PRICING CREDIT DEFAULT SWAPS WITH COUNTERPARTY RISKS

by

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Submitted to the Graduate Faculty of

the Kenneth P. Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2017

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

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A credit default swap, or CDS, is a financial agreement between two parties about an exchange of cash flows that depend on the occurrence of a credit default or in general a credit event. A CDS may terminate earlier than the expiration or the occurrence of the credit event when one party of the contract defaults, this is called counterparty risk. This thesis studies the price of CDS with counterparty risks. The credit default and counterparty risks are modeled by the first arrival times of Poisson processes with stochastic intensities depending on interest rates. The popular CIR and Vasicek models are used for interest rates in this thesis. The prices of CDS are derived as solutions of different partial differential equations with respect to the CIR and Vasicek models, respectively. For the CIR model, the volatility for the interest rate vanishes as interest rate approaches zero. New techniques are introduced here to deal with this degeneracy and cover the full parameter range, thereby allowing the usability of any empirical calibrated CIR models. For the Vasicek model, the allowance of negative interest rate can produce an arbitrarily large discount factor, instead of typically smaller than one; this poses difficulties in mathematical analysis and financial predictions. This thesis solves the mathematical well-posedness problem and more importantly, produces accurate bounds of the CDS price. For the CDS with long time to expiry, the corresponding infinite horizon problems are studied. As time to expiry goes to infinity, the price of CDS being the asymptotic limit of the solution of the infinite horizon problem is verified.

Keywords: Counterparty risk, CIR, Vasicek, negative interest rate, infinite horizon problems.

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1.0 MATHEMATICAL FORMULATION FOR THE PRICE OF CDS

1.1 INTRODUCTION

A swap is a derivative in which two parties exchange cash flows, where the cash flows depend on underlying securities or financial events. Swaps can be used to hedge financial risks such as credit risk, interest rate risk and various kinds of risks due to financial events. In 1981, swaps were first introduced to the public when the world Bank and IBM entered into a swap agreement ([1], 2010). Today, swaps are one of the most heavily and importantly traded contracts in financial market.

There are variety of swaps. Here in this thesis, we study credit default swaps. A credit default swap, or CDS, is a bilateral contract between two parties, buyer and seller, with respect to a credit event. More precisely, the buyer of the CDS pays a periodic fee to the seller of the CDS. If a certain specified credit event occurs, the seller is required to make a compensation to the buyer by means of either physical settlement or cash settlement. For example, Two Sigma Investments (TSI), one of the largest hedge funds in the world, wants to make a profit through a speculative "bet" on stock price of Apple Inc.. In order to achieve this goal, TSI signed a ten-year CDS contract with American International Group, Inc., also known as AIG, one of largest American multinational insurance corporations. Under the CDS contract, TSI makes an annual premium to AIG which is equal to \$200,000. If stock price of Apple is under \$100 a share (currently \$132 a share) during these ten years, AIG need pay TSI \$5 million compensation in return. In order to put the situation that AIG is bankrupt (so is unable to pay) or TSI is bankrupt (so AIG cannot receive the premium), additional amendment should be added.

In the example above, we perceive that a CDS contract is similar to an insurance because the buyer (TSI) pays a premium with protection against the occurrence of a credit event (stock price of Apple under $100), which may have negative impact on the buyer's other financial activities; in return, receives a compensation from the seller (AIG) at the occurrence of such an event. However, each characteristic of CDS compared to the characteristics of insurance produces a gap leading to the conclusion that CDS indeed is not insurance. First, insurance is designed to insure risk that only includes something bad happening or nothing happening. It is unlikely that any measurable benefit will arise from insurance. However, the risk taker may "gamble" on something good happening (make a profit) through CDS contract. In the example above, TSI tries to make a profit through the bet on stock price of Apple going down. From macroeconomics perspective, the CDS notional value was $47 trillion in total by the end of 2007, but the total amount of the assets on which CDSs are based was less than $25 trillion ([2], 2008). Therefore, there was at least $20 trillion in speculative "bets" on the possibility of credit events. Secondly, CDS lacks the element of a requirement of insurable interest which is an essential element of insurance. Specifically, the buyer of CDS does not need to own the underlying security and does not have to suffer any loss form the default event. This is known as a "naked short". In our example, instead of suffering any loss from the drop of Apple 's stock price, TSI will make profit through CDS contract when the stock price is under $100. Moreover, the obligation of payment is triggered upon a credit event, regardless of any loss, which is also different from insurance.

Although CDS is different from insurance, nowadays CDS is one of the most widely used type of credit derivatives and a powerful force in world market. Forms of CDS had been existed since the early 1990s, with early trades carried out by Bankers Trust in 1991 ([3], 2009). The modern CDS was first introduced from J.P. Morgan in 1994, and was widely used after 2003. From International Swaps and Derivatives Association, the amount of CDS was $ 26.3 trillion by the middle of 2010, and was reportedly $25.5 trillion in early 2012. CDS data can be used by financial regulators and the media to monitor how the market views credit risk of any credit event on which a CDS is available; CDS data can also be compared to that provided by the Credit Rating Agencies.

A CDS depends on the underlying credit event, but the credit event is not necessarily related to any party of the CDS contract. Since CDS is a credit derivative contract between two parties, before the expiration or the occurrence of the credit event, the buyer or the seller may default; e.g., the buyer of the CDS is unable to pay the premium or the seller of the CDS is unable to pay the compensation. In this thesis, we take both buyer's and seller's default risks into our consideration. These risks are called counterparty risks.

So far, we have two default risks: reference risk and counterparty risk. Naturally, we use default time to measure default risks, so it is important to calculate default time properly. In general, there are two types of models attempt to describe default time: the structural model and the intensity model. Structural models determine default times by evaluating firms structural data, such as debt and asset values. Merton introduced the first structural model that a firm defaults if its asset is below its outstanding debt ([4], 1974). Then Black and Cox ([5], 1976) used a structural model to describe default time as the moment that firms asset value hits a lower barrier. Thus, structural models link the credit of a firm with its value and debt. Intensity models describe default times by means of exogenous jump processes. Therefore, default is not triggered by firms value but an exogenous component that governed by the parameters inferred from macroeconomic data. In general, a default time is modeled as the first arrival of a Poisson process with variable intensity which depends on macroeconomic data; see, for example, Jarrow and Turnbell ([6], 1995). We shall take the macroeconomic data as the instantaneous interest rate (short rate). That is, the variable intensity of Poisson process is a function of short rate. The two models are connected first by Hu, Jiang, Liang, and Wei ([13], 2012).

A short rate model describes the change of the instantaneous interest rate as a stochastic process. We shall pay attention on two tractable Gaussian models: the Vasicek ([9], 1977) model and the CIR model ([8], 1985). They are regarded as very important equilibrium models. They have the features of mean-reverting ([12], 2008) and affine term structure, essential characteristics that distinguish these two models from other models. Therefore, we formulate CDS under these two widely used short rate models in this thesis.

Vasicek and CIR models are very different, although they are both equilibrium short rate models. The biggest difference is that the Vasicek model allows negative interest rate, whereas the CIR model does not. In the past, Vasicek model was criticized for allowing negative interest rate, but nowadays these opinions should be dropped. In 2014, the European Central Bank, which oversees monetary policy for those countries that use the euro, introduced negative interest rates; Sweden and Switzerland, who do not use the euro also have negative interest rates; Janpan's central bank also allow negative interest rates in 2016. The disadvantage of allowing negative interest rate in the Vasicek model becomes an advantage. Thus, it is meaningful to use Vasicek model to price CDS. After taking both CIR and Vasicek models into our consideration, the study in this thesis becomes more complete and in-depth.

In this thesis, we introduce a pricing model for CDS with counterparty risks. We model default times by intensity model and model the intensity of the corresponding Poisson process by known functions of short rates, respectively. We model interest rates by the CIR and the Vasicek models. As a result, the prices of CDS are derived as the solutions of two different partial differential equations (PDE). For the resulting PDEs, we shall investigate the existence, uniqueness and properties of their solutions. It is important to point out that Hu, Jiang, Liang, and Wei ([13], 2012) have already used the CIR model to price the CDS. However, for the CIR model, they need a restriction on the range of the parameters, which does not fall into the empirical calibration in certain cases; see, for example, Peng ([18], 2016). To mend this deficiency, we provide a theory that covers the full parameter range. In order to remove the restriction on the range of the parameters, we need introduce brand new techniques. For the Vasicek model, large negative interest rate causes technical difficulties for mathematical analysis, although it does not happen in reality. Therefore, our analysis under Vasicek model becomes more of a mathematical interest. Moreover, we study the corresponding infinite horizon problems and connect their solutions with the asymptotic behavior of the price of CDS as time to expiry tends to infinity.

This thesis is divided into 6 chapters: In the remaining of this chapter, we derive mathematical formulations for the price of CDS. In chapters 2 and 3, we study the pricing problem under the CIR model. In chapters 4 and 5, we study the pricing problem under the Vasicek model. Chapter 6 is our conclusion. It is important for me to attest here that most results of chapters 2 and 3 are joint work with Peng He. As a result, chapters 2 and 3 contain a large portion that is similar to that of Peng's thesis ([18], 2016).

1.2 A STANDARD CDS

A swap is a derivative in which two counterparties exchange cash flows of one party's financial instrument for those of the other party's financial instrument. There are variety of swaps. Here in this thesis, we study credit default swaps. A credit default swap, or CDS, is a financial agreement between two parties, A and B, about an exchange of cash flows that depend on the occurrence of a credit default or in general a credit event. More precisely, the cash flows could be follows:

- the party A makes a series of payments at a predetermined rate to party B before the occurrence of the credit event or before the expiration date, whichever is earlier.
- the party B pays the party A a fixed compensation at the occurrence of the credit event, if the occurrence is on or before the expiry.

The credit events in our consideration are indeed credit defaults, such as an overdue mortgage payment. Thus, the underlying swap is called CDS. We call A the buyer of the CDS and B the seller of the CDS. Hence, A buys an "insurance" from B for the occurrence of a credit default or event. The following example illustrates a typical CDS in our study.

On 2017/01/01, the following is signed between two parties:

(1) Agree on a certain indubious definition of the occurrence of the credit event, e.g., the exchange rate of Chinese Yuan to US Dollar is at or above 6.6 (CNY/USD).
(2) A pays B $1 daily up to 2018/01/01 or up to the time when the event occurs, whenever is earlier.
(3) B pays A a compensation $200 at the time when the event occurs, provided that the time is on or before 2018/01/01.

Suppose today is 2017/09/07, the exchange rate is 6.55 (CNY/USD). It is reasonable to believe that the contract favors the buyer. How to quantify this favor? More specifically, we want to know the price of CDS. To do this, we first introduce the following notation:

K: the agreed CDS compensation (strike);
q: the agreed CDS premium (spread= $\frac{q}{K}$);

T: expiration date;

τ: a stoppping time to designate the occurrence of the underlying credit event.

Also, we denote $\min\{x, y\}$ by $x \wedge y$. With these mathematical notation, we can describe a CDS contract in a more precise manner as follows. The contract bonds the buyer and the seller by the following provisions:

1. *The contract terminates after $\tau \wedge T$.*

2. *Before $\tau \wedge T$, the buyer pays the seller the insurance premium being a cash flow of continuous rate q (\$/year).*

3. *If $\tau \leqslant T$, the seller pays the buyer at time τ the insurance compensation being a lump sum of K (\$).*

1.3 CDS WITH COUNTERPARTY RISKS

Due to financial difficulties, either the buyer or the seller may default on certain circumstance, i.e, unable to fulfill the obligation. For example, A is unable to pay the premium or is bankrupt due to some other financial activities; B is unable to pay the compensation. Therefore, the contract may terminate before the regular termination of the CDS. The CDS contract should include this kind of risk, which is called a counterparty risk. To take the counterparty risk into consideration, we need to specify how to settle down the contract at time when one of the parties is unable to comply with the contract. For this reason and also for simplicity, we add the following amendment to the above CDS contract.

4 (a). Agree on a precise definition of what is called A or B default.

(b). If A or B defaults before the regular termination of the CDS, the contract terminates at the default time with no further rights and obligations from each other.

Other amendments can also be considered. For example, 4(b) can be replaced by a provision that allows the default party to sell the contract in auction at its default time to a third party. Different provisions lead to different CDS models. For definiteness here in this thesis, we consider the above amendment 4(a)(b). To be more precise, we introduce stopping times τ_1 and τ_2 as follows:

τ_1: default time of the seller;

τ_2: default time of the buyer.

Now the CDS with counterparty risk that is considered in this study can be described in a mathematical manner as follows:

<u>The CDS Contract</u>

Let τ, τ_1, and τ_2 be certain precisely defined stopping times that agreed by both parties.

1. *This contract terminates after $\tau \wedge \tau_1 \wedge \tau_2 \wedge T$.*
2. *Before $\tau \wedge \tau_1 \wedge \tau_2 \wedge T$, the buyer pays the seller a cash flow of continuous rate q.*
3. *If $\tau \leqslant T \wedge \tau_1 \wedge \tau_2$, the seller pays the buyer at time τ a lump sum of K.*
4. *If $\tau_1 \wedge \tau_2 < \tau \wedge T$, the contract terminates at $\tau_1 \wedge \tau_2$, with no further rights and obligations between the buyer and the seller.*

1.4 MODELING DEFAULT TIMES BY INTENSITIES

Recall that τ, τ_1 and τ_2 are stoppping times of the designated occurrence of the underlying credit event, default time of the seller, and default time of the buyer, respectively. In this thesis, we model them by an intensity framework. To be more precise, we model τ, τ_1 and τ_2 as the first arrival times of Poisson processes with intensities $\{\lambda_t\}_{t\geq 0}$, $\{\lambda_{1t}\}_{t\geq 0}$ and $\{\lambda_{2t}\}_{t\geq 0}$, respectively. Mathematically, τ, τ_1 and τ_2 are stoppping times on a probability space $\big\{\Omega, \{\mathcal{F}_t \times G_t\}_{t\geq 0}, \mathbb{P}\big\}$, where $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{G_t\}_{t\geq 0}$ are filtrations, and $\mathbb{P}$ is the probability measure. Here $\{\lambda_t\}$, $\{\lambda_{1t}\}$ and $\{\lambda_{2t}\}$ are adapted to $\{\mathcal{F}_t\}$, and the conditional probabilities of relevant events are

$$\begin{aligned} \mathbb{P}(\tau \in (t-dt,t] \,\big|\, \mathcal{F}_t, \tau > s) &= \lambda_t e^{-\int_s^t \lambda_\theta \, d\theta}\, dt \qquad \forall\, t > s \geqslant 0, & (1.4.1)\\ \mathbb{P}(\tau_i \in (t-dt,t] \,\big|\, \mathcal{F}_t, \tau_i > s) &= \lambda_{it} e^{-\int_s^t \lambda_{i\theta} \, d\theta}\, dt \qquad \forall\, t > s \geqslant 0, i=1,2. & (1.4.2) \end{aligned}$$

After integrating both sides of (1.4.1) and (1.4.2), we have:

$$\begin{aligned} \mathbb{P}(\tau > t \mid \mathcal{F}_t, \tau > s) &= e^{-\int_s^t \lambda_\theta d\theta} \qquad \forall\, t \geqslant s \geqslant 0, & (1.4.3)\\ \mathbb{P}(\tau_i > t \mid \mathcal{F}_t, \tau_i > s) &= e^{-\int_s^t \lambda_{i\theta} d\theta} \qquad \forall\, t \geqslant s \geqslant 0, i=1,2. & (1.4.4) \end{aligned}$$

For simplicity, we assume that τ, τ_1 and τ_2 are independent in the sense that

$$\begin{aligned} \mathbb{P}(\tau > t,\ \tau_1 > t,\ \tau_2 > t \,\big|\tau \wedge \tau_1 \wedge \tau_2 > s) &= \mathbb{P}(\tau > t \,\big|\tau > s)\mathbb{P}(\tau_1 > t \,\big|\tau_1 > s)\mathbb{P}(\tau_2 > t \,\big|\tau_2 > s)\\ &= e^{-\int_0^t (\lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta}) d\theta}. & (1.4.5) \end{aligned}$$

Let $\{r_t\}_{t \geq 0}$ be the short interest rate and assume that the discount factor is $e^{-\int_0^t r_\theta d\theta}$. Further assume that $\{r_t\}$ is adapted to $\{\mathcal{F}_t\}$. From buyer's point of view, the present value of all payments received from the seller is:

$$p = K e^{-\int_0^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_1 \wedge \tau_2 \wedge T\}} - q \int_0^{\tau \wedge \tau_1 \wedge \tau_2 \wedge T} e^{-\int_0^t r_\theta d\theta}\, dt. \tag{1.4.6}$$

Naturally, we define buyer's value of the CDS as the expection of the present value of payments received from the seller. Setting $\Omega_0 = \{\tau \wedge \tau_1 \wedge \tau_2 > 0\}$, we then define the value of CDS by

$$\begin{aligned} & u := \mathbb{E}\left[p \,\big|\, \Omega_0\right]\\ =& K\mathbb{E}\left[\int_0^T \mathbb{P}\left(\tau \in [s, s+ds), \tau_1 \wedge \tau_2 > s \,\Big|\, \mathcal{F}_s,\, \Omega_0\right) e^{-\int_0^s r_\theta d\theta} ds \,\Big|\, \Omega_0\right]\\ & - q\mathbb{E}\left[\int_0^\infty \mathbb{P}\left(\tau \wedge \tau_1 \wedge \tau_2 \in [s, s+ds) \,\Big|\, \mathcal{F}_s,\, \Omega_0\right)\left(\int_0^{s \wedge T} e^{-\int_0^t r_\theta d\theta} dt\right) ds \,\Big|\, \Omega_0\right]\\ =& \mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta + \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta}) d\theta}\Big\{K\lambda_s - q\Big\} ds \,\Big|\, \Omega_0\right], & (1.4.7) \end{aligned}$$

where the last equation is obtained from (1.4.1)-(1.4.5) and an integration by parts.

In this thesis, we study the case that the intensites depend only on the short interest rate $\{r_t\}$. That is, we assume that there exist functions $\Lambda(\cdot)$, $\Lambda_1(\cdot)$,and $\Lambda_2(\cdot)$ such that

$$\lambda_t = \Lambda(r_t), \quad \lambda_{1t} = \Lambda_1(r_t), \quad \lambda_{2t} = \Lambda_2(r_t). \tag{1.4.8}$$

To evaluate u, we need to model the interest rate. We shall focus on two interest rate models, as discussed in the next section.

1.5 STOCHASTIC MODELS FOR INTEREST RATE

There are many mathematical models for interest rate, e.g., the Vasicek ([9], 1977) model, the Ho-Lee model ([14], 1986), the Hull-White model ([15], 1996), the CIR model ([8], 1985), the Heath-Jarrow-Morton model ([10], 1990), etc. Among many interest interest rate models, we shall use CIR and Vasicek models because they are important and widely used short rate models. The first advantage is that the value functions of these two models are solutions of some certain partial differential equations, which can be numerically computed. Another advantage of Vasicek model and CIR model is that they are mean-reverting process ([12], 2008), an essential characteristic of the interest rate that sets these two models apart from other models. Furthermore, both CIR and Vasicek models are canonical examples of the affine term structure model that has closed formulas for many important quantites. Therefore, we formulate our CDS model under these two interest rate models in this thesis.

1.5.1 The CIR Model

The CIR [Cox-Ingersoll-Ross, 1985] model specifies that the instantaneous interest rate r_t at time t follows the stochastic differential equation:

$$dr_t = \big(\kappa - \beta\, r_t\big)\, dt + \sigma\, \sqrt{r_t}\, dW_t,$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion and κ, β, σ are positive constants. The constant β, $\frac{\kappa}{\beta}$, and σ are referred to as the speed of adjustment, the mean and volatility, respectively.

The drift term $(\kappa - \beta\, r_t)$ represents the expected instantaneous change in the interest rate at time t. Since the standard deviation factor is $\sigma\sqrt{r_t}$, the interest rate is always non-negative. More generally, when the interest rate is close to zero, $\sigma\sqrt{r_t}$ becomes very small, which makes the stochastic process be dominated by $\left(\kappa - \beta\, r_t\right) dt$. In other words, the interest rate will move towards $\frac{\kappa}{\beta}$.

The interest rate will never touch zero when $2\kappa \geq \sigma^2$. However, when $0 < \kappa < \frac{\sigma^2}{2}$, there is a positive probability that the interest rate becomes zero. To continue the process after interest rate becomes zero, we modify the CIR model by the following:

$$dr_t = (\kappa - \beta r_t)dt + \sigma\sqrt{\max\{r_t, 0\}}dW_t, \quad r_0 \geq 0. \tag{1.5.1}$$

Then (1.5.1) is well-defined in the full parameter range, i.e, $\kappa > 0$, $\beta > 0$, and $\sigma > 0$. If $r_0 > 0$, one can show that the process satisfies the following: for each $t > 0$,

$$\mathbb{P}(r_t > 0) = 1 \;\text{ and }\; \begin{cases} \mathbb{P}(\min\limits_{0\leq s\leq t} r_s = 0) = 0 & \text{if} \quad 2\kappa \geq \sigma^2, \\ \\ \mathbb{P}(\min\limits_{0\leq s\leq t} r_s = 0) > 0 & \text{if} \quad 0 < 2\kappa < \sigma^2. \end{cases}$$

1.5.2 The Vasicek Model

The Vasicek [Oldrich Vasicek, 1977] model specifies that the instantaneous interest rate r_t at time t follows the stochastic differential equation:

$$dr_t = (\kappa - \beta\, r_t)\, dt + \sigma\, dW_t \tag{1.5.2}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion and κ, β, σ are positive constants. The drift term, $(\kappa - \beta\, r_t)\, dt$ is the same as in the CIR model, which ensures to generate a mean-reverting process.

By multiplying r_t by the integrating factor $e^{\beta t}$ and applying the Itô product rule on $r_t e^{\beta t}$, we can solve the stochastic differential equation (1.5.2) and obtain the explicit formula:

$$r_t = r_0 e^{-\beta t} + \frac{\kappa}{\beta}(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)} dW_s. \tag{1.5.3}$$

Therefore, the interest rate r_t follows a Normal distribution:

$$r_t \sim \mathbf{N}\Big(r_0 e^{-\beta t} + \frac{\kappa}{\beta}(1 - e^{-\beta t}),\ \ \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})\Big).$$

Since normally distributed random variable can become negative with positive probability, r_t can be negative, which is considered as a weakness of the Vasicek model. However, nowadays many countries like Sweden, Switzerland, and Japan have already introduced negative interest rates, so it is meaningful to take the Vasicek model into our consideration. Mathematically, the existence of negativity of interest rate makes all analysis remarkably different from the analysis under CIR model.

1.6 MATHEMATICAL FORMULATIONS OF CDS

Under the intensity model for the default times, the CIR or Vasicek short rate model, and the connection (1.4.8) between intensity and interest rate, the value in (1.4.7) is a function of r_0 and T. Hence, we define the value of CDS of time-to-expiry T and interest rate $r_0 = r$ by

$$u(r,T) := \mathbb{E}\left[\int_0^T e^{-\int_0^s \big(r_\theta + \Lambda(r_\theta) + \Lambda_1(r_\theta) + \Lambda_2(r_\theta)\big)d\theta} \Big\{K\lambda_s - q\Big\} ds \,\Big|\, r_0 = r,\ \Omega_0\right]. \tag{1.6.1}$$

1.6.1 Under CIR Model

Under the CIR model, from (1.5.1), (1.6.1) and the famous Feynman-Kac formula [17] we can deduce that u is the solution of the following PDE problem:

$$\Big(\frac{\partial}{\partial T} + \mathcal{L}_c + \lambda(r)\Big)u = K\Lambda(r) - q \quad \text{in } (0,\infty)^2,$$

where

$$\mathcal{L}_c = -\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r, \quad \lambda(r) = \Lambda(r) + \Lambda_1(r) + \Lambda_2(r).$$

When $\kappa > \frac{\sigma^2}{2}$, $\beta > 0$, $\sigma > 0$, the problem was solved by Bei Hu, Lishang Jiang, Jin Liang and Wei Wei in [13] under the following "boundary condition":

$$u \in L^\infty((0,\infty)^2). \tag{1.6.2}$$

In this thesis, we extend the theory of Hu-Jiang-Liang-Wei in [13] to general parameter range: $\kappa > 0$, $\beta > 0$, $\sigma > 0$. Instead of using (1.6.2), we impose the following "boundary condition":

$$\frac{\partial u}{\partial r} \in L^\infty((0,\infty)^2). \tag{1.6.3}$$

Thus, we study the following problem: Find u such that

$$\begin{cases} \Big(\frac{\partial}{\partial T} + \mathcal{L}_c + \lambda(r)\Big)u = K\Lambda(r) - q & \text{in } (0,\infty)^2, \\ u(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u}{\partial r} \in L^\infty((0,\infty)^2). \end{cases} \tag{1.6.4}$$

We shall show the well-posedness of this problem in Chapter 2.

1.6.2 Under Vasicek Model

Under the Vasicek model, from (1.5.2), (1.6.1) and the famous Feynman-Kac formula [17] we can deduce that u is the solution of the following:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L}_v + \lambda(r)\right)u = K\Lambda(r) - q & \text{in } \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = 0 & \text{in } \mathbb{R}, \end{cases} \tag{1.6.5}$$

where

$$\mathcal{L}_v = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r, \quad \lambda(r) = \Lambda(r) + \Lambda_1(r) + \Lambda_2(r).$$

Under Vasicek model, the operator $\mathcal{L} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r$, which is different from the operator under CIR model. Another difference is that the domain of r becomes $(-\infty, \infty)$ instead of $(0, \infty)$. To study CDS under Vasicek model, we mainly have the following difficulty:

- Due to allowing the negativity and unboundedness of r in the elliptic operator

  $$\mathcal{L} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r \ ,$$

  how to impose the "boundary condition", i.e., the asymptotic behavior of the solution as $|r| \to \infty$?

For the well-posedness of the PDE problem we impose the following standard boundary condition for heat equation:

$$|u(r, T)| \le e^{A(N)(r^2+1)} \ \forall r \in \mathbb{R}, \ T \in [0, N], \ N > 0, \tag{1.6.6}$$

where $A(\cdot)$ is any increasing function defined on $[0, \infty)$.

We shall show the well-posedness of (1.6.5) supplemented with (1.6.6) in Chapter 4.

1.6.3 The Fair Spread

Notice that the solution of problem (1.6.4) or (1.6.5) can be decomposed as $u = Ku_1 - qu_2$, $i = 1, 2$, where u_i is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right) u_i = f_i \\ \\ u_i(\cdot, 0) = 0 \end{cases} \tag{1.6.7}$$

with $f_1 = \Lambda$ and $f_2 = 1$, and $\mathcal{L}$ is the corresponding partial differential operator for the CIR model and Vasicek model, respectively.

Suppose the contract is signed at time $t = 0$. As a fair contract, its value should be zero. Assume that this is the case. Then for fixed date of expiry, spread at $t = 0$ is

$$S_0 := \frac{q}{K} := \frac{u_1(r_0, T)}{u_2(r_0, T)}.$$

Therefore, at time $t \in (0, T)$, the value of the contract is

$$u(r_t, T - t) = K\big[u_1(r_t, T - t) - S_0\, u_2(r_t, T - t)\big].$$

2.0 WELL-POSEDNESS OF THE PDE PROBLEM UNDER CIR MODEL

In this chapter, we study the price of CDS under an intensity model for the credit event where the intensity depends only on short interest rate modeled by the CIR model. We use one theorem to establishes the existence and uniqueness of the solution of problem (1.6.4). In section one we present the main result. In section two we present the existence of the solution. In section three we present the uniqueness of the solution.

2.1 MAIN RESULT

The main result can be stated as follows.

Theorem 2.1.1. (**Existence and Uniqueness**) *Let σ, κ and β be positive constans and $\mathcal{L}$ be the CIR differential operator defined by*

$$\mathcal{L}u = \Big(- \frac{\sigma^2 r}{2} \frac{\partial^2}{\partial r^2} - (\kappa - \beta r) \frac{\partial}{\partial r} + r \Big) u.$$

Assume that

$$\Lambda(r) = ar + b, \quad \Lambda_1(r) = hH(r - B), \quad \Lambda(r) = cr + d, \tag{2.1.1}$$

where a, b, c, d, h and B are positive constants. Let $\lambda(\cdot)$ and $f(\cdot)$ be functions on $(0, \infty)$ that satisfy

$$\lambda(r) := \Lambda(r) + \Lambda_1(r) + \Lambda_2(r), \quad f(r) = K\Lambda(r) - q. \tag{2.1.2}$$

Then the following problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right)u = f & \text{in } (0,\infty)^2, \\ u(\cdot, 0) = 0 & \text{on } (0,\infty), \\ \frac{\partial u}{\partial r} \in L^\infty((0,\infty)^2), \end{cases} \tag{2.1.3}$$

admits an unique solution in

$$\mathbb{X} = C^{1,\frac{1}{2}}((0,\infty)\times[0,\infty)) \cap C^\infty((0,B]\times[0,\infty)) \cap C^\infty([B,\infty)\times[0,\infty)). \tag{2.1.4}$$

Remark: *The problem* (2.1.3) *is essentially different from the problem solved by Bei Hu, Lishang Jiang, Jin Liang and Wei Wei in [13], although the basic formulations are similar. In [13], it is assumed that* $\Lambda_2(\cdot) \equiv 0$ *so the buyer never default. Another difference is in [13] it is assumed that* $2\kappa > \sigma^2$*, but here we remove this condition. Our theory covers the full parameter range. We actually extend the theory of Hu-Jiang-Liang-Wei in [13] to general parameter range:* $\kappa > 0$*,* $\beta > 0$*,* $\sigma > 0$*.*

In this chapter and next chapter, we assume κ, β and σ are positive constants.

2.2 EXISTENCE

In the sequel, we always assume that conditions and notations of Theorem 2.1.1 hold. In this section, we establish the existence of a solution of problem (2.1.3). Since $\mathcal{L}$ degenerates at $r = 0$ and both λ and f are unbounded as $r \to \infty$, we begin with approximating problem (2.1.3) by the following truncation:

$$
\begin{cases}
\left(\frac{\partial}{\partial T}+\mathcal{L}+\lambda\right)u_n = f & \text{in } (\frac{1}{n},n)\times(0,n^2], \\
u_n(\cdot,0)=0 & \text{on } (\frac{1}{n},n), \\
\frac{\partial u_n}{\partial r}=0 & \text{on } \{\frac{1}{n},n\}\times(0,n^2].
\end{cases}
\tag{2.2.1}
$$

Lemma 2.2.1. *For every $n > B$ and $\alpha \in (0,1)$, problem (2.2.1) admits a unique solution in*

$$
W_p^{2,1}\Big((\frac{1}{n},n)\times(0,n^2]\Big)\cap C^{2+\alpha,1+\frac{\alpha}{2}}\Big((\frac{1}{n},B]\times(0,n^2]\Big)\cap C^{2+\alpha,1+\frac{\alpha}{2}}\Big([B,n)\times(0,n^2]\Big).
$$

Problem (2.2.1) is a standard Cauchy boundary value problem. The assertion of the Lemma 2.2.1 follows from a standard theory of parabolic equations; see, for example, Friedman [19]. We omit the proof here.

Next we provide some a priori estimates, which will allow us to obtain a limit when we send n to infinity.

Lemma 2.2.2. (L^∞**-estimate of** u_n) *There exists a positive constant C, which does not depend on n, such that*

$$
|u_n| \le C. \tag{2.2.2}
$$

Proof. Recall that K in (2.1.2) is the agreed CDS compensation. Since $K>0$ and

$$
\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda\Big)K \ge \lambda K \ge K\Lambda \ge K\Lambda - q = f,
$$

K is a super-solution of problem (2.2.1).

Set $\Lambda_0(r)=r$ and $\alpha = \|\frac{f}{\Lambda_0+\lambda}\|_{L^\infty((0,\infty))}$. Since $-\alpha<0$ and

$$
\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda\Big)(-\alpha) = (\Lambda_0+\lambda)(-\alpha) \le (\Lambda_0+\lambda)\frac{f}{\Lambda_0+\lambda} = f,
$$

$-\alpha$ is a sub-solution of problem (2.2.1).

By comparison principle, $u_n \le K$ and $u_n \ge -\alpha$. Therefore, $|u_n| \le C := \max\{K,\alpha\}$. $\square$

Lemma 2.2.3. (L^∞**-estimate of** $\frac{\partial u_n}{\partial T}$) *There exist positive constants c_1 and c_2, which do not depend on n, such that*

$$\left|\frac{\partial u_n}{\partial T}\right| \le c_1 + c_2 r. \tag{2.2.3}$$

Proof. Denote $v_n = \frac{\partial u_n}{\partial T}$. Then v_n satisfies the following:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right) v_n = 0 & \text{in } \left(\frac{1}{n}, n\right) \times (0, n^2], \\ v_n(\cdot, 0) = f & \text{on } \left(\frac{1}{n}, n\right), \\ \frac{\partial v_n}{\partial r} = 0 & \text{on } \left\{\frac{1}{n}, n\right\} \times (0, n^2]. \end{cases} \tag{2.2.4}$$

Set

$$w(r) = \sqrt{A^2 + \left(r - \frac{1}{n}\right)^2} \quad \forall r > 0, \text{ where } A = \sqrt{\kappa + \frac{\sigma^2}{2}}.$$

Let

$$\bar{v}(r) := \left\|\frac{f}{w}\right\|_{L^\infty((0,\infty))} w(r).$$

Then one can varify $\bar{v}$ satisfies the following:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right) \bar{v} \ge 0 & \text{in } \left(\frac{1}{n}, n\right) \times (0, n^2], \\ \bar{v}(\cdot, 0) \ge f & \text{on } \left(\frac{1}{n}, n\right), \\ \frac{\partial \bar{v}}{\partial \mathbf{n}} \ge 0 & \text{on } \left\{\frac{1}{n}, n\right\} \times (0, n^2]. \end{cases}$$

Hence, $\bar{v}$ is a super-solution of problem (2.2.4). Similarly, $-\bar{v}$ is a sub-solution. By comparison principle, $v_n \le \bar{v}$ and $v_n \ge -\bar{v}$, i.e., $|v_n| \le \bar{v}$.

Since $\sqrt{A^2 + \left(r - \frac{1}{n}\right)^2} \le A + (r - \frac{1}{n})$, we can easily find positive constants c_1 and c_2 such that

$$\left|\frac{\partial u_n}{\partial T}\right| \le \bar{v} \le c_1 + c_2 r.$$

□

Lemma 2.2.4. (L^∞**-estimate of** $\frac{\partial u_n}{\partial r}$) *There exists a positive constant* c_3*, which does not depend on* n*, such that*

$$\left|\frac{\partial u_n}{\partial r}\right| \le c_3. \tag{2.2.5}$$

Proof. By definition of $\mathcal{L}$, we can obtain the following equation:

$$-\frac{\sigma^2 r}{2}\frac{\partial^2 u_n}{\partial r^2} - (\kappa - \beta r)\frac{\partial u_n}{\partial r} = f - \frac{\partial u_n}{\partial T} - \big[r + \lambda(r)\big]u_n. \tag{2.2.6}$$

Set $\mu := \frac{2\kappa}{\sigma^2}$, $\nu := \frac{2\beta}{\sigma^2}$ and multiply the integrating factor $r^{\mu-1}e^{-\nu r}$ on both sides of (2.2.6) to obtain

$$-\Big(r^\mu e^{-\nu r}\frac{\partial u_n}{\partial r}\Big)_r = r^{\mu-1}e^{-\nu r}F, \tag{2.2.7}$$

where $F(r,T) = \frac{2}{\sigma^2}\big[f(r) - \frac{\partial u_n(r,T)}{\partial T} - \big(r+\lambda(r)\big)u_n(r,T)\big]$.

To estimate the bound of $\frac{\partial u_n}{\partial r}$, we note from Lemma 2.2.2 and Lemma 2.2.3 that there exists a positive C_1 such that

$$|F(r,T)| \le \frac{2}{\sigma^2}\Big[|f(r)| + \Big|\frac{\partial u_n(r,T)}{\partial T}\Big| + \big(r+\lambda(r)\big)|u_n(r,T)|\Big] \le C_1(1+r)\ \forall (r,T) \in \Big[\frac{1}{n}, n\Big] \times [0, n^2].$$

To continue estimating $\frac{\partial u_n}{\partial r}$, we consider the following two cases:

(1) The case $r \in (\frac{1}{n}, 1 + \frac{2\mu}{\nu}]$.

Integrating both sides of (2.2.7) over $[\frac{1}{n}, r]$, we obtain

$$|r^\mu e^{-\nu r}\tfrac{\partial u_n}{\partial r}| \le \textstyle\int_{\frac{1}{n}}^r \rho^{\mu-1}e^{-\nu\rho}|F|\,d\rho \le C_1\int_{\frac{1}{n}}^r(\rho^{\mu-1}+\rho^\mu)d\rho \le C_1\int_0^r(\rho^{\mu-1}+\rho^\mu)d\rho = C_1(\frac{r^\mu}{\mu} + \frac{r^{\mu+1}}{\mu+1}).$$

Thus,

$$\Big|\frac{\partial u_n}{\partial r}\Big| \le C_1 r^{-\mu}e^{\nu r}\Big(\frac{r^\mu}{\mu} + \frac{r^{\mu+1}}{\mu+1}\Big) \le C_1 e^{\nu(1+\frac{2\mu}{\nu})}\Big(\frac{1}{\mu} + \frac{1+\frac{2\mu}{\nu}}{\mu+1}\Big) \quad \forall r \in \Big(\frac{1}{n}, 1 + \frac{2\mu}{\nu}\Big].$$

(2) The case $r \in (1 + \frac{2\mu}{\nu}, n)$**.**

Integrating both sides of (2.2.7) over $[r, n]$ gives

$$\Big|r^\mu e^{-\nu r}\frac{\partial u_n}{\partial r}\Big| \le \int_r^n \rho^{\mu-1}e^{-\nu\rho}|F|\,d\rho \le 2C_1\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho. \tag{2.2.8}$$

Applying integrate by parts to $\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho$, we obtain

$$\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho = \frac{1}{\nu}e^{-\nu r}r^\mu + \frac{\mu}{\nu}\int_r^\infty e^{-\nu\rho}\rho^{\mu-1}d\rho \le \frac{1}{\nu}e^{-\nu r}r^\mu + \frac{\mu}{\nu r}\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho.$$

Thus, $\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho \le \frac{r^\mu e^{-\nu r}}{\nu - \frac{\mu}{r}}$.

From (2.2.8), we obtain

$$\Big| r^\mu e^{-\nu r}\frac{\partial u_n}{\partial r}\Big| \le 2C_1\int_r^\infty e^{-\nu\rho}\rho^\mu d\rho \le 2C_1\frac{r^\mu e^{-\nu r}}{\nu - \frac{\mu}{r}}.$$

Thus,

$$\Big|\frac{\partial u_n}{\partial r}\Big| \le \frac{2C_1}{\nu - \frac{\mu}{r}} \le \frac{4C_1}{\nu} \quad \forall r \in \Big(1 + \frac{2\mu}{\nu}, n\Big).$$

Combining these two cases, we complete the proof. □

Lemma 2.2.5. *Problem* (2.1.3) *admits at least one solution in* $\mathbb{X}$*.*

Proof. Fix $p > 1$. For each $\delta \in (0,1)$ and $n > \frac{2}{\delta}$, by interior estimate for parabolic equations [22], the solution u_n of problem (2.2.1) satisfies

$$\|u_n\|_{W_p^{2,1}([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}])} \le C(\delta,p)\big\{\|f\|_{L^p([\frac{\delta}{2},\frac{2}{\delta}])} + \|u_n\|_{L^\infty([\frac{\delta}{2},\frac{2}{\delta}]\times[0,\frac{1}{\delta^2}])}\big\} \le \tilde{C}(\delta,p),$$

where $\tilde{C}(\delta,p)$ is a positive constant depends on δ and p.

Since

$$W_p^{2,1} \text{ is compactly embedded in } C^{1,\frac{1}{2}},$$

by using a diagonal process, we can find a function u and a subsequence $\{n_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} n_k = \infty$ and

$$\lim_{k\to\infty}\|u_{n_k} - u\|_{C^{1,\frac{1}{2}}([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}])} = 0 \quad \forall\ \delta \in (0,1).$$

From the equation of u_{n_k}, we find that u satisfies (2.1.3). In addition, from Lemma 2.2.4, we can obtain $u_r \in L^\infty((0,\infty)^2)$.

Therefore, u is a solution of problem (2.1.3). □

2.3 UNIQUENESS

In this section, we establish the uniqueness of a solution of problem (2.1.3). The uniqueness of the solution is proven by using a comparison principle and constructing an auxiliary function. We begin with introducing an auxiliary function by Laplace transform.

Lemma 2.3.1. *For $i = 1, 2$, the following are linearly independent solutions of $(\mathcal{L} + \Lambda)\varphi = 0$ in $(0, \infty)$:*

$$\varphi_i(r) = \frac{\Gamma(2 - \alpha_1 - \alpha_2)}{2\pi i} \int_{C_i} (s - \lambda_1)^{\alpha_1 - 1}(s - \lambda_2)^{\alpha_2 - 1} e^{sr}\, ds \quad \forall r > 0, \tag{2.3.1}$$

where C_i is a contour on a complex plane starts from $-\infty - \mathbf{i}$, goes horizontally right to $(\lambda_i + \frac{\lambda_2 - \lambda_1}{2}) - \mathbf{i}$, then goes vertically up to $(\lambda_i + \frac{\lambda_2 - \lambda_1}{2}) + \mathbf{i}$, then goes horizontally left to $-\infty + \mathbf{i}$. $\Gamma(z)$ stands for the Gamma function and $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ are

$$\begin{cases} \lambda_1 = \frac{\beta - \sqrt{\beta^2 + 2(a+1)\sigma^2}}{\sigma^2}, \\ \lambda_2 = \frac{\beta + \sqrt{\beta^2 + 2(a+1)\sigma^2}}{\sigma^2}, \\ \alpha_1 = \frac{1}{\lambda_2 - \lambda_1}[\frac{2b}{\sigma^2} - \frac{2\kappa}{\sigma^2}\lambda_1], \\ \alpha_2 = \frac{1}{\lambda_1 - \lambda_2}[\frac{2b}{\sigma^2} - \frac{2\kappa}{\sigma^2}\lambda_2]. \end{cases} \tag{2.3.2}$$

Proof. Here for simplicity of presentation, we assume that $1 - \alpha_1 - \alpha_2$ is not a negative integer.

Note that λ_1 and λ_2 are roots of $\frac{\sigma^2}{2}\lambda^2 - \beta\lambda - (a+1) = 0$. Hence

$$\frac{\sigma^2}{2}\lambda^2 - \beta\lambda - (a+1) = \frac{\sigma^2}{2}(\lambda - \lambda_1)(\lambda - \lambda_2).$$

Denote $\mu := \alpha_1 + \alpha_2 = \frac{2\kappa}{\sigma^2}$. Then

$$\begin{aligned} \frac{2\pi i}{\Gamma(2-\mu)}(\mathcal{L} + \lambda)\varphi_i(r) &= -\frac{\sigma^2}{2} r \int_{C_i} s^2 (s - \lambda_1)^{\alpha_1 - 1}(s - \lambda_2)^{\alpha_2 - 1} e^{sr}\, ds + (\kappa - \beta r) \int_{C_i} s(s - \lambda_1)^{\alpha_1 - 1} \\ &\quad (s - \lambda_2)^{\alpha_2 - 1} e^{sr}\, ds - [(a+1)r + b] \int_{C_i} (s - \lambda_1)^{\alpha_1 - 1}(s - \lambda_2)^{\alpha_2 - 1} e^{sr}\, ds. \end{aligned}$$

After the simplification of above expression and integration by parts,we obtain :

$$\begin{aligned}
\frac{2\pi i}{\Gamma(2-\mu)}(\mathcal{L}+\lambda)\varphi_i(r) &= -\int_{C_i}\big[(\frac{\sigma^2}{2}s^2-\beta s-(a+1))r+(\kappa s-b)\big](s-\lambda_1)^{\alpha_1-1}(s-\lambda_2)^{\alpha_2-1}e^{sr}\ ds \\
&= \int_{C_i}\frac{\sigma^2}{2}(s-\lambda_1)^{\alpha_1}(s-\lambda_2)^{\alpha_2}\ d\ e^{sr}+\int_{C_i}(\kappa s-b)(s-\lambda_1)^{\alpha_1-1}(s-\lambda_2)^{\alpha_2-1} \\
&\quad e^{sr}\ ds \\
&= \int_{C_i}(s-\lambda_1)^{\alpha_1-1}(s-\lambda_2)^{\alpha_2-1}e^{sr}\big\{-\frac{\sigma^2}{2}[(\alpha_1+\alpha_2)s-(\alpha_1\lambda_2+\alpha_2\lambda_1)]+ \\
&\quad \kappa s-b\big\}\ ds \\
&= 0.
\end{aligned}$$

Since the definition of (α_1,α_2) implies that

$$\begin{cases} -\frac{\sigma^2}{2}(\alpha_1+\alpha_2)+\kappa=0, \\ \frac{\sigma^2}{2}(\alpha_1\lambda_2+\alpha_2\lambda_1)-b=0, \end{cases}$$

φ_i in (2.3.1) are two independent solutions of homogeneous ODE problem $(\mathcal{L}+\Lambda)\varphi=0$. $\square$

Remark: *In* (2.3.1)*, we assume assume that* $1-\alpha_1-\alpha_2$ *is not a negative integer. Indeed we can show that* φ_i *in* (2.3.1) *are analytic functions of* $\alpha_1\in\mathbb{Z}$ *and* $\alpha_2\in\mathbb{Z}$*.*

Then we shall use the kernel of $(\mathcal{L}+\Lambda)\varphi=0$ to complete the proof of uniqueness of the solution of problem (2.1.3).

Lemma 2.3.2. *Problem* (2.1.3) *admits at most one solution in* $\mathbb{X}$*.*

Proof. Let

$$\psi(r)=\int_{-\infty}^{\lambda_2}|\lambda_1-s|^{\alpha_1-1}(\lambda_2-s)^{\alpha_2-1}e^{sr}\,ds\quad\forall\, r>0, \tag{2.3.3}$$

where

$$\begin{cases} \lambda_1=\frac{\beta-\sqrt{\beta^2+2(a+c+1)\sigma^2}}{\sigma^2}, \\ \lambda_2=\frac{\beta+\sqrt{\beta^2+2(a+c+1)\sigma^2}}{\sigma^2}, \\ \alpha_1=\frac{\kappa}{\sigma^2}\Big(1-\frac{\beta}{\sqrt{\beta^2+2\sigma^2(a+c+1)}}\Big), \\ \alpha_2=\frac{\kappa}{\sigma^2}\Big(1+\frac{\beta}{\sqrt{\beta^2+2\sigma^2(a+c+1)}}\Big). \end{cases} \tag{2.3.4}$$

By Lemma 2.3.1, one can easily show that ψ satisfies

$$\begin{cases} \big[\mathcal{L} + (a+c)r\big]\psi = 0 & \text{in } (0,\infty), \\ \psi > 0 & \text{in } (0,\infty), \\ \lim\limits_{r\to\infty} \psi'(r) = \infty, \quad \lim\limits_{r\to 0^+} \psi'(r) = -\infty. \end{cases} \tag{2.3.5}$$

Suppose problem (2.1.3) has two solutions u_1 and u_2. Set

$$w := (u_1 - u_2)e^{-T}$$

Then w satisfies

$$\begin{cases} \big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda + 1\big) w = 0 & \text{in } (0,\infty)^2, \\ w(\cdot, 0) = 0 & \text{in } (0,\infty), \\ w_r \in L^\infty((0,\infty)^2) \end{cases} \tag{2.3.6}$$

It suffices to show $w \equiv 0$ in $(0,\infty)^2$. Fix $\epsilon > 0$. Let

$$v_\epsilon(r,T) := \epsilon\psi(r) \pm w(r,T) \quad \text{for all } (r,T) \in (0,\infty)^2.$$

We shall show that $v_\epsilon \geq 0$. If $v_\epsilon \geq 0$ is not true, then there exists $(r_0, t_0) \in (0,\infty)^2$ such that $v_\epsilon(r_0, t_0) < 0$.

Since $\lim\limits_{r\to\infty} \psi'(r) = \infty$ and $\lim\limits_{r\to 0^+} \psi'(r) = -\infty$, there exist positive constants δ_1 and δ_2 such that $r_0 \in [\delta_2, \delta_1]$ and

$$\frac{\partial v_\epsilon}{\partial r} > 1 \text{ in } [\delta_1,\infty)\times[0,\infty); \quad \frac{\partial v_\epsilon}{\partial r} < -1 \quad \text{in } (0,\delta_2]\times[0,\infty). \tag{2.3.7}$$

Therefore, there exists $(x^*, t^*) \in [\delta_2, \delta_1] \times [0, t_0]$ such that

$$v_\epsilon(r^*, t^*) = \min_{[\delta_2,\delta_1]\times[0,t_0]} v_\epsilon \leq v_\epsilon(r_0, t_0) < 0.$$

To continue proving the uniqueness of a solution of problem (2.1.3), we consider the following two cases:

(1) **The case** $r^* \neq B$.

Note $t^* > 0$ since $v_\epsilon(r,0) = \epsilon\psi(r) \pm w(r,0) > 0$. Thus

$$\frac{\partial v_\epsilon}{\partial T}(r^*,t^*) \leq 0, \ \ \frac{\partial v_\epsilon}{\partial r}(r^*,t^*) = 0, \ \ \frac{\partial^2 v_\epsilon}{\partial r^2}(r^*,t^*) \geq 0.$$

Therefore,

$$\Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda + 1\Big)v_\epsilon\Big|_{(r^*,t^*)} \leq 0. \tag{2.3.8}$$

However,

$$\begin{aligned}\Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda + 1\Big)v_\epsilon\Big|_{(r^*,t^*)} &= \Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda + 1\Big)(\epsilon\psi \pm w)\Big|_{(r^*,t^*)} \\ &= \epsilon\Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda + 1\Big)\psi\Big|_{(r^*,t^*)} \\ &= \epsilon\Big(\frac{\partial}{\partial T} + \mathcal{L} + (a+c)r\Big)\psi\Big|_{(r^*,t^*)} + \epsilon\Big(\lambda - (a+c)r + 1\Big)\psi\Big|_{(r^*,t^*)} \\ &\geq \epsilon\psi\big|_{(r^*,t^*)} > 0,\end{aligned}$$

which contradicts (2.3.8).

(2) **The case** $r^* = B$.

Since $v_\epsilon = \epsilon\psi \pm w$ and $w \in \mathbb{X}$, $v_\epsilon \in \mathbb{X}$. Then we can find the left and right limit in a small neighborhood of (B,t^*) such that

$$\frac{\partial v_\epsilon}{\partial T}(B_\pm,t^*) \leq 0; \ \ \frac{\partial^2 v_\epsilon}{\partial r^2}(B_\pm,t^*) \geq 0.$$

Since $\frac{\partial v_\epsilon}{\partial r}(B,t^*) = 0$ and $v_\epsilon \in \mathbb{X}$., we can obtain

$$\frac{\partial v_\epsilon}{\partial r}(B_\pm,t^*) = 0$$

Therefore,

$$(\frac{\partial}{\partial T} + \mathcal{L} + \lambda + 1)v_\epsilon\Big|_{(B_\pm,t^*)} \leq 0. \tag{2.3.9}$$

However,

$$\begin{aligned}
\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda+1\Big)v_\epsilon\Big|_{(B_\pm,t^*)} &= \Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda+1\Big)(\epsilon\psi\pm w)\Big|_{(B_\pm,t^*)}\\
&=\epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda+1\Big)\psi\Big|_{(B_\pm,t^*)}\\
&=\epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}+(a+c)r\Big)\psi\Big|_{(B_\pm,t^*)}+\epsilon\Big(\lambda-(a+c)r+1\Big)\psi\Big|_{(B_\pm,t^*)}\\
&\geq \epsilon\psi\big|_{(B_\pm,t^*)}>0,
\end{aligned}$$

which contradicts with (2.3.9).

Combining these two cases, $v_\epsilon(r,T)\geq 0$ for all $(r,T)\in(0,\infty)^2$, which implies

$$|w|\leq\epsilon\psi.$$

Sending $\epsilon\to 0$, we obtain $w\equiv 0\ \text{ in }\ (0,\infty)^2$. This completes the proof. □

Proof of Theorem 2.1.1

Theorem 2.1.1 then follows from Lemmas 2.2.5 and 2.3.2.

3.0 INFINITE HORIZON PROBLEM AND ASYMPTOTIC BEHAVIOR OF SOLUTION UNDER CIR MODEL

In this chapter, we study the behavior of the price, $u(r,T)$, of CDS as the time to expiration, T, approaches infinity, under the CIR model. Since the CIR model is an affine term structure model, the bond price P, being the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} - \frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r\right)P = 0 & \text{in } (0,\infty)\times[0,\infty), \\ P(\cdot,0) = 1 & \text{on } (0,\infty), \end{cases} \tag{3.0.1}$$

admits an explict formula

$$P(r,T) = A(T)e^{-B(T)r}, \tag{3.0.2}$$

where

$$\begin{cases} A(T) = \left(\frac{2h\exp\left(\frac{(\beta+h)T}{2}\right)}{2h+(\beta+h)[\exp(hT)-1]}\right)^{\frac{2\kappa}{\sigma^2}}, \\ B(T) = \frac{2[\exp(hT)-1]}{2h+(\beta+h)[\exp(hT)-1]}, \\ h = \sqrt{\beta^2+2\sigma^2}. \end{cases} \tag{3.0.3}$$

From (3.0.2) and (3.0.3), we know that as $T \to \infty$, the bond price approaches zero. Thus it is reasonable to expect that the price of CDS approaches, $T \to \infty$, the solution of the corresponding infinite horizon problem which is derived from PDE problem (2.1.3).

3.1 MAIN RESULTS

We shall establish two theorems. The first theorem establishes the existence and uniqueness of the solution of the infinite horizon problem. The asymptotic behavior of the CDS price follows closely to that of the bond price given in (3.0.2)-(3.0.3). Since the bond price admits a limit, we show the same behavior happen to the CDS price in the second theorem.

Theorem 3.1.1. *Let σ, κ and β be positive constans and $\mathcal{L}$ be the CIR differential operator defined by*

$$\mathcal{L}u = \Big(- \frac{\sigma^2 r}{2} \frac{\partial^2}{\partial r^2} - (\kappa - \beta r) \frac{\partial}{\partial r} + r \Big) u.$$

Assume that

$$\Lambda(r) = ar + b, \quad \Lambda_1(r) = hH(r - B), \quad \Lambda(r) = cr + d, \tag{3.1.1}$$

where a, b, c, d, p and B are positive constants. Let $\lambda(\cdot)$ and $f(\cdot)$ be functions on $(0, \infty)$ that satisfy

$$\lambda(r) := \Lambda(r) + \Lambda_1(r) + \Lambda_2(r), \quad f(r) = K\Lambda(r) - q \quad \forall r \in (0, \infty). \tag{3.1.2}$$

Then the infinite horizon problem, for u_*,

$$\begin{cases} \big(\mathcal{L} + \lambda\big) u_* = f \qquad \text{in } (0, \infty), \\ \\ u_*' \in L^\infty(0, \infty), \end{cases} \tag{3.1.3}$$

admits an unique solution.

Theorem 3.1.2. *Assume the conditions of Theorem 3.1.1. Let $u(r, T)$ be the unique solution of problem (2.1.3) given by Theorem 2.1.1 and $u_*(r)$ be the unique solution of problem (3.1.3) given by Theorem 3.1.1. There exist positive constants M and ν such that*

$$\big| u(r, T) - u_*(r) \big| \le M e^{-\nu T} \quad \forall \ T > 0, \ r \in (0, \infty).$$

Consequently,

$$\lim_{T \to \infty} u(r, T) = u_*(r) \quad \forall \ r \in (0, \infty).$$

3.2 WELL-POSEDNESS OF INFINITE PROBLEM

The infinite horizon problem is obtained by removing the $\frac{\partial}{\partial T}$ operator in (2.1.3). Thus, the infinite horizon problem is to find u_* such that

$$\big(\mathcal{L}+\lambda\big)u_* = f \quad \text{in } (0,\infty).$$

In this section, we show that this problem is well-posed under the following boundary condition:

$$u'_* \in L^\infty(0,\infty).$$

For this we begin with considering the corresponding homogeneous equation:

$$\big(\mathcal{L}+\lambda\big)u_* = 0 \quad \text{in } (0,\infty).$$

3.2.1 Asymptotic Behavior of Solutions of the Homogeneous Equation

We use the next lemma to illustrate the solution and its properties.

Lemma 3.2.1. *Let* $\mathcal{L} = -\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r$, $\sigma > 0, \kappa > 0$, *and* $\beta > 0$. *Then the homogeneous ODE problem*

$$\big(\mathcal{L}+\lambda\big)\phi = 0 \quad \text{in } (0,\infty), \tag{3.2.1}$$

admits two linear independent solutions ϕ_1 *and* ϕ_2 *satisfying:*

$$\phi_1' < 0,\ \phi_2' > 0,\ \phi_1 > 0,\ \phi_2 > 0 \quad \text{in } (0,\infty), \tag{3.2.2}$$

$$\lim_{r\to\infty} \phi_1(r) = 0, \tag{3.2.3}$$

$$\lim_{r\to 0} r^{\frac{2\kappa}{\sigma^2}} \phi_1'(r) = -1, \tag{3.2.4}$$

$$\lim_{r\to\infty} \phi_1'(r) = 0, \tag{3.2.5}$$

$$\lim_{r\to 0} \phi_2'(r) \le D \quad \text{for some constant } D > 0, \tag{3.2.6}$$

$$\liminf_{r\to\infty} \frac{\phi_2'(r)}{\phi_2(r)} > \frac{2\beta}{\sigma^2}. \tag{3.2.7}$$

Proof. Dividing $\frac{\sigma^2}{2} r$ on both sides of equation(3.2.1), we obtain the following:

$$-\phi'' + p\,\phi' + q\,\phi = 0 \qquad \text{in } (0,\infty), \tag{3.2.8}$$

where $p = \frac{2}{\sigma^2}\big(\beta - \frac{\kappa}{r}\big)$, $q = \frac{2}{\sigma^2}\big(1 + \frac{\lambda(r)}{r}\big)$.

Denote $W(r)$ as the Wronskian of ϕ_1 and ϕ_2. Since $W(r)$ satisies $W' = pW$, there exists a nonzero constant D_1 such that

$$W(r) = \phi_1(r)\,\phi_2'(r) - \phi_1'(r)\,\phi_2(r) = D_1 e^{\nu r} r^{-\mu}, \tag{3.2.9}$$

where $\nu = \frac{2\beta}{\sigma^2}$ and $\mu = \frac{2\kappa}{\sigma^2}$.

WLOG set $D_1 = 1$. Consider the solution ϕ_2 with initial conditions

$$\phi_2(0) = 1, \quad \phi_2'(0) = 0.$$

Multiplying the integrating factor $e^{-A(r)} := r^{\mu} e^{-\nu r}$, we obtain the following:

$$(e^{-A}\phi_2')' = e^{-A}(-p\phi_2' + \phi_2'') = e^{-A} q \phi_2. \tag{3.2.10}$$

Integrating both sides of (3.2.10) over $[0, r]$, we obtain

$$\begin{aligned} e^{-A(r)}\phi_2'(r) &= \int_0^r e^{-A(\rho)} q\,\phi_2(\rho)\,d\rho \\ &= \frac{2}{\sigma^2}\int_0^r \rho^{\mu} e^{-\nu\rho}\Big[1 + \frac{\lambda(\rho)}{\rho}\Big]\phi_2(\rho)\,d\rho. \end{aligned}$$

By Picard's method, we know that

$$\phi_2(r) = 1 + \int_0^r \phi_2(\rho)\,\mathcal{K}(r,\rho)\,d\rho > 0, \ \ 0 < \rho < r,$$

where

$$\mathcal{K}(r,\rho) = \frac{2}{\sigma^2}\rho^{\mu}e^{-\nu\rho}\Big[1+\frac{\lambda(\rho)}{\rho}\Big]\int_{\rho}^{r} s^{-\mu}e^{\nu s}\,ds.$$

Thus

$$\begin{aligned}\phi_2'(r) &= \int_0^r \phi_2(\rho)\,\frac{\partial\mathcal{K}(r,\rho)}{\partial r}\,d\rho \\ &= \frac{2}{\sigma^2}r^{-\mu}e^{\nu r}\int_0^r \phi_2(\rho)\rho^{\mu}e^{-\nu\rho}\big[1+\frac{\lambda(\rho)}{\rho}\big]\,d\rho > 0 \quad \forall r\in(0,\infty).\end{aligned}$$

Note that $\lim\limits_{r\to 0}\phi_2(r)=\phi_2(0)=1$. By L'Hospital's Rule, we have

$$\lim_{r\to 0}\phi_2'(r) = \frac{2}{\sigma^2}\lim_{r\to 0}\frac{\phi_2(r)r^{\mu}e^{-\nu r}[1+\frac{\lambda(r)}{r}]}{\mu r^{\mu-1}e^{-\nu r}-\nu r^{\mu}e^{-\nu r}} = \frac{2}{\sigma^2}\lim_{r\to 0}\frac{\phi_2(r)[r+\lambda(r)]}{\mu-\nu r} = \frac{1}{\kappa}\lim_{r\to 0}\lambda(r)\le D,$$

where $D := \frac{1}{\kappa}\|\lambda(r)\|_{L^\infty([0,1])}$ is a positive constant.

Next we study the asymptotic behaviour of $\phi_2'(r)$ as $r\to\infty$. Since $\phi_2'(r)>0$, for some positive function $k(r)$, we can write $\phi_2(r)=e^{\int_0^r k(\rho)\,d\rho}$. Then we have

$$\begin{cases}\phi_2'(r)=\phi_2(r)\,k(r), \\ \phi_2''(r)=k'(r)\,\phi_2(r)+k^2(r)\,\phi_2(r).\end{cases}$$

Plugging $\phi_2''(r)$ and $\phi_2'(r)$ into equation (3.2.1), we obtain

$$k'(r) = -k^2(r)+p\,k(r)+q \ge -k^2(r)+\Big(\nu-\frac{\mu}{r}\Big)\,k(r)+\frac{2}{\sigma^2}.$$

Fix $\varepsilon>0$. Then for all $r>\frac{\mu}{\epsilon}$, we have

$$k'(r)\ge -k^2(r)+(\nu-\epsilon)\,k(r)+\frac{2}{\sigma^2}.$$

Therefore,

$$k(r)\ge\frac{(\nu-\epsilon)+\sqrt{(\nu-\epsilon)^2+\frac{8}{\sigma^2}}}{2},$$

i.e., $\liminf_{r\to\infty} k(r) \geq \frac{(\nu-\epsilon)+\sqrt{(\nu-\epsilon)^2+\frac{8}{\sigma^2}}}{2}$. Sending $\epsilon \to 0$, we obtain

$$\liminf_{r\to\infty} \frac{\phi_2'(r)}{\phi_2(r)} = \liminf_{r\to\infty} k(r) \geq \frac{\nu+\sqrt{\nu^2+\frac{8}{\sigma^2}}}{2} > \nu.$$

Thus

$$\phi_2(r) > e^{\nu r} \text{ and } \phi_2'(r) > \nu e^{\nu r} \quad \text{for } r \text{ large enough.} \tag{3.2.11}$$

Now we use the Wronkian to find another solution $\phi_1(r)$. Since

$$W(r) = \phi_1(r)\,\phi_2'(r) - \phi_1'(r)\,\phi_2(r) = -\phi_2^2(r)\Big(\frac{\phi_1(r)}{\phi_2(r)}\Big)' = e^{\nu r} r^{-\mu},$$

we have

$$\phi_1(r) = \phi_2(r) \int_r^\infty \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)}\, d\rho, \quad \forall r \in (0,\infty). \tag{3.2.12}$$

Note that $\phi_1 > 0$ since $\phi_2 > 0$. By (3.2.11) and L'Hospital Rule, we have

$$\lim_{r\to\infty} \phi_1(r) = \lim_{r\to\infty} \frac{-\frac{e^{\nu r} r^{-\mu}}{\phi_2^2(r)}}{-\frac{\phi_2'(r)}{\phi_2^2(r)}} = \lim_{r\to\infty} \frac{e^{\nu r} r^{-\mu}}{\phi_2'(r)} = 0.$$

By the Mean Value Theorem, there exists ρ_n such that

$$\begin{cases} \lim_{n\to\infty} [\phi_1(n+1) - \phi_1(n)] = \lim_{n\to\infty} \phi_1'(\rho_n) = 0, \\ \\ \lim_{n\to\infty} \rho_n = \infty. \end{cases}$$

Since $\phi_1 > 0$, we have

$$\int_r^{\rho_n} e^{-A(s)} q\phi_1(s)\, ds > 0.$$

By (3.2.10), we similarly have

$$e^{-A} q\phi_1 = (e^{-A}\phi_1')'.$$

Thus,

$$\int_r^{\rho_n} e^{-A(s)} q\phi_1(s)\, ds = \int_r^{\rho_n} (e^{-A(s)}\phi(s)_1')'\, ds = e^{-A(\rho_n)}\phi_1'(\rho_n) - e^{-A(r)}\phi_1'(r) > 0.$$

Sending $n \to \infty$, we have $e^{-A(\rho_n)}\phi_1'(\rho_n) \to 0$, which implies $\phi_1'(r) < 0$ for all $r \in (0, \infty)$.

By (3.2.11), $\phi_2' > 0$ and $\phi_2(0) = 1$, we have

$$0 \le \lim_{r\to 0} r^{\mu} \int_r^{\infty} \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)} d\rho \le \lim_{r\to 0} \left\{ r^{\mu} \frac{e^{\nu}}{\phi_2^2(r)} \int_r^1 \rho^{-\mu} d\rho + r^{\mu} \int_1^{\infty} \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)} d\rho \right\}$$
$$= \begin{cases} \lim\limits_{r\to 0} \left\{ \frac{(r^{\mu}-r)e^{\nu}}{(1-\mu)\phi_2^2(r)} + r^{\mu} \int_1^{\infty} \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)} d\rho \right\} = 0 & \text{if } \mu \ne 1, \\ \lim\limits_{r\to 0} \left\{ \frac{-e^{\nu} r \ln(r)}{\phi_2^2(r)} + r \int_1^{\infty} \frac{e^{\nu\rho}\rho^{-1}}{\phi_2^2(\rho)} d\rho \right\} = 0 & \text{if } \mu = 1. \end{cases}$$

Therefore,

$$\lim_{r\to 0} r^{\mu}\phi_1'(r) = \lim_{r\to 0} \left(r^{\mu}\phi_2'(r) \int_r^{\infty} \frac{e^{\nu\rho}\rho^{-\mu}}{\phi_2^2(\rho)} d\rho - \frac{e^{\nu r}}{\phi_2(r)} \right) = -1.$$

Since $\phi_1\phi_2' > 0$ and $W(r) = \phi_1(r)\,\phi_2'(r) - \phi_1'(r)\,\phi_2(r) = e^{\nu r} r^{-\mu}$, we know that

$$-\frac{e^{\nu r} r^{-\mu}}{\phi_2(r)} \le \phi_1'(r) \le 0.$$

By (3.2.11), we know that $-r^{-\mu} \le \phi_1'(r) \le 0$ for r large enough. Therefore, by Squeeze Theorem,

$$\lim_{r\to\infty} \phi_1'(r) = 0.$$

This completes the proof of Lemma 3.2.1. $\square$

3.2.2 Existence of Solution of Infinite Horizon Problem

Lemma 3.2.2. (**Existence**) *Problem (3.1.3) admits at least one solution.*

Proof. Let

$$\begin{cases} C_1(r) := \int_0^r \frac{\phi_2(\rho)\bar{f}(\rho)}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho, \\ \\ C_2(r) := \int_r^\infty \frac{\phi_1(\rho)\bar{f}(\rho)}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho, \end{cases} \tag{3.2.13}$$

where $\bar{f} = \frac{2f}{\sigma^2 r}$; $\phi_1(r)$ and $\phi_2(r)$ are functions defined as in Lemma 3.2.1. Then we claim

$$\varphi(r) = C_1(r)\phi_1(r) + C_2(r)\phi_2(r) \tag{3.2.14}$$

is a solution of inhomogeneous ODE problem (3.1.3).

In order to show this claim, first we shall show that $C_1(r)$ and $C_2(r)$ are well-defined. Since ϕ_2 is bounded as $r \to 0$ and ϕ_1 is bounded as $r \to \infty$, we have

$$\begin{cases} \left|\frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\right| = O(1)\, r^{\mu-1}, \\ \\ \left|\frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2}\right| = O(1)e^{-\nu r} r^{\mu}. \end{cases} \tag{3.2.15}$$

Thus, the integrals defining $C_1(r)$ and $C_2(r)$ are convergent, i.e, $C_1(r)$ and $C_2(r)$ are well-defined.

It is straightforward to verify that $\varphi(r)$ satisfies

$$\big(\mathcal{L} + \lambda\big)\varphi = 0 \quad \text{in} \;\; (0, \infty).$$

Lastly, we shall show that $\varphi(r)$ defined as in (3.2.14) satisfies $\varphi' \in L^\infty(0,\infty)$. By differentiating (3.2.14), we have

$$\varphi'(r) \;\; = \;\; \phi_1'(r)\, C_1(r) + \phi_2'(r)\, C_2(r).$$

As $r \to 0$, by (3.2.4) and (3.2.15), we have

$$\phi_1'(r)\, C_1(r) = O(1). \tag{3.2.16}$$

As $r \to 0$, by (3.2.3), we have

$$C_2(r) = \int_r^1 \frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2} d\rho + \int_1^\infty \frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2} d\rho = \int_r^1 O(1)\, \phi_1(\rho)\, \rho^{\mu-1}\, d\rho + O(1);$$

by (3.2.4), we have

$$\phi_1(r) = \begin{cases} O(1)\, r^{1-\mu} & \text{if } \mu \neq 1, \\ \\ O(1)\, \ln r & \text{if } \mu = 1. \end{cases}$$

Therefore,

$$C_2(r) = \begin{cases} O(1) \int_r^1 \rho^{1-\mu}\, \rho^{\mu-1}\, d\rho + O(1) = O(1) & \text{if } \mu \neq 1, \\ \\ O(1) \int_r^1 \ln \rho\, d\rho + O(1) = O(1) & \text{if } \mu = 1. \end{cases}$$

By (3.2.6), we obtain

$$\phi_2'(r)\, C_2(r) = O(1). \tag{3.2.17}$$

Combining (3.2.16) and (3.2.17), we have

$$\varphi'(r) = \phi_1'(r)\, C_1(r) + \phi_2'(r)\, C_2(r) = O(1) \qquad \text{as } r \to 0. \tag{3.2.18}$$

As $r \to \infty$, since $\phi_2(0) = 1$, we obtain

$$\begin{aligned} C_1(r) &= \int_0^r \frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho = \int_0^1 \frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho + \int_1^r \frac{\bar{f}\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho \\ &= O(1) + \frac{\phi_2(r)}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_1^r \frac{\bar{f}\, r^{-\mu} e^{\nu r} \phi_2(\rho)}{\rho^{-\mu} e^{\nu\rho} \phi_2(r)}\, d\rho. \end{aligned}$$

By (3.2.7), we have

$$k^* := \ln \frac{\phi_2(\rho)}{\phi_2(r)} < -\nu(r - \rho);$$

by (3.2.5), we have

$$\begin{aligned} \left|\phi_1'(r) C_1(r)\right| &= O(1) + \frac{\left|\phi_1'(r)\phi_2(r)\right|}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_1^r \left(\frac{\rho}{r}\right)^\mu \bar{f}\, e^{\nu(r-\rho) + \ln \frac{\phi_2(\rho)}{\phi_2(r)}}\, d\rho \\ &\leq O(1) + O(1) \int_1^r e^{\nu(r-\rho) + k^*}\, d\rho. \end{aligned}$$

Therefore,

$$\left|\phi_1'(r)C_1(r)\right| = O(1) \qquad \text{as } \; r \to \infty. \tag{3.2.19}$$

Next we shall consider $\phi_2'(r)\, C_2(r)$ as $r \to \infty$. Since $\phi_1'(r) < 0$,

$$\ln \frac{\phi_1(\rho)}{\phi_1(r)} < 0 \quad \forall \rho > r.$$

Therefore,

$$\begin{aligned}
\left|\phi_2'(r)C_2(r)\right| &= \phi_2'(r) \int_r^\infty \frac{\bar{f}\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2}\, d\rho \\
&= \frac{\phi_2'(r)\phi_1(r)}{\phi_1(r)\phi_2'(r) - \phi_1'(r)\phi_2(r)} \int_r^\infty \big(\frac{\rho}{r}\big)^{\mu} \bar{f}\, e^{-\nu(\rho-r)+\ln\frac{\phi_1(\rho)}{\phi_1(r)}}\, d\rho \\
&\leq O(1) \int_r^\infty \big(\frac{\rho}{r}\big)^{\mu} e^{-\nu(\rho-r)}\, d\rho
\end{aligned}$$

Hence,

$$\left|\phi_2'(r)C_2(r)\right| = O(1) \qquad \text{as } r \to \infty. \tag{3.2.20}$$

Based on (3.2.19) and (3.2.20), we have

$$\varphi'(r) = \phi_1'(r)\, C_1(r) + \phi_2'(r)\, C_2(r) = O(1) \qquad \text{as } \; r \to \infty. \tag{3.2.21}$$

Combining (3.2.18) and (3.2.21), we proved that $\varphi' \in L^\infty(0,\infty)$. This completes the proof of Lemma 3.2.2. $\square$

3.2.3 Uniqueness of Solution of Infinite Horizon Problem

Lemma 3.2.3. (**Uniqueness**) *Problem (3.1.3) admits at most one solution.*

Proof. Suppose problem (3.1.3) has two solutions u_1 and u_2. Set $\phi := u_1 - u_2$. Then ϕ satisfies

$$\begin{cases} (\mathcal{L} + \lambda)\phi = 0 \quad \text{in } (0, \infty), \\ \phi' \in L^\infty(0, \infty). \end{cases} \tag{3.2.22}$$

It suffices to show $\phi \equiv 0$ in $(0, \infty)$.

The general solution of above ODE problem (3.2.22) is

$$\phi(r) = C_1\phi_1(r) + C_2\phi_2(r) \tag{3.2.23}$$

By Lemma 3.2.1, we know that $\phi_1'(r) \to -\infty$ and $\phi_2'(r)$ is bounded as $r \to 0$. Since $\phi'(r) \in L^\infty(0, \infty)$, we have $C_1 = 0$. Since $\phi_1'(r) \to 0$ and $\phi_2'(r) \to \infty$ as $r \to \infty$, we have $C_2 = 0$.

Therefore, $\phi \equiv 0$ is the only solution of problem (3.2.22). This completes the proof. □

Proof of Theorem 3.1.1

Theorem 3.1.1 then follows from Lemmas 3.2.2 and 3.2.3.

3.3 ASYMPTOTIC BEHAVIOR

In this section, we shall prove Theorem 3.1.2. More precisely, we show the price of CDS, as $T \to \infty$, approaches the solution of the infinite horizon problem (3.1.3).

Proof of Theorem 3.1.2

Proof. Let $u(r,T)$ be the only solution of problem (2.1.3) and $u_*(r)$ be the only solution of problem (3.1.3). Then $v(r,T) := u(r,T) - u_*(r)$ satisfies the following:

$$\begin{cases} (\frac{\partial}{\partial T} + \mathcal{L} + \lambda)v = 0 & \text{in } (0,\infty)^2, \\ v(\cdot,0) = -u_*(\cdot) & \text{on } (0,\infty), \\ v_r \in L^\infty((0,\infty)^2). \end{cases} \tag{3.3.1}$$

Since

$$\begin{cases} (\frac{\partial}{\partial T} + \mathcal{L} + \lambda)\|u_*\|_\infty \, e^{-(b+d)T} = \big[(a+c+1)r + pH(r-B)\big]||u_*||_\infty \, e^{-(b+d)T} \geq 0, \\ ||u_*||_\infty \, e^{-(b+d)T}\big|_{T=0} \geq -u_*, \end{cases}$$

$||u_*||_\infty e^{-(b+d)T}$ is a super-solution of problem (3.3.1).

Similarly, we can show $-||u_*||_\infty e^{-(b+d)T}$ is a sub-solution of problem (3.3.1). Therefore, by comparison principle, $v \leq ||u_*||_\infty e^{-(b+d)T}$ and $v \geq -||u_*||_\infty e^{-(b+d)T}$, i.e., $|v| \leq ||u_*||_\infty e^{-(b+d)T}$. This completes the proof of Theorem 3.1.2. □

So far, we have done the proof of all main results under the CIR model. We shall use the next two chapters to prove the main results under Vasicek model.

4.0 WELL-POSEDNESS OF THE PDE PROBLEM UNDER VASICEK MODEL

In this chapter, we study the price of CDS under an intensity model for the credit event where the intensity depends only on short interest rate modeled by the Vasicek model. In section one we present the main results. In section two we present the existence of the solution. In section three we present the uniqueness of the solution. In section four we present an a priori bound of the solution.

4.1 MAIN RESULTS

We shall state two theorems in this section. The first theorem establishes the existence and uniqueness of the solution of problem (1.6.5) under the standard boundary condition (1.6.6). The second theorem provides an a priori bound for the solution which is the price of CDS.

For convenience, we use $\lambda(\cdot)$ to denote $\Lambda(\cdot) + \Lambda_1(\cdot) + \Lambda_2(\cdot)$.

Theorem 4.1.1. (**Existence and Uniqueness**) *Let σ, κ and β be positive constans and $\mathcal{L}$ be the Vasicek differential operator defined by*

$$\mathcal{L}u = \Big(- \frac{\sigma^2}{2} \frac{\partial^2}{\partial r^2} - (\kappa - \beta r) \frac{\partial}{\partial r} + r \Big) u.$$

Let $\lambda(\cdot)$ and $f(\cdot)$ be functions on $\mathbb{R}$ that satisfy, for some positive constant m,

$$\lambda(r) \geq 0, \quad \lambda(r) + |f(r)| \leq m(1 + |r|) \quad \forall r \in \mathbb{R}. \tag{4.1.1}$$

Then there exists a unique solution u to the following problem:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right)u = f & \text{in } \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = 0 & \text{on } \mathbb{R}, \\ |u(r,T)| \le e^{A(N)(r^2+1)} & \text{for all } r \in \mathbb{R},\ T \in [0, N],\ N > 0, \end{cases} \tag{4.1.2}$$

where $A(\cdot)$ *is any increasing function defined on* $[0, \infty)$.

Theorem 4.1.2. (**A Priori Bound**) *Assume the conditions of Theorem 4.1.1. There exists a positive constant c depending only on* κ, β, *and* σ *such that the unique solution of problem* (4.1.2) *in Theorem 4.1.1 has the following bound: for each* $(r, T) \in \mathbb{R} \times (0, \infty)$,

$$|u(r,T)| \le c \begin{cases} e^{-\frac{r}{\beta}} + \mu & \text{if} \quad 2\kappa\beta > \sigma^2, \\ \beta T e^{-\frac{r}{\beta}} + 1 & \text{if} \quad 2\kappa\beta = \sigma^2, \\ e^{-\frac{r}{\beta} + 2\beta|\mu|T} + |\mu| & \text{if} \quad 2\kappa\beta < \sigma^2, \end{cases} \tag{4.1.3}$$

where $\mu := \frac{2\kappa\beta - \sigma^2}{2\beta^3}$.

Remark: *Since the Vasicek model is an affine term structure model, the bond price P, being the solution of*

$$\begin{cases} \left(\frac{\partial}{\partial T} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r\right)P = 0 & \text{in } \mathbb{R} \times [0, \infty), \\ P(\cdot, 0) = 1 & \text{on } \mathbb{R}, \end{cases} \tag{4.1.4}$$

admits an explict formula

$$P(r,T) = a(T)e^{-b(T)r}, \tag{4.1.5}$$

where

$$\begin{cases} b(T) = \frac{1}{\beta}(1 - e^{-\beta T}), \\ a(T) = \exp\left[\beta\mu\left(b(T) - T\right) - \frac{\sigma^2}{4\beta}b^2(T)\right]. \end{cases} \tag{4.1.6}$$

Comparing the size of the bond price (4.1.5) *and the upper bound of* u *in* (4.1.3), *we can see that the a priori estimate in* (4.1.3) *is reasonably accurate.*

In the rest of this chapter, we prove Theorem 4.1.1 and Theorem 4.1.2; we always assume that the conditions of Theorem 4.1.1 hold.

4.2 EXISTENCE

In this section, we establish the existence of a solution of problem (4.1.2). We begin with introducing an auxiliary function.

Lemma 4.2.1. *There exist a positive constant* M_1 *and a function* $\psi \in C^\infty(\mathbb{R})$ *such that*

$$\begin{cases} \mathcal{L}\psi \geq -M_1\psi & \text{in } \mathbb{R}, \\ \psi > 0 & \text{in } \mathbb{R}, \\ \lim\limits_{|r|\to\infty} \psi(r) = \infty. \end{cases} \tag{4.2.1}$$

Proof. We set

$$\psi := e^{\frac{r}{\beta}} + e^{-\frac{3r}{\beta}} \ \text{ and } \ M_1 := \max\Big\{\frac{\sigma^2}{2\beta^2} + \frac{\kappa}{\beta}, \frac{9\sigma^2}{2\beta^2} - \frac{3\kappa}{\beta}\Big\}. \tag{4.2.2}$$

Since $2r\big(e^{\frac{r}{\beta}} - e^{-\frac{3r}{\beta}}\big) \geq 0$ for every $r \in \mathbb{R}$, we have

$$\begin{aligned} \mathcal{L}\psi &= 2r\big(e^{\frac{r}{\beta}} - e^{-\frac{3r}{\beta}}\big) - e^{\frac{r}{\beta}}\big(\frac{\sigma^2}{2\beta^2} + \frac{\kappa}{\beta}\big) - e^{-\frac{3r}{\beta}}\big(\frac{9\sigma^2}{2\beta^2} - \frac{3\kappa}{\beta}\big) \\ &\geq -e^{\frac{r}{\beta}}\big(\frac{\sigma^2}{2\beta^2} + \frac{\kappa}{\beta}\big) - e^{-\frac{3r}{\beta}}\big(\frac{9\sigma^2}{2\beta^2} - \frac{3\kappa}{\beta}\big) \\ &\geq -\max\Big\{\frac{\sigma^2}{2\beta^2} + \frac{\kappa}{\beta}, \frac{9\sigma^2}{2\beta^2} - \frac{3\kappa}{\beta}\Big\}\psi \\ &= -M_1\psi. \end{aligned}$$

Obviously, $\psi > 0$ and $\lim\limits_{|r|\to\infty} \psi(r) = \infty$. $\square$

We approximate problem (4.1.2) by the following truncation:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right) u_n = f & \text{in } (-n, n) \times (0, n^2], \\ u_n(\cdot, 0) = 0 & \text{on } (-n, n), \\ u_n = 0 & \text{on } \{-n, n\} \times (0, n^2]. \end{cases} \tag{4.2.3}$$

Lemma 4.2.2. *For every $n \in \mathbb{N}$, problem (4.2.3) admits a unique classical solution.*

Problem (4.2.3) is a standard initial boundary value problem. The assertion of the Lemma 4.2.2 follows from a standard theory of parabolic equations; see, for example, Friedman [19]. We omit the proof here.

Next we provide an a priori estimate, which will allow us to obtain a limit when we send n to infinity.

Lemma 4.2.3. (L^∞**-estimate of** u_n) *There exists a positive constant K, which does not depend on n, such that the solution u_n of problem (4.2.3) satisfies*

$$|u_n| \leq K e^{(M_1+1)T} \psi, \tag{4.2.4}$$

where M_1 and ψ are defined in (4.2.2).

Proof. From assumption (4.1.1), we have

$$|f(r)| \leq m(1 + |r|) \quad \forall r \in \mathbb{R}.$$

Let $v := K e^{(M_1+1)T} \psi$, where $K = \max\{m, m\beta\}$. Note that $v > 0$ in $[-n, n] \times [0, n^2]$ and

$$\begin{aligned} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right) v &\geq \left(\frac{\partial}{\partial T} + \mathcal{L}\right) v \\ &= \left[\mathcal{L} + (M_1 + 1)\right] v \geq v \\ &= K e^{(M_1+1)T} \left(e^{\frac{r}{\beta}} + e^{-\frac{3r}{\beta}}\right) \\ &\geq K \left[1 + \frac{|r|}{\beta}\right] \geq |f|. \end{aligned}$$

Thus, v is a super-solution and $-v$ is a sub-solution of problem (4.2.3). By comparison principle, $u_n \leq v$ and $u_n \geq -v$, i.e., $|u_n| \leq v$. $\square$

Lemma 4.2.4. *Problem* (4.1.2) *admits at least one solution.*

Proof. Fix $\alpha \in (0,1)$. For each $\delta \in (0,1)$ and $n > \frac{2}{\delta}$, by interior estimate for parabolic equations [22], the solution u_n of problem (4.2.3) satisfies

$$\|u_n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}([-\frac{1}{\delta},\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}])} \le C(\alpha,\delta)\big\{\|f\|_{C^\alpha([-\frac{2}{\delta},\frac{2}{\delta}])} + \|u_n\|_{L^\infty([-\frac{2}{\delta},\frac{2}{\delta}]\times[0,\frac{1}{\delta^2}])}\big\} \le \tilde{C}(\alpha,\delta),$$

where $\tilde{C}(\alpha,\delta)$ is a positive constant depends on α and δ.

Since

$$C^{2+\alpha,1+\frac{\alpha}{2}} \text{ is compactly embedded in } C^{2,1},$$

by using a diagonal process, we can find a function u and a subsequence $\{n_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} n_k = \infty$ and

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{C^{2,1}([-\frac{1}{\delta},\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}])} = 0 \quad \forall\ \delta \in (0,1).$$

From the equation of u_{n_k}, we find that u satisfies (4.1.2). In addition, from the a priori estimate $|u_{n_k}| \le Ke^{(M_1+1)T}\psi$ estalish in Lemma 4.2.3, we can obtain $|u| \le Ke^{(M_1+1)T}\psi$.

Therefore, u is a solution of problem (4.1.2). □

4.3 UNIQUENESS

In this section, we establish the uniqueness of a solution of problem (4.1.2). The uniqueness of the solution is proven by constructing an auxiliary function and using a comparison principle. We begin with introducing an auxiliary function.

Lemma 4.3.1. *For each $\eta > 0$, let $M(\eta) = \max\{\frac{\sigma^2\eta}{2\beta}, \frac{\kappa^2}{4\beta\sigma^2\eta}\}$ and*

$$\varphi(\eta; r, T) = (2\eta - T)^{-\frac{1}{2}} \exp\Big[\frac{r^2}{2\sigma^2(2\eta - T)} + M(\eta)T\Big]\ \forall\ (r,T) \in \mathbb{R}\times[0,\eta].$$

Then

$$\Big(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\Big)\varphi > 0 \ \text{ in } \ \mathbb{R}\times[0,\eta].$$

Proof. Since $\varphi(\eta;\,\cdot,\cdot)$ is positive on $\mathbb{R}\times[0,\eta]$, we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda\Big)\,\varphi &\geq \Big(\frac{\partial}{\partial T}+\mathcal{L}\Big)\,\varphi \\
&= \Big(M(\eta)+\frac{\beta r^2}{\sigma^2(2\eta-T)}-\frac{\kappa r}{\sigma^2(2\eta-T)}+r\Big)\varphi \\
&\geq \Big[M(\eta)-\frac{\big(\sigma^2(2\eta-T)-\kappa\big)^2}{4\beta\sigma^2(2\eta-T)}\Big]\varphi.
\end{aligned}$$

If $0\leq\kappa\leq\sigma^2(2\eta-T)$, we have

$$\frac{\big(\sigma^2(2\eta-T)-\kappa\big)^2}{4\beta\sigma^2(2\eta-T)}\leq\frac{\big(\sigma^2(2\eta-T)\big)^2}{4\beta\sigma^2(2\eta-T)}=\frac{\sigma^2(2\eta-T)}{4\beta}\leq\frac{\sigma^2\eta}{2\beta}\leq M(\eta).$$

If $\kappa\geq\sigma^2(2\eta-T)$, we have

$$\frac{\big(\sigma^2(2\eta-T)-\kappa\big)^2}{4\beta\sigma^2(2\eta-T)}\leq\frac{\kappa^2}{4\beta\sigma^2(2\eta-T)}\leq\frac{\kappa^2}{4\beta\sigma^2\eta}\leq M(\eta).$$

Therefore, $\big(\frac{\partial}{\partial T}+\mathcal{L}+\lambda\big)\varphi>0$ in $\mathbb{R}\times[0,\eta]$. □

Lemma 4.3.2. *Problem (4.1.2) admits at most one solution.*

Proof. Suppose the problem (4.1.2) has two solutions u_1 and u_2. Set

$$w(r,T):=e^{-A(r,T)}[u_1(r,T)-u_2(r,T)],$$

where $A(r,T)=(\beta+\frac{\sigma^2}{2\beta^2})T+\frac{\beta}{\sigma^2}r^2+(\frac{1}{\beta}-\frac{2\kappa}{\sigma^2})r$.

Then w satisfies

$$\begin{cases}
\big(\frac{\partial}{\partial T}+\mathcal{L}_1+\lambda\big)w=0 & \text{in } \mathbb{R}\times(0,\infty),\\
w(\cdot,0)=0 & \text{on } \mathbb{R},\\
|w(r,T)|\leq e^{B(N)(r^2+1)} & \text{for all } r\in\mathbb{R},\ T\in[0,N],\ N>0,
\end{cases}$$

where $B(\cdot)$ is some increasing function defined on $[0,\infty)$ and $\mathcal{L}_1$ is a second order elliptic operator defined by

$$\mathcal{L}_1 w=-\frac{\sigma^2}{2}w_{rr}+(\kappa-\beta r-\frac{\sigma^2}{\beta})w_r+\frac{\kappa}{\beta}w.$$

We shall show $w \equiv 0$ in $\mathbb{R} \times [0, \infty)$ which is equivalent to show $w \equiv 0$ in $\mathbb{R} \times [0, N]$ for every $N > 0$. Now fix $N > 0$. Set $B = B(N)$. Then we have

$$|w(r, T)| \leq e^{B(r^2+1)} \text{ for all } r \in \mathbb{R},\ T \in [0, N].$$

Let $k = [4\sigma^2 BN] + 1$ ($[x]$ is the greatest integer part of x), $\varsigma = \frac{N}{k}$, $\eta = \min\{\varsigma, \frac{1}{8\beta}\}$, and $\varphi(\cdot;\ \cdot, \cdot)$ be the function defined in Lemma (4.3.1). Then

$$\Psi(r, T) := e^{-A(r,T)}\varphi(\eta; r, T)$$

satisfies

$$\Big(\frac{\partial}{\partial T} + \mathcal{L}_1 + \lambda\Big)\Psi > 0 \qquad \text{in } \mathbb{R} \times [0, \eta].$$

Fix $\epsilon > 0$. Let

$$\phi_\epsilon(r, T) := \epsilon\Psi(r, T) \pm w(r, T) \text{ for all } (r, T) \in \mathbb{R} \times [0, \eta].$$

We shall show that $\phi_\epsilon(r, T) \geq 0$ for all $(r, T) \in \mathbb{R} \times [0, \eta]$. If $\phi_\epsilon \geq 0$ is not true, then there exists $(r_0, t_0) \in \mathbb{R} \times [0, \eta]$ such that $\phi_\epsilon(r_0, t_0) < 0$.

Since $\frac{1}{4\sigma^2\eta} > B$, $\lim_{|r|\to\infty} \phi_\epsilon(r, T) = \infty$ uniformly in $T \in [0, \eta]$. Therefore, there exists $(r^*, t^*) \in \mathbb{R} \times [0, \eta]$ such that

$$\phi_\epsilon(r^*, t^*) = \min_{\mathbb{R}\times[0,\eta]} \phi_\epsilon \leq \phi_\epsilon(r_0, t_0) < 0.$$

Note that $t^* > 0$ since $\phi_\epsilon(r, 0) = \epsilon\Psi(r, 0) \pm w(r, 0) > 0$. Thus

$$\frac{\partial\phi_\epsilon}{\partial T}(r^*, t^*) \leq 0,\ \ \frac{\phi_\epsilon}{\partial r}(r^*, t^*) = 0,\ \ \frac{\partial^2\phi_\epsilon}{\partial r^2}(r^*, t^*) \geq 0.$$

Therefore,

$$\Big(\frac{\partial}{\partial T} + \mathcal{L}_1 + \lambda\Big)\phi_\epsilon\Big|_{(r^*,t^*)} \leq 0. \tag{4.3.1}$$

However,

$$\begin{aligned}\Big(\frac{\partial}{\partial T}+\mathcal{L}_1+\lambda\Big)\phi_\epsilon\Big|_{(r^*,t^*)} &= \Big(\frac{\partial}{\partial T}+\mathcal{L}_1+\lambda\Big)(\epsilon\Psi \pm w)\Big|_{(r^*,t^*)}\\ &= \epsilon\Big(\frac{\partial}{\partial T}+\mathcal{L}_1+\lambda\Big)\Psi\Big|_{(r^*,t^*)}\\ &> 0,\end{aligned}$$

which contradicts (4.3.1).

Therefore, $\phi_\epsilon(r,T) \geq 0$ for all $(r,T) \in \mathbb{R}\times[0,\eta]$, which implies $|w| \leq \epsilon\Psi$. Sending $\epsilon \to 0$, we obtain $w \equiv 0$ in $\mathbb{R}\times[0,\eta]$.

Similarly, we can inductively show that

$$w \equiv 0 \ \text{ in } \ \mathbb{R}\times[\eta,\, 2\eta], \ \ \mathbb{R}\times[2\eta,\, 3\eta], \ \cdots, \ \ \mathbb{R}\times[(k-1)\eta,\, k\eta].$$

Thus $w \equiv 0$ in $\mathbb{R}\times[0,\, N]$.

Since N is arbitrary, $w \equiv 0$ in $\mathbb{R}\times[0,\, \infty)$, i.e., $u_1 \equiv u_2$. This completes the proof. □

Proof of Theorem 4.1.1

Theorem 4.1.1 follows from Lemmas 4.2.4 and 4.3.2.

4.4 A PRIORI ESTIMATES

Lemma 4.2.4 gives the bound $|u| \leq Ke^{(M_1+1)T}[e^{\frac{r}{\beta}} + e^{-\frac{3r}{\beta}}]$. We wish to improve the bound, especially for the case $r > 0$. Thus, in this section we prove Theorem 4.1.2. To do this, we consider three cases: (i) $2\kappa\beta > \sigma^2$, (ii) $2\kappa\beta = \sigma^2$, and (iii) $2\kappa\beta < \sigma^2$. For convenience, we set $\mu := \frac{2\kappa\beta-\sigma^2}{2\beta^3}$.

4.4.1 The case $2\kappa\beta > \sigma^2$

Lemma 4.4.1. *Assume $2\kappa\beta > \sigma^2$. There exists a positive constant c such that the solution u of problem (4.1.2) satisfies*

$$|u(r,T)| \leq c(e^{-\frac{r}{\beta}} + \mu) \quad \text{in } \mathbb{R} \times (0,\infty).$$

Proof. For a positive constant c to be determined later, define $\bar{v} = c(e^{-\frac{r}{\beta}} + \mu)$. Since $\lambda \geq 0$, we have

$$\big(\mathcal{L} + \lambda\big)\bar{v} \geq \mathcal{L}\bar{v} = c\mu\big(\beta e^{-\frac{r}{\beta}} + r\big).$$

Since $\min_{r\in\mathbb{R}}[\beta e^{-\frac{r}{\beta}} + r] = [\beta e^{-\frac{r}{\beta}} + r]\Big|_{r=0} = \beta$, we have $\beta e^{-\frac{r}{\beta}} + r \geq \beta$ for each $r \in \mathbb{R}$. Thus,

$$c := \sup_{r\in\mathbb{R}} \frac{m(1+|r|)}{\mu(\beta e^{-\frac{r}{\beta}} + r)} \tag{4.4.1}$$

is well-defined positive constant.

Thus we have $\big(\mathcal{L} + \lambda\big)\bar{v} \geq |f|$ in $\mathbb{R}$.

Since $\bar{v} > 0$, by comparison principle, we obtain $|u_n| \leq \bar{v}$, where u_n is the solution of problem of (4.2.3). Letting $n \to \infty$, we obtain $|u| \leq \bar{v}$. This completes the proof. $\square$

4.4.2 The case $2\kappa\beta = \sigma^2$

Lemma 4.4.2. *Assume $2\kappa\beta = \sigma^2$. There exists a positive constant c such that the solution u of problem (4.1.2) satisfies*

$$|u(r,T)| \leq c(\beta T e^{-\frac{r}{\beta}} + 1) \quad \text{in } \mathbb{R} \times (0,\infty).$$

Proof. Set

$$\bar{v} = c(\beta T e^{-\frac{r}{\beta}} + 1),$$

where $c = \sup\limits_{r\in\mathbb{R}} \frac{m(1+|r|)}{\beta e^{-\frac{r}{\beta}}+r}$.

Since $\lambda \geq 0$, we have

$$\big(\mathcal{L} + \lambda\big)\bar{v} \geq \mathcal{L}\bar{v} = c\big(\beta e^{-\frac{r}{\beta}} + r\big) \geq |f|.$$

Since $\bar{v} > 0$, by comparison principle, we obtain $|u_n| \leq \bar{v}$, where u_n is the solution of problem of (4.2.3). Letting $n \to \infty$, we obtain $|u| \leq \bar{v}$. This completes the proof. □

4.4.3 The case $2\kappa\beta < \sigma^2$

Lemma 4.4.3. *Assume $2\kappa\beta < \sigma^2$. There exists a positive constant c such that the solution u of problem (4.1.2) satisfies*

$$|u(r,T)| \leq c(e^{-\frac{r}{\beta}+2\beta|\mu|T} + |\mu|) \quad \text{in } \mathbb{R}\times(0,\infty).$$

Proof. Set

$$\bar{v} = c(e^{-\frac{r}{\beta}+2\beta|\mu|T} + |\mu|),$$

where $c = \sup\limits_{r\in\mathbb{R}} \frac{m(1+|r|)}{|\mu|(\beta e^{-\frac{r}{\beta}}+r)}$.

Since $\lambda \geq 0$,we have

$$\big(\mathcal{L} + \lambda\big)\bar{v} \geq \mathcal{L}\bar{v} = c|\mu|\big(\beta e^{-\frac{r}{\beta}+2\beta|\mu|T} + r\big) \geq c|\mu|\big(\beta e^{-\frac{r}{\beta}} + r\big) \geq |f|.$$

Since $\bar{v} > 0$, by comparison principle, we obtain $|u_n| \leq \bar{v}$, where u_n is the solution of problem of (4.2.3). Letting $n \to \infty$, we obtain $|u| \leq \bar{v}$. This completes the proof. □

Proof of Theorem 4.1.2

Theorem 4.1.2 then follows from Lemmas 4.4.1, 4.4.2 and 4.4.3.

5.0 LARGE EXPIRATION AND INFINITE HORIZON PROBLEM UNDER VASICEK MODEL

In this chapter, we study the behavior of the price, $u(r, T)$, of CDS as the time to expiration, T, approaches infinity, under the Vasicek model. From (4.1.5) and (4.1.6), we know that as $T \to \infty$, the bond price approaches (i) zero when $2\kappa\beta > \sigma^2$ and (ii) infinity when $2\kappa\beta < \sigma^2$. Thus it is reasonable to expect that the long-expiration behaviors of the CDS price depend on parameter ranges. More precisely, when $2\kappa\beta > \sigma^2$, the price of CDS approaches, as $T \to \infty$, the solution of the corresponding infinite horizon problem; when $2\kappa\beta \leq \sigma^2$, the price may tend to infinity, and here we only consider a special case when the intensities are constants, i.e., $\lambda = \Lambda + \Lambda_1 + \Lambda_2$ and $f = K\Lambda - q$ are constants.

5.1 MAIN RESULTS

The asymptotic behavior of the CDS price follows closely to that of the bond price given in (4.1.5)-(4.1.6), which depends on the sign of $2\kappa\beta - \sigma^2$. Hence, we consider two cases: (i) $2\kappa\beta - \sigma^2 > 0$, (ii) $2\kappa\beta - \sigma^2 \leq 0$. When $2\kappa\beta - \sigma^2 > 0$, the bond price admits a limit. We show the same behavior happen to the CDS price.

Theorem 5.1.1. *Let σ, κ, and β be positive constans. Assume that $2\kappa\beta > \sigma^2$. Let $\mathcal{L}$ be the differential operator defined by*

$$\mathcal{L}u = \big(-\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r \big)u.$$

Let $\lambda(\cdot)$ and $f(\cdot)$ be functions on $\mathbb{R}$ that satisfy, for some positive constant m,

$$\lambda(r) \geq 0, \quad \lambda(r) + |f(r)| \leq m(1+|r|).$$

The following hold:

(1) The infinite horizon problem, for u_*,

$$\begin{cases} \big(\mathcal{L}+\lambda\big)u_* = f & \text{in } \mathbb{R}, \\ \\ u_* = O(e^{ar^2}) & \text{for some } a \in (0, \frac{\beta}{\sigma^2}) \text{ as } |r| \to \infty, \end{cases} \tag{5.1.1}$$

admits a unique solution.

(2) Let $u(r,T)$ be the unique solution of problem (4.1.2) given by Theorem 4.1.1 and $u_*(r)$ be the unique solution of problem (5.1.1). There exist positive constants p and k such that

$$\big|u(r,T) - u_*(r)\big| \leq k\big[e^{-\frac{r}{\beta}} + \mu\big]e^{-pT} \quad \forall\ T > 0,\ r \in \mathbb{R}.$$

Consequently,

$$\lim_{T\to\infty} u(r,T) = u_*(r) \quad \forall\ r \in \mathbb{R}.$$

When $2\kappa\beta \leq \sigma^2$, without detailed information on f and λ, it is difficult to find an asymptotic behavior of the solution of problem (4.1.2) as $T \to \infty$. Here we consider only a specific case: λ and f are positive constant functions. For this we have the following:

Theorem 5.1.2. *Let σ, κ, and β be positive constans. Assume that $2\kappa\beta \leq \sigma^2$. Suppose that $f(\cdot) \equiv 1$ and $\lambda(\cdot) \equiv \lambda$ are constant functions. Set $c := \frac{\kappa}{\beta} - \frac{\sigma^2}{\beta^2}$ and $\nu := \frac{\sigma^2 - 2\kappa\beta}{4\beta^3}$. Let $u(r,T)$ be the unique solution of problem (4.1.2) given by Theorem 4.1.1. Then for every $r \in \mathbb{R}$,*

$$\begin{cases} \lim\limits_{T\to\infty} u(r,T) = \frac{1}{2\beta}\exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right)\int_1^\infty s^{\nu - \frac{\lambda}{2\beta} - 1}\exp\left(\frac{r-c}{\beta\sqrt{s}} - \frac{\sigma^2}{4\beta^3 s}\right)ds & \text{if } \lambda > 2\beta\nu; \\ \\ \lim\limits_{T\to\infty} \frac{u(r,T)}{T} = \exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right) & \text{if } \lambda = 2\beta\nu; \\ \\ \lim\limits_{T\to\infty} \frac{u(r,T)}{e^{(2\beta\nu-\lambda)T}} = \frac{1}{2\beta\nu - \lambda}\exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right) & \text{if } \lambda < 2\beta\nu. \end{cases}$$

Remark: *Theorem 5.1.2 implies that when $\sigma^2 - 2\kappa\beta \geq 2\beta^2\lambda$, $\lim\limits_{T\to\infty} u(r,T) \to \infty$. This complements Theorem 5.1.1.*

5.2 THE INFINITE HORIZON PROBLEM AND ASYMPTOTIC BEHAVIOR WHEN $2\kappa\beta > \sigma^2$

The infinite horizon problem is obtained by removing the $\frac{\partial}{\partial T}$ operator in (4.1.2). Thus, the infinite horizon problem is to find u_* such that

$$\big(\mathcal{L} + \lambda\big)u_* = f \quad \text{in } \mathbb{R}.$$

In this section, we show that this problem is well-posed when $2\kappa\beta > \sigma^2$. For this we begin with considering the corresponding homogeneous equation with $\lambda = 0$:

$$\mathcal{L}\varphi = 0 \quad \text{in } \mathbb{R}.$$

Throughout this section, we always assume $2\kappa\beta > \sigma^2$.

5.2.1 Exact Solution of the Homogeneous Equation with $\lambda = 0$

We use the next two lemmas to illustrate the solution and its properties.

Lemma 5.2.1. *Let $\mathcal{L} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r$, $\sigma > 0, \kappa > 0, \beta > 0$ and $\mu := \frac{1}{\beta^2}(\kappa - \frac{\sigma^2}{2\beta}) > 0$. Then ϕ_+ and ϕ_- defined by*

$$\phi_\pm(r) = \int_0^\infty t^{\mu-1} e^{-\frac{\sigma^2}{4\beta}t^2 \pm (\frac{\sigma^2}{\beta^2} - \frac{\kappa}{\beta} + r)t - \frac{r}{\beta}}\, dt \tag{5.2.1}$$

are two linearly independent positive solutions of $\mathcal{L}\phi = 0$ in $\mathbb{R}$.

Proof. Let $g_\pm(r,t) = -\frac{\sigma^2}{4\beta}t^2 \pm (\frac{\sigma^2}{\beta^2} - \frac{\kappa}{\beta} + r)t - \frac{r}{\beta}$. Then for every $r \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{L}\phi_\pm(r) &= \int_0^\infty t^{\mu-1}e^{g_\pm(r,t)}\Big\{-\frac{\sigma^2}{2}\Big(\pm t - \frac{1}{\beta}\Big)^2 - (\kappa - \beta r)\Big(\pm t - \frac{1}{\beta}\Big) + r\Big\}dt \\
&= \int_0^\infty t^{\mu-1}e^{g_\pm(r,t)}\Big\{-\frac{\sigma^2}{2}t^2 \pm \Big(\frac{\sigma^2}{\beta} - \kappa + \beta r\Big)t - \frac{\sigma^2}{2\beta^2} + \frac{\kappa}{\beta}\Big\}dt \\
&= \beta\int_0^\infty \frac{d}{dt}[t^\mu e^{g_\pm(r,t)}]dt \\
&= 0.
\end{aligned}$$

Hence ϕ_+ and ϕ_- are positive solutions of $\mathcal{L}\phi = 0$ in $\mathbb{R}$. It remains to show that ϕ_+ and ϕ_- are linearly independent. Indeed note that

$$\pm\big[e^{\frac{r}{\beta}}\phi_\pm(r)\big]' = \int_0^\infty t^\mu e^{g_\pm(r,t)+\frac{r}{\beta}}dt > 0 \quad \forall r \in \mathbb{R}.$$

We find that

$$W\big[\phi_-,\phi_+\big] = e^{-\frac{2r}{\beta}}\begin{vmatrix} e^{\frac{r}{\beta}}\phi_- & \big(e^{\frac{r}{\beta}}\phi_-\big)' \\ e^{\frac{r}{\beta}}\phi_+ & \big(e^{\frac{r}{\beta}}\phi_+\big)' \end{vmatrix} > 0 \quad \forall r \in \mathbb{R}.$$

Thus, $\phi_+(r)$ and $\phi_-(r)$ are two linearly independent positive solutions of $\mathcal{L}\phi = 0$ in $\mathbb{R}$. $\square$

Lemma 5.2.2. (**Asymptotic behavior of** $\phi_\pm$) *Let $\phi_\pm(\cdot)$ be defined in Lemma 5.2.1. Then*

$$\begin{aligned}
\phi_\mp(r) &= (\pm r)^{-\mu}\Big\{\Gamma(\mu) + \frac{O(1)}{r}\Big\}e^{-\frac{r}{\beta}} \quad as\ r \to \pm\infty, \\
\phi_\mp'(r) &= \Big\{-\frac{1}{\beta} - \frac{\mu}{r} + \frac{O(1)}{r^2}\Big\}\phi_\mp(r) \quad as\ r \to \pm\infty, \\
\phi_\pm(r) &= \Big(\frac{2\beta}{\sigma^2}\Big)^{\mu-\frac{1}{2}}(\pm r)^{\mu-1}e^{\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2-\frac{r}{\beta}}\Big\{\sqrt{2\pi} + \frac{O(1)}{r}\Big\} \quad as\ r \to \pm\infty, \\
\phi_\pm'(r) &= \Big\{\frac{2\beta}{\sigma^2}r + O(1)\Big\}\phi_\pm(r) \quad as\ r \to \pm\infty.
\end{aligned}$$

Proof. As $r \to \pm\infty$, using the change of variable $s = \pm rt$ for the integral in (5.2.1), we obtain

$$\begin{aligned}
e^{\frac{r}{\beta}}\phi_\mp(r) &= \int_0^\infty t^{\mu-1}e^{-\frac{\sigma^2}{4\beta}t^2\mp(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)t}dt \\
&= \int_0^\infty \Big(\frac{s}{\pm r}\Big)^{\mu-1}e^{-\frac{\sigma^2 s^2}{4\beta r^2}-[\frac{\sigma^2}{\beta^2 r}-\frac{\kappa}{\beta r}+1]s}d\frac{s}{\pm r} \\
&= (\pm r)^{-\mu}\int_0^\infty s^{\mu-1}e^{-s-\frac{\sigma^2 s^2}{4\beta r^2}-(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta})\frac{s}{r}}ds \\
&= (\pm r)^{-\mu}\Big\{\Gamma(\mu) + \frac{O(1)}{r}\Big\}.
\end{aligned}$$

In addition, by differentiation, we obtain

$$
\begin{aligned}
\frac{d}{dr}(e^{\frac{r}{\beta}}\phi_{\mp}(r)) &= \mp\int_0^\infty t^{\mu}e^{-\frac{\sigma^2}{4\beta}t^2\mp(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)t}dt \\
&= \mp(\pm r)^{-\mu-1}\Big\{\Gamma(\mu+1)+\frac{O(1)}{r}\Big\} \\
&= -\frac{1}{r}\Big(\mu+\frac{O(1)}{r}\Big)\big[e^{\frac{r}{\beta}}\phi_{\mp}(r)\big].
\end{aligned}
$$

Thus, $\phi'_{\mp}(r)=\big\{-\frac{1}{\beta}-\frac{\mu}{r}+\frac{O(1)}{r^2}\big\}\phi_{\mp}(r) \quad as \;\; r\to\pm\infty.$

As $r\to\pm\infty$, using the change of variable $s=t\mp\frac{2\beta}{\sigma^2}(r+\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta})$ for the integral in (5.2.1), we obtain

$$
\begin{aligned}
e^{\frac{r}{\beta}}\phi_{\pm}(r) &= \int_0^\infty t^{\mu-1}e^{-\frac{\sigma^2}{4\beta}[t\mp(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)\frac{2\beta}{\sigma^2}]^2+\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2}dt \\
&= e^{\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2}\int_{\mp\frac{2\beta}{\sigma^2}(r+\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta})}^{\infty} e^{-\frac{\sigma^2}{4\beta}s^2}\Big\{s\pm\frac{2\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)\Big\}^{\mu-1}ds \\
&= \Big(\frac{\pm2\beta r}{\sigma^2}\Big)^{\mu-1}e^{\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2}\Big(\int_{-\infty}^{\infty}e^{\frac{-\sigma^2}{4\beta}s^2}ds+\frac{O(1)}{r}\Big) \\
&= \Big(\sqrt{\frac{2\beta}{\sigma^2}}\sqrt{2\pi}+\frac{O(1)}{r}\Big)\Big(\frac{\pm2\beta r}{\sigma^2}\Big)^{\mu-1}e^{\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2} \\
&= \Big(\frac{\pm2\beta}{\sigma^2}\Big)^{\mu-\frac{1}{2}}\Big(\sqrt{2\pi}+\frac{O(1)}{r}\Big)|r|^{\mu-1}e^{\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{d}{dr}(e^{\frac{r}{\beta}}\phi_{\pm}(r)) &= \int_0^\infty \pm t^{\mu}e^{-\frac{\sigma^2}{4\beta}[t\mp(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)\frac{2\beta}{\sigma^2}]^2+\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2}dt \\
&= \Big(\frac{\pm2\beta}{\sigma^2}\Big)^{\mu+\frac{1}{2}}\Big(\sqrt{2\pi}+\frac{O(1)}{r}\Big)|r|^{\mu}e^{\frac{\beta}{\sigma^2}(\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)^2}.
\end{aligned}
$$

Thus,

$$
\phi'_{\pm}(r)=\Big\{\frac{2\beta}{\sigma^2}r+O(1)\Big\}\phi_{\pm}(r) \quad as \;\; r\to\pm\infty.
$$

This completes the proof of the lemma. $\square$

5.2.2 Existence of Solution of Infinite Horizon Problem

We approximate problem (5.1.1) by the following truncation:

$$\begin{cases} \big(\mathcal{L}+\lambda\big)\, u_n = f & \text{in } (-n,n), \\ u_n = 0 & \text{on } \{-n,n\}. \end{cases} \tag{5.2.2}$$

Lemma 5.2.3. *For each $n \in \mathbb{N}$, problem (5.2.2) admits an unique solution.*

Proof. From the proof of Lemma 5.2.1, actually we can also find the kernel of $(\mathcal{L}-\epsilon)\phi_\epsilon = 0$ for $\epsilon \in (0, \beta\mu)$. One can show that, for every $\epsilon \in (0, \beta\mu)$,

$$\phi_{\pm}^{\epsilon}(r) = \int_0^\infty t^{\mu-(1+\frac{\epsilon}{\beta})} e^{-\frac{\sigma^2}{4\beta}t^2 \pm (\frac{\sigma^2}{\beta^2}-\frac{\kappa}{\beta}+r)t-\frac{r}{\beta}}\, dt \tag{5.2.3}$$

are two linearly independent solutions of $(\mathcal{L}-\epsilon)\phi_\epsilon = 0$ in $\mathbb{R}$. The proof is very similar to 5.2.1, so we omit it here.

Since proving the well-posedness of the problem (5.2.2) is equivalent to showing the uniqueness of the problem (5.2.2), it suffices to show $v_n \equiv 0$ is the only solution of

$$\begin{cases} \big(\mathcal{L}+\lambda\big)\, v_n = 0 & \text{in } (-n,n) \\ v_n = 0 & \text{on } \{-n,n\} \end{cases} \tag{5.2.4}$$

We use a contradiction argument. Suppose $v_n \neq 0$. WLOG, we can assume that the set $\{r \,|\, v_n(\cdot) > 0\}$ is non empty.

Now fix $\epsilon \in (0, \beta\mu)$. Denote $\phi = \phi_{+}^{\epsilon}(r) + \phi_{-}^{\epsilon}(r)$, where $\phi_{\pm}^{\epsilon}(r)$ are defined in (5.2.3).

Since $v_n = 0$ on $\{-n, n\}$, there exists $r_* \in (-n, n)$ such that

$$C := \max_{[-n,n]} \frac{v_n}{\phi} = \frac{v_n(r_*)}{\phi(r_*)} > 0.$$

It's obvious that

$$\Big(\frac{v_n(r)}{\phi(r)}\Big)'\Big|_{r=r_*} = 0 \quad \text{and} \quad \Big(\frac{v_n(r)}{\phi(r)}\Big)''\Big|_{r=r_*} \leq 0.$$

Set $\psi = C\phi$. Then $\psi \geq v_n$ in $\mathbb{R}$. Thus, by quotient rule, we obtain from $\left(\frac{v_n(r)}{\phi(r)}\right)'\Big|_{r=r_*} = 0$ that

$$v_n^{'}(r_*)\phi(r_*) - \phi^{'}(r_*)v_n(r_*) = 0.$$

Thus, $\psi^{'}(r_*) = C\phi^{'}(r_*) = \frac{v_n(r_*)}{\phi(r_*)}\phi^{'}(r_*) = v_n^{'}(r_*)$. Similarly, from $\left(\frac{v_n(r)}{\phi(r)}\right)''\Big|_{r=r_*} \leq 0$, we obtain $v_n^{''}(r_*) \leq \psi^{''}(r_*)$.

Thus,

$$\mathcal{L}(v_n - \psi)\Big|_{r=r_*} = -\frac{\sigma^2}{2}(v_n^{''} - \psi^{''})\Big|_{r=r_*} \geq 0.$$

Since

$$0 = (\mathcal{L} + \lambda)v_n(r_*) > \mathcal{L}v_n(r_*) \geq \mathcal{L}\psi(r_*) = C\mathcal{L}\phi(r_*) = C\epsilon\phi(r_*) > 0,$$

we get a contradiction. Thus, $v_n \equiv 0$, i.e., problem (5.2.2) admits a unique solution. □

Next we derive an a priori estimate for the solution of the approximation problem (5.2.2). Since the coefficient of u in the Vasicek differential operator

$$\mathcal{L}u = -\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial r^2} - (\kappa - \beta r)\frac{\partial u}{\partial r} + ru,$$

is r which is not always positive, we cannot apply the standard comparison principle. Hence, we introduce a new comparison principle in next Lemma.

Lemma 5.2.4. *Let $\mathscr{L}u = -u'' + pu' + qu$ and u be the solution of*

$$\begin{cases} \mathscr{L}u = f & \text{in } (a,b), \\ u = 0 & \text{on } \{a,b\}. \end{cases}$$

Suppose

$$\begin{cases} \mathscr{L}v \geq |f| & \text{in } (a,b), \\ v > 0 & \text{in } [a,b]. \end{cases}$$

Then $|u| \leq v$ in $[a,b]$.

Proof. Set $w = \frac{u}{v}$. Then w satisfies

$$\begin{cases} -w'' + Pw' + Qw = F & \text{in } [a,b], \\ w = 0 & \text{on } \{a,b\}, \end{cases} \tag{5.2.5}$$

where $P = p - \frac{2v'}{v}$, $Q = \frac{-v''+pv'+qv}{v}$ and $F = \frac{f}{v}$.

It's obvious that 1 is a super-solution and -1 is a sub-solution of problem (5.2.5). Since $Q \geq 0$, we can show $w \leq 1$ and $w \geq -1$ by the standard comparison principle. Thus, $|u| \leq v$. ☐

Lemma 5.2.5. (L^∞**-estimate of** u_n) *Let u_n be the solution of problem* (5.2.2). *Set* $\mu = \frac{2\kappa\beta-\sigma^2}{2\beta^3}$. *Define* $c = \sup\limits_{r\in\mathbb{R}} \frac{m(1+|r|)}{\mu(\beta e^{-\frac{r}{\beta}}+r)}$. *Then*

$$|u_n| \leq c[e^{-\frac{r}{\beta}} + \mu]. \tag{5.2.6}$$

Proof. Set $\bar{v} = c[e^{-\frac{r}{\beta}} + \mu]$. Comparing problem (5.2.2) and

$$\begin{cases} \big(\mathcal{L} + \lambda\big)\, \bar{v} \geq |f| & \text{in } (-n,n), \\ \bar{v} > 0 & \text{on } [-n,n], \end{cases}$$

we obtain $|u_n| \leq \bar{v}$ by Lemma 5.2.4. ☐

Lemma 5.2.6. (**Existence**) *Assume $2\kappa\beta > \sigma^2$. Then problem (5.1.1) admits at least one solution.*

Proof. For each $n \in \mathbb{N}$, let u_n be the solution of problem (5.2.2). Extend u_n by $u_n = 0$ for $|r| > \frac{1}{n}$. Now let's consider the family $\{u_n\}_{n=1}^{\infty}$.

By Lemmas 5.2.3 and 5.2.5, we know that for each fixed $M > 0$, $\{u_n\}_{n=1}^{\infty}$ is a compact family in $C^2([-M, M])$. Thus, there exists subsequence of $\{u_n\}_{n=1}^{\infty}$, i.e., $\{u_{n_k}\}_{k=1}^{\infty}$ converges to $u_* \in C^2([-M, M])$ for any $M > 0$.

Since

$$(\mathcal{L} + \lambda)u_* = \lim_{k\to\infty} (\mathcal{L} + \lambda)u_{n_k} = f \text{ and } |u_*| \le \lim_{k\to\infty} |u_{n_k}| \le \bar{v},$$

u_* is a solution of problem (5.1.1). □

5.2.3 Uniqueness of Solution of Infinite Horizon Problem

Lemma 5.2.7. (**Uniqueness**) *Assume $2\kappa\beta > \sigma^2$. Then problem (5.1.1) admits at most one solution.*

Proof. The proof is very similarly to that of Lemma 5.2.3, we give a brief proof instead of providing all details.

Suppose the problem (5.1.1) has two solutions u_1 and u_2. Set $v = u_1 - u_2$. Then there exists $a \in (0, \frac{\beta}{\sigma^2})$ such that

$$\begin{cases} (\mathcal{L} + \lambda)v = 0 & \text{in } \mathbb{R}, \\ v = O(e^{ar^2}) & \text{as } |r| \to \infty. \end{cases}$$

Suppose $v \neq 0$. WLOG, we can assume that the set $\{r \,|\, v(\cdot) > 0\}$ is non empty.

Now fix $\epsilon \in (0, \beta\mu)$. Denote $\phi^\epsilon = \phi_+^\epsilon(r) + \phi_-^\epsilon(r)$, where $\phi_\pm^\epsilon(r)$ are defined in (5.2.3). Use similar proof of Lemma 5.2.2, it's easy to show that there exist positive constants c_1 and c_2 such that

$$\phi(r) \to c_1 r^{\mu-(1+\frac{\epsilon}{\beta})} e^{(\frac{\beta}{\sigma^2}r^2 + \frac{r}{\beta} - \frac{2\kappa}{\sigma^2}r)} \quad \textit{as } \ r \to \infty,$$

$$\phi(r) \to c_2|r|^{\mu-(1+\frac{\epsilon}{\beta})}e^{(\frac{\beta}{\sigma^2}r^2+\frac{r}{\beta}-\frac{2\kappa}{\sigma^2}r)} \quad as \;\; r \to -\infty.$$

Thus, $\frac{v}{\phi} \to 0$ as $|r| \to \infty$. Then there exists $r_1 \in \mathbb{R}$ such that

$$\frac{v(r_1)}{\phi(r_1)} = \max_{\mathbb{R}} \frac{v}{\phi} \geq \frac{v(r)}{\phi(r)} \quad \forall r \in \mathbb{R}.$$

Then we can use the same trick as that in the proof of Lemma 5.2.3 to get a contradiction. Thus, $v \equiv 0$, i.e., $u_1 \equiv u_2$. □

The first part of Theorem 5.1.1 for the well-posedness of the infinite horizon problem when $2\kappa\beta > \sigma^2$ then follows from Lemma 5.2.6 and Lemma 5.2.7.

5.2.4 Asymptotic Behavior When $2\kappa\beta > \sigma^2$

In this section, we shall prove the second part of Theorem 5.1.1. More precisely, we show the price of CDS, as $T \to \infty$, approaches the solution of the infinite horizon problem (5.1.1).

Proof. Let $u_n(r,T)$ be the unique solution of problem (4.2.3) and $u_{*n}(r)$ be the unique solution of problem (5.2.2). Then $v_n(r,T) := u_n(r,T) - u_{*n}(r)$ satisfies the following truncation:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \lambda\right) v_n = 0 & \text{in } (-n,n)\times(0,n^2], \\ v_n(\cdot,0) = -u_{*n} & \text{on } (-n,n), \\ v_n = 0 & \text{on } \{-n,n\}\times(0,n^2]. \end{cases} \tag{5.2.7}$$

Since $\min_{r\in\mathbb{R}}[\beta e^{-\frac{r}{\beta}} + r] = [\beta e^{-\frac{r}{\beta}} + r]\Big|_{r=0} = \beta$, we have $\beta e^{-\frac{r}{\beta}} + r > \beta$ for each $r \in \mathbb{R}$. Thus,

$$k := \sup_{r\in\mathbb{R}} \frac{m(1+|r|)}{\mu(\beta e^{-\frac{r}{\beta}} + r)} \quad \text{and} \quad p := \inf_{r\in\mathbb{R}} \frac{\beta e^{-\frac{r}{\beta}} + r}{e^{-\frac{r}{\beta}} + \mu}$$

are both well-defined positive contants.

Let $\Phi = k(e^{-\frac{r}{\beta}} + \mu)e^{-pT}$.

Since $\lambda \geq 0$, we have

$$(\frac{\partial}{\partial T} + \mathcal{L} + \lambda)\Phi \geq (\frac{\partial}{\partial T} + \mathcal{L})\Phi = ke^{-pT}\big[\mu(\beta e^{-\frac{r}{\beta}} + r) - p(e^{-\frac{r}{\beta}} + \mu)\big] > 0.$$

From Lemma 5.2.5, we know $|u_{*n}| \leq k(e^{-\frac{r}{\beta}} + \mu)$ in $\mathbb{R}$. Since $\Phi|_{T=0} = k(e^{-\frac{r}{\beta}} + \mu)$, by comparison principle, we obtain $|v_n| \leq \Phi$. Letting $n \to \infty$, we obtain $|v| \leq \Phi$. This completes the proof. $\square$

5.3 ASYMPTOTIC BEHAVIOR WHEN $2\kappa\beta \leq \sigma^2$ FOR THE CASE OF CONSTANT INTENSITY

When $2\kappa\beta > \sigma^2$, we already discussed the asymptotic behavior of solution in previous section. In this section, we study the asymptotic behavior of solution when $0 < 2\kappa\beta \leq \sigma^2$. Considering the case without detailed information on λ and f, we can do little. For this reason, here we consider a special case that both λ and f are constant functions. Hence, we assume the following in this section:

(1) $\mathcal{L} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r,\ \sigma > 0, \kappa > 0, \beta > 0,\ 2\kappa\beta \leq \sigma^2$;

(2) $\lambda(r) = \lambda$ is a constant function;

(3) $f(r) = 1$.

Recall that we model default time of the designated occurrence of the underlying credit event by the first arrival time of Poisson process with intensity $\lambda(r)$. One simple specification is to let the intensity be a constant, that is, let $\lambda(r) = \lambda$. The jump process is then referred to as a homogeneous Poisson process.

When $\lambda(\cdot) \equiv \lambda$ and $f(\cdot) \equiv 1$ are constant functions, we can convert problem (4.1.2) into a heat equation. The transformation is as follows. Let u be the solution of problem (4.1.2). For simplicity, we use subscripts to denote partial derivatives.

Step 1: Let $v(r,T) = e^{-\left(-\frac{1}{\beta}r+\frac{\sigma^2-2\kappa\beta}{2\beta^2}T-\lambda T\right)}u(r,T)$ and $F = e^{\left(\frac{1}{\beta}r-\frac{\sigma^2-2\kappa\beta}{2\beta^2}T+\lambda T\right)}f$. Then v satisfies

$$\begin{cases} v_T - \frac{\sigma^2}{2}v_{rr} + (\beta r + \frac{\sigma^2}{\beta} - \kappa)v_r = F & \text{in } \mathbb{R}\times(0,\infty), \\ v(\cdot,0) = 0 & \text{in } \mathbb{R}. \end{cases}$$

Step 2: Let $y = e^{-\beta T}(r - \frac{\kappa}{\beta} + \frac{\sigma^2}{\beta^2})$, $\tau = -\frac{\sigma^2}{4\beta}(e^{-2\beta T}-1)$, $w(y,\tau) = v(r,T)$ and $F^*(y,\tau) = \frac{2e^{2\beta T}F(r,T)}{\sigma^2}$. Then w satisfies

$$\begin{cases} w_\tau - w_{yy} = F^* & \text{in } \mathbb{R}\times(0,\frac{\sigma^2}{4\beta}), \\ w(\cdot,0) = 0 & \text{in } \mathbb{R}. \end{cases} \tag{5.3.1}$$

It then follows from ([23]) that for every $(y,\tau)\in\mathbb{R}\times(0,\frac{\sigma^2}{4\beta})$,

$$w(y,\tau) = \int_0^\tau\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi(\tau-s)}}e^{-\frac{(y-z)^2}{4(\tau-s)}}F^*(z,s)dzds.$$

Substituting back to the original variables, we then obtain follows:

$$\begin{aligned} u(r,T) =& e^{\left(-\frac{1}{\beta}r+\frac{\sigma^2-2\kappa\beta}{2\beta^2}T-\lambda T\right)}w\big[e^{-\beta T}(r-\frac{\kappa}{\beta}+\frac{\sigma^2}{\beta^2}), -\frac{\sigma^2}{4\beta}(e^{-2\beta T}-1)\big] \\ =& e^{\left(-\frac{1}{\beta}r+\frac{\sigma^2-2\kappa\beta}{2\beta^2}T-\lambda T\right)}\int_0^{-\frac{\sigma^2}{4\beta}(e^{-2\beta T}-1)}\int_{-\infty}^\infty \frac{1}{\sqrt{\frac{\pi\sigma^2}{\beta}(1-e^{-2\beta T})-4\pi s}} \\ & e^{-\frac{\beta\left[e^{-\beta T}(r-\frac{\kappa}{\beta}+\frac{\sigma^2}{\beta^2})-z\right]^2}{\sigma^2(1-e^{-2\beta T})-4\beta s}}F^*(z,s)dzds, \end{aligned} \tag{5.3.2}$$

where $F^*(z,s) = \frac{2\left(1-\frac{4\beta}{\sigma^2}s\right)^{\frac{\sigma^2-2\kappa\beta}{4\beta^3}-\frac{\lambda}{2\beta}}}{\sigma^2-4\beta s}\exp\left\{\frac{z}{\beta\sqrt{1-\frac{4\beta}{\sigma^2}s}}+\frac{\kappa}{\beta^2}-\frac{\sigma^2}{\beta^3}\right\}f\left(\frac{z}{\sqrt{1-\frac{4\beta}{\sigma^2}s}}+\frac{\kappa}{\beta}-\frac{\sigma^2}{\beta^2}\right)$.

Let $R = \frac{z}{\sqrt{1-\frac{4\beta}{\sigma^2}s}}+\frac{\kappa}{\beta}-\frac{\sigma^2}{\beta^2}$, $t = e^{2\beta T}(1-\frac{4\beta}{\sigma^2}s)-1$, $c = \frac{\kappa}{\beta}-\frac{\sigma^2}{\beta^2}$ and $\nu = \frac{\sigma^2-2\kappa\beta}{4\beta^3}$. Then we have

$$u(r,T) = \frac{e^{-\frac{1}{\beta}r}}{2\sqrt{\beta\pi}\sigma}\int_0^{e^{2\beta T}-1}\int_{-\infty}^\infty \frac{(t+1)^{\nu-\frac{\lambda}{2\beta}}}{\sqrt{t(t+1)}}f(R)\exp\left(\frac{R}{\beta}-\frac{\beta[(R-c)\sqrt{t+1}-(r-c)]^2}{4t\sigma^2}\right)dRdt.$$

After a further substitution $s = t+1$, we summarize above calculations as following:

Theorem 5.3.1. *Assume that $\lambda(\cdot) \equiv \lambda$ is a constant function. Then the solution of problem* (4.1.2) *is given by the explict formula*

$$u(r,T) = \frac{e^{-\frac{1}{\beta}r}}{2\sqrt{\beta\pi}\sigma}\int_1^{e^{2\beta T}}\int_{-\infty}^{\infty}\frac{s^{\nu-\frac{\lambda}{2\beta}}}{\sqrt{s(s-1)}}f(R)\exp\Big(\frac{R}{\beta}-\frac{\beta[(R-c)\sqrt{s}-(r-c)]^2}{4(s-1)\sigma^2}\Big)dRds.$$

Now we focus our attention on the case $f \equiv 1$.

Proof of Theorem 5.1.2

Proof. Since $f = 1$, then we have

$$u(r,T) = \frac{e^{-\frac{1}{\beta}r}}{2\sqrt{\beta\pi}\sigma}\int_1^{e^{2\beta T}}\int_{-\infty}^{\infty}\frac{s^{\nu-\frac{\lambda}{2\beta}}}{\sqrt{s(s-1)}}\exp\Big(\frac{R}{\beta}-\frac{\beta[(R-c)\sqrt{s}-(r-c)]^2}{4(s-1)\sigma^2}\Big)dRds.$$

Completing the square for the exponent by

$$\frac{R}{\beta}-\frac{\beta[(R-c)\sqrt{s}-(r-c)]^2}{4(s-1)\sigma^2} \quad = \tfrac{-\beta s}{4(s-1)\sigma^2}\Big[R-\Big(\tfrac{(s-1)\sigma^2}{2\beta^2 s}+\tfrac{(\sqrt{s}-1)\,c+r}{\sqrt{s}}\Big)\Big]^2+\tfrac{(s-1)\sigma^2}{4\beta^3 s}+\tfrac{c\sqrt{s}+(r-c)}{\beta\sqrt{s}}.$$

Thus,

$$u(r,T) \quad = \quad \frac{1}{2\beta}\exp\Big(\frac{c-r}{\beta}+\frac{\sigma^2}{4\beta^3}\Big)\int_1^{e^{2\beta T}} s^{\nu-\frac{\lambda}{2\beta}-1}\exp\Big(\frac{r-c}{\beta\sqrt{s}}-\frac{\sigma^2}{4\beta^3\,s}\Big)ds.$$

To study the asymptotic behavior of u as $T \to \infty$, we consider three cases: (1) $\lambda > 2\beta\nu$, (2) $\lambda = 2\beta\nu$, and (3) $\lambda < 2\beta\nu$.

(1) The case $\lambda > 2\beta\nu$.

In this case, sending T in (5.3.3) to ∞, we obtain

$$\lim_{T\to\infty} u(r,T) = \frac{1}{2\beta}\exp\Big(\frac{c-r}{\beta}+\frac{\sigma^2}{4\beta^3}\Big)\int_1^{\infty} s^{\nu-\frac{\lambda}{2\beta}-1}\exp\Big(\frac{r-c}{\beta\sqrt{s}}-\frac{\sigma^2}{4\beta^3\,s}\Big)ds.$$

(2) The case $\lambda = 2\beta\nu$.

Using the L'Hopital's Rule, we derive that, for each fixed $r \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{u(r,T)}{T} = \lim_{T\to\infty} \frac{\exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right)\,\exp\left(\frac{r-c}{\beta\,e^{\beta T}} - \frac{\tau^*}{\beta^2\,e^{2\beta T}}\right)}{1} = \exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right).$$

(3) The case $\lambda < 2\beta\nu$.

Using the L'Hopital's Rule, we derive that, for each fixed $r \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{u(r,T)}{e^{(2\beta\nu-\lambda)T}} = \lim_{T\to\infty} \frac{\exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right)\,\exp\left(\frac{r-c}{\beta\,e^{\beta T}} - \frac{\tau^*}{\beta^2\,e^{2\beta T}}\right)}{2\beta\nu-\lambda} = \frac{\exp\left(\frac{c-r}{\beta} + \frac{\sigma^2}{4\beta^3}\right)}{2\beta\nu-\lambda}.$$

This completes the proof of the Theorem 5.1.2 $\square$

6.0 CONCLUSION

A credit default swap, or CDS, is a financial agreement between two parties about an exchange of cash flows that depend on the occurrence of a credit default or in general a credit event. A CDS may terminate earlier than the expiration or the occurrence of the credit event when one party of the contract defaults, this is called counterparty risk. In this thesis, we price CDSs with counterparty risks. We model the credit default and counterparty risks by the first arrival times of Poisson processes with intensities depending on short interest rates. For short interest rates, we use the CIR and Vasicek models which are the most widely used. As a result, the prices of CDS are derived as solutions of different partial differential equations with respect to the CIR and Vasicek models, respectively.

The emphasis of this thesis is on the mathematical study of these resulting PDE problems. We investigate the existence, uniqueness and properties of their solutions. Moreover, we study the corresponding infinite horizon problems and connect their solutions with the asymptotic behavior of the price of CDS as time to expiry tends to infinity.

It is important to point out that Hu, Jiang, Liang, and Wei ([13], 2012) have already used the CIR model to price the CDS. They solved the problem under the following boundedness "boundary condition":

$$u \in L^{\infty}. \tag{6.0.1}$$

However, intrinsic to the CIR model, they need a restriction on the range of the parameters, which does not fall into the empirical calibration in certain cases; see, for example, the empirical calibration of Peng ([18], 2016). One purpose of this thesis is to remove the restriction by introducing new techniques. As a result, we extend the theory of Hu-Jiang-Liang-Wei in [13] to the full range of parameters. From financial perspective, this extension makes the application of our pricing model more practical and robust. Our idea is to replace (6.0.1) by the following Lipschitz continuity "boundary condition":

$$\frac{\partial u}{\partial r} \in L^{\infty}. \tag{6.0.2}$$

Typically, for parabolic PDE on unbounded domain, the boundary condition (6.0.1) is very standard. Nevertheless, since the diffusion, $\sigma\sqrt{r}$, of the CIR model degenerates at $r = 0$, our work demonstrates that (6.0.2) is the right condition for the solution class, whereas (6.0.1) may not be appropriate in certain parameter ranges. From this perspective, our results can be regarded as a new development toward the theory of PDEs.

We also study the asymptotic behavior of the price of CDS as time to expiration approaches infinity. We show that the corresponding infinite horizon problem is the limit problem of the pricing model, i.e., the price of CDS for long time to expiry is approximately equal to the solution of the infinite horizon problem.

To make the study more complete and in-depth, we also consider the Vasicek model. The Vasicek and the CIR models are very different, although they are both equilibrium short rate models. The biggest difference is that the Vasicek model allows negative interest rate, whereas the CIR model does not. In the past, Vasicek model was criticized for allowing negative interest rate. Since nowadays many countries like Sweden, Switzerland, and Japan have already introduced negative interest rates, the disadvantage of allowing negative interest rate in the Vasicek model becomes an advantage. Thus, it is meaningful to use Vasicek model to price CDS.

Nevertheless, under the Vasicek model, when the interest rate, r, approaches negative infinity, the elliptic operator

$$\mathcal{L} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r$$

becomes very "stiff", and the meaning of the discount factor becomes meaningless, causing lots of technical difficulties. We show the well-posedness of the problem. In addition, we provide accurate bounds for the CDS prices. We regard the upper bound of the CDS price as an important contribution from both mathematical and financial point of view.

Under the Vasicek model, we can obtain the well-posedness of the PDE problem in full parameter range. In addition, we discover that the asymptotic behavior of the CDS price depends on the sign of $2\kappa\beta - \sigma^2$. More precisely, when $2\kappa\beta > \sigma^2$, the price of CDS approaches, as $T \to \infty$, the finite solution of the corresponding infinite horizon problem. When $2\kappa\beta \leq \sigma^2$, the asymptotic behavior depends on the models of intensities; in particular, under the assumption that the intensities are constants, the price of CDS tends to infinity as $T \to \infty$. Moreover, we find the exact growth rate of the price. No matter from mathematical perspective or from financial perspective, these results can be regarded as benchmarks for the pricing model of CDS.

Since CDS becomes more and more popular, more sophisticated mathematical analysis is needed. We can study the pricing problem by other models such as the structure model. We can also perform the empirical work based on different models. These will be our guidance of future work.

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