THE STRUCTURE AND INTERPRETATION OF QUANTUM FIELD THEORY

by

Michael E. Miller

B. A., The University of Chicago, 2008

M. A., Columbia University, 2011

Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences

in partial fulfilment of

the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2017

UNIVERSITY OF PITTSBURGH
DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Michael E. Miller

It was defended on

24 April 2017

and approved by

Robert Batterman, Ph.D., Philosophy
John D. Norton, Ph.D., History and Philosophy of Science
John Earman, Ph.D., History and Philosophy of Science
Mark Wilson, Ph.D., Philosophy
Gordon Belot, Ph.D., Philosophy (University of Michigan)
Laura Ruetsche, Ph.D., Philosophy (University of Michigan)

Dissertation Director: Robert Batterman

THE STRUCTURE AND INTERPRETATION OF QUANTUM FIELD THEORY

Michael E. Miller, PhD

University of Pittsburgh, 2017

Quantum field theory accurately describes the world on the finest scales to which we have empirical access. There has been significant disagreement, however, about which mathematical structures ought to be taken as constitutive of the theory, and thus over which structures should serve as the basis for its interpretation. Perturbative methods allow for successful empirical prediction but require mathematical manipulations that are at odds with the canonical approach to interpreting physical theories that has been passed down from the logical positivists. Axiomatic characterizations of the theory, on the other hand, have not been shown to admit empirically interesting models. This dissertation shows how to understand the empirical success of quantum field theory by reconsidering widely held commitments about how physical meaning accrues to mathematical structure.

Keywords: Quantum field theory, ultraviolet divergences, infrared divergences, large-order divergences, empirical adequacy, structure of scientific theories

TABLE OF CONTENTS

Free fields are mathematical objects; they are not very physical.

– Günter Scharf

PREFACE

This work is the culmination of many years of study during which I have benefited from the work and teaching of too many physicists and philosophers to name. That said, a number people have been especially important influences and I would like to acknowledge them here.

I am grateful to David Albert for his role in shepherding me into philosophy and in giving me a philosophical conscience. During my time with David I also met Porter Williams who continues to be the best collaborator and friend that one could ask for.

I am fortunate to have been a part of the graduate student community in the Department of History and Philosophy of Science at the University of Pittsburgh. I would like to thank Marina Baldissera-Pacchetti, Trey Boone, Nora Boyd, Julia Bursten, Haixin Dang, Josh Eisenthal, Leif Hancox-Li, Joe McCaffrey, Tom Pashby, Bryan Roberts, Lauren Ross, Katie Tabb, and fellow traveler James Fraser for their friendship and support, academic and otherwise.

I am also fortunate to have had guidance and pointed questioning from Giovanni Valente, Jim Woodward and the members of my dissertation committee; Gordon Belot, John Earman, John Norton, Laura Ruetsche, and Mark Wilson. They have all contributed significantly to the development of my thinking about the problems discussed in this dissertation and in their own way they have each contributed to who I am as a philosopher more generally. My deepest intellectual debt is to Bob Batterman. He gave me the large-order disease, and for that I am grateful. I could not have asked for a more supportive advisor.

My greatest personal debts are owed to Patrick Beukema and Ran Liu, Porter Williams and Ashley Waller, Bob Batterman and Carolyn McMullen, Katie Tabb, and my family. I would not have made it without them. I am deeply grateful to my family in particular for their unceasing patience and support.

1.0 INTRODUCTION

Quantum field theory originally arose from efforts to develop a relativistic and quantum mechanical theory of electromagnetic phenomena. This work resulted in quantum electrodynamics, as well as a general framework for describing systems of interacting quantum fields. The modern version of this perturbative Lagrangian formalism is the one presented in introductory textbooks on quantum field theory.[1] The Standard Model of particle physics, which is cast in the perturbative formalism, provides a description of the strong, weak, and electromagnetic forces and the elementary particles that experience them. In this way, quantum field theory plays a central role in describing the contents and behavior of the physical world on the finest scales to which we have empirical access. Philosophers interested in the fundamental material constituents of the world have thus reasonably turned to quantum field theory for guidance. This philosophical project is complicated, however, by the mathematical structure of the theory.

Despite the widely accepted empirical and theoretical successes achieved using the standard perturbative formalism, the mathematical character of the theory remains, at least in certain respects, poorly understood. Frequently cited deficiencies of the perturbative formalism include the facts that the interaction picture in which scattering theory calculations are carried out provably does not exist, individual orders of perturbation theory yield infinite values due to the ultraviolet and infrared regime of the theory, and the measures of path integrals are not always well-defined. Perhaps most worryingly, even when renormalized to all orders, the perturbative expansions for physical quantities are widely believed, and in

[1] See, for example, (Peskin and Schroeder, 1995).

some cases have been conclusively shown, to be divergent. Mathematical techniques to overcome these deficiencies are available. Regulators are imposed and perturbative expansions are truncated at low-order. However, these techniques produce what seem to be physically significant differences and philosophers have questioned whether the mathematical structure of the theory is sufficiently well-specified to support philosophical interpretation.

The mathematical deficiencies of the perturbative approach have motivated the development of alternative formalisms for characterizing the structure of quantum field theory. Writing in 1959, Irving Segal claimed that: "just what is a quantum field theory ... is a difficult question, since at present what we have after thirty years of intensive effort, is a collection of partially heuristic technical developments in search of a theory; but it is a natural one to examine axiomatically" (Segal, 1959, p. 341). In response to the problems with the standard approach, mathematical physicists have developed several axiomatic systems to capture the physical principles that are assumed to obtain in the standard perturbative formalism. Axiomatizations of quantum field theory provide an explicit characterization of the expected non-perturbative structure of models of the theory. This allows for the proof of theorems, satisfying the standard of rigor accepted by mathematicians, that apply to any quantum field theory that satisfies the axioms. The CPT and spin-statistics theorems are paradigmatic examples. The project of generating such theorems is the central task of axiomatic quantum field theory.

Once a set of axioms has been established, models of those axioms can be constructed. Sophisticated mathematical methods, often inspired by techniques originally developed within the perturbative formalism, have been used to produce increasingly physically realistic models of the axioms. With explicit models constructed, further properties of those models such as their scattering behavior can be explored. These are the central tasks of what is commonly referred to as constructive quantum field theory. Though much progress has been made in the half century since Segal was writing, the models available are all defined either in reduced spacetime dimension or without interactions. The 4-dimensional local gauge theories that make up the empirically successful Standard Model have not been shown to be models of any of the sets of axioms.

Philosophers interested in interpreting quantum field theory thus face the following tension.[2] The empirical successes of the theory are achieved using mathematical structures that do not conform to the philosophical accounts of how theories capture information about the world. Axiomatic and constructive field theory provide structures that do conform to philosophical accounts of the structure of theories, but have not been shown to admit the models that are empirically successful. Moreover, in conceptual analysis based on the axiomatic and constructive approaches, there often is no direct argument available for why the conclusions should also apply in the case of empirically adequate models. For this reason it is not clear how such analysis informs our understanding of the actual world. This dilemma is captured nicely in the following remark of Ruetsche:

> Given a theory T, . . . we confront the exemplary interpretive question of how exactly to establish a correspondence between T's models and worlds possible according to T. That is, we confront that question *if* T is the sort of thing that has models. 'A collection of partially heuristic technical developments' isn't readily attributed a set of models about whose underlying ontology or principles of individuation philosophical questions immediately arise. This isn't to say that 'a collection of partially heuristic technical developments' is unworthy of philosophical attention. It is in itself a philosophically provocative circumstance that such a collection can enjoy stunning empirical success. (Ruetsche, 2011, p. 102-103)

In order to understand what the empirical success of quantum field theory tells us about the fundamental material constituents of the world, an account of how that success is possible despite the mathematical deficiencies of the perturbative formalism is required. This dissertation aims to provide such an account.

The first step toward such an account is to recognize that the characterization of perturbative field theory as a "collection of partially heuristic technical developments" is overly

[2]This tension is the topic of a recent debate between Fraser and Wallace. Their positions nicely capture a gulf that is reflected throughout the literature on the interpretation of quantum field theory (Fraser, 2011; Wallace, 2011).

pessimistic. There are three legitimate problems with the mathematical structure of perturbative field theory. Each is a class of divergences that results in the theory predicting infinite probabilities for physical processes. However, these ultraviolet, infrared, and large-order divergences exhaust the mathematical deficiencies of the theory. If conceptually adequate resolutions to these problems can be found, then the theory can be rendered well-defined. And in fact, practitioners of perturbative field theory have developed methods that lead to conceptually adequate resolutions to each of the classes of divergences. Systematic analysis of these methods provides what is required to take up the philosophically provocative circumstance that Reutsche identifies and to provide an account of why the theory is so successful despite its mathematical shortcomings.

What this analysis shows is that we were wrong to think that the presence of divergences is an obstacle to interpretation. Instead, divergences are actually critical hints about how mathematical structure holds physical meaning: they tell us how we should go about interpreting the theory. When one takes these hints seriously, what one finds is that there are assumptions involved in standard philosophical approaches to interpretation that are untenable in empirically adequate models of quantum field theory. The methods used to insulate such models from the divergences that they contain all result in theoretical characterizations of empirical content that are *inexact*. That is not to say that the theory is ill-defined. The sense in which it makes inexact predictions can be made completely precise. If one is willing to make relatively mild modifications to standard philosophical approaches to interpretation then this inexactness can be incorporated into a new account of interpretation that can be used to treat empirically adequate models of quantum field theory. The central aim of this dissertation is use this strategy to provide an alternative approach to interpretation which allows for coherent attributions of physical meaning to the models of quantum field theory that actually make contact with the world.

1.1 STRUCTURE AND INTERPRETATION

The problem of interpreting a physical theory is the problem of associating physical meaning with the mathematical structure of the theory. Much of the work in the philosophy of physics has been dedicated to providing such interpretations for special and general relativity, thermodynamics, statistical mechanics, non-relativistic quantum mechanics, and quantum field theory. Accounts of how to interpret a physical theory have developed in conjunction with, and in several important cases are attendant to, accounts of the structure of scientific theories. In other cases novel commitments about interpretation are tacitly adopted in the work of those interpreting particular physical theories. There are significant differences between views about how to go about the project of interpretation. However, many approaches to interpretation share a common set of commitments about the relationship between mathematical structure and physical meaning. In particular, they all take the physical meaning of mathematically expressed theories to derive from the existence of a map from the exactly specified mathematical structure of the theory to statements about the world itself. When I refer to *standard intepretation*, I mean to include any approach that adopts this commitment. Going forward it will be helpful to provide some examples of standard approaches to interpretation. I will make no attempt to be exhaustive at this stage. Rather, my aim is simply to show that whether one is committed to a syntactic or semantic view of theories, or realism or anti-realism about physical theories, there still is a shared commitment about the nature of the relationship between mathematical structure and physical meaning.

The first attempt to explicitly reconstruct the structure of scientific theories can be found in the work of the logical positivists. For them, theories were taken to be collections of axioms written in a formal language.[3] This articulation of the content of a theory admits a natural notion of a what it is to be a model of the theory. In particular, the models of the theory are

[3] The formal language, $\mathcal{L}$, was taken to include the logical connectives, constants, functions, and relations. The set of axioms for the theory, Σ, consists of sentences in this language.

just its models in the strictly logical sense.[4] A structure is a model of the theory if each axiom of the theory is satisfied by the structure, where satisfaction is understood in the Tarskian sense. The positivists assumed the vocabulary of the language to be neatly divisible into collections of theoretical and observational terms. On this approach, the characterization of the models as physical occurs intrinsically to the theory through correspondence rules that reduce theoretical terms to sentences involving only the observational part of the language.

The positivists' syntactic view of the content of theories has largely been supplanted by semantic views which identify a theory with its class of models.[5] Proponents of these semantic views have relaxed the requirement that the content of the theory is captured in formal logical language and have instead allowed for theories to be characterized in terms of the mathematical structures that are natural for representing the domain in question. The models of general relativity, for example, consist of differentiable manifolds that represent spacetimes and tensors defined on those manifolds that represent matter and energy. The syntactically expressed laws of the theory determine whether or not a particular manifold and associated collection of tensors are models of the theory. While the structures in question are no longer explicitly set-theoretic, what it is to be a model of the theory is still to stand in the exact satisfaction relation with the syntactic expression of the theory.

Following work of Beth, van Fraassen introduced a semantic approach to interpretation based on the state-space of a theory.[6] According to this proposal, models of the theory are trajectories in state-space that exactly satisfy the syntactic expression of the dynamical equations of the theory. The characterization of the models as physical occurs through rules connecting physically measurable quantities to states of the system represented in state-space. The rules are expressed through a collection of statements, $U(m, r, t)$, ascribing a

[4]Recall that in this context models are pairs $\langle A, \mathcal{I} \rangle$, where A is the domain of objects and $\mathcal{I}$ is an interpretation function that maps symbols in the language to the domain, $\mathcal{I} : \mathcal{L} \rightarrow A$. A structure is a model of the theory, $\mathfrak{A} \in \mathrm{Mod}(T)$, if for each $\sigma \in \Sigma$, $\models_{\mathfrak{A}} \sigma$.

[5]A detailed elaboration of the reasons for abandoning the syntactic view in favor of the semantic view is given in (Suppe, 1974). For important recent clarification of the connection between the syntactic and semantic views see (Halvorson, 2012, 2013; Glymour, 2013; van Fraassen, 2014).

[6]The view is introduced and discussed in (Beth, 1960; van Fraassen, 1967, 1970; Arntzenius, 1991).

physical magnitude, m, a definite value, r, at a specific time, t. The truth values of the $U(m, r, t)$ depend on the state of the physical system. For each U there is a region of the state-space, $h(U)$ such that U is true just in case the physical state of the system is accurately represented by an element of $h(U)$. The map h connects the mathematical model to physical quantities in a way that is compatible with van Fraassen's constructive empiricism.

Unique accounts of interpretation have also developed from attempts to interpret particular theories. A currently popular example is provided in the work of the structural realists.[7] One of the central motivations for this program is to refrain from metaphysical commitment to a view of the world consisting of objects that bear properties. This radical move is motivated by an effort to accommodate quantum phenomena. In place of an ontology of objects, the view promotes the role of mathematical structure to a metaphysical one. This is achieved by starting from the mathematically expressed models of the theory and characterizing them as physical by stipulating that they stand in the relation of isomorphism or partial isomorphism with the structure of the world. Another approach to interpretation that has developed in efforts to address quantum phenomena, originally proposed by Albert, is captured in a view called wave function realism.[8] On this view the role of the state-space of quantum mechanics gets promoted to a metaphysical one.

While the approaches to interpretation introduced here clearly differ in important respects, they can all be seen to be instances of *standard interpretation*. In each case one begins by fixing on a particular collection of structures which are models of the theory that exactly satisfy its syntactically expressed dynamical equations. Once the collection of models is delimited, rules for associating physical meaning to those models are developed, and in each case these rules take the form of maps from the models of the theory to a target. Depending on the account, the target might be the world itself, the phenomena as they present themselves to us, or a model of the data from an experiment. On some of the accounts the

[7]There is an extensive literature elaborating this view, parts of which deviate significantly from the characterization given here. See, for example, (French, 2014; da Costa and French, 2003).

[8]The view is introduced in (Albert, 1996). Additional discussion can be found in (Ney and Albert, 2013).

nature of the map is explicitly stipulated to be an isomorphism, a partial isomorphism, or an embedding. In others the nature of the map is left less explicit. The common assumption adopted by all of these approaches that will be critical for my argument is that the models of the theory must always exactly satisfy the dynamical equations. This turns out to be untenable if one wants to interpret the empirically adequate models of quantum field theory.

If you consult any textbook on quantum field theory, you will find an expression for the n-point functions of a model, or a related expression for the S-matrix, that looks like:

$$\begin{aligned}\langle\Omega|T(\phi(x_1)\phi(x_2)\ldots\phi(x_n))|\Omega\rangle &= \sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\int\langle 0|T(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \\ &\quad \cdot H(y_1)H(y_2)\ldots H(y_j)|0\rangle d^4y_1\ldots d^4y_j.\end{aligned} \tag{1.1}$$

This is a perturbative expansion in powers of the coupling constant. If this equation was exactly satisfied in empirically adequate models, there would be no obstacle to applying any of the standard approaches to interpretation described above. The n-point functions encode the empirical content of the theory and so are the relevant kind of mathematical structure to feed into an interpretation. However, in empirically adequate models this equation is not exactly satisfied for three distinct reasons. Individual terms in the expansion are infinite because of the behavior of the theory in both the short and long distance regimes. These are the ultraviolet and infrared divergences of the theory. The individual terms can be rendered finite through regularization and renormalization procedures, but the resulting formal power series does not converge. This is the large-order divergence of the theory. In order to assign meaningful empirical content to the theory, methods for handling each of these classes of divergences are required. Fortunately, such methods are available and if they are employed carefully they provide everything that is required to define the n-point functions (or the S-matrix) up to a small finite error term. To interpret the empirically adequate models, we simply need to develop the resources to understand this finite level of precision in the empirical content of the theory.

1.2 THE ARGUMENT OF THE THESIS

The remainder of this dissertation consists of five self-contained chapters and brief concluding remarks.[9] The argument proceeds in two parts. The first part, consisting of Chapters Two, Three, and Four, addresses the problem of structure specification for quantum field theory. In particular, these chapters consider the ultraviolet, infrared, and large-order problems with the theory, respectively. The second part, consisting of Chapters Five and Six, addresses issues about how physical meaning can be associated with the kind of mathematical structures that result from insulating the theory from each of these classes of divergences. Chapter Seven suggests some further applications of the argument of the thesis.

Chapter Two considers the problem of ultraviolet divergences. The renormalization group provides a physically motivated method for rendering models ultraviolet finite and studying the scaling behavior of the theory. As this scaling behavior is now a confirmed empirical prediction of the Standard Model of particle physics, it would be reasonable to think that the renormalization of the theory plays an indispensable role in its empirical success. However, by attending to the mathematical details of empirically adequate models, one finds an alternative perspective about the ultraviolet behavior of the theory. In particular, ultraviolet divergences can be seen to be the result of incorrectly handling the multiplication of distributions. When the distributional character of field operators is correctly attended to, an alternative procedure for recovering the empirically adequate scaling behavior presents itself. This clarifies why the renormalization group is able to produce empirically adequate predictions despite deploying mathematical structures that are not always well-defined.

An often cited mathematical problem with perturbative quantum field theory is captured by Haag's theorem. In Chapter Three I consider this theorem which results from the infrared divergences of the theory. The result seems to show that the collection of assumptions required to form the interaction picture and establish the empirical adequacy of

[9] In the interest of making the chapters independently readable some relevant technical background is repeated in several of the chapters.

the theory are mathematically inconsistent. I argue that this does not render the empirical information acquired using the interaction picture unreliable. The regularization and renormalization techniques required to control ultraviolet and infrared divergences render some of the assumptions required to prove the theorem false. The argument of this chapter thus establishes that the output of perturbative renormalization theory is a set of well-defined formal power series. As a result, it motivates consideration of whether or not the formal power series converge and thus capture exact non-perturbative structure.

Chapter Four takes up the problem of specifying how perturbative data is related to the exact non-perturbative structures picked out by axiomatic characterizations of the content of the theory. In many models, the formal power series are demonstrably divergent. However, there are precisely specifiable mathematical conditions under which the divergent expansions of the perturbative formalism can be used to uniquely reconstruct an exact model of one of the axiomatic articulations of the theory. Analysis of these conditions reveals that in every case where a model of axiomatic field theory is available, the perturbative formalism exactly and uniquely determines that model. This shows that it is not correct to view the perturbative and axiomatic formalisms as completely distinct, or as competing research programs.

This analysis also shows why empirically adequate models do not satisfy the currently available axiomatizations of the theory. They contain large-order divergences that prevent unique reconstruction to an exact model. I argue that this does not make their perturbative expansions unrigorous. Truncating the expansions at low-order generates accurate values for observables, but there is not a unique exact model lying behind this empirical success. There is a class of such exact models and truncation introduces a finite level of precision into the empirical content of the theory. This limited precision can be constrained in a completely rigorous manner by finding bounds on the error introduced by the truncation.

In Chapter Five I turn to the problem of assessing how the problematic aspects of structure specification for quantum field theory discussed in Chapters Two, Three, and Four affect the project of associating physical meaning with the structure of the theory. As a first step toward addressing the problem I consider one particular proposal concerning how meaning attaches to mathematical structure, namely, the state-space semantics originally developed by Beth and van Fraassen. I argue that when perturbation theory gives rise to covergent expansions their proposal adequately captures the semantics of the theory. However, their proposal fails to capture the empirical meaning derived from truncations of divergent asymptotic expansions. I then provide a modification of state-space semantics that is able to capture this meaning. I argue that this shows that divergent perturbation theory provides a novel connection between mathematical structure and physical meaning. I provide examples from classical mechanics, non-relativistic quantum mechanics, and quantum field theory. In each case, one obtains divergent asymptotic expansions for important physical observables. I apply the proposal for the assignment of physical content outlined in the chapter to show that it successfully accounts for these cases. In this sense, the argument of the chapter is relevant to the interpretation of mathematically expressed scientific theories in general, and not just quantum field theory in particular.

In Chapter Six I consider one additional respect in which the structure specification provided by perturbative quantum field theory gives rise to a novel challenge for attributions of physical meaning. In particular, I argue that the syntax of perturbative field theory is ambiguous between multiple different structural realizations. I call this the problem of ambiguous structure. There are two standard methods for breaking structural ambiguity that provide possible routes to addressing this problem. The first route is to stipulate that among the class of ambiguous structures, there is one true type of structure that is a candidate for mapping onto the world. The second route is to claim that there is a unique common type of structure shared between the ambiguous class which is a candidate for mapping onto the world. If either of these approaches could be shown to be successful, the core commitments of standard approaches could be preserved. I argue that neither of these routes can underwrite the assignment of meaning to empirically adequate models in a way that reflects the nature of the empirical evidence for the theory. I provide a different solution that reflects the nature of the empirical evidence for the theory and which clarifies the role that exact models play in underwriting the physical meaningfulness of quantum field theory.

Taken together, the chapters of the dissertation resolve the tension between perturbative and axiomatic field theory. That is, they show that we can understand how perturbative quantum field theory achieves its successes, and they show how the lessons learned from the available models of axiomatic field theory bear on empirically adequate models. The concluding remarks in Chapter Seven suggest that the resolution of this tension opens the way for a new approach to interpretive questions about quantum field theory. Establishing perturbative field theory as an adequate mathematical foundation for philosophical analysis provides a new perspective on many of the issues that have been debated in the philosophy of quantum field theory. Moreover, it opens up additional core problems of physical practice to philosophical analysis. This chapter points to some of these additional applications of the perspective advocated in the thesis.

2.0 WHY ARE THERE ULTRAVIOLET DIVERGENCES AT ALL?

One of the central alleged obstacles to the interpretation of perturbative quantum field theory is the presence of ultraviolet divergences in empirically adequate models. While the presence of ultraviolet divergences has been taken to be an inevitable consequence of representing realistic field interactions, I argue that they are in fact an artifact of the failure to correctly handle to distributional character of field operators in standard characterizations of quantum field theory. By appealing to techniques from causal perturbation theory, I show that when the multiplication of distributions is handled correctly, ultraviolet divergences are avoided and hence renormalization is not necessary. Moreover, this solution to the ultraviolet problem can be incorporated into axiomatic approaches to quantum field theory. This analysis shows that what differentiates perturbative and axiomatic field theory is not their treatment of arbitrarily short distances.

2.1 INTRODUCTION

Ultraviolet divergences have plagued quantum field theory since the very beginning of the development of quantum electrodynamics. Their presence was first recognized in Pauli and Heisenberg's calculation of the second order contribution to the self-energy of the electron (Heisenberg and Pauli, 1929). This led to persistent worries throughout the 1930's and 1940's that the theory was not consistent. The development of covariant renormalization theory by Schwinger, Feynman, Tomonaga, and Dyson at the end of the 1940's assuaged these worries to some extent, but many regarded the procedure of infinite subtractions as ad hoc and conceptually unsatisfactory. In the late 1960's, Wilson's development of the renormalization group provided the necessary machinery to give a more conceptually satisfactory approach to renormalization. This led to the prediction of the scaling behavior of asymptotic freedom which was subsequently discovered experimentally. It also led to the interpretation of quantum field theories as effective field theories.

Despite these theoretical and experimental achievements, the philosophy of physics literature still harbors the nagging suspicion that perturbative quantum field theory is insufficiently mathematically well-defined to serve as the basis for philosophical investigation. Ultraviolet divergences and the need for renormalization are often cited as reasons for this suspicion. Recently, however, a number of authors have developed a different attitude toward the significance of renormalization for philosophical investigation.[1] These renormalization group realists have advocated that effective field theory actually holds important philosophical lessons about how theories successfully represent the world. For them, renormalization is not fundamentally about canceling ultraviolet divergences, but rather it is about studying how the structure of the theory changes with the energy scale. To be empirically adequate, a formulation of quantum field theory must have the resources to recover this scaling behavior.

My aim in this chapter is to address the following question: if renormalization is about describing a real physical process, and not about removing divergences, why are the divergences present in the mathematical structure of the theory at all? Causal perturbation theory, an axiomatic approach to quantum field theory, provides the necessary resources to answer this question. I will argue that perturbatively renormalizable theories can be cast in a way where no ultraviolet divergences arise, and which still recovers the correct scaling behavior. It follows that there is no problem with the mathematical characterization of perturbatively renormalizable theories in the ultraviolet region. This conclusion shows that what differentiates perturbative and axiomatic treatments of quantum field theory is not their representation of arbitrarily short distances.

[1] (Wallace, 2011; Fraser, 2016; Williams, 2016)

The argument proceeds as follows. In Section Two I recall the standard method for identifying ultraviolet divergences. I then provide an alternative explanation of their presence: they result from improper multiplications of distributions. The third section shows that by properly accounting for the distributional character of field operators, causal perturbation theory leads to a characterization of perturbatively renormalizable theories in which no ultraviolet divergences arise and the appropriate scaling behavior can be recovered. I also note that this resolution to the ultraviolet problem can also be incorporated into algebraic quantum field theory. The fifth concluding section reiterates that taken together, the arguments of this chapter show that what differentiates axiomatic and perturbative field theory is not their mathematical treatment of arbitrarily short distances.

2.2 WHENCE THE ULTRAVIOLET PROBLEM?

Ultraviolet divergences are typically diagnosed as resulting from integration over arbitrarily large momenta in closed internal loops of Feynman graphs. To take a concrete example, consider the first loop graph contributing to 2-2 scattering in the ϕ^4 model. The contribution to the amplitude from this graph is:

$$\mathcal{I}(p_1, p_2, p_3, p_4) = \int_0^\infty \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + m^2)} \frac{1}{((p_1 + p_2 - k)^2 + m^2)} \tag{2.1}$$

In the large momentum regime where $k \gg m, p_i$, this simplifies to:

$$\mathcal{I}(p_1, p_2, p_3, p_4) \propto \int_0^\Lambda \frac{k^{d-1}}{k^4} dk = \int_0^\Lambda k^{d-5} dk, \tag{2.2}$$

where Λ is an ultraviolet regulator. If $d < 4$, the integral is convergent as $\Lambda \to \infty$, if $d = 4$, the integral diverges like $\ln(\Lambda)$, and if $d > 4$ the integral diverges like powers of Λ. Let $D \equiv d - 4$ be the *superficial degree of divergence.* By counting the powers of the momenta in the numerator, denominator, and the integration measure, this quantity can be used to determine when a Feynman graph contains an ultraviolet divergence. In Minkowski space, where $d = 4$, Equation (2.1) is ultraviolet divergent.

This standard approach to identifying ultraviolet divergences in perturbative field theory is captured with more generality by Weinberg's power counting theorem (Weinberg, 1960).[2] Consider an arbitrary loop Feynman graph with external momenta p_1, p_2, …, p_E, and loop momenta, k_1, k_2, …, k_L, in a spacetime of dimension d. The contribution to the amplitude from such a graph is:

$$\mathcal{I}(p_1, \ldots p_E) = \int \frac{d^d k_1 \ldots d^d k_L}{(2\pi)^{dL}} \cdot \frac{\mathcal{N}(p_1, \ldots p_E, k_1, \ldots, k_L)}{\mathcal{D}(p_1, \ldots p_E, k_1, \ldots, k_L)}, \tag{2.3}$$

where the numerator and the denominator are products of the momenta and propagators. Weinberg was able to show that integrals are ultraviolet finite if and only if the corresponding superficial degree of divergence is negative.

This result is helpful because it indicates when renormalization is required. When $D \geq 0$, the behavior of the theory as $\Lambda \to \infty$ can be studied, and redefinitions of the charges and masses that render the theory ultraviolet finite can be determined. In perturbatively renormalizable theories, only a finite number of redefinitions are required, and these need to be fixed with experimental data. Assuming infrared divergences are also adequately addressed during this process, the output of this procedure is a well-defined formal power series the early terms of which can be compared to experiments.

The power counting explanation of the presence of ultraviolet divergences leads to questions about the interpretation of the ultraviolet regulator. Does it indicate that spacetime is a lattice? Does it express ignorance of physics above some energy scale? Have we done irreparable damage to the spacetime symmetries of the theory? In the remainder of this section my aim is to provide an alternative explanation of the presence of ultraviolet divergences that circumvents these questions and provides a more illuminating account of why

[2] Helpful discussion of the theorem can be found in (Duncan, 2012, p. 613).

some models of quantum field theory contain ultraviolet divergences.

The alternative explanation that I have in mind begins by noting that the Feynman integrals being evaluated contain multiplications of distributions in the integrand. The operation of multiplication of distributions is not always well-defined. Whether or not it can be defined is very sensitive to the supports of the distributions. Recall that distributions, ϕ, take test functions, f, and give back numbers:

$$T_f : \phi[f] \to \int_{-\infty}^{\infty} \phi(x) f(x) dx \tag{2.4}$$

The space of compact support test functions, $\mathcal{D}(\mathbb{R}^n)$, is the space $\mathcal{C}_\Omega^\infty(\mathbb{R}^n)$. The space of regular distributions, $\mathcal{D}'(\mathbb{R}^n)$, is defined to be:

$$\{T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C} \ | \ T \text{ is linear and continuous}\}. \tag{2.5}$$

To be in $\mathcal{D}'(\mathbb{R}^n)$, ϕ must satisfy,

$$\int_{-\infty}^{\infty} \phi(x) f(x) dx < \infty, \tag{2.6}$$

$\forall f \in \mathcal{D}(\mathbb{R}^n)$. This is equivalent to requiring that:

$$\int_\Omega |\phi(x)| dx < \infty, \tag{2.7}$$

$\forall \Omega \subset \mathbb{R}^n$, which is just to say $\phi(x) \in L^1_\Omega(\mathbb{R}^n)$.

This condition shows why pointwise multiplication, $(f \cdot g)(x) = f(x) \cdot g(x)$, does not carry over to regular distributions in general. For it to be the case that $f \in L^1_\Omega$, f must not have any singularities that grow faster than $\frac{1}{x^n}$, for $n \geq 1$. However, if $f, g \in L^1_\Omega$, and both have overlapping singularities like $\frac{1}{x^{n/2}}$, $f \cdot g$ has a singularity that goes like $\frac{1}{x^n}$ so $f \cdot g \notin L^1_\Omega$. In this case, even though f and g are regular distributions, $f \cdot g$ is not a regular distribution.

The moral is that one cannot simply multiply distributions without carefully checking their supports and expect to always get mathematically well-defined results. If you integrate a product of distributions that is not well-defined, you produce a divergence.

The Feynman integrals discussed above arise in perturbative evaluation of expressions for the S-matrix that have the form:

$$\text{S-matrix} = \sum_{n=0}^{\infty} \frac{-i^n}{n!} \int \mathcal{T}(\phi(x_1)\cdots\phi(x_n))d^4x_1\ldots d^4x_n \tag{2.8}$$

However, $\phi(x)$ is a bounded operator valued function nowhere in spacetime and so the $\phi(x)$ only have mathematical meaning in the sense of distributions. Moreover, the time-ordering is executed by multiplication by the discontinuous Heaviside function. Thus, the right hand side of Equation (2.8) contains many products of distributions, but we have not checked that this operation is well-defined for the $\phi(x)$. It turns out that it is not and this, I claim, is *why* ultraviolet divergences are present in standard perturbative field theory. This might seem to make the project of understanding the meaningfulness of perturbative evaluation hopeless, but in fact, identifying the source of the problem is the first step in the direction of a solution.

2.3 CAUSAL PERTURBATION THEORY

Causal perturbation theory is an axiomatic approach to quantum field theory that has grown out of the seminal paper of Epstein and Glaser (Epstein and Glaser, 1973).[3] Scharf explains the motivation for the formalism as follows:

> One must only adopt the following two rules. First, use well-defined quantities only, for example free fields. Second, make justified operations only in the calculations; in particular do not multiply certain distributions by discontinuous step

[3]The formalism is clearly developed in (Scharf, 1989) and (Scharf, 2001). The presentation given here is based on the discussion in (Prange, 1999; Helling, 2012; Bain, 2013).

functions. If one really follows these rules, then no infinity can appear and life is beautiful. (Scharf, 1989, p. V)

In perturbative quantum field theory, the quantity being evaluated is:

$$\text{S-matrix} = \sum_{n=0}^{\infty} \frac{-i^n}{n!} \int \mathcal{T}(\phi(x_1)\cdots\phi(x_n))d^4x_1 \ldots d^4x_n. \tag{2.9}$$

In causal perturbation theory a few modifications are made to this expression:

$$\text{S-matrix} = \sum_{n=0}^{\infty} \frac{-i^n}{n!} \int \mathcal{T}(T(x_1)\cdots T(x_n))g(x_1)\ldots g(x_n)d^4x_1 \ldots d^4x_n \tag{2.10}$$

First, the distributional character of the field operators is explicitly noted. Second, the $g(x_1)\ldots g(x_n)$ are test functions that cure the infrared divergences of the theory.[4]

The aim of causal perturbation theory is to produce an order-by-order construction of the S-matrix where each term, S_n, is a well-defined operator valued distribution corresponding to a linear operator on a separable Hilbert space. What you find when you produce this construction is that

$$S_n \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\}) = \{T : \mathcal{D}(\mathbb{R} \setminus \{0\}) \to \mathbb{C}\} \tag{2.11}$$

where,

$$\mathcal{D}(\mathbb{R} \setminus \{0\}) = \{f \in \mathcal{D}(\mathbb{R}^n) | 0 \notin \text{supp}(f)\} \tag{2.12}$$

In other words, the construction almost results in regular distributions, but not quite. This immediately leads one to wonder if a $T^0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ can be uniquely extended to a $T \in \mathcal{D}'(\mathbb{R}^n)$? That is, can we uniquely reconstruct regular distributions? To answer this question, we need a measure of the singularity of a distribution at the origin. The scaling

[4] In fact, this is really just a stopgap measure to make the infrared region sufficiently well-defined to meaningfully analyze the ultraviolet region. Analysis of the infrared problem is given in Chapter Three.

degree of $T \in \mathcal{D}'(\mathbb{R}^n)$ at $x = 0$ is given by:

$$\mathrm{sd}(T) = \inf\{\omega \in \mathbb{R} | \lambda^\omega T(\lambda) \xrightarrow{\lambda \to 0} 0\} \tag{2.13}$$

The power counting arguments of standard perturbative field theory can be thought of as estimating this scaling degree. If $T^0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is a distribution with $\mathrm{sd}(T^0) < n$, there is a unique distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ with $\mathrm{sd}(T) = \mathrm{sd}(T^0)$ extending T^0. When $\mathrm{sd}(T^0) \geq n$, there is not a unique extension. However, there is a unique extension of:

$$T^0 + \sum_{\alpha \leq \mathrm{sd}(T^0)} C_\alpha \partial^\alpha \delta(x) \tag{2.14}$$

This means that to produce a unique extension we need to fix a finite set of numbers, the C_α. These are analogs of the renormalization corrections of standard perturbative field theory. You can even recover the scaling behavior in this formalism (Prange, 1999). This means that causal perturbation theory recovers everything that is required to treat empirically adequate models, and no ultraviolet divergence appears in the theory.

The criticism of axiomatic field theory in (Wallace, 2011) focuses on the structure of the theory at arbitrarily short distances. However, I have argued that a close analog of the Wightman formalism is capable of treating arbitrarily short distances just as well as standard perturbative quantum field theory. The resolution to the ultraviolet problem provided by causal perturbation theory can also be smoothly incorporated into algebraic quantum field theory.

> Although this work does not provide quantum field models satisfying the HAK or Wightman axioms, it falls comfortably within the framework of AQFT, since the primary objects are, again, nets of local $*$-algebras generated by observables which are Poincaré covariant and satisfy Einstein causality and, again, the work is carried out with complete mathematical rigor. However, when these authors speak of representations in Hilbert spaces, the Hilbert spaces are vector spaces over the field $\mathbb{C}[[\lambda]]$, not over $\mathbb{C}$. So taking expectations of observables in states in this approach results in a formal complex power series, not a complex number.

> Hence, in order to make the connection to experiments one must deliberately consider a partial sum of this series, i.e. consider the perturbation series only to a finite order, as is done in heuristic QFT. Since these series are not convergent, one is returned to the question 'Is there an exact model?' (Summers, 2012, pp. 47-48)

The moral that should be drawn from this analysis is that what differentiates perturbative and axiomatic field theory is not their treatment of arbitrarily short distances.

2.4 CONCLUSION

Causal perturbation theory provides the machinery required to characterize empirically adequate models of quantum field theory in an entirely ultraviolet finite way. This shows that what differentiates perturbative and axiomatic field theory is not their treatment of the very high energy regime. It should be noted that I am not advocating abandoning standard perturbative field theory for causal perturbation theory. Rather, my claim is that causal perturbation theory provides an illuminating account of why perturbative renormalization theory is necessary and effective. The remaining problems with perturbative field theory are infrared and large-order divergences. These problems are discussed in Chapter Three and Chapter Four, respectively.

3.0 HAAG'S THEOREM, APPARENT INCONSISTENCY, AND THE EMPIRICAL ADEQUACY OF QUANTUM FIELD THEORY

Haag's theorem demonstrates the inconsistency of a collection of assumptions adopted in the perturbative approach to quantum field theory. The theorem results from infrared problems with the theory and presents a seemingly intractable problem for perturbative quantum field theory. Earman and Fraser have clarified how it is possible to give mathematically consistent calculations in scattering theory despite the theorem by appealing to results from axiomatic and constructive field theory. However, their analysis does not fully address the worry raised by the result. In particular, I argue that their approach fails to be a complete explanation of why Haag's theorem does not undermine claims about the empirical adequacy of particular quantum field theories. I then show that such empirical adequacy claims are protected from Haag's result by the techniques that are required to obtain theoretical predictions for realistic experimental observables. The regularization and renormalization techniques required to insulate models from their ultraviolet and infrared divergences break the assumptions required to prove the theorem. As a result, the output of perturbative renormalization theory is a collection of well-defined formal power series. Recognizing this motivates analysis of the convergence behavior of these series, a task I take up in Chapter Four.

3.1 INTRODUCTION

Despite the often noted empirical successes of the Standard Model of particle physics, the quantum field theories on which it is based have been shown to be mathematically questionable in a number of respects. One such mathematical problem is captured by a result originally proved by Haag[1], and subsequently generalized by Hall and Wightman[2].[3] Haag's theorem has received significant attention because it raises the specter of inconsistency in the context of interacting quantum field theories. For example, Teller claims that because of the theorem "...there appears to be no known consistent formalism within which interacting quantum field theory can be expressed" (Teller, 1995, p. 115).[4] If this claim was correct, then it would be difficult to understand how so much empirical evidence for the interacting quantum field theories that make up the Standard Model has been accumulated. Roughly, the theorem shows that the assumptions required to form the interaction picture in which scattering theory calculations are carried out are consistent only in the case of non-interacting theories. In this sense, the theorem does establish the inconsistency of a set of assumptions that are sometimes simultaneously assumed to hold in interacting quantum field theories. In this paper I show why this does not undermine the empirical adequacy claims that are taken to support the quantum field theories that make up the Standard Model.

Earman and Fraser have made progress in this direction by arguing that previous attempts to articulate the foundational significance of the theorem tend toward "overstatement" and even "hyperventilation" (Earman and Fraser, 2006, p. 305, p. 323). Their analysis leads them to three central conclusions. First, the theorem emphasizes the importance of unitarily inequivalent representations of the canonical commutation relations, whose existence in quantum field theory distinguish it from non-relativistic quantum mechanics. Second, it makes it clear that non-Fock representations have an important role to play in quantum field theory. Finally they claim that the theorem undermines the standard interaction picture formalism and the approaches to scattering theory that depend on it. In particular they claim that "...while Haag's theorem does *not* show that no quantum

[1] (Haag, 1955)
[2] (Hall and Wightman, 1957)
[3] The complex historical development of the theorem is recounted in (Lupher, 2005).
[4] Earman and Fraser note that similar claims can be found in (Barton, 1963, p. 157), (Huggett and Weingard, 1994, p. 376), and (Sklar, 2000, p. 28).

field theory exists which differs from a free field theory, it does pose problems for some of the techniques used in textbook physics for extracting physical predictions from the theory" (Earman and Fraser, 2006, p. 306). They diagnose the strong reaction to the theorem in the literature as referring to this fact (Earman and Fraser, 2006, pp. 306-307). While I agree with the first two conclusions that they draw concerning the importance of the theorem, this paper provides further analysis of the third. This further analysis is necessary in order to properly understand how Haag's theorem bears on the issues of consistency and empirical adequacy for quantum field theory.[5] The textbook calculations they refer to have played an important role in establishing the empirical adequacy of particular models of the theory. If Haag's theorem shows such calculations to be predicated on an inconsistent set of assumptions, then those empirical adequacy claims are unreliable.

Scattering theory calculations are the basis for comparison between quantum field theories and experiments, and thus some explanation for why field-theoretic scattering theory matches empirical data, despite Haag's result, is required. In order to explain this success Earman and Fraser appeal to a mathematically rigorous formalism for scattering theory due to Haag and Ruelle which circumvents Haag's theorem. While this formalism does demonstrate that scattering theory can be formalized in a mathematically consistent manner, the existence of such a formalism does not fully resolve the worry raised by Haag's theorem because it does not explain why theoretical predictions for realistic experimental observables give empirically adequate results. There is, however, a clear reason why such theoretical calculations are not undermined by Haag's theorem; namely, in those cases where the interaction picture is employed the calculational techniques that are required to extract predictions from empirically adequate field theories violate some of the assumptions required to prove the theorem. In other cases, the theoretical calculations that are used to compare to experiments simply do not use the interaction picture in any way. It is these facts that explain why Haag's theorem does not directly undermine claims about the empirical adequacy

[5] Earman and Fraser agree as they note that their analysis leaves "...unfinished business in explaining why perturbation theory works as well as it does" (Earman and Fraser, 2006, pp. 306-307).

of quantum field theories.

This situation shows that Haag's theorem is illustrative of a general tension which exists in much of the literature that is engaged with the philosophical appraisal of the foundations of quantum field theory. It is often unclear how fully mathematically rigorous models inform claims about the actual world because they are defined in reduced spacetime dimension or do not represent realistic interactions. Conversely, it is often not obvious whether or not one should assign interpretive significance to the changes to the mathematical formalism that are required to render calculations of physical observables well-defined. A complete understanding of the significance of Haag's theorem requires analysis of how it bears on both of these problems. I argue that Haag's theorem should be understood as a constraint on the nature of the relation between results obtained in perturbation theory and exact non-perturbative characterizations of quantum field theories, in the sense that it rules out one particular method for forming the infrared limit of a fully regularized theory.

My argument proceeds as follows. The second section briefly introduces the interaction picture formalism for scattering theory and explains how Haag's theorem shows that it is predicated on an inconsistent set of assumptions. In the third section I consider Earman and Fraser's explanation of the success of scattering theory and show that it does not resolve the worry that empirical adequacy claims are undermined by the result. The fourth section shows how the calculational techniques required to obtain empirical predictions avoid Haag's theorem by considering examples of calculations in quantum electrodynamics and quantum chromodynamics. In the concluding section I address how Haag's theorem bears on the relation between perturbative calculations and exact non-perturbative structure.

3.2 HAAG'S THEOREM AND THE INTERACTION PICTURE

Haag's theorem undermines the interaction picture and the standard approach to scattering theory. It does so by showing that the assumptions required to formulate the interaction picture are inconsistent with the presence of a non-trivial interaction in the theory. Thus, when the interaction picture is used for calculations in theories like quantum electrodynamics which contain interactions, the calculations possess an apparent mathematical inconsistency. Furthermore, there is good reason to worry that this renders empirical adequacy claims for particular field theories unreliable. Scattering theory provides the critical connection between a quantum field theory and experimental observables such as cross-sections. Empirical adequacy claims for quantum field theories are based on the agreement between cross-sections calculated with scattering theory and cross-sections observed in experiments at particle accelerators. When the quantum field theories of the Standard Model are used in such calculations they yield results that closely match the observed values for the quantities. Much of the direct evidence for the empirical adequacy of the Standard Model is derived, either directly or indirectly, from comparisons of this sort. In some cases, these theoretical calculations use the interaction picture formalism which is undermined by Haag's theorem. In this way, the theorem seems to show that the formalism that has produced what can be counted among the most precisely confirmed predictions of any physical theory is mathematically inconsistent.

The interaction picture is an intermediate between the Schrödinger picture, in which states evolve in time under the full Hamiltonian and operators are stationary, and the Heisenberg picture, in which states are stationary and operators evolve under the full Hamiltonian.[6] States and operators in the interaction picture are given the subscript, I. The time evolution of operators in the Heisenberg picture is determined by the Heisenberg equation of motion: $\partial O_H(t)/\partial t = -i\,[O_H(t), H]$. Operators in the Schrödinger picture are related to the Heisenberg picture by the transformation, $O_S = e^{-iHt} O_H(t) e^{iHt}$, and the states are related by, $\psi_S(t) = e^{-iHt}\psi_H$. These transformations leave the matrix elements of corresponding

[6]Throughout, the subscripts, H, and, S, denote the Heisenberg and Schrödinger picture, respectively. The Hamiltonian is the same in the Heisenberg and Schrödinger pictures and thus does not need a subscript.

operators invariant,

$$
\begin{aligned}
{}_H\langle\psi|O_H(t)|\phi\rangle_H &= {}_H\langle\psi|e^{iHt}e^{-iHt}O_H(t)e^{iHt}e^{-iHt}|\psi\rangle_H \\
&= {}_S\langle\psi(t)|O_S|\phi(t)\rangle_S,
\end{aligned}
\tag{3.1}
$$

and in this sense they are empirically equivalent. The interaction picture is formed by writing the full Hamiltonian as $H = H^0 + H^1$, where H^0 is the free Hamiltonian and H^1 characterizes the interaction. The interaction picture is then defined by letting the evolution of the operators be implemented by H^0 and the evolution of the states be implemented by H^1. It is connected to the Schrödinger picture by the transformations, $O_I(t) = e^{iH_S^0 t}O_S e^{-iH_S^0 t}$, and, $\psi_I(t) = e^{iH_S^0 t}\psi_S(t)$. All three pictures agree at $t = 0$, as $\psi_I(0) = \psi_S(O) = \psi_H$ and $O_I(0) = O_H(0) = O_S$.

These relations allow for the perturbative expansion of the time evolution operator which is defined by the relation $\psi(t_1) = U(t_1, t_0)\psi_I(t_0)$. Using the transformations connecting the pictures it can be shown that,

$$
U(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T(H_I^1(t_1) \dots H_I^1(t_n)). \tag{3.2}
$$

The S-matrix can then be defined in terms of the time evolution operator by,

$$
S_{jk} = \lim_{t_2 \to \infty} \lim_{t_1 \to -\infty} \langle\phi_k|U(t_2, t_1)|\phi_j\rangle, \tag{3.3}
$$

and thus, inserting the expansion for the time evolution operator yields the Dyson expansion for the S-matrix,

$$
S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T(H_I^1(t_1) \dots H_I^1(t_n)), \tag{3.4}
$$

where the time ordered product rearranges the operators in the order of descending time argument. In general, H_I^1 is a product of free field operators describing the interaction between the fields. Evaluating the time ordered product of these products of field operators in the Dyson expansion can be simplified through an application of Wick's theorem. This theorem allows for the time ordered products in the expansion to be rewritten as a sum of contracted normal products, which are vacuum expectation values of time ordered interaction picture field operators.[7] This technique allows for the perturbative evaluation of S-matrix elements for processes involving particular initial and final states. The interaction picture is essential for this perturbative evaluation because for $t = \pm\infty$, in the interaction picture the Hilbert space representation is simply the Fock representation for the free field. This makes it possible to explicitly calculate vacuum expectation values of products of interaction picture field operators.[8]

There are three primary obstacles to the well-definedness of this approach to the perturbative evaluation of field theoretic quantities, only one of which is related to Haag's theorem.[9] The first two problems with the perturbative evaluation of Equation (3.4) come from the presence of ultraviolet and infrared divergences, respectively. Both types of divergences render individual terms in the sum infinite and thus the whole expression ill-defined. There are techniques for isolating and controlling these divergences. These methods, and how they restore the validity of perturbative evaluation of Equation (3.4) will be discussed in Section 4. The third problem is that one is considering the sum of an infinite set of terms and it must be determined whether or not that sum converges. There is reason to think that in empirically interesting models it does not.[10] The final section of this paper explains an approach to understanding the meaningfulness of perturbation theory in face of this third problem. Of the three obstacles to assigning meaning to the expression for the S-matrix, only the presence of infrared divergences is related to Haag's theorem. This class of diver-

[7] A detailed explanation can be found in, for example, (Greiner and Reinhardt, 1996).

[8] This is not clear in the other pictures because one does not have an explicit representation of the field operators at asymptotic times.

[9] A clear discussion of all three problems can be found in (Haag, 1992a, pp. 70-71).

[10] There are arguments going back to (Dyson, 1952) that suggest that the expansion in fact diverges in empirically interesting models. This has been confirmed rigorously in some simplified models.

gences prevents the establishment of a global unitary transformation between the free and interacting fields, a critical assumption required for forming the interaction picture.

Earman and Fraser provide a clear exposition of Haag's original argument and explain how Hall and Wightman generalized the theorem.[11] My aim here is to review some of the standard assumptions that go into the proof of the theorem and to show how the theorem undermines the existence of a global unitary transformation connecting the free and interacting fields. As Earman and Fraser correctly note, all of the assumptions required for the proof of the theorem are adopted in the approach to scattering theory based on the interaction picture. Many of these assumptions are also taken as axioms in the Wightman formalism for quantum field theory.[12] Others are introduced specifically for the construction of the interaction picture for the perturbative evaluation of observables. The Wightman formalism consists of a set of statements about the properties of a collection of vacuum expectation values for a theory which together exhaust its physical content. They capture physical principles that are assumed to obtain for the objects described by the perturbative evaluation of field theoretic quantities. As Duncan explains, the proof of Haag's theorem can be understood as proceeding in two stages (Duncan, 2012, p. 366). In the first stage it is shown that if two collections of field operators are globally unitarily equivalent, then the vacuum expectation values of products of those field operators at equal times must be identical. The second step is to show that this equality extends to arbitrary spacetime arguments of the fields. An application of the Wightman reconstruction theorem then ensures that the conclusion for field theories characterized in terms of vacuum expectation values also applies to field theories characterized in terms of operators acting on a Hilbert space.

Consider two neutral scalar fields ϕ_j, $j = 1, 2$, with conjugate momenta π_j, where for each j, (ϕ_j, π_j) is an irreducible representation of the equal time canonical commutation

[11] Haag's original version of the theorem fails to be fully general since it restricts attention to a particular class of Hamiltonians. The generalization due to Hall and Wightman closes this gap by extending Haag's result to cover all Hamiltonians. Additional helpful exposition can be found in (Duncan, 2012).

[12] For the details of this approach see, for example, (Streater and Wightman, 1964, pp. 96-102).

relations,

$$[\phi_j(\vec{x},t), \pi_j(\vec{x}\,',t)] = i\delta(\vec{x}-\vec{x}\,') \;\; j=1,2$$
$$[\phi_j(\vec{x},t), \phi_j(\vec{x}\,',t)] = [\pi_j(\vec{x},t), \pi_j(\vec{x}\,',t)] = 0. \tag{3.5}$$

Suppose further that the Euclidean transformations consisting of translations, $\vec{a}$, and rotations, R, are implemented by unitary operators $U_j(\vec{a},R)$,

$$U_j(\vec{a},R)\phi_j(\vec{x},t)U_j^{-1}(\vec{a},R) = \phi_j(R\vec{x}+\vec{a},t)$$
$$U_j(\vec{a},R)\pi_j(\vec{x},t)U_j^{-1}(\vec{a},R) = \pi_j(R\vec{x}+\vec{a},t). \tag{3.6}$$

These are standard assumptions used in perturbative calculations and in the Wightman formalism. Finally suppose that at some time t the fields are related by a unitary transformation $V(t)$,

$$\phi_2(\vec{x},t) = V(t)\phi_1(\vec{x},t)V^{-1}(t), \quad \pi_2(\vec{x},t) = V(t)\pi_1(\vec{x},t)V^{-1}(t). \tag{3.7}$$

This is an assumption necessary for the construction of the interaction picture. These assumptions are sufficient to show that if there are unique normalizable Euclidean invariant states $|0_j\rangle$,[13] then they must be related by, $c|0_2\rangle = V(t)|0_1\rangle$ where $|c| = \pm 1$. From this, the equality of the vacuum expectation values for products of equal time field operators follows directly.[14] The extension of this equality to arbitrary spacetime arguments requires additional assumptions. Critically, the extension requires the full Poincaré invariance of the theory. Specifically, if $(\vec{a},\Lambda)$ are Poincaré transformations implemented by the unitary

[13] Earman and Fraser note that this assumption follows from the classification of representations of the inhomogeneous Lorentz group (Earman and Fraser, 2006, pp. 321-322).

[14] The details of the calculation are given in (Duncan, 2012, pp. 367-368).

operators $T_j(\vec{a},\Lambda)$, then the fields transform as,

$$T_j(\vec{a},\Lambda)\phi_j(x) \quad = \quad \phi_j(\Lambda\vec{x}+\vec{a}), \tag{3.8}$$

and the $|0_j\rangle$ satisfy,

$$T_j(\vec{a},\Lambda)|0_j\rangle = |0_j\rangle. \tag{3.9}$$

The content of Hall and Wightman's generalization of Haag's argument is that on these assumptions, if ϕ_1 is a free field then its vacuum expectation values are equal to those of ϕ_2. This entails that they will also agree on all of their S-matrix elements.

Another way to state the content of the theorem is that if one assumes that the fields belong to the same Hilbert space representation, then if one of the fields is free, they are both free. It follows that free and interacting fields cannot belong to the same Hilbert space representation, an assumption on which the perturbative evaluation of field theoretic quantities in the interaction picture is predicated. For this reason, Haag's theorem undermines the approach to scattering theory based on the interaction picture in any theory satisfying the conditions of the theorem. Earman and Fraser claim that "…the problem brought to light by Haag's theorem is not directly related to the employment of perturbation theory as an approximation method; all of the assumptions of [Haag's] theorem are embraced before the perturbation series is even introduced" (Earman and Fraser, 2006, p. 322). This is a point which merits further clarification. They are correct that the theorem is not concerned with the expansion of field theoretic quantities in a power series in general. What Haag's theorem undermines is precisely the pertubative evaluation of field theoretic observables in the interaction picture in particular. This is undermined by the theorem because the strategy that this method adopts for perturbative evaluation of observables requires the existence of a global unitary transformation connecting the free and interacting fields that the theorem shows not to exist.

The Hall and Wightman generalization of the theorem holds for any pair of neutral scalar fields fields and any Hamiltonian satisfying the conditions of the theorem. In order to determine whether more physically relevant theories are plagued by an analogous result requires determining whether or not the result applies in the case of theories involving higher spin fields and in theories that couple different kinds of fields together. Generalizations of the theorem show that the interaction picture does not exist in essentially all cases in which the free and interacting Hamiltonians are defined on a continuum spacetime with the full Poincaré group as its spacetime symmetries and differ non-trivially. For the case of uncharged scalar fields, this level of generality is already present in the Hall-Wightman version of the theorem introduced here. Duncan has argued that as the complexity of the interaction in a theory grows, it is increasingly likely that there will fail to be unitary transformations connecting the Fock states of the free and interacting theories, and thus when more physically relevant interactions are considered, there is good reason to expect that an analog of Haag's theorem will obtain.[15] For this reason, the theorem seems to show that empirical adequacy claims based on interaction picture calculations are unreliable.

3.3 EARMAN AND FRASER ON SCATTERING THEORY

This section considers how Earman and Fraser attempt to explain the success of scattering theory despite Haag's theorem. Their two part explanation appeals to techniques from axiomatic and constructive field theory. More specifically, they appeal to Haag-Ruelle scattering theory and theorems which establish the existence of local unitary equivalence between free and interacting theories. It should be made clear that they do not present their explanation as a full answer to the question of why the interaction picture and perturbation theory work. Instead they "point to what [they] believe is a critical piece in the overall scheme" (Earman and Fraser, 2006, p. 322), and later they claim to have "indicated one route to such

[15] For a more detailed discussion of the generalization of the theorem see (Duncan, 2012, pp. 363-369).

an explanation" (Earman and Fraser, 2006, p. 333). They are not explicit about what, in their view, is missing from their account. This section explicitly identifies a critical respect in which their explanation of the success of perturbative calculations in scattering theory is deficient.

The first part of Earman and Fraser's explanation relies on the fact that Haag's theorem spoils global unitary equivalence, but it does not necessarily rule out local unitary equivalence. In some cases local unitary equivalence can be established, and they claim that when this is the case it underwrites a "... a perfectly good sense in which the interaction picture and perturbation theory do work ... at least for physical quantities that matter for explaining experimental outcomes" (Earman and Fraser, 2006, pp. 323-324). They seem to have in mind that what actually gets measured are observables corresponding to localized spacetime regions. To illustrate how this explanation works they consider the example of a theory of two free scalar fields with different masses, ϕ_{m_1}, and ϕ_{m_2}, with the masses related by $m_2 = m_1 + \delta m$. In this case they note that local unitary equivalence can be rigorously established.[16]

For Earman and Fraser, this shows why perturbation theory can be used to explain the results of experiments on local observables using such a theory. They note that this solution to the problem is also a viable one in the case of $(\phi^4)_2$ theory, the theory of a self-interacting neutral scalar field in one space and one time dimension. While this is a more physically relevant interaction than the mass shift in their first example, it is highly simplified in that it is defined in reduced spacetime dimension. Many of the models which have successfully been constructed are defined in fewer than four spacetime dimensions because such models tend to have less severe divergences. Earman and Fraser explain that since $(\phi^4)_2$ theory does not have ultraviolet divergences and the restriction to bounded regions of spacetime involved in the definition of local unitary equivalence removes the possibility of infrared

[16]The precise sense of local unitary equivalence that they appeal to is the following one: "Given any bounded region $B \subset \mathbb{R}^3$ and the free fields ϕ_{m_1}, π_{m_1} and ϕ_{m_2}, π_{m_2} acting on the respective Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, there is a unitary map $V_B : \mathcal{H}_1 \to \mathcal{H}_2$ such that $V_B \phi_{m_1}(f) V_B^{-1} = \phi_{m_2}(f)$ and $V_B \pi_{m_1}(f) V_B^{-1} = \pi_{m_2}(f)$ for all suitable test functions f with support in B" (Reed and Simon, 1975, p. 329).

divergences, it was to be expected that local unitary equivalence could be established in this case. They then note that in higher spacetime dimensions the ultraviolet problems become worse, the full power of renormalization methods are required for the theory to be well-defined, and local unitary equivalence is spoiled. Earman and Fraser then declare that "...Haag's theorem is not responsible for the problems created by ultraviolet divergences, so solving them is beyond the scope of this paper" (Earman and Fraser, 2006, p. 323). It is true that Haag's theorem captures a mathematical problem associated with infrared and not ultraviolet divergences. However, in the next section I will argue that, in order to understand the success of the perturbative evaluation of scattering matrix elements, all three mathematical problems with Equation (3.4) introduced above need to be resolved. In this sense, the solution to the problem of ultraviolet divergences does play a role in restoring the validity of the interaction picture. What should be noted at this point is that the solution of the problem by appeal to local unitary equivalence is only demonstrably valid in the case of a handful of simplified models, and not in the field theories in four-dimensional Minkowski space that make up the Standard Model. Since such realistic theories all contain ultraviolet divergences, there is good reason to expect that local unitary equivalence will be spoiled in those cases as well.

The second part of Earman and Fraser's explanation is an appeal to the formalism for scattering theory developed by Haag and Ruelle.[17] This framework begins by assuming that the theory in question satisfies the Wightman axioms and then stipulates that they also satisfy an additional condition on the spectrum of the Hamiltonian to ensure the existence of a mass gap.[18] The central idea of their framework is to rigorously construct the Hilbert spaces $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ spanned by the states before and after the scattering using elements from the full Hilbert space, $\mathcal{H}$, in the asymptotic limit where $t \to \pm\infty$. Earman and Fraser note that "This formalism is not subject to Haag's theorem because - unlike the interaction

[17](Haag, 1958; Ruelle, 1962)

[18]That is, it is required that the operator $P^2 = P_\mu P^\mu$ has an isolated eigenvalue $m^2 > 0$, corresponding to the single particle states, and the remaining part of the spectrum is continuous, beginning at $(2m)^2$. See, for example, (Iagolnitzer, 1993, p. 72) for further discussion.

picture - it neither posits nor entails the existence of a unitary transformation connecting $\mathcal{H}$ (or $\mathcal{H}_{in}$ or $\mathcal{H}_{out}$) to $\mathcal{H}_F$ that relates the interacting field to a free field" (Earman and Fraser, 2006, p. 326). This approach thus seems to afford the possibility of circumventing the problem raised by Haag's theorem entirely.[19]

With respect to the interpretive significance of this formalism Earman and Fraser note that "…the Haag-Ruelle approach shows how to maneuver around [Haag's theorem] to obtain in QFT analogues for most of the significant features of ordinary scattering theory" (Earman and Fraser, 2006, p. 326). They do not raise any particular features as examples, but it is certainly true that for the models to which the Haag-Ruelle theory applies, the formalism shows how to obtain many of the features of standard scattering theory without running afoul of Haag's theorem. One of the central benefits of the constructive approach to scattering theory is that it goes even further and affords explanations for features of the perturbative treatment of the theory that typically must be taken as assumptions.[20] Moreover, since explicit models can be constructed, it is clear that the Haag-Ruelle theory is based on a mathematically consistent framework.

This part of Earman and Fraser's explanation of the success of scattering theory is limited in very much the same way as the first part. In particular, it can only be shown to be valid in certain simplified models[21] and it is not clear that the explanatory significance can be exported from those models to cases of experimental interest. There is no known model of a field theory with local gauge symmetry defined in four-dimensional Minkowski space that satisfies the Wightman axioms and exhibits a mass gap. The field theories that make up the Standard Model are all, however, local gauge theories. This undermines the ability of

[19] It is not unique in this respect. As Bain has emphasized, the LSZ formalism is also able to escape the force of Haag's result in a related way (Bain, 2000).

[20] These features include the presence of the clustering property. While in perturbative treatments of the theory this property is simply assumed as a phenomenological constraint which rules out dependence between far separated scattering experiments, in the Haag-Ruelle formalism it can be recovered as a consequence of the Wightman framework for quantum field theory. Another such feature is the existence of the asymptotic states. Whereas in the LSZ formalism asymptotic completeness is assumed, this feature is recovered as a theorem in the Haag-Ruelle formalism. See (Strocchi, 2013, p. 123) for further discussion.

[21] The models in which the Haag-Ruelle theory can be shown to apply include weakly coupled $P(\phi)_2$, ϕ^4_3, and sine-Gordon$_2$ theories. See (Summers, 2012, pp. 11-12, 16-17, 24) for discussion.

the Haag-Ruelle theory to explain the success of scattering theory in realistic theories in a straightforward way.

At this stage one might object that none of the discussion up to this point rules out the possibility that more physically relevant theories will be shown to satisfy the Wightman axioms or some modified set of axioms characterizing the non-perturbative content of the theory. This is certainly an open possibility, and if it was accomplished then empirically adequate theories could be treated using Haag-Ruelle theory or some close analog for the new axiomatization. Moreover, if this were achieved then Earman and Fraser would have provided an adequate explanation for how scattering theory can be done in a mathematically consistent manner despite Haag's theorem. However, there remains a clear sense in which their explanation is deficient as a response to the question of why scattering theory works despite Haag's theorem.

In order to show that empirical adequacy claims for particular quantum field theories are safe from the theorem, it must be shown that the theoretical predictions that are actually used to match with data are not affected by Haag's result. For Earman and Fraser's response to the theorem to be helpful for this task, it would need to be the case that the theoretical predictions are calculated using the constructive formalism to which they appeal. I have already noted the reason why this cannot be the case: the techniques they appeal to are not demonstrably applicable in the cases of the theories of experimental interest. In some cases, theoretical predictions have been calculated using the interaction picture formalism whose validity Haag's result casts into doubt. Other techniques for obtaining theoretical predictions are also used, but they are not the constructive techniques to which Earman and Fraser appeal. For these reasons I claim that Earman and Fraser have not provided an explanation of why empirical adequacy claims for quantum field theory are not undermined by the theorem. As a result, a complete explanation for why scattering theory works is still lacking. In the next section I argue that the techniques employed in the calculation of realistic experimental observables render some of the assumptions of Haag's theorem false.

It is on this basis that one can conclude that Haag's theorem does not undermine empirical adequacy claims.

3.4 HAAG'S THEOREM AND EMPIRICAL ADEQUACY

If the theoretical calculations that are used to compare with experiments were in fact shown to be invalid by Haag's theorem, it would undermine much of the direct evidence for the Standard Model. The concern about inconsistency raised by the theorem can, however, be resolved by looking to the techniques that are used in the calculations that are compared with experiments. In some calculations the interaction picture is not used. In other calculations regularization and renormalization techniques render some of the assumptions of Haag's theorem false and thus show how it is possible to obtain meaningful answers using the interaction picture formalism. In both cases I submit that there is no stage in the calculation at which the quantities in question are ill-defined because of Haag's theorem and it is this fact that grounds the reliability of empirical adequacy claims in the face of the theorem.

In theories with strong coupling, such as quantum chromodynamics at low energies, the interaction picture formalism is not used. Since the coupling is strong, the parameter in which one is expanding is large and perturbation theory cannot be expected to give meaningful answers.[22] In this case a different approach to generating predictions is necessary. Strongly coupled theories can be regularized by placement on a Euclidean lattice, and contributions to expressions for physical observables can be approximated numerically. Realistic modern experiments frequently have contributions from quantum chromodynamic processes, and thus empirical adequacy claims are dependent on these calculational techniques.[23] Of course, in

[22]Quantum chromodynamics at high energy can be treated perturbatively because the coupling runs to smaller values. However, this perturbative treatment requires that a lattice regularization is imposed. The significance of Haag's theorem for perturbative evaluation of a fully regularized theory is addressed below.

[23]There is a sense in which these calculations are non-perturbative; namely, numerical values are extracted from the theory by a method other than perturbation theory. Note that this is a different sense of 'non-perturbative' from the one used throughout this dissertation. I have reserved this term for exact continuum models of axiomatic articulations of the content of the theory.

this case, the interaction picture simply is not employed at any point in the calculation, and Haag's theorem provides no obstacle to the calculation of experimental observables.

There are, however, cases in which the interaction picture is used to calculate physical observables. This is the context in which Haag's theorem raises a legitimate concern about empirical adequacy claims. The interaction picture was first introduced by Schwinger in (Schwinger, 1948b). One of the motivations for its introduction was to facilitate the calculation of the anomalous magnetic moment of the electron and thus to provide a critical test of the empirical adequacy of quantum electrodynamics. Since its introduction, the perturbative evaluation of vacuum expectation values and S-matrix elements for weakly coupled theories like quantum electrodynamics has relied on the interaction picture formalism. An adequate explanation of the success of scattering theory must show why such calculations give values that match empirical data despite Haag's theorem.

Rendering the perturbative evaluation of S-matrix elements for interacting quantum field theories well-defined requires that all three problems with Equation (3.4) by addressed. Regularization and renormalization techniques are used to isolate and control the infrared and ultraviolet divergences in the theory. There are several different regularization schemes which can be used to control ultraviolet divergences. The simplest example of such an ultraviolet regularization is the imposition of a short distance, or equivalently large momentum, cutoff.[24] When a long distance cutoff is also imposed to control the infrared divergences, the theory is reduced to a finite number of degrees of freedom. Once a regularization is in place, the theory can be renormalized. At the end of a calculation the regularizations can be removed by taking the limit where the spacetime approaches continuous and infinite Minkowski space, thus restoring the full symmetry properties of the theory.

The full regularization that is imposed to control ultraviolet and infrared divergences breaks the Poincaré invariance of the theory. Recall that this an essential assumption required to prove Haag's theorem. In the fully regularized theory, each contribution to the

[24] The details of regularization and renormalization techniques can be found in most standard texts on quantum field theory. For a more comprehensive presentation see (Collins, 1984).

perturbative expansion is thus well-defined when it is evaluated. With the regularization in place the perturbative expansion for the S-matrix elements defined by Equation (3.4) can proceed order by order. The number of terms that must be summed to obtain the contribution from each order grows rapidly, and thus the state of the art only allows for perturbation theory calculations at a few orders for most important observables. The essential thing to note is that what gets compared to experimental data is the sum of the first few terms of the expansion. Since all of the terms in the sum are well-defined when they are calculated, there simply is no problem caused by Haag's theorem. The perspective that I am advocating has recently been argued for by Duncan.[25] He claims that "...the proper response to Haag's theorem is simply a frank admission that the same regularizations needed to make proper mathematical sense of the dynamics of an interacting field theory at each stage of a perturbative calculation will do double duty in restoring the applicability of the interaction picture at intermediate stages of the calculation" (Duncan, 2012, p. 370).[26] I agree with Duncan that the regularizations used to control the ultraviolet and infrared divergences are what preserves the reliability of perturbative calculations in the face of Haag's theorem, but there is one further concern that must be addressed.

One might worry that this resolution to the problem is not completely general. In particular, there is more than one approach to regularizing and renormalizing field theories. Moreover, each method has different effects on the symmetries of the theory. Some methods break Poincaré invariance and others break gauge invariance. Which technique gets used for a particular calculation depends on which properties of the theory one wants to preserve. Thus, to put the worry precisely, one might wonder if some of these techniques leave the full Poincaré symmetry intact.[27] If this were the case then it would seem that such calculations are still subject to Haag's theorem. Consider, for example, the technique of dimensional regularization. Rather than imposing cutoffs one continues the spacetime dimension to $4 - \epsilon$.

[25] Butterfield's review of Duncan's book draws attention to the importance of this argument (Butterfield, 2015).

[26] A similar perspective can also be found in (Strocchi, 2013, p. 52).

[27] I am grateful to Kerry McKenzie for pressing me on this point.

This has the benefit of preserving gauge invariance. The question of Poincaré invariance is more sensitive as the exact spacetime symmetries of Minkowski space are not restored until the dimension is continued back to 4. However, for the purposes of my argument what is critical to note is that dimensional regularization also affords the capability to control infrared divergences.[28] In order to achieve reliable perturbative results using dimensional regularization, the infrared divergences must be addressed using such techniques.

In practice, empirical adequacy claims often involve sums of contributions to different orders obtained using different regularization techniques. Consider, for example, the calculation of the anomalous magnetic moment of the electron. The best theoretical calculation of this observable matches experimental data to more than 10 decimal places. The first order contribution to this quantity was originally calculated by Schwinger (Schwinger, 1949). During the process of the calculation, he encounters an infrared divergence. To control it he introduces a minimum wave number for the photons in the theory, which is equivalent to the imposition of a maximum wavelength and thus a long distance cutoff.[29] Since this quantity provides such a critical precision test of the theory, significant effort has been dedicated to calculating additional orders of perturbation theory beyond the leading term.[30] Some intermediate orders can be calculated analytically, but this analytic evaluation requires regularizations that break Poincaré invariance. The highest orders require the computation of a very large number of complicated terms and must be computed numerically. This of course requires that the theory be reduced to a finite number of degrees of freedom and so again Haag's theorem is rendered inapplicable.

The real difficulty raised by Haag's theorem then, is to understand why contributions from the first few orders of perturbation theory give empirically adequate results, even though when the full symmetries are restored by taking the infinite volume limit and removing the ultraviolet regularization, the formalism used to obtain those results becomes ill-defined.

[28] For the details of this approach see (Gastmans and Meuldermans, 1973) and (Marciano and Sirlin, 1975).

[29] See equation 1.107 of (Schwinger, 1949, p. 801).

[30] (Kinoshita, 1990; Roskies et al., 1990; Aoyama et al., 2012; Kinoshita, 2014)

The best available explanation of this fact is that the observables that get compared to exeriment are insensitive to the removal of the infrared cutoff.[31] Through regularization and renormalization, perturbation theory provides well-defined formal power-series for such observables. The third problem with Equation (3.4) is the problem of determining whether or not these formal power-series converge and thus correspond to exact non-perturbative objects. Haag's theorem is related to this issue only in that it is an obstacle to the well-definedness of the individual terms of the perturbative expansion, and I have argued that it is an obstacle that is overcome through regularization and renormalization.

In this sense, part of the significance of Haag's theorem is that it complicates the relationship between the perturbative content of the theory and our best available characterizations of its non-perturbative structure. Note however, that the result is not unique in this respect. The question of how numerical data from fully regularized theories is related to exact non-perturbative structure is a very general one, and about which much information is available from sources other than Haag's theorem. Even in quantum chromodynamics, where the interaction picture is not used and there is no problem with Haag's theorem, a similar question arises. Results are calculated on a lattice and in some cases give empirically adequate results. However, the full continuum theory corresponding to the limits in which the regularizations are removed has not been shown to be an exact model of the axioms that characterize the structure of the theory.

3.5 CONCLUSION

I have argued that empirical adequacy claims are not undermined by Haag's theorem because the regularizations and renormalization required to give clear meaning to the perturbative

[31]Such observables are called infrared safe. In quantum electrodynamics, the KLN theorem (Kinoshita, 1962; Lee and Nauenberg, 1964), motivated by work of Bloch and Nordsieck (Bloch and Nordsieck, 1937), ensures that observables are infrared safe. In the case of quantum chromodynamics, determining which observables are infrared safe is more difficult. See (Muta, 2010) for a detailed discussion.

evaluation of vacuum expectation values and S-matrix elements also quell the problems associated with the infrared divergences implicated in Haag's theorem. The constructive approach to field theory takes as its starting point physical assumptions that are believed to obtain in the empirically adequate models that can currently only be treated perturbatively. According to this perspective, axiomatic articulations of the non-perturbative structure of the theory amount to expressions of the basic physical properties that need to be satisfied in the continuum and infinite volume limits in order to have what can properly be counted as a relativistic quantum field theory. However, as I have stressed above, the theories of the Standard Model cannot be shown to satisfy the axioms. Obtaining numerical information from them for comparison with experiment requires that they be regularized in ways that render some of the conclusions that can be reached in the unregularized theory, including Haag's theorem, inapplicable. It follows that achieving a complete understanding of why scattering theory does work requires a resolution to the tension between the mathematical characterization of the non-perturbative structure of the theory and the techniques that are required to obtain successful empirical predictions using that structure.

Reactions to Haag's theorem are illustrative of a general tension which exists among much of the literature that attempts to address the interpretation of quantum field theory. It is not obvious what the rigorous models have to do with the actual world because they are defined in a spacetime with dimension other than four or without realistic interactions. At the same time, the modifications to the mathematical formalism required to render the expressions characterizing empirically relevant models well-defined seem to correspond to physically substantive changes according to standard approaches to interpretation. There are two ways that this tension might be resolved. First, it could be that further work will lead to existence proofs for more physically relevant models. If this were achieved then the Haag-Ruelle formalism that Earman and Fraser appeal to could underwrite the success of scattering theory directly. Much of the literature appraising the philosophical significance of quantum field theory seems to be predicated on the hope that this goal will be achieved. In fact, some authors seem to think that this is a necessary condition for quantum field theory to be a foundationally respectable theory, and that absent such a development claims about perturbative field theory are mathematically unintelligible. However, there is no assurance that physically relevant theories are in fact models of the axioms. If they are not, then one could appropriately view the inability to construct models of the axioms as a source of physical information. In this case, the success of scattering theory would need to be accounted for in a more elaborate way.

Further evidence that a more elaborate account is necessary comes from attempts to address the third, and in my opinion, the most important problem with perturbative expansions for empirically adequate models. Even though the first few terms in the expansion give a result which agrees closely with experiment, when the contributions from higher orders of perturbation theory are included, the series goes on to diverge. The divergence in question is independent from the ultraviolet and infrared divergences that are controlled with regularization and renormalization. The perturbation series itself diverges, even once the theory has been renormalized to render each term in the expansion finite. This problem is not related to Haag's theorem, which I have argued is only a challenge to providing well-defined formal power series. The question is whether these formal power series correspond to exact objects, and the divergence of perturbation suggests that they do not.[32] An explanation of the success of scattering theory should also account for the fact that taking the first few terms of what are widely believed to be divergent expansions give such remarkably accurate results.

The empirical adequacy of the first partial sums despite the eventual divergence when the series is summed to all orders can, in fact, be explained. The explanation is provided by the conjecture that empirically successful perturbative expansions are asymptotic to exact solutions of a theory that generates them. Asymptoticity is a precisely defined relation

[32]The reason divergence is only suggestive is that it does not rule out that the expansion can be summed by a method such as Borel-summation which uniquely assosciates a divergent expansion with an exactly determined object.

between a series expansion and the function being expanded. The condition ensures that the differences between the exact value of the function and the partial sums of the series are appropriately small for each fixed order of perturbation theory. When series satisfying this condition are summed to all orders they typically diverge. However, their first partial sums often approximate the exact value of the function to many decimal places of accuracy. It is in this sense that the conjecture that empirically adequate expansions are asymptotic to some unknow exact theory that generates them accounts for their success despite their divergence.[33]

Many non-perturbative structures can yield the same asymptotic expansion. Thus, the conjecture that perturbation theory for a quantum field theory yields asymptotic expansions does not uniquely fix what non-perturbative structure lies behind the empirical success of the theory. The more elaborate account I have in mind must address how well perturbative data can constrain the non-perturbative structure of the theory, as well as the fact that the empirical information that we glean from experiments seems to exhibit a level of insensitivity to the exact non-perturbative structure. The analysis of this chapter shows that Haag's theorem does not undermine empirical adequacy claims, and it also shows that the theorem does not undermine the use of perturbation theory as a guide to determining non-perturbative structure.

[33]The condition of asymptoticity is explained in detail in Chapter Four.

4.0 ON THE COMMON STRUCTURE OF PERTURBATIVE AND AXIOMATIC FIELD THEORY IN BOREL SUMMABLE MODELS

Chapters Two and Three considered what seemed to be insurmountable obstacles to the interpretation of perturbative quantum field theory. Individual orders of the perturbative expansion are infinite because of the short and long distance regimes of the theory. However, the terms can be regularized and renormalized in a way that produces a collection of well-defined formal power series. This motivates a close examination of the convergence properties of these series. That is the task I take up in this chapter. The analysis shows that the series are divergent, but nevertheless, asymptotic to exact structures. Noting this shows that in every case where a model of axiomatic quantum field theory has been shown to exist, the perturbative treatment of that model exactly and uniquely determines the constructive model. In other words, perturbative and axiomatic field theory share a common structure for a restricted class of models. This class fails to extend, however, to empirically adequate models. The empirically adequate models contain additional large-order singularities that inhibit their unique reconstruction from perturbative data. Despite this I show that a rigorous characterization of the empirical content of the theory can still be recovered from their divergent perturbative expansions. What differentiates this characterization of the empirical content of the theory from those required in standard accounts of interpretation is that it has precisely defined, but limited, precision. What differentiates axiomatic and perturbative field theory is not their respective levels of mathematical rigor, but rather that perturbative field theory avails itself of the expressive resources of statements of empirical content with limited precision that result from truncating perturbative expansions at low-order.

4.1 INTRODUCTION

Efforts to discern what successful theoretical representations reveal about the world are complicated by the presence of structural underdetermination. Such underdetermination can take a variety of forms, each of which pose unique challenges to interpreters of a given theory.[1] The particular form of structural underdetermination that will be addressed in this chapter results from the existence of multiple formalisms for capturing the content of a theory. This situation is common in the practice of physics. The Hamiltonian and Lagrangian formalism provide distinct structural realizations of classical mechanics, and wave mechanics and matrix mechanics provide distinct structural realizations of quantum mechanics, for example. In situations where more than one formalism is available, intepreters have two options: either show that the formalisms give equivalent descriptions of the domain that they represent, or find grounds for privileging one of the formalisms to serve as the basis for interpretation.

The received wisdom in the philosophy of physics literature is that the first option is not available in the case of quantum field theory. Smeenk and Myrvold express this received wisdom when they claim that:

> A gulf separates axiomatic treatments from the methods used by most working physicists. And in this case the gulf is deeper than the usual divide between physicists' relaxed standards of rigor and the sophistication of the mathematicians. History of physics offers several examples where apparently quite different formulations turned out to be equivalent versions of a single theory. But in this case it is clear that such a reconciliation of different approaches cannot be easily achieved. (Smeenk and Myrvold, 2011)

The question of which formalism is more appropriate for serving as the basis for interpretive conclusions is the subject of a recent debate between Fraser (Fraser, 2011) and Wallace (Wallace, 2011). They both contend that the perturbative and axiomatic approaches should

[1] A reasonably comprehensive survey of the various forms of structural underdetermination claims, as well as discussion of how they arise in quantum field theory, can be found in (French, 2014).

be viewed as competing research programs. Wallace, for example, defends the claim that they "...should be understood as rival programs to resolve the mathematical and physical pathologies of renormalization theory, and that [perturbative field theory] has succeeded in this task and AQFT has failed" (Wallace, 2011, p. 116). Fraser agrees with the characterization in terms of rival research programs but argues for favoring axiomatic approaches to characterizing the content of the theory(Fraser, 2011, p. 126).

The second option of privileging one formalism is most attractive when all of the senses of success of the theory can be achieved within the privileged formalism. Quantum field theory is successful in two distinct senses. First, it provides theoretical predictions for cross sections that match experimental data to unprecedented accuracy. Second, it provides a unification of the successful theoretical frameworks on which it is based; namely, quantum mechanics and special relativity. The project of interpreting quantum field theory is complicated by the fact that its empirical and theoretical successes are difficult to achieve simultaneously. The empirical successes of the theory are achieved within the standard perturbative formalism, whereas the theoretical unification is usually best captured by axiomatic approaches to characterizing the content of the theory. What inhibits the simultaneous achievement of both senses of success is that the empirically successful models of the perturbative formalism have not been shown to be models of any of the axiomatic frameworks.

In support of the perturbative formalism Wallace notes that it provides models that have produced the most stringently tested predictions of any physical theory in history. Moreover, he notes that the axiomatic approach has yet to provide any physically relevant models. Wallace claims that the two programs are "conceptually incompatible" approaches to the problem of renormalization (Wallace, 2011, p. 119). In support of the axiomatic approach, Fraser argues that all of its constructions are mathematically well-defined, and it can reasonably be hoped that further progress will yield physically relevant models. On the other hand, Fraser claims that the success of renormalization group methods at the conceptual core of the program Wallace defends "...illuminates the empirical content of

QFT, but not the theoretical content" (Fraser, 2011, p. 126). Moreover she notes that the perturbative approach of physicists requires mathematical operations that are not rigorous. Their debate thus arrives at an alleged forced choice between empirical adequacy on the one hand, and mathematical rigor and conceptual clarity on the other.

In this chapter I will argue against this alleged forced choice by showing that option one can actually be profitably pursued. In section two I introduce a condition of full intertranslatability of formalisms, and I show that a number of axiomatic formalisms for quantum field theory, including the Wightman axioms, satisfy the condition. In section three I introduce a condition of partial intertranslatability and I show that this condition obtains for the perturbative formalism and the Osterwalder-Schrader axioms. The most important consequence of this argument is that it shows that in every case where a model of the Wightman axioms is available, perturbative treatment of that model exactly and uniquely recovers the axiomatically characterized model. This section also clarifies the nature of the connection between algebraic quantum field theory and the Wightman axioms. In the fourth section I explain why perturbative field theory and axiomatic field theory are not fully intertranslatable. In particular, I show that what differentiate the formalisms is not their respective levels of mathematical rigor. Rather what differentiates them is that perturbative field theory avails itself of the expressive resources of statements of empirical content that have precisely defined, but limited, precision. The final section shows how the arguments of the paper restore the possibility of pursuing option one, and thus opens a route to a new approach to interpretation that can be applied to empirically adequate models.

4.2 FULL INTERTRANSLATABILITY

There are several different axiomatic formulations of the non-perturbative content of quantum field theory. This section provides an account of the interrelationships between three axiomatic formalisms; the Wightman-Gårding axioms, the Wightman axioms, and the Osterwalder-

Schrader axioms. The central aim of this section to show that all three of these formalisms provide fully intertranslatable characterizations of the domain of local, relativistic, quantum fields. Two formalisms F_1 and F_2 are *fully intertranslatable* just in case the existence of an arbitrary model of F_1 guarantees the existence of a unique model of F_2.[2] A series of reconstruction theorems underwrite the full intertranslatability of these axiomatic formalisms. The Wightman reconstruction theorem establishes the intertranslatability of the Wightman-Gårding axioms and the Wightman axioms, while the Osterwalder-Schrader reconstruction theorem serves the same purpose for the Wightman axioms and the Osterwalder-Schrader axioms.

Of all of the systems of axioms for quantum field theory, the Wightman-Gårding axioms provide what is most readily identifiable as a relativistic quantum theory.[3] The basic mathematical objects which the axioms characterize are a concrete Hilbert space, a set of field operators that act on a common dense domain of the Hilbert space, and a unitary representation of the Poincaré group. A spectrum condition is imposed along with the requirement of the uniqueness of the vacuum, and microcausality.[4] The Gårding-Wightman axioms and the Wightman axioms are connected by the Wightman reconstruction theorem. The Wightman axioms are not stated directly in terms of a Hilbert space. Rather, they are conditions on a set of Wightman distributions $\{\mathcal{W}_n\}$, which are elements of the space of tempered distributions. These are the basic mathematical objects in this axiomatization of the theory.[5] The reconstruction theorem holds that if $\{\mathcal{W}_n\}_{n=0}^{\infty}$ is a sequence satisfying the Wightman axioms, then there exists a unique model obeying the Wightman-Gårding axioms. This shows that from a set of Wightman distributions one is able to recover the more familiar Hilbert space formalism through the reconstruction theorem. This reconstruction theorem establishes a full intertranslatability between the two formalisms as the existence

[2]This is a slightly weaker notion of intertranslatability than the one given in (Glymour, 1977).

[3]The presentation here follows the one in (Simon, 1974). This article provides explicit statements of the Wightman-Gårding, Wightman, and Osterwalder-Schrader axioms in a consistent notation.

[4]These are conditions GW1-GW6 of (Simon, 1974).

[5]They are given by conditions W1-W6 of (Simon, 1974).

of a model of one formalism guarantees the existence of a model of the other.

The Wightman axioms are also connected to the axiomatization of Osterwalder and Schrader (Osterwalder and Schrader, 1973, 1975). In their formalism, the basic mathematical objects are the Schwinger functions. These functions are related to the Wightman distributions by a transformation which takes the Minkowski spacetime point arguments of the Wightman distributions to points in Euclidean space. If $x = (x_0, x_1, x_2, x_3)$ is a point in Minkowski space, let $x^E = (-ix_0, x_1, x_2, x_3)$ be the corresponding point in Euclidean space. Then the Schwinger functions are given by, $S_n(x_1, \ldots, x_n) = \mathcal{W}_n(x_1^E, \ldots, x_n^E)$. This transformation to Euclidean space is more commonly known in the standard perturbative formalism as Wick rotation. The idea of the Wick rotation is to transform the Minkowski space metric into the Euclidean metric, by allowing the Minkowski time coordinate, x_0, to take on complex values. In this case, the Minkowski metric becomes the Euclidean metric when the time is restricted to the imaginary axis. Problems in Minkowski space are thus transformed into problems in Euclidean space by making the substitution $x_0 \to -ix_0$.

By determining a set of conditions on Schwinger functions,[6] Osterwalder and Schrader were able to prove the following reconstruction theorem. To a given sequence of Wightman distributions satisfying the Wightman axioms, there corresponds a unique sequence of Schwinger functions satisfying the Osterwalder-Schrader axioms. Moreover, to a given sequence of Schwinger functions satisfying the Osterwalder-Schrader axioms, there corresponds a unique sequence of Wightman distributions satisfying the Wightman axioms.[7] Just as in the previous case, the existence of a model of one formalism guarantees the existence of a model of the other and so establishes full intertranslatability.

[6] Conditions OS1-OS5 of (Simon, 1974).

[7] There was an error in the original proof of the second part of this theorem. Osterwalder and Schrader corrected the problem by modifying their condition specifying the temperedness of their functions. Both versions of the condition, OS1 and OS1' are stated in (Simon, 1974).

4.3 PARTIAL INTERTRANSLATABILITY

The condition for full intertranslatability of formalisms introduced in the previous section appealed to arbitrary models, that is, arbitrary collections of mathematical structures satisfying the axioms. A weaker condition of intertranslatability can be given if one allows for appeal to model dependent features. If a model of F_2 can exactly and uniquely be reconstructed from a particular model of F_1, F_1 and F_2 share a common structure for that model. For my purposes, F_1 and F_2 are *partially intertranslatable* if F_1 and F_2 share a common structure for at least one model. In this section I will argue that perturbative quantum field theory and the Osterwalder-Schrader axioms are partially intertranslatable in this sense: they share a common structure for a non-empty set of models. As a consequence of the full intertranslatabilities established in the last section it follows that the perturbative formalism is partially intertranslatable with the Wightman axioms and the Wightman-Gårding axioms as well.

Though it is not often expressed in these terms, in setting up a perturbative calculation one begins with a conjecture about the non-perturbative structure of the theory. The standard perturbative formalism proceeds from an assumed non-perturbative structure much like the one characterized by the Wightman axioms. After Wick rotation the assumed non-perturbative structure is the one characterized by the Osterwalder-Schrader axioms. If the perturbative evaluation of the assumed non-perturbative structure resulted in convergent expansions, then perturbation theory would yield an exact model of the conjectured non-perturbative structure. However, in the case of empirically interesting models perturbation theory yields divergent expansions. Dyson produced the first argument for the large-order divergence of perturbation theory for quantum electrodynamics (Dyson, 1952). Let,

$$F(e^2) = a_0 + a_2e^2 + a_4e^4 + \dots, \tag{4.1}$$

be a formal power series expansion for a quantity in the theory. Dyson's argument proceeds by supposing that this expansion converges for some small positive value of e^2, the coupling determining the strength of the interaction in the theory. Then $F(e^2)$ is an analytic function at $e = 0$, so for sufficiently small values of e, $F(-e^2)$ will also be an analytic function with convergent power series expansion. He then notes that the $-e^2$ case corresponds to a world in which like charges attract. This results in an unphysical instability of the vacuum, as states of lower energy can be created out of the vacuum by pair production with two large like-charge collections moving to far separated regions of the universe. This instability of the vacuum is not compatible with analyticity of the function being expanded, so he concludes that the expansion cannot converge for this negative value of the coupling. Thus, he concludes that the expansion for $F(e^2)$ cannot converge either. Concerning the validity of the argument he notes that "The argument here presented is lacking in mathematical rigor and in physical precision. It is intended only to be suggestive, to serve as a basis for further, discussions" (Dyson, 1952, p. 631). This is of course correct, but the argument is deeply suggestive and it motivated further attempts to understand the convergence properties of quantum field theories.

In the year following the publication of Dyson's argument, Hurst, Thirring, and Petermann all produced additional arguments for the divergence of perturbation theory in a particular simplified field theory models (Hurst, 1952; Thirring, 1953; Petermann, 1953). They each found lower bounds for the contributions from sub-collections of the set of terms in the expansion that were relatively simple to evaluate and showed that those alone were sufficient to make the series divergent. These arguments are also inconclusive because there is no assurance that the terms not considered provide sufficient cancellation to render the whole expansion convergent. This illustrates one of the central difficulties in understanding the large-order convergence properties of perturbation theory. Namely, one needs a precise understanding of the large-order terms in the expansion. In quantum field theory, renormalization often results in changes to the large order terms in a non-uniform way, creating unreliable patterns of alternating signs. This makes it difficult to understand to what extent there is cancellation between the terms, which makes it hard to rigorously establish divergence. Models in reduced spacetime dimension and without interactions have simpler renormalized expansions and so it is in this context that constructive field theorists have been able to rigorously demonstrate that perturbation theory is in fact divergent. It is widely believed that perturbation theory diverges in the case of empirically adequate models as well, though this remains to be proved.

Before providing his argument, Dyson notes that had it been possible to show that the perturbative expansions were all convergent, "then the theoretical framework of quantum electrodynamics could have been considered closed, being within its limits a complete and consistent theory" (Dyson, 1952, p. 631). On the other hand he claims that "The divergence [of perturbation theory in QED] in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built" (Dyson, 1952, p. 631). For Dyson, the divergence of perturbation theory resulted in a lack of clarity about the concepts employed in the theory which undermined its claims to completeness and consistency. Interestingly, he thought that this did not necessarily undermine the predictive success of the theory. This is an exceedingly important point, and one which is neglected in the philosophy of physics literature. The most difficult obstacle to the well-definedness of perturbative field theory is not the ultraviolet divergences that have been the locus of attention, but rather the large-order divergence of perturbation theory.

At the end of his paper, Dyson explains that the success of the first few terms of the expansion at matching with experiment can be explained by the conjecture that the perturbative expansion is asymptotic to some as yet undetermined function. To this day, our best explanation of the success of quantum electrodynamics and quantum chromodynamics remains the asymptotic nature of their perturbative expansions. In the regime of small coupling, as in quantum electrodynamics or quantum chromodynamics at high energy, this would produce the following expected behavior.[8] When a series is asymptotic to a function, the partial sums of the first few terms in the series yield increasingly accurate approximations to the function. After the smallest term in the series the terms begin to grow and the partial sums rapidly deviate from the close approximation as those larger orders are included. When all orders are included the series yields an infinite value and thus an infinitely bad approximation to the function. This is precisely the observed behavior of perturbation theory for empirically adequate models of quantum field theory and it is in this sense that Dyson's conjecture of asympoticity accounts for the empirical success of perturbative field theory.

The property of asymptoticity turns out to be the critical key for understanding how perturbative field theory is related to the Osterwalder-Schrader axioms.[9]

Definition 4.1. *For $f(z)$ a function defined in a sector of the complex plane $S = \{z \mid 0 < |z| < B; |\arg z| \leq \theta\}$, the series $\sum_{n=0}^{\infty} a_n z^n$ is asymptotic to $f(z)$ uniformly in the sector as $|z| \downarrow 0$ iff for every fixed N,*

$$\lim_{\substack{z \downarrow 0 \\ |\arg z| \leq \theta}} \frac{f(z) - \sum_{n=0}^{N} a_n z^n}{z^N} = 0. \tag{4.2}$$

It follows immediately from this definition that each function has a unique asymptotic expansion, since if the limit exists it is unique. However, two different functions can have the same asymptotic expansion. To see this consider $f(z) = \exp(-z^{-1})$ with $z > 0$. In this case $z^{-n} f(z) \to 0$ as $z \downarrow 0$, and thus this $f(z)$ has the series with all zero coefficients as its asymptotic series. Since such small exponentials can be added to a function without changing the asymptotic series, it follows that knowledge of an asymptotic series for a function $g(z)$ gives no information about the exact value of $g(z)$ for any nonzero z.

[8] At low energies or correspondingly long distances, QCD becomes strongly coupled and perturbation theory is known not to be informative. In this case a nonperturbative lattice regularization is necessary to obtain meaningful numerical information from the theory.

[9] The presentation here follows the one given in Reed and Simon (1978).

One might reasonably wonder if there is some condition stronger than that the series is asymptotic to the function, but weaker than that the series is convergent, which determines the function uniquely. The following definition specifies just such a condition.

Definition 4.2. *A function $f(z)$, which is analytic in $S = \{zz \mid 0 < |z| < B; |\arg z| < \frac{1}{2}\pi + \epsilon\}$, is said to obey the strong asymptotic condition and has $\sum_{n=0}^{\infty} a_n z^n$ as its strong asymptotic series if there exist C and σ such that*

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \leq C\sigma^{N+1}(N+1)!|z|^{N+1} \tag{4.3}$$

for all N and $z \in S$.

This definition furnishes the desired theorem.

Theorem 4.3. *Suppose $\sum_{n=0}^{\infty} a_n z^n$ is a strong asymptotic series for analytic functions $f(z)$ and $g(z)$ in S. Then $f(z) = g(z) \ \ \forall z \in S$.*

Moreover, if one knows the strong asymptotic series of a function, then the function can be uniquely reconstructed by Borel resummation. This is captured by the following theorem.

Theorem 4.4. *Suppose that $\sum_{n=0}^{\infty} a_n z^n$ is a strong asymptotic series for $f(z)$ in $S = \{z \mid 0 < |z| < B; |\arg z| < \frac{1}{2}\pi + \epsilon\}$. Consider the Borel transform:*

$$g(z') = \sum_{n=0}^{\infty} \frac{a_n}{n!} z'^n, \tag{4.4}$$

which is analytic in some circle around $z' = 0$ because of the strong asymptotic condition. Then, if $|z| < B$ and $|\arg z| < \epsilon$, then

$$f(z) = \int_0^{\infty} g(z'z) e^{-z'} dz'. \tag{4.5}$$

When a divergent perturbative expansion is strongly asymptotic to a function, the expansion exactly and uniquely determines that function.

This result makes it possible to show that perturbative field theory and the Osterwalder-Schrader axioms are partially intertranslatable. Recall that two formalism are partially intertranslatable if they share a common structure for at least one model. In fact, Borel summation has been used to show that perturbative field theory exactly and uniquely recovers the $P(\phi)_2$ models, the ϕ_3^4 model, and the Y_2 model.[10] One might object that this class of common structure only includes superrenormalizable models and that the real failure of intertranslatability should be expected only in strictly renormalizable models. There are two available replies. The first is to note that Borel summability has been established for GN_2, a strictly renormalizable model.[11] The second, and more pertinent, reply is that the real wedge between axiomatic and perturbative field theory is not in their treatment of arbitrarily short distances as emphasized in Chapter Two. The wedge is how they treat the large-order divergences of perturbation theory. The cases of Borel summability show that in many of the situations where a constructive model is available, perturbative and axiomatic field theory agree on the large-order behavior of the model. In this sense, the perturbative formalism and the Osterwalder-Schrader axioms are partially intertranslatable. The full intertranslatabilities established in the last section show that the results introduced here establish the partial intertranslatability of the perturbative formalism and the Wightman axioms and the Wightman-Gårding axioms as well.

These results provide a significant clarification to the nature of the relationship between perturbative and axiomatic field theory. Moreover, they clarify the mathematical status of the empirically adequate models. I have already noted that the best explanation of the empirical success of the low-orders of perturbation theory is best explained by the conjecture that the perturbative expansion is asymptotic to an exact model. One might reasonably wonder if the expansions are strongly asymptotic and thus can be uniquely associated with an exact model through Borel summation. Interestingly, the answer to this question turns out to be negative. In quantum electrodynamics and quantum chromodynamics there are

[10](Eckmann et al., 1975; Magnen and Sénéor, 1977; Renouard, 1979)
[11](Feldman et al., 1986)

singularities along the positive real axis of the Borel plane which make the unique reconstruction of the function impossible, by disrupting the integration required to reconstruct the function.[12] These singularities result from field configurations corresponding to what are called instantons, the number of which grow like $n!$, and renormalons, individual graphs that contain n bubble insertions, for n the order of perturbation theory. The division by $n!$ in the Borel transform is insufficient to completely tame this large-order divergent behavior.

This does not mean that the empirically adequate models are mathematically ill-defined. Rather, they hold information about the world in a different way than we have grown accustomed to. The situation is captured nicely in the following remark of Magnen and Rivasseau:

> Constructive field theory builds functions whose Taylor expansion *is* perturbative field theory. Any formal power series being asymptotic to infinitely many smooth functions, perturbative field theory alone does not give any well defined mathematical recipe to compute to arbitrary accuracy any physical number, so in a deep sense it is no theory at all. (Magnen and Rivasseau, 2008, p. 403, my emphasis)

The first part of this remark is a restatement of the claim I have argued for in this section. When constructive models are available, perturbative treatment of the model exactly and uniquely recovers the model. When the strong asymptotic condition fails and the model is not uniquely recoverable from the perturbative data, further analysis is required. At this stage I will simply note that the inference from the fact that empirically adequate models do not determine their empirical content to arbitrary precision, to the claim that they cannot be construed as *theories*, strikes me as unnecessarily drastic. In Chapter Five I will develop approach according to which non Borel summable models can be construed as theories, whose characterization of their empirical content is inexact.

[12] Details can be found in 't Hooft's contribution to (Zichichi, 1979).

4.4 CONCLUSION

Clarifying the connections between the different formalisms has important consequences for the project of interpreting the theory. As there are multiple formalisms for quantum field theory, interpreters seem to face a choice of which formalism should serve as the basis for interpretive conclusions. The importance of axiomatic approaches for interpretive questions is typically motivated by the claim that they provide the only available mathematically rigorous characterization of the theory. Consider, for example, the following remark of Halvorson concerning the algebraic approach[13] to axiomatic quantum field theory:

> ...philosophers of physics have taken their object of study to be theories, where theories correspond to mathematical objects (perhaps sets of models). But it is not so clear where "quantum field theory" can be located in the mathematical universe. In the absence of some sort of mathematically intelligible description of QFT, the philosopher of physics has two options: either find a new way to understand the task of interpretation, or remain silent about the interpretation of quantum field theory.
>
> It is for this reason that AQFT is of particular interest for the foundations of quantum field theory. In short, AQFT is our best story about where QFT lives in the mathematical universe, and so is a natural starting point for foundational inquires. (Halvorson and Muger, 2006, pp.731-732)

It should be clear at this stage of my argument that the mathematical situation with the theory is in fact significantly more subtle than Halvorson makes it out to be. First, there are axiomatic formalisms other than the algebraic axioms of Haag and Kastler which are indisputably mathematically intelligible. These include the Wightman-Gårding axioms, the Wightman axioms, and the Osterwalder-Schrader axioms. Second, and more interestingly, the perturbative formalism on which Halvorson is casting aspersions turns out to be equally mathematically intelligible. In those instances where models of axiomatic field theory are

[13]The statement of the algebraic axioms can be found, for example, in (Haag, 1992b). There exists a series of results which capture the conditions under which the existence of a model of the Haag-Kastler axioms guarantees the existence of a model of the Wightman axioms, and vice versa. See, for example, (Borchers and Yngvason, 1975, 1990, 1992; Buchholz, 1990; Driessler et al., 1986; Fredenhagen and Hertel, 1981; Summers, 1987; Wollenberg, 1985, 1986, 1988; Yngvason, 1989).

available, the perturbative treatment of those models exactly and uniquely recovers the models of axiomatic field theory. There remain interesting questions about how to interpret perturbatively characterized empirically adequate models, and those questions are the subject of Chapter Five and Chapter Six.

5.0 MATHEMATICAL STRUCTURE AND EMPIRICAL CONTENT

In this chapter I argue that while standard accounts of interpretation adequately address meaning relations when exact models are available or perturbation theory converges, they do not fare as well for models that give rise to divergent perturbative expansions. Since truncations of divergent perturbative expansions often play a critical role in establishing the empirical adequacy of a theory, this is a serious deficiency. I show how to augment state-space semantics, a view developed by Beth and van Fraassen, to capture perturbatively evaluated observables even in cases where perturbation theory is divergent. This new approach to interpretation establishes a sense in which the calculations that underwrite the empirical adequacy of a theory are both meaningful and true, but requires departure from the assumption that physical meaning is captured entirely by the exact models of a theory.

5.1 INTRODUCTION

Accounts of the interpretation of physical theories have developed in conjunction with, and in several important cases are attendant to, accounts of the structure of scientific theories. In other cases novel commitments about interpretation are tacitly adopted in the work of those interpreting particular physical theories. While there are significant differences between the accounts, many share a common set of commitments. Ruetsche has provided a helpful characterization of what is shared between the standard approaches. She explains that according to them, "...to interpret a theory is to characterize the worlds possible according to it. These worlds are (i) models (in something like the logician's sense) of the theory, and

(ii) characterized as physical" (Ruetsche, 2011, p. 7). Standard approaches to interpretation consist in an account of how physical meaning accrues to the mathematical structure of the theory.

Even amongst standard approaches, commitments about the second step in Ruetsche's schema are widely variegated. Some accounts explicitly take the connection between mathematical structure and empirical content to be a map of a particular sort, such as an isomorphism, a partial isomorphism, or an embedding. On other accounts the connection is specified less formally and consists of a metaphysical articulation of the structure of the worlds picked out by the theory, with the mathematical structure functioning as a guide. Commitments about the first step of Ruetsche's schema are comparatively well-regimented. To specify the models of the theory, one must stipulate the states, dynamics, and kinematics characteristic of its structure. Ruetsche's caveat that these structures are models of the theory "in something like the logician's sense" is required because physicists are more permissive about structure specification than logicians.[1] Axiomatizations in mathematical physics typically are given in terms of the mathematical objects most natural for the description of the domain in question, whether they be those of differential geometry, functional analysis, or some other branch of mathematics.[2] This is what differentiates axiomatically characterized models of mathematical physics from the models of formal logical systems. Standard interpreters all agree about the norms of structure specification in that they all require models to exactly satisfy the dynamical equations of the theory, or the axioms that characterize its content.

The aim of this chapter is to consider a different approach to structure specification. Perturbation theory characterizes models as small deviations from models whose structure can be characterized exactly. This technique is used widely in physical practice, and sometimes it is resorted to as a matter of calculational convenience. For this reason, it is not

[1] (L. Ruetsche, personal communication, 18 September 2016)

[2] For example, triples $\langle M, T_{ab}, G_{ab} \rangle$ that exactly satisfy the Einstein field equations are models of general relativity, and the collection of n-point functions for the ϕ_2^4 field theory is a model of quantum field theory because they can be shown to exactly satisfy the the Wightman axioms.

typically regarded by philosophers as a method of structure specification, but instead as a useful approximation scheme for extracting numerical predictions from exact models. However, in a number of important cases including quantum field theory and string theory, the best available characterization of the structure of empirically adequate models is provided by perturbation theory. This creates tension with the norms of structure specification accepted by standard interpreters because in these cases no exact model is available. The absence of an exact model results from the fact that resorting to perturbation theory often results in divergence. The approximation does not converge to a new exact model. For this reason it is not clear how to apply standard accounts of interpretation to empirically adequate models of quantum field theory. In face of this problem it is typically assumed, though often only tacitly, that there is some exact divergence-free structure to which we currently do not have access lying behind the success of the theory. On this view, structures satisfying the standard interpreters' norms of structure specification are supposed to underly the empirical success and physical meaningfulness of the theory.

This chapter pursues a different response to the divergences of perturbation theory. I argue that perturbation theory should not be regarded as an approximation scheme, but instead that it provides a novel connection between the mathematical structure of a theory and its empirical content. The strategy I pursue to argue for this claim is to fix on one explicitly articulated approach to interpretation, the state-space semantics view developed by Beth and van Fraassen.[3] I argue that state-space semantics exemplifies Ruetsche's characterization of the standard approach to interpretation. While focusing on one particular approach to interpretation limits the generality of the conclusions established, state-space semantics exhibits a core feature of most standard accounts by taking meaning relations to derive from the existence of maps from the exact structure of the theory to statements expressing its empirical content. I argue that while state-space semantics adequately captures cases in which perturbation is convergent, it fails to adequately capture the empirical success

[3](Beth, 1960; van Fraassen, 1970)

resulting from truncations of a large class of divergent perturbative expansions.[4] I show that this class cannot be interpreted as an approximation to an exact model. This observation motivates taking seriously the idea that the empirical content of a theory can have limited precision, that is, that empirical content can be *inexact*. I provide an alternative semantics that does capture the empirical success of truncated divergent expansions by articulating principled limits on their precision. Exact models play a privileged role in attributions of physical meaning to the mathematical structure of theories in standard approaches to interpretation. This has had the adverse effect of preventing physical meaning from accruing to empirically adequate models of quantum field theory. I provide an alternative approach to interpretation which allows for meaningful attributions of physical content to the models of the theory that actually make contact with the world.

The argument proceeds as follows. In the second section I consider an example of divergent perturbation theory in quantum field theory. I then review the necessary elements of the mathematical theory of divergent perturbative expansions. In the third section I argue that state-space semantics adequately captures the case in which perturbation theory yields convergent expansions for observables, but fails to do so for divergent perturbation theory. While truncations of convergent expansions can naturally be interpreted as approximations, the same is not true of divergent expansions. The aim is not to demonstrate a deficiency of state-space semantics in particular. Rather, its role in the argument is simply to provide a concrete target which is explicitly formulated. The assumptions it adopts are also adopted in many other prominent approaches to the attribution of physical meaning to mathematically expressed theories. As such, the criticism I provide of state-space semantics in this chapter applies to many standard accounts.[5] The third section also illustrates the limitations of state-space semantics using examples from classical mechanics and non-relativistic quantum mechanics.[6] These are cases in which truncations of divergent perturbative expansions have

[4]A truncation of an infinite expansion is the sum of a finite initial segment of the series.

[5]The details of the argument for that claim will be left to future work.

[6]The large-order divergences I am concerned with have mistakenly been thought to be a special feature of quantum field theory. The argument is actually applicable to a wide variety of physical theories.

been an important factor in establishing the empirical adequacy of a theory. Such cases require a semantics different from state-space semantics and I provide such an alternative in Section Four. The final section concludes by emphasizing that dismissing divergent perturbation theory as merely an approximation scheme has the pernicious effect of divorcing physical meaning from empirically adequate models of quantum field theory.

5.2 THE DIVERGENCE OF PERTURBATION THEORY

Perturbation theory is the predominant method for deriving empirical information from physical theories.[7] Suppose one wants to evaluate an expression in a theory involving a function, $f(x)$, whose exact structure is not necessarily known.[8] The function can be perturbatively evaluated by substituting $f(x) \to \sum_{n=0}^{\infty} c_n x^n$. For this procedure to be effective, x must be a small parameter describing a deviation from a model that can be characterized exactly. Perturbation theory converges if the sequence of partial sums $S_N = \sum_{n=0}^{N} c_n x^n$ converges to a finite limit, that is, $\lim_{N\to\infty} S_N < \infty$. Of course, if the limit exists it is unique and so when perturbation theory converges it uniquely determines the exact value of the function it is being used to characterize. Perturbation theory is said to diverge if the sequence of partial sums diverges, that is, $\lim_{N\to\infty} S_N = \infty$. In either case, the series can be truncated at some order N, and early terms in the series can be summed, $\sum_{n=0}^{N} c_n x^n$, to give a finite estimate of the value of the function.

While convergent expansions are considered mathematically well-understood, divergent expansions have widely been regarded with suspicion since their discovery. This suspicion is often motivated by appeal to the following remark of Abel: "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever

[7]Comparatively little philosophical attention has been dedicated to this method. Noteable exceptions include (Batterman, 1997, 2002, 2007).

[8]The object being perturbatively evaluated need not be a function. Whether it is a function, an eigenvalue, an S-matrix element, an n-point function, or some other structure, the discussion below applies equally well.

. . . ” (Abel and Holmboe, 1839). This section addresses how convergent and divergent perturbation theory differ as attempts to specify the structure of a theory.

Before turning directly to this task, it is instructive to consider the following example. The magnetic moment of the electron, g, is a property of electrons when they are exposed to an external magnetic field. Its value according to the Dirac equation, which treats electrons as relativistic particles, is exactly 2. In 1947, experimental evidence revealed that the value of g deviated just slightly from this exact value.[9] This evidence motivated physicists to search for a theoretical prediction of the anomalous magnetic moment of the electron, $a_e = (g-2)/2$. As noted above, the value of this observable can be calculated in quantum electrodynamics, a perturbatively characterized model of quantum field theory. Schwinger calculated the first term in the perturbative expansion which generated a value that correctly predicted the value of a_e within experimental error.[10] This successful empirical prediction played a critical role in convincing physicists to pursue relativistic quantum field theory as the framework for describing elementary particle physics.[11] It is important to note that what provided the theoretical prediction in this case was the truncation of an infinite perturbative expansion at its *very first term*.

The calculation of terms beyond the first order of perturbation theory becomes increasingly difficult.[12] The anomalous electron magnetic moment continues to function as a precision test of quantum electrodynamics and the Standard Model of particle physics, and so considerable effort has been dedicated to calculating additional orders.[13] The difficulty of the calculation is so great that the current state of the art only allows for the determination of five orders.[14] When these five terms are summed and compared to the experimentally

[9](Kusch and Foley, 1947)

[10](Schwinger, 1948a, 1949)

[11]The history of the role of this calculation in demonstrating the empirical adequacy of quantum electrodynamics is recounted in detail in (Schweber, 1994).

[12]This is the case because the number of Feynman graphs contributing to an individual order grows approximately factorially in the order. The complexity of the integral corresponding to each graph also grows with the order.

[13](Kinoshita, 2014)

[14]More specifically, it is fifth-order in the fine structure constant which is proportional to the square of the coupling, so it is tenth-order in the coupling.

measured value one finds that:

$$\begin{aligned} a_e \text{ (theory)} &= 0.001159652180(73)^{15} \\ a_e \text{ (experiment)} &= 0.001159652181(78).^{16} \end{aligned}$$

Agreement between theory and experiment to ten decimal places is an indication that the theory gets something about the world profoundly correct. This agreement is often cited as evidence that quantum electrodynamics is the most empirically successful physical theory ever created, and for good reason. Feynman famously explained that the success of this calculation is tantamount to theoretically determining the distance between Los Angeles and New York to the width of a human hair (Feynman, 1985, p. 7).

There is, however, a serious problem lurking in the background. There are compelling arguments going back to (Dyson, 1952) that suggest that the perturbative calculation diverges if summed to all orders.[17] This means that as additional orders of perturbation theory are added to the sum, eventually the theoretical prediction will not only deviate from the experimentally determined value, but it will become infinite. The theory, it seems, is not empirically adequate at all: it is infinitely wrong about the value of a_e. Abel's caution about using such series thus seems to have been warranted. Trusting truncations of divergent series at low orders seems to lead to serious errors. In fact, immediately following his cautionary note, Abel remarks that: "...That most of these things [truncation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question" Abel and Holmboe (1839). The sense in which the truncations work is that the sum of the low orders of divergent perturbation theory often yield values in close agreement with empirically determined values. In the century following

[15](Aoyama et al., 2012)

[16](Hanneke et al., 2008)

[17]In particular, the arguments suggest that it diverges even after regularization and renormalization techniques are employed to render every individual term of the series finite. This additional complication is discussed in more detail in Section 4.

Abel's remarks, mathematicians developed a completely rigorous theory of divergent expansions. While this mathematical theory provides many critical pieces of the explanation of their empirical success, below I argue there is an additional philosophical problem which needs to be resolved in order to understand how they hold empirical content.

When divergent perturbation theory is empirically successful, it is usually an indication that the perturbative series in question is asymptotic to some exact structure.[18] Recall that a function $f(x)$ is asymptotic to the series $\sum_{n=0}^{\infty} a_n x^n$, $f(x) \sim \sum_{n=0}^{\infty} a_n x^n$, as $x \to 0$ if and only if for every fixed N,

$$\lim_{x \to 0} \frac{f(x) - \sum_{n=0}^{N} a_n x^n}{x^N} = 0. \tag{5.1}$$

Asymptoticity is a condition that requires that the difference between the exact value of the function and the partial sum of the series that is asymptotic to it is appropriately small, but not necessarily zero, for every order of perturbation theory. This is a different type of condition than the one required for convergence to the exact value of a function. Convergence requires that in the limit where all orders of perturbation theory are included, the difference between the partial sums of the series and the exact value of the function becomes exactly zero.

The order-by-order representation of the function provided by an asymptotic series exhibits qualitatively different behavior from that provided by a convergent series. When a series is convergent to a function, typically the first few partial sums give a rather poor approximation to the function. As additional orders are included, the approximation eventually become better, and as the series is convergent, in the limit where all of the terms are included the approximation becomes an exact representation of the function. Asymptotic series behave differently. Their first few partial sums typically give very close agreement with the exact value of the function. However, when additional orders are included eventually the partial sums begin to exhibit increasingly poor agreement with the exact value. In the

[18]This explanation of the success of the anomalous electron magnetic moment calculation was offered by Dyson immediately upon arguing for the divergence of the calculation (Dyson, 1952).

limit where all of the terms are included, since the asymptotic series is typically divergent, it gives a value that is infinitely different from the exact value. This is precisely the behavior of the series representation of the anomalous electron magnetic moment. It is in this sense that the conjecture that the first few partial sums come from an asymptotic series explains the empirical success of the ultimately divergent expansion.

The effectiveness of the truncation of a convergent expansion at approximating a function is accounted for by the fact that the expansion uniquely determines the function when summed to all orders. It is natural to wonder if asymptotic expansions similarly constrain the exact structure lying behind their success. To answer this question, the first relevant observation to make is that each function has a unique asymptotic expansion: if the limit in the definition of asymptoticity exists, it follows immediately that it is unique. However, two different functions can have the same asymptotic expansion. To see why this is the case, consider $f(x) = e^{-1/x}$ for $x > 0$. Then $f(x) \cdot x^{-N} \to 0$ as $x \to 0$, so $f(x)$ is asymptotic to the series that has all zero coefficients. It follows that $f(x)$ can be added to another function $g(x)$ with non-trivial asymptotic expansion to generate a new function $h(x) = f(x) + g(x)$, which has the same non-trivial asymptotic expansion as $g(x)$. Repeated application of this argument can, of course, generate an infinite collection of functions which do not agree on their exact value anywhere, but which all share the same asymptotic expansion. Thus, the conjecture that a perturbative expansion is asymptotic to an exact structure does not uniquely specify what structure that happens to be.

Recall, however, that in chapter four we saw that there is a condition stronger than that the series is asymptotic to the function, but weaker than that the series is convergent, that does determine the function uniquely. This is the strong asymptotic condition. A function $f(z)$, which is analytic in a sector of the complex plane,

$$S = \{z \mid 0 < |z| < B; |\arg z| < \frac{1}{2}\pi + \epsilon\},$$

is said to obey the *strong asymptotic condition* and have $\sum_{n=0}^{\infty} a_n z^n$ as its *strong asymptotic series* if there exist C and σ such that,

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \leq C\sigma^{N+1}(N+1)!|z|^{N+1}, \tag{5.2}$$

for all N and $z \in S$.[19] Strong asymptoticity, like asymptoticity, is a condition that requires the differences between the exact value of the function and its series representation be appropriately small for every order of perturbation theory. This condition is important because if $\sum_{n=0}^{\infty} a_n z^n$ is a strong asymptotic series for analytic functions $f(z)$ and $g(z)$ in a sector S, it follows that $f(z) = g(z)$ for all $z \in S$. The strong asymptotic series of a function uniquely determines the function, just as a convergent series does. If one knows the strong asymptotic series of a function, then the function can be uniquely reconstructed by Borel summation. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ is a strong asymptotic series for $f(z)$ in the sector S, as defined above. The Borel transform,

$$g(z') = \sum_{n=0}^{\infty} \frac{a_n}{n!} z'^n, \tag{5.3}$$

can be used to produce the unique reconstruction of $f(z)$ from its strong asymptotic series, because for all z such that $|z| < B$ and $|\arg z| < \epsilon$,

$$f(z) = \int_0^{\infty} g(z'z) e^{-z'} dz'. \tag{5.4}$$

A series is Borel summable if and only if it is strongly asymptotic to a function. When a series is Borel summable in this manner, the series uniquely determines the function just as in the case of convergent expansions.

These results are of central importance for assessing the extent to which structure specification is possible using perturbation theory. When perturbation theory is convergent, the perturbative expansion provides what is necessary for exact structure specification. The dis-

[19]Discussion of the origin of the motivation for this bound can be found in (Reed and Simon, 1978).

cussion of this section shows that when perturbation theory diverges, this question is more subtle. If the series resulting from the perturbative characterization of a model satisfies the strong asymptotic condition, the series suffices to exactly specify the structure of the theory. One is naturally led to wonder whether all instances of empirically successful truncations of divergent perturbation theory, including the calculation of a_e, can be explained using this fact. If the strong asymptotic condition is not satisfied, the series by itself does not suffice to pin down the exact structure of the theory. If this is the case for the expansion for a_e, it is not at all clear what the truncated series tells us about the exact model underwriting this empirically successful calculation.

5.3 STATE-SPACE SEMANTICS

State-space semantics, a view due to Beth and van Fraassen, exemplifies the standard account of how mathematical structure supports the meaning of physical claims.[20] This section investigates whether or not state-space semantics adequately captures the truth conditions for models that are characterized perturbatively. I show that the answer is negative in the case of models that are not Borel-summable. It should be noted though that this is not intended as a critique of only state-space semantics. My aim is to make plausible that a similar problem faces all approaches to interpretation that rely on a map from an exact mathematical structure to empirical content. The role of state-space semantics is simply to provide a concrete proposal in which exact and perturbative models can be directly compared.

The semantics consists of three ingredients; a state-space, a collection of elementary physical statements, and a satisfaction function. Many physical theories introduce an abstract

[20] The view is introduced and discussed in (Beth, 1960; van Fraassen, 1967, 1970; Arntzenius, 1991). Debates concerning the semantics of scientific theories in general have largely been supplanted by debates about the interpretation of particular theories, but as I noted, van Fraassen's state-space semantics view exemplifies Ruetsche's characterization of the standard approach to interpretation.

mathematical state-space, S, to represent physical quantities in space and time. Models of the theory are trajectories in the state-space that exactly satisfy the syntactic expression of the dynamical equations of the theory. The elementary physical statements are a collection of sentences, $U(m, r, t)$, expressing the empirical content of the theory. Each $U(m, r, t)$ ascribes a physical magnitude, m, a definite value, r, at a specific time, t. The truth values of the $U(m, r, t)$ depend on the state of the actual physical system being represented. This dependence is captured by a map h, the satisfaction function, which connects the mathematical model to the expression of the empirical content of the theory. van Fraassen explains that: "For each elementary statement U there is a region $h(U)$ of the state-space [S] such that U is true if and only if the system's actual state is represented by an element of $h(U)$" (van Fraassen, 1970, p. 328).[21] The system's actual state is represented by U if measurement of the quantity m at time t would yield precisely the value r ascribed to it by U. In other words, if measurement would yield a value, $\bar{r}$, the satisfaction function is the one that assigns "true" to those $U(m; r; t)$ with $r = \bar{r}$ and "false" to the others. The region $h(U)$ is the collection of states that yield $r = \bar{r}$. Note that this abstract characterization of the semantics straightforwardly exemplifies both stages of Ruetsche's account of standard approaches to interpretation. Structure specification consists of fixing on the trajectories in state-space that exactly satisfy the relevant dynamical equations of motion, and the kinematic constraints. The characterization of the models as physical occurs through a rule connecting statements expressing the empirical content of the theory to states of the system represented in state-space.

Applying this abstract characterization of the semantics to a particular theory requires choosing the appropriate state-space, elementary physical statements, and satisfaction function for the theory in question. To represent the motions of masses interacting through forces, classical mechanics employs an abstract state-space which encodes the positions, $q_k = (q_x, q_y, q_z)$, and momenta, $p_k = (p_x, p_y, p_z)$, of each mass. The state-space is thus $\mathbb{R}^{6n}$

[21] I have changed van Fraassen's notation for the state-space to avoid confusion below.

for n the number of masses. The dynamics of the theory are defined by a Hamiltonian, H, which expresses the nature of the interaction between the masses. Models of the theory are trajectories in state-space that exactly satisfy the canonical equations of motion:

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \qquad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}. \tag{5.5}$$

All of the physical observables described by the theory are functions of the q_k and p_k. The elementary physical statements ascribe values to these observables. For example, one particular $U(m, r, t)$ is the sentence ascribing to a particular physical mass a particular position in space at a particular time. The satisfaction function is the one that assigns "true" to those states in state-space that yield that exact value at the appropriate time, and "false" to the other states.

This semantics adequately captures truth conditions for truncations of convergent perturbation theory. Consider a classical mechanical system of two masses interacting gravitationally. In this case an exact solution to the dynamical equations of the theory can be found, and state-space semantics can be straightforwardly applied. Moreover, perturbative treatment of such models yields convergent expansions whose limits agree by necessity with the exact solutions to the dynamical equations. This makes it possible to treat truncations of the series representation of the exact solutions as approximations to those solutions. For my purposes, a mathematical object O_1, can be treated as an approximation of another mathematical object O_2 if O_1 is appropriately close to O_2 in some sense that is appropriate for the context. It is a relation that obtains between two mathematical objects, independent of their physical interpretation.[22] When the objects in question are a function and a truncation of a

[22] There exists a vast literature on approximation and idealization which does not always carefully distinguish between the terms. (McMullin, 1985) provides an account of many of the relevant distinctions. The notion of approximation I employ agrees with the one articulated in (Norton, 2012). It also agrees with the one articulated by Frigg and Hartmann, who hold up truncations of series expansions as a paradigmatic example of approximation: "One mathematical item is an approximation of another one if it is close to it in some relevant sense. What this item is may vary. Sometimes we want to approximate one curve with another one. This happens when we expand a function into a power series and only keep the first two or three terms. . . . The salient point is that the issue of physical interpretation need not arise. Unlike Galilean idealization, which involves a distortion of a real system, approximation is a purely formal matter." (Frigg

series representation of that function, a measure of the relevant notion of closeness is given by $|f(x) - \sum_{n=0}^{N} a_n x^n|$. Truncations of perturbative expansions for solutions of two-body gravitational systems are approximations of the exact solutions that they converge to when summed to all orders. When they are interpreted as approximations, the empirical success of such truncations is accounted for by state-space semantics. The exactly true physical claim with respect to the semantics is the one generated by the mapping from the exact solution. The numerical value provided by the truncation of the perturbative expansion approximates this exact numerical value, and the empirical content of this truncation is underwritten by the exact solution.

More, as usual, is different. When an additional mass is added to the system being represented, an exact solution to the dynamical equations for all time is not currently available. Under certain conditions, the existence of an exact solution can be shown to exist, but its exact form has not been explicitly constructed.[23] This is an obstacle to the application of state-space semantics: in the absence of an explicitly constructed exact solution, it is not possible to explicitly construct a satisfaction function to connect the mathematical structure of the theory with its empirical content. This is not a mere mathematical curiosity, but rather a critical problem of physical practice. Three-body gravitational systems such as the Sun, Earth, and Moon, and the Sun, Saturn, and Jupiter played an important role in investigations of celestial mechanics in the 19th century. In the absence of explicitly constructed exact solutions, the perturbative treatment of the three-body problem took on a role of increased importance. It was the only method available to generate numerical information to compare with the available empirical data. The addition of the third mass can be treated as a perturbation of the exact solution for two gravitating bodies. Perturbative evaluation of the first few partial sums for the trajectories of the planets generated empirical values that

and Hartmann, 2012).

[23]This is an over-simplification. No closed form analytic solution is available, but Sundman was able to construct an exact solution in terms of convergent infinite series. For discussion of Sundman's results see (Barrow-Green, 2010; Saari, 1990; Siegel, 1941). I am grateful to Gordon Belot for bringing this work to my attention. Unfortunately, Sundman's solution requires far too many terms to generate sufficient accuracy to be of any use for comparison with empirical data.

matched closely with the available astronomical data. However, it also became clear that when summed to all orders, the expansions diverge.[24]

This case reveals a potentially serious problem for state-space semantics. The perturbative calculation yields infinity for the value of a measurable physical observable and so the semantics regards the theoretical value generated by the perturbative calculation as false. It does not have the resources to assign meaningfulness to the empirically adequate early partial sums. But the problem is actually worse than assigning "false" to the calculations that demonstrated the empirical adequacy of the theory: an argument can be made that it actually treats such calculations as meaningless. Recall that divergent perturbation series do not uniquely correspond to a function. Even when a series can be shown to be asymptotic to a function, there are an infinite collection of other functions to which the series is also asymptotic. This inhibits interpreting the perturbative expansion as an approximation, as the object to which it is supposed to be an approximation is indeterminate. Put simply, the problem for state-space semantics is that it renders the statements that express the empirical adequacy of the theory as at best false, and at worst meaningless. There is, however, an escape option available to the defender of state-space semantics that is very much worth considering. In particular, if the relevant perturbative expansions could be shown to satisfy the strong asymptotic condition and be uniquely associated with a function, the interpretation as an approximation would once again become viable.[25]

An example from non-relativistic quantum mechanics serves to illustrate how this escape option for the state-space semanticist might proceed. The state-space of non-relativistic quantum mechanics is a Hilbert space, and the dynamics of the theory is given by the

[24]The second volume of Poincaré's *The New Methods of Celestial Mechanics* is largely a collection of theorems establishing the divergence of all of the different perturbative methods in use at the time (Poincaré, 1993). In fact, this analysis led Poincaré to develop the notion of asymptoticity introduced in the previous section.

[25]Unfortunately, the strong asymptotic condition was not developed until long after the period during which celestial mechanics was developed, and as a consequence the Borel summability of the relevant expansions remain, to my knowledge, unchecked.

Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \tag{5.6}$$

where H is the Hamiltonian. The physically measurable quantities described by the theory are represented by Hermitian operators on the Hilbert space in which the wavefunction, Ψ, is defined. The $U(m, r, t)$ are thus sentences assigning a particular eigenvalue r to a particular Hermitian operator m at particular time. The satisfaction function $h(U)$ assigns "true" to the collection of states yielding the correct eigenvalue for the operator, and "false" to the others. As in the classical case, this semantics can be straightforwardly applied when exact solutions to the Schrödinger equation are available.

The difficulties for state-space semantics arise when exact solutions are not available and appeal must be made to perturbation theory to gain information about systems of interest. An interesting example is provided by the Stark effect which describes the splitting of atomic energy levels in the presence of an external electric field. It is described naturally by the standard formalism for non-relativistic quantum mechanics, with the Hamiltonian $H = -\Delta - Z/r + 2Fx_3$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \qquad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \tag{5.7}$$

Z is the atomic number, and F defines the strength of the field along the x_3 direction. The problem can be treated as a perturbation around the exact solution for the case where the external field is zero. Calculation of the first few orders of perturbation theory yields close agreement with the experimentally observed splittings in atomic spectra. The measurement of this effect played an important role in motivating the transition from the old quantum theory to its modern formulation.[26] But just as in the previous cases I have discussed in this chapter, it can be shown that when summed to all orders the perturbative expansions that provide empirical success at low orders ultimately diverge. State-space semantics thus

[26] The role of the Stark effect in the history of this transition is recounted in detail in (Duncan and Janssen, 2014).

seems to face the same problem in this case as it did in the case of three-body classical gravitating systems. That is, the statements that express the empirical success of the theory come out as either false or meaningless with respect to the semantics. What differentiates this case from the previous one is that perturbation theory for eigenvalues of the Stark effect Hamiltonian have been rigorously shown to satisfy the strong asymptotic condition. They can be Borel summed and uniquely associated with an exact value.[27] As noted above, this restores the viability of treating the empirically successful truncations as approximations to the exact eigenvalues. On this view, the existence of the exact eigenvalues underwrites the physical meaningfulness of the perturbative calculation.

The analysis of this section shows that state-space semantics is well-suited for capturing truth conditions for statements expressing the empirical content of a theory in some circumstances. Specifically, it is completely adequate when the structure specification of the theory conforms to the norms of structure specification insisted upon by Ruetsche's standard interpreter. However, when structure is specified perturbatively I have argued that the status of state-space semantics requires careful scrutiny. Analysis of the case of the Stark effect raises the hope that all perturbative expansions that generate empirical success satisfy the strong asymptotic condition. If this were the case, then the success resulting from the truncation of divergent series would always be underwritten by an exact model which the truncation approximates.

Unfortunately, this hope is dashed in the case of empirically adequate models of quantum field theory.[28] The cause of the failure of Borel summability in quantum field theory is the presence of singularities in the Borel transform due to instantons and renormalons.[29] These singularities result from Feynman graphs which produce growth in the amplitude

[27]Borel summability was originally shown in (Graffi and Grecchi, 1978). Additional discussion and references can be found in (Simon, 1982).

[28]For discussion see (Duncan, 2012). He explains that "...the property of Borel summability is an extremely fragile one, and one which we can hardly ever expect to be present in interesting relativistic field theories" (Duncan, 2012, p. 403).

[29]Detailed discussion of the significance of these singularities for the problem of structure specification is provided in (Miller, 2016c).

proportional to $n!$ for n the order of perturbation theory. The presence of such singularities on the positive real axis of the Borel transform inhibits the integration required to Borel sum a divergent asymptotic series.[30] The retreat to treating truncations of perturbative expansions as approximations is thus not available in empirically adequate models of quantum field theory. This means that the calculations that establish the empirical adequacy of quantum electrodynamics, such as the determination of a_e, cannot be treated as approximations. Accounting for their success requires departure from standard accounts of how mathematical structure underwrites physical meaning.

5.4 SEMANTICS FOR DIVERGENT PERTURBATION THEORY

The task of interpreting quantum field theory has been thought to be especially difficult because in its empirically adequate formulation, the norms of structure specification accepted by standard interpreters are not satisfied. In the previous chapter we saw that Halvorson claim that:

> ...philosophers of physics have taken their object of study to be theories, where theories correspond to mathematical objects (perhaps sets of models). But it is not so clear where "quantum field theory" can be located in the mathematical universe. In the absence of some sort of mathematically intelligible description of QFT, the philosopher of physics has two options: either find a new way to understand the task of interpretation, or remain silent about the interpretation of quantum field theory. (Halvorson and Muger, 2006, pp. 731-732)

In Halvorson's view, the only available mathematically intelligible characterization of the structure of quantum field theory is provided by axiomatization. The particular axiomatization that he prefers enjoys the property that its models satisfy the norms of structure specification accepted by standard interpreters. However, the models that generate empirical success when characterized perturbatively have not been shown to satisfy the axioms. If

[30]Preliminary investigation of simplified models of string theory suggest that the situation is similar in that context (Pasquetti and Schiappa, 2010).

one wants to interpret these perturbatively characterized empirically adequate models, one must depart from standard approaches to interpretation. The previous chapter also noted a similar sentiment expressed by Ruetsche:

> Given a theory T, ... we confront the exemplary interpretive question of how exactly to establish a correspondence between T's models and worlds possible according to T. That is, we confront that question *if* T is the sort of thing that has models. 'A collection of partially heuristic technical developments' isn't readily attributed a set of models about whose underlying ontology or principles of individuation philosophical questions immediately arise. This isn't to say that 'a collection of partially heuristic technical developments' is unworthy of philosophical attention. It is in itself a philosophically provocative circumstance that such a collection can enjoy stunning empirical success.[31] (Ruetsche, 2011, p. 102-103)

The remarks of both Halvorson and Ruetsche amount to insistence that structure specification meets the norms of standard interpreters.[32] *Pace* Halvorson and Ruetsche, in my view an adequate approach to interpretation *must* show how a theory's expression of physical content underwrites its stunning empirical success. The interpretation of physical theories is of philosophical interest because it informs our understanding of the actual world, not possible worlds that differ essentially from our own. That standard approaches to interpretation are unable to accommodate this success is a sign of their inadequacy for establishing how physical meaning attaches to mathematical structure.

There is a response available to defenders of standard interpretation: they can attribute the empirical success of perturbative field theory to the existence of an exact mathematical structure to which we simply do not currently have access. One might reasonably hope to gain access to such a structure in one of the following ways. First, it could turn out that additional work by constructive field theorists will show empirically adequate perturbatively characterized models to satisfy an existing axiomatization of quantum field theory. There exists evidence that they do not, but none of it is definitive and so this remains an open pos-

[31] The characterization of perturbative field theory as "a collection of partially heuristic technical developments" is a reference to a remark of Segal aimed at motivating axiomatic approaches to the theory (Segal, 1959, p. 341).

[32] A similar perspective has also been articulated by Fraser in (Fraser, 2009, 2011).

sibility. Alternatively, a new axiomatization of quantum field theory could be developed and empirically adequate models could be shown to exactly satisfy this new axiomatization.[33] For standard interpreters, showing that quantum field theory underwrites the greatest empirical successes ever achieved with a physical theory is predicated on the hope that such a structure will be discovered.

The argument of the previous section lays the foundation for a departure from standard interpretation according to which physical meaning can be associated with empirically successful perturbatively characterized models, without appealing to hoped for exact mathematical structures. Pursuing this alternative approach requires first recognizing that perturbatively characterized models are entirely mathematically intelligible, and are not merely "a collection of partially heuristic technical developments." The formalism for perturbative evaluation is subject to a number of well-known mathematical problems that have led to pessimism about its mathematical meaningfulness. These problems including infrared and ultraviolet divergences, which render individual orders of perturbation theory infinite. However, it is essential to note that for renormalizable theories like quantum electrodynamics, perturbative renormalization theory provides a conceptually clear and physically motivated procedure for rendering every individual order of perturbation theory finite.[34] Furthermore, the output of this procedure is a mathematically well-defined formal power series which one can attempt to sum.[35] The convergence properties of the series when summed to all orders are thus the only legitimate challenge to the structure specification of perturbatively characterized models.[36]

[33]Realizing either of these possibilities would likely amount to solving the Clay Mathematics Institute Millenium Problem on quantum Yang-Mills theory with a mass gap. The problem is stated in (Jaffe and Witten, 2000), and additional discussion can be found in (Douglas, 2004).

[34]For discussion of infrared divergences and how they are related to the problem of structure specification see (Earman and Fraser, 2006) and (Miller, 2016a). Ultraviolet divergences are addressed in (Wallace, 2006), (Wallace, 2011), (**?**), and (Fraser, 2016). The approach of Wallace, Williams, and Fraser resolves the problem of ultraviolet divergences by treating empirically adequate field theories as effective theories whose empirical content is confined to a limited scale. I develop an approach to resolving the problem for structure specification raised by ultraviolet divergences in a manner which is compatible with an effective field theory interpretation, but which does not require one, in (Miller, 2016b).

[35]If the reader doubts this claim I encourage them to consult (Wightman, 1986).

[36]Note that I am not claiming that all calculations found in the physics literature on quantum field theory

Accounting for how mathematical structure holds empirical content in face of large-order divergences that are not Borel summable requires departure from the norms of structure specification accepted by standard interpreters. In my view, divergent perturbation theory provides a novel connection between mathematical structure and physical meaning that should be incorporated into the semantics of a theory.[37] State-space semantics can be modified in a way that captures this novel connection. The new semantics retains the core aspects of state-space semantics. The role of the state-space and the elementary physical statements remain unchanged. Modifications are only required for the definition of the satisfaction function.

The first observation at the heart of the modification is that van Fraassen's articulation of state-space semantics relies on an unrealistic characterization of measurements.[38] Actual measurements never determine the exact value of a quantity. Agreement between theory and experiment instead takes the form of comparisons of a theoretically determined value, r, and an experimentally determined value, $\bar{r}$, with some associated measurement error ϵ_m. Theoretically determined values are empirically adequate not just when $r = \bar{r}$, but when $r \in (\bar{r} - \epsilon_m, \bar{r} + \epsilon_m)$. The first required modification to the semantics is to redefine the satisfaction function so that it returns "true" not just when $r = \bar{r}$, but also when $r \in (\bar{r} - \epsilon_m, \bar{r} + \epsilon_m)$.

The core issue for state-space semantics raised by the divergence of perturbation theory concerns the role of the theoretically determined value in the semantics. The nature of the modification to the measured value suggests that there is some freedom in the theoretical values that are compatible with measurement results. This freedom can be exploited to

are rigorous. The claim is that there is a subset of such calculations that suffice to run my argument that can be made rigorous.

[37]A number of other authors have recently called into question standard assumptions about how physical content accrues to mathematical structure (Curiel, 2011; Ruetsche, 2011; Wilson, 2006). The departures from standard interpretation that they argue for are not directly connected to the one I advocate in this chapter, however.

[38]I believe van Fraassen would agree. He explains that "The exact relation between $U(m; r; t)$ and the outcome of an actual experiment is the subject of an auxiliary theory of measurement, of which the notion of 'correspondence rule' gives only the shallowest characterization (van Fraassen, 1970, p. 329)."

provide a new semantics that adequately captures the empirical success of truncations of perturbative expansions that are convergent, strongly asymptotic, and even those that are not Borel summable. The details are different for each case, but the central idea is to provide a well-defined bound on the error introduced by truncating the expansion, ϵ_t. If that bound can be shown to be compatible with the freedom resulting from the presence of the measurement error, ϵ_m, the truncation error can be seamlessly incorporated into the semantics.

For convergent and strongly asymptotic series, finding such a principled bound is typically straightforward. In the case of convergent Taylor series for example, the relevant bounds can be provided by results related to Taylor's theorem. Consider the Taylor series of $f(x)$ about the point a, $\sum_{n=0}^{\infty} c_n(x-a)^n$ for $c_n = f^{(n)}/n!$. If $|f^{(n+1)}(x)| \leq M$ for all $x \in (a-r, a+r)$ with some $r > 0$, then the error from truncation at the Nth term, $f(x)-\sum_{n=0}^{N} c_n(x-a)^n \equiv R_N(x)$, is bounded by

$$|R_N(x)| \leq M\frac{|x-a|^{N+1}}{(N+1)!}. \tag{5.8}$$

The truncation error, ϵ_t, can be taken to be,

$$\epsilon_t = M\frac{|x-a|^{N+1}}{(N+1)!}. \tag{5.9}$$

Similarly, it has already been noted in the Section Two that for a series that is strongly asymptotic to a function in the sector S, there exist C and σ such that,

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \leq C\sigma^{N+1}(N+1)!|z|^{N+1}, \tag{5.10}$$

for all N and $z \in S$. In this case we can take,

$$\epsilon_t = C\sigma^{N+1}(N+1)!|z|^{N+1}. \tag{5.11}$$

There are also principled methods for assigning error bounds to truncations of divergent asymptotic expansions that are not Borel summable. Suppose $f(x) \sim \sum_{n=0}^{\infty} a_n x^n$. The optimal truncation rule dictates that the series should be truncated at the smallest term of the series, $N_{\min}$, so that the value of the truncation is $\sum_{n=0}^{N_{\min}} a_n x^n$. This rule is justified by the fact that the minimal error is typically achieved with this truncation. Moreover, the error is typically bounded by the magnitude of the value of the least term, $|f(x) - \sum_{n=0}^{N_{\min}} a_n x^n| \leq |a_{N_{\min}} x^{N_{\min}}|$. The caveat "typically" is important, and both properties must be rigorously confirmed for each individual case. But in those situations where they can be confirmed we can take,

$$\epsilon_t = a_{N_{\min}} x^{N_{\min}}. \tag{5.12}$$

These rigorously established bounds on the truncation error provide the critical ingredient to complete the modified semantics. The satisfaction function must be redefined so that in addition to accounting for the measurement error, it is also ensured that the truncation error is not greater than the freedom allowed in the theoretical value by the measurement error. This is the case when $(r - \epsilon_t, r + \epsilon_t) \subset (\bar{r} - \epsilon_m, \bar{r} + \epsilon_m)$. The satisfaction function thus needs to be redefined so that the $U(m, r, t)$ are true when $(r - \epsilon_t, r + \epsilon_t) \subset (\bar{r} - \epsilon_m, \bar{r} + \epsilon_m)$ and false otherwise. A new feature of the view is that the empirical content of a theory can come with precisely defined, but limited precision.

A number of remarks are in order. In the previous section I argued that truncations of convergent and strongly asymptotic expansions can be interpreted as approximations. When this approach is taken, all of the physical meaning derives from the exact model, and the truncation simply approximates the exact value. But, the proposal of this section makes it clear that it is not necessary to view such truncations as approximations. On the alternative view developed here, convergent and strongly asymptotic expansions are not interpreted as approximations. They are to be interpreted in just the same way as truncations of series that are not Borel summable for which the interpretation as an approximation is not available. It is these limited precision comparisons to experiments that convinced physicists of the truth of the theory in each of the cases introduced above. For this reason I believe this is the best way to capture the semantics of divergent perturbation theory, even in cases where Borel summability is retroactively established. Another important thing to note is that nothing about this proposed modification to state-space semantics involves a lack of mathematical rigor. When the bounds used for ϵ_t are established by the means discussed above, their existence is proved by the standards of rigor accepted by mathematicians.

The most important advantage of this proposal is that it allows us to treat the most empirically successful theories as providing meaningful claims about the world. Moreover, it does so by making minimal, and, in my view, natural modifications to an approach to interpretation that exemplifies the core commitments of standard interpretation. The modifications are minimal as all that is involved is a redefinition of the satisfaction function. They are natural in the sense that the modification that is made is directly motivated by the nature of the empirical support for the theory being interpreted. Rather than adhering to philosophical commitments about how theories ought to hold physical meaning in their mathematical expression, the account captures how they actually do hold empirical content in physical practice.

There are two counterintuitive consequences of the modified state-space semantics that I have developed in this section which must be weighed against the advantages just articulated. The first is explained in the following remark of Magnen and Rivasseau:

> Constructive field theory builds functions whose Taylor expansion is perturbative field theory. Any formal power series being asymptotic to infinitely many smooth functions, perturbative field theory alone does not give any well defined mathematical recipe to compute to arbitrary accuracy any physical number, so in a deep sense it is no theory at all. (Magnen and Rivasseau, 2008, p. 403)

Perturbative field theory is not a theory in the sense that it cannot be given a state-space semantics, or any interpretation that requires that there be physical facts of the matter about the exact value of physical observables. On my view, the empirical content of the theory simply has a limited, but rigorously established, precision. This is counterintuitive, but I think it is worth asking why we default to the assumption that there is a physical fact of the matter about the trillionth decimal place of physical observables, let alone the 10^{1000}th decimal place. I am not aware of any physical observable for which there are empirical grounds for commitment to this level of precision. Defenders of standard interpretation owe an explanation for their tacit commitment to this view, and it should be one based on something other than philosophical preconceptions about the structure of scientific theories.

The second counterintuitive consequence is that the truth values vary with experimental precision. The modified satisfaction function requires that $(r - \epsilon_t, r + \epsilon_t) \subset (\bar{r} - \epsilon_m, \bar{r} + \epsilon_m)$, and over the course of time, ϵ_m can be made smaller with improvements in experimental techniques. This points to another important difference between convergent and divergent expansions. For convergent expansions, ϵ_t can always be made arbitrarily small by summing additional orders of perturbation theory. This is not the case for divergent asymptotic expansions, whether they are strongly asymptotic to a function or not. Asymptoticity and strong asymptoticity only assure that the error induced by truncation at a particular order is small, and there is some order for which this error is minimized. If the measurement error is eventually reduced beyond this minimum truncation error, on my view the theory no longer expresses the empirical content of the theory sufficiently precisely to be confirmed by experiments. I am not aware of any case in actual physical practice where this possibility has been realized, but such a case would certainly warrant careful analysis. While both of these counterintuitive consequences merit further discussion, I believe the advantages of the modified semantics developed here outweigh any negative considerations they bring to bear on my view.

5.5 CONCLUSION

Consider once more Abel's question: why do truncations of perturbative expansions generate empirically adequate values? Defenders of standard interpretation naturally resort to treating truncations as approximations to exact models. However, this route is not available in cases where perturbation theory is not Borel summable. To account for the empirical success of quantum electrodynamics, for example, they have no choice but to wait for some new exact structure to underwrite the success of calculations of observables like a_e. If the norms of structure specification accepted by standard interpreters are to be met, this is the only option.

I have offered an alternative approach to answering Abel's question. On my view, perturbation theory presents a genuinely novel connection between mathematical structure and physical meaning. By incorporating this connection directly into the semantics for physical theories, we can meaningfully account for the empirical successes of quantum field theory. Rather than hoped-for structure, I have advocated that we look to the methodologies used in physical practice. In one sense this is conservative. The modification to state-space semantics that I advocate is minimal in the sense that it preserves most of the features of the view as expressed by van Fraassen. It is natural in the sense that the modifications to state-space semantics are motivated by the nature of the empirical support for the theory in question. But in another sense it is radical. It requires that we accept that expressions of the empirical content of physical theories can have in principle limits on their precision, and that the truth values of statements expressing that empirical content might vary with the precision of measurements. In my view, the benefits of having a firm sense of how theories as we actually have them make contact with the world far outweigh the luxury of rigidly maintaining philosophical commitments about the structure of theories.

6.0 EXACT MODELS AND AMBIGUOUS STRUCTURE

The previous chapter provided a concrete proposal concerning how physical meaning can be associated with the mathematical structure of perturbative field theory. In this chapter I identify one additional puzzling feature of the structure of perturbative field theory which has led some to question whether or not quantum field theory really describes an ontology based on quantum fields. In particular, I show that the syntax of the theory does not unambiguously delimit its possible structural realizations. It is well known that field operators cannot be defined at points, but this leaves open the question of what kind of mathematical object can possibly be used to represent quantum fields. The Wightman axioms provide one answer by taking the fields to be represented by operator-valued distributions. This answer, however, turns out not to be unique. There are other viable options, and the nature of the empirical evidence for the theory does not decide between the different structural realizations of the field operator syntax. I call this the problem of ambiguous structure. I suggest a resolution to this problem based on the nature of the empirical evidence for those models that do make contact with the world. I argue that this resolution clarifies the role that exact models play in underwriting the physical meaningfulness of empirically adequate models, and provides a sense in which quantum field theory can be understood to be a theory about quantum fields after all.

6.1 INTRODUCTION

I have argued that meaningful empirical content can be extracted from perturbatively characterized empirically adequate models of quantum field theory. This raises the question of how we should understand the role of axiomatic characterizations of the theory. We have seen that one attractive feature of the axiomatic approach to the theory is that it provides structures that fit naturally with standard approaches to interpretation. However, empirically adequate models have not been shown to satisfy the axioms and so one might reasonably wonder if I am advocating that we abandon the axiomatic approach entirely. In fact, I think that to do so would be a serious mistake. Axiomatic treatments provide important information about why perturbative field theory works as well as it does, but they do so in different way than is assumed in the philosophy of quantum field theory literature focused on axiomatic approaches.

A reasonable place to look for an account of how to understand the role of axiomatization in specifying the content of quantum field theories is in the work of the mathematical physicists that use the axiomatic approach. For example, at the outset of one of the seminal textbooks on axiomatic and constructive field theory, Bogolubov, Logunov, and Todorov remark that:

> It is widely believed that axiomatization is a kind of polishing, which is applied to an area of science after it has been, for all practical purposes, completed. This is not true, even in pure mathematics. Admittedly, the modern axiomatization of arithmetic and Euclidean geometry marked the completion of these disciplines (although at the same time it stimulated a new science – mathematical logic, or metamathematics). For most areas of contemporary mathematics, however, such as functional analysis, axiomatization is a fundamental method of exploration, a starting point (of course, the system of axioms may be modified as the subject develops.) In theoretical physics, since the time of Newton, the axiomatic method has served not only for the systematization of results previously obtained, but also in the discovery of new results. (Bogolubov et al., 1975, p. 1)

They go on to note that the principle motivation for the axiomatization of quantum field theory was to give an unambiguous articulation of the core principles of empirically successful perturbative field theories. This attitude can be seen, for example, they claim that:

> The original problem of axiomatic field theory, which has not yet been fully solved, was to pick out and to formulate unambiguously the more trustworthy features of the formal apparatus associated with the Lagrangian or Hamiltonian formalism. (Bogolubov et al., 1975, pp. 7-8)

A similar reaction to the limitations of the axiomatic approach is articulated in Horuzhy's book on algebraic quantum field theory where he claims that:

> The whole complex of the results of Chapter 1 is still too poor for the needs of describing concrete models and processes of interaction of elementary particles. It lacks a wide range of notions and properties inherent in quantum field systems .... As a result, the Haag-Araki or Haag-Kastler theory, like other axiomatic approaches, provides an essentially incomplete formulation of quantum field theory and should not be considered as a self-contained 'axiomatic theory' in the strict sense of axiomatic theories in mathematics. It is rather a starting ground, a base set of firmly established facts, which still needs to be expanded and complemented (obviously, on some other principles, not purely axiomatic any more). (Horuzhy, 1990, p. 121)

Both Bogolubov, Logunov, and Todorov, and Horuzhy suggest that the axiomatization of the theory functions as a starting point which stands in need of supplementation with additional information. If this is the proper role for axiomatic articulations of the theory, then it seems like we would be mistaken to press the models of the theory into the service of functioning as the basis for interpretation. To do so is to use those models for a purpose other than the one for which they were designed.

In this chapter I develop an account of the role of axiomatic articulations of the theory according to which their purpose really is to break an ambiguity in the perturbative formalism as Bogolubov, Logunov, and Todorov claim. In the second section I identify an ambiguity associated with standard field operator syntax. This gives rise to what I call the problem of ambiguous structure. In the third section I consider two standard approaches to breaking structural ambiguities that are common in the philosophy of quantum field theory literature and show that they are unfit for resolving the ambiguity of the field operator syntax. The fourth section provides my resolution to the problem and the final section contains concluding remarks.

6.2 THE PROBLEM OF AMBIGUOUS STRUCTURE

According to standard approaches to interpretation, physical meaning derives from the existence of a map from the mathematical structure of the theory to the world. As a result, standard approaches require a structurally unambiguous characterization of the models of the theory. This is necessary in order for the models to serve as the domain of the map from the theory to the world. In this section I argue that the syntax of perturbative field theory does not unambiguously delimit its models. This demonstrates an additional reason for the inapplicability of standard interpretation to perturbative field theory.

A quantum field is typically presented as an association of a quantum mechanical operator to each point in Minkowski space, $\{\phi(x_i) \mid x_i \in \mathcal{M}\}$. The operators are intended to act on a Hilbert space, $\mathcal{H}$. The physical content of a quantum field theory is encoded in a collection of quantities defined in terms of the field operators, namely the vacuum expectation values of products of field operators $\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle$. From these quantities it is possible to calculate the probability of a transition between any arbitrary quantum state and any other arbitrary quantum state.

One of the earliest results in axiomatic field theory showed that the idea that field operators can defined for every spacetime point cannot be exactly correct. In fact, by assuming other principles critical to the perturbative formalism, Wightman showed that there is no point in Minkowski space for which $\phi(x_i)$ is a bounded operator valued function (Wightman, 1964). This makes it impossible to give a straightforward reading to the syntax for quantum fields and the vacuum expectation values, $\langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle$, defined in terms of those fields. The initially intended structural realization of the field operator syntax is demonstrably untenable. This has lead interpreters to question the physical meaningfulness of claims about quantum fields.[1]

Wightman found a way to modify the basic content of the theory so that the field operator syntax does have a concrete structural realization. By using Schwartz's theory of distributions, a mathematical development that he judged to be "providential" for the development of quantum field theory, he showed that the $\phi(x_i)$ have a concrete structural realization when their values are smeared over small regions of spacetime (Wightman, 1981). His idea was to treat the $\phi(x_i)$ as operator valued distributions:

$$\phi(f) = \int f(x)\phi(x)\, d^4x. \tag{6.1}$$

By modifying the basic objects in this way, the vacuum expectation values are reconceived as a collection of Wightman distributions:

$$W_N(f) = \int f(x_1, \dots, x_n)\langle 0|\phi(x_1)\phi(x_2)\dots\phi(x_n)|0\rangle\, d^4x_1 \cdots d^4x_n, \tag{6.2}$$

where f denotes an arbitrary test function. These became the basic objects of the Wightman axiomatization of quantum field theory. Once this modification is made to the perturbative formalism it is possible to construct exact Wightman distributions for some simple models in reduced spacetime dimension or without interactions. These models show the Wightman axioms to be consistent as they admit a class of models that exactly satisfy their syntactic expression.

Empirically interesting models have not, however, been shown to be exact models of the Wightman axioms. That the expansions for empirically adequate models do not seem to satisfy the Wightman axioms has led to pessimism about their mathematical meaningfulness. This pessimism is, however, premature. There are spaces of generalized functions other than the space of Schwartz distributions that Wightman initially employed, and one can

[1] Claims of this type can be found in both (Arntzenius, 2003) and (Halvorson and Muger, 2006).

reasonably hope that they contain exact structures that underwrite the meaningfulness of perturbative field theory. Different classes of generalized functions are defined by what class of test functions they are required to be integrable against. For different classes of test functions, one can define the space of tempered distributions, ultradistributions, Fourier hyperfunctions, or Solovev distributions. And in fact, axiomatizations of quantum field theory based on these spaces have been developed.[2] In each of the axiomatizations, the fields are realized as structurally distinct objects. The axiomatizations that capture these alternative objects amount to what count, on either the syntactic or semantic view, as distinct theories. One can reasonably hope that the field operator syntax of empirically successful field theories works because it is picking out an exact structure in one of these spaces.

Recall that the perturbative treatment of empirically adequate models gives rise to large-order divergent expansions. The only non-conspiratorial explanation of the agreement of the low-orders of perturbation theory with experiment, despite its eventual divergence, is that perturbation theory is asymptotic to some exact structure. This raises the question: in what space are those exact structures defined? It is an intriguing fact that the low-order perturbative data that generates the empirical success of the theory does not seem to resolve this question. To see this note that:

> Until recently, the principal source of information about quantum field theory lay in the renormalized perturbation series for Lagrangean field theories with polynomial interactions. These are formal power series in some coupling constant, and it is an elementary consequence of the polynomial character of the interaction that the terms of the series for the basic quantities of the theories [the vacuum expectation values] are tempered distributions in the space–time variables. Needless to say, this does not imply that the solutions to which the the formal power series are asymptotic, supposing they exist, are also tempered distributions. (Wightman, 1981, p. 774)

The syntax of perturbative field theory is ambiguous between different exact structural realizations of the content of the theory, and the empirical support for the theory does not

[2](Wightman, 1956, 1977; Wightman and Garding, 1965; Wightman, 1986, 1996; Moschella and Strocchi, 1992; Nagamachi and Bruning, 2003; Nagamachi and Mugibayashi, 1977, 1976c,b,a; Constantinescu and Thalheimer, 1979; Solovev, 1995; Schmidt, 1997; Wightman, 1981)

decide between these different structural realizations. This is the *problem of ambiguous structure.*

6.3 BREAKING STRUCTURAL AMBIGUITY

Structural ambiguity has arisen before in attempts to interpret particular theories. In fact, several forms of structural ambiguity distinct from the ambiguity introduced in the previous section have already presented themselves in efforts to interpret quantum field theory. In particular, the presence of gauge freedom[3] and unitarily inequivalent representations of the canonical commutation relations[4] can be thought of as providing instances of ambiguous structure in quantum field theory. In the face of these problems, interpreters have developed two standard approaches to resolving structural ambiguity. One might reasonably wonder if one of these solutions can be applied to the ambiguity introduced in the previous section, thus preserving the core commitments of standard approaches to interpretation.

According to the first method, there is one unique common structure shared between the alternative structures. If this common structure can be identified, then it can serve as a candidate for mapping onto the world. This strategy has been adopted explicitly in theories with gauge freedom which is often viewed as a descriptive redundancy in the structure of the theory. Earman exemplifies commitment to this approach in his discussion of gauge freedom and the Higgs mechanism by claiming that in a theory with gauge freedom, only the gauge invariant observables are acceptable candidates for mapping onto the world.[5]

The second method reifies one of the structures and adopts it as the unique correct type of structure which is a candidate for mapping onto the world. As a consequence of the infinite

[3]An overview can be found in (Earman, 2002).

[4]See (Ruetsche, 2011, Chapter 9 and 10) for a philosophical discussion.

[5]In particular, he claims that "...a genuine property like mass cannot be gained by eating descriptive fluff, which is just what gauge is. Philosophers of science should be asking the Nozick question: What is the objective (i.e., gauge invariant) structure of the world corresponding to the gauge theory presented in the Higgs mechanism" (Earman, 2004).

number of degrees of freedom in models of quantum field theory, quantization does not yield one unique particle concept. Rather, there are unitarily inequivalent representations of the canonical commutation relations that each generate incommensurable particle notions. If one is committed to the second method for resolving ambiguity then they will favor one representation of the commutation relations, and hence one particle notion, as the unique correct one. The others, according to this perspective, can be dismissed as spurious.

Both of these solutions seem as though they may be of use in resolving the structural ambiguity introduced in the previous section, however, this initial promise turns out to be illusory. I argued that the syntax $\phi(x)$ is ambiguous between a number of structural realizations of the domain of quantum fields. It seems possible that one of the structures is the correct one, or that they all share a common structure, but there are straightforward reasons to think that these approaches do not help to resolve the problem of ambiguous structure for quantum field theory. This is because if there is an articulation of the theory according to which empirically adequate models are exact, nothing about the empirical support for quantum field theory indicates which articulation that is. Similarly, even if there is a common structure shared between all of the structures that the empirical data is compatible with, this common structure has not been explicitly articulated and so it has played no role in the empirical success of the theory. Nothing about having access to the invariant or one true structure has been important for generating the empirical success of quantum field theory. I am not advancing an argument against the in principle realizability of either of these resolutions to the problem, as I doubt such an argument exists. Rather, I am emphasizing that if the physical meaning of the terms in our best theories is supposed to reflect the extent of our empirical support for the structures we take to represent the physical world, then neither of these approaches serve to underwrite the physical meaning of terms like "quantum field" in empirically adequate models.

6.4 SEMANTIC CONTENT ACCRUES TO CLASSES OF STRUCTURES

Standard approaches to interpretation presume there is a collection of models that stand in the exact satisfaction relation to the syntactic expression of the theory. I have argued that standard interpretation of quantum field theory is inhibited by the fact that the syntax leading to empirically adequate models of quantum field theory is ambiguous. The syntax can be given concrete mathematical meaning in more than one way, each of which delimits its own class of models. The empirical support for the theory, however, only warrants commitment to the claim that empirically confirmed perturbative expansions are asymptotic to one of the models in this structurally heterogeneous class. Interpretation, therefore, should not proceed only from an axiomatic articulation of the theory and the exact models of those axioms alone. Doing so makes it impossible to use the information about the world provided by the divergent, asymptotic expansions of empirically adequate models. I also have argued that to do so is to press the axiomatic approach into a service for which is was not designed, as indicated in the introduction.

The analysis up to this point suggests a natural resolution to the problem of assigning physical meaning to the ambiguous field operator syntax. This resolution shares with standard approaches the commitment to identifying those elements in the formalism that are candidates for corresponding to stable entities in the world. It differs, however, in that one must allow semantic content to accrue to classes of structures delimited by an appropriate constraint. In the case of the term "quantum field," the appropriate constraint is agreement on low-order perturbative data. One can proceed by finding a representative of the class of structures: the representative in the case in question can be taken to be the Wightman axioms. Having such a representative ensures that the class includes non-trivial models. The other members of the class, $\{T_2\}$, $\{T_3\}$, $\ldots$, are the other axiom systems that treat fields as elements of spaces of more singular generalized functions. Each collection of axioms delimit distinct classes of mathematical models which enter into the exact satisfaction relation with the syntactic expression of the axioms.

Once the relevant empirical constraint is identified, those parts of the theory that have analogs in each of its structural realizations must be identified. In each of the structural realizations of quantum field theory there are natural analogs of the fields, $\phi_{\{T_1\}}$, $\phi_{\{T_2\}}$, ... The central difference between this method and standard approaches is that truth conditions for physical statements are given to the syntax that generates empirically adequate expansions. While the syntax for quantum fields is ambiguous between different structural realizations, this syntax remains a perfectly respectable candidate to which physical meaning can accrue. This is accomplished by treating the physical meaning of "quantum field" as insensitive to exact mathematical structure in a precisely constrained way. Once the analogs are identified, they are lumped together, $[\phi_{\{T_i\}}]$, into a class of structural realizations of the term "quantum field" that are all taken to have the same physical meaning. On this view, the way to understand the success of ambiguous quantum field syntax is that it is ambiguous between structural representations of the phenomena that should have been thought of all along as referring to the same stable entity in the world. I believe that this practice is already implicitly adopted when physicists refer to quantum fields in the case of empirically adequate models like quantum electrodynamics. These claims can be construed as physically meaningful if they are understood in the way that I have reconstructed them here.

This procedure has features common with one of the approaches discussed in the last section. It may seem, in particular, that the idea is to find in some sense a common structure. However, this is a misreading of what I am proposing. My proposal does not require the expression, or even the existence, of a syntax that admits a class of models that share the common structure. Rather, the idea is that physical meaning attaches uniformly to analogous terms in all structures that satisfy the relevant empirical constraint, some of which have been explicitly articulated, and some of which have not. It is also worth reiterating that the asymptotic nature of perturbation theory for empirically adequate models is conjectured, not proved. The conjecture suggests that there is an element of the delimited space that is the exact model. While this is conjectural, it is principled, and it accurately reflects the nature of the empirical support for the theory.

Different axiomatic articulations of quantum field theory have distinct structural realizations. According to the canonical approach to interpretation, the differences between these structurally distinct realizations typically amount to physically significant differences. On the approach outlined here, however, these differences should be treated as physically insignificant. In this sense the account provides a strict criterion for when we have good reason to think that differences in mathematical structure amount to physical differences in the world. It will have the consequence that interpretive conclusions established in some particular axiomatic formalism may come out counting as physically insignificant. Given that the proposal provides a way for physical meaning to attach to the actual world, I think that this is a consequence that may well be worth this cost.

6.5 CONCLUSION

Standard approaches to the interpretation of theories are overly rigid in their demands on the mathematical structure in which physical meaning is supposed to inhere. If one insists on this approach, it has the consequence of restricting interpretations to exact models, and thus does not allow for interpretation of empirically adequate models of quantum field theory. I have provided an alternative approach to delimiting structure which is less rigid and can allow for attributions of physical meaning in empirically adequate models. I have shown that this alternative approach is necessary if the physical meaning of terms like "quantum field" are to derive from the empirical success of our theories. Mathematics has the capacity to make sharper distinctions than those required to represent physical phenomena. This has important consequences for attributions of physical meaning to mathematical structures that have not been appreciated by extant approaches to interpretation. What I have argued in this paper is that the meaning of the theoretical term "quantum field" can profitably be understood as having this feature. The success of the syntax $\phi(x)$ results from physical meaning that is insensitive to differences between classes of distinct mathematical structures, all of which agree on their low-order perturbative data. In this way, axiomatic articulations of the content of the theory function as a guide for the attribution of physical meaning to the syntax in which empirically adequate models are cast. The solution shows how ambiguous mathematical syntax, when interpreted correctly, can have genuine physical meaning. It thus allows the project of interpreting quantum field theory to inform our understanding of the meaning of claims about the fundamental constituents of matter in the actual world.

7.0 CONCLUDING REMARKS

Recall that perturbative quantum field theory attempts to evaluate expressions for n-point functions that take the form of a perturbative expansion in powers of the coupling constant:

$$\begin{aligned}\langle\Omega|T(\phi(x_1)\phi(x_2)\ldots\phi(x_n))|\Omega\rangle &= \sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\int\langle 0|T(\phi(x_1)\phi(x_2)\ldots\phi(x_n)) \\ &\quad \cdot H(y_1)H(y_2)\ldots H(y_j)|0\rangle d^4y_1\ldots d^4y_j.\end{aligned} \tag{7.1}$$

This procedure fails to provide what is required to generate a standard interpretation of the theory because the right hand side of the equation contains ultraviolet, infrared, and large-order divergences. I have argued that these obstacles to the interpretation can be overcome. The meaningful empirical content of the theory can be isolated from the ultraviolet and infrared divergences through the use of regularization and renormalization schemes, and the restriction to infrared-safe observables. The large-order divergences can be overcome by developing principled justifications for truncating the expansion at a particular order of perturbation theory.

The result of these procedures is a rigorous characterization of the empirical content of empirically adequate models of quantum field theory. Establishing perturbative field theory as a sufficiently structurally well-specified foundation for philosophical interpretation has significant consequences for debates in the philosophy of quantum field theory. Many of the questions about the interpretation of quantum field theory have previously only been carried out in contexts that have not been shown to admit empirically adequate models. These debates include the interpretation of gauge symmetry, unitarily inequivalent representations of the canonical commutation relations, and spontaneous symmetry breaking.

As one example of how debates look different when considered from the perspective of perturbatively characterized empirically adequate models, consider whether the theory underwrites commitment to a particle ontology. A number of authors have argued that particle interpretations are untenable, thus rendering opaque the connection between quantum field theory and "particle physics". These arguments are based on axiomatic treatments of the theory. The question that has resulted from the debate is the question of how much localization is required to appropriately recover particle phenomenology. The axiomatic results do not, however, provide adequate resources to answer this question. If one consults the perturbative field theory literature on how to identify the long distance stable structures that are measured in particle physics experiments, one can find methodologically motivated guidance. The restriction to infrared safe observables introduces the energy resolution of the detector as an appropriate measure of localization. Attention to perturbative field theory can show what ought to count as an adequate resolution to the problem.

In addition to providing a novel perspective on existing debates in the philosophy of quantum field theory, the perspective articulated in this dissertation opens up core problems of physical practice to philosophical analysis. Some such problems include, the proton spin crisis, the significance of Landau poles, the triviality of the Higgs field, and the nature of quark confinement. Establishing perturbative field theory as a sufficiently well-defined basis for philosophical interpretation brings these issues into the realm of those that can be subject to analysis by philosophers. Of course, as I have indicated, taking perturbative field theory requires departure from standard interpretation and the assumption that exact models alone underwrite the meaningfulness of the theory. But the advantage of bring philosophical attention to quantum field theory back into contact with the actual world seems to indicate that the alternative proposal advocated here at the very least warrants further consideration.

BIBLIOGRAPHY

Abel, N. and B. Holmboe (1839). *Oeuvres complètes de N.H. Abel, mathématicien, avec des notes et développements: rédigées par ordre du Roi*, Volume 2. C. Gröndahl.

Albert, D. Z. (1996). Elementary quantum metaphysics. In J. T. Cushing, A. Fine, and S. Goldstein (Eds.), *Bohmian Mechanics and Quantum Theory: An Appraisal*, Volume 184 of *Boston Studies in the Philosophy of Science*, pp. 277–284. Springer Netherlands.

Aoyama, T., M. Hayakawa, T. Kinoshita, and M. Nio (2012). Quantum electrodynamics calculation of lepton anomalous magnetic moments: Numerical approach to the perturbation theory of QED. *Progress of Theoretical and Experimental Physics 2012*, 01A107.

Arntzenius, F. (1991). State-Spaces and Meaning Relations Among Predicates. *Topoi 10*(1), 35–42.

Arntzenius, F. (2003). Is quantum mechanics pointless? *Philosophy of Science 70*(5), 1447–1457.

Bain, J. (2000, December). Against Particle/Field Duality: Asymptotic Particle States And Interpolating Fields In Interacting Qft (Or: Who's Afraid Of Haag's Theorem?). *Erkenntnis 53*(3), 375–406.

Bain, J. (2013). Pragmatists and purists on cpt invariance in relativistic quantum field theories.

Barrow-Green, J. (2010). The dramatic episode of sundman. *Historia Mathematica 37*(2), 164 – 203.

Barton, G. (1963). *Introduction to advanced field theory.* Interscience Publishers.

Batterman, R. W. (1997). 'Into a Mist': Asymptotic theories on a caustic. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 28*(3), 395–413.

Batterman, R. W. (2002). *The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence.* Oxford University Press.

Batterman, R. W. (2007). On the Specialness of Special Functions (The Nonrandom Effusions of the Divine Mathematician). *The British Journal for the Philosophy of Science 58*(2), 263–286.

Beth, E. W. (1960). Semantics of Physical Theories. *Synthese 12*(2-3), 172–175.

Bloch, F. and A. Nordsieck (1937). Note on the Radiation Field of the electron. *Phys.Rev. 52*, 54–59.

Bogolubov, N., A. Logunov, and I. Todorov (1975). *Introduction to axiomatic quantum field theory.* W. A. Benjamin Inc.

Borchers, H. and J. Yngvason (1992). From quantum fields to local von Neumann algebras. *Reviews in Mathematical Physics 4*, 15–47.

Borchers, H. J. and J. Yngvason (1975). On the algebra of field operators. The weak commutant and integral decompositions of states. *Communications in Mathematical Physics 42*(3), 231–252.

Borchers, H.-J. and J. Yngvason (1990). Positivity of Wightman functionals and the existence of local nets. *Communications in Mathematical Physics 127*(3), 607–615.

Brunetti, R. and K. Fredenhagen (2000). Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Commun. Math. Phys. 208*, 623–661.

Buchholz, D. (1990). On quantum fields that generate local algebras. *Journal of Mathematical Physics 31*(8), 18–39.

Butterfield, J. (2015). Review: Anthony Duncan. The Conceptual Framework of Quantum Field Theory. *Philosophy of Science 82*(2), 326–330.

Collins, J. C. (1984). *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*. Cambridge University Press.

Constantinescu, F. and W. Thalheimer (1979). Ultradistributions and quantum fields: Fourier-Laplace transforms and boundary values of analytic functions. *Reports on Mathematical Physics 16*, 167–180.

Curiel, E. (2011). On the propriety of physical theories as a basis for their semantics. http://philsci-archive.pitt.edu/8702/.

da Costa, N. C. A. and S. French (2003). *Science and Partial Truth a Unitary Approach to Models and Scientific Reasoning*. Oxford University Press.

Douglas, M. (2004). Report on the status of the Yang-Mills millenium prize problem. http://www.claymath.org/sites/default/files/ym2.pdf.

Driessler, W., S. Summers, and E. Wichmann (1986). On the connection between quantum fields and von Neumann algebras of local operators. *Communications in Mathematical Physics 105*, 49–84.

Duncan, A. (2012). *The Conceptual Framework of Quantum Field Theory*. Oxford University Press.

Duncan, A. and M. Janssen (2014). The trouble with orbits: The Stark effect in the old and the new quantum theory. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 48*, 68–83.

Dyson, F. J. (1952). Divergence of Perturbation Theory in Quantum Electrodynamics. *Physical Review 85*(4), 631–632.

Earman, J. (2002). Gauge Matters. *Proceedings of the Philosophy of Science Association 2002*(3), 209–20.

Earman, J. (2004). Laws, Symmetry, and Symmetry Breaking: Invariance, Conservation Principles, and Objectivity. *Philosophy of Science 71*(5), 1227–1241.

Earman, J. and D. Fraser (2006). Haag's Theorem and its Implications for the Foundations of Quantum Field Theory. *Erkenntnis 64*(3), 305–344.

Eckmann, J. P., J. Magnen, and R. Sénéor (1975, December). Decay properties and borel summability for the Schwinger functions in $P(\phi)_2$ theories. *Communications in Mathematical Physics 39*(4), 251–271.

Epstein, H. and V. Glaser (1973). The Role of locality in perturbation theory. *Annales Poincare Phys. Theor. A19*, 211–295.

Feldman, J., J. Magnen, V. Rivasseau, and R. Seneor (1986). A Renormalizable Field Theory: The Massive Gross-Neveu Model in Two-dimensions. *Commun. Math. Phys. 103*, 67–103.

Feynman, R. (1985). *QED: The Strange Theory of Light and Matter*. Penguin Books.

Fraser, D. (2009). Quantum Field Theory : Underdetermination, Inconsistency, and Idealization. *Philosophy of Science 76*(4), 536–567.

Fraser, D. (2011). How to take particle physics seriously: A further defence of axiomatic quantum field theory. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 42*(2), 126–135.

Fraser, J. D. (2016). *What is Quantum Field Theory?: Idealisation, Explanation, and Realism in High Energy Physics*. Ph. D. thesis, University of Leeds.

Fredenhagen, K. and J. Hertel (1981). Local algebras of observables and pointlike localized fields. *Communications in Mathematical Physics 80*(4), 555–561.

French, S. (2014). *The Structure of the World: Metaphysics and Representation.* OUP Oxford.

Frigg, R. and S. Hartmann (2012). Models in science. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Fall 2012 ed.).

Gastmans, R. and R. Meuldermans (1973). Dimensional regularization of the infrared problem. *Nucl.Phys. B63*, 277–284.

Glymour, C. (1977). The epistemology of geometry. *Noûs 11*(3), 227–251.

Glymour, C. (2013). Theoretical Equivalence and the Semantic View of Theories. *Philosophy of Science 80*(2), 286–297.

Graffi, S. and V. Grecchi (1978). Resonances in Stark effect and perturbation theory. *Communications in Mathematical Physics 62*(1), 83–96.

Greiner, W. and J. Reinhardt (1996). *Field quantization.* Springer.

Haag, R. (1955). On quantum field theories. *Det Kongelige Danske Videnskabernes Selskab, Matematisk-fysiske Meddelelser 29*(12), 1–37.

Haag, R. (1958). Quantum Field Theories with Composite Particles and Asymptotic Conditions. *Physical Review 112*(2), 669–673.

Haag, R. (1992a). *Local quantum physics: Fields, particles, algebras.*

Haag, R. (1992b). *Local quantum physics: Fields, particles, algebras.* Springer-Verlag.

Hall, D. and A. Wightman (1957). A theorem on invariant analytic functions with applications to relativistic quantum field theory. *Det Kongelige Danske Videnskabernes Selskab, Matematisk-fysiske Meddelelser 31*(5), 1–41.

Halvorson, H. (2012). What Scientific Theories Could Not Be. *Philosophy of Science 79*(2), 183–206.

Halvorson, H. (2013). The Semantic View, If Plausible, Is Syntactic. *Philosophy of Science 80*(3), 475–478.

Halvorson, H. and M. Muger (2006). Algebraic quantum field theory. In J. Butterfield and J. Earman (Eds.), *Handbook of the Philosophy of Science*, pp. 731–922.

Hanneke, D., S. Fogwell, and G. Gabrielse (2008). New measurement of the electron magnetic moment and the fine structure constant. *Physical Review Letters 100*, 120801.

Heisenberg, W. and W. Pauli (1929). Zur Quantendynamik der Wellenfelder. *Zeitschrift fur Physik 56*, 1–61.

Helling, R. C. (2012). How I Learned to Stop Worrying and Love QFT.

Horuzhy, S. (1990). *Introduction to Algebraic Quantum Field Theory.* Springer.

Huggett, N. and R. Weingard (1994). Interpretations of quantum field theory.

Hurst, C. a. (1952). An example of a divergent perturbation expansion in field theory. *Mathematical Proceedings of the Cambridge Philosophical Society 48*(04), 625–639.

Iagolnitzer, D. (1993). *Scattering in quantum field theories: the axiomatic and constructive approaches.* Princeton University Press.

Jaffe, A. and E. Witten (2000). Quantum Yang-Mills theory.

Kinoshita, T. (1962). Mass singularities of Feynman amplitudes. *J.Math.Phys. 3*, 650–677.

Kinoshita, T. (1990). Theory of the anomalous magnetic moment of the electron - numerical approach. *Adv.Ser.Direct.High Energy Phys. 7*, 218–321.

Kinoshita, T. (2014). Tenth-order QED contribution to the electron g-2 and high precision test of quantum electrodynamics. *International Journal of Modern Physics A29*, 1430003.

Kusch, P. and H. M. Foley (1947). Precision measurement of the ratio of the atomic 'g values' in the $^2p_{\frac{3}{2}}$ and $^2p_{\frac{1}{2}}$ states of gallium. *Physical Review 72*, 1256–1257.

Lee, T. and M. Nauenberg (1964). Degenerate Systems and Mass Singularities. *Phys.Rev. 133*, B1549–B1562.

Lupher, T. (2005). Who Proved Haag's Theorem? *International Journal of Theoretical Physics 44*(11), 1995–2005.

Magnen, J. and V. Rivasseau (2008). Constructive ϕ^4 Field Theory without Tears. *Annales Henri Poincaré 9*, 403–424.

Magnen, J. and R. Sénéor (1977). Phase space cell expansion and Borel summability for the Euclidean ϕ_3^4 theory. *Communications in Mathematical Physics 276*(1977), 237–276.

Marciano, W. and A. Sirlin (1975). Dimensional Regularization of Infrared Divergences. *Nucl.Phys. B88*, 86.

McMullin, E. (1985). Galilean idealization. *Studies in History and Philosophy of Science Part A 16*(3), 247–273.

Miller, M. E. (2016a). Haag's theorem, apparent inconsistency, and the empirical adequacy of quantum field theory. *The British Journal for the Philosophy of Science (forthcoming)*.

Miller, M. E. (2016b). Mathematical structure and emprical content. *(In Preparation)*.

Miller, M. E. (2016c). On the common structure of perturbative and axiomatic field theory in Borel summable models. *(In Preparation)*.

Moschella, U. and F. Strocchi (1992). The Choice of test functions in gauge quantum field theories. *Lett. Math. Phys. 24*, 103–113.

Muta, T. (2010). Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories, (3rd ed.).

Nagamachi, S. and E. Bruning (2003). Hyperfunction quantum field theory: Localized fields without localized test functions. *Lett. Math. Phys. 63*, 141–155.

Nagamachi, S. and N. Mugibayashi (1976a). Hyperfunction Quantum Field Theory. *Commun. Math. Phys. 46*, 119–134.

Nagamachi, S. and N. Mugibayashi (1976b). Hyperfunction Quantum Field Theory. 2. Euclidean Green's Functions. Part A. *Commun. Math. Phys. 49*, 257–275.

Nagamachi, S. and N. Mugibayashi (1976c). Theory of Fourier Hyperfunctions and Its Applications to Quantum Field Theory. *Lett. Math. Phys. 1*, 259–264.

Nagamachi, S. and N. Mugibayashi (1977). Quantum Field Theory in Terms of Fourier Hyperfunctions. *Publ. Res. Inst. Math. Sci. Kyoto 12*, 309–341.

Ney, A. and D. Z. Albert (2013). *The Wave Function: Essays in the Metaphysics of Quantum Mechanics*. Oxford University Press.

Norton, J. D. (2012). Approximation and idealization: Why the difference matters. *Philosophy of Science 79*(2), 207–232.

Osterwalder, K. and R. Schrader (1973, June). Axioms for Euclidean Green's functions. *Communications in Mathematical Physics 31*(2), 83–112.

Osterwalder, K. and R. Schrader (1975, October). Axioms for Euclidean Green's functions II. *Communications in Mathematical Physics 42*(3), 281–305.

Pasquetti, S. and R. Schiappa (2010). Borel and Stokes nonperturbative phenomena in topological string theory and c = 1 matrix models. *Annales Henri Poincaré 11*(3), 351–431.

Peskin, M. E. and D. V. Schroeder (1995). *An Introduction to quantum field theory*. Addison-Wesley.

Petermann, A. (1953). Une série divergente en représentation intermédiaire. *Helvetica Physica Acta 26*, 731–742.

Poincaré, H. (1993). *New Methods of Celestial Mechanics: Approximations by series*. History of modern physics and astronomy.

Prange, D. (1999). Epstein-Glaser renormalization and differential renormalization. *J. Phys. A32*, 2225–2238.

Reed, M. and B. Simon (1975). *Methods of Modern Mathematical Physics Volume 2: Fourier Analysis, Self-Adjointness*. Academic Press.

Reed, M. and B. Simon (1978). *Methods of Modern Mathematical Physics Volume 4: Analysis of Operators*. Academic Press.

Renouard, P. (1979). Analyticité et sommabilité de borel des fonctions de schwinger du modèle de yukawa en dimension $d = 2$. ii. la limite adiabatique. *Annales de l'I.H.P. Physique théorique 31*(3), 235–318.

Roskies, R., M. J. Levine, and E. Remiddi (1990). Analytic evaluation of sixth order contributions to the electron's g factor. *Advanced Series on Directions in High Energy Physics 7*, 162–217.

Ruelle, D. (1962). On the Asymptotic Condition in Quantum Field Theory. *Helvetica Physica Acta 35*, 147–163.

Ruetsche, L. (2011). *Interpreting Quantum Theories*. Oxford University Press.

Saari, D. G. (1990, February). A visit to the newtonian n-body problem via elementary complex variables. *Am. Math. Monthly 97*(2), 105–119.

Scharf, G. (1989). *Finite quantum electrodynamics.*

Scharf, G. (2001). *Quantum gauge theories: A true ghost story.*

Schmidt, A. U. (1997). Euclidean reconstruction in quantum field theory: Between tempered distributions and Fourier hyperfunctions. In *2nd Frankfurt - Cracow Operator Theory Seminar Crakow, Poland, April 7-15, 1997.*

Schweber, S. S. (1994). *QED and the men who made it: Dyson, Feynman, Schwinger, and Tomonaga.* Princeton University Press.

Schwinger, J. (1948a). On quantum-electrodynamics and the magnetic moment of the electron. *Physical Review 73*, 416–417.

Schwinger, J. (1949). On the classical radiation of accelerated electrons. *Physical Review 75*, 1912–1925.

Schwinger, J. S. (1948b). Quantum electrodynamics. I A covariant formulation. *Phys.Rev. 74*, 1439.

Segal, I. E. (1959). The Mathematical Meaning of Operationalism in Quantum Mechanics. In L. Henkin, P. Suppes, and A. Tarski (Eds.), *The Axiomatic Method With Special Reference to Geometry and Physics*, pp. 341–352. Amsterdam: North Holland Publishing Company.

Siegel, C. L. (1941). On the modern development of celestial mechanics. *The American Mathematical Monthly 48*(7), 430–435.

Simon, B. (1974). *The $P(\phi)_2$ Euclidean (quantum) Field Theory.* Princeton University Press.

Simon, B. (1982). Large orders and summability of eigenvalue perturbation theory: A mathematical overview. *International Journal of Quantum Chemistry 21*(1), 3–25.

Sklar, L. (2000). *Theory and truth: philosophical critique within foundational science.* Oxford University Press.

Smeenk, C. and W. Myrvold (2011). Introduction: philosophy of quantum field theory. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 42*(2), 77–80.

Solovev, M. A. (1995). Towards a generalized distribution formalism for gauge quantum fields. *Lett. Math. Phys. 33*, 49–59.

Streater, R. F. and A. S. Wightman (1964). *PCT, Spin and Statistics, and All That*. Princeton University Press.

Strocchi, F. (2013). *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*. Oxford University Press.

Summers, S. (1987). From algebras of local observables to quantum fields: Generalized H-bounds. *Helvetica Physica Acta 60*, 1004–1023.

Summers, S. J. (2012). A Perspective on Constructive Quantum Field Introduction. *Arxiv preprint math-ph/1203.3991*, 1–59.

Suppe, F. (1974). *The Structure of Scientific Theories*. Urbana,University of Illinois Press.

Teller, P. (1995). *An interpretive introduction to quantum field theory*. Princeton University Press.

Thirring, W. (1953). On the divergence of perturbation theory for quantized fields. *Helvetica Physica Acta 26*, 33–52.

van Fraassen, B. C. (1967). Meaning Relations Among Predicates. *Noûs 1*(2), 161–179.

van Fraassen, B. C. (1970). On the Extension of Beth's Semantics of Physical Theories. *Philosophy of Science 37*(3), 325–339.

van Fraassen, B. C. (2014). One or Two Gentle Remarks About Hans Halvorson's Critique of the Semantic View. *Philosophy of Science 81*(2), 276–283.

Wallace, D. (2006). In defence of naiveté: The conceptual status of Lagrangian quantum field theory. *Synthese 151*(1), 33–80.

Wallace, D. (2011). Taking particle physics seriously: A critique of the algebraic approach to quantum field theory. *Studies in History and Philosophy of Science Part B 42*(2), 116–125.

Weinberg, S. (1960). High-energy behavior in quantum field theory. *Phys. Rev. 118*, 838–849.

Wightman, A. (1956). Quantum Field Theory in Terms of Vacuum Expectation Values. *Physical Review 101*(2), 860–866.

Wightman, A. (1977). Should We Believe in Quantum Field Theory? In *The Whys of Subnuclear Physics*, pp. 983–1025. Springer.

Wightman, A. (1981). The choice of test functions in quantum field theory. *Journal of Mathematical Analysis and Applications B*.

Wightman, A. (1986). Some lessons of renormalization theory. In *Copenhagen1985 Proceedings, The lessons of quantum theory*, pp. 201–226.

Wightman, A. and L. Garding (1965). Fields as operator valued distributions in relativistic quantum theory. *Arkiv för Fysik 28*(13), 129–184.

Wightman, A. S. (1964). La théorie quantique locale et la théorie quantique des champs. *Annales de l'institut Henri Poincaré (A) Physique théorique 1*(4), 403–420.

Wightman, A. S. (1996). How it was learned that quantized fields are operator-valued distributions. *Fortsch. Phys. 44*, 143–178.

Williams, P. (2016). *Philosophy of Science Made Effective: Realism, Reduction, and the Renormalization Group in Quantum Field Theory.* Ph. D. thesis, Columbia University.

Wilson, M. (2006). *Wandering significance: An essay on conceptual behavior*. Oxford University Press.

Wollenberg, M. (1985). On the relations between quantum fields and local algebras of observables. *Reports on Mathematical Physics 22*(3), 409–417.

Wollenberg, M. (1986). Quantum Fields as Pointlike Localized Objects, II. *Mathematische Nachrichten 128*(1), 287–298.

Wollenberg, M. (1988). The existence of quantum fields for local nets of algebras of observables. *Journal of Mathematical Physics 29*(9), 2106.

Yngvason, J. (1989). Bounded and Unbounded Realizations of Locality. In B. Simon, A. Truman, and I. M. Davies (Eds.), *IXth International Congress on Mathematical Physics*, Bristol and New York, pp. 475–478. Adam Hilger.

Zichichi, A. (1979). *The Whys of Subnuclear Physics. Proceedings of the 1977 International School of Subnuclear Physics.*