VORTEX SHEETS IN ELASTIC FLUIDS

by

Jilong Hu

B.S., Xiamen University, 2010

Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2017

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Jilong Hu

It was defended on

January 6th 2017

and approved by

Prof. Dehua Wang, Department of Mathematics

Prof. Ming Chen, Department of Mathematics

Prof. Huiqiang Jiang, Department of Mathematics

Prof. Ian Tice, Department of Mathematical Sciences, Carnegie Mellon University

Dissertation Director: Prof. Dehua Wang, Department of Mathematics

VORTEX SHEETS IN ELASTIC FLUIDS

Jilong Hu, PhD

University of Pittsburgh, 2017

The stability and existence of compressible vortex sheets is studied for two-dimensional isentropic elastic flows. This problem has a free boundary with extra difficulties that the boundary is characteristic and the Kreiss-Lopatinskii condition holds only in a weak sense. A necessary and sufficient condition is obtained for the linear stability of the rectilinear vortex sheets. More precisely, it is shown that, besides the stable supersonic zone, the elasticity exerts an additional stable subsonic zone. Moreover we also obtain the linear stability of the variable states and the local in time existence of the vortex sheets near the stable rectilinear vortex sheets.

For the linear stability, we employ the Fourier transform and para-differential calculus to perform the spectrum analysis. Since only the weak Kreiss-Lopatinskii condition holds, the a priori estimates for the linearized system exhibit the loss of derivatives. Thus the existence of vortex sheets is proved by a suitable variation of Nash-Moser iteration scheme.

Keywords: Vortex sheets, Elastodynamics, Contact discontinuities, Linear stability, Loss of derivatives, Para-differential calculus, Nash-Moser iteration.

TABLE OF CONTENTS

PREFACE . . . viii
1.0 INTRODUCTION . . . 1
2.0 PRELIMINARY . . . 8
2.1 FUNCTIONAL SPACES . . . 8
2.2 PARA-DIFFERENTIAL OPERATOR . . . 11
2.3 NONLINEAR ESTIMATES . . . 16
2.4 KREISS-LOPATINSKII CONDITION AND KREISS SYSMETRIZER . . . 19
3.0 LINEAR ANALYSIS . . . 22
3.1 CONSTANT COEFFICIENT CASE . . . 25
3.1.1 Linearized system and stability results . . . 25
3.1.2 Decomposition of the system and elimination of the front . . . 34
3.1.2.1 Decomposition of the system . . . 34
3.1.2.2 Elimination of the front . . . 36
3.1.3 Normal mode analysis . . . 39
3.1.3.1 Normal modes . . . 39
3.1.3.2 Separation of modes . . . 42
3.1.4 Lopatinskii determinant . . . 45
3.1.5 Energy estimates . . . 55
3.2 VARIABLE COEFFICIENT CASE . . . 60
3.2.1 Linearization and main results . . . 60
3.2.2 Reduction of the system . . . 65
3.2.2.1 Para-linearization . . . 68

3.2.3 Microlocalization . . . 74
3.2.3.1 Poles . . . 75
3.2.3.2 Roots of the Lopatinskii determinant . . . 77
3.2.4 Estimate in each case . . . 80
3.2.4.1 Case 1–poles and roots . . . 80
3.2.4.2 Case 2–roots . . . 97
3.2.4.3 Case 3–poles . . . 100
3.2.4.4 Case 4–other . . . 102
3.2.5 Proof of the main theorem . . . 103
3.3 LINEARIZED PROBLEM: EXISTENCE AND TAME ESTIMATE . . . 104
3.3.1 Well-posedness of the linearized problem . . . 104
3.3.2 Tame estimate . . . 106
3.3.2.1 Tangential derivatives . . . 106
3.3.2.2 Weighted normal derivatives . . . 109
3.3.2.3 Unweighted normal derivatives . . . 112
3.3.2.4 All deriviatives . . . 113
4.0 NONLINEAR ANALYSIS . . . 115
4.1 MAIN RESULTS ON EXISTENCE AND NONLINEAR STABILITY . . . 115
4.2 APPROXIMATE SOLUTION . . . 116
4.2.1 Compatibility condition for the initial data . . . 116
4.2.2 Construction of an approximate solution . . . 118
4.3 DESCRIPTION OF THE ITERATIVE SCHEME . . . 121
4.3.1 The smoothing operators . . . 121
4.3.2 Iterative scheme . . . 121
4.3.3 Basic estimates . . . 127
4.4 CONVERGENCE OF THE ITERATIVE SCHEME . . . 129
4.4.1 Induction scheme . . . 130
4.4.2 Quadratic errors . . . 131
4.4.3 First substitution errors . . . 132
4.4.4 Estimate of the modified states . . . 134

4.4.5 Second substitution errors 136
4.4.6 Estimate of the left error terms 137
4.4.7 Proof of the inductive argument 140
4.4.8 Proof of the main theorem 144
BIBLIOGRAPHY 145

PREFACE

This dissertation is about the vortex sheets in compressible elastic fluids, which is a challenging and fascinating problem. It took me many days and nights in figuring out all the arguments. This experience is really a mix of joy and frustrating. This work would not have been possible without the kind help and support from many individuals and organizations. Thus I want to express my gratitude to all the people who made my graduate study and dissertation successful.

First and foremost, I would like to thank my advisors Prof. Dehua Wang and Prof. Ming Chen for their invaluable guidance and constant support in my graduate study. Six years ago, it was Prof. Wang who introduced me into this delicate and elegant world of fluid dynamics. Without Prof. Wang's invaluable advisory opinions in these years, I would definitely not be able to finish this project. Prof. Chen not only guided me on how to solve mathematical problems but also provided me many practical suggestions on professional writing and presentation. Thus please let me express my deepest appreciation again to my advisors.

Next, I would also like to extend my appreciation to my committee members Prof. Huiqiang Jiang and Prof. Ian Tice for their valuable time and effort for serving in the dissertation committee with many valuable comments.

Moreover, I owe a great deal to the Department of Mathematics and the University of Pittsburgh for providing me such a great place to learn, do and enjoy mathematics. I also want to thank all my friends for their help and encouragement in my graduate study.

Last but not least, I could not close these acknowledgement without thanking my family for their support. Special thanks to my girlfriend, Rongfang Zhang, for her support and understanding in my life. It is an amazing experience for having someone who can tolerate

and appreciate my speech of mathematics from time to time.

1.0 INTRODUCTION

Vortex sheets are interfaces between two incompressible or compressible flows passing along each other. They arise in a broad range of physical problems in fluid mechanics, aerodynamics, oceanography and astrophysical plasma. Some typical examples include the sharp interface between two parallel shear flows, and vortex flows where the vortices are concentrated within a thin layer. In particular, for compressible flows, vortex sheets are fundamental waves which play an important role in the study of general entropy solutions to multi-dimensional hyperbolic systems of conservation laws. Analyzing the existence and stability of compressible vortex sheets may shed light on the understanding of fluid dynamics and the behavior of entropy solutions.

In this dissertation, we are concerned with the vortex sheets in the following two-dimensional compressible inviscid flow in elastodynamics ([22, 27, 37]):

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.0.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}(\rho\mathtt{F}\mathtt{F}^\top), \tag{1.0.2}$$

$$\mathtt{F}_t + \mathbf{u}\cdot\nabla\mathtt{F} = \nabla\mathbf{u}\mathtt{F}, \tag{1.0.3}$$

where ρ stands for the density, $\mathbf{u} = (v, u) \in \mathbb{R}^2$ the velocity, $\mathtt{F} = (F_{ij}) \in \mathbf{M}^{2\times 2}$ the deformation gradient, and p the pressure with $p = p(\rho)$ a smooth strictly increasing function on $(0, +\infty)$. This model arises from the viscoelastic fluids with negligible viscosity, and describes the features of both fluids and solids.

Before going into further discussions of the elastodynamical model, we would like to first review some known results in the study of the compressible vortex sheets. For vortex sheets in the compressible Euler flow, the study of stability in the linear regime dates back to 1950's by Miles [52, 53] and Fejer-Miles [23]. It is a classical result that, in two or three spatial dimensions with Mach number $M < \sqrt{2}$, the vortex sheets are violently unstable (c.f. [53, 67]). These instabilities are the analogue of the Kelvin-Helmholtz instability for incompressible fluids. For the theory of the incompressible vortex sheets, we refer the readers to [2, 11, 35, 36, 45, 49, 69, 73, 77] and the references therein. In a series of papers, Artola-Majda [3, 4, 5] investigated the interaction of the vortex sheets and highly oscillatory waves in the two-dimensional compressible Euler flow, indicating the global-in-time nonlinear instability of vortex sheets for Mach number $M > \sqrt{2}$. These instability results show that one can not expect the global existence of the vortex sheets in multi-dimensional spaces. In the pioneer works [20, 21], Coulombel and Secchi used a micro-local analysis and Nash-Moser method to establish the linear and local-in-time nonlinear stability in two dimensions with March number $M > \sqrt{2}$. In their setup, the initial data is a small perturbation of a rectilinear (i.e. piecewise constant) vortex sheet and their notion of linear stability is in a similar sense to that of shock waves by Majda [46, 47] and Coulombel [16, 17]. Their results imply the local-in-time existence of such vortex sheets. For two-dimensional non-isentropic Euler flows, Morando-Trebeschi [54] obtained a weaker linear stability of the vortex sheets, that is, there is some additional loss of derivatives in their estimates. Moreover, there have been a lot of studies on the vortex sheets in the multi-dimensional steady flows; see [12, 15, 74, 76] and the references therein.

In complex fluids, the situation becomes more complicated. Ruan-Wang-Weng-Zhu [60] considered the linear stability of compressible vortex sheets in two-dimensional inviscid liquid-gas two-phase flows. They showed that the linear stability, compared with the Euler flow, may be weaker when the Mach number satisfies a certain condition. For the two-dimensional isentropic magnetohydrodynamics (MHD), Wang-Yu [75] showed that the magnetic fields will lower the critical Mach number and exert a small subsonic zone where some linear stability holds for the rectilinear current-vortex sheets. Moreover, for the three-dimensional compressible MHD, Blokhin-Trakhinin [8], Trakhinin [70, 71, 72] and Chen-Wang [13, 14] adopted a different symmetrization approach to obtain certain linear and nonlinear stability, and the existence of the current-vortex sheets. The results of MHD indicate the stabilization effects of the magnetic fields on the current-vortex sheets.

For the viscoelastic fluids, there have been extensive studies on various aspects from the modeling and analysis point of view [22, 30, 38, 44, 55, 59], as well as on their applications [25, 41, 79]. In the two important examples, namely, the shear flows and the vortex flows, both numerical experiments and theoretical analysis indicate that the viscoelasticity plays a stabilization role (see [6, 34, 39, 55] and the references therein). Moreover, vortex sheets in viscoelastic fluids have also been discussed by Huilgol [33, 34], where the author considered the Rayleigh problem in viscoelastic fluids and showed by constructing an example that the unsteady shearing motions can lead to vortex sheets in some viscoelastic liquids. On the other hand, Hu-Wang [32] showed the formation of singularity and the breakdown of classical solutions to system (1.0.1)-(1.0.3) for certain initial data. These results motivate us to investigate the stabilization effects of the elasticity on the vortex sheets in inviscid elastic fluids. Specifically, we first derive the necessary and sufficient conditions for the linear stability of rectilinear vortex sheets in a two-dimensional compressible isentropic inviscid elastic fluid. Then we obtain the linear stability of the variable states and the existence of the vortex sheets near some of the stable rectilinear vortex sheets.

As in the aforementioned works of the Euler flow, two-phase flow and MHD flow, a common challenge of the vortex sheet problem is that the system has a free boundary, and the free boundary is characteristic. The characteristic boundary leads to some loss of control on the trace of the characteristic parts of the solutions [20, 42, 48]. Moreover, the Kreiss-Lopatinskii condition does not hold uniformly, which implies some loss of the tangential derivatives in the estimates of the solutions in terms of the sources on the right side of the linearized problem [16, 17, 20].

In addition to the above difficulties, for the elastic flow, the appearance of the elasticity leads to some extra difficulties. For the linear stability of rectilinear vortex sheets, the distribution of the roots in the Lopatinskii determinant is more complicated. More precisely, the non-differentiable points of the eigenvalues may coincide with the roots of the Lopatinskii determinant, which does not happen for the Euler flow (c.f. [20]). In the standard arguments [16, 20, 40], one needs to construct the Kreiss symmetrizer and work along with the Lopatinskii determinant at each point in the frequency space. At the non-differentiable points of the eigenvalues, the usual Kreiss symmetrizer requires no loss of derivatives with respect to the source term; but the degeneracy at the roots of the Lopatinskii determinant implies the loss of the tangential derivatives, which leads to some serious obstacles to apply the Kreiss symmetrizer argument.

To overcome this difficulty, we deal with this problem in a different way. More precisely, instead of using Kreiss symmetrization to separate every single mode, we first perform an upper triangularization of the system to separate only the outgoing modes from the system at all points in the frequency space. Then we establish an exact estimate on the outgoing modes from the equation, and use the L^2-regularity of solutions to conclude that the outgoing modes in the homogeneous system are zero. As a result, we only need to estimate the incoming modes, which can be derived directly from the Lopantiskii determinant. Therefore combining the estimates for the outgoing and incoming modes we achieve the linear stability of the rectilinear vortex sheets. In conclusion, we find that with the added elasticity, a new stability region can be generated in the subsonic zone, indicating the stabilization effect of elasticity as expected. We also find that within the stable subsonic region there exist a class of states where the stability of such states is weaker than the stability of other states in the sense that there is an extra loss of tangential derivatives, due to the fact that the Lopatinskii determinant exhibits higher order of degeneracy at such states. This is a new feature which Euler flows do not possess. We further remark that our upper triangularization simply tells us that the outgoing modes are all zero. Thus we can avoid the lengthy computation and estimates for the outgoing modes when the Kreiss symmetrization is applied, and hence the arguments are greatly simplified.

For the linear stability of the variable states near the stable rectilinear vortex sheets, the main difficulty is that some of the roots in the Lopatinskii determinant coincide with the singular points (poles) of the systems which is introduced by characteristic boundary. Since the Lopatinskii determinant vanishes at some points in frequency space, it is necessary to introduce a weight function on the boundary to characterize this degeneracy. In the argument in deriving the energy estimate, one needs to extend this weight function into the interior of the domain. In the standard approach [16, 20], one needs to extend the weight function in the frequency space along the bicharacteristic curves which originate from the boundary. However, to guarantee the above argument can be developed, one expect that the roots of Lopatinskii determinant do not coincide with the poles of the system. Because in dealing with the poles, people need to construct a special symmetrizer along the poles in the interior of the domain. To combine the special symmetrizer with the above weight function argument in dealing with roots of Lopatinskii determinant, people need to guarantee that poles in the interior of the domain are distributed along the bicharacteristic curves to keep a uniform situation. However, the bicharacteristic curves of the system generally does not propagate in the same way as the poles of the system into the interior of the domain. So the above extension of weight function would leads to some discrepency in the argument which cause some serious obstacles for us to apply this argument if some roots of the Lopatinskii determinant coincide with the poles of the system on the boundary.

To overcome this difficulty, we consider a different approach in dealing with the roots of Lopatinskii determinant. Instead of extending the weight function along the bicharacteristic curves, we consider the relation between the coefficients in the weight function and background states. We observed that the coefficients of the weight function only depends on the value of background states, not the positions in the physical domain. So we can extend the weight function by using the value of background state. Indeed, the poles of the system are also determined by an algebraic equation whose coefficients only depend on the value of background states. This actually guarantees that the poles of the system agree with the weight function we construct. Hence, we can combine the special symmetrizer with the weight function we construct together through the upper triangularization method and yield the expected estimates.

However, the complicated distribution of the roots in the Lopatinskii determinant as in the linear stability of the rectilinear vortex sheets really cause serious obstacles in proving the linear stability of the variable states near every stable rectilinear vortex sheets. Since the variable states lead to the variable coefficient linear systems, one needs to utilize the para-differential calculus to perform the spectrum analysis. However, the para-differential calculus usually generates errors after each operations. These errors can not be controlled if two roots of the Lopatinskii determinant coincide. Moreover, the eigenvalues of the system is not compatible with the para-differential calculus at the non-differentiable points, since the para-differential calculus requires the functions to keep a uniform homogeneity. So one could only apply the standard Kreiss symmetrization method at these points, which may require the non-differentiable points of the eigenvalues cannot coincide with the roots of the Lopatinskii determinant and poles of the system. Therefore, we have to exclude the stable rectilinear vortex sheets where the above situations happen in discussing the linear stability of the variable states

For the existence of the vortex sheets, the appearance of the elasticity changed the structure of the system in the Euler flow. The classical approach [13, 21, 72] is to apply the Nash-Moser iterative scheme to compensate the loss of derivatives in the linear stability. This requires one to derive the tame estimates on the high order derivatives of the solutions to the linear system. In the Euler flow, one can obtains the tame estimates on the normal derivatives through investigating the energy estimates of a vorticity type equation in Euler flow. This equation is a transport type equation with source terms containing only lower order derivatives of the unknowns than the terms in the transport structure. However, the appearance of the elasticity leads to some serious obstacles on this method, because of the interactions between the elasticity and the velocity.

To overcome this difficulty, instead of the standard Sobolev space, we use the anisotropic Sobolev space as the space in which we look for the solution to the nonlinear system. Since the weight function on normal derivatives and inhomogeneity in counting the orders of derivatives, in the transport type equation, we can still close the energy estimates even the source terms containing higher order derivatives than the terms in the transport structure. In fact, this approach can be applied to general hyperbolic systems with characteristic boundary as in [13, 26, 62, 63, 72].

The rest of the dissertation is organized as follows. In Chapter 2, we present some preliminary information on functional spaces, para-differential calculus, nonlinear estimates and Kreiss-Lopatinskii condition. In Chapter 3, we studied the linearized problem for the governing dynamics of vortex sheets. Section 3.1 and Section 3.2 studied the stability of the linearized system with constant coefficients and variable coefficients respectively. In Section 3.3, we obtain the existence and tame estimate of the solutions to the linearized system with the stability results obtained in previous two sections. In Chapter 4, we apply the information from the linear analysis to show the nonlinear stability and local in time existence of the solutions to the nonlinear governing equations of vortex sheets by Nash-Moser iterative scheme.

2.0 PRELIMINARY

In this chapter, we introduce the some notations and mathematical tools needed in our linear and nonlinear analysis.

2.1 FUNCTIONAL SPACES

In this dissertation, we consider the following spatial domain

$$\mathbb{R}^2_+ := \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\right\}.$$

For the sake of our linear stability analysis, we define the domain Ω and its boundary ω,

$$\Omega := \left\{(t, x_1, x_2) \in \mathbb{R}^3 : \ x_2 > 0\right\}, \quad \omega := \{(t, x_1, x_2) \in \Omega : \ x_2 = 0\}.$$

For the well-posedness of linear problem and nonlinear analysis, we consider the following domain Ω_T and its spatial boundary ω_T

$$\Omega_T := \left\{(t, x_1, x_2) \in \mathbb{R}^3 : \ x_2 > 0 \text{ and } t \leq T\right\}, \quad \omega_T := \{(t, x_1, x_2) \in \Omega : \ x_2 = 0 \text{ and } t \leq T\}.$$

On these domains, we introduce the following functional space. First we define the Sobolev space with the weight on the time variable as in [20],

$$H^s_\gamma(\omega) := \{u(t, x_1) \in \mathcal{D}'(\mathbb{R}^2) \ : \ e^{-\gamma t} u(t, x_1) \in H^s(\omega)\},$$

for $s \in \mathbb{N}, \ \gamma \geq 1$, equipped with the norms

$$\|u\|_{H^s_\gamma(\omega)} := \|e^{-\gamma t} u\|_{H^s(\omega)}.$$

Similarly, we can define the space $H^s_\gamma(\omega_T)$ and its norm $\|\cdot\|_{H^s_\gamma(\omega_T)}$. We notice that the above norm has the following equivalent form of definition by Fourier transform:

$$\|\tilde{u}\|^2_{s,\gamma} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}(\gamma^2+|\xi|^2)^s|\hat{\tilde{u}}(\xi)|^2 d\xi, \quad \forall\, u \in H^s(\omega),$$

where $\hat{\tilde{u}}(\xi)$ is the Fourier transform of $\tilde{u}$ and $\tilde{u} := e^{-\gamma t}u$. We denote the above statement by $\|u\|_{H^s_\gamma(\omega)} \simeq \|\tilde{u}\|_{s,\gamma}$, where $\simeq$ is the equivalence relation of norms. Now we define the space $L^2(\mathbb{R}_+; H^s_\gamma(\omega))$, equipped with the norm

$$|||v|||^2_{L^2(H^s_\gamma(\omega))} = \int_0^{+\infty}\|v(\cdot,x_2)\|^2_{H^s_\gamma(\omega)}dx_2.$$

Again we have

$$|||v|||^2_{L^2(H^s_\gamma(\omega))} \simeq |||\tilde{v}|||^2_{s,\gamma} := \int_0^{+\infty}\|\tilde{v}(\cdot,x_2)\|^2_{s,\gamma}dx_2.$$

Note that $\|\cdot\|_{0,\gamma}$ is actually the usual norm on $L^2(\omega)$ and $|||\cdot|||_{0,\gamma}$ is the usual norm on $L^2(\Omega)$. Similarly, we can also define $L^2(\mathbb{R}_+; H^s_\gamma(\omega_T))$ and its norm $|||\cdot|||_{L^2(H^s_\gamma(\omega_T))}$.

Next we introduce the anisotropic Sobolev space as in [1]. For all integer $s > 0$, $\gamma > 1$ and multiindex $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, we define

$$H^{s,\gamma}_*(\Omega_T) := \{u(t,x_1,x_2) \in \mathcal{D}'(\Omega_T) \ : \ e^{-\gamma t}\partial^\alpha_*\partial^k_2 u(t,x_1) \in L^2(\Omega_T) \text{ for } |\alpha|+2k \le s\},$$

imposed with the norms

$$[u]_{s,\gamma,T} := \sum_{|a|+2k\le r, r\le s}\gamma^{r-|\alpha|-2k}\|e^{-\gamma t}\partial^\alpha_*\partial^k_2 u\|_{L^2(\Omega_T)},$$

where $\partial^\alpha_* := \partial^{\alpha_0}_t\partial^{\alpha_1}_1(\sigma(x_2)\partial_2)^{\alpha_2}$ with σ being a fixed smooth weight function such that $\sigma(0) = 0$ and $\sigma(x) = 1$ if $x > 1$. For the convenience of proof, we point out the following trace property and extension property in this anisotropic Sobolev space

Proposition 2.1.1. *If $s > 1$ and $u \in H_*^{s,\gamma}(\Omega_T)$, then $u|_{x_2=0} \in H_\gamma^{s-1}(\omega_T)$ and*

$$\|u|_{x_2=0}\|_{H_\gamma^{s-1}(\omega_T)} \leq C[u]_{s,\gamma,T},$$

where C is some constant. Moreover if there is $v \in H_\gamma^s(\omega_T)$ for $s > 0$, then there is $u \in H_*^{s+1,\gamma}(\Omega_T)$ such that $u|_{x_2=0} = v$ and

$$[u]_{s+1,\gamma,T} \leq C\|u|_{x_2=0}\|_{H_\gamma^s(\omega_T)},$$

where C is some constant. The same result holds with $\mathbb{R}_+^2$ and $\mathbb{R}$ instead of Ω_T and ω_T.

We note that if γ is fixed, the above space $H_*^{s,\gamma}(\Omega_T)$ is equivalent to the classic anisotropic space which has been widely studied in [56, 57, 58, 61, 63, 64, 65, 66, 68, 78]:

$$H_*^s(\Omega_T) := \{u(t,x_1,x_2) \in \mathcal{D}'(\Omega_T) \ : \ \partial_*^\alpha \partial_2^k u(t,x_1) \in L^2(\Omega_T) \text{ for } |\alpha| + 2k \leq s\},$$

imposed with the norms

$$[u]_{s,*,T} := \sum_{|a|+2k\leq s} \|\partial_*^\alpha \partial_2^k u\|_{L^2(\Omega_T)}.$$

Also it is straight forward to check that

$$[u]_{s,\gamma,T} = [e^{-\gamma t}u]_{s,*,T}.$$

Similarly, we can define $H_*^s(\mathbb{R}_+^2)$ and its norm $[\cdot]_{s,*}$.

Finally we define

$$W^{1,\tan}(\Omega_T) := \{u(t,x_1,x_2) \in \mathcal{D}'(\Omega_T) \ : \ \|u\|_{L^\infty(\Omega_T)} + \sum_{|\alpha|=1} \|\partial_*^\alpha u\|_{L^\infty(\Omega_T)} \leq \infty\},$$

$$W^{2,\tan}(\Omega_T) := \{u(t,x_1,x_2) \in \mathcal{D}'(\Omega_T) \ : \ \|u\|_{W^{1,\infty}(\Omega_T)} + \sum_{|\alpha|=1} \|\partial_*^\alpha u\|_{W^{1,\infty}(\Omega_T)} \leq \infty\},$$

with the norms

$$\|u\|_{W^{1,\tan}(\Omega_T)} := \|u\|_{L^\infty(\Omega_T)} + \sum_{|\alpha|=1} \|\partial_*^\alpha u\|_{L^\infty(\Omega_T)},$$

$$\|u\|_{W^{2,\tan}(\Omega_T)} := \|u\|_{W^{1,\infty}(\Omega_T)} + \sum_{|\alpha|=1} \|\partial_*^\alpha u\|_{W^{1,\infty}(\Omega_T)}.$$

respectively.

For the convenience in the proof, we introduce the following differential operators,

$$\nabla := (\partial_t, \partial_1, \partial_2), \quad \nabla^{\tan} := (\partial_t, \partial_1), \quad \nabla_*^{\tan} := (\partial_t, \partial_1, \sigma\partial_2),$$

and, for the multi-index $\beta = (\alpha_0, \alpha_1, \alpha_2, k)$, we denote

$$D^\beta = \partial_t^{\alpha_0} \partial_1^{\alpha_1} (\sigma\partial_2)^{\alpha_2} \partial_2^k.$$

2.2 PARA-DIFFERENTIAL OPERATOR

In this section, we collect the main results of the para-differential operator in [9, 50, 51], which is the main tool in dealing with the variable coefficient cases of linear analysis. In this section α and β are both multi-indices in $\mathbb{N}^2$.

Definition 2.2.1. A paradifferential symbol of degree $m \in \mathbb{R}$ and regularity k ($k \in \mathbb{N}$) is a function $a(x, \xi, \gamma) \in: \mathbb{R}^2 \times \mathbb{R}^2 \times [0, \infty) \to \mathbb{C}^{N\times N}$ such that a is C^∞ with respect to ξ and for all $\alpha \in \mathbb{N}^2$, there exists a constant C_α verifying

$$\forall(\xi, \gamma), \quad \|\partial_\xi^\alpha a(\cdot, \xi, \gamma)\|_{W^{k,\infty}(\mathbb{R}^2)} \leq C_\alpha \lambda^{m-|\alpha|,\gamma}(\xi) = C_\alpha(\gamma^2 + |\xi|^2)^{(m-|\alpha|)/2}.$$

We denote by Γ_k^m the set of paradifferential symbols whose degree is m and regularity is k and by Σ_k^m the subset of paradifferential symbols $a \in \Gamma_k^m$ such that for a suitable $\varepsilon \in (0, 1)$ one has

$$\forall(\xi, \gamma), \quad \mathrm{Supp}\mathcal{F}_x a(\cdot, \xi, \gamma) \subset \{\zeta \in \mathbb{R}^2 : |\zeta| \leq \varepsilon(\gamma^2 + |\xi|^2)^{1/2}\}.$$

We note that the functions a in the symbol class Σ_k^m are C^∞ with respect to both x and ξ, and

$$\forall(x, \xi, \gamma), \quad |\partial_x^\beta \partial_\xi^\alpha a(x, \xi, \gamma)| \leq C_{\alpha,\beta} \lambda^{m-|\alpha|+|\beta|,\gamma}(\xi).$$

This suggests that $\forall a \in \Sigma_k^m$ belongs to Hörmander's class $S_{1,1}^m$ [29] and defines an operator $\mathrm{Op}^\gamma(a)$ on the Schwartz' class S as

$$\forall u \in S, \quad \mathrm{Op}^\gamma(a)u(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} a(x, \xi, \gamma)\hat{u}(\xi) d\xi.$$

Now we introduce the order of a family of operators on Sobolev space.

Definition 2.2.2. A family of operators $\{P^\gamma\}$ defined for $\gamma \geq 1$ will be said of order $\leq m$ ($m \in \mathbb{R}$) if the operators P^γ are uniformly bounded from H^{s+m} to H^s and

$$\forall \gamma \geq 1, \quad \forall u \in H^{s+m}, \quad \|P^\gamma u\|_{s,\gamma} \leq C(s,m)\|u\|_{s+m,\gamma}.$$

With the above definition, we have

Theorem 2.2.1. *If $a \in \Sigma_k^m$, the family $\{Op^\gamma(a)\}$ is of order $\leq m$.*

In order to define a family of order $\leq m$ from the symbol class Γ_k^m, we consider the following cut-off function to regularize the symbol by convolution.

Definition 2.2.3. Let $\mathbf{\Psi} : \mathbb{R}^2 \times \mathbb{R}^2 \times [1,\infty) \to [0,\infty)$ be a C^∞ function such that the following estimates hold

$$\forall (\zeta,\xi,\gamma), \quad |\partial_\zeta^\alpha \partial_\xi^\beta \mathbf{\Psi}(\zeta,\xi,\gamma)| \leq C_{\alpha,\beta}\lambda^{-|\alpha|-|\beta|,\gamma}(\xi).$$

Then $\mathbf{\Psi}$ is an admissible cut-off function if there exist real numbers $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\begin{aligned} \mathbf{\Psi}(\zeta,\xi,\gamma) = 1 \quad & \text{if } |\zeta| \leq \varepsilon_1(\gamma^2 + |\xi|^2)^{1/2}, \\ \mathbf{\Psi}(\zeta,\xi,\gamma) = 0 \quad & \text{if } |\zeta| \geq \varepsilon_2(\gamma^2 + |\xi|^2)^{1/2}. \end{aligned}$$

An simple example of the admissible cut-off function is the following: let χ be a C^∞ function defined on $\mathbb{R}^2 \times \mathbb{R}$ such that

$$\begin{cases} \chi(\xi_1,\gamma_1) \leq \chi(\xi_2,\gamma_2) & \text{if } \gamma_1^2 + |\xi_1|^2 \geq \gamma^2 + |\xi_2|^2, \\ \chi(\xi,\gamma) = 1 & \text{if } (\gamma_1^2 + |\xi_1|^2)^{1/2} \leq 1/2, \\ \chi(\xi,\gamma) = 0 & \text{if } (\gamma_1^2 + |\xi_1|^2)^{1/2} \geq 1. \end{cases}$$

Then we define $\mathbf{\Phi} := \chi(\xi/2,\gamma/2) - \chi(\xi,\gamma)$ and

$$\mathbf{\Psi}_0(\zeta,\xi,\gamma) := \sum_{p\geq 0} \chi(2^{2-p}\zeta,0)\mathbf{\Phi}(2^{-p}\xi,2^{-p}\gamma).$$

Here $\mathbf{\Psi}_0$ is an admissible cut-off function.

Now we assume $\mathbf{\Psi}$ is an admissible cut-off function and $K^{\mathbf{\Psi}}$ is the inverse Fourier transform of $\mathbf{\Psi}$ on ζ variable. It is easy to verify that

$$\forall(\xi,\gamma), \quad \|\partial_\xi^\alpha K^{\mathbf{\Psi}}(\cdot,\xi,\gamma)\|_{L^1(\mathbb{R}^2)} \leq C_\alpha \lambda^{-|\alpha|,\gamma}(\xi),$$

which allow us to obtain

Proposition 2.2.1. *Let $\mathbf{\Psi}$ be an admissible cut-off function. The mapping*

$$a \to \sigma_a^{\mathbf{\Psi}}(x,\xi,\gamma) := \int_{\mathbb{R}^2} K^{\mathbf{\Psi}}(x-y,\xi,\gamma)a(y,\xi,\gamma)dy$$

is continuous from Γ_k^m to Σ_k^m for all m. Moreover, if $a \in \Gamma_1^m$, then $a - \sigma_a^{\mathbf{\Psi}} \in \Gamma_0^{m-1}$. In particular, if $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ are two admissible cut-off functions and $a \in \Gamma_1^m$, then $\sigma_a^{\mathbf{\Psi}_1} - \sigma_a^{\mathbf{\Psi}_2} \in \Sigma_0^{m-1}$.

For an admissible cut-off function $\mathbf{\Psi}$, we can define the paradifferential operator $T_a^{\mathbf{\Psi},\gamma}$ by the formula

$$T_a^{\mathbf{\Psi},\gamma} := \mathrm{Op}^\gamma(\sigma_a^{\mathbf{\Psi}}).$$

If $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ are two admissible cut-off functions and $a \in \Gamma_1^m$, it is easy to show that the family $\{T_a^{\mathbf{\Psi}_1,\gamma} - T_a^{\mathbf{\Psi}_2,\gamma}\}$ is of order $\leq m-1$ from Proposition 2.2.1 and Theorem 2.2.1.

Moreover, we can obtain the following theorems in symbolic calculus.

Theorem 2.2.2. *Let $a \in \Gamma_1^m$ and $b \in \Gamma_1^{m'}$. Then $ab \in \Gamma_1^{m+m'}$ and the family*

$$\{T_a^{\mathbf{\Psi},\gamma} \circ T_b^{\mathbf{\Psi},\gamma} - T_{ab}^{\mathbf{\Psi},\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m+m'-1$ for all admissible cut-off function $\mathbf{\Psi}$.

Let $a \in \Gamma_1^m$. Then the family

$$\{(T_a^{\mathbf{\Psi},\gamma})^* - T_{a^*}^{\mathbf{\Psi},\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m-1$ for all admissible cut-off function $\mathbf{\Psi}$.

Let $a \in \Gamma_2^m$ and $b \in \Gamma_2^{m'}$. Then $ab \in \Gamma_2^{m+m'}$ and the family

$$\{T_a^{\mathbf{\Psi},\gamma} \circ T_b^{\mathbf{\Psi},\gamma} - T_{ab}^{\mathbf{\Psi},\gamma} - T_{-i\Sigma_j \partial_{\xi_j} a \partial_{x_j} b}^{\mathbf{\Psi},\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m+m'-2$ for all admissible cut-off function $\mathbf{\Psi}$.

Let $a \in \Gamma_2^m$. Then the family

$$\{(T_a^{\mathbf{\Psi},\gamma})^* - T_{a^*}^{\mathbf{\Psi},\gamma} - T_{-i\Sigma_j \partial_{\xi_j}\partial_{x_j}a^*}^{\mathbf{\Psi},\gamma}\}_{\gamma\geq 1}$$

is of order $\leq m-2$ for all admissible cut-off function $\mathbf{\Psi}$.

Next we introduce the Gårding's inequality.

Theorem 2.2.3. *Let $a \in \Gamma_1^{2m}$ and $\mathbf{\Psi}$ be an admissible cut-off function. Assume there exists a constant $c > 0$ such that*

$$\forall (x,\xi,\gamma), \quad \Re a(x,\xi,\gamma) \geq c\lambda^{2m,\gamma}(\xi) I.$$

Then there exists $\gamma_0 \geq 1$ such that

$$\forall \gamma \geq \gamma_0, \quad \forall u \in H^m, \quad \Re\langle T_a^{\mathbf{\Psi},\gamma} u, u\rangle_{H^{-m},H^m} \geq \frac{c}{2}\|u\|_{m,\gamma}^2.$$

Here we have a microlocalized version of Gåding's inequality.

Theorem 2.2.4. *Let $a \in \Gamma_1^{2m}$, $\chi \in \Gamma_1^0$ and $\mathbf{\Psi}$ be an admissible cut-off function. Assume there exists $\tilde{\chi} \in \Gamma_1^0$ and a constant $c > 0$ such that $\tilde{\chi} \geq 0$, $\tilde{\chi}\chi = \chi$ and*

$$\forall (x,\xi,\gamma), \quad \tilde{\chi}^2(x,\xi,\gamma)\Re a(x,\xi,\gamma) \geq c\tilde{\chi}^2(x,\chi,\gamma)\lambda^{2m,\gamma}(\xi) I.$$

Then there exists $\gamma_0 \geq 1$ and $C > 0$ such that

$$\forall \gamma \geq \gamma_0, \quad \forall u \in H^m, \quad \Re\langle T_a^{\mathbf{\Psi},\gamma} T_\chi^{\mathbf{\Psi},\gamma} u, T_\chi^{\mathbf{\Psi},\gamma} u\rangle_{H^{-m},H^m} \geq \frac{c}{2}\|T_\chi^{\mathbf{\Psi},\gamma} u\|_{m,\gamma}^2 - C\|u\|_{m-1,\gamma}^2.$$

We now consider the case of paraproducts, which are defined by the particular choice of $\mathbf{\Psi}_0$ as cut-off function. From now on, we fixed the cut-off function $\mathbf{\Psi}_0$ and denote $T_a^{\mathbf{\Psi}_0,\gamma}$ by T_a^γ. The followings are the important results about paraproducts.

Theorem 2.2.5. *Let $a \in W^{1,\infty}(\mathbb{R}^2)$, $u \in L^2(\mathbb{R}^2)$, and $\gamma \geq 1$. Then we ahve*

$$\|au - T_a^\gamma u\|_0 \leq \frac{C}{\gamma}\|a\|_{W^{1,\infty}(\mathbb{R}^2)}\|u\|_0,$$
$$\|a\partial_{x_j} u - T_a^\gamma \partial_{x_j} u\|_0 \leq C\|a\|_{W^{1,\infty}(\mathbb{R}^2)}\|u\|_0,$$

for a some constant C which is independent of a, u and γ.

If $a \in W^{2,\infty}(\mathbb{R}^2)$, we have

$$\|au - T_a^\gamma u\|_{1,\gamma} \leq \frac{C}{\gamma}\|a\|_{W^{2,\infty}(\mathbb{R}^2)}\|u\|_0,$$
$$\|a\partial_{x_j} u - T_a^\gamma \partial_{x_j} u\|_{1,\gamma} \leq C\|a\|_{W^{2,\infty}(\mathbb{R}^2)}\|u\|_0,$$

for a some constant C which is independent of a, u and γ.

At last, we point out that the above definitions and propositions of para-differential calculus can be extended for the symbols $a(x_0, x_1, x_2, \xi, \gamma)$ defined on $\Omega \times \mathbb{R}^2 \times (0, \infty)$ as in [20] such that the mapping $x_2 \to a(\cdot, x_2, \cdot)$ is bounded in Γ_k^m. Then we can define T_a^γ as

$$\forall u \in C_c^\infty(\bar{\Omega}), \quad \forall x_2 \geq 0, \quad (T_a^\gamma u)(\cdot, x_2) := T_{a(x_2)}^\gamma u(\cdot, x_2).$$

Similarly as in Theorem 2.2.5, for all $a \in W^{2,\infty}(\Omega)$ and all $u \in L^2(\Omega)$ we have

$$\|au - T_a^\gamma u\|_0 \leq \frac{C}{\gamma}\|a\|_{W^{1,\infty}(\Omega)}\|u\|_0,$$
$$\|a\partial_{x_j} u - T_a^\gamma \partial_{x_j} u\|_0 \leq C\|a\|_{W^{1,\infty}(\Omega)}\|u\|_0, \quad j = 0, 1.$$

2.3 NONLINEAR ESTIMATES

In this section, we recall some nonlinear tame estimates in weighted Sobolev space $H^s_\gamma(\omega_T)$ and anisotropic Sobolev space $H^{s,\gamma}_*(\Omega_T)$. First we consider weighted Sobolev space $H^s_\gamma(\omega_T)$ as in [16, 17, 20].

Theorem 2.3.1 (Gagliardo-Nirenberg)**.** *Let $s > 1$ be an interger, $\gamma > 1$ and $T \in \mathbb{R}$. There is a constant C which is independent of γ and T such that for all $u \in H^s_\gamma(\omega_T) \cap L^\infty(\omega_T)$ and all multi-index $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq s$, one has*

$$\|\partial^\alpha u\|_{L^{2p}_\gamma(\omega_T)} \leq C\|u\|^{1-1/p}_{L^\infty(\omega_T)}\|u\|^{1/p}_{H^s_\gamma(\omega_T)}, \quad \frac{1}{p} = \frac{|\alpha|}{s}.$$

This results can be used to prove the following tame estimates for products of functions in $H^s_\gamma(\omega_T)$.

Theorem 2.3.2. *Let $s \geq 1$ be an integer, $\gamma \geq 1$ and $T \in \mathbb{R}$. There is a constant C which is independent of γ and T such that for all functions $u, v \in H^s_\gamma(\omega_T) \cap L^\infty(\omega_T)$, the product uv belongs to $H^s_\gamma(\omega_T)$ and satisfies the estimate*

$$\|uv\|_{H^s_\gamma(\omega_T)} \leq C\left(\|u\|_{L^\infty(\omega_T)}\|v\|_{H^m_\gamma(\omega_T)} + \|v\|_{L^\infty(\omega_T)}\|u\|_{H^m_\gamma(\omega_T)}\right).$$

Moreover for all multi-indices $\alpha, \beta \in \mathbb{N}^2$ such that $|\alpha| + |\beta| \leq s$, one has

$$\|\partial^\alpha u\partial^\beta v\|_{L^2_\gamma(\omega_T)} \leq C\left(\|u\|_{L^\infty(\omega_T)}\|v\|_{H^m_\gamma(\omega_T)} + \|v\|_{L^\infty(\omega_T)}\|u\|_{H^m_\gamma(\omega_T)}\right).$$

Further there is a tame estimate for composed functions

Theorem 2.3.3. *Let $s \geq 1$ be an integer, $\gamma \geq 1$ and $T \in \mathbb{R}$. Assume F is a C^∞ function such that $F(0) = 0$. Then there is an increasing fucntion $C(\cdot)$ which is independent of γ and T such that for all $u \in H^s_\gamma(\omega_T) \cap L^\infty(\omega_T)$, the composed function $F(u)$ belongs to $H^s_\gamma(\omega_T)$ and satisfies*

$$\|F(u)\|_{H^s_\gamma(\omega_T)} \leq C\left(\|u\|_{L^\infty(\omega_T)}\right)\|u\|_{H^s_\gamma(\omega_T)}.$$

At last we note the following embedding estimates in $H^s_\gamma(\omega_T)$.

Theorem 2.3.4. *The following inequalities hold with a constant C which is independent of $\gamma \geq 1$*

$$\|e^{-\gamma T} u\|_{L^\infty(\omega_T)} \leq \frac{C}{\gamma}\|u\|_{H^2_\gamma(\omega_T)} \qquad \forall u \in H^2_\gamma(\omega_T),$$
$$\|e^{-\gamma T} u\|_{W^{1,\infty}(\omega_T)} \leq C\|u\|_{H^3_\gamma(\omega_T)} \qquad \forall u \in H^3_\gamma(\omega_T).$$

Next we focus on the anisotropic Sobolev space $H^{s,\gamma}_*(\Omega_T)$ in [13, 72]. We note that the estimates in $H^{s,\gamma}_*(\Omega_T)$ are different for s to be odd and even. First we recall the estimates when s is even.

Theorem 2.3.5 (Gagliardo-Nirenberg)**.** *Let $s > 1$ be an even interger, $\gamma > 1$ and $T \in \mathbb{R}$. There is a constant C which is independent of γ and T such that for all $u \in H^{s,\gamma}_*(\Omega_T) \cap L^\infty(\Omega_T)$ and all multi-index $\alpha \in \mathbb{N}^3$ and $k \in \mathbb{N}$ with $|\alpha| + 2k \leq s$, one has*

$$\|\partial^\alpha_* \partial^k_2 u\|_{L^{2p}_\gamma(\Omega_T)} \leq C\|u\|^{1-1/p}_{L^\infty(\Omega_T)}[u]^{1/p}_{s,\gamma,T}, \quad \frac{1}{p} = \frac{|\alpha| + 2k}{s}.$$

Similarly as in $H^s_\gamma(\omega_T)$, we have the following estimates for the products and composed function.

Theorem 2.3.6. *Let $s \geq 1$ be an even integer, $\gamma \geq 1$ and $T \in \mathbb{R}$. Then for all functions $u, v \in H^{s,\gamma}_*(\Omega_T) \cap L^\infty(\Omega_T)$ and C^∞ function F of u, one has*

$$[uv]_{s,\gamma,T} \leq C_1 \left(\|u\|_{L^\infty(\Omega_T)}[v]_{s,\gamma,T} + \|v\|_{L^\infty(\Omega_T)}[u]_{s,\gamma,T}\right).$$
$$[F(u)]_{s,\gamma,T} \leq C_2(\|u\|_{L^\infty(\Omega_T)})\left(1 + [u]_{s,\gamma,T}\right),$$

where C_1 is a constant and C_2 is an increasing function. They are both independent of γ and T. Moreover if $F(0) = 0$, one has

$$[F(u)]_{s,\gamma,T} \leq C_2(\|u\|_{L^\infty(\Omega_T)})[u]_{s,\gamma,T}.$$

For the embedding theorem, we have

Theorem 2.3.7. *The following inequalities hold with a constant C which is independent of $\gamma \geq 1$*

$$\|e^{-\gamma T} u\|_{L^\infty(\Omega_T)} \leq C[u]_{4,\gamma,T} \qquad \forall u \in H_*^{4,\gamma}(\Omega_T),$$
$$\|e^{-\gamma T} u\|_{W^{1,\infty}(\Omega_T)} \leq C[u]_{6,\gamma,T} \qquad \forall u \in H_*^{6,\gamma}(\Omega_T).$$

For the case when s is odd, we note that

$$[u]_{s,\gamma,T} \leq C \left([u]_{s-1,\gamma,T} + \sum_{|\alpha|=1} [\partial_*^\alpha u]_{s-1,\gamma,T} \right).$$

Thus we have

Theorem 2.3.8. *Let $s \geq 1$ be an odd integer, $\gamma \geq 1$ and $T \in \mathbb{R}$. Then for all functions $u, v \in H_*^{s,\gamma}(\Omega_T) \cap W^{1,tan}(\Omega_T)$ and C^∞ function F of u, one has*

$$[uv]_{s,\gamma,T} \leq C_1 \left(\|u\|_{W^{1,tan}(\Omega_T)} [v]_{s,\gamma,T} + \|v\|_{W^{1,tan}(\Omega_T)} [u]_{s,\gamma,T} \right).$$
$$[F(u)]_{s,\gamma,T} \leq C_2(W^{1,tan}(\Omega_T)) \left(1 + [u]_{s,\gamma,T}\right),$$

where C_1 is a constant and C_2 is an increasing function. They are both independent of γ and T. Moreover if $F(0) = 0$, one has

$$[F(u)]_{s,\gamma,T} \leq C_2(\|u\|_{W^{1,tan}(\Omega_T)})[u]_{s,\gamma,T}.$$

and

Theorem 2.3.9. *The following inequalities hold with a constant C which is independent of $\gamma \geq 1$*

$$\|e^{-\gamma T} u\|_{W^{1,tan}(\Omega_T)} \leq C[u]_{5,\gamma,T} \qquad \forall u \in H_*^{5,\gamma}(\Omega_T),$$
$$\|e^{-\gamma T} u\|_{W^{2,tan}(\Omega_T)} \leq C[u]_{7,\gamma,T} \qquad \forall u \in H_*^{7,\gamma}(\Omega_T).$$

2.4 KREISS-LOPATINSKII CONDITION AND KREISS SYSMETRIZER

The purpose of this section is to explain what is the Kreiss-Lopatinskii condition and why it is important in our analysis. All the discussion in this section can be found in [7].

For this section, we forget the notations defined before and redefine its own notations. First we assume

$$L := \partial_t + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha,$$

be a hyperbolic operator, with A^α being a $n \times n$ matrix, B is a constant real-valued $q \times n$ matrix and Ω be the half space in $\mathbb{R}^d$ defined by $x_d > 0$. Denote the tangential variable $(x_1, x_2, ..., x_{d-1})$ by y. Then we can define a hyperbolic initial boundary value problem on the half space as follows

$$Lu(x,t) = f(x,t), \qquad x_d, t > 0, y \in \mathbb{R}^{d-1}, \tag{2.4.1}$$

$$Bu(y,0,t) = g(y,t), \qquad t > 0, y \in \mathbb{R}^{d-1}, \tag{2.4.2}$$

$$u(x,0) = u_0(x), \qquad x_d > 0, y \in \mathbb{R}^{d-1}. \tag{2.4.3}$$

For simplicity of the exhibition of the idea, we assume the boundary matrix A^d to be invertible which implies the problem is non-characteristic.

Assuming $f(x,t) = 0$ and applying the Laplace transform on time t and Fourier transform on tangential variable y to (2.4.1), we can obtain an ODE as

$$\frac{dU}{dx_d} = \mathcal{A}(\tau, \eta) U, \tag{2.4.4}$$

where U is the unknown after those transform on u and $\mathcal{A}(\tau, \eta)$ is a matrix-valued function of Laplace variable τ with respect to t and Fourier variable η with respect to y. Moreover we define the stable subspace of $\mathcal{A}(\tau, \eta)$ by $E^-(\tau, \eta)$. Then the Kreiss-Lopatinskii condition can be stated as follows

Definition 2.4.1 (KL)**.** We say the hyperbolic IBVP (2.4.1)-(2.4.3) satisfies the Kreiss-Lopatinskii condition if

$$E^{-}(\tau,\eta)\cap KerB=\{0\},$$

for all $\eta\in\mathbb{R}^{d-1}$ and $\Re\tau>0$.

We remark that the definition above guarantees the boundary conditions contain all the information about the unknown on stable subspace of $\mathcal{A}(\tau,\eta)$ for the IBVP satisfies the Kreiss-Lopatinskii condition. Thus we can obtain the energy estimate from (2.4.4).

Now we introduce a stronger condition than (KL).

Definition 2.4.2 (UKL)**.** We say the hyperbolic IBVP (2.4.1)-(2.4.3) satisfies the uniform Kreiss-Lopatinskii condition in the domain $x_d>0$, $t>0$ if

(i) The number of independent boundary conditions in B equals

the number of positive eigenvalues of A^d.

(ii) There is constant $C>0$, such that

$$|V|\le C|BV|,\qquad \forall\eta\in\mathbb{R}^{d-1},\ \forall\Re\tau>0,\ \forall V\in E^{-}(\tau,\eta).$$

Comparing (KL) and (UKL), we see there are some uniformness in the strength of the control from boundary condition on the components of unknown in stable subspace of $\mathcal{A}(\tau,\eta)$ for (UKL). Moreover we have the following theorem about the well-posedness of the IBVP (2.4.1)-(2.4.3).

Theorem 2.4.1. *The IBVP* (2.4.1)-(2.4.3) *is well-posed in* L^2 *if and only if (UKL) holds.*

Thus it is clear that Kreiss-Lopatinskii condition plays an important roles in the well-posedness of the hyperbolic initial boundary value problems on half space. To guarantee a priori estimates, it is necessary to verify the Kreiss-Lopatinskii condition. In our case, we check the conditions and derive the a priori estimates from the Lopatinskii determinant which is actually the projection of the boundary operator on the stable subspace of the corresponding ODE. Similarly if Lopatinskii determinant is invertible, the boundary conditions provide enough information of the components of the unknown in the stable subspace of the ODE.

With the setup above, we can roughly explain the Kreiss symmetrization method, which is an classical argument to obtain the a priori estimate from (2.4.4). We define the Kreiss symmetrizer as a Hermitian matrix r such that $r + CB^*B$ and $r\mathcal{A}$ are both positive definite for some constant $C > 0$. The positivity of $r + CB^*B$ and (2.4.2) imply

$$< U, rU > |_{x_2=0} + C|g|^2 \geq \kappa \left| U|_{x_2=0} \right|^2 ,$$

for some $\kappa > 0$, where $< \cdot, \cdot >$ is the Euclidean inner product. Therefore if one can further choose r such that $< U, rU > |_{x_2=0} < 0$, then $U|_{x_2=0}$ can be controlled by g. To check $< U, rU > |_{x_2=0} < 0$, we multiply the differential equations in (2.4.4) by r^*U and integrate with respect to x_2 from 0 to ∞ to obtain

$$-\frac{1}{2} < U, rU > |_{x_2=0} = \int_0^\infty < U, r\mathcal{A}U > dx_2.$$

Thus by the positivity of $r\mathcal{A}$, we achieve the a priori estimate.

3.0 LINEAR ANALYSIS

In this chapter, we will discuss the stability and existence results of a particular kind of linear problem derived from the nonlinear system (1.0.1)-(1.0.3), which plays the key role in the nonlinear analysis. We will first consider the linearization around a constant background state. This allow us to determine the possible constant background states around which there are stable variable background states. Then we show the well-posedness of the linear problem corresponding to these variable background states. At last, we will further develop the high order a priori tame estimates on the solutions of these linear problem.

Before the discussion of the linear analysis, we derive the governing equations of the vortex sheets. We first reformulate the system (1.0.1)-(1.0.3) into a divergence form. Notice that the intrinsic property $\operatorname{div}(\rho\mathtt{F}^\top) = 0$ holds at any time throughout the flow if it is satisfied initially [31]. We can rewrite the system as the following form:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p - \operatorname{div}(\rho\mathtt{F}\mathtt{F}^\top) = 0, \\ (\rho\mathtt{F}_j)_t + \operatorname{div}(\rho\mathtt{F}_j\otimes\mathbf{u} - \mathbf{u}\otimes\rho\mathtt{F}_j) = 0, \end{cases} \tag{3.0.1}$$

where $\mathtt{F}_j$ is the jth column of the deformation gradient $\mathtt{F} = (F_{ij})$, $i, j = 1, 2$. In column-wise components, the intrinsic property actually means

$$\operatorname{div}(\rho\mathtt{F}_j) = 0 \quad \text{for } j = 1, 2. \tag{3.0.2}$$

Consider $U(t, x_1, x_2) = (\rho, \mathbf{u}, \mathtt{F})(t, x_1, x_2)$ to be a solution to the system (3.0.1) which is smooth on each side of a smooth hypersurface $\Gamma = \{x_2 = \varphi(t, x_1)\}$. We denote by $\nu =$

$(-\partial_1\varphi, 1)$ a normal vector on Γ and

$$U(t,x_1,x_2) = \begin{cases} U^+(t,x_1,x_2), & \text{when } x_2 > \varphi(t,x_1), \\ U^-(t,x_1,x_2), & \text{when } x_2 < \varphi(t,x_1), \end{cases}$$

where $U^\pm = (\rho^\pm, \mathbf{u}^\pm, \mathtt{F}^\pm)$. It follows that U satisfies the Rankine-Hugoniot conditions at each point of Γ:

$$\partial_t\varphi[\rho] - [\rho\mathbf{u}\cdot\nu] = 0, \tag{3.0.3a}$$

$$\partial_t\varphi[\rho\mathbf{u}] - [(\rho\mathbf{u}\cdot\nu)\mathbf{u}] - [p]\nu + [\rho\mathtt{F}\mathtt{F}^\top\nu] = 0, \tag{3.0.3b}$$

$$\partial_t\varphi[\rho\mathtt{F}_j] - [(\mathbf{u}\cdot\nu)\rho\mathtt{F}_j] + [(\rho\mathtt{F}_j\cdot\nu)\mathbf{u}] = 0, \tag{3.0.3c}$$

$$[\rho\mathtt{F}_j\cdot\nu] = 0, \tag{3.0.3d}$$

where $[f]$ denotes the jump of the function f across the hypersuface Γ. Here we have used (3.0.2) to derive the last jump condition. Now, we denote $m_\nu = \rho(\varphi_t - \mathbf{u}\cdot\nu)$. By (3.0.3a), we have $[m_\nu] = 0$. Thus combining (3.0.3d) and (3.0.3c), we can obtain

$$m_\nu[\mathtt{F}_j] + (\rho\mathtt{F}_j\cdot\nu)[\mathbf{u}] = 0. \tag{3.0.4}$$

Since we consider the contact discontinuity, we assume $\partial_t\varphi = \mathbf{u}^+\cdot\nu = \mathbf{u}^-\cdot\nu$. By (3.0.3a) and (3.0.4), we obtain $[p] = 0$, $m_\nu = 0$, $[\mathtt{F}_j\cdot\nu] = 0$ and $\mathtt{F}_j\cdot\nu = 0$, for $j = 1, 2$. Therefore for a vortex sheet in the elastic flow, the jump conditions (3.0.3a)-(3.0.3d) become

$$\rho^+ = \rho^-, \quad \varphi_t = \mathbf{u}^+\cdot\nu = \mathbf{u}^-\cdot\nu \text{ and } \mathtt{F}_j^+\cdot\nu = \mathtt{F}_j^-\cdot\nu = 0, \quad j = 1,2 \quad \text{on } \Gamma. \tag{3.0.5}$$

We now introduce the transformations $\Phi^\pm(t,x_1,x_2)$ to straighten the free boundary Γ as follows. We first consider the class of functions $\Phi(t,x_1,x_2)$ such that $\inf\{\partial_2\Phi\} > 0$ and $\Phi(t,x_1,0) = \varphi(t,x_1)$. Then we define

$$U^\pm_\# = (\rho^\pm_\#, \mathbf{u}^\pm_\#, \mathtt{F}^\pm_\#)(t,x_1,x_2) = (\rho, \mathbf{u}, \mathtt{F})(t, x_1, \Phi(t,x_1,\pm x_2))$$

for $x_2 \geq 0$. For simplicity of notation, we drop the # index in the rest of the paper. Define $\Phi^{\pm}(t, x_1, x_2) = \Phi(t, x_1, \pm x_2)$. With this change of variables, we can rewrite the equations (1.0.1)-(1.0.3) in the following form:

$$\begin{aligned} \mathbb{L}(U^{\pm}, \Phi^{\pm}) := \partial_t U^{\pm} + A_1(U^{\pm})\partial_1 U^{\pm} \\ + \frac{1}{\partial_2 \Phi^{\pm}} \left[A_2(U^{\pm}) - \partial_t \Phi^{\pm} I - \partial_1 \Phi^{\pm} A_3(U^{\pm}) \right] \partial_2 U^{\pm} = 0, \end{aligned} \tag{3.0.6}$$

for $x_2 > 0$ with the fixed boundary $x_2 = 0$, where

$$A_1(U) = A_3(U) = \begin{pmatrix} v & \rho & 0 & 0 & 0 & 0 & 0 \\ \frac{p'}{\rho} & v & 0 & -F_{11} & 0 & -F_{12} & 0 \\ 0 & 0 & v & 0 & -F_{11} & 0 & -F_{12} \\ 0 & -F_{11} & 0 & v & 0 & 0 & 0 \\ 0 & 0 & -F_{11} & 0 & v & 0 & 0 \\ 0 & -F_{12} & 0 & 0 & 0 & v & 0 \\ 0 & 0 & -F_{12} & 0 & 0 & 0 & v \end{pmatrix}$$

and

$$A_2(U) = \begin{pmatrix} u & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & u & 0 & -F_{21} & 0 & -F_{22} & 0 \\ \frac{p'}{\rho} & 0 & u & 0 & -F_{21} & 0 & -F_{22} \\ 0 & -F_{21} & 0 & u & 0 & 0 & 0 \\ 0 & 0 & -F_{21} & 0 & u & 0 & 0 \\ 0 & -F_{22} & 0 & 0 & 0 & u & 0 \\ 0 & 0 & -F_{22} & 0 & 0 & 0 & u \end{pmatrix}.$$

Moreover, from (3.0.5), we obtain the boundary conditions at $x_2 = 0$:

$$\mathbb{B}(U|_{x_2=0}, \varphi) := \begin{cases} (v^+ - v^-)\partial_1\varphi - (u^+ - u^-) = 0, \\ \partial_t\varphi + v^+\partial_1\varphi - u^+ = 0, \\ (F_{11}^+ - F_{11}^-)\partial_1\varphi - (F_{21}^+ - F_{21}^-) = 0, \\ F_{11}^+\partial_1\varphi - F_{21}^+ = 0, \\ (F_{12}^+ - F_{12}^-)\partial_1\varphi - (F_{22}^+ - F_{22}^-) = 0, \\ F_{12}^+\partial_1\varphi - F_{22}^+ = 0, \\ \rho^+ - \rho^- = 0, \end{cases} \tag{3.0.7}$$

where we note $\Phi^\pm = \varphi$ at $x_2 = 0$.

3.1 CONSTANT COEFFICIENT CASE

3.1.1 Linearized system and stability results

To obtain the constant coefficient linear system, we linearize (3.0.6)-(3.0.7) around its rectilinear solutions. Moreover, all the rectilinear solutions can be transformed under the Galilean transformation and change of the scale of measurement to the following form:

$$\mathring{U}^+ := \begin{pmatrix} \mathring{\rho} \\ v^r \\ 0 \\ F_{11}^r \\ 0 \\ F_{12}^r \\ 0 \end{pmatrix}, \quad \mathring{U}^- := \begin{pmatrix} \mathring{\rho} \\ v^l \\ 0 \\ F_{11}^l \\ 0 \\ F_{12}^l \\ 0 \end{pmatrix}, \quad \mathring{\Phi}^\pm(t, x_1, x_2) = \pm x_2, \tag{3.1.1}$$

where the constants $\mathring{\rho}$, v^r, v^l, F_{11}^r, F_{11}^l, F_{12}^r and F_{12}^l satisfy

$$v^r + v^l = F_{11}^r + F_{11}^l = F_{12}^r + F_{12}^l = 0 \text{ and } v^r > 0,\ F_{11}^r,\ F_{12}^r \neq 0.$$

Now we linearize the system (3.0.6)-(3.0.7) around the above constant states (3.1.1). Let $V^\pm = (\dot\rho^\pm, \dot{\mathbf{u}}^\pm, \dot{\mathtt{F}}^\pm) = U^\pm - \mathring{U}^\pm$ and $\dot\Phi^\pm = \Phi^\pm - \mathring{\Phi}^\pm$ be the small perturbations of the constant solution, and consider the following linearized problem:

$$\partial_t V^\pm + A_1(\mathring{U}^\pm)\partial_1 V^\pm \pm A_2(\mathring{U}^\pm)\partial_2 V^\pm = 0,$$

in $x_2 > 0$, with the boundary condition at $x_2 = 0$:

$$\begin{cases} (v^r - v^l)\partial_1\psi - (\dot u^+ - \dot u^-) = 0, \\ \partial_t\psi + v^r\partial_1\psi - \dot u^+ = 0, \\ (F^r_{11} - F^l_{11})\partial_1\psi - (\dot F^+_{21} - \dot F^-_{21}) = 0, \\ F^r_{11}\partial_1\psi - \dot F^+_{21} = 0, \\ (F^r_{12} - F^l_{12})\partial_1\psi - (\dot F^+_{22} - \dot F^-_{22}) = 0, \\ F^r_{12}\partial_1\psi - \dot F^+_{22} = 0, \\ (\dot\rho^+ - \dot\rho^-) = 0, \end{cases}$$

where $\psi = (\Phi^\pm - \mathring{\Phi}^\pm)|_{x_2=0} = \varphi$ at $x_2 = 0$. In short, we have

$$\begin{cases} L'V = 0, & \text{if } x_2 > 0, \\ B(V,\psi) = 0, & \text{if } x_2 = 0, \end{cases} \tag{3.1.2}$$

where

$$L'V = \partial_t \begin{pmatrix} V^+ \\ V^- \end{pmatrix} + \begin{pmatrix} A_1(\mathring{U}^+) & 0 \\ 0 & A_1(\mathring{U}^-) \end{pmatrix} \partial_1 \begin{pmatrix} V^+ \\ V^- \end{pmatrix} + \begin{pmatrix} A_2(\mathring{U}^+) & 0 \\ 0 & -A_2(\mathring{U}^-) \end{pmatrix} \partial_2 \begin{pmatrix} V^+ \\ V^- \end{pmatrix},$$

$$B(V,\psi) = \begin{pmatrix} (v^r - v^l)\partial_1\psi - (\dot u^+ - \dot u^-) \\ \partial_t\psi + v^r\partial_1\psi - \dot u^+ \\ (F^r_{11} - F^l_{11})\partial_1\psi - (\dot F^+_{21} - \dot F^-_{21}) \\ F^r_{11}\partial_1\psi - \dot F^+_{21} \\ (F^r_{12} - F^l_{12})\partial_1\psi - (\dot F^+_{22} - \dot F^-_{22}) \\ F^r_{12}\partial_1\psi - \dot F^+_{22} \\ (\dot\rho^+ - \dot\rho^-) \end{pmatrix}.$$

Next we symmetrize the system (3.1.2). Consider the following change of variables,

$$W = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} V^+ \\ V^- \end{pmatrix}, \tag{3.1.3}$$

where T is a matrix of the following form:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\mathring{\rho}} & 0 & \frac{1}{2c} & 0 & 0 & 0 & 0 \\ \frac{1}{2\mathring{\rho}} & 0 & \frac{1}{2c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $c = \sqrt{p'(\mathring{\rho})}$ is the sound speed of the constant solution. For convenience, we denote the components of the new variable by

$$W = (W_1, W_2, W_3, ..., W_{14})^\top,$$

and the tangential, normal, characteristic and non-characteristic parts of W by

$$\begin{aligned} W^{\tan} &= (W_1, W_4, W_6, W_8, W_{11}, W_{13})^\top, \\ W^{\mathrm{n}} &= (W_2, W_3, W_5, W_7, W_9, W_{10}, W_{12}, W_{14})^\top, \\ W^{\mathrm{c}} &= (W_1, W_4, W_5, W_6, W_7, W_8, W_{11}, W_{12}, W_{13}, W_{14})^\top, \\ W^{\mathrm{nc}} &= (W_2, W_3, W_9, W_{10})^\top. \end{aligned} \tag{3.1.4}$$

With the above change of variables, we multiply the system (3.1.2) by a symmetrizer

$$\mathcal{A}_0 = \mathrm{diag}\{1, 2c^2, 2c^2, 1, 1, 1, 1, 1, 2c^2, 2c^2, 1, 1, 1, 1\}.$$

Thus we can obtain the following equation

$$\begin{cases} \mathcal{L}W := \mathcal{A}_0 \partial_t W + \mathcal{A}_1 \partial_1 W + \mathcal{A}_2 \partial_2 W = 0, & x_2 > 0, \\ \mathcal{B}(W^{\mathrm{n}}, \psi) := \underline{M} W^{\mathrm{n}} + \underline{b} \begin{pmatrix} \partial_t \psi \\ \partial_1 \psi \end{pmatrix} = 0, & x_2 = 0, \end{cases} \tag{3.1.5}$$

where

$$\mathcal{A}_1 = \begin{pmatrix} \mathcal{A}_1^r & 0 \\ 0 & \mathcal{A}_1^l \end{pmatrix} \quad \text{with}$$

$$\mathcal{A}_1^{r,l} = \begin{pmatrix} v^{r,l} & -c^2 & c^2 & -F_{11}^{r,l} & 0 & -F_{12}^{r,l} & 0 \\ -c^2 & 2c^2 v^{r,l} & 0 & 0 & -cF_{11}^{r,l} & 0 & -cF_{12}^{r,l} \\ c^2 & 0 & 2c^2 v^{r,l} & 0 & -cF_{11}^{r,l} & 0 & -cF_{12}^{r,l} \\ -F_{11}^{r,l} & 0 & 0 & v^{r,l} & 0 & 0 & 0 \\ 0 & -cF_{11}^{r,l} & -cF_{11}^{r,l} & 0 & v^{r,l} & 0 & 0 \\ -F_{12}^{r,l} & 0 & 0 & 0 & 0 & v^{r,l} & 0 \\ 0 & -cF_{12}^{r,l} & -cF_{12}^{r,l} & 0 & 0 & 0 & v^{r,l} \end{pmatrix},$$

$$\mathcal{A}_2 = \mathrm{diag}\{0, -2c^3, 2c^3, 0, 0, 0, 0, 0, 2c^3, -2c^3, 0, 0, 0, 0\},$$

$$\underline{M} = \begin{pmatrix} -c & -c & 0 & 0 & c & c & 0 & 0 \\ -c & -c & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 & 2v^r \\ 1 & v^r \\ 0 & 0 \\ 0 & 2F_{11}^r \\ 0 & F_{11}^r \\ 0 & 2F_{12}^r \\ 0 & F_{12}^r \end{pmatrix}.$$

Now we can state our stability result as follows.

Theorem 3.1.1. *(1) If the particular solution defined by* (3.1.1) *satisfies*

$$\begin{aligned} &(v^r)^2 > 2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2, \qquad \textit{or} \\ &(v^r)^2 < (F_{11}^r)^2 + (F_{12}^r)^2 \ \textit{and} \ (v^r)^2 \neq \frac{\left((F_{11}^r)^2 + (F_{12}^r)^2\right)\left(2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2\right)}{4\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)}, \end{aligned} \tag{3.1.6}$$

then there is a positive constant C such that for all $\gamma \geq 1$, $W \in H^2_\gamma(\mathbb{R}^3_+)$ and $\psi \in H^2_\gamma(\mathbb{R}^2)$, the following estimate holds:

$$\begin{aligned}&\gamma |||W|||^2_{L^2(H^0_\gamma)} + \|W^{\mathrm{n}}|_{x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)} \\ &\qquad \leq C\left(\frac{1}{\gamma^3}|||\mathcal{L}W|||^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^2}\|\mathcal{B}(W^{\mathrm{n}}|_{x_2=0},\psi)\|^2_{H^1_\gamma(\mathbb{R}^2)}\right).\end{aligned} \tag{3.1.7}$$

(2) If the particular solution defined by (3.1.1) satisfies

$$\begin{aligned}&(v^r)^2 = \frac{((F^r_{11})^2 + (F^r_{12})^2)\,(2c^2 + (F^r_{11})^2 + (F^r_{12})^2)}{4\,((F^r_{11})^2 + (F^r_{12})^2 + c^2)}, \qquad \textit{or} \\ &(v^r)^2 = (F^r_{11})^2 + (F^r_{12})^2,\end{aligned} \tag{3.1.8}$$

then there is a positive constant C such that for all $\gamma \geq 1$, $W \in H^3_\gamma(\mathbb{R}^3_+)$ and $\psi \in H^3_\gamma(\mathbb{R}^2)$, the following estimate holds:

$$\begin{aligned}&\gamma |||W|||^2_{L^2(H^0_\gamma)} + \|W^{\mathrm{n}}|_{x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)} \\ &\qquad \leq C\left(\frac{1}{\gamma^5}|||\mathcal{L}W|||^2_{L^2(H^2_\gamma)} + \frac{1}{\gamma^4}\|\mathcal{B}(W^{\mathrm{n}}|_{x_2=0},\psi)\|^2_{H^2_\gamma(\mathbb{R}^2)}\right).\end{aligned} \tag{3.1.9}$$

(3) If the particular solution defined by (3.1.1) satisfies

$$(v^r)^2 = (F^r_{11})^2 + (F^r_{12})^2 + 2c^2, \tag{3.1.10}$$

then there is a positive constant C such that for all $\gamma \geq 1$, $W \in H^4_\gamma(\mathbb{R}^3_+)$ and $\psi \in H^4_\gamma(\mathbb{R}^2)$, the following estimate holds:

$$\begin{aligned}&\gamma |||W|||^2_{L^2(H^0_\gamma)} + \|W^{\mathrm{n}}|_{x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)} \\ &\qquad \leq C\left(\frac{1}{\gamma^7}|||\mathcal{L}W|||^2_{L^2(H^3_\gamma)} + \frac{1}{\gamma^6}\|\mathcal{B}(W^{\mathrm{n}}|_{x_2=0},\psi)\|^2_{H^3_\gamma(\mathbb{R}^2)}\right).\end{aligned} \tag{3.1.11}$$

(4) If the particular solution defined by (3.1.1) satisfies

$$(F^r_{11})^2 + (F^r_{12})^2 < (v^r)^2 < 2c^2 + (F^r_{11})^2 + (F^r_{12})^2, \tag{3.1.12}$$

the constant vortex sheets (3.1.1) is linearly unstable, in the sense that the Lopatinskii condition is violated.

Remark 3.1.1. The above theorem gives all the situations (1)-(3) where the linear stability holds. On the other hand, for the remaining situation (4), we can show that the linearized problem is unstable due to the failure of the Lopatinskii condition. The details will be discussed in Section 3.1.4. Therefore this theorem actually gives us a sufficient and necessary condition on the stability of the linearized problem.

Remark 3.1.2. (i) From the second condition of (3.1.6) we observe that the elasticity term $\mathtt{F}$ produces a stable subsonic region, and hence provides a stabilization effect on the vortex sheets. It is easily seen that in absence of the elasticity $\mathtt{F} \equiv 0$, our results recover the stability theory for the vortex sheets in the Euler flow.

(ii) Another feature of the elastic flow which is different from the case in the Euler flow, can be inferred from the first condition in (3.1.8). It follows from (3.1.8) that there is a class of states in the interior of the subsonic region where the stability holds in a weaker sense, i.e. there is a two-order loss of derivatives in the estimates (3.1.9).

(iii) The second condition in (3.1.8) and the condition (3.1.10) give the transition states across which the stability property changes. This property is similar to the Euler and MHD flows (c.f. [19, 75]). At these two classes of states the stability holds in a weaker sense.

Now we sketch the idea of the proof of the main theorem. We follow the standard argument (c.f. [20]) to first remove the source term from the equations, eliminate the wave front ψ from the resulting system, and then single out the non-characteristic part W^{nc} of the unknown W to arrive at a reduced system in the Fourier space of the form:

$$\begin{cases} \frac{d}{dx_2}\widehat{W}^{\mathrm{nc}} = A\widehat{W}^{\mathrm{nc}}, \\ \\ \beta\widehat{W}^{\mathrm{nc}}|_{x_2=0} = h, \end{cases} \tag{3.1.13}$$

where A is a 4×4 block diagonal matrix and β is a 2×4 matrix (the explicit forms are given in (3.1.36) and (3.1.33)). It turns out that all the desired estimates in Theorem 3.1.1 can be achieved by estimating $\widehat{W}^{\mathrm{nc}}|_{x_2=0}$.

By counting the number of boundary conditions one can only hope to control at most two components of $\widehat{W}^{\mathrm{nc}}|_{x_2=0}$. Thus in order to obtain the full estimate on $\widehat{W}^{\mathrm{nc}}|_{x_2=0}$ one has to utilize the differential equation in (3.1.13). The conventional way to do so is to use the Kreiss symmetrization. Roughly speaking, this argument is to first find a 4×4 Hermitian matrix r and a number $C > 0$ such that $r + C\beta^*\beta$ and rA are both positive definite, where β^* is the Hermitian transpose of β. The positivity of $r + C\beta^*\beta$ implies

$$< \widehat{W}^{\mathrm{nc}}, r\widehat{W}^{\mathrm{nc}} > |_{x_2=0} + C|h|^2 \geq \kappa \left| \widehat{W}^{\mathrm{nc}}|_{x_2=0} \right|^2, \tag{3.1.14}$$

for some $\kappa > 0$, where $< \cdot, \cdot >$ is the Euclidean inner product in $\mathbb{C}^4$. Therefore if one can further choose r such that $< \widehat{W}^{\mathrm{nc}}, r\widehat{W}^{\mathrm{nc}} > |_{x_2=0} < 0$, then $\widehat{W}^{\mathrm{nc}}|_{x_2=0}$ can be controlled by h. To check $< \widehat{W}^{\mathrm{nc}}, r\widehat{W}^{\mathrm{nc}} > |_{x_2=0} < 0$, we multiply the differential equations in (3.1.13) by $r^*\widehat{W}^{\mathrm{nc}}$ and integrate with respect to x_2 from 0 to ∞ to obtain

$$-\frac{1}{2} < \widehat{W}^{\mathrm{nc}}, r\widehat{W}^{\mathrm{nc}} > |_{x_2=0} = \int_0^\infty < \widehat{W}^{\mathrm{nc}}, rA\widehat{W}^{\mathrm{nc}} > dx_2. \tag{3.1.15}$$

Thus a sufficient condition for stability is that rA is positive definite. However at the non-differetiable point of the eigenvalues of A, the usual symmetrizer r can not make $r + C\beta^*\beta$ positive definite for any $C > 0$ when the Lopatinskii determinant is zero there, and hence the Kreiss symmetrization method seems hard to apply. It turns out that this situation can happen for the vortex sheets in elastodynamics. More precisely, if the particular solution defined by (3.1.1) satisfies

$$(v^r)^2 = \frac{1}{4}\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right) \tag{3.1.16}$$

with $c \leq \sqrt{3\left((F_{11}^r)^2 + (F_{12}^r)^2\right)}$, some non-differentiable points of A will coincide with some roots of the Lopatinskii determinant (c.f. Remark 3.1.5), and hence the Kreiss symmetrization cannot be applied directly. We will develop some new idea to overcome this difficulty which we now describe as follows.

Our approach is to first perform an upper triangularization of A to obtain a closed differential system of the two components of $\widehat{W}^{\mathrm{nc}}$ which are not in the stable subspace of A (we refer to these two components as *outgoing modes* and the other two components as *incoming modes*). This way the differential system (3.1.13) is transformed to

$$\frac{d}{dx_2}\begin{pmatrix} \widehat{W}^{in} \\ \widehat{W}^{out} \end{pmatrix} = \begin{pmatrix} G & * \\ 0 & H \end{pmatrix}\begin{pmatrix} \widehat{W}^{in} \\ \widehat{W}^{out} \end{pmatrix}, \tag{3.1.17}$$

where $\widehat{W}^{in}$ and $\widehat{W}^{out}$ are both two-dimensional vectors corresponding to the incoming and outgoing modes of $\widehat{W}^{\rm nc}$, G and H are both 2×2 matrices. The closed differential system of $\widehat{W}^{out}$ is

$$\frac{d}{dx_2}\widehat{W}^{out} = H\widehat{W}^{out}.$$

This upper triangularization ensures that all the eigenvalues of H have positive real parts (in fact for our case, H can actually be a diagonal matrix whose diagonal entries both have positive real parts). Hence an exact estimates of $\widehat{W}^{out}$ can be obtained; furthermore by the L^2-regularity of $\widehat{W}^{\rm nc}$, one has that $\widehat{W}^{out} = 0$ for $x_2 \geq 0$. Therefore to obtain the full estimates for $\widehat{W}^{\rm nc}|_{x_2=0}$, it remains to estimate the other two components of $\widehat{W}^{\rm nc}|_{x_2=0}$, i.e. $\widehat{W}^{in}|_{x_2=0}$, which are in the stable subspace of A. Instead of integrating the first two rows of (3.1.17) to derive the estimates of $\widehat{W}^{in}|_{x_2=0}$, we only use the boundary conditions in (3.1.13). In fact, from $\widehat{W}^{out}|_{x_2=0} = 0$, the boundary conditions are reduced to

$$P\widehat{W}^{in} = h, \quad \text{at } x_2 = 0, \tag{3.1.18}$$

where P is a 2×2 matrix whose determinant is exactly the Lopatinskii determinant. Thus, if the Lopatinskii determinant is not zero, P is invertible and the matrix norm of P^{-1} can be estimated. Therefore $\widehat{W}^{in}|_{x_2=0} = P^{-1}h$, and $\widehat{W}^{in}|_{x_2=0}$ is controlled by h. This together with the fact that $\widehat{W}^{out}|_{x_2=0} = 0$ lead to the estimates of $\widehat{W}^{\rm nc}|_{x_2=0}$ and hence complete the proof of stability. We further point out that our new approach can be applied to other fluid models including the Euler and MHD flows, which will be illustrated in the last section.

Before we conclude this section, we introduce the following change of unknowns in order to simplify the notations in our proof of Theorem 3.1.1:

$$\widetilde{W} = \exp(-\gamma t)W, \quad \widetilde{\psi} = \exp(-\gamma t)\psi$$

with $\gamma \geq 1$. Denote two new operators $\mathcal{L}^\gamma$ and $\mathcal{B}^\gamma$ by

$$\mathcal{L}^\gamma\widetilde{W} = e^{-\gamma t}\mathcal{L}W = \gamma\mathcal{A}_0\widetilde{W} + \mathcal{A}_0\partial_t\widetilde{W} + \mathcal{A}_1\partial_1\widetilde{W} + \mathcal{A}_2\partial_2\widetilde{W},$$

$$\mathcal{B}^\gamma(\widetilde{W}^{\rm n}, \widetilde{\psi}) = e^{-\gamma t}\mathcal{B}(W^{\rm n}, \psi) = \underline{M}\widetilde{W}^{\rm n} + \underline{b}\begin{pmatrix}\gamma\widetilde{\psi} + \partial_t\widetilde{\psi} \\ \partial_1\widetilde{\psi}\end{pmatrix}.$$

We note that $|||e^{-\gamma t}v|||_{s,\gamma}$ and $\|e^{-\gamma t}u\|_{s,\gamma}$ are equivalent to the norms $|||v|||_{L^2(H^s_\gamma)}$ and $\|u\|_{H^s_\gamma}$ respectively. Then we have the following equivalent formulation of Theorem 3.1.1. In the rest of the paper, we aim to prove the following theorem.

Theorem 3.1.2. *(1) If the particular solution defined by* (3.1.1) *satisfies* (3.1.6), *there is a positive constant C such that for all $\gamma \geq 1$, $\widetilde{W} \in H^2(\mathbb{R}^3_+)$ and $\widetilde{\psi} \in H^2(\mathbb{R}^3_+)$, the following estimate holds*

$$\begin{aligned} &\gamma\left|\left|\left|\widetilde{W}\right|\right|\right|^2_{0,\gamma} + \left\|\widetilde{W}^{\mathrm{n}}|_{x_2=0}\right\|^2_{0,\gamma} + \left\|\widetilde{\psi}\right\|^2_{1,\gamma} \\ &\quad \leq C\left(\frac{1}{\gamma^3}\left|\left|\left|\mathcal{L}^\gamma\widetilde{W}\right|\right|\right|^2_{1,\gamma} + \frac{1}{\gamma^2}\left\|\mathcal{B}^\gamma(\widetilde{W}^{\mathrm{n}}|_{x_2=0},\widetilde{\psi})\right\|^2_{1,\gamma}\right). \end{aligned} \tag{3.1.19}$$

(2) If the particular solution defined by (3.1.1) *satisfies* (3.1.8) *Then there is a positive constant C such that for all $\gamma \geq 1$, $\widetilde{W} \in H^3(\mathbb{R}^3_+)$ and $\widetilde{\psi} \in H^3(\mathbb{R}^3_+)$, the following estimate holds*

$$\begin{aligned} &\gamma\left|\left|\left|\widetilde{W}\right|\right|\right|^2_{0,\gamma} + \left\|\widetilde{W}^{\mathrm{n}}|_{x_2=0}\right\|^2_{0,\gamma} + \left\|\widetilde{\psi}\right\|^2_{1,\gamma} \\ &\quad \leq C\left(\frac{1}{\gamma^5}\left|\left|\left|\mathcal{L}^\gamma\widetilde{W}\right|\right|\right|^2_{2,\gamma} + \frac{1}{\gamma^4}\left\|\mathcal{B}^\gamma(\widetilde{W}^{\mathrm{n}}|_{x_2=0},\widetilde{\psi})\right\|^2_{2,\gamma}\right). \end{aligned} \tag{3.1.20}$$

(3) If the particular solution defined by (3.1.1) *satisfies* (3.1.10) *Then there is a positive constant C such that for all $\gamma \geq 1$, $\widetilde{W} \in H^4(\mathbb{R}^3_+)$ and $\widetilde{\psi} \in H^4(\mathbb{R}^3_+)$, the following estimate holds*

$$\begin{aligned} &\gamma\left|\left|\left|\widetilde{W}\right|\right|\right|^2_{0,\gamma} + \left\|\widetilde{W}^{\mathrm{n}}|_{x_2=0}\right\|^2_{0,\gamma} + \left\|\widetilde{\psi}\right\|^2_{1,\gamma} \\ &\quad \leq C\left(\frac{1}{\gamma^7}\left|\left|\left|\mathcal{L}^\gamma\widetilde{W}\right|\right|\right|^2_{3,\gamma} + \frac{1}{\gamma^6}\left\|\mathcal{B}^\gamma(\widetilde{W}^{\mathrm{n}}|_{x_2=0},\widetilde{\psi})\right\|^2_{3,\gamma}\right). \end{aligned} \tag{3.1.21}$$

(4) If the particular solution defined by (3.1.1) *satisfies*

$$(F^r_{11})^2 + (F^r_{12})^2 < (v^r)^2 < 2c^2 + (F^r_{11})^2 + (F^r_{12})^2, \tag{3.1.22}$$

the constant vortex sheets (3.1.1) *is linearly unstable, in the sense that the Lopatinskii condition is violated.*

3.1.2 Decomposition of the system and elimination of the front

In this part, as a first step for the proof of Theorem 3.1.2 (and hence Theorem 3.1.1) we perform some preliminary reductions and eliminate the front $\widetilde{\psi}$ from the linearized system.

Consider the following inhomogeneous differential system for $\widetilde{W}$ and $\widetilde{\psi}$ on $\mathbb{R}^3_+$:

$$\begin{cases} \mathcal{L}^\gamma \widetilde{W} = \widetilde{f}, & \text{if } x_2 > 0, \\ \mathcal{B}^\gamma(\widetilde{W}^{\mathrm{n}}, \widetilde{\psi}) = \widetilde{g}, & \text{if } x_2 = 0, \end{cases} \tag{3.1.23}$$

where $\widetilde{f}$ and $\widetilde{g}$ are given sources.

3.1.2.1 Decomposition of the system Due to the linearity of the system (3.1.23), we can decompose it into two subsystems as follows. First we consider the following problem for V_* with the homogeneous boundary conditions:

$$\begin{cases} \mathcal{L}^\gamma V_* = \widetilde{f}, & \text{if } x_2 > 0, \\ M_1 V_*^{\mathrm{n}} = 0, & \text{if } x_2 = 0, \end{cases} \tag{3.1.24}$$

where

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

From the classical hyperbolic theory [7], the boundary condition is strictly dissipative, and thus (3.1.24) admits a solution satisfying following estimates:

$$\gamma |||V_*|||_0^2 \leq \frac{C}{\gamma} |||f|||_0^2, \qquad \|V_*^{\mathrm{n}}|_{x_2=0}\|_{k,\gamma}^2 \leq \frac{C}{\gamma} |||f|||_{k,\gamma}^2,$$

for any integer $k \geq 0$.

Then we consider the second problem for the difference $W = \widetilde{W} - V_*$. In fact W satisfies the following homogeneous differential equations with inhomogeneous boundary conditions:

$$\begin{cases} \mathcal{L}^\gamma W = 0, & \text{if } x_2 > 0, \\ \mathcal{B}^\gamma(W^{\mathrm{n}}, \psi) = g := \widetilde{g} - \underline{M} V^{\mathrm{n}}, & \text{if } x_2 = 0. \end{cases} \tag{3.1.25}$$

Remark 3.1.3. Here we are slightly abusing notations, since W was previously defined to be the transformed perturbation of the rectilinear vortex sheets, c.f. (3.1.4). From now on we will consider W as a solution to (3.1.25).

Multiplying the equations in system (3.1.25) by W and integrating, we have

$$\gamma \|\!|W|\!\|_0^2 \le C\|W^{\mathrm{nc}}|_{x_2=0}\|_0^2 \le C\|W^{\mathrm{n}}|_{x_2=0}\|_0^2.$$

Thus it suffices to derive the following estimate on W:

$$\|W^{\mathrm{n}}|_{x_2=0}\|_0^2 + \|\psi\|_{1,\gamma}^2 \le \frac{C}{\gamma^{2k}}\|g\|_{k,\gamma}^2 \tag{3.1.26}$$

to obtain all the estimates in Theorem 3.1.2, where k will be determined accordingly.

Now we perform the Fourier transform to system (3.1.25) with respect to tangential variables (t, x_1) and denote the variables in the frequency space by (δ, η). Let $\tau = \gamma + i\delta$. Then $\widehat{W}$, the Fourier transform of W, satisfies the following system:

$$\begin{cases} (\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\widehat{W} + \mathcal{A}_2 \dfrac{d\widehat{W}}{dx_2} = 0, & \text{if } x_2 > 0, \\ b(\tau,\eta)\widehat{\varphi} + \underline{M}\widehat{W}^{\mathrm{n}} = \widehat{g}, & \text{if } x_2 = 0, \end{cases} \tag{3.1.27}$$

where

$$b(\tau,\eta) = \underline{b} \cdot \begin{pmatrix} \tau \\ i\eta \end{pmatrix} = \begin{pmatrix} 2iv^r\eta \\ \tau + iv^r\eta \\ 0 \\ 2iF_{11}^r\eta \\ iF_{11}^r\eta \\ 2iF_{12}^r\eta \\ iF_{12}^r\eta \end{pmatrix}.$$

To utilize the homogeneity structure of the system, we define the hemisphere

$$\Sigma = \{(\tau, \eta) : \ |\tau|^2 + (v^r)^2\eta^2 = 1 \text{ and } \Re\tau \geq 0\}$$

in the whole frequency space $\Pi := \{(\tau, \eta) : \ \tau \in \mathbb{C}, \ \eta \in \mathbb{R}, \ |\tau|^2 + \eta^2 \neq 0, \ \Re\tau \geq 0\}$. It is easily seen that $\Pi = \{s \cdot (\tau, \eta) : \ s > 0, \ (\tau, \eta) \in \Sigma\}$. We will carry out our argument on the hemisphere Σ and use homogeneity property to extend it to the whole frequency space Π.

3.1.2.2 Elimination of the front An important observation in the vortex sheet system is that the wave front ψ is only involved in the boundary conditions. Thus we can estimate the wave front by using the ellipticity of the boundary conditions of (3.1.27) in the sense of the following lemma.

Lemma 3.1.1. *There is a C^∞ map $Q : \Pi \to GL_7(\mathbb{C})$ which is homogeneous of degree 0, such that*

$$Q(\tau, \eta)b(\tau, \eta) = \begin{pmatrix} 0 \\ 0 \\ \theta(\tau, \eta) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall \ (\tau, \eta) \in \Pi,$$

where θ is homogeneous of degree 1 and

$$\min_{(\tau,\eta)\in\Sigma} |\theta(\tau, \eta)| > 0.$$

Proof. We only sketch the proof of the lemma. The idea is to consider the map

$$
Q = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\tau + iv^r\eta & -2iv^r\eta & 0 & 0 & 0 & 0 & 0 \\
-2iv^r\eta & \bar{\tau} - iv^r\eta & 0 & -2iF_{11}^r\eta & -iF_{11}^r\eta & -2iF_{12}^r\eta & -iF_{12}^r\eta \\
-F_{11}^r & 0 & 0 & v^r & 0 & 0 & 0 \\
-F_{11}^r & 0 & 0 & 0 & 2v^r & 0 & 0 \\
-F_{12}^r & 0 & 0 & 0 & 0 & v^r & 0 \\
-F_{12}^r & 0 & 0 & 0 & 0 & 0 & 2v^r
\end{pmatrix}
$$

on Σ and extend it by homogeneity of degree 0 to the whole frequency space. Then the lemma can be proved by a direct computation. $\square$

Now we multiply the boundary conditions in (3.1.27) from the left by Q and obtain the new boundary conditions:

$$
Qb\widehat{\varphi} + Q\underline{M}\widehat{W}^{\mathrm{n}} = Q\widehat{g}, \quad \text{at} \quad x_2 = 0. \tag{3.1.28}
$$

With this choice of Q, we have $Q\underline{M} =$

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
-c(\tau - iv^r\eta) & -c(\tau - iv^r\eta) & 0 & 0 & c(\tau + iv^r\eta) & c(\tau + iv^r\eta) & 0 & 0 \\
-c(\bar{\tau} - 3iv^r\eta) & -c(\bar{\tau} - 3iv^r\eta) & 3iF_{11}^r\eta & 3iF_{12}^r\eta & -2civ^r\eta & -2civ^r\eta & -2iF_{11}^r\eta & -2iF_{12}^r\eta \\
cF_{11}^r & cF_{11}^r & -v^r & 0 & -cF_{11}^r & -cF_{11}^r & v^r & 0 \\
cF_{11}^r & cF_{11}^r & -2v^r & 0 & -cF_{11}^r & -cF_{11}^r & 0 & 0 \\
cF_{12}^r & cF_{12}^r & 0 & -v^r & -cF_{12}^r & -cF_{12}^r & 0 & v^r \\
cF_{12}^r & cF_{12}^r & 0 & -2v^r & -cF_{12}^r & -cF_{12}^r & 0 & 0
\end{pmatrix}
$$

on Σ which is homogeneous of degree 0. Denote the third row of the above matrix by ℓ. Hence the third equation of the new boundary conditions (3.1.28) is

$$
\theta(\tau, \eta)\widehat{\varphi} + \ell(\theta, \eta)\widehat{W}^{\mathrm{n}}|_{x_2=0} = b^*(\theta, \eta)\widehat{g}, \quad \text{on } \Sigma.
$$

Notice that ℓ and b^* are homogeneous of degree 0. From the compactness of Σ and continuity of ℓ and b^*, we know that they are bounded on Π. By Lemma 3.1.1 and a direct integration of the above equation, we have

$$
\|\psi\|_{1,\gamma}^2 \le C(\|\widehat{W}^{\mathrm{n}}|_{x_2=0}\|_0^2 + \|g\|_0^2). \tag{3.1.29}
$$

Moreover, the last four equations in the new boundary conditions (3.1.28) are

$$\begin{pmatrix} cF_{11}^r & cF_{11}^r & -v^r & 0 & -cF_{11}^r & -cF_{11}^r & v^r & 0 \\ cF_{11}^r & cF_{11}^r & -2v^r & 0 & -cF_{11}^r & -cF_{11}^r & 0 & 0 \\ cF_{12}^r & cF_{12}^r & 0 & -v^r & -cF_{12}^r & -cF_{12}^r & 0 & v^r \\ cF_{12}^r & cF_{12}^r & 0 & -2v^r & -cF_{12}^r & -cF_{12}^r & 0 & 0 \end{pmatrix} \widehat{W}^{\mathrm{n}}|_{x_2=0} = \tilde{Q}\widehat{g},$$

for all $(\tau,\eta) \in \Pi$, where $\tilde{Q}$ is the matrix consisting of the last four rows of Q. Isolating the third, forth, seventh and eighth columns of the left matrix in the above equations, we can obtain

$$\begin{pmatrix} -v^r & 0 & v^r & 0 \\ -2v^r & 0 & 0 & 0 \\ 0 & -v^r & 0 & v^r \\ 0 & -2v^r & 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_5 \\ \widehat{W}_7 \\ \widehat{W}_{12} \\ \widehat{W}_{14} \end{pmatrix} \Bigg|_{x_2=0} = \tilde{Q}\widehat{g} - c \begin{pmatrix} F_{11}^r & F_{11}^r & -F_{11}^r & -F_{11}^r \\ F_{11}^r & F_{11}^r & -F_{11}^r & -F_{11}^r \\ F_{12}^r & F_{12}^r & -F_{12}^r & -F_{12}^r \\ F_{12}^r & F_{12}^r & -F_{12}^r & -F_{12}^r \end{pmatrix} \widehat{W}^{\mathrm{nc}}|_{x_2=0},$$

on Π. Since now the left matrix above is constant and invertible, we can obtain

$$\left(\widehat{W}_5^2 + \widehat{W}_7^2 + \widehat{W}_{12}^2 + \widehat{W}_{14}^2\right)|_{x_2=0} \le C\left(\left|\widehat{W}^{\mathrm{nc}}|_{x_2=0}\right|^2 + |\widehat{g}|^2\right). \tag{3.1.30}$$

Recall the definition of W^{n} in (3.1.3). We know from (3.1.30) that an estimate of $\|\widehat{W}^{\mathrm{nc}}|_{x_2=0}\|_0^2$ leads to the estimate of $\|\widehat{W}^{\mathrm{n}}|_{x_2=0}\|_0^2$, and hence the estimate of $\|\psi\|_{1,\gamma}^2$ from (3.1.29). Therefore by (3.1.26) we know that it is sufficient to obtain the estimate of $\|\widehat{W}^{\mathrm{nc}}|_{x_2=0}\|_0^2$.

Now consider the first two rows in the new boundary conditions (3.1.28) at $x_2 = 0$ and the equations of (3.1.27) for $x_2 > 0$. We arrived at the following system where the wave front ψ is eliminated:

$$(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\widehat{W} + \mathcal{A}_2\frac{d\widehat{W}}{dx_2} = 0, \tag{3.1.31}$$

$$\beta\widehat{W}^{\mathrm{nc}}|_{x_2=0} = h, \tag{3.1.32}$$

where h is the product of the first two rows of Q and $\widehat{g}$, and

$$\beta = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -c(\tau - iv^r\eta) & -c(\tau - iv^r\eta) & c(\tau + iv^r\eta) & c(\tau + iv^r\eta) \end{pmatrix} \tag{3.1.33}$$

on Σ and is extended to all of Π by homogeneity of degree 0. Note that from the homogeneity of Q we have the following bounds

$$|h|^2 \leq C|\hat{g}|^2, \tag{3.1.34}$$

for some positive constant C which is independent of (τ, η). The rest of this paper is devoted to the estimate of $\|\widehat{W}^{\mathrm{nc}}|_{x_2=0}\|_0^2$ from the system (3.1.31)-(3.1.32).

3.1.3 Normal mode analysis

In this subsection, we consider the normal modes and want to separate the outgoing modes from the system microlocally. This separation gives us the exact estimates on the outgoing modes. Moreover we will show in Section 3.1.5 that these outgoing modes are all zero.

3.1.3.1 Normal modes To obtain an estimate of $\|\widehat{W}^{\mathrm{nc}}|_{x_2=0}\|_0^2$, we are led to derive a closed differential system of $\widehat{W}^{\mathrm{nc}}$. For this purpose, we single out the following ten algebraic equations from (3.1.31):

$$\begin{aligned}
&(\tau + iv^r\eta)\widehat{W}_1 - ic^2\eta\widehat{W}_2 + ic^2\eta\widehat{W}_3 - iF_{11}^r\eta\widehat{W}_4 - iF_{12}^r\eta\widehat{W}_6 = 0,\\
&-iF_{11}^r\eta\widehat{W}_1 + (\tau + iv^r\eta)\widehat{W}_4 = 0,\\
&-icF_{11}^r\eta\widehat{W}_2 - icF_{11}^r\eta\widehat{W}_3 + (\tau + iv^r\eta)\widehat{W}_5 = 0,\\
&-iF_{12}^r\eta\widehat{W}_1 + (\tau + iv^r\eta)\widehat{W}_6 = 0,\\
&-icF_{12}^r\eta\widehat{W}_2 - icF_{12}^r\eta\widehat{W}_3 + (\tau + iv^r\eta)\widehat{W}_7 = 0,\\
&(\tau + iv^l\eta)\widehat{W}_8 - ic^2\eta\widehat{W}_9 + ic^2\eta\widehat{W}_{10} - iF_{11}^l\eta\widehat{W}_{11} - iF_{12}^l\eta\widehat{W}_{13} = 0,\\
&-iF_{11}^l\eta\widehat{W}_8 + (\tau + iv^l\eta)\widehat{W}_{11} = 0,\\
&-icF_{11}^1\eta\widehat{W}_9 - icF_{11}^l\eta\widehat{W}_{10} + (\tau + iv^l\eta)\widehat{W}_{12} = 0,\\
&-iF_{12}^l\eta\widehat{W}_8 + (\tau + iv^l\eta)\widehat{W}_{13} = 0,\\
&-icF_{12}^l\eta\widehat{W}_9 - icF_{12}^l\eta\widehat{W}_{10} + (\tau + iv^l\eta)\widehat{W}_{14} = 0.
\end{aligned}$$

Recall definition (3.1.4), we know that the above equations can be explicitly solved in terms of $\widehat{W}^{\mathrm{nc}} = (\widehat{W}_2, \widehat{W}_3, \widehat{W}_9, \widehat{W}_{10})^\top$ as the following:

$$
\begin{aligned}
\widehat{W}_1 &= \frac{ic^2\eta(\tau + iv^r\eta)}{(\tau+iv^r\eta)^2 + ((F_{11}^r)^2 + (F_{12}^r)^2)\,\eta^2}(\widehat{W}_2 - \widehat{W}_3),\\
\widehat{W}_4 &= \frac{-c^2F_{11}^r\eta^2}{(\tau+iv^r\eta)^2 + ((F_{11}^r)^2 + (F_{12}^r)^2)\,\eta^2}(\widehat{W}_2 - \widehat{W}_3), \quad \widehat{W}_5 = \frac{icF_{11}^r\eta}{\tau+iv^r\eta}(\widehat{W}_2 + \widehat{W}_3),\\
\widehat{W}_6 &= \frac{-c^2F_{12}^r\eta^2}{(\tau+iv^r\eta)^2 + ((F_{11}^r)^2 + (F_{12}^r)^2)\,\eta^2}(\widehat{W}_2 - \widehat{W}_3), \quad \widehat{W}_7 = \frac{icF_{12}^r\eta}{\tau+iv^r\eta}(\widehat{W}_2 + \widehat{W}_3),\\
\widehat{W}_8 &= \frac{ic^2\eta(\tau + iv^l\eta)}{(\tau+iv^l\eta)^2 + \big((F_{11}^l)^2 + (F_{12}^l)^2\big)\,\eta^2}(\widehat{W}_9 - \widehat{W}_{10}),\\
\widehat{W}_{11} &= \frac{-c^2F_{11}^l\eta^2}{(\tau+iv^l\eta)^2 + \big((F_{11}^l)^2 + (F_{12}^l)^2\big)\,\eta^2}(\widehat{W}_9 - \widehat{W}_{10}), \quad \widehat{W}_{12} = \frac{icF_{11}^l\eta}{\tau+iv^l\eta}(\widehat{W}_9 + \widehat{W}_{10}),\\
\widehat{W}_{13} &= \frac{-c^2F_{12}^l\eta^2}{(\tau+iv^l\eta)^2 + \big((F_{11}^l)^2 + (F_{12}^l)^2\big)\,\eta^2}(\widehat{W}_9 - \widehat{W}_{10}), \quad \widehat{W}_{14} = \frac{icF_{12}^l\eta}{\tau+iv^l\eta}(\widehat{W}_9 + \widehat{W}_{10}).
\end{aligned}
$$

Then using the differential equations in (3.1.31), we obtain the following differential equations for $\widehat{W}^{\mathrm{nc}}$ only:

$$
\frac{d}{dx_2}\widehat{W}^{\mathrm{nc}} = A\widehat{W}^{\mathrm{nc}}, \tag{3.1.35}
$$

where

$$
A = \begin{pmatrix} n^r & -m^r & 0 & 0 \\ m^r & -n^r & 0 & 0 \\ 0 & 0 & -n^l & m^l \\ 0 & 0 & -m^l & n^l \end{pmatrix}, \tag{3.1.36}
$$

and

$$
\begin{aligned}
n^{r,l} &= \frac{2(\tau+iv^{r,l}\eta)^2 + \Big((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\Big)\,\eta^2}{2c(\tau+iv^{r,l}\eta)} + \frac{c}{2}\frac{(\tau+iv^{r,l}\eta)\eta^2}{(\tau+iv^{r,l}\eta)^2 + \Big((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\Big)\,\eta^2},\\
m^{r,l} &= \frac{c}{2}\frac{(\tau+iv^{r,l}\eta)\eta^2}{(\tau+iv^{r,l}\eta)^2 + \Big((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\Big)\,\eta^2} - \frac{\Big((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\Big)\,\eta^2}{2c(\tau+iv^{r,l}\eta)}.
\end{aligned}
$$

From the classical hyperbolic theory (see, for example, [7]), it follows that one of the key issues in the estimates of $\|\widehat{W}^{\mathrm{nc}}|_{x_2=0}\|_0^2$ is to bound the components of $\widehat{W}^{\mathrm{nc}}$ on the stable subspace of A. For this reason, we first derive the following lemma of Hersh-type [28] on the explicit description of the stable subspace of A on Σ. The proof can be done by a direct computation, and hence we omit it.

Lemma 3.1.2. *For $(\tau,\eta)\in\Sigma$ and $\Re\tau>0$, the matrix A defined in (3.1.36) admits four eigenvalues $\pm\omega^r$ and $\pm\omega^l$, where $\Re\omega^r$ and $\Re\omega^l$ are negative. Moreover, the following dispersion relations hold:*

$$\begin{aligned}(\omega^r)^2 = (n^r)^2-(m^r)^2 = \frac{1}{c^2}\left[(\tau+iv^r\eta)^2+\left((F^r_{11})^2+(F^r_{12})^2\right)\eta^2\right]+\eta^2,\\ (\omega^l)^2 = (n^l)^2-(m^l)^2 = \frac{1}{c^2}\left[(\tau+iv^l\eta)^2+\left((F^l_{11})^2+(F^l_{12})^2\right)\eta^2\right]+\eta^2.\end{aligned} \tag{3.1.37}$$

The eigenvectors of ω^r, $-\omega^r$, ω^l and $-\omega^l$ take the following forms respectively:

$$\begin{aligned} E^r_- = (a^r,b^r,0,0)^\top,\quad E^r_+ = (a^r,c^r,0,0)^\top,\\ E^l_- = (0,0,b^l,a^l)^\top,\quad E^l_+ = (0,0,c^l,a^l)^\top,\end{aligned} \tag{3.1.38}$$

where

$$\begin{aligned} a^{r,l} = m^{r,l}\alpha^{r,l},\quad b^{r,l} = (n^{r,l}-\omega^{r,l})\alpha^{r,l},\quad c^{r,l} = (n^{r,l}+\omega^{r,l})\alpha^{r,l},\\ \alpha^{r,l} = (\tau+iv^{r,l}\eta)\left[(\tau+iv^{r,l}\eta)^2+\left((F^{r,l}_{11})^2+(F^{r,l}_{12})^2\right)\eta^2\right].\end{aligned}$$

Both ω^r and ω^l can be extended continuously to all points $(\tau,\eta)\in\Sigma$ such that $\Re\tau=0$, so can $E^r_\pm$ and $E^l_\pm$. The two vectors E^r_- and E^l_- remain linearly independent for all $(\tau,\eta)\in\Sigma$.

By the definition of A, (3.1.37) actually holds on Π. Moreover, from (3.1.38), we can see that A can not be smoothly diagonalized near the points $(\tau,\eta)\in\Sigma$ satisfying one of the following: $m^{r,l}=0$, or $\omega^{r,l}=0$, or $\tau=\pm iv^r\eta$, or $\tau = i\left(\pm v^r\pm\sqrt{(F^r_{11})^2+(F^l_{12})^2}\right)\eta$, because E^r_- and E^r_+ (or E^l_- and E^l_+) become parallel at those points. In fact, $(\tau,\eta)\in\Sigma$ are poles of A when $\tau=\pm iv^r\eta$ or $\tau = i\left(\pm v^r\pm\sqrt{(F^r_{11})^2+(F^l_{12})^2}\right)\eta$. Therefore instead of looking for a diagonalization of A, we perform an upper triangularization of A, which we refer to as *separation of modes.*

3.1.3.2 Separation of modes

Since the eigenbasis of A may degenerate at some points on Σ, we need to treat A microlocally. This means that for each point $(\tau_0, \eta_0) \in \Sigma$, we will separate the outgoing modes of A from the system (3.1.35) in some open neighborhood $\mathcal{V} \subset \Sigma$ of that point (τ_0, η_0). Here we refer to the *outgoing modes* of A as all the components of $\widehat{W}^{\text{nc}}$ which do not belong to the stable subspace of A. By using this separation, we can show later in Section 3.1.5 that for every $(\tau_0, \eta_0) \in \Sigma$, these outgoing modes in fact vanish in $\mathcal{V} \cap \{(\tau, \eta) : \Re\tau > 0\}$. Thus from the compactness of Σ, we can extract a finite covering $\{\mathcal{V}_i\}_{i=1}^N$ of Σ to show that the outgoing modes of A vanish in the entire $\Sigma \cap \{(\tau, \eta) : \Re\tau > 0\}$.

The following proposition guarantees that the separation of modes can always be achieved for all points on Σ.

Proposition 3.1.1. *For $\omega^{r,l}$ defined in Lemma 3.1.2, we have*

$$(\tau + iv^{r,l}\eta)\omega^{r,l} - c\left((\omega^{r,l})^2 - \eta^2\right) \neq 0$$

for all $(\tau, \eta) \in \Sigma$.

Proof. The key idea of the proof is to examine the signs of the real and imaginary parts of $\omega^{r,l}$. For this purpose, we consider the general situation where $(x + iy)^2 = p + iq$ for x, y, p, $q \in \mathbb{R}$ and $x \leq 0$. By a direct computation, we can express x and y in terms of p and q as

$$x = -\sqrt{\frac{p + \sqrt{p^2 + q^2}}{2}}, \quad y = -\text{sgn}(q)\sqrt{\frac{\sqrt{p^2 + q^2} - p}{2}} \tag{3.1.39}$$

for all $(p, q) \in \mathbb{R}^2 \backslash \{p < 0, q = 0\}$.

Next we apply the above relations to $\omega^{r,l}$ at the point $(\tau, \eta) \in \Sigma$ with $\Re\tau > 0$. Let $\omega^{r,l} = x^{r,l} + iy^{r,l}$ and $(\omega^{r,l})^2 = p^{r,l} + iq^{r,l}$, where $x^{r,l}$, $y^{r,l}$, $p^{r,l}$, $q^{r,l} \in \mathbb{R}$. From the definition (3.1.37) of $\omega^{r,l}$ we know that $x^{r,l} \leq 0$ and

$$p^{r,l} = \frac{\gamma^2 - (\delta + v^{r,l}\eta)^2 + \left((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\right)\eta^2}{c^2} + \eta^2, \tag{3.1.40}$$

$$q^{r,l} = \frac{2\gamma(\delta + v^{r,l}\eta)}{c^2}. \tag{3.1.41}$$

With this setup it is easy to see from (3.1.39) that when $(p^{r,l}, q^{r,l}) \notin \{p < 0, q = 0\}$ and $\delta + v^{r,l}\eta \neq 0$ the sign of $y^{r,l}$ is opposite to the sign of $\delta + v^{r,l}\eta$ respectively. Here we recall the fact $\gamma = \Re\tau > 0$. On the other hand, when $(p^{r,l}, q^{r,l}) \in \{p < 0, q = 0\}$, (3.1.39) fails to hold. More precisely, those points correspond exactly to the case when

$$\gamma = 0, \quad \delta + v^{r,l}\eta \neq 0 \quad \text{and} \quad p^{r,l} < 0.$$

Therefore they all lie on the boundary of Σ. By Lemma 3.1.2, the values of $\omega^{r,l}$ at the boundary of Σ are defined as the continuous limits of the interior values of $\omega^{r,l}$. This way, we can still determine the signs of $x^{r,l}$ and $y^{r,l}$ by continuity. Thus by the continuous extension of $\omega^{r,l}$ along the path where the ratio of δ and η is fixed, the sign of $y^{r,l}$ is opposite to the sign of $\delta + v^{r,l}\eta$ respectively at those exceptional points $(p^{r,l}, q^{r,l}) \in \{p < 0, q = 0\}$. Hence we conclude that

$$\text{if } \delta + v^{r,l}\eta \neq 0, \text{ the sign of } y^{r,l} \text{ is opposite to the sign of } \delta + v^{r,l}\eta \text{ respectively.} \tag{3.1.42}$$

Now we return to the proof of the proposition. We will only prove that $(\tau + iv^r\eta)\omega^r - c\left((\omega^r)^2 - \eta^2\right) \neq 0$. The case that $c\left((w^l)^2 - \eta^2\right) - (\tau + iv^l\eta)\omega^l \neq 0$ can be treated similarly. Assume on the contrary that

$$(\tau + iv^r\eta)\omega^r - c\left((\omega^r)^2 - \eta^2\right) = 0. \tag{3.1.43}$$

If $\tau + iv^r\eta = 0$, the above equation becomes $(\omega^r)^2 - \eta^2 = 0$. Combining this with (3.1.37), we obtain $\left((F^r_{11})^2 + (F^r_{12})^2\right)\eta^2 = 0$, and hence $\eta = 0$, which in turn implies $\tau = 0$. This contradicts the fact that $(\tau, \eta) \in \Sigma$.

Thus we can assume $\tau + iv^r\eta \neq 0$. From this it follows that

$$\omega^r = \frac{c\left((\omega^r)^2 - \eta^2\right)}{(\tau + iv^r\eta)} = \frac{1}{c}\left[(\tau + iv^r\eta) + \frac{\left((F^r_{11})^2 + (F^r_{12})^2\right)\eta^2}{\tau + iv^r\eta}\right]. \tag{3.1.44}$$

If γ (i.e. $\Re\tau$) is positive, it is easy to check that the real part of the right hand side of the above formula is positive, which contradicts the definition that $\Re\omega^r < 0$.

Thus we only need to check the situation when $\gamma = 0$. In this case, we have $\tau + iv^r\eta = i(\delta + v^r\eta) \neq 0$. By (3.1.44), we know that $\Re\omega^r = 0$, and therefore $q^r = 0$ and $p^r \leq 0$. We further claim that $p^r \neq 0$. Otherwise if $p^r = 0$, from (3.1.37) and the fact that $q^r = 0$ we have $\omega^r = 0$. Then from (3.1.43) we must have $\eta = 0$. By (3.1.40) it follows that $\delta = 0$, and hence $\tau = \eta = 0$, which contradicts the fact that $(\tau, \eta) \in \Sigma$.

Therefore we only need to consider $(\tau, \eta) \in \Sigma$ when $\tau + iv^r\eta \neq 0$, $\gamma = 0$ and $p^r < 0$. This immediately implies that $\delta + v^r\eta \neq 0$. From (3.1.42) we know that the sign of $y^r = \Im\omega^r$ is opposite to that of $\delta + v^r\eta$. However by (3.1.44) and the fact that $\Re\tau = 0$,

$$\Im\omega^r = \frac{(\delta + v^r\eta)^2 - ((F_{11}^r)^2 + (F_{12}^r)^2)\,\eta^2}{c(\delta + v^r\eta)}.$$

Since $p^r < 0$, from (3.1.40) we know that $(\delta + v^r\eta)^2 - ((F_{11}^r)^2 + (F_{12}^r)^2)\,\eta^2 > 0$. Thus the sign of $\Im\omega^r$ is the same as the sign of $\delta + v^r\eta$, which is a contradiction. Hence we conclude that $(\tau + iv^r\eta)\omega^r - c\left((\omega^r)^2 - \eta^2\right)$ never vanishes in Σ, which completes the proof of the proposition. □

With this proposition, we can show that $E_-^{r,l}$ do not vanish at any point on Σ. Because if $E^{r,l} = 0$, we have $m^{r,l}\alpha^{r,l} = 0$ and $(n^{r,l} - \omega^{r,l})\alpha^{r,l} = 0$. By a direct computation we have that $\alpha^{r,l} \neq 0$. Then $m^{r,l}\alpha^{r,l} = 0$ implies that $m^{r,l} = 0$. From the definition of $m^{r,l}$ it follows that

$$\frac{c}{2} \cdot \frac{(\tau + iv^{r,l}\eta)\eta^2}{(\tau + iv^{r,l}\eta)^2 + \left((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\right)\eta^2} = \frac{\left((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\right)\eta^2}{2c(\tau + iv^{r,l}\eta)}.$$

Together with $(n^{r,l} - \omega^{r,l})\alpha^{r,l} = 0$ and (3.1.37), we have

$$(\tau + iv^{r,l}\eta)\omega^{r,l} - c\left((\omega^{r,l})^2 - \eta^2\right) = 0,$$

which contradicts Proposition 3.1.1.

The non-degeneracy of $E_-^{r,l}$ allows us to construct the following transformation matrix

$$T = \{E_-^r, F^r, E_-^l, F^l\}$$

in a neighborhood of $(\tau_0, \eta_0) \in \Sigma$ with

$$F^r = \begin{cases} (0,1,0,0)^\top, & \text{if } m^r\alpha^r \neq 0 \text{ at } (\tau_0, \eta_0), \\ (1,0,0,0)^\top, & \text{if } (n^r - \omega^r)\alpha^r \neq 0 \text{ at } (\tau_0, \eta_0), \end{cases}$$

and similarly,

$$F^l = \begin{cases} (0,0,1,0)^\top, & \text{if } m^l\alpha^l \neq 0 \text{ at } (\tau_0,\eta_0), \\ (0,0,0,1)^\top, & \text{if } (n^l-\omega^l)\alpha^l \neq 0 \text{ at } (\tau_0,\eta_0). \end{cases}$$

Obviously, from the above argument, for any point $(\tau_0,\eta_0)\in\Sigma$, there is an open neighborhood $\mathcal{V}$ of (τ_0,η_0) where T is continuously invertible. Then we can obtain

$$T^{-1}AT = \begin{pmatrix} \omega^r & z^r & 0 & 0 \\ 0 & -\omega^r & 0 & 0 \\ 0 & 0 & \omega^l & z^l \\ 0 & 0 & 0 & -\omega^l \end{pmatrix} \tag{3.1.45}$$

on $\mathcal{V}$ where A is given in (3.1.36) and

$$z^{r,l} = \begin{cases} -\dfrac{1}{\alpha^{r,l}}, & \text{if } m^{r,l}\alpha^{r,l}\neq 0 \text{ at } (\tau_0,\eta_0), \\ \dfrac{m^{r,l}}{(n^{r,l}-\omega^{r,l})\alpha^{r,l}}, & \text{if } (n^{r,l}-\omega^{r,l})\alpha^{r,l}\neq 0 \text{ at } (\tau_0,\eta_0). \end{cases}$$

Remark 3.1.4. By the definition of $m^{r,l}$, $n^{r,l}$ and $\alpha^{r,l}$, we know that $z^{r,l}$ may not be well defined at the poles of A which are all located on the boundary of Σ. However, the estimates of $\widehat{W}^{\rm nc}|_{x_2=0}$ only involves the interior points of Σ. Hence it sufficies to obtain a uniform esitamtes of $\widehat{W}^{\rm nc}|_{x_2=0}$ in the interior of Σ, which corresponds to $\Sigma\cap\{\Re\tau>0\}$.

3.1.4 Lopatinskii determinant

In this section, we want to estimate the components of $\widehat{W}^{\rm nc}|_{x_2=0}$ in the stable subspace of A through the boundary conditions, which requires us to investigate the invertibility of the matrix $\beta(E^r_-,E^l_-)$. This can be done by computing $\det(\beta(E^r_-,E^l_-))$, which is known as the Lopatinskii determinant. By a direct computation, we can simplify the Lopatinskii determinant to be

$$\begin{aligned}\Delta = \det(\beta(E^r_-,E^l_-)) = c^4(\tau+iv^r\eta)(\tau+iv^l\eta)\left((\tau+iv^r\eta)\omega^r - c\left((\omega^r)^2-\eta^2\right)\right) \\ \times\left(c\left((w^l)^2-\eta^2\right)-(\tau+iv^l\eta)\omega^l\right)(\omega^l\omega^r-\eta^2)(\omega^r+\omega^l),\end{aligned} \tag{3.1.46}$$

from which we see that the Lopatinskii determinant Δ can vanish at multiple places in Σ, which indicates that one can not expect the uniform Lopatinskii condition to hold. In the following lemma we detail the root distribution of Δ, which is important for later discussion.

Lemma 3.1.3. *The roots of the Lopatinskii determinant Δ are distributed in the following ways.*

Case 1. *If $v^r > \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$, then all the roots are simple and on the boundary of Σ. The Lopatinskii condition holds. More precisely, the roots are $(\tau, \eta) \in \Sigma$ such that*

(1) $\tau = \pm i v^r \eta$, or

(2) $\tau = 0$, or

(3) $\tau = \pm i V_1 \eta$,

where $V_1^2 = (v^r)^2 + (F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - \sqrt{c^4 + 4\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)(v^r)^2}$.

Case 2. *If $0 < v^r < \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$, but $v^r \neq \sqrt{\frac{\left((F_{11}^r)^2+(F_{12}^r)^2\right)\left(2c^2+(F_{11}^r)^2+(F_{12}^r)^2\right)}{4\left((F_{11}^r)^2+(F_{12}^r)^2+c^2\right)}}$, then all roots are also simple and on the boundary of Σ. The Lopatinskii condition still holds. More precisely, the roots are $(\tau, \eta) \in \Sigma$ such that*

(1) $\tau = \pm i v^r \eta$, or

(2) $\tau = \pm i V_1 \eta$.

Case 3. *If $v^r = \sqrt{\frac{\left((F_{11}^r)^2+(F_{12}^r)^2\right)\left(2c^2+(F_{11}^r)^2+(F_{12}^r)^2\right)}{4\left((F_{11}^r)^2+(F_{12}^r)^2+c^2\right)}}$, then all roots are on the boundary of Σ. The Lopatinskii condition still holds. More precisely, the roots are $(\tau, \eta) \in \Sigma$ such that*

$\tau = \pm i v^r \eta$ (double roots).

Case 4. *If $v^r = \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$, then all roots are on the boundary of Σ. The Lopatinskii condition still holds. More precisely, the roots are $(\tau, \eta) \in \Sigma$ such that*

(1) $\tau = \pm i v^r \eta$ (simple roots), or

(2) $\tau = 0$ (triple root).

Case 5. *If $v^r = \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$, then all roots are on the boundary of Σ. The Lopatinskii condition still holds. More precisely, the roots are $(\tau, \eta) \in \Sigma$ such that*

(1) $\tau = \pm i v^r \eta$ (simple roots), or

(2) $\tau = 0$ (double root).

Case 6. *If $\sqrt{(F_{11}^r)^2 + (F_{12}^r)^2} < v^r < \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$, then some roots are in the interior of Σ, and hence the Lopatinskii condition fails.*

Moreover for the roots above, if the degree of a root $(\tau,\eta)\in\Sigma$, with $\tau=i\vartheta\eta$ for some real number ϑ, is k, then we have $\Delta=(\tau-i\vartheta\eta)^k h(\tau,\eta)$ for some continuous $h(\tau,\eta)$ satisfying $h(\tau,\eta)\neq 0$ near the root.

Remark 3.1.5. It is easily seen that if

$$(v^r)^2=\frac{1}{4}\left((F_{11}^r)^2+(F_{12}^r)^2+c^2\right) \tag{3.1.47}$$

with $c\leq\sqrt{3\left((F_{11}^r)^2+(F_{12}^r)^2\right)}$ (this corresponds to the second case of the Lemma 3.1.3), then one of the non-differentiable points of A, namely $\tau=i\left(-v^r+\sqrt{(F_{11}^r)^2+(F_{12}^l)^2+c^2}\right)\eta$, coincides with the root $\tau=iv^r\eta$ of the Lopatinskii determinant. As we have pointed out in the Introduction, this is a new phenomenon which does not appear in the compressible Euler flow.

Proof. The proof of the above lemma depends on a detailed analysis on each factor of the Lopatinskii determinant. Firstly, the factors $(\tau+iv^{r,l}\eta)\omega^{r,l}-c\left((\omega^{r,l})^2-\eta^2\right)$ are exactly the expression in Proposition 3.1.1. Thus we know they are never zero.

Secondly, we consider the factors $\tau+iv^{r,l}\eta$. Obviously $\tau=-iv^{r,l}\eta$ are the only simple roots to $\tau+iv^{r,l}\eta=0$, respectively.

Thirdly, we assume that

$$\omega^r\omega^l-\eta^2=0. \tag{3.1.48}$$

If $\eta=0$, we have $\omega^r=\omega^l=-\frac{\tau}{c}$,which means $\omega^r\omega^l\neq 0$, and hence $\omega^r\omega^l-\eta^2\neq 0$. Moreover, if $\delta+v^{r,l}\eta=0$, for example $\delta+v^r\eta=0$, then from (3.1.37) ω^r is real and negative. However, since $\eta\neq 0$, we know that $\delta+v^l\eta\neq 0$, which implies $\Im\omega^l\neq 0$. Thus $\omega^r\omega^l$ can not be a real number, which violates (3.1.48). Therefore (3.1.48) can not happen for $\eta=0$ or $\delta+v^{r,l}\eta=0$.

This leads us to focus only on the points where $\eta\neq 0$ and $\delta+v^{r,l}\eta\neq 0$. Introduce the following two variables,

$$V=\frac{\tau}{i\eta},\quad \Omega^{r,l}=\frac{\omega^{r,l}}{i\eta}. \tag{3.1.49}$$

From (3.1.48) we have $\Omega^r\Omega^l=-1$, and hence $(\Omega^r)^2(\Omega^l)^2=1$. By the (3.1.37) we know

$$(\Omega^r)^2=\frac{1}{c^2}[(V+v^r)^2-(F_{11}^r)^2-(F_{12}^r)^2]-1, \tag{3.1.50}$$

$$(\Omega^l)^2 = \frac{1}{c^2}[(V+v^l)^2 - (F_{11}^l)^2 - (F_{12}^l)^2] - 1. \tag{3.1.51}$$

Hence we have

$$[(V+v^r)^2 - (F_{11}^r)^2 - (F_{12}^r)^2 - c^2][(V+v^l)^2 - (F_{11}^l)^2 - (F_{12}^l)^2 - c^2] = c^4,$$

which leads to the following equation for V^2:

$$\begin{aligned} V^4 &- 2\left((v^r)^2 + (F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right) V^2 + v^{r4} - 2\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)(v^r)^2 \\ &+ \left((F_{11}^r)^2 + (F_{12}^r)^2\right)^2 + 2c^2\left((F_{11}^r)^2 + (F_{12}^r)^2\right) = 0. \end{aligned}$$

Using the quadratic formula, the two roots of the above equation are

$$V_1^2 = (v^r)^2 + (F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - \sqrt{c^4 + 4\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)(v^r)^2}, \tag{3.1.52}$$

$$V_2^2 = (v^r)^2 + (F_{11}^r)^2 + (F_{12}^r)^2 + c^2 + \sqrt{c^4 + 4\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)(v^r)^2}. \tag{3.1.53}$$

We claim that the points $(\tau, \eta) \in \Sigma$ with $\tau = \pm iV_2\eta$ are not the roots of (3.1.48). Without loss of generality, we can assume V_2 is positive. By a direct computation, we have $V_2 + v^{r,l} > \sqrt{c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$ and $-V_2 + v^{r,l} < -\sqrt{c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$. Now, by (3.1.39), (3.1.41) and (3.1.42), $\Im\omega^{r,l}$ have opposite signs to $\delta + v^{r,l}\eta$ respectively. If $\tau = iV_2\eta$, we have $\gamma = \Re\tau = 0$ and $\delta + v^{r,l}\eta = (V_2 + v^{r,l})\eta$. Therefore ω^r and ω^l are purely imaginary, and

$$\Omega^{r,l} = \frac{\Im\omega^{r,l}}{\eta} \in \mathbb{R},$$

from which we deduce that

$$\operatorname{sgn}\left(\Omega^{r,l}\right) = -\operatorname{sgn}\left(\frac{\delta + v^{r,l}\eta}{\eta}\right) = -\operatorname{sgn}\left(V_2 + v^{r,l}\right) = -1.$$

Therefore $\Omega^r\Omega^l \neq -1$ and (3.1.48) is not satisfied. Similarly, we can show that $(\tau, \eta) \in \Sigma$ with $\tau = -iV_2\eta$ are also not the roots of (3.1.48).

Now we focus on V_1^2. Obviously, by (3.1.52), we have

$$\begin{aligned} &V_1^2 > 0, \text{ if } v^r > \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2} \text{ or } 0 < v^r < \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}; \\ &V_1^2 = 0, \text{ if } v^r = \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2} \text{ or } v^r = \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}; \\ &V_1^2 < 0, \text{ if } \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2} > v^r > \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}. \end{aligned}$$

If $\sqrt{2c^2+(F_{11}^r)^2+(F_{12}^r)^2} > v^r > \sqrt{(F_{11}^r)^2+(F_{12}^r)^2}$, by (3.1.49), we obtain that $\tau = \pm iV_1\eta$ are real. Thus $\delta = 0$, but $\eta \neq 0$ and $\Re\tau \neq 0$. This implies that $\delta + v^{r,l}\eta \neq 0$. By (3.1.40) and (3.1.41), we know that $p^r = p^l$ and $q^r = -q^l \neq 0$. Using (3.1.39), we can have $x^r = x^l$ and $y^r = -y^l$, i.e. ω^r is the complex conjugate of ω^l. Then $\omega^r\omega^l > 0$, which implies that $\tau = \pm iV_1\eta$ are the roots of (3.1.48). This way we are able to find a root (τ,η) with $\Re\tau > 0$, which violates the Lopatinskii condition, and hence the vortex sheets are unstable. This proves **Case 6** in the lemma.

For the rest cases we will consider $V_1^2 \geq 0$ and when taking the square root we always use the positive branch, that is $V_1 \geq 0$.

If $v^r > \sqrt{2c^2+(F_{11}^r)^2+(F_{12}^r)^2}$, we have $\tau = \pm iV_1\eta$ is purely imaginary. Without loss of generality, we only consider the case when $\tau = iV_1\eta$. Then $\Re\tau = 0$, but $\delta \neq 0$ and $\eta \neq 0$. By a direct computation, we can obtain

$$\left|V_1 + v^{r,l}\right| > \sqrt{(F_{11}^r)^2+(F_{12}^r)^2+c^2}. \tag{3.1.54}$$

Thus $\delta + v^{r,l}\eta = (V_1 + v^{r,l})\eta \neq 0$. By (3.1.37) and (3.1.54), we have that $\left(\omega^{r,l}\right)^2$ are both real and negative. This means $\omega^{r,l}$ are purely imaginary, and from (3.1.42) the signs of $\Im\omega^{r,l}$ are opposite to those of $\delta + v^{r,l}\eta$, respectively. Hence

$$\operatorname{sgn}\left(\omega^r\omega^l\right) = -\operatorname{sgn}\left((\delta+v^r\eta)(\delta+v^l\eta)\right) = -\operatorname{sgn}\left((V_1+v^r)(V_1+v^l)\eta^2\right) = 1.$$

Therefore $\omega^r\omega^l > 0$, and $(\tau,\eta) \in \Sigma$ with $\tau = iV_1\eta$ are roots of (3.1.48). The other case when $\tau = -iV_1\eta$ can be treated exactly the same way.

If $v^r < \sqrt{(F_{11}^r)^2+(F_{12}^r)^2}$, we also have that $\tau = \pm iV_1\eta$ is purely imaginary. Similarly as before we only treat the case $\tau = iV_1\eta$. Then $\Re\tau = 0$, but $\delta \neq 0$ and $\eta \neq 0$. Now, we have

$$\left|V_1 + v^{r,l}\right| < \sqrt{(F_{11}^r)^2+(F_{12}^r)^2+c^2}. \tag{3.1.55}$$

By (3.1.37) and (3.1.55), we have $\left(\omega^{r,l}\right)^2$ are both real and positive. Thus $\omega^{r,l}$ are both real and negative, which implies $\omega^r\omega^l > 0$. Hence $(\tau,\eta) \in \Sigma$ with $\tau = \pm iV_1\eta$ are roots of (3.1.48).

Then we want to show that under the condition $v^r > \sqrt{2c^2+(F_{11}^r)^2+(F_{12}^r)^2}$ or $0 < v^r < \sqrt{(F_{11}^r)^2+(F_{12}^r)^2}$ the roots to (3.1.48) are simple. Since (3.1.48) does not admit a root at $\eta = 0$, the points $(\tau,\eta) \in \Sigma$ satisfying $\omega^{r,l} = 0$ are not the roots of $\omega^r\omega^l - \eta^2 = 0$ From

(3.1.37), $\omega^{r,l}$ are analytic near the points where $\omega^{r,l}$ do not vanish. We can differentiate (3.1.50) and (3.1.51) with respect to V at $V = V_1$ to obtain

$$\left.\frac{d\Omega^{r,l}}{dV}\right|_{V=V_1} = \frac{V_1 + v^{r,l}}{\Omega^{r,l} c^2}.$$

Thus

$$\left.\frac{d(\Omega^r\Omega^l + 1)}{dV}\right|_{V=V_1} = \frac{(V_1 + v^r)(\Omega^l)^2 + (V_1 + v^l)(\Omega^r)^2}{c^2\Omega^r\Omega^l}.$$

Plugging in (3.1.50) and (3.1.51), we obtain

$$\left.\frac{d(\Omega^r\Omega^l + 1)}{dV}\right|_{V=V_1} = \frac{2V_1\left(V_1^2 - (v^r)^2 - (F_{11}^r)^2 - (F_{12}^r)^2 - c^2\right)}{c^4\Omega^r\Omega^l}.$$

Using (3.1.52), we have

$$\left.\frac{d(\Omega^r\Omega^l + 1)}{dV}\right|_{V=V_1} \neq 0.$$

Hence we have proved $(\tau, \eta) \in \Sigma$ with $\tau = \pm iV_1\eta$ are all simple roots of (3.1.48) provided $v^r > \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$ or $0 < v^r < \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$. More precisely, near $\tau = \pm iV_1\eta$, we have $\omega^r\omega^l - \eta^2 = (\tau \pm iV_1\eta)h^{\pm}(\tau, \eta)$ for some continuous $h^{\pm}(\tau, \eta) \neq 0$ respectively.

If $v^r = \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$, we obtain $\tau = \pm iV_1\eta = 0$. In this case $\Re\tau = \delta = 0$ but $\eta \neq 0$. By (3.1.40), we have $p^{r,l} = -c^2 < 0$. Together with the fact that $\Re\tau = 0$, $\delta + v^{r,l}\eta = v^{r,l}\eta \neq 0$ and (3.1.42), we infer that $\omega^{r,l}$ are purely imaginary and $\Im\omega^{r,l}$ have opposite signs to $v^{r,l}\eta$ respectively. This implies $\omega^r\omega^l > 0$. Hence $(\tau, \eta) \in \Sigma$ with $\tau = 0$ are roots of (3.1.48).

If $v^r = \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$, we also obtain $\tau = \pm iV_1\eta = 0$. Then $\Re\tau = \delta = 0$ but $\eta \neq 0$. By (3.1.40) and (3.1.41), we have $p^{r,l} = c^2 > 0$ and $q^{r,l} = 0$. This implies $\omega^{r,l}$ are both real and negative. Thus $\omega^r\omega^l > 0$. Hence $(\tau, \eta) \in \Sigma$ with $\tau = 0$ are roots of (3.1.48).

Now we want to check the multiplicity of the root when $\tau = 0$ under the condition $v^r = \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$ or $v^r = \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$. Similarly as in the previous case, we obtain the following first derivative of $\Omega^r\Omega^l + 1$:

$$\frac{d(\Omega^r\Omega^l + 1)}{dV} = \frac{2V\left(V^2 - (v^r)^2 - (F_{11}^r)^2 - (F_{12}^r)^2 - c^2\right)}{c^4\Omega^r\Omega^l}.$$

Further differentiation yields the following second derivative:

$$\frac{d^2(\Omega^r\Omega^l+1)}{(dV)^2} = \frac{6V^2 - \left((v^r)^2 + (F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)}{c^4\Omega^r\Omega^l} - \frac{\left(2V\left(V^2 - (v^r)^2 - (F_{11}^r)^2 - (F_{12}^r)^2 - c^2\right)\right)^2}{c^8(\Omega^r\Omega^l)^3}.$$

Thus

$$\left.\frac{d(\Omega^r\Omega^l+1)}{dV}\right|_{V=V_1} = 0, \quad \left.\frac{d^2(\Omega^r\Omega^l+1)}{(dV)^2}\right|_{V=V_1} \neq 0.$$

Hence $(\tau,\eta)\in\Sigma$ with $\tau=0$ are all double roots of (3.1.48) if $v^r = \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$ or $v^r = \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$. We can also conclude that $\omega^r\omega^l - \eta^2 = \tau^2 h(\tau,\eta)$ for some continuous $h(\tau,\eta)\neq 0$ near $\tau = 0$.

Finally we are left to consider the last factor of (3.1.46)

$$\omega^r + \omega^l = 0. \tag{3.1.56}$$

It is obvious from the (3.1.37) that if $\Re\tau > 0$ then $\Re\omega^r < 0$ and $\Re\omega^l < 0$, and hence $\omega^r + \omega^l \neq 0$. Thus we focus on the case when $\Re\tau = 0$. By (3.1.41), we have $q^{r,l} = 0$. From (3.1.56) and using the definition (3.1.49), we have $(\Omega^r)^2 = (\Omega^l)^2$, which implies $p^r = p^l$. Using (3.1.40), we have

$$2v^r\delta\eta = 2v^l\delta\eta.$$

This implies $\delta\eta = 0$. If $\eta = 0$, we obtain $\delta = 1$. Thus $p^r = p^l = -\frac{1}{c^2} < 0$ and $\delta + v^{r,l}\eta = 1 > 0$. From (3.1.42), we obtain $\Im\omega^r = \Im\omega^l < 0$, which contradicts (3.1.56). Hence it must be $\delta = 0$. Since $(\tau,\eta)\in\Sigma$, we have $\eta = \pm\frac{1}{v^r}$. In this case,

$$p^r = p^l = \frac{(F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - (v^r)^2}{c^2}\eta^2.$$

If $(F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - (v^r)^2 > 0$, we obtain that $\omega^{r,l}$ are both real and negative, which contradicts (3.1.56). Otherwise if $(F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - (v^r)^2 = 0$, we have $\omega^{r,l} = 0$, which means $(0, \pm\frac{1}{v^r})$ are roots of (3.1.56). On the other hand, this situation belongs to **Case 6** which we have already concluded the emergence of the instability. Therefore in the sequel, we only consider the case $(F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - (v^r)^2 < 0$. In this case $\omega^{r,l}$ are purely imaginary.

Since $\Re\tau = \delta = 0$ and $\eta = \pm\frac{1}{v^r}$, we have $\delta + v^r\eta = -(\delta + v^l\eta) \neq 0$. Again a use of (3.1.42) implies that the signs of $\Im\omega^{r,l}$ are opposite to those of $\delta + v^{r,l}\eta$, which implies $\Im\omega^r = -\Im\omega^l$. Hence $\omega^r + \omega^l = 0$, and $(0, \pm\frac{1}{v^r}) \in \Sigma$ are roots of (3.1.56).

Next we turn to the multiplicity of these roots. From (3.1.37) we know ω^r and ω^l can not vanish simultaneously. Thus $\omega^r + \omega^l$ is analytic near these roots. From (3.1.49), (3.1.50) and (3.1.51), we have

$$\frac{d\Omega^{r,l}}{dV} = \frac{V + v^{r,l}}{\Omega^{r,l}c^2}.$$

This implies

$$\frac{d(\Omega^r + \Omega^l)}{dV} = \frac{V + v^r}{c^2\Omega^r} + \frac{V + v^l}{c^2\Omega^l}.$$

Since in this case $\omega^r = -\omega^l \neq 0$, we have $\Omega^r = -\Omega^l \neq 0$ at $(0, \pm\frac{1}{v^r})$. At these roots, we have

$$\frac{d(\Omega^r + \Omega^l)}{dV} = 2\frac{v^r}{c^2\Omega^r} \neq 0.$$

Hence, $(0, \pm\frac{1}{v^r})$ are all simple roots to (3.1.56), if $(F_{11}^r)^2 + (F_{12}^r)^2 + c^2 - (v^r)^2 < 0$. Therefore we have $\omega^r + \omega^l = \tau h(\tau, \eta)$ for some continuous $h(\tau, \eta) \neq 0$ near $\tau = 0$.

To summarize, we have derived all possible roots (τ, η) of the Lopatinskii determinant, namely,

$$\tau = -iv^{r,l}\eta, \quad \tau = \pm iV_1\eta \quad \text{or} \quad \tau = 0, \tag{3.1.57}$$

where we have assumed that $v^r > 0$. In general, some of the roots may coincide, and we have already discussed the possibility that $V_1 = 0$ in the study of (3.1.48). Now we are left to check whether $v^r = V_1$ when $V_1 > 0$. By a direct computation, $v^r = V_1$ if and only if

$$v^r = \sqrt{\frac{((F_{11}^r)^2 + (F_{12}^r)^2)\left(2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2\right)}{4\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)}}.$$

Obivously, $\sqrt{\frac{\left((F_{11}^r)^2+(F_{12}^r)^2\right)\left(2c^2+(F_{11}^r)^2+(F_{12}^r)^2\right)}{4\left((F_{11}^r)^2+(F_{12}^r)^2+c^2\right)}} < \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$.

This way, we can identify the root distribution (3.1.57) with the six cases in the lemma:

$V_1^2 > 0$ and $V_1 \neq v^r \Longrightarrow$ **Case 1, Case 2**,

$V_1^2 > 0$ and $V_1 = v^r \Longrightarrow$ **Case 3**,

$V_1^2 = 0 \Longrightarrow$ **Case 4, Case 5**,

$$V_1^2 < 0 \Longrightarrow \textbf{ Case 6}.$$

We now finish the proof of the lemma. □

With this lemma, we can obtain the following estimates on the stable subspace of A near the roots of the Lopatinskii determinant.

Lemma 3.1.4. *Let $(\tau_0, \eta_0) \in \partial\Sigma$ be a root of the Lopatinskii determinant Δ. Then there is a neighborhood $\mathcal{V}$ of (τ_0, η_0) which does not contain any other roots of Δ and a constant κ_0, such that, for $\forall\ (\tau, \eta) \in \mathcal{V}$ and $\forall\ Z^- \in \mathbb{R}^2$,*

(1) If $v^r > \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^2 |Z^-|^2.$$

(2) If $0 < v^r < \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$, but $v^r \neq \sqrt{\frac{\left((F_{11}^r)^2+(F_{12}^r)^2\right)\left(2c^2+(F_{11}^r)^2+(F_{12}^r)^2\right)}{4\left((F_{11}^r)^2+(F_{12}^r)^2+c^2\right)}}$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^2 |Z^-|^2.$$

(3) If $v^r = \sqrt{\frac{\left((F_{11}^r)^2+(F_{12}^r)^2\right)\left(2c^2+(F_{11}^r)^2+(F_{12}^r)^2\right)}{4\left((F_{11}^r)^2+(F_{12}^r)^2+c^2\right)}} < \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^4 |Z^-|^2.$$

(4) If $v^r = \sqrt{2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2}$,
when $\tau_0 = \pm i v^r \eta_0$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^2 |Z^-|^2;$$

when $\tau_0 = 0$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^6 |Z^-|^2.$$

(5) If $v^r = \sqrt{(F_{11}^r)^2 + (F_{12}^r)^2}$,
when $\tau_0 = \pm i v^r \eta_0$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^2 |Z^-|^2;$$

when $\tau_0 = 0$,

$$|\beta(E^r, E^l)Z^-|^2 \geq \kappa_0 \gamma^4 |Z^-|^2.$$

Proof. We denote the elements in the Lopatinskii matrix by

$$\beta(E^r, E^l) = \begin{pmatrix} -a^r + b^r & a^l - b^l \\ -c(\tau - iv^r\eta)(a^r + b^r) & c(\tau + iv^r\eta)(a^l + b^l) \end{pmatrix} =: \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Since every element of $\beta(E^{,}E^l)$ is continuous, we know that if there is an element of $\beta(E^r, E^l)$ which is nonzero at (τ_0, η_0), then there is an open neighborhood $\mathcal{V}$ of (τ_0, η_0) such that $\beta(E^{,}E^l)$ can be transformed to a diagonal matrix in $\mathcal{V}$, that is

$$P\beta(E^r_-, E^l_-)Q = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \tag{3.1.58}$$

For some P, Q continuously invertible in $\mathcal{V}$. For example, if $d_{11} \neq 0$, then we have the following identity

$$\begin{pmatrix} 1/d_{11} & 0 \\ -d_{21}/d_{11} & 1 \end{pmatrix} \beta(E^r, E^l) \begin{pmatrix} 1 & -d_{12} \\ 0 & d_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}.$$

Now we claim that there is always an element in $\beta(E^r, E^l)$ which is not zero for all (τ, η) in Σ. First we consider

$$\begin{aligned} d_{11} &= -\left[(\tau + iv^r\eta)^2 + \left((F^r_{11})^2 + (F^r_{12})^2\right)\eta^2\right]\left[(\tau + iv^r\eta)\omega^r - c((\omega^r)^2 - \eta^2)\right], \\ d_{12} &= -\left[(\tau + iv^l\eta)^2 + \left((F^l_{11})^2 + (F^l_{12})^2\right)\eta^2\right]\left[(\tau + iv^l\eta)\omega^l - c((\omega^l)^2 - \eta^2)\right]. \end{aligned}$$

From Proposition 3.1.1, $(\tau + iv^r\eta)\omega^r - c((\omega^r)^2 - \eta^2)$ and $(\tau + iv^l\eta)\omega^l - c((\omega^l)^2 - \eta^2)$ are never zero. Thus d_{11} only vanishes when $\tau = -iv^r\eta \pm i\sqrt{(F^r_{11})^2 + (F^r_{12})^2}\eta$, and d_{12} only vanishes when $\tau = iv^r\eta \pm i\sqrt{(F^r_{11})^2 + (F^r_{12})^2}\eta$. If $d_{11} = d_{12} = 0$, we can obtain $\tau = 0$ and $v^r = \sqrt{(F^r_{11})^2 + (F^r_{12})^2}$. Since $(\tau, \eta) \in \Sigma$, we have $\eta \neq 0$, which impiles that $\tau \pm iv^r\eta \neq 0$ and $\omega^{r,l} \neq 0$. From the expression of d_{21} and d_{22}, we know that in this case $d_{21} \neq 0$ and $d_{22} \neq 0$. Hence for any (τ_0, η_0), there is an open neighborhood $\mathcal{V}$ of (τ_0, η_0) such that $\beta(E^r, E^l)$ can always be locally continuously transformed to diag$\{1, \Delta\}$. Therefore by the continuity and boundness of d_{ij}, equation (3.1.58) implies that

$$|\beta(E^r_-, E^l_-)Z^-|^2 \geq \kappa \min(1, |\Delta|^2)|Z^-|^2,$$

in $\mathcal{V}$, where $\kappa > 0$ depends on (τ_0, η_0). From the fact that there are only finitely many roots of Δ we know that $\mathcal{V}$ can be chosen so that it only contains one root of Δ, which is (τ_0, η_0). Now combining the result in Lemma 3.1.3, we finish the proof. □

Remark 3.1.6. For the points (τ_0, η_0) where the Lopatinskii determinant is not zero, by the continuity of Δ, we can also obtain an open neighborhood $\mathcal{V}$ of (τ_0, η_0) in which $\Delta \neq 0$ such that

$$|\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_0 |Z^-|^2,$$

for all $(\tau, \eta) \in \mathcal{V}$ and $Z^- \in \mathbb{R}^2$, where κ_0 is a positive constant depending on (τ_0, η_0).

3.1.5 Energy estimates

Now we want to combine the argument in the previous two sections and obtain the energy estimates. For a point $(\tau_0, \eta_0) \in \Sigma$, shrinking the neighborhood if necessary, we obtain a new neighborhood $\mathcal{V}$ of (τ_0, η_0) where we have the separation of modes of A (c.f. (3.1.45)) and the estimate of the Lopatinskii determinant (c.f. Lemma 3.1.4 and Remark 3.1.6). We call such a point (τ_0, η_0) a “generating point” of $\mathcal{V}$. Notice that if $\Delta(\tau_0, \eta_0) \neq 0$ then $\Delta \neq 0$ at every point of $\mathcal{V}$. Repeating this process for all points on Σ forms a covering of Σ. Then by the compactness of the Σ, there is a finite subcovering $\{\mathcal{V}_i\}_{i=1}^N$ of Σ with the corresponding generating points denoted by $\{(\tau_i, \eta_i)\}_{i=1}^N$. Obviously this subcovering contains all the neighborhoods $\mathcal{V}$ of (τ_0, η_0) such that $\Delta(\tau_0, \eta_0) = 0$. Then we can construct a partition of unity of Σ according to this finite subcovering, i.e., we can find $\chi_i \in C_c^\infty(\mathcal{V}_i)$ for $i = 1, \cdots, N$ such that $\sum_{i=1}^N \chi_i^2 = 1$ on Σ.

Now we derive an energy estimate in each conic zone $\Pi_i = \{(\tau, \eta) : s \cdot (\tau, \eta) \in \mathcal{V}_i, \text{ for some } s > 0\}$. In each neighborhood $\mathcal{V}_i$ of (τ_i, η_i), we denote T_i the transformation matrix of the separation of modes in this neighborhood and extend χ_i and T_i by homogeneity of degree 0 to the conic zone Π_i. Then we consider

$$Z = \chi_i T_i^{-1} \widehat{W}^{\text{nc}} \tag{3.1.59}$$

for all $(\tau, \eta) \in \Pi_i$. The system that $Z = (Z_1, Z_2, Z_3, Z_4)^\top$ satisfies now becomes

$$\frac{dZ}{dx_2} = \left(T_i^{-1} A T_i\right) Z.$$

Recalling Remark 3.1.4, we know that it suffices to obtain uniform estimates of Z for $\Re\tau > 0$. Therefore we only consider $\Re\tau > 0$ below.

From (3.1.45), the second and fourth equations are

$$\frac{dZ_2}{dx_2} = -\omega^r Z_2,$$
$$\frac{dZ_4}{dx_2} = -\omega^l Z_4,$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$. By (3.1.37) we have $\Re\omega^{r,l}(\tau, \eta) < 0$ provided $\Re\tau > 0$. Moreover, since $\widehat{W}(\tau, \eta, \cdot)$ is in L^2 and T_i^{-1} is continuous and bounded from above in Π_i, we know that $Z(\tau, \eta, \cdot)$ is also in L^2 for every $(\tau, \eta) \in \Pi_i$. Hence, the above ODEs implies that

$$Z_2 = 0 \quad \text{and} \quad Z_4 = 0, \tag{3.1.60}$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$.

Next for Z_1 and Z_3, from (3.1.59) and (3.1.60) we have

$$\chi_i \widehat{W}^{\text{nc}} = T_i Z = (E_-^r, E_-^l) \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix},$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$. Then the boundary condition becomes

$$\chi_i h = \chi_i \beta \widehat{W}^{\text{nc}}|_{x_2=0} = \beta(E_-^r, E_-^l) \left. \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} \right|_{x_2=0}, \tag{3.1.61}$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$. In the above equation, $\det\big(\beta(E_-^r, E_-^l)\big)$ is the Lopatinskii determinant Δ. From Remark 3.1.6, if $\det\big(\beta(E_-^r, E_-^l)\big)$ is not zero at (τ_i, η_i) we have

$$|\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i |Z^-|^2,$$

$(\tau, \eta) \in \mathcal{V}_i$ and $Z^- \in \mathbb{R}^2$, where κ_i is a positive constant depending on (τ_i, η_i). Since β is homogeneous of degree 0, we have

$$|\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i |Z^-|^2,$$

for all $(\tau, \eta) \in \Pi_i$ and $Z^- \in \mathbb{R}^2$. By (3.1.61), we have

$$\left| \left. \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} \right|_{x_2=0} \right|^2 \leq \frac{\chi_i^2}{\kappa_i} |h|^2 \tag{3.1.62}$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$.

If (τ_i, η_i) is a simple root of Δ, by Lemma 3.1.4, we have

$$|\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i \gamma^2 |Z^-|^2,$$

for all $(\tau, \eta) \in \mathcal{V}_i$ and $Z^- \in \mathbb{R}^2$. Since β is homogeneous of degree 0, we have

$$\left(|\tau|^2 + (v^r)^2\eta^2\right) |\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i \gamma^2 |Z^-|^2,$$

for all $(\tau, \eta) \in \Pi_i$ and $Z^- \in \mathbb{R}^2$. By (3.1.61), we have

$$\left| \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} \Bigg|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)}{\kappa_i \gamma^2} |h|^2 \tag{3.1.63}$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$.

If (τ_i, η_i) is a double root of Δ, from Lemma 3.1.4, we have

$$|\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i \gamma^4 |Z^-|^2,$$

for all $(\tau, \eta) \in \mathcal{V}_i$ and $Z^- \in \mathbb{R}^2$. It follows from the homogeneity of β that

$$\left(|\tau|^2 + (v^r)^2\eta^2\right)^2 |\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i \gamma^4 |Z^-|^2,$$

for all $(\tau, \eta) \in \Pi_i$ and $Z^- \in \mathbb{R}^2$. By (3.1.61), we have

$$\left| \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} \Bigg|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^2}{\kappa_i \gamma^4} |h|^2 \tag{3.1.64}$$

for all $(\tau, \eta) \in \Pi_i$.

Similarly if (τ_i, η_i) is a triple root of Δ, we have

$$|\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i \gamma^6 |Z^-|^2,$$

for all $(\tau, \eta) \in \mathcal{V}_i$ and $Z^- \in \mathbb{R}^2$. Using homogeneity of β, once again we have

$$\left(|\tau|^2 + (v^r)^2\eta^2\right)^3 |\beta(E_-^r, E_-^l)Z^-|^2 \geq \kappa_i \gamma^6 |Z^-|^2,$$

for all $(\tau, \eta) \in \Pi_i$ and $Z^- \in \mathbb{R}^2$. By (3.1.61), we have

$$\left| \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} \Bigg|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^3}{\kappa_i \gamma^6} |h|^2 \tag{3.1.65}$$

for all $(\tau, \eta) \in \Pi_i$.

Putting together (3.1.60), (3.1.62)-(3.1.65) we obtain the following estimate for Z in Π_i:

$$\left| Z|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^j}{\kappa_i \gamma^{2j}} |h|^2 , \tag{3.1.66}$$

where $j = 0$ corresponds to the case when $\beta(E_-^r, E_-^l)$ is invertible at (τ_i, η_i), and $j = 1, 2, 3$ corresponds to the multiplicity of (τ_i, η_i) as a root of Δ.

Now we proceed to the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. When (3.1.6) holds, from Lemma 3.1.3 we know either $\beta(E_-^r, E_-^l)$ is invertible at (τ_i, η_i) or (τ_i, η_i) is a simple root of Δ. Then from (3.1.66) we can always have

$$\left| Z|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)}{\kappa_i \gamma^2} |h|^2 ,$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$ and all $i = 1, \cdots, N$. From (3.1.59), we have

$$\chi_i^2 \left| T_i^{-1} \widehat{W}^{\mathrm{nc}} \Big|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)}{\kappa_i \gamma^2} |h|^2 .$$

Using the boundedness of T_i in Π_i and summing up the estimates over all the conic zones $\{\Pi_i\}_{i=1}^N$, we obtain

$$\left| \widehat{W}^{\mathrm{nc}} \Big|_{x_2=0} \right|^2 \leq C \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)}{\gamma^2} |h|^2$$

for all $(\tau, \eta) \in \Pi$ with $\Re\tau > 0$. Then integrating the above inequality with respect to (δ, η) over $\mathbb{R}^2$ and recalling (3.1.34) we have

$$\left\| \widehat{W}^{\mathrm{nc}} \right\|_0^2 \leq \frac{C}{\gamma^2} \|g\|_{1,\gamma}^2,$$

which implies (3.1.19).

The other two cases can be treated the same way. For the sake of completeness we provide the details. When (3.1.8) holds, Lemma 3.1.3 indicates that (τ_i, η_i) may be a nonzero point, simple root or double root of Δ. From (3.1.66) we always have

$$\left| Z|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^2}{\kappa_i \gamma^4} |h|^2 ,$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$ and all $i = 1, \cdots, N$. Thus from (3.1.59), we have

$$\chi_i^2 \left| T_i^{-1} \widehat{W}^{\mathrm{nc}} \Big|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^2}{\kappa_i \gamma^4} |h|^2 .$$

Patching up the estimates over all the conic zones $\{\Pi_i\}_{i=1}^N$ we obtain

$$\left| \widehat{W}^{\mathrm{nc}} \right|^2 \leq C \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^2}{\gamma^4} |h|^2$$

for all $(\tau, \eta) \in \Pi$ with $\Re\tau > 0$. Integration in (δ, η) over $\mathbb{R}^2$ implies

$$\left\| \widehat{W}^{\mathrm{nc}} \right\|_0^2 \leq \frac{C}{\gamma^4} \|g\|_{2,\gamma}^2,$$

which proves (3.1.20).

The last case when (3.1.10) is satisfied, from Lemma 3.1.3 we know (τ_i, η_i) may be a nonzero point, simple root or triple root of Δ, and hence we have

$$\left| Z|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^3}{\kappa_i \gamma^6} |h|^2 ,$$

for all $(\tau, \eta) \in \Pi_i$ with $\Re\tau > 0$ and all $i = 1, \cdots, N$. Converting into $\widehat{W}^{\mathrm{nc}}$, we have

$$\chi_i^2 \left| T_i^{-1} \widehat{W}^{\mathrm{nc}} \Big|_{x_2=0} \right|^2 \leq \chi_i^2 \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^3}{\kappa_i \gamma^6} |h|^2 .$$

Therefore

$$\left| \widehat{W}^{\mathrm{nc}} \right|^2 \leq C \frac{\left(|\tau|^2 + (v^r)^2\eta^2\right)^3}{\gamma^6} |h|^2$$

for all $(\tau, \eta) \in \Pi$ with $\Re\tau > 0$, which shows that

$$\left\| \widehat{W}^{\mathrm{nc}} \right\|_0^2 \leq \frac{C}{\gamma^6} \|g\|_{3,\gamma}^2,$$

proving estimate (3.1.21). Thus we finish the proof of our main theorem. □

3.2 VARIABLE COEFFICIENT CASE

3.2.1 Linearization and main results

For the linear analysis in the variable coefficient case, our goal is to show the stability of the linear system around the variable states which are near the stable constant states. To obtain the expected estimate, we need to utilize the para-differential calculus to help us in the spectrum analysis. However unlike Fourier transform in the constant coefficient case, each operation in para-differential calculus generates some errors. This cause some serious troubles in dealing with the loss of more than one order of derivatives. So in our linear analysis for the variable coefficient case, we only focus on the case where there is only one loss of derivative in the energy estimate. As we will see later, this case corresponding to the variable states near some (not all) of the constant states in case (1) of Theorem 3.1.1.

First we denote the variable coefficient background states as

$$
\begin{aligned}
U^{r,l} &= \begin{pmatrix} \rho^{r,l} \\ v^{r,l} \\ u^{r,l} \\ F_{11}^{r,l} \\ F_{21}^{r,l} \\ F_{12}^{r,l} \\ F_{22}^{r,l} \end{pmatrix} := \bar{U}^{r,l} + \dot{U}^{r,l} = \begin{pmatrix} \bar{\rho} \\ \pm\bar{v} \\ 0 \\ \pm\bar{F}_{11} \\ 0 \\ \pm\bar{F}_{12} \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\rho}^{r,l} \\ \dot{v}^{r,l} \\ \dot{u}^{r,l} \\ \dot{F}_{11}^{r,l} \\ \dot{F}_{21}^{r,l} \\ \dot{F}_{12}^{r,l} \\ \dot{F}_{22}^{r,l} \end{pmatrix}, \\
\Phi^{r,l}(t, x_1, x_2) &:= \pm x_2 + \dot{\Phi}^{r,l},
\end{aligned}
\tag{3.2.1}
$$

where $U^{r,l}$ and $\Phi^{r,l}$ are states and changes of variable on each side of the vortex sheet respectively. $\bar{\rho} > 0$, $\bar{v} > 0$, $\bar{F}_{11}$ and $\bar{F}_{12}$ are constants. $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ are functions which represent the small perturbation around the constant states.

Remark 3.2.1. When the perturbations $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ are zero, the states reduce to the rectilinear vortex sheets, and the linear stability around this kind of states has been discussed in the previous section.

Here we assume the following smallness condition on the perturbation of the background states:

$$\dot{U}^{r,l} \in W^{2,\infty}(\Omega), \dot{\Phi}^{r,l} \in W^{3,\infty}(\Omega), \quad \|(\dot{U}^r, \dot{U}^l)\|_{W^{2,\infty}(\Omega)} + \|(\dot{\Phi}^r, \dot{\Phi}^l)\|_{W^{3,\infty}(\Omega)} \leq K, \tag{3.2.2}$$

where K is a suitable positive constant, $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ have compact support in the domain. By the Rankine-Hugoniot condition, we require the perturbed states (3.2.1) satisfying the following on $x_2 = 0$:

$$\begin{cases} (v^r - v^l)\partial_1\varphi - (u^r - u^l) = 0, \\ \partial_t\varphi + v^r\partial_1\varphi - u^r = 0, \\ (F_{11}^r - F_{11}^l)\partial_1\varphi - (F_{21}^r - F_{21}^l) = 0, \\ F_{11}^r\partial_1\varphi - F_{21}^l = 0, \\ (F_{12}^r - F_{12}^l)\partial_1\varphi - (F_{22}^r - F_{22}^l) = 0, \\ F_{12}^r\partial_1\varphi - F_{22}^r = 0, \\ \rho^r - \rho^l = 0, \end{cases} \tag{3.2.3}$$

where $\varphi = \Phi^r|_{x_2=0} = \Phi^l|_{x_2=0}$. To keep the constant rank of boundary matrix in the whole domain and have a simpler formulation, inspired by [20, 24], we assume the following conditions on the perturbed states (3.2.1) hold:

$$\begin{cases} \partial_t\Phi^{r,l} + v^{r,l}\partial_1\Phi^{r,l} - u^{r,l} = 0, \\ F_{11}^{r,l}\partial_1\Phi^{r,l} - F_{21}^{r,l} = 0, \\ F_{12}^{r,l}\partial_1\Phi^{r,l} - F_{22}^{r,l} = 0, \end{cases} \tag{3.2.4}$$

and

$$\partial_2\Phi^r \geq \kappa_0 \text{ and } \partial_2\Phi^l \leq -\kappa_0, \tag{3.2.5}$$

for all $(t, x) \in \Omega$ and some positive constant κ_0.

Now we linearize the differential equation (3.0.6) around the states (3.2.1). We denote the perturbation $(V^\pm, \Psi^\pm)$ of the states $(U^{r,l}, \Phi^{r,l})$. Then the linearized equation is

$$\begin{aligned}
&\partial_t V^\pm + A_1(U^{r,l})\partial_1 V^\pm + \frac{1}{\partial_2 \Phi^{r,l}}(A_2(U^{r,l}) - \partial_t \Phi^{r,l} - \partial_1 \Phi^{r,l} A_1(U^{r,l}))\partial_2 V^\pm \\
&+ [dA_1(U^{r,l})V^\pm]\partial_1 U^{r,l} - \frac{\partial_2 \Psi^\pm}{(\partial_2 \Phi^{r,l})^2}(A_2(U^{r,l}) - \partial_t \Phi^{r,l} - \partial_1 \Phi^{r,l} A_1(U^{r,l}))\partial_2 U^{r,l} \\
&+ \frac{1}{\partial_2 \Phi^{r,l}}[dA_2(U^{r,l})V^\pm - \partial_t \Psi^\pm - \partial_1 \Psi^\pm A_1(U^{r,l}) - \partial_1 \Phi^{r,l} dA_1(U^{r,l})V^\pm]\partial_2 U^{r,l} = f,
\end{aligned}$$

for $x_2 > 0$. We define the first order linear operator

$$\begin{aligned}
&L(U^{r,l}, \nabla \Phi^{r,l})V^\pm := \\
&\qquad \partial_t V^\pm + A_1(U^{r,l})\partial_1 V^\pm + \frac{1}{\partial_2 \Phi^{r,l}}(A_2(U^{r,l}) - \partial_t \Phi^{r,l} - \partial_1 \Phi^{r,l} A_1(U^{r,l}))\partial_2 V^\pm,
\end{aligned}$$

and introduce the Alinhac's 'good unknown'

$$\dot{V}^\pm = \begin{pmatrix} \dot{\rho}^\pm \\ \dot{v}^\pm \\ \dot{u}^\pm \\ \dot{F}^\pm_{11} \\ \dot{F}^\pm_{21} \\ \dot{F}^\pm_{12} \\ \dot{F}^\pm_{22} \end{pmatrix} := V^\pm - \frac{\Psi^\pm}{\partial_2 \Phi^{r,l}} \partial_2 U^{r,l}.$$

Then we can rewrite the above equations to be

$$L(U^{r,l}, \nabla \Phi^{r,l})\dot{V}^\pm + C(U^{r,l}, \nabla U^{r,l}, \nabla \Phi^{r,l})\dot{V}^\pm + \frac{\Psi^\pm}{\partial_2 \Phi^{r,l}} \partial_2 [L(U^{r,l}, \nabla \Phi^{r,l})U^{r,l}] = f^{r,l},$$

where

$$\begin{aligned}
&C(U^{r,l}, \nabla U^{r,l}, \nabla \Phi^{r,l})\dot{V}^\pm := \\
&\qquad [dA_1(U^{r,l})\dot{V}^\pm]\partial_1 U^{r,l} + \frac{1}{\partial_2 \Phi^{r,l}}[dA_2(U^{r,l})\dot{V}^\pm - \partial_1 \Phi^{r,l} dA_1(U^{r,l})\dot{V}^\pm]\partial_2 U^{r,l}.
\end{aligned}$$

By the spirit of the previous work, we can neglect the zeroth order terms of $\Psi^\pm$ and only consider the following differential equations:

$$L'_{r,l}\dot{V}^\pm := L(U^{r,l}, \nabla \Phi^{r,l})\dot{V}^\pm + C(U^{r,l}, \nabla U^{r,l}, \nabla \Phi^{r,l})\dot{V}^\pm = f^{r,l}. \tag{3.2.6}$$

Since $U^{r,l}$ are in $W^{2,\infty}(\Omega)$, we know the coefficients in $L(U^{r,l}, \nabla\Phi^{r,l})$ are in $W^{2,\infty}(\Omega)$ and the coefficients in $C(U^{r,l}, \nabla U^{r,l}, \nabla\Phi^{r,l})$ are in $W^{1,\infty}(\Omega)$.

Now we linearize the boundary conditions around the same perturbed states and obtain

$$
\begin{aligned}
&(v^r - v^l)\partial_1\psi + (v^+ - v^-)\partial_1\varphi - (u^+ - u^-) = g_1, \\
&\partial_t\psi + v^r\partial_1\psi + v^+\partial_1\varphi - u^+ = g_2, \\
&\rho^+ - \rho^- = g_3 \\
&(F^r_{11} - F^l_{11})\partial_1\psi + (F^+_{11} - F^-_{11})\partial_1\varphi - (F^+_{21} - F^-_{21}) = g_4, \\
&F^r_{11}\partial_1\psi + F^+_{11}\partial_1\varphi - F^+_{21} = g_5, \\
&(F^r_{12} - F^l_{12})\partial_1\psi + (F^+_{12} - F^-_{12})\partial_1\varphi - (F^+_{22} - F^-_{22}) = g_6, \\
&F^r_{12}\partial_1\psi + F^+_{12}\partial_1\varphi - F^+_{22} = g_7,
\end{aligned}
$$

at $x_2 = 0$, where $\psi = \Psi^+|_{x_2=0} = \Psi^-|_{x_2=0}$. We denote the above equations in the following form:

$$\underline{b}\nabla\psi + \underline{M}V|_{x_2=0} = g,$$

where $V = (V^+, V^-)^\top$, $\nabla\psi = (\partial_t\psi, \partial_1\psi)^\top$, $g = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^\top$,

$$
\underline{b}(t, x_1) := \begin{pmatrix}
0 & (v^r - v^l)|_{x_2=0} \\
1 & v^r|_{x_2=0} \\
0 & 0 \\
0 & (F^r_{11} - F^l_{11})|_{x_2=0} \\
0 & F^r_{11}|_{x_2=0} \\
0 & (F^r_{12} - F^l_{12})|_{x_2=0} \\
0 & F^r_{12}|_{x_2=0}
\end{pmatrix},
$$

and

$$
\underline{M}(t,x_1) :=
\begin{pmatrix}
0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 & 0 & -\partial_1\varphi & 1 & 0 & 0 & 0 & 0 \\
0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 & 0 & -\partial_1\varphi & 1 & 0 & 0 \\
0 & 0 & 0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 & 0 & -\partial_1\varphi & 1 \\
0 & 0 & 0 & 0 & 0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Also, by using Alinhac's 'good unknown', we obtain

$$
B'(\dot{V},\psi) := \underline{b}\nabla\psi + \underline{M}\begin{pmatrix}\partial_2 U^r/\partial_2\Phi^r \\ \partial_2 U^l/\partial_2\Phi^l\end{pmatrix}\psi + \underline{M}\dot{V}|_{x_2=0} = g \tag{3.2.7}
$$

Hence our problem is reduced to consider the following linear problem:

$$
\begin{cases}
L'_{r,l}\dot{V}^{\pm} = f^{r,l}, & x_2 > 0, \\
B'(\dot{V},\psi) = g, & x_2 = 0.
\end{cases} \tag{3.2.8}
$$

We notice that the boundary condition do not invloves the tangential components of $\dot{V}|_{x_2=0}$. Thus we define the components of $\dot{V}|_{x_2=0}$ involved in the boundary condition by $\dot{V}^{\mathrm{n}}|_{x_2=0}$ as

$$
\begin{aligned}
\dot{V}^{\mathrm{n}}|_{x_2=0} = \Big(&\dot{\rho}^+, \dot{u}^+ - \dot{v}^+\partial_1\Phi^r, \dot{F}^+_{21} - \dot{F}^+_{11}\partial_1\Phi^r, \dot{F}^+_{22} - \dot{F}^+_{12}\partial_1\Phi^r, \\
&\dot{\rho}^-, \dot{u}^- - \dot{v}^-\partial_1\Phi^l, \dot{F}^-_{21} - \dot{F}^-_{11}\partial_1\Phi^l, \dot{F}^-_{22} - \dot{F}^-_{12}\partial_1\Phi^l\Big)^{\top}|_{x_2=0}.
\end{aligned}
$$

Hence our main theorem in variable coefficient case is

Theorem 3.2.1. *If the particular solution defined by (3.2.1) satisfying one of the following two conditions*

$$(i) \qquad \bar{v}^2 > 2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2$$

$$(ii) \qquad \bar{v}^2 < \bar{F}_{11}^2 + \bar{F}_{12}^2 \quad but$$

$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2}{4},\ \bar{v}^2 \neq \frac{\left(\sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2} - \sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2}\right)^2}{4},$$

$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2}{4},\ \bar{v}^2 \neq \frac{\left(\bar{F}_{11}^2 + \bar{F}_{12}^2\right)\left(2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2\right)}{4\left(\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2\right)},$$

the perturbation $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ have compact support and K in (3.2.2) is small enough, there are two constant C_0 and γ_0 which are determined by the particular solution, such that for all $\dot{U}$ and ψ and all $\gamma \geq \gamma_0$ the follwing estiamte holds:

$$\gamma \left\|\left\|\dot{V}\right\|\right\|^2_{L^2(H_0^\gamma)} + \left\|\dot{V}^{\mathrm{n}}|_{x_2=0}\right\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)} \leq C_0 \left(\frac{1}{\gamma^3}\left\|\left\|L'\dot{V}\right\|\right\|^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^2}\left\|B'(\dot{V},\psi)\right\|^2_{H^1_\gamma(\mathbb{R}^2)}\right)$$

Where $L'\dot{V} = (L'_r\dot{V}^+, L'_l\dot{V}^-)$ and B' is defined as above.

3.2.2 Reduction of the system

For the system (3.2.8), it is not difficult to find out a symmetrizer

$$S^{r,l} := \mathrm{diag}\{p'(\rho^{r,l})/\rho^{r,l}, \rho^{r,l}, \rho^{r,l}, 1, 1, 1, 1\}.$$

We multiply $(S^{r,l}\dot{V}^\pm)^\top$ to the interior equations of (3.2.8) and integrate by parts we have the following lemma

Lemma 3.2.1. *There are two positive constant C and $\gamma_1 \geq 1$ that for any $\gamma \geq \gamma_1$, we have the following holds*

$$\gamma \left\|\left\|\dot{V}^\pm\right\|\right\|^2_{L^2(H^0_\gamma)} \leq C\left(\frac{1}{\gamma}\left\|\left\|L'_{r,l}\dot{V}^\pm\right\|\right\|^2_{L^2_\gamma(\mathbb{R}^2)} + \left\|\dot{V}^n|_{x_2=0}\right\|^2_{L^2_\gamma(\mathbb{R}^2)}\right)$$

respectively.

With the help of this Lemma, our object is switched to only estimate $\dot{V}^{\mathrm{n}}|_{x_2=0}$ and ψ through the system (3.2.8). To achieve this we need to investigate this system into more detail by using para-linearization. Before we do that, we need to transform the linear operator $L(U^{r,l}, \nabla\Phi^{r,l})$ to make the boundary matrix

$$\tilde{A}_2^{r,l} := \frac{1}{\partial_2\Phi^{r,l}}(A_2(U^{r,l}) - \partial_t\Phi^{r,l} - \partial_1\Phi^{r,l}A_1(U^{r,l})),$$

to be a constant diagonal matrix. This transformation can always be expected by observing that $\tilde{A}_2^{r,l}$ has a constant rank in the closure of the domian $\mathbb{R}^3_+$. In particular, we consider the following transformation matrix

$$T(U^{r,l}, \nabla\Phi^{r,l}) := \begin{pmatrix} 0 & \langle\partial_1\Phi^{r,l}\rangle & \langle\partial_1\Phi^{r,l}\rangle & 0 & 0 & 0 & 0 \\ 1 & -\frac{c(\rho^{r,l})}{\rho^{r,l}}\partial_1\Phi^{r,l} & \frac{c(\rho^{r,l})}{\rho^{r,l}}\partial_1\Phi^{r,l} & 0 & 0 & 0 & 0 \\ \partial_1\Phi^{r,l} & \frac{c(\rho^{r,l})}{\rho^{r,l}} & -\frac{c(\rho^{r,l})}{\rho^{r,l}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\partial_1\Phi^{r,l} & 0 & 0 \\ 0 & 0 & 0 & \partial_1\Phi^{r,l} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\partial_1\Phi^{r,l} \\ 0 & 0 & 0 & 0 & 0 & \partial_1\Phi^{r,l} & 1 \end{pmatrix},$$

where $\langle\partial_1\Phi^{r,l}\rangle = \sqrt{1 + \partial_1\Phi^{r,l2}}$. This diagonalizes the boundary matrix $\tilde{A}^{r,l}$ to be

$$T^{-1}(U^{r,l}, \nabla\Phi^{r,l})\tilde{A}_2^{r,l}T(U^{r,l}, \nabla\Phi^{r,l}) = \mathrm{diag}\{0, \frac{c(\rho^{r,l})\langle\partial_1\Phi^{r,l}\rangle}{\partial_2\Phi^{r,l}}, -\frac{c(\rho^{r,l})\langle\partial_1\Phi^{r,l}\rangle}{\partial_2\Phi^{r,l}}, 0, 0, 0, 0\}.$$

Once multiplying above by the following matrix:

$$A_0^{r,l} = \mathrm{diag}\{1, \frac{\partial_2\Phi^{r,l}}{c(\rho^{r,l})\langle\partial_1\Phi^{r,l}\rangle}, -\frac{\partial_2\Phi^{r,l}}{c(\rho^{r,l})\langle\partial_1\Phi^{r,l}\rangle}, 1, 1, 1, 1\},$$

we can make the boundary matrix to be a constant matrix. In particular, it will take the following form:

$$I_2 = \mathrm{diag}\{0, 1, 1, 0, 0, 0, 0\}$$

With the above transformation matrix, the interior equations of (3.2.8) for the new unknowns $W^{\pm} := T^{-1}(U^{r,l}, \nabla\Phi^{r,l})\dot{V}^{\pm}$ as

$$A_0^{r,l}\partial_t W^{\pm} + A_1^{r,l}\partial_1 W^{\pm} + I_2\partial_2 W^{\pm} + A_0^{r,l}C^{r,l}W^{\pm} = F^{r,l}, \tag{3.2.9}$$

Where

$$\begin{aligned}
A_1^{r,l} =& A_0^{r,l}T^{-1}(U^{r,l}, \nabla\Phi^{r,l})A_1(U^{r,l})T(U^{r,l}, \nabla\Phi^{r,l}),\\
C^{r,l} =& T^{-1}(U^{r,l}, \nabla\Phi^{r,l})\partial_t T(U^{r,l}, \nabla\Phi^{r,l}) + T^{-1}(U^{r,l}, \nabla\Phi^{r,l})A_1(U^{r,l})\partial_1 T(U^{r,l}, \nabla\Phi^{r,l})\\
&+ T^{-1}(U^{r,l}, \nabla\Phi^{r,l})C(U^{r,l}, \nabla U^{r,l}, \nabla\Phi^{r,l})T(U^{r,l}, \nabla\Phi^{r,l})\\
&+ T^{-1}(U^{r,l}, \nabla\Phi^{r,l})\tilde{A}_2^{r,l}\partial_2 T(U^{r,l}, \nabla\Phi^{r,l}),\\
F^{r,l} =& A_0^{r,l}T^{-1}(U^{r,l}, \nabla\Phi^{r,l})f^{r,l}.
\end{aligned}$$

To simplify the argument in obtaining the estimates in weighted Sobolev norm, we consider the weighted unknown $\widetilde{W}^{\pm} = e^{-\gamma t}W^{\pm}$ and rewrite the (3.2.9) into

$$\mathcal{L}_{r,l}^{\gamma}\widetilde{W}^{\pm} := \gamma A_0^{r,l}\widetilde{W}^{\pm} + A_0^{r,l}\partial_t\widetilde{W}^{\pm} + A_1^{r,l}\partial_1\widetilde{W}^{\pm} + I_2\partial_2\widetilde{W}^{\pm} + A_0^{r,l}C^{r,l}\widetilde{W}^{\pm} = e^{-\gamma t}F^{r,l},$$

where $A_j^{r,l} \in W^{2,\infty}(\Omega)$ and $C^{r,l} \in W^{1,\infty}(\Omega)$. Similarly, considering the transformation T, we rewrite the boundary condition of (3.2.8) as

$$\underline{b}\nabla\psi + \underline{M}\begin{pmatrix}\partial_2 U^r/\partial_2\Phi^r\\ \partial_2 U^l/\partial_2\Phi^l\end{pmatrix}\psi + \underline{M}\begin{pmatrix}T(U^r, \nabla\Phi^r) & 0\\ 0 & T(U^l, \nabla\Phi^l)\end{pmatrix}W|_{x_2=0} = g. \tag{3.2.10}$$

In terms of $\widetilde{W} = e^{-\gamma t}W$ and $\widetilde{\psi} = e^{-\gamma t}\psi$, we obtain

$$\mathcal{B}^{\gamma}(\widetilde{W}|_{x_2=0}, \widetilde{\psi}) := \gamma b_0\widetilde{\psi} + \underline{b}\nabla\widetilde{\psi} + \underline{M}\begin{pmatrix}\partial_2 U^r/\partial_2\Phi^r\\ \partial_2 U^l/\partial_2\Phi^l\end{pmatrix}\widetilde{\psi} + \underline{M}\begin{pmatrix}T^r & 0\\ 0 & T^l\end{pmatrix}\widetilde{W}|_{x_2=0} = e^{-\gamma t}g,$$

where b_0 is the first column of $\underline{b}$. Obviously, we have $\underline{b}$, $\underline{M}$ and T are all in $W^{2,\infty}(\Omega)$, and

$$\check{b} := \underline{M}\begin{pmatrix}\partial_2 U^r/\partial_2\Phi^r\\ \partial_2 U^l/\partial_2\Phi^l\end{pmatrix} \in W^{1,\infty}(\Omega).$$

As we pointed out just before Theorem 3.2.1, the boundary condition $\mathcal{B}^\gamma(\widetilde{W}|_{x_2=0}, \widetilde{\psi})$ only involves part of components of $\widetilde{W}$. By examining the matrix coefficient in front of $\widetilde{W}$ in $\mathcal{B}^\gamma(\widetilde{W}|_{x_2=0}, \widetilde{\psi})$, we denote the involved components by $\widetilde{W}^{\mathrm{n}}$ and obtain

$$\widetilde{W}^{\mathrm{n}} = (\widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_5, \widetilde{W}_7, \widetilde{W}_9, \widetilde{W}_{10}, \widetilde{W}_{12}, \widetilde{W}_{14}), \tag{3.2.11}$$

which are actually the normal components of $\widetilde{W}$ on the vortex sheets. Thus we rewrite the boundary condition as

$$\mathcal{B}^\gamma(\widetilde{W}^{\mathrm{n}}|_{x_2=0}, \widetilde{\psi}) = e^{-\gamma t} g.$$

Then by denoting $\widetilde{F}^{r,l} = e^{-\gamma t} F^{r,l}$ and $\widetilde{g} = e^{-\gamma t} g$, we rewrite the system (3.2.8) as

$$\begin{cases} \mathcal{L}^\gamma_{r,l} \widetilde{W}^{\pm} = \widetilde{F}^{r,l}, \\ \mathcal{B}^\gamma(\widetilde{W}^{\mathrm{n}}|_{x_2=0}, \widetilde{\psi}) = \widetilde{g}. \end{cases} \tag{3.2.12}$$

3.2.2.1 Para-linearization Similarly as in the constant coefficient case, the key idea in proving Theorem 3.2.1 is to transform the variable coefficient linear system (3.2.12) into an ODE. Instead of the Fourier transformation in constant coefficient case, we consider the para-linearization to (3.2.12). Thus in the rest of this subsection, we derive the para-linearized system of (3.2.12) and estimate the errors in replacing the system (3.2.12) by its para-linearization.

To simplifed the notation, we will drop the tilde in the system in the rest of this section. First, we consider the boundary conditions and denote

$$b_0 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and } b_1(t, x_1) := \begin{pmatrix} v^r - v^l \\ v^r \\ 0 \\ F_{11}^r - F_{11}^l \\ F_{11}^r \\ F_{12}^r - F_{12}^l \\ F_{12}^r \end{pmatrix} (t, x_1, 0), \text{ and } b = \tau b_0 + i\eta b_1.$$

By the Theorem 2.2.5 in para-linearization, we have

$$
\begin{aligned}
&\gamma b_0\psi + b_0\partial_t\psi = T^\gamma_{\tau b_0}\psi,\\
&\|b_1\partial_1\psi - T^\gamma_{i\eta b_1}\psi\|_{1,\gamma} \le C\|b_1\|_{W^{2,\infty}(\mathbb{R}^2)}\|\psi\|_0 \le \frac{C}{\gamma}\|\psi\|_{1,\gamma},\\
&\|\check{b}\psi - T^\gamma_{\check{b}}\psi\|_{1,\gamma} \le C\|\check{b}\|_{W^{1,\infty}(\mathbb{R}^2)}\|\psi\|_0 \le \frac{C}{\gamma}\|\psi\|_{1,\gamma},\\
&\|T^\gamma_{\check{b}}\psi\|_{1,\gamma} \le C\|\check{b}\|_{L^\infty(\mathbb{R}^2)}\|\psi\|_{1,\gamma} \le C\|\psi\|_{1,\gamma},
\end{aligned}
$$

for some positive constant C. Then we consider the coefficients of W^{n}

$$
\underline{M}\mathrm{diag}\{T^r, T^l\}W =: \mathbf{M}W^{\mathrm{n}} =
$$
$$
\begin{pmatrix}
-\frac{c^r}{\rho^r}\langle\partial_1\varphi\rangle^2 & \frac{c^r}{\rho^r}\langle\partial_1\varphi\rangle^2 & 0 & 0 & \frac{c^l}{\rho^l}\langle\partial_1\varphi\rangle^2 & -\frac{c^l}{\rho^l}\langle\partial_1\varphi\rangle^2 & 0 & 0\\
-\frac{c^r}{\rho^r}\langle\partial_1\varphi\rangle^2 & \frac{c^r}{\rho^r}\langle\partial_1\varphi\rangle^2 & 0 & 0 & 0 & 0 & 0 & 0\\
\langle\partial_1\varphi\rangle & \langle\partial_1\varphi\rangle & 0 & 0 & -\langle\partial_1\varphi\rangle & -\langle\partial_1\varphi\rangle & 0 & 0\\
0 & 0 & -\langle\partial_1\varphi\rangle^2 & 0 & 0 & 0 & \langle\partial_1\varphi\rangle^2 & 0\\
0 & 0 & -\langle\partial_1\varphi\rangle^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\langle\partial_1\varphi\rangle^2 & 0 & 0 & 0 & \langle\partial_1\varphi\rangle^2\\
0 & 0 & 0 & -\langle\partial_1\varphi\rangle^2 & 0 & 0 & 0 & 0
\end{pmatrix} W^{\mathrm{n}},
$$

and

$$
\|\mathbf{M}W^{\mathrm{n}}|_{x_2=0} - T^\gamma_{\mathbf{M}}W^{\mathrm{n}}|_{x_2=0}\|_{1,\gamma} \le \frac{C}{\gamma}\|\mathbf{M}\|_{W^{2,\infty}(\mathbb{R}^2)}\|W^{\mathrm{n}}|_{x_2=0}\|_0 \le \frac{C}{\gamma}\|W^{\mathrm{n}}|_{x_2=0}\|_0,
$$

where C is some positive constant. Adding all the estimate above, we have

$$
\|\mathcal{B}^\gamma(W^{\mathrm{n}}|_{x_2=0},\psi) - T^\gamma_{\check{b}}\psi - T^\gamma_{\mathbf{M}}W^{\mathrm{n}}|_{x_2=0}\|_{1,\gamma} \le C\left(\|\psi\|_{1,\gamma} + \frac{1}{\gamma}\|W^{\mathrm{n}}|_{x_2=0}\|_0\right). \tag{3.2.13}
$$

Next we consider the interior differential equations.

$$
\begin{aligned}
&\left|\left|\left|\gamma A_0^r W^+ - T^\gamma_{\gamma A_0^r}W^+\right|\right|\right|^2_{1,\gamma} = \int_0^{+\infty}\gamma^2\|A_0^r W^+(\cdot,x_2) - T^\gamma_{\gamma A_0^r}W^+(\cdot,x_2)\|^2_{1,\gamma}dx_2\\
&\le C\|A_0^r\|^2_{W^{2,\infty}(\Omega)}|||W^+|||^2_0 \le C|||W^+|||^2_0.\\
&\left|\left|\left|A_0^r\partial_t W^+ - T^\gamma_{i\delta A_0^r}W^+\right|\right|\right|_{1,\gamma} \le C|||W^+|||_0,\\
&\left|\left|\left|A_1^r\partial_1 W^+ - T^\gamma_{i\eta A_1^r}W^+\right|\right|\right|_{1,\gamma} \le C|||W^+|||_0,\\
&\left|\left|\left|A_0^r C^r W^+ - T^\gamma_{A_0^r C^r}W^+\right|\right|\right|_{1,\gamma} \le C|||W^+|||_0.
\end{aligned}
$$

Again we have

$$\left|\left|\left|\mathcal{L}^{\gamma}_{r,l}W^{\pm}-T^{\gamma}_{\tau A_0^{r,l}+i\eta A_1^{r,l}+A_0^{r,l}C^{r,l}}W^{\pm}-I_2\partial_2 W^{\pm}\right|\right|\right|_{1,\gamma}\leq C|||W^{\pm}|||_0. \tag{3.2.14}$$

The estimates (3.2.13) and (3.2.14) guarantee that if we can estimate the terms on the right hand side, we can just use the para-linearized system in our following proof. We will discuss this in details in the following subsections. Now we derive the specific expression of the para-linearized system. First we need to eliminate the wave front ψ by observing that

$$|b(t,x_1,\delta,\eta,\gamma)|^2\geq c(\gamma^2+\delta^2+\eta^2).$$

So by Gårding's inequality (Theorem 2.2.3), we can obtain

$$\Re\langle T^{\gamma}_{b^*b}\psi,\psi\rangle_{L^2(\mathbb{R}^2)}\geq\frac{c}{2}\|\psi\|^2_{1,\gamma},$$

for all $\gamma\geq\gamma_0$, where γ_0 only depends on K_0. By basic properties of para-differential operators, we have $T^{\gamma}_{b^*b}=(T^{\gamma}_b)^*T^{\gamma}_b+R^{\gamma}_1$ where R^{γ}_1 is an operator of degree 1, which implies

$$\|\psi\|_{1,\gamma}\leq C\|T^{\gamma}_b\psi\|_0,$$

for all $\gamma\geq\gamma_0$. By taking (3.2.13) into account, we have

$$\begin{aligned}
&\|\psi\|_{1,\gamma}\leq C(\|T^{\gamma}_b\psi+T^{\gamma}_{\mathbf{M}}W^{\mathrm{n}}|_{x_2=0}\|_0+\|T^{\gamma}_{\mathbf{M}}W^{\mathrm{n}}|_{x_2=0}\|_0)\\
&\leq C(\frac{1}{\gamma}\|T^{\gamma}_b\psi+T^{\gamma}_{\mathbf{M}}W^{\mathrm{n}}|_{x_2=0}\|_{1,\gamma}+\|W^{\mathrm{n}}|_{x_2=0}\|_0)\\
&\leq C(\frac{1}{\gamma}\|\mathcal{B}^{\gamma}(W^{\mathrm{n}},\psi)-T^{\gamma}_b\psi-T^{\gamma}_{\mathbf{M}}W^{\mathrm{n}}|_{x_2=0}\|_{1,\gamma}+\frac{1}{\gamma}\|\mathcal{B}^{\gamma}(W^{\mathrm{n}},\psi)\|_{1,\gamma}+\|W^{\mathrm{n}}|_{x_2=0}\|_0)\\
&\leq C(\frac{1}{\gamma}\|\mathcal{B}^{\gamma}(W^{\mathrm{n}},\psi)\|_{1,\gamma}+\|W^{\mathrm{n}}|_{x_2=0}\|_0)
\end{aligned}\tag{3.2.15}$$

Then we want to identify the part of the boundary condition where the wave front ψ is not involved. So we consider the following matrix:

$$\Pi(t,x_1,\delta,\eta,\gamma):=$$

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\tau + iv^r\eta & -i\eta(v^r - v^l) & 0 & 0 & 0 & 0 & 0 \\
-(F_{11}^r - F_{11}^l) & 0 & 0 & v^r - v^l & 0 & 0 & 0 \\
-F_{11}^r & 0 & 0 & 0 & v^r - v^l & 0 & 0 \\
-(F_{12}^r - F_{12}^l) & 0 & 0 & 0 & 0 & v^r - v^l & 0 \\
-F_{12}^r & 0 & 0 & 0 & 0 & 0 & v^r - v^l
\end{pmatrix},
$$

for $(\tau, \eta) \in \Sigma$ and is homogeneous of degree 0 with respect to (τ, η). In particular $\Pi b = 0$, and

$$
\Pi \mathbf{M} =
$$

$$
\begin{pmatrix}
\langle \partial_1 \varphi \rangle & \langle \partial_1 \varphi \rangle & 0 & 0 & -\langle \partial_1 \varphi \rangle & -\langle \partial_1 \varphi \rangle & 0 & 0 \\
-\frac{c^r \langle \partial_1 \varphi \rangle^2}{\rho^r}(\tau + iv^l\eta) & \frac{c^r \langle \partial_1 \varphi \rangle^2}{\rho^r}(\tau + iv^l\eta) & 0 & 0 & \frac{c^l \langle \partial_1 \varphi \rangle^2}{\rho^l}(\tau + iv^r\eta) & -\frac{c^l \langle \partial_1 \varphi \rangle^2}{\rho^l}(\tau + iv^r\eta) & 0 & 0 \\
* & * & -a & 0 & * & * & a & 0 \\
* & * & -a & 0 & * & * & 0 & 0 \\
* & * & 0 & -a & * & * & 0 & a \\
* & * & 0 & -a & * & * & 0 & 0
\end{pmatrix},
$$

for $(\tau, \eta) \in \Sigma$ and is homogeneous of degree 0 with respect to (τ, η), where $a = \langle \partial_1 \varphi \rangle^2 (v^r - v^l)$ and $*$ are some nonzero elements whose exact expressions are not important. By the condition in the theorem, we can easily obtain $a \neq 0$. So the last four row in $\Pi \mathbf{M}$ are linearly independent, and we can expect the estimates of $(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}$ in terms of $W^{\mathrm{nc}}|_{x_2=0} := (W_2, W_3, W_9, W_{10})^\top|_{x_2=0}$. We denote the last four rows in Π as Π_4. Then by the exact expression of $\Pi \mathbf{M}$, we have

$$
T^\gamma_{\Pi_4 \mathbf{M}} W^{\mathrm{n}}|_{x_2=0} = T^\gamma_A W^{\mathrm{nc}}|_{x_2=0} + T^\gamma_B (W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}
$$

Where B is an invertible matrix in the whole domain and is homogeneous of degree 0. It is easy to check that

$$
|B(t, x_1)|^2 \geq c,
$$

for some constant $c > 0$. By Gårding's inquality, we have

$$
\Re \langle T^\gamma_{B^*B} (W_5, W_7, W_{12}, W_{14})^\top, (W_5, W_7, W_{12}, W_{14})^\top \rangle|_{x_2=0} \geq \frac{c}{2} \|(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}\|_0^2,
$$

for all $\gamma \geq \gamma_0$. This implies

$$\|(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}\|_0 \leq C\|T_B^\gamma(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}\|_0 \\ \leq C(\|T_{\Pi_4\mathbf{M}}^\gamma W^{\mathrm{n}}|_{x_2=0}\|_0 + \|T_A^\gamma W^{\mathrm{nc}}|_{x_2=0}\|_0).$$

Similarly as the wave front ψ, we have

$$\|T_{\Pi_4\mathbf{M}}^\gamma W^{\mathrm{n}}|_{x_2=0}\|_0 \leq \|T_{\Pi_4}^\gamma \mathcal{B}^\gamma(W^{\mathrm{n}}, \psi) - T_{\Pi_4}^\gamma T_{\mathbf{M}}^\gamma W^{\mathrm{n}}|_{x_2=0} - T_{\Pi_4}^\gamma T_b^\gamma \psi\|_0 \\ + \|T_{\Pi_4}^\gamma \mathcal{B}^\gamma(W^{\mathrm{n}}, \psi)\|_0 + \|W^{\mathrm{n}}|_{x_2=0}\|_{-1,\gamma} + \|\psi\|_0 \\ \leq \frac{1}{\gamma}\|\mathcal{B}^\gamma(W^{\mathrm{n}}, \psi) - T_{\mathbf{M}}^\gamma W^{\mathrm{n}}|_{x_2=0} - T_b^\gamma \psi\|_{1,\gamma} + \frac{1}{\gamma}\|\mathcal{B}^\gamma(W^{\mathrm{n}}, \psi)\|_{1,\gamma} + \|W^{\mathrm{n}}|_{x_2=0}\|_{-1,\gamma} + \|\psi\|_0 \\ \leq \frac{1}{\gamma}\|\psi\|_{1,\gamma} + \frac{1}{\gamma}\|\mathcal{B}^\gamma(W^{\mathrm{n}}, \psi)\|_{1,\gamma} + \frac{1}{\gamma}\|W^{\mathrm{n}}|_{x_2=0}\|_0$$

So by taking γ large enough, we have

$$\|(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}\|_0 \leq C(\frac{1}{\gamma}\|\psi\|_{1,\gamma} + \frac{1}{\gamma}\|\mathcal{B}^\gamma(W^{\mathrm{n}}, \psi)\|_{1,\gamma} + \|W^{\mathrm{nc}}|_{x_2=0}\|_0). \quad (3.2.16)$$

Combining (3.2.15) and (3.2.16) together, we have

$$\|(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}\|_0 + \|\psi\|_{1,\gamma} \leq C(\frac{1}{\gamma}\|\mathcal{B}^\gamma(W^{\mathrm{n}}, \psi)\|_{1,\gamma} + \|W^{\mathrm{nc}}|_{x_2=0}\|_0),$$

which suggests that we can obtain the estimate of $(W_5, W_7, W_{12}, W_{14})^\top|_{x_2=0}$ and ψ through the source terms and $W^{\mathrm{nc}}|_{x_2=0}$. To obtain the estimate $W^{\mathrm{nc}}|_{x_2=0}$, we need to utilize the other part of the boundary conditions which we denote as

$$T_\beta^\gamma W^{\mathrm{nc}}|_{x_2=0} = \tilde{G},$$

where

$$\beta = \begin{pmatrix} \langle\partial_1\varphi\rangle & \langle\partial_1\varphi\rangle & -\langle\partial_1\varphi\rangle & -\langle\partial_1\varphi\rangle \\ -\frac{c^r\langle\partial_1\varphi\rangle^2}{\rho^r}(\tau + iv^l\eta) & \frac{c^r\langle\partial_1\varphi\rangle^2}{\rho^r}(\tau + iv^l\eta) & \frac{c^l\langle\partial_1\varphi\rangle^2}{\rho^l}(\tau + iv^r\eta) & -\frac{c^l\langle\partial_1\varphi\rangle^2}{\rho^l}(\tau + iv^r\eta) \end{pmatrix},$$

for $(\tau, \eta) \in \Sigma$ and is homogeneous of degree 0 with respect to (τ, η). The above matrix β is from the first two rows of $\Pi\mathbf{M}$ and in the symbol class Γ_2^0.

Finally, with the help of the above estimates, we can define the para-linearized system as

$$\begin{cases} T^\gamma_{\tau A_0^r + i\eta A_1^r + A_0^r C^r} W^+ + I_2 \partial_2 W^+ = \tilde{F}^+, \\ T^\gamma_{\tau A_0^l + i\eta A_1^l + A_0^l C^l} W^- + I_2 \partial_2 W^- = \tilde{F}^-, \\ T^\gamma_\beta W^{\rm nc}|_{x_2=0} = \tilde{G}. \end{cases} \tag{3.2.17}$$

Moreover, our object is to derive the following estimate:

$$\|W^{\rm nc}|_{x_2=0}\|_0^2 \le C_0 \left(\frac{1}{\gamma^3} \left|\!\left|\!\left| \tilde{F} \right|\!\right|\!\right|^2_{1,\gamma} + \frac{1}{\gamma^2} \|\tilde{G}\|^2_{1,\gamma} \right), \tag{3.2.18}$$

where $\tilde{F} = (\tilde{F}^+, \tilde{F}^-)^\top$. To illustrate this, we show that (3.2.15), (3.2.16) and (3.2.18) imply Theorem 3.2.1.

First, by the above disscussion, we have

$$\left|\!\left|\!\left| \mathcal{L}^\gamma_{r,l} W^\pm - T^\gamma_{\tau A_0^{r,l} + i\eta A_1^{r,l} + A_0^{r,l} C^{r,l}} W^\pm - I_2 \partial_2 W^\pm \right|\!\right|\!\right|_{1,\gamma} \le C |||W^\pm|||_0,$$

$$\|\mathcal{B}^\gamma(W^{\rm n}, \psi) - T^\gamma_b \psi - T^\gamma_{\mathbf{M}} W^{\rm n}|_{x_2=0}\|_{1,\gamma} \le C \left(\|\psi\|_{1,\gamma} + \frac{1}{\gamma} \|W^{\rm n}|_{x_2=0}\|_0 \right).$$

Then

$$\|\tilde{G}\|_{1,\gamma} = \|T^\gamma_\beta W^{\rm nc}|_{x_2=0}\|_{1,\gamma} = \|T^\gamma_{\Pi_2 \mathbf{M}} W^{\rm nc}|_{x_2=0}\|_{1,\gamma} = \|T^\gamma_{\Pi_2 b} \psi + T^\gamma_{\Pi_2 \mathbf{M}} W^{\rm nc}|_{x_2=0}\|_{1,\gamma},$$

where Π_2 is the first two rows of Π. So we have

$$\begin{aligned} \|\tilde{G}\|_{1,\gamma} &\le \|T^\gamma_{\Pi_2}(T^\gamma_b \psi + T^\gamma_{\mathbf{M}} W^{\rm nc}|_{x_2=0})\|_{1,\gamma} + \|\psi\|_{1,\gamma} + \|W^{\rm nc}|_{x_2=0}\|_0 \\ &\le \|\mathcal{B}^\gamma(W^{\rm n}, \psi) - T^\gamma_b \psi - T^\gamma_{\mathbf{M}} W^{\rm n}|_{x_2=0}\|_{1,\gamma} + \|\mathcal{B}^\gamma(W^{\rm n}, \psi)\|_{1,\gamma} + \|\psi\|_{1,\gamma} + \|W^{\rm nc}|_{x_2=0}\|_0 \\ &\le \|\psi\|_{1,\gamma} + \frac{1}{\gamma} \|W^{\rm n}|_{x_2=0}\|_0 + \|W^{\rm nc}|_{x_2=0}\|_0 + \|\mathcal{B}^\gamma(W^{\rm n}, \psi)\|_{1,\gamma}. \end{aligned}$$

Moreover,

$$\left|\!\left|\!\left| \tilde{F}^\pm \right|\!\right|\!\right|_{1,\gamma} = |||\mathcal{L}^\gamma_{r,l} W^\pm|||_{1,\gamma} + \left|\!\left|\!\left| \tilde{F}^\pm - \mathcal{L}^\gamma_{r,l} W^\pm \right|\!\right|\!\right|_{1,\gamma} \le |||\mathcal{L}^\gamma_{r,l} W^\pm|||_{1,\gamma} + C |||W^\pm|||_0$$

So (3.2.18) implies

$$\|W^{\rm nc}|_{x_2=0}\|_0^2 \le C_0 \left(\frac{1}{\gamma^3} |||\mathcal{L}^\gamma_r W^+|||^2_{1,\gamma} + \frac{1}{\gamma^3} |||W^+|||_0^2 + \frac{1}{\gamma^3} |||\mathcal{L}^\gamma_l W^-|||^2_{1,\gamma} + \frac{1}{\gamma^3} |||W^-|||_0^2 \right.$$

$$+\frac{1}{\gamma^2}\|\mathcal{B}^\gamma(W^{\mathrm{n}},\psi)\|_{1,\gamma}^2+\frac{1}{\gamma^4}\|W^{\mathrm{n}}|_{x_2=0}\|_0^2+\frac{1}{\gamma^2}\|\psi\|_{1,\gamma}^2\Big),$$

Combining with (3.2.15) and (3.2.16), we have

$$\begin{aligned}
&\|W^{\mathrm{n}}|_{x_2=0}\|_0^2+\|\psi\|_{1,\gamma}^2\leq C(\frac{1}{\gamma^3}|||\mathcal{L}_r^\gamma W^+|||_{1,\gamma}^2+\frac{1}{\gamma^3}|||\mathcal{L}_l^\gamma W^-|||_{1,\gamma}^2+\frac{1}{\gamma^2}\|\mathcal{B}^\gamma(W^{\mathrm{n}},\psi)\|_{1,\gamma}^2\\
&+\frac{1}{\gamma^3}|||W^+|||_0^2+\frac{1}{\gamma^3}|||W^-|||_0^2+\frac{1}{\gamma^4}\|W^{\mathrm{n}}|_{x_2=0}\|_0^2+\frac{1}{\gamma^2}\|\psi\|_{1,\gamma}^2),\\
&\leq C(\frac{1}{\gamma^3}|||\mathcal{L}_r^\gamma W^+|||_{1,\gamma}^2+\frac{1}{\gamma^3}|||\mathcal{L}_l^\gamma W^-|||_{1,\gamma}^2+\frac{1}{\gamma^2}\|\mathcal{B}^\gamma(W^{\mathrm{n}},\psi)\|_{1,\gamma}^2+\frac{1}{\gamma^3}|||W^+|||_0^2+\frac{1}{\gamma^3}|||W^-|||_0^2)
\end{aligned}$$

Then by Lemma 3.2.1, we obtain the Theorem 3.2.1. So the only thing left to prove is the estimate (3.2.18) from the para-linearized system (3.2.17).

3.2.3 Microlocalization

From the previous subsection, we obtain a system of ODE in (3.2.17). We note that the para-linearization is an analogue of the Fourier transform in the constant coefficient case. To derive the estimate (3.2.18), we need to investigate at each frequency point $(\tau,\eta)\in\Pi$ if the boundary conditions in (3.2.17) give enough information on the incoming mode of the ODE. As in the constant coefficient case, we want to estimate the rate of vanishing for the Lopatinskii determinant near its roots.

In this subsection, we want to analyze and classify the situations which we will need to deal with in the system (3.2.17) for each frequency point $(\tau,\eta)\in\Pi$. Specifically, we need to identify those frequencies where the standard Kreiss symmetrizer method can not be applied as in [16, 20]. To be specific, those frequency points are where the (3.2.17) can not be reduced to only involving non-characteristic part of the unknown W^{nc} and where the Lopatinskii determinant vanishes. For simplicity in the argument, we only focus on the case on the unit hemisphere $\Sigma=\{(\tau,\eta):\ |\tau|^2+\eta^2=1 \text{ and } \Re\tau\geq 0\}$ in the frequency space Π. Then we can extend the results to the whole frequency space Π by their appropriate homogeneities.

3.2.3.1 Poles In this part, we identify where the (3.2.17) can not be reduced to only involving non-characteristic part of the unknown W^{nc}. As in the constant coefficient case, we call these points the poles of the system. For this purpose, we just formally consider the first order symbols $\tau A_0^r + i\eta A_1^r$ and $\tau A_0^l + i\eta A_1^l$ and boundary symbol β in (3.2.17). In particular, we focus on the following differential system

$$\begin{cases} (\tau A_0^r + i\eta A_1^r)W^+ + I_2\partial_2 W^+ = 0, \\ (\tau A_0^l + i\eta A_1^l)W^- + I_2\partial_2 W^- = 0, \\ \beta W^{\mathrm{nc}}|_{x_2=0} = 0, \end{cases} \tag{3.2.19}$$

where $A_0^{r,l}$, $A_1^{r,l}$, I_2 and β are the matrices we specified in the previous subsection. We note that I_2 is a diagonal matrix with only two elements in diagonal being 1 and others being 0. So there are two differential equations and five algebraic equations for W^+ and W^- respectively. This suggest that our system is characteristic. However, for most of the points (τ, η) in the frequency space, we can use the algebraic equations to reduce the system to only involving differential equations, which is a non-characteristic system. To find out those points, we consider the algebraic equations for W^+. The algebraic equation for W^- can be treated similarly.

$$\begin{pmatrix} \tau + iv^r\eta & \frac{i\eta c^{r\,2}}{\langle\partial_1\Phi^r\rangle\rho^r} & \frac{i\eta c^{r\,2}}{\langle\partial_1\Phi^r\rangle\rho^r} & -i\eta F_{11}^r & 0 & -i\eta F_{12}^r & 0 \\ -iF_{11}^r\eta & 0 & 0 & \tau + iv^r\eta & 0 & 0 & 0 \\ 0 & -\frac{ic^r F_{11}^r\eta}{\rho^r} & \frac{ic^r F_{11}^r\eta}{\rho^r} & 0 & \tau + iv^r\eta & 0 & 0 \\ -iF_{12}^r\eta & 0 & 0 & 0 & 0 & \tau + iv^r\eta & 0 \\ 0 & -\frac{ic^r F_{12}^r\eta}{\rho^r} & \frac{ic^r F_{12}^r\eta}{\rho^r} & 0 & 0 & 0 & \tau + iv^r\eta \end{pmatrix} W^+ = 0.$$

It is obvious that, as in the constant coefficient case, if $(\tau + iv^r\eta)((\tau + iv^r\eta)^2 + \big({F_{11}^r}^2 + {F_{12}^r}^2\big)$ $\eta^2) \neq 0$, we can uniquely determine W_1, W_4, W_5, W_6, W_7 from W_2, W_3. Then by using the differential equations in (3.2.19), we can obtain the differential equations only involving W_2 and W_3, which are

$$\mathbb{A}^r \begin{pmatrix} W_2 \\ W_3 \end{pmatrix} + \partial_2 \begin{pmatrix} W_2 \\ W_3 \end{pmatrix} = 0,$$

where

$$\mathbb{A}^r := \begin{pmatrix} \mu^r & -m^r \\ m^r & -\mu^r \end{pmatrix} + i\frac{\partial_1\Phi^r\partial_2\Phi^r}{\langle\partial_1\Phi^r\rangle^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} \mu^r &= -\frac{\partial_2\Phi^r(\tau+iv^r\eta)}{c^r\langle\partial_1\Phi^r\rangle} - \frac{\partial_2\Phi^r({F_{11}^r}^2+{F_{12}^r}^2)\eta^2}{2\langle\partial_1\Phi^r\rangle c^r(\tau+iv^r\eta)} - \frac{\partial_2\Phi^r c^r(\tau+iv^r\eta)\eta^2}{2\langle\partial_1\Phi^r\rangle^3\left((\tau+iv^r\eta)^2+({F_{11}^r}^2+{F_{12}^r}^2)\eta^2\right)}, \\ m^r &= -\frac{\partial_2\Phi^r({F_{11}^r}^2+{F_{12}^r}^2)\eta^2}{2\langle\partial_1\Phi^r\rangle c^r(\tau+iv^r\eta)} + \frac{\partial_2\Phi^r c^r(\tau+iv^r\eta)\eta^2}{2\langle\partial_1\Phi^r\rangle^3\left((\tau+iv^r\eta)^2+({F_{11}^r}^2+{F_{12}^r}^2)\eta^2\right)}, \end{aligned}$$

Combining the discussion on equations for W^-, we obtain that the points in frequency space where we can not reduce the system into non-characteristic form are the points where $(\tau+iv^{r,l}\eta)\left((\tau+iv^{r,l}\eta)^2+\left({F_{11}^{r,l}}^2+{F_{12}^{r,l}}^2\right)\eta^2\right)=0$. We call this kind of points the poles of the system. We define the sets of poles as

$$\Upsilon_p := \{(t,x_1,x_2,\tau,\eta)\in\mathbb{R}^3_+\times\Sigma : (\tau+iv^{r,l}\eta)\left((\tau+iv^{r,l}\eta)^2+\left({F_{11}^{r,l}}^2+{F_{12}^{r,l}}^2\right)\eta^2\right)=0\}.$$

Here for each fixed $(t,x_1,x_2)\in\Omega$, there are six kinds of frequency in Υ_p. Each one corresponds to the root of one factor in the above definition. For the root of each factor in the definition, fixing $(t,x_1)\in\mathbb{R}^2$, we can view it as a curve in in the frequency space parametrized by x_2 which originates from the boundary $x_2=0$ and propagate into the interior of the domain $x_2>0$.

3.2.3.2 Roots of the Lopatinskii determinant

Now we identify where the Lopatinskii determinant vanishes. First we need to derive the Lopatinskii determinant. This require us to find out the eigenvectors of $\mathbb{A}^{r,l}$ corresponding to the eigenvalue with negative real part. By a direct computation, we denote the eigenvalue of $\mathbb{A}^r$ with negative real part by $\omega^r + i\frac{\partial_1\Phi^r\partial_2\Phi^r}{\langle\partial_1\Phi^r\rangle^2}$ where

$$\omega^{r2} = \mu^{r2} - m^{r2} = \frac{\partial_2\Phi^{r2}}{c^{r2}\langle\partial_1\Phi^r\rangle^4}\left[\langle\partial_1\Phi^r\rangle^2\left((\tau+iv^r\eta)^2 + ({F^r_{11}}^2 + {F^r_{12}}^2)\eta^2\right) + c^{r2}\eta^2\right].$$

The corresponding eigenvector is

$$E^r = \begin{pmatrix} -\alpha^r(\mu^r+\omega^r) \\ -\alpha^r m^r \end{pmatrix}, \tag{3.2.20}$$

for $(\tau,\eta)\in\Sigma$ and is homogeneous of degree 0 with respect to (τ,η), where $\alpha^r = (\tau + iv^r\eta)\left((\tau+iv^r\eta)^2 + ({F^r_{11}}^2 + {F^r_{12}}^2)\eta^2\right)$ on Σ. The case for W^- is similar. We obtain the eigenvalue of negative real part is $\omega^l + i\frac{\partial_1\Phi^l\partial_2\Phi^l}{\langle\partial_1\Phi^l\rangle^2}$ where

$$\omega^{l^2} = \mu^{l^2} - m^{l^2} = \frac{\partial_2\Phi^{l^2}}{c^{l^2}\langle\partial_1\Phi^l\rangle^4}\left[\langle\partial_1\Phi^l\rangle^2\left((\tau+iv^l\eta)^2 + ({F^l_{11}}^2 + {F^l_{12}}^2)\eta^2\right) + c^{l^2}\eta^2\right].$$

The corresponding eigenvector is

$$E^l = \begin{pmatrix} \alpha^l m^l \\ \alpha^l(\mu^l-\omega^l) \end{pmatrix}, \tag{3.2.21}$$

for $(\tau,\eta)\in\Sigma$ and is homogeneous of degree 0 with respect to (τ,η), where $\alpha^l = (\tau + iv^l\eta)\left((\tau+iv^l\eta)^2 + ({F^l_{11}}^2 + {F^l_{12}}^2)\eta^2\right)$ on Σ. We note that by a similar computation in the constant coefficient cases, the above eigenvalues and eigenvectors are all well-defined and smooth on the whole space $\mathbb{R}^3\times\Pi$. So the Lopantiskii determinant is well-defined for all the points in frequency space, which is

$$\begin{aligned}\det\left(\beta\begin{pmatrix} E^r & 0 \\ 0 & E^l\end{pmatrix}\right)\Bigg|_{x_2=0} &= \frac{c^4a_1^2}{\rho}(\tau+iv^r\eta)(\tau+iv^l\eta)\left(\frac{a_1^4}{a_2^ra_2^l}\omega^r\omega^l+\eta^2\right)\left(\frac{\omega^r}{a_2^r}-\frac{\omega^l}{a_2^l}\right)\\ &\times\left(\frac{a_2^r}{a_1c}\left((\tau+iv^r\eta)^2+({F^r_{11}}^2+{F^r_{12}}^2)\eta^2\right)-(\tau+iv^r\eta)\omega^r\right)\\ &\times\left(\frac{a_2^l}{a_1c}\left((\tau+iv^l\eta)^2+({F^l_{11}}^2+{F^l_{12}}^2)\eta^2\right)+(\tau+iv^l\eta)\omega^l\right),\end{aligned} \tag{3.2.22}$$

for $(\tau,\eta)\in\Sigma$ and is homogeneous of degree 0 with respect to (τ,η), where $a_1=\langle\partial_1\varphi\rangle$, $a_2^{r,l}=\partial_2\Phi^{r,l}|_{x_2=0}$, $c=c^r|_{x_2=0}=c^l|_{x_2=0}$. The last equality for c is from the fact that $c^{r,l}$ only depends on the density $\rho^{r,l}$ which is continuous at $x_2=0$ by (3.2.3).

By analyzing each factor in Lopatinskii determinant we can easily obtain that the last two factors in (3.2.22) never equal zeros, and the first two factors corresponds to the roots $\tau=-iv^{r,l}\eta$ respectively. Thus we only need to discuss the third and forth factors.

We note that all the coefficient in the factors of the Lopatinskii determinant are coninuous with respect to the background state $U|_{x_2=0}:=(U^r|_{x_2=0},U^l|_{x_2=0})$ and $\Phi|_{x_2=0}:=(\Phi^r|_{x_2=0},\Phi^l|_{x_2=0})$, and those factors reduce to the corresponding factors in the constant coefficient case, see (3.1.46), if the perturbation in (3.2.1) is zero. By assuming K is sufficient small and the continuity argument, we obtain the number of the roots in the third and forth factors in (3.2.22) are the same as the number of roots in corresponding factors in the constant coefficient case. Hence there are two roots of $\frac{a_1^4}{a_2^r a_2^l}\omega^r\omega^l+\eta^2$ which we denote by $\tau=iV_1\eta$ and $\tau=iV_2\eta$, and there is one root of $\frac{\omega^r}{a_2^r}-\frac{\omega^l}{a_2^l}$, which we denote as $\tau=iV_3\eta$, if $\bar{v}^2>2c(\bar{\rho})^2+\bar{F}_{11}^2+\bar{F}_{12}^2$.

We can verify that $V_1(U|_{x_2=0},\nabla\Phi|_{x_2=0})$, $V_2(U|_{x_2=0},\nabla\Phi|_{x_2=0})$ and $V_3(U|_{x_2=0},\nabla\Phi|_{x_2=0})$ are all real and continuously depends on the background state $U|_{x_2=0}$ and $\nabla\Phi|_{x_2=0}$. Further, all the roots above are simple if they don't coincide. So the set of roots of Lopatinskii determinant can be represented by the following set on the boundary of domian:

$$\Upsilon_r^0:=\{(t,x_1,\tau,\eta)\in\mathbb{R}^2\times\Sigma:\Re\tau=0 \text{ and } \sigma=0\},\tag{3.2.23}$$

where $\sigma=(\delta+v^r|_{x_2=0}\eta)(\delta+v^l|_{x_2=0}\eta)(\delta-V_1\eta)(\delta-V_2\eta)(\delta-V_3\eta)$ if $\bar{v}^2>2c(\bar{\rho})^2+\bar{F}_{11}^2+\bar{F}_{12}^2$ and $\sigma=(\delta+v^r|_{x_2=0}\eta)(\delta+v^l|_{x_2=0}\eta)(\delta-V_1\eta)(\delta-V_2\eta)$ if $\bar{v}^2\le\bar{F}_{11}^2+\bar{F}_{12}^2$ on Σ. Here the set Υ_r^0 and function σ can be naturally extended into the interior of physical domain where $x_2>0$. More precisely, we can define the coefficients in σ for $x_2>0$ by the continuous dependency of V_1, V_2 and V_3 on the background state U and $\nabla\Phi$. We denote the extended set by Υ_r

$$\Upsilon_r:=\{(t,x_1,x_2,\tau,\eta)\in\mathbb{R}_+^3\times\Sigma:\Re\tau=0 \text{ and } \sigma=0\}.$$

Just like Υ_p, for fixed $(t,x_1)\in\mathbb{R}^2$, the root of each factor of σ can be considered as a curve in frequency space Σ parametrized by x_2, which originates from the boundary $x_2=0$ and propagates into the interior of the domain $x_2>0$.

Remark 3.2.2. In fact, there are one more kind of points we need to pay attention due to the restraint of para-differential calculus. Those are the points where the eigenvalues ω^r and ω^l vanish. When we deal with these points in deriving the energy estimates, we can not perform the upper triangularization method for poles and roots of Lopatinskii determinant. Because the upper triangularization method will introduce ω^r and ω^l as the symbol. But ω^r and ω^l do not satisfy the requirements in the Definition 2.2.1 of the symbol class near the points where $\omega^r = 0$ and $\omega^l = 0$ respectively. Hence we require the roots of the eigenvalues ω^r and ω^l do not coincide with poles and roots of Lopatinskii determinant. Then we can perform the classical Kreiss symmetrizer method to deal with the zeros of the eigenvalues, which is the same way as in dealing with the points other than poles and roots of Lopatinskii determinant.

In general from the definition of Υ_p, Υ_r and $\omega^{r,l}$, the poles, the roots of Lopatinskii determinant and the zeros of eigenvalues $\omega^{r,l}$ may intersect with each other, which may cause extreme difficulties in deriving the energy estimates through the symbolic cut-off functions in the frequency space. Thus we prescribe conditions on the variable background states (3.2.1) which guarantee that all these three types of points do not intersect with each other in the whole domain $\mathbb{R}^3_+ \times \Sigma$. By the continuity of these three kinds of points with respect to the background states, we only need to require that the poles, the roots of Lopatinskii determinant and the zeros of eigenvalues $\omega^{r,l}$ for the constant background states $\bar{U}$ in (3.2.1) do not coincide with each other and K in (3.2.2) is small enough. Plus the requirement on the constant background states such that the stability result only allows one loss of derivative, we require $\bar{U}$ of (3.2.1) as in Theorem 3.2.1 to obtain the stability results. So we have the following frequencies (τ, η) on Σ such that the standard Kreiss symmetrizer method can not be applied.

$$
\begin{aligned}
&\text{Case 1:} \quad \tau = -iv^{r,l}\eta,\\
&\text{Case 2:} \quad \tau = -i\left(v^{r,l} \pm \sqrt{F_{11}^{r,l^2} + F_{12}^{r,l^2}}\right)\eta,\\
&\text{Case 3:} \quad \tau = iV_1\eta, \quad \tau = iV_2\eta \text{ and}\\
&\qquad\qquad \tau = iV_3\eta \quad \text{if } \bar{v}^2 > 2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2.
\end{aligned}
$$

Here the first case corresponds to the frequencies which are both poles of the system and roots of Lopatinskii determinant. The second case is only the poles of the system, and the third one is only the roots of Lopatinskii determinant. For the rest points in frequency space, we can always treat them by standard Kreiss symmetrizer method. This is our fourth case.

3.2.4 Estimate in each case

Now we want to derive the estimate for each case and eventually obtain the expected estimate from the para-linearized system. From the discuss above, we have three isolated cases where we can not apply Kreiss symmeterizer method. Each relation of τ and η in one case corresponds to a curve on Σ with fixed (t, x_1). In previous section, we explained that we only focus on the situation where all these curves do not intersect with others. So for each curve, we can construct an open neighborhood around it. Up to shrink those open neighborhood, we can guarantee that those neighborhood do no intersect with others and all do not contain any point such that $\omega^r = 0$ or $\omega^l = 0$. We denote, on $\mathbb{R}^3 \times \Sigma$, the open neighborhood around $\tau = -iv^r\eta$ by $\mathcal{V}^r_{p_1}$, around $\tau = -iv^l\eta$ by $\mathcal{V}^l_{p_1}$, around $\tau = -i\left(v^r + \sqrt{{F^r_{11}}^2 + {F^r_{12}}^2}\right)\eta$ by $\mathcal{V}^1_{p_2}$, around $\tau = -i\left(v^r - \sqrt{{F^r_{11}}^2 + {F^r_{12}}^2}\right)\eta$ by $\mathcal{V}^2_{p_2}$, around $\tau = -i\left(v^l + \sqrt{{F^l_{11}}^2 + {F^l_{12}}^2}\right)\eta$ by $\mathcal{V}^3_{p_2}$, around $\tau = -i\left(v^l - \sqrt{{F^l_{11}}^2 + {F^l_{12}}^2}\right)\eta$ by $\mathcal{V}^4_{p_2}$, around $\tau = iV_1\eta$ by $\mathcal{V}^1_r$, around $\tau = iV_2\eta$ by $\mathcal{V}^2_r$ and around $\tau = iV_3\eta$ by $\mathcal{V}^3_r$.

In the following argument we always consider $\tau = iV_3\eta$ is a root of Lopatinskii determinant. For the background state (3.2.1) satisfying $\bar{v}^2 \leq \bar{F}^2_{11} + \bar{F}^2_{12}$, we can just drop the part of argument about $\tau = iV_3\eta$.

3.2.4.1 Case 1–poles and roots In this case, we consider the kind of frequencies which are both poles and roots of Lopatinskii determinant. More precisely, we consider the frequencies in $\mathcal{V}^r_{p_1}$ and $\mathcal{V}^l_{p_1}$. Without loss of generality, we just take $\mathcal{V}^r_{p_1}$ as an example. The other can be treat exactly the same.

We note that $\mathcal{V}^r_{p_1}$ only contains the poles of the equations for W^+ in (3.2.17), but not contains the poles of the equations for W^- in (3.2.17). So the estimates of W^- can be obtained in the exact the same way as the estimate of W^+ in the Case 2. In this part we only details how to derive the estimate for W^+.

To isolate the objective frequency, we introduce the smooth cut-off function χ_{p_1} whose range is the closed interval $[0,1]$. In particular, on $\mathbb{R}^3_+ \times \Sigma$, support of χ_{p_1} contained in $\mathcal{V}^r_{p_1}$ and equals 1 on a smaller neighborhood of the curve satisfying $\tau = -iv^r\eta$. Then we extend χ_{p_1} by homogeneity of degree 0 with respect to (τ, η) into the whole space $\mathbb{R}^3_+ \times \Pi$. As in [20], χ_{p_1} is in the symbol class Γ^0_k for any integer k. With this cut-off function, we define

$$W^+_{p_1} := T^\gamma_{\chi_{p_1}} W^+.$$

From (3.2.17), we can obtain

$$I_2\partial_2 W^+_{p_1} = I_2 T^\gamma_{\partial_2\chi_{p_1}} W^+ + T^\gamma_{\chi_{p_1}} F^+ - T^\gamma_{\chi_{p_1}} T^\gamma_{\tau A^r_0 + i\eta A^r_1 + A^r_0 C^r} W^+.$$

By para-differential calculus, we have

$$T^\gamma_{\mathcal{A}^r} W^+_{p_1} + T^\gamma_{A^r_0 C^r} W^+_{p_1} + T^\gamma_r W^+ + I_2\partial_2 W^+_{p_1} = T^\gamma_{\chi_{p_1}} F + R_{-1} W^+, \tag{3.2.24}$$

where we denote $\mathcal{A}^r := \tau A^r_0 + i\eta A^r_1$ and r is a symbol in the class Γ^0_1 whose support is in the area where $\chi_{p_1} \in (0,1)$. To estimate $W^+_{p_1}$, we need to consider the expression of the differential equations in $\mathcal{V}^r_{p_1}$. Moreover, we recall that ω^r is not in the symbol class Γ^1_2 because the homogeneity of ω^r near the point where $\omega^r = 0$ does not satisfies the definition of this symbol class. To get around this we consider two more cut-off functions χ_1 and χ_2 in Γ^0_1, such that both of their support are in

$$\mathcal{V}^r_{p_1} \cdot \mathbb{R}_+ := \{(t, x_1, x_2, \tau, \eta) \in \Omega \times \Pi : (t, x_1, x_2, \tau/\sqrt{|\tau|^2 + \eta^2}, \eta/\sqrt{|\tau|^2 + \eta^2}) \in \mathcal{V}^r_{p_1}\},$$

$\chi_1 = 1$ on the support of χ_{p_1} and $\chi_2 = 1$ on the support of χ_1. Now we multiply the symbol χ_2 to the equation and obtain

$$T^\gamma_{\chi_2\mathcal{A}^r} W^+_{p_1} + T^\gamma_{\chi_2 A^r_0 C^r} W^+_{p_1} + T^\gamma_r W^+ + I_2\partial_2 W^+_{p_1} = R_0 F + R_{-1} W^+. \tag{3.2.25}$$

Here the support of the first order symbol $\chi_2\mathcal{A}^r$ are contained in the support of χ_2 which is a subset of $\mathcal{V}^r_{p_1} \cdot \mathbb{R}_+$. So we actually exclude the part of the frequency where ω^r vanishes. Now we can construct the transformation matrix to upper triangularize the first order symbol $\chi_2\mathcal{A}^r$ on the support of χ_2. With the inspiration from the eigenvector E^r of $\mathbb{A}^r$ in previous subsection, we consider transformation matrix Q_0^r which is homogeneous of degree 0 with respect to (τ, η) and takes the following form on $\mathcal{V}_{p_1}^r$

$$Q_0^r := \begin{pmatrix} 1 & \widehat{W}_1^r & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha^r(\mu^r + \omega^r) & U_2^r & 0 & 0 & 0 & 0 \\ 0 & -\alpha^r m^r & U_3^r & 0 & 0 & 0 & 0 \\ 0 & \widehat{W}_4^r & 0 & 1 & 0 & 0 & 0 \\ 0 & \widehat{W}_5^r & 0 & 0 & 1 & 0 & 0 \\ 0 & \widehat{W}_6^r & 0 & 0 & 0 & 1 & 0 \\ 0 & \widehat{W}_7^r & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the second and third rows of the second column is from the eigenvector E^r and the choice of $\widehat{W}_i^r$ for $i = 1, 4, 5, 6, 7$ is to garauntee that the second column of $\mathcal{A}^r Q_0^r$ to be zeros expect for the second and third rows, that is

$$\begin{pmatrix} \tau + iv^r\eta & \frac{i\eta c^{r2}}{\langle \partial_1 \Phi^r \rangle \rho} & \frac{i\eta c^{r2}}{\langle \partial_1 \Phi^r \rangle \rho} & -i\eta F_{11}^r & 0 & -i\eta F_{12}^r & 0 \\ -iF_{11}^r\eta & 0 & 0 & \tau + iv^r\eta & 0 & 0 & 0 \\ 0 & -\frac{ic^r F_{11}^r \eta}{\rho^r} & \frac{ic^r F_{11}^r \eta}{\rho^r} & 0 & \tau + iv^r\eta & 0 & 0 \\ -iF_{12}^r\eta & 0 & 0 & 0 & 0 & \tau + iv^r\eta & 0 \\ 0 & -\frac{ic^r F_{12}^r \eta}{\rho^r} & \frac{ic^r F_{12}^r \eta}{\rho^r} & 0 & 0 & 0 & \tau + iv^r\eta \end{pmatrix} \begin{pmatrix} \widehat{W}_1^r \\ -\alpha^r(\mu^r + \omega^r) \\ -\alpha^r m^r \\ \widehat{W}_4^r \\ \widehat{W}_5^r \\ \widehat{W}_6^r \\ \widehat{W}_7^r \end{pmatrix} = 0.$$

Note that we can solve the above equations for $\widehat{W}_i^r$ with $i = 1, 4, 5, 6, 7$ at all points in $\mathbb{R}_+^3 \times \Pi$. Because all the terms in each equation above have factors of $\tau + iv^r\eta$ or $(\tau + iv^r\eta)^2 + ({F_{11}^r}^2 + {F_{12}^r}^2)\eta^2$. Then by cancelling the common factors, we can obtain a linear algebraic system which never degenerated at any point in $\mathbb{R}_+^3 \times \Pi$. However $\widehat{W}_i^r$ defined in this way is not homogeneous of degree 0, because ω^r degenerates near its zeros. We will use $\chi_1 Q_0^r$ in following operations of the para-differential calculus, to exclude the frequencies where ω^r degenerates. Moreover U_2 and U_3 are free to choose, given the matrix Q_0^r invertible. Here to make the following argument simpler, we take $U_2 = 1$ and $U_3 = 0$ in this case. Thus $\chi_1 Q_0^r$ is in the symbol class Γ_2^0.

To successively upper triangularize the first order operator A^r in $\mathcal{V}^r_{p_1}\cdot\mathbb{R}_+$, we construct the symmetrizer R^r_0 which is homogeneous of degree 0 and take the following form on $\mathcal{V}^r_{p_1}$

$$R^r_0=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{\xi} & 0 & 0 & 0 & 0\\ \bar{W}^r_1 & \frac{\alpha^r m^r}{\xi} & -\frac{\alpha^r(\mu^r+\omega^r)}{\xi} & \bar{W}^r_4 & \bar{W}^r_5 & \bar{W}^r_6 & \bar{W}^r_7\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where ξ equals the determinant of Q^r_0. $\bar{W}^r_1$, $\bar{W}^r_4$, $\bar{W}^r_5$, $\bar{W}^r_6$ and $\bar{W}^r_7$ is chosen to be homogeneous of degree 0 and make third row of $R^r_0\mathcal{A}^rQ^r_0$ to be zero except the second and third columns in this row, that is

$$\begin{pmatrix} \bar{W}^r_1\\ \frac{\alpha^r m^r}{\xi}\\ -\frac{\alpha^r(\mu^r+\omega^r)}{\xi}\\ \bar{W}^r_4\\ \bar{W}^r_5\\ \bar{W}^r_6\\ \bar{W}^r_7 \end{pmatrix}^{\top}\begin{pmatrix} \tau+iv^r\eta & -i\eta F^r_{11} & 0 & -i\eta F^r_{12} & 0\\ \frac{i\eta\partial_2\Phi^r\rho^r}{2c^r\langle\partial_1\Phi^r\rangle^2} & 0 & -\frac{i\eta\partial_2\Phi^r\rho^rF^r_{11}}{2c^{r2}\langle\partial_1\Phi^r\rangle} & 0 & -\frac{i\eta\partial_2\Phi^r\rho^rF^r_{12}}{2c^{r2}\langle\partial_1\Phi^r\rangle}\\ -\frac{i\eta\partial_2\Phi^r\rho^r}{2c^r\langle\partial_1\Phi^r\rangle^2} & 0 & -\frac{i\eta\partial_2\Phi^r\rho^rF^r_{11}}{2c^{r2}\langle\partial_1\Phi^r\rangle} & 0 & -\frac{i\eta\partial_2\Phi^r\rho^rF^r_{12}}{2c^{r2}\langle\partial_1\Phi^r\rangle}\\ -iF^r_{11}\eta & \tau+iv^r\eta & 0 & 0 & 0\\ 0 & 0 & \tau+iv^r\eta & 0 & 0\\ -iF^r_{12}\eta & 0 & 0 & \tau+iv^r\eta & 0\\ 0 & 0 & 0 & 0 & \tau+iv^r\eta \end{pmatrix}=0.$$

Similarly as in $\chi_1Q^r_0$, $\chi_1R^r_0$ can be easily verified that it is in the symbol class Γ^0_2. With the above choice of Q^r_0 and R^r_0, we can obtain our upper triangularized first order symbol

$$\widetilde{A}^r:=R^r_0\mathcal{A}^rQ^r_0=$$
$$\begin{pmatrix} \tau+iv^r\eta & 0 & \Theta_1 & -i\eta F^r_{11} & 0 & -i\eta F^r_{12} & 0\\ \Theta_1 & -\omega^r-\frac{i\partial_2\Phi^r\partial_1\Phi^r\eta}{\langle\partial_1\Phi^r\rangle^2} & 0 & 0 & \Theta_1 & 0 & \Theta_1\\ 0 & 0 & \omega^r-\frac{i\partial_2\Phi^r\partial_1\Phi^r\eta}{\langle\partial_1\Phi^r\rangle^2} & 0 & 0 & 0 & 0\\ -i\eta F^r_{11} & 0 & 0 & \tau+iv^r\eta & 0 & 0 & 0\\ 0 & 0 & \Theta_1 & 0 & \tau+iv^r\eta & 0 & 0\\ -i\eta F^r_{12} & 0 & 0 & 0 & 0 & \tau+iv^r\eta & 0\\ 0 & 0 & \Theta_1 & 0 & 0 & 0 & \tau+iv^r\eta \end{pmatrix},$$

where Θ_1 is some symbol in the class of Γ^1_2, whose detailed expression are not important.

Before we apply the transformation matrix Q^r_0 and symmetrizer R^r_0 to (3.2.17), It is necessary to construct the matrix symbol Q^r_{-1} and R^r_{-1} in Γ^{-1}_1 to decouple the incoming modes and outgoing mode in zero order terms. To be specific, we obtain the following lemma

Lemma 3.2.2. *With appropriate choice of Q^r_{-1} and R^r_{-1} in Γ^{-1}_1, there is a symbol $D_0 = (d_{i,j})_{7\times 7}$ in Γ^0_1 satisfying $d_{2,3} = d_{3,2} = 0$ and*

$$R^r_{-1}R^{r-1}_0\widetilde{A}^r - \widetilde{A}^r Q^r_{-1}Q^r_0 + (\partial_2 Q^{r-1}_0 - R^r A^r_0 C^r - [R^r, \chi_2\mathcal{A}^r] + [\chi_2\mathcal{A}^r, Q^r])Q^r_0 - D_0,$$

is a symbol in homogeneous of degree -1 and regularity 1 on $\chi_2 = 1$. Moreover

$$R^r_{-1}I_2 = I_2 Q^r_{-1},$$

where $[\cdot,\cdot]$ above is defined as

$$[A, B] := \frac{1}{i}\left(\frac{\partial A}{\partial \delta}\frac{\partial B}{\partial t} + \frac{\partial A}{\partial \eta}\frac{\partial B}{\partial x_1}\right),$$

for any symbols A and B.

The proof of this Lemma is in the same spirit as the Lemma 5.5 in [20], we just omit the proof here.

Remark 3.2.3. Actually the above lemma is also true for the neighborhoods $\mathcal{V}^l_{p_1}$, $\mathcal{V}^i_r$, $\mathcal{V}^j_{p_2}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, if we construct transformation matrices Q^r_0, Q^r_{-1} and symmetrizers R^r_0, R^r_{-1} in the same way as above.

Then we consider the new unknown

$$Z^+ = T^\gamma_{\chi_1(Q^{r-1}_0 + Q^r_{-1})} W^+_{p_1},$$

and denote $Q^r = Q^{r-1}_0 + Q^r_{-1}$ and $R^r = R^r_0 + R^r_{-1}$. Then we can obtain

$$\begin{aligned} I_2\partial_2 Z^+ &= I_2 T^\gamma_{(\partial_2\chi_1)Q^r} W^+_{p_1} + I_2 T^\gamma_{\chi_1\partial_2 Q^r} W^+_{p_1} + I_2 T^\gamma_{\chi_1 Q^r}\partial_2 W^+_{p_1} \\ &= I_2 T^\gamma_{(\partial_2\chi_1)Q^r} W^+_{p_1} + I_2 T^\gamma_{\chi_1\partial_2 Q^r} W^+_{p_1} + T^\gamma_{\chi_1 R^r} I_2\partial_2 W^+_{p_1} \end{aligned}$$

In the last expression, we know $\partial_2\chi_1$ only supports in the place where $\chi_1 \in (0, 1)$ and by definition this place is disjoint with the support χ_{p_1}. By the asymptotic expansion of the symbols, we obtain that

$$T^\gamma_{(\partial_2\chi_1)Q} W^+_{p_1} = R_{-1}W^+.$$

Recalling (3.2.25) and utilizing the property of para-differential calculus, we can obtain

$$\begin{aligned} I_2\partial_2 Z^+ &= I_2 T^\gamma_{\chi_1\partial_2 Q_0^{r-1}} W^+_{p_1} - T^\gamma_{\chi_1 R^r \mathcal{A}^r} W^+_{p_1} - T^\gamma_{[\chi_1 R^r, \chi_2\mathcal{A}^r]} W^+_{p_1} - T^\gamma_{\chi_1 R^r A_0^r C^r} W^+_{p_1} \\ &\qquad + T^\gamma_r W^+ + R_0 F + R_{-1} W^+ \\ &= I_2 T^\gamma_{\chi_1\partial_2 Q_0^{r-1}} W^+_{p_1} - T^\gamma_{\chi_1 R^r \mathcal{A}^r} W^+_{p_1} + T^\gamma_{\chi_2 \widetilde{A}^r} T^\gamma_{\chi_1 Q^r} W^+_{p_1} - T^\gamma_{\chi_2 \widetilde{A}^r} T^\gamma_{\chi_1 Q^r} W^+_{p_1} - T^\gamma_{[\chi_1 R^r, \chi_2\mathcal{A}^r]} W^+_{p_1} \\ &\qquad - T^\gamma_{\chi_1 R^r A_0^r C^r} W^+_{p_1} + T^\gamma_r W^+ + R_0 F + R_{-1} W^+ \\ &= -T^\gamma_{\chi_1(R_{-1}R_0{}^{-1}\widetilde{A}^r - \widetilde{A}^r Q_{-1} Q_0)} Z^+ - T^\gamma_{\chi_2\widetilde{A}^r} Z^+ + I_2 T^\gamma_{\chi_1\partial_2 Q_0^{r-1}} W^+_{p_1} - T^\gamma_{\chi_1 R^r A_0^r C^r} W^+_{p_1} \\ &\qquad - T^\gamma_{[\chi_1 R^r, \chi_2\mathcal{A}^r] - [\chi_2\widetilde{A}^r, \chi_1 Q^r]} W^+_{p_1} + T^\gamma_r W^+ + R_0 F + R_{-1} W^+ \end{aligned}$$

Then by the asymptotic expansion and the support of each cut-off function we can obtain

$$\begin{aligned} I_2\partial_2 Z^+ &= -T^\gamma_{\chi_1(R_{-1}R_0{}^{-1}\widetilde{A}^r - \widetilde{A}^r Q_{-1} Q_0)} Z^+ - T^\gamma_{\chi_2\widetilde{A}^r} Z^+ + I_2 T^\gamma_{\chi_1\partial_2 Q_0^{r-1} Q_0^r} Z^+ - T^\gamma_{\chi_1 R^r A_0^r C^r Q_0^r} Z^+ \\ &\qquad - T^\gamma_{\chi_1([R^r, \chi_2\mathcal{A}^r] - [\chi_2\widetilde{A}^r, Q^r]) Q_0^r} Z^+ + T^\gamma_r W^+ + R_0 F + R_{-1} W^+ \end{aligned}$$

Then by using Lemma 3.2.2, we can obtain

$$I_2\partial_2 Z^+ = -T^\gamma_{\chi_2\widetilde{A}^r} Z^+ + T^\gamma_{D_0} Z^+ + T^\gamma_r W^+ + R_0 F + R_{-1} W^+. \tag{3.2.26}$$

By definition of Z^+, it is obvious that the support of the Fourier transform of Z^+ is in the support of χ_{p_1}. So the above equation can be rewrite into

$$I_2\partial_2 Z^+ = -T^\gamma_{\widetilde{D_1}} Z^+ + T^\gamma_{\widetilde{D_0}} Z^+ + T^\gamma_r W^+ + R_0 F + R_{-1} W^+, \tag{3.2.27}$$

where $\widetilde{D_1}$ is the same as $\widetilde{A}^r$ except replacing ω^r in each element by $\tilde{\omega}^r$, where $\tilde{\omega}^r$ is a symbol of degree 1 with regularity 2 which equals ω^r on the support of χ_2, and $\widetilde{D_0}$ is an extension of D_0 with $d_{2,3} = d_{3,2} = 0$ in the whole space. Moreover, by a direct computation, we can easily obtain that $\omega^r \geq c\Lambda := c\sqrt{\gamma^2 + \delta^2 + \eta^2}$ in $\mathcal{V}^r_{p_1}$. This suggests that we can choose the extension $\tilde{\omega}^r$ such that $\tilde{\omega}^r \geq c\Lambda$ in the whole space. In the following context, for simplicity of the expression, we will drop the tilde on $\tilde{\omega}^r$.

Then we are going to apply the appropriate symmetrizer to the (3.2.27) and obtain the estimate. By the definition of $\widetilde{D_1}$, we note $\widetilde{D_1}$ will degenerate near the pole we look into in this case. It weaken the estimate in characteristic part of the unknown, because at the pole the leading terms in homogeneity is of degree 0 and we can not neglect the effect of the zero order term. To overcome this difficulty, we consider different symmetrizers for each equation in the system.

We denote $Z^+ = (Z_1, Z_2, ..., Z_7)^\top$ and start from the third equation in (3.2.27), which corresponding to the outgoing mode of the system,

$$\partial_2 Z_3 = T^\gamma_{-\omega^r + i\varpi^r} Z_3 + T^\gamma_{\Theta_0} Z_3 + \Sigma_{i\neq 2,3} T^\gamma_{\Theta_0} Z_i + T^\gamma_r W^+ + R_0 F + R_{-1} W^+,$$

where $\varpi^r = \frac{\partial_2\Phi^r\partial_1\Phi^r\eta}{\langle\partial_1\Phi^r\rangle^2}$. For this equation, we consider the following two operators $(T^\gamma_\sigma)^* T^\gamma_\Lambda T^\gamma_\sigma$ and $(T^\gamma_\Lambda)^* T^\gamma_\Lambda$ as symmetrizer, where σ is defined on $\mathbb{R}^3_+ \times \Sigma$ by (3.2.23) and extend the domain to the whole frequency space $\mathbb{R}^3_+ \times \Pi$ by homogeneity of degree 1. Thus σ is in the symbol class Γ^1_2. We obtain

$$\begin{aligned}
&\Re\langle T^\gamma_\sigma \partial_2 Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle = \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma \partial_2 Z_3\rangle = \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_{-\omega^r+i\varpi^r} Z_3\rangle \\
&+ \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_{\Theta_0} Z_3\rangle + \Sigma_{i\neq 2,3}\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_{\Theta_0} Z_i\rangle \\
&+ \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_r W^+\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma R_{-1} W^+\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma F\rangle
\end{aligned} \tag{3.2.28}$$

For the first term on the right, by the para-differential calculus, we have

$$\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_{-\omega^r+i\varpi^r} Z_3\rangle = \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_{\frac{-\omega^r+i\varpi^r}{\Lambda}} T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, R_1 Z_3\rangle$$

By the way extending the ω^r, we have

$$\Re\frac{-\omega^r + i\varpi^r}{\Lambda} \geq c$$

for some positive c only depends on the background state. By Gårding's inequality, we have

$$\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_{\frac{-\omega^r+i\varpi^r}{\Lambda}} T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle \geq c\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|^2_0 = c\|T^\gamma_\sigma Z_3\|^2_{1,\gamma}$$

For the other terms on the right handside of (3.2.28), we have

$$\begin{aligned}
&\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, R_1 Z_3\rangle \leq \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|^2_0 + \frac{1}{\varepsilon}\|Z_3\|^2_{1,\gamma} \\
&\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_{\Theta_0} Z_3\rangle \leq \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|^2_0 + \frac{1}{\varepsilon}\|Z_3\|^2_{1,\gamma} \\
&\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_{\Theta_0} Z_i\rangle = \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_{\Theta_0} T^\gamma_\sigma Z_i\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, R_0 Z_i\rangle
\end{aligned}$$

$$\le \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon}\|T^\gamma_\sigma Z_i\|_0^2 + \frac{1}{\varepsilon}\|Z_i\|_0^2$$

$$\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma T^\gamma_r W^+\rangle \le \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon}\|T^\gamma_r W^+\|_{1,\gamma}^2$$

$$\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma R_{-1} W^+\rangle \le \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon}\|W^+\|_0^2$$

$$\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma F\rangle \le \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon}\|F\|_{1,\gamma}^2,$$

where ε is a positive number small enough. To achieve the estimate, we need to deal with the term with x_2 derivative. We note

$$\begin{aligned}\partial_2\Re\langle Z_3, (T^\gamma_\sigma)^* T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle &= \Re\partial_2\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle = \Re\langle T^\gamma_{\partial_2\sigma} Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle \\ &+ \Re\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_{\partial_2\sigma} Z_3\rangle + \Re\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_\sigma \partial_2 Z_3\rangle + \Re\langle T^\gamma_\sigma \partial_2 Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle.\end{aligned} \tag{3.2.29}$$

The last term on the right is what we want to estimate. For first three terms on the right handside of (3.2.29), we have

$$\Re\langle T^\gamma_{\partial_2\sigma} Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle \le \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon}\|Z_3\|_{1,\gamma}^2$$

$$\Re\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_{\partial_2\sigma} Z_3\rangle \le \varepsilon\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon}\|Z_3\|_{1,\gamma}^2$$

$$\Re\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_\sigma \partial_2 Z_3\rangle = \Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma \partial_2 Z_3\rangle + \Re\langle T^\gamma_\sigma Z_3, R_0 T^\gamma_\sigma \partial_2 Z_3\rangle.$$

For $\Re\langle T^\gamma_\sigma Z_3, R_0 T^\gamma_\sigma \partial_2 Z_3\rangle$ above, we can deal it in a similar way as $\Re\langle T^\gamma_\Lambda T^\gamma_\sigma Z_3, T^\gamma_\sigma \partial_2 Z_3\rangle$. Adding (3.2.28) and (3.2.29) and integrating with respect to x_2, we can obtain

$$\begin{aligned}|||T^\gamma_\Lambda T^\gamma_\sigma Z_3|||_0^2 + \Re\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle|_{x_2=0} &\lesssim (C+\frac{1}{\varepsilon})|||Z_3|||_{1,\gamma}^2 + \Sigma_{i\ne 2,3}\frac{1}{\varepsilon}\left(|||T^\gamma_\sigma Z_i|||_0^2 + |||Z_i|||_0^2\right) \\ &+ \frac{1}{\varepsilon}\left(\left|\left|\left|T_r W^+\right|\right|\right|_{1,\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_0^2 + |||F|||_{1,\gamma}^2\right)\end{aligned}$$

By considering

$$\Re\langle T^\gamma_\sigma Z_3, T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle|_{x_2=0} = \Re\langle T^\gamma_{\Lambda^{\frac12}} T^\gamma_\sigma Z_3, T^\gamma_{\Lambda^{\frac12}} T^\gamma_\sigma Z_3\rangle|_{x_2=0} + \langle T^\gamma_\sigma Z_3, R_0 T^\gamma_\sigma Z_3\rangle|_{x_2=0},$$

we obtain

$$\begin{aligned}&|||T^\gamma_\Lambda T^\gamma_\sigma Z_3|||_0^2 + \|T^\gamma_{\Lambda^{\frac12}} T^\gamma_\sigma Z_3|_{x_2=0}\|_0^2 \lesssim \|T^\gamma_\sigma Z_3|_{x_2=0}\|_0^2 + (C+\frac{1}{\varepsilon})|||Z_3|||_{1,\gamma}^2 \\ &+ \Sigma_{i\ne 2,3}\frac{1}{\varepsilon}\left(|||T^\gamma_\sigma Z_i|||_0^2 + |||Z_i|||_0^2\right) + \frac{1}{\varepsilon}\left(\left|\left|\left|T_r W^+\right|\right|\right|_{1,\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_0^2 + |||F|||_{1,\gamma}^2\right).\end{aligned} \tag{3.2.30}$$

For the second symmetrizer $(T^\gamma_\Lambda)^* T^\gamma_\Lambda$, we have

$$
\begin{aligned}
&\partial_2 \Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda Z_3\rangle = 2\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda \partial_2 Z_3\rangle \\
&= 2\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_{-\omega^r+i\varpi^r} Z_3\rangle + 2\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_{\Theta_0} Z_3\rangle + 2\Sigma_{i\neq 2,3}\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_{\Theta_0} Z_i\rangle \\
&+ 2\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_r W^+\rangle + 2\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda R_{-1} W^+\rangle + 2\Re\langle T^\gamma_\Lambda Z_3, T^\gamma_\Lambda F\rangle
\end{aligned}
$$

By a similar argument as above, we have

$$
\begin{aligned}
|||Z_3|||^2_{\frac{3}{2},\gamma} + \|Z_3|_{x_2=0}\|^2_{1,\gamma} \lesssim C|||Z_3|||^2_{1,\gamma} + \frac{1}{\varepsilon}|||Z_3|||^2_{\frac{1}{2},\gamma} + \Sigma_{i\neq 2,3}(\frac{1}{\varepsilon}|||Z_i|||^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma}|||Z_i|||^2_0) \\
+ \frac{1}{\varepsilon\gamma}\left(|||T_r W^+|||^2_{1,\gamma} + |||W^+|||^2_0 + |||F|||^2_{1,\gamma}\right)
\end{aligned}
\tag{3.2.31}
$$

Then we consider the 1st, 4th and 6th equations of the system (3.2.27) which can be denoted as

$$
T^\gamma_{\mathbf{a}}(Z_1, Z_4, Z_6)^\top + T^\gamma_{\mathbf{\Theta}_1} Z_3 + T^\gamma_{\mathbf{\Theta}_0} Z^+ + T^\gamma_r W^+ + R_{-1} W^+ = R_0 F,
$$

where $\mathbf{\Theta}_1$ is a 3×1 matrix symbol of degree 1 regularity 2, $\mathbf{\Theta}_0$ is a 3×7 matrix symbol of degree 0 regularity of 1 and a is a 3×3 matrix symbol of degree 1 regularity 2 taking the following form

$$
\mathbf{a} = \begin{pmatrix} \tau + iv^r\eta & -i\eta F^r_{11} & -i\eta F^r_{12} \\ -i\eta F^r_{11} & \tau + iv^r\eta & 0 \\ -i\eta F^r_{12} & 0 & \tau + iv^r\eta \end{pmatrix}.
$$

We first apply the symbol $\frac{\mathbf{a}^*}{\Lambda^2} \in \Gamma^0_2$ to the above equation, where $\mathbf{a}^*$ is the adjoint matrix of $\mathbf{a}$, and obtain

$$
T^\gamma_a Z_j + T^\gamma_{\Theta_1} Z_3 + \Sigma T^\gamma_{\Theta_0} Z_i + T^\gamma_r W^+ + R_{-1} W^+ = R_0 F,
$$

where $j = 1, 4, 6$ and $a = (\tau + iv^r\eta)\left((\tau + iv^r\eta)^2 + ({F^r_{11}}^2 + {F^r_{12}}^2)\eta^2\right)/\Lambda^2$. By the way we construct the cut-off function, $\left((\tau + iv^r\eta)^2 + ({F^r_{11}}^2 + {F^r_{12}}^2)\eta^2\right)/\Lambda^2$ is not zero on the support of χ_2. So we can rewrite

$$
a = (1-\chi_2)a + \chi_2(\tau + iv^r\eta)\left((\tau + iv^r\eta)^2 + ({F^r_{11}}^2 + {F^r_{12}}^2)\eta^2\right)/\Lambda^2.
$$

Because $Z^+ = T^\gamma_{\chi_1(Q_0+Q_{-1})} W^+_{p_1}$, we have

$$T^\gamma_{(1-\chi_2)a} Z_j = T^\gamma_{(1-\chi_2)a} T^\gamma_{\chi_1 Q_j^r} T^\gamma_{\chi_{p_2}} W^+ = R_{-1} T^\gamma_{\chi_{p_2}} W^+.$$

The above is from the fact that the support of $(1-\chi_2)a$ is disjoint with $\chi_1 Q_j^r$'s. Since $\sigma = 0$ only at the points wherer $\tau + iv^r\eta = 0$ in the support of χ_2, we have $\chi_2(\tau + iv^r\eta)\left((\tau+iv^r\eta)^2 + ({F^r_{11}}^2 + {F^r_{12}}^2)\eta^2\right)/\Lambda^2 = \chi_2\Theta_0 \times (\gamma + i\sigma)$. Hence we obtain

$$T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)} Z_j + T^\gamma_{\Theta_1} Z_3 + \Sigma T^\gamma_{\Theta_0} Z_i + T^\gamma_r W^+ + R_{-1} W^+ = R_0 F \tag{3.2.32}$$

For the above equation, we consider the symmetrizer $(T^\gamma_\sigma)^* T^\gamma_\sigma$ and obtain

$$\begin{aligned}&\Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)} Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\Theta_1} Z_3\rangle + \Sigma\Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\Theta_0} Z_i\rangle \\ &+ \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_r W^+\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma R_{-1} W^+\rangle = \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma F\rangle.\end{aligned}$$

In the above equality, all the terms except the first one can easily be estimated as follows

$$\begin{aligned}\Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\Theta_1} Z_3\rangle &= \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\frac{\Theta_1}{\Lambda}} T^\gamma_\Lambda T^\gamma_\sigma Z_3\rangle + \Re\langle T^\gamma_\sigma Z_j, R_1 Z_3\rangle \\ &\le \varepsilon\gamma \|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \frac{1}{\varepsilon\gamma}\|Z_3\|^2_{1,\gamma}, \\ \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\Theta_0} Z_i\rangle &= \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\Theta_0} T^\gamma_\sigma Z_i\rangle + \Re\langle T^\gamma_\sigma Z_j, R_0 Z_i\rangle \\ &\le \varepsilon\gamma \|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|T^\gamma_\sigma Z_i\|_0^2 + \frac{1}{\varepsilon\gamma}\|Z_i\|^2_0, \\ \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_r W^+\rangle &\le \varepsilon\gamma \|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|T^\gamma_r W^+\|^2_{1,\gamma}, \\ \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma R_{-1} W^+\rangle &\le \varepsilon\gamma \|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|W^+\|^2_0, \\ \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma F\rangle &\le \varepsilon\gamma \|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|F\|^2_{1,\gamma},\end{aligned}$$

for the first term, we have

$$\begin{aligned}&\Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)} Z_j\rangle \\ &\quad = \Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\chi_2\Theta_0} T^\gamma_{\gamma+i\sigma} Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\Theta_0} T^\gamma_\sigma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, R_0 Z_j\rangle.\end{aligned}$$

The last two terms above can be estimated as

$$\Re\langle T^\gamma_\sigma Z_j, R_0 Z_j\rangle \le \varepsilon\gamma \|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|Z_j\|_0^2,$$

$$\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\Theta_0} T^\gamma_\sigma Z_j\rangle \le C\|T^\gamma_\sigma Z_j\|_0^2,$$

and the first one can be further write into

$$\Re\langle T^\gamma_\sigma Z_j, T^\gamma_\sigma T^\gamma_{\chi_2\Theta_0} T^\gamma_{\gamma+i\sigma} Z_j\rangle = \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_\sigma T^\gamma_{\gamma+i\sigma} Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, R_0 T^\gamma_{\gamma+i\sigma} Z_j\rangle.$$

The last term above implies

$$\begin{aligned}\Re\langle T^\gamma_\sigma Z_j, R_0 T^\gamma_{\gamma+i\sigma} Z_j\rangle &= \Re\langle T^\gamma_\sigma Z_j, R_0 T^\gamma_\gamma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, R_0 T^\gamma_{i\sigma} Z_j\rangle \\ &\le \varepsilon\gamma\|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|T^\gamma_\gamma Z_j\|_0^2 + C\|T^\gamma_\sigma Z_j\|_0^2,\end{aligned}$$

and the first term leads to

$$\begin{aligned}&\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_\sigma T^\gamma_{\gamma+i\sigma} Z_j\rangle = \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_\sigma T^\gamma_\gamma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_\sigma T^\gamma_{i\sigma} Z_j\rangle \\ &= \gamma\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} R_0 Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_\gamma T^\gamma_\sigma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_{i\sigma} T^\gamma_\sigma Z_j\rangle \\ &= \gamma\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} R_0 Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} T^\gamma_{\gamma+i\sigma} T^\gamma_\sigma Z_j\rangle \\ &= \gamma\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0} R_0 Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0(\gamma+i\sigma)} T^\gamma_\sigma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, R_0 T^\gamma_\sigma Z_j\rangle\end{aligned}$$

In the above equality, the first and the last terms can be estimate directly by Cauchy inequality. For the second terms, we have

$$\begin{aligned}\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2\Theta_0(\gamma+i\sigma)} T^\gamma_\sigma Z_j\rangle &= \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\chi_2 a} T^\gamma_\sigma Z_j\rangle \\ &= \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\tilde a} T^\gamma_\sigma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_j, T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_\sigma Z_j\rangle,\end{aligned} \tag{3.2.33}$$

where $\tilde a$ is an extension of $\chi_2 a$ to the whole space with $|\Re\tilde a| \ge c\gamma$ for some generic positive c and in the last term

$$\begin{aligned}&T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_\sigma Z_j = T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_\sigma T^\gamma_{\chi_1 Q^r_j} T^\gamma_{\chi_{p_2}} W^+ = T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_{\chi_1 Q^r_j} T^\gamma_\sigma T^\gamma_{\chi_{p_2}} W^+ + T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_{O_0} T^\gamma_{\chi_{p_2}} W^+ \\ &+ T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_{O_{-1}} T^\gamma_{\chi_{p_2}} W^+ + T^\gamma_{(\chi_2-1)\tilde a} R_{-2} T^\gamma_{\chi_{p_2}} W^+.\end{aligned}$$

Note that O_0 and O_{-1} only support on the support of χ_1 which is disjoint with the support of $(\chi_2-1)\tilde a$. So it is easy to obtain

$$\begin{aligned}T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_{O_0} T^\gamma_{\chi_{p_2}} W^+ &= R_{-1} W^+, \\ T^\gamma_{(\chi_2-1)\tilde a} T^\gamma_{O_{-1}} T^\gamma_{\chi_{p_2}} W^+ &= R_{-1} W^+,\end{aligned}$$

$$T^\gamma_{(\chi_2-1)\tilde{a}}T^\gamma_{\chi_1 Q^j}T^\gamma_\sigma T^\gamma_{\chi_{p_2}}W^+ = R_{-1}W^+.$$

Then we obtain

$$\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{(\chi_2-1)\tilde{a}}T^\gamma_\sigma Z_j\rangle \leq \varepsilon\gamma\|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\|W^+\|_0^2. \tag{3.2.34}$$

Moreover, since $|\Re\tilde{a}| \geq c\gamma$, we have

$$|\Re\langle T^\gamma_\sigma Z_j, T^\gamma_{\tilde{a}}T^\gamma_\sigma Z_j\rangle| \geq c\gamma\|T^\gamma_\sigma Z_j\|_0^2 \tag{3.2.35}$$

Combining (3.2.33), (3.2.34) and (3.2.35), we obtain

$$\begin{aligned}
&\gamma\|T^\gamma_\sigma Z_j\|_0^2 \leq \frac{1}{\varepsilon\gamma}\|Z_j\|_0^2 + C\|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon}\|Z_j\|_0^2 + \frac{1}{\varepsilon\gamma}\left(\|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|_0^2 + \|Z_3\|_{1,\gamma}^2\right) \\
&\Sigma\frac{1}{\varepsilon\gamma}\left(\|T^\gamma_\sigma Z_i\|_0^2 + \|Z_i\|_0^2\right) + \frac{1}{\varepsilon\gamma}\left(\|T^\gamma_r W^+\|_{1,\gamma}^2 + \|W^+\|_0^2 + \|F\|_{1,\gamma}^2\right),
\end{aligned} \tag{3.2.36}$$

for $j = 1, 4, 6$. Similar as the outgoing mode Z_3, we apply another symmetrizer T^γ_Λ to (3.2.32) and obtain

$$\begin{aligned}
&\Re\langle Z_j, T^\gamma_\Lambda T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}Z_j\rangle + \Re\langle Z_j, T^\gamma_\Lambda T^\gamma_{\Theta_1}Z_3\rangle + \Sigma\Re\langle Z_j, T^\gamma_\Lambda T^\gamma_{\Theta_0}Z_i\rangle \\
&+ \Re\langle Z_j, T^\gamma_\Lambda T^\gamma_r W^+\rangle + \Re\langle Z_j, T^\gamma_\Lambda R_{-1}W^+\rangle = \Re\langle Z_j, T^\gamma_\Lambda F\rangle
\end{aligned}$$

In the above equality, the first term

$$\begin{aligned}
&\Re\langle Z_j, T^\gamma_\Lambda T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}Z_j\rangle = \Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{\Lambda^{\frac{1}{2}}}T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}Z_j\rangle + \Re\langle Z_j, R_0 T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}Z_j\rangle \\
&= \Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle + \Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, R_{\frac{1}{2}}Z_j\rangle + \Re\langle Z_j, R_0 T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}Z_j\rangle
\end{aligned}$$

For $\Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle$, similarly as we did above, we have

$$\Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{\chi_2\Theta_0\times(\gamma+i\sigma)}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle = \Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{\tilde{a}}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle + \Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{(\chi_2-1)\tilde{a}}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle,$$

and

$$\begin{aligned}
&\Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{(\chi_2-1)\tilde{a}}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle \leq C\|Z_j\|_{\frac{1}{2},\gamma}^2 + \varepsilon\gamma\|Z_j\|_{\frac{1}{2},\gamma}^2 + \frac{1}{\varepsilon\gamma}\|W^+\|_{-1,\gamma}^2 \\
&|\Re\langle T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j, T^\gamma_{\tilde{a}}T^\gamma_{\Lambda^{\frac{1}{2}}}Z_j\rangle| \geq c\gamma\|Z_j\|_{\frac{1}{2},\gamma}^2.
\end{aligned}$$

Together with the estimate of other terms, which is straightforward, we just list here

$$\begin{aligned}
&\Re\langle Z_j, R_0 T^{\gamma}_{\chi_2\Theta_0\times(\gamma+i\sigma)} Z_j\rangle \le C\|Z_j\|^2_{\frac{1}{2},\gamma}\\
&\Re\langle Z_j, T^{\gamma}_{\Lambda} T^{\gamma}_{\Theta_1} Z_3\rangle \le \varepsilon\gamma\|Z_j\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma}\|Z_3\|^2_{\frac{3}{2},\gamma}\\
&\Re\langle Z_j, T^{\gamma}_{\Lambda} T^{\gamma}_{\Theta_0} Z_i\rangle \le \varepsilon\gamma\|Z_j\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma}\|Z_i\|^2_{\frac{1}{2},\gamma}\\
&\Re\langle Z_j, T^{\gamma}_{\Lambda} T^{\gamma}_{r} W^+\rangle \le \varepsilon\gamma^2\|Z_j\|^2_{0} + \frac{1}{\varepsilon\gamma^2}\|T^{\gamma}_{r} W^+\|^2_{1,\gamma}\\
&\Re\langle Z_j, T^{\gamma}_{\Lambda} R_{-1} W^+\rangle \le \varepsilon\gamma^2\|Z_j\|^2_{0} + \frac{1}{\varepsilon\gamma^2}\|W^+\|^2_{0}\\
&\Re\langle Z_j, T^{\gamma}_{\Lambda} F\rangle \le \varepsilon\gamma^2\|Z_j\|^2_{0} + \frac{1}{\varepsilon\gamma^2}\|F\|^2_{1,\gamma},
\end{aligned}$$

and obtain

$$\begin{aligned}
\gamma\|Z_j\|^2_{\frac{1}{2},\gamma} \le C\|Z_j\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma}\|Z_3\|^2_{\frac{3}{2},\gamma}&\\
+ \frac{1}{\varepsilon\gamma}\|Z_i\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma^2}(\|T^{\gamma}_{r} W^+\|^2_{1,\gamma} + \|W^+\|^2_{0} + \|F\|^2_{1,\gamma}),&
\end{aligned}\tag{3.2.37}$$

for $j = 1, 4, 6$, by taking ε small enough.

For the fifth and seventh equation in (3.2.27), we have

$$T^{\gamma}_{\tau+iv^r\eta} Z_j + T^{\gamma}_{\Theta_1} Z_3 + \Sigma T^{\gamma}_{\Theta_0} Z_i + T^{\gamma}_{r} W^+ + R_{-1} W^+ = R_0 F,$$

where $j = 5, 7$. We just follow the similar argument for Z_j with $j = 1, 4, 6$ and obtain

$$\begin{aligned}
\gamma\|T^{\gamma}_{\sigma} Z_j\|^2_0 \le \frac{1}{\varepsilon\gamma}\|Z_j\|^2_0 + C\|T^{\gamma}_{\sigma} Z_j\|^2_0 + \frac{1}{\varepsilon}\|Z_j\|^2_0 + \frac{1}{\varepsilon\gamma}\left(\|T^{\gamma}_{\Lambda} T^{\gamma}_{\sigma} Z_3\|^2_0 + \|Z_3\|^2_{1,\gamma}\right)&\\
\Sigma\frac{1}{\varepsilon\gamma}\left(\|T^{\gamma}_{\sigma} Z_i\|^2_0 + \|Z_i\|^2_0\right) + \frac{1}{\varepsilon\gamma}\left(\|T^{\gamma}_{r} W^+\|^2_{1,\gamma} + \|W^+\|^2_0 + \|F\|^2_{1,\gamma}\right),&
\end{aligned}\tag{3.2.38}$$

$$\begin{aligned}
\gamma\|Z_j\|^2_{\frac{1}{2},\gamma} \le C\|Z_j\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma}\|Z_3\|^2_{\frac{3}{2},\gamma}&\\
+ \frac{1}{\varepsilon\gamma}\|Z_i\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma^2}(\|T^{\gamma}_{r} W^+\|^2_{1,\gamma} + \|W^+\|^2_{0} + \|F\|^2_{1,\gamma}),&
\end{aligned}\tag{3.2.39}$$

for $j = 5, 7$. Note that the estimate for Z_j where $j = 1, 4, 5, 6, 7$ are the same. Now we come to discuss the equation of incoming mode Z_2 in (3.2.27)

$$\begin{aligned}
\partial_2 Z_2 = T^{\gamma}_{\omega^r} Z_2 + T^{\gamma}_{\Theta_1} Z_1 + T^{\gamma}_{\Theta_1} Z_5&\\
+ T^{\gamma}_{\Theta_1} Z_7 + T^{\gamma}_{\Theta_0} Z_2 + \Sigma_{i\neq 2,3} T^{\gamma}_{\Theta_0} Z_i + T^{\gamma}_{r} W^+ + R_{-1} W^+ + F.&
\end{aligned}\tag{3.2.40}$$

We apply the two symmetrizers $(T^\gamma_\sigma)^* T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma$ and 1 to (3.2.40). For $(T^\gamma_\sigma)^* T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma$, we have

$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma \partial_2 Z_2\rangle = \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\omega^r+i\varpi^r} Z_2\rangle + \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\Theta_1} Z_j\rangle$$
$$+ \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\Theta_0} Z_2\rangle + \Sigma_{i\neq 2,3}\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\Theta_0} Z_i\rangle + \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_r W^+\rangle$$
$$+ \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma R_{-1} W^+\rangle + \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma F\rangle$$

Similarly for outgoing modes, we have

$$\partial_2\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle = \Re\langle T^\gamma_{\partial_2\sigma} Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle + \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_{\partial_2\sigma} Z_2\rangle$$
$$+ \Re\langle T^\gamma_\sigma \partial_2 Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle + \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma \partial_2 Z_2\rangle$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\omega^r+i\varpi^r} Z_2\rangle = \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{\omega^r+i\varpi^r}{\Lambda}} T^\gamma_\sigma Z_2\rangle + \Re\langle T^\gamma_\sigma Z_2, R_0 Z_2\rangle,$$

with the following estimate

$$\Re\langle T^\gamma_{\partial_2\sigma} Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|Z_2\|_0^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_{\partial_2\sigma} Z_2\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|Z_2\|_0^2$$
$$\Re\langle T^\gamma_\sigma \partial_2 Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle = \Re\langle T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma \partial_2 Z_2, T^\gamma_\sigma Z_2\rangle + \Re\langle T^\gamma_\sigma \partial_2 Z_2, R_{-2} T^\gamma_\sigma Z_2\rangle$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{\omega^r+i\varpi^r}{\Lambda}} T^\gamma_\sigma Z_2\rangle \le -c\|T^\gamma_\sigma Z_2\|_0^2$$
$$\Re\langle T^\gamma_\sigma Z_2, R_0 Z_2\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|Z_2\|_0^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\Theta_1} Z_j\rangle = \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_{\Theta_1} T^\gamma_\sigma Z_j\rangle + \Re\langle T^\gamma_\sigma Z_2, R_0 Z_j\rangle$$
$$\le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|T^\gamma_\sigma Z_j\|_0^2 + \frac{1}{\varepsilon}\|Z_j\|_0^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\Theta_0} Z_2\rangle = \Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_{\Theta_0} T^\gamma_\sigma Z_2\rangle + \Re\langle T^\gamma_\sigma Z_2, R_{-1} Z_2\rangle$$
$$\le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|T^\gamma_\sigma Z_2\|_{-1,\gamma}^2 + \frac{1}{\varepsilon}\|Z_2\|_{-1,\gamma}^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_{\Theta_0} Z_i\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|T^\gamma_\sigma Z_i\|_{-1,\gamma}^2 + \frac{1}{\varepsilon}\|Z_i\|_{-1,\gamma}^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma T^\gamma_r W^+\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|T^\gamma_r W^+\|_0^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma R_{-1} W^+\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|W^+\|_{-1,\gamma}^2$$
$$\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma F\rangle \le \varepsilon\|T^\gamma_\sigma Z_2\|_0^2 + \frac{1}{\varepsilon}\|F\|_0^2.$$

This implies

$$\begin{aligned}
&|||T^{\gamma}_{\sigma}Z_2|||_0^2 \le \Re\langle T^{\gamma}_{\sigma}Z_2, T^{\gamma}_{\frac{1}{\Lambda}}T^{\gamma}_{\sigma}Z_2\rangle|_{x_2=0} + \frac{1}{\varepsilon}|||Z_2|||_0^2 + \frac{1}{\varepsilon}\left(|||T^{\gamma}_{\sigma}Z_j|||_0^2 + |||Z_j|||_0^2\right) \\
&+ \frac{1}{\varepsilon}\left(|||T^{\gamma}_{\sigma}Z_2|||_{-1,\gamma}^2 + |||Z_2|||_{-1,\gamma}^2\right) + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}\left(|||T^{\gamma}_{\sigma}Z_i|||_{-1,\gamma}^2 + |||Z_i|||_{-1,\gamma}^2\right) \\
&+ \frac{1}{\varepsilon}\left(\left|\left|\left|T^{\gamma}_{r}W^+\right|\right|\right|_0^2 + \left|\left|\left|W^+\right|\right|\right|_{-1,\gamma}^2 + |||F|||_0^2\right).
\end{aligned} \tag{3.2.41}$$

At last we consider the symmetrizer 1 and obtain

$$\begin{aligned}
&\partial_2\Re\langle Z_2, Z_2\rangle = 2\Re\langle Z_2, \partial_2 Z_2\rangle = 2\Re\langle Z_2, T^{\gamma}_{\omega^r+i\varpi^r}Z_2\rangle + 2\Re\langle Z_2, T^{\gamma}_{\Theta_1}Z_j\rangle + 2\Re\langle Z_2, T^{\gamma}_{\Theta_0}Z_2\rangle \\
&+ \Sigma_{i\neq 2,3}2\Re\langle Z_2, T^{\gamma}_{\Theta_0}Z_i\rangle + 2\Re\langle Z_2, T^{\gamma}_{r}W^+\rangle + 2\Re\langle Z_2, R_{-1}W^+\rangle + 2\Re\langle Z_2, F\rangle.
\end{aligned}$$

Also we have

$$\begin{aligned}
2\Re\langle Z_2, T^{\gamma}_{\omega^r+i\varpi^r}Z_2\rangle &= 2\Re\langle Z_2, (T^{\gamma}_{\Lambda^{\frac{1}{2}}})^* T^{\gamma}_{\frac{\omega^r+i\varpi^r}{\Lambda^{\frac{1}{2}}}}Z_2\rangle + 2\Re\langle Z_2, R_0 Z_2\rangle \\
&= 2\Re\langle T^{\gamma}_{\Lambda^{\frac{1}{2}}}Z_2, T^{\gamma}_{\frac{\omega^r+i\varpi^r}{\Lambda}}T^{\gamma}_{\Lambda^{\frac{1}{2}}}Z_2\rangle + 2\Re\langle T^{\gamma}_{\Lambda^{\frac{1}{2}}}Z_2, R_{-\frac{1}{2}}Z_2\rangle + 2\Re\langle Z_2, R_0 Z_2\rangle.
\end{aligned}$$

and

$$\begin{aligned}
&\Re\langle T^{\gamma}_{\Lambda^{\frac{1}{2}}}Z_2, T^{\gamma}_{\frac{\omega^r+i\varpi^r}{\Lambda}}T^{\gamma}_{\Lambda^{\frac{1}{2}}}Z_2\rangle \le -c\|Z_2\|_{\frac{1}{2},\gamma}^2 \\
&\Re\langle T^{\gamma}_{\Lambda^{\frac{1}{2}}}Z_2, R_{-\frac{1}{2}}Z_2\rangle \le \varepsilon\|Z_2\|_{\frac{1}{2},\gamma}^2 + \frac{1}{\varepsilon}\|Z_2\|_{-\frac{1}{2},\gamma}^2 \\
&\Re\langle Z_2, R_0 Z_2\rangle \le C\|Z_2\|_0^2
\end{aligned}$$

The estimate of the rest terms are straightforward, and we obtain

$$\begin{aligned}
|||Z_2|||_{\frac{1}{2},\gamma}^2 \le \|Z_2|_{x_2=0}\|_0^2 + (C+\frac{1}{\varepsilon})|||Z_2|||_0^2 + \frac{1}{\varepsilon}|||Z_j|||_{\frac{1}{2},\gamma}^2 + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}|||Z_i|||_{-\frac{1}{2},\gamma}^2 \\
+ \frac{1}{\varepsilon}\left(\left|\left|\left|T^{\gamma}_{r}W^+\right|\right|\right|_{-\frac{1}{2},\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_{-\frac{3}{2},\gamma}^2 + |||F|||_{-\frac{1}{2},\gamma}^2\right).
\end{aligned} \tag{3.2.42}$$

We consider (3.2.30), (3.2.31), (3.2.36), (3.2.37), (3.2.38), (3.2.39), (3.2.41) and (3.2.42) and divided them by appropriate power of γ to obatin

$$\begin{aligned}
&\frac{1}{\gamma}|||T^{\gamma}_{\Lambda}T^{\gamma}_{\sigma}Z_3|||_0^2 + \frac{1}{\gamma}\|T^{\gamma}_{\Lambda^{\frac{1}{2}}}T^{\gamma}_{\sigma}Z_3|_{x_2=0}\|_0^2 \lesssim \frac{1}{\gamma}\|T^{\gamma}_{\sigma}Z_3|_{x_2=0}\|_0^2 + (C+\frac{1}{\varepsilon})\frac{1}{\gamma}|||Z_3|||_{1,\gamma}^2 \\
&\qquad + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon\gamma}\left(|||T^{\gamma}_{\sigma}Z_i|||_0^2 + |||Z_i|||_0^2\right) + \frac{1}{\varepsilon\gamma}\left(\left|\left|\left|T_r W^+\right|\right|\right|_{1,\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_0^2 + |||F|||_{1,\gamma}^2\right) \\
&|||Z_3|||_{\frac{3}{2},\gamma}^2 + \|Z_3|_{x_2=0}\|_{1,\gamma}^2 \lesssim C|||Z_3|||_{1,\gamma}^2 + \frac{1}{\varepsilon}|||Z_3|||_{\frac{1}{2},\gamma}^2 + \Sigma_{i\neq 2,3}(\frac{1}{\varepsilon}|||Z_i|||_{\frac{1}{2},\gamma}^2 + \frac{1}{\varepsilon\gamma}|||Z_i|||_0^2)
\end{aligned}$$

$$+\frac{1}{\varepsilon\gamma}\left(\left|\left|\left|T_rW^+\right|\right|\right|_{1,\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_0^2+|||F|||_{1,\gamma}^2\right)$$

$$\gamma|||T_\sigma^\gamma Z_j|||_0^2\le\frac{1}{\varepsilon\gamma}|||Z_j|||_0^2+C|||T_\sigma^\gamma Z_j|||_0^2+\frac{1}{\varepsilon}|||Z_j|||_0^2+\frac{1}{\varepsilon\gamma}\left(|||T_\Lambda^\gamma T_\sigma^\gamma Z_3|||_0^2+|||Z_3|||_{1,\gamma}^2\right)$$
$$+\Sigma\frac{1}{\varepsilon\gamma}\left(|||T_\sigma^\gamma Z_i|||_0^2+|||Z_i|||_0^2\right)+\frac{1}{\varepsilon\gamma}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_{1,\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_0^2+|||F|||_{1,\gamma}^2\right)$$

$$\gamma^2|||Z_j|||_{\frac{1}{2},\gamma}^2\le C\gamma|||Z_j|||_{\frac{1}{2},\gamma}^2+\frac{1}{\varepsilon}|||Z_3|||_{\frac{3}{2},\gamma}^2+\frac{1}{\varepsilon}|||Z_i|||_{\frac{1}{2},\gamma}^2$$
$$+\frac{1}{\varepsilon\gamma}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_{1,\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_0^2+|||F|||_{1,\gamma}^2\right)$$

$$\gamma|||T_\sigma^\gamma Z_2|||_0^2\le\gamma\Re\langle T_\sigma^\gamma Z_2,T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_2\rangle|_{x_2=0}+\frac{\gamma}{\varepsilon}|||Z_2|||_0^2+\frac{\gamma}{\varepsilon}\left(|||T_\sigma^\gamma Z_j|||_0^2+|||Z_j|||_0^2\right)$$
$$+\frac{\gamma}{\varepsilon}\left(|||T_\sigma^\gamma Z_2|||_{-1,\gamma}^2+|||Z_2|||_{-1,\gamma}^2\right)+\Sigma_{i\ne2,3}\frac{\gamma}{\varepsilon}\left(|||T_\sigma^\gamma Z_i|||_{-1,\gamma}^2+|||Z_i|||_{-1,\gamma}^2\right)$$
$$+\frac{\gamma}{\varepsilon}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_0^2+\left|\left|\left|W^+\right|\right|\right|_{-1,\gamma}^2+|||F|||_0^2\right)$$

$$\gamma^2|||Z_2|||_{\frac{1}{2},\gamma}^2\le\gamma^2\|Z_2|_{x_2=0}\|_0^2+(C+\frac{1}{\varepsilon})\gamma^2|||Z_2|||_0^2+\frac{\gamma^2}{\varepsilon}|||Z_j|||_{\frac{1}{2},\gamma}^2+\Sigma_{i\ne2,3}\frac{\gamma^2}{\varepsilon}|||Z_i|||_{-\frac{1}{2},\gamma}^2$$
$$+\frac{\gamma^2}{\varepsilon}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_{-\frac{1}{2},\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_{-\frac{3}{2},\gamma}^2+|||F|||_{-\frac{1}{2},\gamma}^2\right)$$

Then adding all the equations above together and taking γ large enough, we can obtain

$$\begin{aligned}&\frac{1}{\gamma}\left|\left|\left|T_\Lambda^\gamma T_\sigma^\gamma Z_3\right|\right|\right|_0^2+|||Z_3|||_{\frac{3}{2},\gamma}^2+\gamma|||T_\sigma^\gamma Z_j|||_0^2+\gamma^2|||Z_j|||_{\frac{1}{2},\gamma}^2+\gamma|||T_\sigma^\gamma Z_2|||_0^2+\gamma^2|||Z_2|||_{\frac{1}{2},\gamma}^2\\&+\frac{1}{\gamma}\|T_{\Lambda^{\frac{1}{2}}}^\gamma T_\sigma^\gamma Z_3|_{x_2=0}\|_0^2+\|Z_3|_{x_2=0}\|_{1,\gamma}^2\lesssim\gamma\Re\langle T_\sigma^\gamma Z_2,T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_2\rangle|_{x_2=0}+\gamma^2\|Z_2|_{x_2=0}\|_0^2\\&+\frac{1}{\gamma}|||Z_3|||_{1,\gamma}^2+\Sigma\left(|||Z_i|||_{\frac{1}{2},\gamma}^2+\frac{1}{\gamma}|||T_\sigma^\gamma Z_i|||_0^2\right)+\frac{1}{\gamma}\left(\left|\left|\left|T_rW^+\right|\right|\right|_{1,\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_0^2+|||F|||_{1,\gamma}^2\right)\\&\lesssim\gamma\Re\langle T_\sigma^\gamma Z_2,T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_2\rangle|_{x_2=0}+\gamma^2\|Z_2|_{x_2=0}\|_0^2+\frac{1}{\gamma}\left(\left|\left|\left|T_rW^+\right|\right|\right|_{1,\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_0^2+|||F|||_{1,\gamma}^2\right).\end{aligned}\tag{3.2.43}$$

Since, in our setting, $v^r\ne v^l$, the point where $\tau=-iv^r\eta$ is not a pole of differential equation for W^- in (3.2.17). So it can be treated in the same way as in Case 2. Here we just give the estimate

$$\begin{aligned}&\frac{1}{\gamma}\left|\left|\left|T_\Lambda^\gamma T_\sigma^\gamma Z_{10}\right|\right|\right|_0^2+|||Z_{10}|||_{\frac{3}{2},\gamma}^2+|||T_\sigma^\gamma Z_j|||_{\frac{1}{2},\gamma}^2+\gamma|||Z_j|||_{1,\gamma}^2+\gamma|||T_\sigma^\gamma Z_9|||_0^2+\gamma^2|||Z_9|||_{\frac{1}{2},\gamma}^2\\&+\frac{1}{\gamma}\|T_{\Lambda^{\frac{1}{2}}}^\gamma T_\sigma^\gamma Z_{10}|_{x_2=0}\|_0^2+\|Z_{10}|_{x_2=0}\|_{1,\gamma}\\&\lesssim\gamma\Re\langle T_\sigma^\gamma Z_9,T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_9\rangle|_{x_2=0}+\gamma^2\|Z_9|_{x_2=0}\|_0^2+\frac{1}{\gamma}\left(\left|\left|\left|T_rW^+\right|\right|\right|_{1,\gamma}^2+\left|\left|\left|W^+\right|\right|\right|_0^2+|||F|||_{1,\gamma}^2\right),\end{aligned}\tag{3.2.44}$$

where $Z^- = (Z_9, Z_{10}, ..., Z_{14})^\top := T^\gamma_{\chi_1 Q^l} T^\gamma_{\chi_{p_1}} W^-$ and Q^l is the transformation matrix for W^- which is defined in a similar way as Q^r. In the above notation, Z_3 and Z_{10} are outgoing modes of the system and Z_2 and Z_9 are incoming modes. So we denote

$$Z_{in} = (Z_2, Z_9)^\top, \text{ and } Z_{out} = (Z_3, Z_{10})^\top.$$

At last, we need to use the boundary conditions in (3.2.17) to estimate $\|Z_2|_{x_2=0}\|_0^2$, $\|Z_9|_{x_2=0}\|_0^2$, $\gamma\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle|_{x_2=0}$ and $\gamma\Re\langle T^\gamma_\sigma Z_9, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_9\rangle|_{x_2=0}$. However, it is obvious that

$$\begin{aligned}
&\gamma\Re\langle T^\gamma_\sigma Z_2, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_2\rangle|_{x_2=0} \lesssim \|T^\gamma_\sigma Z_2|_{x_2=0}\|_0^2 \\
&\gamma\Re\langle T^\gamma_\sigma Z_9, T^\gamma_{\frac{1}{\Lambda}} T^\gamma_\sigma Z_9\rangle|_{x_2=0} \lesssim \|T^\gamma_\sigma Z_9|_{x_2=0}\|_0^2.
\end{aligned}$$

So to estimate these boundary terms are exactly to estimate $T^\gamma_\sigma Z_{in}|_{x_2=0}$ and $Z_{in}|_{x_2=0}$, which is the same situation as in [20]. So by the same method there and utilize the boundary conditions in (3.2.17), we obtain

$$\gamma^2\|Z_{in}|_{x_2=0}\|_0^2 + \|T^\gamma_\sigma Z_{in}|_{x_2=0}\|_0^2 \lesssim \|G\|_{1,\gamma}^2 + \|Z_{out}|_{x_2=0}\|_{1,\gamma}^2 + \|W^{\text{nc}}|_{x_2=0}\|_0^2. \tag{3.2.45}$$

Moreover, from (3.2.31), we have

$$\|Z_{out}|_{x_2=0}\|_{1,\gamma}^2 \lesssim \Sigma_{i\neq 2,3}|||Z_i|||_{\frac{1}{2},\gamma}^2 + \frac{1}{\gamma}\left(|||T_r W^+|||_{1,\gamma}^2 + |||W^+|||_0^2 + |||F|||_{1,\gamma}^2\right) \tag{3.2.46}$$

Adding (3.2.43), (3.2.44), (3.2.45) and (3.2.46) together, we obtain

$$\begin{aligned}
&\frac{1}{\gamma}|||T^\gamma_\Lambda T^\gamma_\sigma Z_{out}|||_0^2 + |||Z_{out}|||_{\frac{3}{2},\gamma}^2 + \gamma|||T^\gamma_\sigma Z_c|||_0^2 + \gamma^2|||Z_c|||_{\frac{1}{2},\gamma}^2 + \gamma|||T^\gamma_\sigma Z_{in}|||_0^2 + \gamma^2|||Z_{in}|||_{\frac{1}{2},\gamma}^2 \\
&+ \frac{1}{\gamma}\|T^\gamma_{\Lambda^{\frac{1}{2}}} T^\gamma_\sigma Z_{out}|_{x_2=0}\|_0^2 + \|Z_{out}|_{x_2=0}\|_{1,\gamma} + \gamma^2\|Z_{in}|_{x_2=0}\|_0^2 + \|T^\gamma_\sigma Z_{in}|_{x_2=0}\|_0^2 \\
&\lesssim \|G\|_{1,\gamma}^2 + \|W^{\text{nc}}|_{x_2=0}\|_0^2 + \frac{1}{\gamma}\left(|||T_r W|||_{1,\gamma}^2 + |||W|||_0^2 + |||F|||_{1,\gamma}^2\right)
\end{aligned} \tag{3.2.47}$$

where $Z_c = (Z_1, Z_4, Z_5, Z_6, Z_7, Z_8, Z_{11}, Z_{12}, Z_{13}, Z_{14})^\top$.

3.2.4.2 Case 2–roots We want to estimate the part of $W^{\pm}$ corresponding to $\mathcal{V}_r^1$, $\mathcal{V}_r^2$ and $\mathcal{V}_r^3$. Here we will just show the details for the differential equations of W^+ in $\mathcal{V}_r^1$. For the other neighborhoods $\mathcal{V}_r^2$ and $\mathcal{V}_r^3$ and unkown W^-, we can obtain the estimate in exact the same way. Now we consider the cut-off function χ_{rt} in Γ_k^0 for any integer k, whose support on $\mathbb{R}_+^3 \times \Sigma$ is contained in the neighborhood $\mathcal{V}_r^1$ and equals 1 in a smaller neighborhood of the curve where $\tau = iV_1\eta$ and denote

$$W_{rt}^+ := T_{\chi_{rt}}^{\gamma} W^+.$$

Similarly, as the previous case, we can obtain

$$T_{\tau A_0^r + i\eta A_1^r}^{\gamma} W_{rt}^+ + T_{A_0^r C^r}^{\gamma} W_{rt}^+ + T_r^{\gamma} W^+ + I_2 \partial_2 W_{rt}^+ = T_{\chi_{rt}}^{\gamma} F + R_{-1} W^+,$$

where r is in the class of Γ_1^0, bounded and supported only in the set where $\chi_{rt} \in (0,1)$. Then we take two cut-off functions χ_1 and χ_2 in the class Γ_k^0 for any integer k and both with support in $\mathcal{V}_{rt}$. $\chi_1 = 1$ on the support of χ_{rt} and $\chi_2 = 1$ on the support of χ_1. Similarly, as the previous case, after applying the cut-off symbol to the differential equation, we can find the transform matrix Q_0^r and Q_{-1}^r, and symmetrizer R_0^r and R_{-1}^r such that

$$I_2 \partial_2 Z^+ = -T_{\chi_2 \widetilde{A}^r}^{\gamma} Z^+ + T_{D_0}^{\gamma} Z^+ + T_r^{\gamma} W^+ + R_0 F + R_{-1} W^+,$$

where $\widetilde{A}^r$ is the same as (3.2.26) in the Case 1, $\chi_1 Q_0^r$ and $\chi_1 R_0^r$ are invertible symbols in Γ_2^0, Q_{-1}^r and R_{-1}^r are symbols in Γ_1^{-1} and Z^+ is defined by

$$Z^+ = T_{\chi_1 (Q_0^{r\,-1} + Q_{-1}^r)}^{\gamma} W_{rt}^+,$$

just as the previous case. Then we do the same extension as the previous case for $\chi_2 \widetilde{A}^r$ and D_0, and obtain

$$I_2 \partial_2 Z^+ = -T_{\widetilde{D_1}}^{\gamma} Z^+ + T_{\widetilde{D_0}}^{\gamma} Z^+ + T_r^{\gamma} W^+ + R_0 F + R_{-1} W^+, \tag{3.2.48}$$

where, in $\widetilde{D_1}$, ω^r is a symbol in Γ_2^1 and $\omega^r \geq c\Lambda^{1,\gamma}$, and in $\widetilde{D_0}$, $d_{2,3} = d_{3,2} = 0$. Then we consider each differential equation and construct appropriate symmetrizers to obtain the expected estimate. Still we denote $Z^+ = (Z_1, Z_2, ..., Z_7)^\top$. For the third equation in (3.2.48), we have

$$\partial_2 Z_3 = T^\gamma_{-\omega^r + i\varpi^r} Z_3 + T^\gamma_{\Theta_0} Z_3 + \Sigma_{i\neq 2,3} T^\gamma_{\Theta_0} Z_i + T^\gamma_r W^+ + R_0 F + R_{-1} W^+,$$

which is the same equation as the previous case. So we can still take $(T^\gamma_\sigma)^* T^\gamma_\Lambda T^\gamma_\sigma$ and $(T^\gamma_\Lambda)^* T^\gamma_\Lambda$ as symmetrizer. After the same argument, we can have

$$\begin{aligned}&\|\!|T^\gamma_\Lambda T^\gamma_\sigma Z_3\|\!|^2_0 + \|T^\gamma_{\Lambda^{\frac{1}{2}}} T^\gamma_\sigma Z_3|_{x_2=0}\|^2_0 \lesssim \|T^\gamma_\sigma Z_3|_{x_2=0}\|^2_0 + (C + \frac{1}{\varepsilon})\|\!|Z_3\|\!|^2_{1,\gamma}\\ &\quad + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}\left(\|\!|T^\gamma_\sigma Z_i\|\!|^2_0 + \|\!|Z_i\|\!|^2_0\right) + \frac{1}{\varepsilon}\left(\left|\!\left|\!\left|T_r W^+\right|\!\right|\!\right|^2_{1,\gamma} + \left|\!\left|\!\left|W^+\right|\!\right|\!\right|^2_0 + \|\!|F\|\!|^2_{1,\gamma}\right),\end{aligned}$$

and

$$\begin{aligned}&\|\!|Z_3\|\!|^2_{\frac{3}{2},\gamma} + \|Z_3|_{x_2=0}\|^2_{1,\gamma} \lesssim C\|\!|Z_3\|\!|^2_{1,\gamma} + \frac{1}{\varepsilon}\|\!|Z_3\|\!|^2_{\frac{1}{2},\gamma}\\ &+ \Sigma_{i\neq 2,3}(\frac{1}{\varepsilon}\|\!|Z_i\|\!|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon\gamma}\|\!|Z_i\|\!|^2_0) + \frac{1}{\varepsilon\gamma}\left(\left|\!\left|\!\left|T_r W^+\right|\!\right|\!\right|^2_{1,\gamma} + \left|\!\left|\!\left|W^+\right|\!\right|\!\right|^2_0 + \|\!|F\|\!|^2_{1,\gamma}\right)\end{aligned}$$

Since in this case the root of Lopatinskii determinant does not coincide with any pole of the differential equation, we can estimate Z_j for $j = 1, 4, 5, 6, 7$ in the same strategy. In particular, we multiply the 1st, 4th, 5th, 6th and 7th equations in (3.2.48) by some appropriate matrix symbol in Γ^0_1 and obtain

$$T^\gamma_a Z_j + T^\gamma_{\Theta_1} Z_3 + \Sigma T^\gamma_{\Theta_0} Z_i + T^\gamma_r W^+ + R_{-1} W^+ = R_0 F, \tag{3.2.49}$$

where $|\Re a| \geq c\Lambda$ on the support of χ_2. So we can do the same extension argument of a as in Case 1 and obtain a new symbol $\tilde{a}$ such that $|\Re\tilde{a}| \geq c\Lambda$. Then in this case, we consider the symmetrizers $(T^\gamma_\sigma)^* T^\gamma_\sigma$ and T^γ_Λ, and by a similar argument, we have

$$\begin{aligned}&\|T^\gamma_\sigma Z_j\|^2_{\frac{1}{2},\gamma} \leq C\|Z_j\|^2_{1,\gamma}\\ &+ \frac{1}{\varepsilon}\|Z_3\|^2_{\frac{3}{2},\gamma} + \Sigma\frac{1}{\varepsilon}\|Z_i\|^2_{\frac{1}{2},\gamma} + \frac{1}{\varepsilon}\left(\|T^\gamma_r W^+\|^2_{\frac{1}{2},\gamma} + \|W^+\|^2_{-\frac{1}{2},\gamma} + \|F\|^2_{\frac{1}{2},\gamma}\right)\end{aligned}$$

and

$$\|Z_j\|^2_{1,\gamma} \leq \frac{1}{\varepsilon}\|Z_j\|^2_0 + \frac{1}{\varepsilon}\|Z_3\|^2_{1,\gamma} + \Sigma\frac{1}{\varepsilon}\|Z_i\|^2_0 + \frac{1}{\varepsilon}\left(\|T^\gamma_r W^+\|^2_0 + \|W^+\|^2_{-1,\gamma} + \|F\|^2_0\right)$$

At last for the second equation in (3.2.48), which is corresponding to the incoming mode of the system, we can still follow the previous argument and obtain

$$\begin{aligned}|||T_\sigma^\gamma Z_2|||_0^2 \leq \Re\langle T_\sigma^\gamma Z_2, T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_2\rangle|_{x_2=0} + \frac{1}{\varepsilon}|||Z_2|||_0^2 + \frac{1}{\varepsilon}\left(|||T_\sigma^\gamma Z_j|||_0^2 + |||Z_j|||_0^2\right) \\ + \frac{1}{\varepsilon}\left(|||T_\sigma^\gamma Z_2|||_{-1,\gamma}^2 + |||Z_2|||_{-1,\gamma}^2\right) + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}\left(|||T_\sigma^\gamma Z_i|||_{-1,\gamma}^2 + |||Z_i|||_{-1,\gamma}^2\right) \\ + \frac{1}{\varepsilon}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_0^2 + \left|\left|\left|W^+\right|\right|\right|_{-1,\gamma}^2 + |||F|||_0^2\right)\end{aligned}$$

and

$$\begin{aligned}|||Z_2|||_{\frac{1}{2},\gamma}^2 \leq \|Z_2|_{x_2=0}\|_0^2 + (C+\frac{1}{\varepsilon})|||Z_2|||_0^2 + \frac{1}{\varepsilon}|||Z_j|||_{\frac{1}{2},\gamma}^2 + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}|||Z_i|||_{-\frac{1}{2},\gamma}^2 \\ + \frac{1}{\varepsilon}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_{-\frac{1}{2},\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_{-\frac{3}{2},\gamma}^2 + |||F|||_{-\frac{1}{2},\gamma}^2\right).\end{aligned}$$

Again, combining all the estimate above, dividing them by suitable power of γ and taking γ large enough, we have

$$\begin{aligned}\frac{1}{\gamma}|||T_\Lambda^\gamma T_\sigma^\gamma Z_3|||_0^2 + |||Z_3|||_{\frac{3}{2},\gamma}^2 + |||T_\sigma^\gamma Z_j|||_{\frac{1}{2},\gamma}^2 + \gamma|||Z_j|||_{1,\gamma}^2 + \gamma|||T_\sigma^\gamma Z_2|||_0^2 + \gamma^2|||Z_2|||_{\frac{1}{2},\gamma}^2 \\ + \frac{1}{\gamma}\|T_{\Lambda^{\frac{1}{2}}}^\gamma T_\sigma^\gamma Z_3|_{x_2=0}\|_0^2 + \|Z_3|_{x_2=0}\|_{1,\gamma}^2 \lesssim \gamma\Re\langle T_\sigma^\gamma Z_2, T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_2\rangle|_{x_2=0} \\ + \gamma^2\|Z_2|_{x_2=0}\|_0^2 + \frac{1}{\gamma}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_{1,\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_0^2 + |||F|||_{1,\gamma}^2\right)\end{aligned} \tag{3.2.50}$$

Similarly, for $Z^- = (Z_8, Z_9, ..., Z_{14})^\top := T_{\chi_1(Q_0^l+Q_{-1}^l)}^\gamma T_{\chi_{rt}}^\gamma W^-$, we have

$$\begin{aligned}\frac{1}{\gamma}|||T_\Lambda^\gamma T_\sigma^\gamma Z_{10}|||_0^2 + |||Z_{10}|||_{\frac{3}{2},\gamma}^2 + |||T_\sigma^\gamma Z_j|||_{\frac{1}{2},\gamma}^2 + \gamma|||Z_j|||_{1,\gamma}^2 + \gamma|||T_\sigma^\gamma Z_9|||_0^2 + \gamma^2|||Z_9|||_{\frac{1}{2},\gamma}^2 \\ + \frac{1}{\gamma}\|T_{\Lambda^{\frac{1}{2}}}^\gamma T_\sigma^\gamma Z_{10}|_{x_2=0}\|_0^2 + \|Z_{10}|_{x_2=0}\|_{1,\gamma}^2 \lesssim \gamma\Re\langle T_\sigma^\gamma Z_9, T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_9\rangle|_{x_2=0} \\ + \gamma^2\|Z_9|_{x_2=0}\|_0^2 + \frac{1}{\gamma}\left(\left|\left|\left|T_r^\gamma W^+\right|\right|\right|_{1,\gamma}^2 + \left|\left|\left|W^+\right|\right|\right|_0^2 + |||F|||_{1,\gamma}^2\right)\end{aligned} \tag{3.2.51}$$

We note that just the same as the Case 1, the boundary terms in (3.2.17) can be used to estimate $\gamma\Re\langle T_\sigma^\gamma Z_2, T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_2\rangle|_{x_2=0}$, $\|Z_2|_{x_2=0}\|_0^2$, $\gamma\Re\langle T_\sigma^\gamma Z_9, T_{\frac{1}{\Lambda}}^\gamma T_\sigma^\gamma Z_9\rangle|_{x_2=0}$ and $\|Z_9|_{x_2=0}\|_0^2$. So by the same method in estimating $T_\sigma^\gamma Z_{in}|_{x_2=0}$ and $Z_{in}|_{x_2=0}$ and combining (3.2.50) and (3.2.51), we have

$$\begin{aligned}\frac{1}{\gamma}|||T_\Lambda^\gamma T_\sigma^\gamma Z_{out}|||_0^2 + |||Z_{out}|||_{\frac{3}{2},\gamma}^2 + |||T_\sigma^\gamma Z_c|||_{\frac{1}{2},\gamma}^2 + \gamma|||Z_c|||_{1,\gamma}^2 + \gamma|||T_\sigma^\gamma Z_{in}|||_0^2 + \gamma^2|||Z_{in}|||_{\frac{1}{2},\gamma}^2 \\ + \frac{1}{\gamma}\|T_{\Lambda^{\frac{1}{2}}}^\gamma T_\sigma^\gamma Z_{out}|_{x_2=0}\|_0^2 + \|Z_{out}|_{x_2=0}\|_{1,\gamma} + \gamma^2\|Z_{in}|_{x_2=0}\|_0^2 + \|T_\sigma^\gamma Z_{in}|_{x_2=0}\|_0^2 \\ \lesssim \|G\|_{1,\gamma}^2 + \|W^{\mathrm{nc}}|_{x_2=0}\|_0^2 + \frac{1}{\gamma}\left(|||T_r W|||_{1,\gamma}^2 + |||W|||_0^2 + |||F|||_{1,\gamma}^2\right)\end{aligned} \tag{3.2.52}$$

where $Z_c = (Z_1, Z_4, Z_5, Z_6, Z_7, Z_8, Z_{11}, Z_{12}, Z_{13}, Z_{14})^\top$.

3.2.4.3 Case 3–poles Here we will discuss the case of the pole which is not a root of Lopatinskii determinant. That is the part of $W^{\pm}$ whose frequencies are in $\mathcal{V}^1_{p_2}$, $\mathcal{V}^2_{p_2}$, $\mathcal{V}^3_{p_2}$ and $\mathcal{V}^4_{p_2}$. Since those frequencies are alway from the root of Lopatinskii determinant, the boundary conditions in (3.2.17) provide stronger control on the incoming modes on $\mathcal{V}^1_{p_2}$, $\mathcal{V}^2_{p_2}$, $\mathcal{V}^3_{p_2}$ and $\mathcal{V}^4_{p_2}$ than on the neighborhoods in previous two cases. So in this case, we only need to construct one symmetrizer for each equation in the system. We take the neighborhood $\mathcal{V}^1_{p_2}$ as an example. This neighborhood contains the curve in frequency space such that $\tau = -i\left(v^r + \sqrt{{F^r_{11}}^2 + {F^r_{12}}^2}\right)\eta$ which is a pole of the differential equations for W^+ but not for W^-. So actually the equations of W^- can be reduced to non-characteristic case. However, the strategy we discuss here works for both equations of W^+ and W^-, only if the neighborhood does not contain the points such that $\omega^r = 0$ or $\omega^l = 0$. We just show the details for W^+.

First we take cut-off functions χ_{p_2}, χ_1 and χ_2 as the last pole case, pick the transform matrix Q^r_0 andQ^r_{-1}, and symmetrizer R^r_0 and R^r_{-1} and do the appropriate adjustment of $\chi_2\widetilde{A}^r$ and D_0 to obtain

$$I_2\partial_2 Z^+ = -T^\gamma_{\widetilde{D_1}} Z^+ + T^\gamma_{\widetilde{D_0}} Z^+ + T^\gamma_r W^+ + R_0 F + R_{-1} W^+, \tag{3.2.53}$$

where

$$Z^+ = T^\gamma_{\chi_1(Q^{r\,-1}_0 + Q^r_{-1})} T^\gamma_{\chi_{p_2}} W^+,$$

and the symbols in the equations are the same as previous two cases. For the third equation

$$\partial_2 Z_3 = T^\gamma_{-\omega^r + i\varpi^r} Z_3 + T^\gamma_{\Theta_0} Z_3 + \Sigma_{i\neq 2,3} T^\gamma_{\Theta_0} Z_i + T^\gamma_r W^+ + R_0 F + R_{-1} W^+,$$

we consider the symmetrizer $(T^\gamma_\Lambda)^* T^\gamma_\Lambda T^\gamma_\Lambda$ and obtain

$$\begin{aligned}\Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda \partial_2 Z_3\rangle =& \Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_{-\omega^r + i\varpi^r} Z_3\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_{\Theta_0} Z_3\rangle \\ &+ \Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_{\Theta_0} Z_i\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda T^\gamma_r W^+\rangle \\ &+ \Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda R_{-1} W^+\rangle + \Re\langle T^\gamma_\Lambda T^\gamma_\Lambda Z_3, T^\gamma_\Lambda R_0 F\rangle\end{aligned}$$

Using the similar argument and taking ε small enough, we have

$$\begin{aligned}&|||Z_3|||^2_{2,\gamma} + \|Z_3|_{x_2=0}\|^2_{\frac{3}{2},\gamma} \lesssim \|Z_3|_{x_2=0}\|^2_{1,\gamma} + (\frac{1}{\varepsilon} + C)|||Z_3|||^2_{1,\gamma} + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}|||Z_i|||^2_{1,\gamma}\\ &\frac{1}{\varepsilon}\left(|||T_r^\gamma W^+|||^2_{1,\gamma} + |||W^+|||^2_0 + |||F|||^2_{1,\gamma}\right)\end{aligned}$$

For the 1st, 4th, 5th, 6th and 7th equations in (3.2.53) , we consider the symmetrizer $(T_\Lambda^\gamma)^* T_\Lambda^\gamma$. The argument are quite similar as the previous pole case. We can obtain

$$\gamma\|Z_j\|^2_{1,\gamma} \lesssim C\|Z_j\|^2_{1,\gamma} + \frac{1}{\gamma}\|Z_3\|^2_{2,\gamma} + \Sigma\frac{1}{\gamma}\|Z_i\|^2_{1,\gamma} + \frac{1}{\gamma}\left(\|T_r^\gamma W^+\|^2_{1,\gamma} + \|W^+\|^2_0 + \|F\|^2_{1,\gamma}\right),$$

for $j = 1, 4, 5, 6, 7$.

At last for the differential equation of the incoming mode Z_2, we take the symmetrizer T_Λ^γ, use the similar strategy and obtain

$$\begin{aligned}&|||Z_2|||^2_{1,\gamma} \lesssim \Re\langle Z_2, T_\Lambda^\gamma Z_2\rangle|_{x_2=0} + \frac{1}{\varepsilon}|||Z_2|||^2_0 + \frac{1}{\varepsilon}|||Z_j|||^2_{1,\gamma} + \Sigma_{i\neq 2,3}\frac{1}{\varepsilon}|||Z_i|||^2_0\\ &\frac{1}{\varepsilon}\left(|||T_r^\gamma W^+|||^2_0 + |||W^+|||^2_{-1,\gamma} + |||F|||^2_0\right)\end{aligned}$$

Combining the above estimates together, we can have

$$\begin{aligned}&\frac{1}{\gamma}|||Z_3|||^2_{2,\gamma} + \gamma|||Z_j|||^2_{1,\gamma} + \gamma|||Z_2|||^2_{1,\gamma} + \frac{1}{\gamma}\|Z_3|_{x_2=0}\|^2_{\frac{3}{2},\gamma}\\ &\lesssim \gamma\Re\langle Z_2, T_\Lambda^\gamma Z_2\rangle|_{x_2=0} + \frac{1}{\gamma}\left(|||T_r^\gamma W^+|||^2_{1,\gamma} + |||W^+|||^2_0 + |||F|||^2_{1,\gamma}\right)\end{aligned}$$

Similarly, for $Z^- = (Z_8, Z_9, ..., Z_14) := T^\gamma_{\chi_1 Q^l} T^\gamma_{\chi_{p_1}} W^-$, we have

$$\begin{aligned}&\frac{1}{\gamma}|||Z_{10}|||^2_{2,\gamma} + \gamma|||Z_j|||^2_{1,\gamma} + \gamma|||Z_9|||^2_{1,\gamma} + \frac{1}{\gamma}\|Z_{10}|_{x_2=0}\|^2_{\frac{3}{2},\gamma}\\ &\lesssim \gamma\Re\langle Z_9, T_\Lambda^\gamma Z_9\rangle|_{x_2=0} + \frac{1}{\gamma}\left(|||T_r^\gamma W^+|||^2_{1,\gamma} + |||W^+|||^2_0 + |||F|||^2_{1,\gamma}\right)\end{aligned}$$

Therefore, we only need to estimate $\gamma\Re\langle Z_2, T_\Lambda^\gamma Z_2\rangle|_{x_2=0}$ and $\gamma\Re\langle Z_9, T_\Lambda^\gamma Z_9\rangle|_{x_2=0}$. We denote that those two terms can be controlled by $\|Z_{in}|_{x_2=0}\|^2_{1,\gamma}$. The estimate of $\|Z_{in}|_{x_2=0}\|^2_{1,\gamma}$ can be obtained through the boundary condition, using the fact that the Lopatinskii determinant has a positive lower bound in the open neighborhood $\mathcal{V}^1_{p_1}$. We obtained

$$\|Z_{in}|_{x_2=0}\|^2_{1,\gamma} \lesssim \|G\|^2_{1,\gamma} + \|Z_{out}|_{x_2=0}\|^2_{1,\gamma} + \|W^{\rm nc}|_{x_2=0}\|^2_0.$$

Combining all together, we have

$$\begin{aligned}&\frac{1}{\gamma}|||Z_{out}|||_{2,\gamma}^2+\gamma|||Z_c|||_{1,\gamma}^2+\gamma|||Z_{in}|||_{1,\gamma}^2+\frac{1}{\gamma}\|Z_{out}|_{x_2=0}\|_{\frac{3}{2},\gamma}^2+\|Z_{in}|_{x_2=0}\|_{1,\gamma}^2\\&\lesssim\|G\|_{1,\gamma}^2+\|W^{\mathrm{nc}}|_{x_2=0}\|_0^2+\frac{1}{\gamma}\left(|||T_r^\gamma W^+|||_{1,\gamma}^2+|||W^+|||_0^2+|||F|||_{1,\gamma}^2\right)\end{aligned}\tag{3.2.54}$$

where $Z_c=(Z_1,Z_4,Z_5,Z_6,Z_7,Z_8,Z_{11},Z_{12},Z_{13},Z_{14})^\top$.

3.2.4.4 Case 4–other At the rest of the points, the Lopatinskii determinant does not vanish and system can be reduced to the non-characteristic form. This is exactly the good frequency case in [20]. By the same treatment as in those papers, we can construct the Kreiss symmetrizer. In particular, starting from the original system (3.2.17), we consider the cut-off symbol $\chi_{re}=1-\bar\chi_{p_1}-\bar\chi_{p_2}-\bar\chi_{rt}$ in Γ_k^0 for any integer k, where $\bar\chi_{p_1}$ is the sum of all four cut-off functions χ_{p_1} for the four neighborhood $\mathcal{V}_{p_1}^i, i=1,2,3,4$, $\bar\chi_{p_2}$ is the sum of the two cut-off functions χ_{p_2} for the two neighborhood $\mathcal{V}_{p_2}^1$ and $\mathcal{V}_{p_2}^2$, and $\bar\chi_{rt}$ is the sum of three cut-off functions χ_{rt} for the three neighborhood $\mathcal{V}_{rt}^i, i=1,2,3$. Since all the neighborhood above do not overlap, we obtain χ_{re} is also a cut-off function which equals 0 near the root of Lopatinskii determinant Υ_r and poles of the system Υ_p. Then we can construct an open neighborhood $\mathcal{V}_{re}$ which contains the support of χ_{re} but do not contain a small neighborhood of Υ_r and Υ_p and denote

$$W_{re}^\pm:=T_{\chi_{re}}^\gamma W^\pm,\ \text{ and } W_{re}:=(W_{re}^+,W_{re}^-)^\top.$$

By the same idea in [16, 20], we can eliminate all the components of $W_{re}^\pm$ in the kernal of I_2 and obtain a differential equation in $W_{re}^{\mathrm{nc}}:=T_{\chi_{re}}^\gamma W^{\mathrm{nc}}$. The equation taking the following form

$$\partial_2 W_{re}^{\mathrm{nc}}=T_{\chi_2\mathbb{A}}^\gamma W^{\mathrm{nc}}+T_{\mathbb{E}}^\gamma W^{\mathrm{nc}}+T_r^\gamma W+R_0F+R_{-1}W,\tag{3.2.55}$$

where $\mathbb{A}=\mathrm{diag}\{\mathbb{A}^r,\mathbb{A}^l\}$ and $\mathbb{E}$ is a symbol in Γ_1^0. Same as before, r is a symbol in Γ_1^0 which supports only in the place where $\chi_{re}\in(0,1)$. Combining with the boundary conditions in (3.2.17), we can construct the Kreiss symmetrizer as in Proposition 3.1 [16]. By a standard argument [16, 20] for a boundary value problem on half space, we can obtain the following estimate

$$\begin{aligned} &\gamma |||W_{re}|||_{1,\gamma}^2 + \|W_{re}^{\mathrm{nc}}|_{x_2=0}\|_{1,\gamma}^2 \\ &\quad \lesssim \|G\|_{1,\gamma}^2 + \|W^{\mathrm{nc}}|_{x_2=0}\|_0^2 + \frac{1}{\gamma}\left(|||F|||_{1,\gamma}^2 + |||W|||_0^2 + |||T_r^\gamma W|||_{1,\gamma}^2\right) \end{aligned} \tag{3.2.56}$$

3.2.5 Proof of the main theorem

Now we combine all the estimates in the above four cases. First we point out that the left hand side of (3.2.47), (3.2.52), (3.2.54) and (3.2.56) dominate $\gamma^3 \left|\left|\left|W_{\bar{\chi}_{p_1}}\right|\right|\right|_0^2 + \gamma^2 \|W_{\bar{\chi}_{p_1}}^{\mathrm{nc}}|_{x_2=0}\|_0^2$, $\gamma^3 |||W_{\bar{\chi}_{rt}}|||_0^2 + \gamma^2 \|W_{\bar{\chi}_{rt}}^{\mathrm{nc}}|_{x_2=0}\|_0^2$, $\gamma^3 \left|\left|\left|W_{\bar{\chi}_{p_2}}\right|\right|\right|_0^2 + \gamma^2 \|W_{\bar{\chi}_{p_2}}^{\mathrm{nc}}|_{x_2=0}\|_0^2$ and $\gamma^3 |||W_{\bar{\chi}_{re}}|||_0^2 + \gamma^2 \|W_{\bar{\chi}_{re}}^{\mathrm{nc}}|_{x_2=0}\|_0^2$ respectively. We note that

$$\begin{aligned} &\gamma^3 \left|\left|\left|W_{\bar{\chi}_{p_1}}\right|\right|\right|_0^2 + \gamma^2 \|W_{\bar{\chi}_{p_1}}^{\mathrm{nc}}|_{x_2=0}\|_0^2 + \gamma^3 |||W_{\bar{\chi}_{rt}}|||_0^2 + \gamma^2 \|W_{\bar{\chi}_{rt}}^{\mathrm{nc}}|_{x_2=0}\|_0^2 + \gamma^3 \left|\left|\left|W_{\bar{\chi}_{p_2}}\right|\right|\right|_0^2 \\ &+ \gamma^2 \|W_{\bar{\chi}_{p_2}}^{\mathrm{nc}}|_{x_2=0}\|_0^2 + \gamma^3 |||W_{\bar{\chi}_{re}}|||_0^2 + \gamma^2 \|W_{\bar{\chi}_{re}}^{\mathrm{nc}}|_{x_2=0}\|_0^2 = \gamma^3 |||W|||_0^2 + \gamma^2 \|W^{\mathrm{nc}}|_{x_2=0}\|_0^2 \end{aligned}$$

Adding all these equations and taking γ large enough, we can absorb $\|W^{\mathrm{nc}}|_{x_2=0}\|_0^2$ and $\frac{1}{\gamma}|||W|||_0^2$ into the terms on the left hand side of the sum of equations (3.2.47), (3.2.52), (3.2.54) and (3.2.56). So the only thing left with us is to absorb $\frac{1}{\gamma}|||T_r^\gamma W|||_{1,\gamma}^2$ into the terms on the left hand side of the sum of equations (3.2.47), (3.2.52), (3.2.54) and (3.2.56). Follow the same argument in [20, 16], we observe that the support of r in each case is contained in the following set

$$\left\{(t, x_1, x_2, \delta, \eta) \in \mathbb{R}_+^3 \times \Pi; \bar{\chi}_{p_1} \in (0,1) \text{ or } \bar{\chi}_{p_2} \in (0,1) \text{ or } \bar{\chi}_{rt} \in (0,1) \text{ or } \chi_{re} \in (0,1)\right\}.$$

Since $\bar{\chi}_{p_1} + \bar{\chi}_{p_2} + \bar{\chi}_{rt} + \chi_{re} = 1$, $r = 0$ if one of $\bar{\chi}_{p_1}$, $\bar{\chi}_{p_2}$, $\bar{\chi}_{rt}$ and χ_{re} equals 1. Moreover σ only vanishes at some points in the set where $\bar{\chi}_{p_1} = 1$ and $\bar{\chi}_{rt} = 1$. Thus σ has a positive lower bound on the support of r and r can be decomposed into

$$r = a_{p_2}\bar{\chi}_{p_2} + a_{re}\chi_{re} + a_{p_1}\sigma\chi_1^{p_1}\begin{pmatrix} Q^r & 0 \\ 0 & Q^l \end{pmatrix}\bar{\chi}_{p_1} + a_{rt}\sigma\chi_1^{rt}\begin{pmatrix} Q^r & 0 \\ 0 & Q^l \end{pmatrix}\bar{\chi}_{rt},$$

where a_{p_2}, a_{re}, a_{p_1} and a_{rt} are some block diagonal symbol in the class Γ_1^0, $\chi_1^{p_1}$ and χ_1^{rt} are the corresponding cut-off functions χ_1 in the Case 1 and Case 2 respectively. This suggests that $\frac{1}{\gamma}|||T_r^\gamma W|||_{1,\gamma}^2$ can be absorbed by

$$\gamma\left|\left|\left|T_\sigma^\gamma T_{\bar{\chi}_{p_1}}^\gamma W\right|\right|\right|_0^2 + \gamma|||T_\sigma^\gamma T_{\bar{\chi}_{rt}}^\gamma W|||_0^2 + \gamma\left|\left|\left|T_{\bar{\chi}_{p_2}}^\gamma W\right|\right|\right|_{1,\gamma}^2 + \gamma|||T_{\chi_{re}}^\gamma W|||_{1,\gamma}^2,$$

with γ large enough. The above sum can also be dominated by the sum of the left hand sides of equations (3.2.47), (3.2.52), (3.2.54) and (3.2.56) with γ large enough. Therefore by adding the equations (3.2.47), (3.2.52), (3.2.54) and (3.2.56), we obtain

$$\|W^{\mathrm{nc}}|_{x_2=0}\|_0^2 \leq C_0\left(\frac{1}{\gamma^3}\left|\left|\left|\tilde{F}\right|\right|\right|_{1,\gamma}^2 + \frac{1}{\gamma^2}\|\tilde{G}\|_{1,\gamma}^2\right),$$

and finish the proof of Theorem 3.2.1.

3.3 LINEARIZED PROBLEM: EXISTENCE AND TAME ESTIMATE

3.3.1 Well-posedness of the linearized problem

In this subsection, we show the well-posedness of the linear system which we need in the nonlinear analysis. First we clarify the details of the linear system we consider. For the background states $U^{r,l}$ and $\Phi^{r,l}$ (3.2.1) defined in the previous section, We assume (3.2.2), (3.2.3), (3.2.4), (3.2.5) and the compactness of the support of the perturbation $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$. Starting from this section, we specify the compact support of $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ as, for some fixed $T > 0$,

$$\mathrm{Supp}(\dot{U}^{r,l}, \dot{\Phi}^{r,l}) \subset \{t \in [-T, 2T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + 2\lambda_{\max} T\}, \tag{3.3.1}$$

where $\lambda_{\max}$ is the largest characteristic speed of the system (1.0.1)-(1.0.3) on the constant state $\bar{U}^{\pm}$. As in previous section, by standard method of linearization of the system, changing the unkown to Alinhac's 'good unknown'and neglecting the zero order derivatives of Ψ, we obtain

$$\begin{cases} \mathbb{L}_e'(U^{r,l},\Phi^{r,l})\dot{V}^{\pm} = f^{r,l}, & x_2 > 0, \\ \mathbb{B}_e'(\dot{V},\psi) = g, & x_2 = 0. \end{cases} \tag{3.3.2}$$

Where $\mathbb{L}_e'(U^{r,l},\Phi^{r,l})$ and $\mathbb{B}_e'$ are $L_{r,l}'$ and B' defined in (3.2.8) respectively. Then we formulate the linear problem and its well-posedness result in the followng Theorem.

Theorem 3.3.1. *Let $T > 0$, the stationary solution $\bar{U}^{r,l}$ satisfy the condition in Theorem 3.2.1, perturbations $\dot{U}^{r,l}, \dot{\Phi}^{r,l}$ satisfy (3.3.1) and (3.2.2) and the perturbed states $U^{r,l}, \Phi^{r,l}$ satisfy (3.2.3) and (3.2.4). There are constants $K_0 > 0$, $\gamma_0 \geq 1$ and $C_0 > 0$ which do not depend on T, such that if $K \leq K_0$ in (3.2.2) and $\gamma \geq \gamma_0$, then for all $f^{r,l} \in L^2(\mathbb{R}_+; H^1(\omega_T))$ and $g \in H^1_\gamma(\omega_T)$ which vanish for $t < 0$, there exists a unique $(\dot{V},\psi) \in L^2(\Omega_T) \times H^1(\omega_T)$ vanishing for $t < 0$ and satisfying*

$$\begin{cases} L_e'(U^{r,l},\Phi^{r,l})\dot{V}^{\pm} = f^{r,l}, & t < T,\ x_2 > 0, \\ B_e'(\dot{V},\psi) = g, & t < T,\ x_2 = 0. \end{cases}$$

In addition $\dot{V} \in \mathcal{C}([0,T]; L^2(\mathbb{R}^2_+))$ and we have the following inequality for all $\gamma \geq \gamma_0$ and all $t \in [0,T]$:

$$e^{-\gamma t}\|\dot{V}\|_{L^2(\mathbb{R}^2)} + \sqrt{\gamma}\|\dot{V}\|_{L^2(H^0_\gamma)} + \|\dot{V}^n|_{x_2=0}\|_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|_{H^1_\gamma(\omega_T)} \leq C\left(\frac{1}{\gamma^{3/2}}\|f^r, f^l\|_{L^2(H^1_\gamma)} + \frac{1}{\gamma}\|g\|_{H^1_\gamma(\omega_T)}\right).$$

With the result of linear stability (Theorem 3.2.1), the proof of this theorem is standard and can be found in [18, 21], where the key part is to find out a dual problem of (3.3.2) and verify that it also holds the a priori estimate with one loss of derivative. As as in [10, 21], the dual problem is

$$\begin{cases} \mathbb{L}_e'(U^{r,l},\Phi^{r,l})^*\dot{V}_*^{\pm} = f_*^{r,l} & \text{for } x_2 > 0, \\ N_1\dot{V}_*|_{x_2=0} = 0, & \\ \operatorname{div}\left(\underline{b}^\top M_1\dot{V}_*|_{x_2=0}\right) - \begin{pmatrix} \partial_2 U^r/\partial_2\Phi^r \\ \partial_2 U^l/\partial_2\Phi^l \end{pmatrix}^\top \underline{M}^\top M_1\dot{V}_*|_{x_2=0} = 0, & \end{cases} \tag{3.3.3}$$

where $\mathbb{L}'_e(U^{r,l}, \Phi^{r,l})^*$ is the adjoint operator of $\mathbb{L}'_e(U^{r,l}, \Phi^{r,l})$ and

$$N_1 = \left(0, \frac{-c^2\partial_1\varphi}{2\rho\partial_2\Phi_r}, \frac{c^2}{2\rho\partial_2\Phi_r}, 0, 0, 0, 0, 0, \frac{-c^2\partial_1\varphi}{2\rho\partial_2\Phi_l}, \frac{c^2}{2\rho\partial_2\Phi_l}, 0, 0, 0, 0\right)|_{x_2=0},$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho}{\partial_2\Phi_l} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\rho}{\partial_2\Phi_r} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\rho}{\partial_2\Phi_l} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-c^2\partial_1\varphi}{2\rho\partial_2\Phi_r} & \frac{c^2}{2\rho\partial_2\Phi_r} & 0 & 0 & 0 & 0 & 0 & \frac{c^2\partial_1\varphi}{2\rho\partial_2\Phi_l} & \frac{c^2}{2\rho\partial_2\Phi_l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since our system (3.3.2) is symmetrizable, we can easily verify the a priori estimate by the same calculation as in the previous section.

3.3.2 Tame estimate

Since the estimate we obtained lose at least one order of tangential derivative in the linear stability, we need to employ the Nash-Moser iterative scheme in the nonlinear analysis. In this subsection, we derive the a priori tame estimate. To guarantee the argument, we strengthen the smallness assumption on the perturbed background states as

$$[\dot{U}, \dot{\Phi}]_{10,\gamma,T} \leq K. \tag{3.3.4}$$

We note that $[\cdot]_{s,\gamma,T}$ is the sth order norm of anisotropic Sobolev space $H^{s,\gamma}_*(\Omega_T)$ and the above condition implies (3.2.2) up to some trivial extension in time variable.

3.3.2.1 Tangential derivatives

In this part, we want to derive the estimate on the tangential derivatives and obtain

Proposition 3.3.1. *Let $s \in \mathbb{N}$, $s \geq 1$, and $T > 0$. There are two constant $C_s > 0$ and $\gamma_s \geq 1$ which does not depend on T, such that for the system (3.2.9)-(3.2.10) and all $\gamma \geq \gamma_s$, the following a priori estimate satisfied*

$$\begin{aligned}
&\sqrt{\gamma}\|W\|_{L^2(H^s_\gamma(\omega_T))} + \|W^{nc}|_{x_2=0}\|_{H^s_\gamma(\omega_T)} + \|\varphi\|_{H^{s+1}_\gamma(\omega_T)} \\
&\leq C_s\left\{\frac{1}{\gamma^{3/2}}\|F\|_{L^2(H^{s+1}_\gamma(\omega_T))} + \frac{1}{\gamma}\|g\|_{H^{s+1}_\gamma(\omega_T)} + \frac{1}{\gamma^{3/2}}\|W\|_{W^{1,tan}(\Omega_T)}\|(\dot{U}, \nabla\dot{\Phi})\|_{H^{s+2}_\gamma(\Omega_T)}\right. \\
&\qquad \left. + \frac{1}{\gamma}\left(\|W^{nc}|_{x_2=0}\|_{L^\infty(\omega_T)} + \|\psi\|_{W^{1,\infty}(\omega_T)}\right)\|(\dot{U}, \partial_2\dot{U}, \nabla\dot{\Phi})|_{x_2=0}\|_{H^{s+1}_\gamma(\omega_T)}\right\}
\end{aligned} \tag{3.3.5}$$

Proof. To prove this a priori estimate, we consider the lth order the tangential derivative $D^\beta = \partial_t^{\alpha_0}\partial_1^{\alpha_1}$, with multiindex $\beta = (\alpha_0, \alpha_1, 0, 0)$, $l = \alpha_0 + \alpha_1$ and $0 \le l \le s$, of the (3.2.9) and obtain

$$
\begin{aligned}
&A_0\partial_t D^\beta W + A_1\partial_1 D^\beta W + I_2\partial_2 D^\beta W + CD^\beta W \\
&\qquad + \sum_{\langle\beta'\rangle=1,\beta'\le\beta} c_{\beta'}\left[D^{\beta'}A_0 D^{\beta-\beta'}\partial_t W + D^{\beta'}A_1 D^{\beta-\beta'}\partial_1 W\right] = D^\beta F \\
&+ \sum_{\langle\beta'\rangle\ge2,\beta'\le\beta} c_{\beta'}\left[D^{\beta'}A_0 D^{\beta-\beta'}\partial_t W + D^{\beta'}A_1 D^{\beta-\beta'}\partial_1 W\right] + \sum_{\langle\beta'\rangle\ge1,\beta'\le\beta} c_{\beta'}\left[D^{\beta'}CD^{\beta-\beta'}W\right],
\end{aligned}
$$

where $c_{\beta'}$ denotes a constant number only depends on β and β'. We remark that all the W terms on the left hand side of the above equation are of the derivative order l and $l+1$, and all the W terms on the right hand side of the equation are of the derivative order less that l. Then by denoting the vector

$$
W^{(l)} = \{\partial_t^{\alpha_0}\partial_1^{\alpha_1}W, \quad \alpha_0 + \alpha_1 = l\},
$$

we can combine all the lth order derivatives of the interior equations (3.2.9) into one system as

$$
\mathcal{A}_0\partial_t W^{(l)} + \mathcal{A}_1\partial_1 W^{(l)} + \mathcal{I}\partial_2 W^{(l)} + \mathcal{C}W^{(l)} = \mathcal{F}^{(l)}, \tag{3.3.6}
$$

where $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{I}$ are block diagonal matrices with blocks to be A_0, A_1 and I_2 respectively, $\mathcal{F}^{(l)}$ denotes all the terms on the right of the above equation. For the boundary condition, we also applied the $l$the order tangential derivatives to the boundary condition to obtain

$$
\begin{aligned}
&\underline{b}\nabla D^\beta\psi + b_\sharp D^\beta\psi + \mathbf{M}D^\beta W^{\mathrm{nc}}|_{x_2=0} = D^\beta g + \\
&\qquad + \sum_{|\beta'|\ge1,\beta'\le\beta} c_{\beta'}\left[D^{\beta'}\mathbf{M}D^{\beta-\beta'}W^{\mathrm{nc}}|_{x_2=0} + D^{\beta'}\underline{b}\nabla D^{\beta-\beta'}\psi + D^{\beta'}b_\sharp D^{\beta-\beta'}\psi\right].
\end{aligned}
$$

Then combining all the derivatives of the boundary conditions, we have

$$
\mathcal{B}\nabla\psi^{(l)} + \mathcal{B}_\sharp\psi^{(l)} + \mathcal{M}W^{(l),\mathrm{nc}}|_{x_2=0} = \mathcal{G}^{(l)} \tag{3.3.7}
$$

Similarly as in [21], since the block diagonal structure of $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{I}$, we can easily verify that the new system have the same leading order derivatives as (3.2.9) and (3.2.10). Hence it is straightforward to obtain that the the similar estimate as in Theorem 3.2.1, which is

$$\begin{aligned} &\sqrt{\gamma}\|W^{(l)}\|_{L^2_\gamma(\Omega_T)} + \|W^{(l),\text{nc}}|_{x_2=0}\|_{L^2_\gamma(\Omega_T)} + \|\psi^{(l)}\|_{H^1_\gamma(\omega_T)} \\ &\qquad \le C_l\big(\frac{1}{\gamma^{3/2}}\|\mathcal{F}^{(l)}\|_{L^2(H^1_\gamma(\omega_T))} + \frac{1}{\gamma}\|\mathcal{G}^{(l)}\|_{H^1_\gamma(\omega_T)}\big). \end{aligned} \tag{3.3.8}$$

Now it is left to estimate $\mathcal{F}^{(l)}$ and $\mathcal{G}^{(l)}$ which are the right hand side of (3.3.6) and (3.3.7). First, we have

$$\begin{aligned} &\|D^\beta F\|_{L^2(H^1_\gamma(\omega_T))} \le \gamma\|D^\beta F\|_{L^2_\gamma(\Omega_T)} + \|\nabla^{\tan} D^\beta F\|_{L^2_\gamma(\Omega_T)} \le \|F\|_{L^2(H^{l+1}_\gamma(\omega_T))}, \\ &\|D^\beta g\|_{H^1_\gamma(\omega_T)} \le \|g\|_{H^{l+1}_\gamma(\omega_T)}. \end{aligned}$$

For other terms, we need first to fix x_2, apply Theroem 2.3.2 and integrate with respect to x_2. We take $\|D^{\beta'} A_0 D^{\beta-\beta'}\partial_t W\|_{L^2(H^1_\gamma(\omega_T))}$ with $|\beta'| \ge 2$ as an example.

$$\begin{aligned} &\|D^{\beta'} A_0(x_2) D^{\beta-\beta'}\partial_t W(x_2)\|_{H^1_\gamma(\omega_T)} \le c\|(\nabla^{\tan})^2 A_0(x_2)\partial_t W(x_2)\|_{H^{l-1}_\gamma(\omega_T)} \\ &\le C\left\{\|A_0(x_2)\|_{W^{2,\infty}(\omega_T)}\|W(x_2)\|_{H^l_\gamma(\omega_T)} + \|A_0(x_2)\|_{H^{l+1}_\gamma(\omega_T)}\|\partial_t W(x_2)\|_{L^\infty(\omega_T)}\right\}. \end{aligned}$$

Since A_0 is a C^∞ function of $(\dot{U}\ ,\ \nabla\dot{\Phi})$ and $\|(\dot{U}^r, \dot{U}^l)\|_{W^{2,\infty}(\Omega_T)} + \|(\dot{\Phi}^r, \dot{\Phi}^l)\|_{W^{3,\infty}(\Omega_T)} \le [\dot{U}, \dot{\Phi}]_{10,\gamma,T} \le K$, by integrating the above inequality with respect to x_2, we obtain

$$\begin{aligned} &\|D^{\beta'} A_0 D^{\beta-\beta'}\partial_t W\|_{L^2(H^1_\gamma(\omega_T))} \\ &\qquad \le C(K)\left\{\|W\|_{L^2(H^l_\gamma(\omega_T))} + \|W\|_{W^{1,\tan}(\Omega_T)}\|(\dot{U}, \nabla\dot{\Phi})\|_{L^2(H^{l+1}_\gamma(\omega_T))}\right\}. \end{aligned}$$

Similarly we can show

$$\begin{aligned} &\|D^{\beta'} A_1 D^{\beta-\beta'}\partial_1 W\|_{L^2(H^1_\gamma(\omega_T))} \\ &\qquad \le C(K)\left\{\|W\|_{L^2(H^l_\gamma(\omega_T))} + \|W\|_{W^{1,\tan}(\Omega_T)}\|(\dot{U}, \nabla\dot{\Phi})\|_{L^2(H^{l+1}_\gamma(\omega_T))}\right\}, \\ &\|D^{\beta'} C D^{\beta-\beta'} W\|_{L^2(H^1_\gamma(\omega_T))} \le C(K)\left\{\|W\|_{L^2(H^l_\gamma(\omega_T))}\right. \\ &\qquad\qquad \left. + \|W\|_{L^\infty(\Omega_T)}\|(\dot{U}, \nabla\dot{U}, \nabla\dot{\Phi}, \nabla^{\tan}\nabla\dot{\Phi})\|_{L^2(H^{l+1}_\gamma(\omega_T))}\right\} \end{aligned}$$

Combining the above three estimates, we have

$$\|\mathcal{F}^{(l)}\|_{L^2(H^1_\gamma(\omega_T))} \leq C(K)\Big\{\|F\|_{L^2(H^{l+1}_\gamma(\omega_T))} + \|W\|_{L^2(H^l_\gamma(\omega_T))} + \|W\|_{W^{1,\tan}(\Omega_T)}\|(\dot{U}, \nabla\dot{U}, \nabla\dot{\Phi}, \nabla^{\tan}\nabla\dot{\Phi})\|_{L^2(H^{l+1}_\gamma(\omega_T))}\Big\}$$

Similarly, for $\mathcal{G}^{(l)}$, we can apply the same technique above to obtain

$$\|\mathcal{G}^{(l)}\|_{H^1_\gamma(\omega_T)} \leq C(K)\Big\{\|g\|_{H^{l+1}_\gamma(\omega_T)} + \|W^{\mathrm{nc}}|_{x_2=0}\|_{H^l_\gamma(\omega_T)} + \|\psi\|_{H^{l+1}_\gamma(\omega_T)} + \|W^{\mathrm{nc}}|_{x_2=0}\|_{L^\infty(\omega_T)}\|(\dot{U}|_{x_2=0}, \nabla\psi)\|_{H^{l+1}_\gamma(\omega_T)} + \|\psi\|_{W^{1,\infty}(\omega_T)}\|(\dot{U}, \partial_2\dot{U}, \nabla\dot{\Phi})|_{x_2=0}\|_{H^{l+1}_\gamma(\omega_T)}\Big\}.$$

At last, combining above with (3.3.8), multiplying the inequality by γ^{s-l}, summing over $l = 0, ..., s$, and choosing γ large enough to absorb the terms $\|W\|_{L^2(H^s_\gamma(\omega_T))}$, $\|W^{\mathrm{nc}}|_{x_2=0}\|_{H^s_\gamma(\omega_T)}$ and $\|\psi\|_{H^{s+1}_\gamma(\omega_T)}$ on the right hand side of the estimate by the corresponding terms on the left hand side of the inequality, we obtain the result in this proposition. □

3.3.2.2 Weighted normal derivatives To obtain the estimate on the normal derivatives in anisotropic Sobolev space, we first need estimate $[\partial_2 W^{\mathrm{nc}}]_{s-1,\gamma,T}$. So we rewrite (3.2.9) as

$$I_2\partial_2 W^\pm = F^{r,l} - A_0^{r,l}\partial_t W^\pm - A_1^{r,l}\partial_1 W^\pm - A_0^{r,l}C^{r,l}W^\pm.$$

Since $I_2 = \mathrm{diag}\{0, 1, 1, 0, 0, 0, 0\}$, we have

$$[\partial_2 W^{\mathrm{nc}}]_{s-1,\gamma,T} \leq C\left\{[F]_{s-1,\gamma,T} + [A_0\partial_t W]_{s-1,\gamma,T} + [A_1\partial_1 W]_{s-1,\gamma,T} + [A_0 CW]_{s-1,\gamma,T}\right\}.$$

Then by the Theorem 2.3.6 and Theorem 2.3.8, we have

$$[A_0\partial_t W]_{s-1,\gamma,T} \leq [A_0 W]_{s,\gamma,T} \leq C\left\{\|A_0\|_{W^{1,\tan}}[W]_{s,\gamma,T} + [A_0]_{s,\gamma,T}\|W\|_{W^{1,\tan}}\right\},$$

$$[A_1\partial_1 W]_{s-1,\gamma,T} \leq [A_1 W]_{s,\gamma,T} \leq C\left\{\|A_1\|_{W^{1,\tan}}[W]_{s,\gamma,T} + [A_1]_{s,\gamma,T}\|W\|_{W^{1,\tan}}\right\},$$

$$[A_0 CW]_{s-1,\gamma,T} \leq C\left\{\|A_0 C\|_{W^{1,\tan}}[W]_{s-1,\gamma,T} + [A_0 C]_{s-1,\gamma,T}\|W\|_{W^{1,\tan}}\right\}.$$

Since we know $A_0^{r,l}$ and $A_1^{r,l}$ are C^∞ functions of $(\dot{U}^{r,l}, \nabla\dot{\Phi}^{r,l})$. $C^{r,l}$ is a C^∞ function of $(\dot{U}^{r,l}, \nabla\dot{U}^{r,l}, \nabla\dot{\Phi}^{r,l}, \nabla^{\tan}\nabla\dot{\Phi}^{r,l})$ which vanishes at origin. By assuming (3.3.4), we have

$$[A_0\partial_t W]_{s-1,\gamma,T} \leq C\Big\{[W]_{s,\gamma,T} + [\dot{U}, \nabla\dot{\Phi}]_{s,\gamma,T}\|W\|_{W^{1,\tan}}\Big\},$$

$$[A_1\partial_1 W]_{s-1,\gamma,T} \le C\left\{[W]_{s,\gamma,T} + [\dot{U},\nabla\dot{\Phi}]_{s,\gamma,T}\|W\|_{W^{1,\tan}}\right\},$$
$$[A_0 C W]_{s-1,\gamma,T} \le C\left\{[W]_{s-1,\gamma,T} + ([\dot{U}]_{s+1,\gamma,T} + [\nabla\dot{\Phi}]_{s,\gamma,T})\|W\|_{W^{1,\tan}}\right\}.$$

So in all we can obtain

$$[\partial_2 W^{\mathrm{nc}}]_{s-1,\gamma,T} \le C\left\{[F]_{s-1,\gamma,T} + [W]_{s,\gamma,T} + [\dot{U},\dot{\Phi}]_{s+2,\gamma,T}\|W\|_{W^{1,\tan}}\right\}. \tag{3.3.9}$$

Second we want to estimate the sth order derivatives in anisotropic Sobolev space with some weight on x_2 derivative. First for the convenience of the proof, we multiply A_0^{-1} to (3.2.9) and rewrite this equation in terms of the new variable $\widetilde{W} := e^{-\gamma t}W$. Hence we have

$$\gamma\widetilde{W} + \partial_t\widetilde{W} + A_0^{-1}A_1\partial_1\widetilde{W} + A_0^{-1}I_2\partial_2\widetilde{W} + C\widetilde{W} = A_0^{-1}\widetilde{F}, \tag{3.3.10}$$

where $\widetilde{F} = e^{-\gamma t}F$. Now we consider D^β with $\beta = (\alpha_0,\alpha_1,\alpha_2,k)$, where we require $\alpha_2 \ge 1$ and $1 \le \langle\beta\rangle \le s$. We apply D^β to (3.3.10), multiply it by $D^\beta W$ and integrate to obtain

$$\begin{aligned}
\gamma\langle D^\beta\widetilde{W}, D^\beta\widetilde{W}\rangle + \langle D^\beta\widetilde{W}, \partial_t D^\beta\widetilde{W}\rangle + \langle D^\beta\widetilde{W}, A_0^{-1}A_1\partial_1 D^\beta\widetilde{W}\rangle + \langle D^\beta\widetilde{W}, A_0^{-1}I_2\partial_2 D^\beta\widetilde{W}\rangle \\
+ \langle D^\beta\widetilde{W}, D^\beta(C\widetilde{W})\rangle + \langle D^\beta\widetilde{W}, [D^\beta, A_0^{-1}A_1]\partial_1\widetilde{W})\rangle + \langle D^\beta\widetilde{W}, [D^\beta, A_0^{-1}I_2]\partial_2\widetilde{W})\rangle \\
- \langle D^\beta\widetilde{W}, \alpha_2\sigma' A_0^{-1}I_2\partial_t^{\alpha_0}\partial_1^{\alpha_1}(\sigma\partial_2)^{\alpha_2-1}\partial_2^{k+1}\widetilde{W}\rangle = \langle D^\beta\widetilde{W}, D^\beta\widetilde{F}\rangle,
\end{aligned}$$

where $[D^\beta,\cdot]$ is the commutator. For the terms above, we have

$$\begin{aligned}
&\gamma\langle D^\beta\widetilde{W}, D^\beta\widetilde{W}\rangle = \gamma\|D^\beta\widetilde{W}\|^2_{L^2(\Omega_T)},\\
&\langle D^\beta\widetilde{W}, \partial_t D^\beta\widetilde{W}\rangle = \frac{1}{2}\int_{-\infty}^T\int_{\mathbb{R}^2_+}\partial_t\left((D^\beta\widetilde{W})^\top D^\beta\widetilde{W}\right)dxdt = \frac{1}{2}\|D^\beta\widetilde{W}(T)\|^2_{\mathbb{R}^2_+},\\
&\langle D^\beta\widetilde{W}, A_0^{-1}A_1\partial_1 D^\beta\widetilde{W}\rangle = \frac{1}{2}\int_{-\infty}^T\int_{\mathbb{R}^2_+}\partial_1\left((D^\beta\widetilde{W})^\top A_0^{-1}A_1 D^\beta\widetilde{W}\right)dxdt\\
&\qquad\qquad - \frac{1}{2}\langle D^\beta\widetilde{W}, \partial_1(A_0^{-1}A_1)D^\beta\widetilde{W}\rangle \le C(K)\|D^\beta\widetilde{W}\|^2_{L^2(\Omega_T)},\\
&\langle D^\beta\widetilde{W}, D^\beta(C\widetilde{W})\rangle \le \|D^\beta\widetilde{W}\|^2_{L^2(\Omega_T)} + [CW]^2_{s,\gamma,T}\\
&\qquad\qquad \le C(K)\left\{[W]^2_{s,\gamma,T} + \|W\|^2_{W^{1,\tan}}(1 + [\dot{U},\nabla\dot{\Phi}]^2_{s+2,\gamma,T})\right\},\\
&\langle D^\beta\widetilde{W}, D^\beta\widetilde{F}\rangle \le \|D^\beta\widetilde{W}\|^2_{L^2(\Omega_T)} + \|D^\beta\widetilde{F}\|^2_{L^2(\Omega_T)}.
\end{aligned}$$

Since $\alpha_2 \neq 0$, we have $D^\beta \widetilde{W}|_{x_2=0} = 0$. So we have

$$\langle D^\beta \widetilde{W}, A_0^{-1} I_2 \partial_2 D^\beta \widetilde{W} \rangle = \frac{1}{2} \int_{-\infty}^{T} \int_{\mathbb{R}^2_+} \partial_2 \left((D^\beta \widetilde{W})^\top A_0^{-1} I_2 D^\beta \widetilde{W} \right) dxdt \\ - \frac{1}{2} \langle D^\beta \widetilde{W}, \partial_2 (A_0^{-1} I_2) D^\beta \widetilde{W} \rangle \leq C(K) \| D^\beta \widetilde{W} \|^2_{L^2(\Omega_T)}$$

For the fifth term, we have

$$\begin{aligned} &\langle D^\beta \widetilde{W}, \alpha_2 \sigma' A_0^{-1} I_2 \partial_t^{\alpha_0} \partial_1^{\alpha_1} (\sigma \partial_2)^{\alpha_2 - 1} \partial_2^{k+1} \widetilde{W} \rangle \\ &\qquad \leq \| D^\beta \widetilde{W} \|^2_{L^2(\Omega_T)} + C(K) \| \partial_t^{\alpha_0} \partial_1^{\alpha_1} (\sigma \partial_2)^{\alpha_2 - 1} \partial_2^{k+1} \widetilde{W}^{\mathrm{nc}} \|^2_{L^2(\Omega_T)} \\ &\qquad \leq \| D^\beta \widetilde{W} \|^2_{L^2(\Omega_T)} + C(K) [\partial_2 W^{\mathrm{nc}}]^2_{s-1,\gamma,T}, \end{aligned}$$

where $[\partial_2 W^{\mathrm{nc}}]_{s-1,\gamma,T}$ can be estimate by (3.3.9). Next we consider the terms involving commutators.

$$\begin{aligned} &\langle D^\beta \widetilde{W}, [D^\beta, A_0^{-1} A_1] \partial_1 \widetilde{W}) \rangle \\ &\leq \| D^\beta \widetilde{W} \|^2_{L^2(\Omega_T)} + \sum_{|\alpha'|=1} \| \partial_*^{\alpha'} (A_0^{-1} A_1) D^{\beta - \alpha'} \partial_1 \widetilde{W} \|^2_{L^2(\Omega_T)} + \sum_{\langle \beta'' \rangle = 2} [D^{\beta''} (A_0^{-1} A_1) \partial_1 \widetilde{W}]^2_{s-2,T} \\ &\leq C(K) \left\{ [W]^2_{s,\gamma,T} + \| \widetilde{W} \|^2_{W^{1,\tan}} (1 + [\dot{U}, \nabla^{\tan} \dot{\Phi}]^2_{s+1,\gamma,T}) \right\}, \\ &\langle D^\beta \widetilde{W}, [D^\beta, A_0^{-1} I_2] \partial_2 \widetilde{W} \rangle \\ &\leq \| D^\beta \widetilde{W} \|^2_{L^2(\Omega_T)} + \sum_{|\alpha'|=1} \| \partial_*^{\alpha'} (A_0^{-1} I_2) D^{\beta - \alpha'} \partial_2 \widetilde{W} \|^2_{L^2(\Omega_T)} + \sum_{\langle \beta'' \rangle = 2} [D^{\beta''} (A_0^{-1} I_2) \partial_2 \widetilde{W}]^2_{s-2,T} \\ &\leq C(K) \left\{ [W]^2_{s,\gamma,T} + [\partial_2 W^{\mathrm{nc}}]^2_{s-1,\gamma,T} + \| \widetilde{W} \|^2_{W^{1,\tan}} (1 + [\dot{U}, \nabla \dot{\Phi}]^2_{s+2,\gamma,T}) \right\}, \end{aligned}$$

So in all, we have

$$\begin{aligned} &\gamma \| D^\beta \widetilde{W} \|^2_{L^2(\Omega_T)} + \frac{1}{2} \| D^\beta \widetilde{W}(T) \|^2_{L^2(\mathbb{R}^2_+)} \\ &\qquad \leq C(K) \{ [W]^2_{s,\gamma,T} + \| W \|^2_{W^{1,\tan}} (1 + [\dot{U}, \nabla \dot{\Phi}]^2_{s+2,\gamma,T}) + [F]_{s,\gamma,T} \}. \end{aligned} \tag{3.3.11}$$

3.3.2.3 Unweighted normal derivatives In this part, we want to estimate the sth order derivatives in anisotropic Sobolev space with no weight on x_2 derivative. That is the case where $\alpha_2 = 0$ and $k > 1$ in β. Then we apply $D^\beta = \partial_{\tan}^\alpha \partial_2^k$ to the (3.3.10) and obtain

$$\begin{aligned}
&\gamma\langle D^\beta \widetilde{W}, D^\beta \widetilde{W}\rangle + \langle D^\beta \widetilde{W}, \partial_t D^\beta \widetilde{W}\rangle + \langle D^\beta \widetilde{W}, A_0^{-1} A_1 \partial_1 D^\beta \widetilde{W}\rangle + \langle D^\beta \widetilde{W}, A_0^{-1} I_2 \partial_2 D^\beta \widetilde{W}\rangle \\
&+ \langle D^\beta \widetilde{W}, D^\beta (C\widetilde{W})\rangle + \langle D^\beta \widetilde{W}, [D^\beta, A_0^{-1} A_1]\partial_1 \widetilde{W})\rangle + \langle D^\beta \widetilde{W}, [D^\beta, A_0^{-1} I_2]\partial_2 \widetilde{W})\rangle \\
&= \langle D^\beta \widetilde{W}, D^\beta \widetilde{F}\rangle,
\end{aligned}$$

Here we can use the same technique as in estimating the derivatives with weight on x_2 derivatives, except the fourth term.

$$\begin{aligned}
&\langle D^\beta \widetilde{W}, A_0^{-1} I_2 \partial_2 D^\beta \widetilde{W}\rangle \\
&\qquad = \frac{1}{2}\int_{-\infty}^{T}\int_{\mathbb{R}^2_+} \partial_2 \left((D^\beta \widetilde{W})^\top A_0^{-1} I_2 D^\beta \widetilde{W}\right) dx dt - \frac{1}{2}\langle D^\beta \widetilde{W}, \partial_2 (A_0^{-1} I_2) D^\beta \widetilde{W}\rangle \\
&\qquad = -\frac{1}{2}\int_{\omega_T} (D^\beta \widetilde{W})^\top A_0^{-1} I_2 D^\beta \widetilde{W}|_{x_2=0} dx_1 dt - \frac{1}{2}\langle D^\beta \widetilde{W}, \partial_2 (A_0^{-1} I_2) D^\beta \widetilde{W}\rangle.
\end{aligned}$$

Again, we know

$$\frac{1}{2}\langle D^\beta \widetilde{W}, \partial_2 (A_0^{-1} I_2) D^\beta \widetilde{W}\rangle \le C(K)\|D^\beta \widetilde{W}\|^2_{L^2(\Omega_T)}.$$

We only left to estimate the integral on ω_T.

$$\begin{aligned}
\frac{1}{2}\int_{\omega_T} (D^\beta \widetilde{W})^\top A_0^{-1} I_2 D^\beta \widetilde{W}|_{x_2=0} dx_1 dt &\le C(K)\|D^\beta \widetilde{W}^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2(\omega_T)} \\
&\le C(K)\|\partial_{\tan}^\alpha \partial_2^{k-1}(\widetilde{F} - \gamma A_0 \widetilde{W} - A_0 \partial_t \widetilde{W} - A_1 \partial_1 \widetilde{W} - C\widetilde{W})|_{x_2=0}\|^2_{L^2(\omega_T)} \\
&\le C\int_0^\infty \int_{\omega_T} \partial_2 |\partial_{\tan}^\alpha \partial_2^{k-1}(\widetilde{F} - \gamma A_0 \widetilde{W} - A_0 \partial_t \widetilde{W} - A_1 \partial_1 \widetilde{W} - C\widetilde{W})|^2 dx_1 dt dx_2.
\end{aligned}$$

We notice there are at most $s+1$th order derivatives of anisotropic Sobolev space in the above integral. So by apply the similar strategy as in previous case, we can obtain

$$\begin{aligned}
&\gamma\|D^\beta \widetilde{W}\|^2_{L^2(\Omega_T)} + \frac{1}{2}\|D^\beta \widetilde{W}(T)\|^2_{L^2(\mathbb{R}^2_+)} \\
&\qquad \le C(K)\{[W]^2_{s,\gamma,T} + \|W\|^2_{W^{1,\tan}}(1 + [\dot{U}, \nabla\dot{\Phi}]^2_{s+2,\gamma,T}) + [F]_{s,\gamma,T}\}.
\end{aligned} \tag{3.3.12}$$

3.3.2.4 All deriviatives Combining (3.3.5), (3.3.11) and (3.3.12) and the definition of the anisotropic Sobolev space, we obtain the following theorem about the tame estimate.

Theorem 3.3.2. *Let $s \in \mathbb{N}$ and $T > 0$, the stationary solution $\bar{U}^{r,l}$ satisfy the condition in Theorem 3.2.1, perturbations $\dot{U}^{r,l}, \dot{\Phi}^{r,l}$ satisfy (3.3.1) and (3.3.4) and the perturbed states $U^{r,l}, \Phi^{r,l}$ satisfy (3.2.3) and (3.2.4). There are constants $K_0 > 0$, $\gamma_s \geq 1$ and $C_s > 0$ which do not depend on T, such that if $K \leq K_0$ in (3.3.4) and $\gamma \geq \gamma_s$, the following a priori estimate satisfies*

$$\begin{aligned}&\sqrt{\gamma}[\dot{V}]_{s,\gamma,T} + \|\dot{V}^{nc}|_{x_2=0}\|_{H^s_\gamma(\omega_T)} + \|\psi\|_{H^{s+1}_\gamma(\omega_T)} \\ &\leq C_s \left\{ [f^r, f^l]_{s+1,\gamma,T} + \|g\|_{H^{s+1}_\gamma(\omega_T)} + \left([f^r, f^l]_{6,\gamma,T} + \|g\|_{H^6_\gamma(\omega_T)} \right) [\dot{U}, \dot{\Phi}]_{s+4,\gamma,T} \right\},\end{aligned} \tag{3.3.13}$$

where $\dot{V}^{\pm}$ and ψ satisfy the linear problem (3.3.2).

Proof. By adding (3.3.5), (3.3.11) and (3.3.12) together, we have

$$\begin{aligned}&\sqrt{\gamma}[W]_{s,\gamma,T} + \|W^{\mathrm{nc}}|_{x_2=0}\|_{H^s_\gamma(\omega_T)} + \|\psi\|_{H^{s+1}_\gamma(\omega_T)} \\ &\quad \leq C(K) \Big\{ [F]_{s+1,\gamma,T} + \|g\|_{H^{s+1}_\gamma(\omega_T)} + \|W\|_{W^{1,\tan}} [\dot{U}, \nabla\dot{\Phi}]_{s+2,\gamma,T} \\ &\quad + \frac{1}{\gamma} \left(\|W^{\mathrm{nc}}|_{x_2=0}\|_{L^\infty(\omega_T)} + \|\psi\|_{W^{1,\infty}(\omega_T)} \right) \|\dot{U}, \partial_2 \dot{U}, \nabla\dot{\Phi}|_{x_2=0}\|_{H^{s+1}_\gamma(\omega_T)} \Big\}\end{aligned}$$

To absorb $\|W\|_{W^{1,\tan}}, \|W^{\mathrm{nc}}|_{x_2=0}\|_{L^\infty(\omega_T)}$ and $\|\psi\|_{W^{1,\infty}(\omega_T)}$ on the right, we take $s = 5$ and obtain

$$\begin{aligned}&\sqrt{\gamma}[W]_{5,\gamma,T} + \|W^{\mathrm{nc}}|_{x_2=0}\|_{H^5_\gamma(\omega_T)} + \|\psi\|_{H^6_\gamma(\omega_T)} \leq C(K) \Big\{ [F]_{6,\gamma,T} + \|g\|_{H^6_\gamma(\omega_T)} \\ &+ \|W\|_{W^{1,\tan}} [\dot{U}, \nabla\dot{\Phi}]_{7,\gamma,T} + \frac{1}{\gamma} \left(\|W^{\mathrm{nc}}|_{x_2=0}\|_{L^\infty(\omega_T)} + \|\psi\|_{W^{1,\infty}(\omega_T)} \right) \|\dot{U}, \partial_2 \dot{U}, \nabla\dot{\Phi}|_{x_2=0}\|_{H^6_\gamma(\omega_T)} \Big\}.\end{aligned}$$

Since $\|W\|_{W^{1,\tan}} \leq [W]_{5,\gamma,T}$, and we assume $\|\dot{U}, \partial_2 \dot{U}, \nabla\dot{\Phi}|_{x_2=0}\|_{H^6_\gamma(\omega_T)} + [\dot{U}, \nabla\dot{\Phi}]_{7,\gamma,T} \leq [\dot{U}, \nabla\dot{\Phi}]_{10,\gamma,T} \leq K_0$, we have

$$\sqrt{\gamma}[W]_{5,\gamma,T} + \|W^{\mathrm{nc}}|_{x_2=0}\|_{H^5_\gamma(\omega_T)} + \|\psi\|_{H^6_\gamma(\omega_T)} \leq C(K) \left\{ [F]_{6,\gamma,T} + \|g\|_{H^6_\gamma(\omega_T)} \right\}$$

Putting together, we have for $s \geq 5$,

$$\begin{aligned}&\sqrt{\gamma}[W]_{s,\gamma,T} + \|W^{\mathrm{nc}}|_{x_2=0}\|_{H^s_\gamma(\omega_T)} + \|\psi\|_{H^{s+1}_\gamma(\omega_T)} \leq C(K) \Big\{ [F]_{s+1,\gamma,T} + \|g\|_{H^{s+1}_\gamma(\omega_T)} \\ &+ \left([F]_{6,\gamma,T} + \|g\|_{H^6_\gamma(\omega_T)} \right) \left([\dot{U}, \nabla\dot{\Phi}]_{s+2,\gamma,T} + \|\dot{U}, \partial_2 \dot{U}, \nabla\dot{\Phi}|_{x_2=0}\|_{H^{s+1}_\gamma(\omega_T)} \right) \Big\}\end{aligned}$$

$$\leq C(K)\left\{[F]_{s+1,\gamma,T}+\|g\|_{H^{s+1}_\gamma(\omega_T)}+\left([F]_{6,\gamma,T}+\|g\|_{H^6_\gamma(\omega_T)}\right)[\dot{U},\dot{\Phi}]_{s+4,\gamma,T}\right\}$$

Since from (3.2.9), we know $W^\pm := T^{-1}\dot{V}^\pm$ and $F^{r,l}=A_0^{r,l}T^{-1}f^{r,l}$. Thus we obtain

$$\begin{aligned}
[\dot{V}]_{s,\gamma,T} &\leq C(K)\left\{\|T\|_{W^{1,\tan}}[W]_{s,\gamma,T}+[T]_{s,\gamma,T}\|W\|_{W^{1,\tan}}\right\}\\
&\leq C(K)\left\{[W]_{s,\gamma,T}+\|W\|_{W^{1,\tan}}[\dot{U},\nabla^{\tan}\dot{\Phi}]_{s,\gamma,T}\right\}\\
\|\dot{V}^{\mathrm{n}}|_{x_2=0}\|_{H^s_\gamma(\omega_T)} &\leq C(K)\left\{\|W^{\mathrm{nc}}|_{x_2=0}\|_{H^s_\gamma(\omega_T)}+\|W^{\mathrm{nc}}|_{x_2=0}\|_{L^\infty(\omega_T)}\|\dot{U},\nabla^{\tan}\dot{\Phi}|_{x_2=0}\|_{H^s_\gamma(\omega_T)}\right\},\\
[F]_{s+1,\gamma,T} &\leq C(K)\left\{[f]_{s+1,\gamma,T}+\|f\|_{W^{1,\tan}}[\dot{U},\nabla\dot{\Phi}]_{s+1,\gamma,T}\right\}
\end{aligned}$$

Combining all above, we obtained (3.3.13) in the theorem. □

Remark 3.3.1. The estimates in Theorem 3.3.2 is an a priori estimates, in the sense that given a smooth solution (3.3.13) holds. Morevoer by Theorem 3.3.1, if the sources terms (f,g) are in $L^2(H^1(\omega_T))\times H^1(\omega_T)$ and vanish in the past, the linearized problem (3.3.2) is well-posed. Following the argument in [10], Theorem 3.3.1 can be extended such that given sources terms (f,g) are in $H^{s+1,\gamma}_*(\Omega_T)\times H^{s+1}_\gamma(\omega_T)$ and vanish in the past, there is solution $(\dot{V},\psi)\in H^{s,\gamma}_*(\Omega_T)\times H^{s+1}_\gamma(\omega_T)$ to the linearized problem (3.3.2). Thus (3.3.13) is satisfied for this solution.

4.0 NONLINEAR ANALYSIS

In this chapter, we will construct nontrivial solutions to the nonlinear system (3.0.6)-(3.0.7), from the linear analysis we did in the previous chapter. First we will construct approximate solutions from a particular class of initial datas. With these approximate solutions, we can formulate an nonlinear problem with zero initial data. By iteratively solving the linearization of these nonlinear problems as in Theorem 3.3.1 and updating the approximate solutions, we can obtain a sequence of solutions. At last we show the sequence of solutions converge to a solution to the nonlinear system (3.0.6)-(3.0.7).

4.1 MAIN RESULTS ON EXISTENCE AND NONLINEAR STABILITY

Before we go further into the discussion, we give the existence results as follows:

Theorem 4.1.1. *Let $T > 0$, $\alpha \in \mathbb{N}$ with $\alpha \geq 15$ and the stationary solution $\bar{U}^{r,l}$ satisfying one of the following two conditions*

$$(i) \qquad \bar{v}^2 > 2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2,$$

$$(ii) \qquad \bar{v}^2 < \bar{F}_{11}^2 + \bar{F}_{12}^2 \qquad but$$

$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2}{4},\ \bar{v}^2 \neq \frac{\left(\sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2} - \sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2}\right)^2}{4},$$

$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2}{4},\ \bar{v}^2 \neq \frac{\left(\bar{F}_{11}^2 + \bar{F}_{12}^2\right)\left(2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2\right)}{4\left(\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2\right)}.$$

Assume that the initial data $(U_0^{\pm}, \varphi_0)$ have the form

$$U_0^{\pm} = \bar{U}^{\pm} + \dot{U}_0^{\pm},$$

with $\dot{U}_0 \in H_*^{2\alpha+15,\gamma}(\Omega_T)$, $\varphi_0 \in H_\gamma^{2\alpha+16}(\omega_T)$ *and being compatible up to order* $\alpha+7$ *in the sense of Definition 4.2.1 (see the next subsection). Assume also that* $(\dot{U}_0^\pm, \varphi_0^\pm)$ *have compact support. Then there exists* $\delta > 0$ *such that if* $[\dot{U}_0]_{2m+1,*} + \|\varphi_0\|_{H^{2m+2}} \le \delta$, *then there exists a solution* $U^\pm = \bar{U}^\pm + \dot{U}^\pm$, $\Phi^\pm = \pm x_2 + \dot{\Phi}^\pm$, φ *satisfying* (3.0.6), (3.0.7),

$$
\begin{aligned}
&U^\pm|_{t=0} = U_0^\pm, \quad \varphi|_{t=0} = \varphi_0,\\
&\partial_2\Phi^+ \ge \kappa, \quad \partial_2\Phi^- \le \kappa, \quad \forall (t, x_1, x_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+,
\end{aligned}
$$

on $[0,T]$, *where* κ *is some suitable positive constant. This solution satisfies* $(\dot{U}^\pm, \dot{\Phi}^\pm) \in H_*^{\alpha-1}((0,T)\times\mathbb{R}_+^2)$ *and* $\varphi \in H^\alpha((0,T)\times\mathbb{R})$.

4.2 APPROXIMATE SOLUTION

4.2.1 Compatibility condition for the initial data

Let $m \in \mathbb{N}$. We will start from the initial data $U_0^\pm = (\rho_0^\pm, v_0^\pm, u_0^\pm, F_{11,0}^\pm, F_{21,0}^\pm, F_{12,0}^\pm, F_{22,0}^\pm)^\top$ such that $U_0^\pm = \bar{U}^\pm + \dot{U}_0^\pm$, and φ_0. Our object is to prescribe necessary conditions on the initial data for the existence of smooth solutions in nonlinear system. First we assume $\dot{U}_0^\pm \in H_*^{2m+1}(\mathbb{R}_+^2)$ and $\varphi_0 \in H^{2m+2}(\mathbb{R})$ such that

$$
\text{Supp } \dot{U}_0^\pm \in \left\{ x_2 \ge 0, \sqrt{x_1^2 + x_2^2} \le 1 \right\}, \quad \text{Supp } \varphi_0 \subset [-1, 1]. \tag{4.2.1}
$$

For the initial change of the variable $\Phi_0^\pm = \pm x_2 + \dot{\Phi}_0^\pm$, we can construct $\dot{\Phi}_0^\pm$ by extending φ_0 to the whole spatial damain $\mathbb{R}_+^2$ by Proposition 2.1.1 such that $\dot{\Phi}_0^+ = \dot{\Phi}_0^- \in H_*^{2m+3}(\mathbb{R}_+^2)$, $\dot{\Phi}_0^+|_{x_2=0} = \dot{\Phi}_0^-|_{x_2=0} = \varphi_0$ and

$$
[\dot{\Phi}_0^\pm]_{2m+3,*} \le C\|\varphi_0\|_{H^{2m+2}(\mathbb{R})}. \tag{4.2.2}
$$

Up to multiplying the extension of $\dot{\Phi}_0$ by a suitable C^∞ function with compact support, we may additionally assume

$$
\text{Supp } \dot{\Phi}_0^\pm \in \left\{ x_2 \ge 0, \sqrt{x_1^2 + x_2^2} \le 1 + \frac{1}{2}\lambda_{\max} T \right\}. \tag{4.2.3}
$$

Moreover, by taking $m \geq 2$ and φ_0 small enough, we have

$$\partial_2 \Phi_0^+ \geq \frac{7}{8}, \quad \partial_2 \Phi_0^- \leq -\frac{7}{8}. \tag{4.2.4}$$

Now we are going to introduce the compatibility conditions on $\dot{U}_0^\pm$, $\dot{\Phi}_0^\pm$ and ψ_0. As in the classic approach [13, 21], we define

$$\begin{aligned}
&\partial_t \dot{\Phi}^\pm|_{t=0} := -v_0^\pm \partial_1 \dot{\Phi}_0^\pm + \dot{u}_0^\pm, \\
&\partial_t \dot{U}^\pm|_{t=0} := -A_1(U_0^\pm)\partial_1 \dot{U}_0^\pm - \frac{1}{\partial_2 \Phi_0^\pm}\left(A_2(U_0^\pm) + v_0^\pm \partial_1 \dot{\Phi}_0^\pm - \dot{u}_0^\pm - \partial_1 \dot{\Phi}_0^\pm A_1(U_0^\pm)\right)\partial_2 \dot{U}_0^\pm.
\end{aligned}$$

Then we could iteratively define $\partial_t^l \dot{U}^\pm|_{t=0}$ and $\partial_t^l \dot{\Phi}^\pm|_{t=0}$ for $l \in \mathbb{N}$. We denote

$$\dot{U}_l^\pm := \partial_t^l \dot{U}^\pm|_{t=0}, \quad \dot{\Phi}_l^\pm := \partial_t^l \dot{\Phi}^\pm|_{t=0}.$$

So as in [21], we could rewrite $\dot{U}_l^\pm$ and $\dot{\Phi}_l^\pm$ into the following forms:

$$\dot{\Phi}_l^\pm = \mathbf{F}_{l-1}(\dot{U}_0^\pm, \partial_1 \dot{\Phi}_0^\pm, \partial_2 \dot{\Phi}_0^\pm), \tag{4.2.5}$$

$$\dot{U}_l^\pm = \mathbf{G}_l(\dot{U}_0^\pm, \partial_1 \dot{\Phi}_0^\pm, \partial_2 \dot{\Phi}_0^\pm), \tag{4.2.6}$$

$$\dot{U}_l^\pm = \mathbf{H}_{l-1}(\dot{U}_0^\pm, \partial_1 \dot{U}_0^\pm, \partial_2 \dot{U}_0^\pm, \partial_1 \dot{\Phi}_0^\pm, \partial_2 \dot{\Phi}_0^\pm), \tag{4.2.7}$$

where $\mathbf{F}_{l-1}$, $\mathbf{G}_l$ and $\mathbf{H}_{l-1}$ are nonlinear functions of order $\leq l-1$, $\leq l$ and $\leq l-1$ respectively in the sense of Definition 2 in [21]. Hence we could treat $\mathbf{F}_{l-1}$, $\mathbf{G}_l$ and $\mathbf{H}_{l-1}$ as parts of the lth, $l-1$th and lth order derivatives of some C^∞ nonlinear functions respectively. By the estimate in anisotropic Sobolev norm for C^∞ nonlinear functions, we obtain:

$$\sum_{l=1}^{m} [\dot{U}_l]_{2(m-l)+1,*} + \sum_{l=1}^{m+1} [\dot{\Phi}_l]_{2(m-l)+3,*} \leq C\left([\dot{U}_0]_{2m+1,*} + \|\varphi\|_{H^{2m+2}}\right). \tag{4.2.8}$$

Moreover we have

$$\text{Supp } \dot{\Phi}_l^\pm \in \left\{x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + \frac{1}{2}\lambda_{\max} T\right\}, \quad \text{Supp } \dot{U}_l^\pm \in \left\{x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1\right\}$$

We introduce the definition of the compatibility condition of our initial data as follows:

Definition 4.2.1. Let $m \in \mathbb{N}$ with $m \geq 2$, $U_0^\pm = (\rho_0^\pm, v_0^\pm, u_0^\pm, F_{11,0}^\pm, F_{21,0}^\pm, F_{12,0}^\pm, F_{22,0}^\pm)^\top$ such that $U_0^\pm = \bar{U}^\pm + \dot{U}_0^\pm$ with $\dot{U}^\pm \in H_*^{2m+1}(\mathbb{R}_+^2)$ and $\varphi_0 \in H^{2m+2}(\mathbb{R})$ satisfying (4.2.1). Moreover $\Phi_0^\pm = \pm x_2 + \dot{\Phi}_0^\pm$ constructed as above satisfy (4.2.2), (4.2.3) and (4.2.4) when φ_0 sufficient small. At last let

$$\partial_1 \dot{\Phi}_0^\pm F_{11,0}^\pm - \dot{F}_{21,0}^\pm = 0, \quad \partial_1 \dot{\Phi}_0^\pm F_{12,0}^\pm - \dot{F}_{22,0}^\pm = 0 \tag{4.2.9}$$

Then the initial data is compatible up to order m if the traces of the functions $\dot{U}_1^\pm$,..., $\dot{U}_m^\pm$, $\dot{\Phi}_1^\pm$,..., $\dot{\Phi}_{m+1}^\pm$ which is defined as above satisfy

$$\partial_2^j(\dot{\Phi}_l^+ - \dot{\Phi}_l^-)|_{x_2=0} = 0 \text{ for } l = 0,1,2,...,m,\ j = 0,1,...,m-l,$$
$$\partial_2^j(\dot{\rho}_l^+ - \dot{\rho}_l^-)|_{x_2=0} = 0 \text{ for } l = 0,1,2,...,m-1,\ j = 0,1,...,m-1-l,$$

and

$$\int_{\mathbb{R}_+^2} |\partial_2^{m+1-j}(\dot{\Phi}_l^+ - \dot{\Phi}_l^-)|^2 dx_1 \frac{dx_2}{x_2} \leq \infty \quad \text{ for } j = 0,1,...,m+1,$$
$$\int_{\mathbb{R}_+^2} |\partial_2^{m-j}(\dot{\rho}_l^+ - \dot{\rho}_l^-)|^2 dx_1 \frac{dx_2}{x_2} \leq \infty \quad \text{ for } j = 0,1,...,m.$$

4.2.2 Construction of an approximate solution

Now we are going to construct an approximate solution. To simplify the notation, we denote the pair $(\mathbb{L}(U^{a+}, \Phi^{a+}), \mathbb{L}(U^{a-}, \Phi^{a-}))$ by $\mathbb{L}(U^a, \Phi^a)$. The properties of the approximate solutions have been listed in the following lemma.

Lemma 4.2.1. *For the intial data $(\dot{U}_0, \dot{\Phi}_0, \varphi_0)$ given in Definition 4.2.1 with $\dot{U}_0$ and φ_0 sufficient small , there are some functions U^a, Φ^a and φ^a such that $U^a - \bar{U} = \dot{U}^a \in H_*^{m+1}(\Omega_T)$, $\Phi^{a\pm} \mp x_2 = \dot{\Phi}^{a\pm} \in H_*^{m+2}(\Omega_T)$ and $\varphi^a \in H^{m+1}(\omega_T)$ and such that*

$$\partial_t^j \mathbb{L}(U^a, \Phi^a)|_{t=0} = 0, \quad \textit{for } j = 0,...,m-1, \tag{4.2.10}$$

$$\partial_t \Phi^a + v^a \partial_1 \Phi^a - u^a = 0, \tag{4.2.11}$$

$$F_{11}^a \partial_1 \Phi^a - F_{21}^a = 0, \tag{4.2.12}$$

$$F_{12}^a \partial_1 \Phi^a - F_{22}^a = 0, \tag{4.2.13}$$

$$\varphi^a = \Phi^{a+}|_{x_2=0} = \Phi^{a-}|_{x_2=0}, \tag{4.2.14}$$

$$\mathbb{B}(U^a|_{x_2=0}, \varphi^a) = 0. \tag{4.2.15}$$

Moreover,

$$[\dot{U}^a]_{m+1,*,T} + [\dot{\Phi}^a]_{m+2,*,T} + \|\varphi^a\|_{H^{m+1}(\omega_T)} \leq \varepsilon \left([\dot{U}_0]_{2m+1,*} + \|\varphi_0\|_{H^{2m+2}} \right), \tag{4.2.16}$$

$$\forall (t,x) \in \Omega, \quad \partial_2 \Phi^{a+}(t,x) \geq \frac{3}{4}, \quad \partial_2 \Phi^{a-}(t,x) \leq -\frac{3}{4}, \tag{4.2.17}$$

and the supports of these functions:

$$\begin{aligned} &\mathit{Supp}\ (\dot{U}^a, \dot{\Phi}^a) \subset \left\{ t \in [-T,T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + \lambda_{max} T \right\}, \\ &\mathit{Supp}\ \varphi^a \subset \{ t \in [-T,T], x_1 \in [-1 - \lambda_{max} T, 1 + \lambda_{max} T] \}. \end{aligned} \tag{4.2.18}$$

Proof. Since the compatibility condition, following the standard argument as in [21, 43, 72], we can choose $(\dot{\rho}^a, \dot{v}^a, \dot{F}^a_{11}, \dot{F}^a_{12}) \in H^{m+1}_*(\Omega_T)$, $\dot{\Phi}^a \in H^{m+2}_*(\Omega_T)$ such that

$$\begin{aligned} &\partial_t^j (\dot{\rho}^a, \dot{v}^a, \dot{F}^a_{11}, \dot{F}^a_{12})|_{t=0} = (\dot{\rho}_j, \dot{v}_j, \dot{F}_{11,j}, \dot{F}_{12,j}) \in H^{2(m-j)+1}_*(\mathbb{R}^2_+), && \text{for } j = 0, ..., m, \\ &\partial_t^j (\dot{\Phi}^a)|_{t=0} = (\dot{\Phi}_j) \in H^{2(m-j)+3}_*(\mathbb{R}^2_+), && \text{for } j = 0, ..., m+1, \end{aligned}$$

and

$$(\dot{\rho}^{a+}, \dot{v}^{a+}, \dot{F}^{a+}_{11}, \dot{F}^{a+}_{12})|_{x_2=0} = (\dot{\rho}^{a-}, \dot{v}^{a-}, \dot{F}^{a-}_{11}, \dot{F}^{a-}_{12})|_{x_2=0}, \quad \dot{\Phi}^{a+}|_{x_2=0} = \dot{\Phi}^{a-}|_{x_2=0},$$

Up to multiply a C_0^∞ fucntion, $(\dot{\rho}^a, \dot{v}^a, \dot{F}^a_{11}, \dot{F}^a_{12})$ satisfy (4.2.18). Next we define

$$\varphi^a := \dot{\Phi}^{a+}|_{x_2=0} = \dot{\Phi}^{a-}|_{x_2=0} \in H^{m+1}(\mathbb{R}^2), \tag{4.2.19}$$

$$\dot{u}^a := \partial_t \dot{\Phi}^a + v^a \partial_1 \dot{\Phi}^a \in H^{m+1}(\Omega), \tag{4.2.20}$$

$$\dot{F}^a_{21} := \partial_t \dot{\Phi}^a + F^a_{11} \partial_1 \dot{\Phi}^a \in H^{m+1}(\Omega), \tag{4.2.21}$$

$$\dot{F}^a_{22} := \partial_t \dot{\Phi}^a + F^a_{12} \partial_1 \dot{\Phi}^a \in H^{m+1}(\Omega). \tag{4.2.22}$$

Obviously φ^a and $(\dot{u}^a, \dot{F}^a_{21}, \dot{F}^a_{22})$ also satisfy (4.2.18). Moreover we can verify (4.2.11), (4.2.12), (4.2.13), (4.2.14), (4.2.15). From the definition of $\dot{\Phi}_j$ and (4.2.9), we have

$$\partial_t^j (\dot{u}^a, \dot{F}^a_{21}, \dot{F}^a_{22})|_{t=0} = (\dot{u}_j, \dot{F}_{21,j}, \dot{F}_{22,j}) \in H^{2(m-j)+1}_*(\mathbb{R}^2_+), \quad \text{for } j = 0, ..., m \tag{4.2.23}$$

Combining with definition of $\dot{U}_j$, we obtain (4.2.10). At last (4.2.16) and (4.2.17) is following by the continuity of the lifting operator in [72], Sobolev imbedding theorem and smallness of the initial data. □

At last, we use the approximate solution to reformulate the original problem into a nonlinear problem with zero initial data. First we define

$$f^a := \begin{cases} -\mathbb{L}(U^a, \Phi^a), & t > 0 \\ 0, & t < 0 \end{cases}. \tag{4.2.24}$$

From the above Lemma, we have $f^a \in H_*^{m-1}(\Omega_T)$ and

$$\mathrm{Supp} f^a \subset \left\{ t \in [0,T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + \lambda_{\max} T \right\}, \tag{4.2.25}$$

$$[f^a]_{m-1,*} \leq \varepsilon \left([\dot{U}_0]_{2m+1,*} + \|\varphi_0\|_{H^{2m+2}} \right), \tag{4.2.26}$$

where $\varepsilon(\cdot)$ is an increasing function vanishing at origin.

It is obvious that $(U, \Phi) = (U^a, \Phi^a) + (V, \Psi)$ is a solution to the nonlinear system, if $V = (V^+, V^-)$, $\Psi = (\Psi^+, \Psi^-)$ satisfying the following equations

$$\begin{cases} \mathcal{L}(V, \Psi) = f^a, & \text{in } \Omega_T, \\ \mathcal{E}(V, \Psi) := \partial_t \Psi + (v^a + v)\partial_1 \Psi - u + v\partial_1 \Phi^a = 0, & \text{in } \Omega_T, \\ (F_{11}^a + F_{11})\partial_1 \Psi - F_{21} + F_{11}\partial_1 \Phi^a = 0, & \text{in } \Omega_T, \\ (F_{12}^a + F_{12})\partial_1 \Psi - F_{22} + F_{12}\partial_1 \Phi^a = 0, & \text{in } \Omega_T, \\ \Psi^+|_{x_2=0} = \Psi^-|_{x_2=0} =: \psi, & \text{on } \omega_T, \\ \mathcal{B}(V|_{x_2=0}, \psi) = 0, & \text{on } \omega_T, \\ (V, \Psi) = 0, & \text{for } t < 0, \end{cases} \tag{4.2.27}$$

where

$$\mathcal{L}(V, \Psi) := \mathbb{L}(U^a + V, \Phi^a + \Psi) - \mathbb{L}(U^a, \Phi^a),$$

$$\mathcal{B}(V|_{x_2=0}, \psi) := \mathbb{B}(U^a|_{x_2=0} + V|_{x_2=0}, \varphi^a + \psi).$$

4.3 DESCRIPTION OF THE ITERATIVE SCHEME

As we showed in the last section, with the approximate solution, we actually just need to solve a nonlinear system with zero initial data. Now we illustrate how to solve this nonlinear system.

4.3.1 The smoothing operators

Before we go into the details of iterative scheme, we introduce the smoothing operator we need to use. As in [13, 21, 72], we have

Proposition 4.3.1. *There exists a family of smoothing operators $\{S_\theta\}_{\theta\geq 1}$ acting on the class of functions in $H^s_*(\Omega_T)$ and vanishing in the past such that*

$$
\begin{aligned}
&[S_\theta u]_{\beta,\gamma,T} \leq C\theta^{(\beta-\alpha)_+}[u]_{\alpha,\gamma,T}, && \alpha,\beta\geq 0,\\
&[S_\theta u - u]_{\beta,\gamma,T} \leq C\theta^{\beta-\alpha}[u]_{\alpha,\gamma,T}, && 0\leq\beta\leq\alpha,\\
&[\frac{d}{d\theta}S_\theta u]_{\beta,\gamma,T} \leq C\theta^{\beta-\alpha-1}[u]_{\alpha,\gamma,T}, && \alpha,\beta\geq 0,
\end{aligned}
$$

where $C>0$ is a constant depending on α and β, and $(\beta-\alpha)_+ := max\,(0,\beta-\alpha)$. Moreover, (i) if $U=(u^+,u^-)$ satisfies $u^+=u^-$ on ω_T, then $S_\theta u^+ = S_\theta u^-$ on ω_T, (ii) the following estimate holds:

$$\|(S_\theta u^+ - S_\theta u^-)|_{x_2=0}\|_{H^\beta_\gamma(\omega_T)} \leq C\theta^{(\beta+1-\alpha)_+}\|(u^+-u^-)|_{x_2=0}\|_{H^\alpha_\gamma(\omega_T)}$$

4.3.2 Iterative scheme

Now we are going to describe the iterative scheme for solving the nonlinear system (4.2.27). We start from

$$V_0 = 0,\ \Psi_0 = 0,\ \psi_0 = 0.$$

Then we want construct V_{n+1}, Ψ_{n+1} and ψ_{n+1} from V_n, Ψ_n and ψ_n, for $n \geq 0$. This can be done by considering

$$V_{n+1} = V_n + \delta V_n,\ \Psi_{n+1} = \Psi_n + \delta\Psi_n,\ \psi_{n+1} = \psi_n + \delta\psi_n,$$

where we require

$$\begin{aligned} F_{21,k} &= (F_{11}^a + S_{\theta_k} F_{11,k})\partial_1 S_{\theta_k}\Psi_k + S_{\theta_k} F_{11,k}\partial_1\Phi^a, \\ F_{22,k} &= (F_{12}^a + S_{\theta_k} F_{12,k})\partial_1 S_{\theta_k}\Psi_k + S_{\theta_k} F_{12,k}\partial_1\Phi^a, \end{aligned} \tag{4.3.1}$$

for any positive integer k.

Specifically, we decompose

$$\begin{aligned} \mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) &= \mathbb{L}(U^a + V_{n+1}, \Phi^a + \Psi_{n+1}) - \mathbb{L}(U^a + V_n, \Phi^a + \Psi_n) \\ &= \mathbb{L}'(U^a + V_n, \Phi^a + \Psi_n)(\delta V_n, \delta\Psi_n) + e_n' \end{aligned}$$

where e_n' is the quadratic error. Similarly, for the boundary operator we have

$$\begin{aligned} &\mathcal{B}((V_{n+1})|_{x_2=0}, \psi_{n+1}) - \mathcal{B}((V_n)|_{x_2=0}, \psi_n) \\ &= \mathbb{B}'((U^a + V_n)|_{x_2=0}, \varphi^a + \psi_n)((\delta V_n)|_{x_2=0}, \delta\psi_n) + \tilde{e}_n' \end{aligned}$$

where $\tilde{e}_n'$ is the quadratic error. To solve the linear system, we expect the coefficients of linear operator to be smooth and satisfy (3.2.4). So we would modify the system as follows

$$\begin{aligned} \mathcal{L}(V_{n+1}, \Psi_{n+1}) - &\mathcal{L}(V_n, \Psi_n) \\ &= \mathbb{L}(U^a + V_{n+1}, \Phi^a + \Psi_{n+1}) - \mathbb{L}(U^a + V_n, \Phi^a + \Psi_n) \\ &= \mathbb{L}'(U^a + V_{n+1/4}, \Phi^a + S_{\theta_n}\Psi_n)(\delta V_n, \delta\Psi_n) + e_n' + e_n'' \\ &= \mathbb{L}'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})(\delta V_n, \delta\Psi_n) + e_n' + e_n'' + e_n''', \\ \mathcal{B}((V_{n+1})|_{x_2=0}, \psi_{n+1}) &- \mathcal{B}((V_n)|_{x_2=0}, \psi_n) \\ &= \mathbb{B}'((U^a + V_{n+1/4})|_{x_2=0}, \varphi^a + S_{\theta_n}\Psi_n|_{x_2=0})((\delta V_n)|_{x_2=0}, \delta\psi_n) + \tilde{e}_n' + \tilde{e}_n'' \\ &= \mathbb{B}'((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2})((\delta V_n)|_{x_2=0}, \delta\psi_n) + \tilde{e}_n' + \tilde{e}_n'' + \tilde{e}_n''', \end{aligned}$$

where $V_{n+1/2}$, $\Psi_{n+1/2}$ and $\psi_{n+1/2}$ are the qualified states whose details are included in the last section and

$$V^{\pm}_{n+1/4} = (S_{\theta_n}\rho^{\pm}_n, S_{\theta_n}v^{\pm}_n, S_{\theta_n}u^{\pm}_n, S_{\theta_n}F^{\pm}_{11,n}, F^{\pm}_{21,n}, S_{\theta_n}F^{\pm}_{12,n}, F^{\pm}_{22,n})$$

Moreover, we rewrite the system into the "good unknown":

$$\delta\dot{V}_n = \delta V_n - \delta\Psi_n \frac{\partial_2\left(U^a + V_{n+1/2}\right)}{\partial_2\left(\Phi^a + \Psi_{n+1/2}\right)}, \tag{4.3.2}$$

and obtain

$$\begin{aligned}
\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) &= \mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\delta\dot{V}_n \\
&\quad + e'_n + e''_n + e'''_n + \frac{\delta\Psi_n}{\partial_2(\Phi^a + \Psi_{n+1/2})}\partial_2\{\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\},\\
\mathcal{B}((V_{n+1})|_{x_2=0}, \psi_{n+1}) &- \mathcal{B}((V_n)|_{x_2=0}, \psi_n)\\
&= \mathbb{B}'_e((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2})(\delta\dot{V}_n|_{x_2=0}, \delta\psi_n) + \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n.
\end{aligned}$$

To guarantee the convergence, we require

$$\begin{cases}
\mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\delta\dot{V}_n = f_n & \text{in } \Omega_T,\\
\mathbb{B}'_{n+1/2}(\delta\dot{V}_n|_{x_2=0}, \delta\psi_n) = g_n & \text{on } \omega_T,\\
\delta\dot{V}_n = 0, \quad \delta\psi_n = 0 & \text{for } t < 0,
\end{cases}$$

for some f_n and g_n we will determine later. However because of (4.3.1), the above system cannot holds in general. So we consider

$$\begin{aligned}
&\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = \mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})X_n\\
&- \mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})X_n + \mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\delta\dot{V}_n\\
&+ e'_n + e''_n + e'''_n + \frac{\delta\Psi_n}{\partial_2(\Phi^a + \Psi_{n+1/2})}\partial_2\{\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\},\\
&\mathcal{B}((V_{n+1})|_{x_2=0}, \psi_{n+1}) - \mathcal{B}((V_n)|_{x_2=0}, \psi_n)\\
&= \mathbb{B}'_e((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2})(X_n|_{x_2=0}, x_n)\\
&- \mathbb{B}'_e((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2})(X_n|_{x_2=0}, x_n)\\
&+ \mathbb{B}'_e((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2})(\delta\dot{V}_n|_{x_2=0}, \delta\psi_n) + \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n.
\end{aligned}$$

Here (X_n, x_n) will be solved from

$$\begin{cases} \mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})X_n = f_n & \text{in } \Omega_T, \\ \mathbb{B}'_{n+1/2}(X_n|_{x_2=0}, x_n) = g_n & \text{on } \omega_T, \\ X_n = 0, \quad x_n = 0 & \text{for } t < 0. \end{cases} \tag{4.3.3}$$

Then we choose $(\delta \dot{V}_n, \delta\psi_n)$ which is close enough to (X_n, x_n) such that (4.3.1) holds. Specifically we take $(\delta \dot{V}_n, \delta\psi_n)$ equals (X_n, x_n) except for the components $\delta \dot{F}_{21,n}$ and $\delta \dot{F}_{22,n}$

Now we specify the choice of f_n and g_n. we shorten the notation as

$$D_{n+1/2}\delta\Psi_n := \frac{\delta\Psi_n}{\partial_2(\Phi^a + \Psi_{n+1/2})}\partial_2\{\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\},$$

$$\mathbb{L}'_{n+1/2} := \mathbb{L}'_e(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}),$$

$$\mathbb{B}'_{n+1/2} := \mathbb{B}'_e((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2}),$$

$$e_n := e'_n + e''_n + e'''_n + D_{n+1/2}\delta\Psi_n + \mathbb{L}'_{n+1/2}(\delta\dot{V}_n - X_n),$$

$$\tilde{e}_n := \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n + \mathbb{B}'_{n+1/2}(\delta\dot{V}_n - X_n, \delta\psi_n - x_n).$$

To correct the errors between linear and nonlinear system, we iterates as following. Assuming we have

$$V_0 = 0, \quad \Psi_0 = 0, \quad \psi_0 = 0,$$

$$f_0 := S_{\theta_0} f^a, \quad g_0 := 0, \quad E_0 := 0, \quad \tilde{E}_0 := 0,$$

$$V_1, ..., V_n, \quad \Psi_1, ..., \Psi_n, \quad \psi_1, ..., \psi_n,$$

$$f_1, ..., f_{n-1}, \quad g_1, ..., g_{n-1},$$

$$e_0, ..., e_{n-1}, \quad \tilde{e}_0, ..., \tilde{e}_{n-1}.$$

Then we compute for $n \geq 1$

$$E_n := \sum_{k=0}^{n-1} e_k, \quad \tilde{E}_n := \sum_{k=0}^{n-1} \tilde{e}_k$$

Next we take the source terms f_n and g_n for (4.3.3) to satisfy

$$\sum_{k=0}^{n} f_k + S_{\theta_n} E_n = S_{\theta_n} f^a, \quad \sum_{k=0}^{n} g_k + S_{\theta_n} \tilde{E}_n = 0. \tag{4.3.4}$$

and solve the linear system

Next we need to construct $\delta\Psi_n$, which allows us to compute V_{n+1}, Ψ_{n+1} and ψ_{n+1}. From the boundary condition we know

$$\begin{aligned}
&\partial_t\delta\psi_n + (v^{a+} + v^+_{n+1/2})|_{x_2=0}\partial_1\delta\psi_n \\
&+ \left\{\partial_1(\varphi^a + \psi_{n+1/2})\frac{\partial_2(v^{a+} + v^+_{n+1/2})|_{x_2=0}}{\partial_2(\Phi^{a+} + \Psi^+_{n+1/2})|_{x_2=0}} - \frac{\partial_2(u^{a+} + u^+_{n+1/2})|_{x_2=0}}{\partial_2(\Phi^{a+} + \Psi^+_{n+1/2})|_{x_2=0}}\right\}\delta\psi_n \\
&+ \partial_1(\varphi^a + \psi_{n+1/2})(\delta\dot{v}^+_n)|_{x_2=0} - (\delta\dot{u}^+_n)|_{x_2=0} = g_{n,2}, \\
&\partial_t\delta\psi_n + (v^{a-} + v^-_{n+1/2})|_{x_2=0}\partial_1\delta\psi_n \\
&+ \left\{\partial_1(\varphi^a + \psi_{n+1/2})\frac{\partial_2(v^{a-} + v^-_{n+1/2})|_{x_2=0}}{\partial_2(\Phi^{a-} + \Psi^-_{n+1/2})|_{x_2=0}} - \frac{\partial_2(u^{a-} + u^-_{n+1/2})|_{x_2=0}}{\partial_2(\Phi^{a-} + \Psi^-_{n+1/2})|_{x_2=0}}\right\}\delta\psi_n \\
&+ \partial_1(\varphi^a + \psi_{n+1/2})(\delta\dot{v}^-_n)|_{x_2=0} - (\delta\dot{u}^-_n)|_{x_2=0} = g_{n,2} - g_{n,1},
\end{aligned}$$

where $g_{n,i}$ is the ith component in vector g_n. We always expect expect the change of the variables and the background states satisfying the eikonal equation, which is consistent with the boundary condition. So it is natural to construct $\delta\Psi_n$ by solving

$$\begin{aligned}
&\partial_t\delta\Psi^+_n + (v^{a+} + v^+_{n+1/2})\partial_1\delta\Psi^+_n \\
&+ \left\{\partial_1(\varphi^a + \psi_{n+1/2})\frac{\partial_2(v^{a+} + v^+_{n+1/2})}{\partial_2(\Phi^{a+} + \Psi^+_{n+1/2})} - \frac{\partial_2(u^{a+} + u^+_{n+1/2})}{\partial_2(\Phi^{a+} + \Psi^+_{n+1/2})}\right\}\delta\Psi^+_n \\
&+ \partial_1(\varphi^a + \psi_{n+1/2})\delta\dot{v}^+_n - \delta\dot{u}^+_n = \mathcal{R}_T g_{n,2} + h^+_n,
\end{aligned} \tag{4.3.5}$$

$$\begin{aligned}
&\partial_t\delta\Psi^-_n + (v^{a-} + v^-_{n+1/2})\partial_1\delta\Psi^-_n \\
&+ \left\{\partial_1(\varphi^a + \psi_{n+1/2})\frac{\partial_2(v^{a-} + v^-_{n+1/2})}{\partial_2(\Phi^{a-} + \Psi^-_{n+1/2})} - \frac{\partial_2(u^{a-} + u^-_{n+1/2})}{\partial_2(\Phi^{a-} + \Psi^-_{n+1/2})}\right\}\delta\Psi^-_n \\
&+ \partial_1(\varphi^a + \psi_{n+1/2})\delta\dot{v}^-_n - \delta\dot{u}^-_n = \mathcal{R}_T(g_{n,2} - g_{n,1}) + h^-_n.
\end{aligned} \tag{4.3.6}$$

Since the extension operator $\mathcal{R}_T$ is chosen arbitrary, we add $h^\pm_n$ to correct the system. Now we determine $h^\pm_n$. First we decompose the second equation in (4.2.27) as

$$\mathcal{E}(V_{n+1}, \Psi_{n+1}) - \mathcal{E}(V_n, \Psi_n) = \mathcal{E}'(V_{n+1/2}, \Psi_{n+1/2})(\delta U_n, \delta\Psi_n) + \hat{e}'_n + \hat{e}''_n + \hat{e}'''_n.$$

Similarly as the interior equations and boundary conditions, we denote

$$\hat{e}_n := \hat{e}'_n + \hat{e}''_n + \hat{e}'''_n, \quad \hat{E}_n := \sum_{k=0}^{n-1} \hat{e}_k.$$

With the good unknown, we have

$$\begin{aligned}
&\mathcal{E}'(V_{n+1/2}, \Psi_{n+1/2})(\delta \dot{U}_n, \delta\Psi_n) = \partial_t \delta\Psi_n + (v^a + v_{n+1/2})\partial_1 \delta\Psi_n \\
&+ \left\{ \partial_1(\varphi^a + \psi_{n+1/2}) \frac{\partial_2(v^a + v_{n+1/2})}{\partial_2(\Phi^a + \Psi_{n+1/2})} - \frac{\partial_2(u^a + u_{n+1/2})}{\partial_2(\Phi^a + \Psi_{n+1/2})} \right\} \delta\Psi_n \\
&+ \partial_1(\varphi^a + \psi_{n+1/2})\delta\dot{v}_n - \delta\dot{u}_n.
\end{aligned}$$

Then we have

$$\mathcal{E}(V_{n+1}^+, \Psi_{n+1}^+) - \mathcal{E}(V_n^+, \Psi_n^+) = \mathcal{R}_T g_{n,2} + h_n^+ + \hat{e}_n^+$$

Adding from 0 to $n+1$ and keeping in mind $\mathcal{E}(V_0^+, \Psi_0^+) = 0$, we have

$$\begin{aligned}
\mathcal{E}(V_{n+1}^+, \Psi_{n+1}^+) &= \mathcal{R}_T \left(\sum_{k=0}^{n} g_{k,2} \right) + \sum_{k=0}^{n} h_n^+ + \hat{E}_{n+1}^+ \\
&= \mathcal{R}_T \left(\mathcal{B}(V_{n+1}^+|_{x_2=0}, \psi_{n+1})_2 - \tilde{E}_{n+1,2} \right) + \sum_{k=0}^{n} h_n^+ + \hat{E}_{n+1}^+,
\end{aligned}$$

where $\mathcal{B}(V_{n+1}^+|_{x_2=0}, \psi_{n+1})_2$ is the second component of the vector $\mathcal{B}(V_{n+1}^+|_{x_2=0}, \psi_{n+1})$. Since we need the eikonal equation $\mathcal{E}(V_{n+1}^+, \Psi_{n+1}^+)$ and the boundary conditions $\mathcal{B}(V_{n+1}^+|_{x_2=0}, \psi_{n+1})_2$ are consistent, we just need to assume

$$\sum_{k=0}^{n} h_k^+ + S_{\theta_n}(\hat{E}_n^+ - \mathcal{R}_T \tilde{E}_{n,2}) = 0 \tag{4.3.7}$$

Similar, for h_n^-, we have

$$\sum_{k=0}^{n} h_k^- + S_{\theta_n}(\hat{E}_n^- - \mathcal{R}_T \tilde{E}_{n,2} + \mathcal{R}_T \tilde{E}_{n,1}) = 0 \tag{4.3.8}$$

We could check that the source term $h_k^\pm$ vanish in the past and on the boundary of the domain ω_T. With this choice of $h_n^\pm$, we could solve the change of variables $\delta\Psi_n^\pm$.

To end this step and start the next step in the iteration, we only need to estimate e_n, $\tilde{e}_n$ and $\hat{e}_n$. These error terms can be computed from

$$\begin{cases} \mathcal{L}(V_{n+1},\Psi_{n+1}) - \mathcal{L}(V_n,\Psi_n) = f_n + e_n, \\ \mathcal{E}(V^+_{n+1},\Psi^+_{n+1}) - \mathcal{E}(V^+_n,\Psi^+_n) = \mathcal{R}_T g_{n,2} + h^+_n + \hat{e}^+_n, \\ \mathcal{E}(V^-_{n+1},\Psi^-_{n+1}) - \mathcal{E}(V^-_n,\Psi^-_n) = \mathcal{R}_T (g_{n,2} - g_{n,1}) + h^-_n + \hat{e}^-_n, \\ \mathcal{B}(V_{n+1}|_{x_2=0},\psi_{n+1}) - \mathcal{B}(V_n|_{x_2=0},\psi_n) = g_n + \tilde{e}_n. \end{cases} \tag{4.3.9}$$

If we sum the above systems from $n=0$ to N, we have

$$\begin{aligned} &\mathcal{L}(V_{N+1},\Psi_{N+1}) - f^a = (S_{\theta_N} - I) f^a + (I - S_{\theta_N}) E_N + e_N, \\ &\mathcal{E}(V^+_{N+1},\Psi^+_{N+1}) = \mathcal{R}_T \mathcal{B}(V^+_{N+1}|_{x_2=0},\psi_{N+1})_2 + (I - S_{\theta_N})(\hat{E}^+_n - \mathcal{R}_T \tilde{E}_{N,2}) + \hat{e}^+_N - \mathcal{R}_T \tilde{e}_{N,2}, \\ &\mathcal{E}(V^-_{N+1},\Psi^-_{N+1}) = \mathcal{R}_T \mathcal{B}(V^-_{N+1}|_{x_2=0},\psi_{N+1})_2 + (I - S_{\theta_N})(\hat{E}^-_n - \mathcal{R}_T (\tilde{E}_{N,2} - \tilde{E}_{N,1})) + \hat{e}^-_N \\ &\qquad\qquad - \mathcal{R}_T (\tilde{e}_{N,2} - \tilde{e}_{N,1}), \\ &\mathcal{B}(V_{N+1}|_{x_2=0},\psi_{N+1}) = (I - S_{\theta_N}) \tilde{E}_N + \tilde{e}_N. \end{aligned}$$

Since $S_{\theta_N} \to I$ and we expect $(e_N, \hat{e}_N, \tilde{e}_N) \to 0$ as $N \to +\infty$, we can formally obtain $\mathcal{L}(V_{N+1},\Psi_{N+1}) \to f^a$, $\mathcal{B}(V_{N+1}|_{x_2=0},\psi_{N+1}) \to 0$ and $\mathcal{E}(V_{N+1},\Psi_{N+1}) \to 0$ as $N \to +\infty$.

4.3.3 Basic estimates

In this subsection we present some necessary results in proving the convergence of the iterative scheme. First, we introduce the following estimates in the second derivative of the system.

Proposition 4.3.2. *Let $s \in \mathbb{N}$ and $T > 0$. We assume the perturbations $\dot{U}$ and $\dot{\Phi}$ satisfying*

$$[\dot{U}, \dot{\Phi}]_{7,\gamma,T} \le K,$$

where K is a fixed constant.Then ,for any $\gamma > 1$ and (V', Ψ'), $(V'', \Psi'') \in H^{s+2}_\gamma(\Omega_T)$ we have the following a priori estimates

$$[\mathbb{L}'(U,\Phi)(V',\Psi')]_{s,\gamma,T} \le C(K) \left\{ [V',\Psi']_{s+2,\gamma,T} + [\dot{U},\dot{\Phi}]_{s+2,\gamma,T} [V',\Psi']_{7,\gamma,T} \right\},$$

$$[\mathbb{L}''(U,\Phi)(V',\Psi')(V'',\Psi'')]_{s,\gamma,T} \le C(K)\Big\{[\dot U,\dot\Phi]_{s+2,\gamma,T}[V',\Psi']_{7,\gamma,T}[V'',\Psi'']_{7,\gamma,T}$$
$$+[V',\Psi']_{s+2,\gamma,T}[V'',\Psi'']_{7,\gamma,T}+[V',\Psi']_{7,\gamma,T}[V'',\Psi'']_{s+2,\gamma,T}\},$$
$$[\mathcal{E}''(V',\Psi')(V'',\Psi'')]_{s,\gamma,T} \le C(K)\{[V']_{s,\gamma,T}[\Psi'']_{7,\gamma,T}+[V']_{5,\gamma,T}[\Psi'']_{s+1,\gamma,T}$$
$$+[\Psi']_{s+1,\gamma,T}[V'']_{5,\gamma,T}+[\Psi']_{7,\gamma,T}[V'']_{s,\gamma,T}\}.$$

If (W',ψ'),$(W'',\psi'')\in H^s_\gamma(\omega_T)\times H^{s+1}_\gamma(\omega_T)$, *we have*

$$\|\mathbb{B}''(W',\psi')(W'',\psi'')\|_{H^s_\gamma(\omega_T)} \le C(K)\Big\{\|W'\|_{H^s_\gamma(\omega_T)}\|\psi''\|_{H^3_\gamma(\omega_T)}$$
$$+\|W'\|_{H^2_\gamma(\omega_T)}\|\psi''\|_{H^{s+1}_\gamma(\omega_T)}+\|W''\|_{H^s_\gamma(\omega_T)}\|\psi'\|_{H^3_\gamma(\omega_T)}+\|W''\|_{H^2_\gamma(\omega_T)}\|\psi'\|_{H^{s+1}_\gamma(\omega_T)}\Big\}.$$

Here $C(K)$ is some constants, only depending on K.

The proof of this proposition is a direct use of Theorem 2.3.2, Theorem 2.3.6 and Theorem 2.3.8 and embedding theorems. We omit the proof.

Second we want to derive a priori estimate $\delta\Psi_n$ which we constructed in (4.3.5) and (4.3.6). We take (4.3.5) as an example and denote it in terms of $\delta\widetilde{\Psi}_n := e^{-\gamma t}\delta\Psi_n$ as following

$$\gamma\delta\widetilde{\Psi}_n+\partial_t\delta\widetilde{\Psi}_n+a_1\partial_1\delta\widetilde{\Psi}_n+a_2\delta\widetilde{\Psi}_n+a_3e^{-\gamma t}\delta\dot V_n = e^{-\gamma t}\mathcal{R}_T g_{n,2}+e^{-\gamma t}h_n^+,$$

where $a_1=v^a+v_{n+1/2}$, a_2 and a_3 are smooth functions of $\nabla(\Phi^a+\Psi_{n+1/2})$ and $\nabla(U^a+V_{n+1/2})$. For multi-index β, we apply D^β to the above equation, multiply it by $\gamma D^\beta\delta\widetilde{\Psi}_n$ and integrate on Ω_T to obtain

$$\gamma^2\langle D^\beta\delta\widetilde{\Psi}_n, D^\beta\delta\widetilde{\Psi}_n\rangle+\gamma\langle D^\beta\delta\widetilde{\Psi}_n,\partial_t D^\beta\delta\widetilde{\Psi}_n\rangle+\gamma\langle D^\beta\delta\widetilde{\Psi}_n, a_1\partial_1 D^\beta\delta\widetilde{\Psi}_n\rangle$$
$$+\gamma\langle D^\beta\delta\widetilde{\Psi}_n, a_2D^\beta\delta\widetilde{\Psi}_n\rangle+\gamma\langle D^\beta\delta\widetilde{\Psi}_n, a_3D^\beta(e^{-\gamma t}\delta\dot V_n)\rangle+\gamma\langle D^\beta\delta\widetilde{\Psi}_n,[D^\beta,a_1]\partial_1\delta\widetilde{\Psi}_n\rangle$$
$$+\gamma\langle D^\beta\delta\widetilde{\Psi}_n,[D^\beta,a_2]\delta\widetilde{\Psi}_n\rangle+\gamma\langle D^\beta\delta\widetilde{\Psi}_n,[D^\beta,a_3]e^{-\gamma t}\delta\dot V_n\rangle$$
$$=\gamma\langle D^\beta\delta\widetilde{\Psi}_n, D^\beta e^{-\gamma t}\mathcal{R}_T g_{n,2}\rangle+\gamma\langle D^\beta\delta\widetilde{\Psi}_n, D^\beta e^{-\gamma t}h_n^+\rangle.$$

By a similar treatment as for the a priori tame estimate in linear system, we can easily obtain, for $s\ge 5$ and $\gamma>1$ large enough,

$$\gamma^2[\delta\Psi_n]^2_{s,\gamma,T}\le C(K)\Big\{[\delta\dot V_n]^2_{s,\gamma,T}+[\delta\Psi_n]^2_{5,\gamma,T}[\dot U^a+V_{n+1/2},\dot\Phi^a+\Psi_{n+1/2}]^2_{s+2,\gamma,T}$$
$$+[\delta\dot V_n]^2_{5,\gamma,T}[\dot\Phi^a+\Psi_{n+1/2}]^2_{s+1,\gamma,T}+\|g\|^2_{H^{s-1}_\gamma(\omega_T)}+[h]^2_{s,\gamma,T}\Big\}$$

Taking $s = 5$, using suitable embedding and estimate of $\delta \dot{V}_n$, we have

$$\gamma[\delta\Psi_n]_{5,\gamma,T} \leq C(K) \left\{ [f]_{6,\gamma,T} + \|g\|^2_{H^6_\gamma(\omega_T)} + [h]^2_{5,\gamma,T} \right\} \tag{4.3.10}$$

So combine with the a priori tame estimate we obtained before, we have

$$\begin{aligned} \sqrt{\gamma}[\delta\dot{V}_n]_{s,\gamma,T} + \gamma[\delta\Psi_n]_{s,\gamma,T} + \|\delta\psi_n\|_{H^{s+1}_\gamma(\omega_T)} \leq C(K) \Big\{ [f]_{s+1,\gamma,T} + \|g\|_{H^{s+1}_\gamma(\omega_T)} \\ + [h]_{s,\gamma,T} + \Big([f]_{6,\gamma,T} + \|g\|_{H^6_\gamma(\omega_T)} + [h]_{5,\gamma,T} \Big) [\dot{U}, \dot{\Phi}]_{s+4,\gamma,T} \Big\} \end{aligned} \tag{4.3.11}$$

4.4 CONVERGENCE OF THE ITERATIVE SCHEME

In this section, we want to show the convergence of the iterative scheme by prescribe some statement on each step. Then we will show that the statement of one step is implied by the statement of the previous step, which eventually leads to the completeness of our main result.

Before we introduce the induction scheme, we define a sequence of parameters (θ_n) as follows:

$$\theta_0, \quad \theta_n := \sqrt{\theta_0^2 + n}.$$

Then we denote $\Delta_n := \theta_{n+1} - \theta_n$. It is obvious that the sequence (Δ_n) is decreasing and tends to zero. Moreover

$$\forall n \in \mathbb{N}, \quad \frac{1}{3\theta_n} \leq \Delta_n \leq \frac{1}{2\theta_n}.$$

4.4.1 Induction scheme

By takeing the initial data small enough, for any $\delta > 0$, we can obtain

$$[\dot{U}^a]_{m+1,*} + [\dot{\Phi}^a]_{m+2,*} + \|\varphi^a\|_{H^{m+1}} + [f^a]_{m-1,*} \le \delta. \tag{4.4.1}$$

Here we can choose m large enough to guarantee the proof can be justified. Specifically, we choose $m = \tilde{\alpha} + 3$, where $\tilde{\alpha}$ is the parameter in the following inductive statement.

When we have solved the linear system for $(\delta V_{n-1}, \delta\Psi_{n-1}, \delta\psi_{n-1})$, we claim:

$$(H_{n-1})\begin{cases} a) & \forall k = 0, ..., n-1, \forall s \in [7, \tilde{\alpha}] \cap \mathbb{N}, \\ & [\delta V_k, \delta \Psi_k]_{s,\gamma,T} + \|\delta\psi_k\|_{H^{s+1}_\gamma} \le \delta\theta_k^{s-\alpha-1}\Delta_k, \\ b) & \forall k = 0, ..., n-1, \forall s \in [7, \tilde{\alpha}-2] \cap \mathbb{N}, \\ & [\mathcal{L}(V_k, \Psi_k) - f^a]_{s,\gamma,T} \le 2\delta\theta_k^{s-\alpha-1}, \\ c) & \forall k = 0, ..., n-1, \forall s \in [7, \alpha] \cap \mathbb{N}, \\ & \|\mathcal{B}(V_k|_{x_2=0}, \psi_k)\|_{H^s_\gamma(\omega_T)} \le \delta\theta_k^{s-\alpha-1}, \end{cases}$$

where δ, α and $\tilde{\alpha}$ are parameters we will determine later. Right now, we only require $\alpha < \tilde{\alpha}$.

We want to show with the assumption, H_{n-1} implies H_n. Then By showing H_0 is true, we can conclude H_n is true for all n. So in the following we assume H_{n-1}, which gives us the following lemma.

Lemma 4.4.1. *If θ_0 is big enough and H_{n-1} is true, then for every $k = 0, 1, ..., n$ and for every integer $s \in [7, \tilde{\alpha}]$,we have*

$$\begin{cases} [V_k, \Psi_k]_{s,\gamma,T} + \|\psi_k\|_{H^{s+1}_\gamma} \le \delta\theta_k^{(s-\alpha)_+}, & \alpha \ne s, \\ [V_k, \Psi_k]_{s,\gamma,T} + \|\psi_k\|_{H^{s+1}_\gamma} \le \delta\log\theta_k, & \alpha = s. \end{cases}$$

Moreover for every for every $k = 0, 1, ..., n$ and for every integer $s \in [7, \tilde{\alpha}+5]$,we have

$$\begin{cases} [S_{\theta_k}V_k, S_{\theta_k}\Psi_k]_{s,\gamma,T} \le C\delta\theta_k^{(s-\alpha)_+}, & \alpha \ne s, \\ [S_{\theta_k}V_k, S_{\theta_k}\Psi_k]_{s,\gamma,T} \le C\delta\log\theta_k, & \alpha = s. \end{cases}$$

For every for every $k = 0, 1, ..., n$ and for every integer $s \in [7, \tilde{\alpha}]$,we have

$$[(I - S_{\theta_k})V_k, (I - S_{\theta_k})\Psi_k]_{s,\gamma,T} \le C\delta\theta_k^{s-\alpha}$$

The proof of this lemma is base on the classical comparison between series and integrals, and Proposition 4.3.1.

4.4.2 Quadratic errors

To obtain H_n from H_{n-1}, we need to solve the linear system for the source term f_n and g_n. First we estimate e'_k, $\hat{e}'_k$ and $\tilde{e}'_k$ for $k = 0, 1, ..., n-1$, where

$$\begin{aligned}
e'_k &:= \mathcal{L}(V_{k+1}, \Psi_{k+1}) - \mathcal{L}(V_k, \Psi_k) - \mathcal{L}'(V_k, \Psi_k)(\delta V_k, \delta\Psi_k),\\
\hat{e}'_k &:= \mathcal{E}(V_{k+1}, \Psi_{k+1}) - \mathcal{E}(V_k, \Psi_k) - \mathcal{E}'(V_k, \Psi_k)(\delta V_k, \delta\Psi_k),\\
\tilde{e}'_k &:= \mathcal{B}(V_{k+1}|_{x_2=0}, \psi_{k+1}) - \mathcal{B}(V_k|_{x_2=0}, \psi_k) - \mathcal{B}'(V_k|_{x_2=0}, \psi_k)(\delta V_k|_{x_2=0}, \delta\psi_k).
\end{aligned}$$

We have the following lemma

Lemma 4.4.2. *Let $\alpha \geq 8$, $\delta > 0$ sufficiently small, and θ_0 sufficiently large. Then for $k = 0, ..., n-1$ and for all integer $s \in [7, \tilde{\alpha} - 2]$, the following estimate holds*

$$\begin{aligned}
&[e'_k]_{s,\gamma,T} \leq C\delta^2\theta_k^{L_1(s)-1}\Delta_k,\\
&[\hat{e}'_k]_{s,\gamma,T} \leq C\delta^2\theta_k^{s+5-2\alpha}\Delta_k,\\
&\|\tilde{e}'_k\|_{H^s_\gamma(\omega_T)} \leq C\delta^2\theta_k^{L_1(s)-1}\Delta_k,
\end{aligned}$$

where $L_1(s) = max\{(s+2-\alpha)_+ + 12 - 2\alpha, s + 7 - 2\alpha\}$.

Proof. First we remark that

$$e'_k = \int_0^1 (1-\tau)\mathbb{L}''(U^a + V_k + \tau\delta V_k, \Phi^a + \Psi_k + \tau\delta\Psi_k)(\delta V_k, \delta\Psi_k)(\delta V_k, \delta\Psi_k)d\tau.$$

From (4.4.1), Lemma 4.4.1 and H_{n-1}, we have

$$\sup_{\tau\in[0,1]} [\dot{U}^a + V_k + \tau\delta V_k, \dot{\Phi}^a + \Psi_k + \tau\delta\Psi_k]_{7,\gamma,T} \leq C\delta.$$

So by taking δ small enough and Proposition 4.3.2, we have

$$\begin{aligned}
[e'_k]_{s,\gamma,T} \leq C\Big\{&[\dot{U}^a + V_k + \tau\delta V_k, \dot{\Phi}^a + \Psi_k + \tau\delta\Psi_k]_{s+2,\gamma,T}[\delta V_k, \delta\Psi_k]^2_{7,\gamma,T}\\
&+2[\delta V_k, \delta\Psi_k]_{s+2,\gamma,T}[\delta V_k, \delta\Psi_k]_{7,\gamma,T}\Big\}.
\end{aligned}$$

Since in Lemma 4.4.1 the estimates of V_k and Ψ_k take two different forms, we need to discuss them separately. If $s+2 \neq \alpha$ and $s+2 \leq \tilde{\alpha}$ we have

$$
\begin{aligned}
[e_k']_{s,\gamma,T} &\leq C\left\{(\delta + \delta\theta_k^{(s+2-\alpha)_+} + \delta\theta_k^{s+2-\alpha-1}\Delta_k)\delta^2\theta_k^{12-2\alpha}\Delta_k^2 + 2\delta^2\Delta_k^2\theta_k^{s+7-2\alpha}\right\} \\
&\leq C\left\{\delta^2\theta_k^{(s+2-\alpha)_+ + 11-2\alpha}\Delta_k + \delta^2\theta_k^{s+6-2\alpha}\Delta\right\} \leq C\delta^2\Delta_k\theta_k^{L_1(s)-1},
\end{aligned}
$$

where $L_1(s) = \max\{(s+2-\alpha)_+ + 12 - 2\alpha, s+7-2\alpha\}$. For $s+2=\alpha$, we have

$$
\begin{aligned}
[e_k']_{s,\gamma,T} &\leq C\left\{(\delta + \delta\log\theta_k + \delta\theta_k^{-1}\Delta_k)\delta^2\theta_k^{12-2\alpha}\Delta_k^2 + 2\delta^2\Delta_k^2\theta_k^{5-\alpha}\right\} \\
&\leq C\left\{\delta^2\theta_k^{12-2\alpha}\Delta_k + \delta^2\theta_k^{4-2\alpha}\Delta\right\} \leq C\delta^2\Delta_k\theta_k^{L_1(\alpha-2)-1}.
\end{aligned}
$$

Similarly we can show the estimate of $\hat{e}_k'$. For $\tilde{e}_k'$, by Proposition 4.3.2 and , we have

$$
\begin{aligned}
\|\tilde{e}_k'\|_{H^s_\gamma(\omega_T)} &\leq C\left\{\|\delta V_k|_{x_2=0}\|_{H^s_\gamma(\omega_T)}\|\delta\psi_k\|_{W^{1,\infty}(\omega_T)} + \|\delta V_k|_{x_2=0}\|_{L^\infty(\omega_T)}\|\delta\psi_k\|_{H^{s+1}_\gamma(\omega_T)}\right\} \\
&\leq C\left\{[\delta V_k]_{s,\gamma,T}\|\delta\psi_k\|_{H^3_\gamma(\omega_T)} + [\delta V_k]_{4,\gamma,T}\|\delta\psi_k\|_{H^{s+1}_\gamma(\omega_T)}\right\} \\
&\leq C\left\{\delta^2\Delta_k^2\theta_k^{s+6-2\alpha} + \delta^2\Delta_k^2\theta_k^{s+5-2\alpha}\right\} \leq C\delta^2\Delta_k\theta_k^{L_1(s)-1}.
\end{aligned}
$$

□

4.4.3 First substitution errors

Second, we estimate the first substitution error e_k'', $\hat{e}_k''$ and $\tilde{e}_k''$ for $k = 0, 1, ..., n-1$, where

$$
\begin{aligned}
e_k'' &:= \mathcal{L}'(V_k, \Psi_k)(\delta V_k, \delta\Psi_k) - \mathcal{L}'(S_{\theta_k}V_k, S_{\theta_k}\Psi_k)(\delta V_k, \delta\Psi_k), \\
\hat{e}_k'' &:= \mathcal{E}'(V_k, \Psi_k)(\delta V_k, \delta\Psi_k) - \mathcal{E}'(S_{\theta_k}V_k, S_{\theta_k}\Psi_k)(\delta V_k, \delta\Psi_k), \\
\tilde{e}_k'' &:= \mathcal{B}'(V_k|_{x_2=0}, \psi_k)(\delta V_k|_{x_2=0}, \delta\psi_k) - \mathcal{B}'(S_{\theta_k}V_k|_{x_2=0}, S_{\theta_k}\Psi_k|_{x_2=0})(\delta V_k|_{x_2=0}, \delta\psi_k).
\end{aligned}
$$

Similarly as in the previous subsection, we have

Lemma 4.4.3. *Let $\alpha \geq 8$, $\delta > 0$ sufficiently small, and θ_0 sufficiently large. Then for $k = 0, ..., n-1$ and for all integer $s \in [7, \tilde{\alpha} - 2]$, the following estimate holds*

$$[e_k'']_{s,T,\gamma} \leq C\delta^2 \theta_k^{L_2(s)-1} \Delta_k,$$
$$[\hat{e}_k'']_{s,T,\gamma} \leq C\delta^2 \theta_k^{s+7-2\alpha} \Delta_k,$$
$$\|\tilde{e}_k''\|_{H^s_\gamma(\omega_T)} \leq C\delta^2 \theta_k^{L_2(s)-1} \Delta_k,$$

where $L_2(s) = max\{(s+2-\alpha)_+ + 14 - 2\alpha, s + 9 - 2\alpha\}$.

Proof. As in the previous lemma, we can rewrite

$$e_k'' = \int_0^1 \mathbb{L}''(U^a + S_{\theta_k} V_k + \tau(I - S_{\theta_k})V_k, \Phi^a + S_{\theta_k}\Psi_k + \tau(I - S_{\theta_k})\Psi_k)$$
$$(\delta V_k, \delta\Psi_k)((I - S_{\theta_k})\delta V_k, (I - S_{\theta_k})\delta\Psi_k) d\tau.$$

Similarly as in previous Lemma, we have

$$\sup_{\tau \in [0,1]} [\dot{U}^a + S_{\theta_k} V_k + \tau(I - S_{\theta_k})V_k, \dot{\Phi}^a + S_{\theta_k}\Psi_k + \tau(I - S_{\theta_k})\Psi_k]_{7,\gamma,T} \leq C\delta.$$

Then we obtain

$$[e_k'']_{s,\gamma,T} \leq C \Big\{ [\dot{U}^a + S_{\theta_k} V_k + \tau(I - S_{\theta_k})V_k, \dot{\Phi}^a + S_{\theta_k}\Psi_k + \tau(I - S_{\theta_k})\Psi_k]_{s+2,\gamma,T} \times$$
$$[\delta V_k, \delta\Psi_k]_{7,\gamma,T} [(I - S_{\theta_k})V_k, (I - S_{\theta_k})\Psi_k]_{7,\gamma,T} + [\delta V_k, \delta\Psi_k]_{s+2,\gamma,T} \times$$
$$[(I - S_{\theta_k})V_k, (I - S_{\theta_k})\Psi_k]_{7,\gamma,T} + [(I - S_{\theta_k})V_k, (I - S_{\theta_k})\Psi_k]_{s+2,\gamma,T} [\delta V_k, \delta\Psi_k]_{7,\gamma,T} \Big\}.$$

Again from (4.4.1), Lemma 4.4.1, for $s + 2 \neq \alpha$ and $s + 2 \leq \tilde{\alpha}$, we have

$$[e_k'']_{s,\gamma,T} \leq C \Big\{ (\delta + \delta\theta_k^{(s+2-\alpha)_+} + \delta\theta_k^{s+2-\alpha})\delta^2 \Delta_k \theta_k^{13-2\alpha} + \delta^2 \Delta_k \theta_k^{s+8-2\alpha} + \delta^2 \Delta_k \theta_k^{s+8-2\alpha} \Big\}$$
$$\leq C\delta^2 \Delta_k \theta_k^{L_2(s)-1},$$

where $L_2(s) = \max\{(s+2-\alpha)_+ + 14 - 2\alpha, s + 9 - 2\alpha\}$. For $s + 2 = \alpha$, we have

$$[e_k'']_{s,\gamma,T} \leq C \left\{ (\delta + \delta \log \theta_k + \delta)\delta^2 \Delta_k \theta_k^{13-2\alpha} + \delta^2 \Delta_k \theta_k^{6-\alpha} + \delta^2 \Delta_k \theta_k^{6-\alpha} \right\}$$
$$\leq C\delta^2 \Delta_k \theta_k^{L_2(\alpha-2)-1}.$$

Similarly as in previous Lemma, we can obtain the estimate of $\hat{e}_k''$ and $\tilde{e}_k''$. □

4.4.4 Estimate of the modified states

Before we estimate the second substitution error, we need to illustrate how our intermediate states been constructed and estimate the deviation from the original states. Thus, in this section we will show the following proposition,

Proposition 4.4.1. *Let $\alpha \geq 8$, There exist some functions $V_{n+1/2}$, $\Psi_{n+1/2}$ and $\psi_{n+1/2}$ which vanish in the past, such that $U^a + V_{n+1/2}$, $\Phi^a + \Psi_{n+1/2}$ and $\varphi^a + \psi_{n+1/2}$ satisfy (3.2.4). Moreover, these functions satisfy*

$$
\begin{aligned}
&\Psi^{\pm}_{n+1/2} = S_{\theta_n}\Psi^{\pm}_n, \quad \psi_{n+1/2} = S_{\theta_n}\Psi^{\pm}_n|_{x_2=0},\\
&v^{\pm}_{n+1/2} = S_{\theta_n}v^{\pm}_n, \quad F^{\pm}_{11,n+1/2} = S_{\theta_n}F^{\pm}_{11,n}, \quad F^{\pm}_{12,n+1/2} = S_{\theta_n}F^{\pm}_{12,n},\\
&F^{\pm}_{21,n+1/2} = F^{\pm}_{21,n}, \quad F^{\pm}_{22,n+1/2} = F^{\pm}_{22,n},\\
&[V_{n+1/2} - S_{\theta_n}V_n]_{s,\gamma,T} \leq C\delta\theta_n^{s+1-\alpha}, \qquad \text{for } s \in [7, \tilde{\alpha}+5].
\end{aligned}
$$

Proof. We want to construct the intermediate states $V_{n+1/2}$, $\Psi_{n+1/2}$ and $\psi_{n+1/2}$ to guarantee (3.2.4) and $\psi_{n+1/2} = \Psi^+_{n+1/2}|_{x_2=0} = \Psi^-_{n+1/2}|_{x_2=0}$. As in the statement of the proposition, we define

$$
\begin{aligned}
&\Psi^{\pm}_{n+1/2} = S_{\theta_n}\Psi^{\pm}_n, \quad \psi_{n+1/2} = S_{\theta_n}\Psi^{\pm}_n|_{x_2=0},\\
&v^{\pm}_{n+1/2} = S_{\theta_n}v^{\pm}_n, \quad F^{\pm}_{11,n+1/2} = S_{\theta_n}F^{\pm}_{11,n}, \quad F^{\pm}_{12,n+1/2} = S_{\theta_n}F^{\pm}_{12,n},\\
&F^{\pm}_{21,n+1/2} = F^{\pm}_{21,n}, \quad F^{\pm}_{22,n+1/2} = F^{\pm}_{22,n}.
\end{aligned}
$$

To achieve (3.2.4), we only need to construct $\rho^{\pm}_{n+1/2}$ and $u^{\pm}_{n+1/2}$. Thus we denote

$$
\begin{aligned}
&\varepsilon^n_1 := (S_{\theta_n}\rho^+_n)|_{x_2=0} - (S_{\theta_n}\rho^-_n)|_{x_2=0},\\
&\varepsilon^n_2 := \mathcal{E}(V_n, \Psi_n),
\end{aligned}
$$

and define the rest variables in the intermediate states as

$$
\begin{aligned}
&\rho^{\pm}_{n+1/2} := S_{\theta_n}\rho^{\pm}_n \mp \frac{1}{2}\mathcal{R}_T\varepsilon^n_1,\\
&u^{\pm}_{n+1/2} := \partial_t\Psi^{\pm}_{n+1/2} + (v^{a,\pm} + v^{\pm}_{n+1/2})\partial_1\Psi^{\pm}_{n+1/2} + v^{\pm}_{n+1/2}\partial_1\Phi^{a\pm}.
\end{aligned}
$$

Then the rest of work for this lemma is to estimate the difference in the modification. First we estimate ε_1^n as follows

$$\begin{aligned}\|\rho_n^+ - \rho_n^-|_{x_2=0}\|_{H^s_\gamma(\omega_T)} &\leq \|\rho_{n-1}^+ - \rho_{n-1}^-|_{x_2=0}\|_{H^s_\gamma(\omega_T)} + \|\delta\rho_{n-1}^+ - \delta\rho_{n-1}^-|_{x_2=0}\|_{H^s_\gamma(\omega_T)}\\ &\leq \|\mathcal{B}(V_{n-1}|_{x_2=0}, \psi_{n-1})\|_{H^s_\gamma(\omega_T)} + C[\delta V_{n-1}]_{s+1,\gamma,T}\\ &\leq C\delta\theta_n^{s-\alpha-1},\end{aligned}$$

for $s \in [8, \alpha]$. From Proposition 4.3.1, we have

$$\|\varepsilon_1^n\|_{H^s_\gamma(\omega_T)} \leq C\theta_n^{s+1-\alpha}\|\rho_n^+ - \rho_n^-|_{x_2=0}\|_{H^\alpha_\gamma(\omega_T)} \leq C\delta\theta_n^{s-\alpha},$$

for $s \in [\alpha, \tilde{\alpha}+5]$. While for $s \in [7, \alpha-1]$ we have

$$\|\varepsilon_1^n\|_{H^s_\gamma(\omega_T)} \leq C\|\rho_n^+ - \rho_n^-|_{x_2=0}\|_{H^{s+1}_\gamma(\omega_T)} \leq C\delta\theta_n^{s-\alpha}.$$

So in all for $s \in [7, \tilde{\alpha}+5]$ we have

$$[\rho_{n+1/2}^\pm - S_{\theta_n}\rho_n^\pm]_{s,\gamma,T} = \frac{1}{2}[\mathcal{R}_T\varepsilon_1^n]_{s,\gamma,T} \leq C\|\varepsilon_1^n\|_{H^s_\gamma(\omega_T)} \leq C\delta\theta_n^{s-\alpha}.$$

Then we need to estimate $u_{n+1/2} - S_{\theta_n}u_n$. Similarly as in [21], we rewrite

$$\begin{aligned}&u_{n+1/2} - S_{\theta_n}u_n = S_{\theta_n}\varepsilon_2^n + [\partial_t, S_{\theta_n}]\Psi_n + \bar{v}[\partial_1, S_{\theta_n}]\Psi_n\\ &+ [(\dot{v}^a + S_{\theta_n}v_n)\partial_1 S_{\theta_n}\Psi_n - S_{\theta_n}(\dot{v}^a + v_n)\partial_1\Psi_n] + (S_{\theta_n v_n}\partial_1\Phi^a) - S_{\theta_n}(v_n\partial_1\Phi^a).\end{aligned}$$

Then we need to estimate every terms on the right hand side of the above equation. We note

$$\varepsilon_2^n = \mathcal{E}(V_{n-1}, \Psi_{n-1}) + \partial_t(\delta\Psi_{n-1}) + (v^a + v_{n-1})\partial_1\delta\Psi_{n-1} + \delta v_{n-1}\partial_1(\Phi^a + \Psi_n) - \delta u_{n-1}$$

By using H_{n-1}, we can obtain $[\varepsilon_2^n]_{7,\gamma,T} \leq nC\delta\theta_n^{6-\alpha} \leq C\delta\theta_n^{8-\alpha}$. Hence

$$[S_{\theta_n}\varepsilon_2^n]_{s,\gamma,T} \leq C\theta_n^{s-7}[\varepsilon_2^n]_{7,\gamma,T} \leq C\delta\theta_n^{s-\alpha+1},$$

for $s \in [7, \tilde{\alpha}+5]$. Next we need to estimate the commutators. We take the third commutator as an example. If $s \in [\alpha, \tilde{\alpha}+5]$ We have

$$[(\dot{v}^a + S_{\theta_n}v_n)\partial_1 S_{\theta_n}\Psi_n]_{s,\gamma,T} \leq [\dot{v}^a + S_{\theta_n}v_n]_{7,\gamma,T}[S_{\theta_n}\Psi_n]_{s+1,\gamma,T} + [\dot{v}^a + S_{\theta_n}v_n]_{s,\gamma,T}[S_{\theta_n}\Psi_n]_{7,\gamma,T}$$

$$\leq C\delta^2\theta_n^{s+1-\alpha},$$

and

$$\begin{aligned}[S_{\theta_n}(\dot{v}^a+v_n)\partial_1\Psi_n]_{s,\gamma,T} &\leq C\theta_n^{s-\alpha}[(\dot{v}^a+v_n)\partial_1\Psi_n]_{\alpha,\gamma,T}\\ &\leq C\theta_n^{s-\alpha}\left\{[\dot{v}^a+v_n]_{7,\gamma,T}[\Psi_n]_{s+1,\gamma,T}+[\dot{v}^a+v_n]_{s,\gamma,T}[\Psi_n]_{7,\gamma,T}\right\}\\ &\leq C\delta^2\theta_n^{s-\alpha+1}.\end{aligned}$$

For $s\in[7,\alpha-1]$, we have

$$\begin{aligned}&[(\dot{v}^a+S_{\theta_n}v_n)\partial_1 S_{\theta_n}\Psi_n-S_{\theta_n}(\dot{v}^a+v_n)\partial_1\Psi_n]_{s,\gamma,T}\leq[(v_n-S_{\theta_n}v_n)\partial_1 S_{\theta_n}\Psi_n]_{s,\gamma,T}\\ &+[(\dot{v}^a+S_{\theta_n}v_n)\partial_1(\Psi_n-S_{\theta_n}\Psi_n)]_{s,\gamma,T}+[(I-S_{\theta_n})(\dot{v}^a+v_n)\partial_1\Psi_n]_{s,\gamma,T}.\end{aligned}$$

All the terms above can be treated similarly as before. So we have

$$[u_{n+1/2}-S_{\theta_n}u_n]_{s,\gamma,T}\leq C\delta\theta_n^{s+1-\alpha},$$

Hence we finished the proof of this proposition. □

4.4.5 Second substitution errors

With the help of the last proposition, we can obtain the following estimate on the second substitution errors e_k''', $\hat{e}_k'''$ and $\tilde{e}_k'''$ for $k=0,1,...,n-1$, where

$$\begin{aligned}e_k'''&:=\mathcal{L}'(S_{\theta_k}V_k,S_{\theta_k}\Psi_k)(\delta V_k,\delta\Psi_k)-\mathcal{L}'(V_{k+1/2},\Psi_{k+1/2})(\delta V_k,\delta\Psi_k),\\ \hat{e}_k'''&:=\mathcal{E}'(S_{\theta_k}V_k,S_{\theta_k}\Psi_k)(\delta V_k,\delta\Psi_k)-\mathcal{E}'(V_{k+1/2},\Psi_{k+1/2})(\delta V_k,\delta\Psi_k),\\ \tilde{e}_k'''&:=\mathcal{B}'(S_{\theta_k}V_k|_{x_2=0},S_{\theta_k}\Psi_k|_{x_2=0})(\delta V_k|_{x_2=0},\delta\psi_k)-\mathcal{B}'(V_{k+1/2}|_{x_2=0},\psi_{k+1/2})(\delta V_k|_{x_2=0},\delta\psi_k).\end{aligned}$$

Lemma 4.4.4. *Let $\alpha\geq 8$, $\delta>0$ sufficiently small, and θ_0 sufficiently large. Then for $k=0,...,n-1$ and for all integer $s\in[7,\tilde{\alpha}-2]$, the following estimate holds*

$$[e_k'']_{s,T,\gamma}\leq C\delta^2\theta_k^{L_3(s)-1}\Delta_k,$$

where $L_3(s)=max\{(s+2-\alpha)_+ +16-2\alpha, s+10-2\alpha\}$, and $\hat{e}_k'''=\tilde{e}_k'''=0$.

Proof. Similarly, we notice

$$e_k''' = \int_0^1 \mathbb{L}''(U^a + V_{k+1/2} + \tau(S_{\theta_k}V_k - V_{k+1/2}), \Phi^a + \Psi_{k+1/2})$$
$$(\delta V_k, \delta\Psi_k)(S_{\theta_k}V_k - V_{k+1/2}, 0)d\tau.$$

From (4.4.1), Proposition 4.4.1 and Lemma 4.4.1, we have

$$\sup_{\tau\in[0,1]}[\dot{U}^a + V_{k+1/2} + \tau(S_{\theta_k}V_k - V_{k+1/2}), \dot{\Phi}^a + \Psi_{k+1/2}]_{7,\gamma,T} \le C\delta.$$

Thus

$$[e_k''']_{s,\gamma,T} \le C\Big\{[\dot{U}^a + V_{k+1/2} + \tau(S_{\theta_k}V_k - V_{k+1/2}), \dot{\Phi}^a + \Psi_{k+1/2}]_{s+2,\gamma,T}[\delta V_k, \delta\Psi_k]_{7,\gamma,T}\times$$
$$[S_{\theta_k}V_k - V_{k+1/2}]_{7,\gamma,T} + [\delta V_k, \delta\Psi_k]_{s+2,\gamma,T}[S_{\theta_k}V_k - V_{k+1/2}]_{7,\gamma,T}$$
$$+[S_{\theta_k}V_k - V_{k+1/2}]_{s+2,\gamma,T}[\delta V_k, \delta\Psi_k]_{7,\gamma,T}\Big\} \le C\delta^2\Delta_k\theta_k^{L_3(s)-1},$$

where $L_3(s) = \max\{(s+2-\alpha)_+ + 16 - 2\alpha, s + 10 - 2\alpha\}$.

For $\hat{e}_k'''$ and $\tilde{e}_k'''$, it is easy to check that $\hat{e}_k''' = \tilde{e}_k''' = 0$. We finished the proof of this lemma. □

4.4.6 Estimate of the left error terms

Now in our iterative scheme, we are left with the last error term to be estimated,

$$D_{n+1/2}\delta\Psi_k, \quad \mathbb{L}'_{n+1/2}(\delta\dot{V}_n - X_n), \quad \mathbb{B}'_{n+1/2}(\delta\dot{V}_n - X_n, \delta\psi_n - x_n).$$

To shorten our computation, we denote

$$R_k = \partial_2\{\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\}.$$

Since U^a and Φ^a do not vanish in the past but $V_{n+1/2}$ and $\Psi_{n+1/2}$ vanish in the past, in general, we can not expect R_k vanishes in the past. However, since $\delta\Psi_n$ vanishes in the past, we can still obtain that $D_{n+1/2}\delta\Psi_k$ vanishes in the past. Hence our existence result in the linear system can still be applied, which require the source term vanishing in the past. In the following, to accordance with this feature of $D_{n+1/2}\delta\Psi_k$, we actually need to work on the part of the domain Ω_T with positive time variable. Since we still have the Gagliardo-Nirenberg inequality on the anisotropic Sobolev space with this domain, we will not distinguish the norms of anisotropic Sobolev space on Ω_T and $\Omega_T^+ = \{(t,x) \in \Omega_T, t > 0\}$. Hence, We have the following estimate

$$\begin{aligned}[D_{k+1/2}\delta\Psi_k]_{s,\gamma,T} \leq C\Big\{&[\delta\Psi_k]_{s,\gamma,T}\|R_k\|_{W^{1,\tan}(\Omega_T^+)}\left\|(\partial_2(\Phi^a+\Psi_{n+1/2}))^{-1}\right\|_{W^{1,\tan}(\Omega_T^+)}\\ &+\|\delta\Psi_k\|_{W^{1,\tan}(\Omega_T^+)}\times\Big([R_k]_{s,\gamma,T}\left\|(\partial_2(\Phi^a+\Psi_{n+1/2}))^{-1}\right\|_{W^{1,\tan}(\Omega_T^+)}\\ &+\|R_k\|_{W^{1,\tan}(\Omega_T^+)}\left[(\partial_2(\Phi^a+\Psi_{n+1/2}))^{-1}\right]_{s,\gamma,T}\Big)\Big\}\end{aligned}$$

From above, it is important to estimate R_k which leads to the following lemma

Lemma 4.4.5. *Let $\alpha \geq 8$, $\tilde{\alpha} \geq \alpha+3$, $\delta > 0$ sufficiently small, and θ_0 sufficiently large. Then for $k = 0, ..., n-1$ and for all integer $s \in [7, \tilde{\alpha}-2]$, the following estimate holds*

$$[R_k]_{s,T,\gamma} \leq C\delta(\theta_k^{s+5-\alpha} + \theta_k^{(s+4-\alpha)_+ + 9-\alpha}).$$

Proof. We notice that

$$[R_k]_{s,\gamma,T} = [\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})]_{s+2,\gamma,T}.$$

Then we rewrite

$$\begin{aligned}&\mathbb{L}(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})\\ &= \mathbb{L}(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) - \mathbb{L}(U^a + V_k, \Phi^a + \Psi_k) + \mathcal{L}(V_k, \Psi_k) - f^a.\end{aligned}$$

If $s + 2 \leq \tilde{\alpha} - 2$, from H_{n-1}, we have

$$[\mathcal{L}(V_k, \Psi_k) - f^a]_{s+2,\gamma,T} \leq 2\delta\theta_k^{s+1-\alpha}.$$

Then we notice

$$\begin{aligned}&\mathbb{L}(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) - \mathbb{L}(U^a + V_k, \Phi^a + \Psi_k)\\ &= \int_0^1 \mathbb{L}'(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k))(V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k)d\tau\end{aligned}$$

Moreover, as before, we have

$$\sup_{\tau\in[0,1]}[\dot{U}^a + V_k + \tau(V_{k+1/2} - V_k), \dot{\Phi}^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k)]_{7,\gamma,T} \leq C\delta.$$

So we can obtain

$$
\begin{aligned}
&[\mathbb{L}(U^a+V_{k+1/2},\Phi^a+\Psi_{k+1/2})-\mathbb{L}(U^a+V_k,\Phi^a+\Psi_k)]_{s+2,\gamma,T}\\
&\le C\left\{[V_{k+1/2}-V_k,\Psi_{k+1/2}-\Psi_k]_{s+4,\gamma,T}+[V_{k+1/2}-V_k,\Psi_{k+1/2}-\Psi_k]_{7,\gamma,T}\times\right.\\
&\left.[\dot{U}^a+V_k+\tau(V_{k+1/2}-V_k),\dot{\Phi}^a+\Psi_k+\tau(\Psi_{k+1/2}-\Psi_k)]_{s+4,\gamma,T}\right\},\\
&\le C\delta\left\{\theta_k^{s+5-\alpha}+\theta_k^{(s+4-\alpha)_++9-\alpha}\right\}
\end{aligned}
$$

Then we consider $s=\tilde{\alpha}-2$ and $s=\tilde{\alpha}-3$ and obtain

$$
\begin{aligned}
[R_k]_{s,\gamma,T}=[\mathbb{L}(U^a+V_{n+1/2},\Phi^a+\Psi_{n+1/2})]_{s+2,\gamma,T}&\le C[\dot{U}^a+V_{n+1/2},\dot{\Phi}^a+\Psi_{n+1/2}]_{s+4,\gamma,T}\\
&\le C\delta\theta_k^{s+5-\alpha}.
\end{aligned}
$$

Hence we proved the above lemma. □

Now we are ready to estimate $D_{n+1/2}\delta\Psi_k$. We remark that by Proposition 4.4.1, Lemma 4.4.1 and H_{n-1}, if we take δ small enough, we have

$$|\partial_2(\Phi^a+\Psi_{n+1/2})|\ge\frac{1}{2}.$$

Then from (4.4.5), we can obtain the following lemma

Lemma 4.4.6. *Let $\alpha\ge 13$, $\tilde{\alpha}\ge\alpha+3$, $\delta>0$ sufficiently small, and θ_0 sufficiently large. Then for $k=0,...,n-1$ and for all integer $s\in[7,\tilde{\alpha}-2]$, the following estimate holds*

$$[D_{n+1/2}\delta\Psi_k]_{s,T,\gamma}\le C\delta^2\Delta_k\theta_k^{L_4(s)-1},$$

where $L_4(s)=max\{s+13-2\alpha,(s+4-\alpha)_++16-2\alpha,(s+2-\alpha)_++19-2\alpha\}$.

The proof of this lemma follows from a direct computation.

4.4.7 Proof of the inductive argument

We are ready to show H_n is implied from H_{n-1}. First we add the above three error together and have the following estimate

Lemma 4.4.7. *Let $\alpha \geq 13$, $\delta > 0$ sufficiently small and θ_0 sufficiently large. Then for $k = 0, ..., n-1$ and for all integer $s \in [7, \tilde{\alpha} - 2]$, the following estimate holds*

$$[e_k]_{s,\gamma,T} + \|\tilde{e}_k\|_{H^s_\gamma(\omega_T)} \leq C\delta^2 \Delta_k \theta_k^{L_4(s)-1},$$
$$[\hat{e}_k]_{s,\gamma,T} \leq C\delta^2 \Delta_k \theta_k^{s+7-2\alpha},$$

where $L_4(s)$ is defined as above.

Then by summing e_k, $\hat{e}_k$ and $\tilde{e}_k$ over $k = 0, ..., n-1$, we have

Lemma 4.4.8. *Let $\alpha \geq 15$, $\tilde{\alpha} \geq \alpha + 4$, $\delta > 0$ sufficiently small and θ_0 sufficiently large. Then the following estimate holds*

$$[E_n]_{\tilde{\alpha}-2,\gamma,T} + \|\tilde{E}_n\|_{H^{\tilde{\alpha}-2}_\gamma(\omega_T)} \leq C\delta^2 \theta_k,$$
$$[\hat{E}_n]_{\tilde{\alpha}-2,\gamma,T} \leq C\delta^2.$$

To obtain the estimate of δV_n and $\delta \Psi_n$ by (4.3.11), we need to estimate the source terms f_n, g_n, $h_n^\pm$ which are defined by (4.3.4), (4.3.7) and (4.3.8). We give the result in the following lemma

Lemma 4.4.9. *Let $\alpha \geq 15$, $\tilde{\alpha} \geq \alpha + 4$, $\delta > 0$ sufficiently small and θ_0 sufficiently large. Then for all integer $s \in [7, \tilde{\alpha} + 1]$, the following estimate holds*

$$[f_n]_{s,\gamma,T} \leq C\Delta_n \left\{ \theta_n^{s-\alpha-2}([f^a]_{\alpha+1,\gamma,T} + \delta^2) + \delta^2 \theta_n^{L_4(s)-1} \right\},$$
$$\|g_n\|_{H^s_\gamma(\omega_T)} \leq C\delta^2 \Delta_n (\theta_n^{s-\alpha-2} + \theta_n^{L_4(s)-1}),$$

and for all integer $s \in [7, \tilde{\alpha}]$,the following estimate holds

$$[h_n]_{s,\gamma,T} \leq C\delta^2 \Delta_n (\theta_n^{L_4(s)-1} + \theta_n^{s-\alpha-2}).$$

Proof. From definition of f_n, g_n, $h_n^{\pm}$, we decompose

$$\begin{aligned}
&f_n = (S_{\theta_n} - S_{\theta_{n-1}})f^a - (S_{\theta_n} - S_{\theta_{n-1}})E_{n-1} - S_{\theta_n} e_{n-1},\\
&g_n = -(S_{\theta_n} - S_{\theta_{n-1}})\tilde{E}_{n-1} - S_{\theta_n}\tilde{e}_{n-1},\\
&h_n^+ = (S_{\theta_n} - S_{\theta_{n-1}})(\mathcal{R}_T\tilde{E}_{n-1,2} - \hat{E}^+_{n-1}) + S_{\theta_n}(\mathcal{R}_T\tilde{e}_{n-1,2} - \hat{e}^+_{n-1}).\\
&h_n^- = (S_{\theta_n} - S_{\theta_{n-1}})(\mathcal{R}_T\tilde{E}_{n-1,2} - \mathcal{R}_T\tilde{E}_{n-1,1} - \hat{E}^-_{n-1}) + S_{\theta_n}(\mathcal{R}_T\tilde{e}_{n-1,2} - \mathcal{R}_T\tilde{e}_{n-1,1} - \hat{e}^-_{n-1}).
\end{aligned}$$

By using the Lemma 4.3.1, Lemma 4.4.7, Lemma 4.4.8 and the equivalence of θ_{n-1} and θ_n, we can obtain the above results. □

Now we are able to prove the first statement of H_n. Specifically, we have the following lemma:

Lemma 4.4.10. *Let $\alpha \geq 15$, $\delta > 0$ and $[f_n]_{\alpha+1,\gamma,T}/\delta$ sufficiently small and θ_0 sufficiently large. Then for all integer $s \in [7, \tilde{\alpha}]$, the following estimate holds*

$$[\delta V_n, \delta\Psi_n]_{s,\gamma,T} + \|\delta\psi_n\|_{H^s_\gamma(\omega_T)} \leq \delta\theta_n^{s-\alpha-1}\Delta_n.$$

Proof. From the above analysis, we verified (4.3.3) satisfied the requirement we pose on our linear system to guarantee the well-posedness and the tame estimates. Moreover by (4.3.2), we obtain

$$[\delta V_n]_{s,\gamma,T} \leq C[\delta\dot{V}_n, \delta\Psi_n]_{s,\gamma,T} + [\delta\Psi_n]_{5,\gamma,T}[\dot{U}^a + V_{n+1/2}, \dot{\Phi}^a + \Psi_{n+1/2}]_{s+2,\gamma,T}. \tag{4.4.2}$$

So from (4.3.10) and (4.3.11), we have

$$\begin{aligned}
&[\delta V_n]_{s,\gamma,T} + [\delta\Psi_n]_{s,\gamma,T} + \|\delta\psi_n\|_{H^{s+1}_\gamma(\omega_T)} \leq C(K)\Big\{[f_n]_{s+1,\gamma,T} + \|g_n\|_{H^{s+1}_\gamma(\omega_T)}\\
&+[h_n]_{s,\gamma,T} + \Big([f_n]_{6,\gamma,T} + \|g_n\|_{H^6_\gamma(\omega_T)} + [h_n]_{5,\gamma,T}\Big)[U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}]_{s+4,\gamma,T}\Big\}.
\end{aligned}$$

By (4.4.1), Lemma 4.4.1, Proposition 4.4.1, Lemma 4.4.9, we have

$$\begin{aligned}
&[\delta V_n]_{s,\gamma,T} + [\delta\Psi_n]_{s,\gamma,T} + \|\delta\psi_n\|_{H^{s+1}_\gamma(\omega_T)}\\
&\leq C\left\{\Delta_n\left(\theta_n^{5-\alpha}([f^a]_{\alpha+1,\gamma,T} + \delta^2\theta_n^{20-2\alpha}\right)(\delta + \delta\theta_n^{(s+4-\alpha)_+} + \delta\theta_n^{s+5-\alpha})\right.\\
&\left.+\Delta_n\left(\theta_n^{s-\alpha-1}([f^a]_{\alpha+1,\gamma,T} + \delta^2) + \delta^2\theta_n^{L_4(s+1)-1}\right)\right\}
\end{aligned}$$

So our object is to show the right hand side of above inequality is less that $\delta\theta_n^{s-\alpha-1}\Delta_n$ for all $s \in [7, \tilde{\alpha}]$. Since $\alpha \geq 15$, we have

$$L_4(s+1) \leq s-\alpha, \qquad (s+4-\alpha)_+ + 5 - \alpha \leq s-\alpha-1,$$
$$s+10-2\alpha \leq s-\alpha-1, \qquad (s+4-\alpha)_+ + 20 - 2\alpha \leq s-\alpha-1,$$
$$s+25-3\alpha \leq s-\alpha-1.$$

Thus we just need to take δ and $[f_n]_{\alpha+1,\gamma,T}/\delta$ small enough, and this lemma is proved. □

For the second statement of H_n, we have

Lemma 4.4.11. *Let $\alpha \geq 15$, $\delta > 0$ and $[f_n]_{\alpha+1,\gamma,T}/\delta$ sufficiently small and θ_0 sufficiently large. Then for all integer $s \in [7, \tilde{\alpha}-2]$, the following estimate holds*

$$[\mathcal{L}(V_n, \Psi_n) - f^a]_{s,\gamma,T} \leq 2\delta\theta_n^{s-\alpha-1}.$$

Proof. From (4.3.9), We decompose

$$\mathcal{L}(V_n, \Psi_n) - f^a = (S_{\theta_{n-1}} - I)f^a + (I - S_{\theta_{n-1}})E_{n-1} + e_{n-1}.$$

First we estimate $[(S_{\theta_{n-1}} - I)f^a]_{s,\gamma,T}$. If $\alpha + 1 \leq s \leq \tilde{\alpha} - 2$,

$$\begin{aligned}[(S_{\theta_{n-1}} - I)f^a]_{s,\gamma,T} &\leq [S_{\theta_{n-1}}f^a]_{s,\gamma,T} + [f^a]_{s,\gamma,T} \\ &\leq C\theta_{n-1}^{s-\alpha-1}[f^a]_{\alpha+1} + [f^a]_{\tilde{\alpha}-2} \leq C\theta_n^{s-\alpha-1}([f^a]_{\alpha+1} + \delta).\end{aligned}$$

If $7 \leq s \leq \alpha + 1$, we have

$$[(S_{\theta_{n-1}} - I)f^a]_{s,\gamma,T} \leq C\theta_{n-1}^{s-\alpha-1}[f^a]_{\alpha+1} \leq C\theta_{n-1}^{s-\alpha-1}[f^a]_{\alpha+1}.$$

For the last two terms above, we have

$$\begin{aligned}&[(I - S_{\theta_{n-1}})E_{n-1}]_{s,\gamma,T} \leq C\theta_{n-1}^{s-\tilde{\alpha}+2}[E_{n-1}]_{\tilde{\alpha}-2,\gamma,T} \leq C\theta_{n-1}^{s-\tilde{\alpha}+2}\delta\theta_{n-1} \leq C\delta^2\theta_n^{s-\alpha-1}, \\ &[e_{n-1}]_{s,\gamma,T} \leq C\delta^2\theta_n^{L(s)-1}\Delta_n \leq C\delta^2\theta_n^{L(s)-2} \leq C\delta^2\theta_n^{s-\alpha-1}.\end{aligned}$$

Combining all terms together and taking δ and $[f_n]_{\alpha+1,\gamma,T}/\delta$ small enough, we obtained this lemma. □

For the third statement of H_n, we have

Lemma 4.4.12. *Let $\alpha \geq 15$, $\delta > 0$ sufficiently small and θ_0 sufficiently large. Then for all integer $s \in [7, \tilde{\alpha} - 2]$, the following estimate holds*

$$\|\mathcal{B}(V_n|_{x_2=0}, \psi_n)\|_{H^s_\gamma(\omega_T)} \leq \delta\theta_n^{s-\alpha-1}.$$

Following the exactly the same argument as above, we decompose $\mathcal{B}(V_n|_{x_2=0}, \psi_n) = (I - S_{\theta_{n-1}})\tilde{E}_{n-1} + \tilde{e}_{n-1}$ and estimate each term on the right hand side. Then the lemma is followed with small enough δ and $[f_n]_{\alpha+1,\gamma,T}/\delta$.

The last step to complete the inductive scheme, we are only left to show H_0.

Lemma 4.4.13. *if $[f^a]_{\alpha+1,\gamma,T}/\delta$ sufficiently small, then the property H_0 is true.*

Proof. To show H_0 is true, we consider (4.3.3) with $n = 0$. We recall that $V_0 = \Psi_0 = \psi_0 = 0$. From the definition of the approximate solution, Lemma 4.2.1, and the construction of intermediate states, Proposition 4.4.1, we have $V_{1/2} = \Psi_{1/2} = \psi_{1/2} = 0$. So (4.3.3) become

$$\begin{cases} \mathbb{L}'_e(U^a, \Phi^a)\delta\dot{V}_0 = S_{\theta_0} f^a & \text{in } \Omega_T, \\ \mathbb{B}'_{n+1/2}(\delta\dot{V}_0|_{x_2=0}, \delta\psi_0) = 0 & \text{on } \omega_T, \\ \delta\dot{V}_0 = 0, \quad \delta\psi_0 = 0 & \text{for } t < 0, \end{cases}$$

Of course the above system satisfy all the condition we need in the well-posedness and the tame estimates results. Moreover for the equations determines $\delta\Psi_0$, we have

$$\partial_t\delta\Psi_0^\pm + v^{a\pm}\partial_1\delta\Psi_0^\pm + \left\{\partial_1\Phi^{a\pm}\frac{\partial_2 v^{a\pm}}{\partial_2\Phi^{a\pm}} - \frac{\partial_2 u^{a+}}{\partial_2\Phi^{a\pm}}\right\}\delta\Psi_0^\pm + \partial_1\Phi^a\delta\dot{v}_0^\pm - \delta\dot{u}_0^\pm = 0,$$

So by (4.3.11), (4.3.10) and (4.4.1), we have

$$\begin{aligned} [\delta V_0, \delta\Psi_0]_{s,\gamma,T} + \|\psi_0\|_{H^s_\gamma(\omega_T)} &\leq [S_{\theta_0} f^a]_{s+1,\gamma,T} + [S_{\theta_0} f^a]_{6,\gamma,T}[\dot{U}^a, \dot{\Phi}^a]_{s+4,\gamma,T}, \\ &\leq C[S_{\theta_0} f^a]_{s+1,\gamma,T} \leq C\theta_0^{(s-\alpha)_+}[f^a]_{\alpha+1,\gamma,T}. \end{aligned}$$

By taking $[f^a]_{\alpha+1,\gamma,T}/\delta$ sufficient small and noticing θ_0 is a fixed constant, we have

$$[\delta V_0, \delta\Psi_0]_{s,\gamma,T} + \|\psi_0\|_{H^s_\gamma(\omega_T)} \leq \delta\theta_0^{s-\alpha-1}\Delta_0,$$

for all $7 \leq s \leq \tilde{\alpha}$. For the last two statements, we just point out that

$$\mathcal{L}(V_0, \Psi_0) - f^a = -f^a$$
$$\mathcal{B}(V_0|_{x_2=0}, \psi_0) = \mathbb{B}(U^a|_{x_2=0}, \varphi^a) = 0$$

So the results follows by taking $[f^a]_{\alpha+1,\gamma,T}/\delta$ sufficient small. □

4.4.8 Proof of the main theorem

Finally, we are able to complete the proof of our main result. We fixed $\alpha \geq 15$ and take $\tilde{\alpha} = \alpha + 4$ and $m = \tilde{\alpha} + 3$. From the Lemma 4.2.1 and (4.2.26), by taking $[\dot{U}_0]_{2m+1,*}$ and $\|\varphi_0\|_{H^{2m+2}}$ sufficient small and satisfying the compatibility conditions up to order m, we can construct an approximate solution $U^a = \bar{U} + \dot{U}^a$ and $\Phi^{a\pm} = \pm x_2 + \dot{\Phi}^{a\pm}$ such that (4.4.1) satisfied with $m = \tilde{\alpha} + 3$ and all the requirements in Lemma 4.4.10 Lemma 4.4.11, Lemma 4.4.12 and Lemma 4.4.13. Hence we have H_n holds for all $n \in \mathbb{N}$. Adding all the increments δV_n, $\delta \Psi_n$ and $\delta \psi_n$ and take $s = \alpha - 1$ in H_n, we have

$$\sum_{n \in \mathbb{N}} [\delta V_n, \delta \Psi_n]_{\alpha-1,\gamma,T} + \|\delta \psi_n\|_{H^\alpha_\gamma(\omega_T)} \leq \delta \sum_{n \in \mathbb{N}} \theta_n^{-2} \Delta_n < \infty.$$

So the sequence (V_n) and (Ψ_n) converges in norms of $H^{\alpha-1}_*(\Omega_T)$ to some function V and Ψ, and (ψ_n) converges in norms of $H^\alpha_\gamma(\omega_T)$ to some function ψ. Moreover passing the limit in $\mathcal{L}(V_n, \Psi_n)$ and $\mathcal{B}(V_n|_{x_2=0}, \psi_n)$ as $n \to \infty$, we obtain $U = U^a + V$ and $\Phi = \Phi^a + \Psi$ is a solution to the nonlinear system. The proof of our main theorem is completed.

BIBLIOGRAPHY

[1] S. Alinhac, *Existence d'ondes de rarfaction pour des systmes quasi-linaires hyperboliques multidimensionnels*. Comm. Partial Differential Equations 14 (1989), no. 2, 173–230.

[2] D. M. Ambrose and N. Masmoudi, *Well-posedness of 3D vortex sheets with surface tension*. Commun. Math. Sci. 5 (2007), no. 2, 391–430.

[3] M. Artola and A. J. Majda, *Nonlinear development of instabilities in supersonic vortex sheets. I. The basic kink modes*. Phys. D 28 (1987), no. 3, 253–281.

[4] M. Artola and A. J. Majda, *Nonlinear development of instabilities in supersonic vortex sheets. II. Resonant interaction among kink modes*. SIAM J. Appl. Math. 49 (1989), no. 5, 1310–1349.

[5] M. Artola and A. J. Majda, *Nonlinear kink modes for supersonic vortex sheets*. Phys. Fluids A 1 (1989), no. 3, 583–596.

[6] J. Azaiez and G. Homsy, *Linear stability of free shear flow of viscoelastic liquids*. J. Fluid Mech. 268 (1994), 37–69.

[7] S. Benzoni-Gavage and D. Serre, *Multidimensional hyperbolic partial differential equations. First-order systems and applications*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

[8] A. Blokhin and Y. Trakhinin, *Stability of strong discontinuities in fluids and MHD*. Handbook of mathematical fluid dynamics, Vol. I, 545–652, North-Holland, Amsterdam, 2002.

[9] J. M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*. Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 2, 209–246.

[10] J. Chazarain and A. Piriou, *Introduction to the theory of linear partial differential equations*. Studies in Mathematics and its Applications, 14. North-Holland Publishing Co., Amsterdam-New York, 1982.

[11] J. Y. Chemin, *Perfect incompressible fluids*. Oxford Lecture Series in Mathematics and its Applications, 14. The Clarendon Press, Oxford University Press, New York, 1998.

[12] G.-Q. Chen, V. Kukreja, and H. Yuan, *Well-posedness of transonic characteristic discontinuities in two-dimensional steady compressible Euler flows.* Z. Angew. Math. Phys. 64 (2013), no. 6, 1711–1727.

[13] G.-Q. Chen and Y.-G. Wang, *Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics.* Arch. Ration. Mech. Anal. 187 (2008), no. 3, 369–408.

[14] G.-Q. Chen and Y.-G. Wang, *Characteristic discontinuities and free boundary problems for hyperbolic conservation laws.* Nonlinear partial differential equations, 53–81, Abel Symp., 7, Springer, Heidelberg, 2012.

[15] G.-Q. Chen, Y. Zhang, and D. Zhu, *Stability of compressible vortex sheets in steady supersonic Euler flows over Lipschitz walls.* SIAM J. Math. Anal. 38 (2006/07), no. 5, 1660–1693 (electronic).

[16] J.-F. Coulombel, *Weak stability of nonuniformly stable multidimensional shocks.* SIAM J. Math. Anal. 34 (2002), no. 1, 142–172 (electronic).

[17] J.-F. Coulombel, *Weakly stable multidimensional shocks.* Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 4, 401–443.

[18] J.-F. Coulombel, *Well-posedness of hyperbolic initial boundary value problems.* J. Math. Pures Appl. (9) 84 (2005), no. 6, 786–818.

[19] J.-F. Coulombel and P. Secchi, *On the transition to instability for compressible vortex sheets.* Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), no. 5, 885–892.

[20] J.-F. Coulombel and P. Secchi, *The stability of compressible vortex sheets in two space dimensions.* Indiana Univ. Math. J. 53 (2004), no. 4, 941–1012.

[21] J.-F. Coulombel and P. Secchi, *Nonlinear compressible vortex sheets in two space dimensions.* Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 1, 85–139.

[22] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics.* Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 325. Springer-Verlag, Berlin, 2010.

[23] J. Fejer and J. W. Miles, *On the stability of a plane vortex sheet with respect to three-dimensional disturbances.* J. Fluid Mech. 15 (1963) 335–336.

[24] J. Francheteau and G. Métivier, *Existence de chocs faibles pour des systemes quasi-linéaires hyperboliques multidimensionnels.* Astérisque No. 268 (2000).

[25] T. G. Goktekin, A. W. Bargteil, and J. F. O'Brien, *A method for animating viscoelastic fluids.* ACM Transactions on Graphics (TOG), 23 (2004), 463–468.

[26] O. Guès, *Problème mixte hyperbolique quasi-linéaire caractéristique*. Comm. Partial Differential Equations 15 (1990), no. 5, 595–645.

[27] M. E. Gurtin, *An introduction to continuum mechanics*. Mathematics in Science and Engineering, 158. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[28] R. Hersh, *Mixed problems in several variables*. J. Math. Mech. 12 1963 317–334.

[29] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*. Mathématiques & Applications (Berlin) [Mathematics & Applications], 26. Springer-Verlag, Berlin, 1997.

[30] X. Hu and D. Wang, *Local strong solution to the compressible viscoelastic flow with large data*. J. Differential Equations 249 (2010), no. 5, 1179–1198.

[31] X. Hu and D. Wang, *Global existence for the multi-dimensional compressible viscoelastic flows*. J. Differential Equations 250 (2011), no. 2, 1200–1231.

[32] X. Hu and D. Wang, *Formation of singularity for compressible viscoelasticity*. Acta Math. Sci. Ser. B Engl. Ed. 32 (2012), no. 1, 109–128.

[33] R. R. Huilgol, *Propagation of a vortex sheet in viscoelastic liquids–the Rayleigh problem.* Journal of Non-Newtonian Fluid Mechanics, 8 (1981), 337–347.

[34] R. R. Huilgol, *Fluid Mechanics of Viscoplasticity*. Springer, 2015.

[35] Q. Jiu and Z. Xin, *On strong convergence to 3-D axisymmetric vortex sheets*. J. Differential Equations 223 (2006), no. 1, 3350.

[36] Q. Jiu and Z. Xin, *On strong convergence to 3D steady vortex sheets*. J. Differential Equations 239 (2007), no. 2, 448470.

[37] D. Joseph, *Fluid dynamics of viscoelastic liquids*. Applied Mathematical Sciences, 84. Springer-Verlag, New York, 1990.

[38] D. Joseph, T. Funada, and J. Wang, *Potential flows of viscous and viscoelastic liquids.* 21, Cambridge University Press, UK, 2007.

[39] A. Kaffel and M. Renardy, *On the stability of plane parallel viscoelastic shear flows in the limit of infinite weissenberg and reynolds numbers*. Journal of Non-Newtonian Fluid Mechanics, 165 (2010), 1670–1676.

[40] H.-O. Kreiss, *Initial boundary value problems for hyperbolic systems*. Comm. Pure Appl. Math. 23 (1970), 277–298.

[41] K. Kunisch and X. Marduel, *Optimal control of non-isothermal viscoelastic fluid flow.* Journal of Non-Newtonian Fluid Mechanics, 88 (2000), 261–301.

[42] P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*. Comm. Pure Appl. Math. 13 (1960) 427–455.

[43] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 2*. Travaux et Recherches Mathématiques, No. 18 Dunod, Paris 1968.

[44] C. Liu and N. J. Walkington, *An Eulerian description of fluids containing visco-elastic particles*. Arch. Ration. Mech. Anal. 159 (2001), no. 3, 229–252.

[45] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Z. Xin, *Existence of vortex sheets with reflection symmetry in two space dimensions*. Arch. Ration. Mech. Anal. 158 (2001), no. 3, 235–257.

[46] A. J. Majda, *The stability of multi-dimensional shock fronts*. Mem. Amer. Math. Soc. 41 (1983), no. 275.

[47] A. J. Majda, *The existence of multi-dimensional shock fronts*. Mem. Amer. Math. Soc. 43 (1983), no. 281.

[48] A. J. Majda and S. Osher, *Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary*. Comm. Pure Appl. Math. 28 (1975), no. 5, 607–675.

[49] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.

[50] G. Métivier, *Stability of multidimensional shocks*. Advances in the theory of shock waves, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001, 25–103.

[51] Y. Meyer, *Remarques sur un théorème de J.-M. Bony*. Rend. Circ. Mat. Palermo (2) 1981, suppl. 1, 1–20.

[52] J. W. Miles, *On the reflection of sound at an interface of relative motion*. J. Acoust. Soc. Amer. 29 (1957), 226–228.

[53] J. W. Miles, *On the disturbed motion of a plane vortex sheet*. J. Fluid Mech. 4 (1958), 538–552.

[54] A. Morando and P. Trebeschi, *Two-dimensional vortex sheets for the nonisentropic Euler equations: linear stability*. J. Hyperbolic Differ. Equ. 5 (2008), no. 3, 487–518.

[55] J. G. Oldroyd, *Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids*. Proc. Roy. Soc. London. Ser. A 245 (1958), 278–297.

[56] M. Ohno and T. Shirota, *On the initial-boundary-value problem for the linearized equations of magnetohydrodynamics*. Arch. Rational Mech. Anal. 144 (1998), no. 3, 259–299.

[57] M. Ohno, Y. Shizuta and T. Yanagisawa,*The trace theorem on anisotropic Sobolev spaces*. Tohoku Math. J. (2) 46 (1994), no. 3, 393–401.

[58] M. Ohno, Y. Shizuta and T. Yanagisawa, *The initial-boundary value problem for linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity*. J. Math. Kyoto Univ. 35 (1995), no. 2, 143–210.

[59] M. Renardy, W. J. Hrusa, and J. A. Nohel, *Mathematical problems in viscoelasticity*. Pitman Monographs and Surveys in Pure and Applied Mathematics, 35. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987.

[60] L. Ruan, D. Wang, S. Weng, and C. Zhu, *Rectilinear vortex sheets of inviscid liquid-gas two-phase flow: linear stability*. To appear in Commun. Math. Sci.

[61] P. Secchi, *On an initial-boundary value problem for the equations of ideal magnetohydrodynamics*. Math. Methods Appl. Sci. 18 (1995), no. 11, 841–853.

[62] P. Secchi, *The initial-boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity*. Differential Integral Equations 9 (1996), no. 4, 671–700

[63] P. Secchi, *Well-posedness of characteristic symmetric hyperbolic systems*. Arch. Rational Mech. Anal. 134 (1996), no. 2, 155–197.

[64] P. Secchi, *Some properties of anisotropic Sobolev spaces*. Arch. Math. (Basel) 75 (2000), no. 3, 207–216.

[65] P. Secchi, *An initial boundary value problem in ideal magneto-hydrodynamics*. NoDEA Nonlinear Differential Equations Appl. 9 (2002), no. 4, 441–458.

[66] P. Secchi and P. Trebeschi, *Non-homogeneous quasi-linear symmetric hyperbolic systems with characteristic boundary*. Int. J. Pure Appl. Math. 23 (2005), no. 1, 39–59.

[67] D. Serre, *Systems of conservation laws. 2. Geometric structures, oscillations, and initial-boundary value problems*. Cambridge University Press, Cambridge, 2000.

[68] Y. Shizuta and K. Yabuta, *The trace theorems in anisotropic Sobolev spaces and their applications to the characteristic initial-boundary value problem for symmetric hyperbolic systems*. Math. Models Methods Appl. Sci. 5 (1995), no. 8, 1079–1092.

[69] C. Sulem, P. Sulem, C. Bardos, and U. Frisch, *Finite time analyticity for the two- and three- dimensional Kelvin-Helmholtz instability*. Comm. Math. Phys. 80 (1981), no. 4, 485–516.

[70] Y. Trakhinin, *Existence of compressible current-vortex sheets: variable coefficients linear analysis*. Arch. Ration. Mech. Anal. 177 (2005), no. 3, 331–366.

[71] Y. Trakhinin, *On the existence of incompressible current–vortex sheets: study of a linearized free boundary value problem*. Math. Methods Appl. Sci. 28 (2005), no. 8, 917–945.

[72] Y. Trakhinin, *The existence of current-vortex sheets in ideal compressible magnetohydrodynamics*. Arch. Ration. Mech. Anal. 191 (2009), no. 2, 245–310.

[73] C. Wang and Z. Zhang, *A new proof of Wu's theorem on vortex sheets*. Sci. China Math. 55 (2012), no. 7, 1449–1462.

[74] Y.-G. Wang and F. Yu, *Stability of contact discontinuities in three-dimensional compressible steady flows*. J. Differential Equations 255 (2013), no. 6, 1278–1356.

[75] Y.-G. Wang and F. Yu, *Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets*. Arch. Ration. Mech. Anal. 208 (2013), no. 2, 341–389.

[76] Y.-G. Wang and F. Yu, *Structural stability of supersonic contact discontinuities in three-dimensional compressible steady flows*. preprint, 2014, arXiv: 1407.1464.

[77] S. Wu, *Mathematical analysis of vortex sheets*. Comm. Pure Appl. Math. 59 (2006), no. 8, 1065–1206.

[78] T. Yanagisawa and A. Matsumura, *The fixed boundary value problems for the equations of ideal magnetohydrodynamics with a perfectly conducting wall condition*. Comm. Math. Phys. 136 (1991), no. 1, 119–140.

[79] J.-D. Yu, S. Sakai, and J. A. Sethian, *Two-phase viscoelastic jetting*. J. Comput. Phys. 220 (2007), no. 2, 568–585.