ANALYSIS AND GEOMETRY IN METRIC SPACES: SOBOLEV MAPPINGS, THE HEISENBERG GROUP, AND THE WHITNEY EXTENSION THEOREM

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This thesis focuses on analysis in and the geometry of the Heisenberg group as well as geometric properties of Sobolev mappings. It begins with a detailed introduction to the Heisenberg group. After, we see a new and elementary proof for the structure of geodesics in the sub-Riemannian Heisenberg group. We also prove that the Carnot-Carathéodory metric is real analytic away from the center of the group.

Next, we prove a version of the classical Whitney Extension Theorem for curves in the Heisenberg group. Given a real valued function defined on a compact set in Euclidean space, the classical Whitney Extension Theorem from 1934 gives necessary and sufficient conditions for the existence of a C^k extension defined on the entire space. We prove a version of the Whitney Extension Theorem for C^1, horizontal curves in the Heisenberg group.

We then turn our attention to Sobolev mappings. In particular, given a Lipschitz map from a compact subset Z of Euclidean space into a Lipschitz connected metric space, we construct a Sobolev extension defined on any bounded domain containing Z.

Finally, we generalize a classical result of Dubovitskiĭ for smooth maps to the case of Sobolev mappings. In 1957, Dubovitskiĭ generalized Sard's classical theorem by establishing a bound on the Hausdorff dimension of the intersection of the critical set of a smooth map and almost every one of its level sets. We show that Dubovitskiĭ's theorem can be generalized to $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ mappings for all positive integers k and $p > n$.

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1.0 INTRODUCTION

This thesis is composed of four original papers created during my time as a graduate student at the University of Pittsburgh. These results are summarized here.

1.1 GEODESICS IN THE HEISENBERG GROUP

The sub-Riemannian Heisenberg group $\mathbb{H}^1$ is $\mathbb{R}^3$ given the structure of a Lie group via the introduction of a smooth group operation. This group operation induces a basis of left invariant vector fields $\{X, Y, T\}$ on $\mathbb{R}^3$. We use this basis of vector fields to define a smooth distribution of 2-dimensional planes in the 3-dimensional tangent space at any point p in $\mathbb{R}^3$ as the span of $X(p)$ and $Y(p)$. An absolutely continuous curve in $\mathbb{R}^3$ is horizontal if it lies tangent to this distribution almost everywhere.

Intuitively, a curve in space is horizontal if its velocity is restricted to a particular 2-dimensional plane at almost every time. That is, the motion of a horizontal curve must almost everywhere be some combination of movement in the direction of X and movement in the direction of Y. Such a restriction is very natural. For example, a person on a bicycle can cycle forward and backward or rotate their front tire at some angle. Any motion of the bicycle must be some combination of these two movements.

This Lie group structure, $2n$-dimensional distribution, and notion of horizontality may be similarly defined in higher dimensions to construct the Heisenberg group $\mathbb{H}^n$ from $\mathbb{R}^{2n+1}$. We give $\mathbb{H}^n$ a metric space structure via the usual Carnot-Carathéodory metric d_{cc} defined as the infimum of lengths of horizontal curves connecting any two given points. Any two points in $\mathbb{H}^n$ may be connected by a horizontal curve (see Theorem 26) so d_{cc} is indeed a metric.

In particular, there is a shortest curve (that is, a geodesic) connecting any two points in $\mathbb{H}^n$. In $\mathbb{H}^1$, it can easily be shown that any geodesic from the origin to a point on the vertical axis must project down to a circle on the xy-plane. Indeed, this result follows from the classical isoperimetric inequality and Green's theorem. Any geodesic in $\mathbb{H}^1$ can be obtained from one described here by applying left multiplication in the group operation. However, such a simple geometric argument does not easily generalize to the higher dimensional case.

In Chapter 4, we determine the structure of geodesics in $\mathbb{H}^n$ via an explicit parameterization as seen in Theorem 35. Such parameterizations have been previously established, but previous proofs rely on the Pontryagin maximum principle [5, 10, 73]. We provide a new and more geometric proof involving a version of the isoperimetric inequality in $\mathbb{R}^{2n}$ (see Theorem 38). The proof is similar to that of the classical isoperimetric inequality in the plane given by Hurwitz [53] and relies strongly on Fourier series.

Moreover, in this chapter we see that the Carnot-Carathéodory metric is real analytic away from the vertical axis (Theorem 42). Monti showed in [72, 73] that the metric is C^∞ smooth away from the vertical axis, but real analyticity of the metric is an original result. From the proof of this result, we find an implicit formula for the CC-metric (Corollary 43). Recent advances regarding geodesics in $\mathbb{H}^n$ and the analyticity of the CC-metric can be found in the article by Ritoré [78].

In $\mathbb{H}^1$, geodesics connecting the origin to a fixed point on the vertical axis can all be mapped onto one another via a rotation about this axis. Chapter 4 concludes with a similar type of result. Proposition 44 allows us to obtain any geodesic connecting the origin in $\mathbb{H}^n$ to a point on the $(2n+1)^{st}$ axis via a rotation in $\mathbb{R}^{2n+1}$ about this axis.

1.2 THE WHITNEY EXTENSION THEOREM FOR HORIZONTAL CURVES IN THE HEISENBERG GROUP

Chapter 5 focuses again on curves in the Heisenberg group. The classical Whitney Extension Theorem from 1934 [94] established a necessary and sufficient condition for the existence of a C^k extension of a real-valued, continuous function defined on a compact set in $\mathbb{R}^m$. Whitney's theorem has found applications in the construction of pathological mappings [93] and in the approximation of Lipschitz mappings by C^1 maps [29, Theorem 3.1.15]. These approximations appear in the study of rectifiability which has had recent popularity in the setting of the Heisenberg group.

In 2001, Franchi, Serapioni, and Serra Cassano [32] proved a C^1 version of the Whitney extension theorem for real valued functions on the Heisenberg group. We would like to prove a version of Whitney's theorem for mappings *into* the Heisenberg group. That is, given a continuous mapping $f : K \to \mathbb{H}^n$ defined on a compact set $K \subset \mathbb{R}^m$, under what conditions does f have a C^k, horizontal extension F defined on all of $\mathbb{R}^m$? Here, differentiability of F is defined in the classical sense as a mapping into $\mathbb{R}^{2n+1}$. Clearly, if a smooth, horizontal extension F of f exists, then f must already satisfy Whitney's classical condition. However, we will see in Proposition 48 that Whitney's condition alone is not enough. Moreover, in Proposition 49, we establish a restriction on vertical growth that any C^1, horizontal curve in $\mathbb{H}^n$ necessarily satisfies.

Theorem 50 in Chapter 5 establishes a version of Whitney's theorem for C^1 curves in the Heisenberg group. More precisely, we find a necessary and sufficient condition for the existence of a C^1, horizontal extension of a continuous mapping into $\mathbb{H}^n$ defined on a compact subset of $\mathbb{R}$. Theorem 52 is a rewording of Theorem 50 in terms of the Pansu derivative [75]. As an application of this result, we prove a version of the Lusin approximation for Lipschitz curves in $\mathbb{H}^n$ proven previously by Speight [81]. That is, we show that any Lipschitz curve in the Heisenberg group is equal to a C^1, horizontal curve up to a set of arbitrarily small measure.

1.3 SOBOLEV EXTENSIONS OF LIPSCHITZ MAPPINGS INTO METRIC SPACES

In Chapter 6, we continue to explore the existence of extensions of mappings into the Heisenberg group. Suppose X and Y are metric spaces and $f : X \supset A \to Y$ is L-Lipschitz. Much work has been done historically to determine when one may extend f to a CL-Lipschitz map defined on X (for a constant $C \geq 1$) [17, 28, 54, 61, 62, 63, 90, 91]. It is known that such an extension always exists when Y is $\mathbb{R}$ [69], when Y is a Banach space and A is a finite set [54], or when X and Y are both Hilbert spaces [86].

We say that a metric space Y is Lipschitz $(n-1)$-connected if there is a constant $\gamma \geq 1$ so that any L-Lipschitz map $f : S^k \to Y$ on the k-dimensional sphere has a γL-Lipschitz extension $F : B^{k+1} \to Y$ on the $(k+1)$-ball for $k = 0, 1, \ldots, n-1$. Lang and Schlichenmeier [61] proved that there is a constant $C \geq 1$ such that any L-Lipschitz map from a closed set $A \subset X$ into a Lipschitz $(n-1)$-connected space Y has a CL-Lipschitz extension as long as X has Assouad-Nagata dimension at most n. (See [6, 61, 91] for more information about AN dimension.) Wenger and Young [91] proved this Lipschitz extension result for Lipschitz mappings into the Heisenberg group $\mathbb{H}^n$ with the same dimension restriction on X.

This Lipschitz extension result fails, however, when the dimension of the domain is too large. Balogh and Fässler [7] constructed a Lipschitz map from the n-sphere into $\mathbb{H}^n$ which has no Lipschitz extension defined on the $(n+1)$-ball. Thus, one cannot hope to construct a Lipschitz extension in general from a subset of $\mathbb{R}^m$ into $\mathbb{H}^n$.

However, we show in Chapter 6 that, if $\Omega \subset \mathbb{R}^m$ is a bounded domain, then any Lipschitz mapping from a compact subset of Ω into the Heisenberg group $\mathbb{H}^n$ has a Sobolev extension in the class $W^{1,p}(\Omega, \mathbb{H}^n)$ for $1 \leq p < n+1$ regardless of the dimension m. Moreover, we have a bound on the Sobolev norm of the first $2n$ components of the mapping in terms of the Lipschitz constant. These are the contents of Theorem 63. Here, $W^{1,p}$ Sobolev regularity means that the extension is in the classical Sobolev space $W^{1,p}(\Omega, \mathbb{R}^{2n+1})$ and satisfies a weak horizontality condition (6.1). The restriction on p is sharp in the sense that the theorem fails when $p = n+1$ (see Proposition 64).

Further, we will see in Theorem 66 that this extension result holds in more generality. That is, any Lipschitz mapping from a compact subset of Ω into any Lipschitz $(n-1)$-connected metric space Y has an extension in the class $AR^{1,p}(\Omega, Y)$ for $1 \leq p < n+1$. Again there is no restriction on the dimension, and we have a bound on the "slope" of the extension. The class $AR^{1,p}(\Omega, Y)$ is the set of Ambrosio-Reshetnyak-Sobolev mappings defined in Section 6.1 and first introduced in [4, 76].

1.4 THE DUBOVITSKII-SARD THEOREM IN SOBOLEV SPACES

Chapter 7 continues the study of Sobolev mappings. Consider a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$. If f is of class C^k for $k > \max(n-m, 0)$, the famous theorem of Sard from 1942 [79] demonstrates that the set of critical values of f has m-dimensional Lebesgue measure zero. The regularity k in Sard's result is optimal [24, 37, 43, 55, 66, 93].

In 1957, Dubovitskiĭ [24] proved a generalization of Sard's result (seemingly without any knowledge of the previous theorem). He showed that the intersection of the critical set of a C^k mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ with the level set $f^{-1}(y)$ has $(n - m - k + 1)$-dimensional Hausdorff measure zero for m-almost every $y \in \mathbb{R}^m$.

The theorem of Sard has been generalized in recent years to apply to the class of Sobolev mappings. See, for example, [2, 12, 13, 14, 21, 31, 59, 87]. One such generalization was proven in 2001 by De Pascale [21]. He showed that the critical values of any map in the class $W^{k,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ has m-dimensional Lebesgue measure zero when $k > \max(n - m, 0)$ and $n < p < \infty$. Notice that, since $p > n$, the Sobolev mapping is actually in the class C^{k-1} as a result of the Morrey inequality.

In Theorem 84, we find a generalization of Dubovitskiĭ's result to the Sobolev class in the sense of De Pascale's result. That is, we show that the intersection of the critical set of $f \in W^{k,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ with the level set $f^{-1}(y)$ has $(n - m - k + 1)$-dimensional Hausdorff measure zero for m-almost every $y \in \mathbb{R}^m$ when $n < p < \infty$.

De Pascale's proof of the Sard theorem for Sobolev maps relies on a version of the classical Morse Theorem [74] which decomposes the critical set of a mapping into pieces on which difference quotients converge quickly. Some proofs of the classical Sard theorem [1, 65, 70, 84] utilize the Kneser-Glaeser Rough Composition theorem [35, 57] (see Theorem 86). This is the method that Figalli used in 2008 to reprove De Pascale's version of the Sard theorem. In the proof of Theorem 84, we also rely on this Kneser-Glaeser result. A proof of this classical theorem is then given at the end of the chapter.

2.0 PRELIMINARY NOTIONS

2.1 METRIC SPACES AND THEIR PROPERTIES

Throughout the thesis, the symbol C will be used to represent a generic constant, and the actual value of C may change in a single string of estimates. By writing $C = C(n, m)$, for example, we indicate that the constant C depends on n and m only.

Definition 1. *A metric space is an ordered pair (X, d) where X is a set and $d : X \times X \to [0, \infty)$ is a function (called a metric) satisfying the following for every $x, y, z \in X$:*

1. $d(x, y) = d(y, x)$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

We simply write the metric space X if the metric d is understood.

For any $x \in X$ and $r > 0$, we will call $B(x, r) = \{y \in X \, : \, d(x, y) < r\}$ the *open ball* in X centered at x with radius r. Given any $A \subset X$, the *diameter* of A equals

$$\operatorname{diam}(A) := \sup\{d(x, y) \, : \, x, y \in A\}.$$

Suppose (X, d_X) and (Y, d_Y) are metric spaces.

Definition 2. *A mapping $f : X \to Y$ is L-Lipschitz continuous for $L \geq 0$ if*

$$d_Y(f(a), f(b)) \leq L \, d_X(a, b) \quad \text{for every } a, b \in X.$$

We say $f : X \to Y$ is α-Hölder continuous for $\alpha > 0$ if there is $C \geq 0$ with

$$d_Y(f(a), f(b)) \leq C \, d_X(a, b)^\alpha \quad \text{for every } a, b \in X.$$

We call the collection $\mathcal{P}(X)$ of all subsets of X the *power set* of X.

Definition 3. *An outer measure μ on X is a mapping $\mu : \mathcal{P}(X) \to [0, \infty]$ satisfying the following:*

1. $\mu(\emptyset) = 0$,
2. *if $A, B \in \mathcal{P}(X)$ with $A \subset B$, then $\mu(A) \leq \mu(B)$,*
3. *for any sequence $\{A_j\}$ in $\mathcal{P}(X)$, we have*

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

.

Throughout the thesis, we will consider a very special outer measure. Define for any $s \geq 0$, $\delta > 0$, and $A \subset X$

$$\mathcal{H}_\delta^s(A) := \inf\left\{\sum_{j=1}^{\infty} (\operatorname{diam} U_j)^s \;:\; A \subset \bigcup_{j=1}^{\infty} U_j,\ \operatorname{diam}(U_j) < \delta\right\}.$$

Notice that $\mathcal{H}_\delta^s(A)$ is non-decreasing as $\delta \to 0$ as we are taking infima over smaller and smaller sets.

Definition 4. *The s-dimensional Hausdorff measure $\mathcal{H}^s$ is defined for any $A \subset X$ as*

$$\mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}_\delta^s(A).$$

The Hausdorff measure is indeed an outer measure. We say that a set is s-null if its s-dimensional Hausdorff measure equals zero. In the special case $s = 0$, $\mathcal{H}^0$ is simply the counting measure. The Hausdorff measure is a natural generalization of the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ in the following sense: for any $A \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A) = 2^{-n} \omega_n \mathcal{H}^n(A)$$

where $\omega_n = \pi^{n/2} \Gamma(\frac{n}{2} + 1)^{-1}$ is the volume of the unit n-ball. Occasionally, we will write $|A|$ to represent the Lebesgue measure of a set $A \subset \mathbb{R}^n$ when the dimension of the space is understood.

The Hausdorff measure allows us to extend the notion of dimension to any metric space. Suppose $A \subset X$. It easily follows from the definition of the Hausdorff measure that, if $\mathcal{H}^s(A)$ is non-zero for some $s > 0$, then $\mathcal{H}^t(A) = \infty$ for $t \in (0, s)$. Similarly, if $\mathcal{H}^s(A)$ is finite, then $\mathcal{H}^t(A) = 0$ for $t \in (s, \infty)$. This allows us to define the following:

Definition 5. *The Hausdorff dimension of $A \subset X$ is*

$$\dim_H A = \inf\{s \geq 0 \,:\, \mathcal{H}^s(A) = 0\}.$$

Here, we use the convention $\inf \emptyset = \infty$.

2.1.1 Curves in metric spaces

In this subsection, we will state some properties of curves in metric spaces without proof. For proofs of these results and more details, see Section 3 of [42] and Chapter 5 of [52].

Definition 6. *A curve in a metric space (X, d) is a continuous map $\Gamma : [a, b] \to X$. The length of a curve Γ is defined to be*

$$\ell_d(\Gamma) = \sup \sum_{i=0}^{n-1} d(\Gamma(s_i), \Gamma(s_{i+1})),$$

where the supremum is taken over all $n \in \mathbb{N}$ and all partitions $a = s_0 \leq s_1 \leq \ldots \leq s_n = b$.

We say that a curve Γ is rectifiable if $\ell_d(\Gamma) < \infty$. The classical notion of "speed" carries over to the metric space case in the following sense:

Definition 7. *For a curve $\Gamma : [a, b] \to X$, the speed of Γ at $t \in (a, b)$ is*

$$|\dot{\Gamma}|_d(t) := \lim_{h \to 0} \frac{d(\Gamma(t + h), \Gamma(t))}{|h|}$$

whenever this limit exists.

Theorem 8. *If a curve $\Gamma : [a, b] \to X$ is Lipschitz, then the speed $|\dot{\Gamma}|_d$ exists almost everywhere and*

$$\ell_d(\Gamma) = \int_a^b |\dot{\Gamma}|_d(t)\, dt.$$

With any rectifiable curve Γ we may associate a length function $s_\Gamma : [a,b] \to [0, \ell_d(\Gamma)]$ defined as

$$s_\Gamma(t) = \ell(\Gamma|_{[a,t]}) \text{ for any } t \in [a,b],$$

and this provides us with the following important reparametrization.

Theorem 9. *If $\Gamma : [a,b] \to X$ is a rectifiable curve, then there is a unique curve $\tilde{\Gamma} : [0, \ell_d(\Gamma)] \to X$ (called the arc-length parameterization of Γ) so that $\Gamma = \tilde{\Gamma} \circ s_\Gamma$. Moreover, $\ell_d(\tilde{\Gamma}|_{[0,t]}) = t$ for any $t \in [0, \ell_d(\Gamma)]$ (so $\tilde{\Gamma}$ is 1-Lipschitz), and $|\dot{\tilde{\Gamma}}|_d = 1$ almost everywhere.*

2.2 SOBOLEV MAPPINGS

Detailed proofs of the results in this section may be found in [27].

2.2.1 Sobolev mappings on Euclidean space

Consider an open subset $\Omega \subset \mathbb{R}^m$. Say $L^1_{loc}(\Omega)$ is the space of locally integrable functions on Ω. That is, $f \in L^1_{loc}(\Omega)$ if $f|_K \in L^1(K)$ for every compact $K \subset \Omega$. Here, $f|_K$ denotes the restriction of f to the set K. Also, denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω.

Definition 10. *For $f \in L^1_{loc}(\Omega)$ and $1 \le i \le m$, we say that $g_i \in L^1_{loc}(\Omega)$ is the weak (or distributional) partial derivative of f with respect to x_i in Ω if*

$$\int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega g_i \phi \, dx$$

for every $\phi \in C_0^\infty(\Omega)$.

Assuming the weak partial derivatives of $f \in L^1_{loc}(\Omega)$ exist, we will denote them as $\partial f/\partial x_i$ for $i = 1, \ldots, m$ and write ∇f for the m-vector consisting of the weak partial derivatives of f. Since the weak partial derivatives are themselves functions, we may consider weak partial derivatives of higher order as well. By $D^\alpha f$ we will denote the weak partial derivative of f with respect to the multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$. In particular $D^{\delta_i} f = \partial f/\partial x_i$ where

$\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is a multiindex with 1 in the i^{th} position. Also $|\alpha| = \alpha_1 + \ldots + \alpha_m$ and $\alpha! = \alpha_1! \cdots \alpha_m!$. $D^k f$ will denote the vector whose components are the derivatives $D^\alpha f$, $|\alpha| = k$. Note that $D^0 f = f$.

Definition 11. *The Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$ consists of those real valued functions $f \in L^p(\Omega)$ whose distributional partial derivatives $\partial f/\partial x_i$ are also functions in $L^p(\Omega)$ for $i = 1, \ldots, m$. Moreover, for any positive integer k and $1 \leq p \leq \infty$, we may define $W^{k,p}(\Omega)$ to be the class of all $f \in L^p(\Omega)$ so that $D^\alpha f \in L^p(\Omega)$ for any n-multiindex with $|\alpha| \leq k$.*

For any $f \in W^{k,p}(\Omega)$, define

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}.$$

Elements of $W^{k,p}_{loc}(\Omega)$ are those functions in $L^p_{loc}(\Omega)$ whose weak partial derivatives up to order k also lie in $L^p_{loc}(\Omega)$.

The Sobolev space $W^{k,p}(\Omega, \mathbb{R}^n)$ consists of mappings $f : \Omega \to \mathbb{R}^n$ whose component functions are members of $W^{k,p}(\Omega)$. Occasionally, Df will be used to denote the matrix composed of the weak partial derivatives of the components of f.

2.2.2 The ACL characterization of Sobolev mappings

Definition 12. *For an interval $I \subset \mathbb{R}$, we say that a function $f : I \to \mathbb{R}$ is absolutely continuous if, for every $\varepsilon > 0$, there is a $\delta > 0$ so that, given any pairwise disjoint, finite collection $\{(a_i, b_i)\}_{i=1}^N$ of subintervals of I satisfying*

$$\sum_{i=1}^N (b_i - a_i) < \delta,$$

we have

$$\sum_{i=1}^N |f(b_i) - f(a_i))| < \epsilon.$$

For an open set $U \subset \mathbb{R}$, say $f : U \to \mathbb{R}$ is *locally absolutely continuous* if f is absolutely continuous on any compact interval in U.

Suppose Ω is a domain in $\mathbb{R}^m$. (That is, $\Omega \subset \mathbb{R}^m$ is open and connected.) Call $ACL(\Omega)$ the space of all measurable real valued functions f on Ω so that, for $(m-1)$-almost every line $\bar{\ell}$ parallel to a coordinate axis, the restriction of f to $\ell = \bar{\ell} \cap \Omega$ is locally absolutely continuous. The notation "ACL" is shorthand for "absolutely continuous on lines." The partial derivatives of $f \in ACL(\Omega)$ exist almost everywhere in Ω in the classical sense.

Definition 13. *Say $f \in ACL^p(\Omega)$ if $f \in ACL(\Omega)$ and if f and $|\nabla f|$ are in $L^p(\Omega)$. Say $f \in ACL^p(\Omega, \mathbb{R}^n)$ if each of the component functions of f is in $ACL^p(\Omega)$.*

The following geometric characterization of Sobolev mappings will be important throughout the thesis.

Lemma 14. *Suppose $1 \leq p < \infty$. Then $W^{1,p}(\Omega) = ACL^p(\Omega)$.*

For a proof, see [95, Theorem 2.1.4]. In particular, if $f \in W^{1,p}(\Omega)$, then there is some representative $\tilde{f}$ of f (i.e. $\tilde{f}$ and f differ on a set of measure zero) for which $\tilde{f} \in ACL^p(\Omega)$. Conversely, if $f \in ACL^p(\Omega)$, then $f \in W^{1,p}(\Omega)$ and the weak partial derivatives of f equal the classical partial derivatives almost everywhere.

2.2.3 Approximation by smooth functions

Several properties of Sobolev maps require the boundary of the domain to be regular in some sense. We will now see one example of this regularity. For any $r > 0$ and $x \in \mathbb{R}^m$, say

$$Q(x, r) = \{y \in \mathbb{R}^m : |x_j - y_j| < r, \, j = 1, \ldots, m\}$$

is the cube around x with side length $2r$.

Definition 15. *Suppose $\Omega \subset \mathbb{R}^m$ is a bounded domain. Say Ω is a Lipschitz domain if, for every $x \in \partial\Omega$, $\partial\Omega$ is the graph of a Lipschitz map near x. That is, there is some $r > 0$ and a Lipschitz $h : \mathbb{R}^{m-1} \to \mathbb{R}$ so that (after rotating and relabeling the coordinate axes)*

$$\Omega \cap Q(x, r) = \{y \in \mathbb{R}^m : h(y_1, \ldots, y_{m-1}) < y_m\} \cap Q(x, r).$$

We now have the following important result which states that the Sobolev space is the completion of smooth maps under the Sobolev norm.

Theorem 16. *Suppose $\Omega \subset \mathbb{R}^m$ is a domain and $f \in W^{1,p}(\Omega)$ for $1 \le p < \infty$. Then there is a sequence $\{f_i\}$ in $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ so that*

$$\|f_i - f\|_{W^{1,p}(\Omega)} \to 0 \qquad \text{as } i \to \infty.$$

If Ω is a bounded Lipschitz domain, then we may choose an approximating sequence in $W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega})$.

2.2.4 Trace and extensions

One considers the restriction of a function to the boundary of its domain when solving boundary value problems. However, since Sobolev functions are defined almost everywhere in their domain (due to the equivalence relation on L^p), we cannot restrict the map to the boundary of the domain, as it may not be defined there. Instead, we consider the trace of the mapping.

Theorem 17. *Assume Ω is a bounded Lipschitz domain and $1 \le p < \infty$. Then there is a bounded, linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega, \mathcal{H}^{m-1})$$

called the trace operator so that

$$Tf = f|_{\partial\Omega} \qquad \text{for every } f \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

The trace function Tf is uniquely defined up to $\mathcal{H}^{m-1}$-null sets on $\partial\Omega$. We also have the following: if $\{f_i\}$ is any sequence in $C^\infty(\overline{\Omega})$ converging to f in $W^{1,p}(\Omega)$, then $\{f_i|_{\partial\Omega}\}$ converges in $L^p(\partial\Omega, \mathcal{H}^{m-1})$, so we can define $Tf := \lim_{i\to\infty} f_i|_{\partial\Omega}$ (and this is independent of the approximating sequence). Thus, while we cannot always describe the boundary values of a Sobolev map f, we may define the trace of f as the limit of the boundary values of an approximating sequence.

Complementary to the restriction of a Sobolev map, we also may consider the extension of a Sobolev map. This will be a focus of Chapter 6.

Theorem 18. *Assume Ω is a bounded Lipschitz domain and $1 \leq p < \infty$. Then there is a bounded, linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^m)$$

called the extension operator so that

$$Ef = f \qquad \text{on } \Omega \text{ for all } f \in W^{1,p}(\Omega).$$

3.0 THE HEISENBERG GROUP

Below, we introduce the sub-Riemannian Heisenberg group $\mathbb{H}^n$ and give many important definitions and geometric properties.

3.1 THE FIRST HEISENBERG GROUP

The Heisenberg group is defined as $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ with the group law

$$(z,t) * (z',t') = (z + z', t + t' + 2\,\mathrm{im} z\overline{z}') = (x + x', t + y', t + t' + 2(yx' - xy')).$$

Here $z = x + iy$, $z' = x' + iy'$. Clearly $0 = (0,0) \in \mathbb{C} \times \mathbb{R}$ is the identity element and $(z,t)^{-1} = (-z,-t)$. The Heisenberg group is an example of a *Lie group.*

Left multiplication by $(z,t) \in \mathbb{H}^1$

$$\ell_{(z,t)} : \mathbb{H}^1 \to \mathbb{H}^1, \qquad \ell_{(z,t)}(z',t') = (z,t) * (z',t')$$

defines a diffeomorphism of $\mathbb{R}^3$.

A vector field X on $\mathbb{H}^1 = \mathbb{R}^3$ is said to be *left invariant* if

$$d\ell_{(z,t)} \circ X = X \circ \ell_{(z,t)} \quad \text{for all } (z,t) \in \mathbb{H}^1.$$

In other words

$$d\ell_p(X(q)) = X(p * q) \quad \text{for all } p, q \in \mathbb{H}^1.$$

Any left invariant vector field is uniquely determined by its value at the identity element which is, in our case, the origin. That means the class of left invariant vector fields can be identified with the tangent space to $\mathbb{H}^1$ at 0

$$\mathfrak{h}^1 = T_0\mathbb{H}^1.$$

The vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}$ form a basis of $\mathfrak{h}^1$. We will show now how to find corresponding left invariant vector fields:

$$X(z,t) = d\ell_{(z,t)}\Big(\frac{\partial}{\partial x}\Big|_0\Big), \quad Y(z,t) = d\ell_{(z,t)}\Big(\frac{\partial}{\partial y}\Big|_0\Big), \quad T(z,t) = d\ell_{(z,t)}\Big(\frac{\partial}{\partial t}\Big|_0\Big).$$

Recall that

$$\ell_{(z,t)}(z',t') = (x+x', y+y', t+t'+2(yx'-xy')).$$

Hence[1]

$$d\ell_{(z,t)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2y & -2x & 1 \end{bmatrix}$$

Thus

$$X(x,y,t) = d\ell_{(z,t)}\Big(\frac{\partial}{\partial x}\Big|_0\Big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2y & -2x & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2y \end{bmatrix} = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t}\,.$$

Similarly we find the other two vector fields. This yields

Lemma 19. *The vector fields*

$$X(x,y,t) = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t}, \quad Y(x,y,t) = \frac{\partial}{\partial y} - 2x\,\frac{\partial}{\partial t}, \quad T(x,y,t) = \frac{\partial}{\partial t}$$

are left invariant and form a basis of left invariant vector fields on the Heisenberg group $\mathbb{H}^1$.

[1] We differentiate with respect to x', y', t' and the point (x,y,t) is fixed.

These left invariant vector fields turn $\mathfrak{h}^1$ into a *Lie algebra* with respect to the *commutator operation* $[X, Y] = XY - YX$. Easy calculations show all of the commutator relations in $\mathfrak{h}^1$:

$$[X, Y] = XY - YX = -4T, \quad [X, T] = 0, \quad [Y, T] = 0. \tag{3.1}$$

Observe that the Jacobian determinant of the left translation diffeomorphism $\ell_{(z,t)}$ equals 1 for all (z, t). That is, volume is preserved under left translation, so the Lebesgue measure in $\mathbb{R}^3$ is the left invariant *Haar measure* in $\mathbb{H}^1$. Similarly, one can check that the Lebesgue measure is also the right invariant Haar measure. Thus we proved

Theorem 20. *The Lebesgue measure on $\mathbb{R}^3$ is a bi-invariant Haar measure on $\mathbb{H}^1$.*

For $r > 0$ we define the dilation

$$\delta_r : \mathbb{H}^1 \to \mathbb{H}^1, \qquad \delta_t(z, t) = (rz, r^2 t).$$

Lemma 21. *The dilations form a group of automorphisms of the group $\mathbb{H}^1$. Since the Jacobian of δ_r equals r^4, we have that*

$$|\delta_r(E)| = r^4|E| \quad \textit{for any measurable set } E \subset \mathbb{R}^3.$$

Here and in what follows $|E|$ stands for the Lebesgue measure of a set E.

3.1.1 Other ways to define the Heisenberg group

Consider the group $\tilde{\mathbb{H}}^1$ of upper triangular matrices

$$\begin{bmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \qquad x, y, t \in \mathbb{R}$$

with respect to matrix multiplication. We will show that the groups $\mathbb{H}^1$ and $\tilde{\mathbb{H}}^1$ are isomorphic. That is, this group of matrices provides an alternate definition of the Heisenberg group.

Since

$$\begin{bmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y' & t' \\ 0 & 1 & x' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y+y' & t'+yx'+t \\ 0 & 1 & x'+x \\ 0 & 0 & 1 \end{bmatrix},$$

the group $\tilde{\mathbb{H}}^1$ can be identified with $\mathbb{R}^3$ equipped with the group law

$$(x, y, t) \otimes (x', y', t') = (x + x', y + y', t + t' + yx').$$

One can easily check that the mapping

$$\phi : \tilde{\mathbb{H}}^1 \to \mathbb{H}^1$$
$$\phi(x, y, t) = (x, y, 4t - 2xy)$$

is the desired group isomorphism.

The left invariant vector fields on $\tilde{\mathbb{H}}^1$ are

$$\tilde{X} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y}, \quad \tilde{T} = \frac{\partial}{\partial t},$$

and we have

$$d\phi(\tilde{X}) = X, \quad d\phi(\tilde{Y}) = Y, \quad d\phi(\tilde{T}) = 4T.$$

Indeed,

$$d\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y & -2x & 4 \end{bmatrix},$$

so

$$d\phi(\tilde{X}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y & -2x & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2y+4y \end{bmatrix} = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} = X$$

$$d\phi(\tilde{Y}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y & -2x & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2x \end{bmatrix} = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t} = Y$$

$$d\phi(\tilde{T}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y & -2x & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 4\frac{\partial}{\partial t} = 4T.$$

That is, $d\phi$ maps a basis of left invariant vector fields $\{\tilde{X}, \tilde{Y}, \tilde{T}\}$ onto a basis of left invariant vector fields $\{X, Y, 4T\}$.

3.1.2 Why we call it the Heisenberg group

Heisenberg originally wrote the bracket relations (3.1) in his formulation of quantum mechanics. In his work, X, Y, and T are self-adjoint operators on a Hilbert space. The operator X corresponds to a measurement of position, Y corresponds to a measurement of momentum, and T is a multiple of the identity operator. Heisenberg did not actually construct the Heisenberg group but rather its Lie algebra. Any Lie algebra uniquely determines a connected, simply connected Lie group. Herman Weyl was the first to construct the Lie group associated with Heisenberg's Lie algebra in order to explain the mathematical equivalence of Schrödinger's and Heisenberg's approaches to quantum mechanics [92]. For this reason, physicists often refer to this Lie group as the Weyl group, while mathematicians call it the Heisenberg group.

3.1.3 Higher dimensional generalizations

For any positive integer n, we define the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$. We may generalize the notions of the group law, inverses, and dilations from $\mathbb{H}^1$ to $\mathbb{H}^n$ as follows for any $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ and $r > 0$:

$$(z,t) * (z',t') = (z+z', t+t' + 2\operatorname{im}\sum_{j=1}^{n} z_j \overline{z}'_j) = (x+x', t+y', t+t' + 2\sum_{j=1}^{n}(y_j x'_j - x_j y'_j))$$

$$(z,t)^{-1} = (-z,-t)$$

$$\delta_r(z,t) = (rz, r^2 t)$$

$$|\delta_r(E)| = r^{2n+2}|E|$$

As before, the dilations δ_r form a group of automorphisms, and the Lebesgue measure is a bi-invariant Haar measure on $\mathbb{H}^n$.

A basis of left invariant vector fields is given at any point $(x_1, y_1, \ldots, x_n, y_n, t) \in \mathbb{H}^n$ by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

for $j = 1, 2, \ldots, n$. As above, these left invariant vector fields determine the Lie algebra $\mathfrak{h}^n$. It is not hard to see that $[X_j, Y_j] = -4T$ for $j = 1 \ldots, n$ and all other commutators vanish.

As before, we may also define the Heisenberg group as $\tilde{\mathbb{H}}^n = \mathbb{R}^{2n+1}$ with the group law

$$(x,y,t) \otimes (x',y',t') = (x+x', y+y', t+t'+y\cdot x')$$

and basis of left invariant vector fields

$$\tilde{X}_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \quad \tilde{Y}_j = \frac{\partial}{\partial y_j}, \quad \tilde{T} = \frac{\partial}{\partial t}$$

for $j = 1, 2, \ldots, n$. This group is isomorphic to $\mathbb{H}^n$ via the group isomorphism

$$\phi : \tilde{\mathbb{H}}^n \to \mathbb{H}^n$$

$$\phi(x,y,t) = (x, y, 4t - 2x \cdot y),$$

and

$$d\phi(\tilde{X}_j) = X_j, \quad d\phi(\tilde{Y}_j) = Y_j, \quad d\phi(\tilde{T}) = 4T.$$

3.2 THE HORIZONTAL DISTRIBUTION

The Heisenberg group is equipped with the so called *horizontal distribution*

$$H_p\mathbb{H}^n = \operatorname{span}\{X_1(p), Y_1(p), \ldots, X_n(p), Y_n(p)\} \quad \text{for all } p \in \mathbb{H}^n.$$

This is a smooth distribution of $2n$-dimensional subspaces in the $(2n+1)$-dimensional tangent space $T_p\mathbb{H}^n = T_p\mathbb{R}^{2n+1}$. A vector $v \in T_p\mathbb{R}^{2n+1}$ is *horizontal* if and only if $v \in H_p\mathbb{H}^n$. That is, if we write

$$v = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j}\Big|_p + b_j \frac{\partial}{\partial y_j}\Big|_p \right) + c\frac{\partial}{\partial t}\Big|_p, \tag{3.2}$$

for $a_j, b_j, c \in \mathbb{R}$, then v is horizontal if and only if we can write

$$v = \sum_{j=1}^{n} a'_j X_j(p) + b'_j Y_j(p)$$

for some $a'_j, b'_j \in \mathbb{R}$. Since

$$X_j(p) = \frac{\partial}{\partial x_j}\Big|_p + 2y_j(p)\,\frac{\partial}{\partial t}\Big|_p \quad \text{and} \quad Y_j(p) = \frac{\partial}{\partial y_j}\Big|_p - 2x_j(p)\,\frac{\partial}{\partial t}\Big|_p,$$

it must be the case that $a_j = a'_j$ and $b_j = b'_j$. Thus

$$c = 2\sum_{j=1}^{n}(a_j y_j(p) - b_j x_j(p)). \tag{3.3}$$

We have shown that a vector v given by (3.2) is horizontal if and only if (3.3) is satisfied.

3.2.1 Contact manifolds

Theorem 22. *The horizontal distribution $H\mathbb{H}^n$ is the kernel of the 1-form*

$$\alpha = dt + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j). \tag{3.4}$$

i.e. $H_p\mathbb{H}^n = ker\,\alpha(p) \subset T_p\mathbb{R}^{2n+1}$.

Proof.

$$\alpha(X_j) = \left(dt + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)\right)\left(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\right) = 2y_j + 2(-y_j) = 0$$

and

$$\alpha(Y_j) = \left(dt + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)\right)\left(\frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}\right) = -2x_j + 2x_j = 0,$$

but $\alpha(T) = 1$. □

In other words, a vector v given by (3.2) is in the kernel of α if and only if (3.3) is satisfied.

Definition 23. *Let M be a manifold of dimension $2n+1$. A contact form is a 1-form α satisfying*

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Given a contact form α, the contact structure on M is defined as the distribution of $2n$-dimensional hyperplanes

$$\xi = ker\ \alpha \subset TM,$$

and M is called a contact manifold.

It is easy to check that α defined as in (3.4) is a contact form. Therefore, the Heisenberg group is an example of a contact manifold. As the following result shows, any contact form on a manifold is locally equivalent to the contact form (3.4) on the Heisenberg group up to a change of coordinates. See [83] for a proof.

Theorem 24 (Darboux)**.** *Let α be a contact form on a $(2n+1)$-dimensional manifold M. For any $p \in M$, there is a neighborhood U of p and a coordinate system $(x_1, y_1, \ldots, x_n, y_n, t)$ in U so that $p = (0, \ldots, 0)$ and*

$$\alpha|_U = dt + 2\sum_{j=1}^{n}(x_j\, dy_j - y_j\, dx_j).$$

One of the problems in contact topology is the study of the global structure of a contact manifold M. For example, one may ask if two given contact manifolds are "the same" in the following sense: is there a diffeomorphism between the manifolds which preserves contact structures? The Darboux theorem shows that any two contact manifolds of dimension $2n+1$ are *locally* diffeomorphic via such a diffeomorphism, but the global question is much more difficult. In our investigation of the Heisenberg group, however, we will only be concerned with the local geometry of the horizontal distribution.

3.2.2 Horizontal curves

An absolutely continuous map $\gamma : [a, b] \to \mathbb{R}^{2n+1}$ is called a *curve* in $\mathbb{R}^{2n+1}$. Recall that γ' exists almost everywhere in $[a, b]$.

Definition 25. *A curve in $\mathbb{R}^{2n+1}$ is a horizontal curve if it is almost everywhere tangent to the horizontal distribution i.e.*

$$\gamma'(t) \in H_{\gamma(t)}\mathbb{H}^n \quad \textit{for almost every } t \in [a, b].$$

In other words, the curve γ is horizontal if γ' is almost everywhere a horizontal vector. In terms of coefficients, this means

$$\gamma'(t) = \sum_{j=1}^{n} a_j(t)X_j(\gamma(t)) + b_j(t)Y_j(\gamma(t)) \qquad \text{a.e. } t \in [a, b]$$

for some real valued coefficient functions a_j, b_j. Recall that a vector $v \in T_p\mathbb{H}^n$ written as

$$v = \sum_{j=1}^{n}\left(a_j \frac{\partial}{\partial x_j}\Big|_p + b_j \frac{\partial}{\partial y_j}\Big|_p\right) + c\frac{\partial}{\partial t}\Big|_p$$

is horizontal if and only if

$$c = 2\sum_{j=1}^{n}(a_j y_j(p) - b_j x_j(p)).$$

Therefore, the curve

$$\gamma(t) = (f_1(t), g_1(t), \ldots, f_n(t), g_n(t), h(t))$$

is horizontal if and only if the vector

$$\gamma'(t) = \sum_{j=1}^{n}\left(f_j'(t)\frac{\partial}{\partial x_j}\Big|_{\gamma(t)} + g_j'(t)\frac{\partial}{\partial y_j}\Big|_{\gamma(t)}\right) + h'(t)\frac{\partial}{\partial t}\Big|_{\gamma(t)}$$

satisfies

$$h'(t) = 2\sum_{j=1}^{n}(f_j'(t)g_j(t) - g_j'(t)f_j(t)) \quad \text{for almost every } t \in [a,b]. \tag{3.5}$$

At every point p in $\mathbb{H}^n$ we have a $2n$-dimensional space of directions in which a horizontal curve may travel. Meanwhile, the tangent space at p has dimension $2n+1$. Thus the condition of horizontality on a curve is a very restrictive one. Surprisingly, any two points $p, q \in \mathbb{H}^n$ can be connected by a horizontal curve. That is, there is a horizontal curve $\gamma : [a,b] \to \mathbb{H}^n$ so that $\gamma(a) = p$ and $\gamma(b) = q$. See Theorem 26.

According to (3.5), we may construct a horizontal curve from any absolutely continuous curve in $\mathbb{R}^{2n}$ as follows: suppose

$$\tilde{\gamma} = (f_1, g_1, \ldots, f_n, g_n) : [a,b] \to \mathbb{R}^{2n}$$

is absolutely continuous, set $h(a) \in \mathbb{R}$ arbitrarily, and define

$$h(t) = h(a) + 2\sum_{j=1}^{n}\int_a^t (f_j'(\tau)g_j(\tau) - g_j'(\tau)f_j(\tau))\,d\tau \quad \text{for every } t \in [a,b]. \tag{3.6}$$

The curve $\gamma = (f_1, g_1, \ldots, f_n, g_n, h)$ is called a *horizontal lift* of $\tilde{\gamma}$. Hence

$$\pi \circ \gamma = \tilde{\gamma}$$

where $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ is the standard orthogonal projection onto the first $2n$ coordinates.

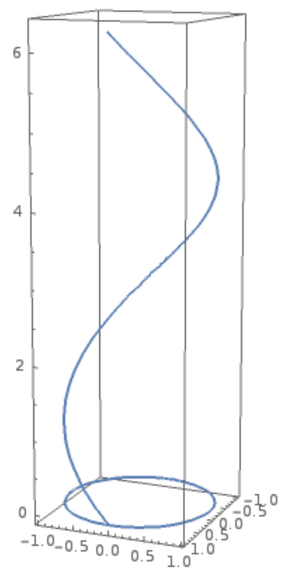

Figure 1: The planar curve $(\cos(t), -\sin(t))$ and its horizontal lift $(\cos(t), -\sin(t), t)$

Given an absolutely continuous curve $\tilde{\gamma}$ in $\mathbb{R}^{2n}$ and a point $p \in \mathbb{R}^{2n+1}$ with $\pi(p) = \tilde{\gamma}(t)$, the curve $\tilde{\gamma}$ has a unique horizontal lift γ satisfying $\gamma(a) = p$. Indeed, (3.6) uniquely defines this lift once a starting height $h(a)$ is chosen.

Observe also that vertical translations along the t-axis remain horizontal. That is, given a horizontal curve $\gamma = (\tilde{\gamma}, h)$ and $c \in \mathbb{R}$, the curve $\beta = (\tilde{\gamma}, h + c)$ is also horizontal.

Assume now that $\gamma : [a, b] \to \mathbb{H}^n$ is horizontal and its projection $\tilde{\gamma} = \pi \circ \gamma$ onto $\mathbb{R}^{2n}$ is a closed curve. Note that

$$\begin{aligned} h(b) - h(a) &= -2 \sum_{j=1}^{n} \int_a^b (f_j(\tau) g_j'(\tau) - g_j(\tau) f_j'(\tau))\, d\tau \\ &= -2 \sum_{j=1}^{n} \int_{\tilde{\gamma}} (x_j\, dy_j - y_j\, dx_j) \\ &= -2 \sum_{j=1}^{n} \int_{\tilde{\gamma}_j} (x_j\, dy_j - y_j\, dx_j) \end{aligned}$$

where $\tilde{\gamma}_j = (f_j, g_j)$ is the orthogonal projection of γ onto the $x_j y_j$-plane. Note that each $\tilde{\gamma}_j$ is a closed curve in $\mathbb{R}^2$. According to Green's Theorem,

$$\frac{1}{2}\int_{\tilde{\gamma}_j} (x_j\,dy_j - y_j\,dx_j) \tag{3.7}$$

equals the area enclosed by the curve $\tilde{\gamma}_j$. More precisely, if the curve $\tilde{\gamma}_j$ has self-intersections in the plane, the integral (3.7) defines an oriented area taking into account the multiplicity of overlaps. Figure 2 illustrates this.

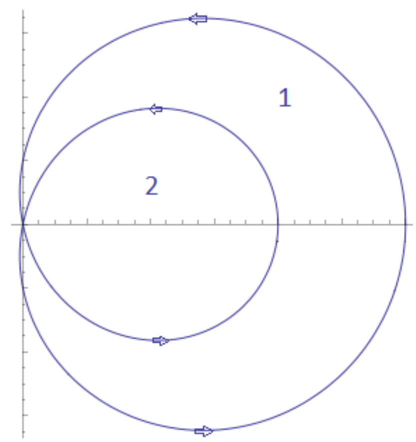

Figure 2: A curve with enclosed multiplicities labeled

Since we calculated

$$h(b) - h(a) = -4\left(\sum_{j=1}^{n} \frac{1}{2}\int_{\tilde{\gamma}_j} (x_j\,dy_j - y_j\,dx_j)\right), \tag{3.8}$$

we may conclude that the change in height $h(b) - h(a)$ of a horizontal curve equals -4 times the sum of oriented areas enclosed by the planar projections $\tilde{\gamma}_j$.

Suppose $q = (0, \ldots, 0, t)$ is any point on the t-axis. We would like to construct a horizontal curve connecting the origin to this point q. We may arbitrarily define n closed curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ in $\mathbb{R}^2$ beginning and ending at the origin $(0, 0)$ so that the sum of their enclosed, oriented areas equals $-t/4$. Then the horizontal lift γ of $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n)$ with starting height $h(a) = 0$ satisfies

$$h(b) = h(b) - h(a) = -4\left(\sum_{j=1}^{n} \frac{1}{2}\int_{\tilde{\gamma}_j} (x_j\,dy_j - y_j\,dx_j)\right) = -4\left(\frac{-t}{4}\right) = t.$$

Thus γ satisfies $\gamma(a) = (0, \ldots, 0, 0)$ and $\gamma(b) = (0, \ldots, 0, t) = q$ as desired.

Fix $p = (a_1, b_1, \ldots, a_n, b_n, t_0) \in \mathbb{H}^n$ and consider the curve $\beta : [0,1] \to \mathbb{H}^n$ defined as

$$\beta(s) = (sa_1, sb_1, \ldots, sa_n, sb_n, t_0).$$

This is simply a line segment in $\mathbb{R}^{2n+1}$ from p to the point $(0, \ldots, 0, t_0)$ on the t-axis. This curve β is horizontal. Indeed,

$$\beta'(s) = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j}\Big|_{\beta(s)} + b_j \frac{\partial}{\partial y_j}\Big|_{\beta(s)} \right) + 0 \frac{\partial}{\partial t}\Big|_{\beta(s)},$$

so the coefficient of $\partial/\partial t$ is

$$0 = 2\sum_{j=1}^{n}(a_j(sb_j) - b_j(sa_j)) = 2\sum_{j=1}^{n}(a_j y_j(\beta(s)) - b_j x_j(\beta(s))),$$

and this satisfies (3.3). We have just seen that any point $p \in \mathbb{H}^n$ may be connected to the t-axis via a horizontal curve (in this case, a line segment). Above, we showed that any point on the t-axis may be connected to the origin via a horizontal curve. Concatenating curves proves the following:

Theorem 26. *Any two points $p, q \in \mathbb{H}^n$ may be connected by a horizontal curve.*

Fix $p \in \mathbb{H}^n$. Connect p to the point $q = (0, \ldots, 0, t)$ on the t-axis via the line segment parallel to $\mathbb{R}^{2n}$ described above. Now define

$$\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n)$$

so that $\tilde{\gamma}_1$ is a negatively oriented circle in the plane starting and ending at the origin with radius $\sqrt{t/4\pi}$, and set $\tilde{\gamma}_j \equiv (0,0)$ for $j = 2, \ldots, n$. By the above arguments, the horizontal lift of $\tilde{\gamma}$ with starting height 0 connects the origin to q. In particular, if p is close to the origin, then there is a short, horizontal curve connecting them. This observation will be used in the proof of Lemma 30 below.

3.3 THE CARNOT-CARATHÈODORY METRIC

Equip $H\mathbb{H}^n$ with a Riemannian metric g so that the vectors $X_1, Y_1, \ldots, X_n, Y_n$ are orthonormal at every point in $\mathbb{H}^n$. The metric g is only defined on the spaces $H_p\mathbb{H}^n$ rather than on the entire tangent space $T_p\mathbb{R}^{2n+1}$. Clearly, the metric g is left-invariant.

3.3.1 Lengths of curves

Let $\gamma : [a, b] \to \mathbb{H}^n$ be a horizontal curve and write

$$\gamma'(t) = \sum_{j=1}^{n} a_j(t) X_j(\gamma(t)) + b_j(t) Y_j(\gamma(t)) \quad \text{a.e. } t \in [a, b].$$

The *horizontal length* of γ is defined by

$$\ell_H(\gamma) := \int_a^b |\gamma'(t)|_H \, dt \tag{3.9}$$

where

$$|\gamma'(t)|_H = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_g} = \sqrt{\sum_{j=1}^{n} (a_j(t))^2 + (b_j(t))^2}.$$

Definition 27. *The* Carnot-Carathèodory metric *in $\mathbb{H}^n$ is defined for all $p, q \in \mathbb{H}^n$ by*

$$d_{cc}(p, q) = \inf_{\gamma} \{\ell_H(\gamma)\}$$

where the infimum is taken over all horizontal curves connecting p and q.

Recall from Theorem 26 that any two points in $\mathbb{H}^n$ may be connected by a horizontal curve, so d_{cc} is a well defined metric. Clearly, the Carnot-Carathèodory metric is left invariant. In what follows, we will always assume that $\mathbb{H}^n$ is equipped with the horizontal distribution and the Carnot-Carathèodory metric.

3.3.2 Properties of the Carnot-Carathèodory metric

Let $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the orthogonal projection

$$\pi(x_1, y_1, \ldots, x_n, y_n, t) = (x_1, y_1, \ldots, x_n, y_n)$$

onto the first $2n$ coordinates. We have for any $p = (x_1, y_1, \ldots, x_n, y_n, t) \in \mathbb{H}^n$

$$d\pi_p(X_j(p)) = d\pi_p \left(\left.\frac{\partial}{\partial x_j}\right|_p + 2y_j \left.\frac{\partial}{\partial t}\right|_p \right) = \left.\frac{\partial}{\partial x_j}\right|_{\pi(p)}$$

and

$$d\pi_p(Y_j(p)) = d\pi_p \left(\left.\frac{\partial}{\partial y_j}\right|_p - 2x_j \left.\frac{\partial}{\partial t}\right|_p \right) = \left.\frac{\partial}{\partial y_j}\right|_{\pi(p)}.$$

Thus $d\pi_p$ maps an orthonormal basis of $(H_p\mathbb{H}^n, g)$ onto an orthonormal basis of $T_{\pi(p)}\mathbb{R}^{2n}$. Hence, the linear map

$$d\pi_p : (H_p\mathbb{H}^n, g) \to T_{\pi(p)}\mathbb{R}^{2n}$$

is an isometry. In particular, for any horizontal curve γ, the horizontal length $\ell_H(\gamma)$ equals the Euclidean length $\ell_E(\tilde{\gamma})$ of the projection

$$\tilde{\gamma} = \pi \circ \gamma : [a, b] \to \mathbb{R}^{2n}.$$

Recall that the dilations δ_r for $r > 0$ are group automorphisms $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$\delta_r(z, t) = (rz, r^2 t).$$

These dilations commute with the lengths of horizontal curves in the following sense.

Theorem 28. *If γ is a horizontal curve in $\mathbb{H}^n$, then $\delta_r \circ \gamma$ is horizontal as well and*

$$\ell_H(\delta_r \circ \gamma) = r\ell_H(\gamma).$$

In particular,

$$d_{cc}(\delta_r(p), \delta_r(q)) = r d_{cc}(p, q) \quad \textit{for all } p, q \in \mathbb{H}^n.$$

Proof. We will prove this result in $\mathbb{H}^1$, but the same argument applies to $\mathbb{H}^n$. We have

$$d\delta_r X(p) = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2y \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 2r^2 y \end{bmatrix} = r\left(\frac{\partial}{\partial x} + 2ry\frac{\partial}{\partial t}\right) = rX(\delta_r(p)).$$

Similarly, $d\delta_r Y(p) = rY(\delta_r(p))$. The result follows easily. $\square$

Theorem 29. *If $E \subset \mathbb{R}^{2n+1}$ is compact, then there is a constant $C = C(E) \geq 1$ such that*

$$C^{-1}|p-q| \leq d_{cc}(p,q) \leq C|p-q|^{1/2} \quad \forall\, p,q \in E. \tag{3.10}$$

In particular, this implies that $(\mathbb{H}^n, d_{cc})$ and $(\mathbb{R}^{2n+1}, |\cdot|)$ are topologically equivalent. In order to prove this result, we introduce a new metric on $\mathbb{H}^n$. For any $(z,t) \in \mathbb{H}^n$, define the *Korányi gauge* $\|\cdot\|_K$ as

$$\|(z,t)\|_K := (|z|^4 + t^2)^{1/4}$$

and the *Korányi metric* d_K as

$$d_K(p,q) := \|q^{-1} * p\|_K.$$

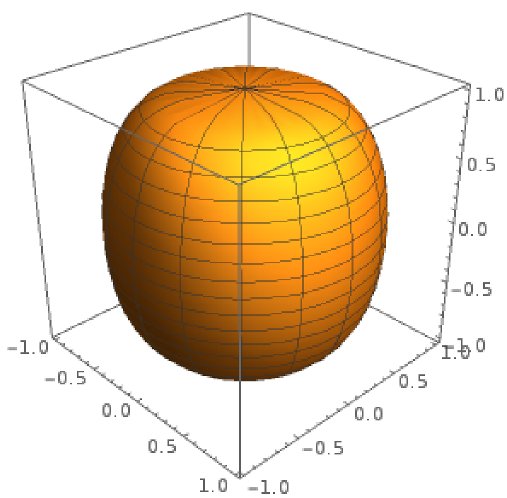

Figure 3: The unit Korányi sphere in $\mathbb{H}^1$

In order to prove that d_K is indeed a metric, one must prove that the triangle inequality holds. While it does indeed hold, we will not prove it as we will not need it.

Lemma 30. *The metrics d_{cc} and d_K are bi-Lipschitz equivalent. That is, there is a constant $C \geq 1$ satisfying*

$$C^{-1} d_K(p,q) \leq d_{cc}(p,q) \leq C d_K(p,q) \quad \forall\, p,q \in \mathbb{H}^n.$$

Proof. We begin with a sketch of a proof that the function $d_0 : \mathbb{R}^{2n+1} \to [0,\infty)$ defined as $d_0(p) := d_{cc}(0,p)$ is continuous. We will prove continuity at the origin, and general continuity will then follow from left multiplication. If $|p|$ is small for some $p \in \mathbb{H}^n$, the horizontal curve γ from the discussion following Theorem 26 connecting p to the origin will have small horizontal length (since the horizontal length of this curve equals the Euclidean length of its projection). Therefore, $d_{cc}(0,p)$ must be small as well.

Observe now that $\|\delta_r(p)\|_K = r\|p\|_K$ for all $r > 0$. Let $S = \{p \in \mathbb{H}^n : \|p\|_K = 1\}$ be the unit sphere in the Korányi metric. Since S is compact in $\mathbb{R}^{2n+1}$ and $p \mapsto d_{cc}(0,p)$ is continuous and non-zero on S, there is a constant $C \geq 1$ so that

$$C^{-1} \leq d_{cc}(0,p) \leq C \quad \forall\, p \in S.$$

Fix $0 \neq p \in \mathbb{H}^n$, and set $r = \|p\|_K^{-1}$. Then $\|\delta_r(p)\|_K = 1$, so $\delta_r(p) \in S$. Hence

$$\|p\|_K^{-1} d_{cc}(0,p) = r d_{cc}(0,p) = d_{cc}(0,\delta_r(p)) \in [C^{-1}, C],$$

so, for any $p \in \mathbb{H}^n$,

$$C^{-1}\|p\|_K \leq d_{cc}(0,p) \leq C\|p\|_K$$

Since left multiplication is an isometry on $(\mathbb{H}^n, d_{cc})$, we have for any $p,q \in \mathbb{H}^n$

$$d_{cc}(p,q) = d_{cc}(q^{-1} * p, 0),$$

and hence

$$C^{-1} d_K(p,q) = C^{-1}\|q^{-1} * p\|_K \leq d_{cc}(p,q) \leq C\|q^{-1} * p\|_K = C d_K(p,q).$$

$\square$

Proof of Theorem 29. We will first prove the left inequality. Notice that, if $E \subset \mathbb{R}^{2n+1}$ is compact, then $d_{cc}(p,q)$ is uniformly bounded for any $p, q \in E$ since the mapping $p \mapsto d_{cc}(0,p)$ is continuous. Fix $p, q \in E$, and suppose $\{\gamma_k\}$ is a sequence of horizontal curves with $\ell_H(\gamma_k) \to d_{cc}(p,q)$. Then there is a some integer N so that, for all $k \geq N$, the x and y components of γ_k are bounded by some constant depending only on the set E. Indeed, the horizontal length of a curve is equal to the Euclidean length of its projection to $\mathbb{R}^{2n}$, and the boundedness of d_{cc} on E implies that these projections cannot stray too far from the projection of E. Hence, for any $\gamma = \gamma_k$ with $k \geq N$, (3.5) gives

$$|\gamma'(t)| = \sqrt{\sum_{j=1}^{n} f_j'(t)^2 + g_j'(t)^2 + h'(t)^2} \leq C\sqrt{\sum_{j=1}^{n} f_j'(t)^2 + g_j'(t)^2} = C|\gamma'(t)|_H \quad \text{a.e. } t.$$

Therefore, $|p-q| \leq \ell_E(\gamma) \leq C\ell_H(\gamma)$ for any horizontal curve connecting p and q. Taking the limit in k gives $|p-q| \leq C d_{cc}(p,q)$.

We will now show that $d_K(p,q) \leq |p-q|^{1/2}$, so the theorem will follow from the bi-Lipschitz equivalence of d_K and d_{cc}. Fix $p, q \in E$. Write $p = (z,t)$ and $q = (z',t')$. One may check that

$$\begin{aligned} d_K(p,q) = \|q^{-1} * p\|_K &= \left(|z-z'|^4 + \left| t - t' + 2\sum_{j=1}^{n}(x_j' y_j - x_j y_j') \right|^2 \right)^{1/4} \\ &\leq C|z-z'| + C\left| t - t' + 2\sum_{j=1}^{n}(x_j' y_j - x_j y_j') \right|^{1/2} \end{aligned}$$

for some $C > 0$. The inequality, and hence the theorem, follows. $\square$

Corollary 31. *$\mathbb{H}^n$ is complete. Closed, bounded sets in $\mathbb{H}^n$ are compact.*

3.3.3 Rectifiable curves in the Heisenberg group

If $\Gamma : [a, b] \to \mathbb{H}^n$ is any continuous (not necessarily horizontal) curve in the metric space $(\mathbb{H}^n, d_{cc})$, then, as in the discussion in Section 2.1.1, its length with respect to the Carnot Carathèodory metric is

$$\ell_{cc}(\Gamma) = \sup \sum_{j=0}^{n-1} d_{cc}(\Gamma(s_i), \Gamma(s_{i+1})).$$

Taking $n = 1$ above, it immediately follows from the definition of d_{cc} that $\ell_{cc}(\Gamma) \leq \ell_H(\Gamma)$. Thus every horizontal curve is rectifiable. However, it is not obvious if $\ell_{cc}(\Gamma) = \ell_H(\Gamma)$. It is also not clear whether every rectifiable curve in $\mathbb{H}^n$ can be reparametrized as a horizontal curve. Both of these facts are true, and this discussion is summarized here.

Proposition 32. *In $\mathbb{H}^n$,*

1. *any horizontal curve Γ is rectifiable and $\ell_{cc}(\Gamma) = \ell_H(\Gamma)$.*
2. *Lipschitz curves in $\mathbb{H}^n$ are horizontal.*
3. *every rectifiable curve admits a 1-Lipschitz parameterization and is horizontal after this reparametrization.*

Condition *2.* in the above result will follow from the following useful proposition.

Proposition 33. *Suppose $\Omega \subset \mathbb{R}^m$ is open. and $f = (f_1, g_1, \ldots, f_n, g_m, h) : \Omega \to \mathbb{H}^n$ is locally Lipschitz. Then f is differentiable almost everywhere, and*

$$Dh(x) = 2 \sum_{j=1}^{n} (g_j(x) Df_j(x) - f_j(x) Dg_j(x)) \quad \text{for almost every } x \in \Omega.$$

In other words, the image of $Df(x)$ lies in $H_{f(x)}\mathbb{H}^n$ almost everywhere.

If α is the contact form (3.4), this result says that $f^*\alpha = 0$ almost everywhere for any locally Lipschitz f.

Proof. Since f is locally Lipschitz as a mapping into $\mathbb{H}^n$, it is also locally Lipschitz as a map into $\mathbb{R}^{2n+1}$ (by Theorem 29). Thus, by the classical Rademacher theorem, Df exists almost everywhere in Ω.

Choose $x \in \Omega$ so that $Df(x)$ exists. Hence the definition of d_K and its bi-Lipschitz equivalence with d_{cc} give

$$\left| h(y) - h(x) + 2\sum_{j=1}^{n} \left(f_j(x)g_j(y) - f_j(y)g_j(x)\right) \right|^{1/2} \leq d_K(f(x), f(y)) \leq C|x-y|$$

for all y close enough to x and some constant $C \geq 1$ (depending on the local Lipschitz constant of f at x). After adding and subtracting $f_j(x)g_j(x)$ in the above sum, we have

$$\begin{aligned}
&\left| h(y) - h(x) - 2\sum_{j=1}^{n} (g_j(x)Df_j(x) - f_j(x)Dg_j(x)) \cdot (y-x) \right| \\
&\quad \leq C^2|x-y|^2 + 2\sum_{j=1}^{n} |f_j(x)|\, |g_j(y) - g_j(x) - Dg_j(x) \cdot (y-x)| \\
&\qquad\qquad + 2\sum_{j=1}^{n} |g_j(x)|\, |f_j(y) - f_j(x) - Df_j(x) \cdot (y-x)| \\
&\quad = o(|x-y|),
\end{aligned}$$

since f_j and g_j are differentiable at x. □

Proof of Proposition 32. Condition *2.* follows from the previous proposition, and *3.* follows from *2.* and the fact that the arc-length parameterization of a rectifiable curve is 1-Lipschitz. We now prove condition *1.*

Recall from the discussion preceding Proposition 32 that $\ell_{cc}(\Gamma) \leq \ell_H(\Gamma)$. Hence it remains to prove that $\ell_{cc}(\Gamma) \geq \ell_H(\Gamma)$. Extend the Riemannian metric defined on the horizontal distribution $H\mathbb{H}^n$ to a Riemannian metric g in $\mathbb{R}^{2n+1}$. For example, we may do so by requiring that the vector fields X_j, Y_j, T are orthonormal at every point of $\mathbb{R}^{2n+1}$. The Riemannian metric g defines a metric d_g in $\mathbb{R}^{2n+1}$ as the infimum of lengths of curves connecting two given points. Here, the length of an absolutely continuous curve $\beta : [a,b] \to \mathbb{R}^{2n+1}$ is defined as the integral

$$\ell_g(\beta) = \int_a^b \sqrt{g\left(\beta'(s), \beta'(s)\right)}\, ds.$$

Since this is the same approach that was used to define the Carnot-Carathéodory metric, we have $\ell_g(\Gamma) = \ell_H(\Gamma)$ for any horizontal curve Γ. Thus it is obvious that $d_g(p,q) \leq d_{cc}(p,q)$ since, in the case of this new Riemannian metric, we take the infimum of lengths over a larger class of curves. It is a well known fact in Riemannian geometry that for an absolutely continuous curve $\beta : [a,b] \to \mathbb{R}^{2n+1}$

$$\ell_g(\beta) = \sup \sum_{i=0}^{n-1} d_g(\beta(s_i), \beta(s_{i+1})),$$

where the supremum is taken over all positive integers n and all partitions $a = s_0 \leq s_1 \leq \ldots \leq s_n = b$ as before. Therefore, if $\Gamma : [a,b] \to \mathbb{R}^{2n+1}$ is horizontal, we have

$$\ell_H(\Gamma) = \ell_g(\Gamma) = \sup \sum_{i=0}^{n-1} d_g(\Gamma(s_i), \Gamma(s_{i+1})) \leq \sup \sum_{i=0}^{n-1} d_{cc}(\Gamma(s_i), \Gamma(s_{i+1})) = \ell_{cc}(\Gamma).$$

$\square$

4.0 GEODESICS IN THE HEISENBERG GROUP

This chapter is based on the paper [48].

Definition 34. *A geodesic from p to q in $\mathbb{H}^n$ is a curve of shortest length connecting the two points.*

As we saw in Theorem 26, any two points in the Heisenberg group can be connected by a horizontal curve. In fact, any two points can be connected by a geodesic, and the structure of every geodesic in the form of an explicit parameterization is known. The proofs in the case of $\mathbb{H}^1$ can be found in [9, 19, 33, 71], and the general case of $\mathbb{H}^n$ is treated in [5, 10, 73].

If $n = 1$, the structure of geodesics can be obtained via the two dimensional isoperimetric inequality (see [9, 19, 71]). Consider a horizontal curve $\Gamma = (\gamma, t)$ in $\mathbb{H}^1$ connecting the origin to some point $q = (0, 0, T)$ with $T \neq 0$. The length of Γ equals the length of its projection γ to $\mathbb{R}^2$ (which is a closed curve). Also, by (3.6), the change T in the height of Γ must equal -4 times the signed area enclosed by γ. Thus the projection of any horizontal curve connecting 0 to q must enclose the same area $|T|/4$, and finding a geodesic which connects 0 to q reduces to a problem of finding a shortest closed curve γ enclosing a fixed area. Thus the classical isoperimetric inequality implies that Γ will have smallest length when γ is a circle. Then the t component of Γ is determined by (3.6) and one obtains an explicit parametrization of the geodesics in $\mathbb{H}^1$ connecting the origin to a point on the t-axis. Such geodesics pass through all points (x_0, y_0, t_0), $t_0 \neq 0$ in $\mathbb{H}^1$. If $q = (x_0, y_0, 0)$, then it is easy to see that the segment $\overline{0q}$ connecting the origin to q is a geodesic. This describes all geodesics connecting the origin to any other point in $\mathbb{H}^1$. Due to the left-invariance of the vector fields X and Y, parameterizations for geodesics between arbitrary points in $\mathbb{H}^1$ may be found by left multiplication of the geodesics discussed above.

This elegant argument, however, does not apply to $\mathbb{H}^n$ when $n > 1$ and known proofs of the structure of geodesics in $\mathbb{H}^n$ are based on the Pontryagin maximum principle [5, 10, 73]. In this chapter we will provide a straightforward and elementary argument leading to an explicit parameterization of geodesics in $\mathbb{H}^n$ (Theorem 35). Our argument is based on Hurwitz's proof [53], of the isoperimetric inequality in $\mathbb{R}^2$ involving Fourier series. The Hurwitz argument is used to prove a version of the isoperimetric inequality for closed curves in $\mathbb{R}^{2n}$ (Theorem 38). This isoperimetric inequality allows us to extend the isoperimetric proof of the structure of geodesics in $\mathbb{H}^1$ to the higher dimensional case $\mathbb{H}^n$ as seen in the proof of Theorem 35. For a related, but different isoperimetric inequality in $\mathbb{R}^{2n}$, see [80].

As an application of our method we also prove that the Carnot-Carathéodory metric is real analytic away from the center of the group (Theorem 42). This improves a result of Monti [72, 73]. He proved that this distance is C^∞ smooth away from the center. We also find a formula for the Carnot-Carathéodory distance (Corollary 43) that, we hope, will find application in the study of geometric properties of the Heisenberg groups.

4.1 THE ISOPERIMETRIC INEQUALITY AND THE STRUCTURE OF GEODESICS

Any horizontal curve Γ is rectifiable, and we may parametrize the curve with respect to arc-length. Under this parameterization, the speed $|\dot{\Gamma}|_{d_{cc}}$ of $\Gamma : [0, \ell_H(\Gamma)] \to \mathbb{H}^n$ (as defined in Section 2.1.1) is equal to 1 almost everywhere. According to Proposition 32, the lengths ℓ_H and ℓ_{cc} of any sub-curve of Γ coincide, so by Theorem 8 and the definition of ℓ_H it follows that $|\Gamma'|_H = |\dot{\Gamma}|_{d_{cc}} = 1$ almost everywhere. Then, we can reparametrize Γ as a curve defined on $[0, 1]$ with $|\Gamma'|_H$ constant almost everywhere, and hence we can assume that $\Gamma : [0, 1] \to \mathbb{H}^n$ satisfies

$$\sum_{j=1}^{n} (x_j'(s))^2 + (y_j'(s))^2 = \ell_H(\Gamma)^2 = \ell_{cc}(\Gamma)^2 \quad \text{for almost all } s \in [0, 1]. \tag{4.1}$$

On the other hand any rectifiable curve in $\mathbb{H}^n$ can be reparametrized as a horizontal curve via the arc length parameterization (Proposition 32), and thus, when looking for length minimizing curves (geodesics), it suffices to restrict our attention to horizontal curves $\Gamma : [0,1] \to \mathbb{H}^n$ satisfying (4.1). When this is satisfied, we say that Γ has *constant speed*.

Since the left translation in $\mathbb{H}^n$ is an isometry, it suffices to investigate geodesics connecting the origin $0 \in \mathbb{H}^n$ to another point in $\mathbb{H}^n$. Indeed, if Γ is a geodesic connecting 0 to $p^{-1} * q$, then $p * \Gamma$ is a geodesic connecting p to q.

If q belongs to the subspace $\mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n+1} = \mathbb{H}^n$, then it is easy to check that the straight line $\Gamma(s) = sq$, $s \in [0,1]$ is a unique geodesic (up to a reparametrization) connecting 0 to q. Indeed, it is easy to check that Γ is horizontal, and its length $\ell_{cc}(\Gamma) = \ell_H(\Gamma)$ equals the Euclidean length $|\overline{0q}|$ of the segment $\overline{0q}$ because Γ is equal to its projection γ. For any other horizontal curve $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{t})$ connecting 0 to q, the projection $\tilde{\gamma}$ on $\mathbb{R}^{2n}$ would not be a segment (since horizontal lifts of curves are unique up to vertical shifts), and hence we would have $\ell_{cc}(\tilde{\Gamma}) = \ell_H(\tilde{\Gamma}) = \ell_E(\tilde{\gamma}) > |\overline{0q}| = \ell_{cc}(\Gamma)$ which proves that $\tilde{\Gamma}$ cannot be a geodesic.

In Theorem 35, we will describe the structure of geodesics in $\mathbb{H}^n$ connecting the origin to a point $(0,0,T) \in \mathbb{R}^{2n} \times \mathbb{R} = \mathbb{H}^n$, $T \neq 0$, lying on the t-axis. Later we will see (Corollary 41) that these curves describe all geodesics in $\mathbb{H}^n$ connecting 0 to $q \notin \mathbb{R}^{2n} \times \{0\}$. The geodesics connecting 0 to $q \in \mathbb{R}^{2n} \times \{0\}$ have been described above.

Theorem 35. *A horizontal curve*

$$\Gamma(s) = (x(s), y(s), t(s)) = (x_1(s), \ldots, x_n(s), y_1(s), \ldots, y_n(s), t(s)) : [0,1] \to \mathbb{H}^n$$

of constant speed, connecting the origin $\Gamma(0) = (0,0,0) \in \mathbb{R}^{2n} \times \mathbb{R} = \mathbb{H}^n$ *to a point* $\Gamma(1) = (0,0,\pm T)$*,* $T > 0$*, on the* t*-axis is a geodesic if and only if*

$$\begin{aligned} x_j(s) &= A_j(1 - \cos(2\pi s)) \mp B_j \sin(2\pi s) \\ y_j(s) &= B_j(1 - \cos(2\pi s)) \pm A_j \sin(2\pi s) \end{aligned} \tag{4.2}$$

for $j = 1, 2, \ldots, n$ *and*

$$t(s) = \pm T\left(s - \frac{\sin(2\pi s)}{2\pi}\right)$$

where $A_1, \ldots, A_n, B_1, \ldots, B_n$ *are any real numbers such that* $4\pi \sum_{j=1}^n (A_j^2 + B_j^2) = T$*.*

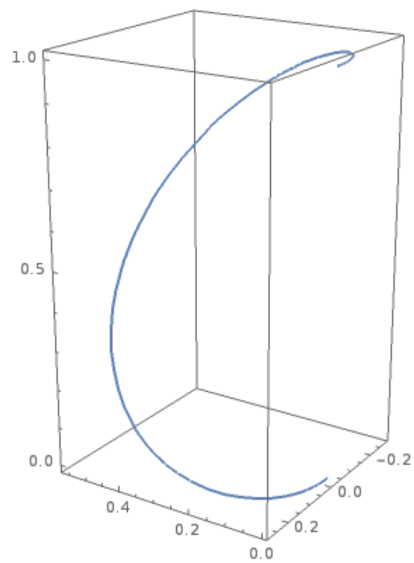

Figure 4: A geodesic in $\mathbb{H}^1$ connecting the origin to $(0, 0, 1)$

Remark 36. *Observe that if* $\Gamma(1) = (0, 0, +T)$*, the equations* (4.2) *give a constant-speed parametrizations of negatively oriented circles in each of the* $x_j y_j$*-planes, centered at* (A_j, B_j)*, and of radius* $\sqrt{A_j^2 + B_j^2}$*. Each circle passes through the origin at* $s = 0$*. The signed area of such a circle equals* $-\pi(A_j^2 + B_j^2)$*. Thus the change in height* $t(1) - t(0)$ *which is* -4 *times the sum of the signed areas of the projections of the curve on the* $x_j y_j$*-planes equals*

$$(-4) \sum_{j=1}^{n} \big(-\pi(A_j^2 + B_j^2) \big) = T.$$

Clearly this must be the case, because Γ *connects the origin to* $(0, 0, T)$*. Any collection of circles in the* $x_j y_j$*-planes passing through the origin and having radii* $r_j \geq 0$ *are projections of a geodesic connecting the origin to the point* $(0, 0, T)$ *where* $T = 4\pi \sum_{j=1}^{n} r_j^2$*. In particular we can find a geodesic for which only one projection is a nontrivial circle (all other radii are zero) and another geodesic for which all projections are non-trivial circles. That suggests that the geodesics connecting* $(0, 0, 0)$ *to* $(0, 0, T)$ *may have many different shapes. This is, however, an incorrect intuition. As we will see in Section 4.3, all such geodesics are obtained from one through a rotation of* $\mathbb{R}^{2n+1}$ *about the* t*-axis. This rotation is also an isometric* *mapping of $\mathbb{H}^n$. The above reasoning applies also to the case when $\Gamma(1) = (0, 0, -T)$ with the only difference being that the circles are positively oriented.*

Remark 37. *The parametric equations for the geodesics can be nicely expressed with the help of complex numbers, see (4.15).*

4.1.1 The isoperimetric inequality

The proof of Theorem 35 is based on the following version of the isoperimetric inequality which is of independent interest. In the theorem below we use identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ given by

$$\mathbb{R}^{2n} \ni (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \leftrightarrow (x_1 + iy_1, \ldots, x_n + iy_n) = x + iy \in \mathbb{C}^n.$$

Every rectifiable curve γ admits the arc-length parametrization. By rescaling it, we may assume that γ is a constant speed curve defined on $[0, 1]$.

Theorem 38. *If $\gamma = (x_1, \ldots, x_n, y_1, \ldots, y_n) : [0, 1] \to \mathbb{R}^{2n}$ is a closed rectifiable curve parametrized by constant speed, then*

$$L^2 \geq 4\pi|\mathfrak{D}|, \tag{4.3}$$

where L is the length of γ and $\mathfrak{D} = \mathfrak{D}_1 + \ldots + \mathfrak{D}_n$ is the sum of signed areas enclosed by the curves $\gamma_j = (x_j, y_j) : [0, 1] \to \mathbb{R}^2$, i.e.

$$\mathfrak{D}_j = \frac{1}{2}\int_0^1 (y_j'(s)x_j(s) - x_j'(s)y_j(s))\, ds.$$

Moreover, equality in (4.3) holds if and only if there are points $A, B, C, D \in \mathbb{R}^n$ such that γ has the form

$$\gamma(s) = (C + iD) + (1 - e^{+2\pi is})(A + iB), \quad \textit{when} \quad L^2 = 4\pi\mathfrak{D} \tag{4.4}$$

and

$$\gamma(s) = (C + iD) + (1 - e^{-2\pi is})(A + iB) \quad \textit{when} \quad L^2 = -4\pi\mathfrak{D}. \tag{4.5}$$

Remark 39. *Let A_j, B_j, C_j and D_j, $j = 1, 2, \dots, n$ be the components of the points A, B, C and D respectively. In terms of real components of γ, (4.4) can be written as*

$$
\begin{aligned}
x_j(s) &= C_j + A_j(1 - \cos(2\pi s)) + B_j \sin(2\pi s) \\
y_j(s) &= D_j + B_j(1 - \cos(2\pi s)) - A_j \sin(2\pi s)
\end{aligned}
\tag{4.6}
$$

and (4.5) as

$$
\begin{aligned}
x_j(s) &= C_j + A_j(1 - \cos(2\pi s)) - B_j \sin(2\pi s) \\
y_j(s) &= D_j + B_j(1 - \cos(2\pi s)) + A_j \sin(2\pi s).
\end{aligned}
\tag{4.7}
$$

That is, the curves $\gamma_j = (x_j, y_j)$ are circles of radius $\sqrt{A_j^2 + B_j^2}$ passing through (C_j, D_j) at $s = 0$. In the case of (4.4) they are all positively oriented and in the case of (4.5) they are all negatively oriented. In either case, they are parametrized with constant angular speed.

Remark 40. *If we have two different circles of the form (4.4) having the same radius, then one can be mapped onto the other one by a composition of translations and a unitary map of $\mathbb{C}^n$. See the proof of Proposition 44. The same comment applies to circles of the form (4.5).*

Proof. Let $\gamma = (x_1, \dots, x_n, y_1, \dots, y_n) : [0, 1] \to \mathbb{R}^{2n}$ be a closed rectifiable curve. By translating the curve, we may assume without loss of generality that $\gamma(0) = 0$. It suffices to prove (4.3) along with equations (4.6) and (4.7) (with $C = D = 0$) which are, as was pointed out in Remark 39, equivalent to (4.4) and (4.5). Since the curve has constant speed, its speed equals the length of the curve, so

$$\sum_{j=1}^{n} (x_j'(s))^2 + (y_j'(s))^2 = L^2.$$

In particular the functions x_j and y_j are L-Lipschitz continuous and $x_j(0) = y_j(0) = x_j(1) = y_j(1) = 0$. Hence the functions x_j, y_j extend to 1-periodic Lipschitz functions on $\mathbb{R}$, and so we can use Fourier series to investigate them. We will follow notation used in [26]. For a 1-periodic function f let

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\, dx, \quad k \in \mathbb{Z}$$

be its kth Fourier coefficient. By Parseval's identity,

$$\begin{aligned} L^2 &= \sum_{j=1}^{n} \int_0^1 |x_j'(s)|^2 + |y_j'(s)|^2\, ds = \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} |\widehat{x_j'}(k)|^2 + |\widehat{y_j'}(k)|^2 \\ &= \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} 4\pi^2 k^2 \left(|\hat{x}_j(k)|^2 + |\hat{y}_j(k)|^2\right) \end{aligned} \tag{4.8}$$

Note that

$$\mathfrak{D} = \mathfrak{D}_1 + \ldots + \mathfrak{D}_n = \frac{1}{2}\sum_{j=1}^{n} \int_0^1 \left(y_j'(s)x_j(s) - x_j'(s)y_j(s)\right)\, ds.$$

Since x_j' and y_j' are real valued, we have $x_j'(s) = \overline{x_j'(s)}$ and $y_j'(s) = \overline{y_j'(s)}$. Thus we may apply Parseval's theorem to this pair of inner products to find

$$\begin{aligned} \mathfrak{D} &= \frac{1}{2}\sum_{j=1}^{n} \left(\sum_{k\in\mathbb{Z}} \overline{\widehat{y_j'}(k)}\hat{x}_j(k) - \sum_{k\in\mathbb{Z}} \overline{\widehat{x_j'}(k)}\hat{y}_j(k)\right) \\ &= \frac{1}{2}\sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} 2\pi k i \left(\overline{\hat{x}_j(k)}\hat{y}_j(k) - \overline{\hat{y}_j(k)}\hat{x}_j(k)\right) \\ &= \pi \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} k \cdot 2\,\mathrm{Im}\Big(\overline{\hat{y}_j(k)}\hat{x}_j(k)\Big), \end{aligned} \tag{4.9}$$

since $i(\bar{z} - z) = 2\,\mathrm{Im}\, z$. Subtracting (4.9) from (4.8) gives

$$\begin{aligned} \frac{L^2}{4\pi^2} - \frac{\mathfrak{D}}{\pi} &= \sum_{j=1}^{n} \Big[\sum_{k\in\mathbb{Z}} k^2 \Big(|\hat{x}_j(k)|^2 + |\hat{y}_j(k)|^2\Big) - k\cdot 2\,\mathrm{Im}\Big(\overline{\hat{y}_j(k)}\hat{x}_j(k)\Big)\Big] \\ &= \sum_{j=1}^{n} \Big[\sum_{k\in\mathbb{Z}} (k^2 - |k|)\Big(|\hat{x}_j(k)|^2 + |\hat{y}_j(k)|^2\Big) \\ &\quad + |k| \left(|\hat{y}_j(k)|^2 - 2\,\mathrm{sgn}(k)\mathrm{Im}\left(\overline{\hat{y}_j(k)}\hat{x}_j(k)\right) + |\hat{x}_j(k)|^2\right)\Big] \\ &= \sum_{j=1}^{n} \Big[\sum_{k\in\mathbb{Z}} (k^2 - |k|)\Big(|\hat{x}_j(k)|^2 + |\hat{y}_j(k)|^2\Big) + \sum_{k\in\mathbb{Z}} |k| \Big|\hat{y}_j(k) + i\,\mathrm{sgn}(k)\hat{x}_j(k)\Big|^2\Big]. \end{aligned} \tag{4.10}$$

The last equality follows from the identity $|a+ib|^2 = |a|^2 - 2\,\mathrm{Im}(\bar{a}b) + |b|^2$ which holds for all $a, b \in \mathbb{C}$. Since every term in this last sum is non-negative, it follows that $\frac{L^2}{4\pi^2} - \frac{\mathfrak{D}}{\pi} \geq 0$. Thus, we have $L^2 \geq 4\pi\mathfrak{D}$. Reversing the orientation of the curve, i.e. applying the above argument to $\tilde{\gamma}(t) = \gamma(1-t)$ gives $L^2 \geq -4\pi\mathfrak{D}$, so (4.3) follows.

Equality in (4.3) holds if and only if either $L^2 = 4\pi\mathfrak{D}$ or $L^2 = -4\pi\mathfrak{D}$. We will first consider the case $L^2 = 4\pi\mathfrak{D}$. This equality will occur if and only if each of the two sums contained inside the brackets in (4.10) equals zero. Since $k^2 - |k| > 0$ for $|k| \geq 2$, the first of the two sums vanishes if and only if $\hat{x}_j(k) = \hat{y}_j(k) = 0$ for every $|k| \geq 2$ and $j = 1, 2, \ldots, n$. Hence nontrivial terms in the second sum correspond to $k = \pm 1$, and thus this sum vanishes if and only if $\hat{y}_j(\pm 1) = -i \operatorname{sgn}(\pm 1) \hat{x}_j(\pm 1)$. That is, for every $j = 1, \ldots, n$,

$$\hat{y}_j(1) = -i\, \hat{x}_j(1) \quad \text{and} \quad \hat{y}_j(-1) = i\, \hat{x}_j(-1). \tag{4.11}$$

Now since each x_j and y_j is Lipschitz, their Fourier series converge uniformly on $[0, 1]$. Note that the only non-zero terms in the Fourier series appear when $|k| \leq 1$. Thus $L^2 = 4\pi\mathfrak{D}$ if and only if (4.11) is satisfied and for every $s \in [0, 1]$ and $j = 1, \ldots, n$

$$\begin{aligned} x_j(s) &= \hat{x}_j(-1)e^{-2\pi i s} + \hat{x}_j(0) + \hat{x}_j(1)e^{2\pi i s} \\ y_j(s) &= \hat{y}_j(-1)e^{-2\pi i s} + \hat{y}_j(0) + \hat{y}_j(1)e^{2\pi i s}. \end{aligned} \tag{4.12}$$

In particular, $0 = x_j(0) = \hat{x}_j(-1) + \hat{x}_j(0) + \hat{x}_j(1)$ and hence $\hat{x}_j(0) = -\hat{x}_j(-1) - \hat{x}_j(1)$ for each $j = 1, \ldots, n$. This together with Euler's formula gives

$$\begin{aligned} x_j(s) &= \hat{x}_j(-1)\big(e^{-2\pi i s} - 1\big) + \hat{x}_j(1)\big(e^{2\pi i s} - 1\big) \\ &= -(\hat{x}_j(-1) + \hat{x}_j(1))(1 - \cos(2\pi s)) + (-i\hat{x}_j(-1) + i\hat{x}_j(1))\sin(2\pi s) \\ &= -(\hat{x}_j(-1) + \hat{x}_j(1))(1 - \cos(2\pi s)) - (\hat{y}_j(-1) + \hat{y}_j(1))\sin(2\pi s). \end{aligned}$$

The last equality follows from (4.11). Similarly, we have

$$y_j(s) = -(\hat{y}_j(-1) + \hat{y}_j(1))(1 - \cos(2\pi s)) + (\hat{x}_j(-1) + \hat{x}_j(1))\sin(2\pi s).$$

If we write $A_j = -(\hat{x}_j(-1) + \hat{x}_j(1))$ and $B_j = -(\hat{y}_j(-1) + \hat{y}_j(1))$, then we have

$$\begin{aligned} x_j(s) &= A_j(1 - \cos(2\pi s)) + B_j \sin(2\pi s) \\ y_j(s) &= B_j(1 - \cos(2\pi s)) - A_j \sin(2\pi s). \end{aligned} \tag{4.13}$$

Note that it follows directly from the definition of Fourier coefficients that the numbers A_j, B_j are real.

The case $L^2 = -4\pi\mathfrak{D}$ is reduced to the above case by reversing the orientation of γ as previously described. In that case the curves γ_j are given by

$$
\begin{aligned}
x_j(s) &= A_j(1-\cos(2\pi s)) - B_j\sin(2\pi s)\\
y_j(s) &= B_j(1-\cos(2\pi s)) + A_j\sin(2\pi s).
\end{aligned}
\tag{4.14}
$$

We proved that if $L^2 = 4\pi\mathfrak{D}$, then γ is of the form (4.6) and if $L^2 = -4\pi\mathfrak{D}$, then it is of the form (4.7). In the other direction, a straightforward calculation shows that any curve of the form (4.6) satisfies $L^2 = 4\pi\mathfrak{D}$ and any curve of the form (4.7) satisfies $L^2 = -4\pi\mathfrak{D}$. This completes the proof. $\square$

4.1.2 The structure of geodesics

Proof of Theorem 35. Suppose first that $\Gamma = (\gamma, t) : [0,1] \to \mathbb{H}^n$ is any horizontal curve of constant speed connecting the origin to the point $(0,0,+T)$, $T>0$. Recall from (4.1) that

$$\sum_{j=1}^n (x_j'(s))^2 + (y_j'(s))^2 = \ell_{cc}(\Gamma)^2 =: L^2.$$

Thus $\gamma : [0,1] \to \mathbb{R}^{2n}$ is a closed curve of length L parametrized by arc-length. Moreover $\gamma(0) = 0$.

If $\mathfrak{D}$ is defined as in Theorem 38, it follows from (3.6) that

$$T = 2\sum_{j=1}^n \int_0^1 \left(x_j'(s)y_j(s) - y_j'(s)x_j(s)\right)\, ds = -4\mathfrak{D}$$

so $\mathfrak{D} < 0$ and $L^2 \geq \pi T$ by Theorem 38. Now Γ is a geodesic if and only if $L^2 = \pi T = -4\pi\mathfrak{D}$ which is the case of the equality in the isoperimetric inequality (4.3). We proved above that this is equivalent to the components of γ satisfying (4.14), and this is the $(0,0,+T)$ case of (4.2). One may also easily check that $4\pi\sum_{j=1}^n (A_j^2 + B_j^2) = -4\mathfrak{D} = T$.

Suppose now that $\Gamma : [0,1] \to \mathbb{H}^n$ is any horizontal curve of constant speed connecting the origin to the point $(0,0,-T)$, $T>0$. Then $\Gamma = (x(s), y(s), t(s))$ is a geodesic if and only if

$$\tilde{\Gamma}(s) = (\tilde{x}(s), \tilde{y}(s), \tilde{t}(s)) = (x(1-s), y(1-s), t(1-s)+T)$$

is a geodesic connecting $\tilde{\Gamma}(0) = (0,0,0)$ and $\tilde{\Gamma}(1) = (0,0,T)$ since reversing a curve's parametrization does not change its length and since the mapping $(x,y,t) \mapsto (x,y,t+T)$ (the vertical lift by T) is an isometry on $\mathbb{H}^n$. Therefore $\tilde{\Gamma} = (\tilde{x},\tilde{y},\tilde{t})$ must have the form (4.14). Hence the $(0,0,-T)$ case of (4.2) follows from (4.14) by replacing s with $1-s$.

The formula for the t component of Γ follows from (3.6); the integral is easy to compute due to numerous cancellations. $\square$

Using the complex notation as in Theorem 38, the geodesics from Theorem 35 connecting the origin to $(0,0,\pm T)$, $T>0$ can be represented as

$$\Gamma(s) = \Big(\big(1-e^{\mp 2\pi i s}\big)(A+iB), t(s)\Big) \tag{4.15}$$

where $A = (A_1,\dots,A_n)$, $B = (B_1,\dots,B_n)$ are such that $4\pi|A+iB|^2 = T$ and

$$t(s) = \pm T\left(s - \frac{\sin(2\pi s)}{2\pi}\right).$$

Theorem 35 and a discussion preceding it describes geodesics connecting the origin to points either on the t-axis $(0,0,\pm T)$, $T>0$ or in $\mathbb{R}^{2n}\times\{0\}$. The question now is how to describe geodesics connecting the origin to a point q which is neither on the t-axis nor in $\mathbb{R}^{2n}\times\{0\}$. It turns out that geodesics described in Theorem 35 cover the entire space $\mathbb{H}^n\setminus(\mathbb{R}^{2n}\times\{0\})$ and we have

Corollary 41. *For any $q\in\mathbb{H}^n$ which is neither in the t-axis nor in the subspace $\mathbb{R}^{2n}\times\{0\}$ there is a unique geodesic connecting the origin to q. This geodesic is a part of a geodesic connecting the origin to a point on the t-axis.*

Proof. Let $q = (c_1,\dots,c_n,d_1,\dots,d_n,h)$ be such that $h\neq 0$ and c_j, d_j are not all zero. We can write $q = (c+id,h)\in\mathbb{C}^n\times\mathbb{R}$. First we will construct a geodesic Γ_q given by (4.15) so that $\Gamma_q(s_0) = q$ for some $s_0\in(0,1)$. Clearly, the curve $\Gamma_q\big|_{[0,s_0]}$ will be part of a geodesic connecting the origin to a point on the t-axis. Then we will prove that this curve is a unique geodesic (up to a reparametrization) connecting the origin to q. Assume that $h>0$ (the case $h<0$ is similar). We will find a geodesic passing through q that connects $(0,0,0)$ to

$(0,0,T)$, for some $T>0$. (If $h<0$ we find Γ that connects $(0,0,0)$ to $(0,0,-T)$.) It suffices to show that there is a point $A+iB\in\mathbb{C}^n$ such that the system of equations

$$(1-e^{-2\pi is})(A+iB)=c+id, \qquad 4\pi|A+iB|^2\left(s-\frac{\sin(2\pi s)}{2\pi}\right)=h \tag{4.16}$$

has a solution $s_0\in(0,1)$. We have $A+iB=(c+id)/(1-e^{-2\pi is})$ and hence

$$2\pi\,\frac{|c+id|^2}{1-\cos(2\pi s)}\left(s-\frac{\sin(2\pi s)}{2\pi}\right)=h. \tag{4.17}$$

This equation has a unique solution $s_0\in(0,1)$ because the function on the left hand side is an increasing diffeomorphism of $(0,1)$ onto $(0,\infty)$. We proved that, among geodesics connecting $(0,0,0)$ to points $(0,0,T)$, $T>0$, there is a unique geodesic Γ_q passing through q. Suppose now that $\tilde{\Gamma}$ is any geodesic connecting $(0,0,0)$ to q. Gluing $\tilde{\Gamma}$ with $\Gamma_q\big|_{[s_0,1]}$ we obtain a geodesic connecting $(0,0,0)$ to $(0,0,T)$ and hence (perhaps after a reparametrization) it must coincide with Γ_q. This proves uniqueness of the geodesic $\Gamma_q\big|_{[0,s_0]}$. $\square$

4.1.3 A formula for the Carnot-Carathéodory distance

We will now use the proof of Corollary 41 to find a formula for the Carnot-Carathéodory distance between 0 and $q=(z,h)$, $z\neq 0$, $h>0$. We will need this formula in the next section. Let

$$H(s)=\frac{2\pi}{1-\cos(2\pi s)}\left(s-\frac{\sin(2\pi s)}{2\pi}\right):(0,1)\to(0,\infty) \tag{4.18}$$

be the diffeomorphism of $(0,1)$ onto $(0,\infty)$ described in (4.17). Let

$$\Gamma(s)=\left(\left(1-e^{-2\pi is}\right)(A+iB),t(s)\right)$$

be the geodesic from the proof of Corollary 41 that passes through q at $s_0\in(0,1)$. We proved that s_0 is a solution to (4.17) and hence s_0 is a function of q given by

$$s_0(q)=H^{-1}(h|z|^{-2}).$$

Note that $A+iB=z/(1-e^{-2\pi is_0})$, so

$$|A+iB|=\frac{|z|}{\sqrt{2(1-\cos(2\pi s_0))}}\,.$$

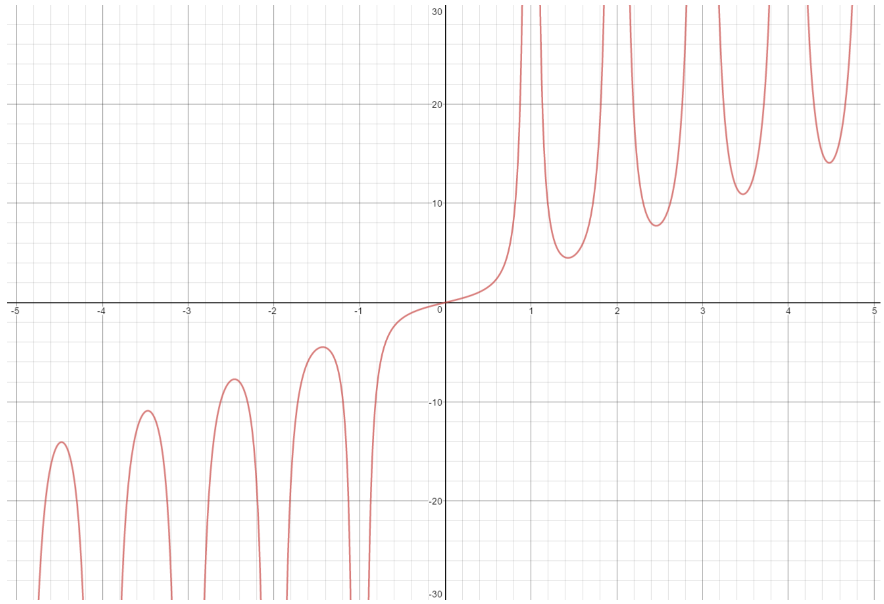

Figure 5: The function H defined on $\bigcup_{n\in\mathbb{Z}}(n, n+1)$

Hence

$$\sqrt{\sum_{j=1}^{n}(x_j'(s))^2 + (y_j'(s))^2} = L = \sqrt{\pi T} = 2\pi|A+iB| = \frac{2\pi|z|}{\sqrt{2(1-\cos(2\pi s_0))}}$$

where L is the length of Γ and $\Gamma(1) = (0,0,T)$. Therefore

$$d_{cc}(0,q) = \int_0^{s_0}\sqrt{\sum_{j=1}^{n}(x_j'(s))^2 + (y_j'(s))^2}\,ds = \frac{2\pi s_0|z|}{\sqrt{2(1-\cos(2\pi s_0))}}. \tag{4.19}$$

4.2 ANALYTICITY OF THE CARNOT-CARATHÉODORY METRIC

The center of the Heisenberg group $\mathbb{H}^n$ is $Z = \{(z,h)\in\mathbb{H}^n : z = 0\}$. It is well known that the distance function in $\mathbb{H}^n$ is C^∞ smooth away from the center [72, 73], but through the use of (4.2), we will now see that this distance function is actually real analytic.

Theorem 42. *The Carnot-Carathéodory distance $d_{cc} : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is real analytic on the set*

$$\left\{(p,q) \in \mathbb{H}^n \times \mathbb{H}^n = \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} : q^{-1} * p \notin Z\right\}.$$

Proof. In the proof we will make a frequent use of a well known fact that a composition of real analytic functions is analytic, [60, Proposition 2.2.8]. It suffices to prove that the function $d_0(p) = d_{cc}(0,p)$ is real analytic on $\mathbb{H}^n \setminus Z$. Indeed, $w(p,q) = q^{-1} * p$ is real analytic as it is a polynomial. Also, $d_{cc}(p,q) = (d_0 \circ w)(p,q)$, so real analyticity of d_0 on $\mathbb{H}^n \setminus Z$ will imply that d_{cc} is real analytic on $w^{-1}(\mathbb{H}^n \setminus Z) = \{(p,q) \in \mathbb{H}^n \times \mathbb{H}^n : q^{-1} * p \notin Z\}$.

Define $H : (-1,1) \to \mathbb{R}$ as

$$H(s) = \frac{2\pi}{1-\cos(2\pi s)}\left(s - \frac{\sin(2\pi s)}{2\pi}\right) = \frac{\frac{2\pi s}{3!} - \frac{(2\pi s)^3}{5!} + \frac{(2\pi s)^5}{7!} - \ldots}{\frac{1}{2!} - \frac{(2\pi s)^2}{4!} + \frac{(2\pi s)^4}{6!} - \ldots}. \tag{4.20}$$

Here, we divided by a common factor of $(2\pi s)^2$ in the two power series on the right hand side. That is, the denominator equals $(1-\cos(2\pi s))(2\pi s)^{-2}$ which does not vanish on $(-1,1)$. This implies that H is real analytic on $(-1,1)$. Indeed, considering s as a complex variable, we see that $H(s)$ is holomorphic (and hence analytic) in an open set containing $(-1,1)$ as a ratio of two holomorphic functions with non-vanishing denominator.

As we pointed out in (4.18), the function H is an increasing diffeomorphism of $(0,1)$ onto $(0,\infty)$. Since it is odd and $H'(0) = 2\pi/3 \neq 0$, H is a real analytic diffeomorphism of $(-1,1)$ onto $\mathbb{R}$. Again, using a holomorphic function argument we see that $H^{-1} : \mathbb{R} \to (-1,1)$ is a real analytic.

The function $z \mapsto |z|^{-2}$ is analytic on $\mathbb{R}^{2n}\setminus\{0\}$ (as a composition of a polynomial $z \mapsto |z|^2$ and an analytic function $1/x$), so the function $(z,h) \mapsto h|z|^{-2}$ is analytic in $\mathbb{H}^n \setminus Z$. Hence also $s_0(q) = H^{-1}(h|z|^{-2})$ is analytic on $\mathbb{H}^n \setminus Z$.

Fix $q = (z,h) \in \mathbb{H}^n \setminus Z$ with $h > 0$. Then by (4.19)

$$d_0(q) = \frac{2\pi s_0|z|}{\sqrt{2(1-\cos(2\pi s_0))}}. \tag{4.21}$$

Since $H(s_0) = h|z|^{-2}$, formula (4.20) yields

$$2\pi s_0 = (1-\cos(2\pi s_0))h|z|^{-2} + \sin(2\pi s_0).$$

Substituting $2\pi s_0$ in the numerator of the right hand side of (4.21) gives

$$d_0(q) = \frac{h\sqrt{1-\cos(2\pi s_0)}}{\sqrt{2}|z|} + \frac{|z|\sin(2\pi s_0)}{\sqrt{2}\sqrt{1-\cos(2\pi s_0)}} = h\sin(\pi s_0)|z|^{-1} + |z|\cos(\pi s_0)$$

where we used the trigonometric identities

$$\sqrt{1-\cos(2\pi s_0)} = \sqrt{2}|\sin(\pi s_0)| = \sqrt{2}\sin(\pi s_0) \quad \text{and} \quad \frac{\sin(2\pi s_0)}{\sin(\pi s_0)} = 2\cos(\pi s_0).$$

To treat the case $h \leq 0$ let us define $s_0(q) = H^{-1}(h|z|^{-2})$ for any $q = (z, h)$, $z \neq 0$. Previously we defined $s_0(q)$ only when $h > 0$. It is easy to check that the mapping

$$q = (x, y, t) = (z, t) \mapsto \bar{q} = (\bar{z}, -t) = (x, -y, -t)$$

is an isometry of the Heisenberg group, so $d_0(q) = d_0(\bar{q})$.

If $h < 0$ and $\bar{q} = (\bar{z}, -h)$, then

$$s_0(q) = H^{-1}(h|z|^{-2}) = -H^{-1}(-h|\bar{z}|^{-2}) = -s_0(\bar{q})$$

and hence

$$d_0(q) = d_0(\bar{q}) = -h\sin(\pi s_0(\bar{q}))|\bar{z}|^{-1} + |\bar{z}|\cos(\pi s_0(\bar{q})) = h\sin(\pi s_0(q)) + |z|\cos(\pi s_0(q)).$$

In the case $h = 0$, Γ is a straight line in $\mathbb{R}^{2n}$ from the origin to q, and so $d_0(q) = |z|$. Therefore

$$d_0(q) = h\sin(\pi s_0(q))|z|^{-1} + |z|\cos(\pi s_0(q)) = h\sin(\pi H^{-1}(h|z|^{-2}))|z|^{-1} + |z|\cos(\pi H^{-1}(h|z|^{-2}))$$

for every $q = (z, h) \in \mathbb{H}^n \setminus Z$, and so d_0 is analytic on $\mathbb{H}^n \setminus Z$. □

We also proved

Corollary 43. *For $z \neq 0$, the Carnot-Carathéodory distance between the origin $(0, 0)$ and (z, h), $z \neq 0$ equals*

$$d_{cc}((0,0),(z,h)) = h\sin(\pi H^{-1}(h|z|^{-2}))|z|^{-1} + |z|\cos(\pi H^{-1}(h|z|^{-2})).$$

4.3 CLASSIFICATION OF NON-UNIQUE GEODESICS

Any point $(0, 0, \pm T)$, $T > 0$ on the t axis can be connected to the origin by infinitely many geodesics. The purpose of this section is to show that all such geodesics are actually obtained from one geodesic by a linear mapping which fixes the t-axis. This map is an isometry of $\mathbb{H}^n$ and also an isometry of $\mathbb{R}^{2n+1}$.

Proposition 44. *If* $\Gamma_1 : [0, 1] \to \mathbb{H}^n$ *and* $\Gamma_2 : [0, 1] \to \mathbb{H}^n$ *are constant-speed geodesics with* $\Gamma_1(0) = \Gamma_2(0) = (0, 0, 0)$ *and* $\Gamma_1(1) = \Gamma_2(1) = (0, 0, \pm T)$ *with* $T > 0$*, then we can write* $\Gamma_2 = V \circ \Gamma_1$ *where* V *is a isometry in* $\mathbb{H}^n$ *which fixes the* t*-coordinate. The map* V *is also an isometry of* $\mathbb{R}^{2n+1}$*, specifically a rotation about the* t*-axis.*

In the following proof, $U(n, \mathbb{C})$ will represent the space of $n \times n$ unitary matrices with complex coefficients.

Proof. Consider geodesics $\Gamma_1 = (\gamma_1, t)$ and $\Gamma_2 = (\gamma_2, t)$ defined in the statement of the proposition. As in the discussion before Corollary 41, we consider γ_1 and γ_2 as functions into $\mathbb{C}^n$ rather than into $\mathbb{R}^{2n}$ and write

$$\gamma_1(s) = \left(1 - e^{\mp 2\pi i s}\right)(A + iB), \qquad \gamma_2(s) = \left(1 - e^{\mp 2\pi i s}\right)(C + iD)$$

where $4\pi|A + iB|^2 = 4\pi|C + iD|^2 = T$. We claim that there is a unitary matrix $U \in \mathrm{U}(n, \mathbb{C})$ such that $U(A + iB) = C + iD$. Indeed, for any $0 \neq z \in \mathbb{C}^n$, use the Gram-Schmidt process to extend $\{z/|z|\}$ to an orthonormal basis of $\mathbb{C}^n$ and define W_z to be the matrix whose columns are these basis vectors. Here, we consider orthogonality with respect to the standard Hermitian inner product $\langle u, v\rangle_{\mathbb{C}} = \sum_{j=1}^n u_j \overline{v}_j$. Then $W_z \in \mathrm{U}(n, \mathbb{C})$ and $W_z e_1 = z/|z|$ where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$. Thus the desired operator is $U = W_{C+iD} \circ W_{A+iB}^{-1}$.

Define the linear map $V : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ by $V(z, t) = (Uz, t)$. Since

$$U\left(\left(1 - e^{\mp 2\pi i s}\right)(A + iB)\right) = \left(1 - e^{\mp 2\pi i s}\right)(C + iD),$$

for every $s \in [0, 1]$ and since V fixes the t-component of $\mathbb{C}^n \times \mathbb{R}$, we have $V \circ \Gamma_1 = \Gamma_2$.

We now prove that V is an isometry on $\mathbb{H}^n$. Indeed, suppose $p, q \in \mathbb{H}^n$ and $\Gamma = (\gamma, t) : [0, 1] \to \mathbb{H}^n$ is a geodesic connecting them. Then $V \circ \Gamma = (U \circ \gamma, t)$. Since Γ is horizontal, it is easy to check that $t'(s) = 2\ \mathrm{Im}\langle \gamma(s), \gamma'(s)\rangle_{\mathbb{C}}$ for almost every $s \in [0,1]$. Unitary operators preserve the standard inner product on $\mathbb{C}^n$, and so

$$\begin{aligned} t'(s) = 2\ \mathrm{Im}\,\langle \gamma(s), \gamma'(s)\rangle_{\mathbb{C}} &= 2\ \mathrm{Im}\,\langle (U\circ\gamma)(s), (U\circ\gamma')(s)\rangle_{\mathbb{C}} \\ &= 2\ \mathrm{Im}\left\langle (U\circ\gamma)(s), \frac{d}{ds}(U\circ\gamma)(s)\right\rangle_{\mathbb{C}} \end{aligned}$$

for almost every $s \in [0,1]$. That is, $V \circ \Gamma$ is horizontal. Also,

$$\ell_H(\Gamma) = \int_0^1 \sqrt{\langle \gamma'(s), \gamma'(s)\rangle_{\mathbb{C}}}\, ds = \int_0^1 \sqrt{\langle (U\circ\gamma')(s), (U\circ\gamma')(s)\rangle_{\mathbb{C}}}\, ds = \ell_H(V\circ\Gamma).$$

Thus $d_{cc}(Vp, Vq) \leq \ell_H(\Gamma) = d_{cc}(p,q)$. Since U is invertible and $U^{-1} \in \mathrm{U}(n,\mathbb{C})$, we may argue similarly to show that $d_{cc}(p,q) = d_{cc}(V^{-1}Vp, V^{-1}Vq) \leq d_{cc}(Vp,Vq)$, and so V is an isometry on $\mathbb{H}^n$. Clearly unitary transformations of $\mathbb{C}^n$ are also orientation preserving isometries of $\mathbb{R}^{2n}$ and hence V is a rotation of $\mathbb{R}^{2n+1}$ about the t-axis. $\square$

5.0 THE WHITNEY EXTENSION THEOREM FOR HORIZONTAL CURVES IN THE HEISENBERG GROUP

This chapter is based on the paper [97]. In 1934, Whitney [94] discovered a necessary and sufficient condition for the existence of an extension $\tilde{f} \in C^k(\mathbb{R}^m)$ of a continuous function $f : K \to \mathbb{R}$ defined on a compact set $K \subset \mathbb{R}^m$. The purpose of this chapter is to prove a version of the Whitney Extension Theorem for mappings from a compact subset of $\mathbb{R}$ into the sub-Riemannian Heisenberg group $\mathbb{H}^n$. Applications of Whitney's extension theorem may be found in the construction of functions with unusual differentiability properties (see [93]) and the existence of C^1 approximations for Lipschitz mappings (see [29, Theorem 3.1.15] or Corollary 60 below). Such approximations are useful in the study of rectifiable sets, and the notion of rectifiability has seen recent activity in the setting of $\mathbb{H}^n$ (see for example [7, 32, 56, 68]). In fact, the authors in [68] indicate that a Whitney type extension theorem into the Heisenberg group would help show the equivalence of two notions of rectifiability in $\mathbb{H}^n$. For a comprehensive summary of the work done on Whitney type questions, see the introduction and references of [30].

Definition 45. *We say that a continuous function $f : K \to \mathbb{R}$ defined on a compact set $K \subset \mathbb{R}^m$ is of Whitney class $\mathfrak{C}^1(K)$ (equivalently $f \in \mathfrak{C}^1(K)$) if there is a continuous function $\mathcal{D}f \in C(K, \mathbb{R}^m)$ such that*

$$\lim_{\substack{|b-a|\to 0\\ a,b\in K}} \frac{|f(b) - f(a) - \mathcal{D}f(a) \cdot (b-a)|}{|b-a|} = 0. \tag{5.1}$$

We will call $\mathcal{D}f$ the derivative of f in the Whitney sense or the Whitney derivative of f.

Note that, a priori, $\mathcal{D}f$ is unrelated to the classical derivative since it is simply a continuous function defined on a compact set.

Condition (5.1) is necessary for the existence of a C^1 extension since any smooth function defined on $\mathbb{R}^m$ will satisfy (5.1) on a compact set $K \subset \mathbb{R}^m$ with Whitney derivative equal to the classical derivative. Whitney proved that (5.1) is also sufficient to guarantee the existence of a C^1 extension. That is, for any compact $K \subset \mathbb{R}^m$ and $f \in \mathfrak{C}^1(K)$, there exists a function $\tilde{f} \in C^1(\mathbb{R}^m)$ such that $\tilde{f}|_K = f$ and $\nabla \tilde{f}|_K = \mathcal{D}f$. See [65, 94] for proofs of this. Whitney actually proved a similar result with higher order regularity of f, but we will focus here only on the first order case. See Theorem 102 below for the statement of this higher order result.

The Whitney class can be defined for mappings between higher dimensional Euclidean spaces in an obvious way. A mapping $F : K \to \mathbb{R}^N$ is said to be of Whitney class $\mathfrak{C}^1(K, \mathbb{R}^N)$ (equivalently $F \in \mathfrak{C}^1(K, \mathbb{R}^N)$) for a compact $K \subset \mathbb{R}^m$ if each component f_j of F is of Whitney class $\mathfrak{C}^1(K)$ with Whitney derivative $\mathcal{D}f_j$. Call $\mathcal{D}F = (\mathcal{D}f_1, \ldots, \mathcal{D}f_N) : K \to (\mathbb{R}^m)^N$ the Whitney derivative of F. Given any $F \in \mathfrak{C}^1(K, \mathbb{R}^N)$, we may construct a C^1 extension of F by applying Whitney's result to each of its components.

5.1 FORMULATING THE PROBLEM

A natural question may be asked: what form would a sort of Whitney extension theorem take in the Heisenberg group? In 2001, Franchi, Serapioni, and Serra Cassano [32] proved a C^1 version of the Whitney extension theorem for mappings from the Heisenberg group $\mathbb{H}^n$ into $\mathbb{R}$. The authors provided a concise proof highlighting the major differences between the Euclidean and Heisenberg cases. For a full exposition of the proof, see [88]. In this theorem, the function defined on a compact $K \subset \mathbb{H}^n$ is extended to C^1_H function. That is, the derivatives of the extension in the horizontal directions exist and are continuous. In 2006, Vodop'yanov and Pupyshev [89] proved a C^k version of Whitney's theorem for real valued functions defined on closed subsets of general Carnot groups.

In 2013, Piotr Hajłasz posed the following two questions:

- (Whitney extension) What are necessary and sufficient conditions for a continuous map $f : K \to \mathbb{R}^{2n+1}$ with $K \subset \mathbb{R}^m$ compact and $m \leq n$ to have a C^1 extension $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^{2n+1}$ satisfying $\mathrm{im}(D\tilde{f}(x)) \subset H_{\tilde{f}(x)}\mathbb{H}^n$ for every $x \in \mathbb{R}^m$?
- (C^1 Luzin property) Is it true that, for every horizontal curve $\Gamma : [a, b] \to \mathbb{H}^n$ and any $\varepsilon > 0$, there is a C^1, horizontal curve $\hat{\Gamma} : [a, b] \to \mathbb{H}^n$ such that

$$|\{s \in [a, b] \,:\, \hat{\Gamma}(s) \neq \Gamma(s)\}| < \varepsilon?$$

Remark 46. Note that the Whitney extension problem stated above is very different from the one solved by Franchi, Serapioni, and Serra Cassano since the nonlinear constraint now lies in the target space. Such a constraint makes the problem much more difficult.

Remark 47. We only consider the Whitney problem in the case when $m \leq n$ since, if $m > n$, we have possible topological obstacles preventing the existence of a smooth extension. For more details, see [7, 22].

Let us consider the Whitney extension question in the case when $m = 1$. For $K \subset \mathbb{R}$ compact, let $\Gamma = (f_1, g_1, \ldots, f_n, g_n, h) : K \to \mathbb{R}^{2n+1}$ be continuous so that there is a C^1, horizontal extension $\tilde{\Gamma} : \mathbb{R} \to \mathbb{R}^{2n+1}$. Then clearly $\Gamma \in \mathfrak{C}^1(K, \mathbb{R}^{2n+1})$ with Whitney derivative $\Gamma' := \tilde{\Gamma}'|_K$. That is,

$$\lim_{\substack{|b-a|\to 0\\ a,b\in K}} \frac{|\Gamma(b) - \Gamma(a) - (b-a)\Gamma'(a)|}{|b-a|} = 0. \tag{5.2}$$

Γ' must also satisfy the horizontality condition

$$h'(s) = 2\sum_{j=1}^{n}(f_j'(s)g_j(s) - f_j(s)g_j'(s)) \tag{5.3}$$

for any $s \in K$ since any C^1, horizontal curve defined on $\mathbb{R}$ satisfies (5.3) for every $s \in \mathbb{R}$. We may ask the following: are conditions (5.3) and (5.2) sufficient to guarantee the existence of a horizontal, C^1 extension $\tilde{\Gamma}$ of Γ? As we see here, the answer to this is, in general, "no".

Proposition 48. *There is a compact $K \subset \mathbb{R}$ and $\Gamma = (f, g, h) \in \mathfrak{C}^1(K, \mathbb{R}^3)$ with Whitney derivative $\Gamma' = (f', g', h')$ satisfying $h' = 2(f'g - fg')$ so that no C^1, horizontal curve $\tilde{\Gamma} : \mathbb{R} \to \mathbb{H}^1$ satisfies $\tilde{\Gamma}|_K = \Gamma$.*

The next natural question to ask is the following: under what additional assumption does there exist a C^1, horizontal extension of $\Gamma \in \mathfrak{C}^1(K, \mathbb{R}^{2n+1})$? The following proposition describes a necessary condition that every C^1, horizontal curve satisfies.

Proposition 49. *Suppose $U \subset \mathbb{R}$ is open and $\Gamma = (f_1, g_1, \ldots, f_n, g_n, h) : U \to \mathbb{H}^n$ is C^1 and horizontal. Then for any compact $K \subset U$*

$$\lim_{\substack{|b-a|\to 0 \\ a,b\in K}} \frac{\left|h(b) - h(a) - 2\sum_{j=1}^n (f_j(b)g_j(a) - f_j(a)g_j(b))\right|}{|b-a|^2} = 0. \tag{5.4}$$

The proofs of these two propositions are presented in Section 5.2.

5.1.1 The Whitney extension theorem for curves

As we will now see, the main result of this chapter shows that assuming condition (5.4) in addition to (5.2) and (5.3) is in fact necessary and sufficient for the existence of a C^1, horizontal extension of a continuous $\Gamma : \mathbb{R} \supset K \to \mathbb{H}^n$. This is summarized as follows:

Theorem 50. *Suppose $K \subset \mathbb{R}$ is compact. Suppose $\Gamma = (f_1, g_1, \ldots, f_n, g_n, h) : K \to \mathbb{H}^n$ is of Whitney class $\mathfrak{C}^1(K, \mathbb{R}^{2n+1})$ with Whitney derivative $\Gamma' = (f_1', g_1', \ldots, f_n', g_n', h')$.*

Then there is a horizontal, C^1 curve $\tilde{\Gamma} : \mathbb{R} \to \mathbb{H}^n$ such that $\tilde{\Gamma}|_K = \Gamma$ and $\tilde{\Gamma}'|_K = \Gamma'$ if and only if

$$\lim_{\substack{|b-a|\to 0 \\ a,b\in K}} \frac{\left|h(b) - h(a) - 2\sum_{j=1}^n (f_j(b)g_j(a) - f_j(a)g_j(b))\right|}{|b-a|^2} = 0 \tag{5.5}$$

and

$$h'(s) = 2\sum_{j=1}^n \left(f_j'(s)g_j(s) - g_j'(s)f_j(s)\right) \quad \textit{for every } s \in K. \tag{5.6}$$

Remark 51. We actually do not need to assume that $h \in \mathfrak{C}^1(K)$ because it is a consequence of (5.5) and the fact that $f_j \in \mathfrak{C}^1(K)$ and $g_j \in \mathfrak{C}^1(K)$ for $j = 1, \ldots, n$. The proof of this is simple, but it is contained at the end of Section 5.2 for completeness.

Theorem 50 can be reformulated using the Lie group structure of $\mathbb{H}^n$ as follows:

Theorem 52. *Suppose $K \subset \mathbb{R}$ is compact. Suppose $\Gamma = (f_1, g_1, \ldots, f_n, g_n, h) : K \to \mathbb{H}^n$ and $\Gamma' = (f_1', g_1', \ldots, f_n', g_n', h') : K \to \mathbb{H}^n$ are continuous.*

Then there is a horizontal, C^1 curve $\tilde{\Gamma} : \mathbb{R} \to \mathbb{H}^n$ such that $\tilde{\Gamma}|_K = \Gamma$ and $\tilde{\Gamma}'|_K = \Gamma'$ if and only if

$$\lim_{\substack{b-a\to 0^+\\ a,b\in K}} \left|\delta_{(b-a)^{-1}}\left(\Gamma(a)^{-1} * \Gamma(b)\right) - \Gamma_0'(a)\right| = 0 \tag{5.7}$$

where $\Gamma_0' = (f_1', g_1', \ldots, f_n', g_n', 0)$, and

$$h'(s) = 2\sum_{j=1}^{n}\left(f_j'(s)g_j(s) - g_j'(s)f_j(s)\right) \quad \textit{for every } s \in K.$$

Here, $\delta_{(b-a)^{-1}}$ is the Heisenberg dilation. After assuming (5.6) and rewriting (5.7) using the definitions of the group law and dilations, we see that (5.7) is satisfied if and only if (5.5) is true and Γ is of Whitney class $\mathfrak{C}^1(K, \mathbb{R}^{2n+1})$ with Whitney derivative Γ'. That is, Theorem 50 and Theorem 52 are indeed equivalent. Notice the similarity between the formulation of (5.7) and the definition of the Pansu derivative (see [72, 75] for information on Pansu differentiation). In fact, Proposition 49 implies that Γ_0' may be viewed as a Whitney-Pansu derivative of Γ. Thus (5.7) acts as a sort of Whitney-Pansu condition for mappings defined on compact subsets of $\mathbb{R}$.

In 2015, Speight [81] showed that a horizontal curve $\Gamma : [a, b] \to \mathbb{H}^n$ coincides with a C^1, horizontal curve $\hat{\Gamma}$ on $[a, b]$ up to a set of arbitrarily small measure. That is, he answered the C^1 Luzin approximation question posed by Hajłasz in the positive. After seeing the paper by Speight, I quickly realized that this C^1 Luzin result follows from Theorem 50. This is summarized at the end of this chapter in Corollary 60. Moreover, Speight showed the surprising result that the Luzin approximation does *not* hold for curves in the Engel group.

5.2 PROOFS OF SOME PROPOSITIONS

We will first present the proof Proposition 49 as this result is used in the proof of Proposition 48.

5.2.1 Necessity of the growth condition

Proof of Proposition 49. Since K is compact, we may assume without loss of generality that U is bounded. It suffices to prove (5.4) when U is an open interval. Indeed, $U = \bigcup_{i=1}^{\infty}(a^i, b^i)$ for disjoint intervals (a^i, b^i). Since K is compact, $K \subset \bigcup_{i=1}^{N}(a^i, b^i)$ for some $N \in \mathbb{N}$, and so we are only required to prove (5.4) on each $(a^i, b^i) \cap K$ with $i \leq N$. We may also replace K by a possibly larger compact interval (also called K) contained in the interval U.

Since Γ is horizontal, we have that $h' = 2\sum_{j=1}^{n}(f_j' g_j - f_j g_j')$ on U. Choose $M > 0$ so that $|f_j'| < M$ and $|g_j'| < M$ on K for every $j = 1, \ldots, n$. Fix $j \in \{1, \ldots, n\}$. For any $a, b \in K$ with $a < b$, we have $(a, b) \subset K$, and so

$$\begin{aligned}
&\int_a^b f_j'(t) g_j(t)\, dt \\
&\quad = \int_a^b f_j'(t)[g_j(a) + g_j'(a)(t-a) + g_j(t) - g_j(a) - g_j'(a)(t-a)]\, dt \\
&\quad = g_j(a)\int_a^b f_j'(t)\, dt + g_j'(a)\int_a^b f_j'(t)(t-a)\, dt + \int_a^b f_j'(t)[g_j(t) - g_j(a) - g_j'(a)(t-a)]\, dt.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{1}{(b-a)^2}\left|\int_a^b f_j'(t)[g_j(t) - g_j(a) - g_j'(a)(t-a)]\, dt\right| \\
&\qquad\qquad \leq \frac{M}{b-a}\int_a^b \frac{|g_j(t) - g_j(a) - g_j'(a)(t-a)|}{|t-a|}\, dt
\end{aligned}$$

which vanishes uniformly on K as $|b-a| \to 0$ since g_j is C^1. In other words, $\int_a^b f_j'(t)[g_j(t) - g_j(a) - g_j'(a)(t-a)]\, dt = o(|b-a|^2)$ uniformly on K as $|b-a| \to 0$. Moreover

$$\int_a^b f_j'(t)(t-a)\, dt = \int_a^b (f_j'(t) - f_j'(a))(t-a)\, dt + \int_a^b f_j'(a)(t-a)\, dt.$$

As above, we have $\int_a^b (f_j'(t) - f_j'(a))(t-a)\, dt = o(|b-a|^2)$ uniformly on K as $|b-a| \to 0$ since f_j is C^1. Thus we can write

$$\int_a^b f_j'(t) g_j(t)\, dt = [f_j(b) - f_j(a)]g_j(a) + g_j'(a) f_j'(a)\int_a^b (t-a)\, dt + o(|b-a|^2).$$

Similar arguments yield

$$\int_a^b g_j'(t)f_j(t)\,dt = [g_j(b)-g_j(a)]f_j(a) + f_j'(a)g_j'(a)\int_a^b (t-a)\,dt + o(|b-a|^2).$$

Hence

$$\int_a^b (f_j'(t)g_j(t) - g_j'(t)f_j(t))\,dt = f_j(b)g_j(a) - f_j(a)g_j(b) + o(|b-a|^2).$$

Therefore,

$$\begin{aligned} h(b)-h(a) = \int_a^b h'(t)\,dt &= 2\sum_{j=1}^n \int_a^b (f_j'(t)g_j(t) - f_j(t)g_j'(t))\,dt \\ &= 2\sum_{j=1}^n (f_j(b)g_j(a) - f_j(a)g_j(b)) + o(|b-a|^2) \end{aligned}$$

uniformly as $|b-a| \to 0$ for $a, b \in K$. This completes the proof. □

5.2.2 Failure of the classical conditions in the Heisenberg group

As implied above, the following counterexample will fail to have a C^1, horizontal extension since any such extension would not satisfy the necessary condition outlined in Proposition 49 on the compact set K.

Proof of Proposition 48. Let

$$K = \bigcup_{n=0}^{\infty} \left[1-\frac{1}{2^n}, 1-\frac{3}{4}\cdot\frac{1}{2^n}\right] \cup \{1\} =: \bigcup_{n=0}^{\infty} [c_n, d_n] \cup \{1\}.$$

Then K is compact. Define $\Gamma = (f, g, h) : K \to \mathbb{H}^1$ so that, for each $n \in \mathbb{N} \cup \{0\}$, $\Gamma(t) = (0, 0, 3^{-n})$ for $t \in [c_n, d_n]$, and set $\Gamma(1) = (0, 0, 0)$ (see Figure 6). Define $\Gamma'(t) = (0, 0, 0)$ for every $t \in K$. We will show that

$$\frac{|\Gamma(b)-\Gamma(a)-(b-a)\Gamma'(a)|}{|b-a|} \tag{5.8}$$

converges uniformly to 0 as $|b-a| \to 0$ on K.

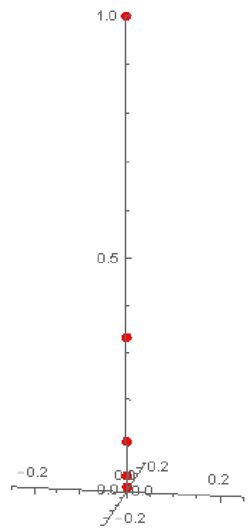

Figure 6: The mapping Γ from the proof of Proposition 48

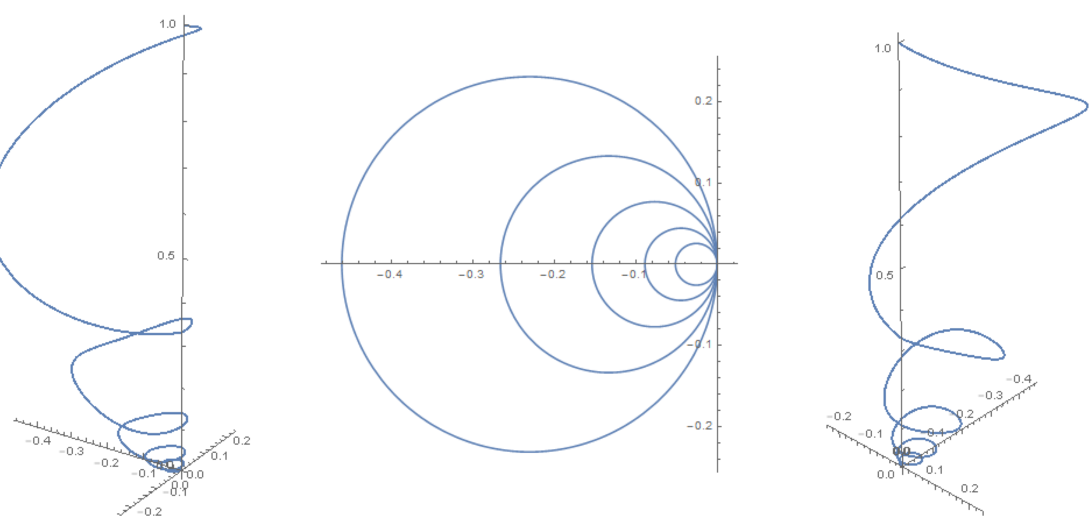

Figure 7: A possible horizontal extension of Γ. This extension is not smooth at the origin.

Let $\varepsilon > 0$ and fix $n \in \mathbb{N}$ with $4(2/3)^n < \varepsilon$. Suppose $a, b \in K$ with $|b-a| < 2^{-(n+2)}$. If a and b lie in the same interval $[c_k, d_k]$, then (5.8) equals 0. If a and b lie in different intervals $[c_k, d_k]$ and $[c_\ell, d_\ell]$ for some $k, \ell \in \mathbb{N} \cup \{0\}$ (say $\ell > k$), then $k \geq n$. Indeed, if $k < n$, then

$$|b-a| \geq c_\ell - d_k \geq c_{k+1} - d_k = 2^{-(k+2)} > 2^{-(n+2)}$$

which is impossible. Hence,

$$\frac{|\Gamma(b) - \Gamma(a) - (b-a)\Gamma'(a)|}{|b-a|} \leq \frac{3^{-k} - 3^{-\ell}}{c_\ell - d_k} \leq \frac{3^{-k}}{c_{k+1} - d_k} = 4\left(\frac{2}{3}\right)^k \leq 4\left(\frac{2}{3}\right)^n < \varepsilon.$$

If either a or b equals 1 and the other point lies in the interval $[c_k, d_k]$ for some $k \in \mathbb{N} \cup \{0\}$, then, as in the above argument, $k \geq n$. In this case, (5.8) is bounded by $\frac{4}{3}\left(\frac{2}{3}\right)^n < \varepsilon$. Thus $\Gamma \in \mathfrak{C}^1(K, \mathbb{R}^3)$, so there exists a C^1 extension of Γ to all of $\mathbb{R}$.

Suppose now that $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{h}) : \mathbb{R} \to \mathbb{H}^1$ is a C^1, horizontal extension of Γ. By Proposition 49, $\tilde{\Gamma}$ must satisfy $|\tilde{h}(b) - \tilde{h}(a)|/|b-a|^2 \to 0$ uniformly on K as $|b-a| \to 0$. However, $|c_{n+1} - d_n| = 2^{-(n+2)} \to 0$ as $n \to \infty$, but

$$\frac{|h(c_{n+1}) - h(d_n)|}{|c_{n+1} - d_n|^2} = \frac{3^{-n} - 3^{-(n+1)}}{4^{-(n+2)}} = \frac{32}{3}\left(\frac{4}{3}\right)^n \to \infty$$

as $n \to \infty$. Thus Γ has no C^1, horizontal extension to all of $\mathbb{R}$. □

5.2.3 Equivalence of the theorem statements

We complete this section with the proof of Remark 51.

Proof. Let $K \subset \mathbb{R}$ be compact. Suppose $\Gamma = (f_1, g_1, \ldots, f_n, g_n, h) : K \to \mathbb{H}^n$ satisfies (4.9). Suppose also that $f_j \in \mathfrak{C}^1(K)$ and $g_j \in \mathfrak{C}^1(K)$ for $j = 1, \ldots, n$ with Whitney derivatives $f_1', g_1', \ldots, f_n', g_n'$. Then

$$\begin{aligned}|h(b) - h(a) - (b-a)h'(a)| \leq &\left| h(b) - h(a) - 2\sum_{j=1}^n (f_j(b)g_j(a) - f_j(a)g_j(b)) \right| \\ &+ 2\sum_{j=1}^n \Big| f_j(b)g_j(a) - f_j(a)g_j(b) \\ &\qquad - (b-a)\left(f_j'(a)g_j(a) - f_j(a)g_j'(a)\right) \Big|\end{aligned}$$

$$= o(|b-a|^2) + 2\sum_{j=1}^{n} \Big| g_j(a)\left(f_j(b) - (b-a)f_j'(a)\right)$$
$$- f_j(a)\left(g_j(b) - (b-a)g_j'(a)\right)\Big|$$
$$= o(|b-a|^2) + o(|b-a|)$$

uniformly as $|b-a| \to 0$ for any $a, b \in K$. That is, $h \in \mathfrak{C}^1(K)$. □

5.3 CONSTRUCTING THE WHITNEY EXTENSION

Proof of Theorem 50. Write $\Gamma = (\gamma, h) = (\gamma_1, \ldots, \gamma_n, h)$ where $\gamma_j = (f_j, g_j) : K \to \mathbb{R}^2$.

The necessity of conditions (4.9) and (5.6) was verified in Proposition 49 and in the discussion preceding Proposition 48. We will now prove that these are sufficient conditions.

Since K is compact, we can define the closed interval $I = [\min\{K\}, \max\{K\}]$. Thus $I \setminus K$ is open, so $I \setminus K = \bigcup_i (a^i, b^i)$ for pairwise disjoint open intervals (a^i, b^i). To construct the extension $\tilde{\Gamma}$ of Γ, we will define a C^1 extension $\tilde{\gamma}$ of γ on each interval $[a^i, b^i]$ so that the horizontal lift of $\tilde{\gamma}$ will coincide with Γ on K.

If the collection $\{(a^i, b^i)\}_i$ is finite, then we can construct the extension directly. On each $[a^i, b^i]$ and for each $j \in \{1, \ldots, n\}$ define $\tilde{\gamma}_j^i = (\tilde{f}_j^i, \tilde{g}_j^i) : [a^i, b^i] \to \mathbb{R}^2$ to be a curve which is C^1 on $[a^i, b^i]$ satisfying

$$\tilde{\gamma}_j^i(a^i) = \gamma_j(a^i) \quad \text{and} \quad \tilde{\gamma}_j^i(b^i) = \gamma_j(b^i), \tag{5.9}$$

$$(\tilde{\gamma}_j^i)'(a^i) = \gamma_j'(a^i) \quad \text{and} \quad (\tilde{\gamma}_j^i)'(b^i) = \gamma_j'(b^i), \tag{5.10}$$

$$2\int_{a^i}^{b^i} \left((\tilde{f}_j^i)'\tilde{g}_j^i - \tilde{f}_j^i(\tilde{g}_j^i)'\right) = \frac{1}{n}\left[h(b^i) - h(a^i)\right]. \tag{5.11}$$

The fact that a curve exists satisfying the first two conditions is obvious. The value on the right hand side of condition (5.11) is fixed, and the integral on the left may be interpreted as an area via Green's theorem. Thus any curve with prescribed values at a^i and b^i as in (5.9) and (5.10) can be adjusted in (a^i, b^i) without disturbing the curve at the endpoints so that this integral condition (5.11) is indeed satisfied.

Now define the curve $\tilde{\gamma} : I \to \mathbb{R}^{2n}$ so that

$$\tilde{\gamma}|_K = \gamma \quad \text{and} \quad \tilde{\gamma}|_{(a^i,b^i)} = (\tilde{\gamma}_1^i, \dots, \tilde{\gamma}_n^i)$$

for every $i \in \mathbb{N}$. The properties (5.9) and (5.10) above ensure that $\tilde{\gamma}$ is C^1 on I. Extend $\tilde{\gamma}$ to a C^1 curve on all of $\mathbb{R}$. Finally, define $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{h})$ to be the unique horizontal lift of $\tilde{\gamma}$ so that $\tilde{h}(\min\{K\}) = h(\min\{K\})$. Property (5.11) ensures that this lift is a C^1 extension of Γ since, on $[a^i, b^i]$, the horizontal lift traverses the distance $h(b^i) - h(a^i)$ in the vertical direction (as described in (3.6)).

Now, consider the case when the collection $\{(a^i, b^i)\}$ is infinite. The simple construction above can not in general be applied directly in this case. Indeed, in the above construction, there was little control on the behavior of the curves. For example, curves filling a small gap from $\gamma_j(a^i)$ to $\gamma_j(b^i)$ could be made arbitrarily long. Thus we must now be more careful when constructing these curves.

Notice that the sequence $\{(a^i, b^i)\}_{i=1}^{\infty}$ satisfies $b^i - a^i \to 0$ as $i \to \infty$ since I is bounded and the intervals are disjoint. Thus, using the fact that each $f_j \in \mathfrak{C}^1(K)$ and $g_j \in \mathfrak{C}^1(K)$ and using (4.9), we can find a non-increasing sequence $\varepsilon^i \to 0$ so that the following conditions hold for each $i \in \mathbb{N}$:

$$b^i - a^i < \varepsilon^i, \qquad |\gamma(b^i) - \gamma(a^i)| < \varepsilon^i,$$

$$\left| \frac{\gamma(b^i) - \gamma(a^i) - (b^i - a^i)\gamma'(a^i)}{b^i - a^i} \right| < \varepsilon^i, \qquad \left| \frac{\gamma(b^i) - \gamma(a^i) - (b^i - a^i)\gamma'(b^i)}{b^i - a^i} \right| < \varepsilon^i,$$

$$\frac{1}{n} \left| \frac{h(b^i) - h(a^i) - \sum_{j=1}^n (f_j(b^i) g_j(a^i) - f_j(a^i) g_j(b^i))}{(b^i - a^i)^2} \right| < \varepsilon^i.$$

Our plan for the proof will be as follows: for each $i \in \mathbb{N}$ we will construct a horizontal curve $\tilde{\Gamma}^i$ in $\mathbb{H}^n$ defined on $[a^i, b^i]$ connecting $\Gamma(a^i)$ to $\Gamma(b^i)$ and satisfying conditions (5.9), (5.10), and (5.11). In addition, the curves will be constructed in a controlled way so that the concatenation of all of these curves creates a C^1, horizontal extension of Γ. To create these curves in $\mathbb{H}^n$, we will first define for each $i \in \mathbb{N}$ curves $\tilde{\gamma}_j^i : [a^i, b^i] \to \mathbb{R}^2$ in each $x_j y_j$-plane connecting $\gamma_j(a^i)$ to $\gamma_j(b^i)$. Horizontally lifting each curve $\tilde{\gamma}^i = (\tilde{\gamma}_1^i, \dots, \tilde{\gamma}_n^i)$ to $\tilde{\Gamma}^i$ will create an extension $\tilde{\Gamma} : I \to \mathbb{H}^n$ of Γ. The controlled construction of the curves $\tilde{\gamma}_j^i$ together with (5.11) will ensure that this extension $\tilde{\Gamma}$ is indeed C^1.

We begin with the following lemma in which we define a curve η_j^i from $[a^i, b^i]$ into the $x_j y_j$-plane sending a^i to the origin and b^i to $(|\gamma_j(b^i) - \gamma_j(a^i)|, 0)$ Later, we will compose the curves in the lemma with planar rotations and translations to create the curves $\tilde{\gamma}_j^i$ connecting $\gamma_j(a^i)$ to $\gamma_j(b^i)$ as described above.

We now introduce some notation. Fix $j \in \{1, \dots, n\}$. For each $i \in \mathbb{N}$, if $|\gamma_j(b^i) - \gamma_j(a^i)| > 0$, let $\mathbf{u}_j^i = \frac{\gamma_j(b^i) - \gamma_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|}$ and let $\mathbf{v}_j^i$ be the unit vector perpendicular to $\mathbf{u}_j^i$ given by a counter-clockwise rotation of $\mathbf{u}_j^i$ in the $x_j y_j$-plane. If $|\gamma_j(b^i) - \gamma_j(a^i)| = 0$, define $\mathbf{u}_j^i$ and $\mathbf{v}_j^i$ to be the unit vectors pointing in the x_j and y_j coordinate directions respectively. Since each γ_j is of Whitney class $\mathfrak{C}^1(K, \mathbb{R}^2)$, we may choose $M > 0$ so that $\frac{|\gamma_j(b^i) - \gamma_j(a^i)|}{|b^i - a^i|} < M$, $|\gamma_j'(a^i)| < M$, and $|\gamma_j'(b^i)| < M$ for every $i \in \mathbb{N}$ and every $j = 1, \dots, n$.

Lemma 53. *Fix $i \in \mathbb{N}$ and $j \in \{1, \dots n\}$. There exists a C^1 curve $\eta_j^i = (x_j^i, y_j^i) : [a^i, b^i] \to \mathbb{R}^2$ satisfying*

$$\eta_j^i(a^i) = (0,0) \quad \textit{and} \quad \eta_j^i(b^i) = \left(|\gamma_j(b^i) - \gamma_j(a^i)|, 0\right), \tag{5.12}$$

$$(\eta_j^i)'(a^i) = (\gamma_j'(a^i) \cdot \mathbf{u}_j^i, \gamma_j'(a^i) \cdot \mathbf{v}_j^i) \quad \textit{and} \quad (\eta_j^i)'(b^i) = (\gamma_j'(b^i) \cdot \mathbf{u}_j^i, \gamma_j'(b^i) \cdot \mathbf{v}_j^i), \tag{5.13}$$

$$||\eta_j^i||_\infty < P(\varepsilon^i) \quad \textit{and} \quad ||(\eta_j^i)' - (\gamma_j'(a^i) \cdot \mathbf{u}_j^i, \gamma_j'(a^i) \cdot \mathbf{v}_j^i)||_\infty < P(\varepsilon^i) \tag{5.14}$$

where $P(t) = C'(t^{1/2} + t^2)$ for every $t \geq 0$ and some constant $C' \geq 0$ depending only on M, and

$$2 \int_{a^i}^{b^i} \left((x_j^i)' y_j^i - x_j^i (y_j^i)'\right) = \frac{1}{n} \left[h(b^i) - h(a^i) - 2 \sum_{m=1}^{n} \left(f_m(b^i) g_m(a^i) - f_m(a^i) g_m(b^i) \right) \right]. \tag{5.15}$$

The proof of this lemma is omitted here for continuity. It is presented in the appendix.

Remark 54. Observe that $\gamma_j'(a^i)\cdot\mathbf{u}_j^i$ and $\gamma_j'(a^i)\cdot\mathbf{v}_j^i$ are the components of the vector $\gamma_j'(a^i)$ in the orthonormal basis $\langle \mathbf{u}_j^i, \mathbf{v}_j^i\rangle$. Soon, we will define the curve $\tilde{\gamma}_j^i$ by moving η_j^i via a rotation and translation. The rotation will map the standard basis in the x_jy_j-plane to $\langle \mathbf{u}_j^i, \mathbf{v}_j^i\rangle$, and hence (5.13) will imply $(\tilde{\gamma}_j^i)'(a^i) = \gamma_j'(a^i)$ and $(\tilde{\gamma}_j^i)'(b^i) = \gamma_j'(b^i)$. The translation will map the segment connecting the origin and $(|\gamma_j(b^i)-\gamma_j(a^i)|, 0)$ to the segment $\overline{\gamma_j(a^i)\gamma_j(b^i)}$, and so (5.12) will give $\tilde{\gamma}_j^i(a^i) = \gamma_j(a^i)$ and $\tilde{\gamma}_j^i(b^i) = \gamma_j(b^i)$. Condition (5.14) exhibits control on the C^1 norm of η_j^i and will thus give us control on the C^1 norm of its isometric image $\tilde{\gamma}_j^i$. Note also that the integral condition (5.15) seems more complicated than (5.11). However, after rotating and translating η_j^i, (5.15) will reduce to (5.11).

Fix $j \in \{1\dots,n\}$ and $i\in\mathbb{N}$. Define the curve $\eta_j^i : [a^i, b^i] \to \mathbb{R}^2$ as in the lemma. Define the isometry $\Phi_j^i : \mathbb{R}^2\to\mathbb{R}^2$ as $\Phi_j^i(p) = A_j^i p + c_j^i$ for $p\in\mathbb{R}^2$ where $c_j^i = (f_j(a^i), g_j(a^i))$ and

$$A_j^i = \frac{1}{|\gamma_j(b^i)-\gamma_j(a^i)|}\begin{pmatrix} f_j(b^i)-f_j(a^i) & -(g_j(b^i)-g_j(a^i)) \\ g_j(b^i)-g_j(a^i) & f_j(b^i)-f_j(a^i)\end{pmatrix}$$

when $|\gamma_j(b^i)-\gamma_j(a^i)|\neq 0$ and $A_j^i = I_{2\times 2}$ if $|\gamma_j(b^i)-\gamma_j(a^i)| = 0$. That is, Φ_j^i is the isometry described in Remark 54. When $\gamma_j(a^i) = \gamma_j(b^i)$, Φ_j^i is simply a translation sending the origin to $\gamma_j(a^i)$ without any rotation. Now define $\tilde{\gamma}_j^i = \Phi_j^i\circ\eta_j^i : [a^i,b^i]\to\mathbb{R}^2$. Hence $\tilde{\gamma}_j^i$ is a C^1 curve in $\mathbb{R}^2$ connecting $\gamma_j(a^i)$ to $\gamma_j(b^i)$.

Write $\tilde{\gamma}^i = (\tilde{\gamma}_1^i,\dots,\tilde{\gamma}_n^i) : [a^i,b^i]\to\mathbb{R}^{2n}$. Now, define $\tilde{\Gamma}^i : [a^i,b^i]\to\mathbb{H}^n$ to be the unique horizontal lift of $\tilde{\gamma}^i$ with starting height $h(a^i)$. The resulting lift is C^1 on $[a^i,b^i]$ by definition. Define $\tilde{\Gamma} : I\to\mathbb{H}^n$ so that

$$\tilde{\Gamma}|_K = \Gamma \quad\text{ and }\quad \tilde{\Gamma}|_{(a^i,b^i)} = \tilde{\Gamma}^i$$

for each $i\in\mathbb{N}$. Write $\tilde{\Gamma} = (\tilde{\gamma},\tilde{h}) = (\tilde{\gamma}_1,\dots,\tilde{\gamma}_n,\tilde{h})$ where $\tilde{\gamma}_j = (\tilde{f}_j,\tilde{g}_j)$ for each $j\in\{1,\dots,n\}$. It remains to show that $\tilde{\Gamma}$ is C^1 on all of I. Notice that we do not yet know if $\tilde{\Gamma}$ is even continuous.

Claim 55. *For each $i\in\mathbb{N}$, $\int_{a^i}^{b^i}\tilde{h}' = h(b^i)-h(a^i)$.*

Fix $j \in \{1, \ldots, n\}$ and suppose $|\gamma_j(b^i) - \gamma_j(a^i)| \neq 0$. We have

$$(\Phi^i_j \circ \eta^i_j) = \begin{pmatrix} x^i_j \frac{f_j(b^i)-f_j(a^i)}{|\gamma_j(b^i)-\gamma_j(a^i)|} - y^i_j \frac{g_j(b^i)-g_j(a^i)}{|\gamma_j(b^i)-\gamma_j(a^i)|} + f_j(a^i) \\ x^i_j \frac{g_j(b^i)-g_j(a^i)}{|\gamma_j(b^i)-\gamma_j(a^i)|} + y^i_j \frac{f_j(b^i)-f_j(a^i)}{|\gamma_j(b^i)-\gamma_j(a^i)|} + g_j(a^i) \end{pmatrix}$$

where $\eta^i_j = (x^i_j, y^i_j)$. This gives

$$\begin{aligned} \omega(\tilde{\gamma}_j', \tilde{\gamma}_j) &= \omega((\Phi^i_j \circ \eta^i_j)', (\Phi^i_j \circ \eta^i_j)) \\ &= g_j(a^i) \left((x^i_j)' \frac{f_j(b^i) - f_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} - (y^i_j)' \frac{g_j(b^i) - g_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right) \\ &- f_j(a^i) \left((x^i_j)' \frac{g_j(b^i) - g_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} + (y^i_j)' \frac{f_j(b^i) - f_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right) \\ &+ \left(\left(\frac{f_j(b^i) - f_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right)^2 + \left(\frac{g_j(b^i) - g_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right)^2 \right) ((x^i_j)' y^i_j - x^i_j (y^i_j)'). \end{aligned}$$

Now since the constructions in the lemma give

$$\int_{a^i}^{b^i} (y^i_j)' = y^i_j(b^i) - y^i_j(a^i) = 0 \text{ and } \int_{a^i}^{b^i} (x^i_j)' = x^i_j(b^i) - x^i_j(a^i) = |\gamma_j(b^i) - \gamma_j(a^i)|$$

and

$$\left(\frac{f_j(b^i) - f_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right)^2 + \left(\frac{g_j(b^i) - g_j(a^i)}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right)^2 = \left(\frac{|\gamma_j(b^i) - \gamma_j(a^i)|}{|\gamma_j(b^i) - \gamma_j(a^i)|} \right)^2 = 1,$$

we have

$$2 \int_{a^i}^{b^i} \omega((\Phi^i_j \circ \eta^i_j)', (\Phi^i_j \circ \eta^i_j)) = 2(f_j(b^i) g_j(a^i) - f_j(a^i) g_j(b^i)) + 2 \int_{a^i}^{b^i} \left((x^i_j)' y^i_j - x^i_j (y^i_j)' \right).$$

By condition (5.15), we have

$$2 \int_{a^i}^{b^i} \left((x^i_j)' y^i_j - x^i_j (y^i_j)' \right) = \frac{1}{n} \left[h(b^i) - h(a^i) - 2 \sum_{m=1}^{n} (f_m(b^i) g_m(a^i) - f_m(a^i) g_m(b^i)) \right],$$

thus

$$\begin{aligned} \int_{a^i}^{b^i} \tilde{h}' &= 2 \sum_{j=1}^{n} \int_{a^i}^{b^i} \omega((\Phi^i_j \circ \eta^i_j)', (\Phi^i_j \circ \eta^i_j)) \\ &= \sum_{j=1}^{n} \Bigg[2 \left(f_j(b^i) g_j(a^i) - f_j(a^i) g_j(b^i) \right) \\ &\qquad + \frac{1}{n} \left[h(b^i) - h(a^i) - 2 \sum_{m=1}^{n} (f_m(b^i) g_m(a^i) - f_m(a^i) g_m(b^i)) \right] \Bigg] \end{aligned}$$

$$= h(b^i) - h(a^i).$$

If $|\gamma_j(b^i) - \gamma_j(a^i)| = 0$, then A^i is the identity. Thus

$$\int_{a^i}^{b^i} \tilde{h}' = 2\sum_{j=1}^{n} \int_{a^i}^{b^i} \left[(x_j^i)'(y_j^i + g_j(a^i)) - (y_j^i)'(x_j^i + f_j(a^i))\right] = 2\sum_{j=1}^{n} \int_{a^i}^{b^i} \left((x_j^i)'y_j^i - (y_j^i)'x_j^i\right)$$
$$= h(b^i) - h(a^i)$$

since $f_j(a^i) = f_j(b^i)$ and $g_j(a^i) = g_j(b^i)$ in this case. This completes the proof of the claim.

Claim 56. $\sup_{s\in[a^i,b^i]} |\tilde{\gamma}(s) - \gamma(a^i)| \to 0$ *as* $i \to \infty$.

We have for any $i \in \mathbb{N}$, any $s \in [a^i, b^i]$, and any $j \in \{1, \ldots, n\}$,

$$|\tilde{\gamma}_j(s) - \gamma_j(a^i)| = |\Phi_j^i(\eta_j^i(s)) - \gamma_j(a^i)| = |\eta_j^i(s)| < P(\varepsilon^i)$$

by (5.14) since $(\Phi_j^i)^{-1}(\gamma_j(a^i)) = (0,0)$ and Φ_j^i is an isometry on $\mathbb{R}^2$. Since $P(\varepsilon^i) \to 0$ as $i \to \infty$, the claim is proven.

Claim 57. $\sup_{s\in[a^i,b^i]} |\tilde{\Gamma}'(s) - \Gamma'(a^i)| \to 0$ *as* $i \to \infty$.

We have for any $i \in \mathbb{N}$, any $s \in [a^i, b^i]$, and any $j \in \{1, \ldots, n\}$,

$$\begin{aligned}|\tilde{\gamma}_j'(s) - \gamma_j'(a^i)| &= |(\Phi_j^i \circ \eta_j^i)'(s)) - \gamma_j'(a^i)| \\ &= |A_j^i((\eta_j^i)'(s)) - A_j^i(\gamma_j'(a^i) \cdot \mathbf{u}_j^i, \gamma_j'(a^i) \cdot \mathbf{v}_j^i)| \\ &= |(\eta_j^i)'(s) - (\gamma_j'(a^i) \cdot \mathbf{u}_j^i, \gamma_j'(a^i) \cdot \mathbf{v}_j^i)| < P(\varepsilon^i)\end{aligned}$$

by (5.14). Finally, by (5.6) and the definition of a horizontal lift

$$\begin{aligned}&\sup_{s\in[a^i,b^i]} |\tilde{h}'(s) - h'(a^i)| \\ &\qquad \le \sup_{s\in[a^i,b^i]} 2\sum_{j=1}^{n} \left|(\tilde{f}_j'(s)\tilde{g}_j(s) - \tilde{g}_j'(s)\tilde{f}_j(s)) - (f_j'(a^i)g_j(a^i) - g_j'(a^i)f_j(a^i))\right|\end{aligned}$$

which can be made arbitrarily small as $i \to \infty$ because of the convergences in Claim 56 and this claim. This proves the claim.

By definition, $\tilde{\Gamma}$ is C^1 on (a^i, b^i) for any $i \in \mathbb{N}$, so it is C^1 on $I \setminus K$. We will now verify the differentiability of $\tilde{\Gamma}$ on K.

Claim 58. *For any $t \in K$, $\tilde{\Gamma}$ is differentiable at t and $\tilde{\Gamma}'(t) = \Gamma'(t)$.*

Suppose $t \in K$ (so $\tilde{\Gamma}(t) = \Gamma(t)$). If $t = a^i$ for some $i \in \mathbb{N}$, then for any $j \in \{1, \ldots, n\}$ and $0 < \delta < b^i - a^i$, we can use the definition of $\tilde{\gamma}^i$ to write

$$\begin{aligned}
&\delta^{-1}|\tilde{\gamma}_j(a^i + \delta) - \tilde{\gamma}_j(a^i) - \delta\gamma_j'(a^i)| \\
&\qquad = \delta^{-1}|A_j^i(\eta_j^i(a^i + \delta)) + c_j^i - (A_j^i(\eta_j^i(a^i)) + c_j^i) - \delta A_j^i((\eta_j^i)'(a^i))| \\
&\qquad = \delta^{-1}|\eta_j^i(a^i + \delta) - \eta_j^i(a^i) - \delta(\eta_j^i)'(a^i)|
\end{aligned}$$

which vanishes as $\delta \to 0$ since η_j^i is differentiable from the right at a^i. Thus $\tilde{\gamma}$ is differentiable from the right at a^i and the right derivative equals $\gamma'(a^i)$. Moreover,

$$\lim_{\delta \to 0^+} \tilde{\gamma}_j'(a^i + \delta) = \lim_{\delta \to 0^+} A_j^i((\eta_j^i)'(a^i + \delta)) = A_j^i((\eta_j^i)'(a^i)) = \gamma_j'(a^i).$$

Thus $\tilde{\gamma}'$ is continuous from the right at a^i. Now $\tilde{\Gamma}$ was constructed on (a^i, b^i) by lifting $\gamma(a^i)$ to the height $h(a^i)$. Hence $\int_{a^i}^c \tilde{h}' = \tilde{h}(c) - h(a^i)$ for any $c \in (a^i, b^i)$. Thus for $0 < \delta < b^i - a^i$,

$$\begin{aligned}
&\delta^{-1}|\tilde{h}(a^i + \delta) - \tilde{h}(a^i) - \delta h'(a^i)| \\
&\qquad \leq 2\delta^{-1} \sum_{j=1}^n \int_{a^i}^{a^i+\delta} \left|(\tilde{f}_j'(s)\tilde{g}_j(s) - \tilde{g}_j'(s)\tilde{f}_j(s)) - (f_j'(a^i)g_j(a^i) - g_j'(a^i)f_j(a^i))\right| \, ds
\end{aligned}$$

which vanishes as $\delta \to 0$ by the right sided continuity of $\tilde{\gamma}$ and $\tilde{\gamma}'$ at a^i. Therefore $\tilde{\Gamma}$ is differentiable from the right at a^i and the right derivative is $\Gamma'(a^i)$.

We can similarly argue to show that $\tilde{\gamma}$ is differentiable from the left at b^i for any $i \in \mathbb{N}$ with left derivative equal to $\gamma'(b^i)$ and that $\tilde{\gamma}'$ is continuous from the left at b^i. Applying Claim 55 with $0 < \delta < b^i - a^i$ gives

$$\begin{aligned}
\delta^{-1}|\tilde{h}(b^i - \delta) - \tilde{h}(b^i) + \delta h'(b^i)| &= \delta^{-1}|\delta h'(b^i) + (\tilde{h}(b^i - \delta) - \tilde{h}(a^i)) - (h(b^i) - h(a^i))| \\
&\leq \delta^{-1} \int_{b^i-\delta}^{b^i} \left|h'(b^i) - \tilde{h}'(s)\right| \, ds
\end{aligned}$$

which vanishes as $\delta \to 0$ as above. Therefore $\tilde{\Gamma}$ is differentiable from the left b^i and the left derivative equals $\Gamma'(b^i)$.

We will now show that $\tilde{\Gamma}$ is differentiable from the right at any $t \in K$. Suppose now that $t \neq a^i$ for any $i \in \mathbb{N}$ since we already proved right hand differentiability at a^i above. (We may also suppose that $t \neq \max\{K\}$.) Fix $\tilde{\varepsilon} > 0$. Let $\{t^k\}$ be any decreasing sequence in K with $t^k \to t$. Since $\Gamma \in \mathfrak{C}^1(K, \mathbb{R}^{2n+1})$, there is some $N > 0$ so that for any $k > N$

$$(t^k - t)^{-1}|\Gamma(t^k) - \Gamma(t) - (t^k - t)\Gamma'(t)| < \tilde{\varepsilon}.$$

Suppose there is a decreasing sequence $\{t^k\}$ in $I \setminus K$ with $t^k \to t$. Then $t^k \in (a^{i_k}, b^{i_k})$ for some $i_k \in \mathbb{N}$ for every $k \in \mathbb{N}$. (Notice that $i_k \to \infty$ as $k \to \infty$ since $t \neq a^{i_k}$ for any $k \in \mathbb{N}$.) Now

$$\begin{aligned}(t^k - t)^{-1}|\tilde{\Gamma}(t^k) - \Gamma(t) - &(t^k - t)\Gamma'(t)| \\ &\leq (t^k - t)^{-1}|\tilde{\Gamma}(t^k) - \Gamma(a^{i_k}) - (t^k - a^{i_k})\Gamma'(a^{i_k})| \\ &\quad + (t^k - t)^{-1}|(t^k - a^{i_k})\Gamma'(a^{i_k}) - (t^k - a^{i_k})\Gamma'(t)| \\ &\quad + (t^k - t)^{-1}|\Gamma(a^{i_k}) - \Gamma(t) - (a^{i_k} - t)\Gamma'(t)|.\end{aligned}$$

We may bound the first term on the right as follows:

$$(t^k - t)^{-1}|\tilde{\Gamma}(t^k) - \Gamma(a^{i_k}) - (t^k - a^{i_k})\Gamma'(a^{i_k})| \leq (t^k - t)^{-1} \int_{a^{i_k}}^{t^k} \left|\tilde{\Gamma}'(s) - \Gamma'(a^{i_k})\right| ds.$$

By Claim 57, this is bounded by $\tilde{\varepsilon}$ for large enough k since $t^k - a^{i_k} < t^k - t$. The second term can be bounded by $|\Gamma'(a^{i_k}) - \Gamma'(t)|$. Since Γ' is continuous on K, this may also be made less than $\tilde{\varepsilon}$ for large k. Finally, the third term can be made smaller than $\tilde{\varepsilon}$ since $\Gamma \in \mathfrak{C}^1(K, \mathbb{R}^{2n+1})$ and since $(a^{i_k} - t)/(t^k - t) \leq 1$.

Since any decreasing sequence $\{t^k\}$ in I with $t^k \to t$ either has a subsequence entirely contained in K or a subsequence entirely contained in $I \setminus K$, we have proven the differentiability of Γ from the right for any $t \in K$ ($t \neq \max\{K\}$) with right derivative equal to $\Gamma'(t)$. By an identical argument involving an increasing sequence $\{t^k\}$ in I with $t^k \to t$ when $t \neq b^i$ and $t \neq \min\{K\}$, we have that Γ is differentiable from the left at any $t \in K$ ($t \neq \min\{K\}$) with left derivative $\Gamma'(t)$. Thus we may conclude the statement of the claim.

Claim 59. *$\tilde{\Gamma}$ is C^1 on I.*

We have already shown that $\tilde{\Gamma}$ is differentiable on I with $\tilde{\Gamma}'|_K = \Gamma'$. Since $\tilde{\Gamma}$ is C^1 on each (a^i, b^i), $\tilde{\Gamma}'$ is continuous on $I \setminus K$. It remains to show that $\tilde{\Gamma}'$ is continuous on K.

Fix $t \in K$. If $t = a^i$ for some $i \in \mathbb{N}$, we showed in the proof of the previous claim that $\tilde{\gamma}'$ is continuous from the right at t. This gives for any $0 < \delta < b^i - a^i$

$$\begin{aligned}
&|\tilde{h}'(a^i + \delta) - \tilde{h}'(a^i)| \\
&\quad \leq 2 \sum_{j=1}^{n} \left| (\tilde{f}_j'(a^i + \delta)\tilde{g}_j(a^i + \delta) - \tilde{g}_j'(a^i + \delta)\tilde{f}_j(a^i + \delta)) - (f_j'(a^i)g_j(a^i) - g_j'(a^i)f_j(a^i)) \right|
\end{aligned}$$

which vanishes as $\delta \to 0$, and so $\tilde{\Gamma}'$ is continuous from the right at a^i. A similar argument gives continuity of $\tilde{\Gamma}'$ from the left at b^i.

Suppose $t \neq a^i$ for any $i \in \mathbb{N}$ and $t \neq \max\{K\}$. Let $\{t^k\}$ be a decreasing sequence in K with $t^k \to t$. Then $|\Gamma'(t) - \Gamma'(t + \delta^k)|$ may be made arbitrarily small when k is large since Γ' is continuous on K. If there is a decreasing sequence $\{t^k\}$ in $I \setminus K$ with $t^k \to t$, then $t^k \in (a^{i_k}, b^{i_k})$ for some $i_k \in \mathbb{N}$ for every $k \in \mathbb{N}$, and so

$$|\Gamma'(t) - \tilde{\Gamma}'(t^k)| \leq |\Gamma'(t) - \Gamma'(a^{i_k})| + |\Gamma'(a^{i_k}) - \tilde{\Gamma}'(t^k)|$$

may be made arbitrarily small for large k by Claim 57. As above, since any decreasing sequence $\{t^k\}$ in I with $t^k \to t$ either has a subsequence entirely contained in K or a subsequence entirely contained in $I \setminus K$, we have shown that $\tilde{\Gamma}'$ is continuous from the right at t. A similar argument when $t \neq b^i$ and $t \neq \min\{K\}$ involving an increasing sequence $\{t^k\}$ gives continuity of $\tilde{\Gamma}'$ from the left on K. This proves the claim

Extending $\tilde{\Gamma}$ from I to $\mathbb{R}$ in a smooth, horizontal way completes the proof of the theorem.

$\square$

5.4 THE LUZIN APPROXIMATION FOR HORIZONTAL CURVES

We will now see that the Luzin approximation of horizontal curves in $\mathbb{H}^n$ follows from the above result as it does in the classical case. As mentioned earlier, this is a new proof of the result of Speight [81].

Corollary 60. *Suppose $\Gamma = (f_1, g_1, \ldots, f_n, g_n, h) : [a,b] \to \mathbb{H}^n$ is horizontal. Then, for every $\varepsilon > 0$, there is a C^1, horizontal curve $\tilde{\Gamma} : \mathbb{R} \to \mathbb{H}^n$ and a compact set $E \subset [a,b]$ with $|[a,b] \setminus E| < \varepsilon$ so that $\tilde{\Gamma}(t) = \Gamma(t)$ and $\tilde{\Gamma}'(t) = \Gamma'(t)$ for every $t \in E$.*

Proof. Since Γ is horizontal, it is absolutely continuous as a mapping into $\mathbb{R}^{2n+1}$. Thus it is differentiable almost everywhere in (a,b) and the derivative Γ' is L^1 on (a,b). Suppose that $t \in (a,b)$ is a point of differentiability of Γ and that t is a Lebesgue point of f_j' and g_j' for $j = 1, \ldots, n$. Define $\bar{\Gamma} = (\bar{f}_1, \bar{g}_1, \ldots, \bar{f}_n, \bar{g}_n, \bar{h}) : [a,b] \to \mathbb{H}^n$ so that $\bar{\Gamma}(s) = \Gamma(t)^{-1} * \Gamma(s)$. Since $\bar{\Gamma}(t) = 0$ and $\bar{\Gamma}$ is horizontal, we have for any $\delta > 0$ with $t + \delta \in [a,b]$

$$\begin{aligned}
\frac{|\bar{h}(t+\delta)|}{\delta^2} &= \frac{1}{\delta^2}\left|\int_t^{t+\delta} \bar{h}'(s)\,ds\right| \le \frac{2}{\delta}\sum_{j=1}^n \int_t^{t+\delta} \frac{1}{\delta}\left|\bar{f}_j'(s)\bar{g}_j(s) - \bar{f}_j(s)\bar{g}_j'(s)\right|\,ds \\
&\le \frac{2}{\delta}\sum_{j=1}^n \int_t^{t+\delta} \left|\bar{f}_j'(s)\frac{\bar{g}_j(s)}{s-t} - \frac{\bar{f}_j(s)}{s-t}\bar{g}_j'(s)\right|\,ds \\
&= \frac{2}{\delta}\sum_{j=1}^n \int_t^{t+\delta} \left|f_j'(s)\frac{g_j(s)-g_j(t)}{s-t} - \frac{f_j(s)-f_j(t)}{s-t}g_j'(s)\right|\,ds,
\end{aligned}$$

and so $\frac{|\bar{h}(t+\delta)|}{\delta^2} \to 0$ as $\delta \to 0$. Similarly, $\frac{|\bar{h}(t-\delta)|}{\delta^2} \to 0$ as $\delta \to 0$

Notice that $\bar{h}(s) = h(s) - h(t) - 2\sum_{j=1}^n (f_j(s)g_j(t) - f_j(t)g_j(s))$ for every $s \in (a,b)$. Thus since almost every point in (a,b) is a point of differentiability of Γ and a Lebesgue point of each f_j' and g_j', we have

$$\lim_{s\to t} \frac{\left|h(s) - h(t) - 2\sum_{j=1}^n (f_j(s)g_j(t) - f_j(t)g_j(s))\right|}{(s-t)^2} = 0 \tag{5.16}$$

for almost every $t \in (a,b)$. Denote by E_1 the set of all $t \in (a,b)$ satisfying both (5.16) and $\Gamma'(t) \in H_{\Gamma(t)}\mathbb{H}^n$. Hence $|[a,b] \setminus E_1| = 0$.

Let $\varepsilon > 0$. By Luzin's theorem, Γ' is continuous on a compact set $E_2 \subset E_1$ with $|E_1 \setminus E_2| < \varepsilon/3$. By applying Egorov's theorem to the pointwise convergent sequence of functions $\{\psi_k\}$ defined on E_2 as

$$\psi_k(t) = \sup_{s\in(t-\frac{1}{k},t+\frac{1}{k})} \left\{ \frac{|\Gamma(s) - \Gamma(t) - (s-t)\Gamma'(t)|}{|s-t|} \right\},$$

we see that $\Gamma \in \mathfrak{C}^1(E_3, \mathbb{R}^{2n+1})$ with Whitney derivative Γ' for a compact set $E_3 \subset E_2$ with $|E_2 \setminus E_3| < \varepsilon/3$. Once again applying Egorov's theorem to the convergent sequence of functions $\{\phi_k\}$ defined on E_3 as

$$\phi_k(t) = \sup_{s\in(t-\frac{1}{k},t+\frac{1}{k})} \left\{ \frac{\left| h(s) - h(t) - 2\sum_{j=1}^n (f_j(s)g_j(t) - f_j(t)g_j(s)) \right|}{(s-t)^2} \right\},$$

we conclude that (4.9) holds on a compact set $E_4 \subset E_3$ with $|E_3 \setminus E_4| < \varepsilon/3$.

Thus Γ is of Whitney class $\mathfrak{C}^1(E_4, \mathbb{R}^{2n+1})$, and conditions (4.9) and (5.6) hold on the compact set E_4. Therefore, by Theorem 50, there is a C^1, horizontal $\tilde{\Gamma} : \mathbb{R} \to \mathbb{H}^n$ so that $\tilde{\Gamma}(t) = \Gamma(t)$ and $\tilde{\Gamma}'(t) = \Gamma'(t)$ for every $t \in E_4$ where $|[a,b] \setminus E_4| < \varepsilon$. □

5.5 THE GAP FILLING ARGUMENT

We now prove Lemma 53 in which we explicitly construct polynomials to fill the gaps in K.

Proof. Fix $i \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$.

To simplify notation, write $\alpha = \gamma_j'(a^i) \cdot \mathbf{u}_j^i$, $\beta = \gamma_j'(b^i) \cdot \mathbf{u}_j^i$, $\mu = \gamma_j'(a^i) \cdot \mathbf{v}_j^i$, $\nu = \gamma_j'(b^i) \cdot \mathbf{v}_j^i$, and

$$\lambda = \frac{1}{n} \left[h(b^i) - h(a^i) - 2\sum_{k=1}^n (f_k(b^i)g_k(a^i) - f_k(a^i)g_k(b^i)) \right], \tag{5.17}$$

and so $|\lambda|/(b^i - a^i)^2 < \varepsilon^i$. In other words, α and β are the components of the mapping γ_j' at a^i and b^i respectively in the direction of the segment $\overline{\gamma_j(a^i)\gamma_j(b^i)}$, and μ and ν are its components in the perpendicular direction.

First, we prove that $|\mu| < \varepsilon^i$ and $|\nu| < \varepsilon^i$. Indeed, the magnitude of $(\gamma_j(b^i) - \gamma_j(a^i) - (b^i - a^i)\gamma_j'(a^i))$ is at least equal to the magnitude of its projection along $\mathbf{v}_j^i$. That is,

$$\varepsilon^i > \frac{|\gamma_j(b^i) - \gamma_j(a^i) - (b^i - a^i)\gamma_j'(a^i)|}{b^i - a^i} \geq \frac{|(\gamma_j(b^i) - \gamma_j(a^i) - (b^i - a^i)\gamma_j'(a^i)) \cdot \mathbf{v}_j^i|}{b^i - a^i} = |\mu|$$

since $\gamma_j(b^i) - \gamma_j(a^i)$ is orthogonal to $\mathbf{v}_j^i$. Replacing $\gamma_j'(a^i)$ with $\gamma_j'(b^i)$ in this argument gives $|\nu| < \varepsilon^i$. We also have

$$\varepsilon^i > \frac{|(\gamma_j(b^i) - \gamma_j(a^i) - (b^i - a^i)\gamma_j'(a^i)) \cdot \mathbf{u}_j^i|}{b^i - a^i} = \left| \frac{|\gamma_j(b^i) - \gamma_j(a^i)|}{b^i - a^i} - \alpha \right| \tag{5.18}$$

since $(\gamma_j(b^i) - \gamma_j(a^i)) \cdot \mathbf{u}_j^i = |\gamma_j(b^i) - \gamma_j(a^i)|$. Replacing $\gamma_j'(a^i)$ with $\gamma_j'(b^i)$ in this argument gives $\left| \frac{|\gamma_j(b^i) - \gamma_j(a^i)|}{b^i - a^i} - \beta \right| < \varepsilon^i$.

Define $P : [0, \infty) \to [0, \infty)$ as $P(t) = C'(t^{1/2} + t^2)$ where C' is a positive constant whose value will be determined by the constructions of η_j^i and will depend only on M. In particular, the value of C' will not depend on i or j.

Case 1. γ_j, γ_j', a^i, and b^i satisfy

$$\left| \alpha + \beta - 9\frac{|\gamma_j(b^i) - \gamma_j(a^i)|}{b^i - a^i} \right| > \sqrt{\varepsilon^i}. \tag{5.19}$$

Write $\ell = |\gamma_j(b^i) - \gamma_j(a^i)|$. By composing with a translation of the real line, we may assume without loss of generality that $a^i = 0$ and write $\delta := b^i$. The bounds before the statement of the lemma give $\delta < \varepsilon^i$, $\ell < \varepsilon^i$, and $\ell/\delta < M$. Define the curve $\eta_j^i = (x_j^i, y_j^i) : [0, \delta] \to \mathbb{R}^2$ as follows:

$$\begin{aligned} x_j^i(t) &= At^3 + Bt^2 + Ct \\ &= \frac{\delta(\alpha + \beta) - 2\ell}{\delta^3}t^3 + \frac{-\delta(2\alpha + \beta) + 3\ell}{\delta^2}t^2 + \alpha t \end{aligned}$$

and

$$\begin{aligned} y_j^i(t) &= Dt^4 + Et^3 + Ft^2 + Gt \\ &= 7\frac{6\delta\ell(\mu - \nu) + \delta^2(\alpha\nu - \beta\mu) - 15\lambda}{2\delta^4(\delta(\alpha + \beta) - 9\ell)}t^4 \\ &+ \frac{\delta\ell(33\nu - 51\mu) + \delta^2(\alpha(\mu - 6\nu) + \beta(8\mu + \nu)) + 105\lambda}{\delta^3(\delta(\alpha + \beta) - 9\ell)}t^3 \end{aligned}$$

$$-\frac{\delta\ell(24\nu-78\mu)+\delta^2(4\alpha\mu+11\beta\mu-5\alpha\nu+2\beta\nu)+105\lambda}{2\delta^2(\delta(\alpha+\beta)-9\ell)}t^2+\mu t.$$

See Figure 8 for a possible construction. One may check (by hand or with Mathematica; I did both) that this curve satisfies conditions (5.12), (5.13), and (5.15). Now

$$\begin{aligned}|x_j^i(t)| &\leq |A|\delta^3+|B|\delta^2+|C|\delta\\ &\leq \delta(|\alpha|+|\beta|)+2\ell+\delta(2|\alpha|+|\beta|)+3\ell+|\alpha|\delta\\ &< \varepsilon^i(M+M)+2\varepsilon^i+\varepsilon^i(2M+M)+3\varepsilon^i+M\varepsilon^i=(6M+5)\varepsilon^i<P(\varepsilon^i)\end{aligned}$$

after choosing C' large enough (since either $\varepsilon^i \leq (\varepsilon^i)^2$ or $\varepsilon^i \leq (\varepsilon^i)^{1/2}$). Also,

$$\begin{aligned}|(x_j^i)'(t)-\alpha| &\leq 3|A|\delta^2+2|B|\delta+|C-\alpha|\\ &\leq 3\left|\alpha+\beta-2\frac{\ell}{\delta}\right|+2\left|-2\alpha-\beta+3\frac{\ell}{\delta}\right|\\ &< 3\varepsilon^i+3\varepsilon^i+4\varepsilon^i+2\varepsilon^i=12\varepsilon^i<P(\varepsilon^i)\end{aligned}$$

for large enough C' since $\left|\alpha-\frac{\ell}{\delta}\right|<\varepsilon^i$ and $\left|\beta-\frac{\ell}{\delta}\right|<\varepsilon^i$ by (5.18).

Now we will consider the sizes of y and y'. First, we examine the size of terms in D. We have

$$\left|\frac{\delta\ell(\mu-\nu)}{\delta^4(\delta(\alpha+\beta)-9\ell)}\right|\delta^3=\frac{\frac{\ell}{\delta}|\mu-\nu|}{|\alpha+\beta-9\frac{\ell}{\delta}|}<\frac{2M\varepsilon^i}{\sqrt{\varepsilon^i}}=2M\sqrt{\varepsilon^i}$$

by (5.19). Similarly,

$$\left|\frac{\delta^2(\alpha\nu-\beta\mu)}{\delta^4(\delta(\alpha+\beta)-9\ell)}\right|\delta^3=\frac{|\alpha\nu|+|\beta\mu|}{|\alpha+\beta-9\frac{\ell}{\delta}|}<2M\sqrt{\varepsilon^i}.$$

Finally,

$$\left|\frac{\lambda}{\delta^4(\delta(\alpha+\beta)-9\ell)}\right|\delta^3=\frac{|\frac{\lambda}{\delta^2}|}{|\alpha+\beta-9\frac{\ell}{\delta}|}<\sqrt{\varepsilon^i}$$

by (5.17). Thus $4|D|\delta^3<P(\varepsilon^i)$ for large enough C'. Identical arguments may be applied to show that $3|E|\delta^2<P(\varepsilon^i)$ and $2|F|\delta<P(\varepsilon^i)$. Therefore,

$$|y_j^i(t)|\leq |D|\delta^4+|E|\delta^3+|F|\delta^2+|G|\delta<P(\varepsilon^i),$$

and

$$|(y_j^i)'(t)-\mu|\leq 4|D|\delta^3+3|E|\delta^2+2|F|\delta+|G-\mu|<P(\varepsilon^i)$$

for any $t \in [0, \delta]$ and large enough C'. This proves (5.14) and completes the proof of the lemma in this case.

Case 2. γ_j, γ_j', a^i, and b^i satisfy

$$\left| \alpha + \beta - 9\frac{|\gamma_j(b^i) - \gamma_j(a^i)|}{b^i - a^i} \right| \leq \sqrt{\varepsilon^i}.$$

By composing with a translation as before, we can again write $[a^i, b^i] = [0, \delta]$. We will first find bounds on α and β in this case. We have

$$\left| \alpha + \beta - 2\frac{\ell}{\delta} \right| - 7\left| \frac{\ell}{\delta} \right| \leq \left| \alpha + \beta - 9\frac{\ell}{\delta} \right| \leq \sqrt{\varepsilon^i}$$

and so we have $|\frac{\ell}{\delta}| < \frac{\sqrt{\varepsilon^i}+2\varepsilon^i}{7}$ since $|\alpha - \frac{\ell}{\delta}| < \varepsilon^i$ and $|\beta - \frac{\ell}{\delta}| < \varepsilon^i$. Thus also

$$|\alpha| < \varepsilon^i + \frac{\sqrt{\varepsilon^i} + 2\varepsilon^i}{7} = \frac{\sqrt{\varepsilon^i} + 9\varepsilon^i}{7} \quad \text{ and } \quad |\beta| < \frac{\sqrt{\varepsilon^i} + 9\varepsilon^i}{7}.$$

We will define $\eta_j^i = (x_j^i, y_j^i) : [0, \delta] \to \mathbb{R}^2$ piecewise on its domain as follows:

$$x_j^i(t) = \begin{cases} \frac{9\delta\alpha - 27\ell}{\delta^3}t^3 + \frac{-12\delta\alpha + 27\ell}{2\delta^2}t^2 + \alpha t & : t \in [0, \frac{\delta}{3}] \\ \frac{9\delta\beta - 27\ell}{\delta^3}t^3 + \frac{-42\delta\beta + 135\ell}{2\delta^2}t^2 + \frac{16\delta\beta - 54\ell}{\delta}t - 4\delta\beta + \frac{29}{2}\ell & : t \in [\frac{2\delta}{3}, \delta] \end{cases}$$

and

$$y_j^i(t) = \begin{cases} \frac{9\mu}{\delta^2}t^3 - \frac{6\mu}{\delta}t^2 + \mu t & : t \in [0, \frac{\delta}{3}] \\ \frac{9\nu}{\delta^2}t^3 - \frac{21\nu}{\delta}t^2 + 16\nu t - 4\delta\nu & : t \in [\frac{2\delta}{3}, \delta] \end{cases}.$$

On $(\frac{\delta}{3}, \frac{2\delta}{3})$, define η_j^i as

$$\begin{aligned} x_j^i(t) &= R\cos(\tau(t)) - R + \frac{\ell}{2} \\ y_j^i(t) &= R\sin(\tau(t)) \end{aligned}$$

where $R = \frac{1}{2\sqrt{\pi}}|H|^{1/2}$ with

$$\begin{aligned} H &:= \lambda - 2\int_0^{\delta/3} (x'y - xy') - 2\int_{2\delta/3}^{\delta} (x'y - xy') \\ &= \lambda - \frac{\delta\ell(\mu - \nu)}{15}, \end{aligned}$$

and $\tau : (\frac{\delta}{3}, \frac{2\delta}{3}) \to \mathbb{R}$ is defined as

$$\tau(t) = \pm \left(-\frac{108\pi}{\delta^3} t^3 + \frac{162\pi}{\delta^2} t^2 - \frac{72\pi}{\delta} t + 10\pi \right)$$

where we choose $+$ if $H \leq 0$ and $-$ if $H > 0$. With this definition, x_j^i and y_j^i are C^1 on $[0, \delta]$ and conditions (5.12) and (5.13) are satisfied. See Figure 9 for a possible construction.

We can argue as we did in the proof of the previous case to show that

$$|x_j^i(t)| < P(\varepsilon^i), \quad |(x_j^i)'(t) - \alpha| < P(\varepsilon^i), \quad |y_j^i(t)| < P(\varepsilon^i), \quad |(y_j^i)'(t) - \mu| < P(\varepsilon^i)$$

for $t \in [0, \frac{\delta}{3}] \cup [\frac{2\delta}{3}, \delta]$ with large enough C'. To prove condition (5.14), it remains to find bounds for x_j^i, y_j^i, and their derivatives on $(\frac{\delta}{3}, \frac{2\delta}{3})$. We have $R \leq \frac{1}{2\sqrt{\pi}} \left(\varepsilon^i + \frac{2}{15}(\varepsilon^i)^3 \right)^{1/2} \leq \frac{1}{2\sqrt{\pi}} \left((\varepsilon^i)^{1/2} + \sqrt{\frac{2}{15}} (\varepsilon^i)^{3/2} \right)$, and so, for any $t \in (\frac{\delta}{3}, \frac{2\delta}{3})$, $|x_j^i(t)| < P(\varepsilon^i)$ and $|y_j^i(t)| < P(\varepsilon^i)$ for large enough C'.

We will now prove bounds for the derivatives. Notice that on $(\frac{\delta}{3}, \frac{2\delta}{3})$ we have $|(x_j^i)'(t)| \leq |R\tau'(t)|$ and $|(y_j^i)'(t)| \leq |R\tau'(t)|$. Now for any $t \in (\frac{\delta}{3}, \frac{2\delta}{3})$

$$|\tau'(t)| \leq 3 \left| \frac{108\pi}{\delta^3} \right| \left(\frac{2\delta}{3} \right)^2 + 2 \left| \frac{162\pi}{\delta^2} \right| \left(\frac{2\delta}{3} \right) + \left| \frac{72\pi}{\delta} \right| = \frac{432\pi}{\delta}.$$

Therefore, $|(x_j^i)'(t)|^2$ and $|(y_j^i)'(t)|^2$ are bounded by

$$|R\tau'(t)|^2 = 46656\pi \left(\frac{|\lambda|}{\delta^2} + \frac{\delta\ell(|\mu| + |\nu|)}{15\delta^2} \right) = 46656\pi \left(\varepsilon^i + \frac{2(\varepsilon^i)^{3/2}}{105} + \frac{4(\varepsilon^i)^2}{105} \right).$$

Choosing large enough C', we have $|(x_j^i)'(t)| < P(\varepsilon^i)$ and $|(y_j^i)'(t)| < P(\varepsilon^i)$. Since $|\alpha| < \frac{\sqrt{\varepsilon^i} + 9\varepsilon^i}{7}$ and $|\mu| < \varepsilon^i$, this proves condition (5.14).

It remains to prove condition (5.15). We have $2\int_{\delta/3}^{2\delta/3} ((x_j^i)' y_j^i - x_j^i (y_j^i)') = \mp 4\pi R^2$ which is negative if $H < 0$ and positive if $H > 0$ (and so $\mp 4\pi R^2 = H$). Thus

$$\begin{aligned} 2\int_0^\delta &((x_j^i)' y_j^i - x_j^i (y_j^i)') \\ &= 2\int_0^{\delta/3} ((x_j^i)' y_j^i - x_j^i (y_j^i)') \mp 4\pi (R)^2 + 2\int_{2\delta/3}^{\delta} ((x_j^i)' y_j^i - x_j^i (y_j^i)') \\ &= \lambda = \frac{1}{n} \left[h(b^i) - h(a^i) - 2\sum_{m=1}^{n} (f_m(b^i) g_m(a^i) - f_m(a^i) g_m(b^i)) \right]. \end{aligned}$$

This completes the proof of the lemma. $\square$

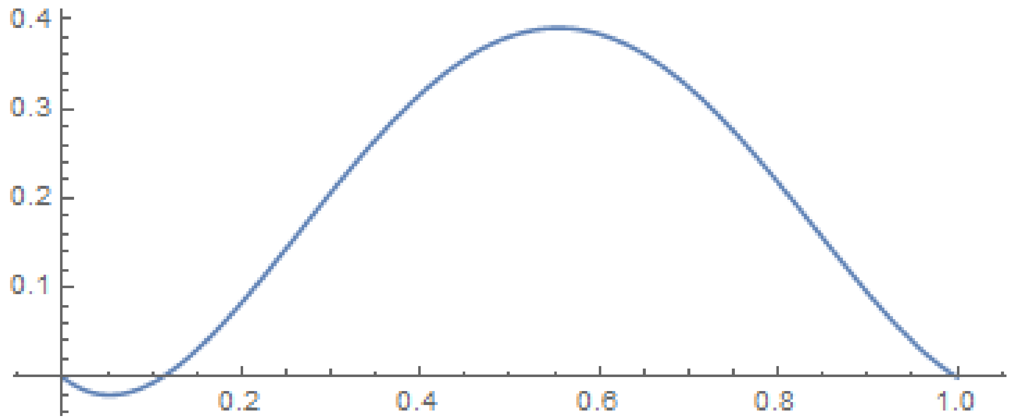

Figure 8: A possible construction of η_j^i in Case 1

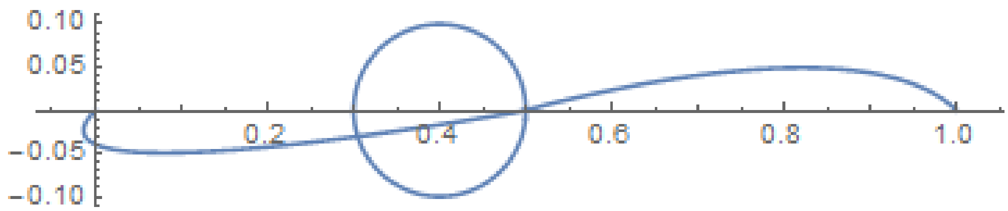

Figure 9: A possible construction of η_j^i in Case 2

6.0 SOBOLEV EXTENSIONS OF LIPSCHITZ MAPPINGS INTO METRIC SPACES

This chapter is based on the paper [96].

Definition 61. *A pair of metric spaces (X, Y) has the Lipschitz extension property if there is a constant $C > 0$ so that any L-Lipschitz mapping $f : A \to Y$, $A \subset X$ has a CL-Lipschitz extension $F : X \to Y$.*

Extensive research has been conducted in the area of Lipschitz extensions. See, for example, [17, 28, 54, 61, 62, 63, 86, 90, 91]. Wenger and Young [91] showed that $(\mathbb{R}^m, \mathbb{H}^n)$ has the Lipschitz extension property for $m \le n$. More generally, the authors proved that $(X, \mathbb{H}^n)$ has the Lipschitz extension property as long as the Assouad-Nagata dimension of X is at most n. See [6, 61, 91] for more information about this notion of dimension. For such metric spaces X, Lang and Schlichenmeier [61] showed that, when Y is any Lipschitz $(n-1)$-connected metric space, there is a constant $C > 0$ so that any L-Lipschitz mapping $f : A \to Y$ defined on a closed subset $A \subset X$ has a CL-Lipschitz extension $F : X \to Y$.

Definition 62. *A metric space Y is* Lipschitz $(n-1)$-connected *if there is a constant $\gamma \ge 1$ so that any L-Lipschitz map $f : S^k \to Y$ $(L > 0)$ on the k-dimensional sphere has a γL-Lipschitz extension $F : B^{k+1} \to Y$ on the $(k+1)$-ball for $k = 0, 1, \dots, n-1$.*

The result of Wenger and Young follows immediately if one proves the Lipschitz $(n-1)$-connectivity of $\mathbb{H}^n$. As Wenger and Young mentioned, however, proving this property for $\mathbb{H}^n$ is difficult, and thus they provided a direct proof of their Lipschitz extension result.

What happens, however, when the dimension of the domain is large? As Balogh and Fässler [7] showed, the pair $(\mathbb{R}^m, \mathbb{H}^n)$ does not have the Lipschitz extension property when $m > n$. Indeed, there is a bi-Lipschitz embedding of the sphere S^n into $\mathbb{H}^n$, and one can show that this embedding does not admit a Lipschitz extension to the ball B^{n+1}. Since B^{n+1} can be regarded as a subset of $\mathbb{R}^m$ for any $m > n$, the result follows. (See also Theorems 1.5 and 1.6 in [45] for a shorter proof.)

In this chapter, we consider Sobolev extensions of Lipschitz mappings $f : A \to \mathbb{H}^n$, $A \subset \mathbb{R}^m$. Since Sobolev mappings form a larger class than Lipschitz mappings, it turns out that, in the Sobolev case, we no longer have any restriction on the dimension of the domain. The first main result of the chapter is stated here.

Theorem 63. *Fix $m, n \in \mathbb{N}$. Suppose $Z \subset \mathbb{R}^m$ is compact and Ω is a bounded domain in $\mathbb{R}^m$ with $Z \subset \Omega$. For $1 \leq p < n+1$ and any L-Lipschitz mapping $f : Z \to \mathbb{H}^n$, $L \geq 0$, there exists $F \in W^{1,p}(\Omega, \mathbb{H}^n)$ with $F(x) = f(x)$ for all $x \in Z$.*
Moreover, there is a constant $C > 0$ depending only on m, n, and p such that, if we write $F = (F_1, \ldots, F_{2n}, F_{2n+1})$, then $\|\partial F_j / \partial x_k\|_{L^p(\Omega)} \leq CL\,(\operatorname{diam}(\Omega))^{m/p}$ for $k = 1, \ldots, m$ and $j = 1, \ldots, 2n$.

If $m \leq n$, then f admits a Lipschitz extension since $\mathbb{H}^n$ is Lipschitz $(n-1)$-connected by the result of Wenger and Young, and this extension belongs to $W^{1,p}(\Omega, \mathbb{H}^n)$ for $1 \leq p \leq \infty$. However, if $m > n$, the result in Theorem 63 does not hold for $p \geq n+1$. Indeed, we have

Proposition 64. *There is a Lipschitz mapping $f : S^n \to \mathbb{H}^n$ which admits no extension $F \in W^{1,n+1}(B^{n+1}, \mathbb{H}^n)$.*

One such mapping $f : S^n \to \mathbb{H}^n$ is the bi-Lipschitz embedding used by Balogh and Fässler [7]. In the proof of Proposition 64, we will see ideas from [40, Theorem 2], [41, Theorem 2.3], and [45, Theorem 1.5].

Note that the bounds in Theorem 63 are given only for $j < 2n+1$. Such a condition follows naturally from the sub-Riemannian geometry of the Heisenberg group. A brief explanation of this follows Definition 73 in Section 6.1.4.

For mappings with Euclidean target, Sobolev extension results like Theorem 63 provide extensions defined on all of $\mathbb{R}^m$ via composition with a cutoff function. However, since we do not have such cutoff functions in $\mathbb{H}^n$, such a simple argument will not work here. However, we have the following

Corollary 65. *Fix $m, n \in \mathbb{N}$. Suppose $Z \subset \mathbb{R}^m$ is compact. For $1 \leq p < n+1$ and any*

L-Lipschitz mapping $f : Z \to \mathbb{H}^n$, $L \geq 0$, there exists $\tilde{F} \in W^{1,p}_{loc}(\mathbb{R}^m, \mathbb{H}^n)$ with $\tilde{F}(x) = f(x)$ for all $x \in Z$.

This follows easily from the theorem. Indeed, suppose Ω is a cube containing Z and $\Phi : \mathbb{R}^m \to \Omega$ is a diffeomorphism which fixes Z. Then, if $F \in W^{1,p}(\Omega, \mathbb{H}^n)$ is the extension from Theorem 63, it follows that $\tilde{F} := F \circ \Phi \in W^{1,p}_{loc}(\mathbb{R}^m, \mathbb{H}^n)$.

It follows from classical Lipschitz extension proofs that there is a constant $C > 0$ so that any L-Lipschitz mapping $f : A \to Y$ defined on a closed subset $A \subset \mathbb{R}^m$ has a CL-Lipschitz extension $F : X \to Y$ when Y is any Lipschitz $(n-1)$-connected metric space and $m \leq n$ (see [3, 54] or the proof of Lemma 78). It turns out that Theorem 63 can be generalized to the case when the target space $\mathbb{H}^n$ is replaced by an arbitrary Lipschitz $(n-1)$-connected metric space Y. In this case, our extension will be in the *Ambrosio-Reshetnyak-Sobolev class* $AR^{1,p}(\Omega, Y)$. For a bounded domain Ω in $\mathbb{R}^m$ and $1 \leq p < \infty$, a mapping $F : \Omega \to Y$ belongs to the class $AR^{1,p}(\Omega, Y)$ if there is a non-negative function $g \in L^p(\Omega)$ satisfying the following: for any K-Lipschitz $\phi : Y \to \mathbb{R}$, we have $\phi \circ F \in W^{1,p}(\Omega)$ and $|\partial(\phi \circ F)/\partial x_k(x)| \leq Kg(x)$ for $k = 1, \ldots, m$ and almost every $x \in \Omega$. This class of mappings was first introduced in [4] and [76].

Theorem 66. *Fix $m, n \in \mathbb{N}$. Suppose $Z \subset \mathbb{R}^m$ is compact, Ω is a bounded domain in $\mathbb{R}^m$ with $Z \subset \Omega$, and Y is a Lipschitz $(n-1)$-connected metric space with constant γ. For $1 \leq p < n+1$ and any L-Lipschitz mapping $f : Z \to Y$, $L \geq 0$, there exists $F \in AR^{1,p}(\Omega, Y)$ with $F(x) = f(x)$ for all $x \in Z$.*

Moreover, there is a constant $C > 0$ depending only on m, n, p, and γ such that we may choose $g \in L^p(\Omega)$ in the definition of $AR^{1,p}(\Omega, Y)$ with $\|g\|_{L^p(\Omega)} \leq CL\,(\mathrm{diam}\,(\Omega))^{m/p}$.

Notice that, as before, there is no restriction on the dimension of the domain. The theory of Sobolev mappings into metric spaces has been studied extensively in [4, 38, 46, 50, 51, 52, 58, 76, 77]. See Section 6.1 below for some details on the topic. In particular, $\mathbb{H}^n$ valued Sobolev mappings have been explored in [8, 20, 22, 46, 64]. One motivation for the study of Sobolev extensions stems from the problem of approximating Sobolev mappings by Lipschitz ones [11, 15, 22, 39, 45, 49]. In fact, the proof of Theorem 66 employs the so called zero degree homogenization discussed in [15, 39].

As we will see in Proposition 74, $W^{1,p}(\Omega, \mathbb{H}^n)$ is contained in $AR^{1,p}(\Omega, \mathbb{H}^n)$. Furthermore, in the case of bounded mappings, the two definitions of the Sobolev class are equivalent. Hence Theorem 63 will be proven as a corollary to Theorem 66.

6.1 SOBOLEV MAPPINGS INTO METRIC SPACES

There are a variety of classes one may consider when defining a Sobolev mapping with a metric space target. Some of these classes are addressed here. Throughout this section, we will consider a mapping $F : \Omega \to Y$ where $\Omega \subset \mathbb{R}^m$ is a bounded domain and Y is a metric space.

6.1.1 Sobolev mappings into Banach spaces

We first consider the case when Y is a Banach space. For more details, see [23].

Definition 67. *We say $F \in L^p(\Omega, Y)$ if*

1. *For some $Z \subset \Omega$ with $|Z| = 0$, the set $F(\Omega \setminus Z)$ is separable,*
2. *$\phi \circ F$ is measurable for every continuous, linear $\phi : Y \to \mathbb{R}$ with $||\phi|| \leq 1$,*
3. *$||F|| \in L^p(\Omega)$.*

In order to define a Sobolev function in this setting, we will introduce a Banach space version of the integration by parts formula. To do this, we first need a new definition of the integral. Suppose F satisfies the first two conditions in the above definition and $\{F_i\}$ is a sequence of simple functions from Ω into Y with $F_i(x) \to F(x)$ pointwise a.e. That is, there are vectors $y_{i1}, \ldots, y_{iN_i}$ in Y and a partition $A_{i1}, \ldots, A_{iN_i}$ of Ω so that

$$F_i(x) = \sum_{j=1}^{N_i} y_{ij} \chi_{A_{ij}}(x).$$

We may always approximate F pointwise a.e. by such simple functions. As in the case of Lebesgue integration, define

$$\int_\Omega F_i(x)\, dx = \sum_{j=1}^{N_i} |A_{ij}| y_{ij}.$$

Definition 68. *The Bochner integral of F is*

$$\int_\Omega F(x)\,dx = \lim_{i\to\infty} \int_\Omega F_i(x)\,dx.$$

Here, the limit is taken over any sequence $\{F_i\}$ of simple functions which converges pointwise a.e. to F.

This integral exists and is unique for any $F \in L^1(\Omega, Y)$. We are now ready to define the Sobolev class.

Definition 69. *For $1 \le p < \infty$, we say $F \in W^{1,p}_B(\Omega, Y)$ if $F \in L^p(\Omega, Y)$ and there are $g_j \in L^p(\Omega, Y)$ for $j = 1, \ldots, m$ with*

$$\int_\Omega \frac{\partial \phi}{\partial x_i} F = -\int_\Omega \phi\, g_j$$

for every $\phi \in C_c^\infty(\Omega)$.

6.1.2 Embedding a metric space in a Banach space

The above definition of Banach space valued Sobolev mappings actually allows us to define Sobolev mappings into any metric space. Suppose Y is now any metric space. Define $C_b(Y)$ to be the set of all bounded, continuous, real-valued functions on Y. This is a Banach space under the supremum norm.

Theorem 70 (Kuratowski)**.** *There is an isometric embedding of Y into $C_b(Y)$.*

Proof. Fix $y_0 \in Y$. The map $\kappa : Y \to C_b(Y)$ defined as $\kappa(y) = d(y, \cdot) - d(y_0, \cdot)$ for any $y \in Y$ is an isometry. □

That is, we may always consider Y as a subset of a Banach space. In fact, if Y is separable, then we may isometrically embed it in the Banach space ℓ^∞ of all bounded sequences of real numbers. We can now define the Sobolev class $W^{1,p}(\Omega, Y)$ in terms of this embedding.

Definition 71. *For $1 \le p < \infty$, define*

$$W^{1,p}_B(\Omega, Y) = \left\{ F \in W^{1,p}_B(\Omega, C_b(Y)) \,:\, F(\Omega) \subset \kappa(Y) \right\}$$

6.1.3 The Ambrosio-Reshetnyak-Sobolev class

Suppose for now that u is in the classical Sobolev space $W^{1,p}(\Omega)$. Suppose also that $\phi : \mathbb{R} \to \mathbb{R}$ is a K-Lipschitz function. According to the ACL characterization of u, there is a representative of u that is locally absolutely continuous along almost every line parallel to a coordinate axis. Thus $\phi \circ u$ is locally absolutely continuous along these lines as well, and so it follows that $\phi \circ u \in W^{1,p}(\Omega)$. In fact, one may check that $|\partial(\phi \circ u)/\partial x_k| \leq K|\partial u/\partial x_k|$ almost everywhere in Ω.

This property of classical Sobolev maps leads to the following definition of the Ambrosio-Reshetnyak-Sobolev class of mappings into metric spaces introduced in [4] and [76].

Definition 72. *For $1 \leq p < \infty$, a mapping F belongs to the class $AR^{1,p}(\Omega, Y)$ if there is a non-negative function $g \in L^p(\Omega)$ satisfying the following: for any K-Lipschitz $\phi : Y \to \mathbb{R}$, we have $\phi \circ F \in W^{1,p}(\Omega)$ and $|\partial(\phi \circ F)/\partial x_j(x)| \leq Kg(x)$ for $j = 1, \ldots, m$ and a.e. $x \in \Omega$.*

For more information about the relationship between these classes of Sobolev mappings, see [46, 51].

6.1.4 Sobolev mappings into the Heisenberg group

The following definition of Sobolev mappings $W^{1,p}(\Omega, \mathbb{H}^n)$ into the Heisenberg group has been discussed in [8, 22, 46, 64]. In these references the Sobolev class is defined in terms of the Banach space embedding as described above. However, in [22, Proposition 6.8], it is proven that $W^{1,p}(\Omega, \mathbb{H}^n) = W^{1,p}_B(\Omega, \mathbb{H}^n)$.

Definition 73. *A mapping $F : \Omega \to \mathbb{H}^n$ is of class $W^{1,p}(\Omega, \mathbb{H}^n)$ if the following two conditions hold:*

1. *$F \in W^{1,p}(\Omega, \mathbb{R}^{2n+1})$, and*
2. *$F = (f_1, g_1, \ldots, f_n, g_n, h)$ satisfies the* weak contact equation

$$\nabla h(x) = 2 \sum_{j=1}^{n} \left(g_j(x) \nabla f_j(x) - f_j(x) \nabla g_j(x)\right) \quad a.e.\ x \in \Omega. \tag{6.1}$$

Say that $F \in W^{1,p}_{loc}(\mathbb{R}^m, \mathbb{H}^n)$ if $F \in W^{1,p}_{loc}(\mathbb{R}^m, \mathbb{R}^{2n+1})$ and the weak contact equation holds for a.e. $x \in \mathbb{R}^m$.

Notice that the weak contact condition (6.1) may also be written as follows:

$$\text{im } DF(x) \subset H_{F(x)}\mathbb{H}^n \quad \text{for a.e. } x \in \Omega$$

where DF is the weak differential of F. Consider the projection mapping π from $\mathbb{R}^{2n+1}$ onto its first $2n$ coordinates. It follows from the definition of the metric on the horizontal space that $d\pi_p : H_p\mathbb{H}^n \to T_{\pi(p)}\mathbb{R}^{2n}$ is an isometry for any $p \in \mathbb{H}^n$. Hence, for almost every $x \in \Omega$, the norm of the linear map $DF(x) : T_x\mathbb{R}^m \to H_{F(x)}\mathbb{H}^n$ is equal to the norm of $D(\pi \circ F)(x) : T_x\mathbb{R}^m \to T_{\pi(F(x))}\mathbb{R}^{2n}$. This is why the quantitative estimates at the end of the statement of Theorem 63 only apply to the partial derivatives of the first $2n$ components of F.

6.1.5 The relationship between the Ambrosio-Reshetnyak and Sobolev classes in the Heisenberg group

As we will now see, this definition gives a sufficient condition for a mapping to be in the class $AR^{1,p}(\Omega, \mathbb{H}^n)$.

Proposition 74. *Suppose Ω is a bounded domain in $\mathbb{R}^m$ and $1 \leq p < \infty$. Then*

$$W^{1,p}(\Omega, \mathbb{H}^n) \subset AR^{1,p}(\Omega, \mathbb{H}^n).$$

Furthermore, if $F \in AR^{1,p}(\Omega, \mathbb{H}^n)$ is bounded, then $F \in W^{1,p}(\Omega, \mathbb{H}^n)$.

A more precise statement of the first inclusion is as follows: any $F \in W^{1,p}(\Omega, \mathbb{H}^n)$ has a representative in $W^{1,p}(\Omega, \mathbb{R}^{2n+1})$ (namely its ACL representative) which is in the class $AR^{1,p}(\Omega, \mathbb{H}^n)$. This representative still satisfies the weak contact equation since (6.1) is only required to hold almost everywhere in Ω.

A result similar to the first inclusion was proven in [8, Proposition 6.1] by embedding $\mathbb{H}^n$ into ℓ^∞ via the Kuratowski embedding. The reverse inclusion for bounded maps is proven in [22, Proposition 6.8] by applying the same embedding and invoking an ACL-type result for Sobolev mappings into Banach spaces. Different, mostly self-contained proofs relying more directly on the geometry of the Heisenberg group are given below.

Proof. Suppose $F \in W^{1,p}(\Omega, \mathbb{R}^{2n+1})$ satisfies (6.1) almost everywhere in Ω. We will consider the $ACL^p(\Omega, \mathbb{R}^{2n+1})$ representative of F. Fix a K-Lipschitz function $\phi : \mathbb{H}^n \to \mathbb{R}$. First, notice for any $x \in \Omega$

$$|\phi(F(x))| \leq K\, d_{cc}(F(x), 0) + |\phi(0)| \leq C\, K\, \|F(x)\|_K + |\phi(0)|$$

for some $C \geq 1$ from the bi-Lipschitz equivalence of d_{cc} and d_K. There is a constant $M \geq 1$ depending only on n so that $\|p\|_K \leq M \max\{1, |p|\}$ for any $p \in \mathbb{H}^n$. Hence, since Ω is bounded and $F \in L^p(\Omega, \mathbb{R}^{2n+1})$, we have $\phi \circ F \in L^p(\Omega)$.

We must now show that $\phi \circ F \in W^{1,p}(\Omega)$ and find a function $g \in L^p(\Omega)$ which dominates the partial derivatives of $\phi \circ F$ and is independent of the choice of ϕ. Fix $k \in \{1, \ldots, m\}$. Choose a line $\bar{\ell}$ parallel to the k^{th} coordinate axis so that F is absolutely continuous along compact intervals in $\ell := \bar{\ell} \cap \Omega$ and so that $\partial F/\partial x_k \in L^p(\ell, \mathbb{R}^{2n+1})$. Suppose also that F satisfies (6.1) almost everywhere along ℓ. (Note that $(m-1)$-almost every $\bar{\ell}$ parallel to the k^{th} coordinate axis satisfies these conditions via Fubini's theorem and Lemma 14.) Choose a compact interval $[a, b] \subset \ell$. (Here, we abuse notation and identify ℓ with a subset of $\mathbb{R}$.) It follows from (6.1) that $\gamma := F|_{[a,b]} : [a, b] \to \mathbb{H}^n$ is a horizontal curve. The definition of the metric in $\mathbb{H}^n$ gives

$$|\phi(F(x)) - \phi(F(y))| \leq K\, d_{cc}(F(x), F(y)) \leq K\, \ell_H(\gamma|_{[x,y]}) \leq K\, \ell_E(\gamma|_{[x,y]})$$

for any $[x, y] \subset [a, b]$. Consider the Euclidean length function $s_\gamma : [a, b] \to [0, \ell_E(\gamma)]$ defined as $s_\gamma(x) = \ell_E(\gamma|_{[a,x]})$. We can write $\ell_E(\gamma|_{[x,y]}) = |s_\gamma(x) - s_\gamma(y)|$ and conclude that

$$|\phi(F(x)) - \phi(F(y))| \leq K\, |s_\gamma(x) - s_\gamma(y)|$$

for any $x, y \in [a, b]$. Since γ is absolutely continuous on $[a, b]$ as a Euclidean curve, s_γ is absolutely continuous as well (see for example [52, Proposition 5.1.5]). Thus $\phi \circ F$ is absolutely continuous on $[a, b]$.

We will now prove the bound on the derivative of $\phi \circ F$ along ℓ. Fix a point $x \in \ell$ where $\partial F/\partial x_k$ and $\partial(\phi \circ F)/\partial x_k$ exist and which is a p-Lebesgue point of each component of $\partial F/\partial x_k$. (Note: almost every point in ℓ satisfies these conditions since the partial derivative of F is p-integrable along ℓ.) For any t small enough so that the interval $(x, x+te_k) \subset \Omega$, we have

$$
\begin{aligned}
\left|\frac{\phi(F(x+te_k)) - \phi(F(x))}{t}\right| &\leq C\,K\,\frac{d_K(F(x+te_k),F(x))}{|t|} \\
&= C\,K\Bigg(\left|\sum_{j=1}^{n}\left(\frac{f_j(x+te_k)-f_j(x)}{t}\right)^2 + \left(\frac{g_j(x+te_k)-g_j(x)}{t}\right)^2\right|^2 \\
&\qquad + \left|\frac{h(x+te_k)-h(x)+2\sum_{j=1}^{n}(f_j(x)g_j(x+te_k)-f_j(x+te_k)g_j(x))}{t^2}\right|^2\Bigg)^{1/4}
\end{aligned}
$$

for a constant $C > 0$ depending only on the bi-Lipschitz equivalence of d_{cc} and d_K. This final fraction above converges to 0 as $t \to 0$. Indeed, the proof of this fact is nearly identical to the proof of Proposition 1.4 in [97] since x is a p-Lebesgue point of the partial derivatives. Therefore,

$$
\left|\frac{\partial(\phi\circ F)}{\partial x_k}(x)\right| \leq C\,K\sqrt{\sum_{j=1}^{n}\left(\frac{\partial f_j}{\partial x_k}(x)\right)^2 + \left(\frac{\partial g_j}{\partial x_k}(x)\right)^2} \leq C\,K\left|\frac{\partial F}{\partial x_k}(x)\right|. \tag{6.2}
$$

Define $g : \Omega \to \mathbb{R}$ as $g(x) = C\sum_{k=1}^{m}\left|\frac{\partial F}{\partial x_k}(x)\right|$. Thus, for any K-Lipschitz $\phi : \mathbb{H}^n \to \mathbb{R}$, we have $|\partial(\phi\circ F)/\partial x_k(x)| \leq K g(x)$ for almost every $x \in \Omega$ and $k = 1, \ldots, m$. Since $g \in L^p(\Omega)$, it follows that $F \in AR^{1,p}(\Omega, \mathbb{H}^n)$.

We will now prove the reverse inclusion for bounded Sobolev mappings. Suppose a mapping $F \in AR^{1,p}(\Omega, \mathbb{H}^n)$ is bounded and say $g \in L^p(\Omega)$ is as in the definition of the Ambrosio-Reshetnyak-Sobolev class. By (3.10), the identity map $\mathrm{id} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$ is Lipschitz on some compact set containing $F(\Omega)$. Thus $F = \mathrm{id}\circ F \in W^{1,p}(\Omega, \mathbb{R}^{2n+1})$. It remains to show that the weak contact equation (6.1) holds almost everywhere. Choose a dense subset $\{p_i\}_{i=1}^{\infty}$ of $\mathbb{H}^n$. (This is possible since $\mathbb{H}^n$ and $\mathbb{R}^{2n+1}$ are topologically equivalent.) Define the 1-Lipschitz maps $\phi_i : \mathbb{H}^n \to \mathbb{R}$ as $\phi_i(x) = d_{cc}(x, p_i)$. Therefore, in Ω along $(m-1)$-almost every line parallel to a coordinate axis, $\phi_i \circ F$ is absolutely continuous (after possibly redefining F on a set of measure zero), g is p-integrable, and $|\partial(\phi_i\circ F)/\partial x_k| \leq g$ almost everywhere for all $i \in \mathbb{N}$. For $k \in \{1, \ldots, m\}$, fix such a line $\bar{\ell}$ parallel to the k^{th} axis and write $\ell = \bar{\ell} \cap \Omega$.

By Fubini's theorem, it suffices to prove that (6.1) holds almost everywhere along ℓ. Choose an interval $[x, x+te_k] \subset \ell$. Fix $s_1, s_2 \in [0,t]$. Let $\varepsilon > 0$ and choose $p_i \in \mathbb{H}^n$ so that $2d_{cc}(F(x+s_1e_k), p_i) < \varepsilon$. Then we have

$$\begin{aligned}
d_{cc}(F(x+s_2e_k), F(x+s_1e_k)) - \varepsilon &\leq d_{cc}(F(x+s_2e_k), F(x+s_1e_k)) - 2d_{cc}(F(x+s_1e_k), p_i) \\
&\leq d_{cc}(F(x+s_2e_k), p_i) - d_{cc}(F(x+s_1e_k), p_i) \\
&= \phi_i(F(x+s_2e_k)) - \phi_i(F(x+s_1e_k)) \\
&= \int_{s_1}^{s_2} \frac{d}{d\tau}\,(\phi_i \circ F)(x+\tau e_k)\,d\tau \\
&\leq \int_{s_1}^{s_2} g(x+\tau e_k)\,d\tau.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, it follows that

$$d_{cc}(F(x+s_2e_k), F(x+s_1e_k)) \leq \int_{s_1}^{s_2} g(x+\tau e_k)\,d\tau$$

for any $s_1, s_2 \in [0,t]$. By the integrability of g along ℓ, the mapping F is absolutely continuous with respect to the metric d along compact intervals in ℓ. Hence (6.1) holds almost everywhere along ℓ as a result of Proposition 4.1 in [75]. We provide the relevant version of this proposition here with its proof for completeness.

Lemma 75. *If $\gamma : [a,b] \to \mathbb{H}^n$ is absolutely continuous with respect to the metric on $\mathbb{H}^n$, then γ is horizontal.*

Proof. By (3.10), γ is absolutely continuous into $\mathbb{R}^{2n+1}$, so γ' exists almost everywhere in the classical sense. Also, by Theorem 3.3 in [25], the metric derivative $md_{cc}(\gamma, \cdot)$ exists almost everywhere. That is, the following limit exists for almost every $t \in [a,b]$:

$$md_{cc}(\gamma, t) := \lim_{s \to 0,\, t+s \in [a,b]} \frac{d_{cc}(\gamma(t+s), \gamma(t))}{|s|}.$$

Choose t so that $\gamma'(t)$ and $md_{cc}(\gamma, t)$ exist. Writing $\gamma = (f_1, g_1, \ldots, f_n, g_n, h)$, we have

$$\begin{aligned}
&\left| h(t+s) - h(t) - 2\sum_{j=1}^{n} \big(g_j(t)f_j'(t) - f_j(t)g_j'(t)\big)\, s \right| \\
&\quad \leq \left| h(t+s) - h(t) + 2\sum_{j=1}^{n} \left(f_j(t)g_j(t+s) - g_j(t)f_j(t+s)\right) \right|
\end{aligned}$$

$$+ 2\sum_{j=1}^{n} |f_j(t)| \left|g_j(t+s) - g_j(t) - sg_j'(t)\right| + 2\sum_{j=1}^{n} |g_j(t)| \left|f_j(t+s) - f_j(t) - sf_j'(t)\right|$$

These last two sums are of order $o(s)$ as $s \to 0$ since $\gamma'(t)$ exists. By the definition of the Korányi metric and its bi-Lipschitz equivalence with the CC-metric on $\mathbb{H}^n$, for some $C \geq 1$, the first term is bounded by

$$C^2 d_{cc}(\gamma(t+s), \gamma(t))^2$$

which is also of order $o(s)$ as $s \to 0$ since $md_{cc}(\gamma, t)$ exists and γ is continuous. Therefore,

$$h'(t) = \lim_{s\to 0} \frac{h(t+s) - h(t)}{s} = 2\sum_{j=1}^{n} \left(g_j(t)f_j'(t) - f_j(t)g_j'(t)\right),$$

so γ satisfies the contact equation (3.4) at t. □

This completes the proof of the proposition. □

Notice in (6.2) that only the first $2n$ components of F appear in the bound of the partial derivatives of $\phi \circ F$. Compare this to the bound in Theorem 63 and to the discussion following Definition 73.

6.1.6 Sharpness of the bounds on p

We will conclude the section with the proof of Proposition 64. Recall that this proposition provides a counterexample to the results in Theorem 63 and Theorem 66 when the upper bound on p is removed. In the proof, we will use the following result from [7]. Another construction is given in [22, Theorem 3.2].

Theorem 76. *For any $n \geq 1$, there is a smooth embedding of the sphere S^n into $\mathbb{R}^{2n+1}$ which is horizontal and bi-Lipschitz as a mapping into $\mathbb{H}^n$ and has no Lipschitz extension $F : B^{n+1} \to \mathbb{H}^n$.*

Proof of Proposition 64. Define $f : S^n \to \mathbb{H}^n$ to be the embedding from Theorem 76. Suppose we have $F \in W^{1,n+1}(B^{n+1}, \mathbb{H}^n)$ with $F(x) = f(x)$ for every $x \in S^n$. By the definition of $W^{1,n+1}(B^{n+1}, \mathbb{H}^n)$ and Theorem 1.4 in [8], $\operatorname{rank} DF(x) \leq n$ for almost every $x \in B^{n+1}$. Since $f^{-1} : f(S^n) \to S^n$ is C^1, we may find a C^1 extension $\Psi : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$ of f^{-1} so that $|D\Psi| \leq M$ for some $M > 0$. Now, choose a sequence $\{F_k\}$ of mappings $F_k : B^{n+1} \to \mathbb{R}^{2n+1}$ which are C^1 up to the boundary and which satisfy the following:

- $\|F_k - F\|_{W^{1,n+1}} \to 0$ as $k \to \infty$,
- $\mathcal{H}^{n+1}(\{F_k \neq F\}) \to 0$ as $k \to \infty$,
- and $F_k = F = f$ on S^n for any $k \in \mathbb{N}$

(see, for example, Theorem 5 and the proof of Theorem 2 in [40].) Fix $k \in \mathbb{N}$. Since $\Psi \circ F_k$ is continuous on B^{n+1} and equals the identity map on S^n, Brouwer's theorem implies $B^{n+1} \subset (\Psi \circ F_k)(B^{n+1})$. Additionally, $|J(\Psi \circ F_k)| \leq M|JF_k|$. Here, the Jacobian $|JF_k|$ is understood in the following sense:

$$|JF_k(x)| = \sqrt{\det\left((DF_k)^T DF_k)(x\right)} \quad \text{for all } x \in B^{n+1}.$$

Thus

$$M \int_{B^{n+1}} |JF_k| \geq \int_{B^{n+1}} |J(\Psi \circ F_k)| \geq \mathcal{H}^{n+1}((\Psi \circ F_k)(B^{n+1})) \geq \mathcal{H}^{n+1}(B^{n+1}).$$

Since $\operatorname{rank} DF(x) \leq n$ for almost every $x \in B^{n+1}$, it follows that $|JF_k| = 0$ almost everywhere on $\{F_k = F\}$. Therefore

$$0 < \frac{\mathcal{H}^{n+1}(B^{n+1})}{M} \leq \int_{B^{n+1}} |JF_k| = \int_{\{F_k \neq F\}} |JF_k|.$$

However, $\mathcal{H}^{n+1}(\{F_k \neq F\}) \to 0$, and $|JF_k|$ converges to $|JF|$ in L^1 due to the convergence of F_k to F in $W^{1,n+1}$ since the Jacobian consists of sums of $(n+1)$-fold products of derivatives. Thus this last integral vanishes as $k \to \infty$. This leads to a contradiction and completes the proof. $\square$

6.2 WHITNEY TRIANGULATION AND LIPSCHITZ EXTENSIONS

Here we introduce important tools that will be used in the proof of Theorem 66.

6.2.1 Whitney triangulation of an open set

Suppose $Z \subset \mathbb{R}^m$ is closed. As in the proof of many extension theorems, we will decompose the complement of Z into Whitney cubes. We will then go one step further and construct the Whitney triangulation of the complement of Z as in [85]. We must first introduce some notation. For any $k \in \{0, 1, \ldots, m\}$, a *(non-degenerate) k-simplex* in $\mathbb{R}^m$ is the convex hull of $k+1$ *vertices* $\{e_0, e_1, \ldots, e_k\} \subset \mathbb{R}^m$ where the vectors $e_1 - e_0, \ldots, e_k - e_0$ are linearly independent. An ℓ-face ω of a k-simplex σ is the convex hull of any subset $\{e_{i_0}, \ldots, e_{i_\ell}\}$ of vertices of σ. Denote by $\partial\omega$ the union of all $(\ell-1)$-faces of ω. Note that, since we define simplices to be non-degenerate, the barycenter of a simplex does not lie in any of its faces. A *simplicial complex* Σ in $\mathbb{R}^m$ is a (possibly infinite) set consisting of simplices in $\mathbb{R}^m$ so that any face of a simplex in Σ is an element of Σ and the intersection of any two simplices in Σ is either empty or is itself an element of Σ. The dimension of Σ is the largest k so that Σ contains a k-simplex. (Notice that the dimension of a simplicial complex in $\mathbb{R}^m$ is at most m.) For any $k \in \{0, 1, \ldots, m\}$, the *$k$-skeleton* of Σ (denoted $\Sigma^{(k)}$) is the subset of $\mathbb{R}^m$ consisting of the union of all k-simplices in Σ. Similarly, the *ℓ-skeleton* $\Sigma^{(\ell)}_\sigma$ of a k-simplex σ, $0 \leq \ell \leq k$, is the union of all ℓ-faces of σ. Finally, we will write $B(k,\ell) := \binom{k+1}{\ell+1}$. This is the number of ℓ-faces of a k-simplex.

Suppose Σ is a simplicial complex in $\mathbb{R}^m$. For each $\ell \in \{1, \ldots, m\}$ and any ℓ-simplex $\omega \in \Sigma$ with barycenter c, say $\beta(\omega)$ is the minimum over all distances $d(c, P)$ where P is an $(\ell-1)$-plane containing an $(\ell-1)$-face of ω. In particular, $\beta(\omega) > 0$. Similarly, say $B(\omega)$ is the maximum over all such distances. For any m-simplex σ, write

$$\beta_\sigma = \min\{\beta(\omega) : \omega \text{ is an } \ell\text{-face of } \sigma \text{ for some } \ell \in \{1, \ldots, m\}\}$$

and

$$B_\sigma = \max\{B(\omega) : \omega \text{ is an } \ell\text{-face of } \sigma \text{ for some } \ell \in \{1, \ldots, m\}\}.$$

That is, β_σ is a lower bound on the "flatness" of σ, and B_σ is an upper bound. We are now ready to define the Whitney triangulation of $\mathbb{R}^m \setminus Z$. This lemma is a minor modification of the results in [85, Section 5.1].

Lemma 77 (Whitney Triangulation)**.** *Suppose $Z \subset \mathbb{R}^m$ is closed. Then there is an m-dimensional simplicial complex Σ in $\mathbb{R}^m$ so that $\Sigma^{(m)} = \mathbb{R}^m \setminus Z$ and the following hold for some constants $D_1, D_2 > 0$ (which depend only on m) and any m-simplex $\sigma \in \Sigma$:*

$$\operatorname{diam}(\sigma) \leq d(\sigma, Z) \leq 12\sqrt{m}\operatorname{diam}(\sigma), \tag{6.3}$$

$$D_1 < \frac{\operatorname{diam}(\sigma)}{B_\sigma} \leq \frac{\operatorname{diam}(\sigma)}{\beta_\sigma} < D_2. \tag{6.4}$$

Intuitively, the second condition here implies that the simplices in Σ are uniformly far from being degenerate.

Proof. As in [36], there is a decomposition of the open set $\mathbb{R}^m \setminus Z$ into a family of closed dyadic cubes $\{Q_i\}$ with pairwise disjoint interiors so that

1. $\bigcup_{i=1}^\infty Q_i = \mathbb{R}^m \setminus Z$,
2. $\operatorname{diam}(Q_i) \leq d(Q_i, Z) \leq 4\operatorname{diam}(Q_i)$ for every $i \in \mathbb{N}$,
3. for any $i \in \mathbb{N}$, at most 12^m cubes Q_j intersect Q_i non-trivially.

From this cubic decomposition, we will construct the Whitney triangulation inductively as in [85]. The collection of the vertices of the cubes is trivially a 0-dimensional simplicial complex Σ_0. We define Σ_1 by dividing each edge of a Whitney cube into two 1-dimensional simplices (segments) at its midpoint. Fix $k \in \{2, \ldots, m\}$, and suppose a simplicial complex Σ_{k-1} has been constructed on the union of the $(k-1)$-cubes by dividing them into simplices. Choose some k-cube Q in the Whitney decomposition. The union of the faces of Q is the k-skeleton of a subcomplex of Σ_{k-1}. (Recall that the k-skeleton is a subset of $\mathbb{R}^m$ rather than a subset of the simplicial complex.) For each $(k-1)$-simplex in this subcomplex, create a k-simplex by appending the center of Q to the set of its vertices. This provides a simplicial subdivision of Q and thus a simplicial complex Σ_k on the union of the k-cubes. Continuing in this way creates $\Sigma = \Sigma_m$.

Condition (6.3) follows immediately from (A2) since, for any m-cube Q, the diameter of an m-simplex in Q is at least half of the side length of Q. We will say that two simplices in Σ are equivalent if one can be obtained from the other via a rotation, translation, and homothetic dilation. There are only finitely many equivalence classes of simplices in Σ as a result of (A3). Since $\operatorname{diam}(\sigma)/B_\sigma$ and $\operatorname{diam}(\sigma)/\beta_\sigma$ are invariant under rotations, translations, and homothetic dilations, we have (6.4). □

6.2.2 Extensions into a Lipschitz connected metric space

The following Lipschitz extension result will be essential to the construction in the proof of Theorem 66. Though the proof of this extension lemma is elementary and similar to classical results (see for example [3, 54]), it is included here for completeness. Recall that a metric space Y is Lipschitz $(n-1)$-connected if there is a constant $\gamma \geq 1$ so that any L-Lipschitz map $f : S^k \to Y$ $(L > 0)$ has a γL-Lipschitz extension $F : B^{k+1} \to Y$ for $k = 0, 1, \ldots, n-1$.

Lemma 78. *Fix positive integers $m > n$. Suppose Y is Lipschitz $(n-1)$-connected with constant γ, and $Z \subset \mathbb{R}^m$ is closed. Say Σ is the Whitney triangulation of $\mathbb{R}^m \setminus Z$ constructed in Lemma 77. Then there is a constant $\tilde{C} \geq 1$ depending only on m, n, and γ such that every L-Lipschitz map $f : Z \to Y$ has an extension $\tilde{f} : Z \cup \Sigma^{(n)} \to Y$ satisfying the following:*

1. *$\tilde{f}$ is $L\tilde{C}$-Lipschitz on any n-simplex in Σ, and*
2. *for any $a \in \Sigma^{(0)}$, $\tilde{f}(a) = f(z_a)$ for some $z_a \in Z$ with $|z_a - a| = d(a, Z)$.*

Proof. Fix an L-Lipschitz map $f : Z \to Y$. For each $a \in \Sigma^{(0)}$ (that is, each vertex of a simplex in Σ), choose a nearest point $z_a \in Z$ i.e. $|z_a - a| = d(a, Z)$. Define the mapping $f^{(0)} : \Sigma^{(0)} \to Y$ as $f^{(0)}(a) := f(z_a)$. Write $C_0 := D_2(12\sqrt{m} + 1) + 1$ where D_2 is the constant from condition (6.4) in Lemma 77. Fix a 1-simplex σ^1 in Σ (that is, an edge of some m-simplex σ). Write $\partial\sigma^1 = \{a, b\}$. Then

$$
\begin{aligned}
d(f^{(0)}(a), f^{(0)}(b)) &= d(f(z_a), f(z_b)) \leq L|z_a - z_b| \leq L(|z_a - a| + |z_b - b| + |a - b|) \\
&= L(d(a, Z) + d(b, Z) + |a - b|) \leq L(2\, d(\sigma, Z) + 2 \operatorname{diam}(\sigma) + |a - b|) \\
&\leq L((24\sqrt{m} + 2) \operatorname{diam}(\sigma) + |a - b|) < L(D_2(12\sqrt{m} + 1) + 1)|a - b|.
\end{aligned}
$$

since $\beta_\sigma \le \frac{1}{2}|a-b|$. That is, $f^{(0)}$ is LC_0-Lipschitz continuous on $\partial\sigma^1$.

By the Lipschitz connectivity of Y, there is a constant $C_1 > 0$ depending only on C_0, n, and γ (and hence only on m, n, and γ) and an LC_1-Lipschitz extension $f^{(1)} : \sigma^1 \to Y$ of $f^{(0)}$. Since the intersection of any two 1-simplices in Σ is a vertex or empty, we can define a map $f^{(1)} : \Sigma^{(1)} \to Y$ which is LC_1-Lipschitz on any 1-simplex in Σ.

Fix $k \in \{2, \ldots, n\}$. Suppose there is a constant C_{k-1} (depending only on m, n, and γ) and a map $f^{(k-1)} : \Sigma^{(k-1)} \to Y$ so that $f^{(k-1)}$ is LC_{k-1}-Lipschitz on any $(k-1)$-simplex in Σ. Choose a k-simplex σ^k in Σ. We will first determine the Lipschitz constant of $f^{(k-1)}$ restricted to $\partial\sigma^k$. Say $x, y \in \partial\sigma^k$. If x and y lie in the same $(k-1)$-face of σ^k, then $d(f^{(k-1)}(x), f^{(k-1)}(y)) \le LC_{k-1}|x-y|$. Suppose x and y lie in different $(k-1)$-faces σ_x^{k-1} and σ_y^{k-1} of σ^k. We have the following simple lemma.

Lemma 79. *Fix $j \in \{1, \ldots, m-1\}$. There is a constant $\mu \ge 1$ depending only on m satisfying the following: suppose ω_1 and ω_2 are j-faces of a $(j+1)$-simplex $\omega \in \Sigma$, and $x \in \omega_1$ and $y \in \omega_2$. Then there is a point $v \in \omega_1 \cap \omega_2$ so that*

$$|x-v| + |v-y| \le \mu|x-y|. \tag{6.5}$$

Proof. Choose v to be the orthogonal projection of x or y onto $\omega_1 \cap \omega_2$. Since there are only finitely many possible angles at which the faces of the simplices in the Whitney triangulation can meet, the law of sines provides a uniform bound for the ratios $|x-v|/|x-y|$ and $|y-v|/|x-y|$. That is, we may choose μ satisfying (6.5) independent of the choice of faces ω_1 and ω_2 and simplex ω. $\square$

By applying the lemma to the faces σ_x^{k-1} and σ_y^{k-1} of σ^k, we have

$$\begin{aligned} d(f^{(k-1)}(x), f^{(k-1)}(y)) &\le d(f^{(k-1)}(x), f^{(k-1)}(v)) + d(f^{(k-1)}(v), f^{(k-1)}(y)) \\ &\le LC_{k-1}|x-v| + LC_{k-1}|v-y| \le \mu LC_{k-1}|x-y| \end{aligned}$$

since $f^{(k-1)}$ is LC_{k-1}-Lipschitz when restricted to each of σ_x^{k-1} and σ_y^{k-1}. Hence $f^{(k-1)}$ is μLC_{k-1}-Lipschitz on $\partial\sigma^k$. Therefore the Lipschitz connectivity of Y gives a constant C_k depending only on m, n, γ, and C_{k-1} and an LC_k-Lipschitz extension $f^{(k)} : \sigma^k \to Y$ of $f^{(k-1)}$. Since the intersection of any two k-simplices is a lower dimensional simplex (or empty), we may define a mapping $f^{(k)} : \Sigma^{(k)} \to Y$ which is LC_k-Lipschitz on each k-simplex in Σ.

Continuing this construction inductively gives a constant C_n (depending only on m, n, and γ) and a map $f^{(n)} : \Sigma^{(n)} \to Y$ so that $f^{(n)}$ is LC_n-Lipschitz on any n-simplex in Σ. Setting $\tilde{f} := f^{(n)}$ and $\tilde{C} := C_n$ completes the proof. $\square$

6.3 CONSTRUCTING THE SOBOLEV EXTENSION

The proof of Theorem 66 is presented here. We will conclude the section with the proof of Theorem 63. It will follow as a simple consequence of Proposition 74 since the extension we construct will be bounded in $\mathbb{H}^n$.

6.3.1 Extensions into a general Lipschitz connected metric space

Proof of Theorem 66. Fix $1 \leq p < n+1$ and let Ω be a bounded domain in $\mathbb{R}^m$. Suppose Y is a Lipschitz $(n-1)$-connected metric space with constant γ. Let $Z \subset \Omega$ be compact and nonempty, and suppose $f : Z \to Y$ is L-Lipschitz.

If $m \leq n$, then it can be seen from classical results [3, 54] that there is a constant $C = C(n,\gamma)$ and a CL-Lipschitz extension $F : \mathbb{R}^m \to Y$ of f. The proof of this fact is similar to the proof of Lemma 78. Hence $\phi \circ F$ is KCL-Lipschitz for any K-Lipschitz function $\phi : Y \to \mathbb{R}$. Moreover, for $k = 1, \ldots, m$, $\partial(\phi \circ F)/\partial x_k$ exists and is bounded by Kg almost everywhere in Ω where $g : \Omega \to \mathbb{R}$ is the constant function $g \equiv CL$. Thus $F \in AR^{1,p}(\Omega, Y)$, and $\|g\|_{L^p(\Omega)} \leq CL|\Omega|^{1/p} \leq CL\,(\mathrm{diam}\,(\Omega))^{m/p}$ for a constant C depending only on m, n, and γ. We may therefore assume for the remainder of the proof that $m > n$.

Define the Whitney triangulation of $\mathbb{R}^m \backslash Z$ as in Lemma 77. We will restrict our attention to the m-dimensional simplicial sub-complex Σ consisting of those simplices in the Whitney triangulation which are contained in a Whitney cube Q with $Q \cap \Omega \neq \emptyset$. We consider this restriction so that $\sup\{\mathrm{diam}\,(\sigma) : \sigma \in \Sigma\} < \infty$ (since Ω is bounded). Note also that $\Omega \setminus Z \subset \Sigma^{(m)}$.

Suppose σ is an m-simplex in Σ. We begin by constructing a sort of radial projection of σ onto its n-skeleton. This is the so called zero degree homogenization mentioned earlier. Denote by c the barycenter of σ. For each $j \in \{1, \ldots, m\}$, say $\{\sigma_i^j\}_{i=1}^{B(m,j)}$ is the collection of j-faces of σ, and say c_i^j is the barycenter of σ_i^j. (Notice $\sigma_1^m = \sigma$ and $c_1^m = c$.) Fix $j \in \{n+1, \ldots, m\}$. For each $i \in \{1, \ldots, B(m,j)\}$, define $P_i^j : \sigma_i^j \setminus \{c_i^j\} \to \partial\sigma_i^j$ to be the projection of $\sigma_i^j \setminus \{c_i^j\}$ onto $\partial\sigma_i^j$ radially out from c_i^j. That is, for $x \in \sigma_i^j \setminus \{c_i^j\}$ if we write $x = c_i^j + t(z - c_i^j)$ with $t \in (0,1]$ and $z \in \partial\sigma_i^j$, then $P_i^j(x) = z$. Fix $x \in \sigma_i^j \setminus \{c_i^j\}$. For all $y \in \sigma_i^j \setminus \{c_i^j\}$ close enough to x, we have by similar triangles

$$\frac{|P_i^j(x) - P_i^j(y)|}{|x-y|} \le \nu \frac{\operatorname{diam}(\sigma)}{|x - c_i^j|}. \tag{6.6}$$

The constant $\nu > 0$ depends only on the dimension m since there are only finitely many equivalence classes of simplices in Σ. In particular, P_i^j is locally Lipschitz on $\sigma_i^j \setminus \{c_i^j\}$. Extend P_i^j to the remaining j-skeleton of σ by the identity map (that is, $P_i^j(x) = x$ for any $x \in \Sigma_\sigma^{(j)} \setminus \sigma_i^j$). Writing $C^j = \{c_1^j, \ldots, c_{B(m,j)}^j\}$, we may define $P^j : \Sigma_\sigma^{(j)} \setminus C^j \to \Sigma_\sigma^{(j-1)}$ as $P^j := P_1^j \circ \cdots \circ P_{B(m,j)}^j$. By arguing in a similar manner to Lemma 79, each P^j is locally Lipschitz on $\Sigma_\sigma^{(j)} \setminus C^j$.

In particular, P^m is locally Lipschitz on $\sigma \setminus c$. Now $P^{m-1} \circ P^m$ is defined and locally Lipschitz on σ away from the 1-dimensional set $\{c\} \cup (P^m)^{-1}(C^{m-1})$. Similarly, $P^{m-2} \circ P^{m-1} \circ P^m$ is locally Lipschitz away from the 2-dimensional set $\{c\} \cup (P^m)^{-1}(C^{m-1}) \cup (P^{m-1} \circ P^m)^{-1}(C^{m-2})$. Continuing in this way, we see that $P_\sigma := P^{n+1} \circ \cdots \circ P^m : \sigma \setminus C_\sigma \to \Sigma_\sigma^{(n)}$ is locally Lipschitz off the closed, $(m-n-1)$-dimensional set of singularities

$$C_\sigma := \{c\} \cup \bigcup_{\ell=1}^{m-(n+1)} (P^{m-\ell+1} \circ \cdots \circ P^m)^{-1}(C^{m-\ell}).$$

We will now build the extension F of f. First, construct the extension $\tilde{f} : Z \cup \Sigma^{(n)} \to Y$ of f given in Lemma 78. Recall that $\tilde{f}$ is $\tilde{C}L$-Lipschitz on any n-simplex in Σ. In particular, $\tilde{f}$ is locally Lipschitz on $\Sigma^{(n)}$. Enumerate the collection of m-simplices $\{\sigma_i\}_{i=1}^\infty$ in Σ, and write $\mathscr{C} = \bigcup_i C_{\sigma_i}$. Define $F : \Sigma^{(m)} \cup Z \to Y$ as

$$F(x) = \begin{cases} \tilde{f}(P_{\sigma_i}(x)) & \text{if } x \in \sigma_i \setminus C_{\sigma_i} \text{ for some } i \in \mathbb{N} \\ f(x) & \text{if } x \in Z \end{cases}$$

and define F to be constant on $\mathscr{C}$. This map is well defined since the intersection $\sigma_i \cap \sigma_j$ is either empty or another simplex in Σ. Moreover, F is locally Lipschitz on each $\sigma_i \setminus C_{\sigma_i}$. We now have the following

Lemma 80. *Suppose* $1 \leq p < n+1$. *Define* $g : \Sigma^{(m)} \setminus \mathscr{C} \to [0, \infty]$ *as*

$$g(x) = \limsup_{y \to x} \frac{d(F(x), F(y))}{|x-y|}.$$

Then $\|g\|_{L^p(\Omega \setminus Z)} \leq CL(\operatorname{diam}(\Omega))^{m/p}$ *for a constant* $C > 0$ *depending only on* m*,* n*,* p*, and* γ*. In particular,* $g \in L^p(\Omega \setminus Z)$*.*

The proof of this lemma is long but elementary. It is contained, therefore, at the end of this section. Extend g to all of Ω so that $g \equiv L(\tilde{C} + 4)$ on $Z \cup \mathscr{C}$. Thus $g \in L^p(\Omega)$ and $\|g\|_{L^p(\Omega)} \leq CL(\operatorname{diam}(\Omega))^{m/p}$ for a constant $C = C(m, n, p, \gamma)$.

It remains to show that F is in the class $AR^{1,p}(\Omega, Y)$. Fix a K-Lipschitz function $\phi : Y \to \mathbb{R}$. We will first show that $\phi \circ F \in L^p(\Omega)$. Let $x \in \Omega \setminus (Z \cup \mathscr{C})$. Then $x \in \sigma_i$ for some $i \in \mathbb{N}$. Choose a vertex a of σ_i so that a and $P_{\sigma_i}(x)$ lie in the same n-face of σ_i. Since $F(a) = \tilde{f}(a) = f(z_a)$ as prescribed in Lemma 78, we have

$$|\phi(F(x))| \leq |\phi(F(x)) - \phi(F(a))| + |\phi(f(z_a))| \leq KL\tilde{C} \operatorname{diam}(\sigma_i) + \|\phi \circ f\|_\infty < M$$

for some $M > 0$. Since Z is compact and Ω is bounded, $\phi \circ F \in L^p(\Omega)$.

Now, we will use the ACL characterization of Sobolev mappings to show that $\phi \circ F \in W^{1,p}(\Omega)$. Fix $k \in \{1, \ldots, m\}$. Notice that $(m-1)$-almost every line parallel to the k^{th} coordinate axis is disjoint from $\mathscr{C}$ since each C_{σ_i} is $(m-n-1)$-dimensional. Also, g and $\phi \circ F$ are p-integrable in Ω along $(m-1)$-almost every such line since g and $\phi \circ F$ are in the class $L^p(\Omega)$.

Choose a line $\bar{\ell}$ parallel to the k^{th} coordinate axis that is disjoint from $\mathscr{C}$ and suppose that $g \in L^p(\bar{\ell} \cap \Omega)$, and $\phi \circ F \in L^p(\bar{\ell} \cap \Omega)$. Write $\ell := \bar{\ell} \cap \Omega$. We will now show that $\phi \circ F$ is locally Lipschitz along $\ell \setminus Z$ and its derivative along $\ell \setminus Z$ is p-integrable. Choose $x \in \ell \setminus Z$. We need only consider the case when $x \in \partial\sigma_i$ for some $i \in \mathbb{N}$ since F is locally Lipschitz on each $\sigma_i \setminus C_{\sigma_i}$. In this case, for some $a, b \in \ell$, the segments $[a, x]$ and $[x, b]$ each lie entirely in some m-simplices σ_a and σ_b respectively. Since F is locally Lipschitz when restricted to each of these simplices, it follows that F is Lipschitz along some segment $I \subset [a,b]$ containing x. Therefore, F is locally Lipschitz on $\ell \setminus Z$, and hence $\phi \circ F$ is as well. Now $\partial(\phi \circ F)/\partial x_k$ exists almost everywhere along $\ell \setminus Z$, and the definition of g gives

$$\left|\frac{\partial(\phi \circ F)}{\partial x_k}(x)\right| \leq K \left[\limsup_{h \to 0} \frac{d(F(x+he_k), F(x))}{|h|}\right] \leq K\, g(x)$$

for every $x \in \ell \setminus Z$ at which the partial derivative exists. In particular, $\partial(\phi \circ F)/\partial x_k \in L^p(\ell \setminus Z)$.

Next, we will see that $\phi \circ F$ is in fact continuous along all of ℓ. By the previous paragraph, F is continuous along ℓ at any $x \in \ell \setminus Z$. Suppose now that $x \in \ell \cap Z$. If $y \in \ell \cap Z$, then $d(F(x), F(y)) \leq L|x-y|$. Suppose instead that $y \in \ell \setminus Z$. Then $y \in \sigma_i$ for some $i \in \mathbb{N}$. Choose a vertex a of σ_i so that a and $P_{\sigma_i}(y)$ lie in the same n-face of σ_i. Then

$$\begin{aligned} d(F(y), F(a)) &= d(\tilde{f}(P_{\sigma_i}(y)), \tilde{f}(a)) \\ &\leq L\tilde{C}|P_{\sigma_i}(y) - a| \leq L\tilde{C}\operatorname{diam}(\sigma_i) \leq L\tilde{C}d(\sigma_i, Z) \leq L\tilde{C}|x-y|. \end{aligned}$$

Also, since $F(a) = f(z_a)$,

$$\begin{aligned} d(F(a), F(x)) &= d(f(z_a), f(x)) \leq L(|z_a - a| + |a-y| + |y-x|) \\ &\leq L(d(a,Z) + \operatorname{diam}(\sigma_i) + |x-y|) \\ &\leq L((d(\sigma_i, Z) + \operatorname{diam}(\sigma_i)) + d(\sigma_i, Z) + |x-y|) < 4L|x-y|. \end{aligned}$$

Therefore,

$$d(F(x), F(y)) \leq L(\tilde{C} + 4)|x-y| \tag{6.7}$$

for any $x \in \ell \cap Z$ and $y \in \ell$. That is, F is continuous on ℓ, and so $\phi \circ F$ is as well.

Finally, we will show that $\phi \circ F$ is absolutely continuous on any compact interval in ℓ as desired. Since $(\phi \circ f)|_{\ell \cap Z}$ is Lipschitz, we may use the classical McShane extension [69] to find a Lipschitz extension $\psi : \ell \to \mathbb{R}$ of $(\phi \circ f)\big|_{\ell \cap Z}$. Set $v := (\phi \circ F) - \psi$ on ℓ. Notice that v' exists almost everywhere on $\ell \setminus Z$, and $v' \in L^p(\ell \setminus Z)$. Moreover, v is continuous on ℓ, is absolutely continuous on compact intervals in $\ell \setminus Z$, and vanishes on $\ell \cap Z$. Therefore, by defining

$$w(x) = \begin{cases} v'(x) & \text{if } x \in \ell \setminus Z \text{ and } v'(x) \text{ exists} \\ 0 & \text{if } x \in \ell \cap Z \text{ or } v'(x) \text{ does not exist,} \end{cases}$$

v is the integral of w over any interval in ℓ. Since w is integrable on ℓ, it follows that v is absolutely continuous on compact intervals in ℓ, and so $\phi \circ F = v + \psi$ is as well. Therefore, $\phi \circ F \in ACL^p(\Omega)$.

Furthermore, the definition of g together with (6.7) gives $|\partial(\phi \circ F)/\partial x_k| \leq Kg$ almost everywhere along ℓ. Hence, given any K-Lipschitz $\phi : Y \to \mathbb{R}$, we have $\phi \circ F \in W^{1,p}(\Omega)$ and $|\partial(\phi \circ F)/\partial x_k| \leq K\, g$ almost everywhere in Ω for $k = 1, \ldots, m$. We may thus conclude that $F \in AR^{1,p}(\Omega, Y)$. □

6.3.2 Extensions into the Heisenberg group

We are now ready for the proof of Theorem 63. Recall from the discussion at the beginning of the chapter that $\mathbb{H}^n$ is Lipschitz $(n-1)$-connected [91]. According to Proposition 74, we need only prove that the extension F constructed in the previous proof is bounded as a mapping into $\mathbb{H}^n$ and then prove the desired quantitative estimates.

Proof of Theorem 63. Suppose $Y = \mathbb{H}^n$. Fix $x \in \Omega$. Notice that $\|F(\cdot)\|_K$ is bounded on Z since $F\big|_Z = f$ is Lipschitz. Also, F is constant on $\mathscr{C}$. It therefore suffices to consider $x \in \Omega \setminus (Z \cup \mathscr{C})$. Hence $x \in \sigma$ for some m-simplex $\sigma \in \Sigma$. Choose a vertex a of σ so that a and $P_\sigma(x)$ lie in the same n-face of σ. Then there is some $M > 0$ independent of x so that

$$\|F(x)\|_K \leq C d_{cc}(F(x), F(a)) + \|F(a)\|_K \leq CL\tilde{C}\mathrm{diam}\,(\sigma) + \|f(z_a)\|_K < M$$

where C is the constant from the bi-Lipschitz equivalence of d_{cc} and d_K. Thus $F \in AR^{1,p}(\Omega, \mathbb{H}^n)$ is bounded, so, by Proposition 74, $F \in W^{1,p}(\Omega, \mathbb{H}^n)$.

We now establish the quantitative estimate. Recall that $\|g\|_{L^p(\Omega)} \leq CL(\mathrm{diam}\,(\Omega))^{m/p}$ where g was defined in the proof of Theorem 66. For $j \in \{1, \ldots, 2n\}$, suppose $\phi_j : \mathbb{H}^n \to \mathbb{R}$ is the projection onto the j^{th} coordinate. Since ϕ_j is 1-Lipschitz, the definition of $AR^{1,p}(\Omega, \mathbb{H}^n)$

gives $|\partial(\phi_j \circ F)/\partial x_k| \le g$ almost everywhere on Ω, so $\|\partial F_j/\partial x_k\|_{L^p(\Omega)} \le CL(\mathrm{diam}\,(\Omega))^{m/p}$ for $k = 1, \ldots, m$ and $j = 1, \ldots, 2n$. $\square$

6.3.3 Integrability of g

We present here a detailed proof of the L^p integrability of the bounding function g. As mentioned above, this proof is technical but elementary.

Proof. Suppose σ is an m-simplex in Σ. For the sake of notation, we will write $\Phi^k := \tilde{f} \circ P^{n+1} \circ \cdots \circ P^k$ for $k \in \{n+1, \ldots, m\}$ where each P^k is the radial projection of $\Sigma_\sigma^{(k)} \setminus C^k$ to $\Sigma_\sigma^{(k-1)}$ as defined earlier. As before, for $j = 1, \ldots, m$, say $\{\sigma_i^j\}_{i=1}^{B(m,j)}$ is the collection of j-faces of σ. We will prove this lemma by induction on the dimensions of the faces of σ. In particular, we will use the Fubini theorem to bound the integral of the "slope" of Φ^k by a bound on the integral of the "slope" of Φ^{k-1}. This will allow us to bound the integral of g (which is the "slope" of $\Phi^m = F$).

We begin with the $(n+1)$-faces of σ. Suppose $x \in \sigma_i^{n+1} \setminus \{c_i^{n+1}\}$ for some $i \in \{1, \ldots, B(m, n+1)\}$. If $x \notin \partial\sigma_i^{n+1}$, then for any $y \in \Sigma_\sigma^{(n+1)}$ close enough to x, in fact $y \in \sigma_i^{n+1}$ and $P^{n+1}(x)$ and $P^{n+1}(y)$ lie in the same n-face of σ_i^{n+1}. In this case (6.6) gives

$$\frac{d(\tilde{f}(P^{n+1}(x)), \tilde{f}(P^{n+1}(y)))}{|x-y|} \le L\tilde{C}\,\frac{|P_i^{n+1}(x) - P_i^{n+1}(y)|}{|x-y|} \le \nu L\tilde{C}\,\frac{\mathrm{diam}\,(\sigma)}{|x - c_i^{n+1}|}.$$

for $y \in \Sigma_\sigma^{(n+1)}$ close enough to x. Since each $\partial\sigma_i^{n+1}$ has $\mathcal{H}^{n+1}$ measure zero,

$$\begin{aligned}
&\int_{\Sigma_\sigma^{(n+1)}} \limsup_{y\to x,\, y\in\Sigma_\sigma^{(n+1)}} \frac{d(\Phi^{n+1}(x), \Phi^{n+1}(y))^p}{|x-y|^p}\, d\mathcal{H}^{n+1}(x) \\
&\qquad = \sum_{i=1}^{B(m,n+1)} \int_{\sigma_i^{n+1}\setminus\partial\sigma_i^{n+1}} \limsup_{y\to x,\, y\in\sigma_i^{n+1}} \frac{d(\tilde{f}(P^{n+1}(x)), \tilde{f}(P^{n+1}(y)))^p}{|x-y|^p}\, d\mathcal{H}^{n+1}(x) \\
&\qquad \le (\nu L\tilde{C})^p \sum_{i=1}^{B(m,n+1)} \int_{\sigma_i^{n+1}} \frac{\mathrm{diam}\,(\sigma)^p}{|x - c_i^{n+1}|^p}\, d\mathcal{H}^{n+1}(x).
\end{aligned}$$

In what follows, the constant C may change value between lines in the inequalities but will depend only on m, n, p, and γ. We first estimate the integral over each $(n+1)$-face of σ. Since $p < n+1$, we have

$$\int_{\sigma_i^{n+1}} \frac{1}{|x - c_i^{n+1}|^p}\, d\mathcal{H}^{n+1}(x) \le C\,\mathcal{H}^{n+1}\left(\sigma_i^{n+1}\right)^{1-\frac{p}{n+1}} \le C\,\mathrm{diam}\,(\sigma)^{n+1-p}.$$

Therefore, on the entire $(n+1)$-skeleton, we have

$$\int_{\Sigma_\sigma^{(n+1)}} \limsup_{y\to x,\, y\in\Sigma_\sigma^{(n+1)}} \frac{d(\Phi^{n+1}(x),\Phi^{n+1}(y))^p}{|x-y|^p}\, d\mathcal{H}^{n+1}(x) \le L^p\, C\, \mathrm{diam}\,(\sigma)^{n+1}.$$

Now suppose $k\in\{n+1,\dots,m-1\}$ satisfies the following for a constant C depending only on m, n, p, and γ:

$$\int_{\Sigma_\sigma^{(k)}} \limsup_{y\to x,\, y\in\Sigma_\sigma^{(k)}} \frac{d(\Phi^{k}(x),\Phi^{k}(y))^p}{|x-y|^p}\, d\mathcal{H}^{k}(x) \le L^p\, C\, \mathrm{diam}\,(\sigma)^{k}.$$

We have as before

$$\begin{aligned}
&\int_{\Sigma_\sigma^{(k+1)}} \limsup_{y\to x,\, y\in\Sigma_\sigma^{(k+1)}} \frac{d(\Phi^{k+1}(x),\Phi^{k+1}(y))^p}{|x-y|^p}\, d\mathcal{H}^{k+1}(x)\\
&\le \sum_{i=1}^{B(m,k+1)} \int_{\sigma_i^{k+1}\setminus\partial\sigma_i^{k+1}} \limsup_{y\to x,\, y\in\sigma_i^{k+1}} \frac{d(\Phi^k(P^{k+1}(x)),\Phi^k(P^{k+1}(y)))^p}{|P^{k+1}(x)-P^{k+1}(y)|^p}\cdot\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\frac{|P^{k+1}(x)-P^{k+1}(y)|^p}{|x-y|^p}\, d\mathcal{H}^{k+1}(x).
\end{aligned}$$

Fix $i\in\{1,\dots,B(m,k+1)\}$. As before, we estimate the integral over each $(k+1)$-face of the simplex σ. Without loss of generality (after a translation), we may assume σ_i^{k+1} is centered at the origin. We thus have by (6.6)

$$\begin{aligned}
&\int_{\sigma_i^{k+1}\setminus\partial\sigma_i^{k+1}} \limsup_{y\to x,\, y\in\sigma_i^{k+1}} \frac{d(\Phi^k(P^{k+1}(x)),\Phi^k(P^{k+1}(y)))^p}{|P^{k+1}(x)-P^{k+1}(y)|^p}\frac{|P^{k+1}(x)-P^{k+1}(y)|^p}{|x-y|^p}\, d\mathcal{H}^{k+1}(x)\\
&\le \nu^p \int_{\sigma_i^{k+1}} \limsup_{y\to x,\, y\in\sigma_i^{k+1}} \frac{d(\Phi^k(P^{k+1}(x)),\Phi^k(P^{k+1}(y)))^p}{|P^{k+1}(x)-P^{k+1}(y)|^p}\,\frac{\mathrm{diam}\,(\sigma)^p}{|x|^p}\, d\mathcal{H}^{k+1}(x)\\
&\le \nu^p \sum_{q=1}^{k+2} \int_{(P_i^{k+1})^{-1}(\sigma_{i_q}^k)} \limsup_{y\to x,\, y\in\sigma_i^{k+1}} \frac{d(\Phi^k(P^{k+1}(x)),\Phi^k(P^{k+1}(y)))^p}{|P^{k+1}(x)-P^{k+1}(y)|^p}\,\frac{\mathrm{diam}\,(\sigma)^p}{|x|^p}\, d\mathcal{H}^{k+1}(x)
\end{aligned}$$

where $\sigma_{i_1}^k,\dots,\sigma_{i_{k+2}}^k$ are the k-dimensional faces of σ_i^{k+1}. We will compute the integral of each summand in the last line. Fix $q\in\{1,\dots,k+2\}$. The integral is invariant up to rotation, so we may assume without loss of generality that $\sigma_{i_q}^k$ is contained in the k-plane $\{b\}\times\mathbb{R}^k$.

Thus we may consider $(P_i^{k+1})^{-1}(\sigma_{i_q}^k)$ a subset of $\mathbb{R}^{k+1}$. Write $\hat{\sigma}_{i_q}^k = \{\hat{z} : (b, \hat{z}) \in \sigma_{i_q}^k\} \subset \mathbb{R}^k$ so that $(P_i^{k+1})^{-1}(\sigma_{i_q}^k) = \{(t, \hat{x}) : \hat{x} \in \frac{t}{b}\hat{\sigma}_{i_q}^k,\, t \in (0, b]\}$. Thus since

$$\limsup_{y\to x,\, y\in\sigma_i^{k+1}} \frac{d(\Phi^k(P^{k+1}(x)), \Phi^k(P^{k+1}(y)))}{|P^{k+1}(x) - P^{k+1}(y)|} \leq \limsup_{z\to P^{k+1}(x),\, z\in\sigma_{i_q}^k} \frac{d(\Phi^k(P^{k+1}(x)), \Phi^k(z))}{|P^{k+1}(x) - z|}$$

for any $x \in (P_i^{k+1})^{-1}(\sigma_{i_q}^k \setminus \partial\sigma_{i_q}^k)$ and since $(P_i^{k+1})^{-1}(\partial\sigma_{i_q}^k)$ has $\mathcal{H}^{k+1}$ measure zero, we have

$$\begin{aligned}
\int_{(P_i^{k+1})^{-1}(\sigma_{i_q}^k)} &\limsup_{y\to x,\, y\in\sigma_i^{k+1}} \frac{d(\Phi^k(P^{k+1}(x)), \Phi^k(P^{k+1}(y)))^p}{|P^{k+1}(x) - P^{k+1}(y)|^p} \frac{\operatorname{diam}(\sigma)^p}{|x|^p}\, d\mathcal{H}^{k+1}(x) \\
&\leq \int_0^b \int_{\frac{t}{b}\hat{\sigma}_{i_q}^k} \limsup_{\hat{z}\to\frac{b}{t}\hat{x},\, \hat{z}\in\hat{\sigma}_{i_q}^k} \frac{d(\Phi^k(b, \frac{b}{t}\hat{x}), \Phi^k(b, \hat{z}))^p}{|(b, \frac{b}{t}\hat{x}) - (b, \hat{z})|^p} \frac{\operatorname{diam}(\sigma)^p}{|(t, \hat{x})|^p}\, d\mathcal{H}^k(\hat{x})\, dt \\
&= \int_0^b \int_{\hat{\sigma}_{i_q}^k} \left(\frac{t}{b}\right)^k \limsup_{\hat{z}\to\hat{x},\, \hat{z}\in\hat{\sigma}_{i_q}^k} \frac{d(\Phi^k(b, \hat{x}), \Phi^k(b, \hat{z}))^p}{|(b, \hat{x}) - (b, \hat{z})|^p} \frac{\operatorname{diam}(\sigma)^p}{\left(\frac{t}{b}\right)^p |(b, \hat{x})|^p}\, d\mathcal{H}^k(\hat{x})\, dt \\
&\leq \int_0^b \left(\frac{t}{b}\right)^{k-p} dt \int_{\sigma_{i_q}^k} \limsup_{z\to x,\, z\in\sigma_{i_q}^k} \frac{d(\Phi^k(x), \Phi^k(z))^p}{|x - z|^p} \frac{\operatorname{diam}(\sigma)^p}{b^p}\, d\mathcal{H}^k(x) \\
&\leq \left(\frac{\operatorname{diam}(\sigma)}{b}\right)^p b\, L^p\, C \operatorname{diam}(\sigma)^k
\end{aligned}$$

since $k - p > 0$. Since $b \geq \beta_\sigma$ and $b \leq \operatorname{diam}(\sigma)$, we may use (6.4) to conclude on the $(k+1)$-skeleton of σ

$$\int_{\Sigma_\sigma^{(k+1)}} \limsup_{y\to x,\, y\in\Sigma_\sigma^{(k+1)}} \frac{d(\Phi^{k+1}(x), \Phi^{k+1}(y))^p}{|x - y|^p}\, d\mathcal{H}^{k+1}(x) \leq L^p\, C \operatorname{diam}(\sigma)^{k+1}.$$

By way of induction, then, we have

$$\int_\sigma g(x)^p\, d\mathcal{H}^m(x) = \int_{\sigma\setminus\partial\sigma} \limsup_{y\to x} \frac{d(\Phi^m(x), \Phi^m(y))^p}{|x - y|^p}\, d\mathcal{H}^m(x) \leq L^p\, C \operatorname{diam}(\sigma)^m$$

since $\Sigma_\sigma^{(m)} = \sigma$ and $\Phi^m = F$ on σ. Therefore, we have

$$\int_{\Omega\setminus Z} g(x)^p\, dx \leq \sum_{i=1}^\infty \int_{\sigma_i} g(x)^p\, dx \leq L^p\, C \sum_{i=1}^\infty \operatorname{diam}(\sigma_i)^m.$$

The number of m-simplices in each cube in the Whitney decomposition of $\mathbb{R}^m \setminus Z$ is bounded by a constant C depending only on m. Hence

$$\sum_{i=1}^\infty \operatorname{diam}(\sigma_i)^m = \sum_Q \sum_{\sigma\subset Q} \operatorname{diam}(\sigma)^m \leq \sum_Q \sum_{\sigma\subset Q} \operatorname{diam}(Q)^m \leq C \sum_Q \operatorname{diam}(Q)^m$$

$$\leq C\mathcal{H}^m(\Sigma^{(m)})$$

where these sums are taken over all cubes Q in the Whitney decomposition that meet Ω. Notice that, for any $x, y \in \Sigma^{(m)}$ and cubes Q_x and Q_y containing them, we have

$$\begin{aligned}|x - y| &\leq \operatorname{diam}(Q_x) + d(Q_x, Q_y) + \operatorname{diam}(Q_y) \\ &\leq d(Q_x, Z) + d(Q_x, Q_y) + d(Q_y, Z) \leq 3\operatorname{diam}(\Omega).\end{aligned}$$

Therefore, $\mathcal{H}^m(\Sigma^{(m)}) \leq C \operatorname{diam}(\Sigma^{(m)})^m \leq C(\operatorname{diam}(\Omega))^m$, and so

$$\|g\|_{L^p(\Omega\setminus Z)} \leq CL(\operatorname{diam}(\Omega))^{m/p}$$

for a constant $C > 0$ depending only on m, n, p, and the Lipschitz connectivity constant γ of Y. In particular, $g \in L^p(\Omega \setminus Z)$. $\square$

7.0 THE DUBOVITSKII-SARD THEOREM IN SOBOLEV SPACES

This chapter is based on the paper [47]. Originally proven in 1942, Arthur Sard's [79] famous theorem asserts that the set of critical values of a sufficiently regular mapping is null. We will use the following notation to represent the *critical set* of a given smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$C_f = \{x \in \mathbb{R}^n \mid \operatorname{rank} Df(x) < m\}.$$

Throughout this chapter, we will assume that m and n are integers at least 1.

Theorem 81 (Sard)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k. If $k > \max(n-m, 0)$, then*

$$\mathcal{H}^m(f(C_f)) = 0.$$

Several results have shown that Sard's result is optimal, see e.g. [24, 37, 43, 55, 66, 93]. In 1957 Dubovitskiĭ [24], extended Sard's theorem to all orders of smoothness k. See [12] for a modernized proof of this result and some generalizations.

Theorem 82 (Dubovitskiĭ)**.** *Fix $n, m, k \in \mathbb{N}$. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k. Write $\ell = \max(n-m-k+1, 0)$. Then*

$$\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0 \quad \textit{for a.e. } y \in \mathbb{R}^m.$$

This result tells us that almost every level set of a smooth mapping intersects with its critical set on an ℓ-null set. Higher regularity of the function implies a reduction in the Hausdorff dimension of the overlap between $f^{-1}(y)$ and C_f for a.e. $y \in \mathbb{R}^m$.

Notice that if $k > \max(n-m, 0)$, then $n-m-k+1 \le 0$, and so $\mathcal{H}^\ell = \mathcal{H}^0$ is simply the counting measure on $\mathbb{R}^n$. That is, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k and additionally $k > \max(n-m, 0)$, Dubovitskiĭ's theorem implies that $f^{-1}(y) \cap C_f$ is empty for almost every $y \in \mathbb{R}^m$. In other words, $\mathcal{H}^m(f(C_f)) = 0$. Thus Sard's theorem is a special case of Dubovitskiĭ's theorem.

7.1 HISTORY IN THE CASE OF SOBOLEV MAPPINGS

Recently, many mathematicians have worked to generalize Sard's result to the class of Sobolev mappings [2, 12, 13, 14, 21, 31, 59, 87]. Specifically, in 2001 De Pascale [21] proved the following version of Sard's theorem for Sobolev mappings.

Theorem 83. *Suppose $k > \max(n-m, 0)$. Suppose $\Omega \subset \mathbb{R}^n$ is open. If $f \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for $n < p < \infty$, then $\mathcal{H}^m(f(C_f)) = 0$.*

The purpose of this chapter is to show that also the Dubovitskiĭ theorem generalizes to the case of $W^{k,p}_{\mathrm{loc}}$ mappings when $n < p < \infty$. We must be very careful when dealing with Sobolev mappings because the set $f^{-1}(y)$ depends on what representative of f we take. If $k \geq 2$, then Morrey's inequality implies that f has a representative of class $C^{k-1,1-\frac{n}{p}}$, so the critical set C_f is well defined. If $k = 1$, then Df is only defined almost everywhere and hence the set C_f is defined up to a set of measure zero. We will say that f is *precisely represented* if each component f_i of f satisfies

$$f_i(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_i(y)\, dy$$

for all $x \in \Omega$ at which this limit exists. The Lebesgue differentiation theorem ensures that this is indeed a well defined representative of f. In what follows, we will always refer to the $C^{k-1,1-\frac{n}{p}}$ representative of f when $k \geq 2$ and a precise representation of f when $k = 1$. (Notice that the precise representative of f and the smooth representative of f are the same for $k \geq 2$.)

The main result of the chapter reads as follows.

Theorem 84. *Fix $n, m, k \in \mathbb{N}$. Suppose $\Omega \subset \mathbb{R}^n$ is open and $f \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for some $n < p < \infty$. If $\ell = \max(n-m-k+1, 0)$, then*

$$\mathcal{H}^{\ell}(C_f \cap f^{-1}(y)) = 0 \quad \textit{for a.e. } y \in \mathbb{R}^m.$$

The proof of this is the subject of Section 7.3. If $m > n$, then since $p > n$ we may apply Morrey's inequality combined with Hölder's inequality to show that $\mathcal{H}^n(f(Q)) < \infty$ for any cube $Q \Subset \Omega$, and so $\mathcal{H}^m(f(\Omega)) = 0$. Thus $f^{-1}(y)$ is empty for almost every $y \in \mathbb{R}^m$, and the theorem follows.

We will now discuss the details behind the argument that $\mathcal{H}^n(f(Q)) < \infty$ for any cube $Q \Subset \Omega$. Fix $\delta > 0$, and cover Q with $2^{n\nu}$ congruent dyadic cubes $\{Q_j\}_{j=1}^{2^{n\nu}}$ with pairwise disjoint interiors. According to Morrey's inequality (see Lemma 89),

$$\operatorname{diam} f(Q_j) \le C(\operatorname{diam} Q_j)^{1-\frac{n}{p}} \left(\int_{Q_j} |Df(z)|^p \, dz \right)^{1/p}$$

for every $1 \le j \le 2^{n\nu}$. Since $\operatorname{diam} Q_j = 2^{-\nu} \operatorname{diam} Q$, choosing ν large enough gives $\sup_j \operatorname{diam} f(Q_j) < \delta$, and so we can estimate the pre-Hausdorff measure

$$\begin{aligned}
\mathcal{H}^n_\delta(f(Q)) &\le C \sum_{j=1}^{2^{n\nu}} (\operatorname{diam} f(Q_j))^n \\
&\le C \sum_{j=1}^{2^{n\nu}} (\operatorname{diam} Q_j)^{n(1-\frac{n}{p})} \left(\int_{Q_j} |Df(z)|^p \, dz \right)^{n/p} \\
&\le C \left(\sum_{j=1}^{2^{n\nu}} (\operatorname{diam} Q_j)^n \right)^{1-\frac{n}{p}} \left(\sum_{j=1}^{2^{n\nu}} \int_{Q_j} |Df(z)|^p \, dz \right)^{n/p} \\
&\le C \mathcal{H}^n(Q)^{1-\frac{n}{p}} \left(\int_Q |Df(z)|^p \, dz \right)^{n/p}.
\end{aligned}$$

We used Hölder's inequality with exponents p/n and $p/(p-n)$ to obtain the third line. Since the right hand estimate does not depend on δ, sending $\delta \to 0^+$ yields $\mathcal{H}^n(f(Q)) < \infty$. This completes the proof of Theorem 84 when $m > n$. Hence we may assume that $m \le n$.

We will now discuss the case $k = 1$ to avoid any confusion involving the definition of C_f. Since $m \le n$, we may apply the following co-area formula due to Malý, Swanson, and Ziemer [66]:

Theorem 85. *Suppose that $1 \leq m \leq n$, $\Omega \subset \mathbb{R}^n$ is open, $p > m$, and $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is precisely represented. Then the following holds for all measurable $E \subset \Omega$:*

$$\int_E |J_m f(x)|\, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(E \cap f^{-1}(y))\, dy$$

where $|J_m f|$ is the square root of the sum of the squares of the determinants of the $m \times m$ minors of Df.

Notice that $|J_m f|$ is equals zero almost everywhere on the set $E = C_f$. Therefore the above equality with $E = C_f$ reads

$$0 = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(C_f \cap f^{-1}(y))\, dy = \int_{\mathbb{R}^m} \mathcal{H}^{\ell}(C_f \cap f^{-1}(y))\, dy.$$

That is, $\mathcal{H}^{\ell}(C_f \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$, and the theorem follows. Therefore, we may assume for the remainder of the chapter that $m \leq n$ and $k \geq 2$.

Most proofs of Sard-type results typically involve some form of a Morse Theorem [74] in which the critical set of a mapping is decomposed into pieces on which the function's difference quotients converge quickly. See [83] for the proof of the classical Sard theorem based on this method. A version of the Morse Theorem was also used by De Pascale [21]. However, there is another approach to the Sard theorem based on the so called Kneser-Glaeser Rough Composition theorem, and this method entirely avoids the use of the Morse theorem. We say that a mapping $f : W \subset \mathbb{R}^r \to \mathbb{R}$ of class C^k is *s-flat* on $A \subset W$ for $1 \leq s \leq k$ if $D^\alpha f = 0$ on A for every $1 \leq |\alpha| \leq s$.

Theorem 86 (Kneser-Glaeser Rough Composition)**.** *Fix positive integers s, k, r, n with $s < k$. Suppose $V \subset \mathbb{R}^r$ and $W \subset \mathbb{R}^n$ are open. Let $g : V \to W$ be of class C^{k-s} and $f : W \to \mathbb{R}$ be of class C^k. Suppose $A^* \subset V$ and $A \subset W$ are compact sets with*

1. *$g(A^*) \subset A$ and*
2. *f is s-flat on A.*

Then there is a function $F : \mathbb{R}^r \to \mathbb{R}$ of class C^k so that $F = f \circ g$ on A^* and F is s-flat on A^*.

This theorem ensures that the composition of two smooth maps will have the same regularity as the second function involved in the composition provided that enough of the derivatives of this second function are zero. After a brief examination of the rule for differentiation of composite functions, such a conclusion seems very natural. Indeed, we can formally compute $D^\alpha(f\circ g)(x)$ for all $|\alpha|\leq k$ and $x\in A^*$ since any "non-existing" derivative $D^\beta g(x)$ with $|\beta|>k-s$ is multiplied by a vanishing $D^\gamma f(g(x))$ term with $|\gamma|=|\alpha|-|\beta|<s$. Thus we can formally set $D^\gamma f(g(x))D^\beta g(x)=0$. However the proof of this theorem is not easy since it is based on the celebrated Whitney extension theorem. That should not be surprising after all. The existence of the extension F is proven by verification that the formal jet of derivatives of $f\circ g$ up to order k defined above satisfies the assumptions of the Whitney extension theorem.

In 1951, Kneser presented a proof of this composition result in [57]. In the same paper, he proved a theorem which may be obtained as an immediate corollary to the theorem of Sard, though he did so without any reference to or influence from Sard's result. The composition theorem is also discussed in a different context in a 1958 paper by Glaeser [35]. A proof is provided in Section 7.4. The reader may also find proofs of this theorem in [1, Theorem 14.1], [65, Chapter 1, Theorem 6.1], and [67, Theorem 8.3.1].

Thom [84], quickly realized that the method of Kneser can be used to prove the Sard theorem. See also [1, 65, 70]. Recently Figalli [31] used this method to provide a simpler proof of Theorem 83. Our proof of Theorem 84 we will also be based on the Kneser-Glaeser result.

7.2 AUXILIARY RESULTS

In this section we will prove some technical results related to the Morrey inequality that will be used in the proof of Theorem 84.

The classes of functions with continuous and α-Hölder continuous derivatives of order up to k will be denoted by C^k and $C^{k,\alpha}$ respectively. The integral average over a set S of positive measure will be denoted by

$$f_S = \fint_S f(x)\,dx = \frac{1}{|S|}\int_S f(x)\,dx.$$

The characteristic function of a set E will be denoted by χ_E. Cubes in $\mathbb{R}^n$ will always have sides parallel to coordinate directions.

We will use the following elementary result several times.

Lemma 87. *Let $E \subset \mathbb{R}^n$ be a bounded measurable set and let $-\infty < a < n$. Then there is a constant $C = C(n, a)$ such that for every $x \in E$*

$$\int_E \frac{dy}{|x-y|^a} \leq \begin{cases} C|E|^{1-\frac{a}{n}} & \text{if } 0 \leq a < n. \\ (\operatorname{diam} E)^{-a}|E| & \text{if } a < 0. \end{cases}$$

Proof. The case $a < 0$ is obvious since then $|x-y|^{-a} \leq (\operatorname{diam} E)^{-a}$. Thus assume that $0 \leq a < n$. In this case the inequality is actually true for all $x \in \mathbb{R}^n$ and not only for $x \in E$. Let $B = B(0,r)$, $|B| = |E|$. We have

$$\int_E \frac{dy}{|x-y|^a} \leq \int_B \frac{dy}{|y|^a} = C\int_0^r t^{-a}t^{n-1}\,dt = Cr^{n-a} = C|E|^{1-\frac{a}{n}}.$$

□

The following result [34, Lemma 7.16] is a basic pointwise estimate for Sobolev functions.

Lemma 88. *Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $S \subset D$ be a measurable set of positive measure. If $f \in W^{1,p}(D)$, $p \geq 1$, then*

$$|f(x) - f_S| \leq C(n)\,\frac{|D|}{|S|}\int_D \frac{|Df(z)|}{|x-z|^{n-1}}\,dz \quad a.e. \tag{7.1}$$

When $p > n$, the triangle inequality $|f(y) - f(x)| \leq |f(y) - f_D| + |f(x) - f_D|$, Hölder inequality, and Lemma 87 applied to the right hand side of (7.1) yield a well known

Lemma 89 (Morrey's inequality). *Suppose $n < p < \infty$ and $f \in W^{1,p}(D)$, where $D \subset \mathbb{R}^n$ is a cube or a ball. Then there is a constant $C = C(n,p)$ such that*

$$|f(y) - f(x)| \leq C(\operatorname{diam} D)^{1-\frac{n}{p}} \left(\int_D |Df(z)|^p \, dz \right)^{1/p} \quad \textit{for all } x, y \in D.$$

In particular,

$$\operatorname{diam} f(D) \leq C(\operatorname{diam} D)^{1-\frac{n}{p}} \left(\int_D |Df(z)|^p \, dz \right)^{1/p}.$$

Since $p > n$, the function f is continuous (Sobolev embedding) and hence the lemma does indeed hold for *all* $x, y \in D$.

From this lemma we can easily deduce a corresponding result for higher order derivatives. The Taylor polynomial and the averaged Taylor polynomial of f will be denoted by

$$T_x^k f(y) = \sum_{|\alpha| \leq k} D^\alpha f(x) \, \frac{(y-x)^\alpha}{\alpha!}, \quad T_S^k f(y) = \fint_S T_x^k f(y) \, dx.$$

Lemma 90. *Suppose $n < p < \infty$, $k \geq 1$ and $f \in W^{k,p}(D)$, where $D \subset \mathbb{R}^n$ is a cube or a ball. Then there is a constant $C = C(n,k,p)$ such that*

$$|f(y) - T_x^{k-1} f(y)| \leq C(\operatorname{diam} D)^{k-\frac{n}{p}} \left(\int_D |D^k f(z)|^p \, dz \right)^{1/p} \quad \textit{for all } x, y \in D.$$

Proof. Given $y \in D$ let

$$\psi(x) := T_x^{k-1} f(y) = \sum_{|\alpha| \leq k-1} D^\alpha f(x) \, \frac{(y-x)^\alpha}{\alpha!} \in W^{1,p}(D).$$

Observe that $\psi(y) = f(y)$ and

$$\frac{\partial \psi}{\partial x_j}(x) = \sum_{|\alpha| = k-1} D^{\alpha + \delta_j} f(x) \, \frac{(y-x)^\alpha}{\alpha!},$$

where $\delta_j = (0, \ldots, 1, \ldots, 0)$. Indeed, after applying the Leibniz rule to $\partial \psi / \partial x_j$ the lower order terms will cancel out. Since

$$|D\psi(z)| \leq C(n,k) |D^k f(z)| |y - z|^{k-1},$$

Lemma 89 applied to ψ yields the result. $\square$

Applying the same argument to Lemma 88 leads to the following result, see [12, Theorem 3.3].

Lemma 91. *Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $S \subset D$ be a measurable set of positive measure. If $f \in W^{k,p}(D)$, $p \geq 1$, $k \geq 1$, then there is constant $C = C(n,k)$ such that*

$$|f(x) - T_S^{k-1} f(x)| \leq C \, \frac{|D|}{|S|} \int_D \frac{|D^k f(z)|}{|x-z|^{n-k}} \, dz \quad \textit{for a.e. } x \in D. \tag{7.2}$$

In the next result we will improve the above estimates under the additional assumption that the derivative Df vanishes on a given subset of D. For a similar result in a different setting see [44, Proposition 2.3].

Lemma 92. *Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $f \in W^{k,p}(D)$, $n < p < \infty$, $k \geq 1$. Let*

$$A = \{x \in D |\, Df(x) = 0\}.$$

Then for any $\varepsilon > 0$ there is $\delta = \delta(n,k,p,\varepsilon) > 0$ such that if

$$\frac{|D \setminus A|}{|D|} < \delta,$$

then

$$\operatorname{diam} f(D) \leq \varepsilon (\operatorname{diam} D)^{k - \frac{n}{p}} \left(\int_D |D^k f(z)|^p \, dz \right)^{1/p}.$$

Remark 93. *It is important that δ does not depend of f. The result applies very well to density points of A. Indeed, it follows immediately that if $x \in A$ is a density point, then for any $\varepsilon > 0$ there is $r_x > 0$ such that*

$$\operatorname{diam} f(B(x, r_x)) \leq \varepsilon r_x^{k - \frac{n}{p}} \left(\int_{B(x,r_x)} |D^k f(z)|^p \, dz \right)^{1/p}.$$

Proof of Lemma 92. Although only the first order derivatives of f are equal zero in A, it easily follows that $D^\alpha f = 0$ a.e. in A for all $1 \le |\alpha| \le k$. Indeed, if a Sobolev function is constant in a set, its derivative equals zero a.e. in the set, [34, Lemma 7.7], and we apply induction. Hence

$$T_A^{k-1} f(x) = f_A \quad \text{for all } x \in \mathbb{R}^n.$$

Let $\varepsilon > 0$. Choose $0 < \delta < 1/2$ with $\max\left\{\delta^{\frac{k}{n}-\frac{1}{p}}, \delta^{1-\frac{1}{p}}\right\} < \varepsilon$. Since $\delta < 1/2$, $|D|/|A| < 2$. Thus Lemma 91 with $S = A$ yields

$$|f(x) - f_A| \le C(n) \int_{D\setminus A} \frac{|D^k f(z)|}{|x-z|^{n-k}}\, dz \le C(n) \|D^k f\|_{L^p(D)} \left(\int_{D\setminus A} \frac{dz}{|x-z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}}.$$

Now the result follows directly from Lemma 87. Indeed, if $k \le n$, Lemma 87 and the estimate

$$|D \setminus A| < \delta |D| \le C(n)\delta (\operatorname{diam} D)^n$$

yield

$$\left(\int_{D\setminus A} \frac{dz}{|x-z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \le C(n,k,p) |D\setminus A|^{\frac{1}{n}(k-\frac{n}{p})} \le C(n,k,p) \delta^{\frac{k}{n}-\frac{1}{p}} (\operatorname{diam} D)^{k-\frac{n}{p}}.$$

If $k > n$, then we have

$$\left(\int_{D\setminus A} \frac{dz}{|x-z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \le (\operatorname{diam} D)^{k-n} |D\setminus A|^{\frac{p-1}{p}} \le C(n,p) \delta^{1-\frac{1}{p}} (\operatorname{diam} D)^{k-\frac{n}{p}}.$$

Hence

$$\operatorname{diam} f(D) = \sup_{x,y\in D} |f(x) - f(y)| \le 2 \sup_{x\in D} |f(x) - f_A| \le C(n,k,p)\varepsilon (\operatorname{diam} D)^{k-\frac{n}{p}} \|D^k f\|_{L^p(D)}.$$

The proof is complete. □

We will also need the following classical Besicovitch covering lemma, see e.g. [95, Theorem 1.3.5]

Lemma 94 (Besicovitch)**.** *Let $E \subset \mathbb{R}^n$ and let $\{B_x\}_{x\in E}$ be a family of closed balls $B_x = \overline{B}(x, r_x)$ so that $\sup_{x\in E}\{r_x\} < \infty$. Then there is a countable (possibly finite) subfamily $\{B_{x_i}\}_{i=1}^\infty$ with the property that*

$$E \subset \bigcup_{i=1}^\infty B_{x_i}$$

and no point of $\mathbb{R}^n$ belongs to more than $C(n)$ balls.

7.3 PROOF OF THE DUBOVITSKII THEOREM FOR SOBOLEV MAPS

As we pointed out above we may assume that $m \leq n$ and $k \geq 2$. It is also easy to see that we can assume that $\Omega = \mathbb{R}^n$ and $f \in W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$. Indeed, it suffices to prove the claim of Theorem 84 on compact subsets of Ω and so we may multiply f by a compactly supported smooth cut-off function to get a function in $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$.

We will prove the result using induction with respect to n. If $n = 1$, then $m = n = 1$. This gives $n - m - k + 1 = 1 - k \leq 0$ for any $k \in \mathbb{N}$, so $\ell = 0$. Thus the theorem is a direct consequence of Theorem 85.

We shall prove now the theorem for $n \geq 2$ assuming that it is true in dimensions less than or equal to $n - 1$. Fix p and integers m and k satisfying $n < p < \infty$, $m \leq n$, and $k \geq 2$. Write $\ell = \max(n - m - k + 1, 0)$. Let $f \in W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$.

We can write

$$C_f = K \cup A_1 \cup \cdots \cup A_{k-1},$$

where

$$K := \{x \in C_f \,:\, 0 < \operatorname{rank} Df(x) < m\}$$

and

$$A_s := \{x \in \mathbb{R}^n \,:\, D^\alpha f(x) = 0 \text{ for all } 1 \leq |\alpha| \leq s\}$$

Note that $A_1 \supset A_2 \supset \ldots \supset A_{k-1}$ is a decreasing sequence of sets.

In the first step, we will show that $A_{k-1} \cap f^{-1}(y)$ is ℓ-null for a.e. $y \in \mathbb{R}^m$. Then we will prove the same for $(A_{s-1} \setminus A_s) \cap f^{-1}(y)$ for $s = 2, 3, \ldots, k - 1$. To do this we will use the Implicit Function and Kneser-Glaeser theorems to reduce our problem to a lower dimensional one and apply the induction hypothesis. Finally, we will consider the set K and use a change of variables to show that we can reduce the dimension in the domain and in the target so that the fact that $\mathcal{H}^\ell(K \cap f^{-1}(y)) = 0$ will follow from the induction hypothesis.

Claim 95. *$\mathcal{H}^\ell(A_{k-1} \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.*

Proof. Suppose $x \in A_{k-1}$. Notice that $T_x^{k-1} f(y) = f(x)$ for any $y \in \mathbb{R}^n$ since $D^\alpha f(x) = 0$ for every $1 \leq |\alpha| \leq k - 1$. By Lemma 90 applied to each coordinate of $f = (f_1, \ldots, f_m)$, we have for any cube $Q \subset \mathbb{R}^n$ containing x and any $y \in Q$,

$$|f(y) - f(x)| \leq C(\operatorname{diam} Q)^{k-\frac{n}{p}} \left(\int_Q |D^k f(z)|^p \, dz \right)^{1/p}. \tag{7.3}$$

Hence

$$\operatorname{diam} f(Q) \leq C(\operatorname{diam} Q)^{k-\frac{n}{p}} \left(\int_Q |D^k f(z)|^p \, dz \right)^{1/p}. \tag{7.4}$$

Let $F_1 := \{x \in A_{k-1} : x \text{ is a density point of } A_{k-1}\}$ and $F_2 := A_{k-1} \setminus F_1$. We will treat the sets $F_1 \cap f^{-1}(y)$ and $F_2 \cap f^{-1}(y)$ separately.

Step 1. First we will prove that $\mathcal{H}^\ell(F_2 \cap f^{-1}(y)) = 0$ for almost every $y \in \mathbb{R}^m$.

Let $0 < \varepsilon < 1$. Since $\mathcal{H}^n(F_2) = 0$, there is an open set $F_2 \subset U \subset \Omega$ such that $\mathcal{H}^n(U) < \varepsilon^{\frac{p}{p-m}}$. For any $j \geq 1$ let $\{Q_{ij}\}_{i=1}^\infty$ be a collection of closed cubes with pairwise disjoint interiors such that

$$Q_{ij} \cap F_2 \neq \emptyset, \quad F_2 \subset \bigcup_{i=1}^\infty Q_{ij} \subset U, \quad \operatorname{diam} Q_{ij} < \frac{1}{j}.$$

Since $F_2 \cap Q_{ij} \neq \emptyset$, (7.4) yields

$$\mathcal{H}^m(f(Q_{ij})) \leq C(\operatorname{diam} f(Q_{ij}))^m \leq C(\operatorname{diam} Q_{ij})^{m(k-\frac{n}{p})} \left(\int_{Q_{ij}} |D^k f(x)|^p \, dx \right)^{m/p}.$$

Case: $n - m - k + 1 \leq 0$ so $\ell = 0$.

This condition easily implies that $mk \geq n$ so we also have $\frac{mp}{p-m}(k - \frac{n}{p}) \geq n$, and by Hölder's inequality,

$$\begin{aligned}
\mathcal{H}^m(f(F_2)) &\leq \sum_{i=1}^\infty \mathcal{H}^m(f(Q_{ij})) \leq C \sum_{i=1}^\infty (\operatorname{diam} Q_{ij})^{m(k-\frac{n}{p})} \left(\int_{Q_{ij}} |D^k f(x)|^p \, dx \right)^{m/p} \\
&\leq C \left(\sum_{i=1}^\infty (\operatorname{diam} Q_{ij})^{\frac{pm}{p-m}(k-\frac{n}{p})} \right)^{\frac{p-m}{p}} \left(\int_{\bigcup_{i=1}^\infty Q_{ij}} |D^k f(x)|^p \, dx \right)^{m/p} \\
&\leq C \mathcal{H}^n(U)^{\frac{p-m}{p}} \left(\int_U |D^k f(x)|^p \, dx \right)^{m/p} < C\varepsilon \|D^k f\|_p. \qquad (7.5)
\end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, $\mathcal{H}^m(f(F_2)) = 0$ and hence $F_2 \cap f^{-1}(y) = \emptyset$, i.e. $\mathcal{H}^\ell(F_2 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Case: $\ell = n - m - k + 1 > 0$.

The sets $\{Q_{ij}\cap f^{-1}(y)\}_{i=1}^\infty$ form a covering of $F_2\cap f^{-1}(y)$ by sets of diameters less than $1/j$. Since

$$\operatorname{diam}(Q_{ij}\cap f^{-1}(y))\leq(\operatorname{diam} Q_{ij})\chi_{f(Q_{ij})}(y)$$

the definition of the Hausdorff measure yields

$$\begin{aligned}\text{(7.6)}\qquad \mathcal{H}^\ell(F_2\cap f^{-1}(y)) &\leq C\liminf_{j\to\infty}\sum_{i=1}^\infty \operatorname{diam}(Q_{ij}\cap f^{-1}(y))^\ell\\ &\leq C\liminf_{j\to\infty}\sum_{i=1}^\infty(\operatorname{diam} Q_{ij})^\ell\chi_{f(Q_{ij})}(y).\end{aligned}$$

We would like to integrate both sides with respect to $y\in\mathbb{R}^m$. Note that the function on the right hand side is measurable since the sets $f(Q_{ij})$ are compact. However measurability of the function $y\mapsto\mathcal{H}^\ell(F_2\cap f^{-1}(y))$ is far from being obvious. To deal with this problem we will use the *upper integral* which for a non-negative function $g:X\to[0,\infty]$ defined μ-a.e. on a measure space (X,μ) is defined as follows:

$$\int_X^* g\,d\mu=\inf\left\{\int_X\phi\,d\mu:\ 0\leq g\leq\phi\text{ and }\phi\text{ is }\mu\text{-measurable.}\right\}.$$

An important property of the upper integral is that if $\int_X^* g\,d\mu=0$, then $g=0$ μ-a.e. Indeed, there is a sequence $\phi_i\geq g\geq 0$ such that $\int_X\phi_i\,d\mu\to 0$. That means $\phi_i\to 0$ in $L^1(\mu)$. Taking a subsequence we get $\phi_{i_j}\to 0$ μ-a.e. which proves that $g=0$ μ-a.e.

Applying the upper integral with respect to $y\in\mathbb{R}^m$ to both sides of (7.6), using Fatou's lemma, and noticing that

$$\frac{p}{p-m}\left(\ell+m\left(k-\frac{n}{p}\right)\right)\geq n$$

gives

$$\begin{aligned}&\int_{\mathbb{R}^m}^*\mathcal{H}^\ell(F_2\cap f^{-1}(y))\,d\mathcal{H}^m(y)\leq C\liminf_{j\to\infty}\sum_{i=1}^\infty(\operatorname{diam} Q_{ij})^\ell\mathcal{H}^m(f(Q_{ij}))\\ &\leq\ C\liminf_{j\to\infty}\sum_{i=1}^\infty(\operatorname{diam} Q_{ij})^{\ell+m(k-\frac{n}{p})}\left(\int_{Q_{ij}}|D^kf(x)|^p\,dx\right)^{m/p}<C\varepsilon\|D^kf\|_p\end{aligned}$$

by the same argument as in (7.5). Again, since $\varepsilon>0$ can be arbitrarily small, we conclude that $\mathcal{H}^\ell(F_2\cap f^{-1}(y))=0$ for a.e. $y\in\mathbb{R}^m$.

Step 2. It remains to prove that $\mathcal{H}^\ell(F_1 \cap f^{-1}(y)) = 0$ for almost every $y \in \mathbb{R}^m$.

The proof is similar to that in Step 1 and the arguments which are almost the same will be presented in a more sketchy form now. In Step 1 it was essential that the set F_2 had measure zero. We will compensate the lack of this property now by the estimates from Remark 93.

It suffices to prove that for any cube $\tilde{Q}$, $\mathcal{H}^\ell(\tilde{Q}\cap F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. Assume that $\tilde{Q}$ is in the interior of a larger cube $\tilde{Q} \Subset Q$.

By Remark 93, for each $x \in \tilde{Q} \cap F_1$ and $j \in \mathbb{N}$ there is $0 < r_{jx} < 1/j$ such that

$$\operatorname{diam} f(B(x, r_{jx})) \leq j^{-1} r_{jx}^{k-\frac{n}{p}} \left(\int_{B(x,r_{jx})} |D^k f(z)|^p \, dz \right)^{1/p}.$$

We may further assume that $B(x, r_{jx}) \subset Q$.

Denote $B_{jx} = \overline{B}(x, r_{jx})$. According to the Besicovitch Lemma 94, there is a countable subcovering $\{B_{jx_i}\}_{i=1}^\infty$ of $\tilde{Q}\cap F_1$ so that no point of $\mathbb{R}^n$ belongs to more than $C(n)$ balls B_{jx_i}.

Case: $n - m - k + 1 \leq 0$ so $\ell = 0$.

We have $\frac{pm}{p-m}(k - \frac{n}{p}) \geq n$ as before, so

$$\begin{aligned} \mathcal{H}^m(f(\tilde{Q} \cap F_1)) &\leq C \sum_{i=1}^\infty \mathcal{H}^m(f(B_{jx_i})) \leq C j^{-m} \sum_{i=1}^\infty r_{jx_i}^{m(k-\frac{n}{p})} \left(\int_{B_{jx_i}} |D^k f(z)|^p \, dz \right)^{m/p} \\ &\leq C j^{-m} \left(\sum_{i=1}^\infty r_{jx_i}^n \right)^{\frac{p-m}{p}} \left(\sum_{i=1}^\infty \int_{B_{jx_i}} |D^k f(z)|^p \, dz \right)^{m/p}. \end{aligned}$$

Since the balls are contained in Q and no point belongs to more than $C(n)$ balls we conclude that

$$\mathcal{H}^m(f(\tilde{Q} \cap F_1)) \leq C j^{-m} \mathcal{H}^n(Q)^{\frac{p-m}{p}} \|D^k f\|_p^m.$$

Since j can be arbitrarily large, $\mathcal{H}^m(f(\tilde{Q} \cap F_1)) = 0$, i.e. $\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$.

Case: $\ell = n - m - k + 1 > 0$.

The sets $\{B_{jx_i} \cap f^{-1}(y)\}_{i=1}^\infty$ form a covering of $\tilde{Q} \cap F_1 \cap f^{-1}(y)$ and

$$\operatorname{diam}(B_{jx_i} \cap f^{-1}(y)) \leq C r_{jx_i} \chi_{f(B_{jx_i})}(y).$$

The definition of the Hausdorff measure yields

$$\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) \le C \liminf_{j\to\infty} \sum_{i=1}^\infty r_{jx_i}^\ell \chi_{f(B_{jx_i})}(y).$$

Thus as above

$$\begin{aligned}
&\int^*_{\mathbb{R}^m} \mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y))\, d\mathcal{H}^m(y) \\
&\le \ C \liminf_{j\to\infty} \sum_{i=1}^\infty r_{jx_i}^\ell \mathcal{H}^m(f(B_{jx_i})) \\
&\le \ C \liminf_{j\to\infty} j^{-m} \sum_{i=1}^\infty r_{jx_i}^{\ell+m(k-\frac{n}{p})} \left(\int_{B_{jx_i}} |D^k f(z)|^p\, dz \right)^{m/p} \\
&\le \ C \liminf_{j\to\infty} j^{-m} \mathcal{H}^n(Q)^{\frac{p-m}{p}} \|D^k f\|_p^m = 0
\end{aligned}$$

since $\frac{p}{p-m}\left(\ell + m\left(k - \frac{n}{p}\right)\right) \ge n$. Therefore $\mathcal{H}^\ell(\tilde{Q} \cap F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. This completes the proof that $\mathcal{H}^\ell(F_1 \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$ and hence that of Claim 95 $\square$

Claim 96. $\mathcal{H}^\ell((A_{s-1} \setminus A_s) \cap f^{-1}(y)) = 0$ *for a.e.* $y \in \mathbb{R}^m$, $s = 2, 3, \dots, k-1$.

In this step, we will use the Kneser-Glaeser composition theorem and the implicit function theorem to apply the induction hypothesis in $\mathbb{R}^{n-1}$.

Fix $s \in \{2, 3, \dots, k-1\}$ and $\bar{x} \in A_{s-1} \setminus A_s$. It suffices to show that the ℓ-Hausdorff measure of $W \cap (A_{s-1} \setminus A_s) \cap f^{-1}(y)$ is zero for some neighborhood W of $\bar{x}$ and a.e. $y \in \mathbb{R}^m$. Indeed, $A_{s-1} \setminus A_s$ can be covered by countably many such neighborhoods.

By the definitions of A_s and A_{s-1}, $D^\gamma f(\bar{x}) = 0$ for all $1 \le |\gamma| \le s-1$, and $D^\beta f(\bar{x}) \ne 0$ for some $|\beta| = s$. That is, for some $|\gamma| = s-1$ and $j \in \{1, \dots, m\}$, $D(D^\gamma f_j)(\bar{x}) \ne 0$ and $D^\gamma f_j \in W^{k-(s-1),p} \subset C^{k-s, 1-\frac{n}{p}}$.

Hence, by the implicit function theorem, there is some neighborhood U of $\bar{x}$ and an open set $V \subset \mathbb{R}^{n-1}$ so that $U \cap \{D^\gamma f_j = 0\} = g(V)$ for some $g : V \to \mathbb{R}^n$ of class C^{k-s}. In particular, $U \cap A_{s-1} \subset g(V)$ since $D^\gamma f_j = 0$ on A_{s-1}.

Choose a neighborhood $W \Subset U$ of $\bar{x}$ and say $A^* := g^{-1}(\overline{W} \cap A_{s-1})$ so that A^* is compact. Since f is $s-1$ flat on the closed set A_{s-1}, f is of class C^{k-1}, g is of class $C^{(k-1)-(s-1)}$, and $g(A^*) \subset A_{s-1}$, we can apply Theorem 86 to each component of f to find a C^{k-1} function

$F:\mathbb{R}^{n-1}\to\mathbb{R}^m$ so that, for every $x\in A^*$, $F(x)=(f\circ g)(x)$ and $D^\lambda F(x)=0$ for all $|\lambda|\le s-1$. That is, $A^*\subset C_F$. Hence

$$\mathcal{H}^\ell(A^*\cap F^{-1}(y))\le\mathcal{H}^\ell(C_F\cap F^{-1}(y))=0.$$

for almost every $y\in\mathbb{R}^m$. In this last equality, we invoked the induction hypothesis on $F\in C^{k-1}(\mathbb{R}^{n-1},\mathbb{R}^m)\subset W^{k-1,p}_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{R}^m)$ with $\ell=\max((n-1)-m-(k-1)+1,0)$. Since g is of class C^1, it is locally Lipschitz, and so $\mathcal{H}^\ell(g(A^*\cap F^{-1}(y)))=0$ for almost every $y\in\mathbb{R}^m$. Since $W\cap A_{s-1}\subset g(A^*)$, we have

$$W\cap A_{s-1}\cap f^{-1}(y)\subset g(A^*\cap F^{-1}(y))$$

for all $y\in\mathbb{R}^m$, and thus

$$\mathcal{H}^\ell(W\cap(A_{s-1}\setminus A_s)\cap f^{-1}(y))\le\mathcal{H}^\ell(W\cap A_{s-1}\cap f^{-1}(y))=0$$

for almost every $y\in\mathbb{R}^m$. The proof of the claim is complete.

Claim 97. *$\mathcal{H}^\ell(K\cap f^{-1}(y))=0$ for a.e. $y\in\mathbb{R}^m$.*

Proof. Write $K=\bigcup_{r=1}^{m-1}K_r$ where $K_r:=\{x\in\mathbb{R}^n\,:\,\operatorname{rank}Df(x)=r\}$. Fix $x_0\in K_r$ for some $r\in\{1,\dots,m-1\}$. For the same reason as in Claim 96 it suffices to show that $\mathcal{H}^\ell((V\cap K_r)\cap f^{-1}(y))=0$ for some neighborhood V of x_0 for a.e. $y\in\mathbb{R}^m$.

Without loss of generality, assume that the submatrix $[\partial f_i/\partial x_j(x_0)]_{i,j=1}^r$ formed by the first r rows and columns of Df has rank r. Let

$$Y(x)=(f_1(x),f_2(x),\dots,f_r(x),x_{r+1},\dots,x_n)\quad\text{for all }x\in\mathbb{R}^n. \tag{7.7}$$

Y is of class C^{k-1} since each component of f is. Also, $\operatorname{rank}DY(x_0)=n$, so by the inverse function theorem Y is a C^{k-1} diffeomorphism of some neighborhood V of x_0 onto an open set $\tilde V\subset\mathbb{R}^n$. From now on we will assume that Y is defined in V only.

Claim 98. $Y^{-1}\in W^{k,p}_{\mathrm{loc}}(\tilde V,\mathbb{R}^n)$.

Proof. In the proof we will need

Lemma 99. *Let $\Omega \subset \mathbb{R}^n$ be open. If $g, h \in W^{\ell,p}_{\mathrm{loc}}(\Omega)$, where $p > n$ and $\ell \geq 1$, then $gh \in W^{\ell,p}_{\mathrm{loc}}(\Omega)$.*

Proof. Since $g, h \in C^{\ell-1}$, it suffices to show that the classical partial derivatives $D^\beta(gh)$, $|\beta| = \ell - 1$ belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$ (when $\ell = 1$, $\beta = 0$ so $D^\beta(gh) = gh$).

The product rule for $C^{\ell-1}$ functions yields

$$D^\beta(gh) = \sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\,\delta!}\, D^\gamma g D^\delta h. \tag{7.8}$$

Each of the functions $D^\gamma g$, $D^\delta h$ is absolutely continuous on almost all lines parallel to coordinate axes, [27, Section 4.9.2], so is their product. Thus $D^\beta(gh)$ is absolutely continuous on almost all lines and hence it has partial derivatives (or order 1) almost everywhere. According to a characterization of $W^{1,p}_{\mathrm{loc}}$ by absolute continuity on lines, [27, Section 4.9.2], it suffices to show that partial derivatives of $D^\beta(gh)$ (of order 1) belong to L^p_{loc}. This will imply that $D^\beta(gh) \in W^{1,p}_{\mathrm{loc}}$ for all β, $|\beta| = \ell - 1$ so $gh \in W^{\ell,p}_{\mathrm{loc}}$.

If $D^\alpha = D^{\delta_i} D^\beta$, then the product rule applied to the right hand side of (7.8) yields

$$D^\alpha(gh) = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\,\delta!}\, D^\gamma g D^\delta h.$$

If $|\gamma| < |\alpha| = \ell$ and $|\delta| < |\alpha| = \ell$, then the function $D^\gamma g D^\delta h$ is continuous and hence in L^p_{loc}. The remaining terms are $hD^\alpha g + gD^\alpha h$. Clearly this function also belongs to L^p_{loc} because the functions g, h are continuous and $D^\alpha g, D^\alpha h \in L^p_{\mathrm{loc}}$. This completes the proof of the lemma. $\square$

Now we can complete the proof of Claim 98. Since Y is a diffeomorphism of class C^{k-1}, we have

$$D(Y^{-1})(y) = [DY(Y^{-1}(y))]^{-1} \quad \text{for every } y \in \tilde{V}. \tag{7.9}$$

It suffices to prove that $D(Y^{-1}) \in W^{k-1,p}_{\mathrm{loc}}$. It follows from (7.9) and a formula for the inverse matrix that

$$D(Y^{-1}) = \left(\frac{P_1(Df)}{P_2(Df)}\right) \circ Y^{-1},$$

where P_1 and P_2 and polynomials whose variables are replaced by partial derivatives of f. The polynomial $P_2(Df)$ is just $\det DY$.

Since $Df \in W^{k-1,p}_{\rm loc}$ and $p > n$, it follows from Lemma 99 that

$$P_1(Df), P_2(Df) \in W^{k-1,p}_{\rm loc}.$$

Note that $P_2(Df) = \det DY$ is continuous and different than zero. Hence

$$\frac{1}{P_2(Df)} \in W^{k-1,p}_{\rm loc}$$

as a composition of a $W^{k-1,p}_{\rm loc}$ function which is locally bounded away from 0 and ∞ with a smooth function $x \mapsto x^{-1}$. Thus Lemma 99 applied one more time implies $P_1(Df)/P_2(Df)$ is in the class $W^{k-1,p}_{\rm loc}$. Finally

$$D(Y^{-1}) = \left(\frac{P_1(Df)}{P_2(Df)}\right) \circ Y^{-1} \in W^{k-1,p}_{\rm loc}$$

because composition with a diffeomorphism Y^{-1} of class C^{k-1} preserves $W^{k-1,p}_{\rm loc}$. The proof of the claim is complete. $\square$

It follows directly from (7.7) that

$$f(Y^{-1}(x)) = (x_1, \ldots, x_r, g(x)) \tag{7.10}$$

for all $x \in \tilde{V}$ and some function $g : \tilde{V} \to \mathbb{R}^{m-r}$.

Claim 100. $g \in W^{k,p}_{\rm loc}(\tilde{V}, \mathbb{R}^{m-r})$.

This statement is a direct consequence of the next

Lemma 101. *Let $\Omega \subset \mathbb{R}^n$ be open, $p > n$ and $k \geq 1$. If $\Phi \in W^{k,p}_{\rm loc}(\Omega, \mathbb{R}^n)$ is a diffeomorphism and $u \in W^{k,p}_{\rm loc}(\Phi(\Omega))$, then $u \circ \Phi \in W^{k,p}_{\rm loc}(\Omega)$.*

Proof. When $k = 1$ the result is obvious because diffeomorphisms preserve $W^{1,p}_{\rm loc}$. Assume thus that $k \geq 2$. Since $p > n$, $\Phi \in C^{k-1}$ so Φ is a diffeomorphism of class C^{k-1}, but also $u \in C^{k-1} \subset C^1$ and hence the classical chain rule gives

$$D(u \circ \Phi) = ((Du) \circ \Phi) \cdot D\Phi. \tag{7.11}$$

Since $Du \in W^{k-1,p}_{\rm loc}$ and Φ is a diffeomorphism of class C^{k-1}, we conclude that $(Du) \circ \Phi \in W^{k-1,p}_{\rm loc}$. Now the fact that $D\Phi \in W^{k-1,p}_{\rm loc}$ combined with (7.11) and Lemma 99 yield that the right hand side of (7.11) belongs to $W^{k-1,p}_{\rm loc}$ so $D(u \circ \Phi) \in W^{k-1,p}_{\rm loc}$ and hence $u \circ \Phi \in W^{k,p}_{\rm loc}$. This completes the proof of Lemma 101 and hence that of Claim 100. $\square$

Now we can complete the proof of Claim 97. Recall that we need to prove that

$$\mathcal{H}^\ell((V \cap K_r) \cap f^{-1}(y)) = 0 \quad \text{for a.e. } y \in \mathbb{R}^m. \tag{7.12}$$

The diffeomorphism Y^{-1} is a change of variables that simplifies the structure of the mapping f because $f \circ Y^{-1}$ fixes the first r coordinates (see (7.10)) and hence it maps $(n-r)$-dimensional slices orthogonal to $\mathbb{R}^r$ to the corresponding $(m-r)$-dimensional slices orthogonal to $\mathbb{R}^r$. Because of this observation it is more convenient to work with $f \circ Y^{-1}$ rather than with f. Translating (7.12) to the case of $f \circ Y^{-1}$ it suffices to show that

$$\mathcal{H}^\ell((\tilde{V} \cap Y(K_r)) \cap (f \circ Y^{-1})^{-1})(y) = 0 \quad \text{for a.e. } y \in \mathbb{R}^m.$$

We used here a simple fact that the diffeomorphism Y preserves ℓ-null sets.

Observe also that

$$\operatorname{rank} D(f \circ Y^{-1})(x) = r \quad \text{for } x \in \tilde{V} \cap Y(K_r). \tag{7.13}$$

For any $\tilde{x} \in \mathbb{R}^r$ and $A \subset \mathbb{R}^n$, we will denote by $A_{\tilde{x}}$ the $(n-r)$–dimensional slice of A with the first r coordinates equal to $\tilde{x}$. That is, $A_{\tilde{x}} := \{z \in \mathbb{R}^{n-r} : (\tilde{x}, z) \in A\}$. Let $g_{\tilde{x}} : \tilde{V}_{\tilde{x}} \to \mathbb{R}^{m-r}$ be defined by $g_{\tilde{x}}(z) = g(\tilde{x}, z)$. With this notation

$$(f \circ Y^{-1})(\tilde{x}, z) = (\tilde{x}, g_{\tilde{x}}(z))$$

and hence for $y = (\tilde{x}, w) \in \mathbb{R}^m$

$$(\tilde{V} \cap Y(K_r)) \cap (f \circ Y^{-1})^{-1}(y) = g_{\tilde{x}}^{-1}(w) \cap (\tilde{V} \cap Y(K_r))_{\tilde{x}}.$$

More precisely the set on the left hand side is contained in an affine $(n-r)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $\mathbb{R}^r$ at $\tilde{x}$ while the set on the right hand side is contained in $\mathbb{R}^{n-r}$ but the two sets are identified through a translation by the vector $(\tilde{x}, 0) \in \mathbb{R}^n$ which identifies $\mathbb{R}^{n-r}$ with the affine subspace orthogonal to $\mathbb{R}^r$ at $\tilde{x}$.

According to the Fubini theorem it suffices to show that for almost all $\tilde{x} \in \mathbb{R}^r$ the following is true: for almost all $w \in \mathbb{R}^{m-r}$

$$\mathcal{H}^\ell(g_{\tilde{x}}^{-1}(w) \cap (\tilde{V} \cap Y(K_r)_{\tilde{x}})) = 0. \tag{7.14}$$

As we will see this is a direct consequence of the induction hypothesis applied to the mapping $g_{\tilde{V}} : \tilde{x}_{\tilde{x}} \to \mathbb{R}^{n-r}$ defined in a set of dimension $n - r \leq n - 1$. We only need to check that $g_{\tilde{x}}$ satisfies the assumptions of the induction hypothesis.

It is easy to see that for each $x = (\tilde{x}, z) \in \tilde{V}$

$$D(f \circ Y^{-1})(x) = \begin{pmatrix} \mathrm{id}_{\,r \times r} & 0 \\ * & D(g_{\tilde{x}})(z) \end{pmatrix}.$$

This and (7.13) imply that for each $\tilde{x} \in \mathbb{R}^r$, $Dg_{\tilde{x}} = 0$ on the slice $(\tilde{V} \cap Y(K_r))_{\tilde{x}}$. Hence the set $(\tilde{V} \cap Y(K_r))_{\tilde{x}}$ is contained in the critical set of $g_{\tilde{x}}$ so

$$\mathcal{H}^\ell(g_{\tilde{x}}^{-1}(w) \cap (\tilde{V} \cap Y(K_r))_{\tilde{x}}) \leq \mathcal{H}^\ell(g_{\tilde{x}}^{-1}(w) \cap C_{g_{\tilde{x}}}). \tag{7.15}$$

It follows from the Fubini theorem applied to Sobolev spaces that for almost all $\tilde{x} \in \mathbb{R}^n$, $g_{\tilde{x}} \in W^{k,p}_{\mathrm{loc}}(\tilde{V}_{\tilde{x}}, \mathbb{R}^{m-r})$ and hence the induction hypothesis is satisfied for such mappings

$$W^{k,p}_{\mathrm{loc}} \ni g_{\tilde{x}} : \tilde{V}_{\tilde{x}} \subset \mathbb{R}^{n-r} \to \mathbb{R}^{m-r}.$$

Since

$$\ell = \max(n - m - k + 1, 0) = \max((n - r) - (m - r) - k + 1, 0),$$

for almost all $w \in \mathbb{R}^{m-n}$ the expression on the right hand side of (7.15) equals zero and (7.14) follows. This completes the proof of Claim 97 and hence that of the theorem. $\square$

7.4 PROOF OF THE KNESER-GLAESER ROUGH COMPOSITION THEOREM

The Kneser-Glaeser theorem follows as a direct application of Whitney's Extension Theorem (Theorem 102). This classical theorem was discussed in the introduction of Chapter 5. The statement of the theorem for higher order derivatives is given here.

Theorem 102 (Whitney's Extension Theorem)**.** *Let $\{h_\alpha\}_{|\alpha|\leq k}$ be a collection of real valued functions defined on a compact set $K \subset \mathbb{R}^r$ satisfying*

$$h_\alpha(x) = \sum_{|\beta|\leq k-|\alpha|} \frac{h_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y) \tag{7.16}$$

for every $x, y \in K$ where R_α is uniformly $o\left(|x-y|^{k-\alpha}\right)$ as $|x-y| \to 0$. Then there is a function $H : \mathbb{R}^r \to \mathbb{R}$ of class C^k so that $D^\alpha H = h_\alpha$ on K for every $|\alpha| \leq k$.

7.4.1 Conditions equivalent to Whitney's Theorem

For simplicity, we will write $h = h_0$. Now define the formal Taylor series of h as

$$T_x^k h(z) = \sum_{|\alpha|\leq k} \frac{h_\alpha(x)}{\alpha!}(z-x)^\alpha$$

for any $x \in K$ and $z \in \mathbb{R}^r$. Notice that, if K was open, h was of class C^k on K, and $h_\alpha = D^\alpha h$ for each $|\alpha| \leq k$, then $T_x^k h$ would simply be the usual Taylor polynomial for h. Using this notation, we now have the following equivalent formulation of Whitney's Extension Theorem:

Proposition 103. *Let $\{h_\alpha\}_{|\alpha|\leq k}$ be a collection of real valued functions defined on a compact set $K \subset \mathbb{R}^r$. Let B be a ball with $K \subset B$. Condition (7.16) is equivalent to the following:*

$$|T_x^k h(z) - T_y^k h(z)| \leq c(|x-y|)\left(|x-z|^k + |x-y|^k\right) \tag{7.17}$$

for every $x, y \in K$ and $z \in B$ where $c : [0,\infty) \to [0,\infty)$ is increasing, continuous, and concave with $c(0) = 0$. (We say that c is a modulus of continuity.)

Throughout the remainder of the section, the notation c will be used to represent any constant multiple of a modulus of continuity. Thus c may change values in the same inequality.

Proof. To begin, we will show that (7.16) implies (7.17). Notice that

$$|R_\alpha(x,y)| \leq c(|x-y|)|x-y|^{k-|\alpha|} \tag{7.18}$$

for every $x, y \in K$, $|\alpha| \leq k$, and some modulus of continuity c. Indeed, define the function $\tilde{c} : [0,\infty) \to [0,\infty)$ as

$$\tilde{c}(t) = \sup\left\{ \frac{|R_\alpha(x,y)|}{|x-y|^{k-|\alpha|}} \quad : \quad |\alpha| \leq k,\ x, y \in K,\ 0 < |x-y| \leq t \right\},$$

and define the modulus of continuity c to be the infimum over all convex functions which are greater than or equal to $\tilde{c}$.

We will now show that

$$\sum_{|\alpha| \leq k} \frac{(z-x)^\alpha}{\alpha!} R_\alpha(x,y) = T_x^k h(z) - T_y^k h(z). \tag{7.19}$$

for every $x, y, \in K$ and $z \in B$. To do so, notice for each $|\alpha| \leq k$

$$R_\alpha(x,y) = h_\alpha(x) - \sum_{|\beta| \leq k-|\alpha|} \frac{h_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta = h_\alpha(x) - D^\alpha\left(\sum_{|\beta| \leq k} \frac{h_\beta(y)}{\beta!}(x-y)^\beta\right)$$

where this derivative is taken with respect to x. That is,

$$\begin{aligned}
\sum_{|\alpha| \leq k} \frac{(z-x)^\alpha}{\alpha!} R_\alpha(x,y) &= \sum_{|\alpha| \leq k} \frac{h_\alpha(x)}{\alpha!}(z-x)^\alpha - \sum_{|\alpha| \leq k} \frac{(z-x)^\alpha}{\alpha!} D^\alpha\left(\sum_{|\beta| \leq k} \frac{h_\beta(y)}{\beta!}(x-y)^\beta\right) \\
&= T_x^k h(z) - \sum_{|\alpha| \leq k} \frac{D^\alpha(T_x^k h(y))}{\alpha!}(z-x)^\alpha \\
&= T_x^k h(z) - T_y^k h(z)
\end{aligned}$$

This last equality holds since $T_x^k h(y)$ is a polynomial in x of degree k, and every polynomial of degree k is equal to its Taylor polynomial of degree k. Combining (7.19) with (7.18) gives

$$|T_x^k h(z) - T_y^k h(z)| \leq c(|x-y|) \sum_{|\alpha| \leq k} \frac{|z-x|^\alpha}{\alpha!} |x-y|^{k-|\alpha|} \leq c(|x-y|)\left(|x-z|^k + |x-y|^k\right)$$

where we used Young's inequality with exponents $\frac{k}{|\alpha|}$ and $\frac{k}{k-|\alpha|}$. This proves that (7.16) implies (7.17).

We will now show that (7.17) implies (7.16). Fix $x, y \in K$. By (7.19),

$$\left|\sum_{|\alpha|\leq k} \frac{(z-x)^\alpha}{\alpha!} R_\alpha(x,y)\right| = |T_x^k h(z) - T_y^k h(z)| \leq c(|x-y|)\left(|x-z|^k + |x-y|^k\right)$$

for every $z \in B$ and some modulus of continuity c. Say $z' \in \mathbb{R}^r$ is defined so that $z - x = |x-y|(z'-x)$. Then

$$\left|\sum_{|\alpha|\leq k} \frac{|x-y|^{|\alpha|}}{\alpha!}(z'-x)^\alpha R_\alpha(x,y)\right| \leq c(|x-y|)|x-y|^k\left(|z'-x|^k + 1\right). \tag{7.20}$$

The left hand side of this inequality is a polynomial in the variable $(z'-x)$ (with x fixed) of degree k. The coefficients of any such polynomial can be written as a linear combination of the value of the polynomial at any collection of points $a_0, \ldots, a_N \in \mathbb{R}^r$ (for some N). Since the value of this polynomial is bounded according to (7.20), it follows that the coefficients $\frac{|x-y|^{|\alpha|}}{\alpha!} R_\alpha(x,y)$ can be bounded in magnitude by $c(|x-y|)|x-y|^k$. That is,

$$|R_\alpha(x,y)| \leq c(|x-y|)|x-y|^{k-|\alpha|}$$

which completes the proof. $\square$

Since every smooth function satisfies condition (7.16), the above proposition implies the following

Lemma 104. *Suppose $f : \Omega \to \mathbb{R}$ is of class C^k for an open $\Omega \subset \mathbb{R}^r$. Then*

$$|T_x^k f(z) - T_y^k f(z)| \leq c(|x-y|)\left(|x-z|^k + |x-y|^k\right)$$

for every $x, y \in \Omega$ and $z \in \mathbb{R}^r$ where c is a modulus of continuity.

Notice that for any $x, y, z \in \mathbb{R}^r$,

$$
\begin{aligned}
|x-z|^k + |x-y|^k &\le \left(|x-z|^k + (|x-z| + |y-z|)^k\right) \\
&= |x-z|^k + \sum_{j=0}^{k} \binom{k}{j} |x-z|^j |y-z|^{k-j} \\
&\le |x-z|^k + C\left(|x-z|^k + |y-z|^k\right) \\
&\le C\left(|x-z|^k + |y-z|^k\right).
\end{aligned}
$$

We used Young's inequality with the exponents $\frac{k}{j}$ and $\frac{k}{k-j}$ in the second-to-last line. Similarly, $|x-z|^k + |y-z|^k \le C\left(|x-z|^k + |x-y|^k\right)$. Thus the statements

$$|T_x^k f(z) - T_y^k f(z)| \le c(|x-y|)\left(|x-z|^k + |x-y|^k\right)$$

and

$$|T_x^k f(z) - T_y^k f(z)| \le c(|x-y|)\left(|x-z|^k + |y-z|^k\right)$$

are equivalent. We will refer to these approximations interchangeably later.

7.4.2 Rough composition of mappings

We now prove the main result of the section.

Theorem 105 (Kneser-Glaeser)**.** *Fix positive integers s, k, m, n with $s < k$. Suppose $V \subset \mathbb{R}^r$ and $W \subset \mathbb{R}^n$ are open and $A^* \subset V$ and $A \subset W$ are compact. Let $g : V \to W$ be of class C^{k-s} with $g(A^*) \subset A$, and let $f : W \to \mathbb{R}$ be of class C^k. Suppose also that f is s-flat on A. Then there is a function $F : W \to \mathbb{R}$ of class C^k so that $F = f \circ g$ on A^* and F is s-flat on A^*.*

Proof. To begin, we will define a collection $\{h_\alpha\}_{|\alpha| \le k}$ of real valued functions on A^* so that

$$T_y^k h(z) + \sum_{|\lambda|=k+1} p_\lambda(x,z)(z-x)^\lambda = T_{g(x)}^k f(T_x^{k-s} g(z))$$

where each p_λ is a polynomial in x and z. Here, the notation $T_x^{k-s} g(z)$ refers to the coordinate-wise Taylor polynomial of g. That is, $T_x^{k-s} g(z)$ is a point in $\mathbb{R}^n$.

To define this collection, choose a ball B with $A^* \subset B$. Notice that we can write for any $x \in A^*$ and $z \in B$

$$T_y^k f(T_x^{k-s} g(z)) = f(y) + \sum_{s+1 \le |\nu| \le k} \frac{D^\nu f(y)}{\nu!} \left[\sum_{1 \le |\beta| \le k-s} \frac{D^\beta g(x)}{\beta!} (z-x)^\beta \right]^\nu \tag{7.21}$$

where $y = g(x)$.

Expand the right hand side of (7.21) and collect all terms of the form $a_\alpha(x)(z-x)^\alpha$ for each $|\alpha| \le k$. All remaining terms will have the form $p_\lambda(x,z)(z-x)^\lambda$ for some $|\lambda| = k+1$ where p_λ is a polynomial in x and z. Write $h_\alpha(x) = \alpha! a_\alpha(x)$.

First, notice that $h_\alpha(x) = 0$ for every $1 \le |\alpha| \le s$ since, in (7.21), $|\nu| \ge s+1$. Thus no terms of the form $(z-x)^\alpha$ appear when $1 \le |\alpha| \le s$. Also, the only term in (7.21) which does not contain any $(z-x)^\alpha$ with $|\alpha| \ge 1$ is $f(y)$, and so $h_0(x) = f(g(x))$. This leaves for every $x \in A^*$ and $z \in B$

$$T_x^k h(z) = \sum_{s+1 \le |\alpha| \le k} \frac{h_\alpha(x)}{\alpha!}(z-x)^\alpha = T_y^k f(T_x^{k-s} g(z)) - \sum_{|\lambda| = k+1} p_\lambda(x,z)(z-x)^\lambda.$$

By way of (7.17), it suffices to show that

$$|T_{x_1}^k h(z) - T_{x_2}^k h(z)| \le c(|x_1 - x_2|)(|z-x_1|^k + |z-x_2|^k)$$

for every $x_1, x_2 \in A^*$ and $z \in B$. In fact, it is enough to show that

$$|T_{y_1}^k f(T_{x_1}^{k-s} g(z)) - T_{y_2}^k f(T_{x_2}^{k-s} g(z))| \le c(|x_1 - x_2|)(|z-x_1|^k + |z-x_2|^k) \tag{7.22}$$

for every $x_1, x_2 \in A^*$ and $z \in B$ where $y_1 = g(x_1)$ and $y_2 = g(x_2)$. Indeed, we have

$$\begin{aligned} |(T_{x_1}^k h(z) - T_{y_1}^k f(T_{x_1}^{k-s} g(z))) - &(T_{x_2}^k h(z) - T_{y_2}^k f(T_{x_2}^{k-s} g(z)))| \\ &\le \sum_{|\lambda|=k+1} |p_\lambda(x_1,z)(z-x_1)^\lambda - p_\lambda(x_2,z)(z-x_2)^\lambda|. \end{aligned}$$

For each $|\lambda| = k+1$, we can rewrite the terms in the last line as

$$\left| [p_\lambda(x_1,z) - p_\lambda(x_2,z)] (z-x_1)^\lambda + p_\lambda(x_2,z) \left[(z-x_1)^\lambda - (z-x_2)^\lambda \right] \right|$$

The first term here is bounded by

$$|p_\lambda(x_1,z)-p_\lambda(x_2,z)|\left(|z-x_1|^k+|z-x_2|^k\right)=c(|x_2-x_1|)\left(|z-x_1|^k+|z-x_2|^k\right),$$

and we can bound the second term as follows by using the mean value theorem:

$$\begin{aligned}\left|(z-x_1)^\lambda-(z-x_2)^\lambda\right| &\le C|x_1-x_2|\left(t|z-x_1|+(1-t)|z-x_2|\right)^{|\lambda|-1}\\ &= C|x_1-x_2|\sum_{j=0}^{|\lambda|-1}\binom{|\lambda|-1}{j}|z-x_1|^j|z-x_2|^{|\lambda|-1-j}\\ &\le C|x_1-x_2|\left(|z-x_1|^{|\lambda|-1}+|z-x_2|^{|\lambda|-1}\right)\\ &= C|x_1-x_2|\left(|z-x_1|^k+|z-x_2|^k\right)\end{aligned}$$

where we used Young's inequality in the second-to-last line. Thus it remains to prove (7.22).

Now, rewrite the left hand side of (7.22) as

$$\left|\left[T^k_{y_1}f(T^{k-s}_{x_1}g(z))-T^k_{y_1}f(T^{k-s}_{x_2}g(z))\right]+\left[T^k_{y_1}f(T^{k-s}_{x_2}g(z))-T^k_{y_2}f(T^{k-s}_{x_2}g(z))\right]\right| \tag{7.23}$$

We can use Lemma 104 to bound the second term in brackets by

$$c(|g(x_1)-g(x_2)|)\left(|T^{k-s}_{x_2}g(z)-g(x_2)|^k+|g(x_1)-g(x_2)|^k\right).$$

Since g is of class C^{k-s}, we have $|g(x_1)-g(x_2)|\le C|x_1-x_2|$ and

$$|T^{k-s}_{x_2}g(z)-g(x_2)|\le\sum_{1\le|\beta|\le k-s}\frac{|D^\beta g(x_2)|}{\beta!}|z-x_2|^{|\beta|}\le C|z-x_2|$$

for a constant $C>0$ independent of the choices of x_2 and z since g is C^{k-s} on U, $A^*\subset U$ is compact, and $x_2,z\in B$. This gives the desired upper bound for the second bracketed term in (7.23).

Write $w_1=T^{k-s}_{x_1}g(z)$ and $w_2=T^{k-s}_{x_2}g(z)$. Applying Taylor's theorem to $T^k_{y_1}f$ centered at w_1, we have

$$T^k_{y_1}f(w_2)=\sum_{|\mu|\le k}\frac{D^\mu\left(T^k_{y_1}f(w_1)\right)}{\mu!}(w_2-w_1)^\mu,$$

and so we can bound the first bracketed term in (7.23) as follows:

$$\left|T^k_{y_1}f(w_1)-T^k_{y_1}f(w_2)\right|\le\sum_{1\le|\mu|\le k}\frac{1}{\mu!}\left|D^\mu\left(T^k_{y_1}f(w_1)\right)\right||w_2-w_1|^{|\mu|}. \tag{7.24}$$

Also, for each component g_j of g, we have by the statement after Lemma 104

$$|T_{x_2}^{k-s}g_j(z) - T_{x_1}^{k-s}g_j(z)| \le c(|x_2 - x_1|)(|z - x_2|^{k-s} + |z - x_1|^{k-s}).$$

Thus

$$|w_2 - w_1| = |T_{x_2}^{k-s}g(z) - T_{x_1}^{k-s}g(z)| \le c(|x_2 - x_1|)(|z - x_2|^{k-s} + |z - x_1|^{k-s}) \tag{7.25}$$

We will now find bounds for each term in the sum on the right hand side of (7.24). Consider first the case when $1 \le |\mu| \le s$. We may write

$$D^\mu\left(T_{y_1}^k f(w_1)\right) = T_{y_1}^{k-|\mu|}\left(D^\mu f(w_1)\right) = \sum_{|\gamma|\le k-|\mu|} \frac{D^{\mu+\gamma}f(y_1)}{\gamma!}(w_1 - y_1)^\gamma.$$

When $|\gamma| \le s - |\mu|$, we have $D^{\mu+\gamma}f(y_1) = 0$, and so

$$|D^\mu\left(T_{y_1}^k f(w_1)\right)| \le \sum_{s-|\mu|<|\gamma|\le k-|\mu|} \frac{|D^{\mu+\gamma}f(y_1)|}{\gamma!}|w_1 - y_1|^{|\gamma|} \le C|w_1 - y_1|^{s-|\mu|+1}$$

for a constant $C > 0$ independent of the choice of y_1 since f is C^k on W, $A \subset W$ is compact, and $g(A^*) \subset A$. As seen above, $|w_1 - y_1| = |T_{x_1}^{k-s}g(z) - g(x_1)| \le C|z - x_1|$, so by (7.25),

$$\begin{aligned}
&\left|D^\mu\left(T_{y_1}^k f(w_1)\right)\right||w_2 - w_1|^{|\mu|} \\
&\quad\le c(|x_2 - x_1|)\,|z - x_1|^{s-|\mu|+1}\left(|z - x_2|^{k-s} + |z - x_1|^{k-s}\right)^{|\mu|} \\
&\quad= c(|x_2 - x_1|)\sum_{j=0}^{|\mu|}\binom{|\mu|}{j}|z - x_2|^{(k-s)j}|z - x_1|^{(k-s)(|\mu|-j)+s-|\mu|+1} \\
&\quad\le c(|x_2 - x_1|)\left(|z - x_2|^{k|\mu|-s|\mu|+s-|\mu|+1} + |z - x_1|^{k|\mu|-s|\mu|+s-|\mu|+1}\right) \\
&\quad\le c(|x_2 - x_1|)\left(|z - x_2|^k + |z - x_1|^k\right).
\end{aligned}$$

In the second-to-last line, we used Young's inequality with the exponents $\frac{k|\mu|-s|\mu|+s-|\mu|+1}{(k-s)j}$ and $\frac{k|\mu|-s|\mu|+s-|\mu|+1}{(k-s)(|\mu|-j)+s-|\mu|+1}$, and in the last line we used the fact that $(k-s-1)(|\mu|-1) \ge 0$. This provides the desired bound for (7.24) when $1 \le |\mu| \le s$.

It remains to bound (7.24) when $s + 1 \le |\mu| \le k$. In this case,

$$|w_2 - w_1|^{|\mu|} \le C|w_2 - w_1|^{s+1} \le c(|x_2 - x_1|)(|z - x_2|^{k-s} + |z - x_1|^{k-s})^{s+1}$$

$$
\begin{aligned}
&\leq c(|x_2 - x_1|) \sum_{j=0}^{s+1} \binom{s+1}{j} |z - x_2|^{(k-s)j} |z - x_1|^{(k-s)(s+1-j)} \\
&\leq c(|x_2 - x_1|) \left(|z - x_2|^{|\mu|(k-s)(s+1)} + |z - x_1|^{(k-s)(s+1)} \right) \\
&\leq c(|x_2 - x_1|) \left(|z - x_2|^{k} + |z - x_1|^{k} \right)
\end{aligned}
$$

As before, we have used Young's inequality with the exponents $\frac{(k-s)(s+1)}{(k-s)j}$ and $\frac{(k-s)(s+1)}{(k-s)(s+1-j)}$, and in the last line we used the fact that $(k-s)(s+1) \geq k$. This provides the desired bound for (7.24) when $s+1 \leq |\mu| \leq k$, and thus we have appropriately bounded (7.23). This completes the proof. $\square$

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