IMPROVING STUDENTS' PROPORTIONAL REASONING ABILITY IN THE CONTEXT OF ALGEBRA I

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Submitted to the Graduate Faculty of

the School of Education in partial fulfillment

of the requirements for the degree of

Doctorate of Education

University of Pittsburgh

2017

UNIVERSITY OF PITTSBURGH

SCHOOL OF EDUCATION

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University of Pittsburgh, 2017

Students enter Algebra 1 with varied proportional reasoning ability and understanding. The ability to reason proportionally can impact understanding in Algebra and beyond. Despite its characterization as "the capstone of elementary school arithmetic" and "the cornerstone of all that is to follow" (Lesh, Post, & Behr, 1998, p. 97), research has shown that even the most basic understanding of proportionality continues to significantly challenge students.

To determine the extent of proportional reasoning ability upon entry into Algebra 1 and whether proportional reasoning ability can be improved in the context of Algebra, students in five sections of honors and academic Algebra 1 classes were evaluated using pre-assessments, followed by engagement in algebraic tasks rooted in proportional reasoning, and then evaluated using post-assessments. The results of this study indicate that students varied ability to reason proportionally correlates with their placement in honors or academic algebra, and that proportional reasoning ability for all students can be improved in the context of Algebra.

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ACKNOWLEDGEMENTS

There are several people that I would like to thank for their support and guidance as I completed this project.

Dr. Peg Smith: Thank you for your continuous support throughout the last eighteen years. I would not be the educator I am today without your guidance. I am forever grateful to call you my professor, my mentor, and my friend.

My students: Thank you for your enthusiastic engagement in the process of learning. Your learning is my learning, and my gratitude for each of you is endless.

Ethan, Noah, Evelyn, and Winnie: You are my greatest accomplishments and I thank you for giving mommy the time to work on this project and for your flexibility for the last three years. I hope that I am a role model that you are proud to call your mom. I love you all to the moon and back.

My parents: Thank you for going above and beyond your duties as Grammy and Pappy to support me during these past few years. I couldn’t have done this without you stepping in to help with taxi duties, sports schedules, and dance classes. I am eternally indebted to you both.

My beloved husband, Jared: Thank you for your unending support of every crazy endeavor that I chose to jump into, especially this one. Thank you for giving me quiet, uninterrupted time to work and endless amounts of coffee! I cannot begin to describe what your support and love mean to me.

1.0 INTRODUCTION

In its final report *Foundations for Success*, the National Mathematics Advisory Panel (2008) put out a call to action for the nation to make substantial changes to the educational system in the United States to "strengthen the American people in this central area of learning [mathematics]" because "success matters to the nation at large" and "to individual students and their families…[to] open doors and create opportunities" (p.xi). According to the Panel, students who are successful in Algebra II are more than twice as likely to graduate from high school when compared to those students that did not have the same preparedness. To improve mathematics in the United States, the Panel details six important elements that stress "to put first things first" (p. xiii). The Panel further emphasizes that the critical foundations of Algebra include fluency with fractions. More specifically, the Panel states that "by the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate and extend this work to proportionality" (p.20). The Panel further recommends that "the curriculum should allow for sufficient time to ensure acquisition of conceptual and procedural knowledge of fractions (including decimals and percents) and proportional reasoning" (p. 29).

The National Council of Teacher of Mathematics (NCTM) (1989) describes proportional reasoning as "one of the hallmarks of the middle grades program" (p. 213). Diane Briars, president of NCTM (2014-2016), explains that the adoption of the Common Core State Standards for Mathematics (CCSSM) has ushered in even higher expectations for developing the conceptual understanding of proportional reasoning in the middle grades in order to facilitate development of the concepts of slope and rate of change in Algebra (Heitin, 2015). Despite its characterization as “the capstone of elementary school arithmetic” and “the cornerstone of all that is to follow” (Lesh, Post, & Behr, 1998, p. 97), proportional reasoning continues to significantly challenge students with even the most basic understanding of proportionality. The 1996 National Assessment of Educational Progress makes this very clear: only about one-fifth of eighth graders and one- fourth of twelfth graders correctly answered proportional reasoning items (Reese, 1997). This poor performance is consistent with results from a multitude of research, and results on other national and international assessments, and little has changed in the last twenty years as evidenced by recent NAEP results available. On the 2013 NAEP assessment, less than half of eighth grade students correctly answered a basic proportional reasoning question when given a ratio and asked to find a missing value in a second ratio (National Center for Educational Statistics, 2013). It is evident that students struggle with proportional reasoning and possess a limited understanding of proportional relationships.

A limited conceptual understanding of proportional reasoning can hinder student progress in algebraic topics. One of the key beginning topics of study in Algebra is linearity, and key to the definition of linearity is the concept of slope. The slope of the line is defined mathematically as the ratio between the vertical and horizontal changes for any two points on a line (or change in output, y, to change in input, x). A common misconception is that slope is a difference (or additive) relationship since it refers to the vertical and horizontal differences when, in fact, slope is the ***ratio*** of the differences and requires a deeper understanding of proportional reasoning (Greenes, Chang, & Ben-Chaim, 2007).

2.0 A REVIEW OF THE LITERATURE

To further understand the impact of student understanding of proportional reasoning on their success within the Algebra 1 content, the literature was examined with the following questions in mind:

1. *What misconceptions do students have about proportional reasoning?*
2. *What is the connection between proportional reasoning and algebraic understanding?*
3. *How does the literature guide the next steps for research?*

In this review, the importance of proportional reasoning is identified as a key factor in the development of mathematical understanding and this chapter outlines what research reveals about misconceptions in proportional reasoning. This information will inform a summary of what implications this understanding has for student performance in algebra.

2.1 THE IMPORTANCE OF PROPORTIONAL REASONING AND TEACHER UNDERSTANDING

In 2006, The National Council of Teachers of Mathematics (NCTM) published *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* to help those engaging in curriculum development to focus on specific areas of emphasis within each grade level. Specifically, the focal points for grade 7 call for student understanding and applying proportionality, with follow-up on this understanding in grade eight's focal point of analyzing and interpreting linear functions. Moreover, NCTM specifically notes that students should recognize a proportion as a special case of a linear function (NCTM, 2006). Realizing the importance of understanding particular conceptual bands within mathematics, NCTM responded by developing the *Developing Essential Understandings* series of resource books to enrich and extend teachers' knowledge of particular concepts in order to support students learning. In *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics: Grade 6-8*, there is emphasis on the importance of teachers' knowledge to refute common misconceptions about ratios as well as the ability to illustrate connections among fractions and ratios within proportional reasoning.

NCTM suggests that it is important for teachers to understand the relationship between proportional reasoning and linearity in order to guide students in choosing high cognitive demand tasks, facilitating meaningful mathematical classroom discourse, as well as evaluating student work to support and challenge students' mathematical thinking. Through this increased understanding, teachers can become better equipped to maintain cognitive demand throughout the implementation of chosen tasks (as depicted by the Mathematical Tasks Framework) (Stein & Smith, 1998). NCTM's commitment to developing resources to support teachers' understanding of proportional reasoning speaks to the importance of depth of knowledge of proportionality that teachers need to further students' understanding of mathematics.

Ma (1999) describes teachers who make connections among ideas and have a deep understanding of the broader mathematics curriculum as having a profound understanding of fundamental mathematics (PUFM). Teachers with PUFM in regards to proportional reasoning can distinguish between proportional and non-proportional relationships, as well as understand that proportional relationships are multiplicative and a special case of a linear function (Cramer, Post & Currier, 1993), and can perform procedures efficiently while understanding the rationale for algorithmic procedural processes (Ma, 1999). For example, a teacher with PUFM of proportional reasoning is able to use cross multiplication to solve for a missing value, but also understands the cross-products method's relationship to the direct variation relationship and the algebraic equation that represents the situation.

Cramer et al. (1993) assert that teachers at all levels of mathematics education have difficulty with understanding proportional reasoning and actually exhibit many of the same misconceptions held by students. For example, an algebra teacher with a limited understanding of proportional reasoning might not see the connection between the slope of a line and the proportional relationship between the vertical change and the horizontal change (Lobato et al., 2010). Without understanding these types of connections, teachers of Algebra do not possess PUFM and they may contribute to the students' perpetuation of misconceptions regarding proportional reasoning. NCTM seeks to counteract this perpetuation of misconception with the publication of books to assist teachers in the essential understandings of a variety of topics, including proportional reasoning.

2.2 STUDENT UNDERSTANDING OF PROPORTIONAL REASONING

Common errors and misconceptions have been characterized in a variety of studies. The type of mistakes that students make is affected by both the context of the problem situation and the numerical content of the task (Karplus et al, 1983), but most students' mistakes tend to share similar characteristics. Karplus et al. (1983) administered four proportional reasoning tasks involving different units of measure for each problem, as well as the difference of discrete quantities versus continuous quantities, to 116 sixth graders and 137 eighth graders in ethnically diverse middle school. The results of this study showed that there was no significant age or sex effect for the qualitative features of proportional reasoning and of incorrect strategies, but rather the context, numerical content of the problem, and the immediately preceding task greatly affected the frequency of the type of comparison and the strategy used. The students who could successfully complete the tasks utilized integral ratios within or between relationships, emphasizing Karplus et al's previous research (Karplus, 1981; Karplus et al, 1979) that found that an approach to teaching proportional reasoning that uses equivalent fractions and cross-multiplication is not an effective method for developing an understanding of proportional reasoning. The authors of this study also reveal that the additive strategy was not as prevalent as they had once believed, though other research is not consistent with this finding and further refines the incorrect additive strategy.

2.2.1 Diagnostic assessment of student understanding

A diagnostic assessment of the proportional reasoning ability of 212 students, ranging in age from 10 to 13, revealed distinct levels within which common mistakes fall (Misailidou & Williams, 2003). In this study, Misailidou and Williams (2003) used Rasch analyses of test results from 303 students, as well as data from student interviews, to calibrate an assessment of diagnostic proportional reasoning items. Assessment items were developed from the related literature, and a variety of problems were used in regards to numerical structure and context, with the overarching criteria that the items must be consistent with the curriculum and high potential for diagnostic purposes. Because the authors' review of literature suggests that models can promote student understanding of proportional reasoning, two versions of the tool were created: one without models and one with models. For example, consider the two versions of the 'Books' Price' task as seen in Figure 2.1. While the text of both versions remains unchanged, the second task (Task B) in the figure provides a concrete image of both objects described in the text: the books and the money.

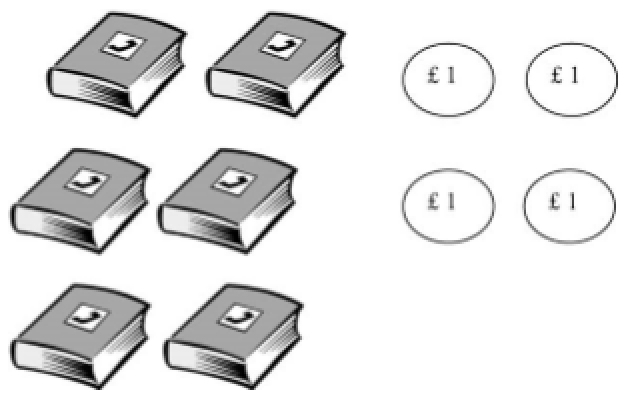
Task A	**6. 'Books' Price'** There is a sale at a bookstore. Every book in this sale costs exactly the same. Mary bought 6 books from the sale and paid 4 pounds. Rosy bought 24 books from the sale. How much did Rosy pay?
Task B	**6. 'Books' Price + Pictorial Representation'** There is a sale at a bookstore. Every book in this sale costs exactly the same. Mary bought 6 books from the sale and paid 4 pounds.  Rosy bought 24 books from the sale. How much did Rosy pay?

Figure 2-1 Task with and without model from Misaladou & Williams (2003)

The assessment without models contains 24 items, while the version with models uses the same items, but 13 items utilize a model in some way. Misailidou and Williams (2003) state the purpose of creating two versions was to "compare the difficulty of the parallel items for children and to spot differences in the strategies used in each mode." For each version, two scales were constructed to measure students' ratio attainment and the tendency for the use of the additive strategy. Others significant errors emerged as well including incorrect build-up, magical halving/doubling, constant sum, and incomplete reasoning. The emergence of these errors was subsequently investigated and then validated with structured clinical interviews with twenty students and small group interviews with 64 students. In these interviews, responses for those students who incorrectly used the additive strategy fell into three categories:

1. *Some students said they just "add" or "take away" the numbers to get the answers, without any further explanation.*
2. *Some students explain their addition process by explaining there are a certain number 'more' or because there is a 'certain difference."*
3. *Some students provide an explanation that evoked the concept of equality (i.e. "doing the same thing to both sets of numbers").*

Based on these three categories, the authors suggest that there must be some underlying conceptual structure that provides the students justification for using the addition strategy. A scale was then developed to assess a student's tendency for using the additive strategy, which upon further investigation reveals that in this sample of students "the 'ability' to make additive errors is, if anything, as stable or more stable and consistent than the 'ability' to reason proportionally."

From their work, Misailidou and Williams (2003) built a hierarchy of performance for proportional reasoning, with levels 0 and 4 implicitly defined. This is further developed into three levels with details regarding what types of questions students at a particular level are successful in completing and what types of common mistakes are prevalent in any particular level (see Table 2.1). The development of this assessment (see Appendix A) allows teachers and researchers to measure and diagnose students' proportional thinking and their additive tendency, as well as identify other significant errors made in proportional reasoning.

Table 2-1 Levels of pupils' proportional reasoning (Misailidou and Williams, 2003)

Level	Typical Performance	Typical Common Errors (Items referenced can be found on the Diagnostic Assessment in Appendix A.)
1	Students are typically successful in answering questions with familiar contexts and single digit numbers (easy numerical structure). Answers can be found mainly through scalar multiplication (by 2, 3 or halving).	• Characterized by 'incomplete reasoning error' • Magical halving/doubling is used on relatively easy items (e.g. 2 Onion Soup) • Use of additive strategies is predominant only on easy items (e.g. 1 Eels) • Incorrect build-up is not used unless attempting a very easy item that due to the context is not prone to additive errors (e.g. Fruits' Price)
2	Students can succeed in problems with familiar contexts where the answers can be found by: i. simple multiplication working on a scalar and functional ratio ii. taking an amount then half as much again working on the scalar and functional ratio In difficult contexts (i.e., Books Price (Appendix A)) the answer can be found with simple multiplication and the answers may be easy fractions.	• Characterized by the use of the additive strategy—most of the significant additive errors that are indicated by the diagnostic test are made at this level • Magical halving/doubling is used at this level to solve difficult items (e.g. 6 Onion Soup) • Constant sum method used at this level that provokes errors due to the context (e.g. 1 Paint or 2 Paint) • Incorrect build up may be used by the highest ability students at this level
3	Students at this level are successful with items that: i. Have a more difficult numerical structure and they need to work either on scalar or functional ratios ii.The context is unfamiliar (i.e. Mr. Short) and the answers can be more complex fractions.	• Characterized by the use of the incorrect application of the build-up method error on items of a difficult context (e.g. 1 Paint or 2 Paint with mixtures). This is the only error made by the higher ability students at this level. • Use of the additive strategy is predominant on the items that were identified as likely to provoke such errors

While the additive strategy is the most common mistake that students make related to proportionality (Inhelder & Piaget, 1958; Hart, 1984), there are other frequent incorrect methods including "the 'incorrect build up' method, the 'magical halving and magical doubling,' the 'constant sum,' and the 'incomplete method'" (Misailidou & Williams, 2003, p. 346), each of which are described in the following paragraphs.

2.2.1.1 The additive strategy

The additive strategy uses the difference between two of the terms in the ratio and then uses that same difference in the second pair of terms. For example, Figure 2.2 lists the "Fruits Price" problem from the diagnostic assessment. A student with a high tendency for the additive strategy may add up from 3 to 7 (an addition of 4) and then employ that added amount to the cost, incorrectly identifying the cost of 7 apples to be 94 pence.

> 5. 'Fruits' Price'
> At a fruit stand, 3 apples cost 90 pence. You want to buy 7 apples. How much will they cost?

Figure 2-2 Task from Misiladou & Williams (2003)

Misailidou & William (2003) found that within the erroneous use of the additive strategy, students use various reasoning in order to justify their strategies. Three common justifications are the "add" or "take away" justification, the use of the idea of equating, and an explanation that reasons "there are a certain number more." In Figure 2.2, a student might reason that since one must add 4 to 3 in order to get 7, an equal amount must be added to 90. This highlights an underlying misconception of equality in regards to proportionality.

2.2.1.2 Incorrect build-up

"Incorrect build up" method errors occurred when the quantities being compared were not multiples of one another. For example, in Figure 2.3, a student employing the incorrect build up strategy might incorrectly reason that since Sue has 3 cans of yellow paint and Jenny has 7, the relationship of (3 cans)·2+1 can be used on the red paint as well, thus incorrectly calculating the relationship as (7 cans)·2+1 = 15 cans of red paint.

> 7. '1 Paint'
> Sue and Jenny want to paint together. They want to use each exactly the same color.
> Sue uses 3 cans of yellow paint and 6 cans of red paint. Jenny uses 7 cans of yellow paint.
> How much red paint does Jenny need?

Figure 2-3 Task from Misailidou & Williams (2003)

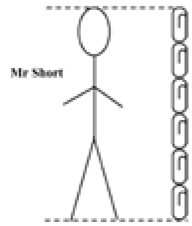
Misailidou and Williams (2003) hypothesize that this method may be a back-up method that students with relatively high proportional reasoning attainment use when dealing with more challenging problems and that this strategy is a blend of the additive and multiplicative strategies.

2.2.1.3 Magical halving and doubling

Magical halving, or magical doubling, happens when students use halving, or doubling, reasoning when it is not appropriate, while the additive strategy describes the strategy used when a student thinks that the sum of a pair should remain constant throughout the proportional relationship (Misailidou & Williams, 2003). For example, a student answered 12 paperclips to problem presented in Figure 2.4. The subsequent interview revealed that the student simply doubled the amount of paperclips and indicated that Mr. Tall was twice as tall as Mr. Short, though this relationship is not stated anywhere in the problem.

11. 'Mr. Short'

You can see the height of Mr. Short measured with paper clips.



Mr. Short has a friend Mr. Tall. When we measure their heights with matchsticks: Mr. Short's height is four matchsticks. Mr. Tall's height is six matchsticks. How many paper clips are needed for Mr. Tall's height?

Figure 2-4 Task to illustrate magical halving/doubling (Misailidou & Williams, 2003)

2.2.1.4 Incomplete reasoning

Incomplete reasoning was described by Karplus, et al (1983) as strategies that are illogical or incomplete, resulting in what they would deem the lowest level (Level 1) on their hierarchy of proportional reasoning strategies. For the Paint problem in Figure 2.3, a student with incomplete reasoning would justify that Jenny needs six cans of red paint because Sue has six cans of red paint and the girls want to have the same. Incomplete reasoning is characterized by both Karplus et al (1983) and Misailidou and Williams (2003) as a common strategy employed by students at the lowest level of proportional reasoning.

Misailidou and Williams (2003) define a hierarchy (Table 2.1) based on their findings and previous research (Hart, 1981; Hart, Brown, Kerslake, Kuchemann, & Ruddock, 1985).

Levels 0 and 4 are implicitly defined, with levels 1, 2, and 3 in between. At Level 1 in the hierarchy, students are successful in solving proportion problems that are in contexts that are familiar to them or with "friendly" numbers, but cannot move into unfamiliar contexts; whereas in Level 2, these students are successful with problems that can be solved with simple multiplication or taking half and then half as much again. Students at Level 3 in the hierarchy can solve problems where the numerical structure is more challenging and the context is not as familiar (Karplus, et al, 1983). Misailidou and Williams suggest that this hierarchy can provide a streamlined approach to the learning trajectory for proportions and help educators focus on and remedy common mistakes. Furthermore, Karplus, et al. (1983) criticize the approach to teaching proportional reasoning that uses equivalent fractions and cross multiplication and recommend using an approach that examines the relationships among variables and distinguishing between a constant ratio relationship and a constant difference relationship. This speaks to the aforementioned need regarding the choice of tasks and implementation that maintains the high cognitive demand necessary to allow students to grapple with problems and develop meaning for proportional relationships.

2.2.2 Curriculum and pedagogy to improve understanding

Curriculum and pedagogy that utilizes high demand tasks and mental sweat is supported by research. Ben-Chaim, Fey, Fitzgerald, Benedetto, and Miller (1998) compared traditional curricular materials and pedagogical methods of teaching middle grades mathematics with reform curriculum to determine the differences in conceptual understanding, computational skills, and problem solving strategies and successes for students in each type of curriculum. Traditional curriculum can be classified as having primarily low level tasks that require the use of learned procedures, while reform curriculum consists of high level tasks that require reasoning and mathematical thinking rather than a rote application of procedure. To do this, data was collected from two seventh grade groups, one using traditional curricula and one a reform curriculum--Connected Mathematics Project (CMP). The main goal of this particular research focused on describing the character and effectiveness of proportional reasoning within and between these two groups, but secondary interests included examining how students learn and assessing proportional reasoning ability. The test sample (those enrolled in classes using the CMP curriculum) consisted of eight seventh grade classes, taught by seven different teachers, from five different geographic areas around the United States; the control sample (those enrolled in classes using traditional mathematics curricula) consisted of six seventh grade classes, taught by six different teachers, from four different geographic areas. Each sample was tested on proportional reasoning through the use of three forms of an assessment that were randomly assigned to each classroom. Following the assessment, about one-fourth of the students were interviewed in order to get some perspective regarding their proportional reasoning understanding in both contextualized problems and with pure computation in proportionality. Students in both groups performed better on numerical comparisons with ratios (i.e., items where two complete ratios or rates are given and no numerical answer is needed but the rates or ratios are compared) than on missing value problems. However, CMP students correctly responded, including supporting work, to both types of questions more often than the traditional students. The same results were true when students were presented with a time-distance proportionality problem and a rate problem dealing with density. Though this study was a comparison of curricula rather than teaching practices, these results provide strong evidence that students afforded the opportunity to develop their own conceptual and procedural knowledge perform better than those students that are taught in a more teacher-centered classroom (Ben-Chaim, et al., 1998).

Further analysis of student work in order to understand students' proportional thinking additionally revealed different common strategies for working on proportional reasoning problems (Ben-Chaim, et al., 1998). Designated as Strategy 1, this method compared the ratio of different variables using a unit rate. This was used by students in both the traditional classrooms and the CMP classrooms, but more often by the CMP students (65% versus 24%). It is important to note that CMP students would not, by design of the curriculum, be taught any specific method for solving rate problems, so it can be interpreted that using unit rates is a natural development in the learning progression of students who develop strategies on their own. Strategy 2 compared ratios of the same variable using a scalar. Though only a few students in the reform group utilized this strategy, Ben-Chaim, et al. state that this illustrates a diversity of thought processes when students develop and apply proportional reasoning. In strategy 3, students compared by using common factors or multiples, again illustrating this diversity of problem solving techniques for proportion problems. Strategies 4 and 5 illustrate the use of the building up strategy, while strategy 6 examines the ratio of differences between the same variables. Both of these methods can lead to some common mistakes as mentioned by Misailidou and Williams (2003), such as incorrect buildup and incomplete reasoning. Strategy 7 involved responding to the numbers but not the context of the proportional reasoning problem. Student work samples employing strategy 7 appear, at first, to be nonsense, however, CMP students appear to be on the trajectory of mathematizing the problem to generalization through their ability to abandon the context. Strategy 8 erroneously ignores part of the data within the problem and only deals with one variable. The simplicity of this method makes it a common strategy for low-level proportional reasoning students and illustrates the importance of understanding that a proportion is a single entity apart from the two quantities that compose it (Lamon, 1993). Strategy 9 was seldom used and would also be characterized as at the lowest level of understanding. Here, students responded affectively rather than dealing with the numerical presentation of the proportion problem. For example, when asked to answer a better deal question, one student responded, "No, Gatorade tastes better," revealing no evidence for proportional understanding. CMP students were more likely to use the most efficient method of utilizing the unit rate than the traditional students, though they were not directly taught this method for efficient solving. Ben-Chaim, et al. (1998) concludes that curriculum that encourages student authority over construction of proportional reasoning understanding will help children discover the methods of optimal efficiency.

Though their study didn't focus solely on proportional reasoning, Reyes, Reyes, Lapan, Holliday, and Wasman (2003) found similar results when examining mathematics achievement of students in three different school districts using National Science Foundation (NSF) funded reform curricula. These students were engaged in either the aforementioned *CMP* or *MATH Thematics* and were compared to students in three other districts using a traditional mathematics curriculum. Achievement was measured using state standardized assessments. It was determined that students who had been using the standards-based (reform) curricula for at least two years reflected a higher level of achievement on the state assessments than students in other districts that were using different, traditional curricula: the students at the standards-based districts scored in the highest two achievement categories, whereas the students in the comparison traditional school districts scored in the bottom two achievement categories.

2.2.3 Connection to algebra

The National Mathematics Advisory Panel (2008) asserts that students who complete Algebra II are more prepared for success in college than peers that have an insufficient understanding of Algebra. Knowing and understanding algebra allows students to tackle difficult problem situations and reason abstractly. Being able to associate steepness of a graph to a ratio opens pathways to understanding other concepts in higher-level mathematics such as trigonometric ratios and geometry. In the 2005 National Assessment of Educational Progress, only 36% of eighth grade students correctly responded to a ratio problem associated with slope or steepness; these same eighth graders also correctly answered a problem involving calculating a rate correctly only 21% of the time. Cheng and Sabinin (2009) address the development of proportional reasoning in association with the steepness of a line and suggest that this approach is consistent with the historical development of algebra. It is often the misconception that the slope is merely a difference relationship, reinforced when students only learn how to find slope using the procedural formula $\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$. Students with a deeper conceptual understanding of proportional reasoning have no need for the slope formula (Lobato et al., 2010). Cheng and Sabinin (2009) administered a written survey containing one experimental task and nine other steepness questions to 194 fifth graders and 256 seventh graders in the Boston-area. The experimental task involved two spiders who shoot straight webs between a horizontal floor and two vertical walls, and students are asked to draw a web of equal steepness through a given point (see Figure 2.5).

9. Nid says that he can shoot a web exactly as steep as Ari's. Where does Nid need to aim his web for the two webs to be exactly the same steepness? Draw Nid's web on the picture below.

How did you decide?

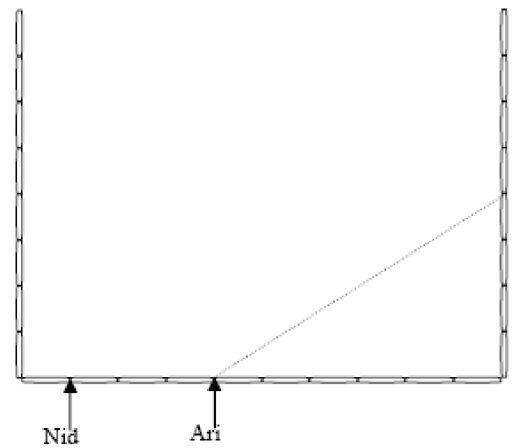


Figure 2-5 Experimental task (Cheng and Sabinin, 2009)

The students worked individually on the experimental task, and their responses were coded as correct, incorrect additive thinking, and other incorrect thinking. Nineteen seventh graders were subsequently interviewed in small groups. These interviews presented students with a problem similar to the one shown in Figure 2-5 and then allowed any student in the group to provide a first response. Following this first response, the other students in the group were able to react to the first responder's work. Interviewers used scripted prompts to ensure equal participation by all students and all interviews were videotaped and transcribed. Correct responses were coded into five categories based on the explanation provided: proportional, geometric, visual, none and other. A greater number of students in grade 7 than grade 5 answered the task correctly, yet a greater number of students in grade 7 incorrectly used additive reasoning than in grade 5. The small group discussion groups were able to sway some of the students to use a more visual or proportional representation to solve the experimental task, revealing some connections that students can make regarding conceptual connections between geometry and algebra specific to the concept of slope. A strong understanding of proportionality is important in developing an understanding of linearity as a relationship with a constant rate (ratio) of change.

Other research shows that students have a tendency to apply proportional reasoning when a situation does not describe a proportional relationship (Modestou & Gagatsis, 2007) and this tendency increases from grade to grade (Van Dooren, DeBock, Hessals, Janssens, & Verschaffel, 2005). The illusion of proportionality often misleads students to the improper application of proportional reasoning for linear problem situations that involve a starting value that is not zero. This illustrates a lack of deep understanding of proportionality. According to Stavy & Tirosh (2000), this is a result of the intuitive rule theory: students are using common intuition when misapplying proportionality. Modestou and Gagatsis (2007) found that incorrectly applying proportions was not due to lack of knowledge: even when given the volume formula students continue to use the linearity model yet were able to produce correct solutions within a particular context. In this study, 307 students in seventh and eighth grade enrolled at six different schools were administered assessments in two phases. The second phase was administered fifteen days after the first (Test A), and students were allotted thirty minutes for each assessment, with the purpose of examining to what extent students would apply proportionality to area and volume tasks that were non-proportional. The second version involved two different assessments: Test B and Test C. Test B was administered to only a portion of the original sample (157 students) and included the same tasks with additional information regarding dimensions provided for each task with the purpose of examining whether or not the inclusion of these dimensions would lead students to a multiplicative relationship. Test C was administered to the other part of the original sample and included the same original tasks from the first assessment, but this version included two alternate responses given by fictional students. One fictional response involved the correct response and the other showed the dominant misconception that area and volume are proportional to length. Test C asked students to choose the response that they agreed with and provide a justification for this response. All responses were categorized into three point values: 1 point for correct responses, 0 points for incorrect responses, and 0.5 points for responses with a false answer but correct mathematical expression. Data analysis revealed that even though statistically significant improvement was evident from Test A to Tests B and C, almost 60% of the students persisted in applying proportional reasoning in problems for which it was not suited (Modestou, Gagatsis, & Pitta-Pantazi, 2004). These results further illustrate the impact of a lack of deep proportional reasoning understand on algebraic understanding. If this lack of understanding is not addressed, misconceptions will continue to plague students in their development of conceptual understanding in algebra.

2.3 CONCEPTUAL UNDERSTANDING

The NCTM Standards (2000) state "facility with proportionality involves much more than setting two ratios equal and solving for the missing term. It involves recognizing quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship" (p. 217). The Common Core State Standards for Mathematics (CCSS-M) call for students to analyze proportional relationships and use them to solve real world problems (Common Core State Standards, 2012). Deep conceptual understanding of proportionality is not simply setting up and solving proportions. Given that research has shown that reform curricula can produce higher achievement than traditional curricula, there are two questions that are raised: 1) How can teachers use high-level tasks (i.e. tasks characteristic of those found in reform curricula) during Algebra instruction to improve understanding of proportionality? 2) Can the selection and implementation of high-level proportional reasoning tasks improve proportional reasoning ability in the context of Algebra I?

3.0 METHODOLOGY

This study was designed to examine the proportional reasoning ability of eighth grade students and the impact of using high cognitive demand tasks as part of the Algebra 1 curriculum on students' ability to reason proportionally. Specifically, it sought to answer the following questions:

R1. Is there a connection between proportional reasoning ability and course placement in eighth grade?

R2. To what extent are students proficient in proportional reasoning upon entering an algebra course?

R3. To what extent does the current curriculum (a) align with essential understandings as related to Essential Understandings of Ratio, Proportion, and Proportional Reasoning and (b) contain high-level mathematical tasks?

R4. To what extent do students enrolled in the honors and academic Algebra 1 classes improve their basic capacity to reason proportionally?

Details regarding the study are provided in the following sections. The first section provides information about the context in which the data was collected. The second section contains information regarding the key stakeholders in this investigation. The third section includes an explanation of the data collection instruments: the proportional reasoning diagnostic tool and the Keystone scores. The fourth section contains explanations of how the data will be coded and analyzed in order to answer each of the research questions, following by a section detailing the tasks that have been selecting to be implemented.

3.1 CONTEXT

3.1.1 The research site

The school district in which this research was conducted lies about seven miles south of a mid-size city in the eastern United States, and encompasses a compact six square miles. It was chosen due to the convenience for the researcher. The town has an urban flair to it, with mostly modest homes close to one another, sidewalks for the children who attend the neighborhood schools, planned green spaces and easy access to public transportation via the trolley system that runs through the town. It is here in the center of town that Appleglen Middle School (AMS) sits, nestled neatly around a neighborhood church. The front entrance of the handsome stone building bears an inscription of the words of Aristotle: "It is by education I learn to do by choice what other men do by the constraint of fear."

The School Performance Profile (State Department of Education) lists enrollment at AMS at 673 students out of the over 5,000 students that the school district serves, of which 4.46% qualify for gifted education services, 11.89% for special education services, and 9.81% are economically disadvantaged. Of the 673 students, 89.44% are White (non-Hispanic), 5.13% are Asian, 2.12% are Hispanic, 1.7% are Multi-racial, and 1.41% are Black. Of the 673 students at AMS, 98.5% met the annual growth expectations in mathematics according to the Pennsylvania Department of Education, with 89.24% scoring above proficiency on the PSSA and

83% scoring above proficiency on the Keystone Algebra 1 exam. The consistently high standardized test scores earned the school district a top ten ranking among the schools in the region according to the local paper (2015).

Given this tradition of success in traditional measures for student achievement, there exists a tension between preparing students for achievement on standardized tests and facilitating rich, rigorous and ambitious instruction in mathematics that builds students foundational understanding of key ideas such as proportional reasoning. Mathematics intervention has been an ongoing conversation in the schools. The investigation of the relationship between proportional reasoning ability and algebraic understanding could shed some light on how the administrators and teacher leaders can facilitate change in the area of proportional reasoning in order to bolster student understanding and achievement in Algebra. As a teaching team in the middle school, there has been much deliberation without much resolution on how to best meet the needs of all children as they move through the middle level mathematics curriculum and transition into Algebra. A deeper understanding of how teachers can influence a student's ability to engage in Algebra topics by increasing proportional reasoning ability can potentially guide decisions that are being made to meet these needs.

This investigation took place in two honors sections and three academic sections of eighth grade Algebra 1 in which the principal investigator is the classroom teacher with eighteen years of middle school experience. At AMS, nearly all students take Algebra 1 in eighth grade. The Algebra course is designed as a discovery-based curriculum in which student engage in tasks that guide them to the discovery of algebraic concepts. The nature of the curriculum gives students the opportunity to make choices and gives the teacher the opportunity to share authority for learning with students (Engle and Conant, 2002). Tasks and classroom activities are selected to engage students in work that is cognitively demanding. Stein, Smith, Silver and Henningsen (2000) categorize high cognitive demand tasks as ‘procedures with connections’ or ‘doing mathematics’ tasks (see Table 3.1). Task selection is also closely aligned with the state standards and the overarching goals of the unit of study. Each of the tasks in the curricular unit of study that is being investigated is described in section 3.3.

Table 3-1 Mathematics tasks framework (Stein, Smith, Henningsen & Silver, 2000)

Low Cognitive Demand Tasks	High Cognitive Demand Tasks
Memorization- The task/solution • Involves reproducing previously learned facts, rules, formulas, or definitions or committing these to memory. • Cannot be solved using procedures because a procedure does not exist or because the rime frame in which the task is being completed is too short to use a procedure. • Is not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated. • Has no connections to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned.	**Procedures with Connections**- The task/solution • Focuses student attention on the use of procedures for the purpose of developing deeper understanding of mathematical concepts and ideas • Suggests explicit and/or implicit pathways to follow that involve the use of broad procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms. • Can usually be represented in multiple ways, including the use of manipulative materials, diagrams, and symbols. Making connections among the representations helps students develop meaning. • Requires some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students are engaged in conceptual ideas that underlie the procedure and develop understanding.
Procedures without Connections- The task/solution • Is algorithmic. The use of a procedure is specifically called for or is evident from prior instruction and/or experience. • Requires limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it. o Is not connected to the concepts or meaning that underlie the procedure being used. o Is focused on producing correct answers. o Requires no explanation or explanations focus solely on describing the procedure that was used.	**Doing Mathematics**- The task/solution • Requires complex, non-algorithmic thinking. • Requires students to explore and understand the nature of mathematical concepts, processes, or relationships. • Demands students do some type of solf-monitoring or self-regulation of their own cognitive processes. • Requires students to access relevant knowledge and experiences and make appropriate uses of them in working through the task. • Requires students to analyze the task constraints that may limit possible solution strategies or solutions. • Requires considerable cognitive effort and may cause some level of anxiety for the students as they are working through the problem.

3.1.2 Key stakeholders

The school district holds high expectations for students and teachers in the area of achievement. Consistently ranked within the top ten school districts in the area, the school district strives to maintain this record of academic success. The school board represents its constituents of the community and the tradition of excellence for the District allows property owners to continue to maintain high values for their homes. Business owners rely on residents to generate revenue. Consequently, the school board, the administration, and the community members have a stake in this inquiry to potentially mitigate factors that could result in lower achievement in the area of mathematics.

Closer to the center of the problem of practice are the teachers and the students. Students are the primary stakeholder for this problem of practice. The National Mathematics Advisory Panel (2008) suggests that "success in mathematics education is important…because it gives them college and career options, and it increases prospects for future income." Furthermore, success in mathematics "correlates powerfully with access to college, graduation from college" and a "gateway to later achievement" (National Mathematics Advisory Panel, 2008).

3.1.3 Problem of practice in context

In the context of this inquiry setting, student achievement and mathematical understanding have influenced student placement in middle school mathematics courses. It is hypothesized that a connection exists between students' proportional reasoning ability and their course placement as an eighth grader. All students situated in this inquiry setting engage in algebraic thinking and reasoning through their eighth grade coursework. It is hoped that this investigation might reveal a connection between proportional reasoning ability and course placement, as well as the extent to which students are reasoning proportionally in the context of algebra. It is of interest to the aforementioned stakeholders of this school district to investigate this problem of practice in the name of improving proportional reasoning ability within the context of algebra concepts and enhancing the mathematics education for the students of this school district and striving to meet the goal of school district mission statement: "to provide the best education possible for each and every student" (School District, 2016).

3.1.4 The participants

Eighth grade students enrolled in the principal investigator's courses – 2 sections of honors algebra and 3 sections of academic algebra - were included in this study. This represents 50% of all eighth grade students at AMS. Background information and demographics, collected from the school district's student database and the historical records for course enrollment, are shown in Table 3-2. This study included 95 students currently taking Algebra 1: 60 honors students and 35 academic students and nearly evenly split between male and female students. The racial/ethnic background was primarily Caucasian with nearly 90% of students identifying as Caucasian, 8% as Asian/Pacific Islander, and around 1% each for African American and Hispanic.

Table 3-2 Demographics of students included in study

Level		Gender		Racial/Ethnic Background				Total
Academic	Honors	Male	Female	Caucasian	Asian/Pacific Islander	African American	Hispanic	95
60	35	47	48	85	8	1	1	

3.2 DATA COLLECTION

Table 3.3 outlines the sources of data collection and the methods for analysis. As shown in the table, there are eight sources of data used in this study: the diagnostic assessment of proportional reasoning administered pre- and post-unit, the “Identifying Proportional Reasoning” and “Snowfall” tasks administered pre- and post-unit, course placement records, and the curricular tasks of the unit.

Table 3-3 Collected Data for Analysis

Research Questions	Data Sources	Analysis
RQ1. Is there a connection between proportional reasoning ability and course placement in eighth grade?	• Proportional reasoning diagnostic assessment (Misialidou & Williams, 2003) (Appendix A) • “Identifying Proportional Reasoning” Task (Appendix C) • 8^{th} grade course placement data	Student results on the diagnostic assessment compared to actual course placement of the students (Honors vs. regular). Mean scores were calculated and disaggregated for honors and academic. A one-way analysis of variance for independent samples (ANOVA) test will indicate statistical significance. The spread of each data set will be compared using a five-number summary and box plot.
RQ2: To what extent are students proficient in proportional reasoning upon entering an algebra course?	• Proportional reasoning diagnostic assessment (Misialidou & Williams, 2003) (Appendix A) • “Identifying Proportional Reasoning” Task (Appendix C) • “Snowfall Task” (Appendix C)	Student work analyzed for correctness and also disaggregated for the two subsets of students (honors and academic). Proportional reasoning ability was also evaluated based on the “Levels of Pupils’ Proportional Reasoning” (Misailidou & Williams, 2003). ANOVA test for statistical significance was used to analyze variance.

Table 3-3 continued

RQ3: To what extent does the current curriculum (a) align with essential understandings as related to Essential Understandings of Ratio, Proportion, and Proportional Reasoning and (b) contain high-level mathematical tasks?	Curricular Tasks from *Discovering Algebra* and supplemental tasks used in the classroom (Appendix B)	Tasks mapped to the Essential Understandings of Ratio, Proportion and Proportional Reasoning (see Table 2-2) and to the cognitive demand using the Task Analysis Guide (see Table 3-1)
RQ4: To what extent do students in the honors and academic Algebra 1 classes improve their basic capacity to reason proportionally?	• Proportional reasoning diagnostic assessment (post unit) scored for correctness and for level of proportional reasoning ability according to the Misailidou & Williams' rubric • Identifying Proportional Relationships Task & Snowfall Task administered pre- and post-unit (Appendix C)	Growth on assessments following the utilization of the tasks implemented as instructional intervention (statistical significance will be evaluated using t-tests).

3.2.1 Scoring and coding

Curricular tasks, both those from the textbook publisher and the supplemental tasks, were mapped to the essential understandings of ratios, proportions and proportional reasoning as outlined in *Developing Essential Understanding of Ratios, Proportions and Proportional*

Reasoning: Grades 6-8 (Table 3-4) and analyzed for the connection to algebra. This mapping does not include tasks that were not considered to be high-level tasks.

Table 3-4 Essential Understandings of Ratios, Proportions, and Proportional Reasoning

	Essential Understanding	**Essential Question**
EU 1	Reasoning with ratios involves attending to and coordinating two quantities.	How does ratio reasoning differ from other types of reasoning?
EU 2	A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.	What is a ratio?
EU 3	Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.	What is a ratio as a measure of an attribute in a real-world situation?
EU 4	A number of mathematical connections link ratios and fractions: • Ratios are often expressed in fraction notation, although ratios and fractions do not have identical meaning. • Ratios are often used to make "part-part" comparisons, but fractions are not. • Ratios and fractions can be thought of as overlapping sets. • Ratios can often be meaningfully reinterpreted as fractions.	How are ratios related to fractions?
EU 5	Ratios can be meaningfully reinterpreted as quotients.	How are ratios related to division?
EU 6	A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.	What is a proportion?
EU 7	Proportional reasoning is complex and involves understanding that- • Equivalent ratios can be created by iterating and/or partitioning a composed unit; • If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and • The two types of ratios-- composed units and multiplicative comparisons—are related.	What are the key aspects of proportional reasoning?

Table 3-4 continued

EU 8	A rate is a set of infinitely many equivalent ratios.	What is a rate and how is it related to proportional reasoning?
EU 9	Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.	What is the relationship between the cross-multiplication algorithm and proportional reasoning?
EU 10	Superficial cues present in the context of a problem do not provide sufficient evidence of proportional relationships between quantities.	When is it appropriate to reason proportionality?

Developing conceptual understanding of proportionality in the middle grades extends to high school topics, particularly understanding slope and linear functions, so it seems natural to make attempts to improve proportional reasoning understanding in the context of Algebra 1 as described later in this paper. The Task Analysis Guide provided a framework for categorizing the level of cognitive demand for each task.

The diagnostic assessment was scored based on correctness of the final solutions, as well as by using Misialidou and Williams' (2003) levels of proportional reasoning understanding. Each student response was analyzed and coded according to the method(s) in which items were solved. Common errors and incorrect methods were noted and used to determine students' level of proportional reasoning as shown in table 2.1. Students answering each item correctly, with no errors, were placed on level 4. Students that did not correctly answer any item were placed on level 0.

The *Identifying Proportional Reasoning Relationships* Task was scored for correctness, but each response was examined individually to assess the reasoning which students used to make their decisions and to determine common mistakes across items. The Snowfall Task was scored on a rubric developed based on the State System of School Assessment, shown in Table 3-5. Student responses were placed on the scale from 0 to 4, with an emphasis on student explanation rather than on providing the correct answer.

Table 3-5 Snowfall Task Rubric

Snowfall Task Scoring Rubric	
4	***The response demonstrates a thorough understanding of the mathematical concepts and procedures required by the task.*** The response provides correct answer(s) with clear and complete mathematical procedures shown and a correct explanation, as required by the task. Response may contain a minor "blemish" (e.g., missing units of measurement) or omission in work or explanation that does not detract from demonstrating a *thorough* understanding. Students correctly identify Cedar Rapids as a proportional relationship and provide ample evidence for this decision. Evidence includes but is not limited to the structure of the graph through the origin, the constant ratio as opposed to the constant rate of Mason City, and/or the structure of the linear equation that models the data. Connections between the mathematical structures and the real world meanings are made.
3	***The response demonstrates a general understanding of the mathematical concepts and procedures required by the task.*** The response and explanation (as required by the task) are mostly complete and correct. The response may have minor omissions that do not detract from demonstrating a general understanding. Students correctly identify Cedar Rapids as a proportional relationship, and provide sufficient evidence to justify this choice.
2	***The response demonstrates a partial understanding of the mathematical concepts and procedures as required by the task.*** The response is partially correct with *partial* understanding of the required mathematical concepts and/or procedures demonstrated and/or explained. The response may contain some work that is incomplete or unclear. Students correctly identify Cedar Rapids as a proportional relationship, but the evidence provided is incomplete or incorrect.
1	***The response demonstrates a minimal understanding of the mathematical concepts and procedures required by the task.*** Students correctly identify both relationships as having a constant rate, but cannot distinguish between constant ratio and constant rate of change.
0	***The response has no correct answer and insufficient evidence to demonstrate any understanding of the mathematical concepts and procedures required by the task.*** Students may correctly identify Cedar Rapids as the proportional relationship but provide no justification for that choice. Students' responses may show only information copied from the question.

The data sources detailed above were collected during the fall semester of the 2016-2017 school year. Students were administered the diagnostic assessment, Identifying Proportional Relationships task, and the Snowfall task pre-assessments in September, prior to the implementation of the Linear Equations unit. Students then engaged in Linear Equations unit, including the curricular tasks and classroom activities as described herein, during a period of approximately eight weeks. Following student engagement in the linear equations unit, the post-assessments were administered in late December of 2016.

3.2.2 Analysis

This section explains how the data was analyzed in order to determine answers for each research questions. The first two research questions rely solely on the initial pre-assessments in order to compare students' understanding of proportionality and the course in which they are enrolled. The third question examined the curricular tasks as they pertained to the essential understandings of proportional reasoning and the task analysis guide. The last question focuses on the growth of students following the implementation of the instructional intervention.

R1: Is there a connection between proportional reasoning and course placement in eighth grade? In order to determine a connection between proportional reasoning ability and course placement, average scores on the initial administration of the diagnostic assessment were compared with student placement in an honors or academic section of Algebra 1. Student scores for correctness, as well as for student understanding based on the levels of pupils' proportional reasoning outlined by Misailidou and Williams (2003) were used. These scores were disaggregated based on student placement in Algebra 1 honors or academic. The mean score on the diagnostic assessment was calculated for the entire group of students included in the study, and then examined as separate honors and academic subsets of the group. Examining the mean score by subgroup (honors versus academic) reveals the typical performance for a student in Algebra 1, a student in Algebra 1 Honors, and a student in Algebra 1 Academic.

R2: To what extent are students proficient in proportional reasoning upon entering an algebra course? Proficiency in proportional reasoning was defined by both the average score for correctness on the diagnostic assessment and the score on the diagnostic rubric for proportional reasoning as presented in table 2-1, as well as by using the scores on the "Identifying Proportional Relationships" task and the "Snowfall Task". Each student was assessed prior to the implementation of the Linear Equations at the beginning of the school year unit tasks with these three pre-assessments. A mean score was calculated for the whole group, as well as for both subgroups (honors and academic). Analysis also included examining the distribution of scores using a five-number summary and box plot.

R3: To what extent does the current curriculum (a) align with the essential understandings as related to _*Essential Understanding of Ratio, Proportion & Proportional*_ *Reasoning and (b) contain high level mathematical tasks?* To answer this question, curricular tasks were mapped to the essential understandings of ratios, proportions, and proportional reasoning (table 4-3), as well as identified as memorization, procedures without connections, procedures with connections, and doing mathematics as described in the Task Analysis Guide (TAG) (table 3.1). By identifying which essential understandings the task were mapped to, it could be ascertained whether or not the curriculum addressed proportional reasoning within the context of the Algebra I curriculum. Use of the TAG made certain that the tasks as represented in the curriculum, as well as those selected as supplemental tasks, met the criteria to be considered high cognitive demand tasks and supported the vision of NCTM's *Principles and Standards for School Mathematics* (NCTM, 2000): "a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction" (p. 3).

R4: To what extent do students in honors and academic Algebra 1 improve their capacity to reason proportionally as it is related to algebra? The diagnostic assessment, “Identifying Proportional Relationships” task, and the Snowfall Task were administered as a pre-assessment prior to the start of the unit of study and following completion of the unit. Scores on these assessments and the level of proportional reasoning understanding were examined for growth and evaluated for statistical significance. Each pair of assessments was given a one-tailed, paired t-test with $p<0.05$ to determine statistical significance.

4.0 RESULTS

The results of the data analysis described in the previous chapter will be presented in this chapter organized by each of the research questions.

4.1 COURSE PLACEMENT AND PROFICIENCY

Research Question 1: Is there a connection between proportional reasoning ability and course placement in eighth grade? Students entering this eighth grade course are placed into one of two levels of algebra, honors or academic, and this placement is based on several factors including previous teacher recommendation, standardized test scores, classroom assessment scores, and overall grades. Students and parents who wish to override the school recommendation may do so by completing a “request to override” form, but typically students enroll in the course that was recommended.

Eighth grade students enrolled in all of the instructor’s sections of algebra, both honors and academic sections, were administered the diagnostic assessment (Misialidou & Williams, 2003) in early September within the first few days of the start of the course. The results of the diagnostic assessment are found in Table 4-3. Honors students, on average, scored 83.47% based on correctness, while academic students scored an average of 62.7% based on correctness. By looking at these simple mean scores on the Diagnostic Assessment, it is clear that, on average, honors students outperformed academic students in correctly solving these proportional reasoning items.

On the Identifying Proportional Relationships Task, when given a relationship expressed as a table, graph or equation (see appendix C), academic students correctly identified a proportional relationship when presented as a table, graph or equation only 58.55% of the time, whereas honors students correctly identified these relationships 89.15% of the time.

Students were also administered the Snowfall Task as performance assessment indicator prior to algebra instruction. When scored on the performance assessment rubric, honors students scored an average of 1.47 on the performance assessment, and academic students scored 0.74 (see Table 4-1). An ANOVA one-way analysis of variance with $p<0.01$ was conducted for all data, resulting in a statistical significance between academic and honors students for each of the three assessments.

Table 4-1 Scores upon entry into Algebra 1

Proportional Reasoning Diagnostic Assessment	Mean (honors students)	83.47%
	Mean (academic students)	62.74%
Identifying Proportional Reasoning	Mean (honors students)	89.15%
	Mean (academic students)	58.55%
Snowfall Task Rubric Score	Mean (honors students)	1.47
	Mean (academic students)	0.74

These results taken together indicate that students with a higher level of ability to reason proportionally are likely to be placed into an honors course, whereas those with a lower proportional reasoning ability are likely enrolled in an academic section of Algebra 1.

4.2 PROPORTIONAL REASONING PROFICIENCY

Research Question 2: To what extent are students proficient in proportional reasoning upon entering an algebra course? In addition to correctness, students were evaluated on their level of proportional reasoning ability based on the diagnostic assessment rubric as outlined in table 2-1(Misailidou & Williams, 2003). Honors students had an average level score of 3.07. This indicates that honors students were successful on items with more difficult numerical structure and unfamiliar contexts, where answers can be more complex fractions. Errors by students at this level were typically characterized by incorrect use of the build-up method on difficult items, such as ‘Paint’ (figure 2-3). This level indicates a relatively strong depth of understanding of proportional reasoning.

Academic students scored an average of 1.9 on the diagnostic levels of understanding. This score indicates that students were successful on problems where answers are found by simple multiplication or by taking an amount than half as much again to work on the scalar or functional ratio. Mistakes at this level are characterized by use of the additive strategy and magical halving or doubling on easy items. An ANOVA test of variance indicates that the differences between academic and honors students’ performances are statistically significant with $p<0.01$.

Table 4-2 Level of proportional reasoning understanding

Mean Proportional Reasoning Diagnostic Level 0-4	Population	Average Score	Typical Performance and Common Errors at this Level
	Honors Students	3.07	Typical success with items that possess a more difficult numerical structure and have unfamiliar contexts. Typical errors are characterized by incorrect application of the build-up method on items of a difficult context. This is the only error made by the higher ability students at this level. Use of the additive strategy is predominant on items that are identified to provoke such errors.
	Academic Students	1.90	Typical success in answering questions with familiar contexts and single digit numbers with items with answers that can mainly be found through scalar multiplication. Common errors are characterized by incomplete reasoning, and magical halving/doubling is used on relatively easy items. Incorrect build-up is not used by students at this level unless attempting a very easy item that due to the context is not prone to additive errors.

This data suggests that students entering an honors algebra class have a fairly strong proportional reasoning ability, are able to work with proportional relationships that are of contexts they are unfamiliar with, and make few errors. The common errors at this level are characterized as incorrect application of the build-up method as shown in Maddie's work in figure 4-3.

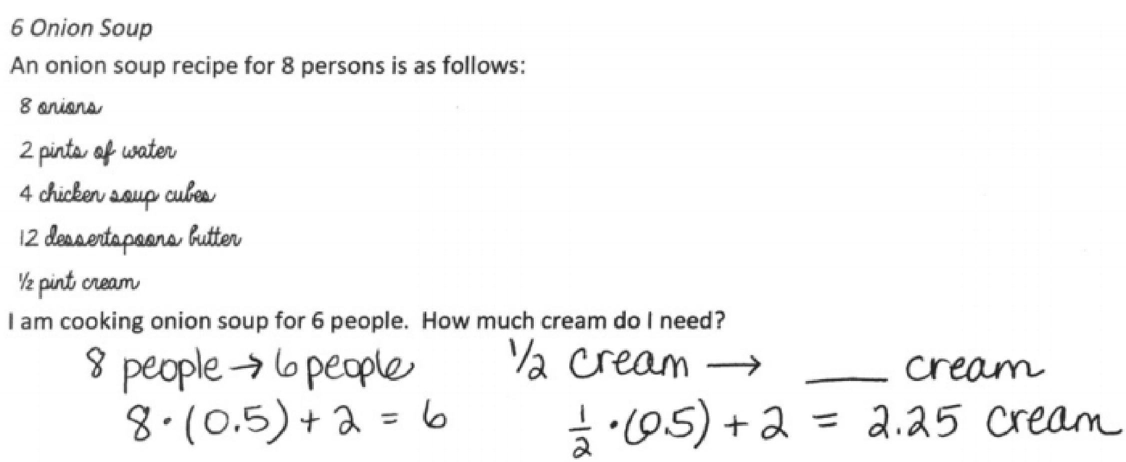


Figure 4-1 Student work sample illustrating incorrect build-up

In this work, Maddie created a relationship between eight people and six people, and then she incorrectly applied this relationship to the amount of cream needed in the recipe. Misailidou and Williams (2003) suggest that students with higher proportional reasoning ability utilize this strategy when dealing with problems that are more challenging. Given the numerical structure of this item, this student blended the additive and multiplicative methods to create an incorrect procedure to relate the sets of numbers to one another and arrived at an incorrect solution.

Academic students' common mistakes were characterized by the repeated use of the additive strategy and magical halving/doubling as shown in figures 4-2 and 4-3 respectively. In 'Mr. Short,' Gavin examined the relationship between Mr. Short's paperclips and matchsticks and surmised that the relationship must be adding two, thus giving Mr. Tall two additional paperclips to his matchsticks. Gavin's use of the additive strategy illustrates a common mistake made by the students in the academic classes.

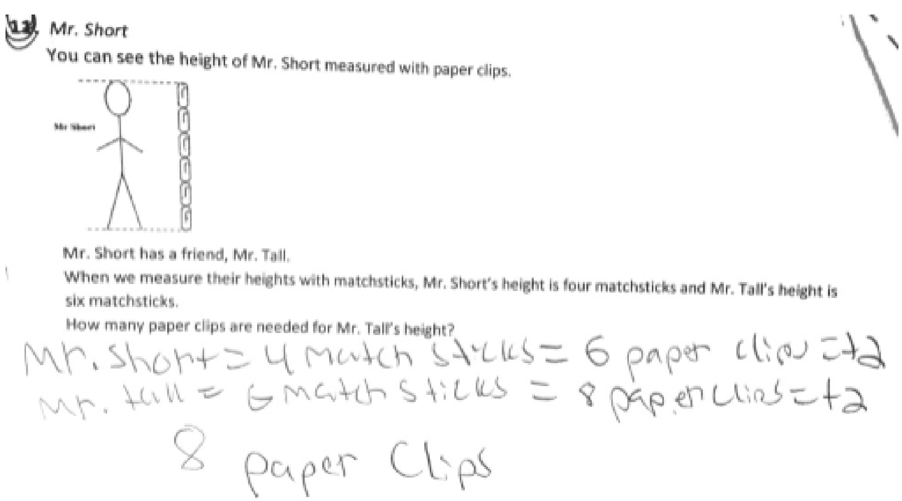
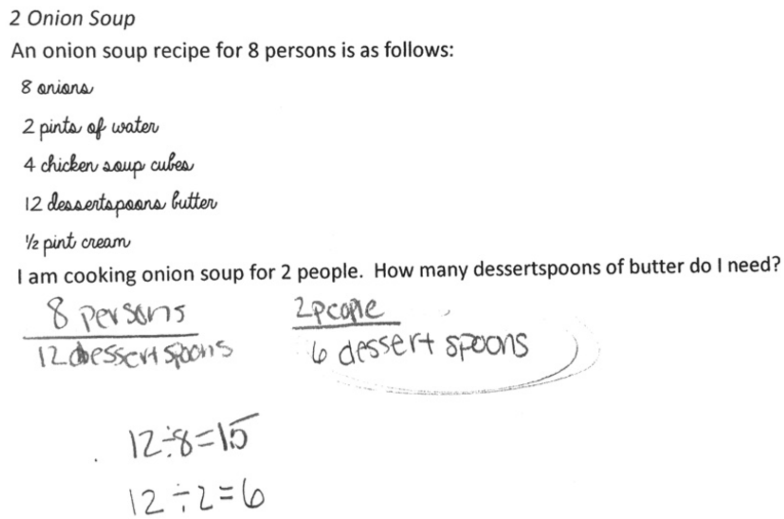


Figure 4-2 Student work illustrating the additive method

Magical halving and doubling was also used by students with low levels of proportional reasoning ability on items that are categorized as relatively easy items, such as '2 Onion Soup.'


2 Onion Soup

An onion soup recipe for 8 persons is as follows:

8 onions

2 pints of water

4 chicken soup cubes

12 dessertspoons butter

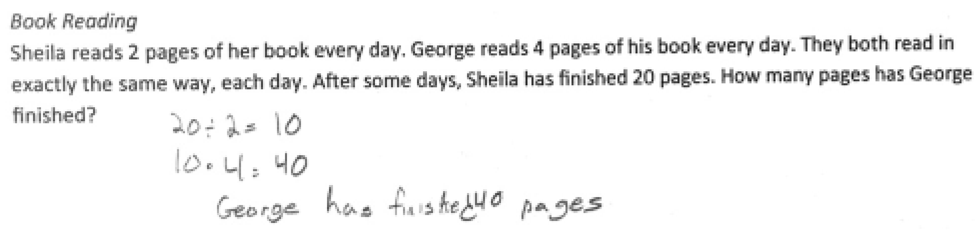
½ pint cream

I am cooking onion soup for 2 people. How many dessertspoons of butter do I need?

Figure 4-3 Student work illustrating magical doubling method

In Figure 4-3, Jenny's work illustrates the "magical halving" of the number of dessertspoons of butter, despite her calculation for the scale of the recipe. Mistakes such as Jenny's were prevalent in the work of students occupying the lower levels of proportional reasoning and were common on other items that Misailidou and Williams (2003) also classified as relatively easy to solve.

In addition to common errors, academic students also illustrated success in correctly answering items with simpler numerical structure and familiar contexts as the illustration in figure 4-4. In 'Book Reading,' Evie is able to utilize the simpler numerical structure to correctly solve the item. Evie first divides twenty pages by two pages per day to arrive at ten days. Using George's rate of four pages per day and the time period of ten days, she arrives at a correct solution of forty pages. The numerical structure of this problem lends itself to simple division and multiplication.



Book Reading
Sheila reads 2 pages of her book every day. George reads 4 pages of his book every day. They both read in exactly the same way, each day. After some days, Sheila has finished 20 pages. How many pages has George finished?

$20 \div 2 = 10$
$10 \cdot 4 = 40$
George has finished 40 pages

Figure 4-4 Item with simple numerical structure

While academic students in this study typically entered algebra nearing a level 2, the typical mistakes characterized at levels 1 and 2 indicate that students in academic Algebra need to improve their proportional reasoning ability. Honors students, on average, outperform academic students and showed success on items that possess more difficult numerical structure than item such as 'Book Reading' (figure 4-4). This would suggest that eighth grade students that can navigate problems with difficult numerical structures may be better poised to tackle Algebra 1.

4.3 CURRENT CURRICULUM

Research Question 3: *To what extent does the current curriculum (a) align with essential understandings as related to* <u>*Essential Understandings of Ratio, Proportion, and Proportional Reasoning*</u> *and (b) contain high-level mathematical tasks?* The curriculum that was implemented in the classroom that is being used in this inquiry is *Discovering Algebra* (Murdock, Kamischke, & Kamischke, 2007), a program designed to cover the topics of a traditional Algebra I course in a way that "encourages [students] to investigate interesting problems (p. xii)" collaboratively, use technology to complement instruction, and come away with an appreciation of mathematics as a tool for science, business and everyday life. The over-arching big idea question for the course is “How can I use my mathematical power to understand my world?” and each unit of study aims to provide students with an arsenal of tools and understandings to help them to address this question.

The unit of study in which this investigation is situated explored the concept of linearity over a period of eight weeks and the high level tasks outlined in table 4-1 engaged students in developing their own logical reasons behind the mathematical ideas and methods. These tasks were used in conjunction with the investigations and examples provided with the curricular materials (which in general were not high level) in order to build a stronger connection to essential understandings of ratio, proportion and proportional reasoning, as well as algebra, and to provide more opportunities for students to engage in thinking and reasoning. The concept of slope is developed only after an understanding of recursion is explored and established. Students engage in a series of tasks in order to help them to develop an understanding that a constant rate of change denotes linearity. Connections are made among different representations of problem situations: the recursive routine, the graph, the table, and the equation. As mentioned previously, the nature of the curriculum naturally lends itself to allowing the teacher and students to share authority of learning (Engle and Conant, 2002).

In order to determine the extent to which implementation of high cognitive demand tasks can improve students’ proportional reasoning ability, tasks were selected and implementation was designed to include certain characteristics. The selection of tasks to include in this study aligned with the following characteristics of high-quality tasks as outlined in the *Putting Essential Understanding into Practice* series:

- Aligns with relevant mathematics content standard(s).
- Encourages the use of multiple representations.

- Provides opportunities for students to develop and demonstrate mathematical practices.
- Involves students in an inquiry-oriented or exploratory approach.
- Allows entry to the mathematics (all students can begin the task) but also has a high ceiling (some students can extend the task to higher-level activities).
- Connects previous knowledge to new learning.
- Allows for multiple solution approaches and strategies.
- Engages students in explaining the meaning of the result.
- Includes a relevant and interesting context (Dougherty, 2015).

Additionally, the tasks were selected to target specific essential understandings of ratio, proportion, and proportional reasoning as outlined in *Developing Essential Understanding of Ratios, Proportions and Proportional Reasoning Grades 6-8* (Lobato, Ellis & Zbiek, 2010) and are included in the unit focused on linearity and linear relationships. This volume of Essential Understandings focused on ratios, proportions and proportional reasoning (2010) outlines key mathematical ideas that are central to mathematical understanding in the middle grades. The Essential Understanding Series aims to develop an understanding of ratios, proportions, and proportional reasoning to help teachers implement the teaching practices promoted in *Principles and Standards for School Mathematics*. The essential understandings outlined in table 3-4 aim to engage teachers in developing a deeper understanding of these ideas to afford them greater ability implementing lessons and assessing students' understanding in a way that reflects the rich intricacy of proportionality.

Table 4.4 provides an explanation of how each task selected aims to target an essential understanding of proportionality and the connection to algebra. All tasks are found in Appendix

B. Each task was coded according to the Task Analysis Guide as memorization, procedures without connections, procedures with connections, or doing mathematics. Additionally, each task was mapped to one of the ten essential understandings of proportionality as outlined in table 2-2. The essential understandings contained within the *Discovering Algebra* curriculum focused on proportions and proportional reasoning (EU 6, 8 and 10 primarily) are prominent, rather than those focused on ratios (EU 1 through 5) which would be included in previous grade levels curriculum where ratios are introduced such as grade six. Some of these essential understandings are addressed through the supplemental tasks that are described in table 4-4.

Table 4-3 Tasks included in the *Discovering Algebra* curriculum (Chapters 3 & 4)

Task	Level of Cognitive Demand of Task	Essential Understanding(s) of Proportionality	Connection to algebra
Walking Graphs	Doing Mathematics- The task requires considerable cognitive effort as students' access relevant previous knowledge regarding starting value and rates of change and self- monitor. Students also self-assess their work by acting out the walks using sonic motion sensors.	EU 8: A rate is a set of infinitely many equivalent ratios (in this case, the rate is the speed of the walker). Superficial cues presented in the context of the problem do not provide sufficient evidence of proportional relationships.	A constant rate of change produces a linear relationship.
Airplane Task	Procedures with Connections- The task suggests a pathway for students to use a rule but it does not specify what the rule must be (i.e. equation, recusive routine, words,etc.).It still requires cognitive effort and it makes connections to underlying conceptual development of reclusiveness and linearity.	EU 8: A rate is a set of infinitely many equivalent ratios (in this case, the rate is the speed of the walker). Superficial cue present in the context of the problem do not provide sufficient evidence of proportional relationships.	A constant rate of change denotes linearity and the constant rate of change can be used to find a starting value (y-intercept).
Internet Use	Doing Mathematics- The task requires students to explore the nature of the linear relationship between time and total fee in a way that has no established solution path but requires that they make use of previous knowledge.	EU 10: Superficial cues present in the context of the problem do not provide sufficient evidence of proportional relationships.	A constant rate of change denotes linearity, but rate of change is not necessarily a proportional relationship. The starting value (y-intercept) plays a role in the linear relationship as well.

Table 4-3 continued

Beth's Birthday	Doing Mathematics- The task requires students to explore and make sense of the mathematical relationships that exist when the rate of population change remains constant. There is no set solution path or suggested solution path so students can use varied approaches (such as a recursive routine or equation).	EU 10: Superficial cues present in the context of the problem do not provide sufficient evidence of proportional relationships.	A linear relationship can be modeled with the rate of change and one point in the linear relationship.
Sam's Swimming	Procedures with Connections- The task suggests pathways by giving a table but it possesses a close connection to underlying conceptual knowledge regarding rate of change and starting value. Connections among the table, graph and equations are elicited.	EU 6: A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change. EU 10: Superficial cues present in the context of the problem do not provide sufficient evidence of proportional relationships.	Intercept form of a line is the translation of the direct variation relationship. Direct variation produces a linear relationship that passes through the origin.

Additional tasks were selected for supplemental classroom activities because they also represented high-demand tasks in essential understandings of proportionality that were not present in the publisher provided tasks and possessed a connection to algebra. Not all essential understandings are present in these or the aforementioned tasks, as those addressed in this study are related to the context of algebra and were not identified in the diagnostic assessment as needing to be addressed within the context of algebra. These tasks are described in Table 4-2 and appear in full in the Appendix.

Table 4-4 Supplemental tasks

Task	Level of Cognitive Demand	Essential Understanding(s) of Proportionality	Connection to algebra
Rabbit and Frog Task (adapted from Lobato and Thanheiser, 2002)	Doing Mathematics- The task requires students to explore and make sense of the mathematical relationships between time and distance. There is no set solution path or suggested solution path so students can use varied approaches (such as diagrams or equations).	EU 2: A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit. EU 7: Equivalent ratios can be created by iterating and/or partitioning a composed unit.	Proportionality produces a linear relationship
Walking Home Task (Lobato, Ellis, & Zbiek, 2010)	Doing Mathematics- The task requires students to explore and make sense of time-distance relationships when the starting distance is not at 0 units. There is no set solution path or suggested solution path so students can use varied approaches.	EU 7: Proportional reasoning is complex and involves understanding that (1) equivalent ratios can be created by iterating and/or partitioning a composed unit, (2) if one quantity is multiplied or divided by a factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship, and (3) the two types of ratios-composed units and multiplicative comparisons- are related.	A constant rate of change produces a linear relationship in the form of $y = a + bx$, essentially statement of proportionality combined with a vertical translation of *a* units.
Revisiting the Walking Home Task	Doing Mathematics- The task requires students to explore and make sense of the relationship between the graphical representation of Rabbit's walk and Rabbit's speed.	EU 8: A rate is a set of infinitely many equivalent ratios	The slope of a line is the rate of change in one quantity relative to the rate of change of another quantity, and the slope will remain constant in a linear relationship.
State Park and Zoo Task (Learning Research & Development Center, University of Pittsburgh, 2012)	Procedures with Connections- The task suggests pathways by giving a table but it possesses a close connection to underlying conceptual knowledge Connections among the table, graph and equations are elicited.	EU 9: Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportions. EU 10: Superficial cues present in the context of the problem do not provide sufficient evidence of a proportional relationship between two quantities.	Not all linear relationships are proportional (direct variation).

From this mapping, the curriculum contains multiple opportunities for students to engage in high-level tasks that are in the procedures with connections or doing mathematics categories of the Task Analysis Guide. Students in this study, both honors and academic students, had the opportunity to engage in tasks that are high-level and are connected to the essential understandings of ratio, proportion and proportional reasoning though they are embedded within an algebra unit focused on linearity over a period of eight weeks of instruction. These opportunities explicitly engaged students in five of the ten essential understandings of ratio, proportion and proportional reasoning, though it is reasonable to expect that there was some overlap with the essential understandings that were not explicitly identified, as well as eleven tasks that are classified by the TAG as high cognitive demand tasks.

4.4 PROPORTIONAL REASONING IMPROVEMENT

Research Question 4: To what extent do students in the honors and academic Algebra 1 classes improve their basic capacity to reason proportionally? Following the implementation of the curricular tasks, students' scores increased on all three assessments (as shown in Table 4-5), though Academic does not illustrate a statistically significant change despite an average improvement of almost 3.5%..

Table 4-5 Pre- and Post-Assessment Scores

Assessments		Pre-score	Post-score	T-test (*demotes significance)
Proportional Reasoning Diagnostic	All students	76.41	84.24	0.0002*
	Honors	83.47	93.58	0.0001*
	Academic	62.74	66.16	0.238
Identifying Proportional Reasoning Task	All students	78.72	91.80	0.001*
	Honors	89.15	94.08	0.005*
	Academic	58.55	87.39	0.001*
Snowfall Task	All students	1.22	3.38	0.001*
	Honors	1.47	3.70	0.001*
	Academic	0.74	2.77	0.001*

Overall, scores for the entire population on the Proportional Reasoning Diagnostic assessment increased from 76.41% correct to 84.24% correct. Within the subgroups, honors students increased the number of items correct by about 10%, whereas academic only increased by about 1.5%, with t-test scores indicating that it is highly unlikely that Honors increases are from chance alone. Prior to the unit, 25% of academic students scored above 82%, but on the post-assessment the top quarter of students were above 92% in the academic classes (see Figure 4-5). Half of the students scored below median mark of 64% and 66% pre- and post-assessment respectively.

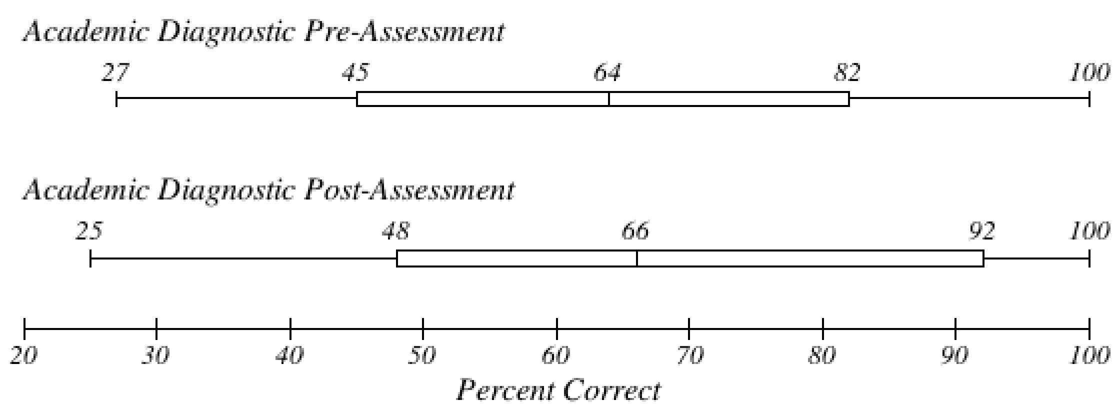


Figure 4-5 Percent correct on diagnostic assessment for academic students

Although the average score for the honors students increased by about 10%, half of the students scored above 92% on both the pre- and post-assessment, indicating that the students with proficient proportional reasoning ability did not change (Figure 4-6). However, the bottom half of students increased the minimum score from 36% to 58%, indicating that the lower half of students experience a fair amount of growth on the proportional reasoning diagnostic.

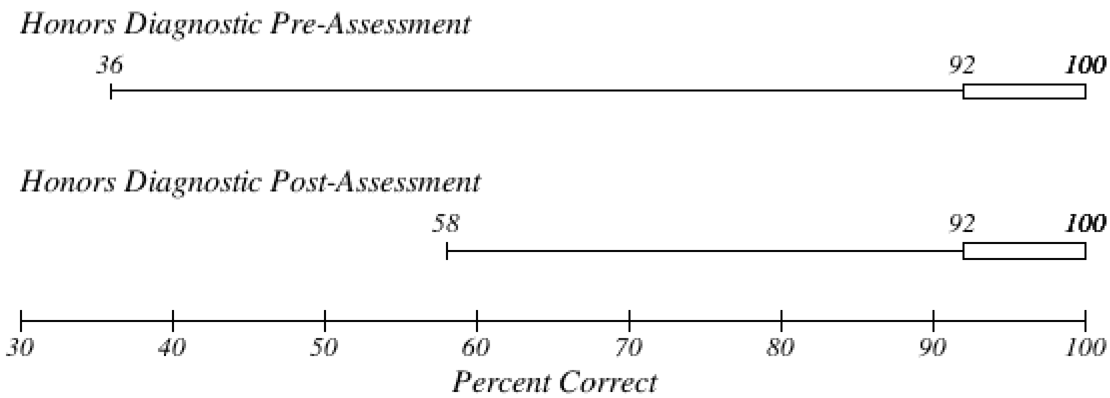


Figure 4-6 Percent correct on diagnostic assessment for honors students

While academics students' overall scores did not increase significantly, there is still some benefit to engaging in the curricular tasks, as seen in the scores for the other two assessments that are described in the paragraphs that follow.

Prior to the unit of study, on the "Identifying Proportional Relationships" task students were able to correctly identify a relationship as proportional 78.72% of the time, and post-tests indicate an increase to 91.8% of the time. Interestingly, honors students increased their scores by about 5%, from 89.15% to 94.08%, and academic students exhibited a much more significant increase of almost 30%, from 58.55% to 87.39%. Again, tests of significance for *Identifying Proportional Relationships* indicate $p<0.05$ that these results were obtained by chance. Similar results are exhibited on the performance assessment, *Snowfall Task,* where all students increased their rubric scores following the implementation of the tasks in the linear equations unit, though the academic students' increase was only slightly less than the growth for honors students.

Taking all three of these pairs of pre- and post-assessments into consideration, it appears that the curricular tasks, rooted in linearity and focused on proportional reasoning, had an impact on students proportional reasoning ability as evidenced by the increase in performance on the post assessments. When examining the two groups separately, honors students increased their performance on the Diagnostic Assessment and the Snowfall Task by a larger percentage, but academic students had a larger growth in identifying proportional relationships on the "Identifying Proportional Relationships" Task.

Academic students began with a score of 1.903 on the pre-assessment and displayed growth to an average score of 2.71 on the Diagnostic Rubric of Pupils' Proportional Reasoning (table 4-6). This would suggest that after engaging in the curricular tasks and discussion, academic students are at about the same level of understanding of proportional reasoning as entering honors students, with an increased ability to successfully solve problems with more difficult numerical structure and unfamiliar contexts. At this level, academic students need to work on improving work with proportional reasoning contexts that are unfamiliar and have more challenging numerical structures.

Table 4-6 Levels 0 – 4 of Proportional Reasoning Understanding

		Pre-	**Post-**	**T-test** (*denotes significance)
Level 0 - 4 of Proportional Reasoning Understanding	**All students**	2.67	3.33	0.0000000005*
	Honors	3.067	3.65	0.0000059*
	Academic	1.903	2.71	0.0000000005*

Prior to the instructional unit, half of all academic students could correctly identify a proportional relationship with an accuracy of less than 50%. Following the unit, three-forths of academic students could correctly identify proportional relationships 83% of the time, and no students scored lower than 50% (figure 4-7). Half of all honors students initially correcly identified proportional relationships 83% of the time, while the other half were between 50% and 83% of the time. Following the instructional unit, the top half of students remained the same, but the bottom half feel between 67% and 83% (figure 4-10).

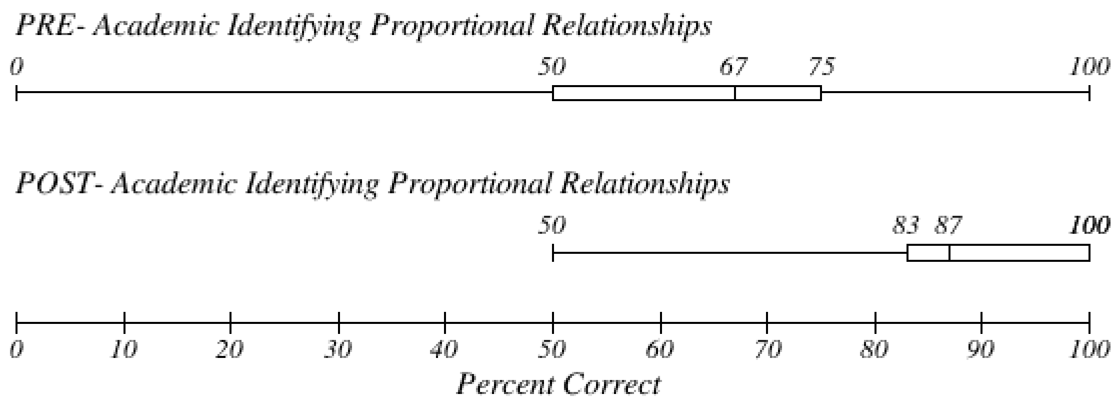


Figure 4-7 Percent correct for academic students when asked to identify a proportional relationship

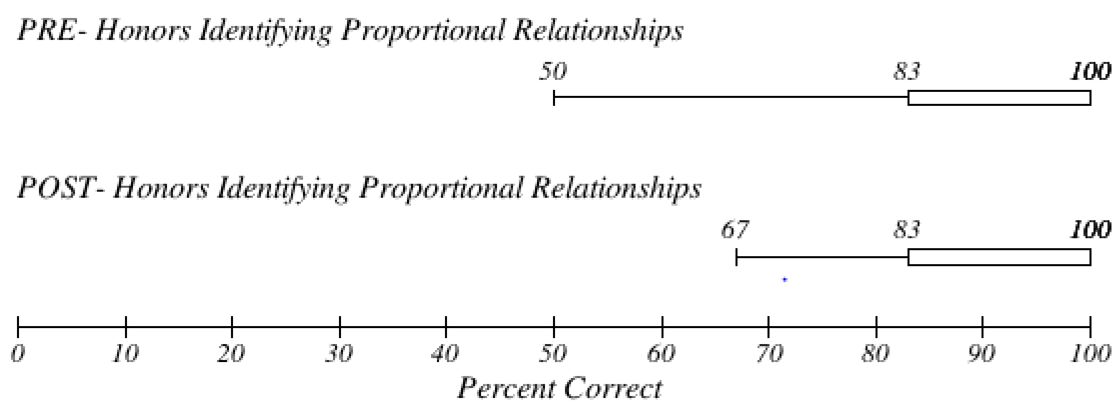


Figure 4-8 Percent correct for honors students when asked to identify a proportional relationship

Students in honors illustrated growth on the diagnostic assessment while students in both groups illustrated growth on their level of proportional reasoning and in identifying proportional relationships following the implementation of tasks in the linearity unit of study, suggesting that proportional reasoning understanding and ability can be improved within the context of Algebra 1.

4.4.1 Atypical patterns of change

There were some students that exhibited patterns of change different than the typical performance illustrated above. A few students demonstrated no change in their performance from the pre-assessment to the post-assessment. In the case of Maya, she answered all of the problems presented in the diagnostic assessment correctly prior to the linear equations unit; her post-test score remained 100% as well, illustrating that she did not regress in proportional reasoning ability according to the diagnostic. However, Maya's post-assessment illustrated a more varied approach to solving proportion problems, whereas she only used unit rates on the pre-assessment. There were no students that began with a low score on the pre-assessments and stay the same on the post-assessments, suggesting that all students benefited in some way from the implementation of the tasks in the linear equations unit. This was the case with Luke: he score 45% and had a diagnostic rubric score of 1 on the diagnostic pre-assessment and was able to improve to 66% with a diagnostic rubric score of 2. While this post-assessment performance might be considered sub-par, it is clear that improvement was made in proportional reasoning. Though some students performed similar to Luke, there were more students that made considerable progress from pre- to post-assessment, illustrating students that made the most improvement. Such was the case for Isabelle and Joseph. Prior to the linear equations unit, Joseph could only correctly identify a proportional relationship (Identifying Proportional Relationships) one-third of the time and he scored 1 on the Snowfall Task rubric. Following engagement in the linear equations unit tasks, Joseph could correctly identify a proportional relationship 100% of the time and increased his score on the Snowfall Task to 4. Isabelle's work illustrated a similar story to Joseph's: she originally scored 36% on the diagnostic assessment and a diagnostic rubric score of 1, but increased her scores to 100% and 4 in the post-assessment to illustrate a considerable amount of progress made on her part. These students' cases illustrate that though the typical performance indicated a statistically significant difference in pre- and post-unit assessments, some students benefited more than others but no students regressed in their proportional reasoning ability.

5.0 CONCLUSIONS AND IMPLICATIONS

The results of this study illustrate three important conclusions: proportional reasoning ability prior to an Algebra 1 class varies among different groups of students, course placement appears to account for differences in proportional reasoning ability in honors and academic students, and proportional reasoning ability can improve in the context of Algebra 1. The results of this study linked proportional reasoning ability to course placement, with honors students exhibiting higher levels of proportional reasoning ability than academic students. However, both groups increased this ability with engagement in high cognitive demand tasks within the Algebra 1 context as evidenced by performance on the “Identifying Proportional Relationships” task and the “Snowfall” task. This raises the question of whether improvement in proportional reasoning will improve overall performance in Algebra, and further research should focus on this question.

Students enter algebra with varied abilities in proportional reasoning: students in this study with higher proportional reasoning understanding and ability are placed into honors levels of Algebra 1. Given the strong alignment with course placement in this study, administering an assessment, such as the diagnostic assessment used in this study, to students can serve as data to assist schools with course placement as students are scheduling for Algebra 1. Having another data point to help decide course placement would help to make sure that students are properly placed in a class that will be appropriately challenging for them. In this study, there were a handful of students that seemed to be improperly placed in an algebra course based on their diagnostic score, either a high score indicative of an honors placement in an academic section or a low score indicative of an academic placement in an honors section.

The diagnostic assessment can also be used to assess the extent of students' proportional reasoning ability, allowing teachers to select tasks to target areas of weakness and foster growth in courses prior to Algebra 1. Identifying areas of weakness and strength prior to entry into Algebra 1 can inform teachers with regards to developing differentiation for students to foster growth in proportional reasoning ability and better position students for engagement in Algebra 1 concepts. There were a couple of academic students, such as the case of Maya described in the previous section, who scored high on the diagnostic pre-assessment. This could indicate that she is incorrectly placed in the academic section of Algebra I and may be better served in an honors section. Or as in the case of Luke: he scored low on the diagnostic pre-assessment despite being enrolled in an honors section. While Luke did show improvement on the post-assessment, his initial score and his post-assessment score suggest that his needs may be better met in an academic section of Algebra 1.

The implications for teachers are that tasks can be embedded in an algebra class that target algebra fundamentals of linearity but can still enhance proportional reasoning ability. Furthermore, the evidence of student growth using such a curriculum supports the assertion that tasks that are mapped to the essential understandings of proportionality within the context of Algebra 1 can be effective in promoting growth in proportional reasoning, while meeting the standards for algebra. The results of this study can inform teachers and teacher leaders when selecting tasks in algebra courses.

It is important to note that this study does not imply that tasks alone can improve proportional reasoning in the context of an Algebra 1 classroom. One limitation of this study is that no data was collected regarding implementation. Prior research (Stein, Grover, & Henningsen, 1996) suggests that high level tasks are necessary but not sufficient condition for ensuring student learning and that high level tasks often decline during implementation. Hence tasks are only one piece of the puzzle: careful steps must be taken to maintain cognitive demand and engage students in productive classroom discourse. As tasks are selected, teachers should not only be cognizant of the cognitive demand of the task, but also the maintenance of this demand through implementation. The Task Analysis Guide (TAG) and the Mathematical Tasks Framework provide a framework for this work, and the five practices for productive mathematical discourse can be used to a guide to facilitate discussion around the selected tasks.

Overwhelmingly, honors students outperformed academic students on the Diagnostic Assessment upon entry into Algebra. The honors students appear to be better poised to tackle algebraic concepts as related to proportionality and continue to exhibit growth. The relatively poor performance of academic students on the diagnostic assessment (both pre and post) suggests that these students are entering eighth grade without a sufficient ability to reason proportionally. In an effort to better prepare academic students for algebra, it is recommended that the curriculum in sixth and seventh grade be evaluated and mapped to the essential understandings of ratio, proportions and proportional reasoning, in addition to identifying tasks in the curriculum that can be classified as high-level tasks.

Given the aforementioned myriad of factors that can influence the maintenance of cognitive demand throughout task implementation (see Stein et al, 1996), it is recommended that teachers engage in professional development regarding implementation and discourse in conjunction with task selection. In addition to implementing professional development to dissect the curriculum, there should be some attention devoted to developing teachers' own understanding of proportionality as well as their ability to draw out more connections between proportional reasoning and algebra. In this study, connections between proportionality and linearity were drawn out extensively through the use of high-level tasks. While many curricula suggest connections to prior knowledge, it may not be drawn out as explicitly as it was done during the course of the linear equations unit in this study. Teachers of both middle and high school can benefit from professional development that facilitates teachers' abilities to make these connections to underlying concepts in meaningful ways.

Given the literature on the relationship between proportional reasoning and the results of this study, further research should seek to examine whether students perform better in Algebra when they can reason proportionally. It is clear that proportional reasoning ability impacts student placement, but more investigation is needed to explore how proportional reasoning ability influences understanding throughout Algebra 1. If proportional reasoning ability can be improved within the context of Algebra 1, can this improvement in proportional reasoning improve performance in Algebra 1? This research suggests that more investigation needs to be done to examine whether students do better in Algebra when they are proficient in proportional reasoning.

APPENDIX A

DIAGNOSTIC ASSESSMENT

Class

Mrs. Green put her students into groups of 5, with 3 girls in each group.

If Mrs. Green has 25 children in her class, how many boys and how many girls does she have?

1 Eels

There are 3 eels, *A, B* and *C* in the tank at the Zoo.

A: 15 cm long

B: 10 cm long

C: 5 cm long

The eels are fed sprats, the number depending on their length.

If *C* is fed 2 sprats, how many sprats should *B* be fed to match?

2 Onion Soup

An onion soup recipe for 8 persons is as follows:

8 onions

2 pints of water

4 chicken soup cubes

12 dessertspoons butter

½ pint cream

I am cooking onion soup for 2 people. How many dessertspoons of butter do I need?

6 Onion Soup

An onion soup recipe for 8 persons is as follows:

8 onions

2 pints of water

4 chicken soup cubes

12 dessertspoons butter

½ pint cream

I am cooking onion soup for 6 people. How much cream do I need?

Fruits' Price

At a fruit stand, 3 apples cost 90 cents.

You want to buy 7 apples. How much will they cost?

Books' Price

There is a sale at a bookstore.

Every book at this sale costs exactly the same.

Mary bought 6 books from the sale and paid $4.

Rosy bought 24 books from the sale. How much did Rosy pay?

1 Paint

Sue and Jenny want to paint together. They want to use exactly the same color.

Sue uses 3 cans of yellow paint and 6 cans of red paint. Jenny uses 7 cans of yellow paint.

How much red paint does Jenny need?

2 Paint

John and George are painting together. They want to use exactly the same color.

John uses 3 cans of yellow paint and 5 cans of green paint. George uses 20 cans of green paint.

How much yellow paint does Greorge need?

1 Campers

10 campers have camped at the "Blue Mountain" camp the previous week.

Each day there are 8 loaves of bread available for them to eat.

The loaves are provided by the camp's cook and the campers have to share the bread equally among the group.

This Monday 15 campers camped at the "Blue Mountain" camp.

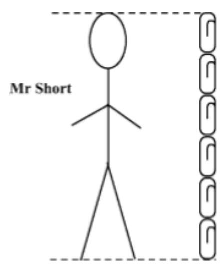
How many loaves are available to them for the day?

2 Campers

10 campers have camped at the "Blue Mountain" camp the previous week.
Each day there are 8 loaves of bread available for them to eat.
The loaves are provided by the camp's cook and the campers have to share the bread equally among the group.
The camp leader told the cook that for next Monday she should prepare 16 loaves of bread.
How many campers will be at the camp next Monday?

Mr. Short

You can see the height of Mr. Short measured with paper clips.



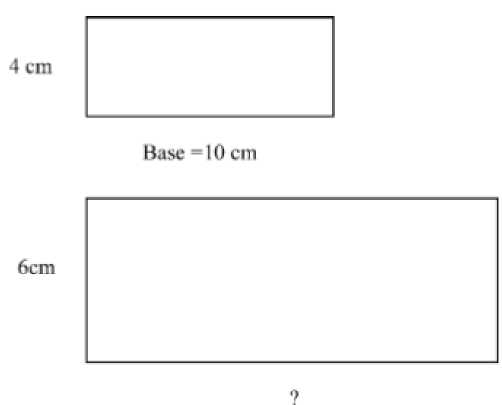
Mr. Short has a friend, Mr. Tall.
When we measure their heights with matchsticks, Mr. Short's height is four matchsticks and Mr. Tall's height is six matchsticks.
How many paper clips are needed for Mr. Tall's height?

Printing Press

A printing press takes exactly 12 minutes to print 14 dictionaries.
How many dictionaries can it print in 30 minutes?

Rectangles

These two rectangles have exactly the same shape, but one is larger than the other.
What is the length of the base of the larger rectangle?

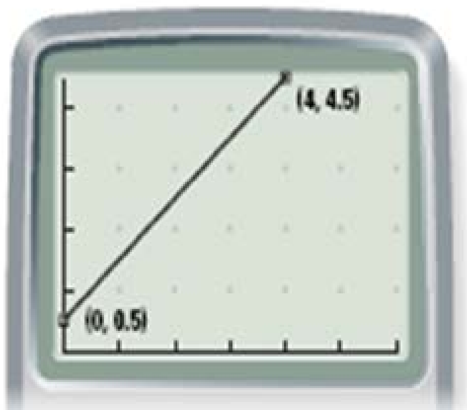


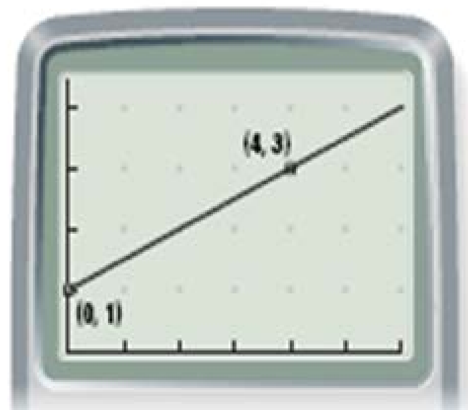
APPENDIX B

CURRICULAR TASKS

WALKING GRAPHS

The (*time, distance*) graphs below provide a lot of information about the "walks" they picture.





Write a set of instructions to that your partners could recreate this graph with the motion sensor.

AIRPLANE TASK

This table shows the temperature of the air outside an airplane at different altitudes.

Input	Output
Altitude (m)	Temperature (°C)
1000	7.7
1500	4.2
2200	−0.7
3000	−6.3
4700	−18.2
6000	−27.3

a. Add three columns to the table and record the change in input values, change in output values, and the rate of change. What do you notice about the rate of change?

b. Write a rule to describe the relationship between the altitude and the air outside the airplane. Describe what each part of your rule represents.

c. Use your rule to find the temperature at 9000 meters.

d. Where will the temperature reach freezing (0°C)?

INTERNET USAGE

Internet Use

Month	Time (h)	Total fee ($)
September	40	16.55
October	50	19.45
November	80	28.15

Is there a linear relationship between the time in hours that Hector uses the Internet and his total fee in dollars? If so, why do you think such a relationship exists?

BETH'S BIRTHDAY

Since the time Beth was born, the population of her town has increased at a rate of approximately 850 people per year. On Beth's 9th birthday the total population was nearly 307,650. If this rate of growth continues, what will be the population on Beth's 16th birthday?

SAM'S SWIMMING

Suppose Sam has already burned 325 calories before he begins to swim for his workout. His swim will burn 7.8 calories per minute.

a. Create a table of values for the calories Sam will burn by swimming 60 minutes and the total calories he will burn after each minute of swimming.

Number of minutes swimming	0	10	20	30	40	50	60
Calories burned by swimming							
Total calories burned							

b. Give a representation in which you could find the total calories for any number of minutes. Show your thinking.

c. Kelly completes the same swimming workout at Sam, but she does not get to burn any calories to start. How is her equation different from Sam's? Is either of the swimmer's in a proportional relationship? Why or why not?

d. On the same set of axes, graph the equation for total calories burned and the direct variation equation for calories burned by swimming. Sketch this graph below. Describe any similarities or differences.

STATE PARK AND ZOO TASK

Task 1: The cost of admission to the state park is $5.00 per person. Complete the table below showing the cost of admission for different sizes of groups.

Number of people in the group	Admission Cost
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

a. Explain how you completed the table. Examine the table. Describe at least three different patterns in the table.

b. Write an equation for the admission cost given any number of people.

Task 2: The cost of admission to the zoo is $1.00 for each person in a vehicle (car or van) plus $3.00 per vehicle. Complete the table below showing how much it will cost for admission based on the people in the vehicle.

Number of people in the group	Admission Cost
1	
2	
3	
4	
5	

6	
7	
8	
9	
10	

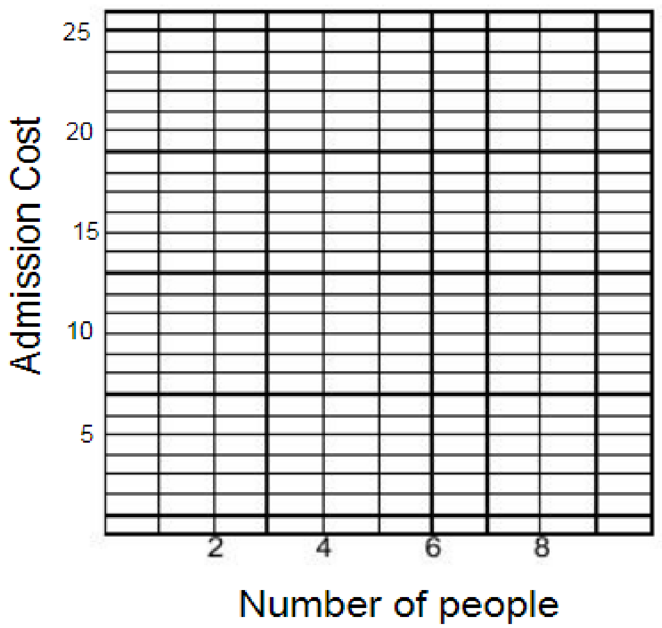
a. Explain how you completed the table. Examine the table. Describe at least three different patterns in the table.

b. Write an equation for the admission cost given any number of people.

Task 3: Graph the data.

a. Use the grid below to make a graph of the data for the cost of admission to the **state park**. Label three points on the graph with an ordered pair.

b. On the same grid, make a graph of the data for the cost of admission to the **zoo**. Label three points on the graph with an ordered pair.

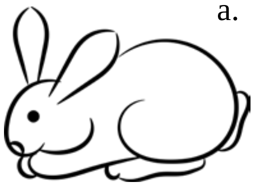


c. How are the graphs for the cost of admission to the state park and the zoo the same? How are they different?

d. Does either graph reflect a proportional relationship? Why or why not? How can you tell if a relationship is proportional from the table and the equation?

RABBIT & FROG TASK

The rabbit below walks 10 cm in 4 seconds.



a. a. Create several *(time, distance)* pairs for his friend, the frog, in which he would walk the same speed as the rabbit. Explain your thinking.

b. Create a rule to determine the frog or rabbit's distance for any time.

c. Create a graph to represent this relationship.

WALKING HOME TASK

Suppose Rabbit is 4 centimeters from home when he begins walking, and after 3 seconds it is 11.5 centimeters from home. Generate several "distance from home" and elapsed time values for other parts of Rabbit's journey so that he travels the same speed throughout his entire journey. Show your thinking.

REVISITING THE WALKING HOME TASK

Create a graph to represent Rabbit's journey home.

a. What is Rabbit's speed?

b. How is this shown on the graph? Show your thinking.

APPENDIX C

ASSESSMENTS

SNOWFALL PROBLEM

Consider the data on a snowstorm that hit both Mason City and Cedar Rapids. What can you tell from the table and the graph about each of these situations? Does either relationship represent a proportional relationship?

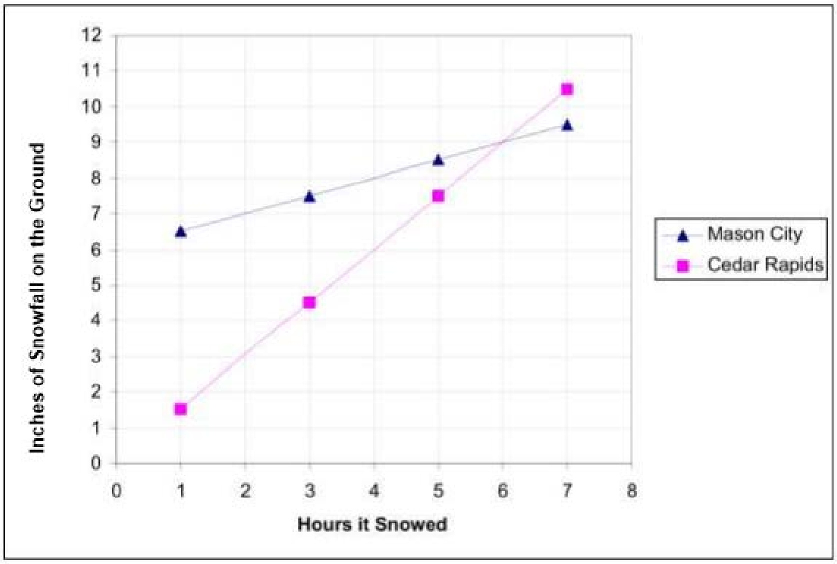
Mason City and Cedar Rapids have both been hit by a snowstorm. Mason City had 6 inches of snow on the ground before it started snowing. This storm brought 1/2 an inch of snow per hour to Mason City. In Cedar Rapids it snowed more heavily, 1.5 inches per hour. Fortunately, Cedar Rapids did not have any snow on the ground when the storm started.

Mason City

Hours it Snowed	Inches of Snow on the Ground
1	6.5
3	7.5
5	8.5
7	9.5

Cedar Rapids

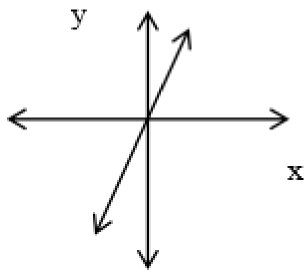
Hours it Snowed	Inches of Snow on the Ground
1	1.5
3	4.5
5	7.5
7	10.5



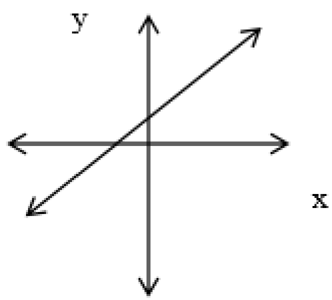
IDENTIFYING PROPORTIONAL RELATIONSHIPS

Indicate whether any of the relationships shown below are proportional and explain how you know.

a.



b.



c. $y = 3x + 4.5$

d. $y = 2.5x$

e.

x	y
4	6
6	9
8	12
10	15
12	18

f.

x	y
4	10
6	14
8	18
10	22
12	26

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