GEOMETRIC ANALYSIS: REGULARITY THEORY FOR SUBELLIPTIC PDES AND INCOMPATIBLE ELASTICITY

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This thesis is divided in two parts, which share a common theme of analysis in non-Euclidean spaces.

The first one focuses on regularity of weak solutions of the p-Laplace equation in the Heisenberg group. In particular, we give a proof of the fact that, for $p > 4$, solutions assumed to be in the horizontal Sobolev space $HW^{1,p}$ (consisting of L^p functions whose horizontal gradient is in L^p), possess Hölder continuous horizontal derivatives. The argument is based on approximation via solutions of regularized problems: estimates independent of a non degeneracy parameter are obtained and passed to the limit. In particular, we show that the horizontal derivatives belong to a weighted De Giorgi space and then employ an alternative argument, not unlike the Euclidean case.

The second part deals with non-Euclidean elasticity. We study incompatibly prestrained thin plates characterized by a prescribed Riemannian metric on their reference configuration. We analyze scaling of the elastic energy E^h of order higher than 2 in plate's thickness h, i.e. $\inf h^{-\beta} E^h$ for $\beta > 2$. We find that, within this range, the only possible non trivial scaling is $\beta = 4$. In this case we identify and study the Γ-limit functional, which consists of a von Kármán-like energy, given in terms of the first order infinitesimal isometries and of the admissible strains on the surface isometrically immersing the prestrain metric on the midplate in $\mathbb{R}^3$.

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PREFACE

I would like to thank my advisor Juan Manfredi, whose guidance throughout my PhD years has shaped me as a mathematician and helped me grow as a person.

I would like to thank my advisor Marta Lewicka, whose passion and dedication for mathematics have been a source of inspiration.

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1.0 INTRODUCTION

The Euclidean setting is not always suitable to address problems arising in physics, engineering and applied sciences in general. Often times, the most natural setup is Riemannian or subRiemannian. This is the case for certain problems in nonlinear elasticity, constrained mechanics, robotics and neuroscience to name a few. It is therefore of interest to study analysis and partial differential equations in such settings.

In the first part of this work we focus on the subRiemannian part, specifically dealing with the Heisenberg group. This is the simplest example of subRiemannian manifold, and as such has attracted a lot of interest as a starting point for the study of more general subRiemannian structures. We refer the reader to Section 2.1 for the definition and basic properties of the Heisenberg group.

The minimization of non quadratic energy functionals leads to equations of p-Laplacian type, which have been extensively studied in the Euclidean case. Our aim is to extend the theory, in particular concerning regularity of weak solutions, to the Heisenberg group setting. In Section 2.2 we present the p-Laplace equation in the Heisenberg group:

$$\sum_{i=1}^{2} X_i(|\nabla_{\mathbb{H}} u|^{p-2} X_i u) = 0 \quad \text{in } \ \Omega, \tag{1.1}$$

where X_i are the horizontal vector fields (see (2.2)) and $\nabla_{\mathbb{H}} u = (X_1 u, X_2 u)$ is the horizontal gradient. A weak solution is to be intended in the integral sense:

$$\int_\Omega \sum_{i=1}^{2} |\nabla_{\mathbb{H}} u|^{p-2} X_i u \, X_i \phi \, \mathrm{d}x = 0 \quad \text{for all } \ \phi \in C_0^\infty(\Omega), \tag{1.2}$$

where u is assumed to be in $HW^{1,p}(\Omega)$, i.e. the space of $L^p(\Omega)$ functions whose horizontal derivatives belong to $L^p(\Omega)$. This equation is degenerate/singular at points where the horizontal gradient vanishes.

In the Euclidean case, weak solutions to the corresponding version of the p-Laplace equation for $1 < p < \infty$ are known to have Hölder continuous derivatives. For $p \geq 2$ this was proved by Uraltseva in [Ura68] and for $1 < p < 2$ (independently) by DiBenedetto in [DiB83] and Lewis in [Lew83].

The Heisenberg group presents new challenges with respect to its Euclidean counterpart, since we only assume that u is in the horizontal Sobolev space $HW^{1,p}$, while the differentiation of the equation produces terms involving the vertical derivative Tu. This is due to the non commutativity of the horizontal vector fields and constitutes the main difficulty. It is useful to approximate the p-Laplace equation (1.1) via the regularized family of equations:

$$\int_\Omega \sum_{i=1}^{2} (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} X_i u \, X_i \phi \, \mathrm{d}x = 0 \quad \text{for all } \ \phi \in C_0^\infty(\Omega), \tag{1.3}$$

for $\delta > 0$, and obtain estimates independent of δ. We will refer to the case $\delta > 0$ as the non degenerate p-Laplace equation.

In the Heisenberg group (or more in general in so called Carnot groups), the regularity theory has been developed by the contributions of many authors. We give a brief overwiev which is by no means exhaustive.

For $p \neq 2$ the Hölder regularity of solutions of equations modeled on (1.1) was established by Capogna and Garofalo [CG03] and Lu [Lu96]. In [Cap97] Capogna obtains C^∞ regularity in the range $p \geq 2$ under the additional non degeneracy hypothesis $M^{-1} \leq |\nabla_{\mathbb{H}} u| \leq M$. Manfredi and Mingione [MM07] were able to prove $C^{1,\alpha}$ regularity in the non degenerate case for $2 \leq p < c(n) < 4$, and by adapting an argument used by Capogna, they achieve C^∞ regularity for this range of values of p. The starting point is the integrability result for the vertical derivative $Tu \in L^p$ established by Domokos for $1 < p < 4$ in [Dom04], where he extends integrability results considered by Marchi for $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$ in [Mar72], [Mar88].

Mingione, Zatorska-Goldstein and Zhong proved in [MZGZ09] that the Euclidean gradient of solutions of the non degenerate equation are $C^{1,\alpha}$ for $2 \leq p < 4$, and also that solutions of the degenerate equation are locally Lipschitz continuous for $2 \leq p < 4$. Zhong in [Zho18] extended the Hilbert-Haar theory to the Heisenberg group setting and proved that solutions of the degenerate equation (1.1) are locally Lipschitz for the full range $1 < p < \infty$. For an account of this theory, further historical details and additional references see [Ric15].

As for the Hölder continuity of the horizontal derivatives for the degenerate equation (1.1) the only published result for $p \neq 2$ had been obtained by Manfredi and Domokos in [DM05b], [DM05a] via the Cordes pertubation technique for p near 2.

In the first part of this work we present a proof of the following:

Theorem 1.0.1. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of the p-Laplace equation (1.1) for $p > 4$. Fix a Carnot-Carathéodory ball $B_{R_0} \subset\subset \Omega$. Then there exists $\beta = \beta(p) \in (0,1)$ such that for every $l \in \{1,2\}$ we have:*

$$\operatorname*{osc}_{B_r} (X_l u) \leq C_p \, \|\nabla_{\mathbb{H}} u\|_{L^\infty(B_{R_0})} \left(\frac{r}{R_0}\right)^{\beta} \quad \text{for all } r \leq \frac{R_0}{2},$$

where C_p is a constant depending only on p.

The proof uses the particular form of the equation in $\mathbb{H}^1$ and new integration by parts for the second derivatives that produce weights of the form $(\delta^2 + |\nabla_{\mathbb{H}} u|)^{\frac{p-4}{2}}$. These are the reasons why our proof is only valid in the first Heisenberg group $\mathbb{H}^1$ and for the range $p > 4$.

More recently, the $C^{1,\alpha}$ regularity result has been established for the full range $1 < p < \infty$. The work [CCDO17] contains a proof for $p \geq 2$ in general settings (which include the case of Heisenberg groups of any dimension), while the preprint [MZv2] (focusing directly on the Heisenberg groups) extends the result to include the case $1 < p < 2$.

The second part of the thesis is related to nonlinear elasticity. Thin elastic bodies are structures in which one dimension is much smaller than the others. In the presence of external forces or given boundary conditions, such bodies undergo deformations that change their geometry and configuration in space. In order to study these phenomena, the main idea is to pass from a three dimensional model to a two dimensional one. Originally, additional a priori assumptions were used to carry out such procedure, sometimes lacking a rigorous justification.

The mathematical framework of Γ-convergence provides a suitable tool to address this question. The idea is to employ the thin dimension of the body, that is its thickness, and study the limiting behaviour of appropriate elastic energy functionals as the thickness tends to zero. In this regard, there are two main questions: one is to determine the appropriate scaling of the energy in terms of the thickness, and the other is to identify the models arising in the identified scaling regime. In this way a hierarchy of limiting theories can be derived. Regarding the case in which the scaling is induced by that of the external forces, we quote the works of Le Dret and Raoult [LDR95] for the case where external forces are of the order $\mathcal{O}(1)$ and Frieseke, James and Muller [FJM06] where forces are of the order $\mathcal{O}(h^\beta)$, $\beta \geq 2$. The preceding papers deal with elastic bodies whose midsurface at rest is planar, i.e. plates. For the case of shells, that is when the midsurface at rest is not planar, we mention [LDR96] for scaling $\beta = 0$, [FJMM03] for $\beta = 2$ and Lewicka, Mora and Pakzad [LMP10] ($\beta \geq 4$), and [LMP11b] ($\beta > 2$, elliptic shells).

Recently there has been an increasing interest in cases where the shape formation is not caused by external forces or imposed boundary conditions, but rather by internal pre-strains. Examples of such phenomena are growing leaves and tissues, torn plastic sheets and engineered polymers to name a few. In these situations, the complicated three dimensional shapes that we can observe arise as a consequence of the mechanisms of growth, shrinkage, plasticity etc...

A model that has been introduced in this regard is that of the incompatible elasticity or non-Euclidean elasticity. This assumes the existence of a target metric G, given on the reference configuration $\Omega \subset \mathbb{R}^3$, that the body seeks to attain through an orientation preserving isometric immersion, i.e. a map $u : \Omega \longrightarrow \mathbb{R}^3$ such that $(\nabla u)^t \nabla u = G$ and $\det(\nabla u) > 0$. If the metric G is not flat, that is if its Riemann curvature tensor is not identically null, then there is no such isometric immersion. Nonetheless, the body accomodates to attain a configuration minimizing its elastic energy, by means of a deformation which is the closest to being a realization of G. This results in the presence of residual stress in the rest configuration.

Given a thin plate $\Omega^h \subset \mathbb{R}^3$, a Riemannian metric G independent and uniform through the thickness, and a deformation $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, the incompatibility is measured through an elastic energy functional of the form:

$$E^h(u^h) = \frac{1}{h} \int_{\Omega^h} W(\nabla u^h G^{-1/2}) \, \mathrm{d}x$$

where $W : \mathbb{R}^{3\times 3} \longrightarrow \bar{\mathbb{R}}_+$ is a given density function that vanishes on $SO(3)$ (see Section 4.1 for the precise assumptions). The functional E^h measures the incompatibility, since $E^h(u^h) = 0$ precisely when u^h is an orientation preserving isometric immersion of G.

In [LP11] Lewicka and Pakzad proved that:

$$\inf_{W^{1,2}(\Omega^h,\mathbb{R}^3)} E^h = 0 \quad \text{if and only if} \quad Riem(G) \equiv 0.$$

In [BLS16], Battacharya, Lewicka and Schaffner study the case when $\inf E^h \sim h^2$. They prove that this happens if and only if the Riemann curvatures R_{1212}, R_{1213} and R_{1223} of G do not vanish identically in Ω^h, and under this assumption they compute the Γ-limit of $h^{-2}E^h$ (see Section 4.2 for more details).

In this work, we continue the analysis in [BLS16], and we study higher order scalings of the elastic energy, i.e. $h^{-\beta}E^h$ for $\beta > 2$. There are two main results: the first is a quantitazion of the scalings leading to non trivial limiting theories:

Theorem 1.0.2. *In the above notations we have:*

- *Assume* $\lim_{h\to 0} h^{-2} \inf E^h = 0$. *Then* $\inf E^h \leq Ch^4$.
- *Assume* $\lim_{h\to 0} h^{-4} \inf E^h = 0$. *Then* $\inf E^h = \min E^h = 0$.

Here the infima are computed in the set $W^{1,2}(\Omega^h, \mathbb{R}^3)$.

The other main result is the identification of the Γ-limit functional in the case when the energy scales like h^4. This is given by a functional $\mathcal{I}_4$ consisting of a bending, stretching and a geometric term due to the possible incompatibility of the metric. Details are contained in Chapter 5.

The results presented in this thesis are based on the following publications:

- D. Ricciotti, *On the $C^{1,\alpha}$ regularity of p-harmonic functions in the Heisenberg group*, Proc. Amer. Math. Soc., electronically published on February 8, 2018, DOI: https://doi.org/10.1090/proc/13961 (to appear in print)
- M. Lewicka, A. Raoult and D. Ricciotti, *Plates with incompatible prestrain of higher order*, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), no. 7, 1883-1912

- D. Ricciotti, *p-Laplace Equation in the Heisenberg Group, Regularity of Solutions*, SpringerBriefs in Mathematics. Springer, [Cham]; BCAM Basque Center for Applied Mathematics, Bilbao, 2015.

2.0 p-LAPLACE EQUATION IN THE HEISENBERG GROUP

In this Chapter we give a brief introduction to the Heisenberg group and the p-Laplace equation, highlighting some of the properties that we will use and fixing the notation.

2.1 THE HEISENBERG GROUP

The first Heisenberg group $\mathbb{H}$ is the Lie group $(\mathbb{R}^3, \star)$, where, indicating points x, $y \in \mathbb{H}$ by $x = (x_1, x_2, z)$ and $y = (y_1, y_2, s)$, the group operation is:

$$x \star y = (x_1, x_2, z) \star (y_1, y_2, s) = \left(x_1 + y_1, x_2 + y_2, z + s + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right) . \tag{2.1}$$

A basis of left-invariant vector fields for the associated Lie algebra $\mathfrak{h}$ is given by:

$$\begin{aligned} X_1 &= \partial_{x_1} - \frac{x_2}{2}\partial_z, \\ X_2 &= \partial_{x_2} + \frac{x_1}{2}\partial_z, \\ T &= \partial_z \ . \end{aligned} \tag{2.2}$$

The only non vanishing commutator is $[X_1, X_2] = T$. We obtain the stratification of the Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \bigoplus \mathfrak{h}_2$, where:

$$\mathfrak{h}_1 = \text{span}\{X_1, X_2\}, \tag{2.3}$$

$$\mathfrak{h}_2 = \text{span}\{T\}, \tag{2.4}$$

with $[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2$.

Let Ω be an open subset of $\mathbb{H}$ and consider a function $u : \Omega \longrightarrow \mathbb{R}$. We will indicate by $\nabla_{\mathbb{H}} u = (X_1 u, X_2 u)$ the horizontal gradient of u and by $\nabla^2_{\mathbb{H}} u = (X_i X_j u)_{i,j=1,2}$ the horizontal Hessian of u. If $X = a_1 X_1 + a_2 X_2$ is a horizontal vector field, we denote by $\mathrm{div}_{\mathbb{H}} X = X_1 a_1 + X_2 a_2$ its horizontal divergence.

Definition 2.1.1. *(Horizontal Curves) An absolutely continuous curve $\Gamma : [0, T] \longrightarrow \mathbb{H}$ is a horizontal curve if:*

$$\Gamma'(t) = \sum_{j=1}^{2} \alpha_j(t) X_j(\Gamma(t))$$

for some real valued functions $\alpha_j(t)$. If $\sum_{j=1}^{2} \alpha_j(t)^2 \leq 1$ we call the horizontal curve subunitary and define its length as $l(\Gamma) = T$. We denote by $S(x, y)$ the set of all horizontal subunitary curves joining x and y.

Definition 2.1.2 (Carnot-Carathéodory distance)**.** *We define the Carnot-Carathéodory distance on $\mathbb{H}$ as:*

$$d_{cc}(x, y) = \inf\{l(\Gamma) \mid \Gamma \in S(x, y)\}\ .$$

The previous definition is well posed thanks to Chow's accessibility Theorem (see [BLU07]) which implies that for all x, $y \in \mathbb{H}$ we have $S(x, y) \neq 0$.

When there is no possibility of confusion, we will denote the Carnot-Carathéodory distance simply as d and the balls:

$$B_r(x) = \{y \in \mathbb{R}^n \mid d(x, y) < r\}$$

will be with respect to the Carnot-Carathéodory distance unless otherwise stated.

We will also use a particular homogenous norm, the Korányi norm:

$$|x|_K = \left((x_1^2 + x_2^2)^2 + 16z^2\right)^{\frac{1}{4}}$$

which is equivalent to the Carnot-Carathéodory norm $|x|_c = d(x, 0)$. We have a family of dilations $(\delta_\lambda)_{\lambda>0}$, that are group homomorphism, given by:

$$\delta_\lambda(x_1, x_2, z) = (\lambda x_1, \lambda x_2, \lambda^2 z).$$

The Lebesgue measure in $\mathbb{R}^3$ is the Haar measure of the group and we will denote the measure of a set A by $|A|$. The homogeneous dimension of $\mathbb{H}$ is $Q = 4$.

The horizontal Sobolev space $HW^{1,p}(\Omega)$ is the space of $L^p(\Omega)$ functions u whose first horizontal weak derivatives X_1u and X_2u are in $L^p(\Omega)$. It is a Banach space if endowed with the Sobolev norm:

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}} u\|_{L^p(\Omega)}$$

and it is reflexive if $1 < p < \infty$. Analogously to the Euclidean case we can approximate $HW^{1,p}$ functions with smooth functions (see for example [HK00], Theorem 11.9).

Theorem 2.1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $1 \le p < \infty$. Then:*

$$C^\infty(\Omega) \cap HW^{1,p}(\Omega) \ \ \textit{is dense in} \ \ HW^{1,p}(\Omega) \ .$$

We also define the space $HW_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ with respect to the norm $\|\cdot\|_{HW^{1,p}(\Omega)}$. We denote the average of a function f over a set B by:

$$f_B = \fint_B f(x)\,\mathrm{d}x = \frac{1}{|B|}\int_B f(x)\,\mathrm{d}x \ .$$

We have the following Sobolev inequality for the Heisenberg group (valid in more general settings, see [Lu96]).

Theorem 2.1.2. *Let $B_r \subset \mathbb{H}$ and $1 \le q < Q$. Then:*

$$\left(\fint_{B_r} |u|^{\frac{Qq}{Q-q}}\right)^{\frac{Q-q}{Qq}} \le C_q r \left(\fint_{B_r} |\nabla_{\mathbb{H}} u|^q\right)^{\frac{1}{q}}$$

for all $u \in HW_0^{1,q}(B_r)$.

We have also a Rellich type Theorem (see [HK00], Remark after Proposition 11.17).

Theorem 2.1.3. *Let Ω be a bounded open subset of $\mathbb{H}$ and $1 < q < Q$. Then $HW_0^{1,q}(\Omega)$ is compactly embedded in $L^{\frac{qQ}{Q-q}}(\Omega)$.*

2.2 p-LAPLACE EQUATION

The p-Laplace equation in $\mathbb{H}$, for $1 < p < \infty$, is:

$$\sum_{i=1}^{2} X_i(|\nabla_{\mathbb{H}} u|^{p-2} X_i u) = 0 \quad \text{in } \Omega, \tag{2.5}$$

where $\Omega \subset \mathbb{H}$ is open. It is the Euler-Lagrange equation of the p-Dirichlet functional:

$$\mathcal{D}_p(u) = \frac{1}{p} \int_\Omega |\nabla_{\mathbb{H}} u|^p \, \mathrm{d}x. \tag{2.6}$$

We introduce and study the regularized equations:

$$\sum_{i=1}^{2} X_i((\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} X_i u) = 0 \quad \text{in } \Omega, \tag{2.7}$$

for $\delta > 0$. We aim at obtaining estimates independent of δ that can be passed to the limit as $\delta \to 0$.

We say that a function $u \in HW^{1,p}(\Omega)$ is a weak solution of (2.7) if:

$$\int_\Omega (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} \varphi \rangle \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in HW_0^{1,p}(\Omega) \, . \tag{2.8}$$

Denoting $z = (z_1, z_2) \in \mathbb{R}^2$ and calling:

$$a_i(z) = (\delta^2 + |z|^2)^{\frac{p-2}{2}} z_i \quad \text{and} \quad w = \delta^2 + |\nabla_{\mathbb{H}} u|^2 \, , \tag{2.9}$$

equation (2.7) rewrites as:

$$\sum_{i=1}^{2} X_i a_i(\nabla_{\mathbb{H}} u) = 0 \quad \text{in } \Omega \tag{2.10}$$

and satisfies the following ellipticity and growth conditions, for all $p > 1$:

$$\begin{gathered} \sum_{i,j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) \xi_i \xi_j \geq c_p w^{\frac{p-2}{2}} |\xi|^2, \\ |a_i(\nabla_{\mathbb{H}} u)| \leq w^{\frac{p-1}{2}}, \\ |\partial_{z_j} a_i(\nabla_{\mathbb{H}} u)| \leq C_p w^{\frac{p-2}{2}}, \\ |\partial_{z_s} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u)| \leq C_p w^{\frac{p-3}{2}}. \end{gathered} \tag{2.11}$$

Also, we have for all $p > 1$:

$$\left|\frac{\partial_{z_j} a_i(z)}{\partial_{z_l} a_l(z)}\right| \leq C_p \quad \text{for all } i, j, l \in \{1, 2\}. \tag{2.12}$$

We remark that the proofs presented in this work depend only on these properties, therefore they extend to more general equations of p-Laplacian type as in (2.10) for a_i of class C^2 satisfying (2.11) and (2.12).

The growth conditions in (2.11) follow from:

$$\begin{aligned}
\partial_{z_j} a_i(z) &= (p-2) z_i z_j (\delta^2 + |z|^2)^{\frac{p-4}{2}} + \delta_{i,j} (\delta^2 + |z|^2)^{\frac{p-2}{2}}, \\
\partial_{z_s} \partial_{z_j} a_i(z) &= (p-2) \delta_{i,s} z_j (\delta^2 + |z|^2)^{\frac{p-4}{2}} + (p-2) \delta_{j,s} z_i (\delta^2 + |z|^2)^{\frac{p-4}{2}} \\
&\quad + (p-2)(p-4) z_i z_j z_s (\delta^2 + |z|^2)^{\frac{p-6}{2}} + (p-2) \delta_{i,j} z_s (\delta^2 + |z|^2)^{\frac{p-4}{2}};
\end{aligned}$$

while (2.12) follows from the fact that for for $p > 2$ we have:

$$\left|\frac{\partial_{z_j} a_i(z)}{\partial_{z_l} a_l(z)}\right| \leq \frac{(p-2)|z_i z_j| + \delta_{i,j}(\delta^2 + |z|^2)}{(p-2) z_l^2 + (\delta^2 + |z|^2)} \leq (p-2) \frac{|z_i z_j|}{\delta^2 + |z|^2} + \delta_{i,j} \leq C_p \quad \text{for all } i, j, l \in \{1, 2\}$$

and for $1 < p < 2$:

$$\left|\frac{\partial_{z_j} a_i(z)}{\partial_{z_l} a_l(z)}\right| \leq \frac{(2-p)|z_i z_j| + \delta_{i,j}(\delta^2 + |z|^2)}{(p-1)(\delta^2 + |z|^2)} \leq \frac{2-p}{p-1} \frac{|z_i z_j|}{\delta^2 + |z|^2} + \frac{\delta_{i,j}}{p-1} \leq C_p \quad \text{for all } i, j, l \in \{1, 2\}.$$

2.3 EXISTENCE, UNIQUENESS AND CONVERGENCE

Equation (2.7) is the Euler-Lagrange equation for the regularized p-Dirichlet functional:

$$\mathcal{D}_p^\delta(u) = \frac{1}{p}\int_\Omega (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p}{2}}\,\mathrm{d}x\ . \tag{2.13}$$

We will use the notation $\mathcal{D}_p = \mathcal{D}_p^0$. We have:

Theorem 2.3.1. *Let $\psi \in HW^{1,p}(\Omega)$ and $\mathcal{A}_\psi = \{f \in HW^{1,p}(\Omega) \mid f - \psi \in HW_0^{1,p}(\Omega)\}$. Then v is a weak solution of the Dirichlet problem:*

$$\begin{cases} \operatorname{div}_{\mathbb{H}}\Big((\delta^2 + |\nabla_{\mathbb{H}} v|^2)^{\frac{p-2}{2}}\nabla_{\mathbb{H}} v\Big) = 0 & \text{in } \Omega \\ v - \psi \in HW_0^{1,p}(\Omega) \end{cases} \tag{2.14}$$

if and only if v is a minimum of the p-Dirichlet functional $\mathcal{D}_{p,\delta}$ in $\mathcal{A}_\psi$.

Using the direct method of the calculus of variations we get the existence and uniqueness for the solution of the Dirichlet problem for the p-Laplace equation by proving existence and uniqueness for the minimun of the p-Dirichlet functional:.

Theorem 2.3.2. *Let $1 < p < \infty$, $\delta \geq 0$. There exists a unique solution of the Dirichlet problem (2.14).*

We now show that solutions to the regularized problem converge to solutions of the (degenerate) p-Laplace equation as the non degeneracy parameter δ tends to 0.

Theorem 2.3.3. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (2.5) for $1 < p < \infty$. Fix $B_{R_0} \subset\subset \Omega$ and let u^δ be the solution of the Dirichlet problem (2.14) with boundary datum $\psi = u$. Then u^δ converges (up to a subsequence) to u in $HW^{1,p}(B_{R_0})$.*

Proof. In the above notation for the p-Dirichlet functional $\mathcal{D}_p^\delta$ on B_{R_0} we have:

$$\int_{B_{R_0}} |\nabla_{\mathbb{H}} u^\delta|^p\,\mathrm{d}x = p\mathcal{D}_p(u^\delta) \leq p\mathcal{D}_p^\delta(u^\delta) \leq p\mathcal{D}_p^\delta(u),$$

which implies u^δ is bounded in $HW^{1,p}(B_{R_0})$ uniformly in δ, for $0 < \delta \leq 1$. Therefore u^δ converges weakly in $HW^{1,p}(B_{R_0})$ to some $v \in HW^{1,p}(B_{R_0})$ (up to a non relabeled subsequence).

By weakly lower semicontinuity, we have:

$$\mathcal{D}_p(v) \leq \liminf_{\delta \to 0} \mathcal{D}_p(u^\delta) \leq \liminf_{\delta \to 0} \mathcal{D}_p^\delta(u^\delta) \leq \liminf_{\delta \to 0} \mathcal{D}_p^\delta(u) = \mathcal{D}_p(u),$$

where we also used the minimization property of u^δ. Since $v - u \in HW_0^{1,p}(B_{R_0})$, by uniqueness of the minimizer of the functional $\mathcal{D}_p$ in $HW^{1,p}(B_{R_0})$ with given boundary value, we get $v = u$.

We now show that the convergence is strong. Adding and subtracting the term $(\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} \nabla_{\mathbb{H}} u \nabla_{\mathbb{H}} \phi$ in the weak formulation of (2.7), and choosing $\phi = u^\delta - u$ we obtain:

$$\begin{aligned} \int_{B_{R_0}} \langle (\delta^2 + |\nabla_{\mathbb{H}} u^\delta|^2)^{\frac{p-2}{2}} \nabla_{\mathbb{H}} u^\delta - (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} u^\delta - \nabla_{\mathbb{H}} u \rangle \, \mathrm{d}x \\ + \int_{B_{R_0}} (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} u^\delta - \nabla_{\mathbb{H}} u \rangle \, \mathrm{d}x = 0. \end{aligned} \tag{2.15}$$

By weak convergence, the second integral tends to 0. Using the inequality:

$$|a - b|^p \leq C_p \langle (\delta^2 + a^2)^{\frac{p-2}{2}} a - (\delta^2 + b^2)^{\frac{p-2}{2}} b, a - b \rangle$$

valid for $p \geq 2$, we get the strong convergence of the gradients.

Consider now $1 < p < 2$ and denote $W = \delta^2 + |\nabla_{\mathbb{H}} u^\delta|^2 + |\nabla_{\mathbb{H}} u|^2$. We get:

$$\begin{aligned} \int_{B_{R_0}} |\nabla_{\mathbb{H}} u^\delta - \nabla_{\mathbb{H}} u|^p \, \mathrm{d}x &= \int_{B_{R_0}} |\nabla_{\mathbb{H}} u^\delta - \nabla_{\mathbb{H}} u|^p W^{\frac{(p-2)p}{4}} W^{\frac{(2-p)p}{4}} \, \mathrm{d}x \\ &\leq \left(\int_{B_{R_0}} |\nabla_{\mathbb{H}} u^\delta - \nabla_{\mathbb{H}} u|^2 W^{\frac{p-2}{2}} \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_{R_0}} W^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{2-p}{2}}. \end{aligned}$$

The last factor is bounded by weak convergence of $\nabla_{\mathbb{H}} u^\delta$, therefore combining (2.15) and the inequality:

$$(\delta^2 + |a|^2 + |b|^2)^{\frac{p-2}{2}} |a - b|^2 \leq C_p \langle (\delta^2 + a^2)^{\frac{p-2}{2}} a - (\delta^2 + b^2)^{\frac{p-2}{2}} b, a - b \rangle$$

valid for all $1 < p < \infty$, we get convergence of the gradients also for the case $1 < p < 2$. □

3.0 REGULARITY

This Chapter develops the theory necessary for our main regularity result and presents its proof.

3.1 NON DEGENERATE EQUATION

We now collect some known results about the non degenerate equation (2.7) that will be used in the following sections. We refer to [Ric15] for a comprehensive presentation and complete proofs.

First we have that solutions of the non degenerate p-Laplace equation (2.7) are smooth. This was proved by Capogna in [Cap97] for $p \geq 2$ under the additional assumption $M^{-1} \leq |\nabla_{\mathbb{H}} u| \leq M$. Without the additional boundedness of the horizontal gradient, it was proved by Manfredi and Mingione in [MM07] for $2 \leq p < c(n) < 4$, by Mingione, Zatorska-Goldstein and Zhong in [MZGZ09] for $2 \leq p < 4$ and finally extended to the full range $1 < p < \infty$ in [Ric15] by adapting techniques of Domokos in [Dom04].

Theorem 3.1.1. *Let $u \in HW^{1,p}(\Omega)$, $1 < p < \infty$ be a weak solution of the non-degenerate p-Laplace equation (2.7). Then $u \in C^{\infty}(\Omega)$.*

As a consequence, we have the pointwise equation:

$$\partial_{z_1} a_1(\nabla_{\mathbb{H}} u) X_1 X_1 u + \partial_{z_2} a_1(\nabla_{\mathbb{H}} u) X_1 X_2 u + \partial_{z_1} a_2(\nabla_{\mathbb{H}} u) X_2 X_1 u + \partial_{z_2} a_2(\nabla_{\mathbb{H}} u) X_2 X_2 u = 0$$

in Ω, hence:

$$
\begin{aligned}
X_1X_1u &= -\frac{1}{\partial_{z_1}a_1(\nabla_{\mathbb{H}}u)}\left(\partial_{z_2}a_2(\nabla_{\mathbb{H}}u)X_2X_2u + \partial_{z_2}a_1(\nabla_{\mathbb{H}}u)X_1X_2u + \partial_{z_1}a_2(\nabla_{\mathbb{H}}u)X_2X_1u\right),\\
X_2X_2u &= -\frac{1}{\partial_{z_2}a_2(\nabla_{\mathbb{H}}u)}\left(\partial_{z_1}a_1(\nabla_{\mathbb{H}}u)X_1X_1u + \partial_{z_2}a_1(\nabla_{\mathbb{H}}u)X_1X_2u + \partial_{z_1}a_2(\nabla_{\mathbb{H}}u)X_2X_1u\right).
\end{aligned}
\tag{3.1}
$$

These will be used in the following, as they allow to express X_1X_1u (respectively X_2X_2u) in terms of X_iX_ju where at least one index is a 2 (respectively a 1).

We now collect the equations satisfied by the horizontal and vertical derivatives:

Lemma 3.1.1. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $1 < p < \infty$. The functions X_1u, X_2u and Tu are weak solutions respectively of the following equations (in Ω):*

$$\sum_{i=1}^{2} X_i\left(\sum_{j=1}^{2}\partial_{z_j}a_i(\nabla_{\mathbb{H}}u)X_jX_1u\right) + \sum_{i=1}^{2} X_i\left(\partial_{z_2}a_i(\nabla_{\mathbb{H}}u)Tu\right) + T\left(a_2(\nabla_{\mathbb{H}}u)\right) = 0, \tag{3.2}$$

$$\sum_{i=1}^{2} X_i\left(\sum_{j=1}^{2}\partial_{z_j}a_i(\nabla_{\mathbb{H}}u)X_jX_2u\right) - \sum_{i=1}^{2} X_i\left(\partial_{z_1}a_i(\nabla_{\mathbb{H}}u)Tu\right) - T\left(a_1(\nabla_{\mathbb{H}}u)\right) = 0, \tag{3.3}$$

$$\sum_{i=1}^{2} X_i\left(\sum_{j=1}^{2}\partial_{z_j}a_i(\nabla_{\mathbb{H}}u)X_jTu\right) = 0. \tag{3.4}$$

Proof. We start by proving (3.2). Consider $\varphi \in C_0^\infty(\Omega)$ and use $\psi = X_1\varphi$ as a test function in (2.8). Since the horizontal vector fields do not commute, terms involving the vertical vector field T appear. We get:

$$\int_\Omega \sum_{i=1}^{2} a_i(\nabla_{\mathbb{H}}u)X_iX_1\varphi\,\mathrm{d}x = 0\,. \tag{3.5}$$

Keeping in mind the commutation relation $X_1X_2 - X_2X_1 = T$ we obtain:

$$\int_\Omega \sum_{i=1}^{2} a_i(\nabla_{\mathbb{H}}u)X_1X_i\varphi\,\mathrm{d}x - \int_\Omega a_2(\nabla_{\mathbb{H}}u)T\varphi\,\mathrm{d}x = 0\,. \tag{3.6}$$

In the first integral we integrate by parts with respect to X_1 to get:

$$
\begin{aligned}
&\int_\Omega \sum_{i=1}^{2} a_i(\nabla_{\mathbb{H}} u) X_1 X_i \varphi \, \mathrm{d}x = -\int_\Omega \sum_{i=1}^{2} X_1 \left(a_i(\nabla_{\mathbb{H}} u)\right) X_i \varphi \, \mathrm{d}x \\
&= -\int_\Omega \sum_{i=1}^{2} \sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_1 X_j u X_i \varphi \, \mathrm{d}x \\
&= -\int_\Omega \sum_{i,j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j X_1 u X_i \varphi \, \mathrm{d}x - \int_\Omega \sum_{i=1}^{2} \partial_{z_2} a_i(\nabla_{\mathbb{H}} u) T u X_i \varphi \, \mathrm{d}x \, .
\end{aligned}
\tag{3.7}
$$

Combining (3.6) and (3.7) we obtain:

$$
\int_\Omega \sum_{i=1}^{2} \left(\sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j X_1 u \right) X_i \varphi + \int_\Omega \sum_{i=1}^{2} \partial_{z_2} a_i(\nabla_{\mathbb{H}} u) T u X_i \varphi + a_2(\nabla_{\mathbb{H}} u) T \varphi = 0, \tag{3.8}
$$

which is the weak formulation of (3.2). Equation (3.3) is obtained in a similar fashion using $\psi = X_2 \varphi$ as a test function.

To prove (3.4) use $\psi = T\varphi$ as a test function in (2.8). This time X_i and T commute, so we can exchange their order and integrate by parts:

$$
\begin{aligned}
0 &= -\int_\Omega \sum_{i=1}^{2} a_i(\nabla_{\mathbb{H}} u) X_i T \varphi \, \mathrm{d}x = -\int_\Omega \sum_{i=1}^{2} a_i(\nabla_{\mathbb{H}} u) T X_i \varphi \, \mathrm{d}x \\
&= \int_\Omega \sum_{i=1}^{2} T \left(a_i(\nabla_{\mathbb{H}} u)\right) X_i \varphi \, \mathrm{d}x = \int_\Omega \sum_{i=1}^{2} \left(\sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) T X_j u \right) X_i \varphi \, \mathrm{d}x \\
&= \int_\Omega \sum_{i=1}^{2} \left(\sum_{j=1}^{2} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j T u \right) X_i \varphi \, \mathrm{d}x
\end{aligned}
\tag{3.9}
$$

which is the weak formulation of (3.4). □

The following estimate will be essential. It was proved in [Zho18] as a consequence of several Caccioppoli-type estimates involving Tu and $\nabla^2_{\mathbb{H}} u$.

Lemma 3.1.2. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $1 < p < \infty$. Then, for all $q \geq 4$ and $\xi \in C_0^\infty(\Omega)$ we have:*

$$
\int_\Omega \xi^q \, w^{\frac{p-2}{2}} \, |Tu|^q \, \mathrm{d}x \leq C(q) \left(\|\nabla_{\mathbb{H}} \xi\|^2_{L^\infty} + \|\xi T \xi\|_{L^\infty} \right)^{\frac{q}{2}} \int_{\operatorname{supp}(\xi)} w^{\frac{p-2+q}{2}} \, \mathrm{d}x, \tag{3.10}
$$

where w is defined in (2.9), $C(q) = C_p^{\frac{q-2}{2}} q^{q+8}$ and C_p depends only on p.

Lemma 3.1.2 is a consequence of the following Lemmas which can be found in [Zho18] (see also [Ric15], Lemmas 5.3 and 5.4).

Lemma 3.1.3. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $1 < p < \infty$. Let $q \geq 4$ and $\xi \in C_0^\infty(\Omega)$. Then:*

$$\int_\Omega \xi^q \, w^{\frac{p-2}{2}} \, |Tu|^{q-2} \, |\nabla_{\mathbb{H}}^2 u|^2 \, \mathrm{d}x \leq C_p^{\frac{q-2}{2}} \, (q-1)^{q-2} \, \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^{q-2} \int_\Omega \xi^2 \, w^{\frac{p+q-4}{2}} \, |\nabla_{\mathbb{H}}^2 u|^2 \, \mathrm{d}x \, , \tag{3.11}$$

where w is defined in (2.9) and C_p depends only on p.

Lemma 3.1.4. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $1 < p < \infty$. Let $q \geq 4$ and $\xi \in C_0^\infty(\Omega)$. Then:*

$$\int_\Omega \xi^2 \, w^{\frac{p+q-4}{2}} \, |\nabla_{\mathbb{H}}^2 u|^2 \, \mathrm{d}x \leq C_p \, \left(\|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^2 + \xi \, \|T\xi\|_{L^\infty} \right) \, (q-1)^{10} \int_{\mathrm{supp}\,(\xi)} w^{\frac{p+q-2}{2}} \, \mathrm{d}x,$$

where w is defined in (2.9) and C_p depends only on p.

Lemma 3.1.3 follows by using $\phi = \xi^q |Tu|^{q-2} X_i u$ as test functions in equations (3.2) and (3.3), while Lemma 3.1.4 follows by using $\phi = \xi^2 w^{\frac{q-2}{2}} X_i u$ and the estimate in Lemma 3.1.3.

Proof of Lemma 3.1.2. Using $|Tu| \leq 2|\nabla_{\mathbb{H}}^2 u|$ and Lemmas 3.1.3 and 3.1.4, we have for $q \geq 4$:

$$\begin{aligned} \int_\Omega \xi^q \, w^{\frac{p-2}{2}} \, |Tu|^q \, \mathrm{d}x &\leq 4 \int_\Omega \xi^q \, w^{\frac{p-2}{2}} \, |Tu|^{q-2} \, |\nabla_{\mathbb{H}}^2 u|^2 \, \mathrm{d}x \\ &\leq C_p^{\frac{q-2}{2}} (q-1)^{q-2} \, \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^{q-2} \int_\Omega \xi^2 \, w^{\frac{p+q-4}{2}} \, |\nabla_{\mathbb{H}}^2 u|^2 \, \mathrm{d}x \\ &\leq C^{\frac{q-2}{q}} (q)^{q+8} \left(\|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^2 + \xi \, \|T\xi\|_{L^\infty} \right)^{\frac{q}{2}} \int_{\mathrm{supp}\,(\xi)} w^{\frac{p+q-2}{2}} \, \mathrm{d}x. \end{aligned} \tag{3.12}$$

□

We have a uniform Lipschitz bound, originally proved in [Zho18]. We remark that this estimate also holds for $\delta = 0$, through the approximation procedure described in [Ric15], Theorem 5.3.

Theorem 3.1.2. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $1 < p < \infty$. Then:*

$$\|\nabla_{\mathbb{H}} u\|_{L^\infty(B_r)} \leq C_p \left(\fint_{B_{2r}} (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{p}} \tag{3.13}$$

for every ball B_r such that the concentric ball $B_{2r} \subset \Omega$.

3.2 DE GIORGI CLASSES IN THE HEISENBERG GROUP

We now describe a type of De Giorgi class in the Heisenberg group. These kind of spaces were introduced and studied by De Giorgi in the Euclidean case (see [DG57]). They can be defined in the Heisenberg groups $\mathbb{H}^n$ and for exponents $p \neq 2$, but for our purposes we just need to consider $p = 2$ and $n = 1$. In order to emphasize the role of the homogeneous dimension, we will keep the general notation Q, which is equal to 4 for $\mathbb{H}$. We will use the standard notation for super- (sub-) level sets of a measurable function:

$$A^+_{k,r} = A^+_{k,r}(f) := B_r \cap \{f > k\},$$
$$A^-_{k,r} = A^-_{k,r}(f) := B_r \cap \{f < k\}.$$

Definition 3.2.1 (De Giorgi class in the Heisenberg group)**.** *Let $\Omega \subset \mathbb{H}$ be open, γ, χ positive real constants and $q > Q = 4$. A function $f \in HW^{1,2}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ belongs to the De Giorgi class $DG^+(\Omega, \gamma, \chi, q)$ if:*

$$\int_{A^+_{h,r'}} |\nabla_{\mathbb{H}} f|^2 \, \mathrm{d}x \leq \frac{\gamma}{(r - r')^2} \sup_{B_r} |(f - h)^+|^2 |A^+_{k,r}| + \chi |A^+_{k,r}|^{1 - \frac{2}{q}} \tag{3.14}$$

for some concentric balls $B_{r'} \subset B_r \subset\subset \Omega$ and levels $h \in \mathbb{R}$.

We say $f \in DG^-(\Omega, \gamma, \chi, q)$ if $-f \in DG^+(\Omega, \gamma, \chi, q)$.

In this section we consider an arbitrary ball $B_R \subset\subset \Omega$ and denote by $M = M(R) = \sup_{B_R} f$ and $m = m(R) = \inf_{B_R} f$. First we recall a technical Lemma about sequences.

Lemma 3.2.2 (Fast Geometric Convergence)**.** *Let μ_i be a sequence of positive real numbers such that:*

$$\mu_{i+1} \leq C\, b^{i+1} \mu_i^{1+\varepsilon} \tag{3.15}$$

for some $C, \varepsilon > 0$ and $b > 1$. If:

$$\mu_0 \leq C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}} \tag{3.16}$$

then $\mu_i \longrightarrow 0$.

We are taking the following Lemma from [KMMP12], Lemma 2.3, where it is proved in more general settings.

Lemma 3.2.3. *Let $\Omega \subset \mathbb{H}$, $l > k$, $f \in HW^{1,1}_{loc}(\Omega)$, $B_r \subset\subset \Omega$. Denote $A^+_{l,r} = A^+_{l,r}(f)$ for simplicity. Then, if $|B_r \setminus A^+_{k,r}| > 0$ we have:*

$$(l-k)|A^+_{l,r}|^{1-\frac{1}{Q}} \leq \frac{C(Q)r^Q}{|B_r \setminus A^+_{k,r}|} \int_{A^+_{k,r} \setminus A^+_{l,r}} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x, \tag{3.17}$$

where $C(Q)$ is a constant depending only on Q.

Proof. Consider $v = (\min\{f, l\} - k)^+$, which belongs to $HW^{1,1}(B_r)$. We use the notation $1^* = \frac{Q}{Q-1}$ for the Sobolev's conjugated exponent. Now, since $v = l - k$ on $A^+_{l,r}$ we get:

$$\int_{B_r} |v|^{1^*} \, \mathrm{d}x \geq \int_{A^+_{l,r}} (l-k)^{1^*} \, \mathrm{d}x = (l-k)^{1^*} |A^+_{l,r}|.$$

Hence:

$$(l-k)|A^+_{l,r}|^{\frac{1}{1^*}} \leq \left(\int_{B_r} |v|^{1^*} \, \mathrm{d}x \right)^{\frac{1}{1^*}} \leq \left(\int_{B_r} |v - v_{B_r}|^{1^*} \, \mathrm{d}x \right)^{\frac{1}{1^*}} + \left(|v_{B_r}|^{1^*} |B_r| \right)^{\frac{1}{1^*}}. \tag{3.18}$$

Towards estimating the second term in the previous inequality consider:

$$\int_{B_r} |v - v_{B_r}|^{1^*} \, \mathrm{d}x \geq \int_{B_r \setminus A^+_{k,r}} |v - v_{B_r}|^{1^*} \, \mathrm{d}x = \int_{B_r \setminus A^+_{k,r}} |v_{B_r}|^{1^*} \, \mathrm{d}x = |v_{B_r}|^{1^*} |B_r \setminus A^+_{k,r}| \tag{3.19}$$

since $\operatorname{supp}(v) \subseteq A^+_{k,r}$. Now from (3.18) and (3.19) we obtain:

$$(l-k)|A^+_{l,r}|^{\frac{1}{1^*}} \leq 2 \left(\frac{|B_r|}{|B_r \setminus A^+_{k,r}|} \int_{B_r} |v - v_{B_r}|^{1^*} \, \mathrm{d}x \right)^{\frac{1}{1^*}}.$$

Now use the Sobolev-Poincaré inequality (2.1.2):

$$\left(\fint_{B_r} |v - v_{B_r}|^{1^*} \, \mathrm{d}x \right)^{\frac{1}{1^*}} \leq Cr \fint_{B_r} |\nabla_{\mathbb{H}} v| \, \mathrm{d}x$$

to get:

$$\begin{aligned}(l-k)|A^+_{l,r}|^{1-\frac{1}{Q}} \leq Cr \left(\frac{|B_r|^2}{|B_r \setminus A^+_{k,r}|} \right)^{1-\frac{1}{Q}} \fint_{B_r} |\nabla_{\mathbb{H}} v| \, \mathrm{d}x &\leq C \frac{r^{Q-1} |B_r \setminus A^+_{k,r}|^{\frac{1}{Q}}}{|B_r \setminus A^+_{k,r}|} \int_{A^+_{k,r} \setminus A^+_{l,r}} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x \\ &\leq \frac{Cr^Q}{|B_r \setminus A^+_{k,r}|} \int_{A^+_{k,r} \setminus A^+_{l,r}} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x,\end{aligned}$$

after observing that $\operatorname{supp}(\nabla_{\mathbb{H}} v) \subseteq A^+_{k,r} \setminus A^+_{l,r}$. □

The next Lemma is adapted from Lemma 2.3 in [Man86] and Lemma 6.1 in [LU68].

Lemma 3.2.4. *Let $\Omega \subset \mathbb{H}$, $B_R \subset\subset \Omega$, $0 < \lambda_0,\ \lambda_1 < 1$, $M \geq \sup_{B_R} f$ and $k < M$. Suppose $f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [k, \lambda_0 k + (1-\lambda_0)M]$ and for $r' < r \in [\lambda_1 R, R]$. Then there exists $\theta = \theta(Q, \gamma, \lambda_0, \lambda_1) \in (0,1)$ such that if:*

$$M - k \geq \chi^{\frac{1}{2}} R^{1-\frac{Q}{q}},$$

then

$$|A^+_{k,R}| \leq \theta |B_R| \quad \textit{implies} \quad f \leq \lambda_0 k + (1-\lambda_0)M \ \ \textit{a.e. in} \ \ B_{\lambda_1 R}.$$

Proof. Consider a sequence of radii decreasing from R to $\lambda_1 R$ and a sequence of level sets increasing from k to $\lambda_0 k + (1-\lambda_0)M$ defined as follows:

$$r_i = \lambda_1 R + (1-\lambda_1)^{i+1} R,$$
$$k_i = k + (1-\lambda_0)(M-k) - (1-\lambda_0)^{i+1}(M-k)$$

for $i \geq 0$. Observe that $k_{i+1} - k_i = (M-k)\lambda_0(1-\lambda_0)^{i+1}$. Using (3.17) with levels k_{i+1} and k_i and radius r_{i+1} we get:

$$(M-k)\lambda_0(1-\lambda_0)^{i+1} |A^+_{k_{i+1},r_{i+1}}|^{1-\frac{1}{Q}} \leq C(Q) \frac{r_{i+1}^Q}{|B_{r_{i+1}} \setminus A^+_{k_i,r_{i+1}}|} \int_{A^+_{k_i,r_{i+1}} \setminus A^+_{k_{i+1},r_{i+1}}} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x.$$

Let $\theta_1 = \frac{|B_{\lambda_1 R}|}{2|B_R|} = \frac{\lambda_1^Q}{2}$. Now if $|A^+_{k,R}| \leq \theta_1 |B_R|$ we have:

$$|B_{r_{i+1}} \setminus A^+_{k_i,r_{i+1}}| = |B_{r_{i+1}}| - |A^+_{k_i,r_{i+1}}| \geq |B_{\lambda_1 R}| - |A^+_{k,R}| \geq |B_{\lambda_1 R}| - \theta_1 |B_R| \geq \frac{1}{2} |B_{\lambda_1 R}|.$$

Therefore, including the dependence of λ_1 in the constant C, we get:

$$\begin{aligned}
(M-k)\lambda_0(1-\lambda_0)^{i+1} |A^+_{k_{i+1},r_{i+1}}|^{1-\frac{1}{Q}} &\leq C(Q,\lambda_1) \frac{R^Q}{|B_{\lambda_1 R}|} \int_{A^+_{k_i,r_{i+1}}} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x \leq C(Q,\lambda_1) \int_{A^+_{k_i,r_{i+1}}} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x \\
&\leq C(Q,\lambda_1) \left(\int_{A^+_{k_i,r_{i+1}}} |\nabla_{\mathbb{H}} f|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} |A^+_{k_i,r_{i+1}}|^{\frac{1}{2}} \\
&\leq C(Q,\lambda_1,\gamma) \left(\frac{1}{r_i - r_{i+1}} \sup_{B_{r_i}} (f-k_i)^+ |A^+_{k_i,r_i}|^{\frac{1}{2}} + \chi^{\frac{1}{2}} \ |A^+_{k_i,r_i}|^{\frac{1}{2}-\frac{1}{q}} \right) |A^+_{k_i,r_i}|^{\frac{1}{2}} \\
&\leq C(Q,\lambda_1,\gamma) \left(\frac{1}{r_i - r_{i+1}} \sup_{B_{r_i}} (f-k_i)^+ |A^+_{k_i,r_i}|^{\frac{1}{q}} + \chi^{\frac{1}{2}} \right) |A^+_{k_i,r_i}|^{1-\frac{1}{q}},
\end{aligned}$$

where we have used the definition of De Giorgi class (3.14) and included the dependence on γ in the constant. Noting that $(f-k_i)^+ \leq M-k$ on B_{r_i} and $r_i - r_{i+1} = \lambda_1(1-\lambda_1)^{i+1}R$ we get (including the dependence on λ_0 and λ_1 in the constant):

$$\begin{aligned} |A^+_{k_{i+1},r_{i+1}}|^{1-\frac{1}{Q}} &\leq \frac{C(Q,\lambda_0,\lambda_1,\gamma)}{(M-k)(1-\lambda_0)^i}\left(\frac{M-k}{\lambda_1(1-\lambda_1)^{i+1}R}|A^+_{k_i,r_i}|^{\frac{1}{q}} + \chi^{\frac{1}{2}}\right)|A^+_{k_i,r_i}|^{1-\frac{1}{q}} \\ &\leq \frac{C(Q,\lambda_0,\lambda_1,\gamma)}{[(1-\lambda_0)(1-\lambda_1)]^i}|A^+_{k_i,r_i}|^{1-\frac{1}{q}}\left(R^{\frac{Q}{q}-1} + \frac{\chi^{\frac{1}{2}}}{M-k}\right), \end{aligned}$$

since $1-\lambda_1 < 1$. Now if $M-k \geq \chi^{\frac{1}{2}}R^{1-\frac{Q}{q}}$ we obtain:

$$|A^+_{k_{i+1},r_{i+1}}|^{1-\frac{1}{Q}} \leq \frac{C(Q,\lambda_0,\lambda_1,\gamma)}{[(1-\lambda_0)(1-\lambda_1)]^i}|A^+_{k_i,r_i}|^{1-\frac{1}{q}}\ R^{\frac{Q}{q}-1}.$$

Raising to the $\frac{Q}{Q-1}$, dividing by R^Q and denoting by $\mu_i = \frac{|A^+_{k_i,r_i}|}{R^Q}$ we get:

$$\mu_{i+1} \leq C(Q,\lambda_0,\lambda_1,\gamma)\{[(1-\lambda_0)(1-\lambda_1)]^{-\frac{Q}{Q-1}}\}^i\ \mu_i^{\left(1-\frac{1}{q}\right)\frac{Q}{Q-1}}.$$

Hence we are in the position to apply the Fast Geometric Convergence Lemma 3.2.2 with $b = [(1-\lambda_0)(1-\lambda_1)]^{-\frac{Q}{Q-1}} > 1$, $\varepsilon = \frac{(q-1)Q}{(Q-1)q} - 1 = \frac{q-Q}{q(Q-1)} > 0$ to conclude $\mu_i \longrightarrow 0$, provided

$$\frac{|A^+_{k,R}|}{R^Q} = \mu_0 \leq C(Q,\lambda_0,\lambda_1,\gamma)[(1-\lambda_0)(1-\lambda_1)]^{Q(Q-1)} = \theta_2(Q,\lambda_0,\lambda_1,\gamma),$$

where we used $\left(\frac{q}{q-Q}\right)^2 > 1$ for $q > Q$ to get rid of the dependence of q in the last exponent. Since μ_i converges to $|A^+_{\lambda_0 k+(1-\lambda_0)M,\lambda_1 R}|R^{-Q}$, taking $\theta = \min\{\theta_1,\theta_2\}$ we have the desired conclusion. □

Remark 3.2.5. *The assumption $q > Q$ is essential in the previous proof in order to apply the Fast Geometric Convergence Lemma (it is necessary to have $\varepsilon > 0$).*

The following Lemma is adapted from Lemma 2.4 in [Man86] and Lemma 6.2 in [LU68].

Lemma 3.2.6. *Let $\Omega \subset \mathbb{H}$, $B_R \subset\subset \Omega$, $0 < \lambda_1 < 1$, $M \geq \sup_{B_R} f$ and $k < M$. Suppose $f \in DG^+(\Omega,\gamma,\chi,q)$ for $h \in [k,M]$ and for $r' = \lambda_1 R$, $r = R$. If there exists a constant $0 < C_0 < 1$ such that $|A^+_{k,\lambda_1 R}| \leq C_0|B_{\lambda_1 R}|$ then, given $0 < \theta < 1$, there exists $s = s(Q,\gamma,\lambda_1,C_0,\theta) \in \mathbb{N}$ such that:*

$$\textit{if} \quad M-k \geq 2^s\chi^{\frac{1}{2}}R^{1-\frac{Q}{q}} \quad \textit{then} \quad |A^+_{k_s,\lambda_1 R}| \leq \theta|B_{\lambda_1 R}|,$$

where $k_s = k + (1-2^{-s})(M-k)$ is a level set between k and M.

Proof. Fix $s \in \mathbb{N}$ that will be chosen to satisfy some constraints during the course of the proof. Define a sequence of levels increasing from k to M as follows:

$$k_i = k + (1 - 2^{-i})(M - k),$$

and denote by $D_i = A^+_{k_i,\lambda_1 R} \setminus A^+_{k_{i+1},\lambda_1 R}$. Noting that $k_{i+1} - k_i = (M-k)2^{-(i+1)}$ and using (3.17) with levels k_{i+1} and k_i and radius $\lambda_1 R$ we get:

$$\frac{M-k}{2^{i+1}} |A^+_{k_{i+1},\lambda_1 R}|^{1-\frac{1}{Q}} \leq C(Q) \frac{\lambda_1^Q R^Q}{|B_{\lambda_1 R} \setminus A^+_{k_i,\lambda_1 R}|} \int_{D_i} |\nabla_{\mathbb{H}} f| \, \mathrm{d}x.$$

Since $A^+_{k_i,\lambda_1 R} \subseteq A^+_{k,\lambda_1 R}$, the hypotheses imply $|A^+_{k_i,\lambda_1 R}| \leq C_0 |B_{\lambda_1 R}|$, hence $|B_{\lambda_1 R} \setminus A^+_{k_i,\lambda_1 R}| \geq (1 - C_0)|B_{\lambda_1 R}|$. Therefore, including the dependences of the data in the constant and using the Cauchy-Schwartz inequality we obtain:

$$\begin{aligned}
\frac{M-k}{2^{i+1}} |A^+_{k_{i+1},\lambda_1 R}|^{1-\frac{1}{Q}} &\leq C(Q, C_0) \left(\int_{D_i} |\nabla_{\mathbb{H}} f|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} |D_i|^{\frac{1}{2}} \\
&\leq C(Q, C_0, \gamma, \lambda_1) \left(\frac{\sup_{B_R} (f - k_i)^+}{R} |A^+_{k_i,R}|^{\frac{1}{2}} + \chi^{\frac{1}{2}} |A^+_{k_i,R}|^{\frac{1}{2}-\frac{1}{q}} \right) |D_i|^{\frac{1}{2}} \\
&\leq C(Q, C_0, \gamma, \lambda_1) \frac{M-k}{2^i} \left(R^{\frac{Q}{q}-1} + \frac{2^i \chi^{\frac{1}{2}}}{M-k} \right) |A^+_{k_i,R}|^{\frac{1}{2}-\frac{1}{q}} |D_i|^{\frac{1}{2}},
\end{aligned}$$

where we used $\sup\limits_{B_R} (f - k_i)^+ = M - k_i = (M-k)2^{-i}$ and $|A^+_{k_i,R}| \leq |B_R| = C(Q)R^Q$.

For $i = 0, 1, ..., s-1$, provided $M - k \geq 2^s \chi^{\frac{1}{2}} R^{1-\frac{Q}{q}}$, we get:

$$\frac{M-k}{2^{i+1}} |A^+_{k_{i+1},\lambda_1 R}|^{1-\frac{1}{Q}} \leq C(Q, C_0, \gamma, \lambda_1) \frac{M-k}{2^i} R^{\frac{Q}{2}-1} |D_i|^{\frac{1}{2}}.$$

Now simplifying $(M-k)2^{-i}$ and squaring:

$$|A^+_{k_{i+1},\lambda_1 R}|^{\frac{2(Q-1)}{Q}} \leq C(Q, C_0, \gamma, \lambda_1) R^{Q-2} |D_i|.$$

Observe that for the specified range of i we have $A^+_{k_s,\lambda_1 R} \subseteq A^+_{k_{i+1},\lambda_1 R}$ hence:

$$|A^+_{k_s,\lambda_1 R}|^{\frac{2(Q-1)}{Q}} \leq C(Q, C_0, \gamma, \lambda_1) R^{Q-2} |D_i| \quad \text{for } \; i = 0, 1, ..., s-1.$$

Adding the previous relations we get:

$$s |A^+_{k_s,\lambda_1 R}|^{\frac{2(Q-1)}{Q}} \leq C(Q, C_0, \gamma, \lambda_1) R^{Q-2} \sum_{i=0}^{s-1} |D_i| \leq C(Q, C_0, \gamma, \lambda_1) R^{2Q-2}$$

since the sum is telescoping and we have estimated $|A_{k,\lambda_1 R}|$ and $|A_{k_s,\lambda_1 R}|$ by $|B_{\lambda_1 R}|$. Hence:

$$|A^+_{k_s,\lambda_1 R}| \leq C(Q, C_0, \gamma, \lambda_1) \frac{|B_{\lambda_1 R}|}{s^{\frac{Q}{2Q-2}}},$$

therefore, choosing $s = s(Q, C_0, \gamma, \lambda_1, \theta)$ so that the left hand side is smaller than $\theta |B_{\lambda_1 R}|$, we get the result. □

Combining the previous Lemmas we get an estimate for the decay of the oscillation of functions in the De Giorgi class. We are adapting it from Lemma 2.5 in [Man86] and from [LU68].

Lemma 3.2.7 (Oscillation estimate)**.** *Let $\Omega \subset \mathbb{H}$, $B_R \subset\subset \Omega$, $M = M(R) = \sup_{B_R} f$, $m = m(R) = \inf_{B_R} f$, $0 < \lambda_1 < 1$ and suppose that for radii $r' < r \in [\lambda_1 R, R]$ we have $f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [\frac{m+M}{2}, M]$ and $-f \in DG^+(\Omega, \gamma, \chi, q)$ for $h \in [-M, -\frac{m+M}{2}]$. Then there exists $A = A(Q, \gamma, \lambda_1) \in (0, 1)$ such that:*

$$\operatorname*{osc}_{B_{\lambda_1 R}} u \leq A \operatorname*{osc}_{B_R} u + B R^{1-\frac{Q}{q}}, \tag{3.20}$$

where:

$$B = \frac{\chi^{\frac{1}{2}}}{4(1-A)}.$$

Proof. Withouth loss of generality, by considering $f - \frac{m+M}{2}$, we can assume $m = -M$. Also we can assume

$$|A^+_{0,\sqrt{\lambda_1}R}(f)| \leq \frac{1}{2} |B_{\sqrt{\lambda_1}R}|,$$

otherwise $-f$ satisfies this condition. Now the idea is to use this to apply Lemma 3.2.6 in order to achieve a greater superlevel set whose measure is sufficently small so that we can use Lemma 3.2.4. We will also have to keep track of the several conditions that will then be combined in the final estimate. To be precise, given $\theta_0 \in (0, 1)$ apply Lemma 3.2.6 with $k = 0$, $C_0 = \frac{1}{2}$, radii $r' = \sqrt{\lambda_1}R$, $r = R$. We get the existence of a natural number $s = s(Q, \gamma, \lambda_1, \theta_0)$ such that either:

$$M(R) \leq 2^s \chi^{\frac{1}{2}} R^{1-\frac{Q}{q}} \tag{3.21}$$

or:

$$|A^+_{k_s,\sqrt{\lambda_1}R}(f)| \leq \theta_0 |B_{\sqrt{\lambda_1}R}| \tag{3.22}$$

where $k_s = \left(1 - \frac{1}{2^s}\right) M(R)$. Now in the case (3.22) we want to use Lemma 3.2.4 for radii $r' < r \in [\lambda_1 R, \sqrt{\lambda_1} R]$, $k = k_s$, $\lambda_0 = \frac{1}{2}$. To do so, first choose θ_0 to be smaller than $\theta = \theta(Q, \gamma, \lambda_1)$ given in Lemma 3.2.4. The Lemma applies if:

$$k_s < M(\sqrt{\lambda_1} R). \tag{3.23}$$

In this case we conclude that we have either

$$\begin{aligned} f &\leq \frac{1}{2} k_s + \frac{1}{2} M(\sqrt{\lambda_1} R) \\ &\leq \frac{1}{2}\left(1 - \frac{1}{2^s}\right) M(R) + \frac{1}{2} M(R) \\ &= \left(1 - \frac{1}{2^{s+1}}\right) M(R) \quad \text{a.e in} \quad B_{\lambda_1 R} \end{aligned} \tag{3.24}$$

or $M(\sqrt{\lambda_1} R) - k_s \leq \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}}$, i.e.

$$M(\sqrt{\lambda_1} R) \leq \left(1 - \frac{1}{2^s}\right) M(R) + \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}}. \tag{3.25}$$

If (3.23) is not true then we get:

$$f \leq M(\sqrt{\lambda_1} R) \leq \left(1 - \frac{1}{2^s}\right) M(R) \quad \text{in} \quad B_{\lambda_1 R} \tag{3.26}$$

since $\lambda_1 < \sqrt{\lambda_1}$. Finally, collecting alternatives (3.21), (3.24), (3.25) and (3.26) we obtain the existence of $s \in \mathbb{N}$ such that:

$$M(\lambda_1 R) \leq \left(1 - \frac{1}{2^{s+1}}\right) M(R) + 2^s \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}}. \tag{3.27}$$

Note that s depends only on Q ,γ , λ_1 and θ, hence only on Q, γ and λ_1. To conclude observe that $\inf_{B_{\lambda_1 R}} f \geq \inf_{B_R} f = -M(R)$, hence:

$$\begin{aligned} \operatorname*{osc}_{B_{\lambda_1 R}} f = \sup_{B_{\lambda_1 R}} f - \inf_{B_{\lambda_1 R}} f &\leq \left(1 - \frac{1}{2^{s+1}}\right) M(R) + 2^s \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}} + M(R) \\ &= \left(2 - \frac{1}{2^{s+1}}\right) M(R) + 2^s \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}} \\ &= \left(1 - \frac{1}{2^{s+2}}\right) \operatorname*{osc}_{B_R} f + 2^s \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}}, \end{aligned}$$

since $\operatorname*{osc}_{B_R} f = 2M(R)$. Take $A = 1 - \frac{1}{2^{s+2}}$ to get the required estimate with the claimed dependencies. $\square$

3.3 $C^{1,\alpha}$ PROOF FOR $p > 4$

3.3.1 Main Estimate

From now on we will fix a ball $B_{R_0} \subset\subset \Omega$ and for a concentric ball $B_R \subset B_{R_0}$ we introduce the notation:

$$\mu(R) = \max_{1\le l\le 2} \|X_l u\|_{L^\infty(B_R)} \quad \text{and} \quad \lambda(R) = \frac{1}{2}\mu(R). \tag{3.28}$$

This Section contains a proof of the main estimate:

Proposition 3.3.1. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $p > 4$ and fix $B_{R_0} \subset\subset \Omega$. For every $0 < r' < r < \frac{R_0}{2}$, $l = 1, 2$ and for every $q > \max\{4\,,\, 2 + \frac{4}{p-4}\}$ we have:*

$$\int_{B_{r'}} w^{\frac{p-2}{2}} \left|\nabla_{\mathbb{H}}(X_l u - k)^+\right|^2 \mathrm{d}x \le \frac{C_p}{(r-r')^2} \int_{B_r} w^{\frac{p-2}{2}} |(X_l u - k)^+|^2 \,\mathrm{d}x + \chi \left|A^+_{k,r}(X_l u)\right|^{1-\frac{2}{q}}, \tag{3.29}$$

$$\int_{B_{r'}} w^{\frac{p-2}{2}} \left|\nabla_{\mathbb{H}}(X_l u - k)^-\right|^2 \mathrm{d}x \le \frac{C_p}{(r-r')^2} \int_{B_r} w^{\frac{p-2}{2}} |(X_l u - k)^-|^2 \,\mathrm{d}x + \chi \left|A^-_{k,r}(X_l u)\right|^{1-\frac{2}{q}}. \tag{3.30}$$

The inequalities (3.29) hold for levels $k \ge -\mu(R_0)$, while (3.30) hold for levels $k \le \mu(R_0)$. The constant C_p depends only on p and the parameter χ is given by:

$$\chi = \frac{C_p q^6}{R_0^2} \left(\delta^2 + \mu(R_0)^2\right)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}}. \tag{3.31}$$

Proof. We will prove (3.29) for $l = 1$, the other estimates follow in a similar fashion. Introduce the notation $v_1 = (X_1 u - k)^+$. Fix $0 < r' < r < \frac{R_0}{2}$ and let $\phi = \xi^2 v_1$, where ξ is a cut-off function between $B_{r'}$ and B_r with $|\nabla_{\mathbb{H}}\xi| \le \frac{C}{(r-r')}$. Denote $A^+_{k,r}(X_1 u)$ for simplicity by $A^+_{k,r}$ and we adopt the usual convention of sum on repeated indices. Test equation (3.2) with ϕ to get:

$$\begin{aligned} J_1 := \int_{B_r} \xi^2\, \partial_{z_j} a_i(\nabla_{\mathbb{H}} u)\, X_j X_1 u\, X_i v_1 \,\mathrm{d}x &= -2 \int_{B_r} \xi\, \partial_{z_j} a_i(\nabla_{\mathbb{H}} u)\, X_j X_1 u\, X_i \xi\, v_1 \,\mathrm{d}x \\ &\quad - \int_{B_r} \xi^2\, \partial_{z_2} a_i(\nabla_{\mathbb{H}} u)\, X_i v_1\, Tu \,\mathrm{d}x \\ &\quad - 2 \int_{B_r} \xi\, \partial_{z_2} a_i(\nabla_{\mathbb{H}} u)\, X_i \xi\, Tu\, v_1 \,\mathrm{d}x \\ &\quad - \int_{B_r} a_2(\nabla_{\mathbb{H}} u)\, T(\xi^2 v_1) \,\mathrm{d}x \\ &:= J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Since on $A^+_{k,r}$ we have $X_j X_1 u = X_j v_1$ and by the ellipticity we get:

$$J_1 \geq c_p \int_{B_r} \xi^2\, w^{\frac{p-2}{2}}\, |\nabla_{\mathbb{H}} v_1|^2 \,\mathrm{d}x.$$

Using the growth estimates on a_i in (2.11) and Young's inequality with a parameter ε to be chosen we get:

$$\begin{aligned} |J_2| &\leq C \int_{B_r} \xi\, |\nabla_{\mathbb{H}} \xi|\, w^{\frac{p-2}{2}}\, v_1\, |\nabla_{\mathbb{H}} v_1| \,\mathrm{d}x \leq C\varepsilon \int_{B_r} \xi^2\, w^{\frac{p-2}{2}}\, |\nabla_{\mathbb{H}} v_1|^2 \,\mathrm{d}x + \frac{C}{\varepsilon} \int_{B_r} |\nabla_{\mathbb{H}} \xi|^2\, w^{\frac{p-2}{2}}\, v_1^2 \,\mathrm{d}x \\ |J_3| &\leq C \int_{A^+_{k,r}} \xi^2\, w^{\frac{p-2}{2}}\, |Tu|\, |\nabla_{\mathbb{H}} v_1| \,\mathrm{d}x \leq C\varepsilon \int_{B_r} \xi^2\, w^{\frac{p-2}{2}}\, |\nabla_{\mathbb{H}} v_1|^2 \,\mathrm{d}x + \frac{C}{\varepsilon} \int_{A^+_{k,r}} \xi^2\, w^{\frac{p-2}{2}}\, |Tu|^2 \,\mathrm{d}x \\ |J_4| &\leq C \int_{A^+_{k,r}} \xi\, |\nabla_{\mathbb{H}} \xi|\, w^{\frac{p-2}{2}}\, |Tu|\, v_1 \,\mathrm{d}x \leq C \int_{B_r} |\nabla_{\mathbb{H}} \xi|^2\, w^{\frac{p-2}{2}}\, v_1^2 \,\mathrm{d}x + C \int_{A^+_{k,r}} \xi^2\, w^{\frac{p-2}{2}}\, |Tu|^2 \,\mathrm{d}x. \end{aligned}$$

The new idea is to integrate by parts twice the term J_5, first with respect to the field T and then with respect to the fields X_j. We get:

$$\begin{aligned} J_5 = \int_{B_r} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, X_j T u\, \xi^2\, v_1 \,\mathrm{d}x &= - \int_{B_r} Tu\, X_j \left(\partial_{z_j} a_2(\nabla_{\mathbb{H}} u) \xi^2 v_1 \right) \,\mathrm{d}x \\ &= - \int_{B_r} Tu\, \partial_{z_s} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, X_j X_s u\, \xi^2\, v_1 \,\mathrm{d}x \\ &\quad - 2 \int_{B_r} Tu\, \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, \xi\, X_j \xi\, v_1 \,\mathrm{d}x \\ &\quad - \int_{B_r} Tu\, \partial_{z_j} a_2(\nabla_{\mathbb{H}} u)\, \xi^2\, X_j v_1 \,\mathrm{d}x \\ &:= J_{5,1} + J_{5,2} + J_{5,3}. \end{aligned}$$

Note that $J_{5,2}$ and $J_{5,3}$ can be estimated respectively as J_4 and J_3.

Denoting $J_{5,1} = \sum_{s,j} J_{5,1}^{s,j}$ we have:

$$
\begin{aligned}
\left| \sum_j J_{5,1}^{1,j} \right| &\le C \int_{B_r} \xi^2 \, w^{\frac{p-3}{2}} \, |\nabla_{\mathbb{H}} v_1| \, v_1 \, |Tu| \, \mathrm{d}x \le C\varepsilon \int_{B_r} \xi^2 \, w^{\frac{p-2}{2}} \, |\nabla_{\mathbb{H}} v_1|^2 \, \mathrm{d}x \\
&\qquad + \frac{C}{\varepsilon} \int_{B_r} \xi^2 \, w^{\frac{p-4}{2}} \, |Tu|^2 \, v_1^2 \, \mathrm{d}x, \\
\left| J_{5,1}^{2,1} \right| &\le \int_{B_r} |\partial_{z_2}\partial_{z_1} a_2(\nabla_{\mathbb{H}} u) \, X_1 X_2 u \, Tu| \, \xi^2 \, v_1 \, \mathrm{d}x \le \int_{B_r} |\partial_{z_2}\partial_{z_1} a_2(\nabla_{\mathbb{H}} u) \, X_2 X_1 u \, Tu| \, \xi^2 \, v_1 \, \mathrm{d}x \\
&\qquad + \int_{B_r} |\partial_{z_2}\partial_{z_1} a_2(\nabla_{\mathbb{H}} u)| \, |Tu|^2 \, \xi^2 \, v_1 \, \mathrm{d}x.
\end{aligned}
$$

The first term of the last inequality has the same estimate as $J_{5,1}^{1,j}$. For the other term we have:

$$
\begin{aligned}
\int_{B_r} |\partial_{z_2}\partial_{z_1} a_2(\nabla_{\mathbb{H}} u)| \, |Tu|^2 \, \xi^2 \, v_1 \, \mathrm{d}x &\le C_p \int_{B_r} \xi^2 \, w^{\frac{p-3}{2}} \, |Tu|^2 \, v_1 \, \mathrm{d}x \le C_p \int_{A_{k,r}^+} \xi^2 \, w^{\frac{p-2}{2}} \, |Tu|^2 \, \mathrm{d}x \\
&\qquad + C_p \int_{B_r} \xi^2 \, w^{\frac{p-4}{2}} \, |Tu|^2 \, v_1^2 \, \mathrm{d}x.
\end{aligned}
$$

Now another key step is to use (2.12) and (3.1) to get:

$$
\begin{aligned}
\left| J_{5,1}^{2,2} \right| = \left| \int_{B_r} \partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u) \, X_2 X_2 u \, \xi^2 \, v_1 \, Tu \, \mathrm{d}x \right| &\le C_p \int_{B_r} |\partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u) \, X_1 X_1 u \, Tu| \, \xi^2 \, v_1 \, \mathrm{d}x \\
&\quad + C_p \int_{B_r} |\partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u) \, X_2 X_1 u \, Tu| \, \xi^2 \, v_1 \, \mathrm{d}x \\
&\quad + C_p \int_{B_r} |\partial_{z_2}\partial_{z_2} a_2(\nabla_{\mathbb{H}} u) \, X_1 X_2 u \, Tu| \, \xi^2 \, v_1 \, \mathrm{d}x \\
&:= F_1 + F_2 + F_3.
\end{aligned}
$$

Note that F_1 and F_2 can be estimated as $J_{5,1}^{1,j}$ while F_3 can be estimated as $J_{5,1}^{2,1}$.

Choosing ε small enough (depending only on the constants that depend only on p) and putting all the previous estimates together we get:

$$
\begin{aligned}
\int_{B_r} \xi^2 \, w^{\frac{p-2}{2}} \, |\nabla_{\mathbb{H}} v_1|^2 \, \mathrm{d}x &\le C \int_{B_r} |\nabla_{\mathbb{H}} \xi|^2 \, w^{\frac{p-2}{2}} \, v_1^2 \, \mathrm{d}x + C \int_{A_{k,r}^+} \xi^2 \, w^{\frac{p-2}{2}} \, |Tu|^2 \, \mathrm{d}x \\
&\quad + C \int_{B_r} \xi^2 \, w^{\frac{p-4}{2}} \, |Tu|^2 \, v_1^2 \, \mathrm{d}x \qquad (3.32) \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

We only need to estimate I_2 and I_3. We use Hölder's inequality with exponent $q/2$ (which is greater than 2) and Lemma (3.1.2):

$$
\begin{aligned}
I_2 &\le \left(\int_{A^+_{k,r}} \xi^q \, w^{\frac{p-2}{2}} \; |Tu|^q \, \mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A^+_{k,r}} w^{\frac{p-2}{2}} \, \mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\int_{B_{\frac{R_0}{2}}} w^{\frac{p-2}{2}} \; |Tu|^q \, \mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A^+_{k,r}} w^{\frac{p-2}{2}} \, \mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\int_{B_{R_0}} \eta^q \, w^{\frac{p-2}{2}} \; |Tu|^q \, \mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A^+_{k,r}} w^{\frac{p-2}{2}} \, \mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\left(\|\nabla_{\mathbb{H}} \eta\|^2_{L^\infty} + \|\eta T \eta\|_{L^\infty} \right)^{\frac{q}{2}} \int_{B_{R_0}} w^{\frac{p-2+q}{2}} \, \mathrm{d}x \right)^{\frac{2}{q}} \left(\delta^2 + \mu(r)^2 \right)^{\frac{p-2}{2}(1-\frac{2}{q})} \left| A^+_{k,r} \right|^{1-\frac{2}{q}} \\
&\le \frac{C_p \, q^6}{R_0^2} \left(\delta^2 + \mu(R_0)^2 \right)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}} \left| A^+_{k,r} \right|^{1-\frac{2}{q}} ,
\end{aligned}
$$

where η is a cut-off function between $B_{\frac{R_0}{2}}$ and B_{R_0} with $|\nabla_{\mathbb{H}} \eta| \le \frac{C}{R_0}$. In a similar way and noting that $v_1^2 \le 2(\delta^2 + \mu(R_0)^2)$ for $k \ge -\mu(R_0)$ we get:

$$
\begin{aligned}
I_3 &\le \left(\delta^2 + \mu(R_0)^2 \right) \left(\int_{A^+_{k,r}} \xi^q \, w^{\frac{p-2}{2}} \; |Tu|^q \, \mathrm{d}x \right)^{\frac{2}{q}} \left(\int_{A^+_{k,r}} w^{\frac{p-4}{2} - \frac{2}{q-2}} \, \mathrm{d}x \right)^{1-\frac{2}{q}} \\
&\le \left(\delta^2 + \mu(R_0)^2 \right) \left(\left(\|\nabla_{\mathbb{H}} \eta\|^2_{L^\infty} + \|\eta T \eta\|_{L^\infty} \right)^{\frac{q}{2}} \int_{B_{R_0}} w^{\frac{p-2+q}{2}} \, \mathrm{d}x \right)^{\frac{2}{q}} \left(\delta^2 + \mu(r)^2 \right)^{\left(\frac{p-4}{2} - \frac{2}{q-2}\right)\left(1-\frac{2}{q}\right)} \left| A^+_{k,r} \right|^{1-\frac{2}{q}} \\
&\le \frac{C_p \, q^6}{R_0^2} \left(\delta^2 + \mu(R_0)^2 \right)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}} \left| A^+_{k,r} \right|^{1-\frac{2}{q}} .
\end{aligned}
$$

□

Remark 3.3.2. *Note that in the last step we need the assumption $q > 2 + \frac{4}{p-4}$ to estimate w. This forces $p > 4$.*

3.3.2 The Alternative

Recall that we fixed a ball $B_{R_0} \subset\subset \Omega$ and we now consider an arbitrary concentric ball $B_R \subset B_{\frac{R_0}{2}}$.

Lemma 3.3.3. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of (2.7) for $p > 4$. Fix $B_{R_0} \subset\subset \Omega$ and $B_R \subset B_{R_0/2}$. If:*

$$\delta \geq \lambda(R) = \frac{1}{2} \max_{j=1,2} \|X_j u\|_{L^\infty(B_R)},$$

for every $\lambda_1 \in (0,1)$ there exists $A = A(p, \lambda_1) \in (0,1)$ such that:

$$\operatorname*{osc}_{B_{\lambda_1 R}} (X_l u) \leq A \operatorname*{osc}_{B_R} (X_l u) + BR^\alpha \quad \text{for every } l \in \{1,2\},$$

where:

$$B = \frac{C_p q^{\frac{6}{p}} (\delta^2 + \mu(R_0)^2)^{\frac{1}{2}}}{4(1-A) R_0^\alpha} \quad \text{and} \quad \alpha = \left(1 - \frac{Q}{q}\right)\frac{2}{p},$$

and $\mu(R_0)$ is defined in (3.28).

Proof. Since $\delta \geq \lambda(R)$ we can get rid of the weight and obtain that $X_l u$ is in a De Giorgi class. Indeed observe that $\lambda(R)^2 \leq \delta^2 \leq w \leq C\lambda(R)^2$, hence from (3.29) we get:

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \, \mathrm{d}x \leq \frac{C_p}{(r-r')^2} \int_{B_r} v_l^2 \, \mathrm{d}x + \frac{2^{p-2}\chi}{\mu(R)^{p-2}} \left|A^+_{k,r}(X_l u)\right|^{1-\frac{2}{q}}$$

for all levels $k > -\mu(R_0)$ and radii $r' < r < R$. Now if:

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \tag{3.33}$$

then:

$$\mu(R)^{p-2} \geq \chi^{\frac{p-2}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}(p-2)},$$

hence:

$$\frac{\chi}{\mu(R)^{p-2}} \leq \frac{\chi^{\frac{2}{p}}}{R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}(p-2)}} = C_p q^{\frac{12}{p}} (\delta^2 + \mu(R_0)^2) \left(\frac{R}{R_0}\right)^{2\left(1-\frac{Q}{q}\right)\frac{2}{p}} R^{2\left(\frac{Q}{q}-1\right)} =: \chi'.$$

Therefore we get that $X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$ for all levels $k > -\mu(R_0)$ and radii $r' < r < R$.

Analogously, from (3.30), we get also $-X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$ for all levels $k < \mu(R_0)$

and radii $r' < r < R$, hence we can apply the Oscillation estimate in Lemma 3.2.7 to get for any $\lambda_1 \in (0,1)$ the existence of $A = A(p, \lambda_1) \in (0,1)$ such that for every $l \in \{1,2\}$ we have:

$$\operatorname*{osc}_{B_{\lambda_1 R}} (X_l u) \leq A \operatorname*{osc}_{B_R} (X_l u) + B' R^{1-\frac{Q}{q}},$$

where $4(1-A)B' = (\chi')^{\frac{1}{2}}$. By the definition of χ', and combining with the case when (3.33) does not hold, we get the result. □

We now consider the case when the equation degenerates, namely $\delta < \lambda(R)$. Here we face an alternative: either the maximum $\mu(R)$ (defined in (3.28)) has the right Hölder decay or the horizontal gradient $\nabla_{\mathbb{H}} u$ is bounded away from zero, and hence the equation behaves like the non degenerate case in Lemma 3.3.3. More precisely we have:

Proposition 3.3.4. *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $p > 4$ and $B_{R_0} \subset\subset \Omega$. Consider $B_R \subset B_{\frac{R_0}{2}}$ and assume:*

$$\delta < \lambda(R) = \frac{1}{2} \max_{j=1,2} \|X_j u\|_{L^\infty(B_R)}.$$

Then there exist $\theta = \theta(p) \in (0,1)$ and $A = A(p) \in (0,1)$ such that:

Case 1. *If for some $l \in \{1,2\}$ we have either:*

$$\left| B_R \cap \left\{ X_l u \geq \frac{1}{2}\mu(R) \right\} \right| \geq \theta |B_R| \tag{3.34}$$

or:

$$\left| B_R \cap \left\{ X_l u \leq -\frac{1}{2}\mu(R) \right\} \right| \geq \theta |B_R|, \tag{3.35}$$

then:

$$\text{either} \quad \mu(R) \leq c_p \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \quad \text{or} \quad |X_l u| \geq \frac{1}{32}\mu(R) \quad \text{in } B_{R/2}$$

where $c_p = 2(4/3)^{\frac{2}{p}}$.

Case 2. *If for every $l \in \{1,2\}$ neither (3.34) nor (3.35) holds, then:*

$$\mu(R/2) \leq A\mu(R) + BR^\alpha, \tag{3.36}$$

where:

$$B = \frac{C_p q^{\frac{6}{p}}}{2(1-A)} \frac{\mu(R_0)}{R_0^\alpha} \quad \text{and} \quad \alpha = \left(1 - \frac{Q}{q}\right)\frac{2}{p}.$$

Proof. **Case 1.**

Consider (3.35). We will show that it implies $X_l u \le -\frac{1}{32}\mu(R)$ provided $\mu(R) \ge c_p \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}}$. Recall that we are considering $\delta < \lambda(R)$. Define the auxiliary function:

$$V_l = |X_l u|^{\frac{p}{2}} \operatorname{sign}(X_l u).$$

Observe that $|V_l| \le (2\lambda(R))^{\frac{p}{2}}$ on B_R. Also:

$$|\nabla_{\mathbb{H}} V_l|^2 = \frac{p^2}{4} |X_l u|^{p-2} |\nabla_{\mathbb{H}} X_l u|^2. \tag{3.37}$$

Denote by $h = |k|^{\frac{p}{2}} \operatorname{sign}(k) = g(k)$ and note that $\{X_l > k\} = \{V_l > h\}$ (since g is a continuous bijection), hence $A^+_{k,r}(X_l u) = A^+_{h,r}(V_l)$. Now (3.37) implies:

$$|\nabla_{\mathbb{H}}(V_l - h)^+|^2 \le C_p w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_l|^2,$$

therefore (3.29) becomes:

$$\begin{aligned}\int_{A^+_{h,r'}(V_l)} |\nabla_{\mathbb{H}} V_l|^2 \,\mathrm{d}x \le C_p \int_{B_{r'}} w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_l|^2 \,\mathrm{d}x &\le \frac{C_p}{(r-r')^2} (\mu(r)-k)^2 (\delta^2 + \mu(r)^2)^{\frac{p-2}{2}} |A^+_{h,r}(V_l)| \\ &\quad + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}} \\ &\le \frac{C_p}{(r-r')^2} (3\lambda(R))^2 (5\lambda(R))^{p-2} |A^+_{h,r}(V_l)| \\ &\quad + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}} \\ &\le \frac{C_p}{(r-r')^2} (\lambda(R))^p |A^+_{h,r}(V_l)| + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}}\end{aligned} \tag{3.38}$$

for $k > -\lambda(R)$ and $r' < r \le R$. Denoting by $H = H(R) = (\lambda(R))^{\frac{p}{2}}$ the inequality (3.38) rewrites as:

$$\int_{A^+_{h,r'}(V_l)} |\nabla_{\mathbb{H}} V_l|^2 \,\mathrm{d}x \le \frac{C_p}{(r-r')^2} (H(R))^2 |A^+_{h,r}(V_l)| + \chi |A^+_{h,r}(V_l)|^{1-\frac{2}{q}} \tag{3.39}$$

for levels $h > -H(R)$ and radii $r' < r \le R$. Now denote $M\left(\frac{R}{2}\right) = \sup_{B_{R/2}} V_l$.

Case a. $M(\frac{R}{2}) < -\frac{H(R)}{4}$.

This means $X_l u < 0$ in $B_{R/2}$. Hence $(-X_l u)^{\frac{p}{2}} > \frac{(\lambda(R))^{\frac{p}{2}}}{4}$, so:

$$X_l u < -\frac{\lambda(R)}{4^{\frac{2}{p}}} < -\frac{\lambda(R)}{16} = -\frac{\mu(R)}{32} \quad \text{on } B_{R/2}.$$

Case b. $M(\frac{R}{2}) \geq -\frac{H(R)}{4}$.

For levels $h \in [-H(R), -H(R)/2]$ we have:

$$\sup_{B_{R/2}} (V_l - h)^+ \geq -\frac{H(R)}{4} + \frac{H(R)}{2} = \frac{H(R)}{4},$$

therefore:

$$\frac{(H(R))^2}{16} \leq \sup_{B_{R/2}} |(V_l - h)^+|^2 \leq \sup_{B_r} |(V_l - h)^+|^2 \quad \text{for } r \in [R/2, R].$$

Hence, from (3.39) we get that $V_l \in DG^+(B_{R_0}, C_p, \chi, q)$ for levels $h \in [-H(R), -H(R)/2]$ and radii $r' < r \in [R/2, R]$. Apply Lemma 3.2.4 with $k = -H(R)$, $\lambda_0 = \frac{2^{\frac{p}{2}+1/2}}{2^{\frac{p}{2}+1}}$, $\lambda_1 = 1/2$ to get the existence of $\theta_1 = \theta_1(p) \in (0,1)$ such that $|A^+_{-H,R}(V_l)| \leq \theta_1 |B_R|$ implies $V_l \leq -\frac{H(R)}{2}$ on $B_{R/2}$, provided $M(R) + H(R) \geq \chi^{\frac{1}{2}} R^{1-\frac{Q}{q}}$. But if:

$$\mu(R) \geq 2 \left(\frac{4}{3}\right)^{\frac{2}{p}} \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \tag{3.40}$$

then:

$$M(R) + H(R) \geq \frac{3}{4} H(R) = \frac{3}{4}\left(\frac{\mu(R)}{2}\right)^{\frac{p}{2}} \geq \chi^{\frac{1}{2}} R^{1-\frac{Q}{q}}$$

so Lemma 3.2.4 applies under condition (3.40). Then, as in Case a, we obtain $X_l u < 0$ on $B_{R/2}$, hence:

$$X_l u < -\frac{\lambda(R)}{2^{\frac{2}{p}}} < -\frac{\lambda(R)}{4} = -\frac{\mu(R)}{8} \quad \text{in } B_{R/2}.$$

Observe that $\{V_l > -H(R)\} = \{X_l u > -\lambda(R)\} = \{X_l u > -\mu(R)/2\}$. Passing to the complements, we have proved that there exists $\theta(p) = 1 - \theta_1(p)$ such that (3.35) implies $X_l u \leq -\mu(R)/32$ on $B_{R/2}$, provided (3.40) holds.

In a similar way, if (3.34) holds then:

$$|B_R \cap \{-X_l u \leq -\mu(R)/2\}| = |B_R \cap \{X_l u \geq \mu(R)/2\}| \geq \theta |B_R|.$$

Apply the previous proof to $-X_l u$ to get $-X_l u \leq -\mu(R)/32$ which means $X_l u \geq \mu(R)/32$, and this concludes the proof of Case 1.

Remark 3.3.5. *In Case b the hypotheses of Lemma 3.2.4 are satisfied since:*

$$\sup_{B_R} V_l = M(R) \geq M(R/2) \geq -H(R)/4 \geq -H(R)$$

and:

$$\lambda_0 k + (1-\lambda_0)M(R) = -\lambda_0 H(R) + (1-\lambda_0)M(R) \leq -H(R)(\lambda_0 - (1-\lambda_0)2^{\frac{p}{2}}) = -H(R)/2,$$

hence $[-H(R), \lambda_0 k + (1-\lambda_0)M(R)] \subset [-H(R), -H(R)/2]$.

Case 2.

If for the θ found in Case 1 neither (3.34) nor (3.35) holds for any $l \in \{1,2\}$, then the inequalities $\left|B_R \cap \left\{X_l u \geq \frac{1}{2}\mu(R)\right\}\right| \leq \theta|B_R|$ and: $\left|B_R \cap \left\{X_l u \leq -\frac{1}{2}\mu(R)\right\}\right| \leq \theta|B_R|$ are satisfied for every $l \in \{1,2\}$. In particular there exist $\frac{1}{2} < \lambda_1 < 1$ and $0 < C_0 < 1$ such that:

$$\left|B_{\lambda_1 R} \cap \left\{X_l u \geq \frac{1}{2}\mu(R)\right\}\right| \leq C_0|B_{\lambda_1 R}| \tag{3.41}$$

and:

$$\left|B_{\lambda_1 R} \cap \left\{X_l u \leq -\frac{1}{2}\mu(R)\right\}\right| \leq C_0|B_{\lambda_1 R}|. \tag{3.42}$$

Otherwise, if (3.41) or (3.42) are not true for any such λ_1 and C_0, then (3.34) or (3.35) are satisfied and we are in Case 1.

Considering levels $k \in [\frac{\mu(R)}{2}, \mu(R)]$, on $\{X_l u > k\} \cap B_R$ we have $k < X_l u \leq |X_l u| \leq \mu(R) \leq 2k$. Then, recalling that we are in the case $\delta \leq \lambda(R) \leq k$ we get:

$$k^{p-2} \leq |\nabla_{\mathbb{H}} u|^{p-2} \leq w^{\frac{p-2}{2}} \leq C_p k^{p-2}.$$

Therefore in (3.29) we can get rid of the weight:

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \,\mathrm{d}x \leq \frac{C}{(r-r')^2}\int_{B_r} v_l^2 \,\mathrm{d}x + \frac{2^{p-2}\chi}{\mu(R)^{p-2}}\left|A^+_{k,r}(X_l u)\right|^{1-\frac{2}{q}}.$$

Now if:

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}}, \tag{3.43}$$

then:

$$\mu(R)^{p-2} \geq \chi^{\frac{p-2}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}(p-2)},$$

hence:

$$\frac{\chi}{\mu(R)^{p-2}} \leq \frac{\chi^{\frac{2}{p}}}{R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}(p-2)}} = C_p q^{\frac{12}{p}} \lambda(R_0)^2 \left(\frac{R}{R_0}\right)^{2\left(1-\frac{Q}{q}\right)\frac{2}{p}} R^{2\left(\frac{Q}{q}-1\right)} =: \chi'.$$

Therefore we get that $X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$ for levels $k \in [\frac{\mu(R)}{2}, \mu(R)]$, radii $r' < r < R$ (under the assumption (3.43)). Apply Lemma 3.2.6 with λ_1 and C_0 as in (3.41), $k = \frac{\mu(R)}{2}$ to conclude that given $\theta_0 \in (0,1)$ there exists a natural number $s = s(p, \lambda_1, C_0, \theta_0)$ such that either:

$$\mu(R) \leq 2^{s+1}(\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}} = 2^{s+1} C_p q^{\frac{6}{p}} \lambda(R_0) \left(\frac{R}{R_0}\right)^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \tag{3.44}$$

or:

$$|A^+_{k_s,\lambda_1 R}| \leq \theta_0 |B_{\lambda_1 R}|, \tag{3.45}$$

where $k_s = \frac{\mu(R)}{2}(2 - 2^{-s}) = \mu(R)(1 - 2^{-s-1})$. Now in the case (3.45) we want to use Lemma 3.2.4 for radii $r' < r \in [R/2, \lambda_1 R]$, $k = k_s = (1 - 2^{-s-1})\mu(R)$, $\lambda_0 = 1/2$. This can be applied if:

$$k_s < \sup_{B_{\lambda_1 R}} (X_l u). \tag{3.46}$$

Then we would conclude that either:

$$\begin{aligned} X_l u \leq \frac{1}{2} k_s + \frac{1}{2}\mu(\lambda_1 R) &\leq \left(1 - \frac{1}{2^{s+1}}\right)\frac{1}{2}\mu(R) + \frac{1}{2}\mu(R) \\ &= \mu(R)\left(1 - \frac{1}{2^{s+2}}\right) \quad \text{a.e. in} \quad B_{\frac{R}{2}} \end{aligned} \tag{3.47}$$

or $\sup_{B_{\lambda_1 R}}(X_l u) - k_s \leq (\chi')^{\frac{1}{2}}(\lambda_1 R)^{1-\frac{Q}{q}} \leq (\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}}$, i.e.

$$\sup_{\lambda_1 R}(X_l u) \leq \left(1 - \frac{1}{2^{s+1}}\right)\mu(R) + (\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}}. \tag{3.48}$$

If (3.46) is not true, we get:

$$\sup_{B_{R/2}} (X_l u) \leq \sup_{\lambda_1 R}(X_l u) \leq k_s = \left(1 - \frac{1}{2^{s+1}}\right)\mu(R). \tag{3.49}$$

Repeating the same steps for $-X_l u$, using assumption (3.42) and the estimate (3.30), we will find the same alternatives except instead of (3.47)-(3.49) we will have:

$$X_l u \geq -\mu(R)\left(1 - \frac{1}{2^{s+2}}\right) - (\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}} \quad \text{a.e. in} \quad B_{\frac{R}{2}}. \tag{3.50}$$

In conclusion, collecting (3.43), (3.44) (3.48), (3.47), (3.50) and (3.49) (which are true for every index l) we obtain:

$$\mu(R/2) \leq \left(1 - \frac{1}{2^{s+2}}\right)\mu(R) + c_p q^{\frac{6}{p}} 2^{s+1}\lambda(R_0)\left(\frac{R}{R_0}\right)^{\left(1-\frac{Q}{q}\right)\frac{2}{p}}.$$

□

Now we need the following technical Lemma:

Lemma 3.3.6. *Let* $0 < A, \lambda, \alpha < 1$ *with* $A \neq \lambda^\alpha$ *and* $B, R_0 \geq 0$. *Let* $\varphi : [0, +\infty[\longrightarrow [0, +\infty[$ *be an increasing function such that:*

$$\varphi(\lambda R) \leq A\varphi(R) + BR^\alpha \quad \textit{for all} \quad R \leq R_0. \tag{3.51}$$

Then, for every $R \leq R_0$, *we have:*

$$\varphi(r) \leq \frac{1}{A}\left(\frac{r}{R}\right)^{\min\{\log_\lambda A, \alpha\}}\left[\varphi(R) + \frac{BR^\alpha}{|A - \lambda^\alpha|}\right] \quad \textit{for all} \quad r \leq R. \tag{3.52}$$

Proof. Consider $r < R$. Then there exists $k \in \mathbb{N}$ such that $\lambda^{k+1} < r/R \leq \lambda^k$. Since φ is increasing, iterating (3.51) we get:

$$\varphi(r) \leq A^k\varphi(R) + BR^\alpha(\lambda^\alpha)^{k-1}\sum_{i=0}^{k-1}\left(\frac{A}{\lambda^\alpha}\right)^i = A^k\varphi(R) + BR^\alpha\frac{(\lambda^\alpha)^k - A^k}{\lambda^\alpha - A}.$$

Now if $A > \lambda^\alpha$ we get:

$$\varphi(r) \leq \frac{1}{A}\left(\frac{r}{R}\right)^{\log_\lambda A}\left[\varphi(R) + \frac{BR^\alpha}{A - \lambda^\alpha}\right]$$

while if $A < \lambda^\alpha$ we obtain:

$$\varphi(r) \leq \frac{1}{A}\left(\frac{r}{R}\right)^{\log_\lambda A}\varphi(R) + \frac{BR^\alpha}{\lambda^\alpha(\lambda^\alpha - A)}\left(\frac{r}{R}\right)^\alpha,$$

and combining the two cases we obtain the estimate of the Lemma. □

We finally prove the main regularity result, in the non degenerate case:

Theorem 3.3.1 (Oscillation Estimate)**.** *Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (2.7) for $p > 4$. Fix $B_{R_0} \subset\subset \Omega$. Then there exists $\beta = \beta(p) \in (0,1)$ such that for every $l \in \{1,2\}$ we have:*

$$\operatorname*{osc}_{B_r}(X_l u) \leq C_p(\delta + \|\nabla_{\mathbb{H}} u\|_{L^\infty(B_{R_0})}) \left(\frac{r}{R_0}\right)^\beta \quad \textit{for all } r \leq \frac{R_0}{2}, \tag{3.53}$$

where C_p is a constant depending only on p.

Proof. We can combine the alternatives in Proposition 3.3.4 in either:

$$\mu(R/2) \leq A\mu(R) + BR^\alpha \tag{3.54}$$

or:

$$|X_l u| \geq \frac{1}{32}\mu(R) \quad \text{in } B_{R/2}. \tag{3.55}$$

In this last case we have:

$$w^{\frac{p-2}{2}} \geq \left(\frac{1}{32}\right)^{p-2} \mu(R)^{p-2} \quad \text{in} \quad B_{R/2}.$$

Since also:

$$w^{\frac{p-2}{2}} \leq \left(\delta^2 + \mu(R)^2\right)^{\frac{p-2}{2}} \leq C_p \mu(R)^{p-2} \quad \text{in} \quad B_R,$$

from the estimate (3.29) we get:

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \, \mathrm{d}x \leq \frac{C_p}{(r-r')^2} \int_{B_r} v_l^2 \, \mathrm{d}x + \frac{\chi}{\mu(R)^{p-2}} \left|A^+_{k,r}(X_l u)\right|^{1-\frac{2}{q}}$$

for every $r' < r \leq R/2$ and for every level $k > -\mu(R_0)$. Now as before, if:

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \tag{3.56}$$

we get $v_l \in DG^+(B_{R_0}, C_p, \chi', q)$ for every $r' < r \leq R/2$ and for every level $k > -\mu(R_0)$. The same is true for $-v_l$, with levels $k < \mu(R_0)$, so proceeding as in the proof of Lemma 3.3.3, we are in the position to apply the Oscillation Lemma 3.2.7 to conclude that there exists $A = A(p) \in (0,1)$ such that:

$$\operatorname*{osc}_{B_{R/4}}(X_l u) \leq A \operatorname*{osc}_{B_{R/2}}(X_l u) + BR^\alpha \leq A \operatorname*{osc}_{B_R}(X_l u) + BR^\alpha \quad \text{for all} \quad R \leq \frac{R_0}{2}, \tag{3.57}$$

where B and α are as in Proposition 3.3.4. Now apply Lemma 3.3.6 to (3.54) and (3.57) with $\lambda = 1/4$, A and B as given in Proposition 3.3.4. Note that $\operatorname*{osc}_{B_r}(X_l u) \leq 2\mu(r)$ and combine the estimates to get:

$$\operatorname*{osc}_{B_r}(X_l u) \leq L\left(\frac{r}{R}\right)^{\beta}$$

for every $l \in \{1,2\}$, where $\beta = \min\{\log_{\frac{1}{4}} A, \alpha\}$, $\alpha = \left(1 - \frac{Q}{q}\right)\frac{2}{p}$ and $L = \frac{1}{A}\left[\mu(R) + \frac{BR^{\alpha}}{|A-\lambda^{\alpha}|}\right]$. Since $R \leq R_0/2$ is arbitrary, fixing a value for q and combining with the estimate in Lemma 3.3.3 we get the result. □

Remark 3.3.7. *From the explicit expression of β and B we see that the estimate blows up when q goes to infinity, hence the Hölder exponent found with this proof satisfies the constraint $0 < \beta < \frac{2}{p}$.*

3.3.3 Passing to the limit $\delta \to 0$

We now extend the validity of the previous estimate to the degenerate case $\delta = 0$.

Theorem 3.3.2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (2.5) for $p > 4$. Fix $B_{R_0} \subset\subset \Omega$. Then there exists $\beta = \beta(p) \in (0,1)$ such that for every $l \in \{1,2\}$ we have:*

$$\operatorname*{osc}_{B_r}(X_l u) \leq C_p \|\nabla_{\mathbb{H}} u\|_{L^{\infty}(B_{R_0})}\left(\frac{r}{R_0}\right)^{\beta} \quad \textit{for all } r \leq \frac{R_0}{2},$$

where C_p is a constant depending only on p.

Proof. Let u^{δ} be the solution of the Dirichlet problem (2.14) with boundary value $u^{\delta} - u \in HW_0^{1,p}(B_{R_0})$. From Theorem 2.3.3 we have that, up to a subsequence, u^{δ} converges to u in $HW^{1,p}(B_{R_0})$. Together with the uniform Lipschitz estimate (3.13) and the oscillation estimate (3.53) we get that $\nabla_{\mathbb{H}} u^{\delta}$ converges to $\nabla_{\mathbb{H}} u$ uniformly in B_{R_0}, and therefore we can pass the estimate to the limit. □

4.0 NON-EUCLIDEAN ELASTICITY

This Chapter contains a description of the non-Euclidean elasticity model we are going to study. First we fix some notation that will be used throughout the rest of the thesis.

Given a matrix $F \in \mathbb{R}^{3\times 3}$, we denote its transpose by F^t, its symmetric part by $\mathrm{sym}F = \frac{1}{2}(F + F^t)$, and its skew part by $\mathrm{skew}F = F - \mathrm{sym}F$. By $SO(n) = \{R \in \mathbb{R}^{n\times n};\ R^t = R^{-1} \text{ and } \det R = 1\}$ we denote the group of special rotations, while $so(n) = \{F \in \mathbb{R}^{n\times n};\ \mathrm{sym}F = 0\}$ is the space of skew-symmetric matrices. We use the matrix norm $|F| = (\mathrm{trace}(F^tF))^{1/2}$, which is induced by the inner product $\langle F_1 : F_2\rangle = \mathrm{trace}(F_1^tF_2)$. The 2×2 principal minor of a matrix $F \in \mathbb{R}^{3\times 3}$ is denoted by $F_{2\times 2}$. Conversely, for a given $F_{2\times 2} \in \mathbb{R}^{2\times 2}$, the 3×3 matrix with principal minor equal $F_{2\times 2}$ and all other entries equal to 0, is denoted by $(F_{2\times 2})^*$.

We will denote by $Riem(G)$ the covariant Riemann curvature tensor, whose components $R_{....}$ and their relation to the contravariant curvatures in $R^{\cdot}_{...}$ are:

$$R_{iklm} = \frac{1}{2}\big(\partial_{kl}G_{im} + \partial_{im}G_{kl} - \partial_{km}G_{il} - \partial_{il}G_{km}\big) + G_{np}\big(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\big)$$
$$R_{iklm} = G_{is}R^s_{klm},$$

where we used the Einstein summation convention and where the Christoffel symbols are:

$$\Gamma^n_{kl} = \frac{1}{2}G^{ns}\big(\partial_k G_{sl} + \partial_l G_{sk} - \partial_s G_{kl}\big). \tag{4.1}$$

Here G^{ij} denotes the (i,j) entry of the inverse G^{-1}.

4.1 DIMENSION REDUCTION: THE SETUP

Let Ω be an open, bounded, smooth and simply connected subset of $\mathbb{R}^2$. For $0 < h \ll 1$ we consider thin films Ω^h around the mid-plate Ω:

$$\Omega^h = \big\{x = (x', x_3);\quad x' \in \Omega,\quad x_3 \in (-h/2, h/2)\big\}. \tag{4.2}$$

Let $G : \bar{\Omega}^h \to \mathbb{R}^{3\times3}$ be a given smooth Riemann metric on Ω^h independent of and uniform through the thickness:

$$G(x', x_3) = G(x') \quad \text{for every } (x', x_3) \in \Omega^h,$$

and let $A = \sqrt{G}$ denote the unique positive definite symmetric square root of G.

Consider the energy functional $E^h : W^{1,2}(\Omega^h, \mathbb{R}^3) \to \bar{\mathbb{R}}_+$ defined as:

$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h A^{-1})\,\mathrm{d}x. \tag{4.3}$$

The nonlinear elastic energy density $W : \mathbb{R}^{3\times3} \to \bar{\mathbb{R}}_+$ is a Borel measurable function, assumed to be $\mathcal{C}^2$ in a neighborhood of $SO(3)$ and to satisfy, for every $F \in \mathbb{R}^{3\times3}$, every $R \in SO(3)$ and with a uniform constant $c > 0$, the conditions:

$$W(R) = 0, \qquad W(RF) = W(F), \qquad W(F) \geq c\,\mathrm{dist}^2\left(F, SO(3)\right). \tag{4.4}$$

The first condition states that the energy of a rigid motion is zero and implies $DW(\mathrm{Id}_3) = 0$ since W is minimized at Id_3. The second is the frame invariance and, together with the previous property, implies that $D^2W(\mathrm{Id}_3) = 0$ on skew symmetric matrices (see Lemma (4.1.1) below). The third assumption above reflects the quadratic growth of the density W away from the energy well $SO(3)$. Note that $E^h(u^h) = 0$ if and only if $\nabla u^h(x) \in SO(3)A(x)$ for a.e. $x \in \Omega$. By the polar decomposition theorem for matrices, this implies $(\nabla u^h)^t\nabla u^h = A^tA = G$ and $\det(\nabla u^h) > 0$, i.e. we have an orientation preserving isometric immersion of the given metric G. Moreover, it was proved in [LP11] that $\inf_{u^h\in W^{1,2}(\Omega^h,\mathbb{R}^3)} E^h(u^h) = 0$ if and only if the Riemann curvature tensor of G vanishes identically in Ω^h, i.e.: $\mathrm{Riem}(G) \equiv 0$, and when (equivalently) the infimum above is achieved through a smooth isometric immersion u^h

of the metric G on Ω^h. Therefore, in the case of a non realizable G, the thin body exhibits residual stress at equilibrium and $\inf E^h > 0$.

For future use, we define the quadratic forms $\mathcal{Q}_2$ and $\mathcal{Q}_3$ as follows:

$$\mathcal{Q}_3(F) = D^2W(\mathrm{Id}_3)(F, F), \tag{4.5}$$

and:

$$\mathcal{Q}_{2,A}(x', F_{2\times2}) = \min\Big\{\mathcal{Q}_3\big(A(x')^{-1}\tilde{F}A(x')^{-1}\big); \quad \tilde{F} \in \mathbb{R}^{3\times3} \text{ with } \tilde{F}_{2\times2} = F_{2\times2}\Big\}. \tag{4.6}$$

Lemma 4.1.1. *Let W satisfy (4.4). Then the quadratic form $\mathcal{Q}_3$ satisfies:*

$$\mathcal{Q}_3(F) = \mathcal{Q}_3(\mathrm{sym}(F))$$

for all $F \in \mathbb{R}^{3\times3}$. Moreover it is non negative definite and positive definite on symmetric matrices.

Proof. Consider $A \in so(3)$, so that $e^{tA} \in SO(3)$. For every $F \in \mathbb{R}^{3\times3}$ we have:

$$W(\mathrm{Id}_3+tF) = W(e^{tA}(\mathrm{Id}_3+tF)) = W((\mathrm{Id}_3+tA+o(t))(\mathrm{Id}_3+tF)) = W(\mathrm{Id}_3+t(A+F)+o(t))$$

for t sufficiently small. By Taylor expansion, since $W(\mathrm{Id}_3) = DW(\mathrm{Id}_3) = 0$, we have:

$$D^2W(\mathrm{Id}_3)(F, F) = D^2W(\mathrm{Id}_3)(A + F + o(1), A + F + o(1)) + o(1)$$

for t small, hence:

$$\mathcal{Q}_3(F) = D^2W(\mathrm{Id}_3)(F, F) = D^2W(\mathrm{Id}_3)(F + A, F + A) = \mathcal{Q}_3(F + A).$$

In particular $\mathcal{Q}_3(F) = \mathcal{Q}_3(\mathrm{sym}(F))$ by choosing $A = -skew(F)$. Now:

$$\begin{aligned}\mathcal{Q}_3(F) = \mathcal{Q}_3(\mathrm{sym}(F)) &= \frac{1}{t^2}\mathcal{Q}_3(t\,\mathrm{sym}(F)) \geq \frac{1}{t^2}W(\mathrm{Id}_3 + t\,\mathrm{sym}(F)) - o(1)|\,\mathrm{sym}(F)|^2 \\ &\geq c\mathrm{dist}^2(\mathrm{Id}_3 + t\,\mathrm{sym}(F), SO(3)) - o(1)|\,\mathrm{sym}(F)|^2 \geq c|\,\mathrm{sym}(F)|^2.\end{aligned} \tag{4.7}$$

Note that we used again Taylor expansion, the growth hypothesis in (4.4) and the fact that for small t we have $\mathrm{dist}(\mathrm{Id}_3 + t\,\mathrm{sym}(F), SO(3)) = t|\,\mathrm{sym}(F)|$, since the tangent space to $SO(3)$ at Id_3 is $so(3)$. □

4.2 PREVIOUS RELATED RESULTS

In this section we collect results on the energy functional $h^{-2}E^h$ that will be needed later. They are taken from [BLS16], to which we refer for detailed proofs. The authors proved that the Γ-limit of the sequence of functionals $h^{-2}E^h$ is given by:

$$\mathcal{I}_2(y) = \frac{1}{24}\int_\Omega \mathcal{Q}_{2,A}\big(x', (\nabla y)^t\nabla\vec{b}\big)\ \mathrm{d}x', \tag{4.8}$$

effectively defined on the set of all $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ such that $(\nabla y)^t\nabla y = G_{2\times 2}$. The quadratic forms $\mathcal{Q}_{2,A}(x', \cdot)$ are given by means of the energy density W as in (4.6). The Cosserat vector $\vec{b} \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^3)$ is uniquely determined from the isometric immersion y by:

$$Q^tQ = G \quad \text{where} \quad Qe_1 = \partial_1 y, \quad Qe_2 = \partial_2 y, \quad Qe_3 = \vec{b}, \quad \text{with } \det Q > 0. \tag{4.9}$$

More precisely we have:

Theorem 4.2.1. *Given a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfying:*

$$E^h(u^h) \leq Ch^2,$$

there exists a sequence of translations $c^h \in \mathbb{R}^3$ such that the normalized deformations:

$$y^h(x', x_3) = u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$$

satisfy the following properties:

(i) there exists an isometric immersion $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ of the midplate metric $G_{2\times 2}$ such that $y^h \to y$ strongly in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ and:

$$\liminf_{h\to 0} \frac{1}{h^2}E^h(u^h) \geq \mathcal{I}_2(y),$$

where $\mathcal{I}_2$ is defined in (4.8).

(ii) Conversely, given an isometric immersion $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ *of the midplate metric* $G_{2\times 2}$*, we can find a sequence of deformations* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ *such that:*

$$y^h(x', x_3) := u^h(x', hx_3) \in W^{1,2}(\Omega^1, \mathbb{R}^3)$$

converges to y *strongly in* $W^{1,2}(\Omega^1, \mathbb{R}^3)$*, and:*

$$\lim_{h\to 0} \frac{1}{h^2} E^h(u^h) = \mathcal{I}_2(y).$$

Corollary 4.2.1. *Existence of a* $W^{2,2}$ *isometric immersion of the midplate metric is equivalent to the upper bound* $\inf E^h \leq Ch^2$.

The authors also give several conditions equivalent to the fact that the infimum of the energy scales as a higher power than h^2. We report some of them in the following:

Theorem 4.2.2. *The following conditions are equivalent:*

- $\lim_{h\to 0} h^{-2} \inf_{W^{1,2}(\Omega^h, \mathbb{R}^3)} E^h = 0$,
- *the following Riemann curvatures of* G *vanish identically:*

$$R_{1212} = R_{1213} = R_{1223} \equiv 0 \quad \text{in } \Omega^h. \tag{4.10}$$

- $\min\{\mathcal{I}_2(y) \mid y \in W^{2,2}(\Omega, \mathbb{R}^3),\ (\nabla y)^t \nabla y = G_{2\times 2}\} = 0$,
- *there exists an isometric immersion* $y_0 : \Omega \to \mathbb{R}^3$ *of* $G_{2\times 2}$*, which in turn is smooth and unique, such that:*

$$\begin{cases} (\nabla y_0)^t \nabla y_0 = G_{2\times 2} \\ \operatorname{sym}\left((\nabla y_0)^t \nabla \vec{b}_0\right) = 0. \end{cases} \tag{4.11}$$

Here the Cosserat vector $\vec{b}_0$ *associated to* y_0 *and the smooth matrix field* Q_0 *are given as in (4.9):*

$$Q_0^t Q_0 = G, \quad Q_0 e_1 = \partial_1 y_0, \quad Q_0 e_2 = \partial_2 y_0 \quad \text{and} \quad Q_0 e_3 = \vec{b}_0 \quad \text{with } \det Q_0 > 0. \tag{4.12}$$

Uniqueness of the immersion y_0 in (4.11) follows from Theorem 5.3 in [BLS16] which shows that the second fundamental form of the surface $y_0(\Omega)$ is given in terms of G. Therefore, both fundamental forms are known. Also, the second equation in (4.11) comes from the fact that the kernel of each quadratic form $\mathcal{Q}_{2,A}$ consists of $so(2)$, as a consequence of Lemma 4.1.1.

5.0 HIGHER ORDER SCALINGS

In this Chapter we investigate higher order scalings, i.e. $h^{-\beta}E^h$ for $\beta > 2$. The first surprising outcome is a quantization of the energy scalings. The first result in this direction is the following:

Lemma 5.0.1. *Assume:*

$$\lim_{h\to 0} h^{-2} \inf_{W^{1,2}(\Omega^h,\mathbb{R}^3)} E^h = 0. \tag{5.1}$$

Then:

$$\inf_{W^{1,2}(\Omega^h,\mathbb{R}^3)} E^h \leq Ch^4.$$

Proof. By the equivalences in Theorem 4.2.2, the hypothesis (5.1) implies the existence of a unique and smooth isometric immersion y_0 of the midplate metric $G_{2\times 2}$. Define the corresponding Cosserat vector $\vec{b}_0$ and the matrix field Q_0 as in (4.12). Also consider the smooth vector field $\vec{d}_0 : \Omega \to \mathbb{R}^3$ given by:

$$\langle Q_0^t \vec{d}_0, e_1\rangle = -\langle \partial_1 \vec{b}_0, \vec{b}_0\rangle, \quad \langle Q_0^t \vec{d}_0, e_2\rangle = -\langle \partial_2 \vec{b}_0, \vec{b}_0\rangle, \quad \langle Q_0^t \vec{d}_0, e_3\rangle = 0. \tag{5.2}$$

We now construct a sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ that has low energy. Let

$$u^h(x', x_3) = y_0(x') + x_3 \vec{b}_0(x') + \frac{x_3^2}{2} \vec{d}_0(x'). \tag{5.3}$$

Note that each u^h is the restriction on its domain Ω^h of the same deformation. Let $B_0(x')$ be the matrix field satisfying:

$$B_0 e_1 = \partial_1 \vec{b}_0, \quad B_0 e_2 = \partial_2 \vec{b}_0 \quad \text{and} \quad B_0 e_3 = \vec{d}_0. \tag{5.4}$$

Observe that in this way $Q_0^t B_0$ is skew symmetric. Indeed, it has the following block form:

$$Q_0^t B_0 = \left[\begin{array}{c|c} (\nabla y_0)^t\, \nabla \vec{b}_0 & (\nabla y_0)^t\, \vec{d}_0 \\ \hline (\vec{b}_0)^t \nabla \vec{b}_0 & \langle \vec{b}_0, \vec{d}_0 \rangle \end{array}\right] \tag{5.5}$$

and by (4.11) we see that $(\nabla y_0)^t\, \nabla \vec{b}_0 \in so(2)$ is skew symmetric, while by (5.2) we have $(\nabla y_0)^t\, \vec{d}_0 = -(\nabla \vec{b}_0)^t \vec{b}_0$ and $\langle \vec{b}_0, \vec{d}_0 \rangle = 0$. Define the matrix field $D_0(x') \in \mathbb{R}^{3\times 3}$ through:

$$D_0(x')e_1 = \partial_1 \vec{d}_0, \quad D_0(x')e_2 = \partial_2 \vec{d}_0, \quad D_0(x')e_3 = 0.$$

We have:

$$\nabla u^h(x', x_3) = Q_0(x') + x_3 B_0(x') + \frac{x_3^2}{2} D_0(x'),$$

and hence:

$$\nabla u^h A^{-1} = Q_0 A^{-1} + x_3 B_0 A^{-1} + \frac{x_3^2}{2} D_0 A^{-1}.$$

For brevity, denote $F^h = \nabla u^h A^{-1}$. Then:

$$F^h(x', x_3) = Q_0 A^{-1}(x')(\mathrm{Id}_3 + x_3 S(x') + x_3^2 T(x')) = (Q_0 A^{-1}(x')) G^h(x', x_3). \tag{5.6}$$

Note that we used $Q_0 A^{-1} \in SO(3)$ and we denoted $S = A^{-1} Q_0^t B_0 A^{-1}$, $T = \frac{1}{2} A^{-1} Q_0^t D_0 A^{-1}$ and $G^h = \mathrm{Id}_3 + x_3 S + x_3^2 T$. Frame indifference and polar decomposition imply that:

$$W(F^h) = W(G^h) = W\left(((G^h)^t G^h)^{1/2}\right).$$

Since $Q_0^t B_0$ is skew symmetric, S is skew symmetric. Therefore, $(G^h)^t G^h$ and the expansion of its square root do not contain terms linear in x_3. Indeed, letting $K = T + T^t - S^2$:

$$((G^h)^t G^h)(x', x_3) = \mathrm{Id}_3 + x_3^2 K(x') + \mathcal{O}(x_3^3)$$

and:

$$((G^h)^t G^h)^{1/2}(x', x_3) = \mathrm{Id}_3 + \frac{x_3^2}{2} K(x') + \mathcal{O}(x_3^3).$$

As a consequence, using $W(\mathrm{Id}_3) = 0$ and $DW(\mathrm{Id}_3) = 0$, we obtain:

$$W(F^h) = W\left(((G^h)^t G^h)^{1/2}\right) = \frac{x_3^4}{8} D^2 W(\mathrm{Id}_3)(K, K) + \mathcal{O}(x_3^5).$$

Using (4.3), we get:

$$E^h(u^h) = \frac{1}{h} \int_{\Omega^h} W(F^h)\, \mathrm{d}x \le Ch^4,$$

which accomplishes the proof of the Lemma. □

We now report the fundamental rigidity estimate due to Friesecke, James and Muller ([FJM06]):

Theorem 5.0.1. *Let $U \subset \mathbb{R}^n$ be a bounded, Lipschitz domain and $n \geq 2$. There exists a contant $C = C(U)$ such that for every $v \in W^{1,2}(U, \mathbb{R}^n)$ there exists $R \in SO(n)$ satisfying:*

$$\int_U |\nabla v - R|^2 \, \mathrm{d}x \leq C \int_U \mathrm{dist}^2(\nabla v, SO(n)) \, \mathrm{d}x. \tag{5.7}$$

The constant C is uniform with respect to biLipschitz equivalent domains with controlled Lipschitz constants.

In Lemma 5.0.1 above, we constructed deformations whose gradient was sufficiently close to $Q_0 + x_3 B_0$, to provide the energy of the order h^4. Conversely, in Corollary 5.0.3 below, we establish that the gradients of deformations u^h whose energy scales like h^4, are close to $Q_0 + x_3 B_0$ modulo local multiplications by $R^h(x') \in SO(3)$. Corollary 5.0.3 makes this statement precise and gives an estimation on ∇R^h as well.

For any $\mathcal{V}$ which is an open subset of Ω, we let $\mathcal{V}^h = \mathcal{V} \times (-h/2, h/2)$ and we define the local energy functional by:

$$E^h(u^h, \mathcal{V}^h) = \frac{1}{h} \int_{\mathcal{V}^h} W(\nabla u^h A^{-1}) \, \mathrm{d}x.$$

Lemma 5.0.2. *Assume (4.10). There exists a constant $C > 0$ with the following property. For any $u^h \in W^{1,2}(\mathcal{V}^h, \mathbb{R}^3)$, there exists $\bar{R}^h \in SO(3)$ such that:*

$$\frac{1}{h} \int_{\mathcal{V}^h} \left| \nabla u^h(x) - \bar{R}^h (Q_0(x') + x_3 B_0(x')) \right|^2 \, \mathrm{d}x \leq C \left(E^h(u^h, \mathcal{V}^h) + h^3 |\mathcal{V}^h| \right). \tag{5.8}$$

The constant C is uniform for all $\mathcal{V}^h$ which are bi-Lipschitz equivalent with controlled Lipschitz constants.

Proof. By the third assumption in (4.4), we have:

$$E^h(u^h, \mathcal{V}^h) \geq \frac{c}{h} \int_{\mathcal{V}^h} \mathrm{dist}^2 \left(\nabla u^h A^{-1}, SO(3)\right) \, \mathrm{d}x. \tag{5.9}$$

This suggests performing a change of variables in order to use the nonlinear geometric rigidity estimate (5.7). For any $u^h \in W^{1,2}(\mathcal{V}^h, \mathbb{R}^3)$, we let $v^h = u^h \circ Y^{-1}$ with $Y : \mathcal{V}^h \to Y(\mathcal{V}^h) = \mathcal{U}^h \subset \mathbb{R}^3$ given as in (5.3), namely:

$$Y(x', x_3) = y_0(x') + x_3\vec{b}_0(x') + \frac{x_3^2}{2}\vec{d}_0(x').$$

We have $v^h \in W^{1,2}(\mathcal{U}^h, \mathbb{R}^3)$ and:

$$\nabla u^h A^{-1}(x', x_3) = \nabla v^h(z)(\nabla Y A^{-1})(x', x_3), \qquad z := Y(x', x_3). \tag{5.10}$$

Let $S' = B_0Q_0^{-1}$ and $T' = \frac{1}{2}D_0Q_0^{-1}$. Note that $S' = B_0Q_0^{-1} = Q_0^{-1,t}(Q_0^tB_0Q_0^{-1}) = -Q_0^{-1,t}B_0^t = -(B_0Q_0^{-1})^t$ in view of $Q_0^tB_0 \in so(3)$. Therefore $S' \in so(3)$. Computations as in Lemma 5.0.1 now give:

$$\nabla Y(x', x_3) = Q_0(x') + x_3B_0(x') + \frac{x_3^2}{2}D_0(x'), \tag{5.11}$$

and:

$$\nabla Y A^{-1} = \left(\mathrm{Id}_3 + x_3S'(x') + x_3^2T'(x')\right)(Q_0A^{-1}).$$

We see that for h small, $\det(\nabla Y A^{-1}) > 0$. Further, the left polar decomposition $\nabla Y A^{-1} = \left(\nabla Y A^{-1}(\nabla Y A^{-1})^t\right)^{1/2} R$, allows us to write:

$$\nabla Y A^{-1} = (\mathrm{Id}_3 + x_3^2M(x', x_3))R(x', x_3),$$

where $M = \mathcal{O}(1)$ is a symmetric matrix field and $R \in SO(3)$. Again, the symmetric term does not contain any term linear in x_3. Therefore:

$$\begin{aligned}
\mathrm{dist}\left(\nabla v^h \nabla Y A^{-1}, SO(3)\right) &= \mathrm{dist}\left(\nabla v^h(\mathrm{Id}_3 + x_3^2M)R, SO(3)\right) \\
&= \mathrm{dist}\left(\nabla v^h(\mathrm{Id}_3 + x_3^2M), SO(3)\right) \geq c\,\mathrm{dist}\left(\nabla v^h, SO(3)(\mathrm{Id}_3 + x_3^2M)^{-1}\right) \\
&\geq c\,\mathrm{dist}\left(\nabla v^h, SO(3)\right) + \mathcal{O}(x_3^2).
\end{aligned}$$

Now, let $J = |\det \nabla Y \circ Y^{-1}|^{-1}$. By (5.10) and the above computation:

$$\begin{aligned}\int_{\mathcal{V}^h} \operatorname{dist}^2\left(\nabla u^h A^{-1}, SO(3)\right)\,\mathrm{d}x &\geq c\int_{\mathcal{U}^h} \operatorname{dist}^2\left(\nabla v^h, SO(3)\right) J\,\mathrm{d}z - c\int_{\mathcal{V}^h} x_3^4\,\mathrm{d}x\\ &\geq c\int_{\mathcal{U}^h} \operatorname{dist}^2\left(\nabla v^h, SO(3)\right) J\,\mathrm{d}z - ch^4|\mathcal{V}^h|.\end{aligned}$$

In other words, since $J \geq c > 0$:

$$\frac{1}{h}\int_{\mathcal{V}^h} \operatorname{dist}^2\left(\nabla u^h A^{-1}, SO(3)\right)\,\mathrm{d}x + h^3|\mathcal{V}^h| \geq \frac{c}{h}\int_{\mathcal{U}^h} \operatorname{dist}^2\left(\nabla v^h, SO(3)\right)\,\mathrm{d}z.$$

By Theorem 5.0.1, there exists $C > 0$ with the following property. For any $v^h \in W^{1,2}(\mathcal{U}^h, \mathbb{R}^3)$, there exists $\bar{R}^h \in SO(3)$ such that:

$$C\int_{\mathcal{U}^h} \operatorname{dist}^2\left(\nabla v^h, SO(3)\right)\,\mathrm{d}z \geq \int_{\mathcal{U}^h} \left|\nabla v^h - \bar{R}^h\right|^2\,\mathrm{d}z.$$

The constant C can be chosen uniformly for domains $\mathcal{U}^h$ which are bi-Lipschitz equivalent with controlled Lipschitz constants. By (5.9) and the reverse change of variables which satisfies $J^{-1} \geq c > 0$ and $|\nabla Y| \leq C$, we obtain:

$$C\left(E^h(u^h, \mathcal{V}^h) + h^3|\mathcal{V}^h|\right) \geq \frac{1}{h}\int_{\mathcal{V}^h} \left|\nabla u^h - \bar{R}^h \nabla Y\right|^2\,\mathrm{d}x,$$

again with a constant C uniform for domains $\mathcal{V}^h$ that are bi-Lipschitz equivalent with controlled Lipschitz constants. This accomplishes the proof in view of (5.11). □

Corollary 5.0.3. *Assume (4.10) and let u^h be a sequence of deformations such that:*

$$\lim_{h\to 0} h^{-2}E^h(u^h) = 0.$$

Then, there exist matrix fields $R^h \in W^{1,2}(\Omega, SO(3))$ such that:

$$\frac{1}{h}\int_{\Omega^h} \left|\nabla u^h(x) - R^h(x')\left(Q_0(x') + x_3 B_0(x')\right)\right|^2\,\mathrm{d}x \leq C\left(E^h(u^h) + h^4\right) \tag{5.12}$$

and:

$$\int_{\Omega} \left|\nabla R^h(x')\right|^2\,\mathrm{d}x' \leq \frac{C}{h^2}\left(E^h(u^h) + h^4\right). \tag{5.13}$$

The proof follows the lines of [FJM06, LP11, LMP11a], with necessary modifications in view of the expected error of the order h^4. For completeness, we present the details.

Proof. **1.** For every $x' \in \Omega$ denote $D_{x',\delta} = B(x',\delta) \cap \Omega$ and $B_{x',\delta,h} = D_{x',\delta} \times (-h/2, h/2)$. For short, we write $B_{x',2h} = B_{x',2h,h}$ and $B_{x',h} = B_{x',h,h}$. Apply Lemma 5.0.2 to the set $\mathcal{V}^h = B_{x',2h}$ to get a rotation $R_{x',2h} \in SO(3)$ such that, with a universal constant C:

$$\begin{aligned} \frac{1}{h}\int_{B_{x',2h}} \left|\nabla u^h(z) - R_{x',2h}\left(Q_0(z') + z_3 B_0(z')\right)\right|^2 \mathrm{d}z \\ \leq C\left(E^h(u^h, B_{x',2h}) + h^3\left|B_{x',2h}\right|\right). \end{aligned} \tag{5.14}$$

Consider a family of mollifiers $\eta_{x'} \in C^\infty(\Omega, \mathbb{R})$, parametrized by $x' \in \Omega$:

$$\int_\Omega \eta_{x'} = \frac{1}{h}, \quad \|\eta_{x'}\|_{L^\infty(\Omega)} \leq \frac{C}{h^3}, \quad \|\nabla_{x'}\eta_{x'}\|_{L^\infty(\Omega)} \leq \frac{C}{h^4} \quad \text{and } (\operatorname{supp}\eta_{x'}) \cap \Omega \subset D_{x',h}.$$

Define $\tilde{R}^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$ as:

$$\tilde{R}^h(x') = \int_{\Omega^h} \eta_{x'}(z')\nabla u^h(z)\left(Q_0(z') + z_3 B_0(z')\right)^{-1} \mathrm{d}z. \tag{5.15}$$

We then have:

$$\begin{aligned} \frac{1}{h}\int_{B_{x',h}} &|\nabla u^h(z) - \tilde{R}^h(z')\left(Q_0(z') + z_3 B_0(z')\right)|^2 \mathrm{d}z \\ &\leq \frac{C}{h}\int_{B_{x',2h}} \left|\nabla u^h(z) - R_{x',2h}\left(Q_0(z') + z_3 B_0(z')\right)\right|^2 \mathrm{d}z \\ &\quad + \frac{C}{h}\int_{B_{x',h}} |\tilde{R}^h(z') - R_{x',2h}|^2 |Q_0(z') + z_3 B_0(z')|^2 \mathrm{d}z \\ &\leq C\left(E^h(u^h, B_{x',2h}) + h^3\left|B_{x',2h}\right|\right) + \frac{C}{h}\int_{B_{x',h}} |\tilde{R}^h(z') - R_{x',2h}|^2 \mathrm{d}z, \end{aligned} \tag{5.16}$$

where we have used (5.14) and $\|Q_0(z') + z_3 B_0(z')\|_{L^\infty} \leq C$. Now, for every $z' \in B_{x',h}$ we have:

$$\begin{aligned} |\tilde{R}^h(z') - R_{x',2h}|^2 &= \left|\int_{\Omega^h} \eta_{z'}(y')\nabla u^h(y)\left(Q_0(y') + y_3 B_0(y')\right)^{-1} \mathrm{d}y - R_{x',2h}\right|^2 \\ &= \left|\int_{\Omega^h} \eta_{z'}(y')\left(\nabla u^h(y) - R_{x',2h}\left(Q_0(y') + y_3 B_0(y')\right)\right)\left(Q_0(z') + y_3 B_0(z')\right)^{-1} \mathrm{d}y\right|^2 \\ &\leq C\left(\int_{B_{z',h}} \eta_{z'}(y')^2 \mathrm{d}y\right)\left(\int_{B_{z',h}} \left|\nabla u^h(y) - R_{x',2h}\left(Q_0(y') + y_3 B_0(y')\right)\right|^2 \mathrm{d}y\right) \\ &\leq \frac{C}{h^3}\int_{B_{x',2h}} \left|\nabla u^h(y) - R_{x',2h}\left(Q_0(y') + y_3 B_0(y')\right)\right|^2 \mathrm{d}y \\ &\leq \frac{C}{h^2}\left(E^h(u^h, B_{x',2h}) + h^3\left|B_{x',2h}\right|\right). \end{aligned} \tag{5.17}$$

In a similar way, in view of $\int_{\Omega^h} \nabla_{z'}\eta_{z'}(y')\ \mathrm{d}y = 0$, it follows that:

$$
\begin{aligned}
|\nabla \tilde{R}^h(z')|^2 &= \left(\int_{\Omega^h} \nabla_{z'}\eta_{z'}(y')\nabla u^h(y)\left(Q_0(y') + y_3 B_0(y')\right)^{-1}\,\mathrm{d}y\right)^2 \\
&= \left(\int_{B_{x',2h}} \nabla_{z'}\eta_{z'}(y')\left(\nabla u^h(y)\left(Q_0(y') + y_3 B_0(y')\right)^{-1} - R_{x',2h}\right)\mathrm{d}y\right)^2 \\
&\le C\int_{\Omega^h} |\nabla_{z'}\eta_{z'}(y')|^2\,\mathrm{d}y \int_{B_{x',2h}} \left|\nabla u^h(y) - R_{x',2h}\left(Q_0(y') + y_3 B_0(y')\right)\right|^2\,\mathrm{d}y \\
&\le \frac{C}{h^4}\left(E^h(u^h, B_{x',2h}) + h^3\,|B_{x',2h}|\right).
\end{aligned}
$$

From (5.17) we obtain:

$$
\begin{aligned}
\int_{B_{x',h}} |\tilde{R}^h(z') - R_{x',2h}|^2\,\mathrm{d}z &\le \frac{C}{h^2}\int_{B_{x',h}} \left(E^h(u^h, B_{x',2h}) + h^3|B_{x',2h}|\right)\mathrm{d}z \\
&\le Ch\left(E^h(u^h, B_{x',2h}) + h^3|B_{x',2h}|\right),
\end{aligned}
$$

and therefore by (5.16) we further see that:

$$
\begin{aligned}
\frac{1}{h}\int_{B_{x',h}} &|\nabla u^h(z) - \tilde{R}^h(z')\left(Q_0(z') + z_3 B_0(z')\right)|^2\,\mathrm{d}z \\
&\le C\left(E^h(u^h, B_{x',2h}) + h^3|B_{x',2h}|\right).
\end{aligned} \tag{5.18}
$$

2. Covering Ω^h by a finite family of sets $\{B_{x',h}\}$, such that the intersection number of the doubled covering $\{B_{x',2h}\}$ is independent of h, applying (5.18) and summing over the covering, it follows that:

$$
\frac{1}{h}\int_{\Omega^h} |\nabla u^h(z) - \tilde{R}^h(z')\left(Q_0(z') + z_3 B_0(z')\right)|^2\,\mathrm{d}z \le C\left(E^h(u^h) + h^4\right).
$$

In a similar fashion we obtain:

$$
\begin{aligned}
\int_{D_{x',h}} |\nabla \tilde{R}^h(z')|^2\,\mathrm{d}z &\le \frac{C}{h^4}\int_{D_{x',h}} \left(E^h(u^h, B_{x',2h}) + h^3|B_{x',2h}|\right)\mathrm{d}z \\
&\le \frac{C}{h^2}\left(E^h(u^h, B_{x',2h}) + h^3|B_{x',2h}|\right),
\end{aligned}
$$

and by the same covering argument:

$$
\int_{\Omega^h} |\nabla \tilde{R}^h(z')|^2\,\mathrm{d}z \le \frac{C}{h^2}\left(E^h(u^h) + h^4\right).
$$

3. Note that, in the above two estimates, we can replace $\tilde{R}^h$ by $R^h = \mathbb{P}_{\mathrm{SO(3)}}\,\tilde{R}^h \in W^{1,2}(\Omega, SO(3))$. Firstly, the projection in question is well defined in view of (5.17), since:

$$\mathrm{dist}^2\left(\tilde{R}^h, SO(3)\right) \leq |\tilde{R}^h - R_{x',2h}| \leq \frac{C}{h^2}\left(E^h(u^h) + h^4\right).$$

Moreover:

$$\begin{aligned}\frac{1}{h}\int_{B_{x',h}} &|\nabla u^h(z) - R^h(z')\left(Q_0(z') + z_3 B_0(z')\right)|^2\,\mathrm{d}z \\ &\leq \frac{C}{h}\int_{B_{x',h}} \left|\nabla u^h(z) - \tilde{R}^h(z')\left(Q_0(z') + z_3 B_0(z')\right)\right|^2\,\mathrm{d}z \\ &\quad + \frac{C}{h}\int_{B_{x',h}} |\tilde{R}^h(z') - R^h(z')|^2 |Q_0(z') + z_3 B_0(z')|^2\,\mathrm{d}z \\ &\leq C\left(E^h(u^h, B_{x',2h}) + h^3|B_{x',2h}|\right)\end{aligned}$$

because of (5.18) and (5.17). Finally, the previous covering argument implies (5.12), and $\int_\Omega |\nabla R^h|^2\,\mathrm{d}z \leq C\int_\Omega |\nabla \tilde{R}^h|^2\,\mathrm{d}z$ yields (5.13). □

5.1 THE LOWER BOUND

Towards the identification of the Γ-limit of the functional $h^{-4}E^h$, we now present a proof of the liminf inequality and some compactness results:

Theorem 5.1.1. *Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ be a sequence of deformations satisfying:*

$$E^h(u^h) \leq Ch^4.$$

Then, there exist a sequence of translations $c^h \in \mathbb{R}^3$ and rotations $\bar{R}^h \in SO(3)$ such that the associated renormalizations:

$$y^h(x', x_3) = (\bar{R}^h)^t u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3) \tag{5.19}$$

have the following properties, where y_0 and $\vec{b}_0$ are the unique solutions respectively of (4.11) and (4.12). All convergences hold up to a subsequence:

(i) $y^h \to y_0$ in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ and $\frac{1}{h}\partial_3 y^h \to \vec{b}_0$ in $L^2(\Omega^1, \mathbb{R}^3)$;

(ii) the scaled average displacements:

$$V^h(x') = \frac{1}{h}\fint_{-\frac{1}{2}}^{\frac{1}{2}} \Big(y^h(x', x_3) - \big(y_0(x') + hx_3\vec{b}_0(x')\big)\Big)\,\mathrm{d}x_3 \tag{5.20}$$

converge in $W^{1,2}(\Omega, \mathbb{R}^3)$ to a limiting field $V \in W^{2,2}(\Omega, \mathbb{R}^3)$, satisfying the constraint:

$$\mathrm{sym}\left((\nabla y_0)^t \nabla V\right) = 0; \tag{5.21}$$

(iii) the scaled tangential strains:

$$\frac{1}{h}\,\mathrm{sym}\left((\nabla y_0)^t \nabla V^h\right)$$

converge weakly in $L^2(\Omega, \mathbb{R}^{2\times 2})$ to some $\mathbb{S} \in L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$.

(iv) Further, defining the quadratic forms $\mathcal{Q}_3$ and $\mathcal{Q}_{2,A}$ as in (4.5) and (4.6) we have:

$$\begin{aligned}\liminf_{h\to 0} \frac{1}{h^4} E^h(u^h) \geq \mathcal{I}_4(V, \mathbb{S}) = \; & \frac{1}{2}\int_\Omega \mathcal{Q}_{2,A}\left(x', \mathbb{S} + \frac{1}{2}(\nabla V)^t \nabla V + \frac{1}{24}(\nabla \vec{b}_0)^t \nabla \vec{b}_0\right)\mathrm{d}x' \\ & + \frac{1}{24}\int_\Omega \mathcal{Q}_{2,A}\left(x', (\nabla y_0)^t \nabla \vec{p} + (\nabla V)^t \nabla \vec{b}_0\right)\mathrm{d}x' \\ & + \frac{1}{1440}\int_\Omega \mathcal{Q}_{2,A}\left(x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0\right)\mathrm{d}x',\end{aligned} \tag{5.22}$$

where the vector field $\vec{p} \in W^{1,2}(\Omega, \mathbb{R}^3)$ is uniquely associated with V by:

$$\begin{cases} (\nabla y_0)^t \vec{p} = -(\nabla V)^t \vec{b}_0 \\ \langle \vec{b}_0\,,\, \vec{p}\rangle = 0. \end{cases} \tag{5.23}$$

Proof. **1.** Corollary 5.0.3 yields existence of $R^h \in W^{1,2}(\Omega, SO(3))$ such that (5.12) and (5.13) hold with Ch^4 and Ch^2 in their right hand sides, respectively. We rewrite these inequalities for the reader's convenience:

$$\frac{1}{h}\int_{\Omega^h} \left|\nabla u^h(x) - R^h(x')\left(Q_0(x') + x_3 B_0(x')\right)\right|^2 \mathrm{d}x \leq Ch^4 \tag{5.24}$$

and:

$$\int_\Omega \left|\nabla R^h(x')\right|^2 \mathrm{d}x' \leq Ch^2. \tag{5.25}$$

To prove the claimed convergence properties for (5.19), it is natural in view of (5.24) to set:

$$\bar{R}^h = \mathbb{P}_{\mathrm{SO}(3)} \fint_{\Omega^h} \nabla u^h(x) Q_0(x')^{-1} \, \mathrm{d}x.$$

This projection is well defined, because for every $x' \in \Omega$, in view of (5.24):

$$\begin{aligned}
\mathrm{dist}^2 \left(\fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x, SO(3) \right) &\le \left| \fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x - R^h(x') \right|^2 \\
&\le C \left| \fint_{\Omega^h} (\nabla u^h Q_0^{-1} - R^h) \, \mathrm{d}x \right|^2 + C \left| \fint_{\Omega^h} R^h \, \mathrm{d}x - R^h(x') \right|^2 \\
&\le C \left| \fint_{\Omega^h} \left(\nabla u^h - R^h (Q_0 + x_3 B_0) \right) Q_0^{-1} \right|^2 \, \mathrm{d}x + C \left| R^h(x') - \fint_{\Omega} R^h \right|^2 \\
&\le C \fint_{\Omega^h} |\nabla u^h - R^h(Q_0 + x_3 B_0)|^2 \, \mathrm{d}x + C |R^h(x') - \fint_{\Omega} R^h|^2 \\
&\le C h^4 + C |R^h(x') - \fint_{\Omega} R^h|^2.
\end{aligned}$$

Now, taking the average on Ω, by the Poincaré-Wirtinger inequality and (5.25), we get:

$$\mathrm{dist}^2 \left(\fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x, SO(3) \right) \le C h^4 + C \int_{\Omega} |\nabla R^h|^2 \le C h^2,$$

which proves that the average $\fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x$ is close to $SO(3)$ and, by definition of $\bar{R}^h$, that:

$$| \fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x - \bar{R}^h |^2 \le C h^2. \tag{5.26}$$

Moreover:

$$\begin{aligned}
&\fint_{\Omega} |R^h - \bar{R}^h|^2 \, \mathrm{d}x = \fint_{\Omega^h} |R^h - \bar{R}^h|^2 \, \mathrm{d}x \\
&\le C \fint_{\Omega^h} \left(|R^h - \fint_{\Omega} R^h|^2 + |(\fint_{\Omega} R^h) - \fint_{\Omega^h} \nabla u^h Q_0^{-1}|^2 \right) + \fint_{\Omega^h} |\bar{R}^h - \fint_{\Omega^h} \nabla u^h Q_0^{-1}|^2 \qquad (5.27) \\
&\le C \fint_{\Omega^h} |\nabla R^h|^2 \, \mathrm{d}x + C \fint_{\Omega^h} |\nabla u^h - R^h(Q_0 + x_3 B_0)|^2 \, \mathrm{d}x + C h^2 \le C h^2,
\end{aligned}$$

where the last estimate follows by (5.24), (5.25) and (5.26).

Let now $c^h \in \mathbb{R}^3$ be such that $\int_{\Omega} V^h = 0$ where V^h is defined as in (5.20). Denote by $\nabla_h y^h$ the matrix whose columns are given by $\partial_1 y^h$, $\partial_2 y^h$ and $\partial_3 y^h / h$. We have:

$$\nabla_h y^h(x', x_3) = (\bar{R}^h)^t \nabla u^h(x', h x_3). \tag{5.28}$$

Observe that:

$$\int_{\Omega^1} |\nabla_h y^h - Q_0|^2 \,\mathrm{d}x \leq C \fint_{\Omega^h} |\nabla u^h - \bar{R}^h Q_0|^2 \,\mathrm{d}x$$
$$\leq C\Big(\fint_{\Omega^h} |\nabla u^h - R^h(Q_0 + x_3 B_0)|^2 \,\mathrm{d}x + \fint_{\Omega^h} |x_3 R^h B_0|^2 \,\mathrm{d}x + \fint_{\Omega^h} |R^h - \bar{R}^h|^2 \,\mathrm{d}x\Big) \leq Ch^2$$

by (5.24) and (5.27). Therefore, $\nabla_h y^h$ converges in $L^2(\Omega^1)$ to Q_0. Further, the sequence $\{y^h\}$ is bounded in $W^{1,2}(\Omega^1)$, by the choice of c^h. Passing to a subsequence, we get that y^h converges weakly in $W^{1,2}(\Omega^1)$ and in view of the strong convergence of ∇y^h we have:

$$y^h \to y_0 \quad \text{in} \quad W^{1,2}(\Omega^1, \mathbb{R}^3) \quad \text{and} \quad \frac{1}{h}\partial_3 y^h \to \vec{b}_0 \quad \text{in} \quad L^2(\Omega^1, \mathbb{R}^3).$$

2. Note that, for every $x' \in \Omega$:

$$\begin{aligned} \nabla V^h(x') &= \frac{1}{h}\left(\fint_{-1/2}^{1/2} \nabla_h y^h(x) - Q_0(x') \,\mathrm{d}x_3\right)_{3\times 2} \\ &= \frac{1}{h}\left(\fint_{-1/2}^{1/2} \nabla_h y^h - (\bar{R}^h)^t R^h(Q_0 + hx_3 B_0) \,\mathrm{d}x_3\right)_{3\times 2} + \frac{1}{h}\big(((\bar{R}^h)^t R^h - \mathrm{Id}_3)Q_0\big)_{3\times 2} \\ &= I_1^h + I_2^h. \end{aligned} \tag{5.29}$$

The first term above converges to 0. Indeed:

$$\begin{aligned} \|I_1^h\|^2_{L^2(\Omega)} &\leq \frac{C}{h^2}\fint_{\Omega^1} |(\bar{R}^h)^t \nabla u^h(x', hx_3) - (\bar{R}^h)^t R^h(Q_0(x') + hx_3 B_0)|^2 \,\mathrm{d}x \\ &\leq \frac{C}{h^2}\fint_{\Omega^h} |\nabla u^h(x', x_3) - R^h(Q_0 + x_3 B_0)|^2 \,\mathrm{d}x \leq Ch^2. \end{aligned} \tag{5.30}$$

Towards estimating the second term in (5.29), denote:

$$S^h = \frac{1}{h}((\bar{R}^h)^t R^h - \mathrm{Id}_3).$$

By (5.27) and (5.25), it follows that:

$$\|S^h\|^2_{L^2(\Omega)} \leq \frac{C}{h^2}\int_\Omega |R^h - \bar{R}^h|^2 \leq C \quad \text{and} \quad \|\nabla S^h\|^2_{L^2(\Omega)} \leq \frac{C}{h^2}\int_\Omega |\nabla R^h|^2 \leq C.$$

Passing to a subsequence, we can assume that:

$$S^h \rightharpoonup S \quad \text{weakly in } W^{1,2}(\Omega), \tag{5.31}$$

which implies:

$$I_2^h \to (SQ_0)_{3\times 2} \quad \text{in } L^2(\Omega, \mathbb{R}^{3\times 2}). \tag{5.32}$$

Consequently, by (5.29):

$$\nabla V^h \to (SQ_0)_{3\times 2} \quad \text{in } L^2(\Omega, \mathbb{R}^{3\times 2}). \tag{5.33}$$

As before, we conclude that V^h converges in $W^{1,2}(\Omega)$ and that its limit V belongs to $W^{2,2}(\Omega, \mathbb{R}^3)$, since $\nabla V = (SQ_0)_{3\times 2} \in W^{1,2}(\Omega)$. We now prove (5.21). By definition of S^h:

$$\operatorname{sym} S^h = -\frac{h}{2}(S^h)^t S^h, \tag{5.34}$$

so in view of the boundedness of $\{S^h\}$ in $W^{1,2}$:

$$\|\operatorname{sym} S^h\|_{L^2(\Omega)} \le Ch\|S^h\|^2_{L^4(\Omega)} \le Ch\|S^h\|^2_{W^{1,2}(\Omega)} \le Ch.$$

Consequently, S is a skew symmetric field. But $(\nabla y_0)^t \nabla V = (Q_0^t S Q_0)_{2\times 2}$, hence (5.21) follows.

For future use, let us define $\vec{p} \in W^{1,2}(\Omega, \mathbb{R}^3)$ by:

$$[\nabla V \mid \vec{p}] = SQ_0. \tag{5.35}$$

Since $Q_0^t[\nabla V \mid p] = Q_0^t S Q_0 \in so(3)$, we have

$$Q_0^t[\nabla V \mid \vec{p}] = \left[\begin{array}{c|c} (\nabla y_0)^t\, \nabla V & (\nabla y_0)^t \vec{p} \\ \hline (\vec{b}_0)^t\, \nabla V & (\vec{b}_0)^t \vec{p} \end{array}\right],$$

hence $\vec{p}$ is given solely in terms of V by:

$$\begin{cases} (\nabla y_0)^t \vec{p} = -(\nabla V)^t \vec{b}_0 \\ \langle \vec{b}_0\,,\, \vec{p}\rangle = 0. \end{cases} \tag{5.36}$$

3. We now want to establish convergence in (iii). From (5.29) and the definition of S^h we have:

$$\begin{aligned} \frac{1}{h}\operatorname{sym}(Q_0^t \nabla V^h)_{2\times 2}(x') &= \frac{1}{h}\operatorname{sym}(Q_0^t I_1^h)_{2\times 2} + \frac{1}{h}\operatorname{sym}\left(Q_0^t S^h Q_0\right)_{2\times 2} \\ &= J_1^h + J_2^h. \end{aligned} \tag{5.37}$$

We first deal with the sequence J_2^h. By (5.31), $S^h \to S$ in $L^4(\Omega)$ and so (5.34) implies:

$$\frac{1}{h}\,\mathrm{sym}\, S^h \to -\frac{1}{2}S^t S = \frac{1}{2}S^2 \quad \text{in } L^2(\Omega).$$

Therefore:

$$J_2^h \to -\frac{1}{2}\left(Q_0^t S^t S Q_0\right)_{2\times 2} = -\frac{1}{2}(\nabla V)^t \nabla V \quad \text{in} \quad L^2(\Omega). \tag{5.38}$$

We now prove that J_1^h converges. Recall that by (5.37), (5.29) and (5.28):

$$J_1^h = \frac{1}{h}\,\mathrm{sym}\,(Q_0^t I_1^h)_{2\times 2} = \mathrm{sym}\left(Q_0^t(\bar{R}^h)^t \fint_{-1/2}^{1/2} Z^h(x', x_3)\,\mathrm{d}x_3\right)_{2\times 2}, \tag{5.39}$$

where the rescaled strains Z^h are defined by:

$$Z^h(x', x_3) = \frac{1}{h^2}\left(\nabla u^h(x', hx_3) - R^h(x')(Q_0(x') + hx_3 B_0(x'))\right). \tag{5.40}$$

By (5.24), the sequence $\{Z^h\}$ is bounded in $L^2(\Omega^1, \mathbb{R}^3)$. Therefore, up to a subsequence:

$$Z^h \rightharpoonup Z \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3). \tag{5.41}$$

It follows that:

$$J_1^h \rightharpoonup J_1 := \mathrm{sym}\left(Q_0^t(\bar{R})^t \fint_{-1/2}^{1/2} Z(x', x_3)\,\mathrm{d}x_3\right)_{2\times 2} \quad \text{weakly in } L^2(\Omega). \tag{5.42}$$

which yields (iii) by (5.37) and (5.38).

4. We now aim at giving the structure of the weak limit $\mathbb{S}$ of $\frac{1}{h}\,\mathrm{sym}\,(Q_0^t \nabla V^h)_{2\times 2}$ in terms of the limiting fields V and Z. We have just seen that:

$$\mathbb{S} = J_1 - \frac{1}{2}(\nabla V)^t \nabla V, \tag{5.43}$$

where J_1 is given by (5.42). As a tool, consider the difference quotients $f^{s,h}$:

$$f^{s,h}(x', x_3) = \frac{1}{h^2 s}\left(y^h(x', x_3 + s) - y^h(x', x_3) - hs\left(\vec{b}_0 + h\big(x_3 + \frac{s}{2}\big)\vec{d}_0\right)\right),$$

and let us study for any s the convergence of $f^{s,h}$ as $h \to 0$. In fact, we will show that $f^{s,h} \rightharpoonup \vec{p}$, weakly in $W^{1,2}(\Omega^1, \mathbb{R}^3)$. Write:

$$f^{s,h}(x', x_3) = \frac{1}{h^2}\fint_0^s \partial_3 y^h(x', x_3 + t) - h(\vec{b}_0 + h(x_3 + t)\vec{d}_0)\,\mathrm{d}t,$$

and observe that:

$$\frac{1}{h^2}\left(\partial_3 y^h - h(\vec{b}_0 + hx_3\vec{d}_0)\right) = \frac{1}{h}\left((\bar{R}^h)^t \nabla u^h(x', hx_3) - (Q_0 + hx_3 B_0)\right) e_3$$
$$= \frac{1}{h}(\bar{R}^h)^t \left(\nabla u^h(x', hx_3) - R^h(Q_0 + hx_3 B_0)\right) e_3 + S^h(Q_0 + hx_3 B_0)e_3$$
$$= h(\bar{R}^h)^t Z^h(x', x_3)e_3 + S^h(Q_0 + hx_3 B_0)e_3.$$

The first term in the right hand side above converges to 0 in $L^2(\Omega^1)$ because $\{Z^h\}$ is bounded in $L^2(\Omega^1, \mathbb{R}^3)$, while the second term converges to $SQ_0e_3 = S\vec{b}_0$ in $L^2(\Omega^1)$ by (5.31). Note that $SQ_0e_3 = \vec{p}$ by (5.35). Therefore, $f^{s,h} \to \vec{p}$ in $L^2(\Omega^1)$.

We now deal with the derivatives of the studied sequence. Firstly:

$$\partial_3 f^{s,h}(x', x_3) = \frac{1}{s}\Big(\frac{1}{h^2}\left(\partial_3 y^h(x', x_3+s) - h(\vec{b}_0 + h(x_3+s)\vec{d}_0)\right)$$
$$- \frac{1}{h^2}\left(\partial_3 y^h(x', x_3) - h(\vec{b}_0 + hx_3\vec{d}_0)\right)\Big)$$

converges to 0 in $L^2(\Omega^1)$. For $i = 1, 2$, the in-plane derivatives read as:

$$\partial_i f^{s,h}(x', x_3) = \frac{1}{h^2 s}\Big((\bar{R}^h)^t \partial_i u^h(x', h(x_3+s))$$
$$- (\bar{R^h})^t \partial_i u^h(x', hx_3) - hs\left(\partial_i \vec{b}_0 + h\left(x_3 + \frac{s}{2}\right)\right)\partial_i \vec{d}_0\Big)$$
$$= \frac{1}{s}\left((\bar{R}^h)^t Z^h(x', x_3+s) - (\bar{R^h})^t Z^h(x', x_3)\right) e_i$$
$$+ \frac{1}{h^2 s}\left((\bar{R}^h)^t R^h(Q_0 + h(x_3+s)B_0) - (\bar{R}^h)^t R^h(Q_0 + hx_3 B_0)\right) e_i$$
$$- \frac{1}{h}\left(B_0 e_i + h\big(x_3 + \frac{s}{2}\big)\partial_i \vec{d}_0\right).$$

The last two terms above can be written as: $S^h B_0 e_i - \left(x_3 + \frac{s}{2}\right)\partial_i \vec{d}_0$, hence by (5.41):

$$\partial_i f^{s,h}(x', x_3) \rightharpoonup \frac{1}{s}(\bar{R})^t \Big(Z(x', x_3+s) - Z(x', x_3)\Big)e_i$$
$$+ SB_0 e_i - \left(x_3 + \frac{s}{2}\right)\partial_i \vec{d}_0 \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^3),$$

where $\bar{R} \in SO(3)$ is an accumulation point of the rotations $\bar{R}^h$.

Consequently, $f^{s,h} \rightharpoonup \vec{p}$ weakly in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ and, for $i = 1, 2$:

$$s\partial_i \vec{p} = (\bar{R})^t \Big(Z(x', x_3+s) - Z(x', x_3)\Big)e_i + sSB_0 e_i - s\left(x_3 + \frac{s}{2}\right)\partial_i \vec{d}_0, \tag{5.44}$$

which proves that $Z(x', \cdot)e_i$ has polynomial form and that:

$$\big(\bar{R}^t Z(x', x_3)\big)_{3\times 2} = \big(\bar{R}^t Z(x', 0)\big)_{3\times 2} + x_3\left(\nabla\vec{p} - (SB_0)_{3\times 2}\right) + \frac{x_3^2}{2}\nabla\vec{d}_0. \tag{5.45}$$

By (5.41), it follows that:

$$J_1 = \operatorname{sym}\big(Q_0^t(\bar{R})^t Z(x', 0)\big)_{2\times 2} + \frac{1}{24}\operatorname{sym}(Q_0^t\nabla\vec{d}_0)_{2\times 2}.$$

With (5.43), we finally arrive at the following identity that links $\mathbb{S}$ and V and Z:

$$\mathbb{S}(x') = \operatorname{sym}\big(Q_0^t(\bar{R})^t Z(x', 0)\big)_{2\times 2} + \frac{1}{24}\operatorname{sym}(Q_0^t\nabla\vec{d}_0)_{2\times 2} - \frac{1}{2}(\nabla V)^t\nabla V. \tag{5.46}$$

5. We now prove the lower bound in (iv). Recall that by (5.40):

$$\nabla u^h(x', hx_3) = R^h(x')(Q_0(x') + hx_3B_0(x')) + h^2Z^h(x', x_3).$$

Since $Q_0A^{-1} \in SO(3)$ we have:

$$W(\nabla u^hA^{-1}) = W\big((Q_0A^{-1})^t(R^h)^t\nabla u^hA^{-1}\big) = W\big(\mathrm{Id}_3 + h\mathcal{J} + h^2\mathcal{G}^h\big),$$

where:

$$\mathcal{J}(x', x_3) = x_3A^{-1}(Q_0^tB_0)A^{-1}(x') \in so(3), \quad \mathcal{G}^h(x', x_3) = A^{-1}Q_0^t(R^h)^tZ^h(x', x_3)A^{-1}. \tag{5.47}$$

Note that by (5.41):

$$\mathcal{G}^h(x', x_3) \rightharpoonup \mathcal{G} = A^{-1}Q_0^t(\bar{R}^t)Z(x', x_3)A^{-1} \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{5.48}$$

Define the "good sets":

$$\Omega_h = \{x \in \Omega^1;\ h|\mathcal{G}^h| < 1\}.$$

By the above, the characteristic functions $\mathbb{1}_{\Omega_h}$ converge to $\mathbb{1}$ in $L^1(\Omega)$. Further, by frame invariance and Taylor expanding W on Ω_h:

$$\begin{aligned} W\big(\mathrm{Id}_3 + h\mathcal{J} + h^2\mathcal{G}^h\big) &= W\big(e^{-h\mathcal{J}}(\mathrm{Id}_3 + h\mathcal{J} + h^2\mathcal{G}^h)\big) \\ &= W(\mathrm{Id}_3 + h^2(\mathcal{G}^h - \frac{1}{2}\mathcal{J}^2) + o(h^2)) \\ &= \frac{1}{2}\mathcal{Q}_3\left(h^2(\mathcal{G}^h - \frac{1}{2}\mathcal{J}^2)\right) + o(h^4). \end{aligned}$$

Therefore:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{h^4}E^h(u^h) &\geq \liminf_{h\to 0}\frac{1}{h^4}\int_{\Omega^1}\mathbb{1}_{\Omega_h}W\big(\mathrm{Id}_3+h\mathcal{J}+h^2\mathcal{G}^h\big)\,\mathrm{d}x\\ &=\liminf_{h\to 0}\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\Big(\mathbb{1}_{\Omega_h}\,\mathrm{sym}\,\Big(\mathcal{G}^h-\frac{1}{2}\mathcal{J}^2\Big)\Big)\,\mathrm{d}x\\ &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3\Big(\,\mathrm{sym}\,\Big(\mathcal{G}-\frac{1}{2}\mathcal{J}^2\Big)\Big)\,\mathrm{d}x,\end{aligned}\tag{5.49}$$

by the weak sequential lower semi-continuity of the quadratic form $\mathcal{Q}_3$ in L^2 and in view of:

$$\mathbb{1}_{\Omega^h}\,\mathrm{sym}\,\Big(\mathcal{G}^h-\frac{1}{2}\mathcal{J}^2\Big)\rightharpoonup \mathrm{sym}\,\mathcal{G}-\frac{1}{2}\mathcal{J}^2\quad\text{weakly in } L^2(\Omega^1).$$

Note that by (5.35) we have: $(Q_0^tSB_0)_{2\times 2}=-(\nabla V)^t\nabla\vec{b}_0$ and that:

$$\mathcal{J}^2=-\mathcal{J}^t\mathcal{J}=-x_3^2A^{-1}B_0^tB_0A^{-1}.$$

Therefore, using (5.45), the right hand side of (5.49) is bounded below by:

$$\begin{aligned}&\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_{2,A}\Big(x',\mathrm{sym}\Big(Q_0^t(\bar{R})^tZ(x',0)+x_3\big(Q_0^t\nabla\vec{p}+(\nabla V)^t\nabla\vec{b}_0\big)+\frac{x_3^2}{2}\big(Q_0^t\nabla\vec{d}_0+(\nabla\vec{b}_0)^t\nabla\vec{b}_0\big)\Big)_{2\times 2}\Big)\,\mathrm{d}x\\ &\qquad=\frac{1}{2}\int_{\Omega^1}\mathcal{Q}_{2,A}\Big(x',I(x')+x_3III(x')+x_3^2II(x')\Big)\,\mathrm{d}x.\end{aligned}$$

Above we used (5.46) and we denoted:

$$\begin{aligned}I(x')&=\mathbb{S}-\frac{1}{24}\,\mathrm{sym}\,((\nabla y_0)^t\nabla\vec{d}_0)+\frac{1}{2}(\nabla V)^t\nabla V\\ II(x')&=\frac{1}{2}\,\mathrm{sym}\,((\nabla y_0)^t\nabla\vec{d}_0)+\frac{1}{2}(\nabla\vec{b}_0)^t\nabla\vec{b}_0\\ III(x')&=\mathrm{sym}((\nabla y_0)^t\nabla\vec{p})+\mathrm{sym}((\nabla V)^t\nabla\vec{b}_0).\end{aligned}\tag{5.50}$$

Let $\mathcal{L}_{2,A}(x')$ be the symmetric bilinear form generating the quadratic form $\mathcal{Q}_{2,A}(x')$. Since the odd powers of x_3 integrate to 0 on the symmetric interval $(-1/2,1/2)$, we get:

$$\begin{aligned}&\int_{\Omega^1}\mathcal{Q}_{2,A}\Big(x',I(x')+x_3III(x')+x_3^2II(x')\Big)\,\mathrm{d}x\\ &\quad=\int_{\Omega}\mathcal{Q}_{2,A}(x',I(x'))\,\mathrm{d}x'+\Big(\int_{-1/2}^{1/2}x_3^2\,\mathrm{d}x_3\Big)\int_{\Omega}\mathcal{Q}_{2,A}(x',III(x'))\,\mathrm{d}x'\\ &\qquad+\Big(\int_{-1/2}^{1/2}x_3^4\,\mathrm{d}x_3\Big)\int_{\Omega}\mathcal{Q}_{2,A}(x',II(x'))\,\mathrm{d}x'+2\Big(\int_{-1/2}^{1/2}x_3^2\,\mathrm{d}x_3\Big)\int_{\Omega}\mathcal{L}_{2,A}(x',I(x'),II(x'))\,\mathrm{d}x'\\ &\quad=\int_{\Omega}\mathcal{Q}_{2,A}(x',I)+\frac{1}{12}\int_{\Omega}\mathcal{Q}_{2,A}(x',III)+\frac{1}{80}\int_{\Omega}\mathcal{Q}_{2,A}(x',II)+\frac{2}{12}\int_{\Omega}\mathcal{L}_{2,A}(x',I,II)\,\mathrm{d}x'\\ &\quad=\int_{\Omega}\mathcal{Q}_{2,A}\Big(x',I+\frac{1}{12}II\Big)\,\mathrm{d}x'+\frac{1}{12}\int_{\Omega}\mathcal{Q}_{2,A}(x',III)\,\mathrm{d}x'+\frac{1}{180}\int_{\Omega}\mathcal{Q}_{2,A}(x',II)\,\mathrm{d}x'\\ &\quad=\mathcal{I}_4(V,\mathbb{S}),\end{aligned}$$

by a direct calculation. This completes the proof of Theorem 5.1.1 in view of (5.49). □

5.2 THE UPPER BOUND

We now prove that the lower bound (5.22) is optimal, in the following sense:

Theorem 5.2.1. *Let $V \in W^{2,2}(\Omega, \mathbb{R}^3)$ and $\mathbb{S} \in L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$ satisfy:*

$$\begin{aligned} &\mathrm{sym}\left((\nabla y_0)^t \nabla V\right) = 0, \\ &\mathbb{S} \in \mathcal{S} := \mathrm{cl}_{L^2}\left\{ \mathrm{sym}\,((\nabla y_0)^t \nabla w);\ w \in W^{1,2}(\Omega, \mathbb{R}^3)\right\}. \end{aligned} \tag{5.51}$$

Then there exists a sequence $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ such that assertions (i), (ii) and (iii) of Theorem 5.1.1 are satisfied with $R^h = \mathrm{Id}$ and $c^h = 0$, and:

$$\limsup_{h\to 0} \frac{1}{h^4} E^h(u^h) \leq \mathcal{I}_4(V, \mathbb{S}). \tag{5.52}$$

Proof. In the construction below, we will use the following notation. In view of (4.6), for every $F_{2\times 2} \in \mathbb{R}^{2\times 2}$ one can write:

$$\mathcal{Q}_{2,A}(x', F_{2\times 2}) = \min_{c\in\mathbb{R}^3} \left\{ \mathcal{Q}_3\left(A^{-1}(F^*_{2\times 2} + \mathrm{sym}(c\otimes e_3))A^{-1}\right)\right\}. \tag{5.53}$$

Recall that $F^*_{2\times 2}$ denotes the $\mathbb{R}^{3\times 3}$ matrix whose principal 2×2 minor equals $F_{2\times 2}$ and all other entries equal to 0. We will denote by $c(x', F_{2\times 2})$ the unique minimizer in (5.53). Note that $c(x', \cdot)$ is a linear function of $F_{2\times 2}$ and it depends only on its symmetric part ($\mathrm{sym}\, F_{2\times 2}$).

1. Since $\mathbb{S} \in \mathcal{S}$, there exists a sequence $w^h \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that:

$$\mathrm{sym}\left((\nabla y_0)^t \nabla (w^h + \frac{1}{24}\vec{d}_0)\right) \to \mathbb{S} \quad \text{in } L^2(\Omega, \mathbb{R}^{2\times 2}) \tag{5.54}$$

and without loss of generality we can assume that each w^h is smooth up to the boundary of Ω (because the domain is sufficiently regular), together with:

$$\lim_{h\to 0} \sqrt{h}\|w^h\|_{W^{2,\infty}} = 0. \tag{5.55}$$

Note that we can always assume (5.55) by passing to a slowed down subsequence: given an arbitrary sequence a_n, consider the subsequence $\tilde{a}_k = a_{n_k}$ where $n_k := \min\{n \mid |a_n| > k\} - 1$. Clearly we only need to consider the case where a_n is unbounded, otherwise there is no need to pass to a subsequence, hence n_k is well defined. Then $|\tilde{a}_k| \leq k$, hence $\tilde{a}_k/\sqrt{k} \to 0$ as

$k \to \infty$.

Fix a small $\epsilon_0 \in (0,1)$ and let $v^h \in W^{2,\infty}(\Omega, \mathbb{R}^3)$ be a sequence of Lipschitz deformations with the properties:

$$\begin{aligned} v^h \to V \quad \text{in } W^{2,2}(\Omega, \mathbb{R}^3), \\ h\|v^h\|_{W^{2,\infty}} \leq \epsilon_0, \\ \lim_{h\to 0} \frac{1}{h^2}\big|\big\{x' \in \Omega;\ v^h(x') \neq V(x')\big\}\big| = 0. \end{aligned} \tag{5.56}$$

We refer to [Liu77] and [FJM06] for the construction of such truncated sequence v^h. Define now $\vec{p}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ by:

$$\vec{p}^h = (Q_0^t)^{-1} \begin{bmatrix} -(\nabla v^h)^t \vec{b}_0 \\ 0 \end{bmatrix}, \tag{5.57}$$

and also define the fields $\vec{q}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$, $\vec{k}_0$ smooth and $\tilde{r}^h \in L^\infty(\Omega, \mathbb{R}^3)$ such that:

$$Q_0^t \vec{q}^h = \frac{1}{2} c\big(x', 2(\nabla y_0)^t \nabla w^h + (\nabla v^h)^t \nabla v^h\big) - \begin{bmatrix} (\nabla w^h)^t \vec{b}_0 \\ 0 \end{bmatrix} - \begin{bmatrix} (\nabla v^h)^t \vec{p}^h \\ \frac{1}{2}|\vec{p}^h|^2 \end{bmatrix},$$

$$Q_0^t \vec{k}_0 = c\big(x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0\big) - \begin{bmatrix} (\nabla \vec{b}_0)^t \vec{d}_0 \\ |\vec{d}_0|^2 \end{bmatrix},$$

$$Q_0^t \tilde{r}^h = c\big(x', (\nabla y_0)^t \nabla \vec{p}^h + (\nabla v^h)^t \nabla \vec{b}_0\big) - \begin{bmatrix} (\nabla v^h)^t \vec{d}_0 \\ \langle \vec{p}^h, \vec{d}_0 \rangle \end{bmatrix}.$$

Finally, let $\vec{r}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ be such that:

$$\lim_{h\to 0} \|\vec{r}^h - \tilde{r}^h\|_{L^2} = 0, \qquad \lim_{h\to 0} \sqrt{h}\ \|\vec{r}^h\|_{W^{1,\infty}} = 0. \tag{5.58}$$

It follows from the definition of the minimizing map c, that:

$$\begin{aligned} &\mathcal{Q}_3\Big(A^{-1}\big(2Q_0^t[\nabla w^h \mid \vec{q}^h] + [\nabla v^h \mid \vec{p}^h]^t[\nabla v^h \mid \vec{p}^h]\big)A^{-1}\Big) \\ &\qquad\qquad = \mathcal{Q}_{2,A}\Big(x', 2(\nabla y_0)^t \nabla w^h + (\nabla v^h)^t \nabla v^h\Big), \\ &\mathcal{Q}_3\Big(A^{-1}\big(Q_0^t[\nabla \vec{d}_0 \mid \vec{k}_0] + [\nabla \vec{b}_0 \mid \vec{d}_0]^t[\nabla \vec{b}_0 \mid \vec{d}_0]\big)A^{-1}\Big) \\ &\qquad\qquad = \mathcal{Q}_{2,A}\Big(x', (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0\Big), \\ &\mathcal{Q}_3\Big(A^{-1}\big(2Q_0^t[\nabla \vec{p}^h \mid \tilde{r}^h] + 2[\nabla v^h \mid \vec{p}^h]^t[\nabla \vec{b}_0 \mid \vec{d}_0]^t\big)A^{-1}\Big) \\ &\qquad\qquad = \mathcal{Q}_{2,A}\Big(x', (\nabla y_0)^t \nabla \vec{p}^h + (\nabla v^h)^t \nabla \vec{b}_0\Big). \end{aligned} \tag{5.59}$$

Moreover, we have the following pointwise bounds:

$$
\begin{aligned}
|\vec{p}^h| &\leq C|\nabla v^h|,\\
|\nabla \vec{p}^h| &\leq C(|\nabla v^h| + |\nabla^2 v^h|),\\
|\vec{q}^h| &\leq C(|\nabla w^h| + |\nabla v^h|^2 + |\nabla v^h||\vec{p}^h| + |\vec{p}^h|^2) \leq C(|\nabla w^h| + |\nabla v^h|^2),\\
|\nabla \vec{q}^h| &\leq C(|\nabla w^h| + |\nabla^2 w^h| + |\nabla^2 v^h||\nabla v^h| + |\nabla v^h|^2).
\end{aligned}
\tag{5.60}
$$

2. Consider the sequence $u^h \in W^{1,\infty}(\Omega^h, \mathbb{R}^3)$ defined as:

$$
\begin{aligned}
u^h(x', x_3) = y_0(x') + hv^h(x') + h^2 w^h(x') + x_3 \vec{b}_0(x') + \frac{x_3^2}{2}\vec{d}_0(x')\\
+ \frac{x_3^3}{6}\vec{k}_0(x') + hx_3\vec{p}^h(x') + h^2 x_3 \vec{q}^h(x') + \frac{hx_3^2}{2}\vec{r}^h(x').
\end{aligned}
$$

For every $(x', x_3) \in \Omega^1$ we write:

$$
\nabla u^h(x', hx_3) = Q_0(x') + Z_1^h(x', x_3) + Z_2^h(x', x_3),
$$

where:

$$
\begin{aligned}
Z_1^h(x', x_3) &= h[\nabla v^h \mid \vec{p}^h] + h^2[\nabla w^h \mid \vec{q}^h] + hx_3[\nabla \vec{b}_0 \mid \vec{d}_0] + \frac{h^2 x_3^2}{2}[\nabla \vec{d}_0 \mid \vec{k}_0] + h^2 x_3[\nabla \vec{p}^h \mid \vec{r}^h],\\
Z_2^h(x', x_3) &= \frac{h^3 x_3^3}{6}[\nabla \vec{k}_0 \mid 0] + h^3 x_3[\nabla \vec{q}^h \mid 0] + \frac{h^3 x_3}{2}[\nabla \vec{r}^h \mid 0].
\end{aligned}
$$

Since $Q_0 A^{-1} \in SO(3)$, we get:

$$
\nabla u^h A^{-1}(x', hx_3) = Q_0 A^{-1}\Big(\mathrm{Id}_3 + A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1}\Big)
$$

and, in view of (5.56), (5.58) and (5.60), there follows for h sufficiently small:

$$
\begin{aligned}
&\|A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1}\|_{L^\infty}\\
&\quad \leq C\Big(h\|\nabla v^h\|_{L^\infty} + h\|\vec{p}^h\|_{L^\infty} + h^2\|\nabla w^h\|_{L^\infty} + h^2\|\vec{q}^h\|_{L^\infty} + h\|\nabla \vec{b}_0\|_{L^\infty} + h\|\vec{d}_0\|_{L^\infty}\\
&\qquad + h^2\|\nabla \vec{d}_0\|_{L^\infty} + h^2\|\vec{k}_0\|_{L^\infty} + h^2\|\nabla \vec{p}^h\|_{L^\infty} + h^2\|\vec{r}^h\|_{L^\infty} + h^3\|\nabla \vec{k}_0\|_{L^\infty}\\
&\qquad + h^3\|\nabla \vec{q}^h\|_{L^\infty} + h^3\|\nabla \vec{r}^h\|_{L^\infty}\Big) \leq C\epsilon_0.
\end{aligned}
$$

By the left polar decomposition, there exists a further rotation $R \in SO(3)$ such that:

$$\begin{aligned}
R\nabla u^h A^{-1} &= \Big((\mathrm{Id}_3 + A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1})^t (\mathrm{Id}_3 + A^{-1}Q_0^t Z_1^h A^{-1} + A^{-1}Q_0^t Z_2^h A^{-1}) \Big)^{1/2} \\
&= \Big(\mathrm{Id}_3 + 2A^{-1}\operatorname{sym}(Q_0^t Z_1^h)A^{-1} + A^{-1}(Z_1^h)^t Z_1^h A^{-1} + \mathcal{O}(|Z_2^h|) \Big)^{1/2} \\
&= \mathrm{Id}_3 + A^{-1}\operatorname{sym}(Q_0^t Z_1^h)A^{-1} + \frac{1}{2}A^{-1}(Z_1^h)^t Z_1^h A^{-1} \\
&\qquad + \mathcal{O}\Big(|\operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h|^2 \Big) + \mathcal{O}(|Z_2^h|).
\end{aligned}$$

3. Consider the set:

$$\Omega_h = \big\{ (x', x_3) \in \Omega;\ v^h(x') = V(x') \big\}.$$

Note that on Ω_h we have: $\vec{p}^h = \vec{p}$ and $Q_0^t[\nabla v^h \mid \vec{p}^h] \in so(3)$. Using Taylor's expansion, it follows that:

$$\frac{1}{h^4}\int_{\Omega_h} W\big(\nabla u(x', hx_3)A^{-1}\big)\,\mathrm{d}x = \frac{1}{2h^4}\int_{\Omega_h} \mathcal{Q}_3\Big(A^{-1}\big(Q_0^t Z_1^h + \frac{1}{2}(Z_1^h)^t Z_1^h\big)A^{-1} \Big)\,\mathrm{d}x + \mathcal{E}_1^h,$$

where the error term $\mathcal{E}_1^h$ can be estimated by:

$$|\mathcal{E}_1^h| \leq \frac{C}{h^4}\int_{\Omega_h} \big|2\operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h\big|^3 + |Z_2^h|^2 + \big|2\operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h\big||Z_2^h|\,\mathrm{d}x.$$

Now on Ω_h we also have, by (5.60):

$$\begin{aligned}
\big|2\operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h\big| &\leq C\Big(h^2|\nabla w^h| + h^2|\nabla v^h|^2 + h^2 + h^2|\nabla v^h| + h^2|\nabla^2 v^h| + h^2|\vec{r}^h| \Big), \\
|Z_2^h| &\leq Ch^3\big(1 + |\nabla \vec{q}^h| + |\nabla \vec{r}^h|\big) \\
&\leq Ch^3\Big(1 + |\nabla w^h| + |\nabla^2 w^h| + |\nabla^2 v^h||\nabla v^h| + |\nabla v^h|^2 + |\nabla \vec{r}^h|\Big),
\end{aligned}$$

and therefore, in view of (5.55), (5.58), (5.56) and $V \in W^{2,2}$:

$$\begin{aligned}
&\frac{1}{h^4}\int_{\Omega_h} \big|2\operatorname{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h\big|^3\,\mathrm{d}x \\
&\qquad \leq \frac{C}{h^4}\int_{\Omega_h} h^6|\nabla w^h|^3 + h^6|\nabla v^h|^6 + h^6 + h^6|\nabla v^h|^3 + h^6|\nabla^2 v^h|^3 + h^6|\vec{r}^h|^3\,\mathrm{d}x \\
&\qquad \leq \frac{C}{h^4}\Big(h^2\|\nabla w^h\|_{L^\infty}(h^2\|\nabla w^h\|_{L^2})^2 + h^6\|\nabla V\|_{L^6}^6 + h^6|\Omega| + h^6\|\nabla V\|_{L^3}^3 \\
&\qquad\qquad + h^6\|\nabla^2 v^h\|_{L^\infty}\|\nabla^2 V\|_{L^2}^2 + (\sqrt{h}\|\vec{r}^h\|_{L^\infty})^3 h^{9/2} \Big) \to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Analogously:

$$
\begin{aligned}
&\frac{1}{h^4}\int_{\Omega_h}|Z_2^h|^2\,\mathrm{d}x \le \frac{C}{h^4}\int_{\Omega_h} h^5 + (h\|\nabla v^h\|_{L^\infty})^2 h^4|\nabla^2 v^h|^2 + h^6|\nabla v^h|^4\,\mathrm{d}x \to 0 \quad \text{as } h\to 0,\\
&\frac{1}{h^4}\int_{\Omega_h}\big|2\,\mathrm{sym}(Q_0^t)Z_1^h + (Z_1^h)^t Z_1^h\big||Z_2^h|\,\mathrm{d}x\\
&\qquad\le \frac{C}{h^4}\int_{\Omega_h}\Big(h^5|\nabla w^h|^2 + h^5|\nabla^2 w^h|^2 + h^5|\nabla v^h|^2 + h^5 + h^5|\nabla V| + h^5|\nabla^2 V| + h^5|\vec r^{\,h}|\\
&\qquad\qquad + h^5|\nabla V|^2|\nabla^2 V| + h^5|\nabla V||\nabla^2 V|^2\Big)\,\mathrm{d}x \le C\epsilon_0.
\end{aligned}
$$

We therefore conclude that:

$$
\limsup_{h\to 0}|\mathcal{E}_1^h| \le C\epsilon_0. \tag{5.61}
$$

4. Consider now the error due to integrating on the residual subdomain:

$$
\mathcal{E}_2^h = \frac{1}{h^4}\int_{\Omega^1\setminus\Omega_h} W\Big(\nabla u^h A^{-1}(x', hx_3)\Big)\,\mathrm{d}x \le \frac{C}{h^4}\int_{\Omega^1\setminus\Omega_h}\big|2\,\mathrm{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h\big|^2 + |Z_2^h|^2\,\mathrm{d}x.
$$

Observe that, since the matrix field $[\nabla v^h \mid \vec p^{\,h}]$ is Lipschitz, we have:

$$
\begin{aligned}
\Big|\mathrm{sym}(Q_0^t[\nabla v^h \mid \vec p^{\,h}])(x')\Big| &\le C\|\nabla v^h\|_{W^{1,\infty}}\,\mathrm{dist}\,\big(x', \{v^h = V\}\big)\\
&\le \frac{C\epsilon_0}{h}\,\mathrm{dist}\,\big(x', \{v^h = V\}\big) \to 0 \quad \text{in } L^\infty(\Omega).
\end{aligned}
$$

The last inequality above follows by a standard argument by contradiction. If there was a sequence $x^h \in \Omega$ such that $\mathrm{dist}(x^h, \{v^h = V\}) \ge ch$, this would imply that: $|\{x';\ v^h(x') \neq V(x')\}| \ge \big|\Omega \cap B(x^h, ch)\big| \ge ch^2$, contradicting (5.56). Consequently, by (5.55), (5.58), (5.56):

$$
\begin{aligned}
|\mathcal{E}_2^h| &\le \frac{C}{h^4}\int_{\Omega^1\setminus\Omega_h} h^2\big|\mathrm{sym}(Q_0^t[\nabla v^h \mid \vec p^{\,h}])\big|\,\mathrm{d}x\\
&\qquad + \frac{C}{h^4}\int_{\Omega^1\setminus\Omega_h} h^4|\nabla w^h|^2 + h^4|\nabla v^h|^4 + h^4|\nabla^2 v^h|^2 + h^4|\vec r^{\,h}|^2 + h^4 + h^6|\nabla v^h|^4\,\mathrm{d}x\\
&\le \frac{C}{h^4}o(h^2)|\Omega^1\setminus\Omega_h| + \frac{C}{h^4}\sqrt{h}\|\nabla w^h\|_{L^\infty} h^{7/2}|\Omega^1\setminus\mathcal{U}^h|^{1/2}\|\nabla w^h\|_{L^2}\\
&\qquad + C|\Omega^1\setminus\mathcal{U}^h|\|\nabla v^h\|_{L^8}^4 + Ch\|\nabla^2 v^h\|_{L^\infty}\frac{1}{h}\|\nabla^2 v^h\|_{L^2}|\Omega^1\setminus\mathcal{U}^h|^{1/2} + \frac{1}{h}(\sqrt{h}\|\vec r^{\,h}\|_{L^\infty})^2|\Omega^1\setminus\mathcal{U}^h|\\
&\qquad + (h\|\nabla^2 v^h\|_{L^\infty})^2\|\nabla v^h\|_{L^4}^2|\Omega^1\setminus\mathcal{U}^h|^{1/2} \qquad \to 0 \quad \text{as } h\to 0.
\end{aligned}
$$

Thus:

$$\limsup_{h\to 0} \frac{1}{h^4} E^h(u^h) \leq \limsup_{h\to 0} \frac{1}{h^4} \int_{\Omega_h} \frac{1}{2} \mathcal{Q}_3\Big(A^{-1}\big(\mathrm{sym}(Q_0^t Z_1^h) + \frac{1}{2}(Z_1^h)^t Z_1^h\big)A^{-1}\Big)\ \mathrm{d}x + C\epsilon_0.$$

Now on Ω_h we have:

$$\begin{aligned}
&2\,\mathrm{sym}(Q_0^t Z_1^h) + (Z_1^h)^t Z_1^h \\
&\quad = 2h^2\Big(\mathrm{sym}(Q_0^t[\nabla w^h \mid \vec{q}^h]) + \frac{x_3^2}{2}\mathrm{sym}(Q_0^t[\nabla \vec{d}_0 \mid \vec{k}_0]) + x_3\,\mathrm{sym}(Q_0^t[\nabla \vec{p} \mid \vec{r}^h])\Big) \\
&\qquad + h^2\Big([\nabla V \mid \vec{p}]^t[\nabla V \mid \vec{p}] + x_3^2[\nabla \vec{b}_0 \mid \vec{d}_0]^t[\nabla \vec{b}_0 \mid \vec{d}_0] + 2x_3\,\mathrm{sym}([\nabla V \mid \vec{p}]^t[\nabla \vec{b}_0 \mid \vec{d}_0])\Big) + \mathcal{E}^h,
\end{aligned}$$

where the present error $\mathcal{E}^h$ is estimated by:

$$\begin{aligned}
|\mathcal{E}^h| \leq C\Big(& h^3|\nabla V||\nabla w^h| + h^3|\nabla V| + h^3|\nabla V||\nabla \vec{p}| + h^3|\nabla V||\vec{r}^h| \\
&+ h^4|\nabla w^h|^2 + h^3|\nabla w^h| + h^4|\nabla w^h||\nabla \vec{p} + h^4|\nabla w^h||\vec{r}^h| + h^3 \\
&+ h^3|\nabla \vec{p}| + h^3|\vec{r}^h| + h^4 + h^4|\nabla \vec{p}| + h^4|\vec{r}^h| + h^4|\nabla \vec{p}|^2 + h^4|\vec{r}^h|^2\Big) \\
\leq Ch^2\Big(& o(1)\sqrt{h}|\nabla V| + \epsilon_0^2|\nabla^2 V| + o(1)\sqrt{h} + o(1)\epsilon_0\sqrt{h}\Big).
\end{aligned} \tag{5.62}$$

Consequently:

$$\begin{aligned}
&\limsup_{h\to 0} \frac{1}{h^4} E^h(u^h) \\
&\quad \leq \limsup_{h\to 0} \frac{1}{2} \int_{\Omega_h} \mathcal{Q}_3\Big(A^{-1}\big(\mathrm{sym}(Q_0^t[\nabla w^h \mid \vec{q}^h]) + \frac{1}{2}x_3^2\,\mathrm{sym}(Q_0^t[\nabla \vec{d}_0 \mid \vec{k}_0]) \\
&\qquad\qquad + x_3\,\mathrm{sym}(Q_0^t[\nabla \vec{p} \mid \vec{r}^h]) + \frac{1}{2}[\nabla V \mid \vec{p}]^t[\nabla V \mid \vec{p}] \\
&\qquad\qquad + \frac{1}{2}x_3^2[\nabla \vec{b}_0 \mid \vec{d}_0]^t[\nabla \vec{b}_0 \mid \vec{d}_0] + x_3\,\mathrm{sym}([\nabla V \mid \vec{p}]^t[\nabla \vec{b}_0 \mid \vec{d}_0])\big)A^{-1}\Big)\ \mathrm{d}x + C\epsilon_0 \\
&\quad = \limsup_{h\to 0} \frac{1}{2} \int_{\Omega_h} \mathcal{Q}_3\Big(A^{-1}\big(\mathrm{sym}(Q_0^t[\nabla w^h \mid \vec{q}^h]) + \frac{1}{2}[\nabla V \mid \vec{p}]^t[\nabla V \mid \vec{p}] \\
&\qquad\qquad + \frac{1}{2}x_3^2\,\mathrm{sym}(Q_0^t[\nabla \vec{d}_0 \mid \vec{k}_0]) + \frac{1}{2}x_3^2[\nabla \vec{b}_0 \mid \vec{d}_0]^t[\nabla \vec{b}_0 \mid \vec{d}_0]\big)A^{-1}\Big) \\
&\qquad + \mathcal{Q}_3\Big(A^{-1}\big(x_3\,\mathrm{sym}(Q_0^t[\nabla \vec{p} \mid \vec{r}^h]) + x_3\,\mathrm{sym}([\nabla V \mid \vec{p}]^t[\nabla \vec{b}_0 \mid \vec{d}_0])\big)A^{-1}\Big)\,\mathrm{d}x + C\epsilon_0.
\end{aligned}$$

Denoting by:

$$I_1(x') = \mathrm{sym}((\nabla y_0)^t \nabla w^h) + \frac{1}{2}(\nabla v^h)^t \nabla v^h, \qquad I_2(x') = \frac{1}{2}\mathrm{sym}((\nabla y_0)^t \nabla \vec{d}_0) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0,$$

we have:

$$
\begin{aligned}
&\mathcal{Q}_3\Big(A^{-1}\big(I_1^*(x') + \mathrm{sym}(c(x', I_1(x')) \otimes e_3) + x_3^2 I_2^*(x') + x_3^2\,\mathrm{sym}(c(x', I_2(x')) \otimes e_3)\big)A^{-1}\Big)\\
&\quad = \mathcal{Q}_3\Big(A^{-1}\big((I_1(x') + x_3^2 I_2(x'))^* + \mathrm{sym}(c(x', I_1(x') + x_3^2 I_2(x')) \otimes e_3)\big)A^{-1}\Big)\\
&\quad = \mathcal{Q}_{2,A}\Big((I_1(x') + x_3^2 I_2(x')\Big),
\end{aligned}
$$

where we have used the definition and linearity of the minimizing map c. Recalling the definitions of the curvature forms $I(x')$, $II(x')$ and $III(x')$ in (5.50), observe that $I_2(x') = 2II(x')$ and that $\frac{1}{2}I_1$ converges to I in L^2 by (5.54). Hence:

$$
\begin{aligned}
\limsup_{h\to 0} \frac{1}{h^4} E^h(u^h) &\le \frac{1}{2}\int_{\Omega^1} \mathcal{Q}_{2,A}\Big(I(x') + x_3^2 II(x')\Big)\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^1} \mathcal{Q}_{2,A}\Big(x_3 III(x')\Big)\,\mathrm{d}x + C\epsilon_0\\
&= \mathcal{I}_4(V, \mathbb{S}) + C\epsilon_0.
\end{aligned}
$$

Since $\epsilon_0 > 0$ was arbitrary, the proof is achieved by a diagonal argument. □

5.3 Γ-CONVERGENCE

We now recall the notion of Γ-convergence and a few basic facts about it.

Let (X, d) be a metric space and $\mathcal{F}_n$, $\mathcal{F} : X \longrightarrow \bar{\mathbb{R}}$. We say that $\mathcal{F}_n$ Γ-converges to $\mathcal{F}$ if:

(i) for every $x \in X$, for every sequence $(x_n)_n \subset X$ converging to x we have:

$$\mathcal{F}(x) \le \liminf_{n\to\infty} \mathcal{F}_n(x_n) \tag{5.63}$$

(ii) for every $x \in X$, there exists a sequence $(x_n)_n \subset X$ converging to x, such that:

$$\mathcal{F}(x) \ge \limsup_{n\to\infty} \mathcal{F}_n(x_n). \tag{5.64}$$

Note that by (5.63), condition (5.64) is equivalent to:

(ii') for every $x \in X$, there exists a sequence $(x_n)_n \subset X$ converging to x, such that:

$$\mathcal{F}(x) = \lim_{n\to\infty} \mathcal{F}_n(x_n). \tag{5.65}$$

The main importance of Γ-convergence is that it implies convergence of minimizers. Precisely we have:

Theorem 5.3.1. *Suppose $\mathcal{F}_n$, $\mathcal{F} : X \longrightarrow \bar{\mathbb{R}}$ are such that $\mathcal{F}_n$ Γ-converges to $\mathcal{F}$ and there exists a compact set $K \subset X$ such that:*

$$\inf_X \mathcal{F}_n = \inf_K \mathcal{F}_n$$

for every n. Then:

$$\exists \min_X \mathcal{F} = \lim_{n\to\infty} \inf_X \mathcal{F}_n.$$

Moreover, if $(y_n)_n$ is a pre-compact almost minimizing sequence, i.e.

$$\lim_{n\to\infty} (\mathcal{F}_n(y_n) - \inf_X \mathcal{F}_n) = 0,$$

any limit point of y_n is a minimizer for $\mathcal{F}$.

Proof. By hypothesis, there exists a sequence $(x_n)_n \subset K$ such that:

$$\liminf_{n\to\infty} \mathcal{F}_n(x_n) = \liminf_{n\to\infty} \inf_X \mathcal{F}_n.$$

Since K is compact, passing to a non relabeled subsequence, we can assume x_n converges to some $x \in K$ and:

$$\lim_{n\to\infty} \mathcal{F}_n(x_n) = \liminf_{n\to\infty} \inf_X \mathcal{F}_n.$$

By the Γ-convergence (liminf inequality) we have:

$$\inf_X \mathcal{F} \le \mathcal{F}(x) \le \liminf_{n\to\infty} \mathcal{F}_n(x_n) = \liminf_{n\to\infty} \inf_X \mathcal{F}_n. \tag{5.66}$$

By the Γ-convergence (limsup inequality), for any $z \in X$ there exists $(z_n)_n \subset X$ such that $\mathcal{F}(z) \ge \limsup_{n\to\infty} \mathcal{F}_n(z_n) \ge \limsup_{n\to\infty} \inf_X \mathcal{F}_n$, and by arbitrariness of z, we have:

$$\inf_X \mathcal{F} \ge \limsup_{n\to\infty} \inf_X \mathcal{F}_n.$$

Combining with (5.66) we obtained the existence of $x \in K$ such that

$$\mathcal{F}(x) = \inf_X \mathcal{F} = \lim_{n\to\infty} \inf_X \mathcal{F}_n.$$

This concludes the proof of the first part of the Theorem.

Assume now that $(y_n)_n$ is a sequence of almost minimizers. From the previous part we know that there exists $\lim_{n\to\infty} \inf_X \mathcal{F}_n = \min_X \mathcal{F}$, hence there exists

$$\lim_{n\to\infty} \mathcal{F}_n(y_n) = \lim_{n\to\infty} (\mathcal{F}_n(y_n) - \inf_X \mathcal{F}_n) + \lim_{n\to\infty} \inf_X \mathcal{F}_n = \min_K \mathcal{F}.$$

Consider $(y_n)_n$ a non relabeled subsequence converging to some $y \in X$. Then, as before, we find:

$$\mathcal{F}(y) \leq \liminf_{n\to\infty} \mathcal{F}_n(y_n) = \min_X \mathcal{F},$$

which concludes the proof. □

We will use the following:

Proposition 5.3.1. *Suppose $\mathcal{F}_n$, $\mathcal{F} : X \longrightarrow \bar{\mathbb{R}}$ are such that $\mathcal{F}_n \geq 0$ for all n and $\mathcal{F}_n$ Γ-converges to $\mathcal{F}$. Moreover, assume that for every sequence $(x_n)_n \subset X$ such that $(\mathcal{F}_n(x_n))_n$ is bounded, we can extract a convergent subsequence $(x_{n_k})_k$. Then,*

$$\exists \min_X \mathcal{F} = \lim_{n\to\infty} \inf_X \mathcal{F}_n.$$

Moreover, if y_n is an almost minimizing sequence, i.e.

$$\lim_{n\to\infty} (\mathcal{F}_n(y_n) - \inf_X \mathcal{F}_n) = 0,$$

then $(y_n)_n$ is precompact, and any limit point is a minimizer for $\mathcal{F}$.

Proof. The proof follows along the lines of the previous one, the only difference being the use of the new hypotesis replacing condition $\inf_X \mathcal{F}_n = \inf_K \mathcal{F}_n$ to extract a convergent subsequence.

We can find a sequence $(x_n)_n \subset X$ such that:

$$\liminf_{n\to\infty} \mathcal{F}_n(x_n) = \liminf_{n\to\infty} \inf_X \mathcal{F}_n \geq 0.$$

Passing to a non relabeled subsequence we can assume:

$$\lim_{n\to\infty} \mathcal{F}_n(x_n) = \liminf_{n\to\infty} \inf_X \mathcal{F}_n.$$

Now if $\mathcal{F}$ is not identically infinite at each point, we must have that $\liminf_{n\to\infty} \inf_X \mathcal{F}_n$ is finite. Indeed, if it were infinite, then for every $x \in X$, there would exists a sequence z_n converging to x such that $\mathcal{F}(x) \geq \limsup_{n\to\infty} \mathcal{F}_n(z_n) \geq +\infty$.

Therefore, by assumption we can extract a convergent subsequence from $(x_n)_n$. From this point the proof follows precisely that of Theorem 5.3.1, in view of the fact that any sequence of almost minimizers $(y_n)_n$ is automatically precompact, since $\lim_{n\to\infty} \inf_X \mathcal{F}_n$ is finite, implying $(\mathcal{F}_n(y_n))_n$ is bounded. □

Theorems 5.1.1 and 5.2.1 can now be rephrased in the language of Γ-convergence and, as a consequence, they imply convergence of almost minimizers. We use the following notation:

$$\mathcal{A}_G = \{y \in W^{2,2}(\Omega, \mathbb{R}^3) \mid (\nabla y)^t \nabla y = G_{2\times 2}, \ \operatorname{sym}((\nabla y)^t \nabla \vec{b}) = 0\},$$

where $\vec{b}$ is the Cosserat vector associated to y as in (4.9). Note that under the assumption $R_{1212} = R_{1213} = R_{1223} = 0$, $\mathcal{A}_G$ consists of one element which we denoted by y_0 (see Theorem 4.2.2). We now introduce the following notation for the spaces $\mathcal{V}$ and $\mathcal{S}$, to emphasize the dependence on y_0:

$$\begin{aligned} \mathcal{V} = \mathcal{V}_{y_0} &= \{V \in W^{2,2}(\Omega, \mathbb{R}^3) \mid \operatorname{sym}((\nabla y_0)^t \nabla V) = 0\}, \\ \mathcal{S} = \mathcal{S}_{y_0} &= \operatorname{cl}_{L^2}\big\{ \operatorname{sym}\left((\nabla y_0)^t \nabla w\right); \ w \in W^{1,2}(\Omega, \mathbb{R}^3)\big\}. \end{aligned} \tag{5.67}$$

Theorem 5.3.2. *Define $\mathcal{F}^h, \mathcal{F} : W^{1,2}(\Omega^1, \mathbb{R}^3) \times W^{1,2}(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^{2\times 2})$ as:*

$$\mathcal{F}^h(u, V, S) = \begin{cases} h^{-4} E^h(u(x', hx_3)) & \text{if} \quad V(x') = \frac{1}{h} \fint_{-1/2}^{1/2} u(x', x_3) - y_0(x') \,\mathrm{d}x_3 \\ & \qquad S = h^{-1} \operatorname{sym}((\nabla y_0)^t \nabla V), \ \ y_0 \in \mathcal{A}_G \\ +\infty & \qquad \text{otherwise} \end{cases} \tag{5.68}$$

and:

$$\mathcal{F}(u, V, S) = \begin{cases} \mathcal{I}_4(V, S) & \text{if } u = y_0, \ V \in \mathcal{V}_{y_0}, \ S \in \mathcal{S}_{y_0}, \ y_0 \in \mathcal{A}_G \\ +\infty & \text{otherwise.} \end{cases} \tag{5.69}$$

Then, $\mathcal{F}^h$ Γ-converges to $\mathcal{F}$.

Proof. We first prove the liminf inequality. Consider sequences $u^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$, $V^h \in W^{1,2}(\Omega, \mathbb{R}^3)$ and $S^h \in L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$ that converge to some u, V, S in their respective spaces. Without loss of generality we can assume $\liminf_{h\to 0} \mathcal{F}^h(u^h, V^h, S^h) < +\infty$ (otherwise there is nothing to prove), hence passing to a non relabeled subsequence we can further assume that the sequence $\{\mathcal{F}^h(u^h, V^h, S^h)\}$ is bounded. In particular, $E^h(u^h(x', hx_3)) \leq Ch^4$ and:

$$V^h(x') = \frac{1}{h} \fint_{-1/2}^{1/2} u^h(x', x_3) - y_0(x') \, \mathrm{d}x_3, \quad S^h = \frac{1}{h} \operatorname{sym}((\nabla y_0)^t \nabla V^h).$$

Applying Theorem 5.1.1 we infer the existence of $\bar{R}^h \in SO(3)$, $c^h \in \mathbb{R}^3$ such that $y^h(x', x_3) = (\bar{R}^h)^t u^h(x', x_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$ converges to y_0 in $W^{1,2}(\Omega^1, \mathbb{R}^3)$. Additionally:

$$\tilde{V}^h(x') = \frac{1}{h} \fint_{-1/2}^{1/2} y^h(x', x_3) - y_0(x') \, \mathrm{d}x_3$$

converges to some $\tilde{V} \in \mathcal{V}_{y_0}$ in $W^{1,2}(\Omega, \mathbb{R}^3)$ and $\tilde{S}^h = \frac{1}{h} \operatorname{sym}((\nabla y_0)^t \nabla \tilde{V}^h)$ converges to some $\tilde{S} \in \mathcal{S}_{y_0}$ weakly in $L^2(\Omega, \mathbb{R}^{2\times 2})$.

Now we claim that we can choose $\bar{R}^h = \mathrm{Id}_3$. Indeed, looking at the proof of Theorem 5.1.1 we see that we only need the rotation $\bar{R}^h$ to satisfy the property:

$$|\fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x - \bar{R}^h|^2 \leq Ch^2.$$

Note that:

$$|\bar{R}^h - \mathrm{Id}_3|^2 \leq h^2 \left\| \nabla V^h - \nabla \tilde{V}^h \right\|^2_{L^2(\Omega, \mathbb{R}^3)} \leq Ch^2$$

since V^h and $\tilde{V}^h$ are bounded in $W^{1,2}(\Omega, \mathbb{R}^3)$. Therefore:

$$|\fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x - \mathrm{Id}_3|^2 \leq |\fint_{\Omega^h} \nabla u^h Q_0^{-1} \, \mathrm{d}x - \bar{R}^h|^2 + |\bar{R}^h - \mathrm{Id}_3|^2 \leq Ch^2,$$

and we can choose $y^h = u^h - c^h$ (possibly modifying the constants c^h). Since $\frac{c^h}{h} = V^h - \tilde{V}^h$ is bounded, we get that c^h converges to 0. On the other hand, $c^h = u^h - y^h$ converges to $u - y_0$, implying $u = y_0$. Moreover $\nabla V^h = \nabla \tilde{V}^h$ and $S^h = \tilde{S}^h$, hence $\tilde{V} = V$ and $\tilde{S} = S$. Finally, by part *(iv)* in Theorem 5.1.1 we get the desired liminf inequality.

For the limsup inequality, it's enough to consider $u = y_0 \in \mathcal{A}_G \neq \emptyset$, $V \in \mathcal{V}_{y_0}$, $S \in \mathcal{S}_{y_0}$ (otherwise $\mathcal{F}(u, V, S) = +\infty$ and the result trivially holds) in which case it follows from Theorem 5.2.1. ☐

Corollary 5.3.2. *If $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ is a minimizing sequence to $h^{-4}E^h$, that is:*

$$\lim_{h\to 0}\left(\frac{1}{h^4}E^h(u^h) - \inf \frac{1}{h^4}E^h\right) = 0,$$

then the appropriate renormalizations $y^h = (\bar{R}^h)^t u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$ obey the convergence statements of Theorem 5.1.1 (i), (ii), (iii). The convergence of $h^{-1}sym\big((\nabla y_0)^t \nabla V^h\big)$ to $\mathbb{S}$ in (iii) is strong in $L^2(\Omega)$. Moreover, any limit $(V, \mathbb{S})$ minimizes the functional $\mathcal{I}_4$.

Proof. The proof follows from the Γ-convergence statement above, Theorems 5.1.1, 5.2.1 and Proposition 5.3.1, except for the strong convergence of the scaled tangential strains in (iii). Recall that, in the notations of (5.37) we defined:

$$S^h = h^{-1}\mathrm{sym}\big((\nabla y_0)^t \nabla V^h\big) = J_1^h + J_2^h,$$

where J_2^h converges strongly in $L^2(\Omega, \mathbb{R}^{2\times 2})$ and:

$$J_1^h = \frac{1}{h}\,\mathrm{sym}\,(Q_0^t I_1^h)_{2\times 2} = \mathrm{sym}\left(Q_0^t (\bar{R}^h)^t \fint_{-1/2}^{1/2} Z^h(x', x_3)\,\mathrm{d}x_3\right)_{2\times 2}$$

with Z^h defined in (5.40). Recall also that:

$$\mathcal{G}^h(x', x_3) = A^{-1} Q_0^t (R^h)^t Z^h(x', x_3) A^{-1}$$

(defined in (5.47)) converges weakly in $L^2(\Omega, \mathbb{R}^{3\times 3})$ to $\mathcal{G}$ (see (5.48)).

Since the sequence u^h is almost minimizing, the inequalities in (5.49) are equalities and the limits inferior are actual limits. As a consequence:

$$\lim_{h\to 0}\int_{\Omega^1} \mathcal{Q}_3(1_{\Omega_h}\,\mathrm{sym}(\mathcal{G}^h - \frac{1}{2}\mathcal{J}^2))\,\mathrm{d}x = \int_{\Omega^1} \mathcal{Q}_3(\mathrm{sym}(\mathcal{G} - \frac{1}{2}\mathcal{J}^2))\,\mathrm{d}x.$$

Since $\mathcal{G}^h$ converges to $\mathcal{G}$ weakly in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$ and $\mathcal{Q}_3$ is a quadratic form which is positive definite on symmetric matrices, we get that $\mathrm{sym}\,\mathcal{G}^h$ converges to $\mathrm{sym}\,\mathcal{G}$ strongly in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$. Therefore, the convergence of $\mathrm{sym}\,Z^h$ is strong, which implies that J_1^h converges strongly, hence so does S^h.

□

5.4 DISCUSSION OF THE Γ-LIMIT FUNCTIONAL $\mathcal{I}_4$

We now give a description and geometric interpretation of the terms appearing in the expression of $\mathcal{I}_4$. The arguments of $\mathcal{I}_4(V, \mathbb{S})$ are:

(i) First order infinitesimal isometries V on $(\Omega, G_{2\times 2})$. These are vector fields $V \in \mathcal{V} = \{W^{2,2}(\Omega, \mathbb{R}^3) \mid \mathrm{sym}((\nabla y_0)^t \nabla V) = 0\}$ and they are such that the transformation $u^h = y_0 + hV$ mapping Ω to $u^h(\Omega)$ preserves the metric $G_{2\times 2}$ up to order h. Indeed we have:

$$(\nabla u^h)^t \nabla u^h = (\nabla y_0)^t \nabla y_0 + 2h\,\mathrm{sym}((\nabla y_0)^t \nabla V) + O(h^2) = G_{2\times 2} + O(h^2);$$

(ii) Finite strains $\mathbb{S}$ on Ω. These are tensor fields $\mathbb{S} \in L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$ such that:

$$\mathbb{S} = L^2 - \lim_{h\to 0} \mathrm{sym}((\nabla y_0)^t \nabla w_h) \quad \text{for some} \quad w_h \in W^{1,2}(\Omega, \mathbb{R}^3). \tag{5.70}$$

Denoting the surface $S = y_0(\Omega)$, V and $\mathbb{S}$ can be pushed to an infinitesimal isometry on S and an element of the space of finite strains on S, respectively. Indeed we can define $\tilde{V} \in W^{2,2}(S, \mathbb{R}^3)$ through $V = \tilde{V} \circ y_0$, and $\tilde{S} \in L^2(S, \mathbb{R}^{2\times 2}_{sym})$ through

$$\langle \tilde{\mathbb{S}}(y_0(x'))\partial_e y_0, \partial_e y_0\rangle = \langle \mathbb{S}(x')e, e\rangle \qquad \forall e \in \mathbb{R}^2.$$

The condition $\mathrm{sym}((\nabla y_0)^t \nabla V) = 0$ can be encoded in a skew-symmetric tensor field $\tilde{A}$ uniquely given by:

$$\tilde{A}(y_0(x'))\partial_e y_0 = \partial_e V(x') \quad \text{and} \quad \tilde{A}\vec{b}_0 = \vec{p} \qquad \forall e \in \mathbb{R}^2. \tag{5.71}$$

Now we compute the amount of stretching of S in the direction $\tau \in T_y S$, induced by the deformation $\tilde{u}^h = id + h\tilde{V} + h^2\tilde{w}$, for some $\tilde{w} \in W^{1,2}(S, \mathbb{R}^3)$. Let $e \in \mathbb{R}^2$ be such that $\partial_e y_0 = \tau$ and consider $u^h = y_0 + hV + h^2 w = \tilde{u}^h \circ y_0$. Stretching is the change in the metric, and we have the following expansion, up to terms of order h^2:

$$\begin{aligned}|\partial_\tau \tilde{u}^h|^2 - |\tau|^2 = |\partial_e u^h|^2 - |\partial_e y_0|^2 &= 2h\langle \partial_e y_0, \partial_e V\rangle + h^2(|\partial_e V|^2 + 2\langle \partial_e y_0, \partial_e w\rangle) + O(h^3)\\ &= h^2(|\partial_e V|^2 + 2\langle \partial_e y_0, \partial_e w\rangle) + O(h^3),\end{aligned}$$

where we used $2\langle \partial_e y_0, \partial_e V\rangle = \langle \mathrm{sym}((\nabla y_0)^t \nabla V)e, e\rangle = 0$. The leading order quantity in the right hand side above coincides with:

$$\langle(\mathrm{sym}((\nabla y_0)^t\nabla w))e, e\rangle + \frac{1}{2}\langle \partial_e V, \partial_e V\rangle = \Big\langle\big(\mathrm{sym}((\nabla y_0)^t\nabla w) + \frac{1}{2}(\nabla V)^t\nabla V\big)e, e\Big\rangle.$$

This is the argument of the first term in $\mathcal{I}_4$, modulo the correction $(\nabla\vec{b}_0)^t\nabla\vec{b}_0$ (equal to the third fundamental form on S in case $\vec{b}_0 = \vec{N}$), due to the incompatibility of the ambient Euclidean metric of S^h with the given prestrain G on Ω^h.

We will now describe, following the calculations in [LMP10], how the second term of $\mathcal{I}_4$ represents bending, which is the change in the second fundamental form. More precisely, it measures the difference of order h, between the shape operator Π on S (whose unit normal vector we denote by $\vec{N}$) and the shape operator Π^h on the deformed surface $S^h = (id+h\tilde{V})(S)$ (whose unit normal we denote by $\vec{N}^h$). Let τ_1, $\tau_2 \in T_yS$ be tangent vectors to S, such that $\vec{N} = \tau_1 \times \tau_2$. Then, a (non unit) normal vector to S^h at $(id + h\tilde{V})(y)$ is given by:

$$\vec{n}^h = (\mathrm{Id}+h\tilde{A})\tau_1 \times (\mathrm{Id}+h\tilde{A})\tau_2 = \tau_1\times\tau_2 + h(\tau_1\times\tilde{A}\tau_2 + \tilde{A}\tau_1\times\tau_2) + O(h^2) = \vec{N} + h\tilde{A}\vec{N} + O(h^2).$$

Observing that $|\vec{n}^h| = 1 + O(h^2)$, we have $\vec{N}^h = \frac{\vec{n}^h}{|\vec{n}^h|} = n^h + O(h^2)$. Note that, by a slight abuse of notation, we are considering $\vec{N}^h$ defined at the point $y \in S$. This way we have $\Pi^h(\mathrm{Id} + h\tilde{A})\tau = \partial_\tau \vec{N}^h$ and it follows that:

$$\begin{aligned}\Pi^h(\mathrm{Id} + h\tilde{A})\tau &= \partial_\tau(N + h\tilde{A}N) + O(h^2) = \partial_\tau N + h\tilde{A}\partial_\tau N + h(\partial_\tau\tilde{A})N + O(h^2)\\ &= (\mathrm{Id} + h\tilde{A})\Pi\tau + h(\partial_\tau\tilde{A})N + O(h^2).\end{aligned}$$

Now the amount of bending of S, in the direction $\tau \in T_yS$, induced by the deformation $id + h\tilde{V}$ can be estimated by :

$$\begin{aligned}(\mathrm{Id} + h\tilde{A})^{-1}\Pi^h(\mathrm{Id} + h\tilde{A})\tau - \Pi\tau &= (\mathrm{Id} + h\tilde{A})^{-1}\Big((\mathrm{Id} + h\tilde{A})\Pi\tau + h(\partial_\tau A)\vec{N} + \mathcal{O}(h^2)\Big) - \Pi\tau\\ &= (\mathrm{Id} - h\tilde{A})h(\partial_\tau\tilde{A})\vec{N} + \mathcal{O}(h^2) = h(\partial_\tau\tilde{A})\vec{N} + \mathcal{O}(h^2).\end{aligned}$$

The leading order term in this expansion coincides with the term $(\nabla y_0)^t\nabla\vec{p} + (\nabla V)^t\nabla\vec{b}_0$ when $\vec{b}_0 = \vec{N}$, because in view of (5.71):

$$\begin{aligned}\langle(\partial_\tau\tilde{A})\vec{b}_0, \tau\rangle &= \langle(\partial_e(\tilde{A}\vec{b}_0), \partial_e y_0\rangle - \langle(\tilde{A}\partial_e\vec{b}_0, \partial_e y_0\rangle = \langle(\partial_e\vec{p}, \partial_e y_0\rangle + \langle(\partial_e\vec{b}_0, \tilde{A}\partial_e y_0\rangle\\ &= \big\langle(\nabla y_0)^t\nabla\vec{p}\; e, e\big\rangle - \big\langle(\nabla V)^t\nabla\vec{b}_0\; e, e\big\rangle,\end{aligned}$$

where we again wrote $\tau = \partial_e y_0 \in T_{y_0(x')}S$, for any $e \in \mathbb{R}^2$. This is precisely the argument in the second term in $\mathcal{I}_4(V, \mathbb{S})$.

In the next section we identify the geometric significance of the last term in (5.22).

5.5 THE SCALING OPTIMALITY

In this section, we prove the following crucial result:

Theorem 5.5.1. *Assume (4.10), together with:*

$$\mathrm{sym}\,\big((\nabla y_0)^t \nabla \vec{d}_0\big) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = 0, \tag{5.72}$$

where y_0, $\vec{b}_0$ and $\vec{d}_0$ are defined in (4.11), (4.12), (5.2). Then the metric G is flat, i.e. $Riem(G) \equiv 0$ in Ω^h. Equivalently: $\min E^h = 0$ for all h.

In view of the symmetries in $Riem(G)$ of a 3-dimensional metric G, its flatness is equivalent to the vanishing of the following curvatures:

$$R_{1212}, \quad R_{1213}, \quad R_{1223}, \quad R_{1313}, \quad R_{1323}, \quad R_{2323}.$$

The proof of Theorem 5.5.1 is a consequence of the following observation.

Theorem 5.5.2. *Assume (4.10) and let y_0, $\vec{b}_0$ and $\vec{d}_0$ be defined as in (4.11), (5.2). Then:*

$$\mathrm{sym}\,\big((\nabla y_0)^t \nabla \vec{d}_0\big) + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323} \end{bmatrix}. \tag{5.73}$$

Proof. **1.** We have:

$$\begin{aligned}
R_{1313} &= -\frac{1}{2}\partial_{11} G_{33} + G_{np}\big(\Gamma^n_{13}\Gamma^p_{13} - \Gamma^n_{11}\Gamma^p_{33}\big), \\
R_{2323} &= -\frac{1}{2}\partial_{22} G_{33} + G_{np}\big(\Gamma^n_{23}\Gamma^p_{23} - \Gamma^n_{22}\Gamma^p_{33}\big), \\
R_{1323} &= -\frac{1}{2}\partial_{12} G_{33} + G_{np}\big(\Gamma^n_{13}\Gamma^p_{23} - \Gamma^n_{12}\Gamma^p_{33}\big).
\end{aligned}$$

On the other hand, in view of (5.2):

$$\forall i,j=1,2 \qquad \frac{1}{2}\Big(\langle\partial_i y_0, \partial_j\vec{d}_0\rangle + \langle\partial_j y_0, \partial_i\vec{d}_0\rangle\Big) = \frac{1}{2}\Big(\partial_j\langle\partial_i y_0, \vec{d}_0\rangle + \partial_i\langle\partial_j y_0, \vec{d}_0\rangle\Big) - \langle\partial_{ij} y_0, \vec{d}_0\rangle$$
$$= -\frac{1}{2}\partial_{ij}G_{33} - \langle\partial_{ij} y_0, \vec{d}_0\rangle$$

because: $\partial_j\langle\partial_i y_0, \vec{d}_0\rangle + \partial_i\langle\partial_j y_0, \vec{d}_0\rangle = -\partial_{ij}|\vec{b}_0|^2 = -\partial_{ij}G_{33}$. Consequently, the formula (5.73) will follow, if we establish:

$$\forall i,j=1,2 \qquad \langle\partial_{ij} y_0, \vec{d}_0\rangle = G_{np}\Gamma^n_{ij}\Gamma^p_{33} \quad \text{and} \quad \langle\partial_i\vec{b}_0, \partial_j\vec{b}_0\rangle = G_{np}\Gamma^n_{i3}\Gamma^p_{j3}. \tag{5.74}$$

2. Before proving (5.74) we gather some useful formulas. Note that $\partial_i G = 2\,\mathrm{sym}\,((\partial_i Q)^t Q)$ for $i=1,2$. Therefore, by direct inspection:

$$\forall i,j,k=1,2 \qquad \langle\partial_{ij} y_0\,,\,\partial_k y_0\rangle = \frac{1}{2}(\partial_i G_{kj} + \partial_j G_{ki} - \partial_k G_{ij}). \tag{5.75}$$

Also, recall that condition (4.11) is equivalent to (see [BLS16], proof of Theorem 5.3, formula (5.8)):

$$\forall i,j=1,2 \qquad \langle\partial_{ij} y_0\,,\,\vec{b}_0\rangle = \frac{1}{2}(\partial_i G_{j3} + \partial_j G_{i3}). \tag{5.76}$$

Therefore, for all $i,j=1,2$:

$$\begin{aligned}
&\langle\partial_j y_0\,,\,\partial_i\vec{b}_0\rangle = \partial_i\langle\partial_j y_0\,,\,\vec{b}_0\rangle - \langle\partial_{ij} y_0\,,\,\vec{b}_0\rangle = \frac{1}{2}(\partial_i G_{j3} - \partial_j G_{i3}),\\
&\langle\partial_i\vec{b}_0, \vec{b}_0\rangle = \frac{1}{2}\partial_i G_{33}.
\end{aligned} \tag{5.77}$$

We now express $\partial_{ij} y_0$, $\partial_i\vec{b}_0$ and $\vec{d}_0$ in the basis $\{\partial_1 y_0, \partial_2 y_0, \vec{b}_0\}$, writing:

$$\begin{aligned}
&\partial_{ij} y_0 = \alpha^1_{ij}\partial_1 y_0 + \alpha^2_{ij}\partial_2 y_0 + \alpha^3_{ij}\vec{b}_0,\\
&\partial_i\vec{b}_0 = \beta^1_i\partial_1 y_0 + \beta^2_i\partial_2 y_0 + \beta^3_i\vec{b}_0,\\
&\vec{d}_0 = \gamma^1\partial_1 y_0 + \gamma^2\partial_2 y_0 + \gamma^3\vec{b}_0.
\end{aligned} \tag{5.78}$$

By (5.75), (5.76), (5.77) and (5.2), it follows that:

$$\begin{aligned}
G\Big(\alpha^1_{ij}, \alpha^2_{ij}, \alpha^3_{ij}\Big)^t &= GQ_0^{-1}\partial_{ij}y_0 = Q_0^t\partial_{ij}y_0 \\
&= \frac{1}{2}\Big(\partial_i G_{1j} + \partial_j G_{1i} - \partial_1 G_{ij}, \partial_i G_{2j} + \partial_j G_{2i} - \partial_2 G_{ij}, \partial_i G_{3j} + \partial_j G_{3i}\Big), \\
G\Big(\beta^1_i, \beta^2_i, \beta^3_i\Big)^t &= GQ_0^{-1}\partial_i \vec{b}_0 = Q_0^t\partial\vec{b}_0 = Q_0^t\partial_i\vec{b}_0 \\
&= \frac{1}{2}\Big(\partial_i G_{13} - \partial_1 G_{i3}, \partial_i G_{23} - \partial_2 G_{i3}, \partial_i G_{33}\Big)^t, \\
G\big(\gamma^1, \gamma^2, \gamma^3\big)^t &= GQ_0^{-1}\vec{d}_0 = Q_0^t\vec{d}_0 = -\frac{1}{2}\Big(\partial_1 G_{33}, \partial_2 G_{33}, 0\Big)^t.
\end{aligned}$$

In view of (4.1) we then obtain, for all $i, j = 1, 2$:

$$(\alpha^1_{ij}, \alpha^2_{ij}, \alpha^3_{ij}) = (\Gamma^1_{ij}, \Gamma^2_{ij}, \Gamma^3_{ij}), \quad (\beta^1_i, \beta^2_i, \beta^3_i) = (\Gamma^1_{i3}, \Gamma^2_{i3}, \Gamma^3_{i3}), \quad (\gamma^1, \gamma^2, \gamma^3)^t = (\Gamma^1_{33}, \Gamma^2_{33}, \Gamma^3_{33}),$$

so that (5.78) becomes:

$$\begin{aligned}
\partial_{ij}y_0 &= \Gamma^1_{ij}\partial_1 y_0 + \Gamma^2_{ij}\partial_2 y_0 + \Gamma^3_{ij}\vec{b}_0, \\
\partial_i\vec{b}_0 &= \Gamma^1_{i3}\partial_1 y_0 + \Gamma^2_{i3}\partial_2 y_0 + \Gamma^3_{i3}\vec{b}_0, \\
\vec{d}_0 &= \Gamma^1_{33}\partial_1 y_0 + \Gamma^2_{33}\partial_2 y_0 + \Gamma^3_{33}\vec{b}_0.
\end{aligned} \tag{5.79}$$

3. We now prove (5.74). Keeping in mind that $Q_0^TQ_0 = G$, the scalar products of expressions in (5.79) are:

$$\begin{aligned}
\langle\partial_{ij}y_0\,,\,\vec{d}_0\rangle &= \big\langle\Gamma^n_{ij}\partial_n y_0 + \Gamma^3_{ij}\vec{b}_0, \Gamma^p_{33}\partial_p y_0 + \Gamma^3_{33}\vec{b}_0\big\rangle = G_{np}\Gamma^n_{ij}\Gamma^p_{33}, \\
\langle\partial_i\vec{b}_0\,,\,\partial_j\vec{b}_0\rangle &= \big\langle\Gamma^n_{i3}\partial_n y_0 + \Gamma^3_{i3}\vec{b}_0, \Gamma^p_{j3}\partial_p y_0 + \Gamma^3_{j3}\vec{b}_0\big\rangle = G_{np}\Gamma^n_{i3}\Gamma^p_{j3},
\end{aligned}$$

exactly as claimed in (5.74). This ends the proof of Theorem 5.5.2 and also of Theorem 5.5.1. □

5.6 TWO EXAMPLES

In this section we compute the energy $\mathcal{I}_4(V, \mathbb{S})$ in the two particular cases:

$$G(x', x_3) = \mathrm{diag}(1, 1, \lambda(x')) \qquad \text{and} \qquad G(x', x_3) = \lambda(x')\mathrm{Id}_3.$$

Let $\vec{p}$ be as in the definition (5.23). Writing: $\vec{p} = \alpha^1\partial_1 y_0 + \alpha^2\partial_2 y_0 + \alpha^3\vec{b}_0$, we obtain:

$$G\big(\alpha^1, \alpha^2, \alpha^3\big)^t = -\big(\langle \partial_1 V\,,\,\vec{b}_0\rangle, \langle \partial_2 V\,,\,\vec{b}_0\rangle, 0\big)^t.$$

Consequently:

$$\vec{p} = -G^{1i}\langle \partial_i V\,,\,\vec{b}_0\rangle\partial_1 y_0 - G^{2i}\langle \partial_i V\,,\,\vec{b}_0\rangle\partial_2 y_0 - G^{3i}\langle \partial_i V\,,\,\vec{b}_0\rangle\vec{b}_0. \tag{5.80}$$

We will also use the following formula for $\vec{b}_0$ given in [BLS16]:

$$\vec{b}_0 = -\frac{1}{G^{33}}(G^{13}\partial_1 y_0 + G^{23}\partial_2 y_0) + \frac{1}{\sqrt{G^{33}}}\vec{N}. \tag{5.81}$$

Lemma 5.6.1. *Let $\lambda : \bar{\Omega} \to \mathbb{R}$ be smooth and strictly positive. Consider the metric of the form $G(x', x_3) = \mathrm{diag}(1, 1, \lambda(x'))$. Then:*

(i) G is immersible in $\mathbb{R}^3$ if and only if:

$$M_\lambda = \nabla^2\lambda - \frac{1}{2\lambda}\nabla\lambda \otimes \nabla\lambda \equiv 0 \quad \text{in } \Omega,$$

while the condition $M_\lambda \not\equiv 0$ is equivalent to $ch^4 \leq \inf E^h \leq Ch^4$.

(ii) The Γ-limit energy functional $\mathcal{I}_4$ in (5.22) becomes:

$$\begin{aligned}
&\forall w \in W^{1,2}(\Omega, \mathbb{R}^2) \quad \forall v \in W^{2,2}(\Omega, \mathbb{R}) \\
&\quad \mathcal{I}_4(v, w) = \frac{1}{2}\int_\Omega \mathcal{Q}_2\big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v + \frac{1}{96\lambda}\nabla\lambda \otimes \nabla\lambda\big)\;\mathrm{d}x' \\
&\qquad\qquad + \frac{1}{24}\int_\Omega \mathcal{Q}_2\big(\sqrt{\lambda}\nabla^2 v\big) + \frac{1}{5760}\int_\Omega \mathcal{Q}_2\big(M_\lambda\big)\;\mathrm{d}x',
\end{aligned}$$

where $\mathcal{Q}_2$ is independent of x' and it is defined by $\mathcal{Q}_{2,Id}$ in (4.6).

Proof. Part (i) of the assertion has been shown in [BLS16]. For (ii), note first that:

$$y_0(x') = x' \quad \text{and} \quad Q_0 = A = \mathrm{diag}(1, 1, \sqrt{\lambda}).$$

Consequently, directly from (4.6) we see that $\mathcal{Q}_{2,A} = \mathcal{Q}_{2,Id}$, which we denote simply by $\mathcal{Q}_2$.

Further, every admissible limiting strain $\mathbb{S} \in \mathcal{S}$ has the form $\mathbb{S} = \mathrm{sym}\nabla w$ for some $w \in W^{1,2}(\Omega, \mathbb{R}^2)$. Indeed

$$\begin{aligned}\mathrm{cl}_{L^2}\{\mathrm{sym}\,(\nabla y_0)^t \nabla \tilde{w} \ \mid \ \tilde{w} \in W^{1,2}(\Omega, \mathbb{R}^3)\} &= \mathrm{cl}_{L^2}\{\mathrm{sym}\,(\nabla \tilde{w})_{2\times 2} \ \mid \ \tilde{w} \in W^{1,2}(\Omega, \mathbb{R}^3)\} \\ &= \{\mathrm{sym}\,(\nabla w) \ \mid \ w \in W^{1,2}(\Omega, \mathbb{R}^2)\}.\end{aligned}$$

This can be seen as follows. Let $w_h \in W^{1,2}(\Omega, \mathbb{R}^3)$ be such that $\mathrm{sym}\,(\nabla w_h)_{2\times 2} \longrightarrow \mathbb{S}$ in $L^2(\Omega)$. Therefore $\mathrm{sym}\,(\nabla w_h)_{2\times 2}$ is bounded in $L^2(\Omega)$. By Korn's inequality, it follows that there exists $T^h \in so(2)$ such that:

$$\left\|(\nabla w_h)_{2\times 2} - T^h\right\|_{L^2(\Omega)}$$

is uniformly bounded in h. By Poincare' inequality, there exist $b^h \in \mathbb{R}^3$ such that $\bar{w_h} = w_h - (T^h x' + b^h)$ is bounded in $W^{1,2}(\Omega)$. Hence, there exists $w \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that, up to a subsequence, $\bar{w_h} \longrightarrow w$ weakly in $W^{1,2}(\Omega)$. In particular:

$$\mathrm{sym}\,(\nabla w_h)_{2\times 2} = \mathrm{sym}\,(\nabla \bar{w_h})_{2\times 2} \longrightarrow \mathrm{sym}\,(\nabla w) \quad \text{weakly in} \quad L^2(\Omega).$$

Therefore $\mathbb{S} = \mathrm{sym}\,(\nabla w)$.

Also, without loss of generality, every admissible limiting displacement V is of the form $V = (0, 0, v)$ for some $v \in W^{2,2}(\Omega, \mathbb{R})$. Indeed, an admissible V satisfies

$$0 = \mathrm{sym}\,(\nabla y_0)^t \nabla V = \mathrm{sym}\,(\nabla V)_{2\times 2}.$$

By Korn's inequality we get $(V_1, V_2)(x') = Tx' + b$ with T skew-symmetric and $b \in \mathbb{R}^3$, therefore up to a rigid motion $V = (0, 0, v)$, $v \in W^{2,2}(\Omega, \mathbb{R})$.

Now, using (5.81), (5.79) and (5.80) we compute:

$$\vec{b}_0 = \sqrt{\lambda} e_3, \qquad \vec{d}_0 = -\frac{1}{2}(\partial_1 \lambda, \partial_2 \lambda, 0), \qquad \vec{p} = -\sqrt{\lambda}(\partial_1 v, \partial_2 v, 0).$$

Therefore:

$$(\nabla \vec{b}_0)^t \nabla \vec{b}_0 = \frac{1}{4\lambda}\nabla\lambda \otimes \nabla\lambda, \qquad (\nabla y_0)^t \nabla \vec{d}_0 = -\frac{1}{2}\nabla^2\lambda,$$
$$(\nabla y_0)^t \nabla \vec{p} = -\frac{1}{2\sqrt{\lambda}}\nabla v \otimes \nabla\lambda - \sqrt{\lambda}\nabla^2 v, \qquad (\nabla V)^t \nabla \vec{b}_0 = \frac{1}{2\sqrt{\lambda}}\nabla v \otimes \nabla \lambda.$$

This ends the proof of Lemma 5.6.1 in view of (5.22). □

Lemma 5.6.2. *Let $\lambda : \bar{\Omega} \to \mathbb{R}$ be smooth and strictly positive. Consider the metric $G(x', x_3) = \lambda(x')\mathrm{Id}_3$. Denote $f = \frac{1}{2}\log\lambda$. Then:*

(i) Condition (4.11) is equivalent to $\Delta f = 0$, which is also equivalent to the immersability of the metric $G_{2\times 2}$ in $\mathbb{R}^2$.

(ii) Under condition (4.11), condition (5.72) can be directly seen as equivalent to $\mathrm{Ric}(G) = 0$ and therefore to the immersability of G.

(iii) The Γ-limit energy functional in (5.22) has the following form:

$$\begin{aligned}\mathcal{I}_4(V, \mathbb{S}) = \frac{1}{2}\int_\Omega e^{-2f}\mathcal{Q}_2\big(\mathbb{S} + \frac{1}{2}(\nabla V)^t\nabla V + \frac{1}{24}e^{2f}\nabla f \otimes \nabla f\big)\,\mathrm{d}x' \\ + \frac{1}{24}\int_\Omega \mathcal{Q}_2\big(2\nabla V_3 \otimes \nabla f - \nabla^2 V_3 - \langle \nabla V_3\,,\,\nabla f\rangle \mathrm{Id}_2\big)\,\mathrm{d}x' \\ + \frac{1}{1440}\int_\Omega \mathcal{Q}_2\big(e^f \mathrm{Ric}(G)_{2\times 2}\big)\,\mathrm{d}x',\end{aligned}$$

where $\mathcal{Q}_2$ is as in Lemma 5.6.1, and where $\mathrm{Ric}(G)_{2\times 2}$ denotes the tangential part of the Ricci curvature tensor of G, i.e.:

$$\mathrm{Ric}(G)_{2\times 2} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22}\end{bmatrix}.$$

Proof. The part (i) has been deduced in [BLS16], together with the expression:

$$\mathrm{Ric}(G) = -(\nabla^2 f - \nabla f \otimes \nabla f)^* - (\Delta f + |\nabla f|^2)\mathrm{Id}_3. \tag{5.82}$$

We now consider the case when (4.11) holds. By (i) the metric $G_{2\times 2}$ is immersible in $\mathbb{R}^2$ and in particular $\vec{N} = e_3$. Writing $V = (V_1, V_2, V_3)$, from (5.81), (5.79) and (5.80) we obtain:

$$\vec{b}_0 = \sqrt{\lambda}e_3, \qquad \vec{d}_0 = -\big(\partial_1 f \partial_1 y_0 + \partial_2 f \partial_2 y_0\big), \qquad \vec{p} = -\frac{1}{\sqrt{\lambda}}\big(\partial_1 V_3 \partial_1 y_0 + \partial_2 V_3 \partial_2 y_0\big).$$

$$(\nabla \vec{b}_0)^t \nabla \vec{b}_0 = e^{2f} \nabla f \otimes \nabla f, \qquad (\nabla V)^t \nabla \vec{b}_0 = e^f \nabla V_3 \otimes \nabla f.$$

Further, observe that: $\partial_i \vec{d}_0 = -(\partial_{1i} f \partial_1 y_0 + \partial_{2i} f \partial_2 y_0 + \partial_1 f \partial_{1i} y_0 + \partial_2 f \partial_{2i} y_0)$, and so:

$$\frac{1}{\lambda}\langle \partial_1 y_0 \, , \, \partial_1 \vec{d}_0 \rangle = -\frac{1}{\lambda}\big(\lambda \partial_{11} f + \frac{1}{2}\partial_1 \lambda \partial_1 f + \frac{1}{2}\partial_2 \lambda \partial_2 f\big) = -(\partial_{11} f + |\nabla f|^2).$$

In the same manner, we arrive at:

$$\frac{1}{\lambda}\langle \partial_2 y_0 \, , \, \partial_2 \vec{d}_0 \rangle = -(\partial_{22} f + |\nabla f|^2), \quad \frac{1}{\lambda}\langle \partial_2 y_0 \, , \, \partial_1 \vec{d}_0 \rangle = -\partial_{12} f, \quad \frac{1}{\lambda}\langle \partial_1 y_0 \, , \, \partial_2 \vec{d}_0 \rangle = -\partial_{21} f.$$

Consequently, $(\nabla y_0)^t \nabla \vec{d}_0$ is already a symmetric matrix, and:

$$(\nabla y_0)^t \nabla \vec{d}_0 = -e^{2f}(\nabla^2 f + |\nabla f|^2 \mathrm{Id}_2).$$

In particular, under condition $\Delta f = 0$, the formula (5.82) yields:

$$\mathrm{sym}\, (\nabla y_0)^t \nabla \vec{d}_0 + (\nabla \vec{b}_0)^t \nabla \vec{b}_0 = e^{2f} \mathrm{Ric}(G)_{2\times 2},$$

which we directly see to be equivalent with $\nabla f = 0$ and hence with $\mathrm{Ric}(G) = 0$. This establishes (ii).

We now compute the remaining quantities appearing in the expression of $\mathcal{I}_4$. Firstly:

$$\nabla \vec{p} = \frac{1}{2\lambda^{3/2}} \nabla y_0 (\nabla V_3 \otimes \nabla \lambda) - \frac{1}{\sqrt{\lambda}} \nabla y_0 \nabla^2 V_3 - \frac{1}{\sqrt{\lambda}}\Big(\partial_1 V_3 (\partial_{11} y_0, \partial_{12} y_0) + \partial_2 V_3 (\partial_{12} y_0, \partial_{22} y_0)\Big).$$

Using the relations between $\langle \partial_{ij} y_0 \, , \, \partial_k y_0 \rangle$ and $\partial_l G$ in (5.75), we obtain:

$$(\nabla y_0)^t \nabla \vec{p} = \frac{1}{2\lambda^{3/2}} G_{2\times 2} \nabla V_3 \otimes \nabla \lambda - \frac{1}{\sqrt{\lambda}} G_{2\times 2} \nabla^2 V_3 - \frac{1}{2\sqrt{\lambda}} \left[\begin{array}{c|c} \langle \nabla V_3 \, , \, \nabla \lambda \rangle & \langle \nabla V_3 \, , \, \nabla \lambda^\perp \rangle \\ \hline -\langle \nabla V_3 \, , \, \nabla \lambda^\perp \rangle & \langle \nabla V_3 \, , \, \nabla \lambda \rangle \end{array}\right],$$

and therefore:

$$\mathrm{sym} (\nabla y_0)^t \nabla \vec{p} = \sqrt{\lambda}\, \mathrm{sym}\big(\nabla V_3 \otimes \nabla f\big) - \sqrt{\lambda} \nabla^2 V_3 - \sqrt{\lambda} \langle \nabla V_3 \, , \, \nabla \lambda \rangle \mathrm{Id}_2.$$

In a similar manner, it follows that:

$$\mathrm{sym} (\nabla y_0)^t \nabla \vec{d}_0 = -\lambda \Big(\nabla^2 f + |\nabla f|^2 \mathrm{Id}_2\Big).$$

Since $\mathcal{Q}_{2,A}(x') = \lambda^{-1} \mathcal{Q}_2$, the formula in (5.22) becomes:

$$\begin{aligned} \mathcal{I}_4(V, \mathbb{S}) =& \frac{1}{2} \int_\Omega e^{-2f} \mathcal{Q}_2 \big(\mathbb{S} + \frac{1}{2}(\nabla V)^t \nabla V + \frac{1}{24} e^{2f} \nabla f \otimes \nabla f\big)\, \mathrm{d}x' \\ &+ \frac{1}{24} \int_\Omega e^{-2f} \mathcal{Q}_2 \big(2 e^f \nabla V_3 \otimes \nabla f - e^f \nabla^2 V_3 - e^f \langle \nabla V_3 \, , \, \nabla f \rangle \mathrm{Id}_2\big)\, \mathrm{d}x' \\ &+ \frac{1}{1440} \int_\Omega e^{-2f} \mathcal{Q}_2 \big(e^{2f} \mathrm{Ric}(G)_{2\times 2}\big)\, \mathrm{d}x', \end{aligned} \tag{5.83}$$

which implies the result. □

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