THE SZEMERÉDI REGULARITY LEMMA

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The Szemerédi Regularity Lemma is a deep result in graph theory which roughly states that large, dense graphs can be approximated by random graphs. The lemma is most helpful in proofs where it may be hard to prove a result for a large graph but could be proven for a smaller random graph. This paper gives an overview of the lemma including relevant definitions and the proof of the theorem. The main importance of the theorem can be found in applications in several disciplines of mathematics such as extremal graph theory, Ramsey theory, and number theory. The main focus of the paper is to demonstrate the use of the lemma in several applications including the Triangle Removal Lemma, Roth's Theorem, the Erdős-Stone theorem and more.

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PREFACE

As with any major undertaking in life, I could not have written this paper without the help and support of numerous people in my life. I would like to specifically thank a few of these people.

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1.0 INTRODUCTION

Szemereédi's Regularity Lemma is one of the most important and powerful results in graph theory. Simply stated, the lemma tells us that for any large enough graph, if we partition the vertices into disjoint subsets of relatively the same size, then the edges between different subsets behave almost randomly.

Given a large, dense graph, that is, a graph with many vertices and the number of edges is close to the number of possible edges, such that the vertices are split into smaller subsets of the same size and one "leftover" set, the Regularity Lemma states that the edges between these subsets are almost random or well-distributed between the subsets; later, the notion of random will be properly defined.

The goal of this paper is to give an introduction to this amazing result in graph theory. After discussing the history of how the Regularity Lemma came to be and the proof, the majority of the paper will be about some of the applications of the Regularity Lemma. It is fitting that the main importance of the lemma comes from how it can be used to prove other results seeing how the lemma was originally just that, a lemma.

We have selected results from several disciplines to show the reach of the Regularity Lemma. We start with some of the earliest applications that were found for the lemma, namely the Triangle Removal Lemma and Roth's Theorem. We will then move our attention to an important result in extremal graph theory, the Erdős-Stone Theorem. Then, we switch focus to a bit of Ramsey Theory - another discipline with many results benefiting from the Regularity Lemma. After discussing embedding graphs into graphs, we continue the paper with a look at embedding trees into graphs before finally discussing the Green-Tao Theorem which is one of the better known applications of the Regularity Lemma (deserving of its own book) and one that we will only briefly discuss.

2.0 HISTORY

As with many results in mathematics, the Regularity Lemma can be traced back to other results. The first is a result from B.L. van der Waerden and is known as van der Waerden's Theorem [52].

Definition 1 (Coloring)**.**
Let $K = \{C_1, C_2, \ldots, C_r\}$ *be a set of* r *colors. A* ***coloring*** *of* $[n] \coloneqq \{1, 2, \ldots, n\}$ *is a function* $C : [n] \to K$*; that is, each integer is assigned a color. A coloring using* r *colors is referred to as an* ***r-coloring****.*

Recall that an *arithmetic progression* of length k is a sequence containing k integers, a, $a + d$, $a + 2d$, $\ldots$, $a + (k - 1)d$, with common difference d. Van der Waerden's Theorem guarantees a monochromatic arithmetic progression, that is, an arithmetic progression with all integers in the sequence colored one color. For example, let $K = \{R, B, G\}$ such that R denotes red, B denotes blue, and G denotes green. We define the coloring from the set [5] to K in the following way:

$$1 \to R$$

$$2 \to B$$

$$3 \to R$$

$$4 \to G$$

$$5 \to R$$

Because 1, 3, and 5 are colored red, there exists a monochromatic (red) arithmetic progression with $d = 2$ of length 3.

Theorem 1 (van der Waerden's Theorem)**.**
For any positive integers r and k, if the positive integers $\mathbb{Z}^+$ are colored with r colors, then there exists a monochromatic arithmetic progression of length k.

There is an equivalent finite version of the above theorem that gives rise to $W(r, k)$ which is known as the *van der Waerden number*.

Theorem 2 (van der Waerden's Theorem - Finite Version)**.**
For any positive integers r and k, there exists a smallest constant $W(r, k) \in \mathbb{Z}^+$ such that, for any $N \geq W(r, k)$, if the the set $\{1, 2, \ldots, N\}$ is colored with r colors, then there exists a monochromatic arithmetic progression of length k.

In other words, $W(r, k)$ is the smallest integer such that in a r-coloring of the integers $\{1, 2, \ldots, W(r, k)\}$, we are guaranteed a monochromatic arithmetic progression of length k and this $W(r, k)$ is only based upon the choice of r and k. There are numerous van der Waerden numbers that have been found, but one that is quite well known and also relatively easy to find is $W(2, 3)$, that is the smallest integer k such that for any $N \geq k$, the integers colored with two colors, there exists a monochromatic arithmetic progression of length 3. Using computing facilities, Vašek Chvátal proved that $W(2, 3) = 9$ as well as some other van der Waerden numbers such as $W(3, 3) = 27$ and $W(2, 4) = 35$ [14].

Next, we look at an example of a 2-coloring of [8] that does not contain a monochromatic arithmetic progression of length 3, but does contain a progression with the addition of a ninth colored integer. Consider the following coloring of the set [8] to $K = \{R, B\}$, where R denotes red and B denotes blue:

RBBRRBBR

There does not currently exist an arithmetic progression of length 3, so this tells us that $W(2, 3) \neq 8$. Now, consider if the next color is red.

RBBRRBBRR

Then, the integers corresponding to the red color are 1,4,5,8, and 9. So, there exists an arithmetic progression of length 3 with $a = 1$ and $d = 4$, namely $\{1, 5, 9\}$. Now, what would happen if we had instead made the next color blue?

RBBRRBBRB

Now, the integers that correspond to the blue color are 2,3,6,7, and 9. So, there exists an arithmetic progression of length 3 with $a = 3$ and $d = 3$, namely $\{3, 6, 9\}$. These are just two examples of a 2-coloring of [9], but because $W(2, 3) = 9$, we know that every possible 2-coloring ($2^9 = 512$ possibilities to be exact) of [9] will contain a monochromatic arithmetic progression of length 3.

The van der Waerden Theorem is seen as the precursor to Szemerédi's Theorem which is discussed next.

The direct history of the regularity lemma starts with a conjecture made by Paul Erdős and Paul Turán. They wanted to know how large a subset of a finite domain could be if it did not contain any arithmetic progression with length k. In 1936, they proposed that a set without an arithmetic progression of length k has a density that approaches 0 as n goes to infinity [28]; here, density is a measure of how large a set is compared to the natural numbers. Endre Szemerédi proved the conjecture in 1975 in what is now known as Szemerédi's Theorem [49]. Szemerédi's Theorem improves upon the aforementioned van der Waerden's Theorem.

In the process of proving his theorem, Szemerédi proved a weaker version of what would eventually become the regularity lemma. In the weaker version, the graphs were restricted to bipartite graphs. Then, in 1978, Szemerédi proved the full version giving us the Regularity Lemma.

Before considering the Regularity Lemma, we first take a look at Szemerédi's Theorem. While we will not go through the entire proof of the theorem, as it is quite complex (a 46 page paper! [49]) for our presentation, we will look specifically at a lemma used in its proof.

The original statement of Szemerédi's Theorem utilizes the following definition.

Definition 2 (Upper Density)**.**
Let A be a subset of the positive integers. Then, the ***upper density*** *of A, denoted $\bar{d}(A)$, is*

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1, N]|}{N}.$$

Note that for any finite set A, $\bar{d}(A) = 0$. For the set of natural numbers $\mathbb{N}$, we have $\bar{d}(\mathbb{N}) = 1$. If we put $E = \{2n \mid n \in \mathbb{Z}^+\}$, i.e. the set of positive even integers, then $\bar{d}(E) = \frac{1}{2}$ and similarly for any arithmetic progression, if $F = \{a + nd \mid n \in \mathbb{Z}^+\}$, then $\bar{d}(F) = \frac{1}{d}$ (note the set of even numbers is an arithmetic progression with $a = 2$ and $d = 2$, so $\bar{d}(E) = \frac{1}{d} = \frac{1}{2}$). If P denotes the set of prime numbers, then by the Prime Number Theorem $\bar{d}(P) = 0$.

Theorem 3 (Szemerédi's Theorem - Original Statement)**.**
Let A be any subset of the positive integers with positive upper density, that is, $\bar{d}(A) > 0$. Then, for all k, A contains infinitely many arithmetic progressions of length k.

Theorem 3 is the theorem that Szemerédi proved in his paper, however, what is currently referenced as Szemerédi's Theorem was originally a corollary in Szemerédi's paper and is stated next.

Theorem 4 (Szemerédi's Theorem)**.**
For all $0 < \epsilon \leq 1$ and $k \in \mathbb{Z}^+$. If there exists $N = N(k, \epsilon)$ such that for all $n \geq N$ and $A \subseteq [n]$, where $|A| \geq \epsilon n$, then A contains an arithmetic progression of length k.

Szemerédi's original proof was combinatorial in nature, however, other mathematicians have proven the theorem using other methods. In 1977, just two years after Szemerédi's proof, a second proof to Szemerédi's Theorem was provided by Harry Furstenburg using ergodic theory [29, 30]. W.T. Gowers offered a third proof in 2001 that utilized Fourier analysis [34]. As well as different proofs, there have also been extensions to Szemerédi's Theorem; most notably is the Green-Tao Theorem (Theorem 15) which we will discuss briefly in Section 5.6.

In a talk given in 1976 (and appearing in a book in 1977) Erdős expanded his original conjecture on arithmetic progressions to the following conjecture:

Conjecture 1.
Let $a_1 < a_2 < \ldots$ be a sequence of integers such that $\sum \frac{1}{a_i} = \infty$, then the sequence contains an arithmetic progression of arbitrary length.

Erdős offered \$3000 for the proof of this conjecture, but never planned on having to pay the reward due to the difficulty of the problem [24]. Erdős later raised the amount to \$5000 and said he would leave money for the prize when he passed to be paid by Ronald Graham, a longtime friend and fellow mathematician [48].

3.0 DEFINITIONS AND EXAMPLES

We begin by providing relevant definitions for our discussion and note that a primer on graph theory is offered in the Appendix. For a more in-depth look at graph theory, refer to [12] or [37]. Before we dive into the formal statement and applications of the Regularity Lemma (Theorem 5), we need to define and understand several terms that our critical for our discussion.

3.1 DENSITY

We begin with a simple definition of the density of a graph, more specifically, the density between a pair of vertex subsets. We only consider simple graphs, that is, graphs with no repeated edges or loops, and the majority of graphs considered in this paper will be *dense graphs* meaning that the density is on the higher side and thus close to one. On the other hand, a graph that is not dense is called *sparse*, and the density of a sparse graph is closer to zero.

To determine whether a graph, $G = (V, E)$, is dense or sparse, we must find the ratio between the number of edges, $|E|$, and the maximum number of edges possible, $\frac{1}{2}|V|(|V|-1)$.

Definition 3 (Density of a Graph)**.**
The ***density of a graph*** *G, denoted $d(G)$, is*

$$d(G) = \frac{|E|}{\frac{|V|(|V|-1)}{2}} = \frac{2|E|}{|V|(|V|-1)}.$$

Complete graphs will always have a density of one, while a graph with no edges (an empty graph) will have a density of zero, hence $0 \leq d(G) \leq 1$.

Our discussion mainly requires the density between sets of vertices, hence the next definition.

Definition 4 (Density of a Pair of Vertex Subsets)**.**
*If $G = (V, E)$ is a graph with X and Y nonempty, disjoint subsets of vertices, then the **density of the pair** (X, Y) is defined as*

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|},$$

where $e(X, Y)$ is the number of edges with one incident vertex in X and the other in Y.

Similar to the density of a graph, if a pair of sets of vertices does not have any edges between them, then the density is 0. However, if every vertex of subset X is connected to every vertex of subset Y, then $e(X, Y) = |X||Y|$ and the density of (X, Y) is 1. Thus for any pair (X, Y), we have $0 \leq d(X, Y) \leq 1$.

Example 1 and Example 2 show how to calculate the density of vertex pairs in a bipartite graph (only one vertex pair to consider) and in a tripartite graph (three different pairs to consider). Example 2 also finds the density of the tripartite graph itself.

Example 1.
To calculate the density of the pair in Figure 1, we note that we have 6 edges between the sets X and Y. In addition, $|X| = 6$ and $|Y| = 4$. Thus,

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|} = \frac{6}{24} = \frac{1}{4}.$$

Example 2.
In the tripartite graph in Figure 2, we consider the pairs (A, B), (A, C), and (B, C). The densities for each pair are:

$$d(A, B) = \frac{e(A, B)}{|A||B|} = \frac{5}{9};\ d(A, C) = \frac{e(A, C)}{|A||C|} = \frac{3}{9} = \frac{1}{3};\ d(B, C) = \frac{e(B, C)}{|B||C|} = \frac{4}{9}.$$

We can also calculate the density of the entire graph G: counting all of the edges gives 12 edges total and $e(K_9) = 36$, so $d(G) = \frac{12}{36} = \frac{1}{3}$.

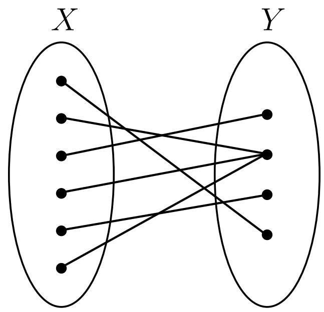


Figure 1: A Bipartite Graph

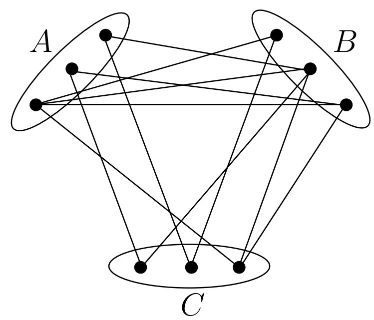


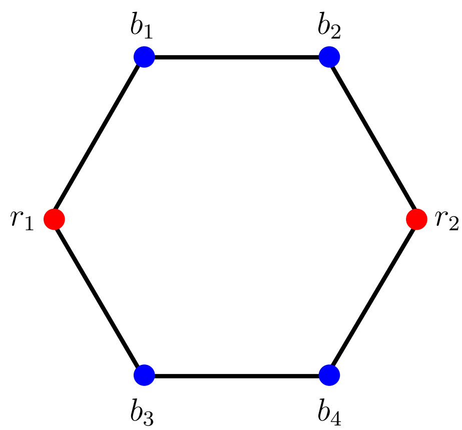
Figure 2: A Tripartite Graph

So far, the density examples have been k-partite graphs already partitioned into k sets. Next, we show how to find the density of a pair of vertex subsets of a cycle graph with less natural selections of X and Y.

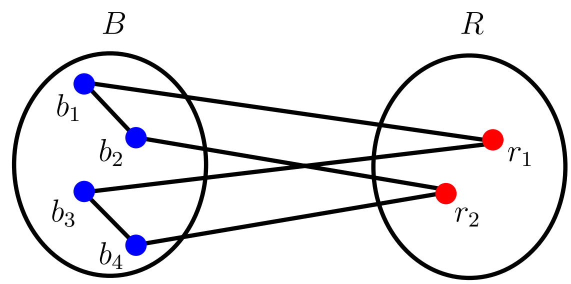
Example 3.

We will compute the density between the vertex pair (B, R) *formed by the vertices of the cycle graph of length six shown in Figure 3. We let* $B = \{b_1, b_2, b_3, b_4\}$ *and* $R = \{r_1, r_2\}$*. The density between the pair* (B, R) *is*

$$d(B, R) = \frac{e(B, R)}{|B||R|} = \frac{4}{4 \cdot 2} = \frac{4}{8} = \frac{1}{2}.$$



(a) The Cycle Graph C_6



(b) Vertex Pair (B, R)

Figure 3: The Cycle Graph C_6 and Vertex Pair (B, R)

3.2 ϵ-REGULARITY

Szemerédi's Regularity Lemma (Theorem 5) centers around vertex pairs that are ϵ-regular, so it will be quite useful to understand what is meant by ϵ-regularity.

Definition 5 (ϵ-Regularity)**.**
*Let $0 < \epsilon \leq 1$ be given and let $G = (V, E)$ be a graph. For two disjoint non-empty sets of vertices X and Y, the pair (X, Y) is said to be **ϵ-regular** if for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$,*

$$|d(A, B) - d(X, Y)| \leq \epsilon.$$

Thus, a pair (X, Y) of subsets of vertices in a graph are ϵ-regular if big enough subsets of X and Y have densities not very different from the density of (X, Y). While we allow $\epsilon \leq 1$, in that case that $\epsilon = 1$, a vertex pair will always be ϵ-regular, so we are most interested in the case when $\epsilon < 1$. Note, if $\epsilon = 1$, then the only subsets to consider are X and Y themselves because we require $A \subseteq X$ and $|A| \geq 1 \cdot |X|$ which implies $A = X$ and likewise $B = Y$.

If a pair does not meet this condition, we say the pair is *ϵ-irregular*. The following examples demonstrate ϵ-regularity and ϵ-irregularity: in Example 4, we look at a pair that does not meet the ϵ-regularity condition for a chosen ϵ; in Example 5, we consider a pair that is ϵ-regular for some values of ϵ and ϵ-irregular for other values.

Example 4.

We will show that the pair (X, Y) in Figure 4 is ϵ-irregular for a chosen positive $\epsilon < 1$. Note, $d(X, Y) = \frac{3}{4}$. In X, we have the three possible subsets as $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, and $A_3 = \{x_1, x_2\}$. Similarly in Y, we have $B_1 = \{y_1\}$, $B_2 = \{y_2\}$, and $B_3 = \{y_1, y_2\}$.

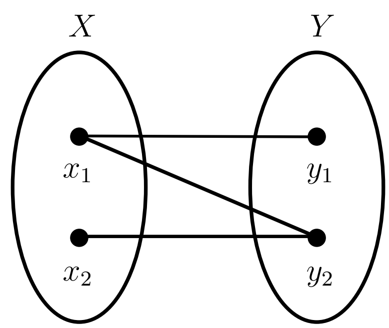


Figure 4: ϵ-Irregular Pair

Following the definition of ϵ-regularity, we find a working ϵ by comparing the cardinalities of the subsets to the cardinalities of X and Y. For $i = 1, 2$, we have

$$|A_i| \geq \epsilon|X| \implies \epsilon \leq \frac{1}{2} \text{ and } |B_i| \geq \epsilon|X| \implies \epsilon \leq \frac{1}{2}.$$

In addition,

$$|A_3| \geq \epsilon|X| \implies \epsilon \leq 1 \text{ and } |B_3| \geq \epsilon|X| \implies \epsilon \leq 1.$$

We will show (X, Y) is ϵ-irregular for $0 < \epsilon \leq \frac{1}{2}$. We must check that for at least one pair, (V_i, V_j) such that $1 \leq i \leq 3$ and $1 \leq j \leq 3$, we have $|d(A_i, B_j) - d(X, Y)| > \frac{1}{2}$.

The densities of the subsets are listed in Table 1. When $i = 2$ and $j = 1$, $d(A_2, B_1) = 0$ and so

$$|d(A_2, B_1) - d(X, Y)| = \frac{3}{4} > \frac{1}{2}$$

	B_1	B_2	B_3
A_1	1	1	1
A_2	0	1	$\frac{1}{2}$
A_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$

Table 1: Densities for subsets of irregular pair (X, Y)

which contradicts our choice of ϵ. Thus, (X, Y) is $\frac{1}{2}$-irregular and in fact, is ϵ-irregular for all $\epsilon \leq \frac{1}{2}$. Note that in general the same graph can be ϵ-regular for one ϵ, but ϵ-irregular for a different ϵ. In this example, if $\epsilon > \frac{1}{2}$, the pair is regular because the only large subset to consider would be the sets themselves and the difference between the densities is zero.

Example 5.

We will show that the pair (X, Y) in Figure 5 is $\frac{1}{2}$-regular; that is, for all $|A_i| \geq \frac{1}{2}|X|$ and $|B_j| \geq \frac{1}{2}|Y|$, we have $|d(A_i, B_j) - d(X, Y)| \leq \frac{1}{2}$. We have the subsets $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, $A_3 = \{x_3\}$, $A_4 = \{x_1, x_2\}$, $A_5 = \{x_1, x_3\}$, $A_6 = \{x_2, x_3\}$, and $A_7 = \{x_1, x_2, x_3\}$ of the set X, and we have the subsets $B_1 = \{y_1\}$, $B_2 = \{y_2\}$, $B_3 = \{y_3\}$, $B_4 = \{y_1, y_2\}$, $B_5 = \{y_1, y_3\}$, $B_6 = \{y_2, y_3\}$, and $B_7 = \{y_1, y_2, y_3\}$ of the set Y.

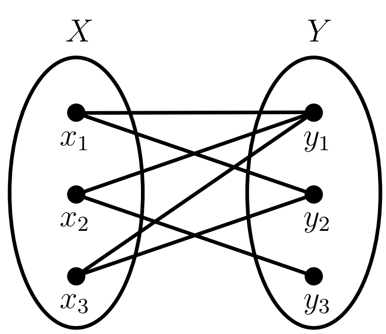


Figure 5: ϵ-Regular Pair

We only consider the subsets with cardinality greater than or equal to $\epsilon|X| = \epsilon|Y| = \frac{3}{2}$, i.e.

subsets with two or three vertices. Note that we have $|A_i| \geq \frac{3}{2}$ *and* $|B_i| \geq \frac{3}{2}$ *for* $4 \leq i, j \leq 7$*. Next, we compute the densities between each pair of subsets of* X *and* Y *which are stated in Table 2. Note,* $d(X,Y) = \frac{6}{9} = \frac{2}{3}$*.*

	B_4	B_5	B_6	B_7
A_4	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{2}{3}$
A_5	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$
A_6	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{2}{3}$
A_7	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

Table 2: Densities for subsets of regular pair (X, Y)

Then we compute the absolute values of the difference between the densities of the subsets and the density of the pair.

$$|d(A_i, B_j) - d(X,Y)| = \begin{cases} 0 & \text{if } j = 7, \text{ and } (i,j) = (7,5) \\ \frac{1}{12} & \text{if } i = 4, 6 \text{ and } j = 4, 5 \\ \frac{1}{6} & \text{if } j = 6 \text{ and } (i,j) = (5,5) \text{ and } (7,4) \\ \frac{1}{3} & \text{if } i = 5 \text{ and } j = 4 \end{cases}$$

We can see that in all cases the absolute value of the difference between densities is less than $\frac{1}{2}$*; hence, the pair in Figure 5 is* $\frac{1}{2}$*-regular as desired.*

Now that we understand what it means for a pair (X, Y) to be ϵ-regular, we only need a few more definitions. The first we consider is a particular partition of the vertices.

Definition 6 (Equipartition)**.**
A vertex partition of disjoint sets $V = V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_k$ *is called an* ***equipartition*** *if* $|V_1| = |V_2| = \ldots = |V_k|$ *with* V_0 *being the* ***exceptional set****.*

Note the size of the exceptional set is allowed to differ from the size of the rest of the sets. After the vertices of a graph are separated into k subsets of the same size, the remaining (or leftover) vertices become elements of the exceptional set; this allows us to create an equipartition of a graph of any size. Furthermore, the exceptional set is allowed to be empty as well which is the case if k divides $|V|$.

Now, we combine the idea of an *equipartition* with the condition of ϵ-*regularity* to define an ϵ-*regular partition*:

Definition 7 (ϵ-Regular Partition)**.**
*Let ϵ be fixed and let $G = (V, E)$ be a graph. An equipartition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ is said to be an **ϵ-regular partition** if $|V_0| \leq \epsilon|V|$ and at most ϵk^2 pairs (V_i, V_j), which satisfy $1 \leq i < j \leq k$, are ϵ-irregular.*

An example of an ϵ-regular partition is included next.

Example 6.
In this example, we will show that the equipartition $V = V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ of a graph G, shown in Figure 6, is a $\frac{2}{5}$-regular partition.

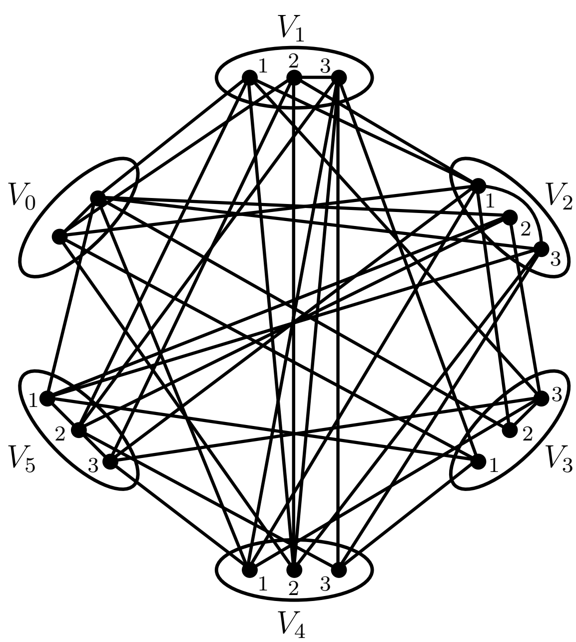


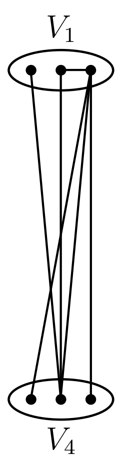
Figure 6: ϵ-Regular Partition

First, we check 2 is less than $\frac{2}{5} \cdot 17$*, which it is, to meet the condition that* $|V_0| \leq \epsilon|V|$*. The only other condition to meet to ensure that the equipartition is a* $\frac{2}{5}$*-regular partition is to determine how many* $\frac{2}{5}$*-regular pairs exist in the partition. The partition must have less than* $\epsilon k^2 = \frac{2}{5} \cdot 5^2 = 10$ *irregular pairs to be a* $\frac{2}{5}$*-regular partition.*

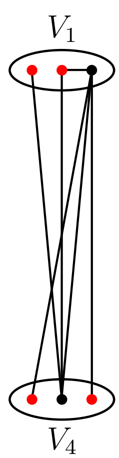
Because $\epsilon = \frac{2}{5}$*, we only consider subsets* A *of* V_j $(1 \leq j \leq 5)$ *such that* $|A| \geq \frac{2}{5}|V_j| = \frac{6}{5}$*. Let* A_i *denote the subsets of* V_1*,* B_i *denote the subsets of* V_2*,* C_i *denote the subsets of* V_3*,* D_i *denote the subsets of* V_4*, and* F_i *denote the subsets of* V_5*. Each* V_i *has seven subsets and the subsets of* V_1 *are the following:*

$$A_1 = \{a_1\} \qquad A_2 = \{a_2\} \qquad A_3 = \{a_3\}$$
$$A_4 = \{a_1, a_2\} \qquad A_5 = \{a_1, a_3\} \qquad A_6 = \{a_2, a_3\}$$
$$A_7 = \{a_1, a_2, a_3\}$$

We only consider the subsets numbered four through seven because the cardinality of those subsets is greater that $\frac{6}{5}$*. Now, we must determine which of the ten pairs* (V_i, V_j)*,* $1 \leq i < j \leq 5$*, are* $\frac{2}{5}$*-regular and which are* $\frac{2}{5}$*-irregular. Table 3 lists the densities of the subsets of* V_1*,* V_2*,* V_3*,* V_4*, and* V_5*.*



(a) (V_1, V_4)



(b) Subset of (V_1, V_4) with density zero.

Figure 7: (V_1, V_4) is an irregular pair

	B_4	B_5	B_6	B_7	C_4	C_5	C_6	C_7	D_4	D_5	D_6	D_7	F_4	F_5	F_6	F_7
A_4	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
A_5	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
A_6	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
A_7	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
B_4					$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
B_5					$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$
B_6					0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
B_7					$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{4}{9}$
C_4									0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$
C_5									$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$
C_6									$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
C_7									$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{9}$
D_4													0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
D_5													$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
D_6													$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
D_7													$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{9}$

Table 3: Densities for subsets of V_1, V_2, V_3, V_4, and V_5

The $\frac{2}{5}$-regular pairs are (V_1, V_2), (V_1, V_3), (V_1, V_5), (V_2, V_3), (V_2, V_4), (V_2, V_5), (V_3, V_4), (V_3, V_5), and (V_4, V_5). The only $\frac{2}{5}$-irregular pair is (V_1, V_4).

From Figure 7a, we find $d(V_1, V_4) = \frac{5}{9}$. If we consider the subsets containing the vertices highlighted in red in Figure 7b (which corresponds to the subsets A_4 and D_5), then from Table 3, $d(A_4, D_5) = 0$. Then

$$|d(A_4, D_5) - d(V_1, V_4)| = \frac{5}{9} > \frac{2}{5}$$

which implies (V_1, V_4) is an irregular pair.

Because the size of the exceptional set and the number of irregulars pairs are both bounded, the equipartition in Figure 6 is a $\frac{2}{5}$-regular partition as desired.

3.3 INDEX OF A PARTITION

We have stated the definitions that are needed for the formal statement of the Regularity Lemma and now state two definitions which are needed for other parts of this paper. The first will be needed for the proof of the Regularity Lemma (Theorem 5). The index is the mean square density of P (as noted in [6]) and is a measure of how regular a partition is.

Definition 8 (Index)**.**
*If $V = V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_k$ is an equipartition of a graph G, then we define the **index of V** as*

$$ind(V) = \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} [d(V_i, V_j)]^2$$

Recall, $d(V_i, V_j) \leq 1$ for any pair of vertex subsets. Therefore, we see that for any V

$$ind(V) \leq \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} 1 = \frac{1}{k^2} \sum_{i=1}^{k} (k-i) = \frac{1}{k^2} \cdot \frac{1}{2} k(k-1) = \frac{k-1}{2k} < \frac{1}{2}.$$

3.4 REDUCED GRAPH

Our last definition is needed in a few of the applications that we will discuss later in the paper. In the chapter of applications (Chapter 5), one technique using the Regularity Lemma is to create a *reduced graph.*

Definition 9 (Reduced Graph)**.**
*Given a graph $G = (V, E)$ and an ϵ-regular partition $P = V_0 \cup V_1 \cup \ldots \cup V_k$ of V, such that $|V_1| = \ldots = |V_k| = \ell$, the **reduced graph**, R, is the graph formed by vertices $V_1, V_2, \ldots, V_k$ such that V_iV_j is an edge if the pair (V_i, V_j) is a ϵ-regular pair with density at least $d \in (0, 1]$. We may also denote the reduced graph R as the **(ϵ, d)-reduced graph**.*

Furthermore, we can take the reduced graph R one step farther and construct the graph R_s. To construct R_s, we first replace each vertex, V_i, of R with a set of s vertices, where s is any positive integer, which can be denoted $V_i(s)$. Then we replace the edges in R (that were formed between ϵ-regular pairs) with complete bipartite graphs between the newly constructed s-sets. This means if V_iV_j was an edge in R, then $(V_i(s), V_j(s))$ becomes a complete bipartite graph in R_s.

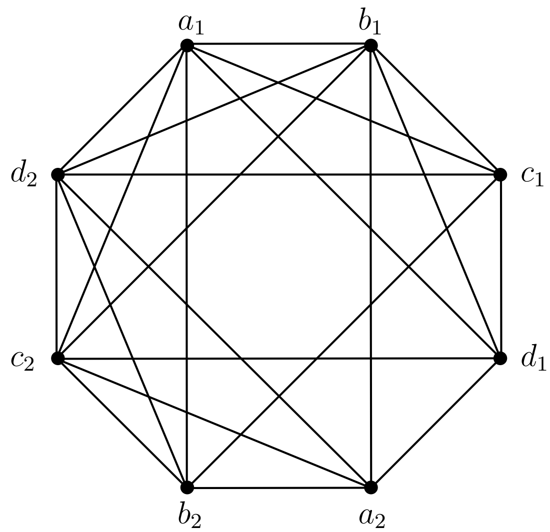
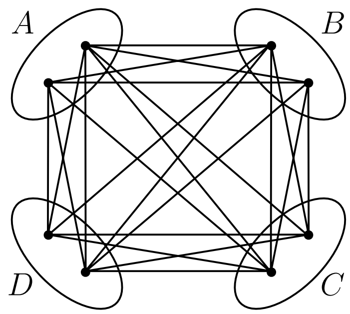


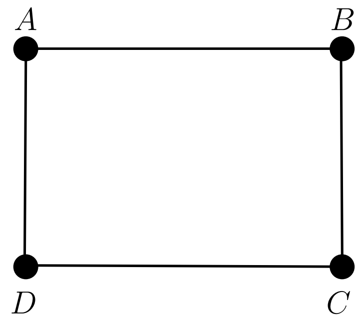
Figure 8: Graph G to construct R

To see how a reduced graph is created, first consider the graph $G = (V, E)$ in Figure 8. G contains eight vertices that we can partition into four disjoint vertex subsets each with cardinality two. For a reduced graph R, consider the partition $P = V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_1 = A = \{a_1, a_2\}$, $V_2 = B = \{b_1, b_2\}$, $V_3 = C = \{c_1, c_2\}$, and $V_4 = D = \{d_1, d_2\}$. So, we have $|V_1| = |V_2| = |V_3| = |V_4| = 2$ and $|V_0| = 0$. The graph with the partition $P = V_1 \cup V_2 \cup V_3 \cup V_4 = A \cup B \cup C \cup D$ is shown in Figure 9a.

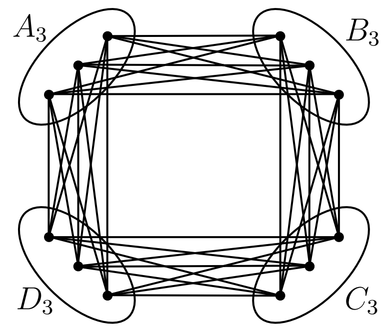
Now, to construct the reduced graph R, we must determine which pairs are ϵ-regular and which ones are not. For A, B, C, and D, we have three possible subsets for each set. We have $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_1, a_2\}$. We have similar subsets for B, C, and D.



(a) Partition $P = A \cup B \cup C \cup D$



(b) Reduced graph R of P



(c) R_3, the graph formed from R with 3 vertices in each subset and each pair is a complete bipartite graph.

Figure 9: Reduced Graph Progression

For $i = 1, 2$, we have

$$|A_i| \geq \epsilon|A| \implies \epsilon \leq \frac{1}{2};\ |B_i| \geq \epsilon|B| \implies \epsilon \leq \frac{1}{2}$$
$$|C_i| \geq \epsilon|C| \implies \epsilon \leq \frac{1}{2};\ |D_i| \geq \epsilon|D| \implies \epsilon \leq \frac{1}{2}$$

Then, for $i = 3$, we have

$$|A_3| \geq \epsilon|A| \implies \epsilon \leq 1;\ |B_3| \geq \epsilon|B| \implies \epsilon \leq 1$$
$$|C_3| \geq \epsilon|C| \implies \epsilon \leq 1;\ |D_3| \geq \epsilon|D| \implies \epsilon \leq 1$$

So, we will choose $\epsilon = \frac{1}{2}$. Below we state the densities of all possible pairs.

$$d(A, B) = 1 \qquad d(B, C) = 1$$
$$d(A, C) = \frac{3}{4} \qquad d(B, D) = \frac{3}{4}$$
$$d(A, D) = 1 \qquad d(C, D) = 1$$

	B_1	B_2	B_3	C_1	C_2	C_3	D_1	D_2	D_3
A_1	1	1	1	1	1	1	1	1	1
A_2	1	1	1	0	1	$\frac{1}{2}$	1	1	1
A_3	1	1	1	$\frac{1}{2}$	1	$\frac{3}{4}$	1	1	1
B_1				1	1	1	1	1	1
B_2				1	1	1	0	1	$\frac{1}{2}$
B_3				1	1	1	$\frac{1}{2}$	1	$\frac{3}{4}$
C_1							1	1	1
C_2							1	1	1
C_3							1	1	1

Table 4: Densities for subsets of A, B, C, and D

Next, we want to determine which pairs are $\frac{1}{2}$-regular and which pairs are $\frac{1}{2}$-irregular. Recall we are checking if the difference between the densities of the sets and densities of the subsets are less than or equal to $\frac{1}{2}$. Utilizing the values in Table 4, for $1 \leq i, j \leq 3$, we have the following:

$$|d(A_i, B_j) - d(A, B)| \leq \frac{1}{2} \qquad |d(A_i, D_j) - d(A, D)| \leq \frac{1}{2}$$
$$|d(B_i, C_j) - d(B, C)| \leq \frac{1}{2} \qquad |d(C_i, D_j) - d(C, D)| \leq \frac{1}{2}$$

However, for the pairs (A, C) and (B, D), we have specific values of i and j such that

$$|d(A_2, C_1) - d(A, C)| > \frac{1}{2} \text{ and } |d(B_3, D_1) - d(B, D)| > \frac{1}{2}.$$

From our calculations, the regular pairs are (A, B), (A, D), (B, C), and (C, D) while the irregular pairs are (A, C) and (B, D).

By the definition of a reduced graph, there must be an ϵ-regular partition which means we need to check that we have at most ϵk^2 irregular pairs. From our calculations above, we have four regular pairs and two irregular pairs. We have $\epsilon = \frac{1}{2}$ and $k = 4$, so $\epsilon k^2 = \frac{1}{2}(4^2) = 8$. So, we meet the condition on the number irregular pairs. Also, we must have $|V_0| \leq \epsilon|V|$ and we meet this condition immediately because $|V_0| = 0$. So, our partition P is an ϵ-regular partition with $\epsilon = \frac{1}{2}$.

We are now ready to construct the reduced graph R, shown in Figure 9b, by replacing each vertex set with a single vertex and constructing an edge wherever there is a regular pair. Because we have 4 vertex sets, we have 4 vertices in R and because the regular pairs are (A, B), (B, C), (C, D), and (A, D), we construct the edges AB, BC, CD, and AD. Furthermore the pairs (A, C) and (B, D) are irregular, so we do not construct edges between these pairs.

The last step is to construct R_s from R for some positive integer s. For our example, we will replace each vertex with a set of three vertices. So, the vertex A in R will become a set of three vertices denoted A_3. We replace B, C, and D similarly with sets of three vertices denoted B_3, C_3, and D_3. The next step is to replace each edge by a complete bipartite graph between the vertex sets. For example, the edge AB becomes a complete bipartite graph between A_3 and B_3 with nine edges. The graph R_3 can been seen in Figure 9c. Going from R to R_s, we could have chosen any positive integer for s and the process would have been the same. However, for demonstration purposes, $s = 3$ was a manageable integer.

4.0 FORMAL STATEMENT AND PROOF

Equipped with the relevant definitions, we are ready to formally state the Szemerédi Regularity Lemma. Recall we stated that the lemma roughly states that dense graphs can be approximated by random graphs. Naturally, we want to see what this looks like mathematically.

Theorem 5 (Szemerédi's Regularity Lemma)**.**
For every $\epsilon \in (0,1]$ and $m > 0$, there exist integers N and $M = M(m,\epsilon)$ such that for every $n \geq N$, every graph $G = (V,E)$ with n vertices has an ϵ-regular partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$, which satisfies $m \leq k \leq M$.

Following Szemerédi's approach in [50], we need one main lemma in order to prove the Regularity Lemma. We will discuss the proof of Lemma 1 after the proof of the Regularity Lemma.

Lemma 1.
Let $G = (V,E)$ be a graph with n vertices. We let P be an equipartition of V into classes $V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_k$, where V_0 is the exceptional class. Let $0 < \epsilon \leq 1$ be given such that $4^k > 600\epsilon^{-5}$. If more than ϵk^2 pairs are ϵ-irregular, then there exists an equipartition Q of V into $1 + k4^k$ classes such that the size of the exceptional class increases by at most $\frac{n}{4^k}$, that is, $|Q_0| \leq |V_0| + \frac{n}{4^k}$ and

$$ind(Q) > ind(P) + \frac{\epsilon^5}{20}.$$

Proof of Theorem 5. Let s be the smallest integer such that $4^s > 600\epsilon^{-5}$, where $s > m$ and

$s > \frac{2}{\epsilon}$. Let us define the sequence $f(t)$ as follows:

$$f(t) = \begin{cases} s & \text{if } t = 0 \\ f(t-1)4^{f(t-1)} & \text{otherwise} \end{cases}$$

We let t be the largest nonnegative integer such that G has an equipartition, P, into $1 + f(t)$ classes such that $ind(P) \geq \frac{t\epsilon^5}{20}$. Also, the size of the exceptional class of the partition is at most $\epsilon n \left(1 - 2^{-(t+1)}\right)$. We can see that this partition exists if $t = 0$, then we have $1 + s$ subsets, the size of the exceptional class is at most $\frac{\epsilon}{2} n < \epsilon n$, and $ind(P) \geq 0$. Previously, we showed $ind(P) < \frac{1}{2}$ for every partition, so t is well defined and in fact, $0 \leq t < \frac{10}{\epsilon^5}$. This implies there are a finite number of refinements that can be made by Lemma 1. By t being the largest integer such that G has an equipartition and Lemma 1, P is ϵ-regular as desired and we can set $M = f\left(\left\lfloor \frac{10}{\epsilon^5} \right\rfloor\right)$.

□

With the use of Lemma 1, the proof of the Regularity Lemma is quite short. In his paper, Szemerédi used a defect form of the Cauchy-Schwarz inequality in his proof of Lemma 1 along with a fact about the continuity of density. The fact states that for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| \geq (1-\delta)|X|$ and $|B| \geq (1-\delta)|Y|$ $\left(0 < \delta < \frac{1}{3}\right)$, then

$$|d(A,B) - d(X,Y)| < 6\delta.$$

The following form of the Cauchy-Schwarz Inequality is another main ingredient in the proof of Lemma 1.

Lemma 2 (Defect Form of Cauchy-Schwarz Inequality)**.**
If, for $m \leq n$,

$$\sum_{k=1}^{m} x_k = \frac{m}{n} \sum_{k=1}^{n} x_k + \delta,$$

then

$$\sum_{k=1}^{n} x_k^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} x_k \right)^2 + \frac{\delta^2 n}{m(n-m)}.$$

While Szemerédi did not include a proof of Lemma 2, we have included one (following the procedure in [6]) to show how it follows nicely from the Cauchy-Schwarz Inequality.

Proof of Lemma 2. Suppose for $m \leq n$,

$$\sum_{k=1}^{m} x_k = \frac{m}{n} \sum_{k=1}^{n} x_k + \delta,$$

. Then define

$$y_n = \frac{1}{n} \sum_{k=1}^{n} x_k \text{ and } y_m = \frac{1}{m} \sum_{k=1}^{m} x_k$$

which implies $y_m = y_n + \frac{\delta}{m}$.

Then,

$$\begin{aligned}
\sum_{k=1}^{n} x_k^2 &= \sum_{k=1}^{m} x_k^2 + \sum_{k=m+1}^{n} x_k^2 \\
&\geq \frac{1}{m} \left(\sum_{k=1}^{m} x_k \right)^2 + \frac{1}{n-m} \left(\sum_{k=m+1}^{n} x_k \right)^2 \text{ by the Cauchy-Schwarz Inequality} \\
&= m \left(\frac{1}{m} \sum_{k=1}^{m} x_k \right)^2 + (n-m) \left(\frac{1}{n-m} \sum_{k=m+1}^{n} x_k \right)^2 \\
&= m y_m^2 + \frac{(n y_n - m y_m)^2}{n-m} \\
&= \frac{n m y_m^2 - m^2 y_m^2 + n^2 y_n^2 - 2 n y_n m y_m + m^2 y_m^2}{n-m} \\
&= \frac{n^2 y_n^2 - n m y_n^2 + n m y_n^2 + n m y_m^2 - m^2 y_m^2 - 2 n y_n m y_m + m^2 y_m^2}{n-m} \\
&= \frac{n^2 y_n^2 - n m y_n^2}{n-m} + \frac{n m y_n^2 + n m y_m^2 - 2 n y_n m y_m}{n-m} \\
&= n y_n^2 + \frac{n m \left(y_n^2 + y_m^2 - 2 y_n y_m \right)}{n-m} \\
&= n y_n^2 + \frac{n m \left(y_n - y_m \right)^2}{n-m} \\
&= n y_n^2 + \frac{n m \left(\frac{\delta}{m} \right)^2}{n-m} \\
&= \frac{1}{n} \left(\sum_{k=1}^{n} x_k \right)^2 + \frac{n \delta^2}{m(n-m)}
\end{aligned}$$

□

We are now ready to prove Lemma 1.

Proof of Lemma 1. Let $P = V_0 \cup V_1 \cup \ldots V_k$ be an equipartition such that more than ϵk^2 pairs are ϵ-irregular. Let (V_i, V_j), $1 \leq i < j \leq k$ and $i \neq j$, be one of the irregular pairs. This means that there exists some subsets $X = X(i,j) \subseteq V_i$ and $Y = Y(i,j) \subseteq V_j$ such that $|X| \geq \epsilon |V_i|$, $|Y| \geq \epsilon |V_j|$ and

$$|d(X,Y) - d(V_i, V_j)| > \epsilon.$$

Furthermore, in each set V_i, define the equivalence relation that

$$\forall x, y \in V_i,\, x \equiv y \iff x \in X(i,j) \text{ when } y \in X(i,j)\ \forall i \neq j.$$

This partitions each V_i into at most 2^{k-1} classes, called atoms by Szemerédi. We denote these atoms as $A_{i,m}$. The partition for each i is $V_i = A_{i,1} \cup A_{i,2} \cup \ldots \cup A_{i,2^{k-1}}$.

Set

$$\ell = \left\lfloor \frac{|V_i|}{4^k} \right\rfloor$$

and create a partition Q of disjoint subsets of V such that

1. Every subset of Q has size ℓ.
2. Every atom $A_{i,m}$ has exactly $\left\lfloor \frac{|A_{i,m}|}{\ell} \right\rfloor$ members of Q.
3. Every set V_i has exactly $\left\lfloor \frac{|V_i|}{\ell} \right\rfloor$ members of Q.

Note that $\left\lfloor \frac{|V_i|}{\ell} \right\rfloor = \left\lfloor \frac{|V_i|}{\frac{|V_i|}{4^k}} \right\rfloor = 4^k$, so every V_i has 4^k members of Q, but recall our partition P has k sets (not including the exceptional set) which implies that Q has a total of exactly $k4^k$ members. We want Q to be an equipartition of V and it almost is but the exceptional class V_0 will not work as is. Now let V_0' be the exceptional class of the partition Q. If we partition the atoms into ℓ subsets, there are at most $\ell - 1$ leftover vertices and we add these leftover vertices from k clusters to the exceptional set. Then

$$|V_0'| \leq |V_0| + k\ell \leq |V_0'| + k\left\lfloor \frac{V_i}{4^k} \right\rfloor \leq |V_0| + k\left\lfloor \frac{\frac{n}{k}}{4^k} \right\rfloor \leq |V_0| + \frac{n}{4^k}.$$

All that is left to show is that $ind(Q) > ind(P) + \frac{\epsilon^5}{20}$. Denote each of the $k4^k$ members of Q as $V_i(s)$ where $1 \leq s \leq q := 4^k$ and $V_i^* := \bigcup_{s=1}^{q} V_i(s)$. From above, we added at most ℓ

vertices from each set to the new exceptional set, so we have to refine V_i to V_i^* by removing at most ℓ vertices from each V_i. Then

$$|V_i^*| > |V_i| - \ell = |V_i| - \left\lfloor \frac{|V_i|}{4^k} \right\rfloor \geq |V_i| \left(1 - \frac{1}{4^k}\right) > |V_i| \left(1 - \frac{\epsilon^5}{600}\right).$$

By the continuity of density fact, for $1 \leq i < j \leq k$,

$$\begin{aligned}
|d^2(V_i^*, V_j^*) - d^2(V_i, V_j)| &= |d(V_i^*, V_j^*) - d(V_i, V_j)||d(V_i^*, V_j^*) + d(V_i, V_j)| \\
&\leq 2|d(V_i^*, V_j^*) - d(V_i, V_j)| \\
&< 2 \cdot 6 \left(\frac{\epsilon^5}{600}\right) \\
&= \frac{\epsilon^5}{50}
\end{aligned}$$

Now by the Cauchy-Schwarz Inequality,

$$\begin{aligned}
\frac{1}{q^2} \sum_{s=1}^{q} \sum_{t=1}^{q} d^2(V_i(s), V_j(t)) &\geq \left(\frac{1}{q^2} \sum_{s=1}^{q} \sum_{t=1}^{q} d(V_i(s), V_j(t))\right)^2 \\
&= \left(\frac{1}{q^2} \sum_{s=1}^{q} \sum_{t=1}^{q} \frac{e(V_i(s), V_j(t))}{|V_i(s)||V_j(t)|}\right)^2 \\
&= \left(\frac{1}{q^2 |V_i^*||V_j^*|} \sum_{s=1}^{q} \sum_{t=1}^{q} e(V_i(s), V_j(t))\right)^2 \\
&\geq \left(d(V_i^*, V_j^*)\right)^2 \\
&> d^2(V_i, V_j) - \frac{\epsilon^5}{50}.
\end{aligned}$$

Now, let us fix $i \neq j$ such that the pair (V_i, V_j) is ϵ-irregular. Recall, we have the sets $X = X(i,j)$ and $Y = Y(i,j)$. Let X_0 denote the largest subset that contains members of Q

and because each X can split into at most 2^k atoms with less than ℓ vertices in each atom, we have

$$\begin{aligned}
|X_0| &\geq |X| - 2^k \ell \\
&> |X| - 2^k \left\lfloor \frac{|V_i|}{4^k} \right\rfloor \\
&\geq |X| - 2^k \frac{|X|/\epsilon}{4^k} \\
&= |X| \left(1 - \frac{1}{\epsilon 2^k}\right) \\
&> |X| \left(1 - \frac{1}{\epsilon \sqrt{\frac{600}{\epsilon^5}}}\right) \\
&= |X| \left(1 - \frac{\epsilon\sqrt{\epsilon}}{10\sqrt{6}}\right) \\
&> |X| \left(1 - \frac{\epsilon}{100}\right)
\end{aligned}$$

as long as $\epsilon < 0.06$.

Define $r = \left\lceil \frac{|X|}{\ell} \left(1 - \frac{\epsilon}{100}\right) \right\rceil$ and without loss of generality,

$$X^* = \bigcup_{s=1}^{r} V_i(s) \subseteq X \text{ and } Y^* = \bigcup_{t=1}^{r} V_j(t) \subseteq Y.$$

Now we have $|X^*| \geq |X| \left(1 - \frac{\epsilon}{100}\right)$ and $|Y^*| \geq |Y| \left(1 - \frac{\epsilon}{100}\right)$ and by the continuity of density

$$|d(X^*, Y^*) - d(X, Y)| \leq \frac{6\epsilon}{100} < \frac{\epsilon}{4}.$$

Recall because (V_i, V_j) is irregular, $|d(X, Y) - d(V_i, V_j)| > \epsilon$. Furthermore,

$$\begin{aligned}
\left|(d(X^*, Y^*) - d(V_i^*, V_j^*)\right| &= |(d(X^*, Y^*) - d(X, Y) + d(X, Y) - d(V_i, V_j) \\
&\quad + d(V_i, V_j) - d(V_i^*, V_j^*)| \\
&\geq |(d(X^*, Y^*) - d(X, Y)| - |d(X, Y) - d(V_i, V_j)| \\
&\quad - \left|d(V_i, V_j) - d(V_i^*, V_j^*)\right| \\
&> \epsilon - \frac{\epsilon}{4} - \frac{\epsilon^5}{50} \\
&> \frac{\epsilon}{2}.
\end{aligned}$$

Now we are ready to use the defect form of the Cauchy Schwarz Inequality (Lemma 2) with $n = q^2 = 4^{2k}$, $m = r^2$, and $\delta = r^2\left(d(X^*, Y^*) - d(V_i^*, V_j^*)\right)$.

$$\begin{aligned}\sum_{s=1}^{q}\sum_{t=1}^{q} d^2(V_i(s), V_j(t)) &\geq \frac{1}{q^2}\left(\sum_{s=1}^{q}\sum_{t=1}^{q} d(V_i(s), V_j(t))\right)^2 + \frac{\left(r^2\left(d(X^*, Y^*) - d(V_i^*, V_j^*)\right)\right)^2 q^2}{r^2(q^2 - r^2)} \\ &> \frac{1}{q^2}\left(q^2 d(V_i^*, V_j^*)\right)^2 + \frac{\epsilon^2}{4}\cdot\frac{r^2 q^2}{q^2 - r^2}\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{q^2}\sum_{s=1}^{q}\sum_{t=1}^{q} d^2(V_i(s), V_j(t)) &> d^2(V_i^*, V_j^*) + \frac{\epsilon^2}{4}\cdot\frac{r^2}{q^2 - r^2} \\ &> d^2(V_i^*, V_j^*) + \frac{\epsilon^2}{4}\cdot\frac{\epsilon^2 4^{2k}\left(1 - \frac{\epsilon}{100}\right)}{4^{2k} - \epsilon^2 4^{2k}\left(1 - \frac{\epsilon}{100}\right)} \\ &= d^2(V_i^*, V_j^*) + \frac{\epsilon^2}{4}\cdot\frac{\epsilon^2\left(1 - \frac{\epsilon}{100}\right)}{1 - \epsilon^2\left(1 - \frac{\epsilon}{100}\right)} \\ &> d^2(V_i, V_j) - \frac{\epsilon^5}{50} + \frac{\epsilon^4}{16}\end{aligned}$$

Now

$$\begin{aligned}ind(Q) &= \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\frac{1}{q^2}\sum_{s=1}^{q}\sum_{t=1}^{q} d^2(V_i(s), V_j(t))\right) \\ &> \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(d^2(V_i, V_j) - \frac{\epsilon^5}{50} + \frac{\epsilon^4}{16}\right) \\ &= \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=i+1}^{k} d^2(V_i, V_j) - \frac{\binom{k}{2}}{k^2}\frac{\epsilon^5}{50} + \frac{\epsilon k^2}{k^2}\cdot\frac{\epsilon^4}{16} \\ &\geq ind(P) - \frac{\epsilon^5}{100} + \frac{\epsilon^5}{16} \\ &> ind(P) + \frac{\epsilon^5}{20}\end{aligned}$$

as desired. □

5.0 APPLICATIONS

Having the relevant definitions and formal statement of the Regularity Lemma, we move on to its applications. The main importance of the Regularity Lemma comes from its use in proving deep results such as the Green-Tao Theorem (Theorem 15). Because the number of applications is vast and continually expanding, we have collected several different applications to showcase how to apply the Regularity Lemma and to show the different disciplines of mathematics that can use the help of the lemma.

5.1 TRIANGLE REMOVAL LEMMA

Let us begin with the Triangle Removal Lemma. We start with this theorem because it was one of the early applications of the Regularity Lemma due to the fact that the solution was found in 1976 and utilized an early form of the Regularity Lemma. The theorem was originally presented as the (6,3)-problem proven by Ruzsa and Szemerédi in 1976 [47] which solved a specific case of a conjecture by Brown, Erdős, and Sós [10]. The full conjecture and generalization of the (6,3) theorem still remains unsolved.

Conjecture 2 (Brown-Erdős-Sós)**.**
Let $k \geq 3$ be an integer. If H is a 3-uniform hypergraph such that no $k+3$ vertices span at least k edges, then $e(H) = o(n^2)$.

The original statement of the theorem was as follows: if H is a 3-uniform hypergraph such that no 6 vertices contain 3 edges, then the number of edges is $o(n^2)$. The (6,3)-problem and the triangle removal lemma are not equivalent, but they are relatively close with similar proofs.

For a more in depth discussion of the graph removal lemma, refer to [18].

Theorem 6 (Triangle Removal Lemma)**.**
For all $0 < \epsilon \leq 1$, there exists a δ such that for large enough n, if G is a n-vertex graph such that at least ϵn^2 edges must be removed for G to be triangle-free, then G has at least δn^3 triangles.

We need one simple lemma in order to prove the Triangle Removal Lemma. In the proof of the Triangle Removal Lemma, the following lemma will aid us in determining the number of triangles in our graph, G, after we have applied the Regularity Lemma and removed less than ϵn^2 edges ensuring that we do indeed have at least one triangle remaining.

Lemma 3.
Let (A, B) be an ϵ-regular pair with density d. For any $B' \subseteq B$ with $|B'| \geq \epsilon|B|$, the number of vertices in A, $a \in A$, with $deg(a, B') < (d - \epsilon)|B'|$ is less than $\epsilon|A|$.

Note, $deg(a, B')$ denotes the number of adjacent vertices that a has in B'.

Proof of Lemma 3. Let (A, B) be an ϵ-regular pair such that $d(A, B) = d$ and let $A' \subseteq A$ be the subset of vertices in A with less than $(d - \epsilon)|B'|$ adjacent vertices in B. We want to prove $|A'| < \epsilon|A|$. From the given statement, $e(A', B') < |A'|(d - \epsilon)|B'|$. This implies

$$d(A', B') = \frac{e(A', B')}{|A'||B'|} < \frac{|A'|(d - \epsilon)|B'|}{|A'||B'|} = d - \epsilon = d(A, B) - \epsilon.$$

Because (A, B) is an ϵ-regular pair, if $|A'| \geq \epsilon|A|$, then we have $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$ (from the given) which implies $|d(A', B') - d(A, B)| \leq \epsilon$ becuase (A, B) is an ϵ-regular pair. However, from our above calculation, we have $|d(A', B') - d(A, B)| > \epsilon$ which implies $|A'| < \epsilon|A|$ as desired. □

Now, we are ready to tackle the proof of the triangle removal lemma utilizing the Regularity Lemma and the lemma we just proved.

Proof of Theorem 6. Let us define ϵ and G as in the statement of the Triangle Removal Lemma (Theorem 6). Let ϵ_0 be dependent upon ϵ such that $\epsilon_0 = \frac{\epsilon}{8}$ and $t = \frac{1}{\epsilon_0}$. We apply the Regularity Lemma (Theorem 5) to find an ϵ_0-regular partition $V_0 \cup V_1 \cup \ldots \cup V_k$ of V

such that $t \leq k \leq T$ for some $T = T(t, \epsilon_0)$. Now, we remove the following edges from our partition:

(i) edges incident to V_0.

(ii) edges inside V_i for $1 \leq i \leq k$.

(iii) edges between an irregular pair, (V_i, V_j).

(iv) edges between a regular pair, (V_i, V_j), with density at most ϵ.

After removing these edges, we will only be left with ϵ_0-regular pairs (V_i, V_j) such that $d(V_i, V_j) \geq \epsilon$.

In Step (i), each vertex, $x_i \in V_0$, can have degree at most n and by the definition of our ϵ_0-regular partition, we know $|V_0| \leq \epsilon_0 |V| = \epsilon_0 n$. So, we have at most $\epsilon_0 n$ vertices with at most n edges per vertex, thus the total number of edges removed from Step (i) is at most $\epsilon_0 n^2$.

In Step (ii), we remove all edges inside the remaining sets. As before, we need the maximum number of vertices in each set. Recall, $|V_0| \leq \epsilon_0 n$, the total number of vertices in G is n, and $|V_1| = \ldots = |V_k|$. Combining these facts, we have $|V_0| + k|V_i| \leq n \implies |V_i| \leq \frac{n}{k}$. Thus, the number of edges is

$$\sum_{i=1}^{k} \binom{|V_i|}{2} \leq \sum_{i=1}^{k} \binom{\frac{n}{k}}{2} = k \cdot \binom{\frac{n}{k}}{2} \leq \frac{n^2}{2k} \leq \frac{n^2}{2t} \leq \epsilon_0 n^2$$

Hence, the total number of edges removed from Step (ii) is at most $\epsilon_0 n^2$.

In Step (iii), we remove all edges between ϵ_0-irregular pairs. By the definition of our ϵ_0-regular partition, we have at most $\epsilon_0 k^2$ irregular pairs. The maximum number of edges between pairs is at most $\left(\frac{n}{k}\right)^2$. This implies that we remove at most $\epsilon_0 k^2 \cdot \left(\frac{n}{k}\right)^2 = \epsilon_0 n^2$.

In Step (iv), we remove all edges between ϵ_0-regular pairs with density at most ϵ. The maximum number of edges between these pairs can be found by the following:

$$d(V_i, V_j) = \frac{e(V_i, V_j)}{|V_i||V_j|} \implies e(V_i, V_j) = d(V_i, V_j)|V_i||V_j| < \epsilon \left(\frac{n}{k}\right)^2.$$

There can only be at most $\binom{k}{2}$ regular pairs with at most $\epsilon \left(\frac{n}{k}\right)^2$ edges which implies we remove at most $\epsilon \binom{k}{2} \left(\frac{n}{k}\right)^2 \leq \frac{\epsilon}{2} n^2$ edges.

Combining all of the edges from each step, there is a total of $\left(\frac{\epsilon}{2}+3\epsilon_0\right)n^2 = \frac{7}{8}\epsilon n^2 < \epsilon n^2$ edges deleted from G. Because we have removed less than ϵn^2 edges, a triangle must still remain in G. Each vertex of the triangle must be contained in a distinct set excluding the exceptional set, say V_1, V_2, and V_3. Let $m = |V_1| = |V_2| = |V_3|$. From above, we know each pair (V_i, V_j), $1 \leq i, j \leq 3$, is an ϵ_0-regular pair such that $d(V_i, V_j) \geq \epsilon$.

By Lemma 3, less than $\epsilon_0|V_1| = \epsilon_0 m$ vertices in V_1 have less than $(\epsilon-\epsilon_0)|V_2| = (\epsilon-\epsilon_0)m$ neighbors in V_2. Similarly, less than $\epsilon_0 m$ vertices in V_1 have less than $(\epsilon-\epsilon_0)m$ neighbors in V_3. This is the same as saying that at least $(1-\epsilon_0)m$ vertices in V_1 have at least $(\epsilon-\epsilon_0)m$ adjacent vertices in V_2 and the same goes for adjacent vertices in V_3. Combining these facts, we have that at least $(1-2\epsilon_0)m$ vertices in V_1 that have at least $(\epsilon-\epsilon_0)m$ adjacent vertices in both V_2 and V_3.

Let x be a vertex in V_1. Let V_2' and V_3' be the two subsets where the adjacent vertices of x lie in V_2 and V_3 respetively. From above, we know $|V_2'| \geq (\epsilon-\epsilon_0)m$ and similarly, $|V_3'| \geq (\epsilon-\epsilon_0)m$. Note

$$|V_2'| \geq (\epsilon-\epsilon_0)m. = \left(\epsilon - \frac{\epsilon}{8}\right)m = \frac{7\epsilon}{8}m > \frac{\epsilon}{8}m = \epsilon_0 m.$$

So, $|V_2'| \geq \epsilon_0 m$ and likewise, $|V_3'| \geq \epsilon_0 m$. Because (V_2, V_3) is an ϵ_0-regular pair along with the fact that $|V_2'| \geq \epsilon_0|V_2|$ and $|V_3'| \geq \epsilon_0|V_3|$, we know

$$|d(V_2', V_3') - d(V_2, V_3)| \leq \epsilon_0 \implies d(V_2', V_3') \geq d(V_2, V_3) - \epsilon_0 \geq \epsilon - \epsilon_0 = \epsilon - \frac{\epsilon}{8} = \frac{7\epsilon}{8}.$$

Then, the number of edges between V_2' and V_3' is at least $\frac{7\epsilon}{8}|V_2'||V_3'|$. Using the fact that $|V_2'|$ and $|V_3'|$ are at least $\epsilon_0 m = \frac{\epsilon}{8}m$, we have the number of edges between V_2' and V_3' is at least $7\left(\frac{\epsilon}{8}\right)^3 m^2$.

To find the number of triangles, we take the number of vertices we have in V_1 and multiply by the number of edges found above which represents the number of adjacent vertices in V_2 and V_3 for any given $x \in V_1$. So, the number of triangles is at least

$$(1-2\epsilon_0)m \cdot 7\left(\frac{\epsilon}{8}\right)^3 m^2 = 7\left(1-\frac{\epsilon}{4}\right)\left(\frac{\epsilon}{8}\right)^3 m^3.$$

We would like the number of triangles to be in terms of δn^3. Recall, we defined m as $|V_1| = |V_2| = |V_3|$. Because $|V_1| = \ldots = |V_k|$, we know

$$|V_0| + k|V_1| = n \implies m = \frac{n - |V_0|}{k} \geq \frac{n - \epsilon_0 n}{k} \geq \frac{n\left(1 - \frac{\epsilon}{8}\right)}{T}.$$

Thus, the number of triangles is at least $\frac{7(1-\frac{\epsilon}{4})\left(1-\frac{\epsilon}{8}\right)^3\left(\frac{\epsilon}{8}\right)^3}{T^3} n^3$. Taking $\delta = \frac{7(1-\frac{\epsilon}{4})\left(1-\frac{\epsilon}{8}\right)^3\left(\frac{\epsilon}{8}\right)^3}{T^3}$, we have at least δn^3 triangles as desired.

□

5.2 ROTH'S THEOREM

Next we consider Roth's Theorem which follows nicely from the Triangle Removal Lemma. It may not be immediately clear how we can prove a theorem regarding arithmetic progressions using triangles, but we will relate triangles in our graph to arithmetic progressions of length 3.

First, a brief history note: in 1953, Klaus Roth proved the first non trivial case of Szemerédi's Theorem (Theorem 4), that is, $k = 3$ [28]. Because Roth proved the following theorem in 1953, clearly the original proof did not include the use of the Regularity Lemma. Our approach will utilize the Triangle Removal Lemma which uses the Regularity Lemma.

Theorem 7 (Roth's Theorem)**.**
For $0 < \beta \leq 1$, if there exists N such that for all $n \geq N$ sufficiently large and $A \subseteq [n]$, where $|A| \geq \beta n$, then A contains an arithmetic progression of length 3.

Proof. Define A such that $A \subseteq [n]$ and $|A| \geq \beta n$ for $0 < \beta \leq 1$. We construct a tripartite graph G with the vertex partition $V = V_1 \cup V_2 \cup V_3$, where V_1 has n vertices labeled from 1 to n, V_2 has $2n$ vertices labeled from 1 to $2n$, and V_3 has $3n$ vertices labeled from 1 to $3n$; this implies $|V| = 6n$. Now we define the edges between the vertex sets in the following way:

- $(x, y) \in (V_1, V_2)$ is adjacent $\iff$ $y - x \in A$.
- $(y, z) \in (V_2, V_3)$ is adjacent $\iff$ $z - y \in A$.
- $(x, z) \in (V_1, V_3)$ is adjacent $\iff$ $\frac{z-x}{2} \in A$.

A triangle is formed by the vertices x, y, z if and only if all three of the following conditions are met: $y - x = a_1 \in A$, $\frac{z-x}{2} = a_2 \in A$, and $z - y = a_3 \in A$. Then, (a_1, a_2, a_3) forms an arithmetic progression with common difference, $d = a_2 - a_1 = \frac{z+x}{2} - y$.

If $y - x$, $z - y$, and $\frac{z-x}{2}$ are all equal to $a \in A$, then we have the arithmetic progression $(a, a, a) \in A$ which corresponds to the triangle $(x, x + a, x + 2a) \in V_1 \times V_2 \times V_3$. We refer to these triangles as trivial. The trivial triangles are formed by the pair $(x, a) \in V_1 \times A$ which is a distinct pair, so the trivial triangles are edges disjoint. Note, there are at least βn^2, but at most n^2 distinct trivial triangles. From above, we must remove at least $\frac{\beta}{36}(6n)^2 = \beta n^2$ edges for G to be triangle free and thus, by the Triangle Removal Lemma (Theorem 6), G has at least $\delta(6n)^3 = 216\delta n^3$ triangles. Because $216\delta n^3 > n^2$, there must be some nontrivial triangles. Thus we can find a nontrivial arithmetic progression of length 3 as desired.

□

5.3 ERDŐS-STONE-SIMONOVITS

The Regularity Lemma is a huge result especially in the area of extremal graph theory. Most credit Turán [51] as the founder of extremal graph theory with his 1941 publication in which he finds for any r and n, the maximum number of edges a graph of n vertices can have if it does not contain a complete graph with $r + 1$ vertices. Extremal graph theory deals with questions such as the one above presented by Turán. Almost all problems in extremal graph theory start with the same basic idea: given certain parameters of a large graph and a given property, what is the maximum number of edges the graph can have such that the graph still has the property desired.

We give a brief overview of some basic definitions in extremal graph theory along with Turán's Theorem before moving onto the result from Erdős and Stone.

Definition 10 (Extremal number)**.**
The ***extremal number****, denoted ex(n,F), is the maximum number of edges in a n-vertex graph that is* F*-free.*

The following theorems in this section were proven before Szemerédi proved the Regularity Lemma (Theorem 5) in 1975. However, the Regularity Lemma gives a nice standard argument to tackle some of these classic extremal graph theorems.

Theorem 8 (Turán's Theorem)**.**
If $G = (V, E)$ is a graph with n vertices, where n is sufficiently large, that does not contain K_{r+1}, then the number of edges of G is at most $\left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$.

The theorem implies that if a graph on n vertices has more than $\left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$ edges, then the graph contains K_{r+1}.

Definition 11 (Turán Graph)**.**
The ***Turán graph****, denoted $T(n, r)$, is a complete r-partite graph with n vertices such that each part has either $\left\lfloor \frac{n}{r} \right\rfloor$ or $\left\lceil \frac{n}{r} \right\rceil$ vertices.*

Note, the restriction put on the number of vertices in each part ensures that the number of vertices between parts differ by no more than 1.

Another way that Turán's Theorem can be stated is that for any graph G with n vertices that does not contain K_{r+1}, then the maximum number of edges is $e[T(n, r)]$. Let us consider an example of the Turán graph $T(7, 3)$ shown in Figure 10. Notice each part either has $\left\lfloor \frac{7}{3} \right\rfloor = 2$ or $\left\lceil \frac{7}{3} \right\rceil = 3$ vertices in each part. Specifically, $|V_1| = |V_2| = 2$ and $|V_3| = 3$.

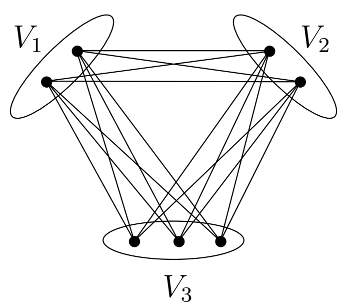


Figure 10: The Turán graph $T(7, 3)$

Notice that $T(7, 3)$ contains a copy of K_3, but does not contain a copy of K_4. We can count the number of edges in $T(7, 3)$ and we find $e[T(7, 3)] = 16$. Using the formula in

Turán's Theorem (Theorem 8), we see the number of edges of a K_4-free graph must be less than

$$\left(1 - \frac{1}{3}\right) \cdot \frac{n^2}{2} = \left(1 - \frac{1}{3}\right) \cdot \frac{7^2}{2} = \frac{2}{3} \cdot \frac{49}{2} = \frac{49}{3} \approx 16.3333.$$

So, $e[T(n,r)]$ and $\left(1 - \frac{1}{r}\right)\frac{n^2}{2}$ give similar upper bounds for the number of edges of a K_{r+1}-free graph.

Now, onto the main theorem we will be discussing in this section.

Theorem 9 (Erdős-Stone)**.**
Let $0 < \epsilon \leq 1$ and $r \in \mathbb{N}$, $r \geq 2$, be given. Then, there exists $n' = n'(\epsilon, r)$ such that for all $n \geq n'$ large enough, if G is a graph with n vertices and greater than $\left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2}$ edges, then G contains $K_r(t)$, the complete r-partite graph with t vertices in each vertex subset.

An example of a complete r-partite graph with t vertices in each subset is shown in Figure 11. Specifically, we show $K_5(2)$, the complete 5-partite graph with 2 vertices in each subset.

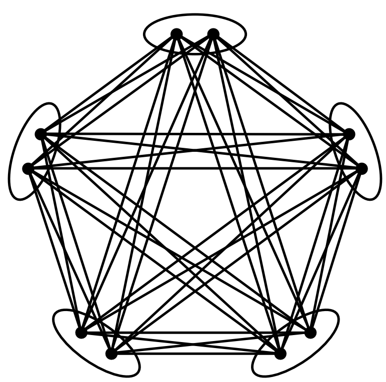


Figure 11: $K_5(2)$

Twenty years after Theorem 9, Erdős and Miklós Simonovits found the upper bound for the number of edges in a n-vertex, F-free graph [26]. Recall, $\chi(G)$ denotes the chromatic number of a graph G. The *chromatic number* is the minimum number of colors needed to color the vertices of a graph G such that no two vertices are colored the same color.

Theorem 10 (Erdős-Stone-Simonovits)**.**

Let F be a graph such that $\chi(F) = r \geq 2$. Then,

$$ex(n, F) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(1).$$

When Erdős and Stone proved Theorem 9, the bound they found on t was that t had to be greater than or equal to $(l_{r-1}(n))^{1/2}$. Note, we use the same notation as in [27] so that $l_1(x) = \ln(x)$, $l_2(x) = \ln(\ln(x)), \ldots, l_r(x) = \ln(l_{r-1}(x))$. Next, Erdős and Béla Bollobás proved in [7] that the bounds of t must be of the form $c_1 \log(n) \leq t \leq c_2 \log(n)$, where c_1 and c_2 are constants such that $c_2 \to 0$ as $\epsilon \to 0$. Expanding upon their findings, Erdős, Bollobás, and Simonovits [8] proved that the lower bound, for a positive constant α, was

$$t = \frac{\alpha \log(n)}{(r-1)\log(\frac{1}{\epsilon})}$$

and the upper bound was

$$t = 5\frac{\log(n)}{\log(\frac{1}{\epsilon})}.$$

Finally, Chvátal and Szemerédi [17] improved the lower bound to

$$\frac{1}{500\log(\frac{1}{\epsilon})}\log(n) \leq t \leq \frac{5}{\log(\frac{1}{\epsilon})}\log(n).$$

To prove the Erdős-Stone Theorem (Theorem 9) using the Regularity Lemma (Theorem 5), we will use both Turán's Theorem (Theorem 8) and the following lemma. The motivation for the proof is first to find an ϵ-regular partition using the Regularity Lemma, remove edges to eventually construct a reduced graph R, then show that $K_r \subseteq R$ by using Turán's Theorem, and finally, show $K_r \subseteq R \implies K_r(t) = T(rt, r) \subseteq G$. For the last step, we need the result stated next found in [22] and [45]. We refer to the lemma as the Key Lemma following Komlós and Szemerédi [45] because it is critical in the proofs of several applications of the Regularity Lemma.

Theorem 11 (Key Lemma)**.**

Let $G = (V, E)$ be any graph. Given $\Delta > 0$ and $s \in \mathbb{N}$. For any $0 < \epsilon_0 \leq 1$, let H be a graph with maximum degree Δ and R be the reduced graph of ϵ-regular partition P with the parameters $\epsilon \leq \epsilon_0$, d, and $\ell \geq \frac{s}{\epsilon_0}$. Then, if H is contained in R_s, then H is a subgraph of G as well.

We are now ready to tackle the Erdős-Stone Theorem.

Proof of Theorem 9. Let $G = (V, E)$ be a graph with n vertices and at least $\left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2}$ edges, where n is large enough. We want a regular subgraph of G, so we apply the Regularity Lemma (Theorem 5) to G with $\epsilon_0 = \frac{\epsilon}{24}$ and $t = (\epsilon_0)^{-1}$. Then, we have a ϵ_0-regular partition of V, say $V_0 \cup V_1 \cup \ldots \cup V_k$. Similar to the proof of the Triangle Removal Lemma (Theorem 6), we remove the following edges in order to create our subgraph, G':

(i) edges incident to V_0.

(ii) edges inside V_i for $1 \leq i \leq k$.

(iii) edges between an irregular pair, (V_i, V_j).

(iv) edges between a regular pair, (V_i, V_j), with density less than $\frac{\epsilon}{2}$.

The total number of edges removed from Step (i) is at most $\epsilon_0 n^2$. The total number of edges removed from Step (ii) is at most $\epsilon_0 n^2$. The total number of edges removed from Step (iii) is at most $\epsilon_0 n^2$. The total number of edges removed from Step (iv) is at most $\frac{\epsilon}{8}n^2$. In total, we remove at most $\left(\frac{\epsilon}{8} + 3\epsilon_0\right)n^2$. So, we have

$$\begin{aligned} e(G') &\geq e(G) - \left[\left(\frac{\epsilon}{8} + 3\epsilon_0\right)n^2\right] \\ &\geq \left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2} - \left(\frac{\epsilon}{2} \cdot \frac{n^2}{2}\right) \\ &= \left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right)\frac{n^2}{2} \end{aligned}$$

Then the number of edges of R, the reduced graph of our partition, is at least $\left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right)\frac{k^2}{2} > \left(1 - \frac{1}{r-1}\right)\frac{k^2}{2}$. Note, the change from n to k is because the number of vertices changes from G' to R. G' has n vertices while R has k vertices because each V_i is a vertex in R. Now, by Turán's Theorem (Theorem 8), R contains K_r. Becuase $K_r \subset R$, we can replace the vertices and edges with t-sets and complete bipartite graphs to find that $K_r(t) \subset R_t$. Finally, we can apply the Key Lemma (Theorem 11) which gives us that $K_r(t) \subset G$ as desired. □

The original proof by Erdős and Stone used an induction argument and while the proof wasn't very long, the use of the Regularity Lemma greatly shortens the proof along with the help of Turán's Theorem and the Key Lemma.

5.4 RAMSEY THEORY

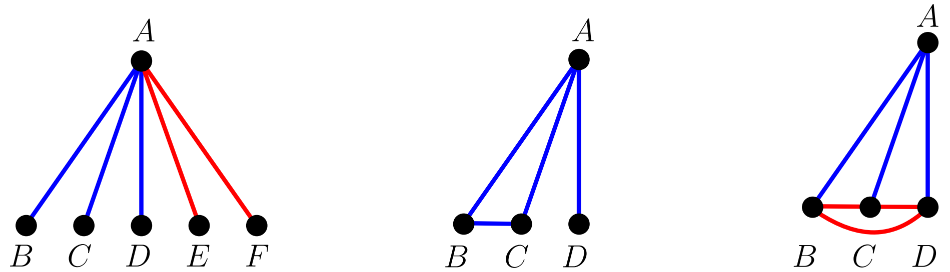
Ramsey theory started with a paper published in 1930 by Frank P. Ramsey [46]. For an extensive look into Ramsey theory, refer to [35].

Theorem 12 (Ramsey's Theorem)**.**
There exists a minimum number $v = r(m, n)$ such that for a blue-red edge coloring, K_v contains either a copy of a blue K_m or a red K_n.

The number v in the above theorem is known as the *Ramsey number*, denoted $r(m, n)$. Another common definition of the Ramsey number is $r(G, H)$, where G and H are graphs, is the minimum number v such that the blue-red coloring of K_v either contains a red copy of G or a blue copy of H. When $G = H$, we can simply write $r(G)$.

Probably the most well known Ramsey number is $r(3, 3)$. However, this particular Ramsey number is typically presented as the theorem on friends and strangers, which is a special case of Ramsey's Theorem (Theorem 12). The theorem is also referred to as the Party Problem. The theorem states: At a party with six people, it is guaranteed that either three are mutual friends or three are mutual strangers. The proof of the theorem relies on the pigeonhole principle. We will quickly give an outline of how to prove this theorem.

Suppose the party goers are Abbey, Brandon, Carl, Dana, Edward, and Frank. We view the guests as the vertices A, B, C, D, E, and F. If any two are friends, color the edge between them blue. For example, if A and B are friends, then AB is blue. On the other hand, if any two are not friends and thus strangers, color the edge between them red. Let us fix a single vertex, say A. We have five edges attached to A to be split into two colors. By the pigeonhole principle, three edges must be colored blue and two must be colored red (or vice versa and the argument would be the same). This is because we have five edges to be put into two colors and because $5 > 2$, one of the colors must have more than one edge assigned to it. Suppose the three blue edges connect A to B, C, and D. If any edge between B, C, or D is blue, we will have a triangle where all edges are colored blue. Then, we want all the edges between B, C, and D to be red, however, that ensures a red triangle. See Figure 12 for a visual representation of how we are guaranteed either a blue triangle or red triangle for a fixed A. Thus, in a party of six people, we are guaranteed that either three of them are mutual friends or three are mutual strangers.



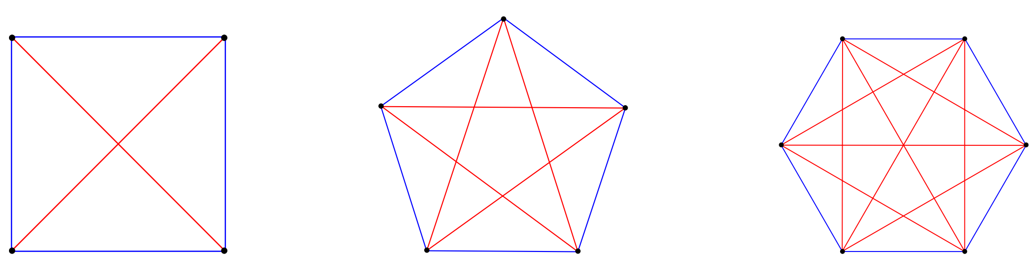
(a) A has three friends and two strangers.

(b) Case where there is a blue triangle.

(c) Case where there is a red triangle.

Figure 12: Friends and Strangers of fixed A

In Figure 13, we see that if we color K_4 and K_5 with blue and red edges, we do not have either a blue K_3 or red K_3. However, in the blue-red coloring of K_6, we have not one, but two red copies of K_3. Obviously, this is not the only possible way to color the edges of K_6 with two colors. Although, no matter the way we color K_6, we are always guaranteed to have either a blue or red copy of K_3. In fact, there are numerous different possibilities for a blue-red coloring of K_6.



(a) K_4 without a blue or red K_3.

(b) K_5 without a blue or red K_3.

(c) K_6 with two red copies of K_3.

Figure 13: Blue-Red Coloring of K_4, K_5, and K_6

Now, onto an example in Ramsey Theory that utilizes the Regularity Lemma. The following theorem was proven by Chvátal, Vojtěch Rödl, Szemerédi, and William T. Trotter, Jr. [16] and demonstrates how the Regularity Lemma can be used within Ramsey Theory. The theorem comes from the Erdős-Burr conjecture [11], and proves the conjecture for graphs with bounded degree.

Theorem 13.

Let G be a graph with n vertices, where n is sufficiently large, and let $d \geq 1$ be any integer. Then, there exists a c depending upon d such that if G has maximum degree d, then $r(G) \leq cn$.

Below, we give a sketch of the proof of Theorem 13. For the formal proof, see the original proof in [16] or the proof in [22]. We do not walk through the entire formal proof because the sketch is more intuitive and still shows how we can apply the Regularity Lemma. Our approach follows the procedure found in [22] which is similar to the original proof in [16]. The main difference between the two approaches is the use of the Key Lemma (Theorem 11) which was not used in the original proof.

Sketch of Proof. Let d be any positive integer. Let G be a graph on n vertices. Next, we consider the graph K_N, where $N \geq cn$ and $N \in \mathbb{Z}^+$, and color the graph red and blue. Furthermore, $K_N = H \cup \overline{H}$, where H is the graph determined by the red edges and $\overline{H}$ is the graph determined by the blue edges. We wish to show that $G \subseteq H$ or $G \subseteq \overline{H}$. Now, consider an ϵ-regular partition of H that is guaranteed by the Regularity Lemma (Theorem 5). The partition is $v(H) = V_0 \cup V_1 \cup \ldots \cup V_k$ such that $|V_1| = \ldots |V_k| = \ell$ and the exceptional set is V_0. Next, consider the reduced graph R of our partition. Because the edges of R are formed by ϵ-regular pairs, R has at least $(1-\epsilon)\binom{k}{2}$ edges. Then, as long as $\epsilon < \frac{1}{r-1}$, K_r is contained in R by Turán's Theorem (Theorem 8). Next, we color the edges of R green and yellow. Let R^* be the graph determined by the green edges and let R^{**} be the graph determined by the yellow edges. We define these two sets of edges such that if $d_H(V_i, V_j) \geq \frac{1}{2}$, then the edge is colored green and if $d_H(V_i, V_j) < \frac{1}{2}$, then the edge is colored yellow. Note, $d_H(V_i, V_j)$ denotes the density of the pair (V_i, V_j) in H. By our construction of R^* and R^{**}, R is the disjoint union of the two graphs, that is, $R = R^* \cup R^{**}$. Let us define $r = r(K_{d+1})$, so r is the Ramsey number of K_{d+1}. Recall, d is the maximum degree of G. Because $K_r \subseteq R$ from above, it follows that K_{d+1} is contained in either R^* or R^{**} which implies $K_{d+1}(n) \subseteq R^*_n$ or R^{**}_n. Our graph G can be partitioned into at most $d+1$ sets so we have $G \subseteq K_{d+1}(n) \subseteq R_n$. By the Key Lemma (Theorem 11), we have $G \subseteq H$ as desired. $\square$

5.5 EMBEDDING TREES

We switch our focus now to a set of conjectures in which approximate forms have been proven using the Regularity Lemma. A brief overview of relevant definitions for this section can be found in the Appendix. So far, we have looked at graph removal lemmas, a theorem regarding arithmetic progressions, and theorems that involve embeddings graphs. We now look at embedding trees into graphs instead of just embedding different types of graphs into a graph G.

The following conjecture from Erdős and Vera T. Sós appeared in [23].

Conjecture 3 (Erdős-Sós Conjecture)**.**
Every graph with an average degree greater than $k-2$ contains all trees on k vertices as subgraphs.

In the paper, the conjecture was originally stated as a graph on n vertices with $\left[\frac{1}{2}(k-1)n+1\right]$ edges contains all trees with k edges.

The next conjecture appeared in a paper on tree discrepancies written by Erdős, Zoltán Füredi, Martin Loebl, and Sós [25].

Conjecture 4 (Loebl Conjecture)**.**
If $G=(V,E)$ is a graph with n vertices for n large enough such that at least $\frac{n}{2}$ vertices have degree at least $\frac{n}{2}$, then G contains every tree on at most $\frac{n}{2}$ vertices.

The above conjecture is sometimes also referred to as the $(n/2-n/2-n/2)$ conjecture. An expansion of the Loebl Conjecture appeared in the same paper from János Komlós and Sós. The conjecture is stated below. Instead of being confined to trees of size $\frac{n}{2}$ and thus dependent upon the vertices of G, Komlós and Sós conjectured that this can be expanded to trees of any size, say k.

Conjecture 5 (Loebl-Komlós-Sós Conjecture).
If $G = (V, E)$ is a graph on n vertices for large enough n and at least $\frac{n}{2}$ vertices have degree greater than k, then G contains all tress with at most k edges as subgraphs.

Conjecture 5 appears very similar to the Erdős-Sós Conjecture (Conjecture 3). While all three conjectures remain unsolved, there have been approximate versions proven that we will discuss.

Theorem 14 (Loebl-Komlós-Sós - Approximate Version).
For every $0 < \alpha \leq 1$, there exists a k_0 such that for all $k > k_0$ each graph G with n vertices for large enough n with at least $\left(\frac{1}{2} + \alpha\right)$ vertices of degree at least $(1+\alpha)k$ contains each tree T of order k.

Theorem 14 was recently proven by Jan Hladkỳ, Komlós, Diana Piguet, Simonovits, Maya Stein, and Szemerédi. They published an article giving an overview of their proof [43] and split the result into four papers ([39], [40], [41], and [42]) that have been accepted to appear in the SIAM Journal of Discrete Mathematics.

The main ingredient in the proof is the Regularity Lemma and the rest of the proof is similar to those previously discussed except now with trees instead of graphs. To embed a tree T into a graph G, the authors partition T and then construct a matching from T to the reduced graph R of G. Recall the use of the Regularity Lemma only holds when the graph G is dense, however the authors use a technique known as sparse decomposition to ensure the Regularity Lemma works for embedding trees (and also for sparse graphs).

5.6 GREEN-TAO THEOREM

One of the most well known applications of the Regularity Lemma is an extension of Szemerédi's Theorem (Theorem 4) proven by Ben Green and Terrence Tao in 2004 and appearing in 2008 [36]. We will not go through the entire proof of theorem due to the complexity and constraints of this paper; instead, we give a brief outline.

Theorem 15 (Green-Tao Theorem)**.**
For any positive integer k, the prime numbers contain infinitely many arithmetic progressions of length k.

The stronger statement of the theorem is stated next using upper density which is similar to the original statement of Szemerédi's Theorem (Theorem 3).

Theorem 16 (Green-Tao Theorem)**.**
Let $\pi(N)$ denote the number of primes less than or equal to N. And let A be any subset of the prime integers with positive upper density, that is,

$$\limsup_{N\to\infty} \frac{|A \cap [1, N]|}{\pi(N)} > 0,$$

then A contains infinitely many arithmetic progressions of length k, for all k.

The proof supplied by Green and Tao includes several main components. The first main component is Szemerédi's Theorem (Theorem 4) which is a logical first component based upon the fact that this theorem is an extension of Szemerédi's. The second component is described by the Green and Tao as a *transference principle.* Szemerédi's Theorem cannot be applied to the prime numbers directly because the set of primes does not have positive upper density (in fact, the primes have density zero), so Green and Tao used this idea of transferring Szemerédi's Theorem to a relative Szemerédi's Theorem for subsets of a pseudorandom set of integers. This relative Szemerédi theorem tells us that if X is a set of integers that is pseudorandom, then for any $A \subset X$ with $\bar{d}(A) > 0$, A contains arbitrarily long arithmetic progressions. The last component of the proof is to construct a set which contains the primes as a dense subset; Green and Tao used results from the work of Goldston and Yildirim ([31], [32], [33]) on gaps in the primes to construct such a set.

Recently (2014 to be exact), David Conlon, Jacob Fox and Yufei Zhao provided simplifications of the proof of the Green-Tao Theorem in [19].

6.0 BIOGRAPHIES OF REFERENCED MATHEMATICIANS

In this chapter, we give brief biographies of the mathematicians referenced in this paper[1].

Béla Bollobás

Béla Bollobás was born on August 3, 1943 in Budapest, Hungary. He began publishing papers at a young age with one of his first papers being written with Paul Erdős while Bollobás was still in high school. He received his first Ph.D. in discrete geometry from Eötvös Loránd University of Budapest in 1967 under the supervision of Erdős and László Fejes Tóth and received his second Ph.D. in functional analysis from the University of Cambridge in 1972. As of 1996, he is the Jabie Hardin Chair of Excellence in Combinatorics at the University of Memphis. His main areas of interest are extremal graph theory and random graph theory. In addition to hundreds of papers published, Bollobás has also published several books including *Extremal Graph Theory* [5] and *Random Graphs* [9]. In 2007, he was awarded the Senior Whitehead Prize from the London Mathematical Society and in 2011, he was named a Fellow of the Royal Society. He was also recently awarded the Széchenyi Prize by the state of Hungary in 2017.

Vašek Chvátal

Vašek Chvátal was born on July 20, 1946 in Prague, Czechoslovakia (now the Czech Republic). He originally studied mathematics at Charles University, but fled the country in 1968 three days after the Soviet Invasion (the beginning of the Prague Spring), and eventually received his Ph.D. from the University of Waterloo in 1970. He taught at numerous institutions including Standford University and Rutgers University before taking a position at

[1]Basic information not directly cited in the biographies comes from https://en.wikipedia.org.

Concordia University in Montreal, Canada as the Canada Research Chair in combinatorial optimization and then discrete mathematics before retiring in 2014. His research interest began in graph theory and combinatorics, spending about 17 years on the Traveling Salesman Problem, then he worked in analysis of algorithms and operations research before developing an interest in computational neurosience. One of his well known results is the art gallery theorem. The original art gallery problem was posed to determine the minimum number of guards needed to guard an art gallery such that the entire art gallery can be observed by the guards. Chvátal gave an upper bound stating that $\left\lfloor \frac{n}{3} \right\rfloor$ will always be a sufficient number of guards [15].

Pál Erdős

Pál (Paul) Erdős was born on March 26, 1913 in Budapest, Austria-Hungary and is considered one of most prolific mathematicians of all time. From an early age, he showed great mathematical ability; by the age of four, he could multiple three digit integers and understand the idea of negative integers. He received his Ph.D. from Péter Pázmány University in Budapest (now Eötvös Loránd University of Budapest) in 1934 at the age of 21.

At the start of his career, Erdős took a postdoctoral position at the University of Manchester for four years before moving to the United States to work at the Institute for Advanced Study at Princeton University for a year. This was the start of his nomadic life; he opted to travel from one university to another to collaborate with whomever he wished without being tied down to just one university. This lead to Erdős collaborating with over 500 mathematicians [2] and publishing over 1500 papers in his lifetime [3]. To pay tribute to the number of mathematicians Erdős collaborated with, his friends created the Erdős number, a number assigned to a mathematician to determine the distance from Erdős to the mathematician through collaborations. Erdős has an Erdős number of 0, the over 500 mathematicians who have co-written a paper with Erdős have an Erdős number of 1, those who have co-written with a co-author of Erdős have an Erdős number of 2, and so on. If there is no path of papers between Erdős and a mathematician, the mathematician has an Erdős number of infinity.

He was awarded the Wolf Prize in Mathematics in 1983/4 for his contributions in number theory, combinatorics, set theory, mathematical analysis, and probability as well as encouraging other mathematicians. One way in which Erdős stimulated other mathematicians was by offering prize money for proofs (or disproofs) of problems that he himself could not solve with the amount of the award being based upon the difficulty of the problem according to Erdős. The amount would range from around $10 to several thousands of dollars. After his death in 1996, his close friends Ronald Graham and Fan Chung (both mathematicians themselves and collaborators of Erdős') announced they would continue to award prize money for the Erdős problems in graph theory which can be found in the book they published detailing the unsolved problems [13].

When referring to someone who had passed away, Erdős would say they had "left" while someone who "died" was someone who had stopped doing mathematics. Erdős lived until the age of 83 when he left on September 20, 1996 due to suffering a heart attack while attending a conference in Warsaw, Poland.

Ben Green

Ben Green was born on February 27, 1977 in Bristol, England and earned his Ph.D. from the University of Cambridge under the supervision of Timothy Gowers. He is currently the Waynflete Professor of Pure Mathematics at the University of Oxford; the Waynflete Professorship is one of four professorial fellowships at the University of Oxford. Green works in the area of additive combinatorics and one of his well known results is his theorem with Terrence Tao on arithmetic progression in the primes (see Section 5.6).

Frank Ramsey

Frank Ramsey was born on February 22, 1903 in Cambridge, England and unfortunately had quite a short life before passing on January 19, 1930. However, he was able to influence mathematics in those short 26 years. He received his Bachelor's degree from Trinity College. Along with his work in mathematics, he was also a philosopher and an economist. He began teaching in 1926 as a lecturer at King's College and later became the Director of Studies in Mathematics. His second paper in mathematics titled *On a problem of formal logic* is what started the area of Ramsey Theory (Section 5.4) [46].

Miklós Simonovits

Miklós Simonovits was born in September 4, 1943 in Budapest, Hungary. He earned his Ph.D. in 1971 from Eötvös Loránd University of Budapest under the supervision of Vera Sós. His main areas of research are extremal graph theory, theoretical computer science, and random graphs. As of 1979, he is a part of the Alfréd Rényi Insitute of Mathematics. He has an Erdős number of 1 having worked with Erdős on twenty one papers. In 2014, he was awarded the Széchenyi Prize, an award given by the government of Hungary to recognize scientists' contributions to academia in the country; in fact, Simonovits' advisor Sós was also awarded the prize in 1997.

Vera T. Sós

Vera Sós was born on September 11, 1930 in Budapest, Hungary. While still an undergraduate at Eötvös Loránd University of Budapest, she began teaching at the school in 1950. Then, she graduated in 1952, married Pal Turán, and began her graduate studies; she was awarded her Ph.D. from the Hungarian Academy of Sciences in 1957. Since 1987, she has been a professor at Alfréd Rényi Institute of Mathematics, Hungarian Academy of Science. Sós is known for her work in number theory and combinatorics. She collaborated with Erdős on thirty five papers giving her an Erdős number of 1.

Arthur Stone

Arthur Harold Stone was born on September 30, 1916 in London, England. He received his Ph.D. from Princeton University in 1941 and worked mostly in the discipline of topology. His wife, Dorothy Maharam, was also a mathematician working in the field of measure theory. They both were professors for many years at the University of Rochester with Stone becoming professor emeritus in 1987 upon his retirement. He continued teaching at Northwestern University as an adjunct professor until his passing in 2000.

Endre Szemerédi

Endre Szemerédi was born on August 21, 1940 in Budapest, Hungary. Before becoming a mathematician, Szemerédi went to medical school because his parents wanted him to be a doctor. He left after several months when he decided he did not want to be a doctor and the profession was not for him. He completed his Master's degree at Eötvös Loránd University of Budapest before earning his Ph.D. in 1970 at Moscow State University. His advisor was Israel Gelfand, but Szemerédi had originally intended on studying under Alexander Gelfond and unfortunately misspelled Gelfond's name on his application which resulted in him becoming Gelfand's student. As of 1986, Szemerédi is a Professor of Computer Science at Rutgers University in New Jersey and he is also a reseach fellow at the Alfréd Rényi Institute of Mathematics. He is well known for his proof of a conjecture from Erdős and Turán known as Szemerédi's Theorem (Theorem 4) [49] and his Regularity Lemma (Theorem 5) [50]. In 2012, he received the Abel Prize, one of the top prizes in mathematics, for his work in discrete mathematics and theoretical computer science.

Terence Tao

Terence Tao was born on July 17, 1975 in Adelaide, Australia and showed extraordinary mathematical ability from an early age, eventually earning the nickname the Mozart of Math. He completed his Bachelor's degree at age 16 and his Master's degree at age 17. Tao applied to Princeton University with a letter of recommendation from Erdős, was accepted, and earned his Ph.D. when he was only 20 years old. He immediately joined the faculty at UCLA before being appointed to full professor when he was 24 years old. While he knew Erdős, they never wrote a paper together, but Tao does have an Erdős number of 2. In 2006, Tao was awarded the Fields Medal: one of the highest honors in mathematics. The Fields Medal is only awarded once every four years to either two, three, or four mathematicians under the age of 40 for their contributions to mathematics and for their potential of future achievements. Tao also received one of the 2015 Breakthrough Prizes in Mathematics worth three million dollars [44]. The Breakthrough Prize in Mathematics was announced in 2013, with the first prize being awarded in 2015, and funded by Yuri Milner and Facebook founder Mark Zuckerberg.

Pál Turán

Pál (Paul) Turán was born on August 18, 1910 in Budapest, Hungary. He received his Ph.D. in 1935 from Eötvös Loránd University of Budapest. Because of his Jewish heritage, it was difficult to find a job, but he was able to take a teaching position in 1938 before being send to labor service between the years 1940 and 1944. In 1945, he was hired at Eötvös Loránd University and became a full professor in 1949. He worked for many years before eventually passing in 1976 as a result of leukemia. His work was mainly in the area of number theory, but he worked in analysis and graph theory as well. In graph theory, he is known for founding extremal graph theory with his theorem (Theorem 8). Some regard his power sum method as his most well known achievement [38]; the power sum method provides lower bounds for power sums and was discovered by Turán while he was investigating zeros of the zeta function.

Bartel Leendert van der Waerden

B.L. van der Waerden was born on February 2, 1903 in Amsterdam, Netherlands. His father was a mathematics teacher, but as a young child, van der Waerden was not allowed to read his father's math books but rather was encouraged to play outside [1]. This only made van der Waerden more curious about mathematics and led him to receiving his Ph.D. from the University of Amsterdam in 1926 at the age of 23. He began teaching at the University of Groningen before going to the University of Leipzig in 1931, but was forced to leave due to the bombing of Leipzig the night of December 4, 1943. Eventually, he went to the University of Zürich for the remainder of his career. He retired in 1973 and passed on January 12, 1996. He is known for his work in abstract algebra, but he also worked in the areas of algebraic geometry, quantum mechanics (he worked with Werner Heisenberg in Leipzig), and more.

7.0 CONCLUSION

We have seen that Szemerédi's Regularity lemma is a deep result with applications in numerous disciplines of mathematics. What began as a lemma in another theorem has become a celebrated result in graph theory.

Whether we are removing edges from a graph, embedding graphs and trees, or proving there exist arithmetic progressions in sets of the integers, the Regularity Lemma gives us a starting point. The lemma allows us to know how the edges of a graph are distributed over a partition of the vertices whereas without the lemma we may not know anything about a given graph.

Szemerédi's Regularity Lemma continues to be a frequently used theorem in graph theory as we saw with the work on the approximate version of the Loebl-Komlós-Sós Conjecture [43]. Furthermore, there is unpublished work that is in preparation of a proof of the Erdős-Sós Conjecture for large trees that utilizes the Regularity Lemma as well [4].

This paper is only an introduction to the complexity and reach of the Regularity Lemma discussing the basic information to understand the lemma and a few of the numerous applications, but now we have a little bit more knowledge of this truly amazing result.

APPENDIX

A GRAPH THEORY PRIMER

For those who may not be as familiar with graph theory, this appendix is provided as a short crash course in basic material needed for our discussion. The first part of the Appendix will go through the basic definitions associated with graphs, then we will have a short discussion on trees.

In order to study graph theory, we must first know how a graph is defined. A *graph* is an ordered pair, which we shall denote as $G = (V, E)$, where V is a nonempty set of objects called vertices and E is a set of objects called edges. We can also denote the set of vertices of G as $v(G)$ and the set of edges of G as $e(G)$. An edge is an unordered pair of distinct vertices. Vertices connected by an edge are called *adjacent vertices* . Furthermore, if vertices a and b are connected by the edge e, we say a and e are *incident* as well as b and e are incident. In addition, we can say that a is an incident vertex of e and similarly for b. One may be interested in the *degree of a vertex* which is the number of edges incident to a particular vertex. Let $v \in V$ be a vertex of G, then the degree of v is denoted $deg(v)$.

If every unique edge connects a pair of distinct vertices, we refer to this graph as *complete*, as seen in Figure 14. We denote a complete graph by K_r, where r is the number of vertices. Another way to think about a complete graph is that if G has n vertices, then for every $v_i \in V$, $i = 1, 2, \ldots, n$, $deg(v_i) = n - 1$.

For our discussion, we will be considering simple graphs unless otherwise stated. A *simple graph* is a graph with no repeated edges or loops. A loop is an edge that only has one vertex, meaning the loop connects a vertex to itself.

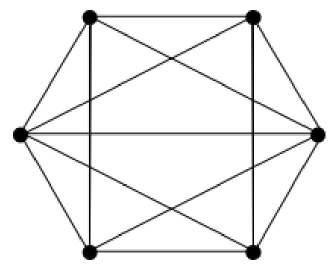


Figure 14: The Complete Graph K_6

We say a graph is *k-partite* ($k \geq 1$) if the set V can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that every edge connects a vertex from V_i to a vertex in V_j such that $i \neq j$. Thus, there does not exist any edges between vertices in the same subset. When $k = 2$ and $k = 3$, we refer to this as a bipartite graph and tripartite graph, respectively.

Now that we have a basic understanding of graphs, we continue with a brief discussion of trees. A *tree* is a connected acyclic graph. See Figure 16 for an example of trees. As a reminder, a connected graph is a graph in which there is a path between any two vertices. This means that we can reach any vertex from any of the remaining vertices by a series of edges. Also, an acyclic graph is a graph that does not contain any graph cycles.

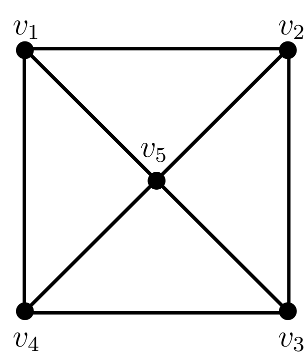


Figure 15: Cyclic Graph

To define a cycle, let us also consider a walk and a path. These terms are used for all graphs, however, for our discussion, it is only relevant for trees. Given two vertices u and v

in a graph G, a u-v *walk* is a finite and alternating sequence of vertices and edges beginning with u and ending with v. Furthermore, if $u = v$, that is, the walk begins and ends with u, then we classify the walk as a *closed walk*. In Figure 15, $W_1 : v_1, V_5, v_3, v_1, v_2$ is a walk and $W_2 : v_5, v_2, v_3, v_5$ is a closed walk. Note, we may repeat vertices and edges in a walk.

A *path*, which can also be called a u-v path, is a walk in which no vertex is repeated. For example, in Figure 15, $W_3 : v_1, v_2, v_5, v_4$ is a path. We can relate walks and paths because every path is a walk but the converse, that every walk is a path, is not necessarily true.

Finally, a *cycle* is a u-v walk in which $u = v$ and no vertex is repeated. Another way to think of a cycle is a closed path, or a closed walk with all distinct vertices. Figure 15 is a cyclic graph because it contains at least one cycle. For example, $W_4 : v_1, v_2, v_3, v_4, v_1$ is a cycle.

Now, back to trees. It may be interesting to note that all acyclic graphs (and therefore, all trees) are bipartite. Figure 17 shows the trees from Figure 16 as bipartite graphs.

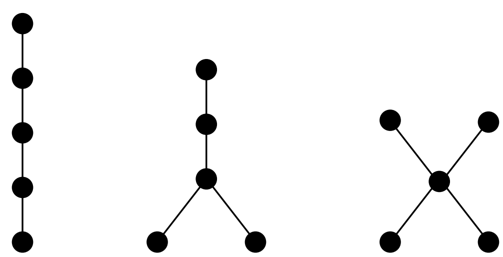


Figure 16: All trees of order 5.

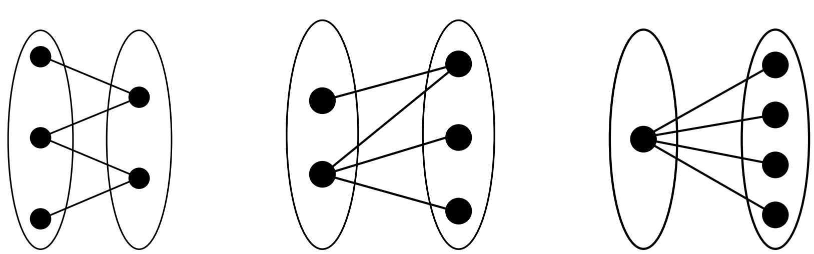


Figure 17: All trees of order 5 shown as bipartite graphs.

If a graph G is acyclic and disconnected, then G is a *forest*. A forest can be thought of as a collection of trees. Figure 18 shows an example of a forest. We can see that we have three trees that are disconnected from each other.

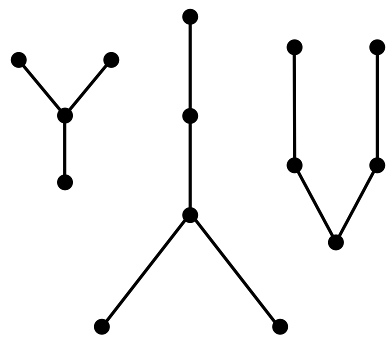


Figure 18: Forest

BIBLIOGRAPHY

[1] http://www-history.mcs.st-andrews.ac.uk/Biographies/Van_der_Waerden.html. Accessed: 2017-04-17.

[2] List of Erdős Coauthors. http://www.ams.org/mathscinet/MRAuthorID/189017. Accessed: 2017-04-20.

[3] List of Erdős Publications. https://oakland.edu/enp/pubinfo/. Accessed: 2017-04-20.

[4] Ajtai, M., Komlós, J., Simonovits, M., and Szemerédi, E. Proof of the Erdős-Sós Conjecture for large trees. In preparation.

[5] Bollobás, B. *Extremal graph theory*. Courier Corporation, 2004.

[6] Bollobás, B. *Modern graph theory*, vol. 184. Springer Science & Business Media, 2013.

[7] Bollobás, B., and Erdős, P. On the Structure of Edge Graphs. *Bulletin of the London Mathematical Society 5*, 3 (1973), 317–321.

[8] Bollobás, B., Erdős, P., and Simonovits, M. On the Structure of Edge Graphs II. *Journal of the London Mathematical Society s2-12*, 2 (1976), 219–224.

[9] Bollobs, B. *Random Graphs*, 2 ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.

[10] Brown, W. G., Erdős, P., and Sós, V. T. Some extremal problems on r-graphs. In *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich, 1971)* (1973), Academic Press, New York, pp. 53–63.

[11] Burr, S. A., and Erdős, P. On the magnitude of generalized Ramsey numbers for graphs. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. 1* (1975), North-Holland, Amsterdam, pp. 215–240. Colloq. Math. Soc. János Bolyai, Vol. 10.

[12] Chartrand, G. *Introduction to graph theory*. Tata McGraw-Hill Education, 2006.

[13] Chung, F., and Graham, R. *Erdős on Graphs: His Legacy of Unsolved Problems.* Ak Peters Series. Taylor & Francis, 1998.

[14] Chvátal, V. Some unknown van der waerden numbers. *Combinatorial Structures and their Applications* (1970), 31–33.

[15] Chvátal, V. A combinatorial theorem in plane geometry. *Journal of Combinatorial Theory, Series B 18*, 1 (1975), 39–41.

[16] Chvatál, V., Rödl, V., Szemerédi, E., and Trotter, W. The Ramsey number of a graph with bounded maximum degree. *Journal of Combinatorial Theory, Series B 34*, 3 (1983), 239 – 243.

[17] Chvátal, V., and Szemerédi, E. On the Erdős-Stone theorem. *Journal of the London Mathematical Society s2-23*, 2 (1981), 207–214.

[18] Conlon, D., and Fox, J. Graph removal lemmas. *Surveys in combinatorics 1*, 2 (2013), 3.

[19] Conlon, D., Fox, J., and Zhao, Y. The green-tao theorem: an exposition. *arXiv preprint arXiv:1403.2957* (2014).

[20] Das, S. Applications of the Szemerédi Regularity Lemma. Available at http://page.mi.fu-berlin.de/shagnik/notes/regularity.pdf (2016/10/06).

[21] DESKUS, C. An Introduction To The Regularity Lemma. Available at http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Deskus.pdf (2016/10/06).

[22] Diestel, R. *Graph Theory*. Graduate Texts in Mathematics. Springer New York, 2000.

[23] Erdős, P. Extremal problems in graph theory. In *Theory of Graphs and its Applications, Proc. Sympos. Smolenice* (1964), pp. 29–36.

[24] Erdős, P. Problems in number theory and combinatorics. In *Proceedings of the sixth Manitoba Conference on Numerical Mathematics* (1977), pp. 35–58.

[25] Erdős, P., Füredi, Z., Loebl, M., and Sós, V. T. Discrepancy of trees. *Studia Sci. Math. Hungar. 30*, 1-2 (1995), 47–57.

[26] Erdős, P., and Simonovits, M. A limit theorem in graph theory. *Studia Scientiarum Mathematicarum Hungarica 1* (1965), 51–57.

[27] Erdős, P., and Stone, A. H. On the Structure of Certain Graphs. *Bulletin of the American Mathematical Society 28* (1946), 1087–1091.

[28] Erdős, P., and Turán, P. On Some Sequences of Integers. *Journal of the London Mathematical Society s1-11*, 4 (1936), 261–264.

[29] Furstenberg, H. Ergodic behavior of diagonal measures and a theorem of szemerédi on arithmetic progressions. *Journal d'Analyse Mathématique 31*, 1 (1977), 204–256.

[30] Furstenberg, H., Katznelson, Y., Ornstein, D., et al. The ergodic theoretical proof of szemerédi's theorem. *Bull. Am. Math. Soc.(NS) 7* (1982).

[31] Goldston, D. A., and Yildi rim, C. Y. Higher correlations of divisor sums related to primes. I. Triple correlations. *Integers 3* (2003), A5, 66.

[32] Goldston, D. A., and Yildirim, C. Higher correlations of divisor sums related to primes III: Small gaps between primes. *Proceedings of the London Mathematical Society 95*, 3 (2007), 653–686.

[33] Goldston, D. A., and Yildirim, C. Y. Small Gaps Between Primes I, 2005.

[34] Gowers, W. A new proof of szemerédi's theorem. *Geometric & Functional Analysis GAFA 11*, 3 (2001), 465–588.

[35] Graham, R. L., Rothschild, B. L., and Spencer, J. H. *Ramsey theory*, vol. 20. John Wiley & Sons, 1990.

[36] Green, B., and Tao, T. The primes contain arbitrarily long arithmetic progressions. *Annals of Mathematics* (2008), 481–547.

[37] Gross, J., and Yellen, J. *Graph Theory and Its Applications, Second Edition*. Textbooks in Mathematics. Taylor & Francis, 2005.

[38] Halász, G. The number-theoretic work of paul turán. *Acta Arithmetica 37*, 1 (1980), 9–19.

[39] Hladkỳ, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M. J., and Szemerédi, E. The approximate Loebl-Komlós-Sós Conjecture I: The sparse decomposition. *arXiv preprint arXiv:1408.3858* (2014).

[40] Hladkỳ, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M. J., and Szemerédi, E. The approximate Loebl-Komlós-Sós Conjecture II: The rough structure of LKS graphs. *arXiv preprint arXiv:1408.3871* (2014).

[41] Hladkỳ, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M. J., and Szemerédi, E. The approximate Loebl-Komlós-Sós Conjecture III: The finer structure of LKS graphs. *arXiv preprint arXiv:1408.3866* (2014).

[42] Hladkỳ, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M. J., and Szemerédi, E. The approximate Loebl-Komlós-Sós Conjecture IV: Embedding techniques and the proof of the main result. *arXiv preprint arXiv:1408.3870* (2014).

[43] Hladký, J., Piguet, D., Simonovits, M., Stein, M., and Szemerédi, E. The approximate Loebl-Komlós-Sós conjecture and embedding trees in sparse graphs. *Electronic Research Announcements in Mathematical Sciences 22*, 0 (2015), 1–11.

[44] Kendall, R. UCLA's Terence Tao awarded inaugural $3 million Breakthrough Prize in Mathematics. http://newsroom.ucla.edu/releases/ucla-s-terence-tao-awarded-inaugural-3-million-breakthrough-prize-in-mathematics, 2014. Accessed: 2017-04-18.

[45] Komlós, J., and Simonovits, M. Szemerédi's regularity lemma and its applications in graph theory.

[46] Ramsey, F. P. On a problem of formal logic. *Proceedings of the London Mathematical Society s2-30*, 1 (1930), 264–286.

[47] Ruzsa, I. Z., and Szemerédi, E. Triple systems with no six points carrying three triangles. *Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18* (1978), 939–945.

[48] Soifer, A. *The mathematical coloring book: mathematics of coloring and the colorful life of its creators*. Springer Science & Business Media, 2008.

[49] Szemerédi, E. On sets of integers containing no k elements in arithmetic progression. *Acta Arithmetica 27* (1975), 199–245.

[50] Szemerédi, E. Regular Partitions of Graphs. *Problémes en Combinatoire et Théorie des Graphes* (1978), 399–401.

[51] Turán, P. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok (in Hungarian) 48* (1941), 436–452.

[52] Van der Waerden, B. L. Beweis einer baudetschen vermutung. *Nieuw Arch. Wisk 15*, 2 (1927), 212–216.

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