the entire set of priors $[\underline{p}, \overline{p}]$.

3.3.3.1 Fully informative voting equilibrium

In a fully informative voting equilibrium, all voters vote according to their private signals. All voters vote for party candidate A (B) if signal a (b) is received. Such an equilibrium is possible if party candidate A is an optimal choice given signal a, while party candidate B is an optimal choice given signal b, as illustrated in Table 2.1. In this equilibrium,

$$\left((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\right) = \left((1,0), (1,0), (1,0)\right).$$

Since the voting behavior is deterministic given signals, the expected number of votes in each state only depends on the signal precision q and the electorate size n:

$$\phi_A = qn = \tau_B, \ \phi_B = (1-q)n = \tau_A.$$

Given the expected number of votes in state α and β, the pivotal ratio always equals one by Lemma C.1 in the Appendix C.

Since party candidate A is an optimal choice given signal a, he is strictly preferred to party candidate B for $\underline{p}$:

$$\Omega = 1 > Q^a_{\underline{p}} = \frac{\left(1-\underline{p}\right)(1-q)}{\underline{p}q} \Leftrightarrow q > 1-\underline{p}.$$

On the other hand, party candidate B is an optimal choice given signal b, therefore, it is strictly preferred to party candidate A for $\overline{p}$:

$$\Omega = 1 < Q^b_{\overline{p}} = \frac{(1-\overline{p})\,q}{\overline{p}\,(1-q)} \Leftrightarrow q > \overline{p}.$$

The following proposition summarizes the necessary and sufficient condition for the existence of a fully informative voting equilibrium under uncertainty.

Proposition 3.2 (Fully Informative Voting Equilibrium)**.** *A fully informative voting equilibrium exists if and only if*

$$q > \max(1-\underline{p}, \overline{p}). \tag{3.6}$$

To support a fully informative voting equilibrium, the signals have to be precise enough to overcome the uncertainty in the prior belief. If q is lower than $\overline{p}$, signal b is not able to persuade partisans of A to vote against their own party. If q is lower than $1-\underline{p}$, no signal can induce a vote for party candidate A from partisans of B. On the other hand, if (3.6) is satisfied, given signal a, voters are reasonably sure that the true state is α. Same for signal b. When the prior is a singleton, i.e., $\underline{p} = \overline{p}$, the condition (3.6) is simply $q > p$. It corresponds to the condition required to support a fully informative voting equilibrium in an environment without uncertainty.[4]

[4]Strictly speaking, in a standard expected utility model, the condition required is $q \geq p$. However, due to the inertia assumption we used, the case $q = p$ could not support a sincere voting equilibrium.

3.3.3.2 Uninformative Voting Equilibrium Uninformative equilibria, in which all voters vote for one party regardless of their own signal, are also possible. In voting games with a fixed electorate size, an uninformative equilibrium may arise because the probability of being pivotal is zero. However, in voting games with unknown electorate size, the probability of being pivotal is always positive.

In the equilibrium where all voters always vote for party candidate A, we have

$$\left((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\right) = \left((1,1), (1,1), (1,1)\right),$$

and in the equilibrium where all voters always vote for party candidate B, we have

$$\left((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\right) = \left((0,0), (0,0), (0,0)\right).$$

With an unknown electorate size, there is always a positive probability that there are less than three voters. When there are three or more voters, a single voter can never be pivotal in the equilibrium where all voters vote for one party. When there are only two voters, a voter can cast a vote to cancel the vote cast by the other. When there is only one voter, the election result is determined solely by his vote.[5] Again, we find that the pivotal ratio Ω is exactly one by Lemma C.1 in Appendix C. This is because the voting profile is independent of the state, $\phi_A = \tau_A$.

Suppose party candidate A is the optimal choice given both signals, party candidate A is preferred to party candidate B for $\underline{p}$ given both signals a and b, we must have

$$\Omega > Q_{\underline{p}}^a \text{ and } \Omega > Q_{\underline{p}}^b.$$

Proposition 3.3 immediately follows.

Proposition 3.3 (Uninformative Voting Equilibrium. A)**.** *An uninformative voting equilibrium with every voter voting for party candidate A exists if and only if*

$$q < \underline{p}. \tag{3.7}$$

[5] If there is no voter, which is still possible, there is no voting problem.

To make partisans of B vote against their own party regardless of their signals, we need a biased prior belief and noisy signals such that $q < \underline{p}$. In that case, a favorable signal alone does not constitute a reason to vote against the population, as there is no information provided by pivotal events.

Similarly, the necessary and sufficient condition for the existence of an uninformative voting equilibrium, in which every voter votes for party candidate B, is $\overline{p} < 1 - q$. But since $q > \frac{1}{2}$, it contradicts our assumption that $\overline{p} > \frac{1}{2}$. Therefore, such an equilibrium does not exist under our assumptions.

Proposition 3.4 (Uninformative Voting Equilibrium. B)**.** *An uninformative voting equilibrium with every voter voting for party candidate B does not exist.*

3.3.3.3 Full Partisan Voting Equilibrium In a full partisan voting equilibrium, partisans of A always vote for party candidate A, and partisans of B always vote for party candidate B. Partisans do not agree on their choices only when no party candidate is an optimal choice. In that case, swing voters are free to use any strategy as both party candidates are maximal choices. Swing voters might or might not respond to their signals. To support a full partisan voting equilibrium, it is necessary that upon receiving a signal, either a or b, neither party candidate is strictly preferred to the other.

In a full partisan voting equilibrium, the partisans' voting strategies are fixed, therefore,

$$\left((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\right) = \left((1,1)\,, (\gamma_a^S, \gamma_b^S), (0,0)\right).$$

To support a full partisan voting equilibrium, party candidate A needs to be weakly preferred to party candidate B for some p and party candidate B weakly preferred to party candidate A for some p, given both signals. Therefore, given signal b, party candidate A is weakly preferred to party candidate B for $\overline{p}$. Also, given signal a, party candidate B is weakly preferred to party candidate A for $\underline{p}$. We must have

$$Q_{\overline{p}}^b \leq \Omega \leq Q_{\underline{p}}^a. \tag{3.8}$$

At this stage, we do not know what condition is required to guarantee that the pivotal ratio Ω falls into this interval. Condition (3.8) merely says that Ω is bounded above and below by some positive constants. Moreover, the bounds are uniquely defined by the triple $\left(\underline{p}, \overline{p}, q\right)$. The interval $\left[Q^b_{\overline{p}}, Q^a_{\underline{p}}\right]$ is nonempty if and only if

$$\left(\frac{q}{1-q}\right)^2 \leq \frac{\left(\frac{\overline{p}}{1-\overline{p}}\right)}{\left(\frac{\underline{p}}{1-\underline{p}}\right)}. \tag{3.9}$$

The left-hand side of (3.9) is strictly increasing in q while the right-hand side is strictly increasing in $\overline{p}$ and strictly decreasing in $\underline{p}$. Intuitively, as information precision grows, larger uncertainty is required to sustain full partisan voting in equilibrium. Our next proposition provides a sufficient condition for the existence of a full partisan voting equilibrium.

Proposition 3.5 (Full Partisan Voting Equilibrium)**.** *A full partisan voting equilibrium exists if*

$$q \leq \min\left(\overline{p}, 1-\underline{p}\right). \tag{3.10}$$

In proving Proposition 3.5, we identify a particular set of full partisan voting equilibria, namely, full partisan voting profiles satisfying $\gamma^S_a = \gamma^S_b$ or $\gamma^S_a + \gamma^S_b = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}$. In Section 3.4.2, we will prove that these are the only full partisan limit voting equilibria in large elections. We also defer the discussion on the properties of these strategy profiles to Section 3.4.2.

3.3.3.4 Partial Partisan Voting Equilibrium The last type of equilibrium is the partial partisan voting equilibrium, where only one type of the partisans vote regardless of their signals, see Table 2.4. To support such an equilibrium, it is necessary that one party candidate is optimal when its corresponding signal is received, and is maximal but not optimal when the other signal is received. As a result, swing voters vote according to the signal upon receiving one of the two signals, but are free to use any strategy upon receiving the other one.

In a partial partisan voting equilibrium, in which partisans of A are not responsive to their signals, party candidate A is optimal given signal a, while both party candidate A and B are maximal given signal b. Thus, the equilibrium voting profile is given by

$$\left((\gamma^A_a, \gamma^A_b), (\gamma^S_a, \gamma^S_b), (\gamma^B_a, \gamma^B_b)\right) = \left((1,1), \left(1, \gamma^S_b\right), (1,0)\right).$$

In equilibrium, party candidate A dominates B given signal a, and both party candidates are maximal choices given signal b. Therefore, party candidate A is strictly preferred to party candidate B for $\underline{p}$ given signal a, and is weakly preferred to party candidate B for $\overline{p}$ given signal b; party candidate B is weakly preferred to party candidate A for $\underline{p}$ given signal b. The required conditions are

$$\Omega > Q^a_{\underline{p}}, \text{ and } Q^b_{\overline{p}} \leq \Omega \leq Q^b_{\underline{p}}. \tag{3.11}$$

Similarly, in a partial partisan voting equilibrium, in which partisans of B are not responsive to their signals, we must have

$$Q^a_{\overline{p}} \leq \Omega \leq Q^a_{\underline{p}}, \text{ and } \Omega < Q^b_{\overline{p}}. \tag{3.12}$$

The equilibrium conditions look similar to those for a full partisan equilibrium. Given any triple of $\left(\underline{p}, \overline{p}, q\right)$, there are only two possible orderings of the Q's: $Q^a_{\overline{p}} < Q^a_{\underline{p}} \leq Q^b_{\overline{p}} < Q^b_{\underline{p}}$ or $Q^a_{\overline{p}} < Q^b_{\overline{p}} < Q^a_{\underline{p}} < Q^b_{\underline{p}}$. Figure 2 illustrates the requirements for the partisan voting equilibrium in these two cases.

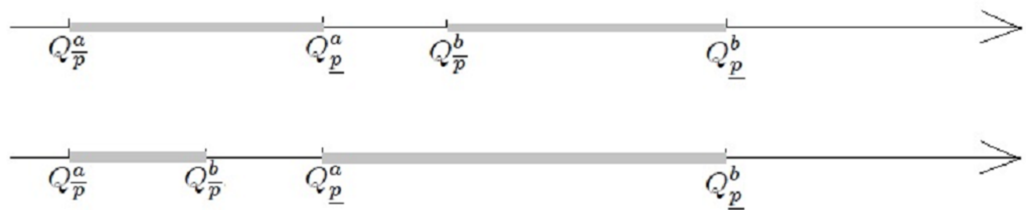

Figure 2: Supports of partisan voting equilibrium

In Figure 2, the grey segments on the left represent the values of Ω that support a partial partisan equilibrium favoring party candidate B, while the grey segments on the right represent the values of Ω that support a partial partisan equilibrium favoring party candidate A.

3.4 LARGE ELECTIONS

In the previous section, we list all the possible forms of equilibria that can arise in the voting game under uncertainty. In the cases of fully informative voting equilibrium and uninformative voting equilibrium, we are able to obtain necessary and sufficient conditions for existence. The specifications of fully informative and uninformative voting equilibrium also pin down the equilibrium voting profiles. In the case of full partisan voting equilibrium, we obtain a set of voting profiles that would constitute a full partisan voting equilibrium if condition (3.10) is met. Condition (3.10) is, therefore, only sufficient. Moreover, we do not have much idea about what conditions are required for a partial partisan voting equilibrium to exist. The difficulty to derive analytical results for these voting equilibria is that the pivotal ratio Ω is determined by the equilibrium voting profile.

Moreover, Knightian uncertainty presents an additional difficulty in a complete analysis of equilibria. Unlike voters in expected utility models, whose equilibrium strategies are determined by the optimal action under a single prior, the swing voters in our model are free to vote for anyone when the candidates are maximal choices. This indeterminacy leads to a larger set of equilibria. The approach in this paper is to limit the equilibrium analysis to a type of equilibrium for large electorate, which we call limit voting equilibrium. A limit voting equilibrium is a voting equilibrium for large electorate with the property that the voting profile is independent of the expected number of voters n. A limit voting equilibrium is thus robust to perturbation of the parameter n. Therefore, it can be viewed as a selection of equilibria for large electorate. The notion of limit voting equilibrium is also related to the notion of sincere voting studied in Austen-Smith and Banks (1996). In Proposition 3.6, we show that any limit voting equilibrium must be sincere.

By studying the large population property of the pivotal ratio Ω of limit voting equilibria, we show that, for each triple $(\underline{p}, \overline{p}, q)$, at most one of the four types of voting equilibrium exists. Thus, based on our selection criterion, we are able to make unique prediction of the *type* of the limit voting equilibrium in large electorate in some cases. In other cases, the selection criterion does not produce any prediction as a limit voting equilibrium does not exist.

3.4.1 Limit Voting Equilibrium

To begin with, we define the concept of limit voting equilibrium. A voting profile is a limit voting equilibrium if it is an equilibrium for electorate size that is sufficiently large.

Definition 3.4 (Limit Voting Equilibrium)**.** *A voting profile* $\big((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\big)$ *is a limit voting equilibrium under uncertainty if and only if there exists an integer* N *such that whenever the expected number of voters* n *exceeds* N*,* $\big((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\big)$ *constitutes a voting equilibrium under uncertainty.*

The following lemma shows that the pivotal ratio Ω of a limit voting equilibrium is always one.

Lemma 3.2. *In any limit voting equilibrium, either*

$$\phi_A = \tau_A \quad or \quad \phi_A = \tau_B. \tag{3.13}$$

Moreover,

$$\Omega = 1. \tag{3.14}$$

This result characterizes the vote shares in all limit voting equilibria. Moreover, all limit voting equilibria are voting equilibria for *all* electorate size n. This is because conditions (3.13) and (3.14) depend on the voting profile but not the electorate size n. If (3.13) and (3.14) hold for a particular n, it also holds for all n. One of the properties of limit voting equilibrium is that voting must be *sincere*, which we define below.

Definition 3.5 (Sincere Voting)**.** *A voting profile* $\big((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\big)$ *is* sincere *if and only if each voter votes as if he were the only decision maker.*

When a single voter decides the election outcome, the pivotal ratio Ω equals 1 as he is always pivotal. By Lemma 3.2, $\Omega = 1$ in any limit voting equilibrium. Thus, the problems faced by the voters in the two cases are identical. We have,

Proposition 3.6. *Any limit voting equilibrium is* sincere*.*

Lemma 3.3 follows immediately from Lemma 3.2 and conditions (3.11) and (3.12).

Lemma 3.3. *A partial partisan limit voting equilibrium favoring party candidate A exists only if*

$$\underline{p} \leq q \leq \overline{p} \text{ and } 1 - \underline{p} < q,$$

and a partial partisan limit voting equilibrium favoring party candidate B exists only if

$$\overline{p} < q \leq 1 - \underline{p}.$$

Since any limit voting equilibrium is also a voting equilibrium for some electorate size, Propositions 3.2 to 3.5 still apply. With Lemma 3.3, we can now illustrate these *necessary* conditions graphically in the $(\underline{p}, \overline{p})$ space. In Figures 3 to 5, the $x-axis$ presents values of $\underline{p}$ while the $y-axis$ presents values of $\overline{p}$. The area above the 45-degree line represents the set of $(\underline{p}, \overline{p})$ such that $\overline{p} > \underline{p}$. The shaded area in each figure represents the set of $(\underline{p}, \overline{p})$ under consideration.

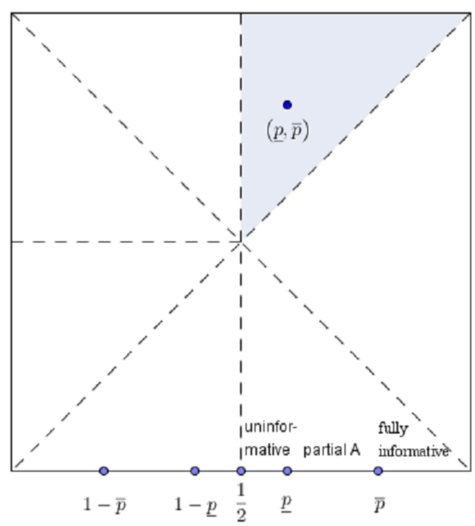

Figure 3: $(\underline{p}, \overline{p}, q)$ and the *potential* types of limit voting equilibrium: $\frac{1}{2} < \underline{p} \leq \overline{p}$

Figure 3 shows the potential limit voting equilibria when $\frac{1}{2} \leq \underline{p} < \overline{p}$. In this case, the voters hold a set of prior beliefs that favors party candidate A. When the signals are precise enough, i.e., $q > \overline{p}$, there exists a fully informative limit voting equilibrium. When the signals are imprecise enough, i.e., $q < \underline{p}$, there exists an uninformative voting equilibrium in which voters vote for party candidate A unanimously. When the signal precision is moderate, i.e.,

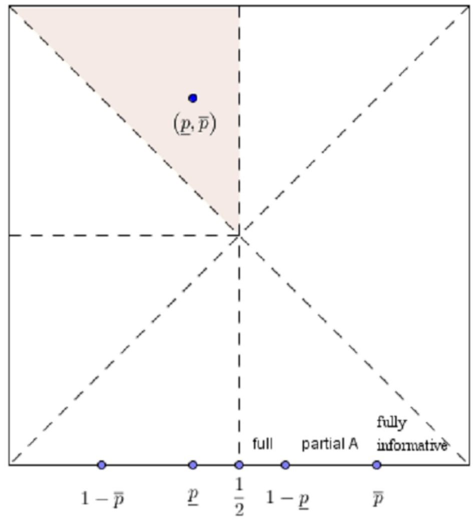

Figure 4: $(\underline{p},\overline{p},q)$ and the *potential* types of limit voting equilibrium: $\underline{p} < \frac{1}{2} < 1-\underline{p} \leq \overline{p}$

$\underline{p} \leq q \leq \overline{p}$, a limit voting equilibrium, if exists, is partial partisan favoring party candidate A.

Figure 4 shows the potential limit voting equilibria when $\underline{p} < \frac{1}{2} < 1-\underline{p} < \overline{p}$. In this case, the voters hold a balanced set of prior beliefs, i.e., $\underline{p} < \frac{1}{2} < \overline{p}$. But the set of priors slightly favors party candidate A, i.e., $\overline{p}+\underline{p} > 1$. Similar to the previous case, when signal quality is high enough, $q > \overline{p}$, there exists a fully informative limit voting equilibrium. When the signals are noisy enough, i.e., $q \leq 1-\underline{p}$, there exists a full partisan limit voting equilibrium, as the private signals are not strong enough to persuade partisans to vote against their own parties. With moderate signal precision, i.e., $1-p < q \leq \overline{p}$, a limit voting equilibrium, if exists, is partial partisan favoring party candidate A.

Figure 5 shows the potential limit voting equilibria when $\underline{p} \leq \frac{1}{2} < \overline{p} \leq 1-\underline{p}$. This case differs from the situation in Figure 4 only when signal precision is moderate, i.e., $\overline{p} < q \leq 1-\underline{p}$. In this case, a partial partisan limit voting equilibrium favors party candidate B, instead of party candidate A, may exist.

To summarize, we have shown that given a triple $(\underline{p},\overline{p},q)$, at most one type of the limit voting equilibria may exist.

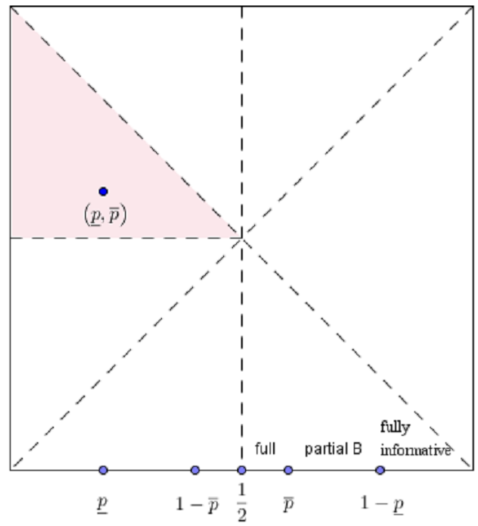

Figure 5: $(\underline{p}, \overline{p}, q)$ and the *potential* types of limit voting equilibrium: $\underline{p} < \frac{1}{2} < \overline{p} \leq 1 - \underline{p}$

1. When the signal precision q is high enough, a fully informative limit voting equilibrium exists by Proposition 3.2.
2. When the signal precision q is moderate, Lemma 3.3 and condition (3.8) implies that limit voting equilibrium, if exists, is partial partisan. In the next section, we will check the equilibrium condition and show that partial partisan limit voting equilibrium *does not* exist. Therefore, the selection criterion of limit voting equilibrium fails to produce a prediction in this case.
3. When the signal precision q is low, a limit voting equilibrium, if exists, is either uninformative or full partisan. If $q < \underline{p}$, the existence of uninformative voting equilibrium is guaranteed by Proposition 3.3. Furthermore, a limit voting equilibrium must be uninformative. At first sight, this seems to contradict the results of Feddersen and Pesendorfer (1997), who show that there always exists a sequence of responsive equilibria in large electrons and that the outcomes of such sequence always converge to the full information case. However, such sequence of responsive equilibria would generally involve voting profiles that vary with the electorate size n, which is ruled out by the definition of limit voting equilibrium. If $q \leq \min\left(\overline{p}, 1 - \underline{p}\right)$, limit voting equilibrium, if exists, must be

full partisan. Unlike partial partisan limit voting equilibrium, we will show in the next section that full partisan limit voting equilibrium *does* exist.

3.4.2 Equilibrium Strategies

In this section, we study partisan voting profiles that satisfy condition (3.13). By doing so, we are able to characterize the entire set of voting profiles that supports a full partisan limit voting equilibrium. On the other hand, it turns out that no voting profile could support a partial partisan limit voting equilibrium.

3.4.2.1 Full partisan voting equilibrium In a full partisan voting equilibrium,

$$\big((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\big) = \big((1,1), (\gamma_a^S, \gamma_b^S), (0,0)\big).$$

In state α,

$$\phi_A = n_A + n_S\left[q\gamma_a^S + (1-q)\gamma_b^S\right],$$

$$\phi_B = n_B + n_S\left[q(1-\gamma_a^S) + (1-q)(1-\gamma_b^S)\right],$$

while in state β,

$$\tau_A = n_A + n_S\left[(1-q)\gamma_a^S + q\gamma_b^S\right],$$

$$\tau_B = n_B + n_S\left[(1-q)(1-\gamma_a^S) + q(1-\gamma_b^S)\right].$$

By imposing condition (3.13) on the vote shares in state α and β, we find that there is a large set of voting profiles that satisfies $\Omega = 1$. Some of the voting profiles are uninformative. Such equilibrium always exists as long as condition (3.10) is satisfied. There are also equilibria that are informative and require a relatively balanced partisan voter population, i.e., $\frac{\lambda_B - \lambda_A}{\lambda_S} \in (-1, 1)$.

Proposition 3.7. *A full partisan limit voting equilibrium exists if and only if*

$$q \leq \min\left(\overline{p}, 1 - \underline{p}\right). \tag{3.15}$$

Moreover, the set of all full partisan limit voting equilibria is

$$\left\{\big((1,1), (\gamma_a^S, \gamma_b^S), (0,0)\big) : \gamma_a^S = \gamma_b^S \textit{ or } \gamma_a^S + \gamma_b^S = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}\right\}. \tag{3.16}$$

In a full partisan limit voting equilibrium, two possible scenarios may occur. In one scenario, $\gamma_a^S = \gamma_b^S$, swing voters may mix or not, but they do not respond to their private signals even though the signals contain valuable information. As a result, information is not aggregated in such equilibrium. It is an uninformative full partisan voting equilibrium, and it differs from an uninformative voting equilibrium by having partisans stick to their own parties. In such equilibrium, partisans express their preferences, while swing voters express their indecisiveness. It happens when there is no useful information revealed by pivotal events, and the private signal is not precise enough to help a voter reach a decision out of a balanced prior belief.

In the other scenario, $\gamma_a^S + \gamma_b^S = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}$, swing voters vote responsively, except when $\gamma_a^S = \gamma_b^S = \frac{1}{2}\left(1 + \frac{\lambda_B - \lambda_A}{\lambda_S}\right)$. When $\gamma_a^S > \gamma_b^S$, voting is informative. The necessary condition for the existence of an informative full partisan limit voting equilibrium is $\frac{\lambda_B - \lambda_A}{\lambda_S} \in [-1, 1]$. Swing voters mix their votes to the extent that it enables others to vote informatively after counter-balancing impact of partisan votes. The terms 1 and $\frac{\lambda_B - \lambda_A}{\lambda_S}$ on the right-hand side represent the pivotal and vote-balancing considerations, respectively. When $\lambda_B = \lambda_A$, the balancing component disappears. It also suggests that the difference between the two partisan populations cannot be too large relative to the population size of the swing voters, otherwise swing voters would not be able to counter-balance the impact of the over-populated party supporters. When voting is informative, indecisiveness remains, but it constitutes an overall informative decision.

To summarize, we do not have a sharp prediction of how a swing voter would vote in a full partisan limit voting equilibrium. The equilibrium requirements impose little restrictions on the relation between γ_a^S and γ_b^S. Knightian decision making introduces the flexibility into our voting model. In Section 3.5, we discuss a way of selecting these equilibria based on the idea of justifiable preferences.

3.4.2.2 Partial partisan voting equilibrium In a partial partisan voting equilibrium favoring party candidate A,

$$\left((\gamma_a^A, \gamma_b^A), (\gamma_a^S, \gamma_b^S), (\gamma_a^B, \gamma_b^B)\right) = \left((1, 1), \left(1, \gamma_b^S\right), (1, 0)\right).$$

In state α,

$$\phi_A = n_A + n_S\left[q + (1-q)\gamma_b^S\right] + n_B q,$$

$$\phi_B = n_B\left(1-q\right) + n_S(1-q)(1-\gamma_b^S);$$

while in state β,

$$\tau_A = n_A + n_S\left[(1-q) + q\gamma_b^S\right] + n_B\left(1-q\right),$$

$$\tau_B = n_B q + n_S q(1-\gamma_b^S).$$

The expressions for the vote shares in a partial partisan voting equilibrium favoring party candidate B are similar. Checking condition (3.13) shows that no feasible voting profile can support a partial partisan limit voting equilibrium. This is our next proposition.

Proposition 3.8. *There does not exist a partial partisan limit voting equilibrium.*

3.4.3 Information Aggregation

In this section, we study the information aggregation of limit voting equilibria. In state α, it is optimal to elect party candidate A, so the probability of an incorrect decision is

$$\begin{aligned}\Pr\left[B \text{ wins}|\alpha\right] &= \frac{1}{2}\Pr\left[T|\alpha\right] + \sum_{m=1}^{\infty}\Pr\left[T_{-m}|\alpha\right] \\ &< \sum_{m=0}^{\infty}\Pr\left[T_{-m}|\alpha\right],\end{aligned}$$

where $T_{-m} = \{(k-m, k) : k \geq m\}$ is the event that B wins by exactly m votes. Using the approximation formulas (3.3) and (3.4), we have

$$\begin{aligned}\sum_{m=0}^{\infty}\Pr\left[T_{-m}|\alpha\right] &\approx \frac{e^{-\left(\sqrt{\phi_A}-\sqrt{\phi_B}\right)^2}}{\sqrt{4\pi\sqrt{\phi_A\phi_B}}}\sum_{m=0}^{\infty}\left(\sqrt{\frac{\phi_B}{\phi_A}}\right)^m \\ &= \frac{e^{-\left(\sqrt{\phi_A}-\sqrt{\phi_B}\right)^2}}{\sqrt{4\pi\sqrt{\phi_A\phi_B}}}\frac{1}{1-\sqrt{\frac{\phi_B}{\phi_A}}}. \qquad (3.17)\end{aligned}$$

We study when the sum (3.17) would tend to zero as the expected number of voters n goes to infinity in the next proposition.

Proposition 3.9. *If $q > \max(\overline{p}, 1 - \underline{p})$, the probability that the right candidate is elected in each state goes to one as the expected number of voters n goes to infinity in all limit voting equilibria. If $q \leq \min\left(\overline{p}, 1 - \underline{p}\right)$ and $|\lambda_A - \lambda_B| < \lambda_S$, there exists a limit voting equilibrium in which the probability that the right candidate is elected in each state goes to one as the expected number of voters n goes to infinity.*

It is straightforward to show that the sum in (3.17) tends to zero under the stated conditions. If $q \leq \min\left(\overline{p}, 1 - \underline{p}\right)$, by Proposition 3.7, there are only two types of full partisan limit voting equilibria. In the first type of equilibria, $\gamma_a^S = \gamma_b^S$, voting is not informative. Thus, no information is aggregated. If the condition $|\lambda_A - \lambda_B| < \lambda_S$ is also satisfied, there exists a limit voting equilibrium satisfying $\gamma_a^S + \gamma_b^S = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}$ and $\gamma_a^S > \gamma_b^S$. In such equilibrium, the probability that the right candidate is elected in each state goes to one. On the other hand, if the condition $|\lambda_A - \lambda_B| < \lambda_S$ is violated, say, $\lambda_A > \lambda_S + \lambda_B$, the partisans of A would be too numerous for information to aggregate in large elections. Even if all the swing voters vote for party candidate B in state β, party candidate A is still expected to win.

To summarize, in our model, when the signal precision is high enough, by Proposition 3.2, the only limit voting equilibrium is fully informative, information aggregation is guaranteed. However, when the signal is only moderately precise, by Proposition 3.8, limit voting equilibrium ceases to exist. The most interesting case is when the signal is very imprecise. In this case, information aggregation depends on the value of $\underline{p}$ and the composition of partisans and swing voters. If $\underline{p} \leq \frac{1}{2}$, the limit voting equilibrium is full partisan, there are always equilibria that do not aggregate information properly. But if $|\lambda_A - \lambda_B| < \lambda_S$ is also satisfied, there also exists an equilibrium that aggregates information properly. If $\underline{p} > \frac{1}{2}$, the limit voting equilibrium is uninformative, information does not aggregate.

3.5 JUSTIFIABLE VOTING EQUILIBRIUM

In Section 3.4.2.1, we have identified the necessary and sufficient condition for the existence of a full partisan limit voting equilibrium. There is, however, always a continuum of such equilibria whenever one exists. The multiplicity arises from the incompleteness of swing voters' preference as the two party candidates may be incomparable. One way to "resolve" such indeterminacy is to consider the completion of the swing voters' Knightian preference to *Knightian-justifiable preference* (Lehrer and Teper, 2011). Given a utility function u and a multiple–prior Π that represent an incomplete Knightian preference $\succ$, consider the *justifiable extension* $\succeq'$, where

$$x \succeq' y \Leftrightarrow \quad \exists p \in \Pi,\ E_p[u(x)] \geq E_p[u(y)].$$

It is clear that the extension agrees with the original preference whenever there is an optimal choice. Such an extension puts additional restrictions on the voters' equilibrium strategies when both party candidates are maximal choices.

Rather than introducing such an extension formally to the model, we take a shortcut to view justifiability as an equilibrium section criterion. We call the selected equilibrium *justifiable voting equilibrium under uncertainty.*

Definition 3.6 (Justifiable Voting Equilibrium under Uncertainty). *A voting equilibrium under uncertainty* $\left(\left(\gamma_a^A,\gamma_b^A\right),\left(\gamma_a^S,\gamma_b^S\right),\left(\gamma_a^B,\gamma_b^B\right)\right)$ *is* justifiable *if and only if for each type of the voters* $i \in \{A,B,S\}$*, there exists a* $p_i \in [\underline{p},\overline{p}]$*, such that* (γ_a^i,γ_b^i) *maximizes voter* i*'s expected utility under* p_i*.*

Notice that the justifiability requirement on the partisans' strategies is always satisfied by any voting equilibrium under uncertainty. Since the ratios Q_p^a and Q_p^b are strictly decreasing in p, the equilibrium strategies of partisans of A and B are justified by $\overline{p}$ and $\underline{p}$, respectively. The justifiability requirement, however, does impose restrictions upon swing voters' behavior. Since, for all $p \in [\underline{p},\overline{p}]$, $Q_p^a > Q_p^b$, the swing voters' equilibrium voting strategy must be monotone in the signal under justifiability. Moreover, the swing voters cannot mix given both signals. If the swing voters mix given one signal, he must vote for the favored party candidate for sure given the other signal. Thus, we have

Lemma 3.4. *In a justifiable voting equilibrium under uncertainty, we must have*

$$\gamma_a^S \geq \gamma_b^S$$

and either

$$\gamma_a^S = 1 \text{ or } \gamma_b^S = 0.$$

Applying Lemma 3.4 to the set of full partisan limit voting equilibria identified in Proposition 3.7, we find that, in a *justifiable* full partisan limit voting equilibrium, the swing voters either always vote for one candidate or vote according to the most informative responsive equilibrium in Proposition 3.7. Our next proposition shows that these are indeed *justifiable* voting equilibrium strategies.

Proposition 3.10. *Suppose* $q \leq \min\left(\overline{p}, 1 - \underline{p}\right)$*. If* $|\lambda_A - \lambda_B| \geq \lambda_S$*, the set of* justifiable *full partisan limit voting equilibria is given by*

$$\left\{\left((1,1),(1,1),(0,0)\right),\left((1,1),(0,0),(0,0)\right)\right\}. \tag{3.18}$$

If $|\lambda_A - \lambda_B| < \lambda_S$*, the set of* justifiable *full partisan limit voting equilibria is given by those in (3.18) and*

$$\left((1,1),(1 - \min\left\{0, \frac{\lambda_B - \lambda_A}{\lambda_S}\right\}, \max\left\{0, \frac{\lambda_B - \lambda_A}{\lambda_S}\right\}),(0,0)\right). \tag{3.19}$$

In the equilibria given by (3.18), the swing voters "choose" to become a partisan by justifying his choice with an extreme prior. In the responsive justifiable equilibrium given by (3.19), the swing voters vote for the underdog for sure after receiving a favorable signal but mix when the opposite signal is received. This strategy is justified by a prior belief that makes him indifferent between two choices upon receiving one signal and strictly prefer to vote for the underdog upon receiving the other signal. Notice that this equilibrium is the most informative among those identified in Proposition 3.7.

3.6 COSTLY INFORMATION

In this section, we consider an extension of the original model, in which voters need to acquire costly information before the election at their own cost. In this environment, we show that partisan behaviors arise naturally from the status quo biases in large elections. Moreover, when partisan population is balanced, there is an informative voting equilibrium with "robust" voting profile and justifiable strategies, in which the swing voters rationally mix between two alternatives: either they acquire information and vote informatively or they do not acquire information and vote to cancel the partisans' votes. However, the swing voters only acquire a vanishing amount of information in large elections.

We assume that the voters have a balanced set of prior, i.e., $\underline{p} < \frac{1}{2} < \overline{p}$. The voting game is the same as in the previous sections, except that voters are endowed with no information. Before signals are revealed, a voter individually decide how much to pay to improve his signal precision. If he decides not to pay, a random noise is generated. The cost of information is a twice differentiable, increasing and strictly convex function of information precision. That is, for all $q \in [\frac{1}{2}, 1)$, $c'(q) \geq 0$, $c''(q) > 0$. Moreover, it satisfies the Inada conditions

$$c\left(\frac{1}{2}\right) = c'\left(\frac{1}{2}\right) = 0 \text{ and } \lim_{q \to 1} c'(q) = \infty.$$

That is, a vanishing amount of information has a vanishing cost, i.e., $c'\left(\frac{1}{2}\right) = 0$,[6] and nobody can afford perfect knowledge, i.e., $c'(1) = \infty$. A pure strategy in this model is a triple $(q, \gamma_a, \gamma_b) \in S$, where $q \in \left[\frac{1}{2}, 1\right]$ specifies an information acquisition level, $\gamma_a \in \{1, 0\}$ specifies which party candidate to vote after receiving signal a, and $\gamma_b \in \{1, 0\}$ specifies which party candidate to vote after receiving signal b. A mixed strategy for voter $i \in \{A, S, B\}$ is a probability distribution σ_i over the set of pure strategies. We also call q the *information acquisition strategy* and the pair (γ_a, γ_b) the *voting strategy*. Thus, a pure strategy consists of an information acquisition strategy and a voting strategy.

[6] Notice that $q = \frac{1}{2}$ corresponds to the zero information acquisition level, as the signal is completely uninformative when $q = \frac{1}{2}$.

For any $p \in [\underline{p}, \overline{p}]$, the expected benefit of a pure strategy $s = (q, \gamma_a, \gamma_b)$ is

$$\begin{aligned} v_p(s) &= p(a,q) \left\{ \begin{array}{c} \gamma_a \left(q(\alpha \mid a) \Pr[piv_A \mid \alpha] - q(\beta \mid a) \Pr[piv_A \mid \beta] \right) \\ +(1-\gamma_a) \left(q(\beta \mid a) \Pr[piv_B \mid \beta] - q(\alpha \mid a) \Pr[piv_B \mid \alpha] \right) \end{array} \right\} \\ &\quad + p(b,q) \left\{ \begin{array}{c} \gamma_b \left(q(\alpha \mid b) \Pr[piv_A \mid \alpha] - q(\beta \mid b) \Pr[piv_A \mid \beta] \right) \\ +(1-\gamma_b) \left(q(\beta \mid b) \Pr[piv_B \mid \beta] - q(\alpha \mid b) \Pr[piv_B \mid \alpha] \right) \end{array} \right\}, \end{aligned}$$

where $p(a, q)$ and $p(b, q)$ are the probabilities of acquiring signals a and b, given signal precision q, respectively. Given the cost of signal $c(q)$, the expected payoff of a pure strategy $s = (q, \gamma_a, \gamma_b)$ given $p \in [\underline{p}, \overline{p}]$ is given by

$$V_p(s) = v_p(s) - c(q).$$

Next, we define the concepts of optimal and maximal strategies. Then, we use these concepts to define an equilibrium in this voting game with endogenous information acquisition. Let $\Sigma = \Delta S$.

Definition 3.7 (Dominance)**.** *Let $\sigma, \sigma' \in \Sigma$, σ dominates σ' if and only if $V_p(\sigma) > V_p(\sigma')$ for all $p \in [\underline{p}, \overline{p}]$.*

As in the previous sections, if σ dominates σ', σ is strictly better than σ' in all circumstances. The concepts of optimal and maximal strategies are defined accordingly.

Definition 3.8 (Maximal and Optimal Strategies)**.** *$\sigma \in \Sigma$ is an optimal strategy if and only if it dominates $\sigma' \neq \sigma$ for all $\sigma' \in \Sigma$. $\sigma \in \Sigma$ is a maximal strategy if and only if it is not dominated by any $\sigma' \in \Sigma$.*

Facing information acquisition decision, voters are also under uncertainty. In such an environment, uncertainty never goes away, but it may shrink or exaggerate. We define the following equilibrium concept with costly information under uncertainty. We assume that status quo choice of the partisan voters is to vote for their own party candidate and not acquire any information. We have,

Definition 3.9 (Voting Equilibrium with Costly Information under Uncertainty)**.** *A strategy profile* $(\sigma_A^*, \sigma_B^*, \sigma_S^*)$ *forms a voting equilibrium with costly information under uncertainty if and only if*

i) partisan voters vote for their own party exclusively and acquire no information if it is a maximal strategy,

ii) if there is an optimal strategy, all voters use it exclusively, and

iii) $\forall i \in \{A, B, S\}, \sigma_i^*$ *is a maximal strategy.*

A *justifiable* voting equilibrium can be defined in a way analogous to Definition 3.6.

Definition 3.10 (Justifiable Voting Equilibrium with Costly Information)**.** *A voting equilibrium with costly information under uncertainty* $(\sigma_A^*, \sigma_B^*, \sigma_S^*)$ *is* justifiable *if and only if for each type of the voters* $i \in \{A, B, S\}$*, there exists a* $p_i \in [\underline{p}, \overline{p}]$*, such that* σ_i^* *maximizes voter i's expected utility under* p_i*.*

Our first observation for this game is that when information cost is out of one's own pocket, a voter pays for it only if he knows he is going to use it. Every piece of information that is acquired is to be used properly. That is,

Lemma 3.5 (Fully Informative Voting with Positive Information Acquisition)**.** *In any voting equilibrium with costly information under uncertainty, for any pure strategy* (q, γ_a, γ_b) *played in equilibrium with positive probability, if* $q > \frac{1}{2}$*, then,*

$$\gamma_a = 1 \text{ and } \gamma_b = 0.$$

Lemma 3.5 depends on neither the electorate size, the party identity of the voter nor the equilibrium voting profile. However, it does not require the voters to acquire a positive amount of information. It may be maximal for some voters to acquire no information at all. Indeed, when a voter does not acquire any information, his signal does not contain any information and *any* voting strategy that conditions on the signal becomes equivalent to a mixed voting strategy.

Our second observation is that the voters' levels of information acquisition must tend to zero as the electorate size increases. The intuition is straightforward. In a large election, the probability of being pivotal must tend to zero. Thus, a voter, whether a swing voter or not, could not find it beneficial to acquire a significant amount of information.

Lemma 3.6 (Vanishing Information Acquisition)**.** *In any sequence of voting equilibria with costly information under uncertainty, for all* $i \in \{A, B, S\}$*, for any pure strategy* (q, γ_a, γ_b) *played in equilibrium with positive probability,*

$$\lim_{n \to \infty} q = \frac{1}{2}.$$

By Lemma 3.6, the only equilibrium information acquisition strategy that is robust to electorate size is no information acquisition. Thus, requiring the equilibrium information acquisition strategy to be robust to electorate size may impose too much restriction on the possible forms of equilibrium. Therefore, to extend the notion of limit voting equilibrium to this environment, we only require condition (3.13) to be satisfied. We define,

Definition 3.11 (Balanced Voting)**.** *A strategy profile* $(\sigma_A, \sigma_B, \sigma_S)$ *for an electorate size* n *is* balanced *if and only if it satisfies condition (3.13).*

The concept of a balanced voting equilibrium with costly information is closely related to the notion of a limit voting equilibrium with exogenous information. We have shown in Lemma 3.2 that any limit voting equilibrium with exogenous information must be balanced. In fact, any sequence of voting equilibria must satisfy (3.13) asymptotically. A balanced voting equilibrium is *balanced* in the sense that a voter's pivotal probabilities are identical across the two states. The concept of balanced voting is also closely related to sincere voting. An application of Lemma C.1 in the Appendix shows that a balanced voting equilibrium must be sincere. This is because when the pivotal ratio Ω equals to one, "pivotal" consideration disappears, voters vote as if they alone decide the outcome of the election.

The next proposition is the main result of this section. It states that when the electorate size is large enough, any voting equilibrium that is both *balanced* and *justifiable* must be full partisan. Moreover, there are always two that are uninformative. If the partisan population is balanced, i.e., $|\lambda_A - \lambda_B| < \lambda_S$, there is an additional equilibrium that is *balanced*, *justifiable*, and yet informative.

Proposition 3.11 (Full Partisan Voting Equilibrium with Costly Information). *Suppose $\underline{p} < \frac{1}{2} < \overline{p}$, when the electorate size n is large enough,*

1. *If $|\lambda_A - \lambda_B| \geq \lambda_S$, the set of* balanced *and* justifiable *voting equilibria is given by*

$$\left\{ \left(\left(\frac{1}{2}, 1, 1 \right), (\frac{1}{2}, 1, 1), \left(\frac{1}{2}, 0, 0 \right) \right), \left(\left(\frac{1}{2}, 1, 1 \right), (\frac{1}{2}, 0, 0), \left(\frac{1}{2}, 0, 0 \right) \right) \right\}. \tag{3.20}$$

2. *If $|\lambda_A - \lambda_B| < \lambda_S$, the set of* balanced *and* justifiable *voting equilibria is given by those in (3.20) and equilibrium $(\sigma_A^*, \sigma_B^*, \sigma_S^*)$ that satisfies*

   a. *$\sigma_A^* = \left(\frac{1}{2}, 1, 1\right)$, and $\sigma_B^* = \left(\frac{1}{2}, 0, 0\right)$,*

   b. *If $\lambda_B \geq \lambda_A$, $\sigma_S^* = \left\{\mu_A, \left(\frac{1}{2}, 1, 1\right); (1 - \mu_A), (q^*, 1, 0)\right\}$, where $\mu_A = \frac{\lambda_B - \lambda_A}{\lambda_S}$. If $\lambda_B < \lambda_A$, $\sigma_S^* = \left\{\mu_B, \left(\frac{1}{2}, 0, 0\right); (1 - \mu_B), (q^*, 1, 0)\right\}$, where $\mu_B = \frac{\lambda_A - \lambda_B}{\lambda_S}$.*

   c. *q^* solves*

$$2e^{-n} \sum_{k=0}^{\infty} \frac{\phi_A^k}{k!} \frac{\phi_B^k}{k!} + e^{-n} \sum_{k=1}^{\infty} \frac{\phi_A^{k-1}}{(k-1)!} \frac{\phi_B^k}{k!} + e^{-n} \sum_{k=1}^{\infty} \frac{\phi_A^k}{k!} \frac{\phi_B^{k-1}}{(k-1)!} = c'(q). \tag{3.21}$$

Notice that R.H.S. of (3.21) is a continuous increasing function with $c'\left(\frac{1}{2}\right) = 0$ and $c'(1) = \infty$ and the L.H.S. of (3.21) is a continuous and bounded positive function. Thus, (3.21) must have at least one solution whenever $|\lambda_A - \lambda_B| < \lambda_S$.

3.7 CONCLUSION

In this paper, we study a common value Knightian voting model. The model has some interesting properties and yields several interesting results.

First, our Knightian voting model offers additional flexibility in the swing voters' behaviors. In the absence of an optimal choice, a Knightian swing voter is free to use any strategy. In contrast, in an expected utility model, maximization under a particular prior often pins down equilibrium strategies and mixed strategies can only be used in the case of indifference. We show that this additional flexibility allows us to construct equilibrium that is simple and robust to small perturbation of parameters. In particular, we are able to find voting equilibria that are robust to the electorate size with a large set of parameters. In addition to being simple, these voting equilibria are also *sincere* in the sense of Austen-Smith and Banks (1996).

Second, the status quo choices of Knightian decision theory provide a rationalization of partisan behaviors. In our model with costly information acquisition, a voter with a status quo party choice rationally "ignores" information in large elections and becomes "partisan". Moreover, a voter without a status quo may also behave like a partisan. The observation of Burden and Klofstad (2005) that a voter may show partisan tendency even if he/she is not identified with a party himself/herself is consistent with our model, in which voters without a status quo party choice may justify their partisan behavior by an extreme prior. On the other hand, voters without a status quo can also contribute to the overall informativeness of the election by actively acquiring information and voting informatively. The prerequisite of such behavior is that the number of partisans of one party does not overwhelm the other's, so that the swing voters may reasonably hope to change the outcome of the election.

APPENDIX A

APPENDIX FOR CHAPTER 1

Proof of Lemma 1.1. Given any information structure F, take any equilibrium (σ_1^*, σ_2^*), the first order conditions of the players imply that for $i \in \{1, 2\}$,

$$\sigma_i^* (x_i; F) = \alpha E \left(\sigma_j^* (x_j; F) | x_i \right) + \beta_i E (\theta | x_i) .$$

Applying the law of iterated expectations, we have

$$E (\sigma_1^* (x_1; F)) = \alpha E (\sigma_2^* (x_2; F)) + \beta_1 E (\theta) , \tag{A.1}$$

$$E (\sigma_2^* (x_2; F)) = \alpha E (\sigma_1^* (x_1; F)) + \beta_2 E (\theta) . \tag{A.2}$$

Our assumption that $|\alpha| < 1$ implies that the solution to (A.1) and (A.2) is unique. Thus, for $i \in \{1, 2\}$,

$$E (\sigma_i^* (x_i; F)) = \overline{a}_i.$$

□

Proof of Lemma 1.2. Proof in text. □

Proof of Lemma 1.3. Proof in text. □

Proof of Lemma 1.4. The first order condition for equilibrium is

$$\sigma_i^*(x_i) = \alpha E\left(\sigma_j^*(x_j)|x_i\right) + \beta_i E(\theta|x_i).$$

Denote $E_i(\theta) = E(\theta|x_i)$, $E_iE_j(\theta) = E(E(\theta|x_j)|x_i)$, $E_{ij}^k(\theta) = E\left(E\left(E_{ij}^{k-1}(\theta)|x_j\right)|x_i\right)$ and $E_{ij}^0(\theta) = \theta$. Then,

$$\begin{aligned}
\sigma_i^*(x_i) &= \beta_i E(\theta|x_i) + \alpha E\left(\sigma_j^*(x_j)|x_i\right) \\
&= \beta_i E_i(\theta) + \alpha\beta_j E_iE_j(\theta) + \alpha^2 E\left(E\left(\sigma_i^*(x_i)|x_j\right)|x_i\right) \\
&= \ldots \\
&= \sum_{k=1}^{\infty}\left(\alpha^2\right)^{k-1}\left[\beta_i E_{ij}^{k-1}E_i(\theta) + \alpha\beta_j E_{ij}^k(\theta)\right] \qquad \text{(A.3)}
\end{aligned}$$

By $(A1)$, there exist $0 \leq A_i, A_j < 1$ and $B_i, B_j \in \mathbb{R}$, such that $E(\theta|x_i) = A_ix_i + B_i$ and $E(\theta|x_j) = A_jx_j + B_j$. Then, by $(A2)$, we have, for all $k \geq 1$,

$$\begin{aligned}
E_{ij}^kE_i(\theta) &= E_{ij}^k(A_ix_i + B_i) \\
&= A_iE_{ij}^{k-1}E_iE_j(x_i) + B_i \\
&= A_iE_{ij}^{k-1}E_i\left(E(x_i) + E(\theta|x_j) - E(\theta)\right) + B_i \\
&= A_iE_{ij}^k(\theta) + \left[A_iE(x_i) - A_iE(\theta) + B_i\right]
\end{aligned}$$

Let $A_iE(x_i) - A_iE(\theta) + B_i = B_i'$, then $E_{ij}^kE_i(\theta) = A_iE_{ij}^k(\theta) + B_i'$. Similarly, all $k \geq 1$, $E_{ij}^k(\theta) = A_jE_{ij}^{k-1}E_i(\theta) + B_j'$, where $B_j' = A_jE(x_j) - A_jE(\theta) + B_j$. Therefore, $\sigma_i^*(x_i)$ is linear in x_i. Since $0 \leq A_i, A_j < 1$, our assumption that $|\alpha| < 1$ implies that the sum (A.3) must converge. Thus, a unique equilibrium exists and is linear. Finally, by $(A1)$, $A_i, A_j \geq 0$ and $\beta_i, \beta_j \geq 0$, if $\alpha \geq 0$, $\sigma_i^*(x_i)$ must be increasing in x_i. □

Proof of Proposition 1.4. Part (1), as noted in the text, we only need to show that

$$Cov\left(x_i, x_j\right) \geq 0.$$

By $(A1)$, there exists $A_i \geq 0$ and $B_i \in \mathbb{R}$ such that

$$E\left(\theta | x_i\right) = A_i x_i + B_i. \tag{A.4}$$

Applying the law of iterated expectations to (A.4), we have $E\left(\theta\right) = A_i E\left(x_i\right) + B_i$. $(A2)$ implies that

$$E\left(x_j | x_i\right) - E\left(x_j\right) = E\left(\theta | x_i\right) - E\left(\theta\right) = A_i\left(x_i - E\left(x_i\right)\right).$$

Thus, $Cov\left(x_i, x_j\right) = E\left(\left(x_i - E\left(x_i\right)\right)\left(E\left(x_j|x_i\right) - E\left(x_j\right)\right)\right) = A_i E\left(\left(x_i - E\left(x_i\right)\right)^2\right) \geq 0$.

Part (2.a), consider the information structure $\left(\overline{F}_i, \phi\right)$, by Lemma 1.1, $\sigma_j^* = \overline{a}_j$. The first order condition of the player i becomes

$$\sigma_i^*\left(x_i; \overline{F}_i, \phi\right) = \alpha \overline{a}_j + \beta_i E\left(\theta | x_i\right).$$

By $(A1)$, $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ must be increasing. By Lemma 1.4, $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ are linear. By assumptions, $\alpha \leq 0$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ is decreasing. By the proof of part (1), $Cov\left(x_i, x_j\right) \geq 0$, thus $2\alpha Cov\left(\sigma_i^*\left(x_i; \overline{F}_i, \phi\right), \sigma_j^*\left(x_j; \overline{F}\right)\right) \geq 0$ and the claim follows from Lemma 1.3 and Proposition 1.3. Similarly, part (2.b) follows from the proof of part (1) and an application of Lemma 1.3 and Proposition 1.3. □

Proof of Lemma 1.5. Given any strategy profile σ, consider the best response mapping $\mathbf{B}$ which takes strategy profile σ and returns the best response strategy profile $\mathbf{B}\left(\sigma\right)$, where, for each $i \in \{1, 2\}$, the i-component of the best response mapping, $\mathbf{B}_i\left(\sigma\right)$ is player i's best response to player j's strategy σ_j. We have

$$\mathbf{B}_i\left(\sigma\right)\left(x_i\right) = \alpha E(\sigma_j | x_i) + \beta_i E\left(\theta | x_i\right).$$

By $(M1)$, given any bounded strategy profile σ, $\mathbf{B}_i\left(\sigma\right)$ is bounded and well-defined. By $(M2)$ and $(M3)$, $\mathbf{B}\left(\sigma\right)$ is also continuous. Thus, it is without loss of generality to focus on the space of bounded continuous functions from X to A, which we denote by S. Consider the metric space (S, d), where d is the metric associated with the p-norm. (S, d) is a complete metric space. Since, by assumption, $|\alpha| < 1$, for all $\sigma, \sigma' \in S$,

$$
\begin{aligned}
& d\left(\mathbf{B}\left(\sigma\right), \mathbf{B}\left(\sigma'\right)\right) \\
= \; & \left\|\mathbf{B}\left(\sigma\right) - \mathbf{B}\left(\sigma'\right)\right\|_p \\
= \; & \sup_{(x_1, x_2) \in X_1 \times X_2} \left\{\left(\alpha\left[E(\sigma_2 - \sigma_2'|x_1)\right]\right)^p + \left(\alpha\left[E(\sigma_1 - \sigma_1'|x_2)\right]\right)^p\right\}^{\frac{1}{p}} \\
< \; & \sup_{(x_1, x_2) \in X_1 \times X_2} \left\{\left[E(\sigma_2 - \sigma_2'|x_1)\right]^p + \left[E(\sigma_1 - \sigma_1'|x_2)\right]^p\right\}^{\frac{1}{p}} \\
\leq \; & \left\{\sup_{x_2 \in X_2} \left[\sigma_2\left(x_2\right) - \sigma_2'\left(x_2\right)\right]^p + \sup_{x_1 \in X_1} \left[\sigma_1\left(x_1\right) - \sigma_1'\left(x_1\right)\right]^p\right\}^{\frac{1}{p}} \\
= \; & \sup_{(x_1, x_2) \in X_1 \times X_2} \left\{\left[\sigma_2\left(x_2\right) - \sigma_2'\left(x_2\right)\right]^p + \left[\sigma_1\left(x_1\right) - \sigma_1'\left(x_1\right)\right]^p\right\}^{\frac{1}{p}} \\
= \; & d\left(\sigma, \sigma'\right).
\end{aligned}
$$

Thus, $\mathbf{B}$ is a contraction. By the contraction mapping theorem, e.g. Theorem 3.2 in Stokey, Lucas, and Prescott (1989), $\mathbf{B}$ has a unique fixed point.

Finally, suppose $\alpha \geq 0$, $(M2)$ and $(M3)$ imply that given any increasing strategy profile σ, $\mathbf{B}\left(\sigma\right)$ is also increasing. Thus, the fixed point of $\mathbf{B}$ must also be increasing. □

Proof of Proposition 1.5. Part *(1)*, by Lemma 1.5, if $\alpha \geq 0$, the equilibrium strategies $\sigma_i^*\left(x_i; \overline{F}_i, \phi\right)$ and $\sigma_j^*\left(x_j; \overline{F}\right)$ are increasing. Since X_i is convex, Lemma 1.1 and continuity imply that there must be an $\hat{x}_i \in X_i$ such that

$$\int_{X_j} \sigma_j^*\left(x_j; \overline{F}\right) dF\left(x_j|\hat{x}_i\right) = \overline{a}_j.$$

Since $\sigma_j^*\left(x_j; \overline{F}\right)$ is an increasing function of x_j and the family $F\left(\cdot|x_i\right)$ is ordered by first order stochastic dominance, we must have

$$\int_{X_j} \sigma_j^*\left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) \geq \overline{a}_j, \text{ if } x_i > \hat{x}_i,$$

and

$$\int_{X_j} \sigma_j^*\left(x_j; \overline{F}\right) dF\left(x_j|x_i\right) \leq \overline{a}_j, \text{ if } x_i < \hat{x}_i.$$

Thus,

$$\begin{aligned}
& Cov\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right)\right) \\
= & \int_{x_i\leq\hat{x}_i}\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\underbrace{\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)}_{\leq 0}dF\left(x_i\right) \\
& +\int_{x_i>\hat{x}_i}\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\underbrace{\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)}_{\geq 0}dF\left(x_i\right) \\
\geq & \left(\sigma_i^*\left(\hat{x}_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\int_{X_i}\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)dF\left(x_i\right) \\
= & \left(\sigma_i^*\left(\hat{x}_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\left(E\left[\sigma_j^*\left(x_j;\overline{F}\right)\right]-\overline{a}_j\right) \\
= & 0,
\end{aligned}$$

where the inequality follows from the fact that $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ is increasing in x_i and the last equality follows from Lemma 1.1.

Part (*2.a*), consider the information structure $\left(\overline{F}_i,\phi\right)$, by Lemma 1.1, $\sigma_j^*=\overline{a}_j$. The first order condition of the player i becomes

$$\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)=\alpha\overline{a}_j+\beta_i E\left(\theta|x_i\right).$$

By $(M2)$, $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ must be increasing. Part (*2.a*) will follow from an application of Lemma 1.3 and Proposition 1.2 if

$$Cov\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right)\right)\leq 0.$$

Since X_i is convex, Lemma 1.1 and continuity imply that there must be an $\hat{x}_i\in X_i$ such that

$$\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|\hat{x}_i\right)=\overline{a}_j.$$

Since $\sigma_j^*\left(x_j;\overline{F}\right)$ is by assumption a decreasing function of x_j and the family $F\left(\cdot|x_i\right)$ is ordered by first order stochastic dominance, we must have

$$\begin{aligned}
& Cov\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right),\sigma_j^*\left(x_j;\overline{F}\right)\right) \\
= & \int_{x_i\leq\hat{x}_i}\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\underbrace{\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)}_{\geq 0}dF\left(x_i\right) \\
& +\int_{x_i>\hat{x}_i}\left(\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\underbrace{\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)}_{\leq 0}dF\left(x_i\right) \\
\leq & \left(\sigma_i^*\left(\hat{x}_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\int_{X_i}\left(\int_{X_j}\sigma_j^*\left(x_j;\overline{F}\right)dF\left(x_j|x_i\right)-\overline{a}_j\right)dF\left(x_i\right) \\
= & \left(\sigma_i^*\left(\hat{x}_i;\overline{F}_i,\phi\right)-\overline{a}_i\right)\left(E\left[\sigma_j^*\left(x_j;\overline{F}\right)\right]-\overline{a}_j\right) \\
= & 0,
\end{aligned}$$

where the inequality follows from the fact that $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ is increasing in x_i and the last equality follows from Lemma 1.1.

Part (*2.b*) follows from the fact that

$$Cov\left(\sigma_i^*\left(x_i;\overline{F}\right),\sigma_j^*\left(x_j;\overline{F}\right)\right)\geq 0,$$

the proof of which follows from replacing $\sigma_i^*\left(x_i;\overline{F}_i,\phi\right)$ with $\sigma_i^*\left(x_i;\overline{F}\right)$ in the proof of part (*1*), and an application of Lemma 1.3 and Proposition 1.3. □

Proof of Lemma 1.6. Given the cut-off strategies τ_i as a function of i, then, by $(MG2)$ and $(MG3)$, for all $\theta'>\theta$,

$$\begin{aligned}
p\left(\theta'\right) &= \int_0^1 \Pr\left(x_i\geq\tau_i|\theta'\right)di \\
&\geq \int_0^1 \Pr\left(x_i\geq\tau_i|\theta\right)di \\
&= p\left(\theta\right).
\end{aligned}$$

□

Proof of Lemma 1.7. Suppose each investor $j \neq i$ is using a cut-off strategy, we need to show that player i's best response is uniquely given by a cut-off strategy, so that one can restrict the strategy space to the set of cut-off strategies to prove the existence of an equilibrium in cut-off strategies. Given any x_i, investor i's expected payoff of investment is

$$(1-T_i)\Pr(p(\theta) > 1-\theta|x_i) - [1-\Pr(p(\theta) > 1-\theta|x_i)]\,T_i = \Pr(p(\theta) > 1-\theta|x_i) - T_i,$$

and the expected payoff of no investment is 0. Thus, our claim follows if $\Pr(p(\theta) > 1-\theta|x_i)$ is a strictly increasing function of x_i. By Lemma 1.6, there exists a $\overline{\theta} \in [0,1]$ such that $p(\theta) \lesseqgtr 1-\theta$ if and only if $\theta \lesseqgtr \overline{\theta}$. Thus, by $(MG4)$, $\Pr(p(\theta) > 1-\theta|x_i)$ is strictly increasing in x_i. As a result, each investor i can be thought of as choosing a cut-off τ_i to maximize his payoff. Suppose τ_i is in the interior of X_i, τ_i is uniquely pinned down by

$$\Pr\left(\theta > \overline{\theta}|\tau_i\right) = T_i. \tag{A.5}$$

Differentiating (A.5), we have

$$\frac{d\tau_i}{di} = \frac{\frac{dT_i}{di}}{\frac{\partial \Pr\left(\theta > \overline{\theta}|x_i\right)}{\partial x_i}\Big|_{x_i=\tau_i}}.$$

By $(MG4)$, the derivative $\frac{d\tau_i}{di}$ is uniformly bounded. Thus, the set of all best response cut-off functions is equicontinuous and thus compact.[1] An application of Glicksberg's fixed point theorem implies that an equilibrium exists. □

Proof of Lemma 1.8. Suppose no investment is an optimal default action in the marginal problem, then

$$\begin{aligned}
& \int_{\Theta} \left[1_{\left\{\theta > \overline{\theta}''\right\}} - 1_{\left\{\theta > \overline{\theta}'\right\}}\right] dF(\theta) \\
= \; & \left[1 - F\left(\overline{\theta}''\right)\right] - \left[1 - F\left(\overline{\theta}'\right)\right] \\
= \; & F\left(\overline{\theta}'\right) - F\left(\overline{\theta}''\right) \le 0,
\end{aligned}$$

which is equivalent to $\overline{\theta}'' \geq \overline{\theta}'$. Similarly, investment is an optimal default action if and only if $\overline{\theta}'' \geq \overline{\theta}'$. □

[1] Each element in this set is a function from the index set $[0,1]$ to X_i.

Proof of Lemma 1.9. Suppose $\mathbf{y}_n^* = 1$, then the DM's first order condition implies that

$$(1-p_2)\int_0^1 \frac{\partial u^{DM}(1,\theta)}{\partial y} f(\theta)\, d\theta \geq 0,$$

which is impossible since $\frac{\partial u^{DM}(1,\theta)}{\partial y} < 0$ for all $\theta < 1$. Similarly, suppose $\mathbf{y}_n^* = 0$, then the DM's first order condition implies that

$$(1-p_1)\int_0^1 \frac{\partial u^{DM}(0,\theta)}{\partial y} f(\theta)\, d\theta \leq 0,$$

which is impossible since $\frac{\partial u^{DM}(0,\theta)}{\partial y} > 0$ for all $\theta > 0$. Thus, $\mathbf{y}_n^* \in (0,1)$ and satisfies

$$\begin{aligned} 0 \;=\;& p_1(1-p_2)\int_0^{\mathbf{y}_n^*} \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta + p_2(1-p_1)\int_{\mathbf{y}_n^*}^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta \\ &+(1-p_1)(1-p_2)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta. \end{aligned}$$

Suppose $\mathbf{y}_n^* \leq \mathbf{y}_o^*$, then

$$\begin{aligned} 0 \;=\;& p_1(1-p_2)\int_0^{\mathbf{y}_n^*} \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta + p_2(1-p_1)\int_{\mathbf{y}_n^*}^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta \\ &+(1-p_1)(1-p_2)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta \\ >\;& (1-p_2)\left(p_1\int_0^{\mathbf{y}_n^*} \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta + (1-p_1)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} f(\theta)\, d\theta\right) \\ \geq\;& (1-p_2)\left(p_1\int_0^{\mathbf{y}_o^*} \frac{\partial u^{DM}(\mathbf{y}_o^*,\theta)}{\partial y} f(\theta)\, d\theta + (1-p_1)\int_0^1 \frac{\partial u^{DM}(\mathbf{y}_o^*,\theta)}{\partial y} f(\theta)\, d\theta\right), \end{aligned}$$

where the first inequality follows from the facts that $\mathbf{y}_n^* < 1$ and $\frac{\partial u^{DM}(\mathbf{y}_n^*,\theta)}{\partial y} > 0$ for all $\theta > \mathbf{y}_n^*$ and the second inequality from the fact that the term inside the bracket is strictly decreasing in the null action $\mathbf{y}$. Thus, the DM's first order condition in the original game implies that $\mathbf{y}_o^* = 0$. But this is impossible since $0 < \mathbf{y}_n^* \leq \mathbf{y}_o^*$. Therefore, we must have $\mathbf{y}_o^* < \mathbf{y}_n^* < 1$. □

Proof of Proposition 1.7. In the original equilibrium, expert 2 has no information and the DM's null action is $\mathbf{y}_o^*$. Expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) in the original problem is given by

$$\begin{aligned} \tilde{u}_1(x_1, x_1, x_2, \theta) &= \begin{cases} u_1(\theta, \theta) & \text{if } x_1 = \theta, \\ u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi, \end{cases} \\ \tilde{u}_1(\varphi, x_1, x_2, \theta) &= u_1(\mathbf{y}_o^*, \theta), \end{aligned} \tag{A.6}$$

In the new equilibrium, expert 2 has information and the DM's null action is $\mathbf{y}_n^*$. Expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) in the new problem is given by

$$\tilde{w}_1(x_1, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) & \text{if } x_1 = \theta, \text{ or} \\ & x_1 = \varphi,\ x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) & \text{if } x_1 = \varphi,\ x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or} \\ & x_1 = x_2 = \varphi. \end{cases}$$

and

$$\tilde{w}_1(\varphi, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) & \text{if } x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) & \text{if } x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or } x_2 = \varphi, \end{cases}$$

as expert 2 only sends message x_2 when $x_2 \leq \mathbf{y}_n^*$ in equilibrium. The marginal problem is thus

$$\tilde{v}_1(x_1, x_1, x_2, \theta) = \begin{cases} 0 & \text{if } x_1 = \theta, \\ u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi,\ x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_1 = \varphi,\ x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or} \\ & x_1 = x_2 = \varphi. \end{cases}$$

and

$$\tilde{v}_1(\varphi, x_1, x_2, \theta) = \begin{cases} u_1(\theta, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \theta \text{ and } \theta \leq \mathbf{y}_n^*, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \theta \text{ and } \theta > \mathbf{y}_n^*, \text{ or } x_2 = \varphi. \end{cases}$$

Given null information, expert 1 can only send $m_1 = \varphi$, therefore, expert 1's default action is identical in all three problems. Applying expert 1's optimal decision rule for the new problem to the marginal problem and comparing the payoff to that obtained by using the default action $m_1 = \varphi$, we have

$$\begin{aligned}
& E\left[\Delta\widehat{v}_1\left(x_1, x_2, \theta\right)\right] \\
= \;& E\left[\tilde{v}_1\left(x_1, x_1, x_2, \theta\right) 1_{\{x_1 \geq \mathbf{y}_n^*\}} + \tilde{v}_1\left(\varphi, x_1, x_2, \theta\right) 1_{\{x_1 < \mathbf{y}_n^*, x_1 = \varphi\}} - \tilde{v}_1\left(\varphi, x_1, x_2, \theta\right)\right] \\
= \;& -p_1 \int_{\mathbf{y}_n^*}^{1} \left\{u_1\left(\mathbf{y}_n^*, \theta\right) - u_1\left(\mathbf{y}_o^*, \theta\right)\right\} f\left(\theta\right) d\theta \\
< \;& 0.
\end{aligned}$$

□

Proof for Footnote 13. It is not difficult to show that $\mathbf{y}_o^* = \mathbf{y}_n^* = \mathbf{y}^*$. The original problem is the same as in the proof of Proposition 1.7. i.e.,

$$\begin{aligned}
\tilde{u}_1\left(x_1, x_1, x_2, \theta\right) &= \begin{cases} u_1\left(\theta, \theta\right) & \text{if } x_1 = \theta, \\ u_1\left(\mathbf{y}_o^*, \theta\right) & \text{if } x_1 = \varphi, \end{cases} \\
\tilde{u}_1\left(\varphi, x_1, x_2, \theta\right) &= u_1\left(\mathbf{y}_o^*, \theta\right),
\end{aligned}$$

With a non-strategic expert 2, in the new problem, expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) is given by

$$\tilde{w}_1\left(x_1, x_1, x_2, \theta\right) = \begin{cases} u_1\left(\theta, \theta\right) & \text{if } x_1 = \theta \text{ or } x_2 = \theta, \\ u_1\left(\mathbf{y}_n^*, \theta\right) & \text{if } x_1 = x_2 = \varphi, \end{cases}$$

and

$$\tilde{w}_1\left(\varphi, x_1, x_2, \theta\right) = \begin{cases} u_1\left(\theta, \theta\right) & \text{if } x_2 = \theta, \\ u_1\left(\mathbf{y}_n^*, \theta\right) & \text{if } x_2 = \varphi. \end{cases}$$

In the marginal problem, expert 1's payoff given the quadruple (m_1, x_1, x_2, θ) is given by

$$\widetilde{v}_1\left(x_1, x_1, x_2, \theta\right) = \begin{cases} 0 & \text{if } x_1 = \theta, \\ u_1\left(\theta, \theta\right) - u_1\left(\mathbf{y}_o^*, \theta\right) & \text{if } x_1 = \varphi, \text{ and } x_2 = \theta, \\ u_1\left(\mathbf{y}_n^*, \theta\right) - u_1\left(\mathbf{y}_o^*, \theta\right) & \text{if } x_1 = x_2 = \varphi, \end{cases}$$

and

$$\widetilde{v}_1(\varphi, x_1, x_2, \theta) = \begin{cases} u_1(\theta,\theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \theta, \\ u_1(\mathbf{y}_n^*, \theta) - u_1(\mathbf{y}_o^*, \theta) & \text{if } x_2 = \varphi. \end{cases}$$

Without any information, expert 1 can only send $m_1 = \varphi$. Therefore, expert 1's default action is identical in all three problems. Applying expert 1's optimal decision rule for the new problem to the marginal problem and comparing the payoff to that obtained by using the default action $m_1 = \varphi$, we have

$$\begin{aligned} & E\left[\Delta\widehat{v}_1(x_1, x_2, \theta)\right] \\ = \; & E\left[\tilde{v}_1(x_1, x_1, x_2, \theta)\, 1_{\{x_1 \geq \mathbf{y}_n^*\}} + \tilde{v}_1(\varphi, x_1, x_2, \theta)\, 1_{\{x_1 < \mathbf{y}_n^*, x_1 = \varphi\}} - \tilde{v}_1(\varphi, x_1, x_2, \theta)\right] \\ = \; & -p_1 \int_{\mathbf{y}_n^*}^1 \left[(1-p_2)\, u_1(\mathbf{y}_n^*, \theta) + p_2 u_1(\theta,\theta) - u_1(\mathbf{y}_o^*, \theta)\right] f(\theta)\, d\theta \\ = \; & -p_1 p_2 \int_{\mathbf{y}^*}^1 \left(u_1(\theta,\theta) - u_1(\mathbf{y}^*, \theta)\right) f(\theta)\, d\theta \\ < \; & 0. \end{aligned}$$

Thus, the marginal problem is not similar to the new problem. Applying Proposition 1.3, we obtain the desired conclusion. □

APPENDIX B

APPENDIX FOR CHAPTER 2

Proof of Proposition 2.1. Suppose the DM take the action $\mathbf{y}^* = y^*(\phi)$ in an equilibrium of the auxiliary game $\Gamma(p)$. As $u^E(\cdot, \theta)$ is strictly increasing, the "nondisclosure" set is simply

$$N(\mathbf{y}^*) = \{\theta \in \Theta : \theta \leq \mathbf{y}^*\}.$$

The belief of the DM must be given by

$$\mu^*(\theta|p,m) = \begin{cases} \mathbf{1}_{\{\theta \geq \theta'\}}(\theta) & \text{if } m = \theta' \\ \dfrac{p\int_0^{\min\{\theta,\mathbf{y}^*\}} f(\theta')d\theta' + (1-p)\int_0^{\theta} f(\theta')d\theta'}{p\int_0^{\mathbf{y}^*} f(\theta')d\theta' + (1-p)} & \text{if } m = \phi \end{cases}, \tag{B.1}$$

where $\mathbf{1}_{\{\theta \geq \theta'\}}(\theta)$ denotes the indicator function for the set $\{\theta \geq \theta'\}$. Optimality of the action $\mathbf{y}^*$ implies that

$$\mathbf{y}^* = \arg\max_{y\in[0,1]} \left\{ p\int_0^{\mathbf{y}^*} u^{DM}(y,\theta) f(\theta) d\theta + (1-p)\int_0^1 u^{DM}(y,\theta) f(\theta) d\theta \right\}.$$

The first-order condition is

$$p\int_0^{\mathbf{y}^*} u_1^{DM}(\mathbf{y}^*,\theta) f(\theta) d\theta + (1-p)\int_0^1 u_1^{DM}(\mathbf{y}^*,\theta) f(\theta) d\theta = 0.$$

By assumption, $u_{11}^{DM}(\cdot,\theta) < 0$. The objective function is strictly concave so that the first order condition is both necessary and sufficient for optimality. To show that an equilibrium always exists and is unique, define the function

$$H(y,p) = p\int_0^y u_1^{DM}(y,\theta)\, f(\theta)\, d\theta + (1-p)\int_0^1 u_1^{DM}(y,\theta)\, f(\theta)\, d\theta.$$

The equilibrium condition is thus

$$H(\mathbf{y}^*, p) = 0. \tag{B.2}$$

As $u^{DM}(\cdot,\theta)$ is strictly concave and maximized at θ, we have, for all $p \in [0,\overline{p}]$,

$$\begin{aligned} H(0,p) &= (1-p)\int_0^1 u_1^{DM}(0,\theta)\, f(\theta)\, d\theta > 0, \\ H(1,p) &= \int_0^1 u_1^{DM}(1,\theta)\, f(\theta)\, d\theta < 0. \end{aligned}$$

Moreover,

$$H_1(y,p) = p\int_0^y u_{11}(y,\theta)\, f(\theta)\, d\theta + (1-p)\int_0^1 u_{11}(y,\theta)\, f(\theta)\, d\theta < 0.$$

Therefore, for each $p \in [0,\overline{p}]$, there is an unique $\mathbf{y}^*(p)$ that solves (B.2). □

Proof of Proposition 2.2. Let $\mu_O^*(\theta|p,m) = \mu^*(\theta|p,m)$, where $\mu^*(\theta|p,m)$ is defined in (B.1). By Proposition 2.1, Conditions 2-4 of Definition 2.1 are uniquely satisfied. $[0,\overline{p}]$ is a compact set and the objective function is continuous. Therefore, (2.4) has a solution. An equilibrium as specified exists. The "only if" part follows the sufficiency of the characterization of Proposition 2.1. □

Proof of Proposition 2.3. Let

$$\mu_C^*(p, \theta|m) = \begin{cases} 0 & \text{if } p < p_C^* \\ \mu^*(\theta|p_C^*, m) & \text{if } p \geq p_C^* \end{cases},$$

where $\mu^*(\theta|p, m)$ is defined in (B.1). By Proposition 2.2, Conditions 2-4 of Definition 2.2 are uniquely satisfied. Continuity of the function $U_1^E(p, \mathbf{y}^*(p))$ and the Inada conditions imply that (2.7) must have a solution. An equilibrium as specified exists. The "only if" part follows the sufficiency of the characterization of Proposition 2.1. □

Lemma B.1. *For all* $p \in [0, \overline{p}]$,

$$\int_{\mathbf{y}^*(p)}^{1} f(\theta)\, d\theta > 0.$$

Proof of Lemma B.1. Suppose by way of contradiction that for some $p \in [0, \overline{p}]$,

$$\int_{\mathbf{y}^*(p)}^{1} f(\theta)\, d\theta = 0.$$

Then, the condition (2.1) can be rewritten as

$$\int_{0}^{\mathbf{y}^*(p)} u_1^{DM}(\mathbf{y}^*(p), \theta) f(\theta)\, d\theta = 0.$$

By assumption, for all $\theta < \mathbf{y}^*(p)$, $u_1^{DM}(\mathbf{y}^*(p), \theta) < 0$, it follows that

$$\int_{0}^{1} f(\theta)\, d\theta = 0,$$

which is a contradiction. □

Proof of Proposition 2.4. Partial-differentiating (2.2) leads to

$$U_1^{DM}\left(p, \mathbf{y}^*(p)\right) = \int_{\mathbf{y}^*(p)}^{1} \left\{u^{DM}(\theta,\theta) - u^{DM}\left(\mathbf{y}^*(p), \theta\right)\right\} f(\theta)\, d\theta$$

and

$$U_2^{DM}\left(p, \mathbf{y}^*(p)\right) = p\int_0^{\mathbf{y}^*(p)} u_1^{DM}\left(\mathbf{y}^*(p), \theta\right) f(\theta)\, d\theta + (1-p)\int_0^1 u_1^{DM}\left(\mathbf{y}^*(p), \theta\right) f(\theta)\, d\theta.$$

Because $u^{DM}(\theta,\theta) > u^{DM}\left(\mathbf{y}^*(p), \theta\right)$ for all $\theta > \mathbf{y}^*(p)$, (2.8) implies that the first equation is strictly positive. The second equation is equal to zero due to (2.1). Therefore,

$$\frac{dU^{DM}(p, \mathbf{y}^*)}{dp} = U_1^{DM}(p, \mathbf{y}^*) > 0.$$

□

Proof for Example 2.1. Suppose that the DM's payoff function is quadratic and the expert's gain function is independent of θ, let $\overline{u}^E(y) \equiv u^E(y,\theta)$. (2.11) becomes

$$\mathbf{y}^{*\prime}(p) = \frac{\int_{\mathbf{y}^*(p)}^{1} \left(\mathbf{y}^*(p) - \theta\right) f(\theta)\, d\theta}{p\int_0^{\mathbf{y}^*(p)} f(\theta)\, d\theta + (1-p)}. \tag{B.3}$$

Multiply (2.10) with (B.3) , we have

$$U_2^E\left(p, \mathbf{y}^*(p)\right)\mathbf{y}^{*\prime}(p) = \left[\int_{\mathbf{y}^*(p)}^{1} \left(\mathbf{y}^*(p) - \theta\right) f(\theta)\, d\theta\right] \overline{u}^{E\prime}\left(\mathbf{y}^*(p)\right). \tag{B.4}$$

(2.6), (2.9), and (B.4) lead to

$$\frac{dU^E\left(p, \mathbf{y}^*(p)\right)}{dp} = \int_{\mathbf{y}^*(p)}^{1} \left\{\overline{u}^E(\theta) - \overline{u}^E\left(\mathbf{y}^*(p)\right) - \overline{u}^{E\prime}\left(\mathbf{y}^*(p)\right)\left(\theta - \mathbf{y}^*(p)\right)\right\} f(\theta)\, d\theta \le 0,$$

where the last inequality follows from the concavity of $\overline{u}^E(\cdot)$. (2.5) implies $p_O^* = 0$. By Theorem 2.1, all equilibria of the covert game involve positive equilibrium effort. □

Proof of Proposition 2.5. Suppose the DM take the action $\mathbf{y}^* = y^*(\phi)$ in an equilibrium of the auxiliary game $\Gamma(p)$. The "nondisclosure" set is simply

$$\begin{aligned} N(\mathbf{y}^*) &= \{\theta \in [0,1] : -b^2 \leq -(\mathbf{y}^* - \theta - b)^2\} \\ &= \{\theta \in [0,1] : \mathbf{y}^* - 2b \leq \theta \leq \mathbf{y}^*\}. \end{aligned}$$

The belief of the DM must be given by

$$\mu^*(\theta|p,m) = \begin{cases} \mathbf{1}_{\{\theta \geq \theta'\}}(\theta) & \text{if } m = \theta' \\ \frac{p \max\left\{\int_{\max\{\mathbf{y}^*-2b,0\}}^{\min\{\theta,\mathbf{y}^*\}} f(\theta')d\theta',0\right\}+(1-p)\theta}{p\min\{\mathbf{y}^*,2b\}+(1-p)} & \text{if } m = \phi \end{cases}, \tag{B.5}$$

where $\mathbf{1}_{\{\theta \geq \theta'\}}(\theta)$ denotes the indicator function. As the DM's payoff function is quadratic, $\mathbf{y}^*$ must be given by

$$\begin{aligned} \mathbf{y}^* &= E(\theta|m=\phi) \\ &= \frac{p\left(\int_{\max\{\mathbf{y}^*-2b,0\}}^{\mathbf{y}^*} \theta d\theta\right) + \frac{1-p}{2}}{p\min\{\mathbf{y}^*,2b\}+1-p} \end{aligned} \tag{B.6}$$

Solving (B.6), we have

$$\mathbf{y}^*(p) = \begin{cases} \frac{1}{2} - 2b^2\left(\frac{p}{1-p}\right) & \text{if } p \leq \check{p} \\ \frac{\sqrt{1-p}}{1+\sqrt{1-p}} & \text{if } p > \check{p} \end{cases}.$$

□

Proof of Theorem 2.2. Our proof of Theorem 2.2 mirrors closely the proof of Theorem 2.1. To begin with, we prove two lemmas that would allow us to sign the default action effect.

Lemma B.2. *For all* $p \in [0, \overline{p}]$,

$$U_2^E(p, \mathbf{y}^*(p)) > 0.$$

Proof of Lemma B.2.

$$\begin{aligned}
& U_2^E\left(p, \mathbf{y}^*(p)\right) \\
= \; & -2p \int_{\max\{\mathbf{y}^*(p)-2b,0\}}^{\mathbf{y}^*(p)} \left(\mathbf{y}^*(p) - \theta - b\right) d\theta - 2(1-p) \int_0^1 \left(\mathbf{y}^*(p) - \theta - b\right) d\theta \\
= \; & -2p \left(\left(\mathbf{y}^*(p) - b\right) \min\left\{\mathbf{y}^*(p), 2b\right\} - \frac{\mathbf{y}^{*2}(p)}{2} + \frac{\max\left\{\mathbf{y}^*(p) - 2b, 0\right\}^2}{2} \right) \\
& -2(1-p)\left(\mathbf{y}^*(p) - b - \frac{1}{2}\right)
\end{aligned}$$

If $\mathbf{y}^*(p) < 2b$, we have

$$U_2^E\left(p, \mathbf{y}^*(p)\right) = -p\mathbf{y}^*(p)\left(\mathbf{y}^*(p) - 2b\right) - 2(1-p)\left(\mathbf{y}^*(p) - b - \frac{1}{2}\right).$$

If $\mathbf{y}^*(p) \geq 2b$, we have

$$U_2^E\left(p, \mathbf{y}^*(p)\right) = -2(1-p)\left(\mathbf{y}^*(p) - b - \frac{1}{2}\right).$$

In either case,

$$U_2^E\left(p, \mathbf{y}^*(p)\right) > 0,$$

as $\mathbf{y}^*(p) \leq \frac{1}{2}$. □

Lemma B.3. *For all* $p \in (0, \overline{p}) \setminus \{\check{p}\}$,

$$\mathbf{y}^{*\prime}(p) < 0.$$

Moreover, the left-hand limit $\mathbf{y}^{*\prime}\left(\check{p}^-\right)$ *and right-hand limit* $\mathbf{y}^{*\prime}\left(\check{p}^+\right)$ *exist and*

$$\mathbf{y}^{*\prime}\left(\check{p}^-\right) < \mathbf{y}^{*\prime}\left(\check{p}^+\right) < 0.$$

Proof of Lemma B.3. Define

$$G\left(p,\mathbf{y}^{*}\left(p\right)\right)\equiv\left(p\min\left\{\mathbf{y}^{*}\left(p\right),2b\right\}+\left(1-p\right)\right)\mathbf{y}^{*}\left(p\right)-p\left(\int\limits_{\max\{\mathbf{y}^{*}(p)-2b,0\}}^{\mathbf{y}^{*}(p)}\theta d\theta\right)-\frac{1-p}{2}.$$

If $p>\check{p}$, then, $\mathbf{y}^{*}\left(p\right)<2b$, we have

$$G_{2}\left(p,\mathbf{y}^{*}\left(p\right)\right)=p\mathbf{y}^{*}\left(p\right)+\left(1-p\right)>0.$$

If $p<\check{p}$, then, $\mathbf{y}^{*}\left(p\right)>2b$, we have

$$G_{2}\left(p,\mathbf{y}^{*}\left(p\right)\right)=1-p>0.$$

Moreover,

$$G_{1}\left(p,\mathbf{y}^{*}\left(p\right)\right)=\left(\frac{1}{2}-\mathbf{y}^{*}\left(p\right)\right)+\left(\min\left\{\mathbf{y}^{*}\left(p\right),2b\right\}\mathbf{y}^{*}\left(p\right)-\int\limits_{\max\{\mathbf{y}^{*}(p)-2b,0\}}^{\mathbf{y}^{*}(p)}\theta d\theta\right)>0.$$

By the implicit function theorem, we have

$$\mathbf{y}^{*\prime}\left(p\right)=-\frac{G_{1}\left(p,\mathbf{y}^{*}\left(p\right)\right)}{G_{2}\left(p,\mathbf{y}^{*}\left(p\right)\right)}>0$$

for all $p\in(0,\overline{p})\setminus\{\check{p}\}$. Moreover, $\mathbf{y}^{*\prime}\left(\check{p}^{-}\right)<\mathbf{y}^{*\prime}\left(\check{p}^{+}\right)<0$. □

Next, we have

$$U_{1}^{E}\left(p,\mathbf{y}^{*}\left(p\right)\right)=\int\limits_{\theta\notin N(\mathbf{y}^{*}(p))}\left\{\left(\mathbf{y}^{*}\left(p\right)-\left(\theta+b\right)\right)^{2}-b^{2}\right\}d\theta>0, \tag{B.7}$$

because $\left|\mathbf{y}^{*}\left(p_{C}^{*}\right)-\left(\theta+b\right)\right|>|b|$ for all $\theta\notin N\left(\mathbf{y}^{*}\left(p_{C}^{*}\right)\right)$. It follows directly from (2.7) and (B.7) that all equilibria of the covert game involve positive equilibrium effort. Next, suppose by way of contradiction that there exists an equilibrium effort level p_{O}^{*} of the overt game and an equilibrium effort level p_{C}^{*} of the covert game such that

$$p_{C}^{*}\leq p_{O}^{*}.$$

There are two cases.

1. Suppose $p_C^* = p_O^*$, by (2.4), we must have

$$\lim_{p\to p_O^{*-}} \frac{dU^E\left(p,\mathbf{y}^*\left(p\right)\right)}{dp} - c'\left(p_O^*\right) \geq 0.$$

By Lemmas B.2 and B.3,

$$\lim_{p\to p_O^{*-}} U_2^E\left(p,\mathbf{y}^*\left(p\right)\right)\mathbf{y}^{*\prime}\left(p\right) < 0.$$

Therefore,

$$U_1^E\left(p_C^*,\mathbf{y}^*\left(p_C^*\right)\right) - c'\left(p_C^*\right) > 0,$$

which implies p_C^* is not an equilibrium effort level of the covert game. Contradiction.

2. Next, suppose $p_C^* < p_O^*$, then by Lemma B.3, we have $\mathbf{y}^*\left(p_C^*\right) > \mathbf{y}^*\left(p_O^*\right)$. By the definitions of p_C^* and p_O^*, we must have

$$U^E\left(p_C^*,\mathbf{y}^*\left(p_C^*\right)\right) - c\left(p_C^*\right) \geq U^E\left(p_O^*,\mathbf{y}^*\left(p_C^*\right)\right) - c\left(p_O^*\right),$$

and

$$U^E\left(p_O^*,\mathbf{y}^*\left(p_O^*\right)\right) - c\left(p_O^*\right) \geq U^E\left(p_C^*,\mathbf{y}^*\left(p_C^*\right)\right) - c\left(p_C^*\right),$$

which imply

$$U^E\left(p_O^*,\mathbf{y}^*\left(p_O^*\right)\right) \geq U^E\left(p_O^*,\mathbf{y}^*\left(p_C^*\right)\right).$$

On the other hand, by Lemma B.2, we must have $\mathbf{y}^*\left(p_C^*\right) \leq \mathbf{y}^*\left(p_O^*\right)$. Again, we have reached a contradiction.

□

Proof of Proposition 2.6. Suppose $p \leq \check{p}$, using (2.14), we have

$$\begin{aligned}
& U^{DM}\left(p, \mathbf{y}^*(p)\right) \\
= & -p \int_{\theta \in N(\mathbf{y}^*(p))} \left(\theta - \mathbf{y}^*(p)\right)^2 d\theta - (1-p) \int_0^1 \left(\theta - \mathbf{y}^*(p)\right)^2 d\theta \\
= & -p \left[\frac{\left(\theta - \mathbf{y}^*(p)\right)^3}{3}\right]_{\mathbf{y}^*(p)-2b}^{\mathbf{y}^*(p)} - (1-p) \left[\frac{\left(\theta - \mathbf{y}^*(p)\right)^3}{3}\right]_0^1 \\
= & -\frac{1}{3}\left(\frac{1-p}{4} + 8pb^3 + 12\left(\frac{p^2}{1-p}\right) b^4\right).
\end{aligned}$$

Suppose $p > \check{p}$, using (2.14), we have

$$\begin{aligned}
& U^{DM}\left(p, \mathbf{y}^*(p)\right) \\
= & -p \int_{\theta \in N(\mathbf{y}^*(p))} \left(\theta - \mathbf{y}^*(p)\right)^2 d\theta - (1-p) \int_0^1 \left(\theta - \mathbf{y}^*(p)\right)^2 d\theta \\
= & -p \left[\frac{\left(\theta - \mathbf{y}^*(p)\right)^3}{3}\right]_0^{\mathbf{y}^*(p)} - (1-p) \left[\frac{\left(\theta - \mathbf{y}^*(p)\right)^3}{3}\right]_0^1 \\
= & -\frac{1}{3}\frac{1-p}{\left(1+\sqrt{1-p}\right)^2}
\end{aligned}$$

Total-differentiate, we have

$$\frac{dU^{DM}\left(p, \mathbf{y}^*(p)\right)}{dp} = \begin{cases} -\frac{1}{3}\left(-\frac{1}{4} + 8b^3 + 12\left(\frac{2p}{1-p} + \frac{p^2}{(1-p)^2}\right) b^4\right) & \text{if } p < \check{p} \\ \frac{1}{3\left(1+\sqrt{1-p}\right)^3} & \text{if } p > \check{p} \end{cases}.$$

By solving the equation

$$-\frac{1}{3}\left(-\frac{1}{4} + 8b^3 + 12\left(\frac{2p}{1-p} + \frac{p^2}{(1-p)^2}\right) b^4\right) = 0, \tag{B.8}$$

we find that $U^{DM}\left(\cdot, \mathbf{y}^*(\cdot)\right)$ has a local maximum at $\hat{p} \in (0, \check{p})$ given by

$$\hat{p} = 1 - 4\sqrt{3}b^2 \frac{\sqrt{12b^2 + 4b + 1}}{-24b^3 + 4b^2 + 2b + 1} \tag{B.9}$$

if and only if $b < \frac{1}{8}$. If $b > \frac{1}{8}$, (B.8) has no solution on $[0, \check{p}]$ and $U^{DM}\left(\cdot, \mathbf{y}^*(\cdot)\right)$ is strictly increasing. □

Proof for Example 2.2. By (B.9), $U^{DM}(\cdot, \mathbf{y}^*(\cdot))$ is maximized at $\hat{p} = 1 - \frac{\sqrt{17}}{85}$. By (2.16), $\hat{p}$ is also the unique maximizer of $U^E(p, \mathbf{y}^*(p)) - c(p)$. Therefore, $\hat{p}$ is the unique equilibrium effort level of the overt game. On the other hand, (B.7) implies that $p^*_C = \overline{p} = \frac{24}{25}$. $\square$

APPENDIX C

APPENDIX FOR CHAPTER 3

Proof of Lemma 3.1. i) follows immediately from the definition of voting equilibrium under uncertainty. To show ii), consider $s \in \{a, b\}$, if $\gamma_s^B = 1$, party candidate A is an optimal choice given signal s, we must have $\gamma_s^A = \gamma_s^S = 1$. Similarly, if $\gamma_s^A = 0$, party candidate B is an optimal choice given signal s, we have $\gamma_s^S = \gamma_s^B = 0$. Finally, if $\left(\gamma_s^A, \gamma_s^B\right) = (1, 0)$, both party candidates are maximal choices. Therefore, swing voters are free to use any strategy $\gamma_s^S \in [0, 1]$. In all cases, we have $\gamma_s^A \geq \gamma_s^S \geq \gamma_s^B$. To show iii), suppose partisan A (B) votes for party candidate A after receiving signal b. (If partisan A (B) votes for party candidate B after receiving signal b, his strategy is trivially monotone in the signal.) Notice that party candidate A is maximal (optimal) given signal a if and only if $\Omega \geq Q_{\overline{p}}^a$ ($\Omega > Q_{\underline{p}}^a$). Since $Q_{\overline{p}}^b > Q_{\overline{p}}^a$ ($Q_{\underline{p}}^b > Q_{\underline{p}}^a$), if partisan A (B) votes for party candidate A after receiving signal b, he must vote for party candidate A after receiving signal a. □

Proof of Proposition 3.1. Proof in text. □

Lemma C.1. *Regardless of the expected number of voters n, if condition (3.13) is satisfied,*

$$\Omega = 1.$$

Proof of Lemma C.1. If $\phi_A = \tau_A$, $\Pr[T \mid \alpha] = \Pr[T \mid \beta]$, $\Pr[T_{-1} \mid \alpha] = \Pr[T_{-1} \mid \beta]$, and $\Pr[T_{+1} \mid \alpha] = \Pr[T_{+1} \mid \beta]$. So

$$\Omega = \frac{2\Pr[T \mid \alpha] + \Pr[T_{-1} \mid \alpha] + \Pr[T_{+1} \mid \alpha]}{2\Pr[T \mid \beta] + \Pr[T_{-1} \mid \beta] + \Pr[T_{+1} \mid \beta]} = 1.$$

If $\phi_A = \tau_B$, then,

$$\begin{aligned} \Pr[T \mid \alpha] &= \sum_{k=0}^{\infty} e^{-n} \frac{\phi_A^k}{k!} \frac{\phi_B^k}{k!} = \sum_{k=0}^{\infty} e^{-n} \frac{\tau_A^k}{k!} \frac{\tau_B^k}{k!} \\ &= \Pr[T \mid \beta], \end{aligned}$$

Moreover,

$$\begin{aligned} \Pr[T_{-1} \mid \alpha] &= e^{-n} \sum_{k=1}^{\infty} \frac{\phi_A^{k-1}}{(k-1)!} \frac{\phi_B^k}{k!} = e^{-n} \sum_{k=1}^{\infty} \frac{\tau_B^{k-1}}{(k-1)!} \frac{\tau_A^k}{k!} \\ &= \Pr[T_{+1} \mid \beta], \end{aligned}$$

and

$$\begin{aligned} \Pr[T_{+1} \mid \alpha] &= e^{-n} \sum_{k=1}^{\infty} \frac{\phi_A^k}{k!} \frac{\phi_B^{k-1}}{(k-1)!} = e^{-n} \sum_{k=1}^{\infty} \frac{\tau_B^k}{k!} \frac{\tau_A^{k-1}}{(k-1)!} \\ &= \Pr[T_{-1} \mid \beta]. \end{aligned}$$

Therefore,

$$\Omega = 1.$$

□

Proof of Proposition 3.2. Consider a fully informative voting profile $((1,0),(1,0),(1,0))$, we have $\phi_A = \tau_B = qn$. By Lemma C.1, $\Omega = 1$. Thus,

$$q > \max\left(1 - \underline{p}, \overline{p}\right) \Leftrightarrow Q^a_{\underline{p}} < \Omega < Q^b_{\overline{p}}.$$

Therefore, a fully informative voting equilibrium exists if and only if $q > \max\left(1 - \underline{p}, \overline{p}\right)$. □

Proof of Proposition 3.3. Consider an uninformative voting profile with every voter voting for party candidate A, $((1,1),(1,1),(1,1))$, we have $\phi_A = \tau_A = n$. By Lemma C.1, $\Omega = 1$. Thus,

$$\underline{p} > \max(q, 1-q) = q \Leftrightarrow \Omega > Q^a_{\underline{p}} \;\&\; \Omega > Q^b_{\underline{p}}$$

Since $q > \frac{1}{2}$, an uninformative voting equilibrium with every voter voting for party candidate A exists if and only if $q < \underline{p}$. □

Proof of Proposition 3.4. Proof in text. □

Proof of Proposition 3.5. Consider a full partisan voting profile $\left((1,1),(\gamma^S_a,\gamma^S_b),(0,0)\right)$ satisfying $\gamma^S_a = \gamma^S_b$ or $\gamma^S_a + \gamma^S_b = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}$, notice that

$$\begin{aligned} & \gamma^S_a = \gamma^S_b \\ \Longleftrightarrow \quad & n_A + n_S\left[q\gamma^S_a + (1-q)\gamma^S_b\right] = n_A + n_S\left[(1-q)\gamma^S_a + q\gamma^S_b\right] \\ \Longleftrightarrow \quad & \phi_A = \tau_A \end{aligned}$$

Moreover,

$$\begin{aligned} & \gamma^S_a + \gamma^S_b = 1 + \tfrac{\lambda_B - \lambda_A}{\lambda_S} \\ \Longleftrightarrow \quad & n_A + n_S\left[q\gamma^S_a + (1-q)\gamma^S_b\right] = n_B + n_S\left[(1-q)(1-\gamma^S_a) + q(1-\gamma^S_b)\right] \\ \Longleftrightarrow \quad & \phi_A = \tau_B \end{aligned}$$

By Lemma C.1, $\Omega = 1$. It constitutes a full partisan voting equilibrium if and only if

$$\frac{(1-\overline{p})q}{\overline{p}(1-q)} \leq 1 \leq \frac{(1-\underline{p})(1-q)}{\underline{p}q} \Leftrightarrow q \leq \min\left(\overline{p}, 1-\underline{p}\right). \tag{C.1}$$

□

Proof of Proposition 3.6. Proof in text. □

Proof of Lemma 3.2. Given any voting equilibrium under uncertainty, by Proposition 3.1, it is one of the four types. If it is a fully informative voting equilibrium, we have $\phi_A = qn = \tau_B$. By Lemma C.1, $\Omega = 1$. If it is an uninformative voting equilibrium, we have $\phi_A = \tau_A = n$. By Lemma C.1, $\Omega = 1$.

Next, suppose the limit voting equilibrium is a partisan voting equilibrium, let $\nu_A = \frac{\phi_A}{n} \in [0,1]$ and $\omega_A = \frac{\tau_A}{n} \in [0,1]$ denote the expected vote share of candidate A in states α and β, respectively. Note that ν_A and ω_A do not depend on the expected population size n. Using the Bessel function approximation, we have

$$\begin{aligned}
\Omega &\approx \frac{\Pr[T \mid \alpha]}{\Pr[T \mid \beta]} \frac{[2 + \left(\frac{\phi_A}{\phi_B}\right)^{\frac{1}{2}} + \left(\frac{\phi_A}{\phi_B}\right)^{-\frac{1}{2}}]}{[2 + \left(\frac{\tau_A}{\tau_B}\right)^{\frac{1}{2}} + \left(\frac{\tau_A}{\tau_B}\right)^{-\frac{1}{2}}]} \\
&= \frac{e^{-n} \frac{e^{2\sqrt{\phi_A \phi_B}}}{\sqrt{2\pi \cdot 2\sqrt{\phi_A \phi_B}}}}{e^{-n} \frac{e^{2\sqrt{\tau_A \tau_B}}}{\sqrt{2\pi \cdot 2\sqrt{\tau_A \tau_B}}}} \frac{[2 + \left(\frac{\phi_A}{\phi_B}\right)^{\frac{1}{2}} + \left(\frac{\phi_A}{\phi_B}\right)^{-\frac{1}{2}}]}{[2 + \left(\frac{\tau_A}{\tau_B}\right)^{\frac{1}{2}} + \left(\frac{\tau_A}{\tau_B}\right)^{-\frac{1}{2}}]} \\
&= \left(e^{2n}\right)^{\sqrt{\nu_A(1-\nu_A)} - \sqrt{\omega_A(1-\omega_A)}} \left(\frac{\nu_A (1-\nu_A)}{\omega_A (1-\omega_A)}\right)^{-\frac{1}{4}} \frac{[2 + \left(\frac{\nu_A}{1-\nu_A}\right)^{\frac{1}{2}} + \left(\frac{\nu_A}{1-\nu_A}\right)^{-\frac{1}{2}}]}{[2 + \left(\frac{\omega_A}{1-\omega_A}\right)^{\frac{1}{2}} + \left(\frac{\omega_A}{1-\omega_A}\right)^{-\frac{1}{2}}]} \\
&= g(n, \nu_A, \omega_A) f(\nu_A, \omega_A).
\end{aligned}$$

where

$$g(n, \nu_A, \omega_A) = \left(e^{2n}\right)^{\sqrt{\nu_A(1-\nu_A)} - \sqrt{\omega_A(1-\omega_A)}},$$

and

$$f(\nu_A, \omega_A) = \left(\frac{\nu_A (1-\nu_A)}{\omega_A (1-\omega_A)}\right)^{-\frac{1}{4}} \frac{[2 + \left(\frac{\nu_A}{1-\nu_A}\right)^{\frac{1}{2}} + \left(\frac{\nu_A}{1-\nu_A}\right)^{-\frac{1}{2}}]}{[2 + \left(\frac{\omega_A}{1-\omega_A}\right)^{\frac{1}{2}} + \left(\frac{\omega_A}{1-\omega_A}\right)^{-\frac{1}{2}}]}.$$

$g(n, \nu_A, \omega_A)$ is a function of the expected population size n, and vote shares of party candidate A in two states, ν_A and ω_A. ν_A and ω_A depend only on the voting profile (γ_a^s, γ_b^s) and signal precision q. $f(\nu_A, \omega_A)$ is a function of ν_A and ω_A. In a partisan voting equilibrium, there is at least a type of the partisans who will always vote for their party candidate when they receive a signal favoring their party candidate. Since $q \in \left(\frac{1}{2}, 1\right)$, this implies that $\nu_A, \omega_A \in (0,1)$. Given a particular partisan voting profile, $f(\nu_A, \omega_A)$ is a positive constant uniquely defined.

Next, $g(n, \nu_A, \omega_A)$ increases exponentially in n, if $\sqrt{\nu_A (1-\nu_A)} \geq \sqrt{\omega_A (1-\omega_A)}$. Otherwise, it decreases exponentially in n. Therefore,

$$\lim_{n\to\infty} g(n, \nu_A, \omega_A) = \begin{cases} \infty & if\ \nu_A (1-\nu_A) > \omega_A (1-\omega_A) \\ 1 & if\ \nu_A (1-\nu_A) = \omega_A (1-\omega_A) \\ 0 & if\ \nu_A (1-\nu_A) < \omega_A (1-\omega_A) \end{cases}.$$

By conditions (3.8), (3.11), and (3.12), there are positive and finite upper and lower bounds for Ω. Therefore, in any limit voting equilibrium, we must have $\nu_A (1-\nu_A) = \omega_A (1-\omega_A)$, which implies $g(n, \nu_A, \omega_A) = 1$. This also implies

$$\nu_A = \omega_A \text{ or } \nu_A = 1 - \omega_A,$$

which is equivalent to

$$\phi_A = \tau_A \text{ or } \phi_A = \tau_B.$$

We can now use Lemma C.1 to conclude that Ω equals one exactly. □

Proof of Lemma 3.3. By Lemma 3.2, in any limit voting equilibrium, $\Omega = 1$. By condition (3.11), a partial partisan limit voting equilibrium favoring party candidate A exists only if

$$\frac{(1-\underline{p})(1-q)}{\underline{p}q} < 1 \text{ and } \frac{(1-\overline{p})q}{\overline{p}(1-q)} \leq 1 \leq \frac{(1-\underline{p})q}{\underline{p}(1-q)},$$

which is equivalent to

$$1 - \underline{p} < q \text{ and } \underline{p} \leq q \leq \overline{p}.$$

Moreover, by (3.12), a partial partisan limit voting equilibrium favoring party candidate B exists only if

$$\frac{(1-\overline{p})(1-q)}{\overline{p}q} \leq 1 \leq \frac{(1-\underline{p})(1-q)}{\underline{p}q} \text{ and } 1 < \frac{(1-\overline{p})q}{\overline{p}(1-q)},$$

which is equivalent to

$$1 - \overline{p} \leq q \leq 1 - \underline{p} \text{ and } \overline{p} < q.$$

Notice that, by assumptions, $q > \frac{1}{2}, \overline{p} > \frac{1}{2}$, so the condition that $1-\overline{p} \leq q$ is not required. □

Proof of Proposition 3.7. By Lemma 3.2, in any limit voting equilibrium, $\Omega = 1$. Since

$$q \leq \min\left(\overline{p}, 1 - \underline{p}\right) \Leftrightarrow Q^b_{\overline{p}} \leq \Omega \leq Q^a_{\underline{p}},$$

full partisan limit voting equilibrium can only occur when $q \leq \min\left(\overline{p}, 1 - \underline{p}\right)$. Moreover, by Lemma 3.2, in any limit voting equilibrium, either $\phi_A = \tau_A$ or $\phi_A = \tau_B$. Consider the full partisan voting profile $\left((1,1), (\gamma_a^S, \gamma_b^S), (0,0)\right)$, we have

$$\begin{aligned}
& \phi_A = \tau_A \\
\iff \quad & n_A + n_S\left[q\gamma_a^S + (1-q)\gamma_b^S\right] = n_A + n_S\left[(1-q)\gamma_a^S + q\gamma_b^S\right] \\
\iff \quad & \gamma_a^S = \gamma_b^S
\end{aligned}$$

and

$$\begin{aligned}
& \phi_A = \tau_B \\
\iff \quad & n_A + n_S\left[q\gamma_a^S + (1-q)\gamma_b^S\right] = n_B + n_S\left[(1-q)(1-\gamma_a^S) + q(1-\gamma_b^S)\right] \\
\iff \quad & \gamma_a^S + \gamma_b^S = 1 + \tfrac{\lambda_B - \lambda_A}{\lambda_S}.
\end{aligned}$$

Thus, the set of all full partisan limit voting equilibria is given by (3.16). □

Proof of Proposition 3.8. Suppose there exists a partial partisan limit voting equilibrium favoring party candidate A, by Lemma 3.2, we only need to consider two cases. If $\phi_A = \tau_A$, then

$$n_A + n_S\left[q + (1-q)\gamma_b\right] + n_B q = n_A + n_S\left[(1-q) + q\gamma_b\right] + n_B\left(1-q\right)$$

$$n_S\left[1 - \gamma_b\right] + n_B = 0.$$

Since $n_S, n_B > 0$, $\gamma_b < 1$, we have reached a contradiction. If $\phi_A = n - \tau_A$, then

$$n_A + n_S\gamma_b = 0.$$

Again we have reached a contradiction. Therefore, there does exist a partial partisan voting profile favoring party candidate A to support $\Omega = 1$. Similarly, one can show that a partial partisan limit voting equilibrium favoring party candidate B does not exist. □

Proof of Proposition 3.9. There are two cases.

1. $q > \max(\overline{p}, 1-\underline{p})$: In this case, the limit voting equilibrium is fully informative, thus, $\sqrt{\phi_A} - \sqrt{\phi_B} = \sqrt{n}\left(\sqrt{q} - \sqrt{(1-q)}\right)$, $\sqrt{\frac{\phi_B}{\phi_A}} = \sqrt{\frac{1-q}{q}}$, $\sqrt{\phi_A \phi_B} = n\sqrt{q\,(1-q)}$.
2. $q \leq \min(\overline{p}, 1-\underline{p})$ and $|\lambda_A - \lambda_B| < \lambda_S$: In this case, there is a full partisan limit voting equilibrium that satisfies $\gamma_a^S > \gamma_b^S$ and $\gamma_a + \gamma_b = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}$, so

$$\begin{aligned} & \sqrt{\phi_A} - \sqrt{\phi_B} \\ = & \sqrt{n}\left\{\sqrt{\lambda_B + \lambda_S\left[(1-q)(1-\gamma_a^S) + q(1-\gamma_b^S)\right]}\right. \\ & \left. - \sqrt{\lambda_B + \lambda_S\left[(1-q)(1-\gamma_b^S) + q(1-\gamma_a^S)\right]}\right\} \end{aligned}$$

Moreover,

$$\begin{aligned} \sqrt{\phi_A \phi_B} &= n\sqrt{\lambda_B + \lambda_S\left[(1-q)(1-\gamma_a^S) + q(1-\gamma_b^S)\right]} \\ & \quad \times \sqrt{\lambda_B + \lambda_S\left[(1-q)(1-\gamma_b^S) + q(1-\gamma_a^S)\right]} \end{aligned}$$

and

$$\sqrt{\frac{\phi_B}{\phi_A}} = \sqrt{\frac{\lambda_B + \lambda_S\left[(1-q)(1-\gamma_b^S) + q(1-\gamma_a^S)\right]}{\lambda_B + \lambda_S\left[(1-q)(1-\gamma_a^S) + q(1-\gamma_b^S)\right]}}.$$

In both cases, the exponential term in (3.17) dominates. As a result, $\Pr[B \text{ wins}|\alpha] \to 0$. A similar argument shows that $\Pr[A \text{ wins}|\beta] \to 0$. □

Proof of Lemma 3.4. Suppose the swing voters' strategy (γ_a^S, γ_b^S) is justified by the prior $p_S \in \left[\underline{p}, \overline{p}\right]$. Notice that for all $p \in \left[\underline{p}, \overline{p}\right]$, $Q_p^b > Q_p^a$. Thus, if $\gamma_b^S \in (0,1]$, then $\Omega \geq Q_{p_S}^b$, $Q_{p_S}^b > Q_{p_S}^a$ implies that $\gamma_a^S = 1$. □

Proof of Proposition 3.10. Consider the set of full partisan limit voting equilibria identified in (3.16), suppose $\gamma_a^S = \gamma_b^S$, the requirements in Lemma 3.4 imply that $\gamma_a^S = \gamma_b^S \in \{0,1\}$. Suppose $\gamma_a^S + \gamma_b^S = 1 + \frac{\lambda_B - \lambda_A}{\lambda_S}$, by Lemma 3.4, either $\gamma_a^S = 1$ or $\gamma_b^S = 0$. Suppose $\gamma_a^S = 1$, then $\gamma_b^S = \frac{\lambda_B - \lambda_A}{\lambda_S}$. Since $\gamma_b^S \in [0,1]$, we must have $\frac{\lambda_B - \lambda_A}{\lambda_S} \in [0,1]$. Suppose $\gamma_b^S = 0$, then $\gamma_a^S = 1 - \frac{\lambda_A - \lambda_B}{\lambda_S}$. Since $\gamma_a^S \in [0,1]$, we must have $\frac{\lambda_A - \lambda_B}{\lambda_S} \in [0,1]$. Thus, the equilibria identified in Proposition 3.10 are the only possible forms of justifiable full partisan limit voting equilibria.

Next, we proceed to show that the equilibria identified in Proposition 3.10 are indeed justifiable. Suppose $\gamma_a^S = \gamma_b^S = 0$, the swing voters' strategy can be justified by $p = \underline{p}$. If $\gamma_a^S = \gamma_b^S = 1$, the swing voters' strategy can be justified by $p = \overline{p}$. Suppose $\gamma_a^S = 1$ and $\gamma_b^S = \frac{\lambda_B - \lambda_A}{\lambda_S}$, the swing voters' strategy can be justified by $p = q$. Suppose $\gamma_a^S = 1 - \frac{\lambda_A - \lambda_B}{\lambda_S}$ and $\gamma_b^S = 0$, the swing voters' strategy can be justified by $p = 1 - q$. □

Proof of Lemma 3.5. It is easy to see that, for all $p \in \left[\underline{p}, \overline{p}\right]$, for any $q > \frac{1}{2}$, we have

$$V_p(\frac{1}{2}, 1, 1) - V_p(q, 1, 1) = V_p(\frac{1}{2}, 0, 0) - V_p(q, 0, 0) = c\,(q) > 0.$$

This is because, with the dominated strategies, $(q, 1, 1)$ and $(q, 0, 0)$, the information cost $c\,(q)$ is always paid, but the information is never utilized. It is thus better to not acquire the information in the first place. Similarly, for all $p \in \left[\underline{p}, \overline{p}\right]$, for any $q > \frac{1}{2}$, we have

$$\begin{aligned} & V_p(\frac{1}{2}, 0, 1) - V_p(q, 0, 1) \\ = \; & \left(q - \frac{1}{2}\right) \left(p \left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) + (1-p) \left(\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]\right)\right) + c\,(q)\,. \end{aligned}$$

Thus, for any $q > \frac{1}{2}$, the pure strategies $(q, 1, 1)$, $(q, 0, 0)$, and $(q, 0, 1)$ are not maximal and the only pure strategy with $q > \frac{1}{2}$ that is not ruled out is $(q, 1, 0)$. □

Lemma C.2 (Vanishing Marginal Benefit of Information)**.** *In any sequence of voting equilibria with costly information under uncertainty, the derivative* $\frac{\partial v_p(\sigma)}{\partial q}$ *converges uniformly to zero as* n *tends to infinity.*

Proof of Lemma C.2. To prove the lemma, we prove the stronger result that all the pivotal probabilities, $\Pr[piv_A \mid \alpha]$, $\Pr[piv_B \mid \alpha]$, $\Pr[piv_A \mid \beta]$, and $\Pr[piv_B \mid \beta]$, converge uniformly to zero as $n \to \infty$. The result follows because

$$\begin{aligned} & \frac{\partial v_p(q,\gamma_a,\gamma_b)}{\partial q} \\ = \;& (\gamma_a - \gamma_b)\left\{p\left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) + (1-p)\left(\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]\right)\right\}. \end{aligned}$$

Consider the pivotal probability

$$\Pr[piv_A \mid \alpha] = e^{-n}\sum_{k=0}^{\infty}\frac{(n\eta_A)^k}{k!}\frac{(n(1-\eta_A))^k}{k!} + e^{-n}\sum_{k=1}^{\infty}\frac{(n\eta_A)^{k-1}}{(k-1)!}\frac{(n(1-\eta_A))^k}{k!}, \tag{C.2}$$

where $\eta_A \in [0,1]$. Let $\tilde{\eta}_A^n$ be the maximizer of (C.2) given n. All the terms in the first summation are maximized at $\eta_A = \frac{1}{2}$ and each term in the second summation is maximized at $\eta_A = \frac{k-1}{2k-1}$. Since all the terms are strictly concave, for each n, $\tilde{\eta}_A^n \in \left(0, \frac{1}{2}\right)$. Moreover, as n increases, the latter terms in the second summation receive more "weights", so $\tilde{\eta}_A^n$ is strictly increasing in n. Thus, $\lim_{n\to\infty}\tilde{\eta}_A^n = \frac{1}{2}$ and $\lim_{n\to\infty}\sqrt{\phi_A\phi_B} = \infty$, so by the approximation formulas (3.3) and (3.4),

$$\Pr[piv_A \mid \alpha] \approx \frac{e^{-\left(\sqrt{\phi_A}-\sqrt{\phi_B}\right)^2}}{\sqrt{4\pi\sqrt{\phi_A\phi_B}}}[1 + \left(\frac{\phi_A}{\phi_B}\right)^{-\frac{1}{2}}].$$

Since $\lim_{n\to\infty}\phi_A > 0$, the first term tends to zero as n increases while the second term in the square bracket is bounded. Hence, along the sequence $\{\tilde{\eta}_A^n\}_{n\geq 1}$,

$$\lim_{n\to\infty}\Pr[piv_A \mid \alpha] = 0.$$

By definition, the sequence $\{\tilde{\eta}_A^n\}_{n\geq 1}$ maximizes (C.2) for each n. The convergence of the probability $\Pr[piv_A \mid \alpha]$ to zero for the sequence $\{\tilde{\eta}_A^n\}_{n\geq 1}$ implies the convergence of the probability $\Pr[piv_A \mid \alpha]$ to zero for all other sequences.

Similarly, we can show that

$$\lim_{n\to\infty}\Pr[piv_B \mid \alpha] = \lim_{n\to\infty}\Pr[piv_A \mid \beta] = \lim_{n\to\infty}\Pr[piv_B \mid \beta] = 0.$$

□

Proof of Lemma 3.6. To prove the lemma, we first identify the set of pure maximal strategies in a given equilibrium for a given n. Then we show that the q-projection of this set converges to the set $\left\{\frac{1}{2}\right\}$ as $n \to \infty$. From Lemma 3.5, we know that a pure maximal strategy must be of the form $(q, 1, 0)$ if $q > 0$. Given that voting is fully informative, the maximizer $q^* \in \left[\frac{1}{2}, 1\right]$ for each $p \in \left[\underline{p}, \overline{p}\right]$ is pinned down by the first–order condition

$$c'\left(q^*\right) = p\left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) + (1-p)\left(\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]\right).$$

Note that the optimal q^* as a function of p satisfies

$$q^{*\prime}\left(p\right)\begin{cases} > 0 & \text{if } \Omega > 1 \\ = 0 & \text{if } \Omega = 1 \\ < 0 & \text{if } \Omega < 1 \end{cases}$$

Thus, the image of the mapping $q^* : \left[\underline{p}, \overline{p}\right] \to \mathbb{R}$ is given by

$$\left[\min\left\{q^*\left(\underline{p}\right), q^*\left(\overline{p}\right)\right\}, \max\left\{q^*\left(\underline{p}\right), q^*\left(\overline{p}\right)\right\}\right].$$

Because the function $V_p(\cdot, 1, 0)$ is strictly concave, any pure strategy of the form $(q, 1, 0)$ with $q \in [\frac{1}{2}, \min\{q^*\left(\underline{p}\right), q^*\left(\overline{p}\right)\})$ is dominated by the strategy $\left(\min\left\{q^*\left(\underline{p}\right), q^*\left(\overline{p}\right)\right\}, 1, 0\right)$. Similarly, any pure strategy of the form $(q, 1, 0)$ with $q \in \left(\max\left\{q^*\left(\underline{p}\right), q^*\left(\overline{p}\right)\right\}, 1\right]$ is dominated by the strategy $\left(\max\left\{q^*\left(\underline{p}\right), q^*\left(\overline{p}\right)\right\}, 1, 0\right)$. By Lemma C.2, for all $p \in \left[\underline{p}, \overline{p}\right]$, $q^*\left(p\right)$ converges uniformly to $\frac{1}{2}$ as $n \to \infty$. Therefore, for any sequence of pure strategies played in equilibrium with positive probability, q converges uniformly to $\frac{1}{2}$ as $n \to \infty$. □

Proof of Proposition 3.11. The proof consists of two parts. In the first part, we show that, for n large enough, the strategy profiles identified in Proposition 3.11 are balanced and justifiable equilibria. In the second part, we show that, for n large enough, any balanced and justifiable voting equilibrium must be one of the equilibria identified in Proposition 3.11.

1. We consider balancedness and justifiability separately.

a. Balancedness: Consider the strategy profile $((\frac{1}{2},1,1),(\frac{1}{2},1,1),(\frac{1}{2},0,0))$, we have

$$\phi_A = n_A + n_S = \tau_A.$$

Similarly, consider the strategy profile $((\frac{1}{2},1,1),(\frac{1}{2},0,0),(\frac{1}{2},0,0))$, we have

$$\phi_A = n_A = \tau_A.$$

Suppose the swing voters use the strategies $(\frac{1}{2},1,1)$ and $(q^*,1,0)$ with probabilities μ_A and $1-\mu_A$, respectively, then

$$\begin{aligned}\phi_A &= n_A + n_S\left[\mu_A + (1-\mu_A)\,q^*\right]\\ &= n_B + n_S\left[(1-\mu_A)\,q^*\right]\\ &= \tau_B.\end{aligned}$$

Similarly, suppose the swing voters use the strategies $(\frac{1}{2},0,0)$ and $(q^{**},1,0)$ with probabilities μ_B and $1-\mu_B$, respectively, then

$$\begin{aligned}\phi_A &= n_A + n_S\left[(1-\mu_B)\,q^*\right]\\ &= n_B + n_S\left[\mu_B + (1-\mu_B)\,q^*\right]\\ &= \tau_B.\end{aligned}$$

Thus, all the identified strategy profiles are balanced.

b. Justifiability: Given that the strategy profile is balanced, by Lemma C.1, $\Omega = 1$. Thus, we have, for all $p \in \left[\underline{p},\overline{p}\right]$,

$$\begin{aligned}&\frac{\partial v_p(q,1,0)}{\partial q}\\ =\;& p\left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) + (1-p)\left(\Pr[piv_A \mid \beta] + \Pr[piv_B \mid\beta]\right)\\ =\;& \Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\\ =\;& 2e^{-n}\sum_{k=0}^{\infty}\frac{\phi_A^k}{k!}\frac{\phi_B^k}{k!} + e^{-n}\sum_{k=1}^{\infty}\frac{\phi_A^{k-1}}{(k-1)!}\frac{\phi_B^k}{k!} + e^{-n}\sum_{k=1}^{\infty}\frac{\phi_A^k}{k!}\frac{\phi_B^{k-1}}{(k-1)!}.\end{aligned}$$

By (3.21) and Lemma C.2, there exists N such that for all $n > N$, $\frac{1}{2} < q^* < \min\left(\overline{p}, 1 - \underline{p}\right)$. Fix $n > N$ and consider the difference between the payoffs of the strategies $(\frac{1}{2}, 1, 1)$ and $(q^*, 1, 0)$. Given that $\Omega = 1$, we have

$$V_p(\frac{1}{2}, 1, 1) - V_p\left(q^*, 1, 0\right) = \left(p - q^*\right)\left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) + c\left(q^*\right), \quad \text{(C.3)}$$

which is positive if $p = \overline{p}$. Moreover, as $\Omega = 1$, when $p = \frac{1}{2}$, the symmetry between the two states implies that

$$V_{\frac{1}{2}}(\frac{1}{2}, 1, 1) = V_{\frac{1}{2}}(\frac{1}{2}, 1, 0) < V_{\frac{1}{2}}\left(q^*, 1, 0\right).$$

Thus, (C.3) is negative if $p = \frac{1}{2}$. Since the difference (C.3) is continuous and strictly increasing in p, for each $n > N$, there is a unique $p_A^n \in \left(\frac{1}{2}, \overline{p}\right)$ such that $V_{p_A^n}(\frac{1}{2}, 1, 1) = V_{p_A^n}\left(q^*, 1, 0\right)$ and for each $p \in (p_A^n, \overline{p}]$, $V_p(\frac{1}{2}, 1, 1) > V_p\left(q^*, 1, 0\right)$. Moreover, since $p_A^n > \frac{1}{2}$, for each $p \in [p_A^n, \overline{p}]$,

$$V_p(\frac{1}{2}, 1, 1) > V_p(\frac{1}{2}, 1, 0) = V_p(\frac{1}{2}, 0, 1) > V_p(\frac{1}{2}, 0, 0).$$

Using the same proof as in Lemma 3.5, we can show that all other pure strategies are dominated by the strategy $(q^*, 1, 0)$. Thus, the pure strategy $(\frac{1}{2}, 1, 1)$ is justified by any prior $p \in [p_A^n, \overline{p}]$ and mixed strategy between $(\frac{1}{2}, 1, 1)$ and $(q^*, 1, 0)$ is justified by the prior p_A^n.

Similarly, there exists a $p_B^n \in \left(\underline{p}, \frac{1}{2}\right)$ such that the pure strategy $(\frac{1}{2}, 0, 0)$ is justified by any prior $p \in \left[\underline{p}, p_B^n\right]$ and mixed strategy between $(\frac{1}{2}, 0, 0)$ and $(q^*, 1, 0)$ is justified by the prior p_B^n.

Finally, since the partisans stay with the status quo, justifiability implies that the strategy profiles are indeed voting equilibria.

2. Consider any sequence of balanced and justifiable voting equilibria $\{(\sigma_A^n, \sigma_B^n, \sigma_S^n)\}_{n \geq 1}$, we would like to show that for n large enough, $(\sigma_A^n, \sigma_B^n, \sigma_S^n)$ must be one of equilibria identified in Proposition 3.11. Since $(\sigma_A^n, \sigma_B^n, \sigma_S^n)$ is a balanced voting equilibrium, by Lemma C.1, $\Omega = 1$. Moreover, the proof of Lemma 3.6 establish that there exists N such that for all $n > N$, any pure strategy played with positive probability in equilibrium must have $q < \min\left(\overline{p}, 1 - \underline{p}\right)$. Fix such N.

a. We would like to show that the partisans must not acquire any information and must vote for their party candidates. Suppose partisans A play a pure strategy with $q > \frac{1}{2}$ with positive probability. By Lemma 3.5, the strategy must be of the form $(q, 1, 0)$. However, $q < \overline{p}$ means that $1 > Q_p^b = \frac{(1-\overline{p})q}{\overline{p}(1-q)}$, we have

$$V_{\overline{p}}\left(q,1,0\right) < V_{\overline{p}}\left(q,1,1\right) < V_{\overline{p}}(\frac{1}{2},1,1).$$

The last inequality follows from $V_{\overline{p}}\left(\frac{1}{2},1,1\right) - V_{\overline{p}}\left(q,1,1\right) = c\left(q\right) > 0$. Thus, the strategy $(q, 1, 0)$ is not optimal. As a result, partisans A cannot use any pure strategy with $q > \frac{1}{2}$. Moreover, $\overline{p} > q = \frac{1}{2}$ implies that $1 > Q_p^b$, so

$$V_{\overline{p}}\left(\frac{1}{2},0,0\right) < V_{\overline{p}}\left(\frac{1}{2},0,1\right) = V_{\overline{p}}\left(\frac{1}{2},1,0\right) < V_{\overline{p}}\left(\frac{1}{2},1,1\right).$$

Thus, partisans of A must use the pure strategy $\left(\frac{1}{2},1,1\right)$ in equilibrium. Similarly, partisans of B must use the pure strategy $\left(\frac{1}{2},0,0\right)$ in equilibrium.

b. Next, we would like to show that the swing voters can only use the strategies identified in Proposition 3.11. Suppose the swing voters use a pure strategy with $q > \frac{1}{2}$ with positive probability in equilibrium, as $\Omega = 1$ in a balanced voting equilibrium, we have

$$\begin{aligned} & \frac{\partial v_p(q,1,0)}{\partial q} \\ = \;& p\left(\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha]\right) + (1-p)\left(\Pr[piv_A \mid \beta] + \Pr[piv_B \mid \beta]\right) \\ = \;& \Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha], \end{aligned}$$

which is independent of the prior p. Thus, given the voting strategy $(1, 0)$, the information acquisition level q^{**} that solves

$$\Pr[piv_A \mid \alpha] + \Pr[piv_B \mid \alpha] = c'\left(q\right),$$

maximizes the expected payoff under all $p \in \left[\underline{p}, \overline{p}\right]$. Thus, any strategy $(q, 1, 0)$ with $q \neq q^{**}$ is dominated by $(q^{**}, 1, 0)$ and cannot be used with positive probability in equilibrium. Notice, however, that q^{**} depends on the equilibrium strategy profile and is not determined at this point. However, in the following steps, we will show that the equilibrium strategy profile must be given by those in 2. of Proposition 3.11, forcing $q^{**} = q^*$.

Next, since $V_p(\frac{1}{2}, 0, 1) = V_p(\frac{1}{2}, 1, 0)$ and $(\frac{1}{2}, 1, 0)$ is dominated by $(q^*, 1, 0)$, $(\frac{1}{2}, 0, 1)$ is also dominated.

At this point, we have shown that the swing voters can only mix between $(q^{**}, 1, 0)$, $(\frac{1}{2}, 1, 1)$, and $(\frac{1}{2}, 0, 0)$. Next, we want to show that the swing voters cannot mix between $(\frac{1}{2}, 1, 1)$ and $(\frac{1}{2}, 0, 0)$. Suppose the mixed strategy is justified by the prior $p_S \in [\underline{p}, \overline{p}]$, then the optimality of the strategies given p_S implies that $Q^a_{p_S} = Q^b_{p_S} = 1$, so $p_S = \frac{1}{2}$. But then

$$V_{\frac{1}{2}}(\frac{1}{2}, 1, 1) = V_{\frac{1}{2}}(\frac{1}{2}, 0, 0) = V_{\frac{1}{2}}(\frac{1}{2}, 1, 0) < V_{\frac{1}{2}}(q^{**}, 1, 0).$$

Thus, mixing between $(\frac{1}{2}, 1, 1)$ and $(\frac{1}{2}, 0, 0)$ cannot be justified.

Next, suppose the swing voters mix between $(\frac{1}{2}, 1, 1)$ and $(q^{**}, 1, 0)$ with probabilities μ and $1-\mu$, respectively. A balanced voting equilibrium requires that either $\phi_A = \tau_A$ or $\phi_A = \tau_B$. Thus, suppose $\phi_A = \tau_A$, then

$$\begin{aligned} & \phi_A = \tau_A \\ \iff & n_A + n_S \left[\mu + (1-\mu)\, q^{**}\right] = n_A + n_S \left[\mu + (1-\mu)\,(1 - q^{**})\right] \\ \iff & q^{**} = 1 - q^{**}, \end{aligned}$$

which is impossible since $q^{**} > \frac{1}{2}$. Suppose $\phi_A = \tau_B$, then

$$\begin{aligned} & \phi_A = \tau_B \\ \iff & n_A + n_S \left[\mu + (1-\mu)\, q^{**}\right] = n_B + n_S \left[(1-\mu)\, q^{**}\right] \\ \iff & \mu = \tfrac{\lambda_B - \lambda_A}{\lambda_S}. \end{aligned}$$

which is the probability μ_A identified in 2. of Proposition 3.11.

Finally, suppose the swing voters mix between $(\frac{1}{2}, 0, 0)$ and $(q^{**}, 1, 0)$ with probabilities μ and $1-\mu$, respectively. A balanced voting equilibrium requires that either $\phi_A = \tau_A$ or $\phi_A = \tau_B$. Suppose $\phi_A = \tau_A$, then

$$\begin{aligned} & \phi_A = \tau_A \\ \iff & n_A + n_S\,(1-\mu)\, q^{**} = n_A + n_S\,(1-\mu)\,(1 - q^{**}) \\ \iff & q^{**} = 1 - q^{**} \end{aligned}$$

which is impossible since $q^{**} > \frac{1}{2}$. Suppose $\phi_A = \tau_B$, then

$$\begin{aligned} & \phi_A = \tau_B \\ \iff \quad & n_A + n_S\left[(1-\mu)\,q^{**}\right] = n_B + n_S\left[\mu + (1-\mu)\,q^{**}\right] \\ \iff \quad & \mu = \tfrac{\lambda_A - \lambda_B}{\lambda_S}. \end{aligned}$$

which is the probability μ_B identified in 2. of Proposition 3.11.

□

BIBLIOGRAPHY

[1] Angeletos, G. M. and Pavan, A., "Transparency of Information and Coordination in Economies with Investment Complementarities", *American Economic Review*, 2004, **94**, 91–98

[2] ———,"Efficient Use of Information and Social value of Information", *Econometrica*, 2007, **75**, 1103–1142

[3] Aragones, E. and Palfrey, T., "Mixed Equilibrium in a Downsian Model with a Favored Candidate", *Journal of Economic Theory*, 2002, **103**, 131-161

[4] Athey, S. and Levin, D., "The Value of Information in Monotone Decision Problems", working paper, 2000

[5] Austen-Smith, D., "Information Transmission in Debate", *American Journal of Political Science*, 1990, **34**, 124–152

[6] ———, "Strategic Transmission of Costly Information", *Econometrica*, 1994, **62**, 955–963

[7] ——— and Banks J., 1996. "Information Aggregation, Rationality and the Condorcet Jury Theorem", *American Political Science Review*, 1996, **90**(1), 34–45

[8] Agenziano, R., Severinov, S. and Squintani, F., "Strategic Information Acquisition and Transmission", working paper, University of Essex, 2015

[9] Bassan, B., Scarsini, M. and Zamir, S.,""I Don't Want to Know!": Can It Be Rational?", working paper, 1997

[10] Battaglini, M., "Multiple Referrals and Multidimensional Cheap Talk", *Econometrica*, 2002, **70**, 1379–1401

[11] Bewley, T. F., "Knightian Decision Theory: Part i", *Decicions in Economics and Finance*, 2002, **25**(2), 79-110

[12] ———, "Knightian Decision Theory: Part ii", technical report, Cowles Foundation, 1987

[13] ———, “Market Innovation and Entrepreneurship: A Knightian View”, technical report, Cowles Foundation, 1989

[14] Bhattacharya, S. and Mukherjee, A., “Strategic Information Revelation When Experts Compete to Influence”, *RAND Journal of Economics*, 2013, **44**, 522–544

[15] Blackwell, D., “Comparisons of Experiments”, *Proceedings of Second Berkeley Symposium on Mathematical Statistics*, 1951, 93–102

[16] ———,“Equivalent Comparison of Experiments”, *Annals of Mathematical Statistics*, 1953, **24**, 265–272

[17] Burden, B. C. and Klofstad, C. A., “Affect and Cognition in Party Identification”, *Political Psychology*, 2005, **26**(6), 869-886

[18] Campbell, A., Converse, P. E., Miller, W. E. and Stokes, D. E., The American Voter, The University of Chicago Press, Chicago, 1960

[19] Che, Y-K, and Kartik, N., “Opinions as Incentives”, *Journal of Political Economy*, 2009, **117**, 815–860

[20] Conlisk, J., “Optimization Cost”, *Journal of Economic Behavior and Organization*, 1988, **9**(3), 213–228

[21] Crawford, V. and Sobel, J., “Strategic Information Transmission”, *Econometrica*, 1982, **50**, 1431–1450

[22] Dewatripont, M. and Tirole, J., “Advocates”, *Journal of Political Economy*, 1999, **107**, 1–39

[23] Ericson, W., “A Note on the Posterior Mean of a Population Mean”, *Journal of the Royal Statistical Society*, 1969, **31**, 332–334

[24] Eső, P. and Szalay, D., “Incomplete Language as an Incentive Device”, working paper, 2010

[25] Feddersen, T. J. and Pesendorfer, W., “The Swing Voter’s Curse”, *The American Economic Review*, 1996, **86**(3), 408–424

[26] ———, “Voting Behavior and Information Aggregation in Large Elections with Private Information”, *Econometrica*, 1997, **65**(5), 1029–1058

[27] ———, “Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting”, *The American Political Science Review*, 1998, **92**(1), 23–35

[28] ———, 1999. “Abstention in Elections with Asymmetric Information and Diverse Preferences,” *The American Political Science Review*, 1999, **93**(2), 381–398

[29] Gendron-Saulnier, C. and Gordon, S., "Information Choice and Diversity: The Role of Strategic complementarities", working paper, 2015

[30] Glicksberg, I. L., "A Further Generalization of the Kakutani Fixed Point Theorem with Application to Nash Equilibrium Points", *Proceedings of the American Mathematical Society*, 1952, **3**, 170–174

[31] Goltsman, M., Hörner, J., Pavlov, G. and Squintani, F., "Mediation, Arbitration, and Negotiation", *Journal of Economic Theory*, 2009, **144**, 1397–1420

[32] Gul, F. and Pesendorfer, W., "Partisan Politics and Election Failure with Ignorant Voters", *Journal of Economic Theory*, 2009, **144**(1),146–174

[33] Hellwig, C., "Heterogeneous Information and the Welfare Effects of Public Information Disclosure", working paper, 2005

[34] ——— and Veldkamp, L., "Knowing What Others Know: Coordination Motives in Information Acquisition", *The Review of Economic Studies*, 2009, **76**(1), 223–251

[35] Henry, E., "Strategic Disclosure of Research Results: The Cost of Proving Your Honesty", *Economic Journal*, 2009, **119**, 1036-1064

[36] Holmström, B., "Moral Hazard and Observability", *Bell Journal of Economics*, 1979, **10**, 74–91

[37] ———, "On Theory of Delegation", In M. Boyer and R. Kihlstrom (eds.), *Bayesian Models in Economic Theory*, Amsterdam: North-Holland, 1984

[38] Jiménez-Martínez, A., "Information Acquisition Interactions in Two-player Quadratic Games", *International Journal of Game Theory*, 2014, **43**(2), 455–485

[39] Kamenica, E. and Gentzkow, M., "Bayesian Persuasion", *American Economic Review*, 2011, **101**, 2590–2615

[40] Kartik, N. and Tercieux, O., "Implementation with Evidence", *Theoretical Economics*, 2012, **7**, 323–355

[41] Kartik, N., Xu Lee, F. and Suen, W., "Investment in Concealable Information by Biased Experts", working paper, University of Hong Kong, 2015

[42] Kováč, E. and Mylovanov, T., "Stochastic Mechanisms in Settings Without Monetary Transfers", *Journal of Economic Theory*, 2009, **144**, 1373–1395

[43] Krishna, V. and Morgan, J., "A Model of Expertise", *Quarterly Journal of Economics*, 2001, **116**, 747–775

[44] ——— and ———, "Overcoming Ideological Bias in Elections", *Journal of Political Economy*, 2011, **119**(2), 183–211

[45] ——— and ———, “Voluntary Voting: Costs and Benefits”, *Journal of Economic Theory,* 2012, **147**(6), 2083–2123

[46] Jovanovic, B., “Truthful Disclosure of Information”, *Bell Journal of Economics*, 1982, **13**, 36–44

[47] Lehman, E. L., “Comparing Location Experiments”, *Annual of Statistics*, 1988, **27**, 272–299

[48] Lehrer, E. and Teper, R., “Justifiable preferences”, *Journal of Economic Theory*, 2011, **146**(2), 762–774

[49] Lopomo, P., Rigotti, L. and Shannon, C., “Uncertainty in Mechanism Design”, working paper, The University of Pittsburgh, 2014

[50] Martinelli, C., “Would rational voters acquire costly information?”, *Journal of Economic Theory*, 2006, **129**(1), 225–251

[51] ———, “Ignorance and Naivete in Large Elections”, working paper, CIE-ITAM, 2011

[52] Mason, R. and Valentinyi, Á., “The Existence and Uniqueness of Monotone Pure Strategy Equilibrium in Bayesian Games”, working paper, 2010

[53] Meirowitz, A., “On the Existence of Equilibria to Bayesian Games with Non-Finite Type and Action Spaces”, *Economics Letters*, 2003, **78**, 213–218

[54] Milgrom, P., “What the Seller Won’t Tell You: Persuasion and Disclosure in Markets”, *Journal of Economic Perspectives*, 2008, **22**, 115–131

[55] Milgrom, P. and Roberts, J., “Relying on the Information of Interested Parties”, *RAND Journal of Economics*, 1986, **17**, 18–32

[56] Mohlin, E., “Optimal Categorization”, *Journal of Economic Theory*, 2014, **152**, 356–381

[57] Molzon, R., “Marginal Value of Information”, working paper, 2015

[58] Morris, S. and Shin, H. S., “Social Value of Public Information”, *American Economic Review*, 2002, **92**, 1521–1534

[59] Myatt, D. P., “On the Theory of Strategic Voting”, *Review of Economic Studies*, 2007, **74**(1), 255–281

[60] Myerson, R., “Extended Poisson Games and the Condorcet Jury Theorem”, *Games and Economic Behavior*, 1998, **25**, 111–131

[61] ———, “Large Poisson Games”, *Journal of Economic Theory*, 2000, **94**, 7–45

[62] Palfrey T. R. and Rosenthal, H., “A Strategic Calculus of Voting”, *Public Choice*, 1983, **41**(1), 7–53

[63] Pei, H. D., "Communication with Endogenous Information Acquisition", *Journal of Economic Theory*, 2015, **160**, 132–149

[64] Persico, N., "Information Acquisition in Auctions", *Econometrica*, 2000, **68**(1), 135–148

[65] Prat, A., "The Wrong Kind of Transparency", *American Economic Review*, 2005, **95**, 862–877

[66] Shavell, S., "Acquisition and Disclosure of Information Prior to Sale", *RAND Journal of Economics*, 1994, **25**, 20–36

[67] Shin, H., "The Burden of Proof in a Game of Persuasion", *Journal of Economic Theory*, 1994, **64**, 253–264

[68] ———, "Adversarial and Inquisitorial Procedures in Arbitration", *RAND Journal of Economics*, 1998, **29**, 378–405

[69] Stokey, N. L., Lucas, R. E. Jr and Prescott, E. C., Recursive Methods in Economic Dynamics, Cambridge, MA: Harvard University Press, 1989

[70] Szalay, D., "The Economics of Clear Advice and Extreme Options", *Review of Economic Studies*, 2005, **72**, 1173–1198

[71] Szkup, M. and Trevino, I., "Information Acquisition and Transparency in Global Games", working paper, 2014

[72] Ui, T. and Yoshizawa, Y., "Characterizing Social Value of Information", *Journal of Economic Theory*, 2015, **158**(B), 507–535

[73] Van Zandt, T. and Vives, X., "Monotone Equilibria in Bayesian Games of Strategic Complementarities", *Journal of Economic Theory*, 2007, **134**, 339–360

[74] Vives, X., "Duopoly Information Equilibrium: Cournot and Bertrand", *Journal of Economic Theory*, 1984, **34**, 71–94

[75] ———, "Aggregation of Information in Large Cournot Markets", *Econometrica*, 1988, **56**, 851–876

[76] Wolinsky, A., "Eliciting Information from Multiple Experts", *Games and Economic Behavior,* 2002, **41**, 141–160