NUMERICAL IMPLEMENTATIONS OF THEORETICAL RESULTS IN PARTIAL DIFFERENTIAL EQUATIONS

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In this work, we present two theoretical results in nonlinear partial differential equations and we exploit both of them to produce novel visualizations of their solutions.

First, we show a proof of the existence and uniqueness of viscosity solutions to the p-Laplace equation in the setting of the double obstacle problem. These solutions are built by adopting the framework provided by so-called random tug-of-war games. Using the theoretical result, in this context we employ a finite elements method to obtain visualizations of various approximate solutions.

Second, we develop a proof of the density of $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère equation in the set of continuous functions. This proof was obtained in the framework provided by the technique of convex integration. The proof is written with all due details, which allowus to give explicit bounds for every involved quantity.

By means of numerical computations, we construct approximations of anomalous solutions to the Monge-Ampère equation, suitably guided by the bounds obtained by our theoretical results.

Finally, we use these approximations to give a graphical description of anomalous solutions. The visualizations and the numerical results provide insight into the approximating constructions.

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1.0 GAME THEORETICAL INTERPRETATION OF THE OBSTACLE PROBLEM FOR THE P-LAPLACE EQUATION

1.1 BACKGROUND IN PDE'S AND THE P-LAPLACE EQUATION

The foundation of modern analysis lies in set theory where we will begin the discussion. In this work we only consider sets defined on the real line $\mathbb{R}$ or finite dimensional spaces $\mathbb{R}^N$. We omit a formal definition of these well known and intuitive sets and their properties. A series of basic definitions and theorems follow.

1.1.1 General theory

Definition 1.1.1 (Measurable space)**.** *A family Σ of subsets of a set X is called a* σ-algebra *if:*

- *it includes the empty set, i.e. $\emptyset \in \Sigma$.*
- *it is closed under taking complement, i.e. $A \in \Sigma \Rightarrow X \backslash A = A^c \in \Sigma$.*
- *it is closed under taking countable union, i.e. $\{A_n\}_{n=1}^{\infty}$ with $A_n \in \Sigma \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Sigma$.*

A pair (X, Σ) of a set and a σ-algebra is called a measurable space*.*

Definition 1.1.2 (Measurable function)**.** *Given measurable spaces (X, Σ) and (Y, Γ), a function $f : X \to Y$ is called* measurable *if for all $A \in \Gamma$ it holds $f^{-1}(A) = \{x \in X;\ f(x) \in A\} \in \Sigma$. In words, f is measurable if the preimage of any measurable set is measurable.*

Definition 1.1.3 (Measure space)**.** *Given a measurable space (X, Σ), a function $\mu : \Sigma \to \mathbb{R}$ is a* measure *if the following conditions hold:*

- *$\mu(\emptyset) = 0$.*
- *$\mu(A) \geq 0$ for every $A \in \Sigma$.*
- *$\{A_n\}_{n=1}^{\infty}$ with $A_n \in \Sigma$ pairwise disjoint $\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$*

A triple (X, Σ, μ) *is called a* measure space*.*

The above definitions were stated in the most general case. We will now turn to the actual examples used in the rest of this work.

Definition 1.1.4 (Borel σ-algebra)**.** *The Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$ on $\mathbb{R}^N$ is formed by all sets that can be obtained as countable unions of sets of the type $\prod_{i=1}^N (a_i, b_i)$ which are all the N-dimensional rectangles.*

Definition 1.1.5 (Borel measurable function)**.** *A function $f : \mathbb{R}^N \to \mathbb{R}^M$ is Borel measurable if it is measurable with respect to the Borel σ-algebra. Note that for this to be true it is sufficient to verify that for any rectangle R in $\mathbb{R}^M$ we have $f^{-1}(R) \in \mathcal{B}(\mathbb{R}^N)$.*

Lemma 1.1.6. *Any Borel measurable function can be written as the increasing pointwise limit of simple functions of the form:*

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \qquad \text{where } A_i \cap A_j = \emptyset \quad \forall i \neq j,$$

where each of the A_i is Borel measurable and $\alpha_i \in \mathbb{R}$.

Proof. Given a positive Borel measurable function $f : \Omega \to \mathbb{R}$ where the domain $\Omega \subset \mathbb{R}^N$ is a Borel set, consider a sequence of increasing sequences of numbers $\{\alpha_i^{(n)}\}_{i=0}^{n^2}$ where $\alpha_i^{(n)} = \frac{i}{n}$. For $i = 0 \ldots n^2 - 1$, define the sets:

$$A_i^{(n)} = \{x \in \Omega;\ f(x) \in [\alpha_i^{(n)}, \alpha_{i+1}^{(n)})\} = f^{-1}\big([\alpha_i^{(n)}, \alpha_{i+1}^{(n)})\big),$$

and the set

$$A_{n^2}^{(n)} = \{x \in \Omega;\ f(x) \geq n^2\}.$$

Since these are the preimage of intervals and f is measurable, each of the $A_i^{(n)}$ must be a Borel set. Define the sequence of functions

$$f_n = \sum_{i=1}^{n^2-1} \alpha_i^{(n)} \chi_{A_i^{(n)}} + \chi_{A_{n^2}^{(n)}} n^2 \leq f.$$

For each $x \in \Omega$, if $f(x)$ is finite we have that eventually $n > f(x)$, and there exists a sequence of $\alpha_{i_n}^{(n)}$ converging from below to $f(x)$ due to the density of rational numbers in $\mathbb{R}$. Thus $f(x) \geq f_n(x) \geq \alpha_{i_n}^{(n)}$ guarantees that f_n converge pointwise on the set where f is finite. If we admit $f(x) = +\infty$ we have that $f_n(x) = n^2 \to \infty$. Thus we have shown pointwise convergence in the positive case and when the image set is one dimensional. The construction can easily be extended to every Borel measurable function $f : \mathbb{R}^N \to \mathbb{R}^M$.

Lemma 1.1.7. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Borel measurable functions pointwise converging to a function u. The limit u is a Borel function.*

Proof. Since the limit exists it is equal to the limit infimum and limit supremum. We note that:

$$\big(\inf_{n\geq l} u_n\big)^{-1}([-\infty,a)) = \bigcup_{n=l}^{\infty}\big(u_n^{-1}([-\infty,a))\big),$$
$$\big(\sup_{n\geq l} u_n\big)^{-1}([-\infty,a)) = \bigcap_{n=l}^{\infty}\big(u_n^{-1}([-\infty,a))\big).$$

Both are Borel measurable sets for all values of l as they are countable unions and intersections of Borel sets obtained as preimages of Borel functions. We conclude by noting that:

$$\liminf_{n\to\infty} u_n = \sup_{l\geq 1}\inf_{n\geq l} f_k,$$

$$\limsup_{n\to\infty} u_n = \inf_{l\geq 1}\sup_{n\geq l} f_k.$$

Definition 1.1.8 (Lebesgue measure)**.** *The Lebesgue measure λ is defined on rectangles as the volume given by the product of the length of sides. For a generic set A we define the inner and outer Lebesgue measures as:*

$$\underline{\lambda}(A) = \inf\Big\{\sum_{i=1}^{n} Vol(R_i);\ R_i \ \ \textit{are } N\text{-dimensional rectangles}, A\subset \bigcup_{i=1}^{n} R_i\Big\}.$$

$$\overline{\lambda}(A) = \sup\Big\{\sum_{i=1}^{n} Vol(R_i);\ R_i \ \ \textit{are } N\text{-dimensional rectangles}, A\supset \bigcup_{i=1}^{n} R_i\Big\}.$$

Clearly for any set A we have that $\overline{\lambda}(A) \leq \underline{\lambda}(A)$. On Borel sets the two will coincide due to the fact that any Borel set may be written as the countable union of rectangles. We may thus define the Lebesgue measure on Borel sets as $\lambda(A) = \overline{\lambda}(A) = \underline{\lambda}(A)$.

Consider any $\Omega \subset \mathbb{R}^N$ open.

Definition 1.1.9 (Lebesque integral)**.** *We begin by defining the integral on simple functions, and will extend this definition to all Borel measurable functions.*

$$f = \sum_{i=1}^{n}\alpha_i\chi_{A_i}, \qquad \int f\mathrm{d}\lambda(x) = \sum_{i=1}^{n}\alpha_i\lambda(A_i).$$

For an arbitrary function f we define the lower and upper Lebesgue integrals:

$$\int^{*} f\mathrm{d}\lambda(x) = \inf\Big\{\int g\mathrm{d}\lambda(x);\ f\leq g \text{ simple function}\Big\},$$
$$\int_{*} f\mathrm{d}\lambda(x) = \sup\Big\{\int g\mathrm{d}\lambda(x);\ f\geq g \text{ simple function}\Big\}.$$

If the upper and lower Lebesgue integrals coincide we define the Lebesgue integral of f as:

$$\int f\mathrm{d}\lambda(x) = \int^{*} f\mathrm{d}\lambda(x) = \int_{*} f\mathrm{d}\lambda(x).$$

Given a Borel measurable function, the Lebesgue integral is always well defined. A function f is called integrable *if* $\int |f|\mathrm{d}\lambda(x) < \infty$.

Lemma 1.1.10 (Fatou's Lemma). *Let u_n be a sequence of positive Borel measurable functions. Then:*

$$\int \liminf_{n\to\infty} u_n \mathrm{d}\lambda \leq \liminf_{n\to\infty} \int u_n \mathrm{d}\lambda.$$

Proof. Consider any simple function $v = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}$ with positive α_i and disjoint Borel measurable A_i such that $u \leq \liminf_{n\to\infty} u_n$. Taking some constant $t \in (0,1)$, we may decompose each set as follows:

$$A_i = \bigcup_{j=1}^{\infty} B_{i,j} \quad \text{where} \quad B_{i,j} = A_i \cap \{x \in \Omega;\ u_n(x) > ta_j\}.$$

Clearly $A_i \supset B_{i,j+1} \supset B_{i,j}$ for all i and j. Thus:

$$\int_{\Omega} u_n \mathrm{d}\lambda \geq \sum_{i=1}^{\infty} \int_{A_i} u_n \mathrm{d}\lambda \geq \sum_{i=1}^{\infty} \int_{B_{i,j}} u_n \mathrm{d}\lambda \geq t \sum_{i=1}^{\infty} a_i \lambda(B_{i,j}).$$

Thus:

$$\liminf_{n\to\infty} \int u_n \mathrm{d}\lambda \geq t \sum_{i=1}^{\infty} a_i \lambda(A_i) = t \int_{\Omega} v \mathrm{d}\lambda.$$

But this holds for all t and v, and thus:

$$\liminf_{n\to\infty} \int u_n \mathrm{d}\lambda \geq \int_{*} \liminf_{n\to\infty} u_n \mathrm{d}\lambda = \int \liminf_{n\to\infty} u_n \mathrm{d}\lambda.$$

Theorem 1.1.11 (Monotone convergence theorem). *Let u_n be a sequence of positive Borel measurable functions such that $u_n \leq u_{n+1}$ for all n. Then:*

$$\lim_{n\to\infty} \int_{\Omega} u_n \mathrm{d}\lambda = \int_{\Omega} \lim_{n\to\infty} u_n \mathrm{d}\lambda.$$

Proof. It is immediate to see that for all n:

$$\int_{\Omega} u_n \mathrm{d}\lambda \leq \int_{\Omega} \lim_{n\to\infty} u_n \mathrm{d}\lambda.$$

It follows that:

$$\lim_{n\to\infty} \int_{\Omega} u_n \mathrm{d}\lambda \leq \int_{\Omega} \lim_{n\to\infty} u_n \mathrm{d}\lambda.$$

The opposite direction inequality:

$$\lim_{n\to\infty}\int_\Omega u_n \mathrm{d}\lambda \geq \int_\Omega \lim_{n\to\infty} u_n \mathrm{d}\lambda,$$

follows immediately from Fatou's Lemma when we note that for monotone sequences the concepts of limit and limit infimum coincide.

Definition 1.1.12 (Norm on a space)**.** *Given a vector space V we define a* norm *on it as any function:*

$$\|\cdot\| : V \to \mathbb{R}$$

which satisfies the following properties for any $u, v \in V$ and any positive constant $c \in \mathbb{R}^+$:

(i) $\|u\| \geq 0$.

(ii) $\|u\| = 0$ *implies* $u = 0$.

(iii) $\|cu\| = c\|u\|$.

(iv) $\|u + v\| \leq \|u\| + \|v\|$ *(the triangle inequality).*

A vector space equipped with a norm will be called a normed space *and denoted by $(V, \|\cdot\|)$ If we drop condition (ii) we have a* seminorm *for which we will use the notation $[\cdot]$.*

Definition 1.1.13 (Convergence in norm)**.** *Given a sequence u_n in a normed space $(V, \|\cdot\|)$ we say that u_n* converges in norm *to u as n goes to ∞, namely:*

$$u_n \xrightarrow{n\to\infty} u,$$

if for all $\epsilon > 0$ there exists an N such that for all $n > N$ we have:

$$\|u_n - u\| < \epsilon.$$

Next we define some crucial function spaces. These are all to be considered as subset of the set of all Borel functions.

Definition 1.1.14 (Continuous functions)**.** *A function u is continuous if for all $x_0 \in \Omega$, given any $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|x_0 - x| < \delta \Rightarrow |u(x_0) - u(x)| < \epsilon.$$

The space of all continuous functions is called $\mathcal{C}^0(\Omega)$, and if Ω is bounded may be equipped with the norm $\|u\|_0 = \sup_{x\in\Omega} |u|$.

Definition 1.1.15 (Differentiable functions)**.** *Given a function $u : \Omega \to \mathbb{R}^M$ where $\Omega \subseteq \mathbb{R}^N$ is an open set, we define its* derivative *at a point $x \in \Omega$ if it exists as the linear operator $\nabla u(x)$ which satisfies:*

$$\lim_{|h|\to 0} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|}{|h|} \to 0.$$

If such a linear operator exists we say that u is differentiable *at the point x. If u is differentiable at every point $x \in \Omega$ and the function $x \mapsto \nabla u(x)$ is a continuous function, we say that u is* continuously differentiable *on Ω. A function u is* continuously differentiable up to the boundary *of Ω if there exists a differentiable function $\bar{u} : \mathbb{R}^N \to \mathbb{R}^M$ such that $u(x) = \bar{u}(x)$ for all $x \in \Omega$. We define the space $\mathcal{C}^1(\Omega)$ as the space of all differentiable functions on Ω. If Ω is bounded we may equip this space with the norm $\|u\|_1 = \sup_{x\in\Omega} |u| + \sup_{x\in\Omega} |\nabla u|$.*

Definition 1.1.16 (Lipschitz continuous functions)**.** *A function u is Lipschitz continuous if there exists a constant $L > 0$ such that:*

$$\forall x, y \in \Omega, \quad |u(x) - u(y)| < L|x - y|.$$

Definition 1.1.17 (L^p spaces)**.** *For any $p \in (0, \infty)$ we define $L^p(\Omega)$ the space of all Lebesgue measurable functions for which $\| \cdot \|_{L^p(\Omega)}$ is finite, where:*

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}.$$

This in fact defines a norm on the space $L^p(\Omega)$.

Definition 1.1.18 (Weak derivative)**.** *Given a measurable function $u : \Omega \to \mathbb{R}$ we define its* weak partial derivative *in the i-th coordinate if it exists as the function $\partial_i u$ which satisfies:*

$$\int_\Omega u(\partial_i \phi) = -\int_\Omega (\partial_i u)\phi \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega).$$

Definition 1.1.19 ($W^{1,p}$ spaces)**.** *$W^{1,p}(\Omega)$ is the subspace of $L^p(\Omega)$ comprised of all functions with weak derivatives which also lie in $L^p(\Omega)$. This space is equipped with the natural norm:*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^p(\Omega)}.$$

1.1.2 The p-Laplace operator

The p-Laplacian and the p-Laplace equation are defined as:

$$\text{the operator:} \quad \Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{and the equation:} \quad \Delta_p u = 0. \tag{1.1}$$

Where $p \in (1, \infty)$ The equation is the Euler-Lagrange equation for the potential energy:

$$E_p(u) = \int_\Omega |\nabla u|^p \, \mathrm{d}x. \tag{1.2}$$

That is, solutions of the p-Laplace equation minimize $E_p(u)$.

In this work we consider three separate concepts of generalized solutions to the p-Laplace equation. As we shall see, these solutions coincide under appropriate conditions.

Definition 1.1.20 (Weak solutions to the p-Laplacian)**.** *A function $u \in W^{1,p}_{loc}(\Omega)$ is a* weak supersolution *to the p-Laplacian if for all positive test functions $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^+)$:*

$$\int_\Omega |\nabla u|^p \le \int_\Omega |\nabla u + \nabla \phi|^p.$$

It is a weak subsolution *to the p-Laplacian if for all negative test functions $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^-)$:*

$$\int_\Omega |\nabla u|^p \le \int_\Omega |\nabla u + \nabla \phi|^p.$$

It is a weak solution *if it is both a subsolution and a supersolution.*

Remark 1.1.21. *It is a classical result in calculus of variations that an equivalent definition of weak supersolution is given by:*

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \ge 0 \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^+).$$

This remark is justified by the following inequality:

$$|\nabla(v + \phi)|^p \ge |\nabla u|^p + p\langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle$$

Definition 1.1.22 (p-superharmonic function)**.** *A function $u : \Omega \to \mathbb{R} \cup \{\infty\}$ is* p-superharmonic *if it is lower-semicontinuous, not identically ∞ on any connected component of Ω and for all subdomains $D \Subset \Omega$ and weak solutions $v \in \mathcal{C}(\bar{D}) \cap W^{1,p}(D)$, it satisfies the comparison principle:*

$$v(x) \le u(x) \; \forall x \in \partial D \Longrightarrow v(x) \le u(x) \; \forall x \in D.$$

It is a p-subharmonic *function if it is upper-semicontinuous and for all subdomains $D \Subset \Omega$ and weak solutions $v \in \mathcal{C}(\bar{D}) \cap W^{1,p}(D)$, it satisfies the comparison principle:*

$$v(x) \le u(x) \; \forall x \in \partial D \Longrightarrow v(x) \le u(x) \; \forall x \in D.$$

A p-harmonic *function is continuous and both p-subharmonic and p-superharmonic.*

Definition 1.1.23 (Viscosity supersolution to the p-Laplacian)**.** *We define a lower-semicontinuous function $u:\bar{\Omega}\to\mathbb{R}$ which is not identically ∞ on any connected component of Ω as a* viscosity supersolution *on the set Ω if it satisfies the following property: for every $x_0\in\Omega$ and every $\phi\in\mathcal{C}^2(\Omega)$ such that:*

$$\phi(x_0)=u(x_0),\quad \phi<u \ \textit{in}\ \Omega\backslash\{x_0\},\quad \nabla\phi(x_0)\neq 0,$$

there holds: $\Delta_p\phi(x_0)\leq 0$.
Similarly, an upper-semicontinuous function $u:\bar{\Omega}\to\mathbb{R}$ which is not identically ∞ on any connected component of Ω is a viscosity subsolution *on the set Ω if it satisfies the following property: for every $x_0\in\Omega$ and every $\phi\in\mathcal{C}^2(\Omega)$ such that:*

$$\phi(x_0)=u(x_0),\quad \phi>u \ \textit{in}\ \Omega\backslash\{x_0\},\quad \nabla\phi(x_0)\neq 0,$$

there holds: $\Delta_p\phi(x_0)\geq 0$.

Finally a viscosity solution *is a function which satisfies both properties.*

Observation 1.1.24 (Averaging property)**.** *A useful averaging property of viscosity solutions to the p-Laplacian is stated without proof. The calculation which provides the proof can be found in the proof of Theorem 1.1.30. Given u a viscosity solution to the p-Laplace equation, we have:*

$$\begin{gathered}u(x)=\frac{\alpha}{2}\sup_{B_\epsilon(x)}u+\frac{\alpha}{2}\inf_{B_\epsilon(x)}u+\beta\fint_{B_\epsilon(x)}u+o(\epsilon^2),\\ \text{with}\quad \alpha=\frac{p-2}{N+p},\ \beta=\frac{2+N}{N+p}.\end{gathered}\tag{1.3}$$

This property provides the inspiration for the connection to tug-of-war games used in this work.

Note that in the case of $p=2$ this becomes the well known averaging property for harmonic functions. In which case the proerty holds for all ϵ without the error term.

Theorem 1.1.25. *Let v be in $\mathcal{C}^0(\Omega)$ and in $W^{1,p}_{loc}(\Omega)$. Then v is p-superharmonic if and only if it is a weak supersolution.*

Proof. Assume:

$$\int_\Omega\langle|\nabla u|^{p-2}\nabla u,\nabla\phi\rangle\geq 0\quad\forall\phi\in\mathcal{C}_0^\infty(\Omega,\mathbb{R}^+).$$

Consider any open set $D\subset\subset\Omega$, and a p-harmonic function $h\in\mathcal{C}^0(\bar{D})$ such that $h\leq u$ on ∂D. Consider the test function $\phi=max\{h-v,0\}$, and evaluate:

$$\int_{v<h}|\nabla v|^p\leq\int_{v<h}\langle|\nabla u|^{p-2}\nabla u,\nabla\phi\rangle\leq\left(\int_{v<h}|\nabla v|^p\right)^{1-\frac{1}{p}}\left(\int_{v<h}|\nabla h|^p\right)^{\frac{1}{p}}.$$

This implies:

$$\int_{v<h} |\nabla v|^p \leq \int_{v<h} |\nabla h|^p.$$

On the boundary of the set though, $v = h$, and by the fact that v is a weak supersolution we have that v is a minimizer. This contradiction implies that the set $v < h$ must have measure zero, but since v and h are both continuous, this set must be empty. Thus $v \geq h$ proving it is p-superharmonic.

The converse statement was proven by Heinonen and Kilpeläinen in 1988 [10].

We finish the discussion of the theory of weak solutions to the p-Laplace equation by stating the following Theorem proven by Juutinen, Lindqvist and Manfredi in 2001.

Theorem 1.1.26. *[12] Let v be a viscosity solution on a domain Ω. Then v is in $W^{1,p}_{loc}(\Omega)$ and v is p-harmonic.*

1.1.3 The two obstacle problem

Definition 1.1.27 (Two obstacle problem)**.** *The two obstacle problem for the p-laplacian is defined by two obstacles $\Psi_1 \leq \Psi_2 : \Omega \to \mathbb{R}$ and a boundary value $F : \partial\Omega \to \mathbb{R}$. The solution to the two obstacle problem is a function u which satisfies:*

$$\begin{cases} \Psi_1 \leq u \leq \Psi_2 & in\ \Omega \\ u = F & on\ \partial\Omega \\ -\Delta_p u = 0 & in\ \{x \in \Omega\ s.t.\ \Psi_2(x) > u(x) > \Psi_1(x)\} \\ -\Delta_p u \geq 0 & in\ \{x \in \Omega\ s.t.\ u(x) = \Psi_1(x)\} \\ -\Delta_p u \leq 0 & in\ \{x \in \Omega\ s.t.\ u(x) = \Psi_2(x)\}. \end{cases} \tag{1.4}$$

Note that this general definition may be weakened to include weaker definitions of solutions.

Definition 1.1.28 (Viscosity solution to the double obstacle problem)**.** *We define a* viscosity solution *to the double obstacle problem on the set Ω defined by the boundary condition $F : \partial\Omega \to \mathbb{R}$, and obstacles $\Psi_1, \Psi_2 : \bar{\Omega} \to \mathbb{R}$ as a continuous function $u : \bar{\Omega} \to \mathbb{R}$ which satisfies the following:*

(i) $u = F$ *on* $\partial\Omega$ *and* $\Psi_1 \leq u \leq \Psi_2$ *in* Ω.

(ii) *For every $x_0 \in \Omega$ such that $u(x_0) < \Psi_2(x_0)$ and every $\phi \in \mathcal{C}^2(\Omega)$ such that:*

$$\phi(x_0) = u(x_0), \quad \phi < u \ in\ \Omega\backslash\{x_0\}, \quad \nabla\phi(x_0) \neq 0,$$

there holds: $\Delta_p \phi(x_0) \leq 0$.

(iii) For every $x_0 \in \Omega$ such that $u(x_0) > \Psi_1(x_0)$ and every $\phi \in \mathcal{C}^2(\Omega)$ such that:

$$\phi(x_0) = u(x_0), \quad \phi > u \text{ in } \Omega \backslash \{x_0\}, \quad \nabla\phi(x_0) \neq 0,$$

there holds: $\Delta_p \phi(x_0) \geq 0$.

By discretizing the averaging property we obtain the following useful definition.

Definition 1.1.29 (ϵ-p-harmonious solution to the double obstacle problem)**.** *Define constants $\alpha + \beta = 1$ as in the averaging property (1.3). Consider the expanded domain $X = \Omega \cup \Gamma$ defined by adding a fattened boundary to Ω. The fattened boundary Γ is defined as a set such that $\Omega + B_\epsilon(0) \setminus \Omega \subset \Gamma$, and X is an open set.*

$$u_\epsilon(x) = \begin{cases} \max\left\{\Psi_1(x), \min\left\{\Psi_2(x), \frac{\alpha}{2}\sup_{B_\epsilon(x)} u_\epsilon + \frac{\alpha}{2}\inf_{B_\epsilon(x)} u_\epsilon + ⨍_{B_\epsilon(x)} u_\epsilon\right\}\right\} & \text{in } \Omega, \\ F(x) & \text{in } \Gamma. \end{cases} \tag{1.5}$$

The main result discussed in this section will regard the definition and uniqueness of viscosity solutions.

Theorem 1.1.30. *Given F, Ψ_1 and Ψ_2 bounded Lipschitz continuous functions on a domain $\Omega \subset \mathbb{R}^N$ open and bounded, there exists $u : \bar{\Omega} \to \mathbb{R}$ the unique viscosity solution to the double obstacle problem (1.4).*

The viscosity solution will be found as the limit of ϵ-p-harmonious solutions to the problem.

Proposition 1.1.31. *Given F, Ψ_1 and Ψ_2 bounded Lipschitz continuous functions on a domain $\Omega \cap \Gamma = X \subset \mathbb{R}^N$ open and bounded, for every $\epsilon > 0$, there exists $u_\epsilon : \bar{\Omega} \to \mathbb{R}$ ϵ-p-harmonious solution to the double obstacle problem. Such a solution is unique.*

The construction from the proof of this Proposition was used in the numerical implementation which allow us to view some ϵ-p-harmonious solutions.

The proof of Theorem 1.1.30 was the work of the author in collaboration with Lewicka and Manfredi, and may be found in [5]. It will be presented in more detail in this work. The proof follows the structure of a similar result for the single obstacle problem found in [14]. The double obstacle problem is an extension of the single obstacle problem, and the proof of the latter was used in a non trivial way to obtain the proof of the former.

In the proof of uniqueness of the viscosity solution we will need to use a similar result for weak solutions to the two obstacle problem. We provide the statement without proof.

Theorem 1.1.32. *Define p, F, Ψ_1, Ψ_2 as in Theorem 1.1.30. Define the set of functions:*

$$\mathcal{K}_{F,\Psi_1,\Psi_2}(\Omega) = \Big\{ u \in W^{1,p}(\Omega) \big| u = F \text{ on } \partial\Omega, \psi_1 \leq u \leq \psi_2 \text{ in } \Omega \Big\}. \tag{1.6}$$

(i) There exists a unique $u \in \mathcal{K}_{F,\Psi_1,\Psi_2}(\Omega)$ such that:

$$\forall v \in \mathcal{K}_{F,\Psi_1,\Psi_2}(\Omega) \qquad \int_\Omega |\nabla u|^p \leq \int_\Omega |\nabla v|^p.$$

(ii) The unique minimizer u is continuous up to the boundary: $u \in \mathcal{C}(\bar{\Omega})$.

(iii) Let $\bar{u}$ be the unique minimizer for a new problem defined by $\bar{F}, \bar{\Psi}_1, \bar{\Psi}_2$, then

$$\bar{F} \geq F, \bar{\Psi}_1 \geq \Psi_1, \bar{\Psi}_2 \geq \Psi_2 \quad \Longrightarrow \quad \bar{u} \geq u$$

This result was proven by Farnana [8] in 2009. In this paper, the existence and uniqueness of the minimizer were proven using the convexity of the operator $\int_\Omega |\nabla u|^p$.

1.2 BACKGROUND IN PROBABILITY

We lay out the fundamental concepts in probability theory used in this work. Note that these definitions and theorems will not be stated in the most general setting possible.

Definition 1.2.1 (Probability measure)**.** *A measure $\mathbb{P}$ on a measurable space (Ω, Σ) is a* probability measure *if $\mathbb{P}(\Omega) = 1$. A triple $(\Omega, \Sigma, \mathbb{P})$ is called a* probability space

We present two important probability measures on a measurable space (Ω, Σ) which will be used in this thesis.

Example 1.2.2 (Uniform probability)**.** *For any $A \subset \Omega \subset \mathbb{R}^N$ define the probability $\mathcal{U}_\Omega$ as:*

$$\mathcal{U}_\Omega(A) = \frac{\int_A 1 \mathrm{d}\lambda}{\int_\Omega 1 \mathrm{d}\lambda} = \frac{\lambda(A)}{\lambda(\Omega)}$$

Example 1.2.3 (Dirac probability)**.** *For any $x \in \Omega$, for $A \subset \Omega$ define the probability $\delta_x(A)$ as:*

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Definition 1.2.4 (Random variable)**.** *We define a* random variable *on the space $(\Omega, \mathcal{F}, \mathbb{P})$ as an $\mathcal{F}$ measurable function $X : \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$.*

Definition 1.2.5 (Expectation). *The* expectation *of a random variable X written $\mathbb{E}[X]$ is the integral in* $\mathrm{d}\mathbb{P}$*:*

$$\mathbb{E}[X] = \int_\Omega X \mathrm{d}\mathbb{P}.$$

A random variable X is called integrable if:

$$\mathbb{E}[|X|] = \int_\Omega |X| \mathrm{d}\mathbb{P} < \infty.$$

Definition 1.2.6 (Conditional Expectation). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub σ-algebra $\mathcal{G} \subset \mathcal{F}$. Given X an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, define the* conditional expectation *of X with respect to $\mathcal{G}$ as the $\mathcal{G}$-measurable random variable $Y = \mathbb{E}[X|\mathcal{G}]$, integrable with respect to $\mathbb{P}_{|\mathcal{G}}$, such that:*

$$\int_A X \mathrm{d}\mathbb{P} = \int_A Y \mathrm{d}\mathbb{P} \qquad \forall A \in \mathcal{G}.$$

The following property is evident from the definition:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X].$$

Definition 1.2.7 (Filtration). *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define a* filtration *on $\mathcal{F}$ as a sequence of σ-algebras $\{\mathcal{F}_n\}_{n=0}^\infty \subset \mathcal{F}$ with the following inclusion property:*

$$\mathcal{F}_{n-1} \subset \mathcal{F}_n.$$

The collection $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^\infty, \mathbb{P})$ is called a Filtered space *if $\mathcal{F}$ is the smallest σ-algebra containing the union of all $\mathcal{F}_n$.*

Definition 1.2.8 (Stopping time). *Given a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^\infty, \mathbb{P})$, define a* stopping time *on the filtration as a function*

$$\tau : \Omega \to \mathbb{N} \cup \{0\},$$

such that the following sets have the property:

$$A_n^\tau = \{\omega \in \Omega;\ \tau(\omega) \leq n\} \in \mathcal{F}_n \quad \forall n \geq 0.$$

Definition 1.2.9 (Random variable on a filtered space). *We define a* random variable *on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^\infty, \mathbb{P})$ as a sequence of of random variables $\{X_n\}_{n=0}^\infty$ where each:*

$$X_n : \Omega \to \mathbb{R}$$

is $\mathcal{F}_n$-measurable. A random variable will be integrable if each of the X_n is integrable in $\mathcal{F}_n$.

Definition 1.2.10 (Martingales)**.** *On a filtered space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{\infty}, \mathbb{P})$*, a random variable* $X = \{X_n\}_{n=0}^{\infty}$ *is a* martingale *if for every* $n > 0$ *we have:*

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] = X_{n-1}.$$

It is a submartingale *if:*

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] \leq X_{n-1}.$$

It is a supermartingale *if:*

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] \geq X_{n-1}.$$

Theorem 1.2.11 (Doob's optional stopping theorem)**.** *This famous Theorem is presented here in the particular case used in this work. Let* $X = \{X_n\}_{n=0}^{\infty}$ *be a random variable on* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{\infty}, \mathbb{P})$ *such that for all* $i \in \mathbb{N}$ *there holds* $|X_i| \leq C$*. Let* $\tau : \Omega \to \mathbb{N}$ *be a stopping time such that* $\mathbb{P}(\{\tau = \infty\}) = 0$*. Then, the random variable* X_τ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *defined by* $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ *is integrable and:*

- *if* $X = \{X_n\}_{n=0}^{\infty}$ *is a submartingale, then* $\mathbb{E}[X_\tau] \geq \mathbb{E}[X_0]$*,*
- *if* $X = \{X_n\}_{n=0}^{\infty}$ *is a martingale, then* $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$*,*
- *if* $X = \{X_n\}_{n=0}^{\infty}$ *is a supermartingale, then* $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_0]$*.*

Proof. The fact that X_τ is integrable follows from Fatou's Lemma applied to the integrable sequences of random variables $X_{\tau\wedge n}$ defined by

$$X_{\tau\wedge n}(\omega) = \begin{cases} X_n(\omega) & \text{if } n < \tau(\omega), \\ X_\tau(\omega) & \text{if } n \geq \tau(\omega). \end{cases}$$

Each of these is bounded and integrable, and for every $\omega \in \Omega$ we have that $X_{\tau\wedge n}(\omega) \to X_\tau(\omega)$.

We now prove the result for submartingales. For a supermartingale $\{X_n\}$ the result will follow by applying the submartingale result to the submartingale $\{-X_n\}$. The result for martingales follows because martingales are both sub and super martingales.

Let $\{X_n\}$ be a submartingale. We show that $X_{\tau\wedge n} \geq \mathbb{E}[X_0]$ for every n. Consider any $A \in \mathcal{F}_n$

$$\begin{aligned} \int_A \mathbb{E}\big[X_{\tau\wedge n+1}\big|\mathbb{F}_n\big]\mathrm{d}\mathbb{P} &= \int_A X_{\tau\wedge n+1}\mathrm{d}\mathbb{P} \\ &= \int_{A\cup\{\tau\leq n\}} X_{\tau\wedge n}\mathrm{d}\mathbb{P} + \int_{A\cup\{\tau>n\}} X_{n+1}\mathrm{d}\mathbb{P} \\ &\geq \int_{A\cup\{\tau\leq n\}} X_{\tau\wedge n}\mathrm{d}\mathbb{P} + \int_{A\cup\{\tau>n\}} X_{n}\mathrm{d}\mathbb{P} \\ &= \int_{A\cup\{\tau\leq n\}} X_{\tau\wedge n}\mathrm{d}\mathbb{P} + \int_{A\cup\{\tau>n\}} X_{\tau\wedge n}\mathrm{d}\mathbb{P} = \int_A X_{\tau\wedge n}\mathrm{d}\mathbb{P}. \end{aligned}$$

We may thus prove by induction that:

$$\begin{aligned}\mathbb{E}[X_{\tau\wedge n+1}] &= \mathbb{E}\big[\dots\mathbb{E}\big[X_{\tau\wedge n+1}\big|\mathbb{F}_n\big]\big|\dots\big|\mathbb{F}_0\big]\\ &\geq \mathbb{E}\big[\dots\mathbb{E}\big[X_{\tau\wedge n}\big|\mathbb{F}_{n-1}\big]\big|\dots\big|\mathbb{F}_0\big] \geq \dots \geq \mathbb{E}\big[X_0\big].\end{aligned}$$

Finally we show that:

$$\lim_{n\to\infty}\mathbb{E}[X_{\tau\wedge n}] = \mathbb{E}[X_\tau].$$

Fix $\epsilon > 0$ and consider the sets:

$$A_i = \bigcup_{n=i}^{\infty}\big\{|X_{\tau\wedge n} - X_\tau| > \epsilon\big\}.$$

We have that:

$$\bigcap_{i=1}^{\infty} A_i \subset \{\tau = \infty\} \quad\Longrightarrow\quad \mathbb{P}\big(\bigcap_{i=1}^{\infty} A_i\big) = 0.$$

Thus for i large enough we may write:

$$\int_\Omega |X_{\tau\wedge n} - X_\tau|\mathrm{d}\mathbb{P} \leq \int_{\Omega\setminus A_i}|X_{\tau\wedge n} - X_\tau|\mathrm{d}\mathbb{P} + \int_{A_i}|X_{\tau\wedge n}| + |X_\tau|\mathrm{d}\mathbb{P} \leq \epsilon + 2C\mathbb{P}(A_i) \leq 2\epsilon.$$

1.3 THE LAPLACE EQUATION AND BROWNIAN MOTION

The connection between the p-Laplace equation and game theory was recently discovered. The field was established in 2008 with the publishing of the seminal paper by Peres, Schramm, Sheffield and Wilson [21]. In this paper they showed the connection of solutions to the ∞-Laplace equation to the expected outcome of a zero-sum-game. Around the same time, a similar game was connected to solutions of the p-Laplace equation in the case of $1 < p < \infty$ [22]. In 2012 in [16] the template for the tug-of-war game used in this result was used to show existence of viscosity solutions in the case of $2 \leq p < \infty$.

1.4 TUG-OF-WAR GAMES WITH NOISE

Consider the following setting for a tug-of-war game with noise. The game is played by two Players whom we will refer to as Player I and Player II, on a game board defined by the set X from 1.1.29.

The set X consists of set Ω and its fattened boundary Γ. No point of the set Ω can be within ϵ of any point outside the set X. The game starts from an initial position $x_0 \in \Omega$, and at each turn a new position for the token is chosen within distance ϵ of the previous position. Depending on the outcome of a random event, the token will be moved by one of the Players or to a random point within the ball of radius ϵ. The probability of the token being moved randomly is $\beta = 1 - \alpha$, the probability of each Player moving the token is $\frac{\alpha}{2}$. The game ends when the token exits Ω or when either of the Players decides to stop the game. When the game ends, Player II pays Player I a certain payoff. This payoff is calculated using three functions: $\Psi_1 \leq \Psi_2 : X \to \mathbb{R}$ and $F : \Gamma \to \mathbb{R}$. If the game ends with the token exiting Ω, the payoff is determined by the value of F at the last position of the token. Otherwise, if Player I ends the game the payoff is given by Ψ_1 at the spot, and by Ψ_2 if Player II ends it.

1.4.1 The probability space

We begin the discussion by defining the set of all possible playthroughs of the game. These are the infinite sequences of points in the set X starting with x_0:

$$X^{\infty,x_0} = \{\omega = (x_0, x_1, x_2, \dots);\ x_i \in X\}\,.$$

On this set we define a filtration of σ-algebras $\{\mathcal{F}_n^{x_0}\}_{n=0}^{\infty}$. Each $\mathcal{F}_n^{x_0}$ is the smallest σ-algebra containing all the sets of the form:

$$A \subset X^{\infty,x_0} \text{ such that } A = A_1 \times \cdots \times A_n \times X \times X \times \dots \text{ with } A_i \in X \text{ Borel measurable.}$$

For simplicity of notation we will refer to sets $A \times X \times X ... \in \mathcal{F}_n^{x_0}$ as $A \in X^n$ and omit the coda. We define $\mathcal{F}^{x_0}$ as the smallest σ-algebra on X^{∞,x_0} containing the union of all $\mathcal{F}_n^{x_0}$. We show that $\mathcal{F}_n^{x_0}$ forms a filtration over $(X^{\infty,x_0}, \mathcal{F}^{x_0})$ by noting that $\mathcal{F}_n^{x_0} \subset \mathcal{F}_{n+1}^{x_0}$. Given a set $A \in \mathcal{F}_n^{x_0}$ consider the set $A \times X \subset X^{n+1}$, clearly this new set is in $\mathcal{F}_{n+1}^{x_0}$, but the two are the same set when considered as subsets of X^{∞,x_0}. We further define a very important set of functions $x_n : X^{\infty,x_0} \to X$. Given $\omega = (x_0, x_1, ...)$ we set $x_n(\omega) = x_n$.

Lemma 1.4.1. *For all n, the function x_n is $\mathcal{F}_n^{x_0}$-measurable.*

Proof. Given a Borel set $B \subset X$, the set $A = X \times \cdots \times X \times B \times X \dots$ with B at the n'th position in the series is $\mathcal{F}_n^{x_0}$-measurable. Given $\omega \in A$, we have that $x_n(\omega) \in B$. On the other hand, if $x_n(\omega) \in B$, then $\omega \in A$. Thus we have that $A = f^{-1}(B)$ is in $\mathcal{F}_n^{x_0}$ and thus x_n is measurable.

1.4.2 The strategies

We formalize the notion of strategies that the two Players follow during the game. We define as strategies the sequences of functions $\sigma_I = \{\sigma_I^n : X^n \to X\}_{n=0}^{\infty}$ and $\sigma_{II} = \{\sigma_{II}^n : X^n \to X\}_{n=0}^{\infty}$. These functions follow the rule:

$$\forall n > 0, \quad \sigma_I^n(x_0, \dots, x_n) \in B_\epsilon(x_n), \text{ and } \sigma_{II}^n(x_0, \dots, x_n) \in B_\epsilon(x_n),$$

that is, every move made by a Player must have length less than ϵ.

Next, we define the relevant stopping times. The first stopping time stops the game at the first exit time from Ω, given $\omega \in X^{\infty, x_0}$:

$$\tau_0(\omega) = \min\{n \geq 0;\ x_n(\omega) \in \Gamma\}$$

The two Players respectively choose stopping times τ_I and τ_{II} subject to the condition that $\tau_I \leq \tau_0$ and $\tau_{II} \leq \tau_0$.

For any stopping time τ and number n, define the sets:

$$A_n^\tau = \{\omega \in X^{\infty, x_0};\ \tau(\omega) \leq n\}.$$

To show that A_n^τ is $\mathcal{F}_n^{x_0}$ measurable we note that given $\omega_1 \in A_n^\tau$, any ω_2 such that $x_i(\omega_1) = x_i(\omega_2)$ for all $i \leq n$ must also be in A_n^τ. Furthermore, we define the sets:

$$A_n^{\tau_I < \tau_{II}} = \bigcup_{k=1}^{n} \left(A_k^{\tau_I} \backslash A_k^{\tau_{II}}\right)$$

These sets are also $\mathcal{F}_n^{x_0}$ measurable, as each A_k^τ is measurable and by definition σ-algebras are closed under countable unions, intersections and complements. Note that $A \backslash B = A \cap B^c$. These sets comprise of all game plays ω for which Player I stops the game before the n'th turn and before Player II stops it. Finally for simplicity of notation we define the total stopping time $\tau = \tau_0 \wedge \tau_I \wedge \tau_{II}$.

1.4.3 The probability measures

Given parameters $\alpha \geq 0$, $\beta > 0$ such that $\alpha + \beta = 1$, and strategies and stopping times as above, we define the following family of probability measures. For every $n \geq 0$, and every finite sequence $(x_0, \dots, x_n)$ of points in X define the transition probabilities:

$$\gamma_n[x_0, \dots, x_n] = \begin{cases} \frac{\alpha}{2}\delta_{\sigma_I^n(x_0,\dots,x_n)} + \frac{\alpha}{2}\delta_{\sigma_{II}^n(x_0,\dots,x_n)} + \beta\mathcal{U}_{B_\epsilon(x_n)} & (x_0, \dots, x_n) \notin A_n^\tau, \\ x_n & (x_0, \dots, x_n) \in A_n^\tau. \end{cases} \tag{1.7}$$

with the Dirac delta δ_x and the uniform probability $\mathcal{U}$ measures as defined in (1.2.3), and (1.2.2).

Lemma 1.4.2. *Given a Borel set $A \subset X$ and $n \geq 1$, the function*

$$(x_0, \dots, x_n) \mapsto \gamma_n[x_0, \dots, x_n](A)$$

is Borel measurable. This function represents the probability that x_{n+1} will lie in A given the history $(x_0, \dots, x_n)$. This property means that this family of probabilities is jointly measurable

Proof. The function $f : (x_0, \dots, x_n) \mapsto \delta_{\sigma_I^n(x_0,\dots,x_n)}(A)$ which only takes values 0 or 1 is Borel measurable as:

$$f^{-1}(1) = (\sigma_I^n)^{-1}(A), \quad f^{-1}(0) = (\sigma_I^n)^{-1}(A^c).$$

Likewise, the function $g : (x_0, \dots, x_n) \mapsto \delta_{\sigma_{II}^n(x_0,\dots,x_n)}(A)$ is Borel measurable as:

$$g^{-1}(1) = (\sigma_{II}^n)^{-1}(A), \quad g^{-1}(0) = (\sigma_{II}^n)^{-1}(A^c).$$

The function $h : x \mapsto \mathcal{U}_{B_\epsilon(x)}(A)$ is Borel measurable as it is continuous:

$$|h(x) - h(y)| = \frac{|\lambda(A \cap B_\epsilon(x)) - \lambda(A \cap B_\epsilon(y))|}{\lambda(B_\epsilon(x))} \leq \frac{|\lambda(B_\epsilon(y) \backslash B_\epsilon(x)) - \lambda(B_\epsilon(x) \backslash B_\epsilon(y))|}{\lambda(B_\epsilon(x))} \leq C|x - y|.$$

The sum of Borel measurable functions is Borel measurable, thus the Lemma is proven.

We may now define for every $n \geq 1$ probability measures $\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}$ on the σ-algebra $\mathcal{F}_n^{x_0}$ by:

$$\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \dots \times A_n) = \int_{A_1} \cdots \int_{A_n} 1 \mathrm{d}\gamma_{n-1}[x_0 \dots x_{n-1}] \dots \mathrm{d}\gamma_0[x_0]$$

This probability measure is the probability that when playing a game with initial position x_0, the first n positions $x_1, \dots, x_n$ each lie in the corresponding $A_1, \dots, A_n$. These probabilities satisfy Kolmogoroff's consistency conditions:

$$\mathbb{P}^{n+k,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \dots \times A_n \times X^k) = \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \dots \times A_n).$$

This holds because:

$$\int_X \cdots \int_X 1 \mathrm{d}\gamma_{n+k-1}[x_0 \dots x_{n+k-1}] \mathrm{d}\gamma_n[x_0 \dots x_{n-1}] = 1.$$

Lemma 1.4.3 (Kolmogoroff's consistency theorem)**.** *The Theorem exists in a more general setting but we will state it and prove it in this particular case. Given the sequence $\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}$, there exists a probability $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}$ such that for every n and $A \in \mathcal{F}_n^{x_0}$:*

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(A) = \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}(A).$$

Proof. Let $A \in \mathcal{F}^{x_0}$, then A is formed by the countable union or intersection of sets of the type

$$A = A_1 \times \cdots \times A_n \times \ldots$$

The probability $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}$ needs only be defined on such sets and may be extended to the whole of $\mathcal{F}^{x_0}$ by the rules of measures. Given the set $A = A_1 \times \cdots \times A_n \times \ldots$ we define:

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(A) = \lim_{n\to\infty} \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \cdots \times A_n).$$

Such a limit always exists as the sequence is bounded below by 0 and decreasing:

$$\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \cdots \times A_n) = \mathbb{P}^{n+1,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \cdots \times A_n \times X) \geq \mathbb{P}^{n+1,x_0}_{\sigma_I,\sigma_{II},\tau}(A_1 \times \cdots \times A_n \times A_{n+1}).$$

We must now show that $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}$ is a probability measure. The first two conditions are easily satisfied:

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(\emptyset) = \lim_{n\to\infty} 0 = 0,$$
$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(X) = \lim_{n\to\infty} 1 = 1.$$

We need to show countable disjoint additivity on a set of generator elements of $\mathcal{F}^{x_0}$. Consider any sequence of pairwise disjoint sets:

$$\left\{A^{(i)} = A^{(i)}_1 \times \cdots \times A^{(i)}_n \times \ldots\right\}_{i=1}^{\infty}$$

Consider the sum of the probabilities of each set:

$$\sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(A^{(i)}) = \lim_{k\to\infty} \sum_{i=1}^{k} \lim_{n\to\infty} \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}\big(A^{(i)}_1 \times \cdots \times A^{(i)}_n\big).$$

Since the argument of the sum is bounded and the sum is finite we may switch the sum and the limit, and since the sum is bounded we may switch the two limits:

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(A^{(i)}) &= \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}\big(A^{(i)}_1 \times \cdots \times A^{(i)}_n\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}\big(A^{(i)}_1 \times \cdots \times A^{(i)}_n\big) \\
&= \lim_{n\to\infty} \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}\Big(\bigcup_{i=1}^{\infty} \big(A^{(i)}_1 \times \cdots \times A^{(i)}_n\big)\Big) = \mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}\Big(\bigcup_{i=1}^{\infty} A^{(i)}\Big)
\end{aligned}$$

The second to last equality is justified by the fact that each of the $\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}$ is in fact a probability measure. Thus $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}$ is well defined and for $A \in \mathcal{F}^{x_0}$ we have that $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}(A) = \mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}(A)$.

Observation 1.4.4. *Let $v : X \to \mathbb{R}$ be a bounded Borel function. Given $n \geq 1$, the conditional expectation $\mathbb{E}^{x_0}_{\sigma_I,\sigma_{II},\tau}\{v \circ x_n | \mathcal{F}^{x_0}_{n-1}\}$ of the random variable $v \circ x_n$ is a $\mathcal{F}^{x_0}_{n-1}$-measurable function on X^{∞,x_0}. Using the definition of conditional expectation, we write:*

$$\mathbb{E}^{x_0}_{\sigma_I,\sigma_{II},\tau}\{v \circ x_n | \mathcal{F}^{x_0}_{n-1}\} = \int_X v \mathrm{d}\gamma_{n-1}[x_0 \dots x_n - 1].$$

This follows from the definition:

$$\int_{X^n} \int_X v \mathrm{d}\gamma_{n-1}[x_0 \dots x_n - 1] \mathrm{d}\mathbb{P}^{n-1,x_0}_{\sigma_I,\sigma_{II},\tau} = \int_{X^{n+1}} v \mathrm{d}\mathbb{P}^{n,x_0}_{\sigma_I,\sigma_{II},\tau}$$

Next we need to prove a crucial result that guarantees that the game will end almost surely. To do so we will need to lay out some preliminary theory.

Definition 1.4.5. *Redefine the game on the extended game board $Y = \mathbb{R}^N$. The initial token position will be $x_0 \in \Omega$ and τ_0 will be the time of first exit from Ω as before. The strategies are modified in the following way:*

$$\bar{\sigma}_I(x_0 \dots x_n) = \begin{cases} \sigma_I(x_0 \dots x_n) & (x_0 \dots x_n) \in X^n, \\ x_n & \textit{otherwise.} \end{cases}$$

$\bar{\sigma}_{II}$ is defined in an analogous manner. Since $\tau_I, \tau_{II} \leq \tau_0$, these need no redefinition. The transition probabilities $\bar{\gamma}_n$ and the probabilities $\overline{\mathbb{P}}^{x_0}_{\tau,\bar{\sigma}_I,\bar{\sigma}_{II}}$ are redefined using these strategies.

Given this definition we have:

$$(x_0 \dots x_n) \in X^n \Longrightarrow \bar{\gamma}_n[x_0 \dots x_n] = \gamma_n[x_0 \dots x_n].$$

In fact this holds for any $(x_0 \dots x_n) \notin A^{\tau_0}_n$. This allows us to state a crucial result.

Lemma 1.4.6. *Let $A \in \mathcal{F}^{x_0}_n$ be such that for any $\omega = (x_0 \dots x_n \dots) \in A$, it holds that $\bar{\gamma}_n[x_0 \dots x_n] = \gamma_n[x_0 \dots x_n]$, then:*

$$\overline{\mathbb{P}}^{x_0}_{\tau,\bar{\sigma}_I,\bar{\sigma}_{II}}(A) = \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(A).$$

Proof. The Lemma is proven by induction on n. The base case is trivial as

$$\mathbb{P}^{x_0}_{1,\sigma_I,\sigma_{II}} = \gamma_0[x_0] = \bar{\gamma}_0[x_0] = \overline{\mathbb{P}}^{x_0}_{1,\bar{\sigma}_I,\bar{\sigma}_{II}}.$$

Let $A \subset \{x_0\} \times Y^n$ be a Borel set. For an arbitrary $\eta > 0$ take a covering $A \subset \bigcup_{i=1}^{\infty}(A^i_1 \times A^i_2)$ where each $A^i_1 \subset \{x_0\} \times Y^{n-1}$ and $A^i_2 \subset Y$ are pairwise disjoint Borel sets, and:

$$0 \leq \left(\sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A^i_1 \times A^i_2) - \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A)\right) + \left(\sum_{i=1}^{\infty} \overline{\mathbb{P}}^{x_0}_{n,\bar{\sigma}_I,\bar{\sigma}_{II}}(A^i_1 \times A^i_2) - \overline{\mathbb{P}}^{x_0}_{n,\bar{\sigma}_I,\bar{\sigma}_{II}}(A)\right) \leq \eta.$$

Define $A^i = A \cap (A_1^i \times A_2^i)$ and $\pi(A_i) = \{(x_0, \dots x_{n-1});\ \exists x_n \in Y, (x_0, \dots x_n) \in A^i\}$ for every i. Note that $\pi(A_i)$ is not necessarily a Borel set, but it is an analytic set. This means that there exist Borel sets B_1^i and C_1^i such that $B_1^i \subset \pi(A_i) \subset C_1^i$ and:

$$\mathbb{P}^{x_0}_{n-1,\sigma_I,\sigma_{II}}(C_1^i \backslash B_1^i) = \overline{\mathbb{P}}^{x_0}_{n-1,\bar{\sigma}_I,\bar{\sigma}_{II}}(C_1^i \backslash B_1^i) = 0$$

Thus by the induction hypothesis we have that $\mathbb{P}^{x_0}_{n-1,\sigma_I,\sigma_{II}}|_{C_1^i} = \overline{\mathbb{P}}^{x_0}_{n-1,\bar{\sigma}_I,\bar{\sigma}_{II}}|_{C_1^i}$, and we conclude:

$$\begin{aligned}\mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(B_1^i \times A_2^i) &= \int_{B_1^i} \gamma_{n-1[x_0 \dots x_n-1]}(A_2^i) \mathrm{d}\mathbb{P}^{x_0}_{n-1,\sigma_I,\sigma_{II}} \\ &= \int_{B_1^i} \bar{\gamma}_{n-1[x_0 \dots x_n-1]}(A_2^i) \mathrm{d}\overline{\mathbb{P}}^{x_0}_{n-1,\bar{\sigma}_I,\bar{\sigma}_{II}} = \overline{\mathbb{P}}^{x_0}_{n,\bar{\sigma}_I,\bar{\sigma}_{II}}(B_1^i \times A_2^i).\end{aligned}$$

Furthermore we have that

$$\mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}((C_1^i \backslash B_1^i) \times A_2^i) = \overline{\mathbb{P}}^{x_0}_{n,\bar{\sigma}_I,\bar{\sigma}_{II}}((C_1^i \backslash B_1^i) \times A_2^i) = 0.$$

Putting these together we obtain that:

$$\begin{aligned}\left|\sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(B_1^i \times A_2^i) - \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A)\right| &= \left|\sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(C_1^i \times A_2^i) - \sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A^i)\right| \\ &\leq \left|\sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A_1^i \times A_2^i) - \sum_{i=1}^{\infty} \mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A^i)\right| \leq \eta.\end{aligned}$$

The same estimate holds with the overline probabilities, and thus:

$$\mathbb{P}^{x_0}_{n,\sigma_I,\sigma_{II}}(A) - \overline{\mathbb{P}}^{x_0}_{n,\bar{\sigma}_I,\bar{\sigma}_{II}}(A) \leq 2\eta,$$

where η was arbitrarily small.

Lemma 1.4.7. *In the given setting the probability that the game will continue for infinite turns is zero:*

$$\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II},\tau}\big(\{\omega \in X^{x_0,\infty};\ \tau_0(\omega) = \infty\}\big) = 0.$$

Proof. **1.** In this first part we redefine the game on the extended game board $Y = \mathbb{R}^N$ as defined in (1.4.5). We apply Lemma 1.4.6 to show that $\overline{\mathbb{P}}^{x_0}_{\tau,\bar{\sigma}_I,\bar{\sigma}_{II}}(\{\tau < \infty\}) = \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau < \infty\})$. This claim is proven by writing $\{\tau < \infty\} = \bigcup_{n=0}^{\infty}\{\tau = n\}$ where the union is disjoint, and we note that $\{\tau = n\} \in \mathcal{F}_n^{x_0}$. Thanks to Lemma 1.4.6 we may write $\overline{\mathbb{P}}^{x_0}_{\tau,\bar{\sigma}_I,\bar{\sigma}_{II}}(\tau = n) = \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\tau = n)$ for all n. Thus we obtain:

$$\overline{\mathbb{P}}^{x_0}_{\tau,\bar{\sigma}_I,\bar{\sigma}_{II}}(\{\tau < \infty\}) = \sum_{n=0}^{\infty} \overline{\mathbb{P}}^{x_0}_{\tau,\bar{\sigma}_I,\bar{\sigma}_{II}}(\{\tau = n\}) = \sum_{n=0}^{\infty} \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau = n\}) = \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau < \infty\}).$$

Having shown this the overline notation will be dropped for simplicity of reading.

2. Next we show that the probability of $\{\tau < \infty\}$ is greater than the probability of an interesting set S_{x_0}. We note that since $\tau \le \tau_0$, it holds that $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau < \infty\}) \ge \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau_0 < \infty\})$. Next we define a subset of the ball

$$A_0 = \Big\{x \in B_\epsilon(0);\ |x| \in \big(\frac{\epsilon}{2}, \epsilon\big),\ x \cdot e_1 \in \big(-\frac{\pi}{8}, \frac{\pi}{8}\big)\Big\}.$$

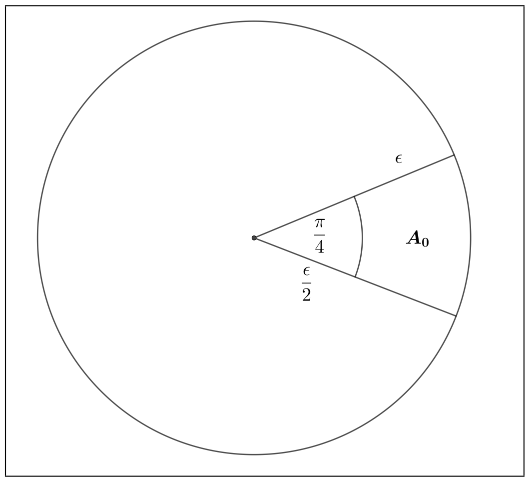


Figure 1: The set A_0 inside the ball of radius ϵ

Clearly there exists K large enough that if the token is moved by a vector in A_0 consecutively K times, it will exit Ω. Define the set of all game plays such that the token is moved in such a manner:

$$S_{x_0} = \{\omega \in Y^{\infty,x_0};\ \exists i_0 \text{ s.t. } \forall i = i_0, \ldots, i_0 + K,\ x_{i+1} - x_i \in A_0\}.$$

Clearly, $S_{x_0} \subset \{\tau_0 < \infty\}$, and thus $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau_0 < \infty\}) \ge \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(S_{x_0})$.

3. All that is left to prove is that $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(S_{x_0}) = 1$.

We begin by noting that for any chosen strategies and any turn n, we have that

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{x_n - x_{n+1} \in A_0\}) \ge \beta \frac{|A_0|}{|B_\epsilon(0)|} = \theta > 0.$$

That is the probability that x_{n+1} is chosen randomly and the random choice lands in A_0.

Considering the set $S^0_{x_0} = \{\omega \in Y^{\infty,x_0};\ \forall i = 0, \ldots, K,\ x_{i+1} - x_i \in A_0\}$, we have

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(S^0_{x_0}) \ge \theta^K.$$

This implies that:

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau > K\}) \leq 1-\theta^K.$$

Taking this as a base case we prove by induction that

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau > nK\}) \leq (1-\theta^K)^n.$$

For the inductive step we calculate:

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau > nK\}) = \mathbb{E}[\chi_{\{\tau>nK\}}] = \mathbb{E}[\mathbb{E}[\chi_{\{\tau>nK\}}|\mathcal{F}_{(n-1)K}]].$$

By the same argument used to prove the base step we have that:

$$\mathbb{E}[\chi_{\{\tau>nK\}}|\mathcal{F}_{(n-1)K}] = \int_X \cdots \int_X \chi_{\{\tau>nK\}} \mathrm{d}\gamma_{(n-1)K}[x_0,\ldots,x_{(n-1)K}]\ldots\gamma_{nK}[x_0,\ldots,x_{nK}]$$

We note that this integral is zero if $x_0,\ldots,x_{(n-1)K}$ are such that the game has already ended, and is equal to the probability of the game ending after the next K turns otherwise. If the game has not ended after $(n-1)K$ the probability of it ending within the next K is greater than θ^K. Thus we obtain that:

$$\mathbb{E}[\chi_{\{\tau>nK\}}|\mathcal{F}_{(n-1)K}] \leq (1-\theta^K)\chi_{\{\tau>(n-1)K\}}.$$

By taking the expectation of this and using the inductive step we obtain:

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau > nK\}) \leq (1-\theta^K)^n.$$

Clearly since $\{\tau=\infty\}\subset\{\tau>n\}$ we have that for all $n\geq 1$:

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau = \infty\}) \leq \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau > nK\}) \leq (1-\theta^K)^n.$$

Thus $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(\{\tau = \infty\}) = 0$.

Lemma 1.4.8. *Let $u : X \to \mathbb{R}$ be a bounded Borel function. Fix $\delta, \epsilon > 0$. Then there exist Borel functions σ_{sup}, $\sigma_{inf} : \Omega \to X$ such that:*

$$\forall x \in \Omega \quad \sigma_{sup}(x), \sigma_{inf}(x) \in B_\epsilon(x) \quad \textit{and} \quad u\left(\sigma_{sup}(x)\right) \geq \sup_{B_\epsilon(x)} u - \delta, \quad u\left(\sigma_{inf}(x)\right) \leq \inf_{B_\epsilon(x)} u + \delta$$

Proof. We show only the existence of σ_{sup} as the other proof is identical. The proof will proceed in three steps where we prove existence for increasingly complex classes of functions.

1. Consider a function $u = \chi_A$ the indicator function for a Borel set $A \subset X$, and assume $\delta < \frac{1}{3}$. Write $A + B_\epsilon(0) = \bigcup_{i=1}^\infty B_\epsilon(x_i)$, the union of countably many balls. This can be done as

$A + B_\epsilon(0) = \bigcup_{x\in A} B_\epsilon(x)$ and any such covering has a countable subcovering. Note that every x_i center of the balls must be a point in A and therefore $u(x_i) = 1$. For every $x \in X$, define:

$$\sigma_{sup}(x) = \begin{cases} x & \text{if } x \notin A + B_\epsilon(0), \\ x_i & \text{if } x \in B_\epsilon(x_i)\setminus \bigcup_{j=i}^\infty B_\epsilon(x_j). \end{cases}$$

This function selects the smallest i for which $x \in B_\epsilon(x_i)$, in this case $\sup_{B_\epsilon(x)} u = 1 = u(x_i) = u(\sigma_{sup}(x))$. If $x \notin B_\epsilon(x_i)$ for any i then

$$\sup_{B_\epsilon(x)} u = 0 = u(x) = u(\sigma_{sup}(x)).$$

2. Now we consider simple functions, that is u of the form $u = \sum_{k=1}^n \alpha_k \chi_{A_k}$ where A_k are pairwise disjoint Borel sets and α_k are increasing numbers. We assume that $\delta < \frac{\alpha_{i+1}-\alpha_i}{3}$ for all i. For each k we write $A_k + B_\epsilon(0) = \bigcup_{i=1}^\infty B_\epsilon(x_i^k)$, as before we have $x_i^k \in A_k$ and $u(x_i^k) = \alpha_k$. Define:

$$\sigma_{sup}(x) = \begin{cases} x & \text{if } x \notin A_k + B_\epsilon(0)\ \forall k, \\ x_i^k & \text{if } x \in B_\epsilon(x_i^k)\setminus \left(\bigcup_{j=i}^\infty B_\epsilon(x_j^k) \bigcup_{l>k} \left(A_l + B_\epsilon(0)\right)\right). \end{cases}$$

Similarly to the earlier argument we have that if $x \notin B_\epsilon(x_i^k)$ for any i, k then

$$\sup_{B_\epsilon(x)} u = 0 = u(x) = u(\sigma_{sup}(x)).$$

Otherwise we take the largest k for which $x \in B_\epsilon(x_i^k)$ for some i, and consider the smallest such i. In this case we have that

$$\sup_{B_\epsilon(x)} u = \alpha_k = u(x_i^k) = u(\sigma_{sup}(x)).$$

3. Finally we consider any Borel measurable function u. Then we have that there exists a simple function u_s as in point **2** such that $|u_s - u| < \frac{\delta}{2}$. Then we consider σ_{sup} chosen for the simple function u_s. We obtain:

$$u(\sigma_{sup}(x)) \geq u_s(\sigma_{sup}(x)) - \frac{\delta}{2} = \sup_{B_\epsilon(x)} u_s - \frac{\delta}{2} \geq \sup_{B_\epsilon(x)} u_s - \delta.$$

1.4.4 The payoff function

Let $\beta > 0$, $\alpha \geq 0$ and $\alpha + \beta = 1$, and let bounded Borel functions $\Psi_1 \leq \Psi_2 : X \to \mathbb{R}$ and $F : \Gamma \to \mathbb{R}$ satisfy $\Psi_1 \leq F \leq \Psi_2$ on Γ. Given two stopping times $\tau_I, \tau_{II} \leq \tau_0$ as above, define the payoff function $G^{\tau_I,\tau_{II}} : X^{\infty,x_0} \to \mathbb{R}$ as a random variable:

$$G^{\tau_I,\tau_{II}}(\omega) = G_\tau^{\tau_I,\tau_{II}}(x_0(\omega), \dots, x_n(\omega)),$$

where we define:

$$G_n^{\tau_I,\tau_{II}}(x_0, \dots, x_n) = \begin{cases} F(x_n) & x_n \in \Gamma \\ \Psi_1(x_n) & x_n \in \Omega,\ (x_0, \dots, x_n) \in A_n^{\tau_I < \tau_{II}} \\ \Psi_2(x_n) & \text{otherwise.} \end{cases}$$

We are interested in this function as a tool to define the actual payoff function for every $\omega = (x_0, \dots) \in X^{x_0,\infty}$. That is, for every ω we evaluate the payoff function at the turn when the game is stopped.

Lemma 1.4.9. *The function $G^{\tau_I,\tau_{II}}$ is $\mathcal{F}^{x_0}$-measurable.*

Proof. Take any interval $I \subset \mathbb{R}$. If the preimage of this set is in $\mathcal{F}^{x_0}$ then the Lemma is proven. We consider the following covering of X^{∞,x_0}:

$$X^{\infty,x_0} = \Big(\bigcup_{n=1}^{\infty} \{n = \tau_0(\omega)\} \Big) \cup \Big(\bigcup_{n=1}^{\infty} \{n = \tau_I(\omega),\ n < \tau_0(\omega)\} \Big) \\ \cup \Big(\bigcup_{n=1}^{\infty} \{n = \tau_{II}(\omega),\ n < \tau_I(\omega),\ n < \tau_0(\omega)\} \Big).$$

Each of the sets in this covering is $\mathcal{F}^{x_0}$-measurable by definition of stopping time. Thus we need only show that the payoff function is $\mathcal{F}^{x_0}$-measurable when restricted to each of these sets. But on these sets $G^{\tau_I,\tau_{II}}$ is equal to one of the Borel measurable functions F, Ψ_1, Ψ_2, and thus is measurable.

1.4.5 Connection to ϵ-p-harmonious solutions

Theorem 1.4.10. *Let u be the unique ϵ-p-harmonious solution to the double obstacle problem defined in 1.1.29. Define two functions:*

$$u_I(x_0) = \sup_{\sigma_I,\tau_I} \inf_{\sigma_{II},\tau_{II}} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II},\tau_I\wedge\tau_{II}} \left[G^{\tau_I,\tau_{II}}_{\tau_I\wedge\tau_{II}} \right], \quad u_{II}(x_0) = \inf_{\sigma_{II},\tau_{II}} \sup_{\sigma_I,\tau_I} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II},\tau_I\wedge\tau_{II}} \left[G^{\tau_I,\tau_{II}}_{\tau_I\wedge\tau_{II}} \right],$$

Then:

$$u_I = u_{II} = u_\epsilon \quad \in \Omega$$

Proof. **1.** We begin by noting that $u_I \leq u_{II}$. This follows from the fact that for any function $f(\sigma_I, \sigma_{II}, \tau_I, \tau_{II})$, if we fix any $\sigma_{II} = \bar{\sigma}_{II}$ and $\tau_{II} = \bar{\tau}_{II}$ we have:

$$\sup_{\sigma_I, \tau_I} f(\sigma_I, \bar{\sigma}_{II}, \tau_I, \bar{\tau}_{II}) \geq \sup_{\sigma_I, \tau_I} \inf_{\sigma_{II}, \tau_{II}} f(\sigma_I, \sigma_{II}, \tau_I, \tau_{II}),$$

and thus when we take the inf of the left hand side over σ_{II}, τ_{II} the inequality will still hold. Thus what we will prove in the next two points will be that $u_{II} \leq u_\epsilon$ and $u_I \geq u_\epsilon$.

2. We now prove the inequality $u_{II} \leq u_\epsilon$. Fix $\eta > 0$ and any admissible strategy σ_I and stopping time τ_I for Player I. Choose a strategy $\bar{\sigma}_{II}$ such that $\bar{\sigma}^n_{II}(x_0, \ldots, x_n) = \bar{\sigma}^n_{II}(x_n)$. That is, the strategy only depends on the last position of the token. Furthermore using Lemma 1.4.8, ensure:

$$\forall n \geq 0 \; \forall x_n \in X \quad u_\epsilon(\bar{\sigma}^n_{II}(x_n)) \leq \inf_{B_\epsilon(x_n)} u + \frac{\eta}{2^{n+1}}. \tag{1.8}$$

Also, choose the following stopping time:

$$\bar{\tau}_{II} = \inf \{n \geq 0; \; u(x_n) = \Psi_2(x_n) \;\text{ or }\; x_n \in \Gamma\}.$$

We will consider the sequence of random variables $\{u \circ x_n + \frac{\eta}{2^n}\}_{n=0}^\infty$ and show that it is a supermartingale with respect to the filtration $\{\mathcal{F}^{x_0}_n\}_{n=1}^\infty$.

Use Observation 1.4.4 along with the definition of γ_n (1.7), the choice from (1.8) and the definition of u_ϵ from (1.1.29) to compute:

$$\begin{aligned}
\forall (x_0, \ldots, x_{n-1}) \notin A^{\tau_I \wedge \bar{\tau}_{II}}_{n-1} \qquad & \mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}_{II}, \tau_I \wedge \bar{\tau}_{II}} \Big\{ u \circ x_n + \frac{\eta}{2^n} \mid \mathcal{F}^{x_0}_{n-1} \Big\} (x_0, \ldots, x_{n-1}) \\
&= \int_X u \, \mathrm{d}\gamma_{n-1}[x_0, \ldots, x_{n-1}] + \frac{\eta}{2^n} \\
&= \frac{\alpha}{2} u(\sigma^{n-1}_I(x_0, \ldots, x_{n-1})) + \frac{\alpha}{2} u(\bar{\sigma}^{n-1}_{II}(x_{n-1})) + \beta \fint_{B_\epsilon(x_{n-1})} u + \frac{\eta}{2^n} \\
&\leq \frac{\alpha}{2} \sup_{B_\epsilon(x_{n-1})} u + \frac{\alpha}{2} \inf_{B_\epsilon(x_{n-1})} u + \beta \fint_{B_\epsilon(x_{n-1})} u + \frac{\eta}{2^n} \frac{\alpha}{2} + 2 \frac{\eta}{2^{n+1}} \\
&\leq \max \Big\{ \Psi_1(x_{n-1}), \frac{\alpha}{2} \sup_{B_\epsilon(x_{n-1})} u + \frac{\alpha}{2} \inf_{B_\epsilon(x_{n-1})} u + \beta \fint_{B_\epsilon(x_{n-1})} u \Big\} + \frac{\eta}{2^n} (\frac{\alpha}{2} + 1) \\
&\leq u(x_{n-1}) + \frac{\eta}{2^{n-1}} = \big(u \circ x_{n-1} + \frac{\eta}{2^{n-1}}\big)(x_0, \ldots, x_{n-1}).
\end{aligned} \tag{1.9}$$

The last inequality follows from the fact that for $(x_0, \ldots, x_{n-1}) \notin A^{\tau_I \wedge \bar{\tau}_{II}}_{n-1}$, we have $u < \Psi_2$. If $(x_0, \ldots, x_{n-1}) \in A^{\tau_I \wedge \bar{\tau}_{II}}_{n-1}$ then

$$\mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}_{II}, \tau_I \wedge \bar{\tau}_{II}} \Big\{ u \circ x_n + \frac{\eta}{2^n} \mid \mathcal{F}^{x_0}_{n-1} \Big\} (x_0, \ldots, x_{n-1}) = u(x_{n-1}) + \frac{\eta}{2^n} \leq u(x_{n-1}) + \frac{\eta}{2^{n-1}}$$

as $\gamma_{n-1}(x_0,\ldots,x_{n-1}) = \delta_{x_{n-1}}$. Thus we have that $\{u\circ x_n + \frac{\eta}{2^n}\}_{n\geq 0}$ is a supermartingale as required. Using Doob's optional stopping time Theorem (1.2.11), with the supermartingale property and uniform boundedness of $u\circ x_{n\wedge\tau} + \frac{\eta}{2^{n\wedge\tau}}$ we evaluate:

$$u_{II} \leq \sup_{\tau_I,\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\bar{\sigma}_{II},\tau_I\wedge\bar{\tau}_{II}}\left\{G^{\tau_I,\bar{\tau}_{II}}_{\tau_I\wedge\bar{\tau}_{II}}\circ x_{\tau_I\wedge\bar{\tau}_{II}} + \frac{\eta}{2^{\tau_I\wedge\bar{\tau}_{II}}}\right\}$$
$$\leq \mathbb{E}^{x_0}_{\sigma_I,\bar{\sigma}_{II},\tau_I\wedge\bar{\tau}_{II}}\left\{u\circ x_{\tau_I\wedge\bar{\tau}_{II}} + \frac{\eta}{2^{\tau_I\wedge\bar{\tau}_{II}}}\right\} \leq u(x_0)+\eta.$$

The second inequality follows from the fact that for every $\omega\in X^{\infty,x_0}$ such that $n=(\tau_0\wedge\tau_I\wedge\bar{\tau}_{II})(\omega)<\infty$, we have:

$$G^{\tau_I,\bar{\tau}_{II}}_{\tau_I\wedge\bar{\tau}_{II}}(\omega) = G^{\tau_I,\bar{\tau}_{II}}_n(x_0,\ldots,x_n) \leq u(x_n).$$

We show this by cases on the stopping time that was activated:

If $n=\tau_0$, then $x_n\in\Gamma$ and thus $G^{\tau_I,\tau_{II}}_{\tau_I\wedge\bar{\tau}_{II}}(\omega) = F(x_n) = u(x_n)$.

If $n=\bar{\tau}_{II}$, then by the choice of $\bar{\tau}_{II}$ we have that x_n is such that $u(x_n)=\Psi_2(x_n)=G^{\tau_I,\bar{\tau}_{II}}_{\tau_I\wedge\tau_{II}}(\omega)$.

Otherwise we have that $G^{\tau_I,\bar{\tau}_{II}}_{\tau_I\wedge\bar{\tau}_{II}}(\omega)=\Psi_1\leq u(x_n)$ by definition of u.

Thus we have shown that $u_{II}\leq u_\epsilon+\eta$ for arbitrarily small η concluding the proof of the inequality.

3. We now reverse the argument to prove the inequality $u_I\geq u_\epsilon$. Fix $\eta>0$ and any admissible strategy σ_{II} and stopping time τ_{II} for Player II. Choose a strategy $\bar{\sigma}_I$ such that $\bar{\sigma}^n_I(x_0,\ldots,x_n) = \bar{\sigma}^n_I(x_n)$. That is, the strategy only depends on the last position of the token. Furthermore using Lemma 1.4.8, ensure:

$$\forall n\geq 0\ \forall x_n\in X \quad u_\epsilon(\bar{\sigma}^n_I(x_n)) \geq \sup_{B_\epsilon(x_n)} u - \frac{\eta}{2^{n+1}}. \tag{1.10}$$

Also, choose the following stopping time:

$$\bar{\tau}_I = \inf\left\{n\geq 0;\ u(x_n)=\Psi_1(x_n) \ \text{ or } \ x_n\in\Gamma\right\}.$$

We will consider the sequence of random variables $\left\{u\circ x_n - \frac{\eta}{2^n}\right\}_{n=0}^{\infty}$ and show that it is a submartingale with respect to the filtration $\{\mathcal{F}^{x_0}_n\}_{n=1}^{\infty}$.

Use Observation 1.4.4 along with the definition of γ_n from (1.7), the choice from (1.10) and the definition of u_ϵ from (1.1.29) to compute:

$$\begin{aligned}
\forall(x_0,\dots,x_{n-1}) \notin A_{n-1}^{\bar\tau_I\wedge\tau_{II}} \qquad & \mathbb{E}^{x_0}_{\bar\sigma_I,\sigma_{II},\bar\tau_I\wedge\tau_{II}}\Big\{u\circ x_n - \frac{\eta}{2^n} \mid \mathcal{F}^{x_0}_{n-1}\Big\}(x_0,\dots,x_{n-1})\\
&= \int_X u \, \mathrm{d}\gamma_{n-1}[x_0,\dots,x_{n-1}] - \frac{\eta}{2^n}\\
&= \frac{\alpha}{2}u(\bar\sigma_I^{n-1}(x_{n-1})) + \frac{\alpha}{2}u(\sigma_{II}^{n-1}(x_0,\dots,x_{n-1})) + \beta \fint_{B_\epsilon(x_{n-1})} u - \frac{\eta}{2^n}\\
&\geq \frac{\alpha}{2}\sup_{B_\epsilon(x_{n-1})} u + \frac{\alpha}{2}\inf_{B_\epsilon(x_{n-1})} u + \beta\fint_{B_\epsilon(x_{n-1})} u - \frac{\eta}{2^n}\frac{\alpha}{2} - 2\frac{\eta}{2^{n+1}}\\
&\geq \min\Big\{\Psi_2(x_{n-1}), \frac{\alpha}{2}\sup_{B_\epsilon(x_{n-1})} u + \frac{\alpha}{2}\inf_{B_\epsilon(x_{n-1})} u + \beta\fint_{B_\epsilon(x_{n-1})} u\Big\} - \frac{\eta}{2^n}(\frac{\alpha}{2}+1)\\
&\geq u(x_{n-1}) - \frac{\eta}{2^{n-1}} = \big(u\circ x_{n-1} - \frac{\eta}{2^{n-1}}\big)(x_0,\dots,x_{n-1}).
\end{aligned} \tag{1.11}$$

If $(x_0,\dots,x_{n-1}) \in A_{n-1}^{\bar\tau_I\wedge\bar\tau_{II}}$ then

$$\mathbb{E}^{x_0}_{\bar\sigma_I,\sigma_{II},\bar\tau_I\wedge\tau_{II}}\Big\{u\circ x_n - \frac{\eta}{2^n} \mid \mathcal{F}^{x_0}_{n-1}\Big\}(x_0,\dots,x_{n-1}) = u(x_{n-1}) - \frac{\eta}{2^n} \geq u(x_{n-1}) - \frac{\eta}{2^{n-1}}$$

as $\gamma_{n-1}(x_0,\dots,x_{n-1}) = \delta_{x_{n-1}}$. Thus we have that $\{u\circ x_n - \frac{\eta}{2^n}\}_{n\geq 0}$ is a submartingale as required. Using Doob's optional stopping time Theorem (1.2.11), with the submartingale property and uniform boundedness of $u\circ x_{n\wedge\tau} - \frac{\eta}{2^{n\wedge\tau}}$ we evaluate:

$$\begin{aligned}
u_I \geq \inf_{\tau_{II},\sigma_{II}} \mathbb{E}^{x_0}_{\bar\sigma_I,\sigma_{II},\bar\tau_I\wedge\tau_{II}}\Big\{G^{\bar\tau_I,\tau_I I}_{\bar\tau_I\wedge\tau_{II}}\circ x_{\bar\tau_I\wedge\tau_{II}} - \frac{\eta}{2^{\bar\tau_I\wedge\tau_{II}}}\Big\}&\\
\geq \mathbb{E}^{x_0}_{\bar\sigma_I,\sigma_{II},\bar\tau_I\wedge\tau_{II}}\Big\{u\circ x_{\bar\tau_I\wedge\tau_{II}} - \frac{\eta}{2^{\bar\tau_I\wedge\tau_{II}}}\Big\} &\geq u(x_0) - \eta.
\end{aligned}$$

The second inequality follows from the fact that for every $\omega \in X^{\infty,x_0}$ such that $n = (\tau_0\wedge\bar\tau_I\wedge\tau_{II})(\omega) < \infty$, we have:

$$G^{\bar\tau_I,\tau_{II}}_{\bar\tau_I\wedge\tau_{II}}(\omega) = G^{\bar\tau_I,\tau_{II}}_n(x_0,\dots,x_n) \geq u(x_n).$$

We show this by cases on the stopping time that was activated:

If $n = \tau_0$, then $x_n \in \Gamma$ and thus $G^{\bar\tau_I,\tau_{II}}_{\bar\tau_I\wedge\tau_{II}}(\omega) = F(x_n) = u(x_n)$.

If $n = \bar\tau_I$, then by the choice of $\bar\tau_I$ we have that x_n is such that $u(x_n) = \Psi_1(x_n) = G^{\bar\tau_I,\tau_{II}}_{\bar\tau_I\wedge\tau_{II}}(\omega)$.

Otherwise we have that $G^{\bar\tau_I,\tau_{II}}_{\bar\tau_I\wedge\tau_{II}}(\omega) = \Psi_2 \geq u(x_n)$ by definition of u.

Thus we have shown that $u_I \geq u_\epsilon - \eta$ for arbitrarily small η concluding the proof.

1.5 THE MAIN ANALYTICAL RESULT

We begin the discussion of the proof of Theorem 1.1.30 by proving the intermediate step of Proposition 1.1.31. The desired solution stated in the Theorem will be obtained by taking the limit of the ϵ-p-harmonious solutions. For simplicity we restate the Proposition:

Proposition 1.5.1. *Given F, Ψ_1 and Ψ_2 bounded Borel functions on a domain $\Omega \subset \mathbb{R}^N$ open and bounded, for every $\epsilon > 0$, there exists $u_\epsilon : \bar{\Omega} \to \mathbb{R}$ ϵ-p-harmonious solution to the double obstacle problem. Such a solution is unique.*

Proof. The proof will be articulated in three parts. First we will construct a sequence of functions which converge pointwise to the solution. Next we show the convergence to be uniform and the limit to be in fact the solution. Finally we will prove uniqueness.

1. Define an operator T on the space of Borel functions of X by defining for any Borel function $v : X \to \mathbb{R}$:

$$Tv(x) = \begin{cases} \max\left\{\Psi_1(x), \min\left\{\psi_2(x), \frac{\alpha}{2}\sup_{B_\epsilon(x)} v + \frac{\alpha}{2}\inf_{B_\epsilon(x)} v + \fint_{B_\epsilon(x)} v\right\}\right\} & \text{in } \Omega, \\ F(x) & \text{in } \Gamma, \end{cases}$$

A crucial property of this operator is that given functions v and w for which $v(x) \le w(x)$ for all x in X, then $Tv(x) \le Tw(x)$ for all x in X. This is easily seen by noting:

$$\begin{aligned} x \in \Gamma \quad &\implies \quad Tv(x) = Tw(x) = F(x), \\ x \in \Omega \quad &\implies \quad \sup_{B_\epsilon(x)} v \le \sup_{B_\epsilon(x)} w, \quad \inf_{B_\epsilon(x)} v \le \inf_{B_\epsilon(x)} w, \quad \fint_{B_\epsilon(x)} v \le \fint_{B_\epsilon(x)} w. \end{aligned}$$

We may now define a sequence of functions $\{u_n\}_{n=0}^{\infty}$ recursively with:

$$u_0 = \chi_\Gamma F + \chi_\Omega \Psi_1 \quad \text{and} \quad u_{n+1} = Tu_n.$$

We show by induction that this sequence of functions is non decreasing. The fact that $u_0 \le u_1$ is given by the fact that the functions coincide on Γ and $\Psi_1(x) \le Tv(x) \le \Psi_2(x)$ in Ω for any function v. The inductive step follows from the statement above giving:

$$u_{n-1}(x) \le u_n(x) \quad \implies \quad Tu_{n-1}(x) \le Tu_n(x) \quad \implies \quad u_n(x) \le u_{n+1}(x).$$

Thus the sequence is non decreasing and bounded above by Ψ_2 and therefore converges pointwise to a function u_ϵ. We have that $\Psi_1(x) \le u_\epsilon(x) \le \Psi_2(x)$ in Ω, $u_\epsilon(x) = F(x)$ and such a function must be a Borel function.

2. We now show that the convergence is uniform by contradiction. Assume the convergence is not uniform and thus $M = \lim_{n\to\infty} \sup_X (u_\epsilon - u_n) > 0$. Fix $\delta > 0$ and take $n > 1$ large enough to guarantee the following two conditions:

$$\begin{aligned}
&\sup_X (u_\epsilon - u_n) < M + \delta, \\
&\frac{\beta}{|B_\epsilon(0)|} \int_{B_\epsilon(x)} (u_\epsilon - u_n) \le \frac{\beta}{|B_\epsilon(0)|} \int_X (u_\epsilon - u_n) \le \delta.
\end{aligned}$$

The existence of such an n is guaranteed by the monotone convergence theorem. Select $x_0 \in \Omega$ such that $u_\epsilon(x_0) - u_{n+1}(x_0) > M - \delta > 0$. Such a point must exist by the hypothesis. Because of monotonicity we may state that $u_\epsilon(x_0) > \Psi_1(x_0)$, and $u_{n+1}(x_0) < \Psi_2(x_0)$ otherwise:

$$\begin{aligned}
&u_\epsilon(x_0) = \Psi_1(x_0) \Longrightarrow u_n(x_0) = \Psi_1(x_0) \quad \forall n > 1, \\
&u_{n+1}(x_0) = \Psi_2(x_0) \Longrightarrow u_m(x_0) = \Psi_2(x_0) = u_\epsilon(x_0) \quad \forall m > n+1.
\end{aligned}$$

Consider $m > n$ such that $u_{m+1}(x_0) - u_{n+1}(x_0) > M - 2\delta$, by similar arguments, $u_m(x_0) > \Psi_1(x_0)$. We may now write:

$$\begin{aligned}
M - 2\delta &< u_{m+1}(x_0) - u_{n+1}(x_0) \\
&\le \frac{\alpha}{2}\Big(\sup_{B_\epsilon(x)} u_m - \sup_{B_\epsilon(x)} u_n + \inf_{B_\epsilon(x)} u_m - \inf_{B_\epsilon(x)} u_n \Big) + \beta \fint_{B_\epsilon(x)} (u_m - u_n) \\
&\le \alpha \sup_{B_\epsilon(x)} (u_m - u_n) + \beta \fint_{B_\epsilon(x)} (u_m - u_n) \\
&\le \alpha \sup_{B_\epsilon(x)} (u_m - u_n) + \beta \fint_{B_\epsilon(x)} (u - u_n) \\
&\le \alpha\big(M + \delta\big) + \delta.
\end{aligned}$$

This implies that $M \le \alpha M + 3\delta$ which is a contradiction due to the fact that $\alpha < 1$ and δ is arbitrarily small. The third inequality in the above calculation follows from the fact that on any domain Ω for any two functions v and w:

$$\sup_\Omega u - \sup_\Omega v \le \sup_\Omega (u - v), \quad \text{and} \quad \inf_\Omega u - \inf_\Omega v \le \sup_\Omega (u - v).$$

Consider sequences of points:

$$\begin{aligned}
&\{x_n\}_{n=1}^\infty \quad \text{such that} \quad u(x_n) \to \sup_\Omega u, \\
&\{y_n\}_{n=1}^\infty \quad \text{such that} \quad v(y_n) \to \sup_\Omega v.
\end{aligned}$$

Then:

$$\sup_\Omega u - \sup_\Omega v = \lim_{n\to\infty} u(x_n) - \lim_{n\to\infty} v(y_n) \le \lim_{n\to\infty} \big(u(x_n) - v(x_n)\big) \le \sup_\Omega (u-v).$$

Next to prove the second statement, consider:

$$\{z_n\}_{n=1}^\infty \quad \text{such that} \quad u(z_n) \to \inf_\Omega u,$$

$$\{t_n\}_{n=1}^\infty \quad \text{such that} \quad v(t_n) \to \inf_\Omega v.$$

Then:

$$\inf_\Omega u - \inf_\Omega v = \lim_{n\to\infty} u(z_n) - \lim_{n\to\infty} v(t_n) \le \lim_{n\to\infty} \big(u(t_n) - v(t_n)\big) \le \sup_\Omega (u-v).$$

3. All that is left is to prove uniqueness. Assume u and $\bar{u}$ to be distinct solutions with:

$$M = \sup_\Omega (u - \bar{u}) > 0.$$

Take a sequence of points $\big\{x_n\big\}_{n=1}^\infty$ such that $\lim_{n\to\infty}(u-\bar{u})(x_n) = M$. By the compactness of the set $\bar{\Omega}$ we therefore have that there exists a subsequence of $\big\{x_n\big\}_{n=1}^\infty$ which converges to some point $x_0 \in \bar{\Omega}$. For n large enough it must hold that $\Psi_1(x_n) < u(x_n)$ and $\Psi_2(x_n) > \bar{u}(x_n)$, otherwise the sequence would not be approaching the supremum. We may therefore compute:

$$\begin{aligned}(u-\bar{u})(x_n) &= \frac{\alpha}{2}\Big(\sup_{B_\epsilon(x)} u - \sup_{B_\epsilon(x)} \bar{u} + \inf_{B_\epsilon(x)} u - \inf_{B_\epsilon(x)} \bar{u}\Big) + \beta \fint_{B_\epsilon(x_n)} (u-\bar{u}) \\ &\le \alpha \sup_\Omega (u-\bar{u}) + \beta \fint_{B_\epsilon(x_n)} (u-\bar{u}).\end{aligned}$$

By passing to the limit we obtain that:

$$M \le \alpha M + \beta \fint_{B_\epsilon(x_0)} (u-\bar{u}) \Rightarrow M \le \fint_{B_\epsilon(x_0)} (u-\bar{u}).$$

Given that $u - \bar{u} \le M$, it must hold that the set where $(u-\bar{u})(x) = M$ is dense in $B_\epsilon(x_0)$. Define the set:

$$G = \big\{x \in X | (u-\bar{u})(x) = M\big\}.$$

Then, $G \cap B_\epsilon(x_0)$ is dense in $B_\epsilon(x_0)$, by the same argument it can be shown that for every $x \in G \cap \Omega$ we have that $G \cap B_\epsilon(x)$ is dense in $B_\epsilon(x)$. By repeating finitely many steps we will obtain that for some $x \in \Gamma$ we have $(u-\bar{u})(x) = M$ which cannot hold as $u(x) = \bar{u}(x) = F(x)$.

We now come to the main result of this section which is the existence of a viscosity solution. Such a solution will be found as the limit of the u_ϵ defined in (1.1.29) as ϵ goes to zero. These functions are not generally continuous and the proof will require showing that the discontinuities of these solutions disappear.

The proof is based on the following alternative version of the Ascoli-Arzelà Lemma.

Lemma 1.5.2. *[16] Given a family of functions $u_\epsilon : \bar{\Omega} \to \mathbb{R}$ which satisfy:*

(i) There exists a constant C such that for all values of ϵ the norm $\|u_\epsilon\|_{L^\infty(\bar{\Omega})} \leq C$.

(ii) For all $\eta > 0$ there exist positive r_0 and ϵ_0 such that for all $\epsilon < \epsilon_0$ and for all $x_0, y_0 \in \bar{\Omega}$:

$$|x_0 - y_0| < r_0 \quad \Longrightarrow \quad |u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta.$$

Then there exists a subsequence of u_ϵ which converges uniformly in $\bar{\Omega}$ to a continuous function u.

Proof. We begin the proof by finding a candidate for the uniform limit u. Consider a countable dense subset $X \in \bar{\Omega}$ written as $X = \{x_i\}_{i=1}^{n}$. We show the existence of a pointwise converging subsequence on X through a standard diagonal argument. Define a subsequence $u_\epsilon^{(1)}$ of u_ϵ such that $u_\epsilon^{(1)}(x_1)$ converges to a value we call $u(x_1)$. Next we recursively define $u_\epsilon^{(n)}$ as a subsequence of $u_\epsilon^{(n-1)}$ such that $u_\epsilon^{(n)}(x_n)$ converges, and call the limit $u(x_n)$. The intersection of such sequences will provide a subsequence which converges pointwise on X to $u : X \to \mathbb{R}$. By hypothesis, for any $\eta > 0$, there exists $r_0, \epsilon_0 > 0$ such that for all $x, y \in X$:

$$\begin{aligned}
|x - y| \leq r_0 \quad &\Longrightarrow \quad |u_\epsilon(x) - u_\epsilon(y)| \leq \frac{\eta}{3}, \\
\exists \bar{\epsilon} \leq \epsilon_0 \quad \text{such that} \quad \epsilon \leq \bar{\epsilon} \quad &\Longrightarrow \quad |u_\epsilon(x) - u(x)|, |u_\epsilon(y) - u(y)| \leq \frac{\eta}{3} \\
&\Longrightarrow \quad |u(x) - u(y)| \leq |u(x) - u_\epsilon(x)| + |u_\epsilon(x) - u_\epsilon(y)| + |u_\epsilon(y) - u(y)| \leq \eta.
\end{aligned}$$

Thus we may continuously extend u to all of $\bar{\Omega}$ by defining:

$$u(z) = \lim_{X \ni x \to z} u(x).$$

We may now prove that the convergence is uniform. Choose a finite covering $\bar{\Omega} \subset \bigcup_{i=1}^{N} B_r(x_i)$ with r small enough that there exists ϵ_0 which guarantees:

$$\forall \epsilon \leq \epsilon_0, \, \forall x \in B_r(x_i), \, \forall i = 1, \ldots, N \quad |u_\epsilon(x) - u_\epsilon(x_i)|, |u(x) - u(x_i)| \leq \frac{\eta}{3}.$$

Furthermore we may request that ϵ_0 be small enough that for $i = 1, \ldots, N$ and $\epsilon < \epsilon_0$ there holds $|u_\epsilon(x_i) - u(x_i)| \leq \frac{\eta}{3}$. This last is guaranteed by the fact that we are considering only finitely many points. Finally we use the triangle inequality to show that there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and for any $x \in \bar{\Omega}$:

$$|u_\epsilon(x) - u(x)| \leq |u_\epsilon(x) - u_\epsilon(x_i)| + |u_\epsilon(x_i) - u(x_i)| + |u(x_i) - u(x)| \leq \eta.$$

Lemma 1.5.3. *Let $\bar{u}_\epsilon$ be the solutions to the ϵ-p-harmonious obstacle problem with lower obstacle Ψ, upper obstacle $\Psi_2(x) = C$ with $C > \max\{\sup_X \Psi, \sup_X F\}$ and boundary values F on X. Then, for all $\eta > 0$ there exist positive r_0 and ϵ_0 such that for all $\epsilon < \epsilon_0$, for all $x_0 \in \bar{\Omega}$ and $y_0 \in \partial\Omega$:*

$$|x_0 - y_0| < r_0 \quad \implies \quad \bar{u}_\epsilon(x_0) - \bar{u}_\epsilon(y_0) < \eta.$$

Proof. This proof follows closely the construction in [14], and uses heavily the identification between the expected value of the tug-of-war game and the ϵ-p-harmonious solutions. There will be several constants used throughout this proof, which are called $C, C_\Psi, C_F, C_{\Psi,F}$. The constant C depends only on the dimension N, the domain Ω, and constants p, α, β, the other constants depend on their subscripts.

1. Take $x_0 \in \bar{\Omega}$, $y_0 \in \partial\Omega$. We begin from the case when $u_\epsilon(x_0) = \Psi(x_0)$. In this case we have:

$$u_\epsilon(x_0) - u_\epsilon(y_0) = \Psi(x_0) - F(y_0) \leq \Psi(x_0) - \Psi(y_0) \leq C_\Psi |x_0 - y_0|$$

We therefore only need to consider the case when $u_\epsilon(x_0) > \Psi(x_0)$. Assume a particular strategy $\sigma_{0,II}$ and stopping time $\tau_{II} = \tau_0$ for Player II has been chosen.

The stopping time for Player II is in fact the optimal stopping time as the upper contact set is empty. In this case we can write $G_n = G = \chi_\Gamma F + \chi_\Omega \Psi$. Then by Proposition 1.4.10 we have:

$$u_\epsilon(x_0) - u_\epsilon(y_0) \leq \sup_{\tau, \sigma_I} \mathbb{E}^{x_0}_{\tau, \sigma_I, \sigma_{0,II}}[G \circ x_\tau - F(y_0)].$$

Furthermore we have that for all $x \in X$:

$$G(x) - F(y_0) \leq \chi_\Gamma(x)(F(x) - F(y_0)) + \chi_\Omega(x)(\Psi(x) - \Psi(y_0)) \leq C_{\Psi,F}|x - y_0|.$$

Combining the two equations we obtain:

$$u_\epsilon(x_0) - u_\epsilon(y_0) \leq C_{\Psi,F} \sup_{\tau, \sigma_I} \mathbb{E}^{x_0}_{\tau, \sigma_I, \sigma_{0,II}}[|x_\tau - y_0|]. \tag{1.12}$$

We need to show

Claim 1.5.3.1. *With an appropriate choice of $\sigma_{0,II}$, for all $0 < \delta \ll 1$ and $\epsilon < \min\left\{\frac{\beta}{2C_\delta}, \frac{\delta}{3}\right\}$:*

$$\sup_{\tau, \sigma_I} \mathbb{E}^{x_0}_{\tau, \sigma_I, \sigma_{0,II}}[|x_\tau - y_0|] \leq C\delta + C_\delta(|x_0 - y_0| + \epsilon).$$

Clearly if this claim is proven the proof of the Lemma is concluded.

2. *Proof of* 1.5.3.1: Take $z_0 \in \mathbb{R}^N \backslash \Omega$ such that $B_\delta(z_0) \cap \bar{\Omega} = \{y_0\}$. Define the strategy for Player II by:

$$\sigma^n_{0,II}(x_0,\ldots,x_n) = \begin{cases} x_n + (\epsilon - \epsilon^2)\frac{z_0 - x_n}{|z_0 - x_n|} & x_n \in \Omega, \\ x_n & x_n \in \Gamma. \end{cases} \tag{1.13}$$

Again, the stopping time for Player II may be taken $\tau_{II} = \tau_0$. Consider any strategy and stopping time for Player I.

Set $\epsilon < \frac{\delta}{3}$, Consider the smooth function $f(x) = |x - z_0|$ on the set $\Omega + B_{\frac{\delta}{2}}(0)$. By writing f as a second order Taylor's polynomial for any $x \in \Omega$ we obtain:

$$\begin{aligned} \fint_{B_\epsilon(x)} f(w)\mathrm{d}w &= \fint_{B_\epsilon(x)} f(x)\mathrm{d}w + \fint_{B_\epsilon(0)} \nabla f(x)\cdot w\mathrm{d}w + \sum_{i,j=1}^N \fint_{B_\epsilon(0)} \partial_i\partial_j f(x) w_i w_j \mathrm{d}w + o(\epsilon^2) \\ &= \fint_{B_\epsilon(x)} f(x)\mathrm{d}w + 0 + 2\sum_{i=1}^N \fint_{B_\epsilon(0)} \partial_i^2 f(x) w_i^2 \mathrm{d}w + o(\epsilon^2) \\ &= f(x) + \frac{\epsilon^2}{2(N+2)}\Delta f(x) + o(\epsilon^2). \end{aligned}$$

This yields:

$$\fint_{B_\epsilon(x)} |w - z_0|\mathrm{d}w \le |x - z_0| + C_\delta \epsilon^2. \tag{1.14}$$

We consider a value $C = C_\delta + 1$. By definition of expected value we write:

$$\begin{aligned} \forall (x_0,\ldots,x_{n-1}) \notin A^\tau_{n-1} \quad & \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[|x_n - z_0| - C\epsilon^2 n | \mathcal{F}^{x_0}_{n-1}\big](x_0,\ldots,x_{n-1}) \\ &\le \frac{\alpha}{2}|\sigma_I^{n-1}(x_0,\ldots,x_{n-1}) - z_0| + \frac{\alpha}{2}|\sigma^{n-1}_{0,II}(x_{n-1}) - z_0| + \beta \fint_{B_\epsilon(x)} |w - z_0|\mathrm{d}w - C\epsilon^2 n \\ &\le \frac{\alpha}{2}\big(|x_{n-1} - z_0| + \epsilon\big) + \frac{\alpha}{2}\big(|x_{n-1} - z_0| - (\epsilon - \epsilon^2)\big) + \beta\big(|x_{n-1} - z_0| + C_\delta\epsilon^2\big) - C\epsilon^2 n \\ &\le |x_{n-1} - z_0| - C\epsilon^2 n - (\frac{\alpha}{2} + C_\delta\beta)\epsilon^2) \le |x_{n-1} - z_0| - C\epsilon^2(n-1). \end{aligned}$$

$$\begin{aligned} \forall (x_0,\ldots,x_{n-1}) \in A^\tau_{n-1} \quad & \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[|x_n - z_0| - C\epsilon^2 n | \mathcal{F}^{x_0}_{n-1}\big](x_0,\ldots,x_{n-1}) \\ &= |x_{n-1} - z_0| - C\epsilon^2 n \le |x_{n-1} - z_0| - C\epsilon^2(n-1). \end{aligned}$$

This shows that the variable $|x_n - z_0| - C\epsilon^2 n$ is in fact a supermartingale, and we may conclude by Doob's optional stopping time theorem:

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[|x_\tau - z_0|\big] - C\epsilon^2 \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[n \wedge \tau\big] \le |x_0 - z_0|.$$

Finally we may write:

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[|x_\tau - y_0|\big] \le \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[|x_\tau - z_0|\big] + \delta \le |x_0 - z_0| + 2\delta + C\epsilon^2 \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[n \wedge \tau\big]. \tag{1.15}$$

3. All that is left in the proof is to find an appropriate bound for $\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}\big[\tau\big]$. To this end we consider a new game board.

Definition 1.5.4. *Redefine the game on a new game board $Y = B_R(z_0)$ with R taken large enough that $X \subset Y$. The initial token position will be $x_0 \in \Omega$ and τ_0 will be the time of first exit from Ω as before. The strategies are modified in the following way:*

$$\bar{\sigma}^n_I(x0,\dots,x_n) = \begin{cases} \sigma^n_I(x0,\dots,x_n) & (x_0,\dots,x_n) \in X^{n+1}, \\ x_n & \textit{otherwise.} \end{cases}$$

$$\bar{\sigma}^n_{0,II}(x_0,\dots,x_n) = \begin{cases} x_n + (\epsilon - \epsilon^2)\frac{z_0 - x_n}{|z_0 - x_n|} & x_n \in \Omega, \\ x_n & x_n \notin \Omega. \end{cases}$$

We consider a new stopping time $\tau_0 \leq \bar{\tau}_0 = \min\big\{n \in \mathbb{N}\big||x_n - z_0| < \delta\big\}$. The transition probabilities $\bar{\gamma}_n$ are redefined:

$$\bar{\gamma}_n[x_0,\dots,x_n] = \begin{cases} \frac{\alpha}{2}\delta_{\bar{\sigma}^n_I(x_0,\dots,x_n)} + \frac{\alpha}{2}\delta_{\bar{\sigma}^n_{0,II}(x_0,\dots,x_n)} + \beta\mathcal{U}_{B_\epsilon(x_n)\cap Y} & (x_0,\dots,x_n) \notin A^\tau_n, \\ x_n & (x_0,\dots,x_n) \in A^\tau_n. \end{cases}$$

The probabilities $\overline{\mathbb{P}}^{x_0}_{\tau\bar{\sigma}_I\bar{\sigma}_{0,II}}$ are defined from the transition probabilities as before.

Given this definition we have:

$$(x_0 \dots x_n) \in X^n \Longrightarrow \bar{\gamma}_n[x_0 \dots x_n] = \gamma_n[x_0 \dots x_n].$$

In fact this holds for any $(x_0 \dots x_n) \notin A^{\tau_0}_n$. We thus extended the game board in a way that satisfies the hypothesis of Lemma 1.4.6, and may state:

$$\forall A \subset X^{\infty,x_0}, \quad \bar{\mathbb{P}}^{x_0}_{\bar{\sigma}_I\bar{\sigma}_{0,II}}(A) = \mathbb{P}^{x_0}_{\sigma_I\sigma_{0,II}}(A).$$

We now evaluate:

$$\begin{aligned} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{0,II}}[\tau] &= \sum_{n=0}^{\infty} n\mathbb{P}^{x_0}_{\sigma_I\sigma_{0,II}}\big(\{\tau(\omega) = n\}\big) = \sum_{n=0}^{\infty} n\overline{\mathbb{P}}^{x_0}_{\sigma_I,\sigma_{0,II}}\big(\{\tau(\omega) = n\}\big) \\ &\leq \sum_{n=0}^{\infty} n\overline{\mathbb{P}}^{x_0}_{\sigma_I,\sigma_{0,II}}\big(\{\bar{\tau}(\omega) = n\}\big) = \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}[\bar{\tau}] = \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}[\bar{\tau}_0]. \end{aligned}$$

We thus need to find an appropriate bound for $\overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}[\bar{\tau}_0]$. To this end we define an auxiliary function $v_0 : (0, +\infty) \to \mathbb{R}$ taken from the theory of fundamental solutions of the Laplace equation.

$$v_0(s) = \begin{cases} -as^2 - bs^{2-N} + c & \text{for } N > 2, \\ -as^2 - b\log(s) + c & \text{for } N = 2. \end{cases}$$

The parameters a, b, c are chosen in such a way that the function $v = v_0(|x_0 - z_0|)$ satisfies the following conditions:

$$\begin{cases} \Delta v = -2(N+2) & \text{in } B_R(z_0) \backslash B_\delta(z_0), \\ v = 0 & \text{on } \partial B_\delta(z_0), \\ \frac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial B_R(z_0) \end{cases}$$

We may rewrite explicitly the system:

$$\text{For } N > 2: \quad \begin{cases} -2aN = -2(N+2), \\ -a\delta^2 - b\delta^{2-N} + c = 0, \\ -2aR + b(N-2)R^{1-N} = 0, \end{cases} \qquad \text{For } N = 2: \quad \begin{cases} -4a = -8, \\ -a\delta^2 - b\log(\delta) + c = 0, \\ -2aR + bR^{-1} = 0, \end{cases}$$

These system have explicit solutions from which we can draw some conclusions. The constants a, b, c are always positive. The function v_0 is increasing and concave in the interval $(\delta, \mathbb{R})$ and achieves it's only maximum for $s = R$. We now want to evaluate for any $x \in Y \backslash B_{\delta - \epsilon}$ the average:

$$\fint_{B_\epsilon(x)\cap Y} v(y)\mathrm{d}y = \fint_{B_\epsilon(x)\cap Y} v(x)\mathrm{d}y + \fint_{B_\epsilon(x)\cap Y} \nabla v(x) \cdot (y-x)\mathrm{d}y$$
$$+ \fint_{B_\epsilon(x)\cap Y} \nabla^2 v(x) \cdot (y-x) \otimes (y-x)\mathrm{d}y + o(\epsilon^2).$$

We look at this integral term by term. Firstly:

$$\fint_{B_\epsilon(x)\cap Y} v(x)\mathrm{d}y = 0.$$

To study the second and third term we refer to figure 2. The ball $B_\epsilon(x)$ is divided into three regions. The first is $A_3 = B_\epsilon(x) \backslash Y$. A_2 will be the reflection of A_3 across the hyperplane through x perpendicular to the line defined by $x - z_0$. Finally $A_1 = B_\epsilon(x) \backslash (A_2 \cup A_3)$ will be the remainder of the ball. We rewrite the second term in the integral with the variable y in coordinates given by $\bar{e}_i$ where the origin is given by x, $\bar{e}_1 = \frac{x - z_0}{|x - z_0|}$, and $\bar{e}_i \perp x - z_0$. In these coordinates we have that $\nabla v(x) = v_0'(|x - z_0|)\bar{e}_1$ is v is constant on the sphere. Furthermore $v_0'(|x - z_0|) > 0$, and $\bar{e}_1 \cdot y < 0$ for any $y \in A_2$. Finally we note that $\nabla v(x) \cdot y$ integrates to zero over the region A_1 due to the fact that it is an odd function integrated over a symmetric domain. By putting these considerations together we obtain:

$$\fint_{B_\epsilon(x)\cap Y} \nabla v(x) : (y - x)\mathrm{d}y = \fint_{A_1} \nabla v(x) \cdot y + \fint_{A_2} \nabla v(x) \cdot y \leq 0.$$

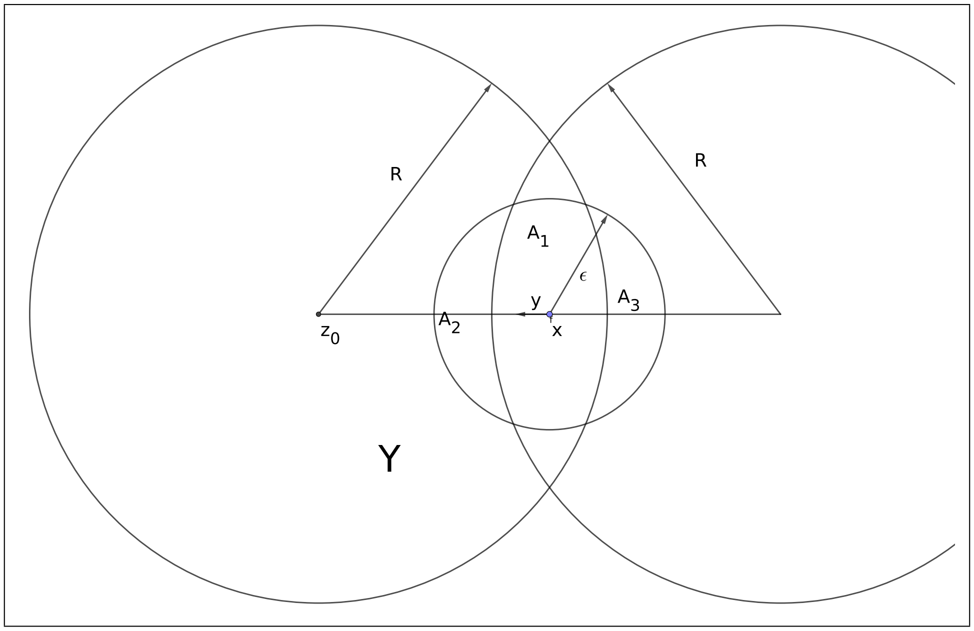


Figure 2: The three ses A_1, A_2 and A_3

Lastly we may write:

$$\fint_{B_\epsilon(x)\cap Y} \nabla^2 v(x)\cdot(y-x)\otimes(y-x)\mathrm{d}y = \sum_{i,j=1}^{N}\fint_{A_1\cup A_2}\partial_i\partial_j v(x) y_i y_j \mathrm{d}y = \sum_{i=1}^{N}\fint_{A_1\cup A_2}\partial_i^2 v(x) y_i^2\mathrm{d}y$$
$$= \Delta v(x)\fint_{A_1\cup A_2}|y|^2\mathrm{d}y = \frac{-2(N+2)}{|B_\epsilon(x)\cap Y|}\Big(\frac{\epsilon^2}{2(N+2)}|B_\epsilon(x)| - \epsilon^2|B_\epsilon(x)\backslash Y|\Big) \leq -\frac{\epsilon}{2}$$

Thus we may write the crucial estimate for ϵ small enough:

$$\fint_{B_\epsilon(x)\cap Y} v(y)\mathrm{d}y \leq v(x) - \frac{\epsilon^2}{2}. \tag{1.16}$$

We may now define a set of auxiliary functions:

$$Q_n(x) = \begin{cases} v(x) + \frac{\beta}{3}n\epsilon^2 & |x-z_0| > \delta-\epsilon, \\ v(x) & \delta-\epsilon \geq |x-z_0| > \delta-2\epsilon, \\ v_0(\delta-2\epsilon) & |x-z_0| \leq \delta-2\epsilon. \end{cases}$$

We want to show that $Q_n\circ x_n$ is in fact a supermartingale. To this end we need to compute:

$$\overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}\big[Q_n\circ x_n\big|\mathcal{F}^{x_0}_{n-1}\big](x_0,\dots,x_{n-1}) = \int_Y Q_n \mathrm{d}\bar{\gamma}_{n-1}[x_0,\dots,x_{n-1}].$$

We define the constant C_δ as the Lipschitz constant of $v_0 \in \mathcal{C}^2$ on the interval $[\frac{\delta}{6}, R+\delta]$. We need to split this calculation in three cases:

Case 1 $x_{n-1} \in Y \backslash \bar{B}_\delta(z_0)$**.** We use the fact that in this case $|\bar{\sigma}_{0,II}(x_{n-1}) - z_0| > \delta - \epsilon$, and the concavity of v_0 to compute:

$$\begin{aligned}
\int_Y Q_n \mathrm{d}\bar{\gamma}_{n-1}[x_0,\dots,x_{n-1}] &= \frac{\alpha}{2} Q_n\big(\bar{\sigma}_I(x_0,\dots,x_{n-1})\big) + \frac{\alpha}{2} Q_n\big(\bar{\sigma}_{0,II}(x_{n-1})\big) + \beta \fint_{B_\epsilon(x)\cap Y} Q_n \\
&\le \frac{\alpha}{2} v_0\big(|x_{n-1}-z_0| + \epsilon\big) + \frac{\alpha}{2} v_0\big(|x_{n-1}-z_0| - \epsilon + \epsilon^3\big) + \beta v_0\Big(|x_{n-1}-z_0| - \frac{\epsilon^2}{2}\Big) + \frac{\beta}{3} n\epsilon^2 \\
&\le \alpha v_0(|x_{n-1}-z_0|) + \beta v_0(|x_{n-1}-z_0|) + C_\delta \epsilon^2 - \beta\frac{\epsilon^2}{2} + \beta n \frac{\epsilon^2}{3} \\
&\le v_0(|x_{n-1}-z_0|) + \frac{\beta}{3}(n-1)\epsilon^2 = Q_{n-1}(x_{n-1}).
\end{aligned}$$

Note that we used the previously discussed bound of $\epsilon \le \frac{\beta}{6C_\delta}$.

Case 2 $x_{n-1} \in B_\delta(z_0) \backslash B_{\delta-\epsilon}(z_0)$**.**

$$\begin{aligned}
\int_Y Q_n \mathrm{d}\bar{\gamma}_{n-1}[x_0,\dots,x_{n-1}] &= \alpha Q_n(x_{n-1}) + \beta \fint_{B_\epsilon(x)\cap Y} Q_n \\
&\le \alpha v_0(|x_{n-1}-z_0|) + \beta v_0(|x_{n-1}-z_0|) - \beta\frac{\epsilon^2}{2} + \beta n\frac{\epsilon^2}{3} \\
&\le v_0(|x_{n-1}-z_0|) + \frac{\beta}{3}(n-1)\epsilon^2 = Q_{n-1}(x_{n-1}).
\end{aligned}$$

Case 3 $x_{n-1} \in B_{\delta-\epsilon}(z_0)$**.**

$$\int_Y Q_n \mathrm{d}\bar{\gamma}_{n-1}[x_0,\dots,x_{n-1}] = Q_{n-1}(x_{n-1}).$$

Thus we may conclude that the random variable $Q = Q_n \circ x_n$ is in fact a supermartingale with respect to the filtration $\mathcal{F}_n^{x_0}$. Applying Doob's optional stopping time Theorem 1.2.11 we obtain:

$$v_0(|x_0 - z_0|) \ge \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}\big[Q_n \circ x_{n\wedge\tau}\big] = \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}\big[v_0(|x_{n\wedge\tau} - z_0|)\big] + \frac{\beta}{3}\epsilon^2 \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}[n\wedge\tau].$$

Taking the limit $n \to \infty$ we obtain:

$$\frac{\beta}{3}\epsilon^2 \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}[\tau] \le v_0(|x_0 - z_0|) + \overline{\mathbb{E}}^{x_0}_{\sigma_I,\sigma_{0,II}}\big[v_0(|x_{\bar{\tau}_0} - z_0|)\big]$$

In light of the fact that $v(\delta) = 0$ we obtain:

$$v_0(|x_0 - z_0|) \le C_\delta(|x_0 - y_0| - \delta) = C_\delta |x_0 - z_0|.$$

Which, together with (1.15), allows us to conclude that for any ϵ small enough we have:

$$\mathbb{E}^{x_0}_{\sigma_I,\sigma_{0,II}}\big[|x_\tau - y_0|\big] \le C\delta + C_\delta(|x_0 - y_0| + \epsilon).$$

By inverting te sign of all the functions we may obtain the inverse inequality for the upper obstacle.

Corollary 1.5.5. *Let $\underline{u}_\epsilon$ be solutions to the ϵ-p-harmonious obstacle problem with upper obstacle Ψ, lower obstacle $\Psi_1(x) = C$ where $C < \min\{\inf_X \Psi, \inf_X F\}$ and boundary values F on X. Then, for all $\eta > 0$ there exist positive r_0 and ϵ_0 such that for all $\epsilon < \epsilon_0$, for all $x_0 \in \bar{\Omega}$ and $y_0 \in \partial\Omega$:*

$$|x_0 - y_0| < r_0 \quad \Longrightarrow \quad \underline{u}_\epsilon(x_0) - \underline{u}_\epsilon(y_0) > -\eta.$$

This allows us to prove the main result which we restate for ease of readership.

Theorem 1.5.6. *Given F, Ψ_1 and Ψ_2 bounded Lipschitz continuous functions on a domain $\Omega \subset \mathbb{R}^N$ open and bounded, there exists $u : \bar{\Omega} \to \mathbb{R}$ the unique viscosity solution to the double obstacle problem (1.4).*

We split the proof of this Theorem into two steps. First we prove the existence of the solution and then we show that it must be unique.

Existence of viscosity solutions. **1.** We begin the proof by providing a uniform continuity property close to the boundary. Consider $\bar{u}_\epsilon$ solutions to the obstacle problem described in Lemma 1.5.3 with $\Psi = \Psi_1$, and $\underline{u}_\epsilon$ solutions to the obstacle problem described in (1.5.5) with $\Psi = \Psi_2$. Then we have $\underline{u}_\epsilon \leq u_\epsilon \leq \bar{u}_\epsilon$, and equality on the boundary. This follows from the fact that the obstacles for $\underline{u}_\epsilon$ are smaller than for the other two, and the obstacles for $\bar{u}_\epsilon$ are larger than the first two. Then by Lemma 1.5.3 and its corollary 1.5.5, we have that for all $\eta > 0$ there exist positive r_0 and ϵ_0 such that for all $\epsilon < \epsilon_0$, for all $x_0 \in \bar{\Omega}$ and $y_0 \in \partial\Omega$:

$$\begin{aligned} u_\epsilon(x_0) - u_\epsilon(y_0) &\leq \bar{u}_\epsilon(x_0) - F(y_0) = \bar{u}_\epsilon(x_0) - \bar{u}_\epsilon(y_0) \leq \eta, \\ u_\epsilon(x_0) - u_\epsilon(y_0) &\geq \underline{u}_\epsilon(x_0) - F(y_0) = \underline{u}_\epsilon(x_0) - \underline{u}_\epsilon(y_0) \geq -\eta. \end{aligned}$$

2. We now extend this property to the whole domain so that we may use Lemma 1.5.2 to prove uniform convergence. Fix $\eta > 0$, use the first part of the proof and the Lipschitz property of the obstacle and boundary value functions to find r_0 and ϵ_0 such that:

$$\begin{aligned} \forall \epsilon < \epsilon_0, \quad \forall x_0 \in \bar{\Omega}, \quad y_0 \in \partial\Omega \qquad & |x_0 - y_0| \leq r_0 \quad \Longrightarrow \quad |u_\epsilon(x_0) - u_\epsilon(y_0)| < \frac{\eta}{4}, \\ \forall x_0, y_0 \in \bar{\Omega} \quad \forall i = 1,2 \qquad & |x_0 - y_0| \leq r_0 \quad \Longrightarrow \quad |\Psi_i(x_0) - \Psi_i(y_0)| < \frac{\eta}{4}, \\ \forall x_0, y_0 \in \Gamma \quad \forall i = 1,2 \qquad & |x_0 - y_0| \leq r_0 \quad \Longrightarrow \quad |F(x_0) - F(y_0)| < \frac{\eta}{4}. \end{aligned} \tag{1.17}$$

We now consider the set

$$\tilde{\Gamma} = \Big\{ x \in \bar{\Omega} \Big| \operatorname{dist}(x, \partial\Omega) \leq \frac{r_0}{2} \Big\}.$$

Using the triangle inequality it is easy to see that for any x_0 in $\tilde{\Gamma}$ and y_0 in $\bar{\Omega}$ such that $|x_0 - y_0| < \frac{r_0}{2}$ there exists $z_0 \in \partial\Omega$ such that $|x_0 - z_0|, |y_0 - z_0| < r_0$. Using (1.17) we see that for every $\epsilon \leq \epsilon_0$:

$$|u_\epsilon(x_0) - u_\epsilon(y_0)| \leq |u_\epsilon(x_0) - u_\epsilon(z_0)| + |u_\epsilon(y_0) - u_\epsilon(z_0)| < \frac{\eta}{2}.$$

Fix $x_0, y_0 \in \Omega$ such that $|x_0 - y_0| \leq \frac{r_0}{2}$. For every $\epsilon < \epsilon_0$ define the following Borel functions:

$$\tilde{F} : \tilde{\Gamma} \to \mathbb{R} \text{ by: } \tilde{F}(z) = u_\epsilon(z - (x_0 - y_0)) + \frac{\eta}{2},$$
$$\tilde{\Psi}_1 : X \to \mathbb{R} \text{ by: } \tilde{\Psi}_1(z) = \Psi_1(z - (x_0 - y_0)) + \frac{\eta}{2},$$
$$\tilde{\Psi}_2 : X \to \mathbb{R} \text{ by: } \tilde{\Psi}_2(z) = \Psi_2(z - (x_0 - y_0)) + \frac{\eta}{2}.$$

Now find the solution $\tilde{u}_\epsilon$ to the ϵ-p-harmonious double obstacle problem defined on $\bar{\Omega}$ with boundary condition $\tilde{F}$ in $\tilde{\Gamma}$ and obstacles $\tilde{\Psi}_1, \tilde{\Psi}_2$. The function $\tilde{u}_\epsilon(z) = u_\epsilon(z - (x_0 - y_0)) + \frac{\eta}{2}$ satisfies:

$$\tilde{u}_\epsilon(z) = \begin{cases} \max\Big\{\tilde{\Psi}_1(z), \min\big\{\Psi_2(z), \frac{\alpha}{2}(\sup_{B_\epsilon(z)} \tilde{u}_\epsilon + \inf_{B_\epsilon(z)} \tilde{u}_\epsilon) + \beta \fint_{B_\epsilon(z)} \tilde{u}_\epsilon \big\}\Big\} & \text{if } x \in \Omega \backslash \tilde{\Gamma}, \\ \tilde{F}(z) & \text{if } x \in \tilde{\Gamma}, \end{cases}$$

and thus it is the unique solution to the equation. Furthermore:

$$\forall z \in \tilde{\Gamma}, \quad \tilde{F}(z) = u_\epsilon(z - (x_0 - y_0)) + \frac{\eta}{2} \geq u_\epsilon(z),$$
$$\forall z \in \bar{\omega}, \quad \tilde{\Psi}_1(z) = \Psi_1(z - (x_0 - y_0)) + \frac{\eta}{2} \geq \Psi_1(z),$$
$$\forall z \in \bar{\omega}, \quad \tilde{\Psi}_2(z) = \Psi_2(z - (x_0 - y_0)) + \frac{\eta}{2} \geq \Psi_2(z).$$

Thus we have that $\tilde{u}_\epsilon(z) \geq u_\epsilon(z)$ in $\bar{\Omega}$. We may further evaluate:

$$u_\epsilon(x_0) - u_\epsilon(y_0) \leq \tilde{u}_\epsilon(x_0) - u_\epsilon(y_0) = u_\epsilon(y_0) + \frac{\eta}{2} - u_\epsilon(y_0) = \frac{\eta}{2} < \eta.$$

By switching x_0 and y_0 we obtain:

$$u_\epsilon(y_0) - u_\epsilon(x_0) < \eta.$$

3. We note that u_ϵ are all bounded by the same constants as they are constrained between the bounded functions Ψ_1, Ψ_2. Thus the family $\{u_\epsilon\}_{\epsilon > 0}$ satisfies the conditions of Lemma 1.5.2, and has a uniformly converging subsequence. We finally show that the limit of any converging subsequence of $\{u_\epsilon\}_{\epsilon > 0}$ must be a viscosity solution as defined in (1.1.28). The fact that the whole sequence converges to the unique solution will follow when uniqueness is proven. Let u be one such limit. Clearly it must hold that $\Psi_1(x) \leq u(x) \leq \Psi_2(x)$ for any $x \in \Omega$, and $u(x) = F(x)$ for any $x \in \partial\Omega$. Thus point (i) in the definition is trivially satisfied. This is because every $u_\epsilon(x)$ satisfies these inequalities. To prove point (ii), take a point x_0 such that $\Psi_2(x_0) > u(x_0)$. Since both u and Ψ_2 are continuous functions there exists $B_\delta(x_0)$ in which $\Psi_2 > u$. By uniform convergence for ϵ below some threshold we also have that $\Psi_2 > u_\epsilon$ in $B_\delta(x_0)$. Let ϕ be a test function as in the definition, then x_0 is the minimum for the function $u - \phi$.

Claim 1.5.6.1. *There exists a sequence of points x_ϵ converging to x_0 as $\epsilon \to 0$, such that:*

$$u_\epsilon(x_\epsilon) - \phi(x_\epsilon) \leq \inf_{\bar{\Omega}}(u_\epsilon - \phi) + \epsilon^3.$$

For every $i \geq 1$, define $a_i = \min\limits_{\bar{\Omega}\setminus B_{1/i}(x_0)}(u - \phi) > 0$, and let $\epsilon_i > 0$ be a threshold for which:

$$\forall \epsilon < \epsilon_i \quad \|u_\epsilon - u\|_{L^\infty(\Omega)} \leq \frac{1}{2}a_i.$$

We may choose the ϵ_i in such a way that they converge to zero. For all $\epsilon \in (\epsilon_{i+1}, \epsilon_i]$ choose $x_\epsilon \in B_{1/i}(x_0)$ such that:

$$u_\epsilon(x_\epsilon) - \phi(x_\epsilon) \leq \inf_{B_{1/i}(x_0)}(u_\epsilon - \phi) + \epsilon^3.$$

We conclude the proof of this claim by noting that for any $x \in \bar{\Omega}\setminus B_{1/i}(x_0)$ we obtain:

$$\begin{aligned} u_\epsilon(x) - \phi(x) \geq u(x) - \phi(x) - \|u_\epsilon - u\|_{L^\infty(\Omega)} \geq \frac{a_i}{2} \geq \|u_\epsilon - u\|_{L^\infty(\Omega)} \\ \geq u_\epsilon(x_0) - u(x_0) = u_\epsilon(x_0) - \phi(x_0) \geq u_\epsilon(x_0) - \phi(x_0) - \epsilon^3. \end{aligned}$$

Using (1.5.6.1) we may therefore state that for any $x \in \Omega$ we have $u_\epsilon(x) \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) + \phi(x) - \epsilon^3$. We may thus wrie:

$$\begin{aligned} u_\epsilon(x_\epsilon) &\geq \frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)} u_\epsilon + \frac{\alpha}{2}\inf_{B_\epsilon(x_\epsilon)} u_\epsilon + \beta \fint_{B_\epsilon(x_\epsilon)} u_\epsilon \\ &\geq (u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \epsilon^3) + \Big(\frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)} \phi + \frac{\alpha}{2}\inf_{B_\epsilon(x_\epsilon)} \phi + \beta \fint_{B_\epsilon(x_\epsilon)} \phi\Big). \end{aligned}$$

We now consider a point $\bar{x}_\epsilon$ to be a point in which ϕ attains minimum on the closed ball $\bar{B}_\epsilon(x_\epsilon)$.

Claim 1.5.6.2.

$$\begin{aligned} \epsilon^3 &\geq \Big(\frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)} \phi + \frac{\alpha}{2}\inf_{B_\epsilon(x_\epsilon)} \phi + \beta \fint_{B_\epsilon(x_\epsilon)} \phi\Big) - \phi(x_\epsilon) \\ &\geq \frac{\beta\epsilon^2}{2(N+2)}\Big((p-2)\Big\langle \nabla^2\phi(x_\epsilon)\frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}, \frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}\Big\rangle + \Delta\phi(x_\epsilon)\Big) + o(\epsilon^2). \end{aligned}$$

The justification for the second inequality in (1.5.6.2) comes from the Taylor expansion of ϕ at the point x_ϵ:

$$\min_{\bar{B}_\epsilon(x_\epsilon)} \phi = \phi(\bar{x}_\epsilon) = \phi(x_\epsilon) + \langle \nabla\phi(x_\epsilon), (\bar{x}_\epsilon - x_\epsilon)\rangle + \frac{1}{2}\langle \nabla^2\phi(x_\epsilon)(\bar{x}_\epsilon - x_\epsilon), (\bar{x}_\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

Similarly we write:

$$\max_{\bar{B}_\epsilon(x_\epsilon)} \phi \geq \phi(x_\epsilon + (x_\epsilon - \bar{x}_\epsilon)) \geq \phi(x_\epsilon) - \langle \nabla\phi(x_\epsilon), (\bar{x}_\epsilon - x_\epsilon)\rangle + \frac{1}{2}\langle \nabla^2\phi(x_\epsilon)(\bar{x}_\epsilon - x_\epsilon), (\bar{x}_\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

Finally we obtain an estimate for the integral term by noting that the second term in the Taylor expansion disappears when averaged as it is linear.

$$\begin{aligned}\fint_{\bar{B}_\epsilon(x_\epsilon)} \phi &= \phi(x_\epsilon) + \fint_{\bar{B}_\epsilon(x_\epsilon)} \langle \nabla\phi(x_\epsilon), (x - x_\epsilon)\rangle \mathrm{d}x + \sum_{i,j=1}^{N} \frac{\partial^2\phi}{\partial_i x \partial_j x}(x_\epsilon) \fint_{\bar{B}_\epsilon(0)} \frac{1}{2} x_i x_j \mathrm{d}x + o(\epsilon^2)\\ &= \phi(x_\epsilon) + \sum_{i=1}^{N} \frac{\partial^2\phi}{\partial_i x^2}(x_\epsilon) \fint_{\bar{B}_\epsilon(x_\epsilon)} |x_i|^2 \mathrm{d}x + o(\epsilon^2) = \Delta\phi(x_\epsilon)\frac{\epsilon^2}{2(N+2)} + o(\epsilon^2).\end{aligned}$$

We may thus write:

$$\begin{aligned}&\Big(\frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)} \phi + \frac{\alpha}{2}\inf_{B_\epsilon(x_\epsilon)} \phi + \beta \fint_{B_\epsilon(x_\epsilon)} \phi\Big) - \phi(x_\epsilon)\\ &\qquad\qquad \geq \frac{\alpha}{2}\langle \nabla^2\phi(x_\epsilon)(\bar{x}_\epsilon - x_\epsilon), (\bar{x}_\epsilon - x_\epsilon)\rangle + \Delta\phi(x_\epsilon)\frac{\beta\epsilon^2}{2(N+2)} + o(\epsilon^2).\end{aligned}$$

The claim is finally proven by noting:

$$\frac{\alpha}{2} = \frac{p-2}{2(N+p)}\beta\frac{N+p}{N+2} = \frac{\beta\epsilon^2}{2(N+2)}\frac{p-2}{\epsilon^2}.$$

4. Dividing by ϵ^2 we obtain:

$$\epsilon \geq \frac{\beta}{2(N+2)}\Big((p-2)\big\langle \nabla^2\phi(x_\epsilon)\frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}, \frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}\big\rangle + \Delta\phi(x_\epsilon)\Big) + \frac{o(\epsilon^2)}{\epsilon^2}.$$

We may thus pass this inequality to the limit to obtain:

$$\limsup_{\epsilon\to 0}\Big((p-2)\big\langle \nabla^2\phi(x_\epsilon)\frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}, \frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}\big\rangle + \Delta\phi(x_\epsilon)\Big) \leq 0.$$

Claim 1.5.6.3.

$$\lim_{\epsilon\to 0}\frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}.$$

To prove (1.5.6.3) we consider the sequence of functions $\phi_\epsilon(z) = \frac{1}{\epsilon}(\phi(x_\epsilon + \epsilon z) - \phi(x_\epsilon))$. This sequence converges uniformly on $B_1(0)$ to the linear map $\langle \nabla\phi(x_0), z\rangle$ by definition of gradient. For every $\epsilon > 0$ the function $z_\epsilon = \frac{1}{\epsilon}(\bar{x}_\epsilon - x_\epsilon)$ will be a minimizer on the ball of radius 1. Thus the limit of any converging subsequence of the z_ϵ must be a minimizer for the limit function $\langle \nabla\phi(x_0), z\rangle$. Since the function is linear the only minimizer is in fact $-\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}$, and the claim is proven.

We may thus write:

$$\frac{1}{|\nabla\phi(x_0)|^{p-2}}\Delta_p\phi(x_0) = (p-2)\Big\langle \nabla^2\phi(x_\epsilon)\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}, \frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}\Big\rangle + \Delta\phi(x_\epsilon) \le 0.$$

Condition (ii) is thus proven.

5. Condition (iii) is obtained analogously to condition (ii) by considering x_0 such that $u(x_0) > \Psi_1(x_0)$. We will have that at x_0 the function $u-\phi$ attains maximum and define a sequence of quasi maximisers x_ϵ. We then invert the direction of every inequality in the argument to obtain:

$$u_\epsilon(x_\epsilon) \le (u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \epsilon^3) + \Big(\frac{\alpha}{2}\sup_{B_\epsilon(x_\epsilon)}\phi + \frac{\alpha}{2}\inf_{B_\epsilon(x_\epsilon)}\phi + \beta \fint_{B_\epsilon(x_\epsilon)}\phi\Big).$$

And finally:

$$-\epsilon \le \frac{\beta}{2(N+2)}\Big((p-2)\big\langle \nabla^2\phi(x_\epsilon)\frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}, \frac{\bar{x}_\epsilon - x_\epsilon}{\epsilon}\big\rangle + \Delta\phi(x_\epsilon)\Big) + \frac{o(\epsilon^2)}{\epsilon^2}.$$

Here $\bar{x}_\epsilon$ is taken to be the maximum over the closed ball and not the minimum. By the same arguments as in step **5** we may conclude that $\nabla_p\phi(x_0) \ge 0$.

We state an intermediate Theorem showing that variational solutions to this problem are unique. This will be used by later proving that viscosity and variational solutions coincide under our assumptions concluding the proof of Theorem 1.1.30.

Uniqueness of viscosity solutions. Consider a viscosity solution to the double obstacle problem u. Consider any open Lipschitz set:

$$\mathcal{U} \subset\subset \big\{x \in \Omega \big| \Psi_1(x) \neq \Psi_2(x)\big\} = A.$$

We show that u is the unique variational solution to the double obstacle problem defined on $\mathcal{K}_{u|\partial\mathcal{U},\Psi_1,\Psi_2}(\mathcal{U})$ as in Theorem 1.1.32. Define the two sets:

$$\mathcal{U}_1 = \big\{x \in \mathcal{U} \big| \Psi_1(x) < u(x)\big\}, \quad \text{and} \quad \mathcal{U}_2 = \big\{x \in \mathcal{U} \big| \Psi_2(x) > u(x)\big\}.$$

By definition we have that on the set $\mathcal{U}_2$ the function u is a viscosity supersolution and a continuous function. By Theorem 1.1.26 we have that u must therefore be p-superharmonic in $\mathcal{U}_2$ and therefore of $W^{1,p}_{loc}(\mathcal{U}_2)$ regularity. Similarly we have that on $\mathcal{U}_1$ the function u is a viscosity subsolution and a continuous function. This implies that u must be p-subharmonic in $\mathcal{U}_1$ and in $W^{1,p}_{loc}(\mathcal{U}_1)$. Thus we have that $u \in W^{1,p}_{loc}(\mathcal{U})$. We may actually apply the same reasoning to any $\tilde{\mathcal{U}} \supset \overline{\mathcal{U}}$ giving that

$u \in W^{1,p}(\mathcal{U})$. By Theorem 1.1.25 we have that p-superharmonic and p-subharmonic are weak super and sub solutions if continuous and in $W^{1,p}$. Thus we may write:

$$\int_{\mathcal{U}_1} |\nabla u|^p \leq \int_{\mathcal{U}_1} |\nabla(u+\phi)|^p \quad \forall \phi \in \mathcal{C}_0^\infty(\mathcal{U}_1, \mathbb{R}^+),$$
$$\int_{\mathcal{U}_2} |\nabla u|^p \leq \int_{\mathcal{U}_1} |\nabla(u+\phi)|^p \quad \forall \phi \in \mathcal{C}_0^\infty(\mathcal{U}_2, \mathbb{R}^-).$$

Now consider any test function $\phi \in \mathcal{C}_0^\infty(\mathcal{U}_2, \mathbb{R}^)$ such that $\Psi_1 \leq u + \phi \leq \Psi_2$. We decompose it into $\phi = \phi^+ + \phi^-$ the sum of its positive and negative parts. Clearly we have that:

$$D^+ = \{x \in \mathcal{U};\ \phi(x) > 0\} \subset \mathcal{U}_1, \quad \text{and} \quad D^- = \{x \in \mathcal{U};\ \phi(x) < 0\} \subset \mathcal{U}_2.$$

Thus we may now compute:

$$\begin{aligned}
\int_{\mathcal{U}} |\nabla(u+\phi)|^p &= \int_{D^+} |\nabla(u+\phi)|^p + \int_{D^-} |\nabla(u+\phi)|^p + \int_{\{\phi=0\}} |\nabla(u+\phi)|^p \\
&= \int_{\mathcal{U}_1} |\nabla(u+\phi^+)|^p - \int_{\mathcal{U}_1\setminus D^+} |\nabla u|^p \int_{\mathcal{U}_2} |\nabla(u+\phi^-)|^p - \int_{\mathcal{U}_1\setminus D^-} |\nabla u|^p + \int_{\{\phi=0\}} |\nabla u|^p \\
&\geq \int_{\mathcal{U}_1} |\nabla u|^p - \int_{\mathcal{U}_1\setminus D^+} |\nabla u|^p \int_{\mathcal{U}_2} |\nabla u|^p - \int_{\mathcal{U}_1\setminus D^-} |\nabla u|^p + \int_{\{\phi=0\}} |\nabla u|^p = \int_{\mathcal{U}} |\nabla u|^p.
\end{aligned}$$

Thus we have proven that u is the unique weak solution to the two obstacle problem defined by $\mathcal{K}_{u_{|\partial\mathcal{U}}, \Psi_1, \Psi_2}(\mathcal{U})$.

We now consider any two viscosity solutions to the double obstacle problem u and $\bar{u}$. We note that for any $x \in A \cup \partial\Omega$ these solutions coincide. Both functions u and $\bar{u}$ must be uniformly continuous as they are continuous on the compact set $\bar{\Omega}$. Fix $\epsilon > 0$, then there exists $\delta > 0$ such that:

$$|u(x) - \bar{u}(x)| < \epsilon \quad \forall x \in \big(A \cup \partial\Omega + B_\delta(0)\big) \cap \bar{\Omega}$$

Consider an arbitrary Lipschitz set $\mathcal{U}$ such that:

$$\Omega \setminus \big(A \cup \partial\Omega + B_\delta(0)\big) \subset\subset \mathcal{U} \subset\subset \Omega \setminus A.$$

Then we have that:

$$u \text{ is the unique weak solution to } \mathcal{K}_{u_{|\partial\mathcal{U}}, \Psi_1, \Psi_2}(\mathcal{U}),$$
$$\bar{u} + \epsilon \text{ is the unique weak solution to } \mathcal{K}_{\bar{u}_{|\partial\mathcal{U}}+\epsilon, \Psi_1, \Psi_2}(\mathcal{U}).$$

By the comparison principle (iii) from Theorem 1.1.32 [8], the boundary condition to define u is less then the boundary condition to define $\bar{u} + \epsilon$ implies that $u \leq \bar{u} + \epsilon$ on the whole set $\mathcal{U}$. If we reverse the argument we have that $u \geq \bar{u} - \epsilon$ on the whole set $\mathcal{U}$. Thus we may conclude that $|\bar{u} - u| \leq \epsilon$ for all $\epsilon > 0$. Thus the two solutions coincide.

1.6 THE DYNAMIC PROGRAMMING PRINCIPLE AND THE APPROXIMATING ALGORITHM

We consider the algorithm used to find the ϵ-p-harmonious solutions to the two obstacle problem. In this section we will discuss how this theoretical algorithm was discretized and used to provide visualizations of these solutions. As was shown in the proof, these solutions eventually converge to the viscosity solutions to the p-Laplace equation. While the proof gives an estimate of the speed with which the discontinuities of the u_ϵ converge to zero, it does not provide an estimate of the convergence of the u_ϵ to u.

In the proof we constructed u_ϵ as the limit of the recursive sequence defined through the operator:

$$Tv(x) = \begin{cases} \max\left\{\Psi_1(x), \min\left\{\Psi_2(x), \frac{\alpha}{2}\sup_{B_\epsilon(x)} v + \frac{\alpha}{2}\inf_{B_\epsilon(x)} v + \fint_{B_\epsilon(x)} v\right\}\right\} & \text{in } \Omega, \\ F(x) & \text{in } \Gamma. \end{cases}$$

This operator is referred to as the dynamic programming principle (DPP) guiding our construction. Such an algorithm is easily discretized over a grid by substituting supremum and infimum with maximum and minimum, and by defining the integral numerically.

We selected a domain $\Omega = (-1,1) \times (-1,1)$, and extended it by $\epsilon_0 = 0.2$ to the extended domain $X = (-1.2, 1.2) \times (-1.2, 1.2)$. Various examples were considered and will be detailed in the next section. The domain was sampled on a square grid of step $h = 1/100$, and the algorithm was studied for varying values of ϵ. The discretized version of the DPP will be referred to as $\bar{T}$ and it works as follows. Let v be a function sampled on all the grid points in X. Given obstacles and boundary function Ψ_1, Ψ_2 and F we sample them on the grid as well. For any point p on the grid inside Ω, let $p_1 \dots p_k$ be all the grid points such that $|p - p_i| < \epsilon$. Then the function $\bar{T}v$ is defined at the point p by:

$$\bar{T}v(p) = \max\left\{\Psi_1(p), \min\left\{\Psi_2(p), \frac{\alpha}{2}\max_{j=1\dots k} v(p_j) + \frac{\alpha}{2}\min_{j=1\dots k} v(p_j) + \frac{\beta}{k}\sum_{j=1}^{k} v(p_j)\right\}\right\}.$$

For any point $p \in X \cap \Omega$, $\bar{T}v$ is defined by:

$$\bar{T}v(p) = F(p).$$

The algorithm works by constructing two sequences of functions on the grid by recursive application of the DPP. We build a lower sequence $\{\underline{u}_n\}_{n=0}^{K}$ by defining $\underline{u}_0 = \Psi_1 \chi_\Omega + F\chi\Gamma$, and recursively by $\underline{u}_n = \bar{T}\underline{u}_{n-1}$. The upper sequence is given by $\{\overline{u}_n\}_{n=0}^{K}$ by defining $\overline{u}_0 = \Psi_2\chi_\Omega + F\chi\Gamma$, and recursively by $\overline{u}_n = \bar{T}\overline{u}_{n-1}$. The first sequence is constructed following the exact construction from the proof of Theorem 1.1.31. The second is constructed by noting that by the same argument that guarantees that the lower sequence is monotone increasing, the upper sequence must be monotone decreasing. Both sequences have been shown theoretically to converge to the solution, therefore we continue the recursive construction until we reach a value K such that $\overline{u}_K(p) - \underline{u}_K(p) < err$ for all $p \in X$ where err is the accepted error tollerance. The value K will be studied in each example and will give a measure of the convergence rate of this algorithm. The algorithm works in the presence of only one obstacle or with no obstacles, one must simply take the upper obstacle to be the supremum of F and the lower obstacle the infimum of F. Once the algorithm converges the solution is taken to be $u(p) = \frac{\overline{u}_K(p)+\underline{u}_K(p)}{2}$ the average of the upper and lower solutions.

Various values of the radius ϵ were studied to choose the best compromise between speed and precision. We considered values of ϵ given by $3h$, $5h$, $10h$ and $15h$. The first test run was using two examples with known solutions. We considered $p = 2$ and boundary values given by $F_1 = e^x \sin(y)$ and $F_2 = x^2 - y^2 - y$. Both these functions are harmonic functions, so we were able to calculate the exact error in the algorithm, given in Table 1 respectively by Error 1 and Error 2. In the same table we record $k =$ points sampled which indicates the number of mesh points within the ball of radius ϵ. Runtime is expressed in seconds, while Iteration No. records the number of applications of the DPP necessary to obtain $\overline{u}_K(p) - \underline{u}_K(p) < 10^{-3}$.

Radius	$k =$ Points Sampled	Runtime	Iteration No.	Error 1	Error 2
15	709	555	335	$8.62 \cdot 10^{-6}$	$8.22 \cdot 10^{-11}$
10	317	617	876	$6.17 \cdot 10^{-6}$	$8.51 \cdot 10^{-11}$
5	81	652	3361	$2.68 \cdot 10^{-6}$	$8.68 \cdot 10^{-11}$
3	29	540	9255	$3.15 \cdot 10^{-7}$	$4.73 \cdot 10^{-7}$

Table 1: Errors in the Δ_2 test cases

It was observed that runtime remained fairly constant in all these examples, due to the fact that the reduced number of iterations was counterbalanced by the fact that larger radii required more computations per iteration. There was however a change in precision moving from $\epsilon = 3h$ to $\epsilon = 5h$. In all following examples, we thus chose $\epsilon = 5h$.

1.7 NUMERICAL RESULTS AND VISUALIZATIONS

Several examples were chosen to obtain visualizations. The visualizations in this section will have two choices of obstacle, one parabolic and thus smooth and one Lipschitz continuous. For each of these examples three boundary conditions were taken, one constant and two parabolic ones. Each example was evaluated for values of p given by 2, 10 and 100. Before we explain the examples we discuss an interesting effect which was noted when creating these visualizations. First, the presence of obstacles significantly reduced the number of iterations needed for convergence. Second, it was noted that as p increased the number of necessary iterations decreased. This second effect is caused by the fact that the min-max averaging method tends to change a function much more than the integral averaging. In Table 2 are the iterations needed for convergence for increasing values of p.

p	3	4	5	10	25	50	100
No Obstacle	5180	4806	4569	3790	3003	2707	209
One Obstacle	1637	1392	1249	975	825	777	166
Two Obstacles	1842	1933	1366	1108	992	967	178

Table 2: Convergence rate for increasing values of p

We may now begin discussing the visualizations, following is a definition of the obstacles and boundary conditions used.

Obstacle type (a), smooth parabolic obstacles:

$$\begin{aligned}\Psi_1(x,y) = \max\Big\{&2-33(x+0.5)^2-27(y+0.1)^2,\ 1.5-40(x+0.3)^2-34(y+0.4)^2,\\ &2.5-36(x-0.6)^2-51(y-0.7)^2,\ \ -3\Big\},\\ \Psi_2(x,y) = \min\Big\{&33(x+0.6)^2+27(y-0.6)^2-3,\ 33(x-0.6)^2+27(y+0.6)^2-3,\ 3\Big\}.\end{aligned}$$

Obstacle type (b), Lipschitz continuous obstacles:

$$\Psi_1(x,y) = \begin{cases} 2-17|x-0.5| & \text{for } y\in[-0.5,0.5] \\ 2-17|x-0.5|-17|y+0.5| & \text{for } y\in(-1,-0.5) \\ 2-17|x-0.5|-17|y-0.5| & \text{for } y\in(0.5,1) \end{cases}$$

$$\Psi_2(x,y) = -4+12|y+0.2|+15|x-0.7|.$$

Boundary condition type (i), constant 0 boundary value: $F = 0$.

Boundary condition type (ii), first parabolic boundary value: $F = 1 - 2y^2$.

Boundary condition type (iii), second parabolic boundary value: $F = 2 - (x + y)^2$.

Each obstacle boundary values pair was used to obtain visualizations for the three values of $p = 2, 10, 100$. Before we show the visualizations it is important to note an expected effect that was observed in these visualizations. From the theory we constructed the u_ϵ as Borel functions with no guarantee of continuity. In fact we may observe that the functions obtained in these visualizations for $p = 10$ and $p = 100$ are discontinuous step functions. This phenomenon is more evident for $p = 100$ than $p = 10$. For $p = 2$ no steps are observed. The discontinuity size seems also to be affected by the slopes of the obstacles, which again is expected from the theory. The case $p = 2$ is a special case as it is well known from the theory of harmonic functions that the averaging property 1.3 holds not as a limit but for every value of ϵ. Thus as we interpolate between $p = 2$ and $p = \infty$ we see the ϵ-p-harmonious functions change behavior.

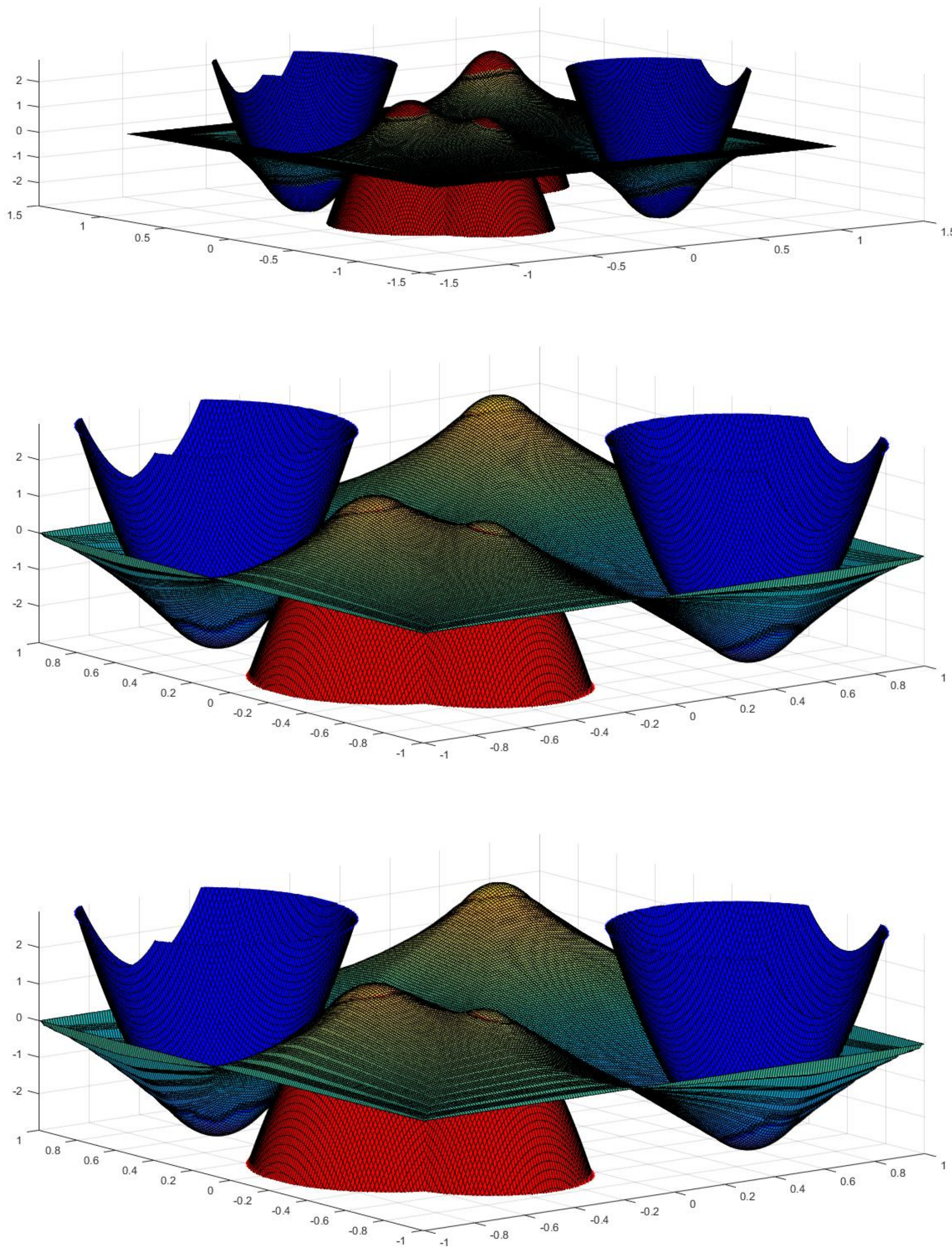


Figure 3: Results of tests for obstacle (a), boundary condition (i), for $p = 2, 10, 100$ respectively

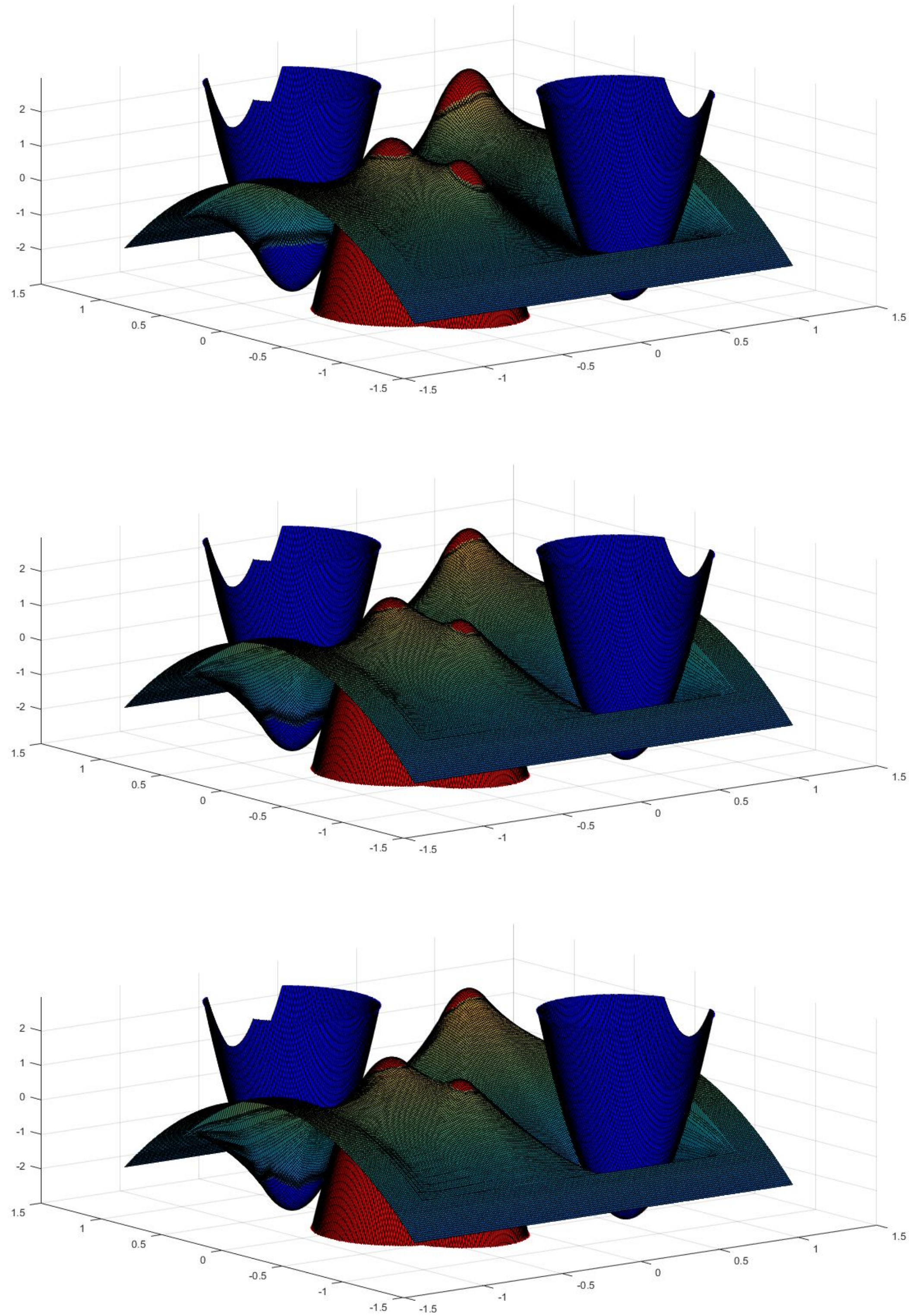


Figure 4: Results of tests for obstacle (a), boundary condition (ii), for $p = 2, 10, 100$ respectively

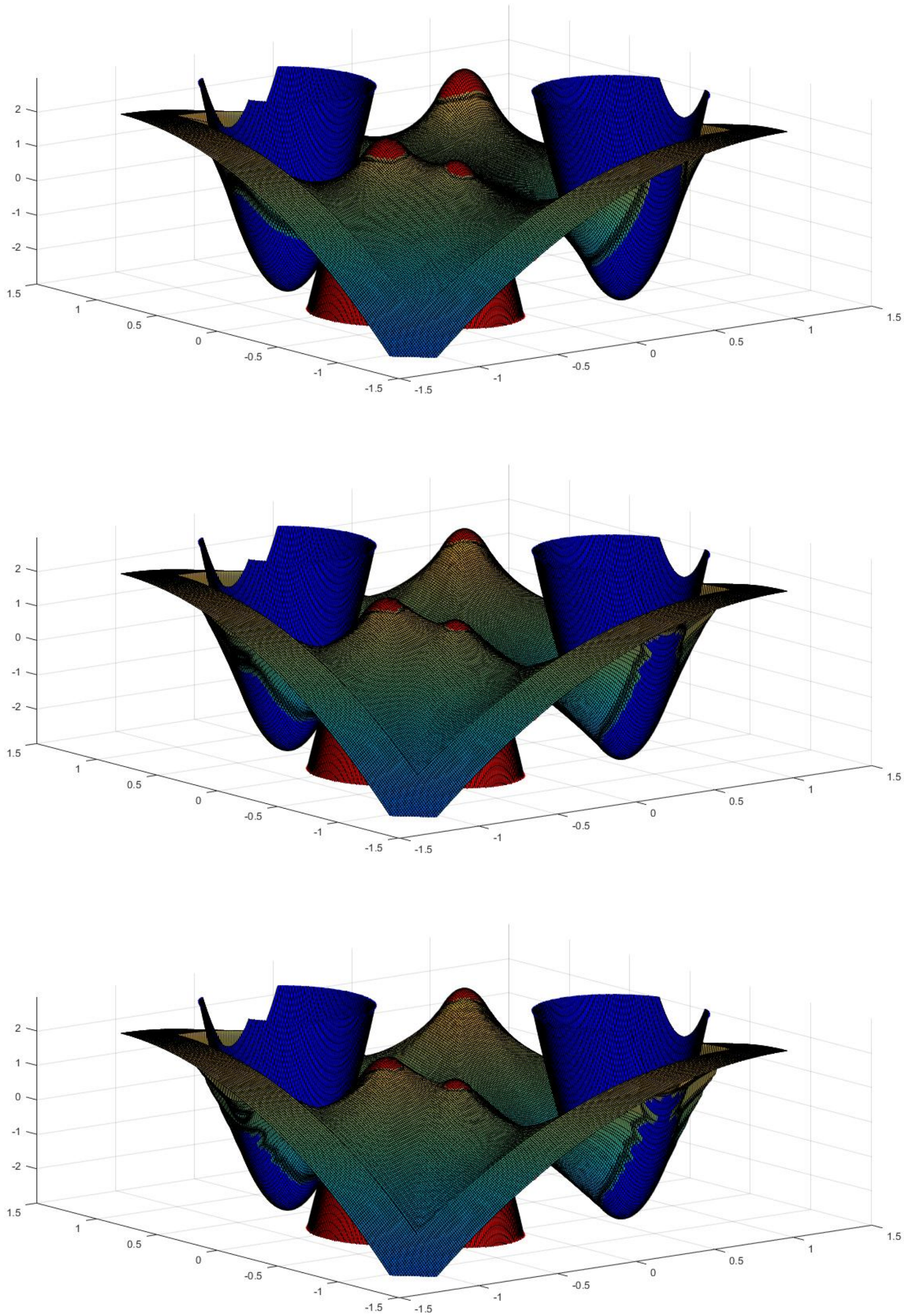


Figure 5: Results of tests for obstacle (a), boundary condition (iii), for $p = 2, 10, 100$ respectively

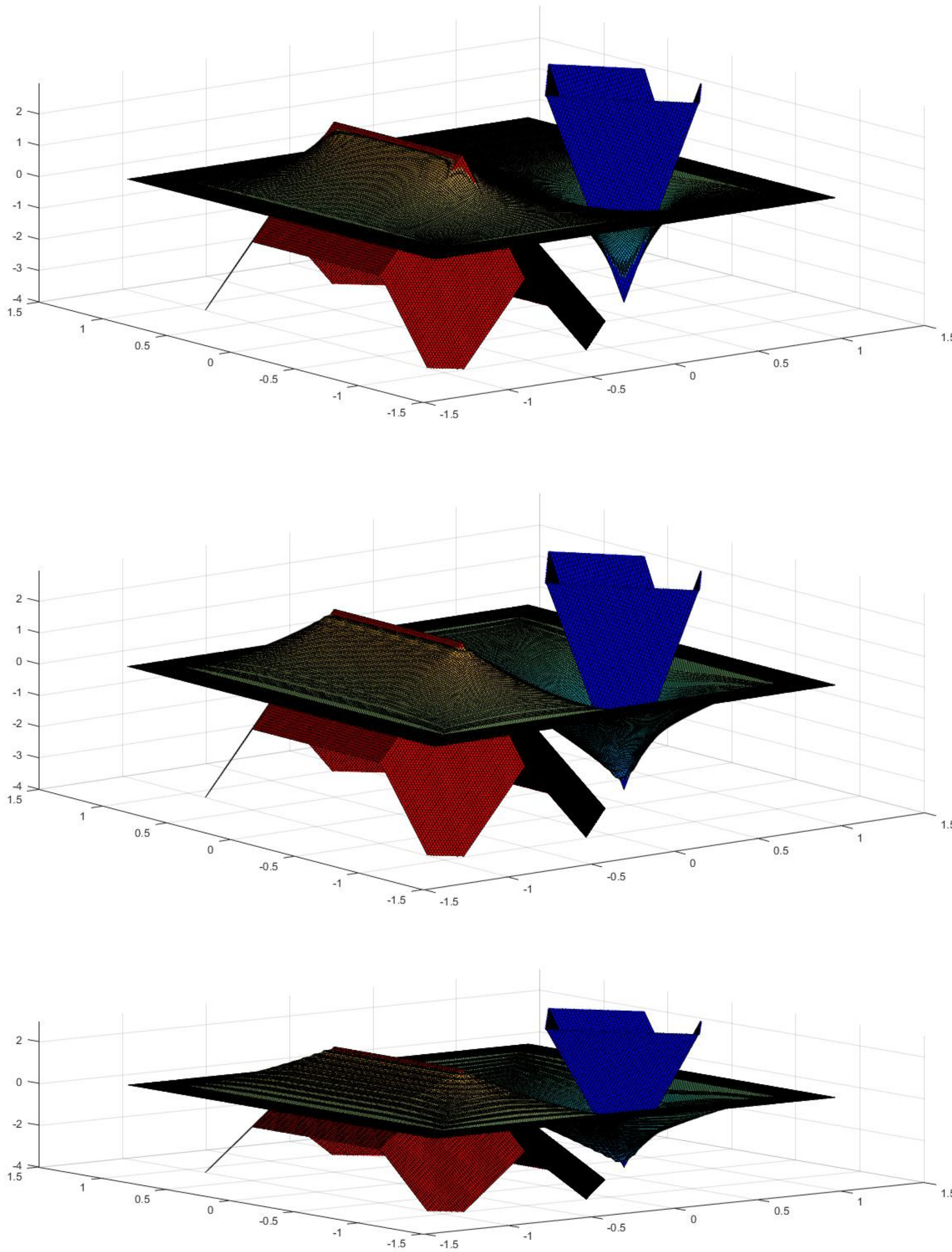


Figure 6: Results of tests for obstacle (b), boundary condition (i), for $p = 2, 10, 100$ respectively

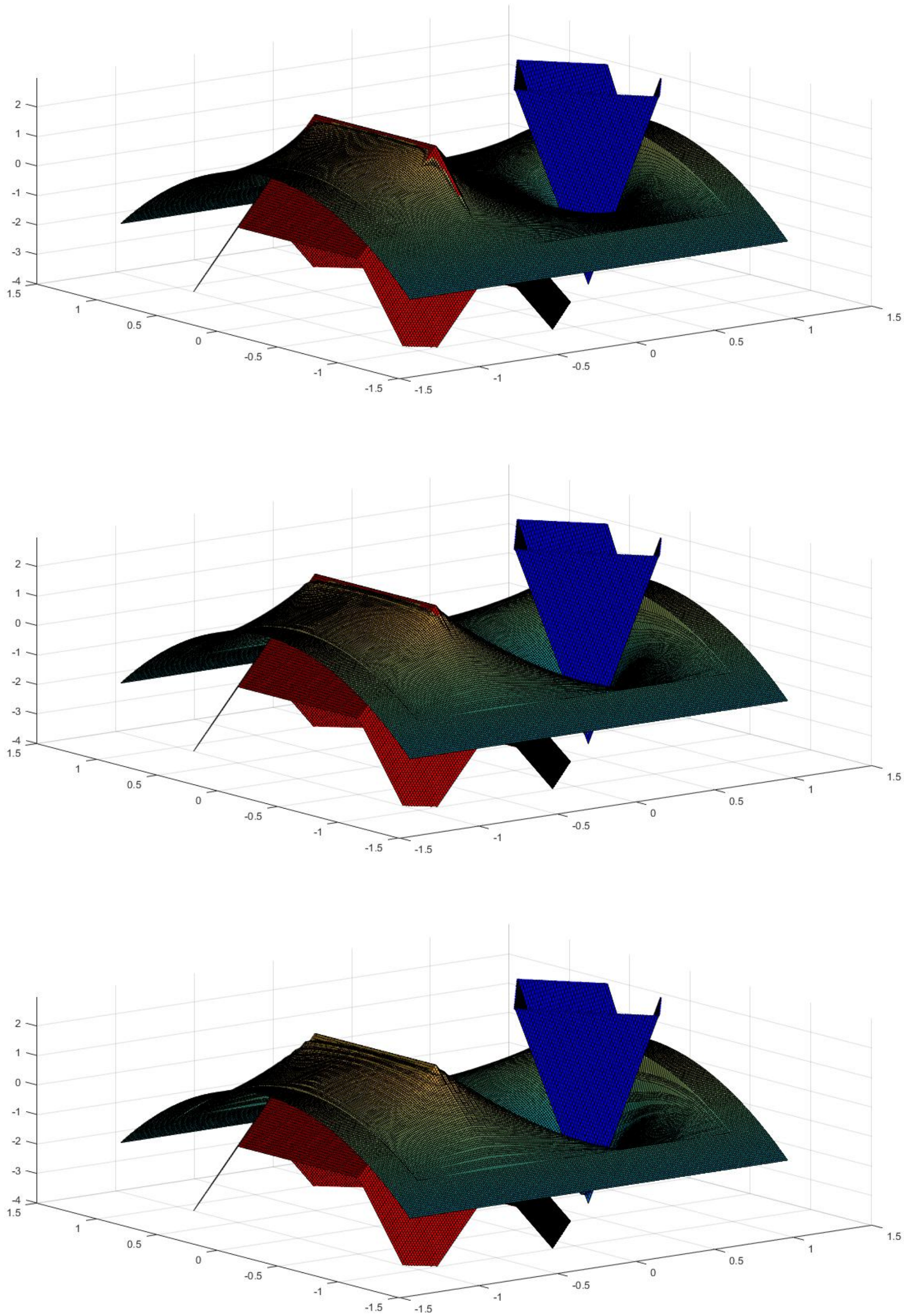


Figure 7: Results of tests for obstacle (b), boundary condition (ii), for $p = 2, 10, 100$ respectively

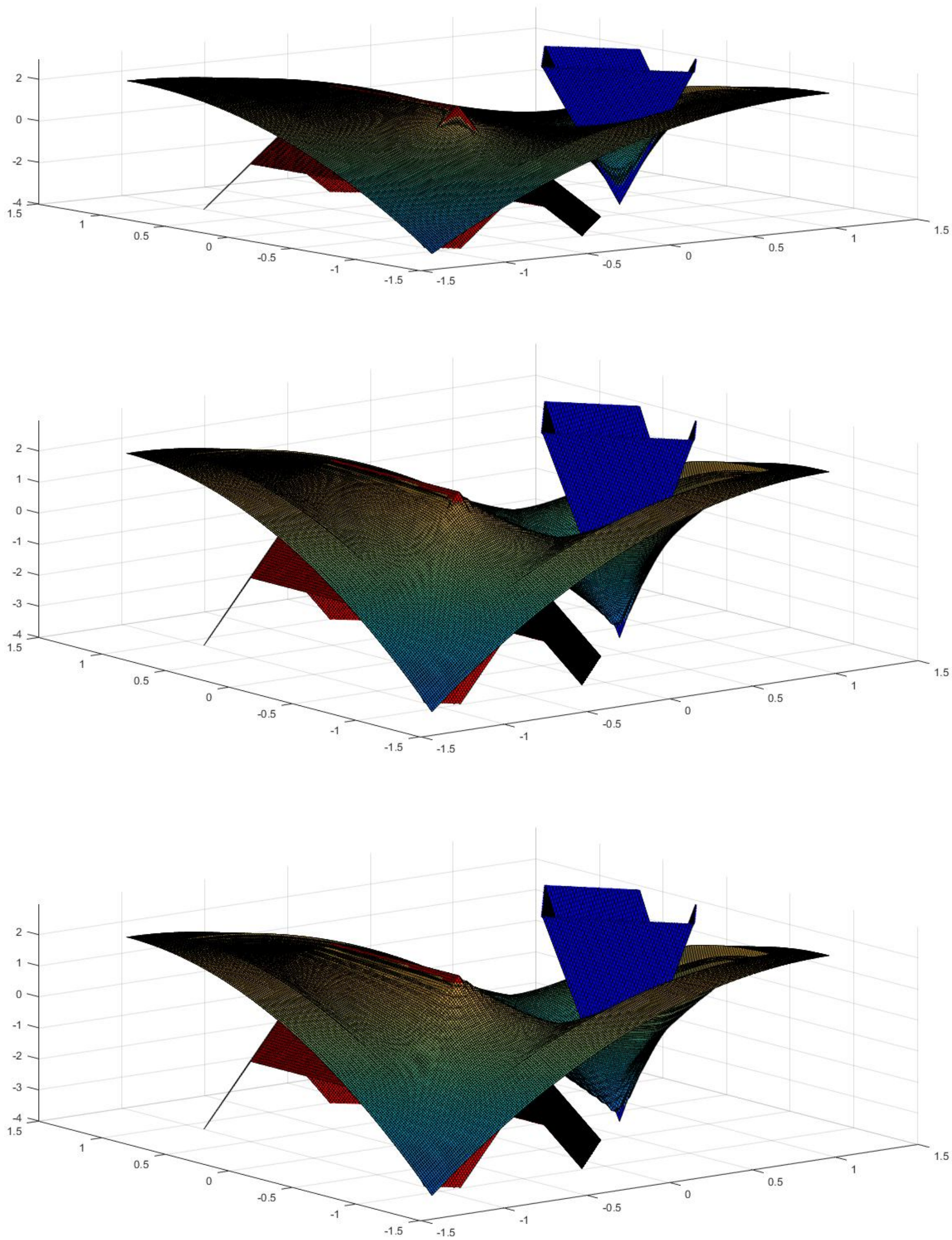


Figure 8: Results of tests for obstacle (b), boundary condition (iii), for $p = 2, 10, 100$ respectively

1.8 OPEN QUESTIONS AND FUTURE RESEARCH

There is value to the algorithm used to find numerical visualizations. It has similarities to certain well established methods for the solution of infinity Laplacian equations developed by Oberman [19]. A thorough study and refinement of our numerical algorithm could yield a very useful tool in the study of the p-Laplace equation.

A first step that could be taken in simplifying the algorithm is to study the effect of changing the shape of the region in which the Players may move the token. Could the algorithm be rewritten using cubes instead of balls? This would simplify the code significantly. If this change is theoretically justifiable, would it affect the rate of convergence? Would this change significantly modify the ϵ-p-solutions?

More refined thechniques of integration could be used, as well as better estimates of the supremum and minimum over the ball. One could envision defining the functions more accurately on the grid by the use of splines or other more refined techniques.

A great point of interest of this study is to observe the behavior of solutions near the contact set and accurately defining the contact set itself. Could adaptive mesh refinement techniques be employed in this algorithm with a focus around the boundary?

The function F is currently defined arbitrarily on the set Γ. How do certain choices of F affect the convergence rate? It should be noted that when testing the algorithm on problems with known solution the ϵ-p-harmonious solution found was within machine error precision of the actual solution. This is due to two facts. Firstly the examples were done with $p = 2$ where the averaging property is true not just as a limiting property. Furthermore F on the boundary was defined as the smooth extension of the actual solution to the equation. When F was defined in other ways, the accuracy of the solution was diminished. Studying this effect could yield clues as to the convergence rate as $\epsilon \to 0$. It could also lead to algorithms to select better choices of F.

Finally it might be of interest to see if a machine could be taught to play the game and learn to estimate the expected value of the payout by playing against itself. This could probably be done for a simple version of the problem. It would only be useful if the knowledge obtained from training on a certain double obstacle problem was transferrable to different setups of the problem. Nevertheless it would be an interesting exercise.

2.0 CONVEX INTEGRATION FOR THE MONGE-AMPÈRE EQUATION

2.1 BACKGROUND IN THE MONGE-AMPÈRE EQUATION

2.1.1 Hölder spaces

Definition 2.1.1 (Hölder continuous functions)**.** *A function u is Hölder continuous with exponent $0 < \alpha < 1$ if there exists a constant $C > 0$ such that:*

$$\forall x, y \in \Omega, \quad |u(x) - u(y)| \leq C|x - y|^\alpha.$$

The space of all Hölder continuous functions with exponent α is called $\mathcal{C}^{0,\alpha}(\Omega)$, and if Ω is bounded it may be equipped with the norm:

$$\|u\|_{0,\alpha} = \sup_{x\in\Omega} |u| + \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Furthermore we define the Hölder seminorm as:

$$[u]_{0,\alpha} = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Lemma 2.1.2. *Given a function $f \in \mathcal{C}^1(\bar{\Omega})$ on a bounded domain, for any $\alpha \in (0, 1)$ we have:*

$$\|f\|_{0,\alpha} \leq C\|f\|_0^{1-\alpha}\|f\|_1^\alpha$$

The constant C depends on the geometry of the domain. In particular, we have $C = 2$ for a convex domain.

Proof. The proof follows simply from the fact that:

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} = |f(x) - f(y)|^{1-\alpha}\frac{|f(x) - f(y)|^\alpha}{|x - y|^\alpha} \leq (2\|f\|_0)^{1-\alpha}\Big(\frac{|f(x) - f(y)|}{|x - y|}\Big)^\alpha.$$

Now consider:

$$\|f\|_{0,\alpha} = \|f\|_0 + [f]_{0,\alpha} \le \|f\|_0 + 2\|f\|_0^{1-\alpha}\left(\sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|}\right)^{\alpha}$$
$$\le 2\|f\|_0^{1-\alpha}\left(\|f\|_0^{\alpha} + C[f]_{0,1}^{\alpha}\right) \le C\|f\|_{0,1}^{1-\alpha}\|f\|_1^{\alpha}.$$

2.1.2 Mollification

Mollification is a standard technique in mathematical analysis which is widely used to provide a smooth approximation to a function. We begin by defining the mollifier used in this work:

$$\varphi(x) = \begin{cases} \frac{1}{A}\exp\left(\frac{-1}{1-|x|^2}\right) & |x| \le 1 \\ 0 & |x| \ge 1. \end{cases} \tag{2.1}$$

The constant A is a parameter needed to ensure that φ integrates to 1, and it may be approximated to any degree of precision. For the purpose of the bounds in Lemma 2.1.4 we took A to lie in the interval between 4.66 and 4.67. For numerical calculations A was calculated to a higher degree of precision. More generally we define:

Definition 2.1.3 (Standard Mollifier)**.** *A fuction $\psi : \mathbb{R}^N \to \mathbb{R}$ is called a standard mollifier if it has the following properties:*

- *ψ is a smooth function.*
- *$\int_{\mathbb{R}^N} \psi = 1$.*
- *$\psi = 0$ outside $B_1(0)$.*
- *ψ is radially symmetric.*
- *$\psi \ge 0$.*

It is clear from the construction that φ is in fact a standard mollifier. We now apply Lemma 4.3 from [15] to this choice in two dimensions to obtain explicit estimates. To evaluate the constants in the Lemma we will need to note that:

$$\|\varphi\|_{L^1} = 1, \quad \|\nabla\varphi\|_{L^1} \le 3.1, \quad \|\nabla^2\varphi\|_{L^1} \le 15.9, \quad \|\nabla^3\varphi\|_{L^1} \le 210.$$

These values were obtained through numerical means, and verified by evaluating the integrals in polar coordinates and evaluating the ensuing one dimensional integral numerically. Different mollifiers will have different estimates on their derivatives.

Lemma 2.1.4. *Taking* φ *defined as in* (2.1) *denote:*

$$\forall l \in (0,1) \quad \varphi_l(x) = \frac{1}{l^2}\varphi\left(\frac{x}{l}\right).$$

Then for every $f, g \in \mathcal{C}^0(\mathbb{R}^2)$ *there holds:*

$$\|\nabla^{j+k} f * \varphi_l\|_0 \leq \frac{1}{l^k}\|\nabla_k \varphi\|_{L^1(\mathbb{R}^2)}\|f\|_j, \quad \text{for all } j,k \geq 0 \tag{2.2}$$

$$\begin{gathered} \|f * \varphi_l - f\|_0 \leq \frac{1}{2}l^2\|f\|_2, \quad \|\nabla(f * \varphi_l - f)\|_0 \leq l\|f\|_2, \\ \|\nabla^2(f * \varphi_l - f)\|_0 \leq 2\|f\|_2 \end{gathered} \tag{2.3}$$

$$\|f * \varphi_l - f\|_0 \leq l^\alpha [f]_{0,\alpha}, \quad \|\nabla(f * \varphi_l)\|_0 \leq \frac{3.1}{l^{1-\alpha}}[f]_{0,\alpha}, \quad \forall \alpha \in (0,1] \tag{2.4}$$

$$\begin{gathered} \forall \alpha \in (0,1] \quad \|(fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\|_0 \leq 2l^{2\alpha}[f]_{0,\alpha}[g]_{0,\alpha}, \\ \forall \alpha \in (0,1] \quad \|\nabla\big((fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\big)\|_0 \leq 9.3 l^{2\alpha-1}[f]_{0,\alpha}[g]_{0,\alpha}, \\ \forall \alpha \in (0,1] \quad \|\nabla^2\big((fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\big)\|_0 \leq 67 l^{2\alpha-2}[f]_{0,\alpha}[g]_{0,\alpha}, \\ \forall \alpha \in (0,1] \quad \|\nabla^3\big((fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\big)\|_0 \leq 925.8 l^{2\alpha-3}[f]_{0,\alpha}[g]_{0,\alpha}. \end{gathered} \tag{2.5}$$

Proof. The proof of (2.2) through (2.5) follows the proof in [15]. We begin by evaluating the following useful equalities which hold for any value of $k \geq 0$:

$$\|\nabla^k \varphi_l\|_{L^1(\mathbb{R}^2)} = l^{-k}\|\nabla^k \varphi\|_{L^1(\mathbb{R}^2)}, \tag{2.6}$$

This is an immediate consequence of the chain rule of derivatives.

To show the (2.2) we prove the well known result that $\|f * g\|_0 \leq \|g\|_{L^1}\|f\|_0$:

$$|f * g| = \left|\int_{\mathbb{R}^N} f(x-y)g(y)\mathrm{d}y\right| \leq \int_{\mathbb{R}^N} \big|f(x-y)g(y)\big|\mathrm{d}y \leq \int_{\mathbb{R}^N} \|f\|_0|g(y)|\mathrm{d}y = \|g\|_{L^1}\|f\|_0.$$

If we set g as the appropriate derivative of φ_l and note that $\nabla f * g = f * \nabla g = \nabla(f * g)$ we obtain the required inequality.

The proofs of inequalities (2.3) are obtained using Taylor's expansion.

- $k = 0$. Consider the fact that:

$$f(x-y) - f(x) = f(x) + \nabla f(x) \cdot (-y) + \frac{1}{2}\nabla^2 f(\xi)(-y) \otimes (-y) \quad \xi = x - ty \text{ for some } t \in [0,1].$$

Thus, considering that the integral of an odd function over all of $\mathbb{R}^2$ is zero we obtain:

$$\int_{\mathbb{R}^2} \phi_l(y)[f(x-y) - f(x)]\mathrm{d}y \leq \int_{\mathbb{R}^2} \phi_l(y)\nabla f(x)(-y)\mathrm{d}y + \frac{1}{2}l^2\|\nabla^2 f\|_0\|\phi_l\|_{L^1} = \frac{1}{2}l^2\|\nabla^2 f\|_0.$$

- $k = 1$. using the intermediate value Theorem we have:

$$\nabla f(x-y) - \nabla f(x) = \frac{1}{2}\nabla^2 f(\xi) \cdot (-y) \quad \xi = x - ty \text{ for some } t \in [0,1].$$

Thus we obtain:

$$\int_{\mathbb{R}^2} \phi_l(y)[\nabla f(x-y) - \nabla f(x)]\mathrm{d}y \leq l\|\nabla^2 f\|_0 \|\phi_l\|_{L^1} = l\|\nabla^2 f\|_0.$$

- $k = 2$. Estimate:

$$\begin{aligned}\int_{\mathbb{R}^2} \phi_l(y)[\nabla^2 f(x-y) - \nabla^2 f(x)]\mathrm{d}y = \int_{\mathbb{R}^2} \phi_l(y)[|\nabla^2 f(x-y)| + |\nabla^2 f(x)|]\mathrm{d}y \\ \leq 2\|\nabla^2 f\|_0 \|\phi_l\|_{L^1} = 2\|\nabla^2 f\|_0.\end{aligned}$$

To prove the first inequality (2.4) we multiply and divide by $|y|^\alpha$ when applying the convolution and by noting that $\varphi_l * f(x) = f(x)$:

$$|(f * \varphi_l - f)(x)| = \int \varphi_l(y)|y|^\alpha \frac{f(x-y) - f(x)}{|y|^\alpha}\mathrm{d}y \leq l^\alpha \|\varphi_l(y)|_{L^1}\|f\|_{0,\alpha}.$$

The same procedure is used to prove the second inequality with some modifications. Note that $\nabla\varphi$ is an odd function and therefore gives zero when convoluted with a constant. We thus obtain:

$$|f * \nabla\varphi_l(x)| = |f * \nabla\varphi_l(x) - f(x) * \nabla\varphi_l| = \int \nabla\varphi_l(y)|y|^\alpha \frac{f(x-y) - f(x)}{|y|^\alpha}\mathrm{d}y \leq l^\alpha \|\nabla\varphi_l(y)\|_{L^1}\|f\|_{0,\alpha}.$$

Finally we come to the inequalities (2.5). We must compute the derivatives of $h = (fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)$:

$$\begin{aligned}\nabla h =& (fg) * \nabla\varphi_l - (f * \nabla\varphi_l)(g * \varphi_l) - (f * \varphi_l)(g * \nabla\varphi_l),\\ \nabla^2 h =& (fg) * \nabla^2\varphi_l - (f * \nabla^2\varphi_l)(g * \varphi_l) - 2(f * \nabla\varphi_l)(g * \nabla\varphi_l) - (f * \varphi_l)(g * \nabla^2\varphi_l),\\ \nabla^3 h =& (fg) * \nabla^3\varphi_l - (f * \nabla^3\varphi_l)(g * \varphi_l) - 3(f * \nabla^2\varphi_l)(g * \nabla\varphi_l) - 3(f * \nabla\varphi_l)(g * \nabla^2\varphi_l)\\ &- (f * \varphi_l)(g * \nabla^3\varphi_l).\end{aligned}$$

From which we evaluate, using triangle inequality and (2.4):

$$\begin{aligned}\|h\|_0 &\leq l^{2\alpha} 2\|f\|_{0,\alpha}\|g\|_{0,\alpha}\\ \|\nabla h\|_0 &\leq 3l^{2\alpha}\|\nabla\varphi_l\|_{L^1}\|f\|_{0,\alpha}\|g\|_{0,\alpha}\\ \|\nabla^2 h\|_0 &\leq l^{2\alpha}\big(3\|\nabla^2\varphi_l\|_{L^1} + 2\|\nabla^2\varphi_l\|_{L^1}\big)\|f\|_{0,\alpha}\|g\|_{0,\alpha}\\ \|\nabla^3 h\|_0 &\leq l^{2\alpha}\big(3\|\nabla^3\varphi_l\|_{L^1} + 6\|\nabla^2\varphi_l\|_{L^1}\|\nabla\varphi_l\|_{L^1}\big)\|f\|_{0,\alpha}\|g\|_{0,\alpha}.\end{aligned}$$

The desired constants are obtained by inserting the appropriate bounds on the derivatives.

The above inequalities are evaluated for functions defined over the whole plane. We now consider two compact sets $\Omega \subset \bar{\Omega}$ such that $\Omega + B_l(0) \subset \bar{\Omega}$. Given a function $f : \bar{\Omega} \to \mathbb{R}$, we may define the convolution $f * \varphi_l : \Omega \to \mathbb{R}$ since φ_l is compactly supported on the ball of radius l. Using this notation every inequality from Lemma 2.1.4 may be rewritten with the norm of the convolution taken over the smaller set being bounded by the norm of the original function taken over the larger set.

2.1.3 The Monge-Ampère equation and its weak formulation

Given $\Omega \subset \mathbb{R}^2$ and a function $f : \Omega \to \mathbb{R}$ the Monge-Ampère equation reads:

$$f = \det\nabla^2 v \quad \text{in } \Omega.$$

This is the classical formulation of the equation for which a solution v will be of $\mathcal{C}^2$ regularity.

A variety of results exist about the existence and flexibility of solutions to this equation, but many questions remain unanswered.

Lemma 2.1.5 (Very weak determinant Hessian)**.** *Given a function $v : \mathbb{R}^2 \to \mathbb{R}$ of regularity $\mathcal{C}^3$ we have that:*

$$\det\nabla^2 v = -\frac{1}{2} \mathit{curl}\ \mathit{curl}(\nabla v \otimes \nabla v).$$

We define the right hand side of this equation as the very weak Hessian of v which will be written as $\mathcal{D}et\nabla^2 v$.

Proof. The proof is obtained through the following straightforward calculation:

$$\begin{aligned}
\text{curl curl}(\nabla v \otimes \nabla v) &= \text{curl curl} \begin{bmatrix} \partial_1 v^2 & \partial_1 v \partial_2 v \\ \partial_1 v \partial_2 v & \partial_2 v^2 \end{bmatrix} = \text{curl} \begin{bmatrix} \partial_2(\partial_1 v^2) - \partial_1(\partial_1 v \partial_2 v) \\ \partial_2(\partial_1 v \partial_2 v) - \partial_1(\partial_2 v^2) \end{bmatrix} \\
&= \partial_2^2\big(\partial_1 v^2\big) - 2\Big[\partial_1\partial_2\big(\partial_1 v \partial_2 v\big)\Big] + \partial_1^2\big(\partial_2 v^2\big) = 2\Big[\partial_1\partial_2^2 v \partial_1 v + \big(\partial_1\partial_2 v\big)^2\Big] \\
&\quad - 2\Big[\partial_1^2\partial_2 v \partial_2 v + \partial_1^2 v \partial_2^2 v + \big(\partial_1\partial_2 v\big)^2 + \partial_1\partial_2^2 v \partial_1 v\Big] + 2\Big[\partial_1^2\partial_2 v \partial_2 v + \big(\partial_1\partial_2 v\big)^2\Big] \\
&= 2\Big[\big(\partial_1\partial_2 v\big)^2 - \partial_1^2 v \partial_2^2 v\Big] = -2\det \begin{bmatrix} \partial_1^2 v & \partial_1\partial_2 v \\ \partial_1\partial_2 v & \partial_2^2 v \end{bmatrix} = -2\det\nabla^2 v.
\end{aligned}$$

By dividing by -2 we obtain the claimed formulation.

Thus we may define the very weak formulation of the Monge-Ampère equation as:

$$f = \mathcal{D}et\nabla^2 v = -\frac{1}{2}\text{curl curl}(\nabla v \otimes \nabla v) \quad \text{in } \Omega. \tag{2.7}$$

Consider an auxiliary matrix valued function $A : \Omega \to \mathbb{R}^{2\times 2}_{\text{Sym}}$ which satisfies $f = -\text{curl curl}(A)$. The solutions to this problem are not unique:

Lemma 2.1.6 (Auxiliary Matrix)**.** *Given a function $f : \Omega \to \mathbb{R}$ let λ be any solution to the problem $-\Delta\lambda = f$. Then the matrix $A(x) = \lambda(x)\mathrm{id}$ is a solution to the equation $f = -curl\ curl(A)$.*

Proof.

$$\text{curl curl}\begin{bmatrix}\lambda(x) & 0\\ 0 & \lambda(x)\end{bmatrix} = \text{curl}\begin{bmatrix}\partial_2\lambda(x) - 0\\ 0 - \partial_1\lambda(x)\end{bmatrix} = \partial_2^2\lambda + \partial_1^2\lambda = \Delta\lambda = -f.$$

If a function v satisfies $\frac{1}{2}$curl curl$(\nabla v \otimes \nabla v) =$ curl curl(A), then clearly it satisfies (2.7).

Lemma 2.1.7 (Symmetric term)**.** *Consider a function $w : \Omega \to \mathbb{R}^2$, then $\frac{1}{2}curl\ curl(\nabla v \otimes \nabla v) = curl\ curl(\frac{1}{2}\nabla v \otimes \nabla v + sym\nabla w)$*

Proof. We show this by first noting that curl curl$(f + g) =$ curl curl(f) + curl curl(g). We now compute:

$$\text{curl curl}\begin{bmatrix}\partial_1 w_1 & \frac{\partial_2 w_1+\partial_1 w_2}{2}\\ \frac{\partial_2 w_1+\partial_1 w_2}{2} & \partial_2 w_2\end{bmatrix} = \partial_1\partial_2^2 w_1 - \partial_1\partial_2(\partial_2 w_1 + \partial_1 w_2) + \partial_1^2\partial_2 w_2 = 0$$

This result has a very interesting physical interpretation. Consider a surface with an out of plane displacement given by ϵv and an in place displacement of $\epsilon^2 w$, then the metric of the surface will be Id$+2\epsilon^2(\frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\nabla w)$. The curvature of the surface will be defined by $-\epsilon^2$curl curl$(\frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\nabla w)$.

We conclude this preliminary discussion by stating the very weak Monge-Ampère equation:

$$A = \frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\nabla w. \tag{2.8}$$

We further note that such an equation requires only one degree of differentiability in v and w to be well defined.

2.1.4 Decomposition

Consider three unit vectors:

$$\eta_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \eta_1 = \frac{1}{\sqrt{(5)}}\begin{bmatrix}1\\2\end{bmatrix}, \quad \eta_1 = \frac{1}{\sqrt{(5)}}5\begin{bmatrix}1\\-2\end{bmatrix}.$$

Lemma 2.1.8. *The matrices $\eta_1 \otimes \eta_1$, $\eta_2 \otimes \eta_2$ and $\eta_3 \otimes \eta_3$ form a basis of $\mathbb{R}^{2\times 2}_{Sym}$.*
Let $B = [b_{ij}] = \sum_{k=1}^{3} \phi_k \eta_k \otimes \eta_k$. Then:

(i) $\phi_1 = b_{11} - \frac{1}{4}b_{22}$ *and* $\phi_2 = \frac{5}{8}(b_{22} + 2b_{12})$, $\phi_3 = \frac{5}{8}(b_{22} - 2b_{12})$.

(ii) $\sum_{k=1}^{3} \phi_k = \mathrm{Tr}\ B$ *and:* $|\phi_k| \le \frac{5\sqrt{3}}{8}|B|$ *for* $k = 1 \ldots 3$.

(iii) *If* $\phi_k \ge d > 0$ *for* $k = 1 \ldots 3$, *then* $B \ge d\mathrm{Id}_2$.

(iv) *The matrix* $B' = B + \alpha diag\left(\frac{\sqrt{2}+9}{4}, (\sqrt{2} + \frac{9}{5})\right)$ *for any* $\alpha \ge \|B\|$ *has* $\|B'\| < 5.05\alpha$ *and if we write* $B' = \sum_{k=1}^{3} \phi'_k$, *then* $\phi'_k \ge \frac{\alpha}{2}$.

Proof. The formula in (i) is obtained through a straightforward calculation:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = \phi_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \phi_2 \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \phi_3 \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}.$$

$$\begin{cases} b_{11} & = \phi_1 + \frac{1}{5}\phi_2 + \frac{1}{5}\phi_3, \\ b_{12} & = \frac{2}{5}\phi_2 - \frac{2}{5}\phi_3, \\ b_{22} & = \frac{4}{5}\phi_2 + \frac{4}{5}\phi_3. \end{cases} \implies \begin{cases} \phi_1 & = b_{11} - \frac{1}{4}b_{22}, \\ \phi_2 & = \frac{5}{8}(b_{22} + 2b_{12}), \\ \phi_3 & = \frac{5}{8}(b_{22} - 2b_{12}). \end{cases}$$

The first part of (ii) follows immediately from adding all the terms. The second part follows from Cauchy-Schwartz inequality if we write B as a 4-dimensional vector:

$$\phi_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{12} \\ b_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{4} \end{bmatrix}, \quad \phi_2 = \frac{5}{8} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{12} \\ b_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \phi_2 = \frac{5}{8} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{12} \\ b_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}.$$

We thus obtain: $|\phi_1| \le \frac{\sqrt{17}}{4}|B|$, $|\phi_2| \le \frac{5\sqrt{3}}{8}|B|$ and $|\phi_3| \le \frac{5\sqrt{3}}{8}|B|$. Consequently, (ii) follows. For (iii), observe that $\sum_{k=1}^{3} \phi_k \eta_k \otimes \eta_k = \sum_{k=1}^{3} (\phi_k - d)\eta_k \otimes \eta_k + d\mathrm{diag}\{\frac{7}{5}, \frac{8}{5}\} \ge d\mathrm{Id}_2$. Lastly, (iv) follows from applying the formulas in (i) to B' and noting that $b_{ii} \le \|B\|$ for $i = 1, 2$ and $b_{12} \le \frac{\sqrt{2}}{2}\|B\|$.

$$\begin{aligned} \phi'_1 &= b_{11} + \alpha\frac{\sqrt{2}+9}{4} - \frac{1}{4}\big(b_{22} + \alpha(\sqrt{2} + \frac{9}{5})\big) \ge -\alpha - \frac{1}{4}\alpha + \frac{9}{5}\alpha \ge \frac{\alpha}{2}, \\ \phi'_{2,3} &= \frac{5}{8}\big(b_{22} + \alpha(\sqrt{2} + \frac{9}{5}) \pm 2b_{12}\big) \ge \frac{\alpha}{2}. \end{aligned}$$

Finally, we evaluate $|B'| \le |B| + \alpha\left(\left(\frac{\sqrt{2}+9}{4}\right)^2 + \left(\sqrt{2} + \frac{9}{5}\right)^2\right) \le 5.05\alpha$.

2.2 BACKGROUND IN CONVEX INTEGRATION

Convex integration originated as an independent field from a generalization of existing perturbation techniques. Such formulation of convex integration is attributed to Gromov, who laid out its foundations in his 1986 seminal work [9]. In the next section We will see some of the perturbation results that were early formulations of what eventually came to be called "convex integration". Convex integration provides the framework for the construction of solutions to partial differential relations with topological constraints. In the field of Partial Differential Equations, these methods have been employed to find useful anomalous solutions.

In simple terms the application of convex integration to find anomalous solutions to a partial differential equation follows the steps below. First the equation is relaxed to an inequality. Solutions of this inequality are called subsolutions. The set of subsolutions must be convex for the method to work. Next a subsolution is chosen. The sought after solution will be required to be "close" to the original subsolution through a topological condition. For example, in the cases we discuss we will find solutions within ϵ to a subsolution in $\mathcal{C}^0$ norm. Finally, from this first subsolution a sequence of subsolutions is constructed which approaches the boundary of the set of subsolutions without breaking the topological condition. The validity of this construction must then be verified through a "Baire category method".

We now give a very simple and common example of an application of convex integration. Consider the equation:

$$u : \mathbb{R} \to \mathbb{R}, \quad \left| \frac{\mathrm{d}u}{\mathrm{d}x}(x) \right| = 1 \quad a.e.\, x \in \mathbb{R}.$$

We want to show some anomalous solutions to this equation. We begin by relaxing the equation into an inequality and choose a subsolution:

$$\tilde{u}(x) = \frac{x}{2} \quad \text{then,} \quad \left| \frac{\mathrm{d}u}{\mathrm{d}x}(x) \right| \leq 1.$$

We next look for a solution u to the equation such that $|u(x) - \tilde{u}(x)| \leq \epsilon$. This is in fact easily done. Let us take for example $\epsilon = 0.1$, Then we build u piecewise linear through:

$$\begin{aligned}
&u(0) = 0, \\
&u(x) = x \text{ until } x - \frac{x}{2} = \frac{1}{10} \text{ thus in } \left(0, \frac{1}{5}\right], \\
&u(x) = \frac{2}{5} - x \text{ until } \frac{2}{5} - x - \frac{x}{2} = -\frac{1}{10} \text{ thus in } \left(\frac{1}{5}, \frac{1}{3}\right], \\
&u(x) = \frac{-4}{15} + x \text{ until } \frac{-4}{15} + x - \frac{x}{2} = 1\frac{1}{10} \text{ thus in } \left(\frac{1}{3}, \frac{11}{15}\right], \\
&u(x) \ldots
\end{aligned}$$

The construction is symmetric for negative values and continued over all of $\mathbb{R}$. It is clear that this construction works for any value of ϵ. In figure 9 we see its implementation.

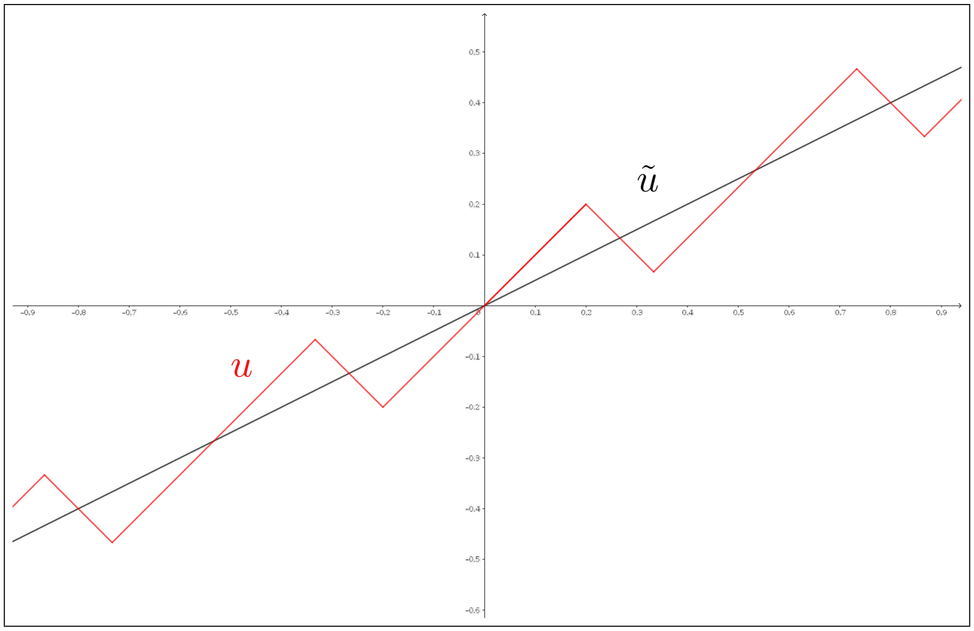


Figure 9: The approximation u of the subsolution $\tilde{u}$

2.3 THREE EXAMPLES OF APPLICATION

2.3.1 Isometric immersion problem

The first and best known application of convex integration arises from differential geometry and regards the existence of isometric embeddings of Riemannian manifolds. Nash and Kuiper in 1954 showed existence of $\mathcal{C}^1$ embeddings of any $N-1$ dimensional manifold in $\mathbb{R}^N$. They used a method of oscillatory perturbations of subsolutions which predates Gromov's formalization of convex integration by decades. We will define the problem and provide an overview of the convex integration schema used in the solution of this problem.

Consider any M dimensional Riemannian manifold (Σ, g). A continuous map $u : \Sigma \to \mathbb{R}^N$ is called isometric if it preserves the length of curves. This condition when written in coordinates of Σ becomes the following set of equations:

$$\partial_i u \cdot \partial_j u = g_{ij} \quad \text{where} \quad g = \sum_{i,j=1}^{M} g_{ij} \mathrm{d}x_i \otimes \mathrm{d}x_j, \quad \text{or} \quad \nabla u^T \nabla u = g.$$

This problem consists of $\frac{M(M+1)}{2}$ equations in N unknowns and becomes easier for larger values of

N. In fact any smooth manifold has smooth isometric embedding for N large enough. Nash and Kuiper provided examples of $\mathcal{C}^1$ solutions for any $N \geq M+1$. Their arguments were then pushed forward using convex integration techniques to show the existence of $\mathcal{C}^{1,\alpha}$ isometric embeddings for $\alpha \leq \frac{1}{5}$ in the case $M=2$, $N=3$. These convex integration techniques have a natural upper boundary, and can never be expected to push this threshold beyond $\mathcal{C}^{1,\frac{1}{3}}$ regularity.

The construction of a solution begins with a short embedding, that is an embedding which shortens the length of any curve. In coordinates this condition is stated as:

$$0 \leq g_{ij} - \partial_i u \cdot \partial_j u \quad \forall i,j = 1,\dots,M.$$

We will be adding a series of oscillatory perturbations to define a sequence of short embeddings u_k which converge in the desired space, and $g_{ij} - \partial_i u \cdot \partial_j u \to 0$.

We illustrate the application of convex integration in the simpler case $N = M+2$ as built by Nash [18]. The construction has been extended to the $N = M+1$ case [13], but the calculations provide little insight on the method itself and include technical details that are beyond the scope of this overview.

We begin by describing what will be called a step of convex integration. Given an embedding $u : \Sigma \to \mathbb{R}^N$, a positive function $a : \Sigma \to \mathbb{R}^+$ and a unit vector $\xi \in \mathbb{R}^M$, we seek a function $u_\lambda : \Sigma \to \mathbb{R}^N$ such that for arbitrarily large λ:

$$\|u - u_\lambda\|_0 = O\Big(\frac{1}{\lambda}\Big) \quad \text{and} \quad \|\nabla u^T \nabla u + a^2 \xi \otimes \xi - \nabla u_\lambda^T \nabla u_\lambda\|_0 = O\Big(\frac{1}{\lambda}\Big).$$

We write $u_\lambda = u + w$ with $\|w\|_0 = O\big(\frac{1}{\lambda}\big)$, then we obtain:

$$\partial_i u_\lambda \cdot \partial_j u_\lambda = \partial_i u \cdot \partial_j u + \partial_i w \cdot \partial_j u + \partial_i u \cdot \partial_j w + \partial_i w \cdot \partial_j w.$$

The second condition thus becomes:

$$\partial_i w \cdot \partial_j u + \partial_i u \cdot \partial_j w + \partial_i w \cdot \partial_j w = a^2 \xi \otimes \xi + O\Big(\frac{1}{\lambda}\Big). \tag{2.9}$$

In the case of $N \geq n+2$ this condition may be greatly simplified by requesting $w \cdot \partial_i u = 0$ for all i obtaining:

$$\partial_i u \cdot \partial_j w = \partial_j(\partial_i \cdot w) - \partial_i \partial_j u \cdot w.$$

Simplifying the condition to:

$$\partial_i w \cdot \partial_j w = a^2 \xi \otimes \xi + O\Big(\frac{1}{\lambda}\Big),$$

this problem actually has closed form solution given by:

$$w(x) = \frac{a(x)}{\lambda}\big(\sin(\lambda x \cdot \xi)\zeta(x) + \cos(\lambda x \cdot \xi)\eta(x)\big).$$

With $\zeta(x), \eta(x)$ two orthogonal vector fields such that at every point and for every i, $\zeta \cdot \eta = \zeta \cdot \partial_i u = \eta \cdot \partial_i u = 0$, we may thus write:

$$\partial_i w = a(x)\xi_i\big(\cos(\lambda x \cdot \xi)\zeta(x) - \sin(\lambda x \cdot \xi)\eta(x)\big) + O\Big(\frac{1}{\lambda}\Big),$$

which gives:

$$\partial_i w \partial_j w = a^2(x)\xi_i\xi_j\big(\cos^2(\lambda x \cdot \xi) + \sin^2(\lambda x \cdot \xi) \\ + \cos(\lambda x \cdot \xi)\sin(\lambda x \cdot \xi)\zeta(x) \cdot \eta(x)\big) + O\Big(\frac{1}{\lambda}\Big) = a^2\xi_i \otimes \xi_j + O\Big(\frac{1}{\lambda}\Big).$$

In the case of $N = M + 1$ the vector fields ζ, η with the same properties do not exist, and thus the costruction is modified. A solution to (2.9) can still be found, but the construction is very complicated, although the structure of w remains similar.

We now want to discuss how to iterate convex integration steps to obtain a solution to the isometric immersion problem. Define metrics:

$$g_n = g - \epsilon_n \mathrm{Id},$$

where $1 > \epsilon_n \to 0$ is some sequence which converges appropriately fast. We will define a sequence of functions u_n recursively starting from the original short embedding u_0, and u_{n+1} is constructed in such a way that it is a short embedding for the metric g_{n+1} by guaranteeing:

$$\|g_n - \nabla u_{n+1}^T \nabla u_{n+1}\|_0 \leq \epsilon_n - \epsilon_{n+1}.$$

This construction is achieved through iterated steps of convex integration. Begin by decomposing:

$$g_n - \nabla u_n^T \nabla u_n = \sum_{i=1}^{k} a_i^2 \eta_i \otimes \eta_i,$$

where η_i are unit vectors. The crucial factor in determining the regularity of the solution found will be the number k of rank one matrices we can decompose the defect into. To go from u_n to u_{n+1} we must apply k steps of convex integration, every step will make the norm of the second derivative of u_{n+1} larger. The convergence of the $\mathcal{C}^{1,\alpha}$ norm of the u_n is dominated by a geometric sequence which converges when:

$$\alpha < \frac{1}{1 + 2k}.$$

The construction that leads to this result is very complex and will be omitted as it parallels closely the one we will detail for the Monge-Ampère equation. In the case of a 2-dimensional manifold embedded in $\mathbb{R}^3$ one may easily decompose any defect into $k = 3$ rank one matrices. This was done in [3] to show the existence of solutions with Hölder exponent less than $\frac{1}{7}$. In fact in [6] it was shown that through appropriate conformal changes of variables the number of convex integration steps can be reduced to $k = 2$ pushing the threshold to $\frac{1}{5}$. The limit of this technique is that it cannot provide solutions of regularity higher than $\frac{1}{3}$ which would require reducing the process to one step of convex integration. This hasn't been achieved for the isometric immersion problem, but it was achieved in the next example we mention.

2.3.2 Onsager's conjecture

Another application of convex integration is found in the study of the Euler equation for incompressible fluid flows. On a domain $\Omega \in \mathbb{R}^N$, and a time interval $[0, T]$ the equations are written as:

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = 0, \\ \text{div } v = 0. \end{cases} \tag{2.10}$$

Solutions to the equations are the velocity of the fluid at each point and time $v(x,t) : \Omega \times [0,T] \to \mathbb{R}^N$ and the pressure at each point and time $p(x,t) : \Omega \times [0,T] \to \mathbb{R}$. In this theory the dimension is taken to be $N = 3$, and the boundary conditions to be periodic. This means the problem is defined on the torus $\Omega = \mathbb{T}^3$.

Classical solutions to this problem are taken to be a pair of differentiable functions:

$$(v, p) \in \mathcal{C}^1\big(\mathbb{T}^3 \times [0,T], \mathbb{R}^3 \times \mathbb{R}\big).$$

Little is known for these solutions, except for the crucial fact that such solutions conserve energy. More formally we define the total energy for the flow at time t:

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \mathrm{d}x. \tag{2.11}$$

For classical solutions of (2.10) such energy will be constant in time.

Many definitions of weak solutions have been studied in connection to the study of turbulent flows. Many of these solutions have been found to have anomalous properties. Of particular interest is the fact that weak solutions do not generally satisfy conservation of energy. This is of relevance in the study of turbulent flows as it may model some anomalous phenomenons that have been observed experimentally.

On the matter of energy conservation Onsager formulated the problem in 1949 [20] asking if there exists a threshold of regularity between $\mathcal{C}^0$ and $\mathcal{C}^1$, above which energy is conserved and below which it is not. He postulated the threshold to be $\frac{1}{3}$ and himself provided a non rigorous proof of energy conservation for solutions of Hölder regularity above $\frac{1}{3}$. This was formally resolved in 1994 in [7, 2]. In the other direction great progress was made in recent years by a group of researchers including Buckmaster, De Lellis, Isset and Székelyhidi. De Lellis and Székelyhidi developed the schema to apply convex integration to this problem.

There are two statements of the concept of dissipation of energy:

1. There exists a nonzero weak solution $v \in L^1\big([0,1], \mathcal{C}^{0,\alpha}(\mathbb{T}^3)\big)$ with compact support in time.
2. For any smooth positive energy function $E(t)$, there exists a nonzero weak solution $v \in L^\infty\big([0,1], \mathcal{C}^{0,\alpha}(\mathbb{T}^3)\big)$ with:
$$\frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2 \mathrm{d}x = E(t).$$

The exponent α has been pushed to $\frac{1}{3} - \epsilon$ for the first statement, and $\frac{1}{5} - \epsilon$ for the second.

We now show the relaxation of (2.10) that allows for the application of convex integration. Consider a matrix valued function
$$u = v \otimes v - \frac{1}{3}|v|^2 \mathrm{Id}.$$
This matrix always has trace zero, and through direct calculation it results that in fact:
$$\text{div } u = (v \cdot \nabla) v.$$
Thus we may rewrite the system of equations as:
$$\begin{cases} \partial_t v + \text{div } u + \nabla p = 0, \\ \text{div } v = 0, \\ v \otimes v - u = \frac{1}{3}|v|^2 \mathrm{Id}. \end{cases} \tag{2.12}$$
We may now define subsolutions to (2.10) by relaxing the last equation in (2.12) to an inequality yielding:
$$\begin{cases} \partial_t v + \text{div } u + \nabla p = 0, \\ \text{div } v = 0, \\ v \otimes v - u \le \frac{1}{3}|v|^2 \mathrm{Id}. \end{cases}$$

The last term will determine a defect which will be decomposed into simple matrices and reduced through the application of convex integration steps. This construction again draws heavily on the construction by Nash. The actual construction of the oscillatory perturbations is however highly technical and will be omitted. We do however note that as before the threshold of regularity is determined by the number of convex integration steps which were reduced to 2 to obtain the $\frac{1}{5}$ bound [1]. Finally, Isset was able to reduce the schema to 1 step of convex integration providing the best threshold of $\frac{1}{3}$ in [11].

2.3.3 The Monge-Ampère equation

In this section we will discuss how convex integration is applied to the Monge-Ampère equation. This application is possible due to the quadratic structure of the very weak formulation of the equation. Given functions $v : \Omega \to \mathbb{R}$, $w : \Omega \to \mathbb{R}^2$ and $A : \Omega \to \mathbb{R}^{2\times 2}_{\text{Sym}}$, we may define the defect as:

$$D = A - \Big(\frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\nabla w\Big).$$

Clearly the Monge-Ampère equation is satisfied when the defect is zero. We consider a rank one defect of the form $a^2\eta \otimes \eta$ where $a : \Omega \to \mathbb{R}^+$ and η is a unit vector of $\mathbb{R}^2$. The goal is to find functions $v_\lambda : \Omega \to \mathbb{R}$ and $w_\lambda : \Omega \to \mathbb{R}^2$ such that:

$$\Big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \text{sym}\nabla w_\lambda\Big) - \Big(\frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\nabla w + a^2\eta \otimes \eta\Big) \approx O\Big(\frac{1}{\lambda}\Big), \tag{2.13}$$

where λ is an arbitrarily large number. Furthermore we require $\|v - v_\lambda\|_0 + \|w - w_\lambda\|_0 \approx O\big(\frac{1}{\lambda}\big)$. We look for solutions to this problem by adding oscillatory functions of frequency λ to both v and w. The ansatz is the following:

$$v_\lambda = v + \frac{1}{\lambda}f(x, \lambda c \cdot \eta), \quad w_\lambda = w + \frac{1}{\lambda}g(x, \lambda c \cdot \eta)\nabla v + \frac{1}{\lambda}h(x, \lambda c \cdot \eta)\eta.$$

We further require that $f(x,t)$, $g(x,t)$ and $h(x,t)$ all be oscillatory functions of period 1 in the variable t. If we evaluate $\big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \text{sym}\nabla w_\lambda\big)$ and ignore order $\frac{1}{\lambda}$ terms we obtain:

$$\begin{aligned}\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \text{sym}\nabla w_\lambda = \frac{1}{2}\nabla v \otimes \nabla v + \text{sym}\nabla w + \frac{1}{2}(f_t)^2\eta \otimes \eta + (f_t)\text{Sym}(\eta \otimes \nabla v)\\ + (g_t)\text{Sym}(\eta \otimes \nabla v) + (h_t)\eta \otimes \eta.\end{aligned}$$

If we want (2.13) to be satisfied we need to guarantee that $f = -g$ and $\frac{1}{2}(f_t)^2 + (h_t) = a^2$. There are closed form solutions to the second equation given by:

$$f(x,t) = \frac{a(x)}{\pi}\sin(2\pi t) \quad \text{and} \quad h(x,t) = -\frac{a(x)^2}{4\pi}\sin(4\pi t).$$

We formalize this in the following Lemma:

Lemma 2.3.1. *Given a positive function $a \in \mathcal{C}^\infty(\bar\Omega)$, define the functions $V, W \in \mathcal{C}^\infty(\bar\Omega \times \mathbb{R}, \mathbb{R})$ as:*

$$V(x,t) := \frac{a(x)}{\pi}\sin(2\pi t), \qquad W(x,t) := -\frac{a(x)^2}{4\pi}\sin(4\pi t).$$

These functions are 1-periodic in t, and satisfy in $\bar\Omega \times \mathbb{R}$:

$$\frac{1}{2}(\partial_t V)^2 + \partial_t W = a^2, \tag{2.14}$$

$$\begin{aligned} &|V| \leq \frac{a}{\pi}, \quad |\partial_t V| \leq 2a, \quad |\nabla_x V| \leq \frac{|\nabla a|}{\pi}, \quad |\nabla_x^2 V| \leq \frac{|\nabla^2 a|}{\pi}, \\ &|W| \leq \frac{a^2}{4\pi}, \quad |\partial_t W| \leq a^2, \quad |\nabla_x W| \leq \frac{a|\nabla a|}{2\pi}. \end{aligned} \tag{2.15}$$

Proof. The proof follows from straightforward derivatives calculations and the periodicity and boundedness of the sin and cos function.

$$\begin{aligned} &\partial_t V(x,t) = 2a(x)\cos(2\pi t) \quad \nabla_x V(x,t) = \frac{\nabla a(x)}{\pi}\sin(2\pi t), \quad \nabla_x^2 V(x,t) = \frac{\nabla^2 a(x)}{\pi}\sin(2\pi t), \\ &\partial_t W(x,t) = -a(x)^2\cos(4\pi t), \quad \nabla_x W(x,t) = -\frac{a(x)\nabla a(x)}{2\pi}\sin(4\pi t). \end{aligned}$$

Using these evaluations we compute:

$$\begin{aligned} \frac{1}{2}(\partial_t V(x,t))^2 + \partial_t W(x,t) &= 2a^2(x)\cos^2(2\pi t) - a(x)^2\cos(2\cdot 2\pi t) \\ &= 2a^2(x)\cos^2(2\pi t) - a(x)^2\cos^2(2\pi t) + a(x)^2\sin^2(2\pi t) = a^2. \end{aligned}$$

We may now provide the Lemma which justifies one step of convex integration. As in other convex integration schemas several steps of convex integration will be applied subsequently to reduce a defect which is not rank one.

Proposition 2.3.2. *Let $v \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R})$, $w \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ and let $a \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R})$ be a positive function. For a unit vector $\eta \in \mathbb{R}^2$ and a frequency $\lambda > 1$, define $v_\lambda \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R})$, $w_\lambda \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ through:*

$$\begin{aligned} v_\lambda(x) &= v(x) + \frac{1}{\lambda}V(x, \lambda x\cdot\eta), \\ w_\lambda(x) &= w(x) - \frac{1}{\lambda}V(x, \lambda x\cdot\eta)\nabla v(x) + \frac{1}{\lambda}W(x, \lambda x\cdot\eta)\eta. \end{aligned}$$

Then we have the following pointwise estimates, valid in $\bar\Omega$:

$$\begin{aligned} &\Big|\big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \mathrm{Sym}\nabla w_\lambda\big) - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w + a^2\eta\otimes\eta\big)\Big| \\ &\qquad\qquad \leq \frac{1}{\lambda}\Big(\frac{a|\nabla a|}{2\pi} + \frac{a|\nabla^2 v|}{\pi}\Big) + \frac{1}{2\lambda^2\pi^2}|\nabla a|^2, \end{aligned} \tag{2.16}$$

$$|v_\lambda - v| \le \frac{a}{\lambda\pi}, \qquad |w_\lambda - w| \le \frac{a}{\lambda\pi}\left(|\nabla v| + \frac{a}{4}\right), \tag{2.17}$$

$$\begin{aligned}
|\nabla v_\lambda - \nabla v| &\le \frac{|\nabla a|}{\lambda\pi} + 2a, \quad \textit{and} \\
|\nabla w_\lambda - \nabla w| &\le 2a|\nabla v| + a^2 + \frac{1}{\lambda}\Big(\frac{1}{\pi}|\nabla v||\nabla a| + \frac{a}{\pi}|\nabla^2 v| + \frac{1}{2\pi}a|\nabla a|\Big),
\end{aligned} \tag{2.18}$$

$$|\nabla^2 v_\lambda - \nabla^2 v| \le \frac{|\nabla^2 a|}{\lambda\pi} + 4|\nabla a| + 4\lambda\pi a. \tag{2.19}$$

Proof. We use (2.15) to show the estimates in (2.17):

$$|v_\lambda - v| = \Big|\frac{1}{\lambda}V\Big| \le \frac{a}{\lambda\pi}, \qquad |w_\lambda - w| = \frac{1}{\lambda}|V\nabla v + W\eta| \le \frac{a}{\lambda\pi}\left(|\nabla v| + \frac{a}{4}\right).$$

We now compute all appropriate derivatives of v_λ and w_λ:

$$\begin{aligned}
\nabla v_\lambda &= \nabla v + \frac{1}{\lambda}\nabla_x V + \eta\partial_t V, \\
\nabla^2 v_\lambda &= \nabla^2 v + \frac{1}{\lambda}\nabla_x^2 V + 2(Sym)(\partial_t\nabla_x V \otimes \eta) + \lambda\partial_t^2 V\eta \otimes \eta, \\
\nabla w_\lambda &= \nabla w - \frac{1}{\lambda}\nabla_x V\nabla v - \partial_t V\nabla v \otimes \eta + \frac{1}{\lambda}\eta \otimes \nabla_x W + \partial_t W\eta \otimes \eta.
\end{aligned}$$

Using these evaluations, a straightforward calculation gives (2.18) and (2.19):

$$\begin{aligned}
|\nabla v_\lambda - \nabla v| &\le \frac{1}{\lambda}|\nabla_x V| + |\partial_t V| \le \frac{|\nabla a|}{\lambda\pi} + 2a, \\
|\nabla w_\lambda - \nabla w| &= \Big| -\frac{1}{\lambda}\nabla v \otimes \nabla_x V - (\partial_t V)\eta \otimes \nabla v - \frac{1}{\lambda}V\nabla^2 v + \frac{1}{\lambda}\eta \otimes \nabla_x W + (\partial_t W)\eta \otimes \eta\Big| \\
&\le 2a|\nabla v| + a^2 + \frac{1}{\lambda}\left(\frac{1}{\pi}|\nabla v||\nabla a| + \frac{a}{\pi}|\nabla^2 v| + \frac{1}{2\pi}a|\nabla a|\right), \\
|\nabla^2 v_\lambda - \nabla^2 v| &\le \frac{1}{\lambda}|\nabla_x^2 V| + 2|\nabla_x\partial_t V| + \lambda|\partial_t^2 V| \le \frac{|\nabla^2 a|}{\lambda\pi} + 4|\nabla a| + 4\lambda\pi a.
\end{aligned}$$

Lastly, using (2.14) and the fact that $(\partial_t V)\nabla_x V + \nabla_x W = \frac{1}{\pi}\sin(2\pi t)a\nabla a$, we evaluate directly:

$$\begin{aligned}
\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \mathrm{Sym}\nabla w_\lambda &= \frac{1}{2}\nabla v \otimes \nabla v + \frac{1}{2}(\partial_t V)^2\eta \otimes \eta + (\partial_t W)\eta \otimes \eta + \mathrm{Sym}\nabla w \\
&\quad + \frac{1}{\lambda}\Big((\partial_t V)\mathrm{Sym}(\nabla_x V \otimes \eta) - V\nabla^2 v + \mathrm{Sym}(\nabla_x W \otimes \eta)\Big) + \frac{1}{2\lambda^2}\nabla_x V \otimes \nabla_x V \\
&= \frac{1}{2}\nabla v \otimes \nabla v + a^2\eta \otimes \eta + \frac{1}{\lambda}\Big(\frac{a}{\pi}\sin(2\pi\lambda x \cdot \eta)\mathrm{Sym}(\nabla a \otimes \eta) - \frac{a}{\pi}\sin(2\pi\lambda x \cdot \eta)\nabla^2 v\Big) \\
&\quad + \mathrm{Sym}\nabla w + \frac{1}{2\pi^2\lambda^2}\sin^2(2\pi\lambda x \cdot \eta)\nabla a \otimes \nabla a.
\end{aligned}$$

We conclude the proof by setting $\tau = 2\pi\lambda x \cdot \eta$ and deriving the following estimate:

$$
\begin{aligned}
&\Big|\big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \mathrm{Sym}\nabla w_\lambda\big) - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w + a^2\eta\otimes\eta\big)\Big| \\
&\qquad \leq \Big|\frac{1}{\lambda}\Big(\frac{a}{\pi}\sin(\tau)\mathrm{Sym}(\nabla a \otimes \eta) - \frac{a}{\pi}\sin(\tau)\nabla^2 v\Big) + \frac{1}{2\pi^2\lambda^2}\sin^2(\tau)\nabla a \otimes \nabla a\Big| \\
&\qquad \leq \frac{1}{\lambda}\Big(\frac{a|\nabla a|}{2\pi} + \frac{a|\nabla^2 v|}{\pi}\Big) + \frac{1}{2\lambda^2\pi^2}|\nabla a|^2.
\end{aligned}
$$

We now present a similar Proposition which will be needed to obtain some stronger inequalities. In particular this framework will provide some control on the second derivatives of v_λ at the cost of stronger assumptions.

Proposition 2.3.3. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set. Let $v \in \mathcal{C}^3(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$, and $a \in \mathcal{C}^3(\bar{\Omega})$ be functions with $a > 0$. Take a unit vector $\eta \in \mathbb{R}^2$ and $\delta, l \in (0,1)$ be two parameter constants such that:*

$$
\|\nabla^m a\|_0 \leq \frac{\delta}{l^m} \quad \forall m = 0 \dots 3, \quad \textit{and} \quad \|\nabla^{m+1} v\|_0 \leq \frac{\delta}{l^m} \quad \forall m = 1, 2. \tag{2.20}
$$

Then for any $\lambda > 1/l$ there exist approximating functions $v_\lambda \in \mathcal{C}^3(\bar{\Omega})$ and $w_\lambda \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ satisfying the following bounds:

$$
\Big|\big(\frac{1}{2}\nabla v_\lambda \otimes \nabla v_\lambda + \mathrm{Sym}\nabla w_\lambda\big) - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w + a^2\eta\otimes\eta\big)\Big| \leq \frac{3}{\pi}\frac{\delta^2}{\lambda l}, \tag{2.21}
$$

$$
\|\nabla^m(v_\lambda - v)\| \leq 2.4^m \pi^{m-1}\delta\lambda^{m-1} \quad \forall m = 0 \dots 3, \tag{2.22}
$$

$$
\|\nabla^m(w_\lambda - w)\| \leq 2.4^m \pi^{m-1}\delta\lambda^{m-1}(1 + \|v\|_0) \quad \forall m = 0 \dots 2. \tag{2.23}
$$

Proof. The first inequality (2.21) follows directly from (2.16) by applying (2.20). We compute each of the norms in (2.22) and (2.23) using (2.20).

$$
\begin{aligned}
\|v - v_\lambda\|_0 &\leq \frac{1}{\lambda}\frac{\|a\|_0}{\pi} \leq \frac{1}{\pi}\frac{1}{\lambda}\delta, \\
\|\nabla(v - v_\lambda)\|_0 &\leq \frac{1}{\lambda}\frac{\|\nabla a\|_0}{\pi} + 2\|a\|_0 \leq 2.4\delta, \\
\|\nabla^2(v - v_\lambda)\|_0 &\leq \frac{1}{\lambda}\frac{\|\nabla^2 a\|_0}{\pi} + 4\|\nabla a\|_0 + 4\pi\lambda\|a\|_0 \leq 16.9\lambda\delta, \\
\|\nabla^3(v - v_\lambda)\|_0 &\leq \frac{1}{\lambda}\frac{\|\nabla^3 a\|_0}{\pi} + 6\|\nabla^2 a\|_0 + 12\pi\lambda\|\nabla a\|_0 + 8\pi^2\lambda^2\|\nabla a\|_0 \leq 123.0\lambda\delta, \\
\|w - w_\lambda\|_0 &\leq \frac{1}{\lambda}\frac{\|a\|_0}{\pi}\|\nabla v\|_0 + \frac{1}{\lambda}\frac{\|a\|_0^2}{4\pi} \leq \frac{1}{\pi}\frac{1}{\lambda}\delta(1 + \|\nabla v\|_0), \\
\|\nabla(w - w_\lambda)\|_0 &\leq \frac{1}{\lambda}\frac{\|\nabla a\|_0}{\pi}\|\nabla v\|_0 + \frac{1}{\lambda}\frac{\|a\|_0}{\pi}\|\nabla^2 v\|_0 + \frac{1}{\lambda}\frac{\|a\|_0\|\nabla a\|_0}{2\pi} + 2\|a\|_0\|\nabla v\|_0 + \|a\|_0^2 \\
&\leq 2.4\delta(1 + \|\nabla v\|_0), \\
\|\nabla^2(w - w_\lambda)\|_0 &\leq \frac{1}{\lambda}\frac{\|\nabla^2 a\|_0}{\pi}\|\nabla v\|_0 + \frac{2}{\lambda}\frac{\|\nabla a\|_0}{\pi}\|\nabla^2 v\|_0 + 4\|\nabla a\|_0\|\nabla v\|_0 + \frac{1}{\lambda}\frac{\|a\|_0}{\pi}\|\nabla^3 v\|_0 \\
&\quad + 4\|a\|_0\|\nabla^2 a\|_0 + 4\pi\lambda\|a\|_0\|\nabla v\|_0 + \frac{1}{\lambda}\frac{\|\nabla a\|_0^2}{2\pi} + \frac{1}{\lambda}\frac{\|a\|_0\|\nabla^2 a\|_0}{2\pi} + 4\|a\|_0\|\nabla a\|_0 \\
&\quad + 4\pi\lambda\|a\|_0^2 \leq 21.9\lambda\delta(1 + \|\nabla v\|_0).
\end{aligned}
$$

2.4 THE MAIN ANALYTICAL RESULTS

We may now state the main analytical result obtained using convex integration for the Monge-Ampère equation. This result was obtained by Lewicka and Pakzad in [15]. The framework of this proof was used by the author and Lewicka with modifications to obtain interesting visualizations of anomalous solutions to the equation in [4].

Theorem 2.4.1. *Let $f \in L^{7/6}(\Omega)$ on an open, bounded, simply connected $\Omega \subset \mathbb{R}^2$. Fix an exponent:*

$$\alpha < \frac{1}{7}.$$

Then the set of $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to:

$$\mathcal{D}et\, \nabla^2 v = f \qquad in\ \Omega \tag{2.24}$$

is dense in the space $\mathcal{C}^0(\bar{\Omega})$. More precisely, for every $v_0 \in \mathcal{C}^0(\bar{\Omega})$ there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$, converging uniformly to v_0 and solving the equation (2.24). When $f \in L^p(\Omega)$ and $p \in (1, \frac{7}{6})$, the same result is true for any $\alpha < 1 - \frac{1}{p}$.

This result follows from two intermediate results which will be proven using convex integration techniques. First we show that there exists a $\mathcal{C}^1$ approximation for any continuous function. Next we show the existence of a $\mathcal{C}^{1,\alpha}$ approximation under stricter conditions. The two intermediate theorems will be used in sequence to prove Theorem 2.4.1.

We first restate the Theorem in terms of the very weak Hessian. The two statements are equivalent in light of the arguments from section 2.1.3.

Theorem 2.4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v_0 \in \mathcal{C}^1(\bar{\Omega})$, $w_0 \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{Sym})$, for some $\beta \in (0,1)$, be such that:*

$$\exists c_0 > 0 \qquad A - \big(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{Sym}\nabla w_0\big) \geq c_0 \mathrm{Id}_2 \qquad \text{in } \bar{\Omega}. \tag{2.25}$$

Then, for every exponent α in the range:

$$0 < \alpha < \min\Big\{\frac{1}{7}, \frac{\beta}{2}\Big\},$$

there exist sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ which converge to v_0 and $w_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to some function $w \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$, and which satisfy:

$$A = \frac{1}{2}\nabla v_n \otimes \nabla v_n + \mathrm{Sym}\nabla w_n \qquad \text{in } \bar{\Omega}. \tag{2.26}$$

Next we consider the existence of a $\mathcal{C}^1$ approximation.

Theorem 2.4.3. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain. Let $v_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R})$, $w_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$, and $A \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2\times 2}_{Sym})$ be functions such that, with some constant $d_0 > 0$, we have:*

$$D_0 = A - \big(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{Sym}\nabla w_0\big) = \sum_{k=1}^{3} \phi_k \eta_k \otimes \eta_k, \qquad \phi_k \geq d_0 \quad \text{in } \bar{\Omega}. \tag{2.27}$$

Then, for any $\epsilon > 0$ there exist sequences $\{v_k\}_{k=1}^\infty \subset \mathcal{C}^\infty(\bar{\Omega})$, and $\{w_k\}_{k=1}^\infty \subset \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ such that: $\|v_0 - v_k\|_0 \leq \epsilon$ for every value of k, $v_k \to v$ in $\mathcal{C}^1(\bar{\Omega})$ and $w_k \to w$ in $\mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$, where v and w satisfy:

$$\|v_0 - v\|_0 \leq \epsilon \quad \text{and} \quad \frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w = A. \tag{2.28}$$

Finally we state the $\mathcal{C}^{1,\alpha}$ result .

Theorem 2.4.4. *Given a small constant $\delta_0 \leq 5.4 \cdot 10^{-16}$, let $\Omega \subset \Omega_{fat} \subset \mathbb{R}^2$ be open bounded domains such that $\Omega + B_{2\delta_0}(0) \subset \Omega_{fat}$. Given three functions $v \in \mathcal{C}^2(\bar{\Omega}_{fat})$, $w \in \mathcal{C}^2(\bar{\Omega}_{fat}, \mathbb{R}^2)$, and $A \in \mathcal{C}^{0,\alpha}(\bar{\Omega}_{fat}, \mathbb{R}^{2\times 2}_{Sym})$ with a constant $\beta \in (0,1)$, assume:*

$$D_0 = A - \left(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w\right), \quad 0 < \|D_0\|_{\mathcal{C}^0(\Omega_{fat})} < \delta_0 \ll 1. \tag{2.29}$$

$$0 < \alpha < \min\left\{\frac{1}{7}, \frac{\beta}{2}\right\}.$$

Then functions $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$, and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ can be found such that:

$$\frac{1}{2}\nabla\bar{v} \otimes \nabla\bar{v} + \mathrm{Sym}\nabla\bar{w} = A, \quad \textit{and} \quad \|v - \bar{v}\|_{\mathcal{C}^0(\Omega)} \leq 0.02\delta_0.$$

Before proving each of the intermediate theorems we show how Theorems 2.4.3 and 2.4.4 are used to prove Theorem 2.4.2.

Having taken a sufficiently small $\epsilon > 0$, we will construct a sequence of functions $v_k \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $w_k \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge in $\mathcal{C}^{1,\alpha}$ sense to functions $\bar{v}$ and $\bar{w}$ such that:

$$A_0 = \frac{1}{2}\nabla\bar{v} \otimes \nabla\bar{v} + \mathrm{Sym}\nabla\bar{w} \quad \text{in } \bar{\Omega} \quad \text{and} \quad \|\bar{v} - v_0\|_0 < \epsilon.$$

To this end we look to apply Theorem 2.4.4, but we have no estimate on the initial deficit. To account for the initial deficit being too large we will utilise the construction in Theorem 2.4.3.

We begin by choosing functions $\tilde{v}_0 \in \mathcal{C}^\infty(\bar{\Omega}_{fat})$, $\tilde{w}_0 \in \mathcal{C}^\infty(\bar{\Omega}_{fat}, \mathbb{R}^2)$ and $\tilde{A}_0 \in \mathcal{C}^\infty(\bar{\Omega}_{fat}, \mathbb{R}^{2\times 2}_{\mathrm{Sym}})$ such that:

$$\begin{aligned} \|\tilde{v}_0 - v_0\|_{\mathcal{C}^0(\Omega)} + \|\tilde{w}_0 - w_0\|_{\mathcal{C}^0(\Omega)} &< \frac{\epsilon}{3}, \\ \|\tilde{A}_0 - A_0\|_{\mathcal{C}^0(\Omega)} &< \frac{\delta_0}{2}. \end{aligned} \tag{2.30}$$

This is done using the density of $\mathcal{C}^\infty$ in $\mathcal{C}^0$ and by finding appropriate smooth extensions of the functions on Ω_{fat} Furthermore we may assume without loss of generality that (2.27) holds on Ω_{fat} by using the arbitrariness of A and w_0. Next we apply the construction from the proof of Theorem 2.4.3.

Following the Theorem we construct sequences $\{v_i\}_{i=1}^\infty \subset \mathcal{C}^\infty(\bar{\Omega}_{fat})$, and $\{w_i\}_{i=1}^\infty \in \mathcal{C}^\infty(\bar{\Omega}_{fat}, \mathbb{R}^2)$ such that: $\|v_i - \tilde{v}_0\|_{\mathcal{C}^0(\Omega_{fat})} < \frac{\epsilon}{3}$. These sequences converge $v_i \to v$ and $w_i \to w$ in the $\mathcal{C}^1(\Omega_{fat})$ sense where v and w satisfy $\tilde{A}_0 = \frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w$. This implies that the sequence of defects defined by $D_i = \tilde{A}_0 - \left(\frac{1}{2}\nabla v_i \otimes \nabla v_i + \mathrm{Sym}\nabla w_i\right)$ converges to zero, and thus there exists a $k \geq 1$ such that the defect $\|\tilde{D}_k\|_{\mathcal{C}^0(\Omega_{fat})} < \frac{\delta_0}{2}$.

We thus define $\tilde{v}_k \in \mathcal{C}^\infty(\bar{\Omega}_{fat})$ and $\tilde{w}_k \in \mathcal{C}^\infty(\bar{\Omega}_{fat}, \mathbb{R}^2)$ to which we may apply Theorem 2.4.4 with the original A_0. We note that these functions satisfy the conditions of the Theorem as:

$$\|A_0 - \frac{1}{2}\nabla\tilde{v}_k \otimes \nabla\tilde{v}_k + \mathrm{Sym}\nabla\tilde{w}_k\|_{\mathcal{C}^0(\Omega)} \leq \|A_0 - \tilde{A}\|_{\mathcal{C}^0(\Omega)} + \|\tilde{D}_k\|_{\mathcal{C}^0(\Omega_{fat})} < \delta_0.$$

Thus we finally obtain $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which obey the desired equality, and such that it holds:

$$\|\bar{v} - v_0\|_{\mathcal{C}^0(\Omega)} \leq \delta_0 + \frac{2\epsilon}{3} < \epsilon,$$

if δ_0 is taken small enough.

2.4.1 $\mathcal{C}^1$ approximation

Proposition 2.4.5. *Let $v \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R})$, $w \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ and $A \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{2\times 2}_{Sym})$ be such that, with some constant $d > 0$, we have:*

$$D := A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w\big) = \sum_{k=1}^{3} \phi_k \eta_k \otimes \eta_k, \qquad \phi_k \geq d \quad \text{in } \bar\Omega. \tag{2.31}$$

Fix $\epsilon > 0$ and $\xi > 0$ such that:

$$\xi \leq \|D\|_0. \tag{2.32}$$

Then, there exist $\tilde v \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R})$, $\tilde w \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ and a constant $\tilde d > 0$ such that:

$$\tilde D := A - \big(\frac{1}{2}\nabla \tilde v \otimes \nabla \tilde v + \mathrm{Sym}\nabla \tilde w\big) = \sum_{k=1}^{3} \tilde\phi_k \eta_k \otimes \eta_k, \qquad \tilde\phi_k \geq \tilde d \quad \text{in } \bar\Omega, \tag{2.33}$$

$$\|\tilde D\|_0 \leq \xi, \qquad \|\tilde v - v\|_0 \leq \epsilon, \qquad \|\tilde w - w\|_0 \leq C\epsilon\big(\|\nabla v\|_0 + \|D\|_0^{1/2}\big), \tag{2.34}$$

$$\|\nabla\tilde v - \nabla v\|_0 \leq C\|D\|_0^{1/2}, \qquad \|\nabla\tilde w - \nabla w\|_0 \leq C\big(\|D\|_0^{1/2}\|\nabla v\|_0 + \|D\|_0\big). \tag{2.35}$$

Proof. **1.** We construct three intermediate fields v_k and w_k, $k = 1\ldots 3$, between the given $v_0 = v$, $w_0 = w$ and the requested $v_3 = \tilde v$, $w_3 = \tilde w$. To this end, define smooth, positive functions $a_k : \bar\Omega \to \mathbb{R}$, $k = 1\ldots 3$, and a constant $\delta \leq \frac{1}{2}$ by:

$$\delta\|D\|_0 = \frac{\xi}{2}, \qquad a_k(x)^2 = (1 - \delta(x))\phi_k(x),$$

so that: $\delta D = D - \sum_{k=1}^{3} a_k^2 \eta_k \otimes \eta_k$.

Given $(v_{k-1}, w_{k-1}) \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^3)$, the successive corrections v_k and w_k are now constructed by applying Proposition 2.3.2 to $v = v_{k-1}$, $w = w_{k-1}$, $a = a_k$, $\eta = \eta_k$ and an appropriate $\lambda = \lambda_k \geq 1$ determined below. Observe that:

$$\begin{aligned}
\tilde D &= A - \big(\frac{1}{2}\nabla \tilde v \otimes \nabla \tilde v + \mathrm{Sym}\nabla \tilde w\big) = D + \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w\big) - \big(\frac{1}{2}\nabla \tilde v \otimes \nabla \tilde v + \mathrm{Sym}\nabla \tilde w\big) \\
&= D - \sum_{k=1}^{3} a_k^2 \eta_k \otimes \eta_k - \sum_{k=1}^{3} \Big(\big(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{Sym}\nabla w_k\big) \\
&\qquad\qquad - \big(\frac{1}{2}\nabla v_{k-1} \otimes \nabla v_{k-1} + \mathrm{Sym}\nabla w_{k-1} + a_k^2 \eta_k \otimes \eta_k\big)\Big) \\
&= \delta D - \sum_{k=1}^{3} B_k,
\end{aligned}$$

where (2.16) yields the following pointwise bound on the error quantities B_k:

$$|B_k| \leq \frac{a_k|\nabla a_k|}{2\pi\lambda_k} + \frac{a_k|\nabla^2 v_{k-1}|}{\pi\lambda_k} + \frac{|\nabla a_k|^2}{2\pi^2\lambda_k^2} \quad \text{in } \bar{\Omega}, \quad k = 1\dots 3. \tag{2.36}$$

To prove positivity of the decomposition in (2.33), we set:

$$\tilde{d} = \frac{\xi d}{4\|D\|_0} = \frac{\delta d}{4}$$

and use Lemma 2.1.8 to:

$$\sum_{k=1}^{3}(\tilde{\phi}_k - \delta\phi_k)\eta_k \otimes \eta_k = \tilde{D} - \delta D = -\sum_{i=1}^{3} B_i \quad \text{in } \bar{\Omega}.$$

Namely, by (2.31) it follows that:

$$\tilde{\phi}_k \geq \delta\phi_k - \frac{5\sqrt{3}}{8}\Big|\sum_{i=1}^{3} B_i\Big| \geq \frac{\delta\phi_k}{2} > \frac{\delta d}{2} \geq \tilde{d} \quad \text{in } \bar{\Omega}, \quad k = 1\dots 3,$$

where the second inequality above is valid when:

$$\frac{5\sqrt{3}}{8}|B_i| \leq \frac{\delta\phi_k}{6} \qquad \text{in } \bar{\Omega}, \quad i, k : 1\dots 3. \tag{2.37}$$

Note that the first estimate in (2.34) holds then as well, because:

$$|\tilde{D}| \leq \delta|D| + \Big|\sum_{i=1}^{3} B_i\Big| \leq \frac{\xi}{2} + \frac{4}{5\sqrt{3}}\delta\phi_i \leq \frac{\xi}{2} + \frac{\delta}{2}|D| = \frac{3}{4}\xi \quad \text{in } \bar{\Omega}.$$

To ensure the validity of (2.37) we must thus request the following three conditions which may be assured by choosing λ_k large enough:

$$\frac{5\sqrt{3}}{8}\frac{a_i|\nabla a_i|}{2\pi\lambda_i} \leq \frac{\delta d}{18} \quad \text{and} \quad \frac{5\sqrt{3}}{8}\frac{a_i|\nabla^2 v_{i-1}|}{\pi\lambda_i} \leq \frac{\delta d}{18} \quad \text{and} \quad \frac{5\sqrt{3}}{8}\frac{|\nabla a_i|^2}{2\pi^2\lambda_i^2} \leq \frac{\delta d}{18} \quad \text{in } \bar{\Omega}, \quad i, k : 1\dots 3. \tag{2.38}$$

2. Observe that, by Lemma 2.1.8, we get:

$$\sum_{i=1}^{3} a_i \leq \sqrt{3}\Big(\sum_{i=1}^{3} a_i^2\Big)^{1/2} = \sqrt{3}\big((1-\delta)\mathrm{Tr}\ D\big)^{1/2} \leq 3|D|^{1/2} \quad \text{in } \bar{\Omega}. \tag{2.39}$$

Consequently, using (2.17), we obtain the second inequality in (2.34):

$$|\tilde{v} - v| \leq \sum_{i=1}^{3} \frac{a_i}{\lambda_i\pi} \leq \frac{3|D|^{1/2}}{\pi \min_{k=1\dots 3}\{\lambda_k\}} < \epsilon \quad \text{in } \bar{\Omega}, \tag{2.40}$$

if only we assume that:

$$\lambda_k \geq \frac{\|D\|_0^{1/2}}{\epsilon} \quad \text{for } k = 1\dots 3. \tag{2.41}$$

Note that (2.38) and Lemma 2.1.8 easily imply that:

$$\frac{|\nabla a_i|}{\pi\lambda_i} \leq \sqrt{\delta\frac{16}{5\sqrt{3}\ 18}\phi_i} \leq \frac{1}{3}|D|^{1/2} \quad \text{in } \bar{\Omega}. \tag{2.42}$$

Thus, by (2.39) and (2.18):

$$\sum_{i=1}^{3} |\nabla v_i - \nabla v_{i-1}| \leq \sum_{i=1}^{3} \big(\frac{|\nabla a_i|}{\lambda_i \pi} + 2a_i\big) \leq 7\|D\|_0^{1/2} \quad \text{in } \bar{\Omega},$$

so that the first bound in (2.35) follows explicitly in: $\|\nabla\tilde{v} - \nabla v\|_0 \leq 7\|D\|_0^{1/2}$. We also observe:

$$|\nabla v_k| \leq |\nabla v_0| + 7\|D\|_0^{1/2} \quad \text{in } \bar{\Omega}, \quad k = 1 \dots 3.$$

Using (2.41) and (2.39) again, we obtain the last inequality in (2.34):

$$\begin{aligned}|\tilde{w} - w| &\leq \sum_{i=1}^{3} \frac{a_i}{\lambda_i \pi}\big(\|\nabla v_{i-1}\|_0 + \frac{a_i}{4}\big) \leq \sum_{i=1}^{3} \frac{a_i}{\lambda_i \pi}\big(\|\nabla v\|_0 + 7\|D\|_0^{1/2} + \frac{a_i}{4}\big) \\ &\leq \frac{\|D\|_0^{1/2}}{\min\limits_{k=1\dots 3}\{\lambda_k\}}\big(\|\nabla v\|_0 + 8\|D\|_0^{1/2}\big) \leq \epsilon\big(\|\nabla v\|_0 + 8\|D\|_0^{1/2}\big) \quad \text{in } \bar{\Omega},\end{aligned}$$

whereas (2.42) is used to obtain the final bound in (2.35):

$$\begin{aligned}|\nabla\tilde{w} - \nabla w| &\leq \sum_{i=1}^{3}\bigg(2a_i|\nabla v_{i-1}| + a_i^2 + \frac{2|\nabla v_{i-1}||\nabla a_i| + 2a_i|\nabla^2 v_{i-1}| + a_i|\nabla a_i|}{2\pi\lambda_i}\bigg) \\ &\leq \sum_{i=1}^{3}\bigg(2a_i(\|\nabla v\|_0 + 7\|D\|_0^{1/2}) + a_i^2 + \frac{2|\nabla a_i|(\|\nabla v\|_0 + 7\|D\|_0^{1/2}) + a_i|\nabla a_i|}{2\pi\lambda_i} + \frac{a_i|\nabla^2 v_{i-1}|}{\pi\lambda_i}\bigg) \\ &\leq 7\|D\|_0^{1/2}\big(\|\nabla v\|_0 + 7\|D\|_0^{1/2}\big) + 11\|D\|_0 \quad \text{in } \bar{\Omega},\end{aligned}$$

where the last term in parentheses above is bounded by $\frac{1}{2}\|D\|_0$ by (2.38).

Proof of Theorem 2.4.3.

We will construct a sequence of approximations $\{v_k\}_{k=0}^{\infty}$ which converge in $\mathcal{C}^1$ to the required solution in (2.28). Starting with v_0, w_0, we define recursively $v_{k+1} \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R})$ and $w_{k+1} \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ by applying Proposition 2.4.5 to $v = v_k \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R})$ and $w = w_k \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$, yielding the corresponding defect $D_k = A - (\frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{Sym}\nabla w_k)$, with $\epsilon = \epsilon_k$ and $\xi = \xi_k$ which satisfy:

$$\sum_{k=1}^{\infty} \epsilon_k \leq \epsilon, \qquad \sum_{k=1}^{\infty} \xi_k^{1/2} < \infty \quad \text{and} \quad \xi_k \leq \|D_k\|_0 \quad \text{for all } k. \tag{2.43}$$

By construction, each D_k can be decomposed in the basis $\eta_1 \otimes \eta_1$, $\eta_2 \otimes \eta_2$, $\eta_3 \otimes \eta_3$, with positive coefficients by (2.33). By (2.34):

$$\|v_k - v_0\|_0 \leq \sum_{i=1}^{k} \|v_i - v_{i-1}\|_0 \leq \sum_{i=1}^{k} \epsilon_i < \epsilon, \tag{2.44}$$

while by (2.35) we get:

$$\|\nabla v_{k+m} - \nabla v_k\|_0 \le \sum_{i=k+1}^{k+m} \|\nabla v_i - \nabla v_{i-1}\|_0 \le C \sum_{i=k+1}^{k+m} \|D_{i-1}\|_0^{1/2} \le C \sum_{i=k+1}^{k+m} \xi_i^{1/2}.$$

We thus see that the sequence $\{v_k\}_{k=1}^\infty$ is Cauchy in $\mathcal{C}^1$, and consequently it converges to some $v \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R})$. By (2.44), the first statement of (2.28) follows. In particular, $\{\|\nabla v_k\|_0\}_{k=1}^\infty$ is a bounded sequence. Similarly we compute:

$$\begin{aligned}
\|w_{k+m} - w_k\|_0 &\le \sum_{i=k+1}^{k+m} \|\nabla w_i - \nabla w_{i-1}\|_0 \le C \sum_{i=k+1}^{k+m} \epsilon_{i-1}\big(\|\nabla v_{i-1}\|_0 + \|D_{i-1}\|_0^{1/2}\big) \\
&\le C \sum_{i=k+1}^{k+m} \epsilon_{i-1}, \\
\|\nabla w_{k+m} - \nabla w_k\|_0 &\le \sum_{i=k+1}^{k+m} \|\nabla w_i - \nabla w_{i-1}\|_0 \le C \sum_{i=k+1}^{k+m} \|D_{i-1}\|_0^{1/2}\big(\|\nabla v_{i-1}\|_0 + \|D_{i-1}\|_0^{1/2}\big) \\
&\le C \sum_{i=k+1}^{k+m} \xi_{i-1}^{1/2}.
\end{aligned}$$

Therefore, $\{w_k\}_{k=0}^\infty$ is Cauchy and hence it converges in $\mathcal{C}^1$ to some $w \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$. Finally:

$$\|A - \big(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w\big)\|_0 = \lim_{k\to\infty} \|D_k\|_0 = 0,$$

proving the second statement in (2.28) by (2.43).

2.4.2 $\mathcal{C}^{1,\alpha}$ convergence

Proposition 2.4.6. *Let $\Omega \subset \Omega_b \subset \mathbb{R}^2$ be open, bounded sets such that $\Omega + B_r(0) \subset \Omega_b$ for some $0 \le r \le 1$. Given three functions $v \in \mathcal{C}^2(\bar{\Omega}_b)$, $w \in \mathcal{C}^2(\bar{\Omega}_b, \mathbb{R}^2)$, and $A \in \mathcal{C}^{0,\alpha}(\bar{\Omega}_b, \mathbb{R}^{2\times 2}_{Sym})$ with a constant $\beta \in (0,1)$, assume:*

$$D = A - \left(\frac{1}{2}\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w\right), \quad 0 < \|D_0\|_{\mathcal{C}^0(\Omega_b)} < \delta_0 \ll 1. \tag{2.45}$$

Then, taking two constants $M, \sigma > 0$ which satisfy:

$$M > \max\left(\frac{\|D\|^{1/2}_{\mathcal{C}^0(\Omega_b)}}{r}, \|\nabla^2 v\|_{\mathcal{C}^0(\Omega_b)}, \|\nabla^2 w\|_{\mathcal{C}^0(\Omega_b)}, 1\right) \quad \text{and} \quad \sigma > 1, \tag{2.46}$$

there exist new functions $\tilde{v} \in \mathcal{C}^2(\bar{\Omega})$ and $\tilde{w} \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ such that:

$$\begin{aligned}
&\|\tilde{D}\|_{\mathcal{C}^0(\Omega)} = \left\|A - \left(\frac{1}{2}\nabla \tilde{v} \otimes \nabla \tilde{v} + \mathrm{Sym}\nabla \tilde{w}\right)\right\|_{\mathcal{C}^0(\Omega)} \\
&\le \frac{\|D\|^{\beta/2}_{\mathcal{C}^0(\Omega_b)}}{M^\beta}\|A\|_{\mathcal{C}^{0,\beta}(\Omega_b)} + 2.7 \cdot 10^{15}\frac{1}{\sigma}\|D\|_{\mathcal{C}^0(\Omega_b)} = \frac{\|D\|^{\beta/2}_{\mathcal{C}^0(\Omega_b)}}{M^\beta}\|A\|_{\mathcal{C}^{0,\beta}(\Omega_b)} + K\frac{1}{\sigma}\|D\|_{\mathcal{C}^0(\Omega_b)},
\end{aligned} \tag{2.47}$$

$$
\begin{aligned}
\|\tilde{v} - v\|_{\mathcal{C}^0(\Omega)} &\leq 1.4 \cdot 10^7 \frac{\|D\|_{\mathcal{C}^0(\Omega_b)}}{M}. \\
\|\tilde{w} - w\|_{\mathcal{C}^0(\Omega)} &\leq 1.4 \cdot 10^7 \frac{\|D\|_{\mathcal{C}^0(\Omega_b)}}{M}(1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_b)}) + 12.6\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} diam(\Omega_b).
\end{aligned} \tag{2.48}
$$

$$
\begin{aligned}
\|\nabla(\tilde{v} - v)\|_{\mathcal{C}^0(\Omega)} &\leq 1.1 \cdot 10^8 \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} = C_1 \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla(\tilde{w} - w)\|_{\mathcal{C}^0(\Omega)} &\leq 1.1 \cdot 10^8 (1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_b)}) \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} = C_2 (1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_b)}) \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)},
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
\|\nabla^2 \tilde{v}\|_{\mathcal{C}^0(\Omega)} &\leq 7.3 \cdot 10^8 M \sigma^3 = C_3 M \sigma^3, \\
\|\nabla^2 \tilde{w}\|_{\mathcal{C}^0(\Omega)} &\leq 9.5 \cdot 10^8 (1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_b)}) M \sigma^3 = C_4 (1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_b)}) M \sigma^3.
\end{aligned} \tag{2.50}
$$

Proof. The proof will proceed in three stages. We will begin with a mollification to control higher derivatives, here the domain will be restricted from Ω_b to Ω. Next we modify the w component of the defect to ensure a positive decomposition of the defect into the desired basis. Finally three consecutive steps of convex integration will be used to reduce the defect.

1. Mollification. Take φ as in (2.1) to be the standard mollifier. We may define the following functions on the domain Ω:

$$
\mathfrak{v} = v * \varphi_l, \quad \mathfrak{w} = w * \varphi_l, \quad \mathfrak{A} = A * \varphi_l, \quad \text{with} \quad l = \frac{\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}}{M} < r < 1.
$$

Finally we define $\mathfrak{D} = \mathfrak{A} - \left(\frac{1}{2}\nabla\mathfrak{v} \otimes \nabla\mathfrak{v} + \mathrm{Sym}\nabla\mathfrak{w}\right)$. We use Lemma 2.1.4 to obtain the following uniform error bounds.

$$
\begin{aligned}
\|\mathfrak{v} - v\|_{\mathcal{C}^0(\Omega)}, \|\mathfrak{w} - w\|_{\mathcal{C}^0(\Omega)} &\leq \frac{l}{2}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla(\mathfrak{v} - v)\|_{\mathcal{C}^0(\Omega)}, \|\nabla(\mathfrak{w} - w)\|_{\mathcal{C}^0(\Omega)} &\leq 1.6\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}, \\
\|\mathfrak{A} - A\|_{\mathcal{C}^0(\Omega)} &\leq l^\beta \|A\|_{\mathcal{C}^{0,\beta}(\Omega_b)}, \\
\|\nabla^m \mathfrak{D}\|_{\mathcal{C}^0(\Omega)} &\leq \|\nabla^m D * \varphi_l\|_{\mathcal{C}^0(\Omega)} + \frac{1}{2}\|\nabla^m \big((\nabla v * \varphi_l) \otimes (\nabla v * \varphi_l) - (\nabla v \otimes \nabla v) * \varphi_l\big)\|_{\mathcal{C}^0(\Omega)}.
\end{aligned} \tag{2.51}
$$

The last inequality must be further developed for each value of m that will be needed in the proof:

$$
\begin{aligned}
\|\mathfrak{D}\|_{\mathcal{C}^0(\Omega)} &\leq \|D\|_{\mathcal{C}^0(\Omega_b)} + l^2 \|\nabla^2 v\|^2_{\mathcal{C}^0(\Omega_b)} \leq 2\|D\|_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla \mathfrak{D}\|_{\mathcal{C}^0(\Omega)} &\leq \frac{3.1}{l}\|D\|_{\mathcal{C}^0(\Omega_b)} + 4.7l\|\nabla^2 v\|^2_{\mathcal{C}^0(\Omega_b)} \leq 7.8\frac{1}{l}\|D\|_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla^2 \mathfrak{D}\|_{\mathcal{C}^0(\Omega)} &\leq \frac{15.9}{l^2}\|D\|_{\mathcal{C}^0(\Omega_b)} + 33.5\|\nabla^2 v\|^2_{\mathcal{C}^0(\Omega_b)} \leq 49.4\frac{1}{l^2}\|D\|_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla^3 \mathfrak{D}\|_{\mathcal{C}^0(\Omega)} &\leq \frac{210}{l^3}\|D\|_{\mathcal{C}^0(\Omega_b)} + 462.9\frac{1}{l}\|\nabla^2 v\|^2_{\mathcal{C}^0(\Omega_b)} \leq 672.9\frac{1}{l^3}\|D\|_{\mathcal{C}^0(\Omega_b)}.
\end{aligned} \tag{2.52}
$$

The following approximations are used to bound the $\mathcal{C}^3$ norm of $\mathfrak{v}$ by the $\mathcal{C}^2$ norm of v, using (2.2) and the definition of M and l:

$$\begin{aligned}
&\|\nabla^2\mathfrak{v}\|_{\mathcal{C}^0(\Omega)} \leq \|\nabla^2 v\|_{\mathcal{C}^0(\Omega_b)} \leq \frac{1}{l}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}, \\
&\|\nabla^3\mathfrak{v}\|_{\mathcal{C}^0(\Omega)} \leq \frac{1}{l}\|\nabla\varphi\|_{L^1(\mathbb{R}^2)}\|\nabla^2 v\|_{\mathcal{C}^0(\Omega_b)} \leq 3.1\frac{1}{l^2}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}.
\end{aligned} \tag{2.53}$$

Finally we may observe one last simple inequality stemming from the Lemma 2.1.4:

$$\|\nabla^2\mathfrak{w}\|_{\mathcal{C}^0(\Omega)} \leq \|\nabla^2 w\|_{\mathcal{C}^0(\Omega_b)} \leq M. \tag{2.54}$$

2. Modification and decomposition. The last manipulation that needs to be done before we apply convex integration steps is to ensure that the deficit may be decomposed positively in the desired basis. We use the last point of Lemma 2.1.8 to define:

$$\mathfrak{w}' = \mathfrak{w} - (\|D\|_{\mathcal{C}^0(\Omega_b)} + \|\mathfrak{D}\|_{\mathcal{C}^0(\Omega)}) \begin{bmatrix} \frac{\sqrt{2}+9}{4}x \\ (\sqrt{2}+\frac{9}{5})y \end{bmatrix}, \quad \mathfrak{D}' = \mathfrak{A} - \left(\frac{1}{2}\nabla\mathfrak{v}\otimes\nabla\mathfrak{v} + \mathrm{Sym}\nabla\mathfrak{w}'\right). \tag{2.55}$$

From this definition and using (2.51) we obtain:

$$\begin{aligned}
\|\nabla\mathfrak{w}' - \nabla\mathfrak{w}\|_{\mathcal{C}^0(\Omega)} &< 4.2\cdot(\|D\|_{\mathcal{C}^0(\Omega_b)} + \|\mathfrak{D}\|_{\mathcal{C}^0(\Omega)}) \leq 12.6\|D\|_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla^2\mathfrak{w}' - \nabla^2\mathfrak{w}\|_{\mathcal{C}^0(\Omega)} &= 0.
\end{aligned} \tag{2.56}$$

Note that $\mathfrak{D}' - \mathfrak{D}$ is a constant matrix and therefore we have that $\|\nabla^m(\mathfrak{D}' - \mathfrak{D})\|_{\mathcal{C}^0(\Omega)} = 0$ for any value of $m \geq 1$. Furthermore this construction guarantees that when we decompose $\mathfrak{D}' = \sum_{k=1}^3 \phi_k\eta_k\otimes\eta_k$ the we have that $\phi_k > (\|D\|_{\mathcal{C}^0(\Omega_b)} + \|\mathfrak{D}\|_{\mathcal{C}^0(\Omega)})/2$. We want to find bounds on the norms of the $a_k = \sqrt{\phi_k}$. Next we proceed to estimating derivatives of a_k. Using the product rule we obtain:

$$\begin{aligned}
\nabla a_k &= \frac{\nabla\phi_k}{2a_k}, \\
\nabla^2 a_k &= \frac{\nabla^2\phi_k}{2a_k} - \frac{\nabla a_k\otimes\nabla a_k}{a_k}, \\
\nabla^3 a_k &= \frac{\nabla^3\phi_k}{2a_k} - \frac{\nabla^2\phi_k\otimes\nabla a_k}{2a_k^2} + \frac{\nabla a_k\otimes\nabla a_k\otimes\nabla a_k}{a_k^2} - \frac{2\mathrm{Sym}(\nabla^2 a_k\otimes\nabla a_k)}{a_k} \\
&= \frac{\nabla^3\phi_k}{2a_k} - \frac{\nabla^2 a_k\otimes\nabla a_k}{a_k} - \frac{2\mathrm{Sym}(\nabla^2 a_k\otimes\nabla a_k)}{a_k}.
\end{aligned}$$

The last equality was obtained by noting that:

$$\nabla^2\phi_k = \nabla^2(a_k^2) = 2\nabla(a_k\nabla a_k) = 2(\nabla a\otimes\nabla a + a\nabla^2 a_k).$$

We may now calculate the desired estimates using the following useful inequalities:

$$a_k > \frac{\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}}{\sqrt{2}} \quad \text{and} \quad \|\nabla^m \phi_k\|_{\mathcal{C}^0(\Omega)} \le \frac{5\sqrt{3}}{8}\|\nabla^m \mathfrak{D}'\|_{\mathcal{C}^0(\Omega)}.$$

$$\begin{aligned}
\|\nabla a_k\|_0 &\le \frac{\|\nabla \phi_k\|_{\mathcal{C}^0(\Omega)}}{2\min_{x\in\Omega} a_k(x)\|_0} \le \frac{5\sqrt{3}}{8\sqrt{2}}\frac{\|\nabla \mathfrak{D}'\|_{\mathcal{C}^0(\Omega)}}{\|D\|^{1/2}_{\mathcal{C}^0(\Omega_b)}} \le 6\frac{1}{l}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)},\\
\|\nabla^2 a_k\|_0 &\le \frac{5\sqrt{3}}{8\sqrt{2}}\frac{\|\nabla^2 \mathfrak{D}'\|_{\mathcal{C}^0(\Omega)}}{\|D\|^{1/2}_{\mathcal{C}^0(\Omega_b)}} + \frac{36\sqrt{2}\|D\|_{\mathcal{C}^0(\Omega_b)}}{l^2\|D\|^{1/2}_{\mathcal{C}^0(\Omega_b)}} \le 88.8\frac{1}{l^2}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)},\\
\|\nabla^3 a_k\|_0 &\le \frac{5\sqrt{3}}{8\sqrt{2}}\frac{\|\nabla^3 \mathfrak{D}'\|_{\mathcal{C}^0(\Omega)}}{\|D\|^{1/2}_{\mathcal{C}^0(\Omega_b)}} + 18\sqrt{2}\frac{88.8}{l^3}\|D\|^{1/2}_{\mathcal{C}^0(\Omega_b)} \le 2775.6\frac{1}{l^3}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}.
\end{aligned}$$

We make note that the last inequality will be the main contributor to the necessity of a small defect.

3. Iteration of convex integration. We define $v_0 = \mathfrak{v}$ and $w_0 = \mathfrak{w}'$ and define recursively $v_k \in \mathcal{C}^3(\bar{\Omega})$, $w_k \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ for $k = 1, 2, 3$. To obtain v_k and w_k we apply Proposition 2.3.3 to v_{k-1} and w_{k-1}. We apply the Proposition with a_k from the decomposition of $\mathfrak{D}'$ into the basis defined by the usual η_k. Lastly, we need to define the parameters:

$$l_k = \frac{l}{\sigma^{k-1}} < 1, \quad \lambda_k = \frac{1}{l_{k+1}} > \frac{1}{l_k},$$

and the nondecreasing parameters δ_k with the choice:

$$\delta_1 = \max_{m=1,2}\left\{l^m\|\nabla^{m+1}\mathfrak{v}\|_{\mathcal{C}^0(\Omega)}\right\} + \max_{m=0\ldots 3, k=1\ldots 3}\left\{l^m\|\nabla^m a_k\|_{\mathcal{C}^0(\Omega)}\right\}. \tag{2.57}$$

We will end the construction by setting $\tilde{v} = v_3$ and $\tilde{w} = w_3$, and claiming that these satisfy the necessary error bounds. First we must check that the assumptions of the Proposition hold at every step. The condition that $l_k \in (0, 1)$ is easily verified as $l < 1$ and $\sigma > 1$. Next we observe that $l^m\|\nabla\mathfrak{v}\|_m \le 3.1\|D\|_0^{\frac{1}{2}}$ and $l^m\|a_k\|_m \le 2775\|D\|_0^{\frac{1}{2}}$, from which we obtain that $\delta_1 \le 2779\|D\|_0^{\frac{1}{2}} < 1$ if δ_0 was taken small enough. The first condition in (2.20) is clearly satisfied for all k by the definition (2.57) given that $l > l_k$. Furthermore, by induction on k and using (2.22), we obtain:

$$\begin{aligned}
\|\nabla^{m+1} v_k\|_{\mathcal{C}^0(\Omega)} &\le \|\nabla^{m+1} v_{k-1}\|_{\mathcal{C}^0(\Omega)} + \|\nabla^{m+1}(v_k - v_{k-1})\|_{\mathcal{C}^0(\Omega)} \le \frac{\delta_k}{l_k^m} + 2.4^{m+1}\pi^m\delta_k\lambda_k^m\\
&\le \delta_k\frac{1 + 2.4^{m+1}\pi^m}{l_{k+1}^m} \le \frac{\delta_{k+1}}{l_{k+1}^m} \quad \forall m, k = 1, 2.
\end{aligned}$$

The last inequality is obtained with the appropriate choice of $\delta_{k+1} = (124)\delta_k$ for each k. This gives us the last condition that $\delta_0 < (2780(124)^2)^{-2} < 5.4 \cdot 10^{-16}$ must hold to ensure that each $\delta_k < 1$.

To prove the estimates in (2.49) and (2.50) we first note the following useful inequalities:

$$\begin{aligned}
&\lambda_1 = \frac{\sigma}{l}, \quad \lambda_1 = \frac{\sigma^2}{l}, \quad \lambda_3 = \frac{\sigma^3}{l},\\
&\delta_1 \leq 2778.7\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)},\\
&\delta_2 \leq 2778.7(124)\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)},\\
&\delta_3 \leq 2778.7(124)^2\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}.
\end{aligned}$$

Using these estimates for the δ_k and λ_k we first calculate some useful bounds:

$$\begin{aligned}
1 + \|\nabla v_0\|_0 &\leq 1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)} + \|\nabla v - \nabla v_0\|_{\mathcal{C}^0(\Omega)}\\
&\leq 1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)} + \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} \leq (1 + 2.32 \cdot 10^{-8})(1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)}),\\
1 + \|\nabla v_1\|_{\mathcal{C}^0(\Omega)} &\leq 1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)} + \|\nabla v - \nabla v_0\|_{\mathcal{C}^0(\Omega)} + \|\nabla v_0 - \nabla v_1\|_{\mathcal{C}^0(\Omega)}\\
&\leq 1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)} + \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + 2.4\delta_1 \leq (1 + 6.5 \cdot 10^{-5})(1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)}),\\
1 + \|\nabla v_2\|_{\mathcal{C}^0(\Omega)} &\leq 1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)} + \|\nabla v - \nabla v_0\|_{\mathcal{C}^0(\Omega)} + \|\nabla v_0 - \nabla v_1\|_{\mathcal{C}^0(\Omega)} + \|\nabla v_1 - \nabla v_2\|_{\mathcal{C}^0(\Omega)}\\
&\leq 1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)} + \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + 2.4\delta_1 + 2.4\delta_2 \leq (1 + 8.1 \cdot 10^{-3})(1 + \|\nabla v\|_{\mathcal{C}^0(\Omega)}.
\end{aligned}$$

Using these we may next evaluate:

$$\begin{aligned}
\|\tilde{v}-v\|_{\mathcal{C}^0(\Omega)} &\leq \|\mathfrak{v}-v\|_{\mathcal{C}^0(\Omega)} + \sum_{i=1}^{3}\|v_i - v_{i-1}\|_{\mathcal{C}^0(\Omega)} \leq \frac{l}{2}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + \frac{1}{\pi}\sum_{i=1}^{3}\frac{\delta_k}{\lambda_k} \\
&\leq 13710444.1 l \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} \leq \frac{13710444.1}{M}\|D\|_{\mathcal{C}^0(\Omega_b)}, \\
\|\tilde{w}-w\|_{\mathcal{C}^0(\Omega)} &\leq \|\mathfrak{w}-w\|_{\mathcal{C}^0(\Omega)} + \|\mathfrak{w}-\mathfrak{w}'\|_{\mathcal{C}^0(\Omega)} + \sum_{i=1}^{3}\|w_i - w_{i-1}\|_{\mathcal{C}^0(\Omega)} \\
&\leq \frac{l}{2}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + 12.6\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} diam(\Omega_b) + \frac{1}{\pi}\sum_{i=1}^{3}\frac{\delta_k}{\lambda_k}(1+\|\nabla v_{k-1}\|_{\mathcal{C}^0(\Omega)}) \\
&\leq 13820610.3 l\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}(1+\|v\|_{\mathcal{C}^0(\Omega_b)}) + 12.6\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} diam(\Omega_b) \\
&\leq \frac{13820610.3}{M}\|D\|_{\mathcal{C}^0(\Omega_b)}(1+\|v\|_{\mathcal{C}^0(\Omega_b)}) + 12.6\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} diam(\Omega_b), \\
\|\nabla(\tilde{v}-v)\|_{\mathcal{C}^0(\Omega)} &\leq |\nabla(\mathfrak{v}-v)\|_{\mathcal{C}^0(\Omega)} + \sum_{i=1}^{3}\|\nabla(v_i - v_{i-1})\|_{\mathcal{C}^0(\Omega)} \\
&\leq \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + \sum_{i=1}^{3} 2.4\delta_k \leq 103374310.3\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}, \\
\|\nabla(\tilde{w}-w)\|_{\mathcal{C}^0(\Omega)} &\leq \|\nabla(w-\mathfrak{w})\|_{\mathcal{C}^0(\Omega)} + \|\nabla(\mathfrak{w}-\mathfrak{w}')\|_{\mathcal{C}^0(\Omega)} + \sum_{i=1}^{3}\|\nabla(w_i - w_{i-1})\|_{\mathcal{C}^0(\Omega)} \\
&\leq \|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + 12.6\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + \sum_{i=1}^{3} 2.4\delta_k(1+\|\nabla v_{k-1}\|_{\mathcal{C}^0(\Omega)}) \\
&\leq 104204955.9\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)}(1+\|v\|_{\mathcal{C}^0(\Omega_b)}), \\
\|\nabla^2\tilde{v}\|_{\mathcal{C}^0(\Omega)} &\leq \|\nabla^2\mathfrak{v}\|_{\mathcal{C}^0(\Omega)} + \sum_{i=1}^{3}\|\nabla^2(v_i - v_{i-1})\|_{\mathcal{C}^0(\Omega)} \\
&\leq \frac{1}{l}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + 16.9\sum_{i=1}^{3}\delta_k\lambda_k \\
&\leq 3M + 16.9\sum_{i=1}^{3} 2778.7(124)^{k-1}M\sigma^k \leq 727927426.1 M\sigma^3, \\
\|\nabla^2\tilde{w}\|_{\mathcal{C}^0(\Omega)} &\leq \|\nabla^2 w\|_{\mathcal{C}^0(\Omega)} + \|\nabla^2(w-\mathfrak{w})\|_{\mathcal{C}^0(\Omega)} + \sum_{i=1}^{3}\|\nabla^2(w_i - w_{i-1})\|_{\mathcal{C}^0(\Omega)} \\
&\leq M + \frac{2}{l}\|D\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_b)} + 21.9\sum_{i=1}^{3}\delta_k\lambda_k(1+\|\nabla v_{k-1}\|_{\mathcal{C}^0(\Omega)}) \\
\leq 3M + 21.9(1+\|v\|_{\mathcal{C}^0(\Omega_b)})\big(&\delta_1\lambda_1(1+2.32\cdot 10^{-8}) + \delta_2\lambda_2(1+6.5\cdot 10^{-5}) + \delta_3\lambda_3(1+8.1\cdot 10^{-3})\big) \\
&\leq 3M\sigma^3 + 950870098.5(1+\|v\|_{\mathcal{C}^0(\Omega_b)})M\sigma^3 \\
&\leq 950870101.5 M\sigma^3(1+\|v\|_{\mathcal{C}^0(\Omega_b)})
\end{aligned}$$

We may finally compute the constant in (2.47). The calculation starts by noting:

$$\begin{aligned}\tilde{D} &= A - \mathfrak{A} + \mathfrak{D}' + \big(\frac{1}{2}\nabla v_0 \otimes \nabla v_0 + \mathrm{Sym}\nabla w_0\big) - \big(\frac{1}{2}\nabla v_3 \otimes \nabla v_3 + \mathrm{Sym}\nabla w_3\big) \\ &= A - \mathfrak{A} - \sum_{k=1}^{3}\Big(\big(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{Sym}\nabla w_k\big) - \big(\frac{1}{2}\nabla v_{k-1} \otimes \nabla v_{k-1} + \mathrm{Sym}\nabla w_{k-1} + a_k\eta_k \otimes \eta_k\big)\Big).\end{aligned}$$

Which gives:

$$\begin{aligned}\|\tilde{D}\|_{\mathcal{C}^0(\Omega)} &\leq \|A - \mathfrak{A}\|_{\mathcal{C}^0(\Omega)} + \sum_{k=1}^{3}\frac{3}{\pi}\cdot\frac{\delta_k^2}{\lambda_k l_k} \\ &\leq l^\beta\|A\|_{\mathcal{C}^{0,\beta}(\Omega_b)} + \frac{3}{\pi}\frac{1}{\sigma}\|D\|_{\mathcal{C}^0(\Omega_b)}(2780^2(1+123^2+123^4)) \\ &\leq \frac{\|D\|^{\beta/2}_{\mathcal{C}^0(\Omega_b)}}{M^\beta}\|A\|_{\mathcal{C}^{0,\beta}(\Omega_b)} + 1.8\cdot 10^{15}\frac{1}{\sigma}\|D\|_{\mathcal{C}^0(\Omega_b)}.\end{aligned}$$

Proof of Theorem 2.4.4. The desired exact solution will be obtained by constructing a sequence of approximations by the recursive application of stages of convex integration. These sequences will be shown to converge in the $\mathcal{C}^{1,\alpha}$ sense to the desired solution. The proof is divided in four sections: in the first the recursive sequence will be defined. In the second we show that the base case of the sequence satisfies the necessary properties. In the third we justify the recursive step. In the fourth we will show that the defect of this sequence converges to zero and that the sequence actually converges in $\mathcal{C}^{1,\alpha}$.

1. The recursive construction. Consider the sequence of sets defined recursively by $\Omega_0 = \Omega_{fat}$ with $\Omega_k = \Omega_{k-1} \setminus \big(\Omega_{k-1}^c + B_{\frac{\delta_0}{2^k}}(0)\big)$. Clearly $\Omega \subset \Omega_k$ for all values of k and thus Ω will lie inside the intersection of the family Ω_k. Set $v_0 = v$ and $w_0 = w$ defined on the set Ω_0. Given $v_k \in \mathcal{C}^2(\bar{\Omega}_k)$, $w_k \in \mathcal{C}^2(\bar{\Omega}_k, \mathbb{R}^2)$ and the restriction of A onto Ω_k, define $v_{k+1} \in \mathcal{C}^2(\bar{\Omega}_{k+1})$ and $w_{k+1} \in \mathcal{C}^2(\bar{\Omega}_{k+1}, \mathbb{R}^2)$ by applying Proposition 2.4.6 taking $\Omega_b = \Omega_k$, $\Omega = \Omega_{k+1}$. We need to define the constants σ_k and M_k as in the Proposition. To do so we must define the constant $s \in (0,1)$, which must satisfy the two following inequalities:

$$s < \frac{6\beta}{2-\beta}, \quad s > \frac{6\alpha}{1-\alpha} \quad \Longrightarrow \quad \alpha(6+s) - s < 0. \tag{2.58}$$

The last inequality will be crucial in the proof of convergence.

We choose the values of σ_k in such a way that they form an increasing sequence converging to a value σ_{max}.

$$\sigma_0^s \geq \frac{16}{9}, \quad \sigma_{max}^{1-s} \geq 3.6\cdot 10^{18} = 2K, \quad \sigma_{max}^{s-\alpha(6+s)} > C_5^\alpha, \tag{2.59}$$

where:

$$C_5 = 2.1\cdot 10^8(1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_0)}). \tag{2.60}$$

We will prove that the sequence $\sigma_k = \sigma_{max}$ for all $k \geq 0$ will guarantee that the inductive hypothesis are satisfied. We however note that lower values of σ_k may be chosen if the sequence of functions they generate still satisfy the inductive hypothesis at every stage. We do however have to guarantee that $\sigma_k \to \sigma_{max}$ to ensure convergence of the sequence of functions. Next define:

$$M_k = M_0 C_5^k \prod_{j=1}^{k} \sigma_j^3, \text{ with } M_0 = \begin{cases} N 2^{\frac{1}{\beta}} \sigma_{max}^{\frac{s}{\beta}} \|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)}^{\frac{1}{\beta}} \|D\|_{\mathcal{C}^0(\Omega_0)}^{\frac{1}{2}-\frac{1}{\beta}} & \text{if } \|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} \neq 0, \\ N & \text{if } \|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} \neq 0 \end{cases} \tag{2.61}$$

Here we take $N \geq 1$ to be some arbitrarily large constant.

We define the intermediate defects on the appropriate Ω_k as:

$$D_k = A - \big(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{Sym}\nabla w_k\big).$$

In the next sections we will prove by induction the following inequalities:

$$M_k > \max\left(\frac{2^{k+1}\|D_k\|_{\mathcal{C}^0(\Omega_k)}^{1/2}}{\delta_0}, \|\nabla^2 v_k\|_{\mathcal{C}^0(\Omega_k)}, \|\nabla^2 w_k\|_{\mathcal{C}^0(\Omega_k)}, 1\right), \quad \forall k \geq 0 \tag{2.62}$$

The following will be shown for all $k \geq 1$:

$$\|D_k\|_{\mathcal{C}^0(\Omega_k)} \leq \frac{\|D_0\|_{\mathcal{C}^0(\Omega_0)}}{\prod_{j=0}^{k-1} \sigma_j^s}. \tag{2.63}$$

$$\begin{aligned} \|v_k - v_{k-1}\|_{\mathcal{C}^0(\Omega_k)} &\leq 1.4 \cdot 10^7 \frac{\|D_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}}{M_{k-1}}, \\ \|w_k - w_{k-1}\|_{\mathcal{C}^0(\Omega_k)} &\leq 1.4 \cdot 10^7 \frac{\|D_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}}{M_{k-1}}(1 + \|\nabla v_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}) \\ &\quad + 12.6\|D_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}^{\frac{1}{2}} diam(\Omega_{k-1}). \end{aligned} \tag{2.64}$$

$$\begin{aligned} &\|\nabla(v_k - v_{k-1})\|_{\mathcal{C}^0(\Omega_k)} \leq C_1 \|D_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}^{\frac{1}{2}}, \\ &\qquad \|\nabla(w_k - w_{k-1})\|_{\mathcal{C}^0(\Omega_k)} \leq C_2(1 + \|\nabla v_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})})\|D_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}^{\frac{1}{2}}, \end{aligned} \tag{2.65}$$

$$C_5 = 2.1 \cdot 10^8 (1 + \|\nabla v\|_{\mathcal{C}^0(\Omega_0)}) \geq \max\Big\{C_3, C_4(1 + \|\nabla v_k\|_{\mathcal{C}^0(\Omega_0)})\Big\} \quad \forall k \geq 0. \tag{2.66}$$

$$\begin{aligned} \|\nabla^2 v_k\|_{\mathcal{C}^0(\Omega_k)} &\leq C_3 M_{k-1}\sigma_{k-1}^3 \leq C_5 M_{k-1}\sigma_{k-1}^3 = M_k, \\ \|\nabla^2 w_k\|_{\mathcal{C}^0(\Omega_k)} &\leq C_4(1 + \|\nabla v_{k-1}\|_{\mathcal{C}^0(\Omega_{k-1})}) M_{k-1}\sigma_{k-1}^3 \leq C_5 M_{k-1}\sigma_{k-1}^3 = M_k. \end{aligned} \tag{2.67}$$

Note that these conditions are not enough to guarantee the hypothesis of Proposition 2.4.6 at every stage. The condition $\|D_k\|_{\mathcal{C}^0(\Omega_k)} > 0$ still needs to be considered. We will simply assume that it holds at every stage as if it does not we have $v_k \in \mathcal{C}^2(\bar{\Omega})$ and $w_k \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ which satisfy:

$$A = \frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{Sym}\nabla w_k.$$

This implies that the Theorem is proven and the recursive construction need not be continued.

2. The base case. To guarantee the $k = 0$ case of (2.62) it suffices to take N large enough. To prove (2.63) we note that we may apply Proposition 2.4.6 to v_0, w_0 and A as by definition they satisfy the Proposition's hypothesis. Furthermore M_0 has also been shown to satisfy the hypothesis. It remains to show that there exists $\sigma_0 \leq \sigma_{max}$ which guarantees the inequality. We have:

$$\sigma_0^s \frac{\|D_1\|_{\mathcal{C}^0(\Omega_1)}}{\|D_0\|_{\mathcal{C}^0(\Omega_0)}} \leq \sigma_0^s \frac{\|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} \|D_0\|^{\beta/2-1}_{\mathcal{C}^0(\Omega_0)}}{M_0^\beta} + K\frac{1}{\sigma_0^{1-s}}.$$

We ask that each of the terms be less than $\frac{1}{2}$. By taking $\sigma_1 = \sigma_{max}$ the second term is verified. For the first term we note that if $\|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} = 0$ it is zero, otherwise:

$$\sigma_0^s \frac{\|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} \|D_0\|^{\beta/2-1}_{\mathcal{C}^0(\Omega_0)}}{M_0^\beta} \leq \frac{1}{2N}\frac{\sigma_0^s}{\sigma_{max}^s} \leq \frac{1}{2}.$$

Next we note that (2.64), (2.65) and the first inequalities in (2.67) also follow from the application of the Proposition. The second inequalities of (2.67) follow from (2.66) with $k = 0$ which is satisfied as $2.1 \cdot 10^8 > C_3, C_4$. The last equalities follow from the definition of M_k.

3. The inductive step. Assume that (2.66), (2.62) and (2.63) hold for a value k. Then we have that:

$$\|\tilde{D}_{k+1}\|_{\mathcal{C}^0(\Omega_{k+1})} \leq \frac{\|D_k\|^{\beta/2}_{\mathcal{C}^0(\Omega_k)}}{M_k^\beta}\|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} + K\frac{1}{\sigma_k}\|D_k\|_{\mathcal{C}^0(\Omega_k)},$$

And thus using the inductive hypothesis we have that:

$$\begin{aligned}\|D_{k+1}\|_{\mathcal{C}^0(\Omega_{k+1})} \leq \frac{\|D_k\|^{\beta/2}_{\mathcal{C}^0(\Omega_k)}}{M_k^\beta}\|A\|_{\mathcal{C}^{0,\beta}(\Omega_0)} + K\frac{1}{\sigma_k}\|D_k\|_{\mathcal{C}^0(\Omega_k)} \\ \leq \frac{1}{C_5^{k\beta}}\frac{1}{\prod_{j=1}^{k-1}\sigma_j^{s\frac{\beta}{2}+3\beta}}\frac{1}{2N}\frac{1}{\sigma_{max}^s}\|D_0\|_{\mathcal{C}^0(\Omega_0)} + \frac{K}{\sigma_k}\frac{\|D_0\|_{\mathcal{C}^0(\Omega_0)}}{\prod_{j=1}^{k-1}\sigma_j^s}.\end{aligned}$$

From this we evaluate:

$$\sigma_k^s \prod_{j=1}^{k-1}\sigma_j^s \frac{\|D_{k+1}\|_{\mathcal{C}^0(\Omega_{k+1})}}{\|D_0\|_{\mathcal{C}^0(\Omega_0)}} \leq \frac{1}{C_5^{k\beta}}\frac{1}{\prod_{j=1}^{k-1}\sigma_j^{s\frac{\beta}{2}+3\beta-s}}\frac{1}{2N}\frac{\sigma_k^s}{\sigma_{max}^s} + \frac{K}{\sigma_k^{1-s}} \leq 1.$$

Similarly to the base case, this holds for $\sigma_k = \sigma_{max}$ although lower values of σ might also satisfy the hypothesis.

Next we consider:

$$M_k > \frac{2^{k+1}\|D_k\|^{1/2}_{\mathcal{C}^0(\Omega_k)}}{\delta_0} \quad \Longrightarrow \quad M_{k+1} > C_5\sigma_k^3 \frac{2^{k+1}\|D_k\|^{1/2}_{\mathcal{C}^0(\Omega_k)}}{\delta_0} > \frac{2^{k+2}\|D_k\|^{1/2}_{\mathcal{C}^0(\Omega_k)}}{\delta_0}.$$

Thus from the construction of Proposition 2.4.6 we obtain (2.64), (2.65) and the first inequalities of (2.67). We may also compute:

$$1 + \|\nabla v_k\|_{\mathcal{C}^0(\Omega_k)} \le 1 + \|\nabla v_0\|_{\mathcal{C}^0(\Omega_0)} + C_1\delta_0^{\frac{1}{2}} \sum_{j=0}^{k} \prod_{i=1}^{j} \frac{1}{\sigma_i^{\frac{s}{2}}} \le 1 + \|\nabla v_0\|_{\mathcal{C}^0(\Omega_0)} + C_1\delta_0^{\frac{1}{2}} \sum_{j=0}^{k} \left(\frac{3}{4}\right)^j$$

$$\le 1 + \|\nabla v_0\|_{\mathcal{C}^0(\Omega_0)} + C_1\delta_0^{\frac{1}{2}} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \le \big(1 + 2.2C_1\big)\big(1 + \|\nabla v_0\|_{\mathcal{C}^0(\Omega_0)}\big) \le 2.1 \cdot 10^8\big(1 + \|\nabla v_0\|_{\mathcal{C}^0(\Omega_0)}\big).$$

This concludes the proof of (2.67) and (2.66).

3. Convergence. All that is left to see is that the sequence of functions actually converges to a solution in the desired space.

We need to show that the sequences $\{v_k\}$ and $\{w_k\}$ are Cauchy in $\mathcal{C}^{1,\alpha}(\Omega)$. We begin by showing that the sequences converge in $\mathcal{C}^0$. Using (2.64) we write:

$$\|v - \bar{v}\|_0 \le 13710445 \sum_{n=0}^{\infty} \frac{\|D_{n-1}\|_{\mathcal{C}^0(\Omega_{n-1})}}{M_n} \le 13710444.1\delta_0 \sum_{n=0}^{\infty} \mathfrak{C}_5^{-n} \le 0.21\delta_0.$$

To show convergence of w_k, ∇v_k and ∇w_k we only need to show summability of $\|D_k\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_k)}$:

$$\sum_{i=0}^{\infty} \|D_i\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_i)} \le \delta_0^{\frac{1}{2}} \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{1}{\sigma_j^{\frac{s}{2}}} \le \delta_0^{\frac{1}{2}} \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i \le 4\delta_0.$$

Finally we must compute $\|\nabla(v_{k+1} - v_k)\|_{\mathcal{C}^{0,\alpha}(\Omega_{k+1})}$ and $\|\nabla(w_{k+1} - w_k)\|_{\mathcal{C}^{0,\alpha}(\Omega_{k+1})}$. We note that being supremum norms the norm taken over a smaller set will be smaller than the norm taken on a larger one. If we can prove that these sequences of norms are summable we will have the desired convergence. The fact that the limit is the desired solution is given by the fact that the defect converges to zero thanks to (2.63).

We will use Lemma 2.1.2 to evaluate the norms:

$$\begin{aligned}
\|\nabla(v_{k+1} - v_k)\|_{\mathcal{C}^{0,\alpha}(\Omega_{k+1})} &\le 2\Big(C_1\|D_k\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_k)}\Big)^{1-\alpha}\Big(C_3 M_k \sigma_k^3\Big)^{\alpha} \\
&\le 2C_1^{1-\alpha}C_3^{\alpha}C_5^{k\alpha}\|D_0\|^{\frac{1-\alpha}{2}}_{\mathcal{C}^0(\Omega_0)} \left(\frac{2}{\prod_{j=1}^{k}\sigma_j^{\frac{s-\alpha s}{2}}}\sigma_k^{3\alpha}\prod_{j=1}^{k}\sigma_j^{3\alpha}M_0^{\alpha}\right) \\
&\le 2\mathfrak{C}_{\alpha}\prod_{j=1}^{k}\sigma_j^{\frac{6\alpha+\alpha s-s}{2}}C_5^{k\alpha},
\end{aligned}$$

and:

$$
\begin{aligned}
\|\nabla(w_{k+1}-w_k)\|_{\mathcal{C}^{0,\alpha}(\Omega_{k+1})} &\le 2(1+\|\nabla v_k\|_{\mathcal{C}^0(\Omega_k)})\Big(C_2\|D_k\|^{\frac{1}{2}}_{\mathcal{C}^0(\Omega_k)}\Big)^{1-\alpha}\Big(C_4 M_k \sigma_k^3\Big)^{\alpha}\\
&\le 2(1+\|\nabla v_k\|_{\mathcal{C}^0(\Omega_k)}) C_2^{1-\alpha} C_4^{\alpha} C_5^{k\alpha}\|D_0\|^{\frac{1-\alpha}{2}}_{\mathcal{C}^0(\Omega_0)}\left(\frac{2}{\prod_{j=1}^k \sigma_j^{\frac{s-\alpha s}{2}}}\sigma_k^{3\alpha}\prod_{j=1}^k \sigma_j^{3\alpha} M_0^{\alpha}\right)\\
&\le 2\mathfrak{K}_\alpha M_0^\alpha \sigma_{max}^{3\alpha}\prod_{j=1}^k \sigma_j^{\frac{6\alpha+\alpha s-s}{2}} C_5^{k\alpha}.
\end{aligned}
$$

With constants:

$$
\begin{aligned}
\mathfrak{C}_\alpha &= M_0^\alpha \sigma_{max}^{3\alpha} C_1^{1-\alpha} C_3^{\alpha}\|D_0\|^{\frac{1-\alpha}{2}}_{\mathcal{C}^0(\Omega_0)},\\
\mathfrak{K}_\alpha &= M_0^\alpha \sigma_{max}^{3\alpha} C_2^{1-\alpha} C_2^{\alpha}\|D_0\|^{\frac{1-\alpha}{2}}_{\mathcal{C}^0(\Omega_0)}(1+\|\nabla v_0\|_{\mathcal{C}^0(\Omega_0)}+2.4\cdot 10^{-7}).
\end{aligned}
$$

Both these sequences converge to zero as geometric series if the exponent $\frac{6\alpha+\alpha s-s}{2}$ is negative and the values of σ_k increase to become larger that C_5^α. For any value $\alpha \le \max\left\{\frac{1}{7},\frac{\beta}{2}\right\}$, there exists $s<1$ and $s<\frac{6\beta}{2-\beta}$ such that the exponent is indeed less that zero. Thus these sequences are summable, and thus the functions are Cauchy in $\mathcal{C}^{0,\alpha}$ and therefore converge. Thus we have a converging sequence of functions with defect decreasing to zero which implies that the limit u, w of these sequences are $\mathcal{C}^{1,\alpha}$ function which solve the very weak formulation of the Monge-Ampére equation.

Note that the exponent $\frac{6\alpha+\alpha s-s}{2}$ may be written as $\frac{2K\alpha+\alpha s-s}{2}$ where K is ne number of steps of convex integration in every stage. If there were only two steps, α could be pushed to $\frac{1}{5}$.

2.5 NUMERICAL RESULTS AND VISUALIZATION

In this section we discuss how the theoretical results were used to construct numerical approximations of stages of convex integration. In certain cases we were able to obtain interesting visualizations of anomalous solutions to the Monge-Ampére equation. This section is articulated in two parts, in the first we implement Proposition 2.4.5 and in the second we implement Proposition 2.4.6.

2.5.1 $\mathcal{C}^1$ stage visualization

In this section we consider two examples. In case study (i) we consider the Monge-Ampére equation with $f(x,y)=1$ and we approximate the non-convex function $v_0(x,y)=x^2-y^2$. In case study (ii)

we will have $f(x,y) = -1$ and approximate the convex function $v_0(x,y) = x^2 + y^2$. In both cases the domain is taken to be $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$. We then look to approximate these initial functions with solutions v such that $\|v - v_0\|_0 < \epsilon$ with $\epsilon = 0.2$. To this end we ask that at the k–th stage we have that $\|\tilde{v} - v\|_0 < \frac{0.1}{2^k}$. This means that in the first stage which we will simulate we ask $\|v_1 - v_0\|_0 < 0.1$.

In each of the examples we construct three corrugations. MatLab was used to perform the necessary computations. All calculations were performed by evaluating functions on fine meshed grids. For the first two corrugations we used a mesh with step size $h = 0.001$, but for the third corrugation it was necessary to consider a mesh with step size $h = 0.0001$. These step sizes were chosen because we required $h < 0.1\frac{1}{\lambda_k}$ to obtain smooth visualizations and to obtain precise numerical derivatives. The derivatives were evaluated numerically using the following formula:

$$\frac{\partial f}{\partial x}(x,\cdot) = \frac{2}{12h}\big(f(x-2h,\cdot) - 8f(x-h,\cdot) + 8f(x+h,\cdot) - f(x+2h,\cdot)\big) + O(h^5).$$

The initial auxiliary function w_0, and matrix A were chosen in both examples in such a way that the defect:

$$D = A - \Big(\nabla v_0 \otimes \nabla v_0 + \mathrm{Sym}\nabla w_0\Big),$$

may be decomposed as in Lemma 2.1.8:

$$D = \sum_{i=1}^{3} \phi_k \eta_k \otimes \eta_k \quad \text{with} \quad \phi_k > 0.4 \quad \forall k = 1,2,3.$$

In the construction from Proposition 2.4.5 we take $\delta = 0.5$ to guarantee that the defect will decrease by a factor of $\frac{3}{4}$. We begin the construction by defining $a_k = \sqrt{\frac{\phi_k}{2}}$, and the quantities:

$$B_k = \frac{1}{2}\nabla v_k \otimes \nabla v_k + \mathrm{Sym}\nabla w_k - \Big(\frac{1}{2}\nabla v_{k-1} \otimes \nabla v_{k-1} + \mathrm{Sym}\nabla w_{k-1} + a_k^2 \eta_k \otimes \eta_k\Big).$$

To guarantee the correctness of the construction these B_k will need to satisfy the following conditions:

$$\|B_k\|_0 \leq \frac{\|D\|_0}{12}, \quad \text{and} \quad |B_k| < \frac{8}{15\sqrt{3}}\big(\phi_i - 0.1\big) \quad \text{in } \Omega \quad \forall i = 1,2,3. \tag{2.68}$$

The first of these conditions will guarantee that:

$$\|\tilde{D}\|_0 \leq \frac{1}{2}\|D\|_0 + \sum_{i=1}^{3}\|B_k\|_0 \leq \frac{3}{4}\|D\|_0.$$

The second condition guarantees that $\tilde{D}$ may be decomposed with $\tilde{\phi}_k > \tilde{d} = 0.1$.

The condition $\|v_3 - v_0\|_0 \leq 0.1$ is in fact satisfied for very small values of λ, and the condition on the gradients is disregarded for the purposes of this visualization as it contains an arbitrarily large constant C.

The choice of the values of λ_k was done in the following way. We used (2.16) to choose an upper bound on the λ_k which would guarantee (2.68). For the λ_1 and λ_2 we then reduced the values and evaluated the B_k until we found the smallest values which satisfied (2.68). In the case of λ_3 we used the values obtained theoretically as we only ran the calculations on a smaller domain and could not justify the smaller values of λ_3 from a theoretical point of view.

Example 2.5.1. *We approximate the function $v_0(x, y) = x^2 - y^2$ with a solution v to:*

$$\mathcal{D}et\, \nabla^2 v = 1.$$

We choose $w_0(x, y) = (xy^2, x^2y)$ to ensure that the defect is diagonal, and take:

$$A(x, y) = \big(c - \frac{x^2 + y^2}{4}\big) Id_2$$

where $c = 5$ is chosen to make the defect positive definite. We then obtain:

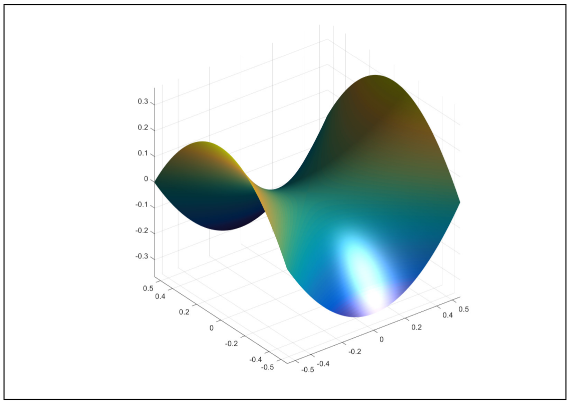
$$\nabla v_0(x, y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix}, \quad \nabla w_0(x, y) = \begin{bmatrix} y^2 & 2xy \\ 2xy & x^2 \end{bmatrix},$$

$$\frac{1}{2}\nabla v_0(x, y) \otimes \nabla v_0(x, y) + \mathrm{Sym}\nabla w_0(x, y) = \begin{bmatrix} 2x^2 + y^2 & 0 \\ 0 & x^2 + 2y^2 \end{bmatrix}.$$

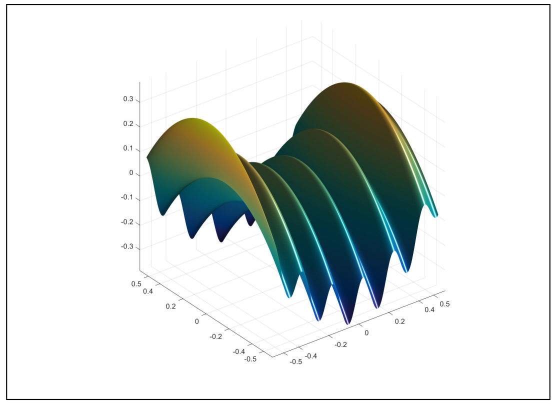
and the defect takes the form:

$$D(x, y) = \begin{bmatrix} 5 - \frac{9x^2+5y^2}{4} & 0 \\ 0 & 5 - \frac{5x^2+9y^2}{4} \end{bmatrix}.$$

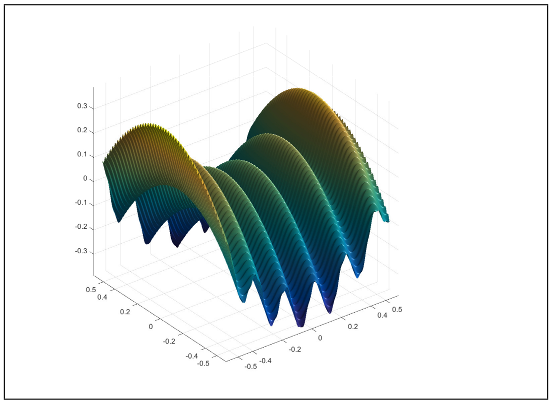
The original function v_0 and the two subsequent corrugations are shown in Figure 10. Next we show a more detailed picture of the second corrugation in Figure 11; the red area is the area on which we applied the third corrugation shown in Figure 12.



(a) The function v_0 on $\bar{\Omega}$



(b) One corrugation



(c) Two corrugations

Figure 10: Construction in Example 2.5.1

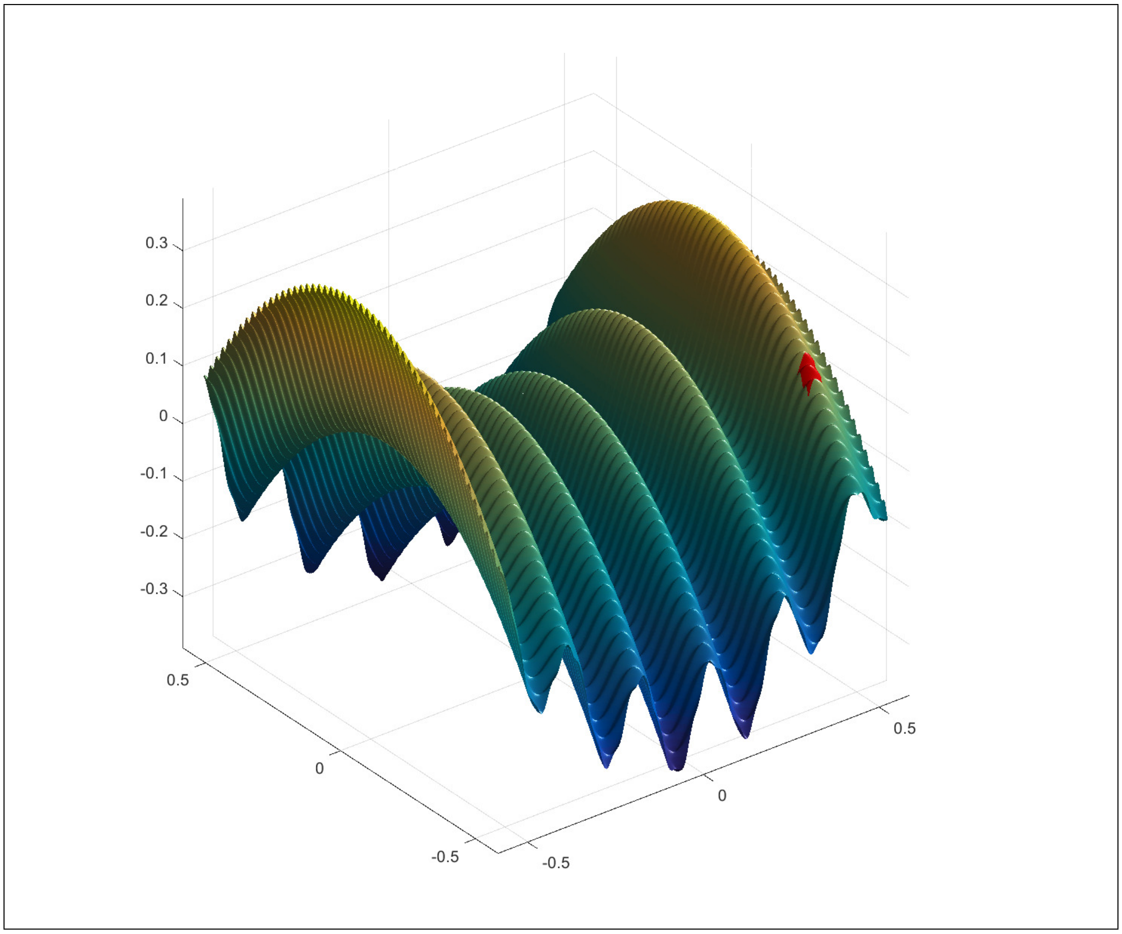


Figure 11: Two corrugations in Example 2.5.1: the red detail is shown in Figure 12

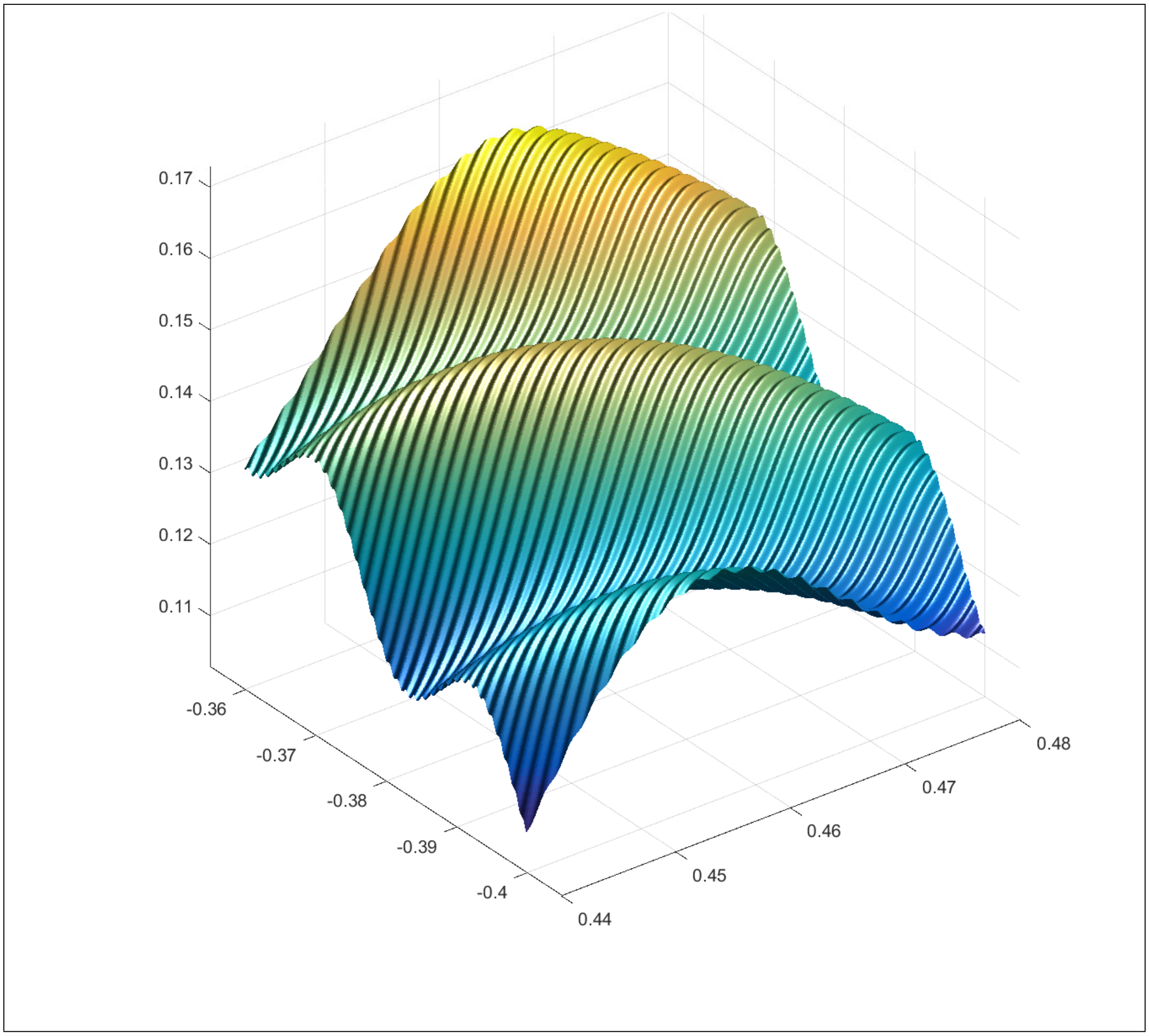


Figure 12: Detail of the three corrugations in Example 2.5.1

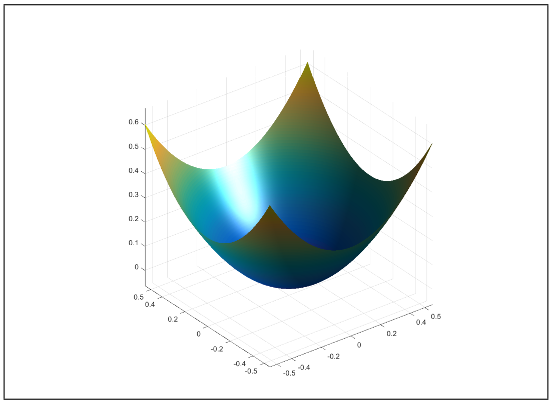
Example 2.5.2. *We approximate the function $v_0(x,y) = x^2 + y^2$ with a solution v to:*

$$\mathcal{D}et\, \nabla^2 v = -1.$$

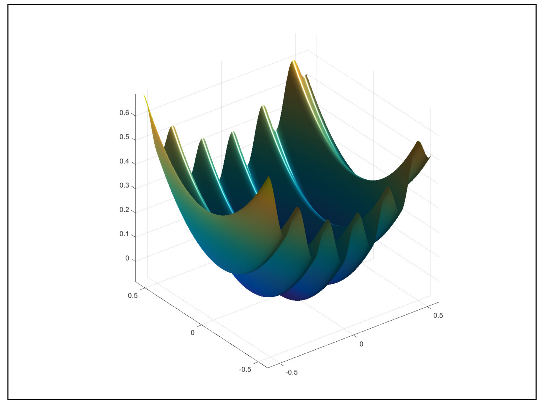
In this example we chose $w_0(x,y) = (-xy^2, -x^2y)$ to ensure that the defect is diagonal and take: $A(x,y) = \big(c + \frac{x^2+y^2}{4}\big) Id_2$, where $c = 5$ is chosen to make the defect positive definite, namely:

$$D(x,y) = \begin{bmatrix} 5 - \frac{7x^2-3y^2}{4} & 0 \\ 0 & 5 + \frac{3x^2-9y^2}{4} \end{bmatrix}.$$

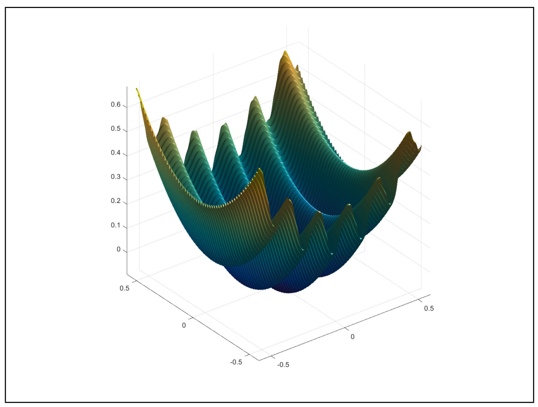
We plot three images starting from v_0 and subsequently adding the first and the second corrugation *in Figure 13. As before, we provide a more detailed picture of the second and third corrugations in Figures 14 and 15.*



(a) Original function v_0



(b) One corrugation



(c) Two corrugations

Figure 13: Construction in Example 2.5.2

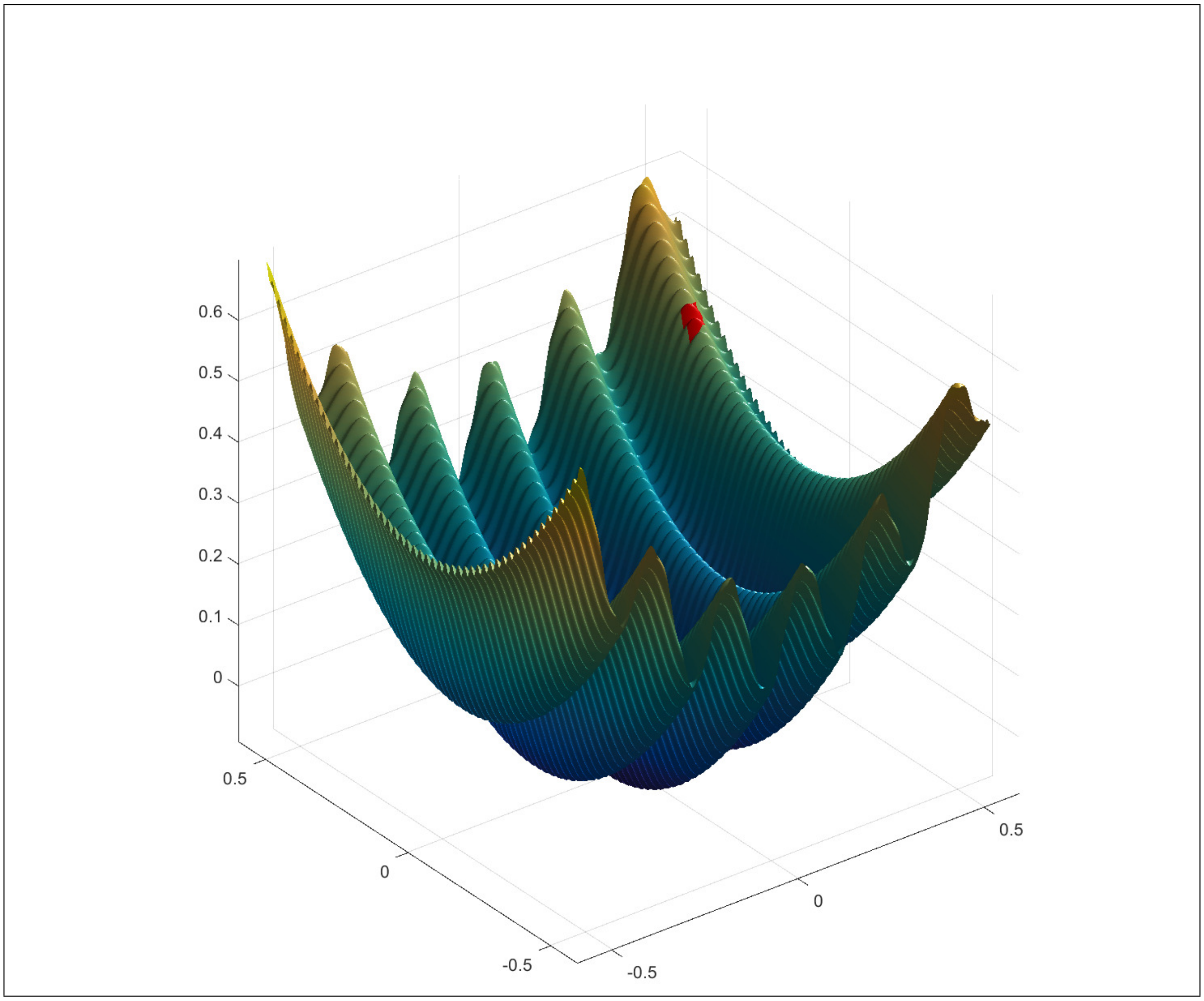


Figure 14: Two corrugations in Figure 13; the red detail shown in Figure 15

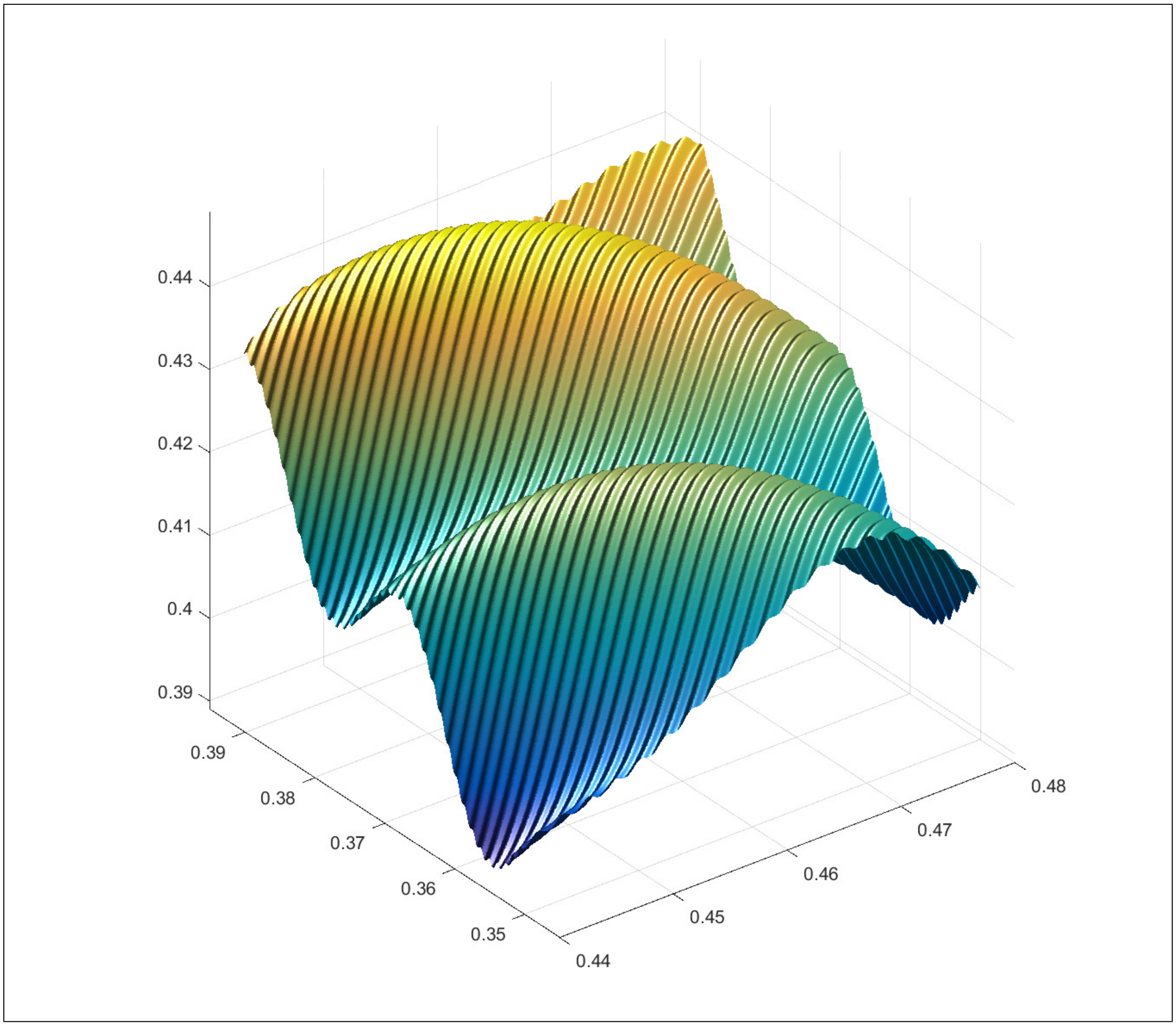


Figure 15: Detail of the three corrugations in Example 2.5.2

We conclude the discussion with a table listing some of the numerical results and implementation choices. The values of $\{\lambda_k\}$ are obtained experimentally. The value $\|v - v_0\|_0$ gives an upper estimate of the uniform distance of the function obtained through three steps of convex integration. The value $(\|B_1\|_0 + \|B_2\|_0)/\|D\|_0$ needs to be below $\frac{1}{6}$ as it does not take the third corrugation into account. The contribution of the last corrugation is guaranteed to be less than $\frac{1}{12}\|D\|_0$ through the a priori estimates. Lastly, $\min \phi_k$ are the minimum of each of the coefficients in $\bar{\Omega}$, in the defect after two corrugations. The a priori estimates once again guarantee that the third step will not make these less than the $\frac{1}{10}$ value required.

	Example 2.5.1	Example 2.5.2
$f(x,y)$	1	-1
$v_0(x,y)$	$x^2 - y^2$	$x^2 + y^2$
$w_0(x,y)$	(xy^2, yx^2)	$(-xy^2, -yx^2)$
λ_1	5	5
λ_2	50	57
λ_3	1000	1100
$\|v - v_0\|_0$	0.0995	0.999
$(\|B_1\|_0 + \|B_2\|_0)/\|D\|_0$	0.1339	0.1246
$\min \phi_1$	0.79	0.94
$\min \phi_2$	1.14	1.29
$\min \phi_3$	1.14	1.28

Table 3: Examples 1, 2 of the $\mathcal{C}^1$ approximation

2.5.2 $\mathcal{C}^{1,\alpha}$ stage numerics

Visualizations relevant to the $\mathcal{C}^{1,\alpha}$ construction are much harder to obtain. The main difficulties come from the smallness of the values in play, and the fact that the ratio σ between the frequency of subsequent corrugation is by necessity very large. The first problem was solved through the use of the Python package mpmath [17] which allows the user to define floating point arithmetics up to an arbitrary precision.

The approach taken in studying this problem numerically was significantly different from the case of $\mathcal{C}^1$ convergence. In each of the examples the problem was solved explicitly through symbolic calculations. This was done by defining symbolic variables and parameters in MatLab and defining functions of these variables. MatLab is capable of evaluating the derivatives of the functions involved in the calculations. The only calculation which could not be done symbolically was the mollification step at the beginning of the calculation. This step was therefore omitted from the computations. The justification for this choice is twofold. Firstly, standard mollifiers are well studied and their theory is well established, and this work does not add to the theory but merely

Test	$\|\tilde{D}\|_0$	$\|v_3\|_0$	$\|\nabla w_3\|_0$
1	$0.64 \cdot 10^{-11}$	$0.114 \cdot 10^{-23}$	$0.53 \cdot 10^{-8}$
2	$0.57 \cdot 10^{-11}$	$0.118 \cdot 10^{-23}$	$0.58 \cdot 10^{-8}$
3	$0.62 \cdot 10^{-11}$	$0.119 \cdot 10^{-23}$	$0.54 \cdot 10^{-8}$
4	$0.61 \cdot 10^{-11}$	$0.115 \cdot 10^{-23}$	$0.49 \cdot 10^{-8}$
5	$0.66 \cdot 10^{-11}$	$0.119 \cdot 10^{-23}$	$0.54 \cdot 10^{-8}$
6	$0.70 \cdot 10^{-11}$	$0.116 \cdot 10^{-23}$	$0.56 \cdot 10^{-8}$
7	$0.65 \cdot 10^{-11}$	$0.104 \cdot 10^{-23}$	$0.62 \cdot 10^{-8}$
8	$0.61 \cdot 10^{-11}$	$0.118 \cdot 10^{-23}$	$0.67 \cdot 10^{-8}$
9	$0.61 \cdot 10^{-11}$	$0.111 \cdot 10^{-23}$	$0.50 \cdot 10^{-8}$
10	$0.58 \cdot 10^{-11}$	$0.108 \cdot 10^{-23}$	$0.76 \cdot 10^{-8}$

Table 4: Repetition of tests

uses them as a tool. Secondly, the functions studied are very smooth, and the inequalities defined in Lemma 2.1.4 thus provide very bad estimations.

Once the symbolic calculations are computed through MatLab, the symbolic functions are saved in text files. These text files are modified through a script to make the syntax compatible with the mpmath package. Then the functions were sampled at 1000 randomly selected points on the square domain $(-1, 1) \times (-1, 1)$, and the maximum value attained by each function was recorded. The main purpose of these calculations was to study how the results of convex integration varied by changing the value of σ. We argue that the maximum sample value, while not being the exact supremum of these functions on the domain, is a good approximation of the order of magnitude. In fact when the process was repeated over different sets of 1000 randomly selected points the variation in the values obtained was negligible when considering the order of magnitude as illustrated in the table below.

In this table we show the results of running our code 10 times on the same example with the same parameters, the only change was the choice of the 1000 random points. In each test we evaluated the maximum defect recorded, the maximum value of v and the maximum gradient of w. As can be seen the order of magnitude of all these functions remains the same. Similar results were obtained with different examples, functions and parameters, but are omitted as they do not add particular insight.

We may now begin discussing the three examples used in this section:

Example 2.5.3. *In this example we considered* $f(x,y) = -10^{-18} < 0$ *and functions* $v_0 = 0$ *and* $w_0 = (0,0)$*. This results in the matrix* $A = -10^{-18}(x^2+y^2)Id_2$ *which is the actual original defect. The initial defect in the domain* $\Omega = (-1,1)\times(-1,1)$ *is thus bounded by* $\|D\|_0 = -10^{-18}\sqrt{2}$*. When we evaluated the defect for* $\sigma = 35$ *we obtained a new defect of* $\|D_3\|_0 \sim 9.7\cdot 10^{-19}$*. Below we show a visualization of* v *on the subdomain* $[1-10^{-19},1]\times[1-10^{-19},1]$

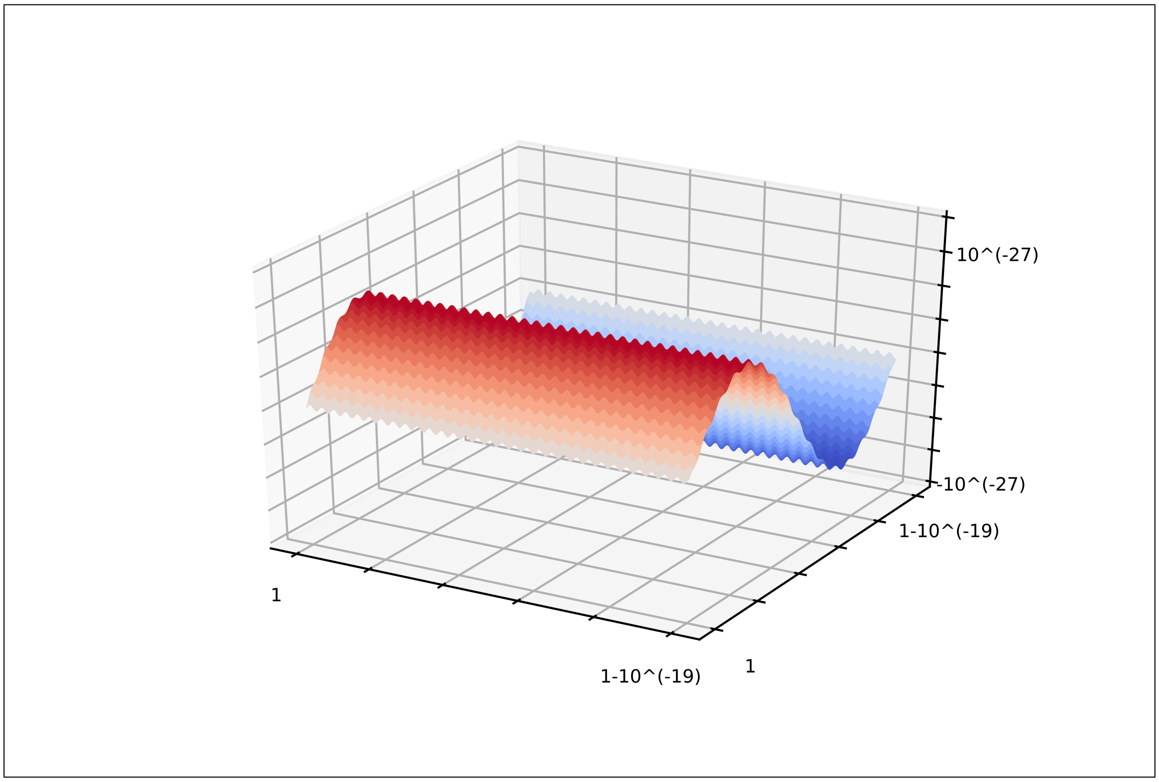


Figure 16: Visualization from Example 2.5.3

Example 2.5.4. *In this example we considered* $f(x,y) = 10^{-18} > 0$ *and functions* $v_0 = 0$ *and* $w_0 = (0,0)$*. This results in the matrix* $A = 10^{-18}(x^2+y^2)Id_2$ *which is the actual original defect. The initial defect in the domain* $\Omega = (-1,1)\times(-1,1)$ *is thus bounded by* $\|D\|_0 = 10^{-18}\sqrt{2}$*. When we evaluated the defect for* $\sigma = 35$ *we obtained a new defect of* $\|D_3\|_0 \sim 9.1 \cdot 10^{-19}$*. Below we show a visualization of* v *on the subdomain* $[1-10^{-19},1]\times[1-10^{-19},1]$

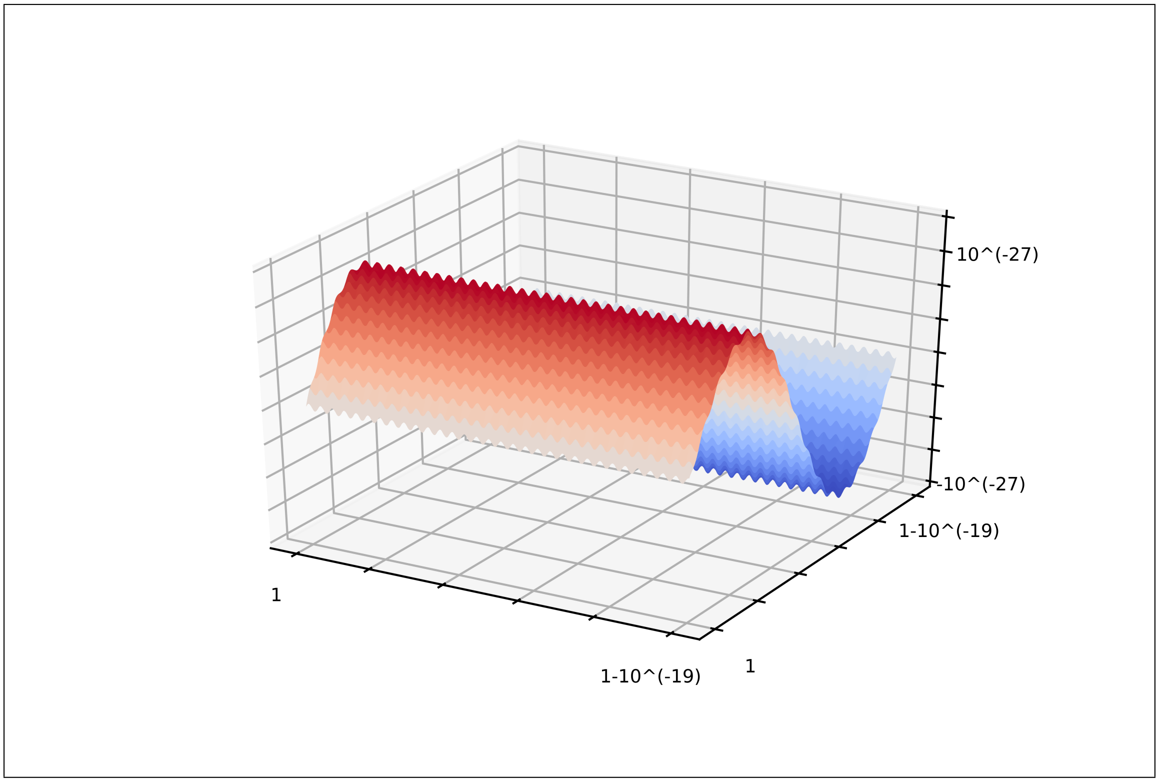


Figure 17: Visualization from Example 2.5.4

Example 2.5.5. *In this example we considered $f(x,y) = 0$ with matrix $A = 0$ and functions $v_0 = 10^{-9}(x^2 + y^2)$ and $w_0 = (0,0)$. The initial defect in the domain $\Omega = (-1,1) \times (-1,1)$ is thus given by:*

$$D = 10^{-18} \begin{bmatrix} 2x^2 & 2xy \\ 2xy & 2y^2 \end{bmatrix} \quad \textit{with:} \quad \|D\|_0 = 4 \cdot 10^{-18}.$$

When we evaluated the defect for $\sigma = 35$ we obtained a new defect of $\|D_3\|_0 \sim 9.5 \cdot 10^{-19}$. Below we show a visualization of v on the subdomain $[1 - 10^{-19}, 1] \times [1 - 10^{-19}, 1]$

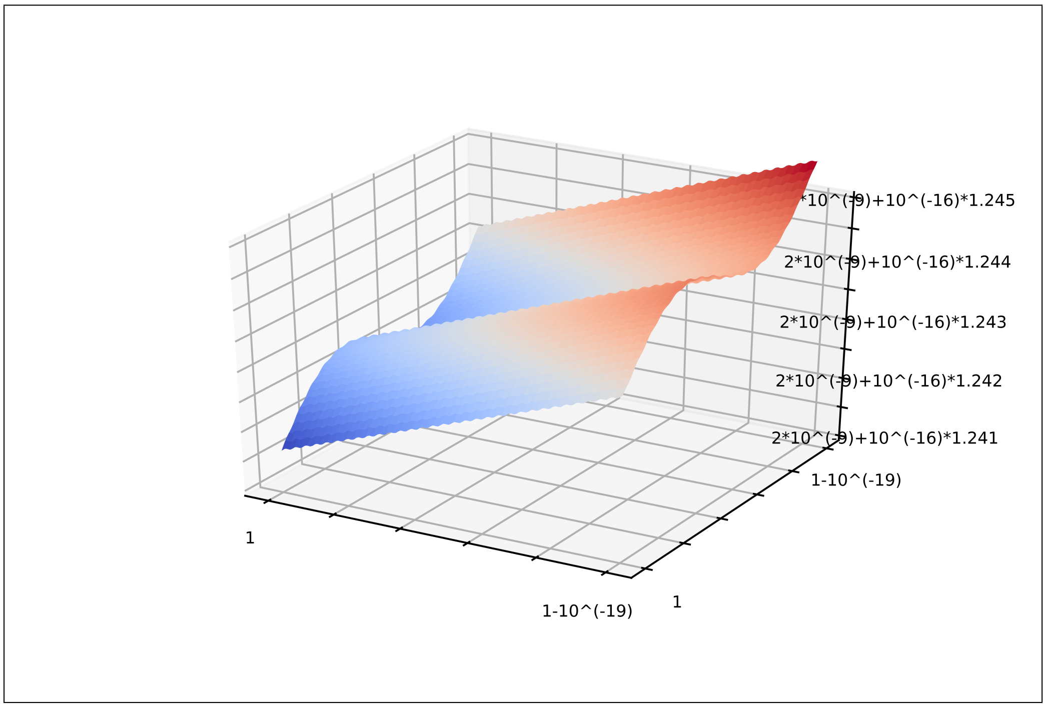


Figure 18: Visualization from Example 2.5.5

As we can see each of these examples satisfies the conditions of Proposition 2.4.6. In each of the examples a value of $\lambda_1 = 10^{19}$ was chosen and sigma was increased exponentially from $10^1 \dots 10^{16}$ which covers the orders of magnitude between the theoretical minimum of σ and σ_{max}. We then applied the modification of w and three steps of convex integration and evaluated the maximum of the new defect, the norm of v, it's gradient and Hessian and did the same for w.

The tables we see confirm the expectations from the theory. The defect decreases with σ, the gradients are indifferent of the choice of sigma and the Hessians increase significantly with sigma.

2.6 OPEN QUESTIONS AND FUTURE RESEARCH

In the context of the Monge-Ampère equation, the next step concerns a possibility of increasing the Hölder exponent. A result might be forthcoming which will increase the threshold exponent to $\frac{1}{5}$ as in the case of the isometric immersion [6]. The construction would be fairly similar to the one presented in [15]. The main change would be in the modification step in the stage construction. Instead of the current choice of $\mathfrak{w}'$, the choice should be made in such a way as to make the defect have rank 2 or 1. This will be done according to the following schema. Given the defect:

$$D = A - \big(\nabla v \otimes \nabla v + \mathrm{Sym}\nabla w\big),$$

one chooses $\mathfrak{w}' = w + \bar{w}$ such that:

$$D - \mathrm{Sym}\nabla \bar{w} = \bar{D},$$

where:

$$\bar{D} = g\mathrm{Id} \qquad \text{or} \qquad \bar{D} = h \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The first case is justified by taking:

$$g \text{ such that } \Delta g = \text{curl curl} D, \quad h \text{ such that } \partial_2 h = \text{curl curl} D.$$

In both cases we have that:

$$\text{curl curl}(D - g\mathrm{Id}) = \text{curl curl}(D - h\eta_1 \otimes \eta_1) = 0.$$

Thus ensuring that an appropriate $\bar{w}$ exists. This reasoning would have to be formalised, and estimates on the norms of $\bar{w}$ should be worked out to verify that this construction works.

On the numerical side, it would be interesting to study subsequent stages of convex integration. There are two approaches which could be taken, symbolic and numerical calculations. The numerical calculations would have to be defined on decreasing domains as stages progress, and thus would lose the global nature of the results. The symbolic approach quickly becomes very complex, and the evaluation of functions very slow. By approaching the problem through random sampling, however, one can still obtain some convincing global estimates.

σ	Example 2.5.3	Example 2.5.4	Example 2.5.5
10^1	$0.332 \cdot 10^{-17}$	$0.393 \cdot 10^{-17}$	$0.327 \cdot 10^{-17}$
10^2	$0.316 \cdot 10^{-18}$	$0.364 \cdot 10^{-18}$	$0.319 \cdot 10^{-18}$
10^3	$0.318 \cdot 10^{-19}$	$0.376 \cdot 10^{-19}$	$0.332 \cdot 10^{-19}$
10^4	$0.318 \cdot 10^{-20}$	$0.396 \cdot 10^{-20}$	$0.318 \cdot 10^{-20}$
10^5	$0.320 \cdot 10^{-21}$	$0.401 \cdot 10^{-21}$	$0.316 \cdot 10^{-21}$
10^6	$0.336 \cdot 10^{-22}$	$0.400 \cdot 10^{-22}$	$0.325 \cdot 10^{-22}$
10^7	$0.326 \cdot 10^{-23}$	$0.375 \cdot 10^{-23}$	$0.330 \cdot 10^{-23}$
10^8	$0.332 \cdot 10^{-24}$	$0.367 \cdot 10^{-24}$	$0.335 \cdot 10^{-24}$
10^9	$0.329 \cdot 10^{-25}$	$0.366 \cdot 10^{-25}$	$0.329 \cdot 10^{-25}$
10^{10}	$0.328 \cdot 10^{-26}$	$0.382 \cdot 10^{-26}$	$0.339 \cdot 10^{-26}$
10^{11}	$0.338 \cdot 10^{-27}$	$0.399 \cdot 10^{-27}$	$0.326 \cdot 10^{-27}$
10^{12}	$0.329 \cdot 10^{-28}$	$0.371 \cdot 10^{-28}$	$0.327 \cdot 10^{-28}$
10^{13}	$0.317 \cdot 10^{-29}$	$0.396 \cdot 10^{-29}$	$0.333 \cdot 10^{-29}$
10^{14}	$0.320 \cdot 10^{-30}$	$0.388 \cdot 10^{-30}$	$0.325 \cdot 10^{-30}$
10^{15}	$0.311 \cdot 10^{-31}$	$0.384 \cdot 10^{-31}$	$0.336 \cdot 10^{-31}$
10^{16}	$0.324 \cdot 10^{-32}$	$0.366 \cdot 10^{-32}$	$0.321 \cdot 10^{-32}$

Table 5: Values of the defect $\|D_3\|_0$

σ	Example 2.5.3	Example 2.5.4	Example 2.5.5
10^1	$0.941 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.106 \cdot 10^{-7}$
10^2	$0.937 \cdot 10^{-8}$	$0.105 \cdot 10^{-7}$	$0.102 \cdot 10^{-7}$
10^3	$0.920 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.102 \cdot 10^{-7}$
10^4	$0.936 \cdot 10^{-8}$	$0.994 \cdot 10^{-8}$	$0.104 \cdot 10^{-7}$
10^5	$0.953 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^6	$0.935 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.105 \cdot 10^{-7}$
10^7	$0.946 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.103 \cdot 10^{-7}$
10^8	$0.936 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.103 \cdot 10^{-7}$
10^9	$0.942 \cdot 10^{-8}$	$0.100 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^{10}	$0.934 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.101 \cdot 10^{-7}$
10^{11}	$0.931 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.103 \cdot 10^{-7}$
10^{12}	$0.926 \cdot 10^{-8}$	$0.101 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^{13}	$0.939 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$
10^{14}	$0.940 \cdot 10^{-8}$	$0.995 \cdot 10^{-8}$	$0.103 \cdot 10^{-7}$
10^{15}	$0.945 \cdot 10^{-8}$	$0.986 \cdot 10^{-8}$	$0.102 \cdot 10^{-7}$
10^{16}	$0.920 \cdot 10^{-8}$	$0.103 \cdot 10^{-7}$	$0.104 \cdot 10^{-7}$

Table 6: Values of the defect $\|\nabla v_3\|_0$

σ	Example 2.5.3	Example 2.5.4	Example 2.5.5
10^1	$0.730 \cdot 10^{-16}$	$0.432 \cdot 10^{-16}$	$0.844 \cdot 10^{-16}$
10^2	$0.728 \cdot 10^{-16}$	$0.445 \cdot 10^{-16}$	$0.857 \cdot 10^{-16}$
10^3	$0.687 \cdot 10^{-16}$	$0.430 \cdot 10^{-16}$	$0.813 \cdot 10^{-16}$
10^4	$0.712 \cdot 10^{-16}$	$0.458 \cdot 10^{-16}$	$0.835 \cdot 10^{-16}$
10^5	$0.754 \cdot 10^{-16}$	$0.424 \cdot 10^{-16}$	$0.848 \cdot 10^{-16}$
10^6	$0.727 \cdot 10^{-16}$	$0.422 \cdot 10^{-16}$	$0.873 \cdot 10^{-16}$
10^7	$0.716 \cdot 10^{-16}$	$0.444 \cdot 10^{-16}$	$0.773 \cdot 10^{-16}$
10^8	$0.723 \cdot 10^{-16}$	$0.444 \cdot 10^{-16}$	$0.859 \cdot 10^{-16}$
10^9	$0.727 \cdot 10^{-16}$	$0.420 \cdot 10^{-16}$	$0.833 \cdot 10^{-16}$
10^{10}	$0.721 \cdot 10^{-16}$	$0.431 \cdot 10^{-16}$	$0.752 \cdot 10^{-16}$
10^{11}	$0.698 \cdot 10^{-16}$	$0.478 \cdot 10^{-16}$	$0.808 \cdot 10^{-16}$
10^{12}	$0.672 \cdot 10^{-16}$	$0.431 \cdot 10^{-16}$	$0.843 \cdot 10^{-16}$
10^{13}	$0.692 \cdot 10^{-16}$	$0.449 \cdot 10^{-16}$	$0.785 \cdot 10^{-16}$
10^{14}	$0.733 \cdot 10^{-16}$	$0.414 \cdot 10^{-16}$	$0.789 \cdot 10^{-16}$
10^{15}	$0.739 \cdot 10^{-16}$	$0.430 \cdot 10^{-16}$	$0.800 \cdot 10^{-16}$
10^{16}	$0.687 \cdot 10^{-16}$	$0.442 \cdot 10^{-16}$	$0.779 \cdot 10^{-16}$

Table 7: Values of the defect $\|\nabla w_3\|_0$

σ	Example 2.5.3	Example 2.5.4	Example 2.5.5
10^1	$3.37 \cdot 10^{13}$	$3.05 \cdot 10^{13}$	$3.26 \cdot 10^{13}$
10^2	$3.30 \cdot 10^{15}$	$3.00 \cdot 10^{15}$	$3.22 \cdot 10^{15}$
10^3	$3.27 \cdot 10^{17}$	$2.98 \cdot 10^{17}$	$3.28 \cdot 10^{17}$
10^4	$3.26 \cdot 10^{19}$	$2.98 \cdot 10^{19}$	$3.27 \cdot 10^{19}$
10^5	$3.27 \cdot 10^{21}$	$2.99 \cdot 10^{21}$	$3.29 \cdot 10^{21}$
10^6	$3.25 \cdot 10^{23}$	$2.99 \cdot 10^{23}$	$3.27 \cdot 10^{23}$
10^7	$3.28 \cdot 10^{25}$	$2.99 \cdot 10^{25}$	$3.23 \cdot 10^{25}$
10^8	$3.21 \cdot 10^{27}$	$2.99 \cdot 10^{27}$	$3.28 \cdot 10^{27}$
10^9	$3.26 \cdot 10^{29}$	$2.98 \cdot 10^{29}$	$3.28 \cdot 10^{29}$
10^{10}	$3.30 \cdot 10^{31}$	$2.99 \cdot 10^{31}$	$3.30 \cdot 10^{31}$
10^{11}	$3.25 \cdot 10^{33}$	$2.99 \cdot 10^{33}$	$3.23 \cdot 10^{33}$
10^{12}	$3.28 \cdot 10^{35}$	$2.99 \cdot 10^{35}$	$3.24 \cdot 10^{35}$
10^{13}	$3.23 \cdot 10^{37}$	$2.99 \cdot 10^{37}$	$3.25 \cdot 10^{37}$
10^{14}	$3.26 \cdot 10^{39}$	$2.98 \cdot 10^{39}$	$3.25 \cdot 10^{39}$
10^{15}	$3.25 \cdot 10^{41}$	$2.99 \cdot 10^{41}$	$3.27 \cdot 10^{41}$
10^{16}	$3.29 \cdot 10^{43}$	$2.98 \cdot 10^{43}$	$3.20 \cdot 10^{43}$

Table 8: Values of the defect $\|\nabla^2 v_3\|_0$

σ	Example 2.5.3	Example 2.5.4	Example 2.5.5
10^1	$3.11 \cdot 10^5$	$2.80 \cdot 10^5$	$2.12 \cdot 10^5$
10^2	$3.12 \cdot 10^7$	$2.82 \cdot 10^7$	$2.05 \cdot 10^7$
10^3	$3.20 \cdot 10^9$	$2.87 \cdot 10^9$	$2.23 \cdot 10^9$
10^4	$3.17 \cdot 10^{11}$	$2.60 \cdot 10^{11}$	$2.18 \cdot 10^{11}$
10^5	$3.12 \cdot 10^{13}$	$2.85 \cdot 10^{13}$	$2.15 \cdot 10^{13}$
10^6	$3.18 \cdot 10^{15}$	$2.80 \cdot 10^{15}$	$2.08 \cdot 10^{15}$
10^7	$3.12 \cdot 10^{17}$	$2.79 \cdot 10^{17}$	$2.18 \cdot 10^{17}$
10^8	$3.17 \cdot 10^{19}$	$2.73 \cdot 10^{19}$	$2.18 \cdot 10^{19}$
10^9	$3.27 \cdot 10^{21}$	$2.74 \cdot 10^{21}$	$2.18 \cdot 10^{21}$
10^{10}	$3.31 \cdot 10^{23}$	$2.75 \cdot 10^{23}$	$2.10 \cdot 10^{23}$
10^{11}	$3.21 \cdot 10^{25}$	$2.65 \cdot 10^{25}$	$2.13 \cdot 10^{25}$
10^{12}	$3.25 \cdot 10^{27}$	$2.69 \cdot 10^{27}$	$2.19 \cdot 10^{27}$
10^{13}	$3.10 \cdot 10^{29}$	$2.79 \cdot 10^{29}$	$2.08 \cdot 10^{29}$
10^{14}	$3.08 \cdot 10^{31}$	$2.71 \cdot 10^{31}$	$2.21 \cdot 10^{31}$
10^{15}	$3.29 \cdot 10^{33}$	$2.79 \cdot 10^{33}$	$2.20 \cdot 10^{33}$
10^{16}	$3.39 \cdot 10^{35}$	$2.72 \cdot 10^{35}$	$2.22 \cdot 10^{35}$

Table 9: Values of the defect $\|\nabla^2 w_3\|_0$

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