# COMPUTATIONS IN TWISTED MORAVA K-THEORY

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**COMPUTATIONS IN TWISTED MORAVA K-THEORY**

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In this dissertation we compute twisted Morava K-theory of all connective covers of the stable orthogonal group and stable unitary group, their classifying spaces, as well as spheres and Eilenberg-MacLane spaces. We employ techniques from [SW15] such as the universal coefficient theorem and Atiyah-Hirzebruch spectral sequence, and develop a similar theory for twists by mod 2 Eilenberg-MacLane spaces.

We establish that in all cases, there are only two possibilities: either the twisted homology vanishes, or it is isomorphic to untwisted homology.

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## 1.0 INTRODUCTION

Twisted cohomology encodes additional data coming from a principal bundle on a space. The earliest incarnation is cohomology with local coefficients which allows, in particular, to define Poincare duality for non-orientable manifolds, as long as we consider the cohomology with coefficients given by the local system of the orientation double cover.

The next prominent example is twisted K-theory introduced by Donovan and Karoubi [DK70], and studied extensively by Atiyah and Segal in [AS04, AS06]. While untwisted K-theory can be viewed global sections of a trivial bundles of Fredholm operators on an infinitely-dimensional space $\mathcal{H}$, twisted K-theory can be interpreted as global sections of a general $PU(\mathcal{H})$ principal bundles (where $PU(\mathcal{H})$ is the projective unitary group of $\mathcal{H}$). Thus we say that K-theory is twisted by $PU(\mathcal{H})$ bundles.

Twisted K-theory has found a lot of applications in physics, mainly in string theory, where Witten has shown that certain topological invariants (charges) of D-branes take values in twisted K-theory of spacetime. More recently, various twisted forms of generalized cohomology theories have appeared in classification of topological insulators from condensed matter physics and nuclear physics [FM13].

On the mathematical side, the Freed-Hopkins-Teleman theorem [FHT08] established a relationship between the equivariant twisted K-theory of a group and its Verlinde algebra. More recently, Ando, Blumberg and Gepner in [ABG10] used the refinement of the Witten genus to String orientation of topological modular forms $tmf$ to show that $tmf$ admit twists by $K(\mathbb{Z}, 3)$ bundles (here $K(\mathbb{Z}, 3)$ denotes the Eilenberg-MacLane space).

One can naturally ask what other cohomology theories admit similar twists. Sati and Westerland in [SW15] show that Morava K-theory $K(n)$, Morava E-theory $E_n$, and some of their variants admit twists by $K(\mathbb{Z}, n+1)$ bundles. The motivation for this theory came from string theory: it was conjectured by [Sa09] that a twisted form of Morava K-theory and E-theory should describe an extension of the untwisted setting in [KS04]. A vast generalization of this conjecture is proved in [SW15]. Morava K-theory $K(n)$ is in some sense an "extension" of K-theory: it is a complex-oriented cohomology theory that defined for every integer $n$ and prime $p$, and $K(1)$ is essentially just mod $p$ K-theory (here $n$ is the height of the corresponding formal group law). Similarly, Morava E-theory at level 2 $E_2$ is related to $tmf$.

We compute twisted Morava K-theory of all the connective covers of $BO$ and $BU$. This requires us to introduce twists by mod 2 Eilenberg-MacLane spaces, and establish a mod 2 analogue of the universal coefficient theorem from [SW15]. We observe that in all cases considered, twisted Morava K-theory is either equal to untwisted Morava K-theory of these covers, or is zero altogether, with transition occurring after height 2. Additionally, we offer a way to obtain Morava K-theory of higher connected covers of $BO$ using twisted homology. Next, we focus of twisted homology of Eilenberg-MacLane spaces and twists by fundamental classes of spheres, and show that at height 2 and above the twisted homology is again zero.

This document is organized as follows. Chapters 1 and 2 establish the language used in the remainder of the dissertation: Chapter 1 concerns vector bundles, classifying spaces, homotopy theory of the orthogonal groups, and Chapter 2 – generalized cohomology and spectra. Chapter 3 provides a very brief overview of complex-oriented cohomology theories and defines Morava K-theory. Chapter 4 focuses on the framework for twisted cohomology, and Chapter 5 describes twisted Morava K-theory and our computations.

## 2.0 VECTOR BUNDLES AND CLASSIFYING SPACES

### 2.1 FIBRATIONS AND FIBER BUNDLES

**Definition 2.1.** A map $p : E \to B$ is a (Hurewicz) *fibration* if it satisfies a homotopy lifting property: Given any homotopy $f_t : X \times [0,1] \to B$, and any map $\widetilde{f} : X \to E$ such that $p \circ \widetilde{f} = f_0$, there exists a homotopy $\widetilde{f_t} : X \times [0,1] \to E$, such that $p \circ \widetilde{f_t} = f_t$ and $\widetilde{f_0} = f$. In other words, there exists a lift of $f$ in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\widetilde{f}_0"] \arrow[d, "X\times\{0\}"'] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{f}"] & B
\end{tikzcd}$$

If the space $B$ is connected then all the fibers $p^{-1}(b)$ are homotopy equivalent and we denote the corresponding homotopy type by $F$, and the fibration by $F \to E \to B$.

A fibration $F \to E \to B$ induces a long exact sequence in homotopy

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots .$$

**Path-loop fibration.** An important example of a fibration is a *path-loop fibration*: let $X$ be a pointed connected topological space, and define $PX$ to be the *space of paths in* $X$, i.e. the set of continuous functions $\gamma : [0,1] \to X$ that start at the base point, topologized with the compact-open topology. Notice that the path space of a connected space is always contractible. There is a projection map $PX \to X$ that sends each path $\gamma$ to $\gamma(1)$. It is immediate that this map satisfies the homotopy lifting property and is, therefore, a fibration.

The fiber over a basepoint then consists of path that start and end at the basepoint, i.e. the loop space $\Omega X$ of $X$. So we have the following fibration:

$$\Omega X \to PX \simeq * \to X.$$

(In this thesis we use $\simeq$ between spaces to mean homotopy equivalent and $\cong$ between spaces to mean homeomorphic. See also the Appendix for notational conventions.)

**Fiber bundles.**

**Definition 2.2.** A *fiber bundle* with fiber $F$ is a map $E \xrightarrow{p} B$ such that for every $b \in B$, there exists an open neighborhood $U \subset B$ of $b$ and a local local trivialization of $E$ on $U$, i.e. a homeomorphism $\phi : p^{-1}(U) \to U \times F$ such that $pr_1 \circ \phi = p$.

If the base space $B$ is paracompact, then a fiber bundle is always a fibration with homeomorphic fibers, and we use the notation $F \to E \to B$ as well.

We will be interested in fiber bundles with additional structure:

**Definition 2.3.** Let $G$ be a topological group. A *$G$-principal bundle* is a fiber bundle $F \to E \to B$ such that:

- $p^{-1}(b) \cong G$ for any $b \in B$;
- $G$ acts on the right on $E$ and the action is continuous;
- The action of $G$ on $E$ preserves the fibers (i.e. $p(y \cdot g) = p(y)$), and restricts to a free and transitive action them.

**Definition 2.4.** A *real vector bundle of rank $n$* is a fiber bundle $E \xrightarrow{p} B$ such that:

- $p^{-1}(b)$ is a real $n$-dimensional vector space for every $b \in B$;
- The local trivializations $\phi : p^{-1}(U) \to U \times F$ restrict to linear isomorphisms on the fibers, i.e. $\pi|_{p^{-1}(b)} : p^{-1}(b) \xrightarrow{x} \{b\} \times F$ is an isomorphism of vector spaces.

A complex vector bundle can be define similarly. Note that every rank $n$ complex vector bundle is a rank $2n$ real vector bundle by forgetting the complex structure.

A *bundle map* between the bundles $\xi_1 : E_1 \to B$ and $\xi_2 : E_2 \to B$ *over the same base space* is a continuous map $f : E_1 \to E_2$ such that $f$ is linear on the fibers and the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\xi_1"] & E_2 \arrow[dl, "\xi_2"] \\ B & \end{tikzcd}$$

Two vector bundles $\xi_1 : E_1 \to B$ and $\xi_2 : E_2 \to B$ are *isomorphic* if there exists a bundle map $f : E_1 \to E_2$ that is a homeomorphism restricts to a linear isomorphism on each fiber.

**Pullback.** Given two vector bundles $\xi_1 : E_1 \to B_1$ and $\xi_2 : E_2 \to B_2$ with *different* base spaces, a map $\phi : E_1 \to E_2$ is a *bundle map* if it is linear on each fiber, and there exists a map $f : B_1 \to B_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "\xi_1"] & E_2 \arrow[d, "\xi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

We denote such a bundle map by $(\phi, f) : \xi_1 \to \xi_2$.

Given a bundle $\xi : E \to B$, and a map $f : A \to B$, the *pullback bundle* $f^*(\xi)$ of $\xi$ along $f$ is a bundle with the total space defined by $E(f^*(\xi)) := \{(x, v) \in A \times E | \xi(v) = f(a)\}$.

Since the fibers of a vector bundle are vector spaces, we can extend operations on vector spaces to vector bundles by performing them fiberwise.

**Whitney sum.** Let $\xi_1 : E_1 \to B$ and $\xi_2 : E_2 \to B$ be two vector bundles over the same base space, and let $d : B \to B \times B$ be the diagonal embedding. Then the pullback $d^*(\xi_1 \times \xi_2)$ is called the *Whitney sum* of $\xi_1$ and $\xi_2$ and denoted $\xi_1 \oplus \xi_2$.

Notice that the fiber of $\xi_1 \oplus \xi_2$ over $b$ is canonically isomorphic to the direct sum of fibers of $\xi_1$ and $\xi_2$ over $B$.

**Tensor product.** Similarly, given two vector bundles $\xi_1 : E_1 \to B$ and $\xi_2 : E_2 \to B$, we can define their *tensor product* $\xi_1 \otimes \xi_2$ with the fiber over each point $b \in B$ is $\xi_1^{-1}(b) \otimes \xi_2^{-1}(b)$. Note that the rank of the tensor product bundle $\xi_1 \otimes \xi_2$ is the product of the ranks of $\xi_1$ and $\xi_2$. In particular, tensor product of line bundles is still a line bundle.

## 2.2 CLASSIFYING SPACES OF BUNDLES AND HOMOTOPY THEORY OF $BO$

As a motivating example, consider the real Grassmannian $Gr_n(\mathbb{R}^\infty)$, which is the space of $n$-dimensional subspaces of $\mathbb{R}^\infty$. We can topologize it as follows. Let $Gr_n\left(R^{n+k}\right)$ be the collection of $n$-dimensional subspaces of $\mathbb{R}^{n+k}$ and let $V_n(\mathbb{R}^{n+k})$ be the Stiefel manifold, i.e. the collection of all orthonormal $k$-frames in $\mathbb{R}^{n+k}$ given the topology of a subspace of $\left(\mathbb{R}^{n+k}\right)^k$. Then there is a map $q : V_k(\mathbb{R}^{n+k}) \to Gr_n(\mathbb{R}^{n+k})$, sending a frame to a subspace that it spans. We can give $Gr_n(\mathbb{R}^{n+k})$ the quotient topology, and topologize $Gr_n(\mathbb{R}^\infty) = \bigcup_{k=0}^{\infty} Gr_n\left(\mathbb{R}^{n+k}\right)$ as a union.

We can define a rank $n$ vector bundle $\gamma_n$ whose fiber over point $V \in Gr_n(\mathbb{R}^\infty)$ is exactly the vector space $V$. Such a bundle is called a *universal* (also sometimes canonical, tautological) rank $n$ bundle.

We will use the notation $BO(n)$ for $Gr_n(\mathbb{R}^\infty)$, and $EO(n)$ for the total space of $\gamma_n$.

**Theorem 2.5.**

*For any rank $n$ vector bundle $\xi : E \to B$ there exists a map $f : B \to BO(n)$ such that $\xi$ is a pullback of $\gamma_n$ along $f$.*

*Moreover, this induces a one-to-one correspondence between rank $n$ bundles up to isomorphism and $[B, BO(n)]$, i.e. homotopy classes of maps $B \to BO(n)$.*

There is an analogous result for classification of principal bundles:

**Theorem 2.6.**

*Let $G$ be an abelian group. Then there exists a space $BG$, called the* classifying space *of $G$, a contractible space $EG$, and a principal $G$-bundle $\gamma_G : EG \to BG$, such that if $\xi : E \to B$ is any other principal bundle, there exists a map $f : B \to BG$ such that $\xi$ is a pullback of $\gamma_G$ along $f$.*

*This induces a one-to-one correspondence between principal $G$-bundles up to isomorphism and homotopy classes of maps $B \to BG$.*

The same result holds for vector bundles by replacing $BG$ by $BO(n)$.

There exist many constructions of the classifying space for a given group $G$, and most of them are functorial (for example, the bar construction [May72, Chapters 9-11]), so from now on, we will refer to the classifying space functor $B$ without specifying a particular construction.

The classifying space $BG$ is also sometimes called the delooping of $G$ because of the following fact:

**Theorem 2.7.**
*If $G$ is an abelian group, then $\Omega BG \simeq G$.*

This determines the homotopy groups of $BG$: $\pi_k(BG) \cong \pi_{k-1}(G)$.

Let $O$ be the stable orthogonal group, $O := \varinjlim O(n)$, and $U$ be the stable unitary group, $U := \varinjlim U(n)$. The following theorem completely determines the homotopy groups of the corresponding classifying spaces:

**Theorem 2.8** (Bott periodicity)**.**
*The classifying spaces of the orthogonal and unitary groups satisfy, respectively, the following*

$$\begin{aligned} \Omega^8 BO &\simeq BO, \\ \Omega^2 BU &\simeq BU. \end{aligned}$$

*Furthermore, the homotopy groups of $BO$ are*

| $\pi_{8k+1}$ | $\pi_{8k+2}$ | $\pi_{8k+3}$ | $\pi_{8k+4}$ | $\pi_{8k+5}$ | $\pi_{8k+6}$ | $\pi_{8k+7}$ | $\pi_{8k+8}$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |

## 2.3 THE WHITEHEAD TOWER OF BO

Whitehead tower is a way of approximating a space by a sequence of spaces of increasing connectivity that under mild assumptions converges to the original space.

**Definition 2.9.** The *Whitehead tower* of a space $X$ is a sequence of fibrations $X\langle n+1\rangle \to X\langle n\rangle \to \cdots \to X$, such that each $X\langle n\rangle$ is $(n-1)$-connected (i.e. the first non-trivial homotopy group occurs in degree $n$), and the induced map $\pi_i(X\langle n\rangle) \to \pi_i(X)$ is an isomorphism for $i \geq n$ (i.e. we are successively killing homotopy groups of $X$, starting from the bottom). In particular, $X\langle 1\rangle$ is the universal cover of $X$. In general $X\langle n\rangle$ is called the *$n$-connected cover* of $X$.

By the long exact sequence of a fibration, the fiber of every $X\langle n-1\rangle \to X\langle n\rangle$ is $K(\pi_n(X), n+1)$. Indeed, if $F$ denotes the fiber of $X\langle n+1\rangle \to X\langle n\rangle$, then we have

$$\cdots \to \pi_i(F) \to \pi_i(X\langle n+1\rangle) \to \pi_i(X\langle n\rangle) \to \pi_{i-1}(F) \to \cdots .$$

When $i \geq n+1$, the second map is an isomorphism, so $\pi_i(F) = 0$ in this range for $i \geq n$. On the other hand, $X\langle n\rangle$ is $(n-1)$-connected and $X\langle n+1\rangle$ is $n$-connected, so for $i \leq n-1$ $\pi_i(X\langle n+1\rangle) = \pi_i(X\langle n\rangle) = 0$, which implies $\pi_i(F) = 0$ for $i \leq n-2$. Therefore, $F$ has at most one non-trivial homotopy group in dimension $n-1$, and it has to be $\pi_n(X\langle n\rangle) \cong \pi_n(X)$, so $F$ is a $K(\pi_n(X), n-1)$. For $n \geq 1$, this can be made into an abelian group, and the fibration can be made into a principal bundle, so we have a fibration sequence

$$K(\pi_n(X), n-1) \to X\langle n+1\rangle \to X\langle n\rangle \to K(\pi_n(X), n)$$

The main example appearing in our work is the connected covers of the stable orthogonal group and its classifying space. Some of the connected covers have distinguished names:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
K(\mathbb{Z},11) \arrow[r] & BO\langle 13\rangle = BNinebrane \arrow[d] & \\
K(\mathbb{Z}/2,9) \arrow[r] & BO\langle 11\rangle = B2\text{-}Spin \arrow[r] \arrow[d] & K(\mathbb{Z},12) \\
K(\mathbb{Z}/2,8) \arrow[r] & BO\langle 10\rangle = B2\text{-}Orient \arrow[r] \arrow[d] & K(\mathbb{Z}/2,10) \\
K(\mathbb{Z},7) \arrow[r] & BO\langle 9\rangle = BFivebrane \arrow[r] \arrow[d] & K(\mathbb{Z}/2,9) \\
K(\mathbb{Z},4) \arrow[r] & BO\langle 8\rangle = BString \arrow[r, "\frac{1}{6}p_2"] \arrow[d] & K(\mathbb{Z},8) \\
K(\mathbb{Z}/2,1) \arrow[r] & BO\langle 4\rangle = BSpin \arrow[r, "\frac{1}{2}p_1"] \arrow[d] & K(\mathbb{Z},3) \\
 & BO\langle 2\rangle = BSO \arrow[r, "w_2"] \arrow[d] & K(\mathbb{Z}/2,2) \\
 & BO \arrow[r, "w_1"] & K(\mathbb{Z}/2,2)
\end{tikzcd} \tag{2.10}$$

Notice that notation $BO\langle n\rangle$ means $(BO)\langle n\rangle$ rather than $B(O\langle n\rangle)$.

Some of the maps in the Whitehead tower of $BO$ are familiar. For example. In above diagram, $w_1$ and $w_2$ are first and second Stiefel-Whitney classes, respectively, and $\frac{1}{2}p_1$ is the fractional first Pontryagin class.

Unlike $SO$ and $Spin$, which have classical descriptions as a Lie groups, $String$ (as well as higher covers) is not a Lie group (in particular, it is infinitely-dimensional) but there are models realizing $String$ as a *2-group*. But we will not need such explicit characterizations in this thesis as we are mainly interested in the homotopy type.

*Remark.* This fibration sequence $K(\pi_n(X), n-1) \to X\langle n+1\rangle \to X\langle n\rangle \to K(\pi_n(X), n)$ gives rise to the homotopy mapping sequence:

$$[Y, K(\pi_n(X), n-1)] \to [Y, X\langle n+1\rangle] \to [Y, X\langle n\rangle] \to [Y, K(\pi_n(X), n)]$$

which is equivalent to

$$H^{n-1}(Y;\pi_n(X)) \to [Y, X\langle n+1\rangle] \to [Y, X\langle n\rangle] \to H^n(Y;\pi_n(X)).$$

A class $\alpha \in [Y, X\langle n\rangle]$ lifts to a class in $[Y, X\langle n+1\rangle]$ iff and only if its image in $H^n(Y;\pi_n(X))$ is zero. This is the starting point of study of geometric structures.

For example, we say that an oriented manifold $M$ is *Spin* if its tangent bundle can be given the structure of a *Spin*-principal bundle, so it defines a map $M \to BSpin$. To determine whether the a given manifold has a Spin structure, we need to study lifts of the map $M \to BSO = BO\langle 2\rangle$ (corresponding to the oriented tangent bundle) to a map $M \to BSpin = BO\langle 4\rangle = BO\langle 3\rangle$. Such a lift exists if and only if the corresponding element of $H^2(M;\pi_2(BSO)) \cong H^2(M;\mathbb{Z}/2)$, i.e., the second Stiefel-Whitney class of the tangent bundle, is zero.

The Whitehead tower can be constructed as a dual to the usual *Postnikov tower*, which approximates the space by *n-truncated* spaces $X_n$.

# 3.0 STABLE HOMOTOPY AND GENERALIZED HOMOLOGY

## 3.1 GENERALIZED (CO)HOMOLOGY THEORIES

For a comprehensive treatment of generalized cohomology theories, the reader is referred to [Ad74] and [Swi75].

### 3.1.1 Axioms and examples

Let $\mathbf{Top_2}$ denote the category of pairs of topological spaces $(X, A)$, where $A$ is a subspace of $X$, and continuous maps $f : (X, A) \to (Y, B)$ such that $f(A) \in B$, and let $\kappa : \mathbf{Top_2} \to \mathbf{Top_2}$ be the restriction defined by $\kappa(X, A) = (A, \varnothing)$. Let $\mathbf{Ab}$ denote the category of abelian groups.

**Definition 3.1.** A *generalized cohomology theory* is a collection of contravariant functors $\{E^n : \mathbf{Top_2} \to \mathbf{Ab}\}$, $n \in \mathbb{Z}$, together with natural transformations $\delta^n : E^n \circ \kappa \to E^{n+1}$, $n \in \mathbb{Z}$, such that the following properties hold:

1. *Homotopy invariance.* If $f, g : (X, A) \to (Y, B)$ are homotopic, then $E^n(f) = E^n(g)$.
2. *Exactness.* For each pair $(X, A)$ the following sequence is exact:

$$\ldots \longrightarrow E^{n-1}(A, \varnothing) \longrightarrow E^n(X, A) \longrightarrow E^n(X, \varnothing) \longrightarrow \ldots .$$

3. *Excision.* If $U$ is such that $\overline{U} \in \mathring{A}$, then

$$E^n(X, A) \cong E^n(X \setminus U, A \setminus U)$$

where the isomorphism is induced by the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$.

*Remarks.* (a) We call $E^n(X, \varnothing)$ the *n-th cohomology group of* $(X.A)$, and $\delta^n$ the coboundary map. We write $E^n(X)$ for $E^n(X, \varnothing)$ and call it the *n-th cohomology group of* $(X)$.

(b) If $X$ is a *pointed* space, we define the *reduced* cohomology of $X$ as

$$\widetilde{E}^n(X) := E^n(X, \{basepoint\})$$

.

(c) Note that, unlike the singular cohomology groups, generalized cohomology groups do not, in general, come from a cochain complex.

We can define generalized *homology* theories as covariant functors $E_n : \mathbf{Top_2} \to \mathbf{Ab}$ where satisfying similar properties.

**Definition 3.2.** The ring $E^*(pt)$ (resp. $E_*(pt)$) is called the *coefficient ring* of the generalized cohomology (resp. homology) theory $E$.

**Definition 3.3.** A generalized cohomology theory $E^*$ is called *additive* if the map

$$E^n\Big(\coprod_\alpha X_\alpha,\ \coprod_\alpha A_\alpha\Big) \xrightarrow{\sim} \prod_\alpha E^n\left(X_\alpha,\ A_\alpha\right),$$

induced by inclusion of summands, is an isomorphism.

A generalized homology theory $E_*$ is called *additive* if the map

$$\bigoplus_\alpha E_n\left(X_\alpha, A_\alpha\right) \xrightarrow{\sim} E_n\Big(\coprod_\alpha X_\alpha, \coprod A_\alpha\Big),$$

is an isomorphism.

We will only be working with additive (co)homology theories.

**Definition 3.4** (Multiplicative cohomology theory)**.** A generalized cohomology theory $E^*$ is called *multiplicative* if, for each $X$, $E^*(X)$ has the structure of a graded ring (i.e. there is an analogue of cup product).

The coefficient ring of an additive cohomology theory essentially determines the entire cohomology theory:

**Theorem 3.5** (Uniqueness theorem)**.**
*Let $T : E^n \Rightarrow F^n$ be a natural transformation of additive generalized cohomology theories, such that $T : E^*(pt) \to F^*(pt)$ is an isomorphism. Then $T : E^*(X, A) \to F^*(X, A)$ is an isomorphism for all CW pairs $(X, A)$.*

*Remarks.* 1. If the cohomology theories in the above theorems are not additive, we still have an isomorphism but only for *finite* CW pairs.

2. Note that the theorem does not say that the cohomology theories with isomorphic coefficient rings are necessarily the same: we need to have an actual natural transformation between cohomology theories that exhibits the isomorphism of coefficient rings.

3. A similar theorem is true for homology theories as well, and is substantially easy to prove, since homology commutes with direct limits.

The following properties of generalized cohomology theories are similar to the familiar properties of singular cohomology.

**Proposition 3.6.** *Let $E^*$ be a generalized cohomology theory.*

1. Long exact sequence of a triple. *Given a triple of spaces $(X, A, B)$ such that $B \subset A \subset X$, then the following sequence is exact:*

$$\ldots \longrightarrow E^n(X, A) \longrightarrow E^n(X, B) \longrightarrow E^n(A, B) \longrightarrow E^{n+1}(X, A) \longrightarrow \ldots.$$

2. Mayer-Vietoris sequence. *If $X = A \cup B$, then*

$$\ldots \longrightarrow E^{n-1}(A \cap B) \longrightarrow E^n(X) \longrightarrow E^n(A) \oplus E^n(B) \longrightarrow E^n(A \cap B) \longrightarrow \ldots$$

Analogous properties hold for homology.

**Examples.** 1. *Singular cohomology.*

Singular cohomology with coefficients in a ring $R$, $H^*(-; R)$, satisfies the axioms of an additive generalized cohomology theory. It is the generalized cohomology theory whose coefficient ring is $R$.

2. *Topological K-theory.*

Denote $K^0(X)$ to be the Grothendieck group of complex vector bundles with respect to Whitney sum, and let $K^{-n}(X) := K(\Sigma^n X)$ for $n \geq 0$. We extend this to all integers by setting $K^n(X) := K^{n-2}(X)$. We can define $K(X, A)$ to be $K(X/A)$. Then the functors $K^n(-, -)$ give a cohomology theory called *Complex K-theory*.

The coefficient ring for complex K-theory is $K_* = \mathbb{Z}[\beta, \beta^{-1}]$ where $\beta \in K^2(pt)$.

3. *Bordism theories.*

   Let $\Omega_n^{un}(X)$ be the ring of singular $n$-manifolds $M \to X$ in $X$ up to cobordism with addition given by disjoint union and multiplication given by Cartesian product. We can also define $\Omega_n^{un}(X, A)$ to be singular manifolds with boundary in $X$ whose boundary lands in $A$. Then $\Omega_n^{un}(-.-)$ defines a generalized homology theory *unoriented cobordism*. Thom in his seminal paper [Th54] showed that

$$\Omega_*^{un} \cong \mathbb{Z}/2[x_1, x_2, \cdots]$$

   with $|x_i| = i$ and $i \neq 2^n - 1$.

4. *Morava K-theory.*

   The subject of this thesis. This theory is surveyed in [JW75] [Ra86] [W91]. Its definition and main properties will be presented in section 4.4.

### 3.1.2 The Atiyah-Hirzebruch spectral sequence

Let $F \to E \to B$ be a fibration, and $R$ a generalized multiplicative cohomology theory. Then there exists a spectral sequence, called the Atiyah-Hirzebruch spectral sequence [AH61] [Swi75] converging to $R^*(E)$, with the $E_2$-page

$$E_2^{p,q} = H^p(F; R^q(B)) \Rightarrow R^{p+q}(E).$$

To understand this spectral sequence, assume for a moment that $R^*$ is singular cohomology with coefficients in a field. If $F \to E \to B$ is a trivial vector bundle, i.e. $E = F \times B$, so $H^{p+q}(E) \cong \bigoplus_{i+j=p+q} H^i(F) \otimes H^i(B) \cong \bigoplus_{i+j=p+q} H^i(F; H^i(B))$. So the spectral sequence measures the failure of the fibration to be trivial and the failure of Künneth isomorphism for $R$ at the same time.

## 3.2 SPECTRA

Recall the following classical result (see e.g. [Ad74][Swi75]):

**Theorem 3.7.**

*If $X$ is an $n$-connected CW-complex, then the suspension homomorphism on homotopy groups*

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

*is an isomorphism $k \leq 2n$.*

In particular, it implies that for large enough $n$, $\pi_{n+k}\left(S^n\right) \cong \pi_{n+1+k} S^{n+1}$. This motivates the following definition:

**Definition 3.8.** The *$n$-th stable homotopy group* is the direct limit

$$\pi_n^S = \varinjlim_k \pi_{n+k}\left(S^k\right) = \varinjlim_k \left[\Sigma^k S^n, S^k\right].$$

A spectrum is a notion that captures stable phenomena (i.e. phenomena that are preserved by the suspension functor) like the above, and the homotopy category of spectra can be viewed as "abelianization" of the homotopy category of topological spaces, thus making it more accessible to algebraic techniques.

**Definition 3.9.** A *spectrum* is a collection of topological spaces $\{E_n\}$, $n \in \mathbb{N}$ together with maps $\sigma_n : \Sigma E_n \to E_{n+1}$.

Recall the *suspension-loop adjunction*: for any pointed topological spaces $X$, $Y$,

$$[\Sigma X, Y] \cong [X, \Omega Y],$$

where $[A, B]$ denotes the set of (pointed) homotopy classes of maps $A \to B$.

Therefore, any map $\sigma_n : \Sigma E_n \to E_{n+1}$ gives rise to a map $\omega_n : E_n \to \Omega E_{n+1}$ and vice versa.

**Definition 3.10.** We say that a collection of topological spaces $E_n$ together with maps $\omega_n : E_n \to \Omega E_{n+1}$ is an *$\Omega$-spectrum* if $\omega_n$ is a homeomorphism for any $n \in \mathbb{N}$.

*Remark.* The terminology is somewhat inconsistent: certain sources reserve the name "spectra" for $\Omega$-spectra, and use the term "prespectra" for spectra (in our sense).

**Examples.** 1. *Sphere spectrum.*

Let $\mathbb{S}$ denote the spectrum with $\mathbb{S}_n = S^n$, where $S^n$ is the $n$-sphere. The maps $\sigma_n$ come from the homeomorphism $\Sigma S^n \cong S^{n+1}$. Note that $S$ is not an omega-spectrum.

2. *Suspension spectrum of a space.*

More generally, let $X$ be a topological space, and define $\Sigma^\infty X$ to be the spectrum with the $n$th level $(\Sigma^\infty X)_n := \Sigma^n X$.

In particular, $\mathbb{S} = \Sigma^\infty S^0$.

3. *Eilenberg-MacLane spectra.*

If $G$ is an abelian group, the *Eilenberg-Maclane spectrum of* $G$, denoted $HG$, is the spectrum with the $n$-th level $(HG)_n = K(G, n)$, the $n$-th Eilenberg-MacLane space, i.e. the space with only one non-trivial homotopy groups in dimension $n$.

Since for any space $X$, $\pi_k(X) \cong \pi_{k-1}(\Omega X)$, loop space $\Omega K(G, k)$ is an Eilenberg-MacLane space $K(G, k-1)$, and therefore $HG$ is an omega-spectrum.

Note that, unlike the two previous examples, $\Sigma HQ_n$ is not, in general, homeomeorphic to $HQ_{n+1}$ (indeed, $K(\mathbb{Z}, 1)$ is a circle, however $\Sigma K(\mathbb{Z}, 2) = \Sigma S^1 = S^2$ is not an Eilenberg-Maclane space $K(\mathbb{Z}, 2)$ – for example, $\pi_3(S^2)$ is non-trivial).

4. *K-theory spectrum.*

The spectrum $K$ with

$$K_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even,} \\ \Omega(\mathbb{Z} \times BU) = U & \text{if } n \text{ is odd} \end{cases}$$

is the *complex $K$-theory spectrum.* Bott periodicity implies that this is an $\Omega$-spectrum.

5. *Thom spectra.*

The reader is referred to chapter 5 for definitions and details of the construction. Briefly, let $\gamma_n$ be the universal real $n$-plane bundle over $BO(n)$, and let $MO_n = Th(\gamma_n)$ be its Thom space.

Recall that the the suspension of a Thom space admits the equivalence $\Sigma(Th(\xi)) \cong Th(\xi \oplus 1)$, where 1 is the trivial bundle on $X$, and $\oplus$ is the Whitney sum. Since $\xi \oplus 1$ admits a bundle map to $\xi_{n+1}$, this induces a map $\Sigma MO_n \to MO_{n+1}$. Therefore, the spaces $MO_n$ assemble into a spectrum $MO$.

Similarly one can define the Thom spectra $MSO$, $MU$, $MSpin$ etc.

Generalized Thom spectra play a crucial role in this dissertation. They will be discussed in more detail in later chapters.

### 3.2.1 Homotopy groups of a spectrum

In analogy to stable homotopy groups of spheres we want to define (stable) homotopy groups of any spectrum.

Note that the structure maps $\sigma_n$ of a spectrum $E$ induce the following homomorphisms:

$$\left[S^{n+k}, E_k\right] \xrightarrow{\Sigma} \left[\Sigma S^{n+k} = S^{n+k+1}, \Sigma E_k\right] \xrightarrow{\sigma_n \circ (-)} \left[S^{n+k+1}, E_{k+1}\right].$$

This allows us to make the following definition:

**Definition 3.11.** The *nth homotopy group of a spectrum* $E$ is defined as the colimit

$$\pi_n(E) := \varinjlim_k \pi_{n+k}\left(E_k\right).$$

Note that $n$ in this definition does not have to be positive.

**Definition 3.12.** A spectrum $E$ is *connective* if $\pi_n E = 0$ for $n < 0$.

**Examples.** 1. The definition directly implies that $\pi_n \mathbb{S} = \pi_n^S$ for $n \geq 0$.

2. Taking $E = HG$, we see that $\pi_0 HG = G$ and $\pi_n HG = 0$ for $n \neq 0$.

3. Homotopy groups of the K-theory spectrum are determined by Bott periodicity.

4. Thom in his seminal paper [Th54] showed that $\pi_*(MO) \cong \Omega_*^{un}$, where $\Omega_*^{un}$ is the coefficient ring of the unoriented cobordism homology.

   Quillen [Qui69] showed that $\pi_*(MU) \cong \Omega_* \cong L$ where $\Omega_*$ is the coefficient ring for complex cobordism, and $L$ is the Lazard ring (the ring carrying the universal formal group law, see section 4.2).

**Maps of spectra**. Let $E$ and $E'$ be spectra with structure maps given by $\sigma_n : \Sigma E_n \to E_{n+1}$ and $\sigma'_n : \Sigma E'_n \to E'_{n+1}$. Then a *map of spectra* $f : E \to E'$ is a collection of maps $f_n : E_n \to E'_n$ such that the following diagram commutes for each $n$:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma E'_n \arrow[r, "\sigma'_n"] & E'_{n+1} \end{tikzcd}$$

where $\Sigma f_n$denotes the suspension of the map $f_n$

### 3.2.2 Brown's representabilty theorem

Let $E$ be an $\Omega$-spectrum, and denote

$$\begin{aligned} E_n(X) &:= \pi_n(X \wedge E), \\ E^n(X) &:= [X, E_n]. \end{aligned}$$

where $(X \wedge E)_n := X \wedge E_n$.

Then it is easy to check that $E_n$ and $E^n$ satisfy Eilenberg-Steenrod axioms for generalized homology and cohomology theory respectively, i.e. any spectrum defined both a generalized homology theory and a generalized homology theory.

On the other hand, it is a classical result that the singular $G$-cohomology classes of $X$ correspond to maps into the Eilenberg-MacLane spaces:

$$H^n(X; G) \cong [X, K(G, n)]$$

i.e. we can say that the singular cohomology functor can be represented by and Eilenberg-Mac Lane spectrum $HG$.

In fact, *any* generalized cohomology can be represented by a spectrum:

**Theorem 3.13** (Brown's representability theorem (corollary))**.**
*Let $E^*$ be a (reduced) generalized cohomology theory. Then there is a spectrum $E = E_n$ such that*

$$E^n(X) = [X, E_n]$$

*for any pointed CW complex $X$.*

**Examples.** 1. As mentioned above, singular cohomology with coefficients in $G$ is represented by the Eilenberg-Mac Lane spectrum $HG$.

2. The homology theory represented by the sphere spectrum $\mathbb{S}$ is stable homotopy $\pi_*^S(X)$.

3. Complex K-theory is represented by (unsurprisingly) the K-theory spectrum $K$.

4. The Thom spectrum $MO(n)$ represents the *unoriented bordism* $\Omega_*^{un}$ and $MU(n)$ represents the *complex bordism* $\Omega_*$.

## 3.3 RING SPECTRA AND STRUCTURED RING SPECTRA

If a cohomology theory in question is multiplicative, we would expect the spectrum representing it to have some additional structure. in particular, we would like to have an analogue of "tensor product" for spectra (usually called smash product and denoted by $\wedge$).

Naively, multiplication $E \wedge E \to E$ should be defined as a collection of maps $E_p \wedge E_q \to E_{p+q}$, but the smash product defined in this way turns out not to have many of the properties that we would want it to have after passage to the stable homotopy category (in particular, we want the smash product to define a symmetric monoidal structure on the stable homotopy category). Finding a correct definition was a big effort by many algebraic topologists over many years. Main models include S-modules [EKMM07], symmetric spectra [HSS00], orthogonal spectra [MMSS01].

Unless working with fine details of multiplicative structure, it is usually not necessary to restrict attention to a particular model, since all of them are Quillen equivalent. This thesis heavily relies on results of [ABGHR14] who use $\mathbb{S}$-modules throughout the paper. In any model, there is always a map $\mu_{\mathbb{S}} : \mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$.

**Definition 3.14** (Ring spectrum)**.** A *ring spectrum* is a spectrum together with the multiplication map $\mu : E \wedge E \to E$ and a unit map $\eta : \mathbb{S} \wedge E \to E$ such that the multiplication is associative up to homotopy.

A *highly structured ring spectrum* is, roughly, a ring spectrum with higher coherence conditions imposed on the multiplication map. Informally, in an $A_\infty$ *ring spectrum*, the multiplication is associative up to some homotopy, and all those homotopies are themselves homotopic, and all those homotopies are....etc. An $E_\infty$ *ring spectrum* is similarly is a ring spectrum with a coherently commutative smash product.

To understand the role that the coherence structure plays in homotopy theory, note that it is possible to similarly define $A_\infty$ and $E_\infty$ *spaces* as *spaces* with a multiplication map that is homotopy coherently associative or associative and commutative respectively (see, for example, [BV73]). If $X$ is an $A_\infty$ space such that $\pi_0(X)$ is group, then $X$ is homotopy equivalent to a loop space [Sta63]. If, in addition, $X$ is an $E_\infty$ space, then $X$ is an infinite loop space, i.e. the zeroth space of an $\Omega$-spectrum [May72].

In the case of $\mathbb{S}$-modules, precise account of highly structured ring spectra and the comparison of definitions is given in [EKMM07, Section II.4].

**Examples.** 1. K-theory and complex cobordism are examples of $E_\infty$ ring spectra [May77, Sections IV, VII].

2. Morava K-theory $K(n)$ is an example of a spectrum which is an $A_\infty$ *ring spectrum* but not an $E_\infty$ *ring spectrum*[Rog08].

## 3.4 SPECTRA WITH COEFFICIENTS

The default coefficients for a (co)homology theory is the integers. We are also interested in other coefficients, mainly the integers mod $p$.

**Definition 3.15.** Given a spectrum $E$ and any abelian group $G$, we can define a *spectrum with coefficients in* $G$ (and, correspondingly, $E$-cohomology with coefficients in $G$) as

$$EG := E \wedge SG$$

where $SG$ is a Moore spectrum of type $G$. [1]

This new spectrum satisfies the following short exact sequences [Ad74, p.200]:

$$0 \to \pi_n(E) \otimes G \to \pi_n(EG) \to \mathrm{Tor}_1(\pi_{n-1}(E), G) \to 0,$$

[1] A Moore spectrum of type $G$ is a spectrum whose homology is concentrated in one dimension and is equal to $G$.

or more generally, for any space $X$

$$0 \to E_n(X) \otimes G \to (EG)_n(X) \to \mathrm{Tor}_1(E_{n-1}, G) \to 0,$$

and if $X$ is a finite $CW$ complex, or $G$ is finitely generated

$$0 \to E^n(X) \otimes G \to (EG)^n(X) \to \mathrm{Tor}_1(E^{n+1}, G) \to 0.$$

where $(EG)_n$ and $(EG)^n$ denote the homology and cohomology theories corresponding to the spectrum $EG$ as in section 3.2.2.

**Example.** 1. Taking $G = \mathbb{Z}/p$, we get "mod $p$" $E$-cohomology. In particular, if $E = K$, the complex K-theory spectrum, the short exact sequence gives

$$0 \to K_n(X) \otimes \mathbb{Z}/p \to (K\mathbb{Z}/p)_n(X) \to \mathrm{Tor}_1(K_{n-1}, \mathbb{Z}/p) \to 0.$$

2. If $G = \mathbb{Z}_{(p)}$ (or, more generally $\mathbb{Z}[J^{-1}]$ for any set of primes $J$), then $EG$ is the *localization* of $E$ at the prime $p$ [Bo79].

## 4.0 COMPLEX ORIENTED COHOMOLOGY AND MORAVA K-THEORY

In this chapter we present overview of complex-oriented cohomology theories and construction of the Morava K-theory spectrum.

### 4.1 COMPLEX-ORIENTED GENERALIZED COHOMOLOGY THEORIES

A complex-oriented cohomology theory is a cohomology theory with a consistent choice of Chern class for any complex vector bundle. More precisely,

**Definition 4.1.** A multiplicative cohomology theory $E$ is called *complex-oriented* if the map $\widetilde{E}^2(\mathbb{CP}^\infty) \to E^2(\mathbb{CP}^1) \cong E^2(S^2)$ induced by inclusion $\mathbb{CP}^1 \simeq S^2 \hookrightarrow \mathbb{CP}^\infty$ is surjective.

This definition implies the existence of an element $c_1 \in \widetilde{E}^2(\mathbb{CP}^\infty)$ that restricts a unit of $\widetilde{E}^2(S^2) \cong \widetilde{E}^0(*)$. Class $c_1$ is often called the *orientation*.

An equivalent definition (using Thom classes) will be given in section 5.1.2.

**Examples.** 1. Singular cohomology $H^*(-;R)$ with coefficients in any commutative ring $R$ is complex-orientable. In fact, the restriction map $H^*(\mathbb{CP}^\infty;R) \to H^*(S^2;R)$ is not only surjective, but an isomorphism.

2. Complex K-theory has a canonical orientation $1-[L] \in K^2(\mathbb{CP}^\infty) \cong K^0(\mathbb{CP}^\infty)$, where $L$ is the universal line bundle.

3. Real K-theory $KO$ is *not* complex-orientable.

The standard fact about complex-oriented cohomology theories is

**Theorem 4.2.**

*If $E$ is a complex-oriented cohomology theory with orientation $c_1$. Then*

$$\begin{aligned}
E^*(\mathbb{CP}^\infty) &= E_*[[c_1]] \\
E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) &= E_*[[1 \otimes c_1, c_1 \otimes 1]] \\
E^*(BU(n)) &= E_*[[c_1, c_2, \dots c_n]].
\end{aligned}$$

*where $c_i$ are* generalized Chern classes.

(for the proof see for example [Ad74] or [Ra86] or any textbook on the topic).

Tensor product of complex line bundles induces a multiplication $\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, and therefore a map $E_*[[c_1]] \to E_*[[1 \otimes c_1, c_1 \otimes 1]]$. In particular, the image of $c_1$ gives a power series. Moreover, the associativity and commutativity of $\mu$ implies that that power series is associative and commutative as well.

## 4.2 FORMAL GROUP LAWS

We will formalize the structure that appeared at the end of the previous section. Throughout this section $R$ will be a commutative right with unity.

**Definition 4.3.** A (one-dimensional, commutative) *formal group law* over a ring $R$ is a power series $F(x, y) \in R[[x, y]]$ such that:

(1) *(commutativity)* $F(x, y) = F(y, x)$

(2) *(associativity)* $F(x, F(y, z)) = F(F(x, y)z)$

(3) *(identity)* $F(x, 0) = F(0, x) = x$

**Proposition 4.4.** *Let $F(x, y) \in R[[x, y]]$ be a formal group law over $R$.*

1. *There exists a power series $i(x) \in R[[x]]$, called the* formal inverse*, such that $F(x, i(x)) = F(i(x), x) = 0$.*
2. *$F(x, y) = x + y \mod (x, y)^2$*

We will sometimes denote $F(x, y)$ as $x +_F y$.

**Examples.** 1. $x +_F y := x + y$, aptly named *additive formal group law.* This is the formal group law of ordinary cohomology.

2. $x +_F y := x + y + xy$, the *multiplicative formal group law* (the name comes from the fact that $1 + x +_F y = (1 + x)(1 + y)$.

3. Every elliptic curve over a field $k$ defines a formal group law over $k$.

4. As described in the end of the previous section, any complex-oriented cohomology theory $E$ has a corresponsing formal group law defined over the coefficient ring $E_*$

A *map of formal group laws* $f : G \to H$ is a power series $f \in R[[x]]$ such that:

1. $f(0) = 0$
2. $f(x +_G y) = f(x) +_H f(y)$

We call $f$ an *isomorphism* if $f$ is invertible, i.e. $f'(0)$ is a unit in $R$, and a *strict isomorphism* if $f'(0) = 1$.

If the underlying ring is torsion-free, we can define a *logarithm* $log_F$ of a formal group law: it is strict isomorphism from a formal group law $F$ to the additive formal group law, i.e. $log_F(F(x, y)) = log_F(x) + log_F(y)$ (see, for example, [Haz78]).

As noted in the previous section, the Chern class of a tensor product of two line bundles of any complex-oriented cohomology theory determines a formal group law. Remarkably, in many cases this process can be reversed: given a formal group law, it might be possible to construct a complex-oriented cohomology theory. Currently we do not know whether it is possible to assign a complex-oriented cohomology theory to *every* formal group law.

First, we will consider a universal formal group law.

**Theorem 4.5.**
*There exists a ring $L$, called the* Lazard ring*, and a formal group law*

$$G(x, y) = \Sigma a_{ij} x^i y^i$$

*defined over it, such that any other formal group law $F$ over any other commutative ring $R$ can be obtained by applying a unique ring homomorphism $\phi : L \to R$, i.e.*

$$F(x, y) = \phi_*(G(x, y)) = \Sigma \phi\left(a_{ij}\right) x^i y^j$$

*Morevover, $L$ can be given the structure of a polynomial ring:*

$$L = \mathbb{Z}[x_1, x_2, \dots], \quad \textit{where } |x_i| = 2i.$$

One could ask if this the universal formal group law corresponds to a complex-oriented cohomology theory, and if so, whether this cohomology theory has similar "universal" properties. The answer to both questions is yes.

## 4.3 COMPLEX COBORDISM AND BROWN-PETERSON SPECTRA

Recall from section 3.2 that the *complex cobordism spectrum $MU$* has $MU_{2n} = Th(\gamma_n)$, the Thom space of the universal complex rank $n$ bundle, and $MU_{2n+1} = \Sigma MU_{2n}$. As we discuss in more detail in the next chapter, Thom space "encodes" orientation, so it would not be entirely unexpected that Thom spaces of universal bundles would encode universal orientation.

**Proposition 4.6.** *If $E$ is a homotopy commutative ring spectrum, then complex orientations of $E$ are in one-to-one correspondence with ring spectra maps $MU \to E$, i.e. $MU$ is the* universal *complex-oriented cohomology theory.*

Moreover, the formal group law of $MU$ corresponds to the universal formal group law:

**Theorem 4.7** (Quillen, [Qui69])**.**
*The map $L \to MU_*$ classifying the formal group law of $MU_*$ is an isomorphism.*

Given a formal group law over a graded ring $R$ we could define a functor $E(-) := MU_*(-) \otimes_{MU_*} R$. If $R$ is flat over $MU_*$, then this is an exact functor, and so the Eilenberg-Steenrod axioms for generalized cohomology are satisfied. This gives a way to obtain a cohomology theory from a formal group law. [1]

As we see from the description of the Lazard ring 4.5, $MU_*$ is very large, and the spectrum $MU$ is too complicated to study on its own. Localization of $MU$ at a prime $p$ gives a spectrum $MU_{(p)}$ that decomposes as a wedge of suspensions of a single spectrum $BP$,

[1] In general, the flatness condition is too strong and can be relaxed *Landweber exactness.*

the Brown-Peterson spectrum. Just like $MU$ is the universal complex-oriented cohomology theory, $BP$ is the universal complex oriented cohomology theory *with a p-typical formal group law.*

**Definition 4.8.** A formal group law over a torsion-free ring $R$ is *p-typical formal group law* if the logarithm has the form $log_F(x) = \Sigma a_i x^{p^i}$.

Now let $[n]_F(x)$ denote the "times $n$" function with respect to the formal group law $F$, i.e. $[1](x) := x$, $[n](x) = [n-1](x) +_F x$. The coefficient ring of $BP$ admits a convenient description:

**Theorem 4.9.**

*[Haz78] For any prime p, there is an isomorphism of $\mathbb{Z}_{(p)}$ algebras*

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

*where $|v_i| = 2(p^i - 1)$, and $v_i$ can be chosen to be the coefficients in the p-series for $BP$:*

$$[p]_{BP}(x) = \Sigma v_i x^{p^i}.$$

The spectrum $BP$ contains essentially the same information (locally) as the spectrum $MU$, and is still hard to compute, so it is usually broken into a smaller pieces – Morava K-theories.

## 4.4 MORAVA K-THEORY

**Definition 4.10.** We say that a formal group law $F$ has *height at least* $n$ if the leading term in the $p$-series $[p](x)$ has the form $a_n x^{p^n}$, and *exactly* $n$ if in addition there are no terms of the form $a_i x^{p^i}$ for $i > n$.

Theorem 4.9 suggests that it might be worthwhile to investigate "parts" of the $BP$ spectrum corresponding to a formal group law of height exactly $n$, for each $n$, and then try to assemble them together.

**Theorem 4.11** ([Lur10]).
*For any prime $p$ and any natural number $n$, there exists a unique up to equivalence homotopy associative spectrum $K(n)$ such that:*

1. *$K(n)$ is a complex-oriented cohomology theory;*
2. *the formal group law corresponding to $K(n)$ has height exactly $n$;*
3. *$K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$.*

Spectra $K(n)$ are called Morava K-theories. The names comes from the fact that $K(1)_*$ is one of $p-1$ summands of mod $p$ complex K-theory. Moreover, $K(0)_*(X) = H_*(X;\mathbb{Q})$, and it is consistent to define $K(\infty)_*(X) := H^*(X;\mathbb{F}_p)$.

We list several important properties of $K(n)$ at any prime $p$:

**Theorem 4.12** ([Ra86]). *1. Every graded module over $K(n)_*$ is free;*

2. *$K(n)$ posses the Künneth isomorphism: $K(n)_*(X \times Y) = K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$;*
3. *$K(n) \wedge K(m) = 0$ for $n \neq m$;*
4. *If $X$ is a $p$-local finite CW complex, then $K(n)_*(X) = 0$ implies $K(n-1)_*(X) = 0$.*
5. *$K(n)$ is an $A_\infty$ (but not $E_\infty$) ring spectrum.*

The construction of $K(n)$ is not relevant to this work so we will omit it (briefly, it is a quotient of the $p$-local complex cobordism). The presence of the Künneth isomorphism makes Morava K-theory "computable" (in fact, the only cohomology theories possessing a Künneth are singular cohomology with coefficients in a field, and wedge products of $K(n)$'s).

Morava K-theories can be viewed as sort of an "approximation" between rational cohomology and mod $p$ cohomology. They also represent higher of K-theory. [2]

[2]Unfortunately, at this point we lack a geometric description of $K(n)$ akin to K-theory.

## 5.0 TWISTED GENERALIZED COHOMOLOGY

In this chapter we provide motivation for the definition and study of twisted cohomology. Twisted cohomology involves the study of spaces that have some sort of local data attached to them (i.e. a bundle). The earliest incarnation of that is cohomology with local coefficients, which in particular can be used to define fundamental class of a nonorientable manifold. If the local information is given by a vector bundle, then the classical Thom space construction provides us with a way to determine which bundles "look" like trivial bundles to the eyes of a cohomology theory – this is what is usually called an orientation. When the local information is given by a principal bundle, a more creative solution is needed. We follow the framework of [ABGHR14] whose roots lie in cohomology with local coefficients and introduce twisted generalized cohomology for principal bundles via generalized Thom spectra.

## 5.1 ORIENTATION IN GENERALIZED COHOMOLOGY

### 5.1.1 Cohomology with local coefficients

One of the most important results of manifold theory is Poincaré duality: the fact that singular homology and cohomology of a compact oriented manifold are dual, via the product with the fundamental class. One could ask, could we relax those restrictions: what happens if manifold is not compact or not oriented?

For non-compact manifolds, one could use *cohomology with compact support* [BT13, Ch 1], and then we can deduce Poincare duality between ordinary homology and cohomology with compact support.

But what about non-orientable manifolds? Recall that a local orientation is a choice of a generator $\mu_x \in H^n(M, M \setminus x; \mathbb{Z}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; \mathbb{Z})$ and a manifold is *orientable* if we can choose local orientations to vary continuously along $M$. We call this orientation the fundamental class of $M$ (denoted $[M]$).

For any ring $R$, we can use the map $\mathbb{Z} \to R$ to define a local orientation (and therefore an $R$-fundamental class) using cohomology with coefficients in $R$. In particular, when $R = \mathbb{Z}/2$, we see that any manifold has a $\mathbb{Z}/2$-fundamental class, i.e. a global orientation with respect to cohomology with $\mathbb{Z}/2$ coefficients, since restricting both generators of $\mathbb{Z}$ gives the same result in $\mathbb{Z}/2$.

If the manifold is not orientable, we cannot define a fundamental class (and so there is no Poincaré duality theorem for regular singular cohomology)). One of the solutions is to use a *local coefficient system* [DK01, Ch 5].

Cohomology with local coefficients is a generalization of cohomology with coefficients in a group where the group is allowed to vary along the points in a consistent way.

**Definition 5.1.** Let $A$ be an abelian group. A *local coefficient system* $A_\rho$ on a manifold $M$ a map

$$\rho : \pi_1(M) \to Aut_{\mathbb{Z}}(A).$$

Equivalently, it is a $\mathbb{Z}[\pi_1(M)]$-module $A_\rho$ with underlying abeliab group $A$.

Alternatively, a local coefficient system can be described as a *bundle of groups*.

Then cohomology with coefficients in local system $H^*(X; A_\rho)$ is defined to be the cohomology of the chain complex given by the twisted tensor product with of the chain complex of the universal cover with $A_\rho$.

How does this help deal with non-orientable manifolds? Instead of considering cohomology with coefficients in $\mathbb{Z}$ and view orientation as picking a generator of $H^n(M, M \setminus x)$, we could allow the coefficient group to vary in a local system. Let $w : \pi_1(M) \to \mathbb{Z}/2 = Aut(\mathbb{Z})$ be such that $w$ returns 1 if a loop preserves orientation and 0 if the loop reverses orientation. This defines a local system $\mathbb{Z}_w$. If the manifold is orientable, this local system is trivial (as a bundle of groups). If the manifold is not orientable, we can still consider $H^n(M; \mathbb{Z}_w)$. For example, this cohomology group will satisfy Poincare duality.

### 5.1.2 Orientation of vector bundles and Thom spaces

Orientation of a real vector bundle generalizes the notion of orienatibily to any vector bundle, not just the tangent bundle of a manifold.

A vector bundle is orientable if we can choose orientations on the fiber in a way compatible with how the bundle is "put together". More precisely,

**Definition 5.2.** A rank $n$ vector bundle $\xi : E \to B$ is called *orientable* if there exists an open cover $\{U_\alpha)\}$ of $B$ and transition functions $\phi_\alpha : U_\alpha \times \mathbb{R}^n \to \xi^{-1}(U_\alpha)$ such that the restriction of $\phi_\alpha$ on each fiber is an orientation-preserving linear isomorphism.

As before, we can view orientation of $\mathbb{R}^n$ as a choice of a generator in $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$. Let $E_0$ denote the total space of a rank $n$ vector bundle $\xi : E \to B$ without the image of the zero section, and $F_0 := F \setminus \{0\}$. Similarly to the definition of the fundamental class of a manifold, one could expect a class in $H^n(E, E_0)$ that restricts to a generator of $H^n(F, F_0) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$, and this is indeed the case, as demonstrated by the Thom isomorphism theorem:

**Theorem 5.3.**
*If a vector bundle is orientable, then there exists a class $u \in H^n(E, E_0)$ that restricts to the generator on the fibers, such that for any $k > 0$,*

$$H^k(B) \cong H^{n+k}(E, E_0)$$

*where the isomorphism is given by the cup product with $u$.*

This suggests the study of pairs of the form $(E, E_0)$ is interesting. The quotient $E/E_0$ can be seen as collapsing all the vectors in the total space that lie outside a small neighborhood of a zero section to a point. The space obtained by this process is called the Thom space.

More precisely, let $\xi : E \to B$ be a rank $n$ vector bundle. We can put a metric on the fibers of $\xi$ and thus define corresponding sphere and disk bundles whose total spaces are denoted usually $S(\xi)$ and $D(\xi)$ (take only vectors in the fiber that have length 1 and less than 1, respectively).

**Definition 5.4.** The *Thom space* of the vector bundle $\xi$ is

$$Th(\xi) := D(\xi)/S(\xi).$$

Alternatively, we perform fiberwise one-point compactification, and then identifying all the added points together. If $B$ is compact, this is the one-point compactification of $E$. It is easy to see that if the bundle $\xi$ is trivial, then $Th(\xi) = \Sigma^n B_+$, where $B_+$ is the space $B$ with adjointed basepoint. So one could think of the Thom space as "twisted suspension" where the "twist" takes into account the local information coming form the bundle.

Notice that the Thom space construction respects Whitney sum of bundles, namely

$$Th(\xi \oplus \eta) \simeq Th(\xi) \wedge Th(\eta),$$

and

$$Th(\xi \oplus \mathbb{1}_n) \simeq \Sigma^n Th(\xi).$$

With this definition in hand, the Thom isomorphism theorem can be rephrased in the following way:

**Theorem 5.5** (Thom isomorphism theorem)**.**
*If $\xi : E \to B$ is an orientable vector bundle of rank $n$, then*

$$H^k(B; \mathbb{Z}) \cong \widetilde{H}^{k+n}(Th(\xi))$$

*where the isomorphism is given by cup product with a class $u \in \widetilde{H}^n(Th(\xi))$, such that $u$ restricts to a generator on each fiber.*

So again, the Thom isomorphism theorem says that to integral cohomology orientable bundles look like trivial bundles, since for trivial vector bundles, $Th(\xi) = \Sigma^n B_+$, so we have $\widetilde{H}^{n+k} Th(\xi) \cong \widetilde{H}^{n+k}(\Sigma^n B_+) \cong H^n(B)$.

We could ask a question: what is the analogue of this phenomenon for a generalized cohomology $R$? Thom isomorphism theorem provides us with a way to generalize the definition:

**Definition 5.6.** Let $R$ be a multiplicative ring spectrum (cohomology theory), and $\xi : E \to B$ be a rank $n$ vector bundle. Then we say that $\xi$ is *$R$-oriented* if there exists a class $u_\xi \in \widetilde{R}^n(Th(\xi))$ that restricts to 1 under the restriction on the fibers map $\widetilde{R}^n(Th(\xi)) \to \widetilde{R}^n(S^n) \cong \widetilde{R}^0(S^0)$.

*Remark.* Notice that the definition of the Thom space uses only the sphere/disk inside the vector bundle. We can generalize the definition slightly by replacing a vector bundle with a spherical fibration and taking the "disk bundle" to be the cone on (i.e. the cofiber of) the projection map. For the Thom isomorphism theorem to hold in this context, we need to restrict our attention to *sectioned* spherical fibrations.

**Examples.** 1. The trivial bundle is oriented for any cohomology theory.

2. The classical Thom isomorphism theorem says that every bundle is $\mathbb{Z}/2$-orientable, and an orientable (in the classical sense) bundle is $H\mathbb{Z}$-orientable.
3. Any complex vector bundle is oriented with respect to complex K-theory $K$.
4. Any $Spin^c$ vector bundle is $KO$-oriented [ABS64].

**Definition 5.7.** A multiplicative cohomology theory $R$ is called *complex-oriented* is every complex vector bundle is $R$-orientable and the choice of the Thom class is natural and multiplicative, i.e.:

1. (naturality) Let $\xi_1 : E_1 \to B_1$ and $\xi_2 : E_2 \to B_2$ be two of rank $n$ vector bundles, and let $(\phi, f)$ be a map between $\xi_1$ and $\xi_2$. Moreover, let $u_{\xi_1} \in \widetilde{R}^n\left(Th(\xi_1)\right)$ and $u_{\xi_2} \in \widetilde{R}^n\left(Th(\xi_2)\right)$ be the Thom classes for $\xi_1$ and $\xi_2$ respectively. Then
$$u_{\xi_1} = f^*(u_{\xi_2}).$$
2. (multiplicativity) Let $\xi_1 : E_1 \to B$ and $\xi_2 : E_2 \to B$ two vector bundles over the same base space $B$. Then $u_{\xi_1 \oplus \xi_2} = u_{\xi_1} u_{\xi_2}$.
(Note that if $\xi_1$ and $\xi_2$ have rank $n$ and $m$ respectively, then $u_{\xi_1\oplus\xi_2} \in \widetilde{R}^{n+m}(Th(\xi_1 \oplus \xi_2)) \cong \widetilde{R}^{n+m}(Th(\xi_1) \wedge Th(\xi_2))$. On the other hand, since $u_{\xi_1} \in \widetilde{R}^n(Th(\xi_1))$ and $u_{\xi_2} \in \widetilde{R}^n(Th(\xi_2))$, $u_{\xi_1}u_{\xi_2} \in \widetilde{R}^{n+m}(Th(\xi_1) \wedge Th(\xi_2))$ as well.)

Such cohomology theories are instrumental in modern algebraic topology. In particular, the subject of study of this thesis – Morava K-theory – is in some sense the "simplest", or "atomic" complex-oriented cohomology. On the other hand, complex cobordism $MU$ is the "universal" complex-oriented cohomology theory, in the sense that any other complex-oriented cohomology theory receives a ring spectrum map from $MU$, which is also sometimes called "orientation".

### 5.1.3 Thom spectra

We will now establish a stable analogue of Thom space. This can be useful, for example, when studying stable normal bundles of manifold, when we do not have the "rank" of a vector bundle given.

There is a standard inclusion $j_n : BO(n) \to BO(n+1)$. Define a $(B, f)$ *system* to be a sequence of spaces $B_0 \to \cdots \to B_n \to B_{n+1} \to \dots$ and mappings $f_n : B_n \to BO(n)$ (each defining a rank $n$ vector bundle) such that

$$\begin{tikzcd} B_n \arrow[r] \arrow[d] & B_{n+1} \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

*Remark.* If the maps $f_n$ are fibrations, then this data is sometimes called a "$(B, f)$ *structure*", where the maps $f$ are sometimes omitted if they are clear from the context. For example, taking $B_n$ to be $BSpin(n)$ and $f_n$ to be the connected cover fibrations $BSpin(n) \to BO(n)$, we obtain the *Spin structure.*

**Definition 5.8.** The *Thom spectrum* $X^f$ corresponding to a system $(X, f)$ is the spectrum with the $n$-th space

$$\left(X^f\right)_n := Th(f_n)$$

Note that since the inclusion $j_n : BO(n) \to BO(n+1)$ corresponds to adding a trivial line bundle to a rank $n$ vector bundle $\xi_n \to \xi_n \oplus \mathbb{1}_n$, and so commutativity of the diagram implies the existence of structure maps $\Sigma \left(X^\xi\right)_n \to \left(X^\xi\right)_{n+1}$, since $\Sigma(Th(\xi_n)) \simeq Th(\xi_n \oplus \mathbb{1}_1)$.

When given a map $X \to BO$ from a CW complex $X$ to the stable orthogonal group $BO = \bigcup n = 1^\infty BO(n)$, we can obtain such a sequence by restricting to $n$-skeleta $X_n \to BO(n)$ (note that $BO(n)$ is the $n$-skeleton of $BO$).

An important class of examples arise when considering maps $BO\langle n \rangle \to BO$ as the universal $O\langle n \rangle$ bundles, and $BU(n) \to BO$ as the universal complex bundle. Thom spectra corresponding to these maps are usually denoted $MO$, $MSO$, $MSpin$, $MU$ etc. A result of Thom is that these spectra represent cobordism theories with corresponding structure on stable normal bundle.

*Remark.* If we are just given a rank $n$ vector bundle $\xi : X \to BO(n)$, then we can define its Thom spectrum as $X^\xi := \Sigma^\infty Th(\xi)$.

## 5.2 TWISTED COHOMOLOGY AFTER [ABGHR14]

One could ask whether we can a similar construction for other types of bundles, for example principal bundles. Our analysis of orientations and the Thom space construction were reliant on the fact that the fibers are vector spaces, or at least spheres. Moreover, in general we do not have an analogue of "dimension" like in the case of $\mathbb{R}^n$ and $S^n$. This suggests that we should attempt to build an analogue of Thom spectrum (as opposed to just Thom space).

### 5.2.1 Units of ring spectra and principal bundles

If $R$ is a connective ring spectrum, then $\pi_0(R)$ is a ring, and we can consider the group of units $\pi_0(R)^\times$, and ask what is the part of the spectrum that corresponds to it (this is a stable analogue of looking for units in $R^0(pt)$ like we did to define Thom classes).

**Definition 5.9.** The space $GL_1R$, defined as the homotopy pullback

$$\begin{tikzcd} GL_1R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty R \arrow[d] \\ (\pi_0 R)^\times \arrow[r, hook] & \pi_0 R \end{tikzcd}$$

is called the *space of units* of the ring spectrum $R$.

(Here $\Omega^\infty R$ is the zeroth space of $R$, and $\pi_0(R)$, $\pi_0(R)^\times$ are taken as a discrete spaces, so the pullback is taken in the category of spaces).

Notice that from the definition if follows that $\pi_0 GL_1 R = \pi_0(R)^\times$, and $\pi_i GL_1 R = \pi_i R$, for $i \geq 1$. Moreover, $[X, GL_1R] \cong (R^0(X))^\times$.

This construction defines a functor $GL_1$ that restricted to $A_\infty$ spectra gives an adjunction [ABGHR14, § 3]:

$$\Sigma^\infty_+ : \text{group-like } A_\infty\text{-spaces} \rightleftarrows A_\infty\text{-spectra} : GL_1R.$$

**Example.** This shows why we restricting our attention to $A_\infty$ ring spectra only: if $R$ is and $A_\infty$ ring spectrum, then $GL_1R$ is a group-like[1] $A_\infty$ space which means that by [Sta63] it is homotopy equivalent to a loop space, and so it is possible to form a delooping $BGL_1R$, and mimic the theory for classification of vector bundles as homotopy classes of maps to $BO$.

**Example.** 1. *Sphere spectrum $R = \mathbb{S}$.*

Since $\pi_0\mathbb{S} = \mathbb{Z}$ and we can write $\Omega^\infty\mathbb{S}$ as $\varinjlim \Omega^n S^n$, the pullback diagram looks like:

$$\begin{tikzcd} GL_1\mathbb{S} \arrow[r] \arrow[d] & \varinjlim \Omega^n S^n \arrow[d] \\ \{\pm 1\} \arrow[r, hook] & \mathbb{Z} \end{tikzcd}$$

Here the left vertical map is the degree map, so we can say that $GL_1\mathbb{S}$ consists of homotopy classes of maps $[S^n, S^n]$ of degree $\pm 1$, i.e. homotopy automorphisms of $S^n$.

2. *K-theory $R = K$* (as defined in 3.2).

Since $\Omega^\infty K \simeq BU \times \mathbb{Z}$, and $\pi_0 BU = 0$, the pullback diagram is

$$\begin{tikzcd} GL_1K \arrow[r] \arrow[d] & BU \times \mathbb{Z} \arrow[d] \\ \{\pm 1\} \arrow[r, hook] & \mathbb{Z} \end{tikzcd}$$

and therefore $GL_1K \simeq BU \times \mathbb{Z}/2 \simeq BSU \times \mathbb{CP}^\infty \times \mathbb{Z}$.

If $R$ is an $A_\infty$ spectrum, then $GL_1R$ is loop space, and we can form the classifying space $BGL_1R$. Homotopically, it is a space that encoder obstructions to orientation with respect to $R$-theory as will be described further. With a little more work, it is possible to view it as a space classifying principal $GL_1R$-bundles and to define the universal bundle $EGL_1R$ (see [ABGHR14, § 3] for the precise construction involving a space-level analogue of $\mathbb{S}$-modules and [ABGHR13, § 2] for the construction via $\infty$-categories).

For example, taking $R = \mathbb{S}$, recall that $GL_1(\mathbb{S}) \simeq \varinjlim hAut\,(S^n)$, so $BGL_1\mathbb{S}$ is the classifying space for stable spherical fibrations from [LMS06], also denoted $BF$, i.e. any map $X \to BGL_1\mathbb{S} \simeq BF$ defines a stable spherical fibration.

[1] A space $X$ is *group-like* if $\pi_0(X)$ is a group and not just a set

### 5.2.2 Orientations

Let $R$ and $A$ be $A_\infty$ ring spectra with a map $A \to R$.

**Definition 5.10.** A $GL_1A$-bundle $\xi : X \to BGL_1A$ is *orientable with respect to* $R$, if the map $X \to BGL_1A \to BGL_1R$ is null-homotopic.

We can view complex orientation in this context. We have a map $BU \to BF \simeq BGL_1\mathbb{S}$ which says "take a complex vector bundle a make a spherical fibration by one-point compactification". Then, for any $n$, we have a map $BU(n) \xrightarrow{id} BU(n) \hookrightarrow BU$ classifying the universal complex $n$-plane bundle. Finally, for any ring spectrum $R$, we have the unit map $\eta : \mathbb{S} \to R$, which induces $BGL_1\mathbb{S} \to BGL_1R$. Then $R$ is a complex-oriented cohomology theory precisely when for any $n$, the composite map

$$BU(n) \xrightarrow{\text{universal bundle}} BU \xrightarrow{\text{compactify}} BGL_1\mathbb{S} \xrightarrow{BGL_1(unit)} BGL_1R$$

is null-homotopic, i.e. the theory is complex-oriented when the universal bundle is oriented in the sense of definition 5.10.

### 5.2.3 Generalized Thom spectra

Given a stable spherical fibration $E \to B$ (or, equivalently, a map $B \to BGL_1\mathbb{S}$), we can define its Thom spectrum as described in previous sections.

**Definition 5.11.** Let $\xi$ be a principal $GL_1R$ bundle on the space $X$, and let $P$ denote its total space. Then the Thom spectrum corresponding to $\xi$ is defined to be

$$X^\xi := \Sigma^\infty P_+ \wedge_{\Sigma^\infty GL_1R_+} R.$$

Here the action of $\Sigma^\infty GL_1R_+$ on $R$ comes from the adjunction

$$\Sigma^\infty_+ : \text{group-like } A_\infty\text{-spaces } \rightleftarrows A_\infty\text{-spectra} : GL_1R.$$

Notice that if we take $R = \mathbb{S}$ and $\xi$ to be a spherical fibration, we recover the classical notion of Thom spectrum associated to a spherical fibration .

Recall the *J-homomorphism*: for each $n$, consider $S^n$ as a one-point compactification of $\mathbb{R}^n$ and let $F(n)$ denote the space of homotopy self-equivalences of the $n$-sphere preserving the point at $\infty$. Then there exists a map $J_n : O(n) \to F(n)$ that extends an orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$ to an automorpshism of $S^n$ fixing the point at $\infty$, and we get a map $J : O \to \varinjlim F(n)$. This map can be delooped to obtain $BJ : BO \to BGL_1\mathbb{S}$ where we used the earlier identification of $GL_1\mathbb{S}$ with the space of homotopy automorphisms of the sphere. Then if $f : BG \to BO$ is the map realizing the existence of a $G$-structure on a vector bundle as in 5.1.3, then the Thom spectrum corresponding to $BJ \circ f$ it is exactly $MG$, the classical spectrum for cobordism with $G$ structure as defined in 5.1.3.

Moreover, if $\xi : X \to BGL_1R$ factors as

$$X \xrightarrow{g} BGL_1\mathbb{S} \to BGL_1R$$

then $X^\xi = X^g \wedge R$, where $X^g$ is now the classical Thom spectrum of $g$ (this is the content of [ABG10, Theorem 4.5]).

So definition 5.11 is indeed a proper generalization of Thom spectrum from section 5.1.3 that allows us to construct a Thom spectrum not only for vector bundles or spherical fibrations, but for also for principal $BGL_1R$ bundles for some $A_\infty$ ring spectrum $R$.

### 5.2.4 Twisted cohomology

Following the pattern that we had from vector bundles, we define twisted (co)homology as (co)homology of the Thom spectrum:

**Definition 5.12.** The *$\xi$-twisted $R$-homology and $R$-cohomology groups* of $X$ are defined, respectively, as

$$\begin{aligned} R_n^\xi(X) &:= \pi_n\left(X^\xi\right) \\ R_\xi^n(X) &:= \pi_{-n}F_R\left(X^\xi, R\right) \end{aligned}$$

where $F_R(A, B)$ is the mapping spectrum of $R$-module maps $A \to B$ as constructed in [EKMM07, Chapters I-III].

For a quick reality check, let us see that happens when the bundle in question is trivial.

The bundle is trivial if and only if the map $X \to BGL_1R$ is constant. The constant map factors through $BGL_1\mathbb{S}$. Then by discussion in the previous section,

$$X^\xi = X^{triv} \wedge R$$

where $X^{triv}$ is the classical Thom spectrum of a trivial vector bundle. But for a trivial bundle, the Thom spectrum is just the suspension spectrum, so

$$X^\xi = X^{triv} \wedge R = \Sigma^\infty_+ X \wedge R.$$

As a consequence,

$$R^\xi_n(X) = \pi_n \Sigma^\infty_+ X \wedge R = R_n(X),$$

i.e. if the bundle is trivial, we recover untwisted homology.

Finally, if we start with a vector bundle, we would like our new twisted (co)homology to reduce to the cohomology of Thom spectrum. But if $X \xrightarrow{V} BO$ is a vector bundle, we can form

$$j(V) : X \xrightarrow{V} BO \xrightarrow{BJ} BGL_1\mathbb{S} \xrightarrow{(-)\wedge R} BGL_1R,$$

where $BJ$ is the delooping of the $J$ homomorphism (we can view $GL_1\mathbb{S}$ as homotopy automorphisms of $\mathbb{S}$) and the second map is induced by the unit $\mathbb{S} \to R$ of the ring spectrum $R$.

Then again $X^{j(V)} \simeq X^V \wedge R$ where $X^V$ is the classical Thom spectrum, and therefore

$$R^\xi_n(X) = \pi_n(X^V \wedge R) = R_n(X^V),$$

so twisted cohomology with twist being a vector bundle is just the cohomology of the classical Thom spectrum.

Finally, for any spectrum $R$, we define

**Definition 5.13.** The *set of twists* of a cohomology theory $R$ is

$$tw_R(Y) := [Y, BGL_1R].$$

## 5.3 COMPUTATIONS IN TWISTED K-HOMOLOGY

In this section we summarize the main results of Khorami [Kh11] which we will need.

Let $K$ be the complex K-theory spectrum. In [ABG10, Section 7.1] Ando, Blumberg and Gepner construct a map $T : K(\mathbb{Z}, 3) \to BGL_1K$. Now consider a space $X$ and twist $\tau \cong [X, K(\mathbb{Z}, 3)]$ (since singular cohomology is represented by Eilenberg-MacLane spaces, this is equivalent to giving a class in $H^3(X; \mathbb{Z})$). We will use $K_*^\tau$ for K-homology twisted by the composite map $T(\tau)$.

Since $K(\mathbb{Z}, n) = BK(\mathbb{Z}, n-1)$, this defines a $K(\mathbb{Z}, 2)$-principal bundle on $X$, and let $P_\tau$ denote the total space of that bundle. Then the K-homology universal coefficient theorem says

**Theorem 5.14** ([Kh11])**.**

$$K_*^\tau(X) \cong K_*(P_\tau) \otimes_{K_*(\mathbb{C}P^\infty)} \hat{K}_*$$

*where $\hat{K}_*$ is just the coefficient ring $K_*$ with the $K_*(\mathbb{C}P^\infty)$-module structure obtained from the action of $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ on K-theory.*

The module structure is important for the purpose of this thesis, and the following is summary from how that works in the case of K-homology [Kh11]. The action of $\mathbb{C}P^\infty$ on $K$ via tensor product with the universal complex line bundle $L$. This is the main reason why twisted K-theory is possible. The bundle $P_\tau$ admits a fiberwise action of $\mathbb{C}P^\infty$

$$\mathbb{C}P^\infty \times P_\tau \to P_\tau.$$

The construction of bundles of spectra gives rise to twisted K-cohomology of $X$, $K^{0,\tau}(X)$ with twist $\tau$ defined to be the set of homotopy classes of sections of the bundle $P \times_{\mathbb{C}P^\infty} K \to X$. The twisted K-homology of $X$ is then the stable homotopy groups of the quotient

$$K_*^\tau(X) = \pi_*\big((P_\tau \times \mathbb{C}P^\infty K)/X\big)\ .$$

The functor $\hat{K}_*$ is the same as $K_*$ with the $K_*(\mathbb{C}P^\infty)$-module structure coming from the action map $K_*(\mathbb{C}P^\infty) \to K_*$ and $K_*(P_\tau)$ is a $K_*(\mathbb{C}P^\infty)$-module via the action $\mathbb{C}P^\infty \times$

$P_\tau \to P_\tau$. The $K_*(\mathbb{C}P^\infty)$-module structure of $K_*$ is obtained by collapsing $\mathbb{C}P^\infty$ to a point: $\mathbb{C}P^\infty \to \mathrm{pt}$. For any principal $\mathbb{C}P^\infty$ bundle $P_\tau \to X$, K-homology $K_*(P_\tau)$ is a $K_*(\mathbb{C}P^\infty)$-module, where the action of $\mathbb{C}P^\infty$ on the total space $P_\tau$ induces a map

$$K_*(\mathbb{C}P^\infty \times P_\tau) \to K_*(P_\tau) \ .$$

Since $K_*(\mathbb{C}P^\infty)$ is free over the coefficients $K_*$, we have an isomorphism

$$K_*(\mathbb{C}P^\infty \times P_\tau) \cong K_*(\mathbb{C}P^\infty) \otimes_{K_*} K_*(P_\tau)$$

which gives the module structure $K_*(\mathbb{C}P^\infty) \otimes_{K_*} K_*(P_\tau) \to K_*(P_\tau)$.

Note that the K-homology of $\mathbb{C}P^\infty$ can be given explicitly as follows (see [Ad74]). From complex orientation, $K^*(\mathbb{C}P^\infty) = K^*(\mathrm{pt})[[x]]$, where $x = L - 1$, where $L$ is the universal complex line bundle over $\mathbb{C}P^\infty$. So there are unique elements $\beta_i \in K_{2i}(\mathbb{C}P^n)$, $1 \le i \le n$ such that $\langle x^k, \beta_i \rangle = \delta_i^k$, $1 \le k \le n$. The collection $\{\beta_0 = 1, \beta_1, \beta_2, \cdots\}$ forms a $K_*$-basis for $K_*(\mathbb{C}P^\infty)$

$$K_*(\mathbb{C}P^\infty) = K_*\{\beta_0, \beta_1, \cdots\} = \mathbb{Z}[t, t^{-1}]\{\beta_0, \beta_1, \cdots\} \ .$$

One can shift the degrees of $\beta$'s to zero by instead requiring $\langle t^{-k} x^k, \beta_i \rangle = \delta_i^k$.

**Examples.**

The main examples presented in [Kh11] are

1. *Degree three integral Eilenberg-MacLane space* $K(\mathbb{Z}, 3)$*:*

   Since $H^3(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \mathbb{Z}$, again any integer gives rise to a twist. For the identity map $id : K(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$, $P_{\mathrm{id}}$ is contractible since it's the total space of the universal $K(\mathbb{Z}, 2)$ principal bundle, and so $K_*(P_{\mathrm{id}}) \cong K_*$, giving the vanishing of $id$-twisted K-homology $K_*^{(\mathrm{id})}(K(\mathbb{Z}, 3)) \cong K_* \otimes_{K_*(\mathbb{C}P^\infty)} \hat{K}_* = 0$. For a nonzero twist $n : K(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$, comparing the bundle $K(\mathbb{Z}, 2) \to P_n \to K(\mathbb{Z}, 3)$ with the path-loop fibration $K(\mathbb{Z}, 3) \to PK(\mathbb{Z}, 2) \simeq * \to K(\mathbb{Z}, 2)$ identities $P_n$ with $K(\mathbb{Z}/n\mathbb{Z}, 2)$. Then, invoking a result of Anderson and Hodgkin [AH68] that $\widetilde{K}_*(K(\mathbb{Z}/n\mathbb{Z}, 2)) = 0$, gives the triviality of twisted K-homology $K_*^{(n)}(K(\mathbb{Z}, 3)) = 0$ for any twist $n$.

2. *Three-Sphere $S^3$:*

   Since $H^3(S^3;\mathbb{Z}) \cong \mathbb{Z}$ any twist corresponds to an integer. The differential in the Atiyah-Hirzebruch-Serre spectral sequence is identified as $d_3(\sigma_3) = n\beta_1$, where $\sigma_3$ is the natural generator of $H_3(S^3, K_0(\mathbb{C}P^\infty))$, corresponding to the natural generator of $H_3(S^3;\mathbb{Z})$, and $\beta_1$ is the degree one generator of K-homology of $\mathbb{C}P^\infty$ above, interpreted cohomologically as a map $S^2 = \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$. This gives that the $\mathbb{Z}/2$-graded twisted K-homology with nonzero twist $n : S^3 \to K(\mathbb{Z}, 3)$ is

   $$K_*^{(n)}(S^3) \cong (K_*(\mathbb{C}P^\infty)/n\beta_1) \otimes_{K_*(\mathbb{C}P^\infty)} \hat{K}_* \cong K_*/n = \mathbb{Z}/n\mathbb{Z}$$

   and vanishes for the basic twist $n = 1$.

We will generalize the above two examples to higher dimensions and higher chromatic levels in section 6.6 and section 6.7, respectively.

## 6.0 COMPUTATIONS IN TWISTED MORAVA K-THEORY

Equipped with the definition of the generalized Thom spectrum, we can define twisted cohomology for any $A_\infty$ ring spectrum. We focus on *Morava K-theory*, a complex-oriented cohomology theory with coefficients $\mathbb{Z}/p[v_n, v_n^{-1}]$ that is a "higher" version of K-theory.

### 6.1 COMPUTATIONS IN MORAVA K-THEORY

We briefly describe some computational results used in our work.

First, just from the fact that $K(n)$ is complex-oriented, we get the result that is true for every complex-oriented cohomology theory 4.2.

**Theorem 6.1.**
*For Morava K-theory cohomology, one has*

$$
\begin{aligned}
K(n)^*(\mathbb{CP}^\infty) &\cong K(n)_*[[x]], \\
K(n)^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) &\cong K(n)_*[[x, y]], \\
K(n)^*(BU(n)) &\cong K(n)_*[c_1, c_2 \dots c_n],
\end{aligned}
$$

*where* $|x| = |y| = 2$, $|c_k| = 2k$.

The main computation is the Morava K-theory of Eilenberg-MacLane spaces by Ravenel and Wilson:

**Theorem 6.2** ([RW80, Theorem 11.1, Theorem 12.1])**.**
*Let* $K(n)$ *be Morava K-theory at prime* $p$. *Then*

*1.* $K(n)_*K(\mathbb{Z}/p^j,q) \cong K(n)_*$ *for* $q > n$.

*2.* $K(n)_*K(\mathbb{Z}/p^j,n) \cong \bigotimes_{i=0}^{j-1} R(a_i)$ *and* $K(n)^*(K(\mathbb{Z}/p^j,n)) \cong K(n)_*[x]/x^{p^j}$,
*where the generator* $x$ *has dimension* $|x| = 2\frac{p^n-1}{p-1}$, *the element* $a_k$ *is dual to* $(-1)^{k(n-1)}x^{p^k}$, *and* $R(a_k) = \mathbb{Z}/p[a_k, v_n^{\pm 1}]/(a_k^p - (-1)^{n-1}v_n^{p^k}a_k)$.

*3.* $K(n)_*K(\mathbb{Z},q+1) \cong K(n)_*$ *for* $q > n$.

*4. Let* $\delta : K(\mathbb{Z}/2^j,q) \to K(\mathbb{Z},q+1)$ *be the Bockstein map, and let* $b_i := \delta_*(a_i)$. *Then*
$K(n)_*K(\mathbb{Z},n+1) \cong \bigotimes_{i=0}^{\infty} R(b_i)$ *and* $K(n)^*(K(\mathbb{Z},n)) \cong K(n)_*[[x]]$.

*Remarks.* 1. Note that difference in notation from [RW80]: our $a_k$ and $b_k$ are originally $a_J$ and $b_J$, with $J = (nk, 1, 2, \ldots, n-1)$, and our $x$ is $x_S$ with $S = (1, 2, \ldots, n-1)$.

2. In [RW80], this theorem is only proven for odd primes, but it was later extended to $p = 2$ in [JW85, Appendix].

Since Morava K-theory possesses a Künneth isomorphism, Morava homology of a loop space will have the structure of a Hopf algebra (for exactly the same reason as in the case of singular cohomology with coefficients in a flat ring $R$, see for example [Hat02, Section 3.C]): briefly, if $X$ is a loop space, i.e. $X = \Omega Y$, there exists a multiplication map $m : X \times X \to X$ which corresponds to concatenation of loops. This induces a map in homology $m_* : K(n)_*(X \times X) \to K(n)_*(X)$, and together with the Künneth isomorphism this gives a map $\mu : K(n)_*(X) \otimes K(n)_*(X) \to K(n)(X)$. On the other hand, the diagonal map $\Delta : X \to X \times X$ induces a map $\delta : K(n)_*(X) \to K(n)_*(X) \otimes K(n)_*(X)$. Then $\mu$ and $\delta$ are the multiplication and comultiplication maps.

The classifying space $BO$ of the stable orthogonal group and all its connected covers are, in fact, *infinite* loop spaces by [May77], so their Morava K homology will have the structure of a Hopf algebra.

When analyzing connected covers of $BO$, we will need several results of Kitchloo, Laures and Wilson [KLW04a] [KLW04b] about Morava K-theory of spaces related to $BO$.

**Theorem 6.3** ([KLW04a, Theorem 1.3])**.**
*Let* $\underline{bo}$, $\underline{BO}$, $\underline{BSO}$, $\underline{BSpin}$ *denote the connective* $\Omega$*-spectra with zeroth spaces* $\mathbb{Z} \times BO$, $BO$, $BSO$, *and* $BSpin$ *respectively. Let* $E \to B$ *be a connected cover with fiber* $F$, *and* $B$ *is one*

*of the following:* $\underline{bo}_i$*, for* $i \geq 4$*,* $\underline{BO}_i$*,* $\underline{BSO}_i$*,* $\underline{BSpin}_i$*, for some* $i \geq 2$*. Then the fibration* $F \to E \to B$ *induces the following short exact sequence of Hopf algebras:*

$$K(n)_* \to K(n)_*(F) \to K(n)_*(E) \to K(n)_*(B) \to K(n)_*,$$

*where* $K(n)$ *is the Morava K-theory at prime* $p = 2$*.*

To deal with base spaces outside of the range specified by this theorem, we will need another exact sequence:

**Theorem 6.4** ([KLW04a, Theorem 1.5])**.**
*Let* $K(n)$ *be the Morava K-theory at* $p = 2$*. There is an exact sequence of Hopf algebras*

$$\begin{aligned} K(n)_* \to & K(n)_*K(\mathbb{Z}/2,2) \xrightarrow{\delta_*} K(n)_*K(\mathbb{Z},3) \to K(n)_*BString \to \\ \to & K(n)_*BSpin \to K(n)_*K(\mathbb{Z},4) \xrightarrow{(\times 2)_*} K(n)_*K(\mathbb{Z},4) \to K(n)_* \end{aligned}$$

*where* $\delta_*$ *is the map induced by Bockstein map, and* $(\times 2)_*$ *is the map induced by multiplication by* $2$ *on* $K(\mathbb{Z}, 4)$*.*

A similar result holds for connected covers of $BU$:

**Theorem 6.5** ([RWY98, Section 2.6],[KLW04a, Theorem 1.2])**.**
*Let* $\underline{bu}$ *denote the connective* $\Omega$*-spectrum with zeroth space* $BU$*, and let* $E \to B$ *is a connected cover with fiber* $F$*, and* $B$ *is* $\underline{bu}_i$*, for some* $i \geq 2$*. Then the fibration* $F \to B \to E$ *induces the following short exact sequence of Hopf algebras:*

$$K(n)_* \to K(n)_*(F) \to K(n)_*(E) \to K(n)_*(B) \to K(n)_*$$

*where* $K(n)$ *be Morava K-theory at any prime* $p$*.*

## 6.2 TWISTED MORAVA K-THEORY

Since $K(n)$ is an $A_\infty$ ring spectrum, one can use the constructions from section 5.2.4 to define twisted Morava K-theory. We will be particularly interested in twists of Morava K-theory by Eilenberg-MacLane spaces.

**Theorem 6.6** ([SW15, Theorem 3.1, 3.3])**.**
*Consider twists of Morava K-theory $K(n)$ by Eilenberg-MacLane spaces $K(\mathbb{Z}, m)$ in the sense of the definition 5.13.*

1. *There are no non-trivial twists for $m > n+2$.*
2. *For $m = n+2$, components of the space $Map(K(\mathbb{Z}, n+2), BGL_1K(n))$ are contractible.*
3. *If $p \neq 2$ then there are no non-trivial twists of $K(n)$ by $K(\mathbb{Z}, n+2)$).*
4. *If $p = 2$, then the set of twists is a group isomorphic to dyadic integers, i.e. one has $tw_{K(n)}(K(\mathbb{Z}, n+2)) \cong \mathbb{Z}_2$.*

Because of this theorem, henceforth we restrict our attention to the case $p = 2$ when dealing with Morava K-theory twisted by integral Eilenberg-MacLane spaces. We will also need the following definitions from [SW15].

**Definition 6.7.** The *universal twist* $u$ is the element of $tw_{K(n)}(K(\mathbb{Z}, n+2)) \cong \mathbb{Z}_2$ that is a topological generator.

**Definition 6.8.** Given a class $H \in H^{n+2}(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, n+2)]$, the *$H$-twisted Morava K-theory* is defined as

$$K(n)_*(X; H) := K(n)_*^{u(H)}(X),$$

and similarly for cohomology.

Different choices of generators will lead to isomorphic groups (as described in [SW15, 3.5]).

Defined this way, twisted Morava K-theory has all the properties that we would like it to have:

**Theorem 6.9** (Properties of twisted Morava K-theory [SW15, Theorem 4.1])**.**
*Let $K(n)^*(X; H)$ be $H$-twisted Morava K-theory of a space $X$ where $H \in H^{n+2}(X; \mathbb{Z})$. Then:*

1. *(*Normalization*) If $H = 0$ then $K(n)^*(X; H) = K(n)^*(X)$.*

2. *(*Module property*)* $K(n)^*(X;H)$ *is a module over* $K^0(n)(X)$.
3. *(*Cup product*)* *There is a cup product homomorphism* $K(n)^i(X;H) \otimes K(n)^j(X;H') \longrightarrow K(n)^{i+j}(X;H+H')$ *which makes* $\bigoplus_H K(n)^*(X;H)$ *into an associative ring.*
4. *(*Naturality*)* *If* $f : Y \to X$ *is a continuous map, then there is a homomorphism* $f^* : K(n)^*(X;H) \to K(n)^*(Y;f^*H)$.

Notice that any cohomology class in $H^{n+2}(X;\mathbb{Z})$ can be interpreted as a $K(\mathbb{Z}, n+1)$-bundle. The main computational tool we employ is the relationship between twisted homology of the base and untwisted cohomology of the total space:

**Theorem 6.10** (Universal coefficient theorem [SW15, Theorem 4.3])**.**
*Let* $H \in H^{n+2}(X)$, *and* $P_H \to X$ *be the principal* $K(\mathbb{Z}, n+1)$ *bundle over* $X$, *classified by* $H$. *Then*

$$K(n)_*(X;H) \cong K(n)_*(P_H) \otimes_{K(n)_*(K(\mathbb{Z},n+1))} K(n)_*$$

*Here* $K(n)_*(P_H)$ *is a* $K(n)_*(K(\mathbb{Z}, n+1))$ *module since* $P_H$ *is a* $K(\mathbb{Z}, n+1)$ *bundle, and* $K(n)_*$ *is made into a* $K(n)_*(K(\mathbb{Z}, n+1))$ *module by sending* $b_0$ *to 1 and* $b_i$ *to 0 for all* $i > 0$ *where* $b_i$, *where we make use of theorem 6.2 for the structure of* $K(n)_*(K(\mathbb{Z}, n+1))$.

Since $K(n)$ of Eilenberg-MacLane spaces is known by Ravenel-Wilson (theorem 6.2 above), this theoretically reduces the problem of computing twisted homology to computing homology of the total space $P_H$.

Another computational tool we will use is the twisted Atiyah-Hirzebruch spectral sequence (AHSS), which approximates a twisted generalized (co)homology theory by usual (co)homology of successive quotients arising from nested filtrations of the underlying space $X$. The construction for twisted Morava K-theory is summarized in the following.

**Theorem 6.11** ([SW15, Theorem 5.1])**.**
*For* $H \in H^{n+2}(X)$, *there is a spectral sequence converging to twisted Morava K-theory*

$$E_2^{p,q} = H^p(X, K(n)^q) \Rightarrow K(n)^*(X;H).$$

*The first possible nontrivial differential is* $d_{2^{n+1}-1}$ *that is given by*

$$d_{2^{n+1}-1}(x) = (Q_n(x) + (-1)^{|x|} x \cup (Q_{n-1} \cdots Q_1(H))).$$

Here $Q_n$ is the cohomology operation known as $n$th Milnor primitive at the prime 2. It may be defined inductively as $Q_0 = Sq^1$, and $Q_{j+1} = Sq^{2^j} Q_j - Q_j Sq^{2^j}$, where $Sq^j : H^n(X;\mathbb{Z}/2) \to H^{n+j}(X;\mathbb{Z}/2)$ is the $j$-th Steenrod square operation in mod 2 cohomology.

### 6.3 TWISTED $K(1)$ AND TWISTED MOD 2 K-THEORY

Morava K-theory $K(n)$ is defined one prime at a time, while K-theory $K$ does not depend on a prime. In order to be able to compare the latter to the former at chromatic level $n = 1$, we need to restrict K-theory to seeing one prime at a time.

Recall from section 3.4 that for we can define K-theory with mod $p$ coefficients as a spectrum $K\mathbb{Z}/p = K \wedge S\mathbb{Z}/p$, where $S\mathbb{Z}/p$ is the Moore spectrum of type $\mathbb{Z}/p$. Moreover, there is a short exact sequence relating $K\mathbb{Z}/p$ homology and $K$-homology of a space $X$:

$$0 \to K_n(X) \otimes \mathbb{Z}/p \to (K\mathbb{Z}/p)_n(X) \to \mathrm{Tor}_1(K_{n-1}(X), \mathbb{Z}/p).$$

On the other hand, by a classical result of Adams [Ad74], mod $p$ K-theory decomposes into a sum of $p-1$ successive suspensions of $K(1)$. In our case of interest, $p = 2$, so mod 2 K-theory coincides with $K(1)$ since there is only one summand, and so $K(1)$ fits into the exact sequence

$$0 \to K_n(X) \otimes \mathbb{Z}/2 \to K(1)_n(X) \to \mathrm{Tor}_1(K_{n-1}(X), \mathbb{Z}/2).$$

We would like to establish a twisted version of this relationship.

**Theorem 6.12.**

*Let $X$ be a topological space and $H_3 \in H^3(X;\mathbb{Z})$. Then we have the following exact sequence*

$$0 \to K_n^{H_3}(X) \otimes \mathbb{Z}/2 \to K(1)_n(X; H_3) \to \mathrm{Tor}_1(K_{n-1}^{H_3}(X), \mathbb{Z}/2)$$

*where $K_*^{H_3}(X)$ is K-theory twisted by $H_3$ as defined in section 5.3.*

*Proof.* Let $P$ be the total space of the principal $K(\mathbb{Z},2)$ bundle classified by a degree three class $H_3 \in H^3(X;\mathbb{Z}) \cong [X, K(\mathbb{Z},3)]$. The Thom spectrum corresponding to $K(1)(X;H)$ is $X^{u(H)} \simeq \Sigma^\infty P_+ \wedge_{\Sigma^\infty K(\mathbb{Z},2)} K(1) \simeq \Sigma^\infty P_+ \wedge_{\Sigma^\infty K(\mathbb{Z},2)} (K \wedge S\mathbb{Z}/2)$, where $S\mathbb{Z}/2$ is the Moore spectrum of type $\mathbb{Z}/2$ as in section 3.4.

The associativity of a "mixed" smash product is established via [EKMM07, Proposition 3.4], so that $X^{u(H_3)} \simeq \left(\Sigma^\infty P_+ \wedge_{K(\mathbb{Z},2)} K\right) \wedge S\mathbb{Z}/2$.

On the other hand, from [ABG10, Section 7.2], the Thom spectrum for K-theory twisted by $H_3$ is exactly $\Sigma^\infty P_+ \wedge_{K(\mathbb{Z},2)} K \simeq X^{T(H_3)}$, where $T : K(\mathbb{Z},3) \to K$ is the map defined in section 5.3. Hence $X^{u(H_3)} \simeq X^{T(H_3)} \wedge S\mathbb{Z}/2$.

Then, using the exact sequence (3.4), we obtain

$$0 \to \pi_n\left(X^{T(H_3)}\right) \otimes \mathbb{Z}/2 \to \pi_n\left(X^{u(H_3)}\right) \to \mathrm{Tor}_1(\pi_{n-1}\left(X^{T(H_3)}\right), \mathbb{Z}/2).$$

But we defined twisted homology precisely as homotopy groups of the Thom spectrum. Therefore, we can rewrite this exact sequence as

$$0 \to K_n^{H_3}(X) \otimes \mathbb{Z}/2 \to K(1)_n(X;H_3) \to \mathrm{Tor}_1(K_{n-1}^{H_3}(X), \mathbb{Z}/2).$$ □

### 6.4 TWISTED HOMOLOGY OF CONNECTED COVERS OF $BO$

Recall that we are working with Morava K-theory at $p = 2$. Our main result in this section is as follows:

**Theorem 6.13.**
*Let $K(\mathbb{Z}, n+1) \to BO\langle n+3\rangle \to BO\langle n+2\rangle$ be a fibration defining a connected cover of $BO$ (so $n = 2 \mod 8$ or $n = 6 \mod 8$ from 2.10) and $n \geq 6$, and let $H_{n+2}$ denote the corresponding class in $H^{n+2}(BO\langle n+2\rangle, \mathbb{Z})$. Then the twisted Morava K-homology of the classifying space $BO\langle n\rangle$ (appearing in the Whitehead tower 2.10) and the corresponding group group $O\langle n\rangle := \Omega BO\langle n\rangle$ are given, respectively, as*

$$\begin{aligned} K(n)_*(BO\langle n+2\rangle; H_{n+2}) &\cong K(n)_*(BO\langle n+2\rangle), \\ K(n-1)_*(O\langle n+1\rangle; H_{n+1}) &\cong K(n-1)_*(O\langle n+1\rangle), \end{aligned}$$

*where $H_{n+2}$ is the twisting class and $H_{n+1}$ is its looping.*

Our main tool in this section, which goes towards proving the above theorem, is the exact sequence of Kitchloo-Laures-Wilson, theorem 6.3,

$$K(n)_* \to K(n)_*(F) \to K(n)_*(E) \to K(n)_*(B) \to K(n)_* \tag{6.14}$$

where $E \to B$ is a connected cover with fiber $F$, and $B$ is one of the following: $\underline{bo}_i$, for $i \geq 4$, $\underline{BO}_i$, $\underline{BSO}_i$, $\underline{BSpin}_i$, for $i \geq 2$.

It is worth noting that the maps in this short exact sequence are precisely the maps induced by maps $F \to E \to B$ defining the connected cover. While it is not explicitly mentioned in the statement of this theorem in [KLW04a], examination of the proof shows that this particular exact sequence (unlike the exact sequence for what [KLW04a] call "transition spaces", e.g. 6.4) relies on repeated use of [RWY98, Proposition 2.0.1] together with [KLW04a, Theorem 4.2]. (compare this to the proof of 6.4 that proceeds by explicitly computing kernels and cokernels). The former proposition explicitly mentions in the statement that the exact sequence is precisely induced by the maps $F \to E \to B$. This fact is crucial to our computations.

First, we will establish the relevance of this result to the thesis.

**Lemma 6.15.** [1] *Let $\underline{bo}$, $\underline{BO}$, $\underline{BSO}$, $\underline{BSpin}$ denote the connective $\Omega$-spectra with zeroth spaces $\mathbb{Z} \times BO$, $BO$, $BSO$, and $BSpin$, respectively. Then we have the following equivalences:*

$$BO\langle 8k\rangle \simeq \underline{bo}_{8k},$$

$$BO\langle 8k+1\rangle \simeq \underline{BO}_{8k},$$

$$BO\langle 8k+2\rangle \simeq \underline{BSO}_{8k},$$

$$BO\langle 8k+4\rangle \simeq \underline{BSpin}_{8k}.$$

*In particular, $BString = BO\langle 8\rangle = \underline{bo}_8$, $BFivebrane = BO\langle 9\rangle = \underline{BO}_8$, and, since the spectra in question are $\Omega$-spectra, $String = \Omega BString \simeq \Omega\underline{bo}_8 \simeq \underline{bo}_7$, $Fivebrane \simeq \underline{BO}_7$.*

[1] We learned this fact from [Lig77] where it was stated without proof

*Proof.* We will prove only the first equivalence, since all the other cases are proved in exactly the same way as $BSO = BO\langle 2\rangle$ and $BSpin = BO\langle 4\rangle$.

Since the spectra in question are $\Omega$-spectra, for any $0 \leq n < 8$, $\Omega^{8k}\underline{bo}_{8k}$ has to be homotopy equivalent to $\underline{bo}_0 = \mathbb{Z} \times BO$. On the other hand, by Bott periodicity (theorem 2.8), $\Omega^{n+8k}(\mathbb{Z}\times BO) \simeq \Omega^n(\mathbb{Z}\times BO)$. So if we were to construct an $\Omega$-spectrum out of $\mathbb{Z}\times BO$, the obvious choice would be to take $\underline{bo}_{8k} := \mathbb{Z} \times BO$, and fill the intermediate spaces by Bott periodicity. However, such a spectrum would not be connective: for any positive $k$, $\pi_{-k}(\underline{bo}) = \varinjlim \pi_{n-k}(\underline{bo}_n)$, so $\underline{bo}_n$ should be at least $n$-connected.

So the spaces $\underline{bo}_n$ have to satisfy following properties: $\Omega^{8k}(\underline{bo}_{8k}) \simeq \mathbb{Z} \times BO$, and $\underline{bo}_n$ is at least $(n-1)$-connected. Therefore, $\underline{bo}_{8k}$ is precisely $BO\langle n\rangle$. $\square$

We will also need to use some basic facts about Hopf algebras. Standard references include [Un11] and [MM65].

**Definition 6.16.** Let $A, B, C$ be commutative Hopf algebras over a field $k$. Suppose $i : A \to B$ is an injection of Hopf algebras, and $j : B \to C$ is a surjection of Hopf algebras. Then if $C \cong B/i(A^+)B$ as Hopf algebras, where $A^+$ denotes the augmentation ideal of $A$, we say that

$$k \to A \xrightarrow{i} B \xrightarrow{j} C \to k$$

is a *short exact sequence of Hopf algebras.*

If we have an injective Hopf algebra morphism $i : A \to B$, we can view $B$ as an $A$-module, and then $B/i(A^+)B \cong B \otimes_A k$ (sometimes also denoted $B//A$). Therefore, in *any* short exact sequence of commutative $k-$Hopf algebras $k \to A \to B \to C \to k$, we always have $C \cong B \otimes_A k$.

In particular, in the exact sequence 6.14, $K(n)_*(B) \cong K(n)_*(E) \otimes_{K(n)_*(F)} K(n)_*$

Recall the following classical result:

**Theorem 6.17** ([MM65, Th. 4.4])**.**
*If $A$ is a connected Hopf algebra over a commutative ring with unity $K$, $B$ is a connected $A$-module coalgebra, $i : A \to B$, $\pi : B \to K \otimes_A B$ are the canonical morphisms, and the sequences $0 \to A \xrightarrow{i} B$, $B \xrightarrow{\pi} K \otimes_A B \to 0$ are split exact as sequences of graded $K$-modules, then there exists $h : B \to A \otimes_K (K \otimes_A B)$, which is an isomorphism of $A$-modules.*

If we take the ring $K$ to be $K(n)_*$, then any module over $K(n)_*$ is free, and so any exact sequence is split automatically as $K(n)_*$-modules.

*Proof. (Theorem 6.13)* Now consider the exact sequence (6.14) again. Applying theorem 6.17 with $A = K(n)_*(F)$ and $B = K(n)_*(E)$, we obtain an isomorphism

$$K(n)_*(E) \cong K(n)_*(F) \otimes_{K(n)_*} \left(K(n)_* \otimes_{K(n)_*(F)} K(n)_*(E)\right).$$

But the latter term is precisely $K(n)_*(B)$ as mentioned above, so

$$K(n)_*(E) \cong K(n)_*(F) \otimes_{K(n)_*} K(n)_*(B).$$

Now if $M$ is any $K(n)_*(F)$-module, we have

$$\begin{aligned} M \otimes_{K(n)_*(F)} K(n)_*(E) &\cong M \otimes_{K(n)_*(F)} K(n)_*(F) \otimes_{K(n)_*} K(n)_*(B) \\ &\cong M \otimes_{K(n)_*} K(n)_*(B) \end{aligned}$$

as $K(n)_*$-modules. Notice that the actual $K(n)_*(F)$-module structure of $M$ is irrelevant!

Now take $F \to E \to B$ to be the connected cover $K(\mathbb{Z}, n+1) \to BO\langle n+3\rangle \to BO\langle n+2\rangle$ for $n = 2 \mod 8$ or $n = 6 \mod 8$. If $n = 2 \mod 8$, then $n+2 = 4 \mod 8$, and by lemma 6.15, $BO\langle n+2\rangle \cong \underline{BSpin}_{n-2}$. If $n = 6 \mod 8$, then $n+2 = 0 \mod 8$, and by the same lemma, $BO\langle n+2\rangle \cong \underline{bo}_{n+2}$. So, as long as $n \geq 6$, $BO\langle n+2\rangle$ is one of the spaces that can serve as the base for the fibration in (6.14). So, in particular, for any $K(n)_*(K(\mathbb{Z}, n+1))$-module $M$,

$$M \otimes_{K(n)_* K(\mathbb{Z},n+1)} K(n)_*(BO\langle n+3\rangle) \cong M \otimes_{K(n)_*} K(n)_*(BO\langle n+2\rangle). \tag{6.18}$$

The universal coefficient theorem 6.10 for the bundle $K(\mathbb{Z}, n+1) \to BO\langle n+3\rangle \to BO\langle n+2\rangle$ states that

$$K(n)_*(BO\langle n+2\rangle; H_{n+2}) \cong K(n)_*(BO\langle n+3\rangle) \otimes_{K(n)_* K(\mathbb{Z},n+1)} K(n)_*$$

with a special $K(n)_* K(\mathbb{Z}, n+1)$-module structure on the latter factor. But taking $M = K(n)_*$ in (6.18), we see that

$$\begin{aligned} K(n)_*(BO\langle n+2\rangle; H_{n+2}) &\cong K(n)_* \otimes_{K(n)_*} K(n)_*(BO\langle n+2\rangle) \\ &\cong K(n)_*(BO\langle n+2\rangle). \end{aligned}$$

□

We highlight that the theorem indicates that for the natural twists associated with connected covers of the orthogonal group and their classifying spaces, the twisted Morava K-homology coincides with the corresponding untwisted Morava K-homology. So, we see a drastic simplification for this family of important spaces which arise often in the literature.

**Example.** For the String group and its classifying space, we have:

$$\begin{aligned} K(5)_* \left(String; H_7\right) &\cong K(5)_*(String) \ , \\ K(6)_* \left(BString; \tfrac{1}{6}p_2\right) &\cong K(6)_* \left(BString\right) \ , \end{aligned}$$

where $\frac{1}{6}p_2$ is the second fractional Pontryagin class, classifying the fibration $K(\mathbb{Z}, 7) \to BFivebrane \to BString$ and $H_7$ is its looping.

*Remark.* Notice that theorem 6.5 provides us with a similar short exact sequence for connective covers of $BU$, and so we can make the same conclusion in that case:

**Theorem 6.19.**

*Let $n$ be an odd natural number, and let $K(\mathbb{Z}, n+1) \to BU\langle n+2\rangle \to BU\langle n+1\rangle$ be a fibration defining a connected cover of $BU$ and $H_{n+2}$ the corresponding class in $H^{n+2}\left(BU\langle n+2\rangle, \mathbb{Z}\right)$. Then*

$$K(n)_*(BU\langle n+1\rangle; H_{n+2}) \cong K(n)_*(BU\langle n+1\rangle).$$

*Proof.* The proof in this complex case follows in exactly the same way as that of the real case in theorem 6.13, with the obvious changes in degrees, as dictated by the Bott periodicity theorem 2.8 and the corresponding Whitehead tower for the unitary group, analogous to (2.10) (but much simpler) . □

A more general class of spaces that will satisfy the same property is provided by [RWY98, Proposition 2.0.1].

Note that the restrictions on the base spaces in the theorem 6.3 prevent us from using the same argument for the fibration $K(\mathbb{Z}, 3) \to BString \to BSpin \to K(\mathbb{Z}, 4)$ , as $BSpin = \underline{BSpin_0}$, and theorem 6.3 requires the base to be $\underline{BSpin}_i$ with $i \geq 2$. However, this case can be handled using a different technique:

**Proposition 6.20.**

$$K(2)_* \left(BSpin; \tfrac{1}{2}p_1\right) = 0$$

*where $\frac{1}{2}p_1 \in H^4(BSpin;\mathbb{Z})$ is the first fractional Pontryagin class, classifying the fibration $K(\mathbb{Z},3) \to BString \to BSpin$.*

*Proof.* By the universal coefficient theorem 6.10, we need to compute the tensor product $K(2)_*\left(BString\right) \otimes_{K(2)_*K(\mathbb{Z},3)} K(2)_*$. Recall also that there the module structure on the second factor $K_*$ is given by mapping $b_0 \in K(2)_*K(\mathbb{Z},3)$ to 1 and $b_i \in K(2)_*K(\mathbb{Z},3)$ to 0 for $i \geq 1$.

Now consider the exact sequence in theorem 6.4:

$$\begin{aligned} K(n)_* \to & K(n)_*K(\mathbb{Z}/2,2) \xrightarrow{\delta_*} K(n)_*K(\mathbb{Z},3) \to \\ & \to K(n)_*BString \to K(n)_*BSpin \to K(n)_*K(\mathbb{Z},4) \xrightarrow{(\times 2)_*} K(n)_*K(\mathbb{Z},4) \to K(n)_* \end{aligned}$$

where $\delta_*$ is the map induced by Bockstein map, and $(\times 2)_*$ is the map induced by multiplication by 2 on $K(\mathbb{Z},4)$.

From the Ravenel-Wilson computations (theorem 6.2), the elements satisfy $b_i = \delta_* a_i$, where $K(n)_*K(\mathbb{Z}/2^j,n) \cong \bigotimes_{i=0}^{j-1} R(a_i)$ and $R(a_k) = \mathbb{Z}/p[a_k, v_n^{\pm 1}]/(a_k^p - (-1)^{n-1}v_n^{p^k}a_k)$ for $k \geq 0$. But since the sequence above is exact, the element $b_i$ has to lie in the kernel of the map $K(n)_*BString \to K(n)_*BSpin$, i.e. it maps to 0 in $K(n)_*BSpin$. Therefore, in the tensor product $K(2)_*\left(BString\right) \otimes_{K(2)_*K(\mathbb{Z},3)} K(2)_*$, the relevant elements multiply as

$$1 \otimes 1 = 1 \otimes b_0 = b_0 \otimes 1 = 0 \otimes 1 = 0,$$

and so the entire product has to be zero. $\square$

Directly tracing the essence of the proofs of the above theorems, this method can be captured in the following vanishing theorem for twisted Morava K-homology.

**Theorem 6.21.**

*If a principal $K(\mathbb{Z}, n+1)$ bundle $\xi : E \to B$ is such that the induced map on Morava homology is a map of Hopf algebras [2], and composition with the Bockstein map gives an exact sequence*

$$K(n)_*\left(K(\mathbb{Z}/2,n)\right) \xrightarrow{\delta_*} K(n)_*K(\mathbb{Z},n+1) \longrightarrow K(n)_*E$$

*then $K(n)_*(B,\xi) = 0$.*

[2] This is, for example, the case when $E \to B$ is a loop space map.

## 6.5 TWISTS BY MOD 2 EILENBERG-MACLANE SPACES

We would like to complete our investigations of connected covers of $BO$. Until now we were focusing solely on those covers which can be viewed as bundles of integral Eilenberg-MacLane spaces, i.e. those levels of the Whitehead tower of $BO$ in diagram 2.10 which have maps to $K(\mathbb{Z}, m)$. We would like to perform similar analysis for the remaining 'non-integral' covers. They can be viewed as $K(\mathbb{Z}/2, m)$-bundles, but we currently lack the definition of Morava K-theory twisted by non-integral Eilenberg-MacLane spaces. The purpose of this section is to fill that gap.

Instead of focusing solely on $p = 2$, we will discuss twists of $K(n)$ by $K(\mathbb{Z}/p^j, m)$ for all primes $p$ and $j \geq 1$.

From the description of Morava K-theory in theorem 6.2 we see $K(n)_*K(\mathbb{Z}/p^j, n)$ is one of the factors of $K(n)_*K(\mathbb{Z}, n+1)$, and $K(n)_*K(\mathbb{Z}/p^j, m) = K(n)_*K(\mathbb{Z}, m) = K(n)_*$ for $m > n$. Therefore, we should expect a similar theory as for twists by integral Eilenberg-MacLane spaces. In fact, the proofs in [SW15] transport to the mod $p$ case with little to no modification, and so we only outline them.

Recall from the definition 5.13 that a twist of theory $R$ by a space $Y$ is an element of $[Y, BGL_1R]$. The following fact provides us with an obstruction-theoretic way to classify these maps:

**Theorem 6.22** ([SW15, Proposition 1.6]).
*Let $R$ be an $A_\infty$ ring spectrum, $Z = \Omega X$ and $R_*(Z)$ is flat over $R_*$. If the obstruction groups*

$$\operatorname{Ext}^{\,k}_{R_*(Z)^{op}}(R_*, \Omega^s R_*) \tag{6.23}$$

*vanish for $s = k-1, k-2$ and any $k \geq 1$, then there is a bijection*

$$tw_R(X) \leftrightarrow Hom_{R_*-alg}(R_*(Z), R_*).$$

*Moreover, the obstruction groups lie in the $E_2$-term of the cobar spectral sequence*

$$\operatorname{Ext}^{\,k}_{R(Z)^{op}}(R_*, \Omega^s R_*) \Rightarrow E^{k-s}(BZ). \tag{6.24}$$

Notice that when $R = K(n)$, the flatness requirement is automatically satisfied for any $Z$, since any $K(n)_*(Z)$ is free over $K(n)_*$.

Now we can establish the following mod 2 analogue of theorem 6.6.

**Theorem 6.25.** *1. There are no non-trivial twists of $K(n)$ by $K(\mathbb{Z}/p^j, m)$ for $m > n+1$;*

*2. There are no non-trivial twists of $K(n)$ by $K(\mathbb{Z}/p^j, n+1)$ at $p \neq 2$;*

*3. $tw_{K(n)}(K\,(\mathbb{Z}/2^j, n+1) \cong \mathbb{Z}/2^j$.*

*Remark.* Notice the shift in degree compared to integral Eilenberg-MacLane spaces. It is the same shift in degree that occurs in theorem 6.2.

*Proof.* (Outline)

1. We will use theorem 6.22 with $X = K(\mathbb{Z}/p^j, m)$, $Z = K(\mathbb{Z}/p^j, m-1)$, and $R = K(n)$. From theorem 6.2, if $m > n+1$ then $K(n)_*(K(\mathbb{Z}/p^j, m-1)) = K(n)_*$. Consequently, the obstruction group is $\mathrm{Ext}^k_{R_*}(R_*, \Omega^s R_*) = 0$, so that the twists are given as $tw_{K(n)}(K(\mathbb{Z}/2, m)) = Hom_{K(n)_*-alg}(K(n)_*, K(n)_*) = \{*\}$.
2. Just like in the integral case, the spectral sequence in theorem 6.24 collapses by the work of Ravenel and Wilson, and the obstruction groups vanish, leading to

$$tw_{K(n)}(K(\mathbb{Z}/p^j, n+1)) = Hom_{K(n)_*-alg}(K(n)_*K(\mathbb{Z}/p^j, n), K(n)_*)$$

   which is a subset of $Hom_{K(n)_*-alg}(K(n)_*K(\mathbb{Z}, n+1), K(n)_*) = tw_{K(n)}(K(\mathbb{Z}, n+2))$ from theorem 6.2. But we know that, for $p > 2$, the latter is trivial.
3. Now fix $p = 2$, and recall from theorem 6.2 that $K(n)_*K(\mathbb{Z}/p^j, n) \cong \bigotimes_{i=0}^{j-1} R(a_i)$. As in the proof of [SW15, Theorem 3.3], $Hom_{K(n)_*-alg}(K(n)_*(K(\mathbb{Z}/2^j, n+1)), K(n)_*)$ is determined by the images of the elements $a_i$, for $0 \leq i \leq j-1$. By degree reasons, there is only one possible target for each $a_i$ in the coefficient ring $K(n)_*$. As a consequence, an element of $Hom_{K(n)_*-alg}(K(n)_*(K(\mathbb{Z}/2^j, n+1)), K(n)_*)$ is determined by the $j$ elements among the $a_i$ which are mapped to zero, and there are $2^j$ elements. By identifying $Hom_{K(n)_*-alg}(K(n)_*(K(\mathbb{Z}/2^j, n+1)), K(n)_*)$ with a subring of $K(n)^*[x]/x^{2^j}$ it is possible to obtain a group structure on it. $\square$

This allows us to seek direct analogues of the constructions in [SW15], as recalled in section 6.2. In particular, since $tw_{K(n)}(K(\mathbb{Z}/2, n+1) \cong \mathbb{Z}/2$, we can present analogues of definition 6.7 and definition 6.8.

**Definition 6.26.** The *universal twist* of $K(n)$ by the mod 2 Eilenberg-MacLane space $K(\mathbb{Z}/2, n+1)$ is the non-zero element of $tw_{K(n)}(K(\mathbb{Z}/2, n+1))$.

**Definition 6.27.** Let $h \in H^n(X; \mathbb{Z}/2)$. Then *Morava K-theory of $X$ twisted by $h$* is defined to be $K_*(X; h) := K_*^{u(h)}(X)$.

The universal coefficient theorem analogue of theorem 6.10 is also true in this case, and the proof follows the proof of that theorem (hence we omit to avoid repetition).

**Theorem 6.28.**

*If $H \in H^{n+1}(X; \mathbb{Z}/2)$, and $P$ denotes the total space of the bundle classified by $H$, then*

$$K(n)_*(X; H) \cong K(n)_*(P) \otimes_{K(n)_* K(\mathbb{Z}/2,n)} K(n)_*.$$

Equipped with this result, we can conclude our investigation of the Whitehead tower of $BO$. Applying theorem 6.3 for $G = BSO_i$ and $G = BO_i$ (with $i \geq 2$), we get the following mod version of theorem 6.13:

**Theorem 6.29.**

*Let $BO\langle n\rangle$ be a connected cover of $BO$ with $n = 1 \mod 8$ or $n = 2 \mod 8$, and let $h_{n+1}$ be the class in $H^{n+1}(BO\langle n\rangle; \mathbb{Z}/2)$ classifying the connected cover fibration. Then:*

$$\begin{aligned} K(n)_*(BO\langle n\rangle; h_{n+1}) &\cong K(n)_*(BO\langle n\rangle) \text{ for any } n \geq 8, \\ K(n)_*(O\langle n\rangle; h_n) &\cong K(n)_*(BO\langle n\rangle) \text{ for any } n \geq 7. \end{aligned}$$

Note that the only connected covers of $O$ and $BO$ that we have not investigated so far are $Spin$, $SO$, and $BSO$. The first two are uninteresting for our purposes: $Spin$ is defined via a map to $K(\mathbb{Z}, 2)$ and $SO$ – via the map to $K(\mathbb{Z}, 1)$, which would mean the corresponding twisted Morava K-theories has to be at height 0, i.e. rational cohomology.

**Example.** The Whitehead tower of the orthogonal group, diagram 2.10, gives us the fibration

$$K(\mathbb{Z}/2, 1) \to BSpin \to BSO \xrightarrow{w_2} K(\mathbb{Z}/2, 2),$$

where $w_2$ is the second Stiefel-Whitney class. However, as shown in [KLW04a, Section 5.3], the induced map $K(n)_*K(\mathbb{Z}/2,1) \to K(n)_*BSpin$ has to be trivial , so it sends $b_0$ to 0 in $K(n)_*BSpin$. Therefore, $K(1)_*(BSO; w_2) \cong 0$, by the same argument as in theorem 6.4.

### 6.6 TWISTED HOMOLOGY OF EILENBERG-MACLANE SPACES

Inspired by Khorami [Kh11], we will now look at bundles of Eilenberg-MacLane spaces with the base space also given by Eilenberg-MacLane spaces. We will generalize the first of the two examples at the end of section 5.3 from $n = 1$ to any natural number $n$. Note that $n$ plays the role of the dimension of the sphere (minus 2) as well as the chromatic level of the Morava K-theory being used. The proof will follow similar strategies to the ones taken in [Kh11] for the case of twisted K-homology.

**Theorem 6.30.**
*Let $k : K(\mathbb{Z}, n+2) \to K(\mathbb{Z}, n+2)$ be the map induced by multiplication by $k$ on $\mathbb{Z}$, for $k \geq 1$. Then*

$$K(n)_*(K(\mathbb{Z}, n+2); k) = 0.$$

*Proof.* Consider the map $1 : K(\mathbb{Z}, n+2) \to K(\mathbb{Z}, n+2)$ and let $P_1$ be the total space of the corresponding $K(\mathbb{Z}, n+1)$ bundle. Notice that $P_1$ is contractible by construction (it is the total space of the universal principal $K(\mathbb{Z}, n+1)$ bundle), therefore $K(n)_*(P_1) \cong K(n)_*$.

Now consider the "multiplication by $k$" map $\mathbb{Z} \xrightarrow{k} \mathbb{Z}$ and let $k : K(\mathbb{Z}, n+2) \to K(\mathbb{Z}, n+2)$ be the induced map on the Eilenberg-MacLane spaces, and $P_k$ the total space of the corresponding $K(\mathbb{Z}, n+1)$ bundle. Then the long exact sequence of the principal fibration $K(\mathbb{Z}, n+1) \to P_k \to K(\mathbb{Z}, n+2)$ reduces to:

$$0 \to \pi_{n+2}(P_k) \to \pi_{n+2}\left(K(\mathbb{Z}, n+2)\right) \cong \mathbb{Z} \to \pi_{n+1}\left(K(\mathbb{Z}, n+1)\right) \cong \mathbb{Z} \to \pi_{n+1}(P_k) \to 0,$$

so that $P_k$ has at most two non-trivial homotopy groups. To see how the multiplication by

$k$ fits into this picture, consider the map between $P_k$ and the universal $K(\mathbb{Z}, n+1)$ bundle:

$$\begin{tikzcd}
K(\mathbb{Z}, n+1) \arrow[r, "id"] \arrow[d] & K(\mathbb{Z}, n+1) \arrow[d] \\
P_k \arrow[r] \arrow[d] & * \arrow[d] \\
K(\mathbb{Z}, n+2) \arrow[r, "k"] & K(\mathbb{Z}, n+2)
\end{tikzcd}$$

Here the map on base spaces is "multiplication by $k$" by definition, and the map on fibers is the identity map. This induces a map of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+2}(P_k) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "k"'] & \mathbb{Z} \arrow[r] \arrow[d, "id"'] & \pi_{n+1}(P_k) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

from which we see that the top map $\mathbb{Z} \to \mathbb{Z}$ has to be multiplication by $k$, and therefore $\pi_{n+1}(P_k) = 0$ and $\pi_{n+1}(P_k) = \mathbb{Z}/k$, so $P_k \simeq K(\mathbb{Z}/k, n+1)$.

Since $K(n)$ at prime 2 is a 2-local theory, $K(n)_*(K(\mathbb{Z}/k, n+1))$ is trivial for $k$ odd. Together with Künneth isomorphism this implies that it is sufficient to look at $K(n)_*(K(\mathbb{Z}/2^j, n+1))$. From theorem 6.2, we see that

$$K(n)_*(K(\mathbb{Z}/2^j, n+1) \cong K(n)_*$$

and, therefore, for all $n \geq 1$

$$K(n)_*(K(\mathbb{Z}, n+2); k) \cong K(n)_* \otimes_{K(n)_*(K(\mathbb{Z},n+1))} K(n)_*.$$

Now recall that in the module structure on the second factor $b_0$ from $K(n)_*K(\mathbb{Z}, n+1)$ is mapped to 1. On the other hand, $b_0$ maps to 0 in $K(n)_*(K(\mathbb{Z}/2^j, n+1) \cong K(n)_*$, so $K(n)_*(K(\mathbb{Z}, n+2); k) = 0$, for any $k > 0$. □

## 6.7 TWISTED (CO)HOMOLOGY OF SPHERES

We now generalize the second of the two examples at the end of section 5.3 from $n = 1$ to any natural number $n$. This time the results will differ and the proof will depart drastically from [Kh11], and instead will use the twisted AHSS of [SW15], i.e., theorem 6.11.

**Theorem 6.31.**
*Fix an integer $n > 1$ and let $\sigma_{n+2}$ be the generator of $H^{n+2}(S^{n+2};\mathbb{Z}) = [S^{n+2}; K(\mathbb{Z}, n+1)] \cong \mathbb{Z}$. Then the twisted nth Morava K-cohomology of the $(n+2)$-sphere with twist given by $\sigma_{n+2}$ is given by the coefficient ring:*

$$K(n)^*\left(S^{n+2};\sigma_{n+2}\right) = H^*(S^{n+2}) \otimes K(n)_*.$$

*Proof.* Let $P$ again denote the total space of the corresponding $K(\mathbb{Z}, n+1)$ bundle. Recall from theorem 6.11 that the first non-trivial differential in the twisted Atiyah-Hirzebruch spectral sequence for Morava K-theory is

$$d_{2^n-1}(x) = Q_n(x) + Q_{n-1}\cdots Q_1(H) \cup x.$$

Because the differential is a module homomorphism, is suffices to consider cases $x = 1$ and $x$ dual to the fundamental class of $S^{n+2}$. In both cases $Q_n(x) = 0$ by dimension, since $H^*(S^{n+2}, K(n)^*)$ concentrated in degree $n+2$, so the target of $Q_n$ is trivial.

When $n > 1$ (so we do have the part $Q_{n-1}\cdots Q_1(H)$ in the second term), by the same dimension argument, $Q_{n-1}\cdots Q_1(H) = 0$ since the target of $Q_{n-1}\cdots Q_1$ is so higher degree cohomology. Therefore, the first non-trivial differential is actually trivial. Also note that all the subsequent differentials must vanish too (they are even longer so will land in even higher cohomology groups). Since this holds for every $x$, the spectral sequence collapses. $\square$

The case of $n = 1$ can be handled separately, by either using non-twisted AHSS to compute $K(n)_*P$, or using the computations of twisted K-theory [Kh11] together with theorem 6.12 : By the computations in 5.3, twisted K-homology for $S^3$ twisted by the generator $\sigma_3 \in H^3(S^3;\mathbb{Z})$ is $K_*^{\sigma_3}(S^3) = 0$. But from theorem 6.12, twisted $K(1)$ fits into the short exact sequence

$$0 \to K_n^{\sigma_3}(S^3) \otimes \mathbb{Z}/2 \to K(1)_n(S^3; \sigma_3) \to \mathrm{Tor}_1(K_{n-1}^{\sigma_3}(X), \mathbb{Z}/2).$$

Since $K_*^{\sigma_3}(S^3) = 0$, both the first and the third term of this exact sequence are zero, and therefore the middle term is zero as well. Therefore, we arrive at the following.

**Theorem 6.32.**

*The Morava K(1)-homology at $p = 2$ (i.e. mod 2 K-homology) of the 3-sphere with a twist $\sigma_3$ vanishes:*

$$K(1)_*(S^3, \sigma_3) = 0.$$

Notice that this is consistent with the pattern we observed before: for 'nice enough' spaces, twisted Morava K-theory groups are either zero or equal to untwisted groups.

## 7.0 CONCLUSIONS

In this dissertation we investigated and computed the twisted Morava K-theory for several key examples: connected covers of the classifying spaces for stable orthogonal and unitary groups, spheres, and Eilenberg-MacLane spaces. In all cases, we discovered that one of the two possibilities occur: either twisted homology is zero, or it is isomorphic to untwisted homology.

At first, it might seems surprising that the twisted homology would ever be zero at all – for untwisted homology, the most trivial case of a point or contractible space gives the homology equal to the coefficient ring. However, *twisted* homology of a point is zero, so vanishing twisted homology just means that the space because like a point in our setting.

The dichotomy between vanishing and untwisted homology is more interesting. From the universal coefficient theorem 6.10 it follows that the "twist" is determined by the image of $b_0 \in K(n)_*(K(\mathbb{Z}, n+1)$ in $K(n)_*$: if the image of $b_0$ is 1, the homology is untwisted, and if the image is 0, the homology vanishes. Our computations suggest that those might be the only two possibilities, which could be a direction of further investigations.

On the other hand, we would also like to obtain a homotopical interpretation of the result. For example, if there exists an orientation of the bundle with respect to Morava K-theory, then the twisted cohomology is isomorphic to non-twisted, so it is worthwhile to investigate the existence of such orientations. Showing that such an orientation does not exist would also be of independent interest: it provide an interesting example of a theory and a bundle without Thom class which still has an isomorphism on homology.

Finally, [SW15] show that twists of Morava K-theory descend from twists of Morava E-theory. Morava E-theory is less computationally accessible, but sometimes might have better homotopical properties. So another avenue of investigation is providing more computational tools for twisted Morava E-theory and integral Morava K-theory, which currently lack the "universal coefficient theorem" relating twisted theory of the base space to untwisted theory of the total space. Finally, we would examine the relationship between twisted Morava K-theory and E-theory to already established twisted K-theory and twisted $tmf$ in the future.

# APPENDIX

## NOTATION

| | |
|---|---|
| $\simeq$ | homotopy equivalent (for spaces) |
| $\cong$ | isomorphic (for groups, rings, algebras, modules, bundles), homeomorphic (for spaces) |
| $[A, B]$ | homotopy classes of maps between $A$ and $B$ |
| $\mathbb{Z}/p$ | integers reduced mod $p$ |
| $\mathbb{Z}_p$ | $p$-adic integers |
| $\mathbb{Z}_{(p)}$ | integers localized at $p$ (invert all primes except $p$) |
| $\gamma_n$ | universal rank $n$ vector bundle |
| $\varepsilon_n$ | trivial rank $n$ vector bundle |
| $X_+$ | space $X$ with adjoined basepoint |
| $\mathbb{S}$ | the sphere spectrum |
| $\Sigma X$ | (reduced) suspension of a space $X$ |
| $\Sigma^\infty X$ | suspension spectrum of $X$ |
| $\Omega X$ | (pointed) loop space of a space $X$ |
| $\Omega^\infty R$ | zeroth space of a connective ring spectrum $R$ |
| $O$ | stable orthogonal group $O = \varinjlim O(n)$ |
| $U$ | stable unitary group $U = \varinjlim U(n)$ |
| $K(G, n)$ | Eilenberg-MacLane space of type $(G, n)$ |
| $HG$ | Eilenberg-MacLane spectrum of type $G$ |
| $SG$ | Moore spectrum of type $G$ |

| | |
|---|---|
| $K$ | complex K-theory spectrum |
| $KO$ | real K-theory spectrum |
| $K(n)$ | Morava K-theory at height $n$ |
| $MG$ | Thom spectrum of cobordism with $G$-structure |

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