GEOMETRIC ANALYSIS ON METRIC SPACES

by

Soheil Malekzadeh

B.Sc. in Mathematics, Shahid Beheshti University, 2009

M.Sc. in Mathematics, Sharif University of Technology, 2011

Submitted to the Graduate Faculty of

the Kenneth P. Dietrich School of Arts and Sciences in partial

fulfillment

of the requirements for the degree of

Doctor of Philosophy in Mathematics

University of Pittsburgh

2016

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Soheil Malekzadeh

It was defended on

November 15th 2016

and approved by

Piotr Hajłasz, Department of Mathematics, University of Pittsburgh

Jason DeBlois, Department of Mathematics, University of Pittsburgh

Christopher J. Lennard, Department of Mathematics, University of Pittsburgh

Jeremy T. Tyson, Department of Mathematics, University of Illinois at Urbana-Champaign

Dissertation Director: Piotr Hajłasz, Department of Mathematics, University of Pittsburgh

GEOMETRIC ANALYSIS ON METRIC SPACES

Soheil Malekzadeh, PhD

University of Pittsburgh, 2016

My research is focused on geometric measure theory and analysis on metric spaces. Over the past fifteen years, these subjects have expanded dramatically with applications far beyond pure mathematics in many branches of natural sciences. The generality of these ideas has reconciled a large number of previously singular attempts to extend tools from differential geometry to a much larger class of spaces that are not necessarily smooth.

This thesis includes three main results that have been the focus of my research at the University of Pittsburgh with the common theme of geometric analysis on metric spaces.

The Lusin's condition (N) plays a significant role in the theory of integration. Because of this importance in applications, we investigate how this condition is related to Sobolev spaces. We focus our attention to the class of $W^{1,n}$ mappings and provide a new proof for the fact that continuous and pseudomonotone mapping of the class $W^{1,n}$ satisfies the condition (N) on open sets. This result is due to Malý and Martio but the original proof makes it difficult to gain insight into the internals of this result. We present a new proof which is based on the Hardy-Littlewood maximal function.

In the second result, we find necessary and sufficient conditions for a Lipschitz map $f : E \subset \mathbb{R}^n \to X$ into a metric space to satisfy $\mathcal{H}^k(f(E)) = 0$. An interesting feature of our approach is that despite the fact that we are dealing with arbitrary metric spaces, we employ a variant of the classical implicit function theorem. Applications include pure unrectifiability of the Heisenberg groups and that of more general Carnot-Carathéodory spaces.

Lastly, we present a new characterization of the mappings of bounded length distortion (BLD for short). In the original geometric definition it is assumed that a BLD mapping is open, discrete and sense preserving. We prove that the first two of the three conditions are redundant and the sense-preserving condition can be replaced by a weaker assumption that the Jacobian is non-negative.

TABLE OF CONTENTS

1.0 INTRODUCTION . . . 1

2.0 PRELIMINARIES . . . 7

- 2.1 Metric and Measure Spaces . . . 7
- 2.2 Curves In Metric Spaces . . . 9
- 2.3 Maximal Functions . . . 14
- 2.4 Sobolev Spaces . . . 20
  - 2.4.1 Definitions and Elementary Properties . . . 20
  - 2.4.2 Sobolev Inequalities . . . 21

3.0 LUSIN CONDITION (N) AND MAPPINGS OF CLASS $W^{1,n}$ . . . 23

- 3.1 Introduction . . . 23
- 3.2 Case $p = n$ and the Main Result . . . 23

4.0 UNRECTIFIABILITY OF METRIC SPACES . . . 31

- 4.1 Introduction . . . 31
- 4.2 Lipschitz mappings into ℓ^∞ . . . 34
- 4.3 Heisenberg groups . . . 44
- 4.4 Generalization of the Main Result . . . 46
- 4.5 Applications . . . 50
  - 4.5.1 Mappings of bounded length distortion . . . 50
  - 4.5.2 Carnot-Carathéodory spaces . . . 51

5.0 MAPPINGS OF BOUNDED LENGTH DISTORTION . . . 54

- 5.1 Introduction . . . 54
- 5.2 Path Lifting for Open and Discrete Mappings . . . 57

5.3 Proof of the Main Result 62
6.0 BLD EMBEDDING CONJECTURE AND RELATED OPEN PROBLEMS 66
6.1 Introduction 66
6.2 BLD Embedding Conjecture 67
BIBLIOGRAPHY 71

1.0 INTRODUCTION

In the 1990s, a large body of independent research was done to generalize results on quasiconformal and quasisymmetric mappings in the setting of Euclidean spaces to a more abstract setting. Doubling metric measure spaces supporting Poincaré inequality was deemed to be the right framework to unify these independent attempts, and as a result, analysis on metric spaces was born. Over the past fifteen years, this branch has expanded dramatically with applications far beyond pure mathematics in many branches of natural sciences. The generality of these ideas has also reconciled a large number of previously singular attempts to extend tools from differential geometry to a much larger class of spaces that are not necessarily smooth, [20].

This thesis includes three main results that have been the focus of my research at the University of Pittsburgh with the common theme of geometric analysis on metric spaces.

In what follows, a summary of these results will be presented.

Lusin condition (N) and mappings of class $W^{1,n}$. A continuous function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfies *Lusin's condition* (N) if $|f(E)| = 0$ whenever $E \subset \Omega$ and $|E| = 0$. Here, $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Lusin's condition (N) plays a significant role in the theory of integration. Because of this importance in applications, we investigate how this condition is related to Sobolev spaces. Smooth mappings, for example C^1, locally Lipschitz, or even continuous mappings in the Sobolev space $W^{1,p}(\Omega)$ with $p > n$, satisfy Lusin's condition (N). This is a simple consequence of Morrey's Inequality. However, it is well-known that for $p < n$, such mappings fail to satisfy the condition (N), (See, for example, [37]). For the case $n = 1$ the condition (N) is completely understood, see [43], but for $n \geq 2$ necessary and sufficient conditions for the condition are not clear.

We focus our attention to the class of $W^{1,n}(\Omega)$ mappings where $\Omega \subset \mathbb{R}^n$ is open and prove the following result. We need a simple definition first.

Definition 1.0.1. A mapping $f : \Omega \to \mathbb{R}^n$ is said to be K*-pseudomonotone* for $K \geq 1$ if for each $x \in \Omega$

$$\operatorname{diam}(f(B(x,r))) \leq K \operatorname{diam}(f(\partial B(x,r)))$$

whenever $\bar{B}(x,r) \subset \Omega$.

Theorem 1.0.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^n$ is a continuous, K-pseudomonotone mapping of the class $W^{1,n}(\Omega)$. Then f satisfies the condition (N).*

This result is due to Malý and Martio [30]. The elegant proof presented by Malý and Martio makes it difficult to gain insight into the internals of this result. In an attempt to gain a better understanding of the condition (N) and due to the important role this condition plays in analysis on metric spaces, in chapter 3, we present a new and elementary proof which is based on the Hardy-Littlewood maximal function and the Hardy-Littlewood-Wiener theorem.

Conditions for Unrectifiability of a Metric Space. The theory of rectifiable sets plays a significant role in geometric measure theory and calculus of variations.

We say that a metric space (X, d) is k*-rectifiable* if there is a family of Lipschitz mappings $f_i : E_i \subset \mathbb{R}^k \to X$ such that

$$\mathcal{H}^k\left(X \setminus \bigcup_{i=1}^{\infty} f_i(E_i)\right) = 0.$$

If $\mathcal{H}^k(f(E)) = 0$ for any Lipschitz map $f : E_i \subset \mathbb{R}^k \to X$, we say that the space X is *purely* k*-unrectifiable*. Here $\mathcal{H}^k$ stands for the Hausdorff measure.

It was proved in [1, 28] that the Heisenberg group $\mathbb{H}^n$ is k-unrectifiable when $k > n$. The original proofs employed the Kirchheim-Rademacher theorem, the Pansu-Rademacher theorem and the area formula for mappings into metric spaces. New and elementary proofs were developed in [3] and in [16]. This new method was used in [3] to solve a problem posed by Magnani [27] about the rank of the derivative of a Sobolev mapping into the Heisenberg group (the problem was solved independently in [29]). Also the same method led us in [16] to new characterizations of pure k-unrectifiability of metric spaces:

Theorem 1.0.3. *Let X be a metric space, let $E \subset \mathbb{R}^k$ be measurable, and let $f : E \to X$ be a Lipschitz mapping. Then the following statements are equivalent:*

1. $\mathcal{H}^k(f(E)) = 0$;
2. *For any Lipschitz mapping $\varphi : X \to \mathbb{R}^k$, we have $\mathcal{H}^k(\varphi(f(E))) = 0$;*
3. *For any collection of distinct points $\{y_1, y_2, \dots, y_k\} \subset X$, the associated projection $g : E \to \mathbb{R}^k$ of f satisfies $\mathcal{H}^k(g(E)) = 0$;*
4. *For any collection of distinct points $\{y_1, y_2, \dots, y_k\} \subset X$, the associated projection $g : E \to \mathbb{R}^k$ of f satisfies $\operatorname{rank}(\operatorname{ap} Dg(x)) < k$ for $\mathcal{H}^k$-a.e. $x \in E$.*

Here, the *associated projection* $g : E \to \mathbb{R}^k$ of f is defined by

$$g(x) = (d(f(x), y_1), \cdots, d(f(x), y_k)).$$

Although conditions (3) and (4) are necessary and sufficient for the validity of (1), often it is not easy to verify them. The problem is that even if X is smooth, the distance function $y \mapsto d(y, y_i)$ is not smooth at y_i and we need to consider such distance functions for y_i from a dense subset of X, thus creating singularities everywhere in X. Actually a collection of such distance functions gives an isometric embedding of X into ℓ^∞. This is a reason why the above theorem is a corollary of the following result.

Theorem 1.0.4. *Let $E \subset \mathbb{R}^k$ be measurable and let $f : E \to \ell^\infty$ be a Lipschitz mapping. Then $\mathcal{H}^k(f(E)) = 0$ if and only if $\operatorname{rank}(\operatorname{ap} Df(x)) < k$, $\mathcal{H}^k$-a.e. in E.*

In applications we often deal with spaces X that have some sort of smoothness (like Heisenberg groups or more general Carnot-Carathéodory spaces) and often for such spaces there is a more natural Lipschitz mapping $\Phi : X \to \mathbb{R}^N$, or even a Lipschitz mapping $\Phi : X \to \ell^\infty$ which is not necessarily an isometric embedding into ℓ^∞, a mapping that takes into account the structure of X. We were able to successfully develop a suitable version of the above theorem and apply it to Carnot-Carathéodory spaces. The following result is much stronger than Theorem 1.0.4.

Theorem 1.0.5. *Suppose that (X, d) is a complete and quasiconvex metric space and that $\Phi : X \to \ell^\infty$ is a Lipschitz map with the property that for some constant $C_\Phi > 0$ and all rectifiable curves γ in X we have*

$$\ell(\gamma) \le C_\Phi \ell(\Phi \circ \gamma). \tag{1.0.1}$$

Then for any $k \ge 1$ and any Lipschitz map $f : E \subset \mathbb{R}^k \to X$ defined on a measurable set $E \subset \mathbb{R}^k$ the following conditions are equivalent.

1. $\mathcal{H}^k(f(E)) = 0$ *in* X,
2. $\mathcal{H}^k(\Phi(f(E))) = 0$ *in* ℓ^∞,
3. $\operatorname{rank}(\operatorname{ap} D(\Phi \circ f)) < k$ *a.e. in* E.

Here, $\ell(\gamma)$ stands for the length of the curve γ.

Corollary 1.0.6. *The above result remains true if we replace ℓ^∞ by $\mathbb{R}^N$ (in two places).*

Indeed, the norm in $\mathbb{R}^N$ is equivalent to the norm in ℓ^∞_N and ℓ^∞_N can be regarded as a subspace in ℓ^∞.

Corollary 1.0.6 was one of the main results in [16] and Theorem 1.0.5 is a generalization of it.

Mappings of Bounded Length Distortion. The class of mappings of bounded length distortion (BLD for short) plays a fundamental role in the contemporary development of geometric analysis and geometric topology, especially in the context of branched coverings of metric spaces.

We say that a continuous map $f : \Omega \to \mathbb{R}^n$ from a domain $\Omega \subset \mathbb{R}^n$ is BLD (of *bounded length distortion*) if it is open, discrete, sense preserving and if there is a constant $M \ge 1$ such that for all curves γ in Ω, we have

$$M^{-1}\ell(\gamma) \le \ell(f \circ \gamma) \le M\ell(\gamma).$$

The BLD class was introduced in [32] where several equivalent characterizations of the class BLD were obtained. In particular, it was proved that f is BLD if and only if it is locally Lipschitz and the Jacobian of f is bounded away from zero, $J(f) \ge C > 0$. Thus BLD mappings form a special class of quasiregular mappings. The class of BLD mappings found numerous applications in geometric topology [5, 17, 21, 22, 23, 25, 26, 36]. The assumption that the mappings are sense preserving is essential: folding of the plane $f(x, y) = (|x|, y)$ preserves the length of curves, but it is not quasiregular. The mapping changes orientation. BLD mappings are locally Lipschitz and hence $J(f)$ is well defined a.e. Thus we necessarily need the condition that $J(f) \geq 0$. The other two assumptions that the mapping is open and discrete are very strong topological requirements. They are satisfied by every quasiregular mapping and they play an important role in most of the proofs involving BLD mappings.

However, we were able to successfully prove in [15] the following result.

Theorem 1.0.7. *A continuous mapping $f : \Omega \to \mathbb{R}^n$ is M-BLD, if and only if*

$$M^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq M\ell(\gamma)$$

for all rectifiable curves γ in Ω and $Jf \geq 0$ a.e. in Ω.

The condition about distortion of curves implies that f is locally Lipschitz and hence $J(f)$ is well defined a.e. Thus the theorem claims that the conditions about openness and discreteness are completely redundant. Here is an explanation how this theorem is related to the Heisenberg group. A mapping $f : X \to Y$ between metric spaces is said to be a *weak BLD* mapping, if there is a constant $M \geq 1$ such that for all rectifiable curves γ in X we have

$$M^{-1}\ell_X(\gamma) \leq \ell_Y(f \circ \gamma) \leq M\ell_X(\gamma).$$

This definition was introduced in [16], see also [25].

Note that the mapping $\Phi : X \to \ell^\infty$ from Theorem 1.0.5 or the mapping $\Phi : X \to \mathbb{R}^N$ from Corollary 1.0.6 are weak BLD. Indeed, since Φ is Lipschitz, (1.0.1) yields

$$C_\Phi^{-1}\ell(\gamma) \leq \ell(\Phi \circ \gamma) \leq \mathrm{Lip}\,(\Phi)\ell(\gamma).$$

It is well known that the identity map $\mathrm{id} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$ has (locally) the weak BLD property. Using this fact and Corollary 1.0.6 we will conclude a new proof of the Ambrosio-Kirchheim-Magnani result about pure k-unrectifiability of Heisenberg group $\mathbb{H}^n$ when $k > n$.

It also follows from Corollary 1.0.6 that if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ has the weak BLD property, then $m \geq n$ and $\operatorname{rank} Df = n$ a.e. In particular weak BLD mappings $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfy $|J(f)| > 0$ a.e. and in order to prove Theorem 1.0.7 it suffices to show a quantitative version of this estimate, $|J(f)| \geq C > 0$ a.e.

This thesis is organized as follows. In Chapter 2, we start by recalling some classical definitions and results from metric and measure spaces followed by a brief discussion of the Hardy-Littlewood maximal function and the Hardy-Littlewood-Wiener theorem and Sobolev spaces. Chapter 3 contains the proof of Theorem 1.0.2 in details and deals with the Lusin condition (N) and mappings of class $W^{1,n}$. Conditions for unrectifiability of metric spaces and the related applications are presented in Chapter 4. Chapter 5 deals with mappings of bounded length distortion and our new characterization of them. And finally, chapter 6 presents a group of open problems related to the BLD embedding conjecture.

2.0 PRELIMINARIES

This chapter will contain the basic definitions and results that will be widely used throughout this thesis. Fundamental concepts such as Sobolev functions and curves in metric spaces will be covered in depth. The main references in this chapter are the books by Rudin [38], Evans and Gariepy [6], and the paper by Hajłasz [13]. Proofs of the theorems without specifications can be found in the above books.

2.1 METRIC AND MEASURE SPACES

We assume the reader has prior familiarity with basic measure theory and topics such as Carathéodory's construction, Lebesgue measure, and the Lebesgue integration theory. However, for the sake of consistency in our terminology, we will present some basic definitions from the theory of metric spaces and more advanced topic of Hausdorff measures in this section.

Definition 2.1.1. A metric d on a set X is a function $d : X \times X \to \mathbb{R}$ such that:

1. $0 \leq d(x, y) < \infty$ for all x and y in X,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$ for all x and y in X, and
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, and z in X.

A metric space (X, d) is a set X with a metric d defined on X.

Property (4) is called the *triangle inequality*.

If $x \in X$ and $r > 0$, the *open ball* with center at x and radius r is the set $B(x,r) = \{y \in X : d(x,y) < r\}$. Sometimes $B(x,r)$ is abbreviated as B. We use λB to denote a concentric ball of B with radius λr.

Definition 2.1.2. Let (X,d) be a metric space. The *diameter* of a set $A \subset X$ is defined as

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}.$$

Definition 2.1.3. A function $f : X \to Y$ from a metric space (X, d_X) to a metric space (Y, d_Y) is said to be *M-Lipschitz* if there exists a constant $M \geq 0$ such that

$$d_Y(f(x), f(y)) \leq M d_X(x,y)$$

for each pair of points $x, y \in X$. We also say that a function is Lipschitz if it is M-Lipschitz for some M.

If $f : X \to Y$ is a Lipschitz bijection whose inverse is also Lipschitz, we say that f is a *biLipschitz* map between X and Y. Also, we say that a function $f : X \to Y$ is *locally Lipschitz* if every point in X has a neighborhood such that the restriction of f to this neighborhood is Lipschitz.

Definition 2.1.4. A function $f : X \to Y$ from a metric space (X, d_X) to a metric space (Y, d_Y) is said to be *α-Hölder*, $\alpha > 0$, if there exists a constant $M \geq 0$ such that

$$d_Y(f(x), f(y)) \leq M d_X(x,y)^\alpha$$

for each pair of points $x, y \in X$. We also say that a function is Hölder if it is α-Hölder for some α.

Definition 2.1.5. 1. Let (X, d_X) be a metric space. Fix a positive real number α. For each $\delta > 0$ and $E \subset X$, set

$$\mathcal{H}^\alpha_\delta(E) = \inf \omega(\alpha) \sum_i (\operatorname{diam}(E_i))^\alpha$$

where the infimum is taken over all countable covers of E by sets $E_i \subset X$ with diameter less than δ. Here, the constant $\omega(\alpha)$ is given by

$$\omega(\alpha) = \frac{2^\alpha \pi^{\alpha/2}}{\Gamma(\frac{\alpha}{2}+1)}, \qquad \Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx.$$

2. The α-*Hausdorff measure* of E is the number

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E).$$

Definition 2.1.6. The *Hausdorff dimension* of a set E in a metric space is the infimum of the numbers $\alpha > 0$ such that $\mathcal{H}^\alpha(E) = 0$. If no such numbers α exist, the Hausdorff dimension of E is infinite.

2.2 CURVES IN METRIC SPACES

Let (X, d) be a metric space. By a *curve* in X we mean any continuous mapping $\gamma : [a, b] \to X$. The *length* of γ is defined as

$$\ell(\gamma) = \sup \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. The curve γ is called *rectifiable* if $\ell(\gamma) < \infty$.

Note that a curve $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable if and only if the coordinate functions are continuous and of bounded variation.

The *length function* associated with a rectifiable curve $\gamma : [a, b] \to X$ is $s_\gamma : [a, b] \to [0, \ell(\gamma)]$, which is defined by $s_\gamma(t) = \ell(\gamma|_{[a,t]})$.

Lemma 2.2.1. *The length function $s_\gamma : [a, b] \to [0, \ell(\gamma)]$ associated with a rectifiable curve $\gamma : [a, b] \to X$ is nondecreasing and continuous.*

Proof. It is clear that s_γ is nondecreasing. We will prove continuity of γ on (a, b). The case of end points is similar and left to the reader. By contradiction suppose that there is a point $\tau \in (a, b)$ with

$$\eta = \lim_{t \to \tau^+} s_\gamma(t) - \lim_{t \to \tau^-} s_\gamma(t) > 0. \tag{2.2.1}$$

Take a partition $a = t_0 < t_1 < \cdots < t_n = b$, such that

$$\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) > \ell(\gamma) - \eta/3 \tag{2.2.2}$$

and $d(\gamma(t_i), \gamma(t_{i+1})) < \eta/3$ for $i = 0, 1, 2, \cdots, n-1$. We can always choose a partition such that τ is not among t_i's i.e. $\tau \in (t_i, t_{i+1})$ for some i. Hence it follows from (2.2.1) that $\ell(\gamma|_{[t_i,t_{i+1}]}) \geq \eta$. Taking a subdivision of $[t_i, t_{i+1}]$, we can replace the summand $d(\gamma(t_i), \gamma(t_{i+1})) < \eta/3$ in (2.2.2) by a sum larger than $2\eta/3$. This implies, however, that the new sum in (2.2.2) is larger than $(\ell(\gamma) - \eta/3) - \eta/3 + 2\eta/3 = \ell(\gamma)$, an obvious contradiction. □

Let $\gamma : [a, b] \to X$ be a curve and let $\alpha : [c, d] \to [a, b]$ be continuous, nondecreasing and surjective, then we say that the curve $\gamma \circ \alpha$ is obtained from γ by a *nondecreasing change of variables*. It is easy to show that

$$\ell(\gamma) = \ell(\gamma \circ \alpha). \tag{2.2.3}$$

If $\gamma : [a, b] \to X$ is an arbitrary mapping which is not necessarily continuous, we can still define the length of γ in the same way we defined the length of a continuous curve. Clearly, it is necessary that there are at most countably many points of discontinuity for rectifiability of such a mapping γ. Note that (2.2.3) holds true for an arbitrary mapping $\gamma : [a, b] \to X$ as well. The only place in this section where we use this observation is in the proof of Theorem 2.2.2 and we will point it out explicitly in that proof. In all other cases throughout this thesis we will consider continuous curves only.

The next result guarantees a useful parametrization by the arc-length for any rectifiable curve.

Theorem 2.2.2. *Let $\gamma : [a, b] \to X$ be a rectifiable curve. Then there is a unique curve $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ such that*

$$\gamma = \tilde{\gamma} \circ s_\gamma. \tag{2.2.4}$$

Moreover, $\ell(\tilde{\gamma}|_{[0,t]}) = t$ for every $t \in [0, \ell(\gamma)]$. Also, in particular, $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ is a 1-Lipschitz mapping.

Proof. Without loss of generality, we can assume that $a = 0$. Let

$$h(t) = \inf s_\gamma^{-1}(t) \quad \text{for } t \in [0, \ell(\gamma)].$$

Note that the sets $s_\gamma^{-1}(t)$ are compact. So, the infimum is attained i.e., $h(t) \in s_\gamma^{-1}(t)$. Therefore

$$s_\gamma(h(t)) = t, \qquad h(s_\gamma(t)) \leq t.$$

The last inequality follows from the observation that $t \in s_\gamma^{-1}(s_\gamma(t))$ and hence infimum of the set $s_\gamma^{-1}(s_\gamma(t))$ which, by definition, equals $h(s_\gamma(t))$ is less than or equal to t. It is important to note that h does not have to be continuous. Actually intervals of constancy of s_γ correspond to jumps of h.

If the curve $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ satisfies (2.2.4), then $\gamma(h(t)) = \tilde{\gamma}(s_\gamma(h(t)))$ for all $t \in [0, \ell(\gamma)]$, and hence

$$\tilde{\gamma}(t) = \gamma(h(t)) \quad \text{for } t \in [0, \ell(\gamma)]. \tag{2.2.5}$$

This shows that $\tilde{\gamma}$ is unique. Hence, it remains to show that if $\tilde{\gamma}$ is defined by formula (2.2.5), then (2.2.4) and $\ell(\tilde{\gamma}|[0,t]) = t$ for all $t \in [0, \ell(\gamma)]$ are true. Note that the last condition will imply that $\tilde{\gamma}$ is a 1-Lipschitz mapping and hence continuous.

Since $h(s_\gamma(t)) \leq t$, we observe that

$$d(\gamma(t), \gamma(h(s_\gamma(t)))) \leq \ell(\gamma|_{[h(s_\gamma(t)),t]}) = s_\gamma(t) - s_\gamma(h(s_\gamma(t))) = s_\gamma(t) - s_\gamma(t) = 0.$$

Therefore, $(\tilde{\gamma} \circ s_\gamma)(t) = \gamma(h(s_\gamma(t))) = \gamma(t)$ which is (2.2.4). The proof of the arc-length parametrization of $\tilde{\gamma}$ is also easy

$$\ell(\tilde{\gamma}|_{[0,t]}) = \ell(\gamma|_{[0,s_\gamma(h(t))]}) = \ell(\tilde{\gamma} \circ s_\gamma|_{[0,h(t)]}) = \ell(\gamma|_{[0,h(t)]}) = s_\gamma(h(t)) = t.$$

Recall that the formula (2.2.3) holds even for discontinuous curves. We used this fact in the last equality because we do not know whether $\tilde{\gamma}$ is continuous or not at this point. However, we can see now that $\tilde{\gamma}$ is 1-Lipschitz

$$d(\tilde{\gamma}(t_1), \tilde{\gamma}(t_2)) \leq \ell(\tilde{\gamma}|_{[t_1,t_2]}) = t_2 - t_1.$$

This completes the proof. $\square$

Remark 2.2.3. Since $\ell(\tilde{\gamma}|_{[0,t]}) = t$ for every $t \in [0, \ell(\gamma)]$, we call $\tilde{\gamma}$ parametrized by the *arc-length*.

In particular, Theorem 2.2.2 shows that every rectifiable curve admits a 1-Lipschitz parametrization.

Definition 2.2.4. For a curve $\gamma : [a, b] \to X$ we define *speed* at a point $t \in (a, b)$ as the limit

$$|\dot{\gamma}|(t) = \lim_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|}$$

provided the limit exists.

Theorem 2.2.5. *Let $\gamma : [a, b] \to X$ be a Lipschitz curve. Then, speed of γ exists a.e. and*

$$\ell(\gamma) = \int_a^b |\dot{\gamma}|(t)\ dt. \tag{2.2.6}$$

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a dense subset of $\gamma([a, b])$. Let $\phi_n(t) = d(\gamma(t), x_n)$. Functions $\phi_n : [a, b] \to \mathbb{R}$ are Lipschitz continuous and therefore differentiable a.e. according to Rademacher's theorem. Let $m(t) = sup_n|\dot{\phi}_n(t)|$. We will show that

$$|\dot{\gamma}|(t) = m(t) \text{ a.e.} \tag{2.2.7}$$

Since each of the functions $x \mapsto d(x, x_n)$ is 1-Lipschitz we conclude that

$$\liminf_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|} \geq \liminf_{h \to 0} \frac{|\phi_n(t+h) - \phi_n(t)|}{|h|} = |\dot{\phi}_n(t)| \text{ a.e.}$$

Taking the supremum over n yields

$$\liminf_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|} \geq m(t) \text{ a.e.} \tag{2.2.8}$$

In particular the function m is bounded by the Lipschitz constant of γ and hence integrable on $[a, b]$. On the other hand for $s \leq t$ we have

$$d(\gamma(t), \gamma(s)) = \sup_n |d(\gamma(t), x_n) - d(\gamma(s), x_n)| \leq \sup_n \int_s^t |\dot{\phi}_n(\tau)|\ d\tau \leq \int_s^t m(\tau)\ d\tau. \tag{2.2.9}$$

Now at a Lebesgue point $t \in (a, b)$ of m we have

$$\limsup_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|} \leq \limsup_{h \to 0} \frac{1}{h} \int_t^{t+h} m(\tau)\ d\tau = m(t).$$

This together with (2.2.8) proves (2.2.7). We are left with the proof of (2.2.6). According to (2.2.9) and (2.2.7), for an arbitrary partition $a = t_0 < t_1 < \cdots < t_n = b$ we have

$$\sum_{i=0}^{n-1} d(\gamma(t_{i+1}), \gamma(t_i)) \leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} m(\tau)\ d\tau = \int_a^b |\dot{\gamma}|(\tau)\ d\tau.$$

Taking the supremum over partitions yields $\ell(\gamma) \leq \int_a^b |\dot{\gamma}|(\tau)\ d\tau$. To prove the opposite inequality, fix $\epsilon > 0$ and split $[a, b]$ into n segments of equal length i.e. $t_i = a + ih_n$, $h_n = (b-a)/n$, $i = 0, 1, 2, \cdots, n$. Take n so that $h_n < \epsilon$. We have

$$\frac{1}{h_n} \int_a^{b-\epsilon} d(\gamma(t+h_n), \gamma(t))\ dt \leq \frac{1}{h_n} \int_0^{h_n} \sum_{i=0}^{n-2} d(\gamma(t+t_{i+1}), \gamma(t+t_i))\ dt$$

$$\leq \frac{1}{h_n} \int_0^{h_n} \ell(\gamma) = \ell(\gamma).$$

Now the definition of speed and Fatou's theorem imply

$$\int_a^{b-\epsilon} |\dot{\gamma}|(t)\ dt \leq \int_a^{b-\epsilon} \lim_{n\to\infty} \frac{d(\gamma(t+h_n), \gamma(t))}{h_n}\ dt$$

$$\leq \liminf_{n\to\infty} \frac{1}{h_n} \int_a^{b-\epsilon} d(\gamma(t+h_n), \gamma(t))\ dt \leq \ell(\gamma).$$

Passing to the limit as $\epsilon \to 0$ yields the desired inequality. □

Corollary 2.2.6. *$|\dot{\tilde{\gamma}}|(t) = 1$ for a.e. $t \in [0, \ell(\gamma)]$.*

Proof. $\ell(\gamma) = \ell(\tilde{\gamma}) = \int_0^{\ell(\gamma)} |\dot{\tilde{\gamma}}|(t)\ dt$. This and $|\dot{\tilde{\gamma}}|(t) \leq 1$ ($\tilde{\gamma}$ is 1-Lipschitz) imply $|\dot{\tilde{\gamma}}|(t) = 1$ a.e. □

Corollary 2.2.7. *Let $\gamma : [a, b] \to X$ be a Lipschitz curve. Then s_γ is Lipschitz and $\dot{s}_\gamma(t) = |\dot{\gamma}|(t)$ for a.e. $t \in (a, b)$.*

Proof. For $a \leq t_1 \leq t_2 \leq b$ we have

$$|s_\gamma(t_1) - s_\gamma(t_2)| = \ell(\gamma|_{[t_1,t_2]}) = \int_{t_1}^{t_2} |\dot{\gamma}|(\tau)\ d\tau \leq L|t_1 - t_2|,$$

where L is a Lipschitz constant of γ, so the function s_γ is Lipschitz. Hence

$$\int_a^b |\dot{\gamma}|(\tau)\ d\tau = \ell(\gamma) = s_\gamma(b) - s_\gamma(a) = \int_a^b \dot{s}_\gamma(\tau)\ d\tau.$$

This and the obvious inequality $\dot{s}_\gamma \geq |\dot{\gamma}|$ prove the result. □

2.3 MAXIMAL FUNCTIONS

For a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ the *Hardy-Littlewood maximal function* is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy,$$

for all $x \in \mathbb{R}^n$. Here,

$$\fint_{B(x,r)} |f(y)|\, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

The operator $\mathcal{M}$ is not linear but it is subadditive. We say that an operator T from a space of measurable functions into a space of measurable functions is *subadditive* if

$$|T(f_1 + f_2)(x)| \le |Tf_1(x)| + |Tf_2(x)| \quad \text{a.e.}$$

and

$$|T(kf)(x)| = |k||Tf(x)| \qquad \text{for } k \in \mathbb{C}.$$

The following integrability result, known also as *the maximal theorem*, plays a fundamental role in many areas of mathematical analysis.

Theorem 2.3.1 (Hardy-Littlewood-Wiener)**.** *If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then $\mathcal{M}f < \infty$ a.e. Moreover*

(a) For $f \in L^1(\mathbb{R}^n)$,

$$|\{x : \mathcal{M}f(x) > t\}| \le \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \quad \text{for all } t > 0. \tag{2.3.1}$$

(b) If $f \in L^p(\mathbb{R}^n)$, $1 < p \le \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ and

$$\|\mathcal{M}f\|_p \le 2 \cdot 5^{n/p} \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p \quad \text{for } 1 < p < \infty,$$

and

$$\|\mathcal{M}f\|_\infty \le \|f\|_\infty.$$

The estimate (2.3.1) is called *weak type estimate*.

Note that if $f \in L^1(\mathbb{R}^n)$ is a nonzero function, then $\mathcal{M}f \notin L^1(\mathbb{R}^n)$. Indeed, if $\lambda = \int_{B(0,R)} |f| > 0$, then for $|x| > R$ we have

$$\mathcal{M}f(x) \geq \fint_{B(x,R+|x|)} |f| \geq \frac{\lambda}{\omega_n (R+|x|)^n},$$

and the function on the right hand side is not integrable on $\mathbb{R}^n$. Thus the statement (b) of the theorem is not true for $p = 1$.

If $g \in L^1(\mathbb{R}^n)$, then the *Chebyschev inequality*

$$|\{x : |g(x)| > t\}| \leq \frac{1}{t} \int_{\mathbb{R}^n} |g| \quad \text{for } t > 0$$

is easy to prove. Hence the inequality at (a) would follow from boundedness of of $\mathcal{M}f$ in L^1. Unfortunately $\mathcal{M}f$ is not integrable and (a) is the best we can get for the case $p = 1$.

Before we prove the theorem we will show that it implies the Lebesgue differentiation theorem.

Theorem 2.3.2 (Lebesgue Differentiation Theorem)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r \to 0} \fint_{B(x,r)} f(y)\, dy = f(x) \quad a.e.$$

Proof. Since the theorem is local in nature we can assume that $f \in L^1(\mathbb{R}^n)$. Let $f_r(x) = \fint_{B(x,r)} f(y)\, dy$ and define

$$\Omega f(x) = \limsup_{r \to 0} f_r(x) - \liminf_{r \to 0} f_r(x).$$

It suffices to prove that $\Omega f = 0$ a.e. and that $f_r \to f$ in L^1. Indeed, the first property means that f_r converges a.e. to a measurable function g while the second one implies that for a subsequence $f_{r_i} \to f$ a.e. and hence $g = f$ a.e.

Observe that $\Omega f \leq 2\mathcal{M}f$ and hence for any $\epsilon > 0$ Theorem 2.3.1 (a) yields

$$|\{x : \Omega f(x) > \epsilon\}| \leq \frac{C}{\epsilon} \int_{\mathbb{R}^n} |f|.$$

Let h be a continuous function such that $\|f - h\|_1 < \epsilon^2$. Continuity of h implies $\Omega h = 0$ everywhere and hence

$$\Omega f \leq \Omega(f - h) + \Omega h = \Omega(f - h),$$

so

$$|\{\Omega f > \epsilon\}| \leq |\{\Omega(f-h) > \epsilon\}| \leq \frac{C}{\epsilon}\int_{\mathbb{R}^n} |f-h| \leq C\epsilon.$$

Since $\epsilon > 0$ can be arbitrarily small we conclude $\Omega f = 0$ a.e. We are left with proving that $f_r \to f$ in L^1. We have

$$\begin{aligned}\int_{\mathbb{R}^n} |f_r(x) - f(x)|\,dx &\leq \int_{\mathbb{R}^n} \fint_{B(x,r)} |f(x) - f(x)|\,dy\,dx \\ &= \int_{\mathbb{R}^n} \fint_{B(0,r)} |f(x+y) - f(x)|\,dy\,dx \\ &= \fint_{B(0,r)} \|f_y - f\|_1\,dy.\end{aligned}$$

where $f_y(x) = f(x+y)$. Since $f_y \to f$ in L^1 as $y \to 0$ the right hand side of the last equality above converges to 0 as $r \to 0$. □

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then we can define f at *every* point by the formula

$$f(x) = \limsup_{r\to 0} \fint_{B(x,r)} f(y)\,dy. \tag{2.3.2}$$

According to the Lebesgue differentiation theorem this is a representative of f in the class of functions that coincide with f a.e.

Definition 2.3.3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $x \in \mathbb{R}^n$ is a *Lebesgue point of f* if

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\,dy = 0,$$

where $f(x)$ is defined by (2.3.2).

Theorem 2.3.4. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then the set of points that are not Lebesgue points of f has measure zero.*

Proof. For $c \in \mathbb{Q}$ let E_c be the set of points for which

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - c| \, dy = |f(x) - c| \tag{2.3.3}$$

does not hold. Clearly $|E_c| = 0$ and hence the set $E = \bigcup_{c\in\mathbb{Q}} E_c$ has measure zero. Thus for $x \in \mathbb{R}^n \setminus E$ and all $c \in \mathbb{Q}$, (2.3.3) is satisfied. If $x \in \mathbb{R}^n \setminus E$ and $f(x) \in \mathbb{R}$, approximating $f(x)$ by rational numbers one can easily check that

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)| \, dy = |f(x) - f(x)| = 0.$$

The proof is complete. □

Definition 2.3.5. Let $E \subset \mathbb{R}^n$ be a measurable set. We say that $x \in \mathbb{R}^n$ is a *density point of E* if

$$\lim_{r\to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 1.$$

Applying the Lebesgue theorem to $f = \chi_E$ we obtain

Theorem 2.3.6. *Almost every point of a measurable set $E \subset \mathbb{R}^n$ is its density point and a.e. point of $\mathbb{R}^n \setminus E$ is not a density point of E.*

In the proof of Theorem 2.3.1 we will need the following two results.

Theorem 2.3.7 (Cavalieri's principle)**.** *If μ is a σ-finite measure on X and $\Phi : [0,\infty) \to [0,\infty)$ is increasing, absolutely continuous and $\Phi(0) = 0$, then*

$$\int_X \Phi(|f|) \, d\mu = \int_0^\infty \Phi'(t)\mu(\{|f| > t\}) \, dt.$$

Proof. The result follows immediately from the equality

$$\int_X \Phi(|f(x)|) \, d\mu(x) = \int_X \int_0^{|f(x)|} \Phi'(t) \, dt \, d\mu(x)$$

and the Fubini theorem. □

Corollary 2.3.8. *If μ is a σ-finite measure on X and $0 < p < \infty$, then*

$$\int_X |f|^p \, d\mu = p \int_0^\infty t^{p-1}\mu(\{|f| > t\}) \, dt.$$

The next result has many applications that go beyond the maximal theorem.

Theorem 2.3.9 (5r-covering lemma)**.** *Let $\mathcal{B}$ be a family of balls in a metric space such that* $\sup\{\operatorname{diam}(B) : B \in \mathcal{B}\} < \infty$*. Then there is a subfamily of pairwise disjoint balls $\mathcal{B}' \subset \mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} 5B.$$

If the metric space is separable, then the family $\mathcal{B}'$ is countable and we can arrange it as a sequence $\mathcal{B}' = \{B_i\}_{i=1}^{\infty}$, so

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Remark 2.3.10. Here $\mathcal{B}$ can be either a family of open balls or closed balls. In both cases proof is the same.

Proof. Let $\sup\{\operatorname{diam}(B) : B \in \mathcal{B}\} = R < \infty$. Divide the family $\mathcal{B}$ according to the diameter of the balls

$$\mathcal{F}_j = \{B \in \mathcal{B} : \frac{R}{2^j} < \operatorname{diam}(B) \le \frac{R}{2^{j-1}}\}.$$

Clearly $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$. Define $\mathcal{B}_1 \subset \mathcal{F}_1$ to be the maximal family of pairwise disjoint balls. Suppose the families $\mathcal{B}_1, \cdots, \mathcal{B}_{j-1}$ are already defined. Then we define $\mathcal{B}_j$ to be the maximal family of pairwise disjoint balls in

$$\mathcal{F}_j \cap \{B : B \cap B' = \emptyset \text{ for all } B' \in \bigcup_{i=1}^{j-1} \mathcal{B}_i\}.$$

Next we define $\mathcal{B}' = \bigcup_{j=1}^{\infty} \mathcal{B}_j$. Observe that every ball $B \in \mathcal{F}_j$ intersects with a ball in $\bigcup_{i=1}^{j} \mathcal{B}_j$. Suppose that $B \cap B_1 \neq \emptyset$ and $B_1 \in \bigcup_{i=1}^{j} \mathcal{B}_j$. Then

$$\operatorname{diam}(B) \le \frac{R}{2^{j-1}} = 2 \cdot \frac{R}{2^j} \le 2\operatorname{diam}(B_1),$$

and hence $B \subset 5B_1$. □

Proof of Theorem 2.3.1. (a) Let $f \in L^1(\mathbb{R}^n)$ and $E_t = \{x : \mathcal{M}f(x) > t\}$. For $x \in E_t$, there is $r_x > 0$ such that

$$⨍_{B(x,r_x)} |f| > t,$$

so

$$|B(x,r_x)| < t^{-1} \int_{B(x,r_x)} |f|.$$

Observe that $\sup_{x\in E_t} r_x < \infty$, because $f \in L^1(\mathbb{R}^n)$. The family of balls $\{B(x, r_x)\}_{x\in E_t}$ forms a covering of the set E_t, so applying the $5r$-covering lemma there is a sequence of pairwise disjoint balls $B(x_i, r_{x_i})$, $i = 1, 2, \cdots$ such that $E_t \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_{x_i})$ and hence

$$|E_t| \leq 5^n \sum_{i=1}^{\infty} |B(x_i, r_{x_i})| \leq \frac{5^n}{t} \sum_{i=1}^{\infty} \int_{B(x_i, r_{x_i})} |f| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f|.$$

The proof is complete. (b) Let $f \in L^p(\mathbb{R}^n)$. Since clearly $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$, we can assume that $1 < p < \infty$. Let $f = f_1 + f_2$ where

$$f_1 = f\chi_{\{|f|>t/2\}}, \qquad f_2 = f\chi_{\{|f|\leq t/2\}}$$

be a decomposition of f into its lower and upper parts. It is easy to check $f_1 \in L^p(\mathbb{R}^n)$. Since $|f| \leq |f_1| + t/2$, we have $\mathcal{M}f \leq \mathcal{M}f_1 + t/2$ and hence

$$\{\mathcal{M}f > t\} \subset \{\mathcal{M}f_1 > t/2\}.$$

Thus

$$|E_t| = |\{\mathcal{M}f > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\mathbb{R}^n} |f_1(x)|\, dx = \frac{2 \cdot 5^n}{t} \int_{\{|f|>t/2\}} |f(x)|\, dx.$$

Cavalieri's principle yields

$$\int_{\mathbb{R}^n} |\mathcal{M}f(x)|^p\, dx = p \int_0^\infty t^{p-1} |\{\mathcal{M}f > t\}|\, dt$$

$$\leq p \int_0^\infty t^{p-1} \left(\frac{2 \cdot 5^n}{t} \int_{\{|f|>t/2\}} |f(x)|\, dx \right) dt$$

$$= 2 \cdot 5^n \cdot p \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} t^{p-2}\, dt\, dx$$

$$= 2 \cdot 5^n \cdot \frac{p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p\, dx.$$

and the results follows. □

2.4 SOBOLEV SPACES

The theory of Sobolev functions has been widely applied in different areas of mathematics including calculus of variations, partial differential equations, and so on. In this section, we will give a brief review of the classical theory of Sobolev functions. We refer to the book by Evans and Gariepy [6] for the proofs of the theorems.

Throughout this section, Ω denotes an open subset of $\mathbb{R}^n$.

2.4.1 Definitions and Elementary Properties

Definition 2.4.1. Let $f \in L^1_{\text{loc}}(\Omega)$ and $1 \leq i \leq n$. We say $g_i \in L^1_{\text{loc}}(\Omega)$ is *the weak partial derivative of f with respect to x_i in Ω* if

$$\int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega g_i \phi\, dx$$

for all $\phi \in C^1_c(\Omega)$ where $C^1_c(\Omega)$ is the space of all compactly supported continuously differentiable functions in Ω.

It is easy to check that if the weak partial derivative with respect to x_i exists, then it is uniquely defined in L^1_{loc}. So, we can write

$$\frac{\partial f}{\partial x_i} = g_i \qquad (i = 1, \cdots, n)$$

and

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right),$$

provided the weak derivatives $\partial f / \partial x_1, \cdots, \partial f / \partial x_n$ all exit.

Definition 2.4.2. Let $1 \leq p \leq \infty$.

1. The function f belongs to the *Sobolev space*

$$W^{1,p}(\Omega)$$

   if $f \in L^p(\Omega)$ and the weak partial derivatives $\partial f / \partial x_i$ exist and belong to $L^p(\Omega), i = 1, \cdots, n$.

2. We say f is a *Sobolev function* if $f \in W^{1,p}(\Omega)$ for some $1 \le p \le \infty$.

Remark 2.4.3. If f is a Sobolev function, then by definition, the integration by parts formula

$$\int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \frac{\partial f}{\partial x_i} \phi\, dx$$

is valid for all $\phi \in C_c^1(\Omega), i = 1, \cdots, n$.

Definition 2.4.4. If $f \in W^{1,p}(\Omega)$, define

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |f|^p + |\nabla f|^p\, dx \right)^{1/p}$$

for $1 \le p < \infty$ and

$$\|f\|_{W^{1,\infty}(\Omega)} = \operatorname*{ess\,sup}_\Omega (|f| + |\nabla f|).$$

2.4.2 Sobolev Inequalities

Definition 2.4.5. For $1 \le p < n$, define

$$p^* = \frac{np}{n-p}.$$

p^* is called the *Sobolev conjugate* of p.

Theorem 2.4.6 (Gagliardo–Nirenberg–Sobolev Inequality)**.** *Let $1 \le p < n$. Then there exists a constant C, depending only on p and n, such that*

$$\left(\int_{\mathbb{R}^n} |f|^{p^*}\, dx \right)^{1/p^*} \le C \left(\int_{\mathbb{R}^n} |\nabla f|^p\, dx \right)^{1/p}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$.

Remark 2.4.7. The Gagliardo–Nirenberg–Sobolev inequality states that if $f \in W^{1,p}(\mathbb{R}^n)$ for some $1 \le p < n$, then in fact f lies in $L^{p^*}(\mathbb{R}^n)$.

Theorem 2.4.8 (Poincaré Inequality)**.** *For each $1 \le p < n$ there exists a constant C, depending only on p and n, such that*

$$\left(\fint_{B(x,r)} |f - (f)_{x,r}|^{p^*}\, dy \right)^{1/p^*} \le Cr \left(\fint_{B(x,r)} |\nabla f|^p\, dy \right)^{1/p}$$

for all $B(x,r) \subset \mathbb{R}^n$ and $f \in W^{1,p}(B(x,r))$. Here, $(f)_{x,r} = \fint_{B(x,r)} f\, dy$.

Theorem 2.4.9 (Morrey's Inequality). *1. For each $n < p < \infty$ there exists a constant C, depending only on p and n, such that*

$$|f(y) - f(z)| \leq Cr \left(\fint_{B(x,r)} |\nabla f|^p \, dw \right)^{1/p}$$

for all $B(x,r) \subset \mathbb{R}^n$ and $f \in W^{1,p}(B(x,r))$, and a.e. $y, z \in B(x,r)$.

2. In particular, if $f \in W^{1,p}(\mathbb{R}^n)$, then the limit

$$\lim_{r \to 0} (f)_{x,r} := f^*(x)$$

exists for all $x \in \mathbb{R}^n$ and f^* is Hölder continuous with exponent $1 - n/p$.

3.0 LUSIN CONDITION (N) AND MAPPINGS OF CLASS $W^{1,n}$

3.1 INTRODUCTION

A continuous function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfies *Lusin's condition* (N) if $|f(E)| = 0$ whenever $E \subset \Omega$ and $|E| = 0$. Here, $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Lusin's condition (N) plays a significant role in the theory of integration. Because of this importance in applications, we investigate how this condition is related to Sobolev spaces. Smooth mappings, for example C^1, locally Lipschitz, or even continuous mappings in the Sobolev space $W^{1,p}(\Omega)$ with $p > n$, satisfy Lusin's condition (N). However, it is well-known that for $p < n$, such mappings fail to satisfy the condition (N), ([37]). For the case $n = 1$ the condition (N) is completely understood, see [43], but for $n \geq 2$ necessary and sufficient conditions for the condition are not clear.

One of the first who studied geometric consequences of the Lusin property was Reshetnyak [39] who proved that quasiregular mappings between Euclidean spaces satisfy the condition (N). He also gave a topological condition that implies condition (N) for continuous mappings in $W^{1,n}(\Omega)$ in [40].

We focus our attention to the class of $W^{1,n}(\Omega)$ mappings where $\Omega \subset \mathbb{R}^n$ is open.

3.2 CASE $p = n$ AND THE MAIN RESULT

Definition 3.2.1. A mapping $f : \Omega \to \mathbb{R}^n$ is said to be *K-pseudomonotone* for $K \geq 1$ if for each $x \in \Omega$

$$\operatorname{diam}(f(B(x,r))) \leq K \operatorname{diam}(f(\partial B(x,r)))$$

whenever $\bar{B}(x,r) \subset \Omega$.

The main result of this section is a new proof of the following theorem due to Malý and Martio.

Theorem 3.2.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^n$ is a continuous, K-pseudomonotone mapping of the class $W^{1,n}(\Omega)$. Then f satisfies the condition (N).*

Lemma 3.2.3. *Let $n \geq 2$ and $n-1 < p < n$. Then under the above assumptions, there exists a constant $C = C(n,p) \geq 0$ such that for all $x \in \Omega$ and for all $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$,*

$$\operatorname{diam}\left(f(B(x,r))\right)^n \leq CK^n \int_{B(x,r)} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \; dy.$$

Proof of Lemma 3.2.3. First, notice $f \in W^{1,n}(\Omega)$ implies that $|\nabla f| \in L^p(B(x,r))$ for all $x \in \Omega$ and all $0 < r < \operatorname{dist}(x, \partial\Omega)$ because $p < n$ and $|B(x,r)| < \infty$. It also implies that $|\nabla f|^p \in L^{n/p}(\Omega)$. Since $n/p > 1$, it follows from the Hardy-Littlewood maximal theorem that $\mathcal{M}|\nabla f|^p \in L^{n/p}(\Omega)$.

We claim that for all $x \in \Omega$ and all $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$ there exists $r \leq \rho \leq 2r$ such that

$$\int_{\partial B(x,\rho)} |\nabla f(y)|^p \; dS(y) \leq \frac{1}{r} \int_{B(x,2r)} |\nabla f(y)|^p \; dy. \tag{3.2.1}$$

Suppose that, contrary to our claim, there exists $x \in \Omega$ and $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$ such that for all $r \leq \rho \leq 2r$,

$$\int_{\partial B(x,\rho)} |\nabla f(y)|^p \; dS(y) > \frac{1}{r} \int_{B(x,2r)} |\nabla f(y)|^p \; dy.$$

Integrating both sides of the above inequality over the interval $[r, 2r]$ we get

$$\int_r^{2r} \int_{\partial B(x,\rho)} |\nabla f(y)|^p \; dS(y) \; d\rho > \int_r^{2r} \frac{1}{r} \int_{B(x,2r)} |\nabla f(y)|^p \; dy \; d\rho.$$

So,

$$\int_{B(x,2r)\setminus B(x,r)} |\nabla f(y)|^p \; dy > \int_{B(x,2r)} |\nabla f(y)|^p \; dy,$$

which is an obvious contradiction.

Next, we claim that there exists a constant $C_1 = C_1(n,p) \geq 0$ such that for all $x \in \Omega$ and for almost every $0 < r < \text{dist}(x, \partial\Omega)$,

$$\text{diam}\left(f(\partial B(x,r))\right) \leq C_1 r^{1-\frac{n-1}{p}} \left(\int_{\partial B(x,r)} |\nabla f(y)|^p \, dS(y) \right)^{1/p}. \tag{3.2.2}$$

Let $x \in \Omega$ and $0 < R < \text{dist}(x, \partial\Omega)$. Then, by the Fubini's theorem,

$$\int_0^R \int_{\partial B(x,\rho)} |f(y)|^n \, dS(y) \, d\rho = \int_{B(x,R)} |f(y)|^n \, dy < \infty$$

and

$$\int_0^R \int_{\partial B(x,\rho)} |\nabla f(y)|^n \, dS(y) \, d\rho = \int_{B(x,R)} |\nabla f(y)|^n \, dy < \infty.$$

Therefore, for almost every $0 < r < R$,

$$\int_{\partial B(x,r)} |f(y)|^n \, dS(y) < \infty$$

and

$$\int_{\partial B(x,r)} |\nabla f(y)|^n \, dS(y) < \infty.$$

Hence, $f \in W^{1,n}(\partial B(x,r))$ for almost every $0 < r < \text{dist}(x, \partial\Omega)$. Consequently, $f \in W^{1,p}(\partial B(x,r))$ for almost every $0 < r < \text{dist}(x, \partial\Omega)$ as $p < n$.

Now, let $0 < r < \text{dist}(x, \partial\Omega)$ such that $f \in W^{1,p}(\partial B(x,r))$. By the Morrey's inequality, there exists $C_2 = C_2(n,p) \geq 0$ such that for all $y, z \in \partial B(x,r)$,

$$|f(y) - f(z)| \leq C_2 |y-z|^{1-\frac{n-1}{p}} \left(\int_{\partial B(x,r)} |\nabla f(y)|^p \, dS(y) \right)^{1/p}.$$

It follows that

$$\text{diam}\left(f(\partial B(x,r))\right) \leq C_2 (2r)^{1-\frac{n-1}{p}} \left(\int_{\partial B(x,r)} |\nabla f(y)|^p \, dS(y) \right)^{1/p}.$$

Letting $C_1 = 2^{1-\frac{n-1}{p}} C_2$, we get

$$\text{diam}\left(f(\partial B(x,r))\right) \leq C_1 r^{1-\frac{n-1}{p}} \left(\int_{\partial B(x,r)} |\nabla f(y)|^p \, dS(y) \right)^{1/p}.$$

Hence, (3.2.2) holds for all $x \in \Omega$ and for almost every $0 < r < \text{dist}(x, \partial\Omega)$.

Since f is K-pseudomonotone, for all $x \in \Omega$ and for all $0 < r < \mathrm{dist}(x, \partial\Omega)$,

$$\mathrm{diam}\,(f(B(x,r))) \leq K \mathrm{diam}\,(f(\partial B(x,r))). \tag{3.2.3}$$

So, the combination of (3.2.2) and (3.2.3) gives us

$$\mathrm{diam}\,(f(B(x,r))) \leq C_1 K r^{1-\frac{n-1}{p}} \left(\int_{\partial B(x,r)} |\nabla f(y)|^p \; dS(y) \right)^{1/p} \tag{3.2.4}$$

for all $x \in \Omega$ and for almost every $0 < r < \mathrm{dist}(x, \partial\Omega)$.

Raising both sides of the above inequality to the power p, we get

$$\mathrm{diam}\,(f(B(x,r)))^p \leq {C_1}^p K^p r^{p-n+1} \int_{\partial B(x,r)} |\nabla f(y)|^p \; dS(y) \tag{3.2.5}$$

for all $x \in \Omega$ and for almost every $0 < r < \mathrm{dist}(x, \partial\Omega)$.

Now, let $x \in \Omega$ and $0 < r < \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega)$. Then, by the first claim, there exists $r \leq \rho \leq 2r$ such that

$$\int_{\partial B(x,\rho)} |\nabla f(y)|^p \; dS(y) \leq \frac{1}{r} \int_{B(x,2r)} |\nabla f(y)|^p \; dy < \infty.$$

Observe also that since f is continuous,

$$\int_{\partial B(x,\rho)} |f(y)|^p \; dS(y) < \infty.$$

Thus, $f \in W^{1,p}(\partial B(x,\rho))$ and (3.2.5) holds for ρ.

Now, by putting (3.2.1) and (3.2.5) together and noticing that $r \leq \rho \leq 2r$, we can conclude that

$$\operatorname{diam}(f(B(x,\rho)))^p \le {C_1}^p K^p \rho^{p-n+1} \int_{\partial B(x,\rho)} |\nabla f(y)|^p \, dS(y)$$

$$\le \frac{{C_1}^p K^p \rho^{p-n+1}}{r} \int_{B(x,2r)} |\nabla f(y)|^p \, dy$$

$$\le 2^{p+1} {C_1}^p K^p r^{p-n} \int_{B(x,2r)} |\nabla f(y)|^p \, dy.$$

Also, $\operatorname{diam}(f(B(x,r)))^p \le \operatorname{diam}(f(B(x,\rho)))^p$ since $B(x,r) \subseteq B(x,\rho)$. Hence,

$$\operatorname{diam}(f(B(x,r)))^p \le 2^{p+1} {C_1}^p K^p r^{p-n} \int_{B(x,2r)} |\nabla f(y)|^p \, dy \tag{3.2.6}$$

holds for all $x \in \Omega$ and $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$.

Therefore, by letting $C_3 = (2^{p+1})^{\frac{n}{p}} {C_1}^n$,

$$\operatorname{diam}(f(B(x,r)))^n \le C_3 K^n r^{(p-n)\frac{n}{p}} \left(\int_{B(x,2r)} |\nabla f(y)|^p \, dy \right)^{n/p} \tag{3.2.7}$$

holds for all $x \in \Omega$ and $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$.

Now, let $x \in \Omega$ and $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$. So,

$$\int_{B(x,2r)} |\nabla f(y)|^p \, dy = \omega_n 2^n r^n ⨍_{B(x,2r)} |\nabla f(y)|^p \, dy$$

$$\le \omega_n 2^n r^n \inf_{y \in B(x,2r)} \mathcal{M}|\nabla f|^p(y)$$

$$\le \omega_n 2^n r^n ⨍_{B(x,r)} \mathcal{M}|\nabla f|^p(y) \, dy$$

$$= 2^n \int_{B(x,r)} \mathcal{M}|\nabla f|^p(y) \, dy.$$

Hence, for all $x \in \Omega$ and all $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$,

$$\int_{B(x,2r)} |\nabla f(y)|^p \, dy \leq 2^n \int_{B(x,r)} \mathcal{M}|\nabla f|^p(y) \, dy. \tag{3.2.8}$$

By combining (3.2.7) and (3.2.8), we conclude that

$$\operatorname{diam}(f(B(x,r)))^n \leq 2^{\frac{n^2}{p}} C_3 K^n r^{(p-n)\frac{n}{p}} \left(\int_{B(x,r)} \mathcal{M}|\nabla f|^p(y) \, dy \right)^{n/p}. \tag{3.2.9}$$

holds for all $x \in \Omega$ and $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$. Now, by applying Hölder inequality to $\frac{n}{p} > 1$, we get

$$\begin{aligned}
&r^{(p-n)\frac{n}{p}} \left(\int_{B(x,r)} \mathcal{M}|\nabla f|^p(y) \, dy \right)^{n/p} \\
&= \left(\int_{B(x,r)} r^{p-n} \mathcal{M}|\nabla f|^p(y) \, dy \right)^{n/p} \\
&\leq \left(\left(\int_{B(x,r)} (r^{p-n})^{\frac{n}{n-p}} \, dy \right)^{\frac{n-p}{n}} \left(\int_{B(x,r)} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \, dy \right)^{\frac{p}{n}} \right)^{n/p} \\
&= \left(\omega_n^{\frac{n-p}{n}} \left(\int_{B(x,r)} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \, dy \right)^{\frac{p}{n}} \right)^{n/p} \\
&= \omega_n^{\frac{n-p}{p}} \int_{B(x,r)} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \, dy.
\end{aligned}$$

Thus, using (3.2.9), we have

$$\begin{aligned}
\operatorname{diam}(f(B(x,r)))^n &\leq 2^{\frac{n^2}{p}} C_3 K^n r^{(p-n)\frac{n}{p}} \left(\int_{B(x,r)} \mathcal{M}|\nabla f|^p(y) \, dy \right)^{n/p} \\
&\leq 2^{\frac{n^2}{p}} \omega_n^{\frac{n-p}{p}} C_3 K^n \int_{B(x,r)} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \, dy.
\end{aligned}$$

So, finally, by letting $C = 2^{\frac{n^2}{p}} \omega_n^{\frac{n-p}{p}} C_3$, we conclude that

$$\operatorname{diam}(f(B(x,r)))^n \leq CK^n \int_{B(x,r)} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \, dy$$

holds for all $x \in \Omega$ and $0 < r < \frac{1}{2} \operatorname{dist}(x, \partial\Omega)$. □

Proof of Theorem 3.2.2. Without loss of generality, we can assume that $n \geq 2$. Let E be a subset of Ω such that $|E| = 0$. We want to prove that $|f(E)| = 0$. Let $0 < \epsilon < |\Omega|$. There exists an open set $E_\epsilon \subseteq \Omega$ that contains E and $|E_\epsilon| < \epsilon$. Since E_ϵ is open, there exists a sequence of open balls $B_i = B(x_i, r_i)$ such that $E_\epsilon = \bigcup_i B(x_i, r_i)$.
Notice that for all $i \in \mathbb{N}$,

$$f(B(x_i, r_i)) \subseteq B(f(x_i), \operatorname{diam}(f(B(x_i, r_i)))). \tag{3.2.10}$$

Now, we can write

$$f(E) \subseteq f(E_\epsilon) = f\left(\bigcup_i B(x_i, r_i)\right) = \bigcup_i f(B(x_i, r_i)) \subseteq \bigcup_i B(f(x_i), \operatorname{diam}(f(B(x_i, r_i)))). \tag{3.2.11}$$

By Lemma 3.2.3,

$$\sup\{\operatorname{diam}(f(B(x_i, r_i))) : i \in \mathbb{N}\} \leq \left(CK^n \int_\Omega |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \, dy\right)^{1/n} < \infty.$$

Therefore, by applying the $5r$-covering lemma to $(B(f(x_i), \operatorname{diam}(f(B(x_i, r_i)))))_i$, we conclude that there exists a subsequence $(B(f(x_{i_k}), \operatorname{diam}(f(B(x_{i_k}, r_{i_k})))))_k$ of $(B(f(x_i), \operatorname{diam}(f(B(x_i, r_i)))))_i$ consisting of disjoint balls such that

$$\bigcup_i B(f(x_i), \operatorname{diam}(f(B(x_i, r_i)))) \subseteq \bigcup_k B(f(x_{i_k}), 5\operatorname{diam}(f(B(x_{i_k}, r_{i_k})))). \tag{3.2.12}$$

Because of (3.2.10) we can conclude that the sequence $(B(x_{i_k}, r_{i_k}))$ consists of disjoint balls. Combining (3.2.11) and (3.2.12), we get

$$f(E) \subseteq \bigcup_k B(f(x_{i_k}), 5\operatorname{diam}(f(B(x_{i_k}, r_{i_k})))).$$

Let $C' = \omega_n 5^n C K^n$. Therefore, Lemma 3.2.3 implies

$$|f(E)| = \left| \bigcup_k B(f(x_{i_k}), 5\text{diam}\,(f(B(x_{i_k}, r_{i_k})))) \right| = \sum_k \omega_n 5^n (\text{diam}\,(f(B(x_{i_k}, r_{i_k}))))^n$$

$$= \omega_n 5^n \sum_k (\text{diam}\,(f(B(x_{i_k}, r_{i_k}))))^n$$

$$\leq \omega_n 5^n C K^n \sum_k \int_{B(x_{i_k}, r_{i_k})} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \;\; dy$$

$$= C' \sum_k \int_{B(x_{i_k}, r_{i_k})} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \;\; dy$$

$$\leq C' \int_{E_\epsilon} |\mathcal{M}|\nabla f|^p(y)|^{\frac{n}{p}} \;\; dy.$$

Notice that in the last inequality, we used the fact that the balls $B(x_{i_k}, r_{i_k})$ are disjoint and their union is a subset of E_ϵ.

Now, since $|\mathcal{M}|\nabla f|^p|^{\frac{n}{p}} \in L^1(\Omega)$, absolute continuity of the integral implies that by sending ϵ to zero, $|f(E)| = 0$.

Hence, f satisfies the condition (N). $\square$

4.0 UNRECTIFIABILITY OF METRIC SPACES

4.1 INTRODUCTION

The theory of rectifiable sets plays a significant role in geometric measure theory and calculus of variations. See e.g. [8, 33] for results in Euclidean spaces. Recent development of analysis on metric spaces extended this theory to metric spaces. See e.g. [1, 2, 4, 24] and references therein. Considering the importance of this theory, it is reasonable to search for simple geometric conditions which would guarantee that the image of a Lipschitz mapping from a subset of a Euclidean space into a metric space would have measure zero. One of the main results of this chapter (Theorem 4.1.1) establishes such conditions.

We say that a metric space (X, d) is *countably k-rectifiable* if there is a family of Lipschitz mappings $f_i : E_i \subset \mathbb{R}^k \to X$ defined on measurable sets $E_i \subset \mathbb{R}^k$ such that

$$\mathcal{H}^k \left(X \setminus \bigcup_{i=1}^{\infty} f_i(E_i) \right) = 0.$$

A metric space (X, d) is said to be *purely k-unrectifiable* if for any Lipschitz mapping $f : E \subset \mathbb{R}^k \to X$, where $E \subset \mathbb{R}^k$ is measurable we have $\mathcal{H}^k(f(E)) = 0$.

Let $f : Z \to (X, d)$ be a mapping between metric spaces and let $\{y_1, \dots, y_k\} \subset X$ be given. The mapping $g : Z \to \mathbb{R}^k$ defined by

$$g(x) = (d(f(x), y_1), \dots, d(f(x), y_k))$$

will be called the *projection of f associated with the points $y_1, \dots, y_k$*.

The mapping $\pi : X \to \mathbb{R}^k$, $\pi(y) = (d(y, y_1), \dots, d(y, y_k))$ is clearly Lipschitz. Since $g = \pi \circ f$, we conclude that if f is Lipschitz, then its projection $g = \pi \circ f$ is Lipschitz too.

A measurable function $g : E \to \mathbb{R}$ defined on a measurable set $E \subset \mathbb{R}^k$ is said to be approximately differentiable at $x \in E$ if there is a measurable set $E_x \subset E$ and a linear function $L : \mathbb{R}^k \to \mathbb{R}$ such that x is a density point of E_x and

$$\lim_{E_x \ni y \to x} \frac{g(y) - g(x) - L(y-x)}{|y-x|} = 0.$$

This definition is equivalent with other definitions that one can find in the literature. The approximate derivative L is unique (if it exists) and it is denoted by $\operatorname{ap} Dg(x)$. Lipschitz functions $g : E \to \mathbb{R}$ are approximately differentiable a.e. (by the McShane extension and the Rademacher theorems). In the case of mappings into $\mathbb{R}^k$, approximate differentiability means approximate differentiability of each component.

Theorem 4.1.1. *Let X be a metric space, let $E \subset \mathbb{R}^k$ be measurable, and let $f : E \to X$ be a Lipschitz mapping. Then the following statements are equivalent:*

1. $\mathcal{H}^k(f(E)) = 0$;
2. *For any Lipschitz mapping $\varphi : X \to \mathbb{R}^k$, we have $\mathcal{H}^k(\varphi(f(E))) = 0$;*
3. *For any collection of distinct points $\{y_1, y_2, \ldots, y_k\} \subset X$, the associated projection $g : E \to \mathbb{R}^k$ of f satisfies $\mathcal{H}^k(g(E)) = 0$;*
4. *For any collection of distinct points $\{y_1, y_2, \ldots, y_k\} \subset X$, the associated projection $g : E \to \mathbb{R}^k$ of f satisfies $\operatorname{rank}(\operatorname{ap} Dg(x)) < k$ for $\mathcal{H}^k$-a.e. $x \in E$.*

Here $\mathcal{H}^k$ stands for the k-dimensional Hausdorff measure.

Remark 4.1.2. It follows from the proof that in conditions (3) and (4) we do not have to consider all families $\{y_1, y_2, \ldots, y_k\} \subset X$ of distinct points, but it suffices to consider such families with points y_i taken from a given countable and dense subset of $f(E)$.

The implications from (1) to (2) and from (2) to (3) are obvious. The equivalence between (3) and (4) easily follows from the classical change of variables formula which states that if $g : E \subset \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz, then

$$\int_E |J_g(x)| \, d\mathcal{H}^k(x) = \int_{g(E)} N_g(y, E) \, d\mathcal{H}^k(y). \tag{4.1.1}$$

Here J_g stands for the Jacobian of g and $N_g(y, E)$ is the number of points in the preimage $g^{-1}(y) \cap E$, see e.g. [6, 8, 12].

Therefore, it remains to prove the implication (4) to (1) which is the most difficult part of the theorem. We will deduce it from another result which deals with Lipschitz mappings into ℓ^∞, see Theorem 4.2.2.

Note that in general it may happen for a subset $A \subset X$ that $\mathcal{H}^k(A) > 0$, but for all Lipschitz mappings $\varphi : X \to \mathbb{R}^k$, $\mathcal{H}^k(\varphi(A)) = 0$. For example, the Heisenberg group $\mathbb{H}^n$ satisfies $\mathcal{H}^{2n+2}(\mathbb{H}^n) = \infty$, but $\mathcal{H}^{2n+2}(\varphi(\mathbb{H}^n)) = 0$ for all Lipschitz mappings $\varphi : \mathbb{H}^n \to \mathbb{R}^{2n+2}$, see [4, Section 11.5]. Hence the implication from (2) to (1) has to use in an essential way the assumption that $A = f(E)$ is a Lipschitz image of a Euclidean set. Since by [4, Section 11.5] the condition (2) is satisfied for $\mathbb{H}^n$ with $k = 2n+2$, we conclude that $\mathbb{H}^n$ is purely $(2n+2)$-unrectifiable. For more general results see Theorem 4.3.2 in Section 4.3 and Theorem 4.5.3 in Section 4.5.

Theorem 4.1.1 is related to the work of Kirchheim [24] and Ambrosio-Kirchheim [1] on metric differentiability and the general area formula for mappings into arbitrary metric spaces. However, our approach in this chapter is elementary and involves neither the Kirchheim-Rademacher theorem [24, Theorem 2] nor any kind of the area formula for mappings into arbitrary metric spaces [1, Theorem 5.1].

Although conditions (3) and (4) are necessary and sufficient for the validity of (1), often it is not easy to verify them. The problem is that even if X is smooth, the distance function $y \mapsto d(y, y_i)$ is not smooth at y_i and we need to consider such distance functions for y_i from a dense subset of X, thus creating singularities everywhere in X. Actually a collection of such distance functions gives an isometric embedding of X into ℓ^∞ (for a more precise statement see Theorem 4.2.2 and the proof of Theorem 4.1.1 which shows how Theorem 4.1.1 follows from Theorem 4.2.2). In applications we often deal with spaces X that have some sort of smoothness (like Heisenberg groups or more general Carnot-Carathéodory spaces) and often for such spaces there is a more natural Lipschitz mapping $\Phi : X \to \mathbb{R}^N$, than the embedding into ℓ^∞, a mapping that takes into account the structure of X. In Section 4.4 we state a suitable version of Theorem 4.1.1 (Theorem 4.4.2) and in Section 4.5 we show how it applies to Carnot-Carathéodory spaces.

This chapter is organized as follows. In Section 4.2 we prove a version of the Sard theorem for Lipschitz mappings into ℓ^∞. We also prove Theorem 4.1.1 as a simple consequence of this result. In Section 4.3 we provide a new proof of the unrectifiability of the Heisenberg group as a consequence of Theorem 4.1.1. In the proof we will encounter a problem with the lack of smoothness of the distance function $y \mapsto d(y, y_i)$. In Section 4.4 we will generalize Theorem 4.1.1 in a way that it will easily apply to general Carnot-Carathéodory spaces (including the Heisenberg groups). This approach will allow us to avoid singularities of the distance function. Applications will be presented in Section 4.5.

4.2 LIPSCHITZ MAPPINGS INTO ℓ^∞

A measurable function coincides with a continuous function outside a set of an arbitrarily small measure. This is the Lusin property of measurable functions. The following result due to Federer shows a similar C^1-Lusin property of a.e. differentiable functions, [46].

Lemma 4.2.1 (Federer)**.** *If $f : \Omega \to \mathbb{R}$ is differentiable a.e. on an open set $\Omega \subset \mathbb{R}^k$, then for any $\epsilon > 0$ there is a function $g \in C^1(\mathbb{R}^k)$ such that*

$$\mathcal{H}^k \left(\{ x \in \Omega : \, f(x) \neq g(x) \} \right) < \epsilon.$$

In particular if $E \subset \mathbb{R}^k$ is measurable and $f : E \to \mathbb{R}$ is Lipschitz, then f can be extended to a Lipschitz function $\tilde{f} : \mathbb{R}^k \to \mathbb{R}$ (McShane) to which the above theorem applies. Hence for any $\epsilon > 0$ there is $g \in C^1(\mathbb{R}^k)$ such that

$$\mathcal{H}^k \left(\{ x \in E : \, f(x) \neq g(x) \} \right) < \epsilon. \tag{4.2.1}$$

Note that at almost all points of the set where $f = g$ we have that $\operatorname{ap} Df(x) = Dg(x)$. This holds true at all density points of the set $\{f = g\}$.

Let now $f = (f_1, f_2, \ldots) : E \subset \mathbb{R}^k \to \ell^\infty$ be an L-Lipschitz mapping. Then the components $f_i : E \to \mathbb{R}$ are also L-Lipschitz. Hence for $\mathcal{H}^k$-almost all points $x \in E$, all functions

f_i, $i \in \mathbb{N}$ are approximately differentiable at $x \in E$. We define the approximate derivative of f componentwise

$$\operatorname{ap} Df(x) = \begin{pmatrix} \operatorname{ap} Df_1(x) \\ \operatorname{ap} Df_2(x) \\ \vdots \end{pmatrix}$$

For each $i \in \mathbb{N}$, $\operatorname{ap} Df_i(x)$ is a vector in $\mathbb{R}^k$ with component bounded by L. Hence $\operatorname{ap} Df(x)$ can be regarded as an $\infty \times k$ matrix of real numbers bounded by L, i.e.

$$\operatorname{ap} Df(x) \in (\ell^\infty)^k, \qquad \|\operatorname{ap} Df\|_\infty \leq L,$$

where the norm in $(\ell^\infty)^k$ is defined as the supremum over all entries in the $\infty \times k$ matrix. It is easy to see that for an $\infty \times k$ matrix the row rank equals the column rank. [1] Hence the rank of such matrix is always less than or equal to k. In particular the rank of the $\infty \times k$ matrix $\operatorname{ap} Df(x)$ equals the dimension of the linear subspace of $\mathbb{R}^k$ spanned by the vectors $\operatorname{ap} Df_i(x)$, $i \in \mathbb{N}$ and $\operatorname{rank}(\operatorname{ap} Df(x)) \leq k$ a.e.

If $f : \Omega \to \ell^\infty$ is Lipschitz, where $\Omega \subset \mathbb{R}^k$ is open, components of f are differentiable a.e. and we will write $Df(x)$ in place of $\operatorname{ap} Df(x)$.

The next theorem is the main result of this section. It is a crucial step in the remaining implication (4) to (1) of Theorem 4.1.1. The proof of Theorem 4.2.2 is based on ideas similar to those developed in [3, Section 7].

Theorem 4.2.2. *Let $E \subset \mathbb{R}^k$ be measurable and let $f : E \to \ell^\infty$ be a Lipschitz mapping. Then $\mathcal{H}^k(f(E)) = 0$ if and only if* $\operatorname{rank}(\operatorname{ap} Df(x)) < k$, *$\mathcal{H}^k$-a.e. in E.*

Before we prove this result we will show how to use it to complete the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. As we already pointed out in Introduction, it remains to prove the implication (4) to (1). Although we do not assume that X is separable, the image $f(E) \subset X$ is separable and hence it can be isometrically embedded into ℓ^∞. More precisely

[1] It is a simple exercise in linear algebra – first we prove that the row rank r (vectors in $\mathbb{R}^k$) equals to the maximum of dimensions of minors with non-zero determinants. Clearly the the column rank (vectors in ℓ^∞) is at least r. However, it cannot be larger than r – it easily follows from the fact that the system of $r \times r$ equations with the non-zero determinant has a unique solution.

let $\{y_i\}_{i=1}^\infty \subset f(E)$ be a dense subset and let $y_0 \in f(E)$. Then it is well-known and easy to prove that the mapping

$$f(E) \ni y \mapsto \kappa(y) = \{d(y, y_i) - d(y_i, y_0)\}_{i=1}^\infty \in \ell^\infty$$

is an isometric embedding of $f(E)$ into ℓ^∞. It is so called the Kuratowski embedding. Clearly

$$\mathcal{H}^k_d(f(E)) = \mathcal{H}^k_{\ell^\infty}((\kappa \circ f)(E)),$$

where subscripts indicate metrics with respect to which we define the Hausdorff measures. It remains to prove that $\mathcal{H}^k_{\ell^\infty}((\kappa \circ f)(E)) = 0$. Since

$$(\kappa \circ f)(x) = \{d(f(x), y_i) - d(y_i, y_0)\}_{i=1}^\infty,$$

it easily follows from the assumptions that

$$\operatorname{rank}(\operatorname{ap} D(\kappa \circ f)) < k \quad \mathcal{H}^k\text{-a.e. in } E.$$

Hence (1) follows from Theorem 4.2.2. □

Thus it remains to prove Theorem 4.2.2. Before doing this, let us make some comments explaining why it is not easy. Theorem 4.2.2 is related to the Sard theorem for Lipschitz mappings which states that if $f : \mathbb{R}^k \to \mathbb{R}^m$, $m \geq k$ is Lipschitz, then

$$\mathcal{H}^k(f(\{x \in \mathbb{R}^k : \operatorname{rank} Df(x) < k\})) = 0.$$

The standard proof of this fact [33, Theorem 7.6] is based on the observation that if $\operatorname{rank} Df(x) < k$, then for any $\epsilon > 0$ there is $r > 0$ such that

$$|f(z) - f(x) - Df(x)(z - x)| < \epsilon r \quad \text{for } z \in B(x, r),$$

and hence

$$\operatorname{dist}(f(z), W_x) \leq \epsilon r \quad \text{for } z \in B(x, r),$$

where $W_x = f(x) + Df(x)(\mathbb{R}^k)$ is an affine subspace of $\mathbb{R}^m$ of dimension less than or equal to $k-1$. That means $f(B(x, r))$ is contained in a thin neighborhood of an ellipsoid of dimension no greater than $k - 1$ and hence we can cover it by $C(L/\epsilon)^{k-1}$ balls of radius $C\epsilon r$, where L

is the Lipschitz constant of f. Now we use covering by these balls with the help of Vitali's lemma to estimate the Hausdorff content of the image of the critical set. For more details, see [33, Theorem 7.6].

The proof described above employs the fact that f is Frechet differentiable and hence this argument *cannot* be applied to the case of mappings into ℓ^∞, because in general Lipschitz mappings into ℓ^∞ are not Frechet differentiable, i.e. in general the image of $f(B(x,r)\cap E)$ is not well approximated by the tangent mapping $\operatorname{ap} Df(x)$. To overcome this difficulty we need to investigate the structure of the set $\{\operatorname{ap} Df(x) < k\}$ using arguments employed in the proof of the general case of the Sard theorem for C^n mappings, [44]. In particular we will need to use a version of the implicit function theorem.

In the proof of Theorem 4.2.2 we will also need the following result which is of independent interest.

Proposition 4.2.3. *Let $D \subset \mathbb{R}^k$ be a bounded and convex set with non-empty interior and let $f : D \to \ell^\infty$ be an L-Lipschitz mapping. Then*

$$\operatorname{diam}(f(D)) \leq C(k)L\frac{(\operatorname{diam} D)^k}{\mathcal{H}^k(D)}\,\mathcal{H}^k(D\setminus A)^{1/k}$$

where

$$A = \{x \in D :\ Df(x) = 0\}.$$

In particular if D is a cube or a ball, then

$$\operatorname{diam}(f(D)) \leq C(k)L\mathcal{H}^k(D\setminus A)^{1/k}. \tag{4.2.2}$$

Proof. We will need two well-known facts.

Lemma 4.2.4. *If $E \subset \mathbb{R}^k$ is measurable, then*

$$\int_E \frac{dy}{|x-y|^{k-1}} \leq C(k)\mathcal{H}^k(E)^{1/k}.$$

Proof. If $\mathcal{H}^k(E) = 0$ or if $\mathcal{H}^k(E) = \infty$, then the result is obvious. So, let $0 < \mathcal{H}^k(E) < \infty$ and let $B = B(x,r) \subset \mathbb{R}^k$ be a ball such that $\mathcal{H}^k(B) = \mathcal{H}^k(E)$. Then

$$\int_E \frac{dy}{|x-y|^{k-1}} \leq \int_B \frac{dy}{|x-y|^{k-1}} = C(k)r = C'(k)\mathcal{H}^k(E)^{1/k}.$$

The inequality follows from the observation that on the part of the set E which lies outside B we integrate a function which is strictly smaller than the function on $B \setminus E$ and $\mathcal{H}^k(E \setminus B) = \mathcal{H}^k(B \setminus E)$. □

Lemma 4.2.5. *If $D \subset \mathbb{R}^k$ is a bounded and convex set with non-empty interior and if $u : D \to \mathbb{R}$ is Lipschitz continuous, then*

$$|u(x) - u_D| \leq \frac{(\operatorname{diam} D)^k}{k\mathcal{H}^k(D)} \int_D \frac{|\nabla u(y)|}{|x-y|^{k-1}}\, dy \quad \text{for all } x \in D,$$

where $u_D = ⨍_D u(x)\, dx$.

Proof. Without loss of generality, we can assume $u \in C^1(D)$. We then have for $x, y \in D$,

$$u(x) - u(y) = -\int_0^{|x-y|} \nabla u(x + r\omega)\ dr,$$

where

$$\omega = \frac{y-x}{|y-x|}.$$

Integrating with respect to y over D, we obtain

$$\mathcal{H}^k(D)\,(u(x) - u_D) = -\int_D dy \int_0^{|x-y|} \nabla u(x + r\omega)\ dr.$$

Writing

$$V(x) = \begin{cases} |\nabla u(x)| & x \in D, \\ 0 & x \notin D, \end{cases}$$

we have

$$
\begin{aligned}
|u(x) - u_D| &\leq \frac{1}{\mathcal{H}^k(D)} \int_{|x-y|<\operatorname{diam}(D)} dy \int_0^\infty V(x + r\omega)\, dr \\
&= \frac{1}{\mathcal{H}^k(D)} \int_0^\infty \int_{|\omega|=1} \int_0^{\operatorname{diam}(D)} V(x + r\omega)\rho^{n-1}\, d\rho\, d\omega\, dr \\
&= \frac{(\operatorname{diam} D)^k}{k\mathcal{H}^k(D)} \int_0^\infty \int_{|\omega|=1} V(x + r\omega)\, d\omega\, dr \\
&= \frac{(\operatorname{diam} D)^k}{k\mathcal{H}^k(D)} \int_D \frac{|\nabla u(y)|}{|x-y|^{k-1}}\, dy.
\end{aligned}
$$

□

Now we can complete the proof of Proposition 4.2.3. If $Df(x) = 0$, then $\nabla f_i(x) = 0$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ we have

$$
\begin{aligned}
|f_i(x) - f_{iD}| &\leq \frac{(\operatorname{diam} D)^k}{k\mathcal{H}^k(D)} \int_D \frac{|\nabla f_i(y)|}{|x-y|^{k-1}}\, dy \leq \frac{L(\operatorname{diam} D)^k}{k\mathcal{H}^k(D)} \int_{D\setminus A} \frac{dy}{|x-y|^{k-1}} \\
&\leq C(k)L\frac{(\operatorname{diam} D)^k}{\mathcal{H}^k(D)} \mathcal{H}^k(D \setminus A)^{1/k}.
\end{aligned}
$$

The last inequality follows from Lemma 4.2.4. Hence for all $x, y \in D$

$$
|f_i(x) - f_i(y)| \leq |f_i(x) - f_{iD}| + |f_i(y) - f_{iD}| \leq 2C(k)L\frac{(\operatorname{diam} D)^k}{\mathcal{H}^k(D)} \mathcal{H}^k(D \setminus A)^{1/k}.
$$

Taking supremum over $i \in \mathbb{N}$ yields

$$
\|f(x) - f(y)\|_\infty \leq 2C(k)L\frac{(\operatorname{diam} D)^k}{\mathcal{H}^k(D)} \mathcal{H}^k(D \setminus A)^{1/k}
$$

and the result follows upon taking supremum over all $x, y \in D$. □

Proof of Theorem 4.2.2. The implication from left to right is easy. Suppose that $\mathcal{H}^k(f(E)) = 0$. For any positive integers $i_1 < i_2 < \ldots < i_k$ the projection

$$\ell^\infty \ni (y_1, y_2, \ldots) \to (y_{i_1}, y_{i_2}, \ldots, y_{i_k}) \in \mathbb{R}^k$$

is Lipschitz continuous and hence the set

$$(f_{i_1}, \ldots, f_{i_k})(E) \subset \mathbb{R}^k$$

has $\mathcal{H}^k$-measure zero. It follows from the change of variables formula (4.1.1) that the matrix $[\partial f_{i_j}/\partial x_\ell]_{j,\ell=1}^k$ of approximate partial derivatives has rank less than k almost everywhere in E. Since this is true for any choice of $i_1 < i_2 < \ldots < i_k$, we conclude that $\operatorname{rank}(\operatorname{ap} Df(x)) < k$ a.e. in E.

Suppose now that $\operatorname{rank}(\operatorname{ap} Df(x)) < k$ a.e. in E. We need to prove that $\mathcal{H}^k(f(E)) = 0$. This implication is more difficult. Since $f_i : E \to \mathbb{R}$ is Lipschitz continuous, by (4.2.1) for any $\epsilon > 0$ there is $g_i \in C^1(\mathbb{R}^k)$ such that

$$\mathcal{H}^k(\{x \in E : \, f_i(x) \neq g_i(x)\}) < \epsilon/2^i.$$

Moreover $\operatorname{ap} Df_i(x) = Dg_i(x)$ for almost all points of the set where $f_i = g_i$. Hence there is a measurable set $F \subset E$ such that $\mathcal{H}^k(E \setminus F) < \epsilon$ and

$$f = g \quad \text{and} \quad \operatorname{ap} Df(x) = Dg(x) \quad \text{in } F,$$

where

$$g = (g_1, g_2, \ldots), \quad Dg = \begin{pmatrix} Dg_1 \\ Dg_2 \\ \vdots \end{pmatrix}.$$

It suffices to prove that $\mathcal{H}^k(f(F)) = 0$, because we can exhaust E with sets F up to a subset of measure zero and f maps sets of measure zero to sets of measure zero. Let

$$\tilde{F} = \{x \in F : \, \operatorname{rank}(\operatorname{ap} Df(x)) = \operatorname{rank} Dg(x) < k\}.$$

Since $\mathcal{H}^k(F \setminus \tilde{F}) = 0$, it suffices to prove that $\mathcal{H}^k(f(\tilde{F})) = 0$. For $0 \leq j \leq k-1$, let

$$K_j = \{x \in \tilde{F} : \operatorname{rank} Dg(x) = j\}.$$

Since $\tilde{F} = \bigcup_{j=0}^{k-1} K_j$, it suffices to prove that $\mathcal{H}^k(f(K_j)) = 0$ for any $0 \le j \le k-1$. Again, by removing a subset of measure zero we can assume that all points of K_j are density points of K_j. To prove that $\mathcal{H}^k(f(K_j)) = 0$ we need to make a change of variables in $\mathbb{R}^k$, but only when $j \ge 1$.

If $x \in \mathbb{R}^k \setminus F$, the sequence $(g_1(x), g_2(x), \ldots)$ is not necessarily bounded. Let V be the linear space of all real sequences $(y_1, y_2, \ldots)$. Clearly $g : \mathbb{R}^k \to V$. We do not equip V with any metric structure. Note that $g|_F : F \to \ell^\infty \subset V$, because g coincides with f on F.

Lemma 4.2.6. *Let $1 \le j \le k-1$ and $x_0 \in K_j$. Then there exists a neighborhood $x_0 \in U \subset \mathbb{R}^k$, a diffeomorphism $\Phi : U \subset \mathbb{R}^k \to \Phi(U) \subset \mathbb{R}^k$, and a composition of a translation (by a vector from ℓ^∞) with a permutation of variables $\Psi \colon V \to V$ such that*

- $\Phi^{-1}(0) = x_0$ *and* $\Psi(g(x_0)) = 0$*;*
- *There is $\epsilon > 0$ such that for $x = (x_1, x_2, \ldots, x_k) \in B(0, \epsilon) \subset \mathbb{R}^k$ and $i = 1, 2, \ldots, j$,*

$$\left(\Psi \circ g \circ \Phi^{-1}\right)_i (x) = x_i,$$

i.e., $\Psi \circ g \circ \Phi^{-1}$ fixes the first j variables in a neighborhood of 0.

Proof. By precomposing g with a translation of $\mathbb{R}^k$ by the vector x_0 and postcomposing it with a translation of V by the vector $-g(x_0) = -f(x_0) \in \ell^\infty$ we may assume that $x_0 = 0$ and $g(x_0) = 0$. A certain $j \times j$ minor of $Dg(x_0)$ has rank j. By precomposing g with a permutation of j variables in $\mathbb{R}^k$ and postcomposing it with a permutation of j variables in V we may assume that

$$\operatorname{rank} \left[\frac{\partial g_m}{\partial x_\ell}(x_0)\right]_{1 \le m, \ell \le j} = j. \tag{4.2.3}$$

Let $H : \mathbb{R}^k \to \mathbb{R}^k$ be defined by

$$H(x) = (g_1(x), \ldots, g_j(x), x_{j+1}, \ldots, x_k).$$

It follows from (4.2.3) that $J_H(x_0) \neq 0$ and hence H is a diffeomorphism in a neighborhood of $x_0 = 0 \in \mathbb{R}^k$. It suffices to observe that for all $i = 1, 2, \ldots, j$,

$$\left(g \circ H^{-1}\right)_i (x) = x_i.$$

□

In what follows, by cubes, we will mean cubes with edges parallel to the coordinate axes in $\mathbb{R}^k$. It suffices to prove that any point $x_0 \in K_j$ has a cubic neighborhood whose intersection with K_j is mapped onto a set of $\mathcal{H}^k$-measure zero. Since we can take cubic neighborhoods to be arbitrarily small, the change of variables from Lemma 4.2.6 allows us to assume that

$$K_j \subset (0,1)^k, \quad g_i(x) = x_i \quad \text{for } i = 1, 2, \ldots, j \text{ and } x \in [0,1]^k. \tag{4.2.4}$$

Indeed, according to Lemma 4.2.6 we can assume that $x_0 = 0$ and that g fixes the first j variables in a neighborhood of 0. The neighborhood can be very small, but a rescaling argument allows us to assume that it contains a unit cube Q around 0. Translating the cube we can assume that $Q = [0,1]^k$. If $x \in K_j$, since $\operatorname{rank} Dg(x) = j$ and g fixes the first j coordinates, the derivative of g in directions orthogonal to the first j coordinates equals zero at x, $\partial g_\ell(x)/\partial x_i = 0$ for $i = j+1, \ldots, k$ and any ℓ.

Lemma 4.2.7. *Under the assumptions (4.2.4) there exists a constant $C = C(k) > 0$ such that for any integer $m \geq 1$, and every $x \in K_j$, there is a closed cube $Q_x \subset [0,1]^k$ with edge length d_x centered at x with the property that $f(K_j \cap Q_x) = g(K_j \cap Q_x)$ can be covered by m^j balls in ℓ^∞, each of radius $CLd_x m^{-1}$, where L is the Lipschitz constant of f.*

The theorem is an easy consequence of this lemma through a standard application of the $5r$-covering lemma, Theorem 2.3.9. First of all observe that cubes with sides parallel to coordinate axes in $\mathbb{R}^k$ are balls with respect to the ℓ^∞_k metric

$$\|x - y\|_\infty = \max_{1 \leq i \leq k} |x_i - y_i|.$$

Hence the $5r$-covering lemma applies to families of cubes in $\mathbb{R}^k$. By $5^{-1}Q$ we will denote a cube concentric with Q and with 5^{-1} times the diameter. The cubes $\{5^{-1}Q_x\}_{x \in K_j}$ form a covering of K_j. Hence we can select disjoint cubes $\{5^{-1}Q_{x_i}\}_{i=1}^\infty$ such that

$$K_j \subset \bigcup_{i=1}^\infty Q_{x_i}.$$

If d_i is the edge length of Q_{x_i}, then $\sum_{i=1}^{\infty}(5^{-1}d_i)^k \leq 1$, because the cubes $5^{-1}Q_{x_i}$ are disjoint and contained in $[0,1]^k$. Hence

$$\mathcal{H}^k_\infty(f(K_j)) \leq \sum_{i=1}^{\infty} \mathcal{H}^k_\infty(f(K_j \cap Q_{x_i})) \leq \sum_{i=1}^{\infty} m^j (CLd_i m^{-1})^k \leq 5^k C^k L^k m^{j-k}.$$

Since the exponent $j-k$ is negative, and m can be arbitrarily large we conclude that $\mathcal{H}^k_\infty(f(K_j)) = 0$ and hence $\mathcal{H}^k(f(K_j)) = 0$. Thus it remains to prove Lemma 4.2.7.

Proof of Lemma 4.2.7. Various constants C in the proof below will depend on k only. Fix an integer $m \geq 1$. Let $x \in K_j$. Since every point in K_j is a density point of K_j, there is a closed cube $Q \subset [0,1]^k$ centered at x of edge length d such that

$$\mathcal{H}^k(Q \setminus K_j) < m^{-k}\mathcal{H}^k(Q) = m^{-k}d^k. \tag{4.2.5}$$

By translating the coordinate system in $\mathbb{R}^k$ we may assume that

$$Q = [0,d]^j \times [0,d]^{k-j}.$$

Each component of $f : Q \cap K_j \to \ell^\infty$ is an L-Lipschitz function. Extending each component to an L-Lipschitz function on Q results in an L-Lipschitz extension $\tilde{f} : Q \to \ell^\infty$. This is well-known and easy to check.

Divide $[0,d]^j$ into m^j cubes with pairwise disjoint interiors, each of edge length $m^{-1}d$. Denote the resulting cubes by Q_ν, $\nu \in \{1,2,\dots,m^j\}$. It remains to prove that

$$f((Q_\nu \times [0,d]^{k-j}) \cap K_j) \subset \tilde{f}(Q_\nu \times [0,d]^{k-j})$$

is contained in a ball (in ℓ^∞) of radius $CLdm^{-1}$. It follows from (4.2.5) that

$$\mathcal{H}^k((Q_\nu \times [0,d]^{k-j}) \setminus K_j) \leq \mathcal{H}^k(Q \setminus K_j) < m^{-k}d^k.$$

Hence

$$\mathcal{H}^k((Q_\nu \times [0,d]^{k-j}) \cap K_j) > (m^{-j} - m^{-k})d^k.$$

This estimate and the Fubini theorem imply that there is $\rho \in Q_\nu$ such that

$$\mathcal{H}^{k-j}((\{\rho\} \times [0,d]^{k-j}) \cap K_j) > (1 - m^{j-k})d^{k-j}.$$

Hence

$$\mathcal{H}^{k-j}((\{\rho\} \times [0,d]^{k-j}) \setminus K_j) < m^{j-k} d^{k-j}.$$

It follows from (4.2.2) with k replaced by $k-j$ that

$$\operatorname{diam}_{\ell^\infty}(\tilde{f}(\{\rho\} \times [0,d]^{k-j})) \leq CL\mathcal{H}^{k-j}((\{\rho\} \times [0,d]^{k-j}) \setminus K_j)^{1/(k-j)} \leq CLm^{-1}d. \quad (4.2.6)$$

Indeed, the rank of the derivative of g restricted to the slice $\{\rho\} \times [0,d]^{k-j}$ equals zero at the points of $(\{\rho\} \times [0,d]^{k-j}) \cap K_j$ and this derivative coincides a.e. with the approximate derivative of $\tilde{f}$ restricted to $\{\rho\} \times [0,d]^{k-j} \cap K_j$ which by the property of g must be zero as well.

Since the distance of any point in $Q_\nu \times [0,d]^{k-j}$ to $\{\rho\} \times [0,d]^{k-j}$ is bounded by $Cm^{-1}d$ and $\tilde{f}$ is L-Lipschitz, (4.2.6) implies that $\tilde{f}(Q_\nu \times [0,d]^{k-j})$ is contained in a ball of radius $CLdm^{-1}$, perhaps with a constant C bigger than that in (4.2.6). The proof of the lemma is complete. □

This also completes the proof of Theorem 4.2.2. □

4.3 HEISENBERG GROUPS

As an application we will show one more proof of the well-known result of Ambrosio-Kircheim [1] and Magnani [28] that the Heisenberg group $\mathbb{H}^n$ is purely k-unrectifiable for $k > n$. Another proof was given in [3] and our argument is related to the one given in [3] in a sense that the proof of Theorem 4.2.2 is based on similar ideas. The following result is well-known, see for example Theorem 1.2 in [3].

Lemma 4.3.1. *Let $k > n$ and let $E \subset \mathbb{R}^k$ be a measurable set. If $f\colon E \to \mathbb{H}^n$ is locally Lipschitz continuous, then for $\mathcal{H}^k$-almost every point $x \in E$,* $\operatorname{rank}(\operatorname{ap} Df(x)) \leq n$.

The Heisenberg group $\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{2n+1}$ and the identity mapping $\operatorname{id} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$ is locally Lipschitz continuous. Hence f is locally Lipschitz as a mapping into $\mathbb{R}^{2n+1}$. The approximate derivative $\operatorname{ap} Df(x)$ is understood as the derivative of the mapping into $\mathbb{R}^{2n+1}$. As an application of Theorem 4.1.1 we will prove unrectifiability of $\mathbb{H}^n$.

Theorem 4.3.2. *Let $k > n$ be positive integers. Let $E \subset \mathbb{R}^k$ be a measurable set, and let $f\colon E \to \mathbb{H}^n$ be a Lipschitz mapping. Then $\mathcal{H}^k(f(E)) = 0$.*

Here the Hausdorff measure in $\mathbb{H}^n$ is with respect to the Carnot-Carathéodory metric or with respect to the Korányi metric d_K which is bi-Lipschitz equivalent to the Carnot-Carathéodory one.

Proof. Let $f : E \subset \mathbb{R}^k \to \mathbb{H}^n$, $k > n$ be Lipschitz. We need to prove that $\mathcal{H}^k(f(E)) = 0$. Recall that by Lemma 4.3.1, $\operatorname{rank}(\operatorname{ap} Df(x)) \leq n$. Fix a collection of k distinct points $y_i, \dots, y_k$ in $\mathbb{H}^n$ and define the mapping $g : E \subset \mathbb{R}^k \to \mathbb{R}^k$ as the projection of f

$$g(x) = (d_K(f(x), y_1), \dots, d_K(f(x), y_k)).$$

The mapping $\pi : \mathbb{H}^n \to \mathbb{R}^k$ defined by $\pi(z) = (d_K(z, y_1), \dots, d_K(z, y_k))$ is Lipschitz continuous, but it is not Lipschitz as a mapping $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^k$. Hence it is not obvious that we can apply the chain rule to $g = \pi \circ f$ and conclude that $\operatorname{rank}(\operatorname{ap} Dg(x)) \leq n < k$ a.e. in E which would imply $\mathcal{H}^k(f(E)) = 0$ by Theorem 4.1.1. To overcome this difficulty we use the fact that the Korányi metric $\mathbb{R}^{2n+1} \ni z \mapsto d_K(z, y) \in \mathbb{R}$ is C^∞ on $\mathbb{R}^{2n+1} \setminus \{y\}$. Hence the chain rule applies to $g = \pi \circ f$ on the set $E \setminus (\bigcup_{i=1}^k E_i)$, where

$$E_i = \{x \in E : \ f(x) = y_i\}$$

and $\operatorname{rank}(\operatorname{ap} Dg(x)) \leq n < k$ a.e. in $E \setminus (\bigcup_{i=1}^k E_i)$. If $x \in E_i$, then $f(x) \neq y_j$ for $j \neq i$ and

$$g(x) = (d_K(f(x), y_1), \dots, d_K(f(x), y_{i-1}), 0, d_K(f(x), y_{i+1}), \dots, d_K(f(x), y_k)), \quad \text{for } x \in E_i.$$

Thus $g = \pi_i \circ f$ on E_i, where

$$\pi_i(z) = (d_K(z, y_1), \dots, d_K(z, y_{i-1}), 0, d_K(z, y_{i+1}), \dots, d_K(z, y_k)).$$

The function π_i is smooth in a neighborhood of $y_i = f(x)$, $x \in E_i$ and hence the chain rule shows that the approximate derivative of $g|_{E_i}$ has rank less than or equal $n < k$ a.e. in E_i. It remains to observe that at almost all points of E_i the approximate derivative of g equals to that of $g|_{E_i}$. □

4.4 GENERALIZATION OF THE MAIN RESULT

Definition 4.4.1. We say that a metric space (X, d) is *quasiconvex* if there is a constant $M \geq 1$ such that any two points $x, y \in X$ can be connected by a curve γ of length $\ell(\gamma) \leq Md(x, y)$.

The next result is a variant of Theorem 4.1.1.

Theorem 4.4.2. *Suppose that (X, d) is a complete and quasiconvex metric space and that $\Phi : X \to \ell^\infty$ is a Lipschitz map with the property that for some constant $C_\Phi > 0$ and all rectifiable curves γ in X we have*

$$\ell(\gamma) \leq C_\Phi \ell(\Phi \circ \gamma). \tag{4.4.1}$$

Then for any $k \geq 1$ and any Lipschitz map $f : E \subset \mathbb{R}^k \to X$ defined on a measurable set $E \subset \mathbb{R}^k$ the following conditions are equivalent.

1. $\mathcal{H}^k(f(E)) = 0$ *in* X,
2. $\mathcal{H}^k(\Phi(f(E))) = 0$ *in* ℓ^∞,
3. $\operatorname{rank}(\operatorname{ap} D(\Phi \circ f)) < k$ *a.e. in* E.

Proof. The implication from (1) to (2) is obvious. The equivalence between (2) and (3) follows immediately from Theorem 4.2.2. So it remains to prove that (3) implies (1). Since $(\Phi \circ f)_i : E \to \mathbb{R}$ is Lipschitz continuous, for any $\epsilon > 0$ there is $g_i \in C^1(\mathbb{R}^k)$ such that

$$\mathcal{H}^k(\{x \in E : (\Phi \circ f)_i(x) \neq g_i(x)\}) < \epsilon/2^i.$$

Moreover $\operatorname{ap} D(\Phi \circ f)_i(x) = Dg_i(x)$ for almost all points of the set where $(\Phi \circ f)_i = g_i$. Hence there is a measurable set $F \subseteq E$ such that $\mathcal{H}^k(E \setminus F) < \epsilon$ and

$$g = \Phi \circ f, \quad Dg = \operatorname{ap} D(\Phi \circ f), \quad \operatorname{rank}(Dg) < k \text{ on } F,$$

where $g = (g_1, g_2, \dots)$.
Since $F = \bigcup_{j=0}^{k-1} K_j$, where

$$K_j = \{x \in F : \operatorname{rank} Dg(x) = j\},$$

it suffices to show tat $\mathcal{H}^k(f(K_j)) = 0$. By removing a subset of measure zero we can assume that all points of K_j are the density points of K_j. Since the problem is local in the nature we can assume that

$$K_j \subseteq (0,1)^k, \quad g_i(x) = x_i \text{ for } i = 1, 2, \ldots, j \text{ and } x \in [0,1]^k. \tag{4.4.2}$$

Lemma 4.4.3. *Let C_Φ be the BLD constant of Φ and let M be the quasiconvexity constant of X. Under the assumption (4.4.2) there is a constant $C = C(k)C_\Phi M > 0$ such that for any integer $m \geq 1$, and any $x \in K_j$, there is a closed cube $Q_x \subseteq [0,1]^k$ centered at x of edge length d_x such that $f(K_j \cap Q_x)$ can be covered by m^j balls in X, each of radius $CLd_x m^{-1}$, where L is the Lipschitz constant of f.*

Proof. To prove the lemma we choose $Q \subseteq [0,1]^k$ with edge length d, centered at x such that $\mathcal{H}^k(Q \setminus K_j) < m^{-k}d^k$. We can assume that $Q = [0,d]^k$. Divide Q into m^j rectangular boxes $Q_\nu \times [0,d]^{k-j}$. We need to show that $f((Q_\nu \times [0,d]^{k-j}) \cap K_j)$ is contained in a ball of radius $CLdm^{-1}$. We find $\rho \in Q_\nu$ such that

$$\mathcal{H}^{k-j}((\{\rho\} \times [0,d]^{k-j}) \setminus K_j) < m^{j-k}d^{k-j}. \tag{4.4.3}$$

By the volume argument every point in $\{\rho\} \times [0,d]^{k-j}$ is at the distance no more than $C(k)m^{-1}d$ to the set $(\{\rho\} \times [0,d]^{k-j}) \cap K_j$. Hence every point in $Q_\nu \times [0,d]^{k-j}$, and thus every point in $(Q_\nu \times [0,d]^{k-j}) \cap K_j$, is at the distance less than or equal to $C(k)m^{-1}d$ from the set $(\{\rho\} \times [0,d]^{k-j}) \cap K_j$. Since f is L-Lipschitz it suffces to show that

$$\operatorname{diam}_X(f((\{\rho\} \times [0,d]^{k-j}) \cap K_j)) < CLdm^{-1}. \tag{4.4.4}$$

□

Lemma 4.4.4. *Let $E \subset Q$ be a measurable subset of a cube $Q \subset \mathbb{R}^n$. For $x, y \in Q$ let $I_x(y)$ be the length of the intersection of the interval $\overline{xy}$ with E, i.e. $I_x(y) = \mathcal{H}^1(\overline{xy} \cap E)$. Then there is a constant $C = C(n) > 0$ such that for any $x \in Q$*

$$\mathcal{H}^n(\{y \in Q : I_x(y) \leq C\mathcal{H}^n(E)^{1/n}\}) > \frac{\mathcal{H}^n(Q)}{2}. \tag{4.4.5}$$

The lemma says that if the measure of E is small, then more than 50% of the intervals $\overline{xy}$ intersect E along a short subset.

Proof. It suffices to show that for some constant $C = C(n)$

$$\fint_Q I_x(y)\,dy \le C\mathcal{H}^n(E)^{1/n}.$$

Then (4.4.5) will be true with C replaced by $2C$. For $z \in S^{n-1}$ let $\delta(z) = \sup\{t > 0 : x + tz \in Q\}$. An integral over Q can be represented in the spherical coordinates centered at x as follows

$$\int_Q f(y)\,dy = \int_{S^{n-1}} \int_0^{\delta(z)} f(x+tz)t^{n-1}\,dt\,d\sigma(z). \tag{4.4.6}$$

If $z \in S^{n-1}$, then

$$I_x(x+tz) \le I_x(x+\delta(z)z) = \int_0^{\delta(z)} \chi_E(x+\tau z)\,d\tau.$$

We have

$$\begin{aligned}
\fint_Q I_x(y)\,dy &= \frac{1}{\mathcal{H}^n(Q)} \int_{S^{n-1}} \int_0^{\delta(z)} t^{n-1} I_x(x+tz)\,dt\,d\sigma(z) \\
&\le \frac{1}{\mathcal{H}^n(Q)} \int_{S^{n-1}} \int_0^{\delta(z)} t^{n-1} \int_0^{\delta(z)} \chi_E(x+\tau z)\,d\tau\,dt\,d\sigma(z) \\
&\le \frac{1}{\mathcal{H}^n(Q)} \int_{S^{n-1}} \int_0^{\operatorname{diam} Q} t^{n-1}\,dt \int_0^{\delta(z)} \chi_E(x+\tau z)\,d\tau\,d\sigma(z) \\
&= C(n) \int_{S^{n-1}} \int_0^{\delta(z)} \frac{\chi_E(x+\tau z)}{\tau^{n-1}}\,\tau^{n-1}\,d\tau\,d\sigma(z) \qquad (4.4.7) \\
&= C \int_Q \frac{\chi_E(y)}{|x-y|^{n-1}}\,dy \le C\mathcal{H}^n(E)^{1/n}
\end{aligned}$$

by Lemma 4.2.4. Equality (4.4.7) follows from (4.4.6). □

Now under the assumptions of the lemma, if $x, y \in Q$, we can find $z \in Q$ such that $I_x(z) + I_y(z) \leq C\mathcal{H}^n(E)^{1/n}$, i.e. the curve $\overline{xz} + \overline{zy}$ connecting x to y has length no bigger than $2\text{diam}\,(Q)$ and it intersects the set E along a subset of length less than or equal to $C\mathcal{H}^n(E)^{1/n}$.

Applying it to $n = k - j$, $Q = \{\rho\} \times [0, d]^{k-j}$, and $E = (\{\rho\} \times [0, d]^{k-j}) \setminus K_j$, every pair of points $x, y \in Q \cap Kj$ can be connected by a curve $\gamma = \overline{xz} + \overline{zy}$ of length $\ell(\gamma) \leq 2d\sqrt{k-j}$ (two times the diameter of the cube) whose intersection with the complement of K_j has length no more than $C(k)m^{-1}d$ by (4.4.3).

We can parametrize γ by arc-length $\gamma : [0, \ell(\gamma)] \to \{\rho\} \times [0, d]^{k-j}$ as a 1-Lipschitz curve. The mapping $f \circ \gamma$ is L-Lipschitz and defined on a subset $\gamma^{-1}(K_j)$. It uniquely extends to the closure of $\gamma^{-1}(K_j)$ (because it is Lipschitz and X is complete). The complement of this set consists of countably many open intervals of total length bounded by $C(k)m^{-1}d$. Since the space X is quasiconvex we can extend $f \circ \gamma$ from the closure of $\gamma^{-1}(K_j)$ to $\widetilde{f \circ \gamma} : [0, \ell(\gamma)] \to X$ as an ML-Lipschitz curve connecting x to y; here M is the quasiconvexity constant of the space X.

The curve

$$\Phi \circ (\widetilde{f \circ \gamma}) : [0, \ell(\gamma))] \to \ell^\infty$$

is $C_\Phi ML$-Lipschitz. Note that on the set $\gamma^{-1}(K_j)$ this curve coincides with $g \circ \gamma$ and hence for a.e. $t \in \gamma^{-1}(K_j)$ we have

$$D(\Phi \circ (\widetilde{f \circ \gamma}))(t) = D(g \circ \gamma)(t) = 0.$$

Hence the length of the curve $\Phi \circ (\widetilde{f \circ \gamma})$ is bounded by

$$\begin{aligned} \ell(\Phi \circ (\widetilde{f \circ \gamma})) &= \int_0^{\ell(\gamma)} \|D(\Phi \circ (\widetilde{f \circ \gamma}))(t)\| dt \\ &\leq C_\Phi ML\mathcal{H}^1([0, \ell(\gamma)] \setminus \gamma^{-1}(K_j)) \\ &\leq C_\Phi MLC(k)m^{-1}d \end{aligned}$$

Now since Φ is weak BLD we have

$$d(f(x), f(y)) \leq \ell(\widetilde{f \circ \gamma}) \leq C_\Phi \ell(\Phi \circ (\widetilde{f \circ \gamma})) \leq C_\Phi^2 MLC(k)m^{-1}d.$$

Since this is true for all $x, y \in (\{\rho\} \times [0, d]^{k-j}) \cap K_j$, (4.4.4) follows. The proof is complete. □

As an immediate consequence of Theorem 4.4.2, we have the following theorem.

Theorem 4.4.5. *Let* (X, d_X) *be a complete and quasiconvex metric space and let* (Y, d_Y) *be any metric space. Let* $\Phi : X \to Y$ *be a weak BLD mapping. Then for any* $k \in \mathbb{N}$ *and any Lipschitz map* $f : E \subset \mathbb{R}^k \to X$ *defined on a measurable set* $E \subset \mathbb{R}^k$ *the following conditions are equivalent.*

1. $\mathcal{H}^k(f(E)) = 0$ *in* X,
2. $\mathcal{H}^k(\Phi(f(E))) = 0$ *in* Y.

Proof. This theorem follows immediately from Theorem 4.4.2 and by embedding $\Phi(f(E))$ into ℓ^∞ using the Kuratowski embedding. □

4.5 APPLICATIONS

4.5.1 Mappings of bounded length distortion

Definition 4.5.1. A mapping $f : X \to Y$ between metric spaces is said to have the *weak bounded length distortion* property (weak BLD) if there is a constant $C \geq 1$ such that for all rectifiable curves γ in X we have

$$C^{-1}\ell_X(\gamma) \leq \ell_Y(f \circ \gamma) \leq C\ell_X(\gamma). \tag{4.5.1}$$

The class of mappings with bounded length distortion (BLD) was introduced in [32] under the assumption that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping on an open domain such that it is open, discrete, sense preserving and satisfies (4.5.1) for all curves γ in Ω. A more general definition without any topological restrictions was given in [25, Definition 2.10]. This definition is almost identical to ours, but it was assumed that (4.5.1) was satisfied for *all* curves γ in X. The two notions are different: it may happen that a mapping has the weak BLD property, but some curves of infinite length in X are mapped onto rectifiable curves and hence such a mapping is not BLD in the sense of [25, Definition 2.10]. For example the identity mapping on the Heisenberg group $\mathrm{id} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$ satisfies the weak BLD condition locally. However, any segment on the t-axis has infinite length in the metric of $\mathbb{H}^n$ (actually its Hausdorff dimension equals 2) and it is mapped by the identity mapping to a segment in the t-axis in $\mathbb{R}^{2n+1}$ of finite Euclidean length.

As a consequence of Theorem 4.4.2 we obtain.

Theorem 4.5.2. *If a mapping $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ defined on an open set $\Omega \subset \mathbb{R}^n$ has the weak BLD property, then f is locally Lipschitz, $m \geq n$ and $\operatorname{rank} Df(x) = n$ a.e. in Ω.*

Proof. For any $y \in B(x,r) \subset \Omega$, the segment $\overline{xy}$ is mapped on a curve of length bounded by $C|x-y|$. Hence $|f(x) - f(y)| \leq C|x-y|$. Let X be a closed ball contained in Ω, equip it with the Euclidean metric and let $\Phi = f|_X : X \to \mathbb{R}^m$. Let $E \subset X$ be the set of points where $\operatorname{rank} Df < n$ and let $\iota : E \to X$ be the identity mapping. According to Theorem 4.4.2, $\mathcal{H}^n(E) = \mathcal{H}^n(\iota(E)) = 0$ if and only if $\operatorname{rank}(\operatorname{ap} D(\Phi \circ \iota)) = \operatorname{rank} Df < n$, a.e. in E. Since the last condition is satisfied by the definition of E, we conclude that $\mathcal{H}^n(E) = 0$, and hence $\operatorname{rank} Df(x) = n$ a.e. in Ω, because Ω is a countable union of closed balls. This however, implies that $m \geq n$. □

Gromov proved in [11, 2.4.11] that any Riemannian manifold of dimension n admits a mapping into $\mathbb{R}^n$ that preserves lengths of curves. It follows from Theorem 4.5.2 that the Jacobian of such mapping is different than zero a.e. and hence there is no such mapping into $\mathbb{R}^m$ for $m < n$ (this result is known).

In [32] it was proved that a mapping $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is BLD (under the topological assumptions: open, discrete, sense preserving) if and only if f is locally Lipschitz and $|J_f| \geq c > 0$ a.e. We proved without any topological assumptions that $|J_f| > 0$ a.e.

4.5.2 Carnot-Carathéodory spaces

Let $X_1, X_2, \ldots, X_m$ be a family of vector fields defined on an open and connected set $\Omega \subset \mathbb{R}^n$ with locally Lipschitz continuous coefficients. Assume that the vector fields are linearly independent at every point of Ω and that for every compact set $K \subset \Omega$

$$\inf_{p \in K} \inf_{i \in \{1,\ldots,m\}} |X_i(p)| > 0.$$

For $v = \sum_i a_i X_i(p) \in \operatorname{span}\{X_i(p), \dots, X_m(p)\}$ we define

$$|v|_H = \Big(\sum_{i=1}^m a_i^2 \Big)^{1/2}.$$

It follows from our assumptions that on compact subsets of Ω, $|v|_H$ is comparable to the Euclidean length $|v|$ of the vector v, i.e. for every compact set $K \subset \Omega$ there is a constant $C \geq 1$ such that

$$C^{-1}|v| \leq |v|_H \leq C|v| \quad \text{for all } p \in K \text{ and all } v \in \operatorname{span}\{X_1(p), \dots, X_m(p)\}. \tag{4.5.2}$$

We say that an absolutely continuous curve $\gamma : [a, b] \to \Omega$ is *horizontal* if there are measurable functions $a_i(t)$, $a \leq t \leq b$, $i = 1, 2, \dots, m$ such that

$$\gamma'(t) = \sum_{i=1}^m a_i(t) X_i(\gamma(t)) \quad \text{for almost all } t \in [a, b].$$

The horizontal length of γ is defined as

$$\ell_H(\gamma) = \int_a^b |\gamma'(t)|_H \, dt.$$

Denoting the Euclidean length of a curve γ by $\ell(\gamma)$, it easily follows from (4.5.2) that if $G \Subset \Omega$, then there is a constant $C \geq 1$ such that for any horizontal curve $\gamma : [a, b] \to G$ we have

$$C^{-1}\ell(\gamma) \leq \ell_H(\gamma) \leq C\ell(\gamma). \tag{4.5.3}$$

Assume that any two points in Ω can be connected by a horizontal curve. This is the case for example if the vector fields satisfy the Hörmander condition [45, Proposition III.4.1]. A Carnot group G is a group structure on $\mathbb{R}^n$ along with a horizontal distribution and the associated Carnot-Carathéodory metric. The Heisenberg group is an example of the Carnot group. All the assumptions about the vector fields given above are satisfied by Carnot groups (and in particular by the Heisenberg groups), [14, Section 11.3], but not by the Grushin type spaces [10]. Namely in general in the Grushin type spaces the inequality $\ell_H(\gamma) \leq C\ell(\gamma)$ need not be satisfied.

The Carnot-Carathéodory distance $d_{cc}(x,y)$ of the points $x,y \in \Omega$ is defined as the infimum of horizontal lengths of horizontal curves connecting x and y. Since we assume that any two points in Ω can be connected by a horizontal curve, (Ω, d_{cc}) is a metric space.

Clearly horizontal curves are rectifiable and it is well-known that every rectifiable curve with the arc-length parametrization is horizontal. Moreover $\ell_H(\gamma)$ equals the length $\ell_{cc}(\gamma)$ of γ with respect to the Carnot-Carathéodory metric. A detailed account on this topic can be found in [34]. $\mathrm{id} : (\Omega, d_{cc}) \to \Omega$ from the Carnot-Carathéodory space onto Ω with Euclidean metric is locally weakly BLD.

The next result follows immediately from a local version of Theorem 4.4.2. It applies to Carnot groups and in particular to the Heisenberg groups.

Theorem 4.5.3. *Let $X_1, \ldots, X_m$ be a family of locally Lipschitz vector fields in an open and connected domain $\Omega \subset \mathbb{R}^n$ such that for every compact set $K \subset \Omega$*

$$\inf_{p \in K} \inf_{i \in \{1,\ldots,m\}} |X_i(p)| > 0. \tag{4.5.4}$$

Assume also that any two points in Ω can be connected by a horizontal curve. Then for $k \geq 1$ and any Lipschitz mapping $f : E \subset \mathbb{R}^k \to (\Omega, d_{cc})$ the following conditions are equivalent.

1. $\mathcal{H}^k_{d_{cc}}(f(E)) = 0$ *in* (Ω, d_{cc})*;*
2. $\mathcal{H}^k(f(E)) = 0$ *with respect to the Euclidean metric in* Ω*;*
3. $\operatorname{rank}(\operatorname{ap} Df) < k$ *a.e. in* E*.*

5.0 MAPPINGS OF BOUNDED LENGTH DISTORTION

5.1 INTRODUCTION

The class of mappings of bounded length distortion (BLD for short) was introduced by Martio and Väisälä [32], and it plays a fundamental role in the contemporary development of geometric analysis and geometric topology, especially in the context of branched coverings of metric spaces, see e.g. [5, 17, 18, 19, 21, 22, 23, 25, 26, 36].

Analytic definition. A mapping $f : \Omega \to \mathbb{R}^n$ defined on a domain $\Omega \subset \mathbb{R}^n$ is said to be of the *M-bounded length distortion (M-BLD)* if it is locally Lipschitz, has non-negative Jacobian $J_f \geq 0$ and

$$M^{-1}|h| \leq |Df(x)h| \leq M|h|, \quad \text{for a.e. } x \in \Omega \text{ and for all } h \in \mathbb{R}^n. \tag{5.1.1}$$

Martio and Väisälä [32], proved that this definition is equivalent to a more geometric one.

Geometric definition. A continuous map $f : \Omega \to \mathbb{R}^n$ defined on a domain $\Omega \subset \mathbb{R}^n$ is M-BLD if it is open, discrete, sense-preserving and for any curve γ in Ω we have

$$M^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq M\ell(\gamma), \tag{5.1.2}$$

where $\ell(\gamma)$ denotes the length of the curve γ.

A mapping is said to be BLD if it is M-BLD for some $M \geq 1$.

Let us recall some topological terminology here. A mapping $f : \Omega \to \mathbb{R}^n$ is called *open* if it maps open subsets of Ω onto open subsets of $\mathbb{R}^n$. It is *discrete* if the inverse image of any point in $\mathbb{R}^n$ is a discrete set in Ω. We say that f is *sense-preserving* (*weakly sense-preserving*) if it is continuous and the topological degree with respect to any subdomain $D \Subset \Omega$ satisfies:

$\deg(f, D, y) > 0$ ($\deg(f, D, y) \geq 0$) for all $y \in f(D) \setminus f(\partial D)$. For more details, see e.g. [42, Section I.4].

The proof of the equivalence of the definitions given in [32] goes as follows.

If f satisfies the analytic definition, then f is locally M-Lipschitz and the right inequality at (5.1.2) follows. Also f is a non-constant quasiregular mapping, and hence by Reshetnyak's theorem [39], [42, Theorems I.4.1 and I.4.5], it is open, discrete and sense-preserving. Then Martio and Väisälä concluded the left inequality at (5.1.2) using the so called path-lifting argument which is available for mappings that are open and discrete. For a detailed account of the proof, see Proposition 5.2.9.

Now suppose that f satisfies the geometric definition. It easily follows from (5.1.2) that f is locally M-Lipschitz and hence $|Df(x)h| \leq M|h|$ for almost all $x \in \Omega$ and all $h \in \mathbb{R}^n$. Since f is locally Lipschitz, open, discrete and sense-preserving it follows from [42, I.4.11] that $J_f \geq 0$. It remains to prove that $|Df(x)h| \geq M^{-1}|h|$. Again, Martio and Väisälä employed the path-lifting argument in the proof. This argument could be used only because it was assumed that the mapping was open and discrete. For the detailed proof, see Proposition 5.2.10.

The above sketch of the proof shows that the path-lifting argument was used in both directions of the proof of the equivalence of the analytic and the geometric definitions. In words of Martio and Väisälä *(path-lifting) is perhaps the most important tool in the theory of BLD maps.*

The main observation in this chapter is that the implication from (5.1.2) to (5.1.1) does not require the path-lifting argument.

Theorem 5.1.1. *If a continuous map $f : \Omega \to \mathbb{R}^n$ defined on a domain $\Omega \subset \mathbb{R}^n$ is such that for any rectifiable curve γ in Ω inequality* (5.1.2) *is satisfied, then f is locally Lipschitz and* (5.1.1) *is true.*

The mappings to which Theorem 5.1.1 applies may change orientation. For example it applies to the folding of the plane $f(x, y) = (|x|, y)$. This map preserves lengths of all curves, yet it changes orientation and hence it is not BLD. In order to apply Theorem 5.1.1 to the class of BLD mappings we need a condition that would eliminate mappings like folding of the plane; it suffices to assume that the Jacobian is non-negative. The following result from [15] which is the main result in this chapter is a straightforward consequence of Theorem 5.1.1.

Theorem 5.1.2. *A continuous mapping* $f : \Omega \to \mathbb{R}^n$ *is* M*-BLD, if and only if*

$$M^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq M\ell(\gamma)$$

for all rectifiable curves γ *in* Ω *and* $Jf \geq 0$ *a.e. in* Ω.

Remark 5.1.3. The condition about non-negative Jacobain is very natural. Indeed, if $f : \Omega \to \mathbb{R}^n$ is a Lipschitz map, that is open, discrete, and sense-preserving, then the Jacobian $Jf \geq 0$ is non-negative a.e. [42, Lemma I.4.11]. On the other hand, a Lipschitz mapping $f : \Omega \to \mathbb{R}^n$ with $Jf \geq 0$ a.e. is weakly sense-preserving, [42, Lemma VI.5.1].

Remark 5.1.4. Theorem 5.1.2 provides a new and a simpler version of the geometric definition by showing that the strong topological assumptions about openness and discreteness in the geometric definition of the BLD mappings are redundant. We want to emphasize that the proof that the analytic definition implies our new geometric definition involves Reshetnyak's theorem and the path-lifting argument as described above. Only the other implication is based on a new argument that avoids topological assumptions.

Remark 5.1.5. The geometric definition of Martio and Väisälä is the base of the independent theory of BLD mappings between generalized metric manifolds, but our result applies to the Euclidean setting only.

Our proof is short and elementary, but we arrived to this simple argument through a rather complicated way by studying unrectifiability of the Heisenberg groups.

A mapping $f : X \to Y$ between metric spaces is said to be a *weak BLD* mapping, if there is a constant $M \geq 1$ such that for all *rectifiable* curves γ in X, we have

$$M^{-1}\ell_X(\gamma) \leq \ell_Y(f \circ \gamma) \leq M\ell_X(\gamma). \tag{5.1.3}$$

This definition was introduced in section 4.5. See also [25], where a stronger condition that (5.1.3) is true for *all* curves γ is required.

It is a well-known fact that the identity map $\mathrm{id} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$ has (locally) the weak BLD property. Here, $\mathbb{H}^n$ is the standard Heisenberg group. One of the results in [16] provides a characterization of pure k-unrectifiability of a metric space X under the assumption that there is a weak BLD mapping $\Phi : X \to \mathbb{R}^N$ for some $N > 0$. Since the identity mapping $\Phi = \mathrm{id} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$ has the weak BLD property, the result provides a new proof of pure k-unrectifiability of $\mathbb{H}^n$ when $k > n$. It also follows from this characterization of pure k-unrectifiability of X that if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ has the weak BLD property, then $m \geq n$ and $\operatorname{rank} Df = n$ a.e. In particular, weak BLD mappings $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfy $|Jf| > 0$ a.e. We quickly realized that a stronger quantitative estimate $|Jf| \geq C > 0$ would imply that weak BLD mappings with $Jf \geq 0$ are BLD which is our Theorem 5.1.2. We could prove this estimate using the methods of [16], but the proof was long and complicated; eventually we discovered a very simple argument which is presented in this chapter.

By S^{n-1} we will denote the unit sphere in $\mathbb{R}^n$ and χ_E will stand for the characteristic function of a set E.

5.2 PATH LIFTING FOR OPEN AND DISCRETE MAPPINGS

In this section, we provide a detailed account of the proof of the equivalence between the analytic and geometric definitions given in [32].

Definition 5.2.1. Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. The *branch set* B_f of f is the set of all points $x \in \Omega$ where f is not a local homeomorphism.

Remark 5.2.2. For open and discrete maps, the topological dimension of B_f is at most $n - 2$. For the basic facts about these properties refer to [31].

Lemma 5.2.3. *Suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the analytic definition. Then, f has the following properties:*

1. *f is discrete, open, and sense-preserving.*
2. *$f|_{\Omega \setminus B_f}$ is locally M-bi-Lipschitz.*
3. *If f is injective, $f^{-1} : f(\Omega) \to \Omega$ satisfies the analytic definition.*

Proof. By a result of Reshetnyak [39], a quasiregular map is either constant or discrete, open, and sens-preserving. Since a BLD map is not constant, (1) follows.

To prove (2), fix an open ball $B \subset \Omega \setminus B_f$ such that $f|_B$ is injective. So, $F|_B$ is quasiconformal, that is, an injective quasiregular map. Hence, $g = (f|_B)^{-1}$ is also quasiconformal and thus absolutely continuous on lines and almost everywhere differentiable. If f is differentiable and satisfies

$$M^{-1}|h| \leq |Df(x)h| \leq M|h|$$

for all $h \in \mathbb{R}^n$ at $x \in B$ and if g is differentiable at $y = f(x)$, then $Dg(y) = (Df(x))^{-1}$ and thus

$$M^{-1}|h| \leq |Dg(y)h| \leq M|h|$$

for all $h \in \mathbb{R}^n$. Since a Lipschitz map preserves the property of being of measure zero, these inequalities are true for almost every $y \in f(B)$. Hence, g is M-BLD and thus locally M-Lipschitz. This proves (2) and (3). □

Lemma 5.2.4. *If $f : \Omega \to \mathbb{R}^n$ satisfies the analytic definition, then $|B_f| = |f(B_f)| = 0$.*

Proof. By [31, Theorems 2.14], $J_f(x) = 0$ whenever f is differentiable at a branch point $x \in B_f$. Hence, it is easy to see that $|B_f| = 0$. Since f is locally Lipschitz, we can conclude that $|f(B_f)| = 0$. □

We will formulate our results for curves on half-open intervals; the obvious modifications for closed intervals can be easily concluded.

Suppose that $f : \Omega \to \mathbb{R}^n$ is discrete, open, and sens-preserving. Let $\beta : [a, b) \to \mathbb{R}^n$ be a curve and let $x \in f^{-1}(\beta(a))$. A *maximal lift of β starting at x* is a curve $\alpha : [a, c) \to \Omega$ such that $\alpha(a) = x$, $f \circ \alpha = \beta|_{[a,c)}$, and α is not a proper subcurve of another curve with these properties. There is always at least one maximal lift α of β starting at x. If $c = b$, then α is called a *total lift of β*.

A domain $D \Subset \Omega$ is a *normal domain of f* if $f(\partial D) = \partial f(D)$. If, in addition, $x \in D$ with $D \cap f^{-1}(f(x)) = \{x\}$, D is called a *normal neighborhood of x*. If $x \in \Omega$, we let $U(x, f, r)$ denote the x-component of $f^{-1}(B(f(x), r))$. If $U = U(x, f, r) \Subset \Omega$, U is a normal domain of f and $f(U) = B(f(x), r)$. There is $r_0 > 0$ such that U is a normal neighborhood of x if for $r \leq r_0$.

If $A \subset \Omega$, we write

$$N(y, f, A) = \text{card}(A \cap f^{-1}(y)),$$

$$N(f, A) = \sup\{N(y, f, A) : y \in \mathbb{R}^n\},$$

and

$$N(f) = N(f, \Omega).$$

If $A \Subset \Omega$, $N(f, A)$ is finite. If U is a normal neighborhood of x, $N(f, V) = i(x, f)$ for every neighborhood $V \subset U$ of x. Here, $i(x, f)$ denotes the local index of f at x. We recall a result of Rickman [41, Theorem 2] which will be used in what follows. We assume that $f : \Omega \to \mathbb{R}^n$ is discrete, open, and sense-preserving.

Lemma 5.2.5. *Suppose that D is a normal domain of f and that $\beta : [a, b) \to f(D)$ is a curve with locus $|\beta|$. Then, $D \cap f^{-1}(|\beta|)$ can be expressed as the union of the loci of $N(f, D)$ total lifts $\alpha_j : [a, b) \to D$ of β such that card$\{j : \alpha_j(t) = x\} = i(x, f)$ for every $x \in D \cap f^{-1}(|\beta|)$.*

Corollary 5.2.6. *Suppose that D is a normal neighborhood of x, that $y \in D$, and that $\beta : [a, b] \to f(D)$ is a curve joining $f(x)$ to $f(y)$. Then there is a lift $\alpha : [a, b] \to D$ of β joining x to y.*

If $f : \Omega \to \mathbb{R}^n$, $x \in \Omega$, and $r > 0$, we set

$$L^*(x, f, r) = \sup\{|y - x| : y \in \partial U\},$$

and

$$l^*(x, f, r) = \inf\{|y - x| : y \in \partial U\},$$

where $U = U(x, f, r)$.

Lemma 5.2.7. *Suppose that $f : \Omega \to \mathbb{R}^n$ satisfies the analytic definition and that $U = U(x, f, r)$ is a normal neighborhood of x. Then*

$$L^*(x, f, r) \le Mr, \qquad l^*(x, f, r) \ge M^{-1}r.$$

Proof. Choose $y \in \partial U$ with $|y - x| = l^*(x, f, r)$. Since the line segment $[x, y]$ lies in Ω, we have

$$r = |f(y) - f(x)| \leq Ml^*(x, f, r).$$

which is the second inequality.

To prove the first inequality, we consider a point $z \in \partial B(f(x), r)$ and the radial curve $\beta_z : [0, r) \to \mathbb{R}^n$ from $f(x)$ to z, defined by

$$\beta_z(t) = f(x) + t\frac{z - f(x)}{r}.$$

By Lemma 5.2.5, $U \cap f^{-1}|\beta_z|$ can be covered by m maximal lifts of β_z starting at x and $m = N(f, U) = i(x, f)$. From [31, Theorem 7.10] it easily follows that for almost every z, each lift α_z is absolutely continuous on every interval $[s, r], s > 0$. By Lemma 5.2.4, almost every $|\beta_z|$ meets $f(B_f)$ in a set of measure zero. Since $f|_{\Omega \setminus B_f}$ is locally M-bi-Lipschitz, we have that $|\alpha'_z(t)| \leq M$ for almost every $t \in [0, r)$. Hence

$$|\alpha_z(t) - x| = |\alpha_z(t) - \alpha_z(0)| \leq Mr$$

for all $t \in [0, r)$ and for almost every $z \in \partial B(f(x), r)$. Thus, $|y - x| \leq Mr$ for a dense set of points $y \in U$. This implies $L^*(x, f, r) \leq Mr$. □

Corollary 5.2.8. *If $f : \Omega \to \mathbb{R}^n$ satisfies the analytic definition and if $U = U(x, f, r)$ is a normal neighborhood of x, then*

$$M^{-1}|y - x| \leq |f(y) - f(x)| \leq M|y - x|,$$

whenever $|y - x| \leq M^{-1}r$.

Proposition 5.2.9. *If $f : \Omega \to \mathbb{R}^n$ satisfies the analytic definition, then for any curve γ in Ω we have*

$$M^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq M\ell(\gamma).$$

Proof. f is locally M-Lipschitz. This implies the second inequality. The proof of the first inequality is somewhat harder. Setting $\beta = f \circ \gamma$, we may assume that $\ell(\beta) < \infty$ and that $\beta : [0, \ell(\beta)] \to \mathbb{R}^n$ is parametrized by arc-length. Let P be a partition of $[0, \ell(\beta)]$. It easily follows from [31, Theorem 2.9] that there is a refinement $P' = \{t_0, \cdots, t_k\}$ of P and numbers $r_i > 0$ such that for each $i \in \{0, \cdots, k\}$, $U(\gamma(t_i), r_i)$ is a normal neighborhood of $\gamma(t_i)$ and $\beta([t_{i-1}, t_i]) \subset B_{i-1} \cup B_i$, where $B_i = B(\beta(t_i), r_i)$. For each $i = 1, \cdots, k$, choose $s_i \in [t_{i-1}, t_i]$ with $\beta(s_i) \in B_{i-1} \cap B_i$. Then, 5.2.8 implies

$$|\gamma(s_i) - \gamma(t_{i-1})| \leq M|s_i - t_{i-1}|, \qquad |\gamma(t_i) - \gamma(s_i)| \leq M|t_i - s_i|.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})| &\leq \sum_{i=1}^{k} \left(|\gamma(t_i) - \gamma(s_i)| + |\gamma(s_i) - \gamma(t_{i-1})|\right) \\ &\leq M \sum_{i=1}^{k} \left((t_i - s_i) + (s_i - t_{i-1})\right) \\ &= M\ell(\beta). \end{aligned}$$

This yields $M^{-1}\ell(\gamma) \leq \ell(f \circ \gamma)$ as required. □

Proposition 5.2.10. *If $f : \Omega \to \mathbb{R}^n$ satisfies the geometric definition, then*

$$|Df(x)h| \geq M^{-1}|h|$$

for almost all $x \in \Omega$ and all $h \in \mathbb{R}^n$.

Proof. Fix $x \in \Omega$ such that f is differentiable at x. It suffices to show that $|Df(x)h| \geq M^{-1}|h|$ for all $h \in \mathbb{R}^n$. Suppose that this is false for some $h = h_0$. Then

$$\frac{|Df(x)h_0|}{|h_0|} = \alpha < M^{-1}. \tag{5.2.1}$$

We may assume that $x = 0 = f(x)$. Let $U = U(0, f, r)$ be a normal neighborhood of 0. Now, f has the expansion

$$f(h) = Df(0)h + \epsilon(h)h, \tag{5.2.2}$$

where $\epsilon(h) \to 0$ as $h \to 0$. Pick $t > 0$ so small that $h = th_0 \in U$ and

$$2|\epsilon(h)| \leq \frac{1}{M - \alpha}. \tag{5.2.3}$$

Let γ be the line segment from 0 to $f(h)$. By Corollary 5.2.6 there exists a lift γ^* of γ from 0 to h. Then, the geometric condition implies

$$|f(h)| = \ell(\gamma) \geq \ell(\gamma^*) \geq \frac{|h|}{M}.$$

On the other hand, (5.2.1), (5.2.2), and (5.2.3) imply

$$\begin{aligned} |f(h)| &\leq |Df(0)h| + |\epsilon(h)||h| \leq (1/M - \alpha)|h|/2 \\ &= (1/M + \alpha)|h|/2 < |h|/M. \end{aligned}$$

This contradicts (5.2.3) and completes the proof. □

5.3 PROOF OF THE MAIN RESULT

Suppose that f satisfies the assumptions of Theorem 5.1.1.

For any $y \in B(x, r) \subset \Omega$, the segment $\overline{xy}$ is mapped onto a curve of length bounded by $M|x - y|$. Hence $|f(x) - f(y)| \leq M|x - y|$ which means f is locally M-Lipschitz in Ω. So, it is differentiable a.e. by Rademacher's theorem and

$$|Df(x)h| \leq M|h| \quad \text{for a.e. } x \in \Omega \text{ and all } h \in \mathbb{R}^n.$$

It remains to prove that $M^{-1}|h| \leq |Df(x)h|$ for a.e. $x \in \Omega$ and all $h \in \mathbb{R}^n$ which is equivalent to proving that

$$|Df(x)h| \geq M^{-1} \tag{5.3.1}$$

for a.e. $x \in \Omega$ and all $h \in S^{n-1}$.

Since Df is a measurable mapping, there exists, for any $m \in \mathbb{N}$, a closed set $K_m \subset \Omega$ such that f is differentiable at all points of K_m, $\mathcal{H}^n(\Omega \setminus K_m) < m^{-1}$ and Df is continuous on K_m. Since we can exhaust Ω with the sets K_m up to a set of measure zero, it suffices to prove (5.3.1) for any given $m \in \mathbb{N}$, almost every $x \in K_m$ and all $h \in S^{n-1}$. Fix $m \in \mathbb{N}$. It suffices to prove inequality (5.3.1) when $x \in K_m$ is a density point of K_m and $h \in S^{n-1}$ is arbitrary. To the contrary suppose that there is a density point $x \in K_m$ and $h_0 \in S^{n-1}$ such that

$$|Df(x)h_0| = \alpha < M^{-1}.$$

Without loss of generality, we may assume that $x = 0$. This is only used to simplify notation.

It suffices to show that there is a rectifiable curve γ passing through x such that $\ell(f\circ\gamma) < M^{-1}\ell(\gamma)$. This will contradict (5.1.2). To this end it suffices to show that there are

$$0 < \alpha_1 < \alpha_2 < M^{-1},\ v \in S^{n-1} \text{ and } R > 0 \tag{5.3.2}$$

such that for the curve

$$\gamma : [0, R] \to \Omega, \quad \gamma(t) = x + tv = tv$$

the following conditions are satisfied

$$|Df(tv)v| < \alpha_1 \quad \text{whenever } tv \in K_m \text{ and } 0 \le t \le R, \tag{5.3.3}$$

$$\mathcal{H}^1(\{t \in [0, R] : tv \notin K_m\}) < R\,\frac{\alpha_2 - \alpha_1}{M}. \tag{5.3.4}$$

Indeed, since f is locally M-Lipschitz and γ is 1-Lipschitz, $|(f\circ\gamma)'(t)| \le M$ for a.e. $t \in [0, R]$. Moreover (5.3.3) implies that

$$|(f\circ\gamma)'(t)| = |Df(\gamma(t))\gamma'(t)| < \alpha_1 \quad \text{whenever } \gamma(t) \in K_m.$$

Hence

$$\begin{aligned} \ell(f\circ\gamma) &= \int_0^R |(f\circ\gamma)'(t)|\,dt < \int_{\{\gamma\notin K_m\}} M\,dt + \int_{\{\gamma\in K_m\}} \alpha_1\,dt \\ &< MR\frac{\alpha_2-\alpha_1}{M} + \alpha_1 R = R\alpha_2 < RM^{-1} = M^{-1}\ell(\gamma). \end{aligned}$$

Therefore it remains to prove that there are α_1, α_2, v and R as in (5.3.2) for which the conditions (5.3.3) and (5.3.4) are satisfied.

Fix any numbers $\beta, \alpha_1, \alpha_2$ such that $\alpha < \beta < \alpha_1 < \alpha_2 < M^{-1}$. Observe that $M \ge 1$.

Let Cone (r, δ) be the set of all vectors $h \in \overline{B}(0, r)$ such that the angle between h and h_0 is less than or equal δ.

Since $Df(0) : \mathbb{R}^n \to \mathbb{R}^n$ is linear and continuous, there exists $\delta > 0$ such that

$$|Df(0)h| < \beta \quad \text{for all } h \in \text{Cone}\,(1, \delta).$$

Since $Df|_{K_m}$ is continuous at $x = 0$, there exists $\tau > 0$ such that

$$|Df(y)h| < \alpha_1, \quad \text{for } y \in B(0, \tau) \cap K_m \text{ and } h \in \text{Cone}\,(1, \delta). \tag{5.3.5}$$

Let $C'(n, \delta) = \mathcal{H}^n(\text{Cone}\,(1, \delta))$. By a scaling argument

$$\mathcal{H}^n(\text{Cone}\,(r, \delta)) = \mathcal{H}^n(\text{Cone}\,(1, \delta))r^n = C'(n, \delta)r^n.$$

Since $x = 0$ is a density point of K_m and $0 < 1 - (\alpha_2 - \alpha_1)^n/M^n < 1$, there is $0 < R < \tau$ such that $B(0, R) = B(x, R) \subset \Omega$ and

$$\mathcal{H}^n(\text{Cone}\,(R, \delta) \cap K_m) > C'(n, \delta)R^n \left(1 - \frac{(\alpha_2 - \alpha_1)^n}{M^n}\right). \tag{5.3.6}$$

Now we claim that there is a vector $v = \bar{h}/|\bar{h}| \in S^{n-1}$ for some $0 \neq \bar{h} \in \text{Cone}\,(R, \delta)$ for which (5.3.4) is satisfied. Indeed, suppose to the contrary that for every $0 \neq \bar{h} \in \text{Cone}\,(R, \delta)$ we have

$$\mathcal{H}^1\big(\big\{t \in [0, R] : t\bar{h}/|\bar{h}| \notin K_m\big\}\big) \geq R\,\frac{\alpha_2 - \alpha_1}{M}\,,$$

i.e.

$$\mathcal{H}^1\big(\big\{t \in [0, R] : t\bar{h}/|\bar{h}| \in K_m\big\}\big) \leq R\left(1 - \frac{\alpha_2 - \alpha_1}{M}\right). \tag{5.3.7}$$

Hence for any $0 \neq \bar{h} \in \text{Cone}\,(R, \delta)$ and $u = \bar{h}/|\bar{h}| \in S^{n-1}$

$$\int_0^R \chi_{K_m}(tu)t^{n-1}\,dt \leq \int_{R(\alpha_2 - \alpha_1)/M}^R t^{n-1}\,dt.$$

Thus integration in spherical coordinates yields

$$\begin{aligned}\mathcal{H}^n(\mathrm{Cone}\,(R,\delta)\cap K_m) &= \int_{S^{n-1}\cap \mathrm{Cone}\,(1,\delta)}\int_0^R \chi_{K_m}(tu)t^{n-1}\,dt\,d\mathcal{H}^{n-1}(u)\\ &\leq \int_{S^{n-1}\cap \mathrm{Cone}\,(1,\delta)}\int_{R(\alpha_2-\alpha_1)/M}^R t^{n-1}\,dt\,d\mathcal{H}^{n-1}(u)\\ &= \mathcal{H}^n\left(\mathrm{Cone}\,(R,\delta)\setminus \mathrm{Cone}\left(R\,\frac{\alpha_2-\alpha_1}{M},\delta\right)\right)\\ &= C'(n,\delta)R^n\left(1-\frac{(\alpha_2-\alpha_1)^n}{M^n}\right)\end{aligned}$$

which clearly contradicts (5.3.6).

We proved that for some $0\neq \bar{h}\in \mathrm{Cone}\,(R,\delta)$ the vector $v=\bar{h}/|\bar{h}|\in S^{n-1}$ satisfies (5.3.4). Now (5.3.3) easily follows from (5.3.5) with $y=tv$ and $h=v$. Indeed, $v\in \mathrm{Cone}\,(1,\delta)$ and $tv\in B(0,R)\subset B(0,\tau)$. The proof is complete. This also completes the proof of Theorem 5.1.2.

6.0 BLD EMBEDDING CONJECTURE AND RELATED OPEN PROBLEMS

6.1 INTRODUCTION

In this section, we will present the basic definitions and results that are needed for the rest of this chapter.

Let (X, d) be a metric space and let C be an open cover of X. The order of the cover C is the smallest number n (if it exists) such that each point of the space belongs to at most n sets in the cover. A refinement of a cover C is another cover, each of whose sets is a subset of a set in C; its order may be smaller than, or possibly larger than, the order of C.

Definition 6.1.1 (Topological Dimension)**.** The *topological dimension* of a metric space (X, d) is defined to be the minimum value of n, such that every open cover C of X has an open refinement with order $n + 1$ or below. If no such minimal n exists, the space is said to be of infinite topological dimension.

As a special case, a topological space is zero-dimensional with respect to the topological dimension if every open cover of the space has a refinement consisting of disjoint open sets so that any point in the space is contained in exactly one open set of this refinement.

Theorem 6.1.2 (Menger and Nòbeling)**.** *Any compact metric space of topological dimension m can be embedded in $\mathbb{R}^k$ for $k = 2m + 1$.*

Definition 6.1.3. A metric space (X, d) endowed with a Borel measure μ is said to be *Ahlfors Q-regular* for some $Q \geq 0$ if there exist constants $0 < c \leq C < \infty$ such that

$$cr^Q \leq \mu(B) \leq Cr^Q$$

for every closed ball B in X with radius $r < \operatorname{diam}(X)$.

Remark 6.1.4. An Ahlfors Q-regular space necessarily has Hausdorff dimension equal to Q.

Definition 6.1.5. A metric space (X, d) is said to be *length metric space* if the distance between any two points $x, y \in X$ is equal to the infimum of length of all curves joining x to y.

6.2 BLD EMBEDDING CONJECTURE

The BLD embedding conjecture which, if proven positively, will yield an extension of the celebrated C^1 embedding theorem due to John Nash, [35]. This conjecture is due to Enrico Le Donne, [25].

Recall that a mapping $f : X \to Y$ between metric spaces is BLD if the condition (4.5.1) holds for *all* curves in X.

Conjecture 6.2.1 (BLD Embedding Conjecture)**.** Any compact length metric space of finite Hausdorff dimension can be embedded in some Euclidean space via a bounded length distortion map.

In what follows, we will outline our attempt to approach the BLD conjecture. The original version of Theorem 4.4.2, as it is stated in [16], is as follows.

Theorem 6.2.2. *Suppose that (X, d) is a complete and quasiconvex metric space and that $\Phi : X \to \mathbb{R}^N$ is a Lipschitz map with the property that for some constant $C_\Phi > 0$ and all rectifiable curves γ in X we have*

$$\ell(\gamma) \leq C_\Phi \ell(\Phi \circ \gamma). \tag{6.2.1}$$

Then for any $k \geq 1$ and any Lipschitz map $f : E \subset \mathbb{R}^k \to X$ defined on a measurable set $E \subset \mathbb{R}^k$ the following conditions are equivalent.

1. $\mathcal{H}^k(f(E)) = 0$ *in* X,

2. $\mathcal{H}^k(\Phi(f(E))) = 0$ *in* $\mathbb{R}^N$,
3. $\operatorname{rank}(\operatorname{ap} D(\Phi \circ f)) < k$ *a.e. in* E.

After we published this result in [16] we realized that the following consequence of the BLD embedding conjecture is true.

Corollary 6.2.3. *Let* (X, d_X) *be a complete and quasiconvex metric space and let* (Y, d_Y) *be a compact length metric space of finite Hausdorff dimension. Also, let* $\phi : X \to Y$ *be a weak BLD mapping. If the conjecture 6.2.1 is true, then for any* $k \in \mathbb{N}$ *and any Lipschitz map* $f : E \subset \mathbb{R}^k \to X$ *defined on a measurable set* $E \subset \mathbb{R}^k$ *the following conditions are equivalent.*

1. $\mathcal{H}^k(f(E)) = 0$ *in* X,
2. $\mathcal{H}^k(\phi(f(E))) = 0$ *in* Y.

Indeed, the implication from 1 to 2 is obvious since ϕ is Lipschitz. Now, assume that $\mathcal{H}^k(\phi(f(E))) = 0$ in Y. Since the BLD conjecture 6.2.1 is assumed to be true, we can find a BLD mapping $\psi : Y \to \mathbb{R}^N$ for some $N \in \mathbb{N}$. Since ψ is Lipschitz, we can conclude that $\mathcal{H}^k((\psi \circ \phi)(f(E))) = 0$ in $\mathbb{R}^N$. Let $\tilde{\phi} = \psi \circ \phi$. Clearly, $\tilde{\phi}$ is a weak BLD mapping, and thus, Theorem 6.2.2 shows that $\mathcal{H}^k((\psi \circ \phi)(f(E))) = 0$ implies $\mathcal{H}^k(f(E)) = 0$ and this completes the proof of the corollary.

The statement of Corollary 6.2.3 is much stronger than that of Theorem 6.2.2. Notice that in this corollary, ϕ is a BLD mappings between metric spaces whereas in Theorem 6.2.2, ϕ is a BLD mapping into the Euclidean space $\mathbb{R}^N$. This is what lead us to Theorem 4.4.5 which is even stronger than the above corollary as Y is a completely arbitrary metric space in that theorem. We still do not know if the BLD embedding conjecture is true or not.

Inspired by the result from K. Menger and G. Nòbeling's (Theorem 6.1.2) and as an attempt to solve the BLD conjecture, Le Donne in [25] provides the following analogue for Lipschitz embeddings of metric spaces.

Theorem 6.2.4. *Any compact metric space of Hausdorff dimension* k *can be embedded in* $\mathbb{R}^n$ *via a Lipschitz map, for* $n = 2k + 1$.

The proof is an application of the Baire Category Theorem as well as the topological version of the theorem, namely the Menger and Nòbeling result. However, due to a technical issue which will be discussed below, the proof is flawed. The following result is taken from [7].

Theorem 6.2.5. *There exists compact subsets A and B of $\mathbb{R}$ of Hausdorff dimension 0 such that $\mathcal{H}^1(A \times B) > 0$.*

Proof. First we construct Borel sets A and B and then we will modify them to be compact. Let $\{s_j\}$ be a sequence of numbers decreasing to 0 and let $0 = m_0 < m_1 < m_2 < \cdots$ be a sequence of integers increasing rapidly enough to ensure that

$$(m_1 - m_0) + (m_3 - m_2) + \cdots + (m_{2j-1} - m_{2j-2}) \le s_j m_{2j}, \tag{6.2.2}$$

and

$$(m_2 - m_1) + (m_4 - m_3) + \cdots + (m_{2j} - m_{2j-1}) \le s_j m_{2j+1}. \tag{6.2.3}$$

Let A be the subset of $[0,1]$ consisting of those numbers with zero in the rth decimal place if $m_j + 1 \le r \le m_{j+1}$ and j is odd. Similarly, take B as the set of numbers which have zeros in the rth decimal place if $m_j + 1 \le r \le m_{j+1}$ and j is even. Taking the obvious covers of A by 10^k intervals of length $10^{-m_{2j}}$, where

$$k = (m_1 - m_0) + (m_3 - m_2) + \cdots + (m_{2j-1} - m_{2j-2}),$$

it follows from (6.2.2) and (6.2.3) that if $s > 0$, then $\mathcal{H}^s(A) = 0$ and, similarly, that $\mathcal{H}^s(B) = 0$.

Let proj denote orthogonal projection from the place onto L, the line $y = x$. Then, $\text{proj}(x,y)$ is the point of L at signed distance $2^{-1/2}(x+y)$ from the origin. If $u \in [0,1]$ we may find $x \in A$ and $y \in B$ such that $u = x + y$. Thus, $\text{proj}(A \times B)$ is a subinterval of L of length $2^{-1/2}$. Using the fact that orthogonal projection does not increase distances, it does not increase Hausdorff measures,

$$2^{-1/2} = \mathcal{H}^1(\text{proj}(A \times B)) \le \mathcal{H}^1(A \times B).$$

Now, A and B may be made into compact sets by the addition of countable sets of points. $\square$

Proposition 5.1 in Le Donne's paper [25] wrongly assumes that if K is a compact set with Hausdorff dimension k, then the Hausdorff dimension of $K \times K$ is $2k$. However, the above Theorem 6.2.5 gives a counter example to this fact.

Le Donne's result becomes true if we replace compact metric spaces of finite Hausdorff dimension with compact Ahlfors Q-regular spaces. Indeed, for Ahlfors Q-regular spaces, Hausdorff dimension of the product of two spaces is actually equal to the sum of the Hausdorff dimensions and Le Donne's argument works flawlessly. However, in that case, existence of a Lipschitz embedding (even with a Hölder continuous inverse) has been already proven by Foias and Olson in [9]. Therefore, as a next step towards approaching the BLD conjecture, a logical direction would be considering Ahlfors regular spaces.

BIBLIOGRAPHY

[1] Ambrosio, L., Kirchheim, B.: Rectifiable sets in metric and Banach spaces. *Math. Ann.* 318 (2000), 527–555.

[2] Ambrosio, L., Kirchheim, B.: Currents in metric spaces. *Acta Math.* 185 (2000),1–80.

[3] Balogh, Z. M., Hajłasz, P., Wildrick, K.: Weak contact equations for mappings into Heisenberg groups. *Indiana Univ. Math. J.* 63 (2014), 1839–1873.

[4] David, G., Semmes, S.: *Fractured fractals and broken dreams. Self-similar geometry through metric and measure.* Oxford Lecture Series in Mathematics and its Applications, 7. The Clarendon Press, Oxford University Press, New York, 1997.

[5] Drasin, D., Pankka, P.: Sharpness of Rickman's Picard theorem in all dimensions. *Acta Math.* 214 (2015), no. 2, 209–306.

[6] Evans, L. C., Gariepy, R. F.: *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[7] Falconer, K. J.: *The Geometry of Fractal Sets* Cambridge University Press, 1986.

[8] Federer, H.: *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969

[9] Foias, C., Olson, E.: Finite fractal dimension and Hölder-Lipschitz parametrization. *Indiana Univ. Math. J.* 45 (1996), no. 3, 603–616.

[10] Franchi, B., Gutiérrez, C. E., Wheeden, R. L.: Weighted Sobolev-Poincaré inequalities for Grushin type operators. *Comm. Partial Differential Equations* 19 (1994), 523–604.

[11] Gromov, M.: *Partial differential relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986.

[12] Hajłasz, P.: Change of variables formula under minimal assumptions. *Colloq. Math.* 64 (1993), 93–101.

[13] Hajłasz, P.: Sobolev spaces on metric-measure spaces. (Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)). 173–218, *Contemp. Math.*, 338, Amer. Math. Soc., Providence, RI, 2003.

[14] Hajłasz, P., Koskela, P.: Sobolev met Poincaré. *Mem. Amer. Math. Soc.* 145 (2000), no. 688, x+101 pp.

[15] Hajłasz, P.; Malekzadeh, S.: A new characterization of the mappings of bounded length distortion. *Int. Math. Res. Not. IMRN* (2015), no. 24, 13238–13244.

[16] Hajłasz, P.; Malekzadeh, S.: On conditions for unrectifiability of a metric space. *Anal. Geom. Metr. Spaces* 3 (2015), 1–14.

[17] Heinonen, J., Keith, S.: Flat forms, bi-Lipschitz parameterizations, and smoothability of manifolds. *Publ. Math. Inst. Hautes Études Sci.* No. 113 (2011), 1–37.

[18] Heinonen, J.; Kilpeläinen, T.: BLD-mappings in $W^{2,2}$ are locally invertible. *Math. Ann.* 318 (2000), 391–396.

[19] Heinonen, J.; Kilpeläinen, T.; Martio, O.: Harmonic morphisms in nonlinear potential theory. *Nagoya Math. J.* 125 (1992), 115–140.

[20] Heinonen, J.; Koskela, P.; Shanmugalingam, N.; Tyson, J. T.: *Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients.* Cambridge University Press, 2015.

[21] Heinonen, J., Rickman, S.: Geometric branched covers between generalized manifolds. *Duke Math. J.* 113 (2002), 465–529.

[22] Heinonen, J., Rickman, S.: Quasiregular maps $S^3 \to S^3$ with wild branch sets. *Topology* 37 (1998), 1–24.

[23] Heinonen, J., Sullivan, D.: On the locally branched Euclidean metric gauge. *Duke Math. J.* 114 (2002), 15–41.

[24] Kirchheim, B.: Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.* 121 (1994), 113–123.

[25] Le Donne, E.: Lipschitz and path isometric embeddings of metric spaces. *Geom. Dedicata* 166 (2013), 47–66.

[26] Le Donne, E., Pankka, P.: Closed BLD-elliptic manifolds have virtually Abelian fundamental groups. *New York J. Math.* 20 (2014), 209–216.

[27] Magnani, V.: Nonexistence of horizontal Sobolev surfaces in the Heisenberg group. *Proc. Amer. Math. Soc.* 138 (2010), 1785–1791.

[28] Magnani, V.: Unrectifiability and rigidity in stratified groups. *Arch. Math. (Basel)* 83 (2004), 568–576.

[29] Magnani, V., Malý, J., Mongodi, S.:: A low rank property and nonexistence of higher dimensional horizontal Sobolev sets. *J. Geom. Anal.* 25 (2015), no. 3, 1444–1458.

[30] Malý, J., Martio, O.: Lusin's condition (N) and mappings of the class $W^{1,n}$ *J. Reine Angew. Math.* 458 (1995), 19–36.

[31] Martio, O., Rickman, S., Väisälä, J.: Definitions for quasiregular mappings *Ann. Acad. Sci. Fenn. Ser. A I 488* (1971), 1–40.

[32] Martio, O., Väisälä, J.: Elliptic equations and maps of bounded length distortion. *Math. Ann.* 282 (1988), 423–443.

[33] Mattila, P.: *Geometry of sets and measures in Euclidean spaces.* Cambridge Studies in Advanced Mathematics, Vol. 44. Cambridge University Press, Cambridge, 1995.

[34] Monti R.: *Distances, boundaries and surface measures in Carnot-Carathéodory spaces*, PhD thesis 2001.

[35] Nash, J.: C^1 Isometric Imbeddings. *Ann. of Math* 2 (1954), 383–396.

[36] Pankka, P.: Slow quasiregular mappings and universal coverings. *Duke Math. J.* 141 (2008), 293–320.

[37] Ponomarev, S. P.: An example of an $ACTL^p$-homeomorphism which is not Banach absolutely continuous *Dokl. Akad. Nauk SSSR [Soviet Math. Dokl.],201, No. 5* (1971), 1053–1054.

[38] Rudin W.: *Real and Complex Analysis.* Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp. ISBN: 0-07-054234-1

[39] Reshetnyak, Yu. G.: Spatial mappings with bounded distortion. (Russian) *Sibirsk. Mat. Zh.* 8 (1967), 629–658.

[40] Reshetnyak, Yu. G.: Property N for the space mappings of class$W^{1,n}_{\text{loc}}$ *Sibirsk. Mat. Zh.* 28 (1987), 149–153.

[41] Rickman, S.: Path lifting for discrete open mappings *Duke Math. J. 40* (1973), 187–191.

[42] Rickman, S.: *Quasiregular mappings.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 26. Springer-Verlag, Berlin, 1993.

[43] Saks, S.: *Theory of the integral* Monografie Matematyczne Vol. VII. Warszawa-Lwow 1937.

[44] Sternberg, S.: *Lectures on differential geometry.* Second edition. With an appendix by Sternberg and Victor W. Guillemin. Chelsea Publishing Co., New York, 1983.

[45] Varopoulos, N. Th., Saloff-Coste, L., Coulhon, T.: *Analysis and geometry on groups.* Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992.

[46] Whitney, H.: On totally differentiable and smooth functions. *Pacific J. Math.* 1 (1951), 143–159.