HÖLDER CONTINUOUS MAPPINGS INTO SUB-RIEMANNIAN MANIFOLDS

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We develop analytic tools with applications to the study of Hölder continuous mappings into manifolds, especially sub-Riemannian manifolds like the Heisenberg Group. The first is a notion of a pullback $f^*\kappa$ of a differential form κ by a Slobodetskiĭ (or fractional Sobolev) mapping $f \in W^{s,p}(M, N)$ between manifolds; the second is Hodge decomposition of these objects $f^*\kappa = \Delta\omega$; the third tool is a notion of generalized Hopf Invariant for mappings $f : \mathbb{S}^{4n-1} \to \mathbb{H}_{2n}$ from spheres into the Heisenberg Group, which relies on this Hodge decomposition. This latter idea was explored in [21] for Lipschitz maps. Here, the definition is extended to Hölder continuous maps. The first tool allows an apparently simpler proof of a slight generalization of Gromov's non Hölder-embedding theorem for maps $f \in C^{0,\gamma}(\mathbb{R}^{n+1}, \mathbb{H}_n)$, $\gamma > \frac{n+1}{n+2}$. The Hopf invariant allows for another rigidity result for γ-Hölder maps, again for sufficiently large γ.

Keywords: sub-Riemannian geometry, Heisenberg group, Hölder mappings, Jacobian, Gromov's conjecture, Hopf invariant.

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1.0 INTRODUCTION

Sub-Riemannian Manifolds are the spaces needed to model systems which evolve in the presence of constraints. In order to parallel park one's car, one needs to navigate their car from point A to point B without violating the constraint that the car must move in the direction its wheels are facing (it cannot slide sideways). Likewise, if a cat falls out of a tree, it must manipulate its body in such a way that it lands on its feet without violating the law of conservation of angular momentum. The parallel parker, the cat, and, loosely speaking, any other controller of a system which obeys laws, are navigating through a sub-Riemannian manifold. A sub-Riemannian manifold is, essentially, a manifold $\mathbb{M}$, a metric g (such as a Riemannian metric), and a sub-bundle $H\mathbb{M} \subseteq T\mathbb{M}$ which is called by sub-Riemannian geometers the *horizontal distribution* of $\mathbb{M}$. This formalizes the notion of "permissible paths" which can be defined as curves $\gamma : [a, b] \to \mathbb{M}$ with $\gamma'(t) \in H\mathbb{M}$ for all t. Since the metric g allows us to find the length of such curves, we obtain the Carnot-Caratheodory metric: the distance between two points in $\mathbb{M}$ is the infimal length among permissible paths between those points. The most famous among these sub-Riemannian manifolds is the family of Heisenberg groups, $\mathbb{H}_n$, $n \in \mathbb{N}$, which are among the simplest possible non-trivial examples. The definitions of sub-Riemannian manifolds and the Heisenberg Group in particular are given in Chapter 6.

As will be seen in Chapter 6, Hölder continuity is a very natural property which arises in sub-Riemannian manifolds. However, the structure of Hölder continuous mappings between sub-Riemannian manifolds is poorly understood, and the questions about them are hard. This dissertation puts a mere dent in the still seemingly-impenetrable question of Mikhael Gromov: *does there exist a γ-Hölder continuous embedding $\mathbb{R}^2 \hookrightarrow \mathbb{H}_1$ for $\gamma > 1/2$?* More generally, Gromov [19, §0.5.C] initiates a study of Hölder continuous maps between sub-Riemannian manifolds (or, as he and others refer to them, *Carnot-Caratheodory spaces*). The study of such questions will no doubt deepen our understanding of the mysterious sub-Riemannian manifolds. In fact, to address Gromov's question, we have developed tools that can likely find applications outside of sub-Riemannian geometry. One fundamental result is the following:

Theorem 1.0.1. *Let $0 \leq k \leq n$ be integers, M a compact oriented n-dimensional Riemannian manifold without boundary, $f \in W^{1-\frac{1}{k+1},k+1}(M,\mathbb{R}^N)$, and $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^N)$ a smooth k-form on $\mathbb{R}^N$. Then the limit*

$$|\langle f^*\kappa, \tau\rangle| = lim_{t\to 0}\int_M f_t^*\kappa \wedge *\tau. \tag{1.1}$$

exists for $\tau \in L^\infty \cap W^{1-\frac{1}{k+1},k+1}(\bigwedge^k M)$, independent of the choice of smooth approximations $f_t \to f$ in $W^{1-\frac{1}{k+1},k+1}(M,\mathbb{R}^N)$. Moreover, $f^*\kappa$, thus defined, can be viewed as a bounded linear functional on $L^\infty \cap W^{1-\frac{1}{k+1},k+1}(\bigwedge^k M)$ with the bound

$$\langle f^*\kappa, \tau\rangle \lesssim \|\kappa\|_{C^1}\Big(\|f\|^k_{W^{1-\frac{1}{k+1},k+1}} + \|f\|^{k+1}_{W^{1-\frac{1}{k+1},k+1}}\Big)(\|\tau\|_{W^{1-\frac{1}{k+1},k+1}} + \|\tau\|_{L^\infty}). \tag{1.2}$$

This theorem allows us to make sense of the expression $f^*\kappa$ even when f is a γ-Hölder continuous function, $\gamma > 1 - \frac{1}{k+1}$. A review of all relevant facts of Slobodetskiĭ spaces $W^{s,p}$—and in fact, a review of all function spaces we use—is contained in Chapter 3. Among these facts is that the Hölder spaces $C^{0,\gamma}$ are contained in $W^{s,p}$ for $\gamma > s$. So any results about Slobodetskiĭ mappings apply to Hölder mappings.

To an analyst trained in Sobolev spaces, it is no surprise that a $W^{1,N}(\mathbb{R}^N,\mathbb{R}^N)$ map has a Jacobian with some nice geometric properties. It has, after all, weak partial derivatives which are sufficiently integrable that their Jacobian determinant is, too, integrable. Fractional Sobolev mappings $W^{1-\frac{1}{N},N}$ do not in general have weak derivatives. They can even be no-where classically differentiable and possess other pathologies not encountered with Sobolev mappings. Indeed, the nowhere differentiable functions of Weierstrass can be constructed to be Hölder continuous, and as we've just mentioned, Hölder continuous functions belong to certain Slobodetskiĭ spaces. This Theorem 1.0.1 (proven as Theorem 8.2.1) exploits the fact that Slobodetskiĭ spaces are the trace spaces of Sobolev spaces. This idea was explored in somewhat different language in the beautiful paper of Brezis and Nguyen [7], without which this research would have been far more difficult. It should be remarked that other authors ([32], [34]) were aware of these types of results, but the paper of Brezis and Nguyen makes them accessible and far less esoteric. Conti, Delellis, and Székelyhidi [10] used similar ideas to define a notion of curvature for $C^{1,\gamma}$ maps for sufficiently large γ and obtain a rigidity result for these maps. Finally, Züst independently discovered a very similar result on the existence of Jacobians specifically for Hölder functions (rather than Slobodetskiĭ spaces as in [7]) using a very different technique [36].

Distributional Hölder pullbacks of differential forms turns out to be very useful to the analysis of Hölder continuous maps between sub-Riemannian manifolds. In many respects, it is as if they do have derivatives. In particular, a version of Stokes' Theorem is true, and we can pull back and integrate a differential form. This notion turns out to be powerful enough to prove a non-embedding theorem of Gromov:

Theorem 1.0.2 (Gromov)**.** *There does not exist a topological embedding $f \in C^{0,\gamma}(\mathbb{R}^k, \mathbb{H}_n)$ for $k \geq n+1$ and $\gamma > 1 - \frac{1}{n+2}$.*

In fact we prove somewhat more than Gromov did; see Theorem 9.2.2. The idea is simple: some routine computations (Section 9.1) show that Hölder mappings f into the Heisenberg Group have a rank which is "essentially less than $n+1$". That is, the Hölder pullbacks $f^*\kappa$ of differential forms κ of dimension $k > n$ define a zero functional in the sense of Theorem 1.0.1. This work is done in Section 9.1. We then consider the restriction of f to the ball $\mathbb{B}^{n+1}$ with boundary $\mathbb{S}^n$. We can pass from the sphere to the ball with Stokes' Theorem

$$\int_{\mathbb{S}^n} f^*\kappa = \int_{\mathbb{B}^{n+1}} f^*(d\kappa) = 0$$

for any n-form κ. In Section 8.2 (Theorem 8.2.4) a simple topological argument shows that f therefore cannot be an embedding on $\mathbb{S}^n$.

As a second application, consider the composition of the Hopf Fibration $\mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ with a horizontal sphere embedding $\mathbb{S}^{2n} \hookrightarrow \mathbb{H}_{2n}$. We show that this map $\mathbb{S}^{4n-1} \to \mathbb{H}_{2n}$ is not homotopic to a constant map via a $C^{0,\gamma}$ homotopy for sufficiently high γ. In this direction, we define a Hopf Invariant for low-rank Hölder continuous maps. It is not defined topologically, but via Hodge decomposition of Hölder pullbacks. To accomplish this we derive Schauder-type estimates for the Poisson equation

$$\Delta\omega = f^*\kappa \tag{1.3}$$

for when f is $C^{0,\gamma}$ and Δ is the Laplace-Beltrami operator on a Riemannian manifold M. The right-hand-side is defined only in the distributinoal sense of equation (1.1). The classical Hodge decomposition is treated in Chapter 5 after a preliminary chapter on manifolds, Chapter 4. In Section 8.3 we prove the actual Schauder estimates for this equation (1.3) to find that the solution ω is in $C^{1,\sigma}$, $\sigma = 1 - k(1-\gamma)$.

Theorem 1.0.3. *Let M be a compact n-dimensional Riemannian manifold without boundary, $f \in C^{0,\gamma}(M, \mathbb{R}^N)$, $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^N)$, $\gamma > 1 - \frac{1}{k+1}$, and $\sigma = 1 - k(1-\gamma)$. Then there exists a unique $\omega \in C^{1,\sigma}(\bigwedge^k M)$ satisfying the* weak *Poisson equation*

$$\Delta\omega = f^*\kappa. \tag{1.4}$$

More precisely, for all $\varphi \in C^\infty(\bigwedge^k M)$ we have

$$\int_M d\omega \wedge *d\varphi + \delta\omega \wedge *\delta\varphi = \int_M f^*\kappa \wedge *\varphi, \tag{1.5}$$

where the right-hand side is understood through the definition (1.1). Moreover, we have the estimate

$$\|\omega\|_{C^{1,\sigma}(\bigwedge^k M)} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}. \tag{1.6}$$

The argument uses the Campanato method [8] as outlined in [18] (see also [33]). Theorem 1.0.3 is new, however, because the right-hand-side is not a Hölder function, but a distributional Jacobian.

In the last sections 9.4 and 9.5, we consider two investigations into what Hölder continuous maps exist from $\mathbb{R}^k$ into $\mathbb{H}_n$. The main result of Section 9.4 is an almost-everywhere horizontal map from $\mathbb{R}^2 n$ into $\mathbb{H}_n$ which is *almost Lipschitz* as a mapping into $\mathbb{R}^{2n+1}$. This notion is made precise in that section. However, it does little to address Gromov's questions about Hölder continuous maps into sub-Riemannian manifolds, because this example is not even 1/2-Hölder continuous as a mapping into $\mathbb{H}_n$. The investigation in Section 9.5 is not a rigorous construction, but merely provides some numerical evidence which allows one to speculate that there may exist non-trivial $C^{0,\gamma}(\mathbb{R}^2, \mathbb{H}_1)$ maps for some values of γ in the interval $1/2 < \gamma \leq 2/3$.

This thesis is based on two joint papers: one with Hajlasz and Schikorra in preparation; and the 2013 paper [20].

1.1 THE LAYOUT OF THE DISSERTATION

In Chapters 2 to 6, we cover all standard results which are necessary to understand the original work, which is contained in Chapters 7 to 9. The topics covered there are best summarized in the table of contents. An attempt was made to prove theorems central to the thesis and to provide a reference for every result used.

This thesis is written in order of logical dependence. Every time a result is proven, be it a lemma or a theorem, the ingredients used have been proven or referenced at a preceding part of the thesis—or else they are standard enough to have been covered in a typical, rigorous undergraduate training in mathematics. The most technically demanding chapter of this thesis is Chapter 8, especially Section 8.3 on Schauder-type estimates for Hölder pullbacks. However, these estimates are mere tools, so, fortunately it seems that minimal understanding should be lost if the reader at first skips these proofs whenever the result feels insufficiently motivated. In fact, the final Chapter 9, which contains all the new results about mappings into the Heisenberg Group, can be understood without reading the preceding chapter, if the tools are taken for granted.

This is, after all, how my research is done, and how, I would surmise, most mathematical research is done: formal computations and intuition are leveraged to decide what is likely to be true, and then, once our path of reasoning seems plausible, we circle back to fill in the details. It is safe to read this dissertation in an analogous way.

1.2 NOTATION AND CONVENTIONS

Given a topological space X, a compact subset $K \subseteq X$, an open set $U \subseteq X$ with $K \subseteq U$, and a continuous function $\psi : X \to \mathbb{R}$, we will write

$$K \prec \psi \prec U$$

if ψ has the properties that $\psi \equiv 1$ on a neighborhood of K, and ψ has compact support inside U. If X is Euclidean space or a manifold—i.e., if X has smooth structure—then we make the additional assumption that ψ is smooth.

$\mathbb{R}^{n+1}_+$ will denote the upper-half space. Unless otherwise specified, we use coordinates $(x_1, \ldots, x_n, t)$ to denote points in $\mathbb{R}^{n+1}_+$. We may also write (x', t) where $x' \in \mathbb{R}^n$ and $t \geq 0$.

For us, a Domain is a connected open subset $\Omega \subseteq \mathbb{R}^n$ with smooth boundary. At no point in this thesis are we concerned with issues of boundary regularity—of the domains themselves or the functions defined on them.

When we say that M is a manifold, we shall mean that it is compact and without boundary, unless we explicitly admonish that M is a manifold with boundary.

When X is a metric space, especially, if it is a Riemannian manifold, we will use $B_r(x)$ to denote the ball of radius $r > 0$ centered at $x \in X$.

When u is a locally integrable function on $\Omega = \mathbb{R}^n$ or a subdomain $\Omega \subseteq \mathbb{R}^n$, we will use the notation $u_{x_0,r}$ as shorthand for the average value

$$u_{x_0,r} := \frac{1}{\mathrm{vol}(\Omega(x_0,r))} \int_{\Omega(x_0,r)} u(x)\,dx =: \fint_{\Omega(x_0,r)} u(x)\,dx$$

where $\Omega(x_0, r) = \Omega \cap B_r(x_0)$.

2.0 VECTOR SPACES

Here we review notions on vector spaces, both finite and infinite dimensional. All vector spaces are over $\mathbb{R}$.

2.1 NORMED LINEAR SPACES AND BOUNDED LINEAR OPERATORS

Definition 2.1.1. A normed linear space is a vector space X together with a *norm* $\|\cdot\|$ satisfying the following properties.

1. $\|\mathbf{0}\| = 0$.
2. $\|x\| > 0 \quad \forall x \in X$ not $\mathbf{0}$.
3. $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$ (triangle inequality).

A norm on X induces a metric structure on X through the formula

$$d(x, y) = \|x - y\|.$$

Definition 2.1.2. A *Banach Space* is a normed linear space X whose norm-metric is complete.

The fundamental tool for checking that a normed space is Banach is the following:

Proposition 2.1.3. *A normed linear space $(X, \|\cdot\|)$ is a Banach space if and only if it has the following property: for all sequences $x_i \in X$ with $\sum_i \|x_i\| < \infty$, the sum $\sum_i x_i$ converges to some $x \in X$.*

Proof. Suppose X is Banach and x_i is such a sequence. Then the partial sums $y_j = \sum_{i=1}^{j} x_i$ are Cauchy, so converge to some $x \in X$. That is to say, the sum $\sum_i x_i$ converges to x.

On the other hand, suppose X has the property that absolutely summable sequences ($\sum_i \|x_i\| < \infty$) are in fact summable, $\sum_i x_i = x \in X$. Let y_i be any Cauchy sequence. From this, extract a subsequence z_i with the property that $\|z_i - z_j\| \leq 2^{-m}$ whenever $i, j > m$. Now the differences $z_i - z_{i-1}$ are absolutely summable since $\sum_i \|z_i - z_{i-1}\| \leq \sum_i 1/2^i = 1$. Consequently the telescoping series $z_1 + \sum_{i=1}^{\infty} (z_{i+1} - z_i)$ converges to a vector z in X, and is the limit of the z_i. But this must also be the limit of the original sequence y_i, since any Cauchy sequence with a convergent subsequence must itself be convergent. □

Theorem 2.1.4. *Given a normed linear space X and a linear functional $f : X \to \mathbb{R}$, the following are equivalent:*

1. *f is continuous on X.*
2. *f is* locally bounded*: $\sup_{x \in B_1} \|f(x)\| < \infty$.*
3. *f is uniformly continuous on X.*

Proof. First suppose f is continuous. Since $f(\mathbf{0}) = 0$, we have $|f(x)| \to 0$ as $x \to \mathbf{0}$. In particular, for some $r > 0$ we have $\sup_{x \in B_r} |f(y)| < 1$. Consequently

$$\sup_{x \in B_1} |f(x)| = \frac{1}{r} \sup_{y \in B_r} |f(y)| < \frac{1}{r} < \infty.$$

Now suppose f is locally bounded with $\sup_{x \in B_{r_0}} |f(x)| = M < \infty$. Then we have, say, for $0 < r < r_0$,

$$\sup_{y \in B_r} |f(y)| = \sup_{y \in B_{r_0}} \left| f\left(\frac{r}{r_0} y\right) \right| \leq \frac{rM}{r_0}.$$

This evidently means $|f(y)| \leq M|y| \to 0$ as $y \to \mathbf{0}$. So f is continuous at $\mathbf{0}$. But in fact, this proves uniform continuity since also $|f(x) - f(y)| = |f(x - y)\| \leq M\|x - y\|$.

Of course 3 implies 1, so the proof is complete. □

Definition 2.1.5. The space of linear functionals $f : X \to \mathbb{R}$, satisfying any (hence all) of the properties above, is denoted X^*. It is endowed with the norm

$$\|f\|_{X^*} = \sup_{x \in B_1} |f(x)|.$$

Also, given two normed linear spaces X and Y, we define the space of bounded linear mappings between them $\mathcal{L}(X, Y)$ and endow that space with the norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{x \in B_1(X)} \|Tx\|_Y.$$

We will interchangeably use the terminology *bounded* and *continuous* when speaking about linear mappings between Banach spaces and linear functionals.

In any finite dimensional normed linear space, it can be shown that any linear functional on V is continuous. In fact, any finite dimensional normed linear space is equivalent to Euclidean space, in the sense that there is a linear homeomorphism between them.

Definition 2.1.6. When $f \in X^*$ is a bounded linear functional on a normed space X and $x \in X$, we will often use the convention of writing (f, x) rather than $f(x)$.

Definition 2.1.7. Let $T : X \to Y$ be a bounded linear operator between normed linear spaces. We define the *adjoint* operator $T^* : Y^* \to X^*$ to be the unique linear operator satisfying

$$(T^* g, x) = (g, Tx) \quad \forall g \in Y^*, x \in X.$$

It is a good exercise to verify that T^* is a bounded linear operator from $Y^* \to X^*$, and that in fact

$$\|T^*\|_{\mathcal{L}(Y^*,X^*)} = \|T\|_{\mathcal{L}(X,Y)} \tag{2.1}$$

Definition 2.1.8. Given a subset S of a normed linear space X, we define $S^\perp$ to be the set of all bounded linear functionals $f \in X^*$ which vanish on S:

$$S^\perp = \{f \in X^* \ : \ (f, x) = 0, \ \forall x \in S\}$$

Definition 2.1.9. Given a normed linear space S and a Banach space X, a linear map $f \in \mathcal{L}(S, X)$ is called *compact* if f sends bounded subsets of S to pre-compact subsets of X. If $V \subseteq X$ is a linear subspace of X, we say that V *compactly embeds* in X if the inclusion map $\iota : V \hookrightarrow X$ is compact.

2.2 HILBERT SPACES AND RIESZ REPRESENTATION

Definition 2.2.1. An inner product space is a vector space X together with an inner product $(\cdot, \cdot)$ – that is, a map from $X \times X \to \mathbb{R}$ which is bilinear, symmetric, and non-degenerate: $(v, v) \geq 0$ with equality if and only if $v = \mathbf{0}$.

An inner product induces a norm via the formula

$$||v|| = \sqrt{(v, v)}. \tag{2.2}$$

Indeed, to verify the triangle inequality, observe that for $t \in \mathbb{R}$,

$$0 \leq (x + ty, x + ty) = (x, x) + 2t(x, y) + t^2(y, y).$$

Let $t = -(x, y)/(y, y)$ to find the Cauchy-Schwarz Inequality

$$|(x, y)| \leq ||x||||y||. \tag{2.3}$$

With this we can obtain

$$\begin{aligned}
||x + y||^2 &= (x + y, x + y) = (x, x) + 2(x, y) + (y, y) \\
&\leq ||x||^2 + 2||x||||y|| + ||y||^2 \\
&= (||x|| + ||y||)^2 \\
||x + y|| &\leq ||x|| + ||y||
\end{aligned}$$

This proves that ever inner product space is a normed linear space via (2.2).

Expanding definitions, observe that we have the so-called *parallelogram law*:

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2. \tag{2.4}$$

This gives an important piece of information about convex sets in an inner product space X.

Definition 2.2.2. A subset $C \subseteq X$ in a vector space X is said to be *convex* if $\lambda x + (1 - \lambda)y \in C$ whenever $x, y \in C$ and $0 \leq \lambda \leq 1$.

Lemma 2.2.3. *If X is an inner product space and $C \subset X$ is convex and $v \in X \backslash C$ is at a distance $m \geq 0$ from C, then*

$$\operatorname{diam}(B_{m+\varepsilon}(v) \cap C) \to 0 \quad \textit{as } \varepsilon \to 0$$

where $B_r(x)$ denotes the open ball of radius r centered at x.

Proof. By translation, we can assume that $v = 0$. We will write B_r instead of $B_r(0)$. Select two points $x, y \in B_{m+\varepsilon} \cap C$. Observe $\frac{x+y}{2} \in C$, so $\|\frac{x+y}{2}\| \geq m$. Now we have thanks to (2.4)

$$\begin{aligned} \|x - y\|^2 &\leq 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x+y}{2}\right\|^2 \\ &\leq 4(m+\varepsilon)^2 - 4m^2 \\ &= 8m\varepsilon + 4\varepsilon^2 \end{aligned}$$

Thus $\operatorname{diam}(B_{m+\varepsilon} \cap C) \leq \sqrt{8m\varepsilon + 4\varepsilon^2}$. □

Definition 2.2.4. A *Hilbert space* is an inner product space H whose induced norm (2.2) makes H complete.

The importance of completeness to the geometry of a Hilbert space is best seen in the following

Proposition 2.2.5. *Let $C \subset H$ be a closed convex subset of a Hilbert space and $v \in H \backslash C$. Then there exists a unique point $\pi_C v \in C$ which is closest to v out of all the points in C.*

Proof. Let m denote the distance from v to C. The sets $B_{m+\varepsilon}(v) \cap C$ are non-empty and have diameters shrinking to zero as $\varepsilon \to 0$, thanks to Lemma 2.2.3. Since H is assumed complete, these sets shrink to a single point p which is evidently the smallest possible distance m to v, as desired. □

Remark 2.2.6. The projection π_V is continuous. If V is a linear subspace, then π_V is linear.

Definition 2.2.7. Let $V \subseteq H$ be a subset of an inner product space H. Define the orthogonal complement $V^\perp$

$$V^\perp = \{x \in H : (x, v) = 0 \;\; \forall v \in H\}.$$

Lemma 2.2.8. *If $V \subsetneq H$ is closed subspace of H, but not all of H, then V has a non-trivial orthogonal complement $V^\perp \neq \{0\}$.*

Proof. Since V is not all of H, select $u \in H \backslash V$ and take $w = u - \pi_V u$. We claim $w \in V^\perp$ which would complete the proof. To see this consider the diagram, where v is an arbitrary vector in V. Here we know that $||w|| \leq ||w - tv||$ by definition of the projection $\pi_V u$. But squaring both sides and

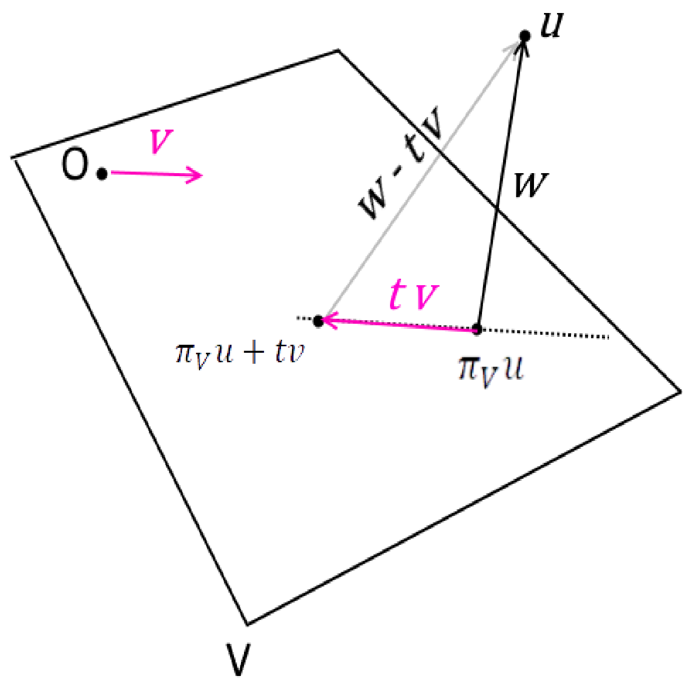


Figure 2.1: Orthogonal Projection

expanding gives

$$||w||^2 \leq ||w||^2 - 2t(w, v) + t^2||v||^2$$

Since this is true for all t, we must have $(w, v) = 0$, which proves our claim. □

Lemma 2.2.9. *Let V be a subspace of a Hilbert space H. Then $V^\perp$ is closed. Also, if V and W are two* closed *subspaces of H, $V + W = \{v + w \,:\, v \in V, w \in W\}$ is closed.*

Proof. If $v_j^\perp \in V^\perp$ is converging to $v \in H$ then for all $u \in V$ we have

$$0 = (v_j^\perp, u) \xrightarrow{j\to\infty} (v, u)$$

so $v \in V^\perp$.

Now suppose $v_i \in V$, $w_i \in W$, and suppose $u_i = v_i + w_i$ is Cauchy. Let $\pi_{V^\perp}$ denote the orthogonal projection onto $V^\perp$. Then $\pi_{V^\perp}(v_i + w_i) = \pi_{V^\perp} w_i$ is a Cauchy sequence. Define $\tilde{w}_i = \pi_{V^\perp} w_i$ and $\tilde{v}_i = v_i + \pi_V w_i$ so that $u_i = v_i + w_i = \tilde{v}_i + \tilde{w}_i$. Then since u_i is Cauchy we have

$$\|u_i - u_j\|^2 = \|\tilde{v}_i - \tilde{v}_j\|^2 + \|\tilde{w}_i - \tilde{w}_j\|^2 \to 0 \quad \text{as } i, j \to \infty$$

and consequently $\tilde{v}_i$ and $\tilde{w}_i$ are Cauchy sequences. Since V and W are closed and H is complete, the limits exist, $\tilde{v}_i \to v \in V$ and $\tilde{w}_i \to w \in W$. So $u_i \to v + w \in V + W$, proving that $V + W$ is closed. □

Proposition 2.2.10. *Let $V \subseteq H$ be a* closed *subspace of H. Then $V \oplus V^\perp = H$. That is, each $x \in H$ can be uniquely decomposed as $x = x^{\|} + x^{\perp}$, where $x^{\|} \in V$ and $x^\perp \in V^\perp$.*

Proof. By Lemma 2.2.9, $V + V^\perp$ is closed. Moreover, $(V + V^\perp)^\perp = \{0\}$. By (the contrapositive of) Proposition 2.2.8, we have $V + V^\perp = H$. For uniqueness of this decomposition, observe that if we can decompose $x = x_2^{\|} + x_2^\perp = x_1^{\|} + x_1^\perp$, then subtracting yields

$$x_2^{\|} - x_1^{\|} = x_1^\perp - x_2^\perp \in V \cap V^\perp = \{0\}.$$

□

Observe that any $v \in H$ can be viewed as an element of H^* through the formula $\varphi \mapsto (v, \varphi)$ for $\varphi \in H$. It turns out every element of H^* can be expressed this way. This has far-reaching consequences, including Hodge decomposition on manifolds, Theorem 5.3.1.

Theorem 2.2.11 (Riesz Representation Theorem)**.** *Let $f \in H^*$ be a linear functional on a Hilbert Space H. Then there exists a unique $v \in H$ with the property that*

$$(v, \varphi) = f(\varphi) \quad \forall \varphi \in H. \tag{2.5}$$

Proof. Assume f is not zero, for that case is trivial. Observe that $V = \ker f$ is a closed subspace of H whose orthogonal complement $V^\perp$ must be one-dimensional. Select $\hat{n} \in V^\perp$ with $\|\hat{n}\| = 1$ and $f(\hat{n}) > 0$. Observe[1] that projection onto $V^\perp$ is given by $\pi_{V^\perp}\varphi = (\hat{n}, \varphi)\hat{n}$. Consequently,

$$f(\varphi) = f(\pi_{V^\perp}\varphi + \pi_V\varphi) = f(\pi_{V^\perp}\varphi) = f((\hat{n}, \varphi)\hat{n}) = (\hat{n}, \varphi)f(\hat{n}) = (f(\hat{n})\hat{n}, \varphi)$$

which completes the proof, with $v = f(\hat{n})\hat{n}$. □

Remark 2.2.12. Observe that if H is a Hilbert space and $x_1, x_2 \in H$ satisfy

$$(x_1, \varphi) = (x_2, \varphi) \quad \forall\varphi \in H$$

then in fact $x_1 = x_2$. Therefore we can conclude that the representation map $\mathrm{rep} : H^* \ni f \xmapsto{\mathrm{rep}} x \in H$ is an isomorphism.

Remark 2.2.13. Given a Hilbert Space H, we can define an inner product on H^* by $(f, g) := (\mathrm{rep}\, f, \mathrm{rep}\, g)$. This naturally makes H^* a Hilbert space, and $\mathrm{rep} : H^* \to H$ an isometry.

We should remark that the path we just took is not the fastest proof of the Riesz Representation Theorem, since we needed the orthogonal compliment machinery. There is a *direct method* which is somewhat less pleasingly geometric, but the idea is important all-the-same.

Direct Proof of Riesz Reprsentation. Define the "energy" functional on H by setting, for $v \in H$

$$\mathcal{F}(v) = \frac{1}{2}\|v\|^2 - f(v).$$

Since f is bounded, $\mathcal{F}$ is bounded below. Let $m = \inf \mathcal{F}(u)$. and define $S_\varepsilon := \{u \in H : \mathcal{F}(u) < m + \varepsilon\}$. Then for $u, v \in S_\varepsilon$ we have

$$\|u - v\|^2 = 4\mathcal{F}(u) + 4\mathcal{F}(v) - 8\mathcal{F}\left(\frac{u+v}{2}\right) \leq 4(m + \varepsilon) + 4(m + \varepsilon) - 8m = 8\varepsilon.$$

Hence $\mathrm{diam}(S_\varepsilon) \leq \sqrt{8\varepsilon}$, and so by the completeness of H, the sets S_ε shrink to a single point $\{v\} = \bigcap_{\varepsilon>0} S_\varepsilon$, and v is a minimizer of $\mathcal{F}$. Consequently, for all $\varphi \in H$,

$$(v, \varphi) - f(\varphi) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(v + t\varphi) = 0 \quad \forall\varphi \in H.$$

□

[1] Minimize the function $t \mapsto \|\varphi - t\hat{n}\|^2$

2.3 TENSOR ALGEBRA AND DETERMINANTS

In the geometric analysis we will be doing, it will be important to measure k-dimensional volume inside n-dimensional space. While Hausdorff measure is up to this task in general, differential forms offer two important advantages. The first is, they allow us to measure volume in specific directions, via projection. The second is that they have an algebraic structure—they can be added, multiplied, pulled back via differentiable maps, differentiated, and integrated; and these operations have all the properties one would desire.

We make several uses of differential forms' dual geometric and algebraic nature, especially to define integration on a manifold. Indeed, a look ahead at Chapter 9 (or back at the introduction) will show that the algebra of differential forms is nothing less than the water in which we swim. This justifies the rather circuitous path we must take to define the operations of tensor algebra precisely. At the end of the day, it is not the definitions of these objects we will care about, but their rich algebraic properties. Once we have those in hand, we make little reference to the actual definition of a differential form.

Definition 2.3.1. Given a set S (finite or infinite), define the *free vector space over* S to be the set of all formal finite sums

$$\sum_i a_i s_i \quad a_i \in \mathbb{R}, \quad s_i \in S.$$

We define addition and scalar multiplication of elements in this set in the obvious way, and call the resulting vector space $\mathcal{F}S$.

Definition 2.3.2. Given two vector spaces V and W, the define the *tensor product* to be the vector space quotient

$$V \otimes W = \mathcal{F}(V \times W)/\mathcal{R}$$

where $\mathcal{R}$ is the subspace generated by elements of the form (where $v_i, v \in V, w_i, w \in W$)

1. $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$
2. $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$
3. $(rv, w) - r(v, w)$
4. $(v, rw) - r(v, w)$

If $v \in V$ and $w \in W$, define $v \otimes w$ to the projection of (v, w) in the quotient $V \otimes W$.

Definition 2.3.3. Given a vector space V, we define the space of k-tensors over V to be the k-fold product $V \otimes \ldots \otimes V$ (see property 3 in the theorem below, which shows that this product is well-defined). The *tensor algebra* over V is defined to be the direct sum

$$\mathcal{T}V = \sum_{k=0}^{\infty} T^k V.$$

The elements of $\mathcal{T}V$ are called tensors over V. Elements of $T^k V$ are called k-tensors over V.

This construction allows us to take products of vectors in a meaningful way. It can be viewed as the "best possible" such construction in the following precise sense:

Theorem 2.3.4. *Let V and W be* finite-dimensional *vector spaces (over $\mathbb{R}$). Denote by $\iota : V \times W \to V \otimes W$ the tensor product map $(v, w) \mapsto v \otimes w$. $V \otimes W$ and ι enjoy the following properties:*

1. *If $\varphi : V \times W \to X$ is any bilinear map from $V \times W$ into a vector space X (see figure 2.2a), then there exists a unique bilinear map $\Phi : V \otimes W \to X$ such that $\varphi = \Phi \circ \iota$ (see figure 2.2b).*
2. *$V \otimes W$ and ι are the unique vector space and bilinear map (up to natural isomorphism) with this property 1.*
3. *$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ via a natural isomorphism. In particular, the k-fold tensor product $V \otimes \ldots \otimes V$ is well-defined, independent of the bracketing of the factors.*
4. *If $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are bases for V and W, then $(v_i \otimes w_j)_{i,j}$ is a basis for $V \otimes W$. Consequently,* $\dim(V \otimes W)) = \dim(V)\dim(W)$*.*
5. *If we define*
$$\xi_1 \otimes \cdots \otimes \xi_k(v_1, \ldots, v_k) = \xi_1(v_1) \cdots \xi_k(v_k) \tag{2.6}$$
we obtain an isomorphism $T^k(V^*) \cong \mathcal{M}^k(V)$ between the space of k-tensors and the space of k-multilinear functionals on V.
6. *If we define*
$$\xi_1 \otimes \cdots \otimes \xi_k(v_1 \otimes \cdots \otimes v_k) = \xi_1(v_1) \cdots \xi_k(v_k) \tag{2.7}$$
we similarly obtain an isomorphism $T^k(V^*) \cong (T^k V)^*$.

$$
\begin{tikzcd}
 & V \otimes W \\
V \times W \arrow[ur, "\iota"] \arrow[r, "\varphi"] & X
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
 & V \otimes W \arrow[d, "\Phi"] \\
V \times W \arrow[ur, "\iota"] \arrow[r, "\varphi"] & X
\end{tikzcd}
$$

(b) The universal arrow

Figure 2.2: The universal property of tensor products

For the proofs, see [13, Chapter 10] and [35, Chapter 2].

$\mathcal{T}V$ is a vector space with *graded algebra* structure—-a ring with a grading structure such that if $u \in T^kV$ and $v \in T^\ell V$ then $u \otimes v \in T^{k+\ell}V$.

Definition 2.3.5. A tensor $v \in T^kV$ shall be called *simple* if it can be written as a product $v = v_1 \otimes \cdots \otimes v_k$ of tensors of degree 1.

It will be useful to apply this construction to the dual of V. And so we get the following generalization:

Definition 2.3.6. For nonnegative integers n, m we define the mixed tensor space

$$T^{n,m} = T^nV \otimes T^mV^*.$$

Remark 2.3.7. In fact, in some references, $\mathcal{T}V$ is taken to be the direct sum of all these mixed tensor spaces, but we will have no need for that usage.

Remark 2.3.8. There is fortunately no ambiguity in the notation T^mV^*, since the spaces $T^m(V^*)$ and $(T^mV)^*$ are naturally isomorphic via (2.7)

Definition 2.3.9. Given a vector space V, we define the *exterior algebra* over V to be the quotient ring

$$\bigwedge V = \mathcal{T}V/C$$

where C is the two-sided ideal in $\mathcal{T}V$ generated by elements of the form $v \otimes v$, for $v \in V$.

It turns out that C is a graded ideal. That is,

$$C = \sum_{k=0}^{\infty} C^k$$

where $C^k = T^k V \cap C$. Consequently, we can define $\bigwedge^k V = T^k V/C^k$ and we have

$$\bigwedge V = \sum_{k=0}^{\infty} \bigwedge^k V.$$

We use the $\wedge$ to denote multiplication in the quotient space $\bigwedge V$. That is, $v \wedge w$ is the image of $v \otimes w$ in $\bigwedge V$. Notice that $\wedge$ is *anti-symmetric* on V. Indeed, we can compute

$$\begin{aligned} 0 &= (v + w) \wedge (v + w) \\ &= v \wedge v + v \wedge w + w \wedge v + w \wedge w \\ &= v \wedge w + w \wedge v \end{aligned}$$

so $v \wedge w = -w \wedge v$. Elements of $\bigwedge V$ are called *forms* over V, and elements of $\bigwedge^k V$ are called k-forms over V.

Theorem 2.3.10. *Let V be an n-dimensional vector space, and let $\iota : V \times \cdots \times V \to \bigwedge^k V$ be the map $(v_1, \ldots, v_k) \mapsto v_1 \wedge \ldots \wedge v_k$. Then ι and $\bigwedge^k V$ enjoy the following properties:*

1. *If $\varphi : V \times \cdots \times V \to X$ is any* alternating *k-linear map from V into X, then there exists a unique linear map $\Phi : \bigwedge^k V \to X$ such that $\varphi = \Phi \circ \iota$.*
2. *$\bigwedge^k V$ and ι are uniquely characterized by property 1, up to isomorphism.*
3. *If $v_1, \ldots, v_n$ is a basis for V, then a basis for $\bigwedge^k V$ is given by $v_I = v_{i_1} \wedge \ldots \wedge v_{i_k}$ where I ranges over all subsets of $\{1, \ldots, n\}$ of size k. Consequently $\dim(\bigwedge^k V) = \binom{n}{k}$ for $0 \leq k \leq n$ and $\bigwedge^k V = \{0\}$ for $k > n$.*

4. *There is a natural isomorphism* $\bigwedge^k(V^*) \cong \operatorname{Alt}^k V$ *between the space of k-forms on* V^* *and the space of alternating k-linear functionals on* V. *The isomorphism is attained by defining*

$$\xi_1 \wedge \ldots \wedge \xi_k(v_1, \ldots, v_k) = \det \xi_i(v_j) \quad \xi_i \in V^*, v_j \in V. \tag{2.8}$$

5. *Similarly,* $\bigwedge^k(V^*)$ *and* $(\bigwedge^k V)^*$ *are isomorphic via*

$$\xi_1 \wedge \ldots \wedge \xi_k(v_1 \wedge \ldots \wedge v_k) = \det \xi_i(v_j) \quad \xi_i \in V^*, v_j \in V. \tag{2.9}$$

(2.8) shows that the k-forms over V^* can be identified with the alternating k-linear maps on V. Many authors take this as the definition of $\bigwedge^k V$. We have chosen our path to emphasize the algebraic structure of $\bigwedge V$.

Definition 2.3.11. Let U and V be finite dimensional vector spaces, $L \in \mathcal{L}(U, V)$ a linear map between then. We define the *pullback* operator $L^* : \mathcal{T} V^* \to \mathcal{T} U^*$ by the formula

$$L^*\xi(u_1, \ldots, u_k) = \xi(L(u_1), \ldots, L(u_k)) \quad \text{whenever } u_1, \ldots, u_k \in U.$$

for k-forms $\xi \in T^k V^*$, where we have used the identification (2.6) to identify tensors with multilinear maps. Notice that this definition coincides with the definition of the adjoint of L for one-forms. Similarly, we define the pullback operator $L^* : \bigwedge V^* \to \bigwedge U^*$ by precisely the same formula when $\xi \in \bigwedge^k V^*$, where we identify ξ with a map in $\operatorname{Alt}^k V$ via Theorem 2.3.10.

A fundamental interface—where the rubber meets the road—between the abstract notion of a differential form, and the geometric applications that we will be considering, is the following:

Lemma 2.3.12. *A linear map* $L : U \to V$ *between finite dimensional vector spaces has* $\operatorname{rank}(L) < k$ *if and only if* $L^*\xi = 0$ *for all* $\xi \in \bigwedge^k V^*$.

Proof. We can choose basis vectors $e_1, \ldots, e_n$ for U and $f_1, \ldots, f_m$ for V, and represent L as an m by n matrix. Let $e^1, \ldots, e^n$ be a dual basis for U^* and $f^1, \ldots, f^n$ be a dual basis for V^*. Recall that $\operatorname{rank}(L) < k$ if and only if all k by k sub-determinants of its matrix representation are zero. But each such sub-determinant is simply

$$L^*(f^{i_1} \wedge \ldots \wedge f^{i_k})(e_{j_1} \wedge \ldots \wedge e_{j_k}) = 0$$

for some index sets $I = (i_1, \ldots, i_k) \subseteq \{1, \ldots, m\}$ and $J = (j_1, \ldots, j_k) \subseteq \{1, \ldots, i\}$, and so the lemma follows. □

3.0 FUNCTION SPACES

We need the tools in Chapter 2 because function spaces are infinite-dimensional. Each section of this chapter introduces (at least) one new function space which will turn out to be a Banach space. Fundamental results like the Hodge Decomposition Theorem follow whenever we can apply the Riesz Representation Theorem 2.2.11 to a linear functional ℓ on a function space H which happens to be a Hilbert space.

3.1 CONTINUOUS FUNCTION SPACES

Definition 3.1.1. Given a metric space X we define $C(X)$ to be the vector space of bounded continuous real-valued functions on X. We make $C(X)$ into a normed linear space by defining

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Recall that when X is compact, continuous functions $f \in C(X)$ are *uniformly continuous*. It will sometimes be useful to be quantitative about this statement. Given $f \in C(X)$ and a non-decreasing continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, we say that φ is a *modulus of continuity* for f if there holds

$$\sup_{d_X(x,y)<t} |f(x) - f(y)| \leq \varphi(t) \quad \forall t > 0.$$

Proposition 3.1.2. *$C(X)$ is a Banach space.*

Definition 3.1.3. We say that a family of functions $\mathcal{F} \subseteq C(X)$ is *equicontinuous* if there is a *common* modulus of continuity φ for all functions $f \in \mathcal{F}$.

The main tool for dealing with the continuous function spaces is the following theorem:

Theorem 3.1.4 (Arzela-Ascoli)**.** *Let X be a compact metric space. Then a family $\mathcal{F}$ of functions in $C(X)$ is compact if and only if it is closed, uniformly bounded, and equicontinuous.*

Proof. First let us suppose $\mathcal{F}$ is closed, uniformly bounded and equicontinuous. Select a sequence (f_n) from $\mathcal{F}$. We will construct a convergent subsequence. The idea is to construct a subsequence f_n^1 of f_n of diameter $1/2$ in $C(X)$, then a further subsequence f_n^2 of diameter $1/4$, etc, and then from this list of subsequences, the diagonal sequence f_n^n will be Cauchy in $C(X)$. Since $C(X)$ is complete, this subsequence converges to a limit in $\mathcal{F}$ (since $\mathcal{F}$ is closed).

Thus, to carry out this argument, it suffices to prove that for any uniformly bounded equicontinuous sequence f_n there is a subsequence of g_n of diameter less than ε for any $\varepsilon > 0$. Let $\varphi(t)$ be the modulus of equicontinuity of the sequence f_n. Let $\delta > 0$ be small enough that $\varphi(\delta) < \varepsilon/3$. Thus for all n and all $x, y \in X$ with $d_X(x, y) < \delta$ we have $|f_n(x) - f_n(y)| < \varepsilon/3$. Now since X is a compact metric space, we are able to find a *finite* set $x_1, \ldots, x_N$ of points which are δ-dense in X. That is, for all $z \in X$, $d_X(z - x_i) < \delta$ for some $i \in \{1, \ldots, N, \}$.

Since f_n is uniformly bounded, the sequences $(f_n(x_i))_{n=1}^{\infty}$ are bounded for each i. Hence, they each have convergent subsequences, and so we can take a further subsequence n_j such that $(f_{n_j}(x_i))_{j=1}^{\infty}$ converges for *each* $i \in \{1, \ldots, N\}$. So for sufficiently large $j, k \geq J_0$, we have $|f_{n_j}(x_i) - f_{n_k}(x_i)| < \varepsilon/3$ for all i. Now, for any $x \in X$ and $j, k > J_0$ we can find x_i such that $|x - x_i| < \delta$, and estimate

$$|f_{n_j}(x) - f_{n_k}(x)| \leq |f_{n_j}(x) - f_{n_j}(x_i)| + |f_{n_j}(x_i) - f_{n_k}(x_i)| + |f_{n_k}(x_i) - f_{n_k}(x)| < \varepsilon$$

and so $\|f_{n_j} - f_{n_k}\|_{C(X)} < \varepsilon$. We have thus constructed a subsequence of f_n of diameter less than ε in $C(X)$, which completes the argument.

The reader may consult, for example, [29]. We never use the converse in this thesis. □

Definition 3.1.5. Given an open set $\Omega \subseteq \mathbb{R}^n$ and $1 \leq k \leq \infty$ any natural number or infinity, we define $C^k(\Omega)$ to be the space of functions $f : \Omega \to \mathbb{R}$ for which all derivatives up to order k exist and are continuous. Naturally, $C^\infty(\Omega)$ functions have derivatives of all orders. We define $C^k(\overline{\Omega})$ to be the space of functions which are the restrictions to Ω of $C^k(\mathbb{R}^n)$ functions. For finite i, $1 \leq i \leq k$ and $U \Subset \Omega$, we define the semi-norms

$$[f]_{C^k(U),i} = \sup_{x \in U} \sum_{|\alpha|=i} |D^\alpha f(x)|.$$

These semi-norms define a topology for $C^k(\Omega)$. For the more restrictive space $C^k(\bar{\Omega})$, we actually have a norm

$$[f]_{C^k(\bar{\Omega})} = \sup_{x \in \Omega} \sum_{|\alpha|=k} |D^\alpha f(x)|,$$

$$\|f\|_{C^k(\bar{\Omega})} = [f]_{C^k(\bar{\Omega})} + \sum_{i=0}^{k-1} [f]_{C^i(\bar{\Omega})}.$$

Details of this construction of topology for $C^k(\Omega)$ from semi-norms can be found in [30]. As for the spaces $C^k(\bar{\Omega})$ we have the following classic theorem, proven for example in [28].

Theorem 3.1.6. *For $\Omega \Subset \mathbb{R}^n$, $C^k(\bar{\Omega})$ is a Banach Space.*

Definition 3.1.7. For $0 < \gamma \leq 1$ and a *metric space* X, we define the Hölder spaces $C^{0,\gamma}(X)$ to be the subspace of $C(X)$ for which the *Hölder semi-norm*

$$[f]_{C^{0,\gamma}(X)} = \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

is finite. We then define the Hölder norm

$$\|f\|_{C^{0,\gamma}(X)} = \|f\|_\infty + [f]_{C^{0,\gamma}(X)}.$$

Remark 3.1.8. Notice that, as we would want from this notation, $C^1(\Omega) \subseteq C^{0,\gamma'}(\Omega) \subseteq C^{0,\gamma}(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$ and $\gamma < \gamma' < 1$. Notice also $C^{0,\gamma}(\Omega)$ functions are uniformly continuous, so they extend to $\bar{\Omega}$. Consequently there is no need to define $C^{0,\gamma}(\bar{\Omega})$ separately.

Definition 3.1.9. For $\Omega \subseteq \mathbb{R}^n$, $0 \leq k < \infty$ we define $C^{k,\gamma}(\Omega)$ to be the subspace of $C^k(\Omega)$ for which the derivatives $D^\alpha f$ of order $|\alpha| = k$ are themselves $C^{0,\gamma}(\Omega)$.

Theorem 3.1.10. *For $\Omega \Subset \mathbb{R}^n$, $k \in \mathbb{N}$, $0 < \gamma < 1$ the spaces $C^{k,\gamma}(\Omega)$ are Banach spaces.*

Proposition 3.1.11. *For a compact metric space X and $0 < \gamma' < \gamma \le 1$, we have the inclusion $C^{0,\gamma}(X) \subseteq C^{0,\gamma'}(X)$, and in fact this is a compact embedding (see Definition 2.1.9). Moreover if f_n is a sequence in $C^{0,\gamma}(X)$ with $f_n \to f$ uniformly, then the limit f is in $C^{0,\gamma}(X)$ with $\|f\|_{C^{0,\gamma}(X)} \le \sup_n \|f_n\|_{C^{0,\gamma}(X)}$*

Proof. Observe, for $h \in C^{0,\gamma}(X)$,

$$\begin{aligned}\frac{|h(x)-h(y)|}{|x-y|^{\gamma'}} &= \left(\frac{|h(x)-h(y)|^{\gamma/\gamma'}}{|x-y|^{\gamma}}\right)^{\gamma'/\gamma} = \left(\frac{|h(x)-h(y)|}{|x-y|^{\gamma}}|h(x)-h(y)|^{\frac{\gamma}{\gamma'}-1}\right)^{\gamma'/\gamma} \\ &= \left(\frac{|h(x)-h(y)|}{|x-y|^{\gamma}}\right)^{\gamma'/\gamma} |h(x)-h(y)|^{1-\frac{\gamma'}{\gamma}} \lesssim [h]_{C^{0,\gamma}(X)}^{\gamma'/\gamma} \|h\|_{\infty}^{1-\frac{\gamma'}{\gamma}}. \end{aligned} \tag{3.1}$$

Now Suppose f_n is a bounded in $C^{0,\gamma}$-norm, say, $\sup \|f_n\|_{C^{0,\gamma}} = M$. In particular, f_n is uniformly bounded (by M) and equicontinuous (with modulus of equicontinuity $\varphi(t) = Mt^{\gamma}$). So, by the Arzela-Ascoli Theorem 3.1.4 a subsequence, which we will again call f_n by abuse of notation, converges *in ∞-norm* to a limit $g \in C(X)$.

Select $\varepsilon > 0$. For sufficiently large n, m we have $\|f_n - f_m\|_\infty < \varepsilon$. Thus, (3.1) implies

$$[f_n - f_m]_{C^{0,\gamma'}(X)} \le 2M^{\gamma'/\gamma}\varepsilon^{1-\frac{\gamma'}{\gamma}}.$$

Thus f_n is a Cauchy sequence in $C^{0,\gamma'}$. This proves simultaneously that $g \in C^{0,\gamma'}$ and that $f_n \to g$ in $C^{0,\gamma'}$-norm.

The proof of the second claim is even simpler: it follows from the observation that, for $x, y \in X$ and $n \in \mathbb{N}$ we have

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \le [f_n]_{C^{0,\gamma}}|x-y|^{\gamma} + 2\|f_n - f\|_\infty$$

□

Corollary 3.1.12. *Suppose $0 < \gamma' < \gamma \le 1$, $f \in C^{0,\gamma}$, and suppose $f_n \in C^{0,\gamma}$ is a sequence which is converging to f* uniformly, *and that $[f_n]_{C^{0,\gamma}}$ is* bounded. *Then $f_n \to f$ in $C^{0,\gamma'}$.*

Exercise 3.1.13. Fix $0 < \gamma' < \gamma < 1$. Verify that the function $f : [-1,1] \ni t \mapsto |t|^{\gamma}$ belongs to $C^{0,\gamma}([-1,1])$. Verify that f can be uniformly approximated by smooth functions f_t in $C^{0,\gamma'}([-1,1])$ for any $\gamma' < \gamma$, but we can never have $f_t \to f$ in $C^{0,\gamma}([-1,1])$.

3.2 LEBESGUE SPACES

We define Lebesgue spaces L^p and then state and prove the Hölder, Minkowski, and Jensen inequalities.

Definition 3.2.1. Let (X, μ) be a measure space. For $1 \leq p < \infty$ we define the Lebesgue space $L^p(X)$ to be the set of μ-measurable functions *up to almost-everywhere equivalence* $f : X \to \mathbb{R}$ with the property that $\int_X |f|^p \, d\mu < \infty$. We endow $L^p(X)$ with the norm $\|\cdot\|_{L^p(X)}$ given by

$$\|f\|_{L^p(X)} = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Definition 3.2.2. We define $L^\infty(X)$ to be the set of (almost-everywhere equivalence classes of) *essentially bounded* functions on X, that is, the set of functions with finite L^∞-norm

$$\|f\|_{L^\infty} = \inf_{\mu(A)=0} \sup_{x \in X \setminus A} |f(x)|$$

where the infimum is taken over all subsets of X of measure zero.

Theorem 3.2.3 (Minkowski Inequality)**.** *For $1 \leq p \leq \infty$, $\|\cdot\|_{L^p(X)}$ is indeed a norm. In particular,*

$$\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}.$$

Proof. First assume $1 \leq p < \infty$. It is a standard fact from measure theory that $\int_X |h| \, d\mu = 0$ if and only if $h = 0$ μ-almost everywhere. Thus $\|f\| \geq 0$ with equality if and only if $f = 0$ almost everywhere. We also clearly have $\|cf\| = |c|\|f\|$ for $c \in \mathbb{R}$. So we are only left to prove triangle inequality. Let $f, g \in L^p(X)$ and we can write $f = \|f\|\hat{f}$ and $g = \|g\|\hat{g}$ where $\|\hat{f}\| = \|\hat{g}\| = 1$. Now observe, by *convexity* of the function $t \mapsto |t|^p$:

$$\begin{aligned}
\|f + g\| &= \left(\int_X |f + g|^p \, d\mu \right)^{1/p} \\
&= (\|f\| + \|g\|) \left(\int_X \left| \frac{\|f\|}{\|f\| + \|g\|} \hat{f} + \frac{\|g\|}{\|g\| + \|g\|} \hat{g} \right|^p \, d\mu \right)^{1/p} \\
&\leq (\|f\| + \|g\|) \left(\int_X \frac{\|f\|}{\|f\| + \|g\|} |\hat{f}|^p + \frac{\|g\|}{\|f\| + \|g\|} |\hat{g}|^p \, d\mu \right)^{1/p} \\
&= \|f\| + \|g\|.
\end{aligned}$$

The case $p = \infty$ is left as an exercise below. $\square$

Exercise 3.2.4. $\|f\|_{L^\infty}$ is the smallest number m such that there exists an *almost-everywhere representative* $\tilde{f}$ of f such that $\sup \tilde{f} = m$.

Exercise 3.2.5. Minkowski's Inequality $\|f + g\|_{L^\infty(X)} \leq \|f\|_{L^\infty(X)} + \|g\|_{L^\infty(X)}$ holds on $L^\infty(X)$ and $L^\infty(X)$ is a Banach space.

Exercise 3.2.6. If $\mu(X) < \infty$ and $f \in L^\infty(X)$ then

$$\|f\|_{L^p(X)} \xrightarrow{p\to\infty} \|f\|_{L^\infty(X)}.$$

Theorem 3.2.7 (Hölder Inequality)**.** *Let* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(X)$, $g \in L^q(X)$. *Then*

$$\int_X fg\, d\mu \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Proof. The proof for $p = \infty$, $q = 1$ is easier, and is left to the reader. Assume $1 < p, q < \infty$. Recall Young's Inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \forall a, b > 0$$

(which is obtained by taking the logarithm of the right side and applying concavity). Then we use this as follows: first let $f = \|f\|_{L^p}\hat{f}$ and $g = \|g\|_{L^q}\hat{g}$. Then

$$\begin{aligned}
\int_X fg\, d\mu &= \|f\|_{L^p}\|g\|_{L^q}\int_X \hat{f}\hat{g}\, d\mu \\
&\leq \|f\|_{L^p}\|g\|_{L^q}\int_X \frac{|\hat{f}|^p}{p} + \frac{|\hat{g}|^q}{q}\, d\mu \\
&= \|f\|_{L^p}\|g\|_{L^q}\left(\frac{1}{p} + \frac{1}{q}\right) \\
&= \|f\|_{L^p}\|g\|_{L^q}.
\end{aligned}$$

□

Exercise 3.2.8. Complete the proof of Hölder's Inequality for $p = 1$, $q = \infty$.

We have seen that $L^p(X)$ is a normed linear space. Now we show that it is actually a Banach space.

Theorem 3.2.9 (Riesz-Fischer)**.** $L^p(X)$ *is complete.*

Proof. Using Proposition 2.1.3, it suffices to check that absolutely summable sequences in $L^p(X)$ are summable in $L^p(X)$. So take an absolutely summable sequence $f_i \in L^p(X)$, that is, with the property that $\sum_i \|f_i\|_{L^p} < \infty$. Consider the partial sums $g_k(x) = \sum_{i=1}^k |f_i(x)|$ and notice that, by the Minkowski Inequality, $\|g_k\|_{L^p}$ remain uniformly bounded, with

$$\|g_k\|_{L^p} \leq \sum_{i=1}^{\infty} \|f_i\|_{L^p}.$$

Apply the Monotone Convergence Theorem (see [31], [29]) to the increasing sequence of functions $(g_k^p)_{k=1}^{\infty}$ to find that g_k^p converges pointwise almost-everywhere to a limit g^p with $\int_X g_k^p \, d\mu \xrightarrow{k\to\infty} \int_X g^p \, d\mu$, and so $g \in L^p$ with $\|g_k\|_{L^p} \to \|g\|_{L^p} \leq \sum_{i=1}^{\infty} \|f_i\|_{L^p}$.

Now consider the partial sums $h_k(x) = \sum_{i=1}^k f_i(x)$. Notice this sum converges *pointwise* almost-everywhere as $k \to \infty$, since it converges absolutely, to a limit $h(x)$. Moreover, $|h_k(x)|^p \leq g_k(x)^p$. Now the Dominated Convergence Theorem implies that the limit $h \in L^p(X)$ and $h_k \to h$ in L^p. $\square$

Definition 3.2.10. We say that $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ if the restriction of f to any $U \Subset \mathbb{R}^n$ is in $L^p(U)$. Given a sequence $f_n \in L^p_{\text{loc}}(\mathbb{R}^n)$, we say that $f_n \to f$ in L^p_{loc} if $\|f_n - f\|_{L^p(U)} \to 0$ for all $U \Subset \mathbb{R}^n$.

Remark 3.2.11. L^p_{loc} is a vector space. Though we will not have any need to make any explicit reference to a metric on this space, it is metrizable with the metric

$$d_{L^p_{\text{loc}}}(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f - g\|_{L^p(U_i)}}{1 + \|f - g\|_{L^p(U_i)}}$$

where U_i is any (fixed) *compact exhaustion* of $\mathbb{R}^n$. This is of course not a norm.

Theorem 3.2.12 (Jensen's Inequality)**.** *Let (X, μ) be a measure space with $\mu X = 1$, $f : X \to \mathbb{R}$ an integrable function, and $\varphi : I \to \mathbb{R} \cup \{\infty\}$ a convex function on an interval I containing $f(X)$. Then Jensen's Inequality holds:*

$$\varphi\left(\int_X f \, d\mu\right) \leq \int_X \varphi \circ f \, d\mu. \tag{3.2}$$

Proof. Let $\bar{f} = \int_X f \, d\mu$. Since φ is convex, we can find an affine linear function $L : \mathbb{R} \to \mathbb{R}$ such that $L(\bar{f}) = \varphi(\bar{f})$ and $L(y) \leq \varphi(y)$ for all $y \in \mathbb{R}$. Take a sequence of simple functions $\psi_k = \sum_{i=1}^{N_k} a_i^k \chi_{A_i^k}$ on X with $\psi_k \to f$ in $L^1(X)$ and we have

$$\varphi(\bar{f}) = L(\bar{f}) \xleftarrow{\infty \leftarrow k} L(\bar{\psi}_k) = L\left(\int_X \psi_k \, d\mu\right) = L\left(\sum_{i=1}^{N_k} a_i^k \mu(A_i^k)\right)$$

$$= \sum_{i=1}^{N_k} L(a_i^k)\mu(A_i^k) = \int_X L \circ \psi_k \, d\mu \xrightarrow{k\to\infty} \int_X L \circ f \, d\mu \leq \int_X \varphi \circ f \, d\mu.$$

□

Remark 3.2.13. The proof can be viewed as verifying Jensen's inequality for measure spaces of *finite* cardinality, and passing to a limit using simple functions. In fact, let us list two common uses as corollaries, the second of which is an application of this observation.

Corollary 3.2.14. *Let* $1 \leq q < p$, f *a measurable function on the measure space* (X, μ)*. Then*

$$\|f\|_{L^q} \leq (\mu X)^{\frac{1}{q} - \frac{1}{p}} \|f\|_{L^p}. \tag{3.3}$$

Proof. Let $\hat{\mu} = \mu/\mu(X)$ be the noramalized measure so that we can apply Jensen's Inequality. We will use the convex function $\varphi(y) = y^{p/q}$. Then

$$\left(\int_X |f|^q \, d\mu\right)^{p/q} = \left(\mu(X) \int_X |f|^q \, d\hat{\mu}\right)^{p/q} \leq \mu(X)^{p/q} \int_X |f|^p \, d\hat{\mu} = \mu(X)^{p/q-1} \int_X |f|^p \, d\mu.$$

Raise both sides to the $1/p$ to recover the corollary. □

Corollary 3.2.15. *For a finite sequence* $\{x_1, \ldots x_N\}$ *of reals, we have*

$$\left|\sum_{i=1}^N x_i\right|^p \leq N^{p-1} \sum_{i=1}^N |x_i|^p.$$

Proof. Apply the previous corollary with $X = \{1, \ldots, N\}$, μ the counting measure, and $q = 1$. □

Remark 3.2.16. These two corollaries are so standard in analytic estimates that they are often used without any further comment than the $\leq$ sign. Corollary 3.2.14 is also often referred to as Hölder's Inequality because it can be proven by applying Hölder's Inequality to the functions $|f|^q$ and $g = 1$. Details are left to the reader.

Of fundamental importance in our work will be the ability to approximate L^p_{loc} functions with smooth functions. For this we will need one tool from measure theory which we will state without proof (see [31]).

Theorem 3.2.17 (Lusin). *Let f a bounded measurable function on $\mathbb{R}^n$ (that is, $f \in L^\infty(\mathbb{R}^n)$) and $\varepsilon > 0$ be given. Then there exists a continuous function g on $\mathbb{R}^n$ with the property that $g = f$ except on a set of measure less than ε, and also $\|g\|_{L^\infty} \leq \|f\|_{L^\infty}$. Moreover, if f has support inside some open set $\Omega \subseteq \mathbb{R}^n$, then g can be taken to have support inside Ω as well.*

Using this we prove

Lemma 3.2.18. *Compactly supported continuous functions are dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. More precisely, for $f \in L^p(\mathbb{R}^n)$, and $\varepsilon > 0$, we can find a continuous, compactly supported function g such that $\|g - f\|_{L^p} < \varepsilon$.*

Proof. Since we can write $f = f^+ - f^-$ and approximate f^+ and f^- separately, we may assume f is non-negative. Define $f_n(x) = \chi_{B_n}(x) \max\{f(x), n\}$ so that we cut off f in its support and its size. By Monotone Convergence Theorem, $f_n \to f$ in L^p. So fix some n_0 such that $\|f_{n_0} - f\|_{L^p} < \varepsilon/2$. Next, since $f_{n_0} \in L^\infty$ with $\|f_{n_0}\|_{L^\infty} \leq n_0 < \infty$, use Lusin's Theorem 3.2.17 to find a continuous function g with $g = f_{n_0}$ except on a set E of measure $mE < (\varepsilon/2n_0)^p$. And g can be taken with compact support since f_n has compact support. Now clearly $\|g - f_{n_0}\|_{L^p} < \varepsilon/2$, and it is also evident that since f_{n_0} has compact support, g can be chosen also with compact support. And so by Minkowski's Inequality 3.2.3, we have

$$\|g - f\|_{L^p} \leq \|g - f_{n_0}\|_{L^p} + \|f_{n_0} - f\|_{L^p} < \varepsilon.$$

□

Remark 3.2.19. Note where in the proof we use the fact that $p < \infty$. The lemma is false when $p = \infty$.

Definition 3.2.20. Let $1 \leq p < \infty$ and $f \in L^p_{\text{loc}}(\mathbb{R}^n)$. For $t > 0$ We define the *standard mollification* of f to be the function

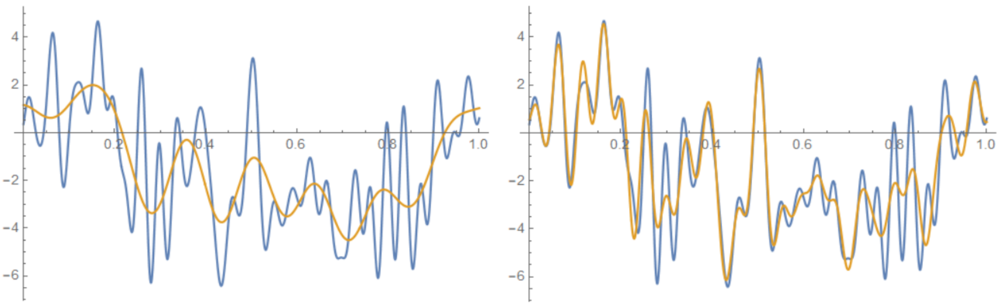
$$f_t(x) = \int_{\mathbb{R}^n} \eta_t(x - y) f(y)\, dy$$

where $\eta(z)$ is the smooth compactly supported function

$$\eta(z) = \begin{cases} 0 & |z| \geq 1 \\ c \exp\left(\frac{1}{|z|^2 - 1}\right) & |z| < 1 \end{cases}$$

and where $c > 0$ is chosen so that $\int_{\mathbb{R}^n} \eta = 1$; and we further define $\eta_t(z) = \frac{1}{t^n}\eta(\frac{z}{t})$.

Note that $\int_{\mathbb{R}^n} \eta_t = 1$ also, for all $t > 0$.



(a) Blue function with an orange mollification for large t

(b) The same function with mollification for smaller t. Approximation gets closer.

Figure 3.1: *Mollify* (v) : *to soothe in temper or disposition* (Merriam Webster)

It is clear that f_t is smooth if, say, f is integrable. Indeed, we have an explicit formula for its derivatives,

$$\frac{\partial f_t}{\partial x_i} = t^{-n-1} \int_{\mathbb{R}^n} [f(x-z) - f(x)] \frac{\partial \eta}{\partial x_i}(t^{-1}z)\, dz. \tag{3.4}$$

Indeed, this formual follows from the fact that $\int_{\mathbb{R}^n} \frac{\partial \eta}{\partial x_i}(t^{-1}z)\, dz = 0$.

Theorem 3.2.21 (Smooth Approximation)**.**

1. *If f is continuous on $\mathbb{R}^n$ then the standard mollification f_t converges to f uniformly on compact $K \Subset \mathbb{R}^n$ (hence in L^p_{loc} for $1 \le p < \infty$).*
2. *If f is uniformly continuous with modulus of continuity φ, then φ is also a modulus of continuity for f_t.*
3. *If $f \in L^p(\mathbb{R}^n)$, then f_t converges to f in $L^p(\mathbb{R}^n)$.*

Proof. [1] We prove the items in order. So we assue that f is continuous. On compact U, f is uniformly continuous with modulus of continuity φ. So, for $x \in U$, we compute using the fact that the support of η_t is B_t, and the fact that $\int_{\mathbb{R}^n} \eta_t = 1$ for all $t > 0$,

$$|f_t(x) - f(x)| = \left| \int_{\mathbb{R}^n} \eta_t(x-y) f(y)\, dy - f(x) \right| = \left| \int_{|x-y| \le t} \eta_t(x-y)(f(y) - f(x))\, dy \right|$$

[1] Actually, the captions of those images were lies. Mollifications are very hard to calculate, so the blue function is simply a Fourier series with 200 terms, and the so-called mollifications are merely truncations of that series. The result is similar, and indeed Fourier series could be used to easily prove density of C^∞ in, say $L^2(0,1)$.

$$
\begin{aligned}
&\leq \int_{|x-y|<t} \eta_t(x-y)|f(y)-f(x)|\,dy \leq \int_{|x-y|<t} \eta_t(x-y)\varphi(t)\,dy \\
&= \varphi(t) \to 0 \quad \text{as } t \to 0.
\end{aligned}
$$

Now we prove the second statement. Suppose f is uniformly continuous with modulus of continuity φ. Then we can compute

$$
\begin{aligned}
|f_t(y)-f_t(x)| &\leq \left|\int_{\mathbb{R}^n} \eta_t(y-z)f(z)dz - \int_{\mathbb{R}^n} \eta_t(x-z)f(z)\,dz\right| = \left|\int_{\mathbb{R}^n} \eta_t(x-z)(f(z+y-x)-f(z))\,dz\right| \\
&\leq \int_{\mathbb{R}^n} \eta_t(x-z)\varphi(|y-x|)\,dz = \varphi(|y-x|).
\end{aligned}
$$

Now we prove (3). Using Lemma 3.2.18 we find compactly supported g with $\|f-g\|_{L^p} < \varepsilon/3$. For convenience let $h = f-g$. Then, since mollification is linear, hence $f_t = g_t + h_t$, we have

$$
\|f_t - f\|_{L^p} = \|g_t - g + h_t - h\|_{L^p} \leq \|g_t - g\|_{L^p} + \|h_t - h\|_{L^p} \leq \|g_t - g\|_{L^p} + \|h\|_{L^p} + \|h_t\|_{L^p}. \tag{3.5}
$$

we have already proven for (1) that the first term approaches zero (so can be made less than $\varepsilon/3$), and we also have $\|h\|_{L^p} < \varepsilon/3$. As for the last term,

$$
\begin{aligned}
\|h_t\|_{L^p}^p &= \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \eta_t(x-y)h(y)\,dy\right|^p\,dx \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \eta_t(x-y)|h(y)|^p\,dy\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \eta_t(x)|h(y-x)|^p\,dx\,dy \\
&= \int_{\mathbb{R}^n} \eta_t(x) \int_{\mathbb{R}^n} |h(y-x)|^p\,dy\,dx = \|h\|_{L^p}^p < \varepsilon/3.
\end{aligned}
$$

In the estimate on the second line, we used Jensen's Inequality 3.2.12 with measure $d\mu(y) = \eta_t(x-y)\,dy$ and convex function $\varphi(s) = |s|^p$, since indeed we have $\mu(\mathbb{R}^n) = 1$. In the following line we used the Fubini Theorem. Thus we have estimated the third term on the right hand side of (3.5), and the Theorem is proven. □

Corollary 3.2.22. *Let $0 < \gamma' < \gamma \leq 1$ and $f \in C^{0,\gamma}(\mathbb{R}^n)$. Then the mollifications f_t converge to f in $C^{0,\gamma'}(\mathbb{R}^n)$, remain bounded in $C^{0,\gamma}(\mathbb{R}^n)$, and converge to f uniformly.*

Proof. Indeed, by Part 1 of Theorem 3.2.21, $f_t \to f$ uniformly, and by Part 2, $[f_t]_{C^{0,\gamma}(\mathbb{R}^n)}$ is bounded. So Corollary 3.1.12 implies $f_t \to f$ in $C^{0,\gamma'}(\mathbb{R}^n)$. □

3.3 CAMPANATO SPACES

Armed with the Jensen Inequality, we are now able to prove a useful characterization of the Hölder Spaces introduced in section 3.1.

Definition 3.3.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with the property that, for some constant $A > 0$

$$|B_R(x) \cap \Omega| \geq AR^n \quad \forall x \in \Omega, R, 0 < R < \operatorname{diam} \Omega. \tag{3.6}$$

For $1 \leq p < \infty$ and $0 < \lambda$, we define the *Campanato* semi-norm for $f \in L^p(\Omega)$,

$$[f]^p_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup_{\substack{x \in \Omega \\ R>0}} \frac{1}{R^\lambda} \int_{B_R(x) \cap \Omega} |f(y) - f(x)|^p \, dy. \tag{3.7}$$

We define the *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ to be the space of all $f \in L^p(\Omega)$ with $[f]_{\mathcal{L}^{p,\lambda}(\Omega)}$ finite, and define the *Campanato norm*

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = [f]_{\mathcal{L}^{p,\lambda}(\Omega)} + \|f\|_{L^p(\Omega)}. \tag{3.8}$$

Theorem 3.3.2 (Campanato)**.** *Let Ω be as in Definition 3.3.1; suppose $n < \lambda < n + p$, and $\gamma = \frac{\lambda - n}{p}$. Then $C^{0,\gamma}(\Omega) = \mathcal{L}^{p,\lambda}(\Omega)$, and for f belonging to these spaces,*

$$[f]_{C^{0,\gamma}} \approx [f]_{\mathcal{L}^{p,\lambda}(\Omega)}$$
$$\|f\|_{C^{0,\gamma}} \approx \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}.$$

More precisely, if $u \in \mathcal{L}^{p,\lambda}(\Omega)$, then u is equal almost everywhere *to a Hölder continuous function on Ω, which we again call u.*

Proof. Let's first check the simpler inclusion $C^{0,\gamma} \subseteq \mathcal{L}^{p,\lambda}$. Let $f \in C^{0,\gamma}(\Omega)$. Take $x \in \Omega$ and $R > 0$ and estimate

$$\begin{aligned}\frac{1}{R^\lambda} \int_{B_R(x) \cap \Omega} |f(y) - f(x)|^p \, dy &\leq \frac{[f]_{C^{0,\gamma}}}{R^\lambda} \int_{B_R(x) \cap \Omega} |y - x|^{\gamma p} \, dy \leq \frac{[f]_{C^{0,\gamma}}}{R^\lambda} \int_{B_R(x)} |y - x|^{\gamma p} \, dy \\ &= \frac{[f]_{C^{0,\gamma}}}{R^\lambda} \int_{B_R(x)} |y - x|^{\lambda - n} \, dy \approx [f]_{C^{0,\gamma}}\end{aligned}$$

where we have integrated in spherical coordinates. This establishes the inequality $[f]_{\mathcal{L}^{p,\lambda}(\Omega)} \lesssim \|f\|_{C^{0,\gamma}}$. Since we assume Ω bounded, we have also $\|f\|_{L^p(\Omega)} \lesssim \|f\|_\infty$ so we also have the inequality for norms.

Now we consider the reverse inequality. Let $u \in \mathcal{L}^{p,\lambda}(\Omega)$. The first step is to show that the average-value functions $x \mapsto u_{x,r}$ converge uniformly to $u(x)$ as $r \to 0$. To establish this, let $0 < r < R$ and estimate, for $y \in \Omega$,

$$|u_{x,R} - u_{x,r}|^p \lesssim |u_{x,R} - u(y)|^p + |u_{x_0,r} - u(y)|^p.$$

Integrate over $y \in \Omega(x,r) := B(x,r) \cap \Omega$ using (3.6)

$$\begin{aligned}
|u_{x,R} - u_{x,r}|^p A r^n &\leq |u_{x,R} - u_{x,r}|^p |B(x,r) \cap \Omega| \\
&= \int_{B(x,r)\cap\Omega} |u_{x,R} - u_{x,r}|^p \, dy \\
&\lesssim \int_{\Omega(x,r)} |u(y) - u_{x,R}|^p \, dy + \int_{\Omega(x,r)} |u(y) - u_{x_0,r}|^p \, dy \\
&\lesssim \int_{\Omega(x,R)} |u(y) - u_{x_0,R}|^p \, dy + \int_{\Omega(x_0,r)} |u(y) - u_{x_0,R}|^p \\
&\leq (R^\lambda + r^\lambda)[u]^p_{\mathcal{L}^{p,\lambda}(\Omega)} \lesssim R^\lambda [u]^p_{\mathcal{L}^{p,\lambda}(\Omega)}.
\end{aligned}$$

Divide by Ar^n and raise to the $1/p$ to obtain

$$|u_{x,R} - u_{x,r}| \lesssim \frac{R^{\lambda/p}}{r^{n/p}} [u]_{\mathcal{L}^{p,\lambda}(\Omega)}. \tag{3.9}$$

Fix $R_0 > 0$. For convenience denote $R_h = \frac{R_0}{2^h}$. Let $R = R_h$ and $r = R_{h+1}$ in (3.9) to obtain

$$|u_{x,R_h} - u_{x,R_{h+1}}| \lesssim \frac{1}{2^{h\gamma}} [u]_{\mathcal{L}^{p,\lambda}(\Omega)}.$$

Sum from h to h' to find that

$$|u_{x,R_h} - u_{x,R_{h'}}| \lesssim \frac{1}{2^{h\gamma}} [u]_{\mathcal{L}^{p,\lambda}(\Omega)} = \left(\frac{R_h}{R_0}\right)^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)}. \tag{3.10}$$

This implies the sequence $(u_{x,R_h})_h$ is a Cauchy sequence *uniformly in x*. Also, it can be shown (see Exercise 3.3.3 below) that the limit is independent of the choice R_0. So call the limit $\tilde{u}(x)$. Letting $h' \to \infty$ in (3.10), we find

$$|u_{x,R} - \tilde{u}(x)| \lesssim R^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)}. \tag{3.11}$$

Since the continuous functions $u_{x,R}$ converge to $\tilde{u}(x)$ uniformly, $\tilde{u}$ is continuous. On the other hand, by the Lebesgue Differentiation Theorem (see [31]), $u_{x,R}$ converges almost everywhere to $u(x)$. Thus $u(x) = \tilde{u}(x)$ almost everywhere, so $u(x)$ is an (essentially) continuous function. After re-defining u on a set of measure zero, it is a continuous function.

Now we establish the Hölder continuity estimate for this continuous representative. Choose $x, y \in \Omega$. We estimate:

$$|u(y) - u(x)| \leq I_1 + I_2 + I_3 \tag{3.12}$$

where we have taken $R = |x - y|$ and

$$\begin{aligned}
I_1 &= |u_{y,2R} - u(y)|,\\
I_2 &= |u_{x,2R} - u(x)|,\\
I_3 &= |u_{y,2R} - u_{x,2R}|.
\end{aligned}$$

By (3.11), we have $I_1, I_2 < |x - y|^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)}$. As for I_3 we have for all z

$$|u_{y,2R} - u_{x,2R}| \leq |u_{y,2R} - u(z)| + |u_{x,2R} - u(z)|. \tag{3.13}$$

Let $U = \Omega(x, 2R) \cap \Omega(y, 2R)$. Notice, since $\Omega(x, R) \subseteq U \subseteq \Omega(x, 2R)$, we have $R^n/A \leq |U| \leq (2R)^n$, which is to say, $|U| \approx R^n$. Thus, integrating over $z \in U$, dividing by R^n, and then applying Jensen's Inequality (Corollary 3.2.14), we have

$$\begin{aligned}
|u_{y,2R} - u_{x,2R}| &\lesssim \frac{1}{R^n}\left(\int_U |u_{y,2R} - u(z)| + \int_U |u_{x,2R} - u(z)|\right)\\
&\lesssim \frac{|U|^{1-\frac{1}{p}}}{R^n}\left(\left(\int_{B_{2R}(x)\cap\Omega} |u_{y,2R} - u(z)|^p \, dz\right)^{1/p} + \left(\int_{B_{2R}(y)\cap\Omega} |u_{x,2R} - u(z)|^p \, dz\right)^{1/p}\right)\\
&\approx R^{n/p}\left(\left(\int_{B_{2R}(x)\cap\Omega} |u_{y,2R} - u(z)|^p \, dz\right)^{1/p} + \left(\int_{B_{2R}(y)\cap\Omega} |u_{x,2R} - u(z)|^p \, dz\right)^{1/p}\right)\\
&\lesssim R^{\frac{n-\lambda}{p}} [u]_{\mathcal{L}^{p,\lambda}(\Omega)}\\
&= |x - y|^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)}.
\end{aligned} \tag{3.14}$$

This completes the estimates of I_1, I_2, I_3, and (3.12) now gives us

$$|u(x) - u(y)| \lesssim |x - y|^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)}$$

which is to say,

$$[u]_{C^{0,\gamma}} \lesssim [u]_{\mathcal{L}^{p,\lambda}(\Omega)}. \tag{3.15}$$

This only leaves us to check that $\|u\|_\infty \lesssim \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)}$. Observe, for x and y in Ω,

$$|u(y) - u(x)| \lesssim [u]_{\mathcal{L}^{p,\lambda}(\Omega)}|x-y|^\gamma \leq \operatorname{diam}(\Omega)^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)}.$$

Since u is uniformly continuous on Ω, it extends to a uniformly continuous function on $\bar{\Omega}$. In particular, it takes on its minimum at some point $x_0 \in \overline{\Omega}$. Thus we have for all $y \in \Omega$,

$$u(y) \leq C \operatorname{diam}(\Omega)^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)} + u(x_0) \leq \operatorname{diam}(\Omega)^\gamma [u]_{\mathcal{L}^{p,\lambda}(\Omega)} + \frac{1}{|\Omega|^{1/p}}\|u\|_{L^p} \lesssim \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)}$$

and similarly we can show $u(y) \gtrsim -\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)}$, so in fact

$$\|u\|_\infty \lesssim \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)}$$

which, together with (3.15), proves the theorem. □

Exercise 3.3.3. Show that, in the above proof, the limit of the sequence u_{x,R_h} is independent of the choice of initial radius R_0. Indeed, suppose R_0' is some other choice, and assume without loss of generality $R_0/2 < R_0' < R_0$. Let $R_h' = R_0'/2^h$ as before, and use (3.9) to estimate $|u_{x,R_h} - u_{x,R_h'}|$.

3.4 SOBOLEV SPACES

This section is a fast introduction to Sobolev Spaces—only the facts we need. These include the definition, smooth approximation, Sobolev embedding, Poincaré inequality, and Rellich-Kondrachov compactness theorem. We will only prove select theorems, leaving most for the reader to either find in the extensive literature ([25], [15], [14], [1]) or attempt proofs of their own (which is a worthwhile exercise). Throughout, we assume $1 \leq p < \infty$.

Definition 3.4.1. For $f \in L^p(\Omega)$ and $g \in L^p(\Omega, \mathbb{R}^n)$, we shall say that g is a *weak gradient* of f if there holds

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega g_i \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{3.16}$$

Exercise 3.4.2. For $f \in L^p(\Omega)$, if the weak gradient $g \in L^p(\Omega, \mathbb{R}^n)$ exists, it is unique. It also coincides with the classical gradient ∇f when f is smooth. Thus it makes sense to denote $\frac{\partial f}{\partial x_i} := g_i$.

Definition 3.4.3. We define the *Sobolev space* $W^{1,p}(\Omega)$ to be the space of all $L^p(\Omega)$ functions with $L^p(\Omega, \mathbb{R}^n)$ weak gradient. Likewise, we define $W^{1,p}_{\text{loc}}(\Omega)$ to be those $L^p_{\text{loc}}(\Omega)$ with $L^p_{\text{loc}}(\Omega, \mathbb{R}^n)$ weak derivatives. We define the Sobolev norm

$$[f]_{W^{1,p}(\Omega)} = \|\nabla f\|_{L^p(\Omega)},$$
$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{1,p}(\Omega)}.$$

Proposition 3.4.4. *$W^{1,p}(\Omega)$ is a Banach space.*

Proposition 3.4.5. *For $f \in W^{1,p}_{loc}(\Omega)$, the mollifications f_t as defined in Definition 3.2.20 converge to f in $W^{1,p}_{loc}(\Omega)$.*

For a given value of $t > 0$ the mollifications are only defined on

$$\Omega_t := \{x \in \Omega \ : \ \operatorname{dist}(x, \mathbb{R}^n \backslash \Omega) > t\};$$

but for fixed $U \Subset \Omega$ the mollifications are defined on all of U, provided $t < \operatorname{dist}(U, \mathbb{R}^n \backslash \Omega)$. It is in this sense that this proposition is true. Global approximation on Ω by functions which are smooth up to the boundary is possible but more delicate, so the result has a name,

Theorem 3.4.6 (Meyers-Serrin)**.** *If Ω is a bounded Lipschitz domain, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

Theorem 3.4.7. *If Ω is a bounded Lipschitz domain, then the trace operator* $\operatorname{tr} : C^\infty(\overline{\Omega}) \to L^p(\partial\Omega)$*, given by restriction to the boundary* $\operatorname{tr} : f \mapsto f|_{\partial\Omega}$*, enjoys the bound*

$$\|\operatorname{tr} f\|_{L^p(\partial\Omega)} \lesssim \|f\|_{W^{1,p}(\Omega)}. \tag{3.17}$$

Therefore, by density $C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega)$, tr *uniquely extends by uniform continuity to an operator*

$$\operatorname{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega). \tag{3.18}$$

Remark 3.4.8. $\operatorname{tr}(W^{1,p}(\Omega))$ is not all of $L^p(\partial\Omega)$. We will show that the image of the trace operator is the Slobodetskiĭ space $W^{1-\frac{1}{p},p}(\Omega)$, to be defined in the next section.

Proposition 3.4.9. *If Ω is a bounded Lipschitz domain, then there is a* bounded *extension operator* ext : $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$*. That is,* ext *has the two properties,*

1. $(\operatorname{ext} f)|_\Omega = f$.
2. $\|\operatorname{ext} f\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|f\|_{W^{1,p}(\Omega)}$.

More generally, when $\Omega \subseteq \mathbb{R}^n$ is any open set, we will say that Ω is an *extension domain* if there is a bounded linear extension operator ext : $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$. By extension operator, we mean that $(\operatorname{ext} f)|_\Omega = f$. The preceding proposition says that Lipschitz domains are extension domains.

Theorem 3.4.10 (Gagliardo, Nirenberg, Sobolev)**.** *For $f \in W^{1,p}(\mathbb{R}^n)$ and $1 \leq p < n$ we have $f \in L^{p^*}(\mathbb{R}^n)$ with*

$$\|f\|_{L^{p^*}} \lesssim \|\nabla f\|_{L^p} \tag{3.19}$$

where p^* is the Sobolev conjugate

$$p^* = \frac{np}{n-p}. \tag{3.20}$$

Theorem 3.4.11 (Kolmogorov-Riesz)**.** *Let $1 < p < \infty$, $\Omega \subseteq \mathbb{R}^n$ a bounded extension domain in $\mathbb{R}^n$, and $\mathcal{F} \subseteq L^p(\mathbb{R}^n)$ a subset. $\mathcal{F}$ is compact if and only if it is closed, bounded, and satisfies*

$$\frac{1}{h}\|u - \tau_h u\|_{L^p(\Omega)} < M$$

for some uniform constant $M > 0$, for all $h > 0$ and all $u \in \mathcal{F}$.

The following three theorems are instrumental to establishing regularity estimates, and later Hodge decomposition and Schauder estimates.

Theorem 3.4.12 (Rellich-Kondrachov)**.** *If Ω is a bounded Lipschitz domain, then the inclusion $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. That is, if f_i is a bounded sequence of functions in $W^{1,p}(\Omega)$, then there is a subsequence which converges in $L^p(\Omega)$-norm.*

Theorem 3.4.13 (Poincaré Inequality)**.** *For $f \in W^{1,p}(B_R(x))$ we have*

$$\|f - f_{x,R}\|_{L^p(B_R(x))} \lesssim R\|\nabla f\|_{L^p(B_R(x))} \tag{3.21}$$

where $f_{x,R} = \fint_{B_R(x)} f(x)\,dx$ is the average value of f on the ball $B_R(x)$. Similarly if $f \in W_0^{1,p}(B_R(x))$ then

$$\|f\|_{L^p(B_R(x))} \lesssim R\|\nabla f\|_{L^p(B_R(x))}. \tag{3.22}$$

Theorem 3.4.14 (Sobolev Embedding)**.** *Let $\Omega \Subset \mathbb{R}^n$ be a smooth bounded domain in $\mathbb{R}^n$. Then the following are continuous embeddings:*

1. *$W^{k,p}(\Omega) \subseteq L^q(\Omega)$ for $k < n/p$ and $q = np/(n-kp)$.*
2. *$W^{k,p}(\Omega) \subseteq C^{k-[n/p]-1,\gamma}(\Omega)$ for $k > n/p$ and $\gamma = 1 - n/p + [n/p]$, except if n/p is an integer, in which case γ can be any number $\gamma \in (0,1)$.*

The proof can be found in [14]. Much more information is in [1]. Our only use of this theorem will be to show that if a function belongs to $W^{k,2}(\Omega)$ for all $k \in \mathbb{N}$, then it is smooth.

3.5 SLOBODETSKIĬ SPACES

In order to prove results about Hölder continuous mappings into sub-riemannian manifolds later, we will need to use the important fact that Hölder continuous functions are *Slobodetskiĭ* functions, a more general *Sobolev-type* class (also called *fractional* Sobolev functions).

Definition 3.5.1. Let $f \in L^p(\Omega)$ be a measurable function on a domain $\Omega \subseteq \mathbb{R}^n$, $1 \leq p < \infty$, and $0 < s < 1$. Define

$$[f]^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy$$

and

$$\|f\|^p_{W^{s,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]^p_{W^{s,p}(\Omega)}.$$

We take $W^{s,p}(\Omega)$ to be the the space of $f \in L^p(\Omega)$ for which this norm is finite.

Proposition 3.5.2. *$C^{0,\gamma}(\Omega)$ compactly embeds in $W^{s,p}(\Omega)$ for bounded domains Ω, $1 \leq p < \infty$ and $\gamma > s$.*

Proof. One can directly check that $C^{0,\gamma}(\Omega) \subseteq W^{s,p}(\Omega)$ with *continuous* embedding. As for compactness, pick γ' with $s < \gamma' < \gamma$, and observe that, by Proposition 3.1.11, $C^{0,\gamma}$ compactly embeds in $C^{0,\gamma'}$ which in turn continuously embeds in $W^{s,p}$. Clearly then the embedding $C^{0,\gamma} \subseteq W^{s,p}$ is compact. □

Lemma 3.5.3. *If $f \in W^{s,p}(\mathbb{R}^n)$ and $\psi \in C^1(\mathbb{R}^n)$ with ψ and $\nabla\psi$ bounded, then $f\psi \in W^{s,p}(\mathbb{R}^n)$ with*

$$[\psi f]_{W^{s,p}(\mathbb{R}^n)} \lesssim \|\psi\|_\infty [f]_{W^{s,p}} + \|\psi\|_{C^1} \|f\|_{L^p}.$$

Proof.

$$
\begin{aligned}
[\psi f]^p_{W^{s,p}} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\psi(x)f(x)-\psi(y)f(y)|}{|x-y|^{n+sp}}\,dx\,dy \\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\psi(x)|^p|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^p|f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\leq \|\psi\|^p_\infty [f]^p_{W^{s,p}} + \|\psi\|^p_{C^1}\int_{y\in\mathbb{R}^n}\int_{x\in\mathbb{R}^n} \frac{|f(y)|^p}{|x-y|^{n-(1-s)p}}\,dx\,dy \\
&\approx \|\psi\|^p_\infty [f]^p_{W^{s,p}} + \|\psi\|^p_{C^1}\|f\|^p_{L^p}.
\end{aligned}
$$

□

Proposition 3.5.4. *Given $f \in W^{s,p}(\mathbb{R}^n)$, it is possible to find compactly supported functions $f_n \in W^{s,p}_c(\mathbb{R}^n)$ converging to f in $W^{s,p}$.*

Proof. Multiply by a smooth cut-off satisfying $B_R \prec \psi_R \prec B_{2R}$. The functions $\psi_R f$ are compactly supported and approach f in $W^{s,p}$ as a direct computation with Lemma 3.5.3 shows. □

Lemma 3.5.5. *Let $f \in W^{s,p}(B_R)$. Define $\tilde{f}(x) = f(Rx)$ for $x \in B_1$. Then*

$$
[\tilde{f}]_{W^{s,p}(B_1)} = R^{\frac{n}{p}-s}[f]_{W^{s,p}(B_R)}.
$$

The proof is immediate from the rescaling change-of-variables $B_1 \to B_R$.

Lemma 3.5.6. *For $f \in W^{s,p}(\mathbb{R}^n)$ we have*

$$
[f]^p_{W^{s,p}(\mathbb{R}^n)} \leq C(n,s,p,R)\|f\|_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\int_{B_R(y)} \frac{|f(y)-f(x)|^p}{|y-x|^{n+sp}}\,dx\,dy.
$$

This lemma can be viewed as a kind of "pseudo-locality" of the fractional Sobolev norm, in that we only need to consider points x which are close to y in the integral.

Proof.

$$\begin{aligned}
[f]^p_{W^{s,p}(\mathbb{R}^n)} &= \int_{y\in\mathbb{R}^n}\int_{x\in\mathbb{R}^n} \frac{|f(y)-f(x)|^p}{|y-x|^{n+sp}}\,dx\,dy \\
&= \int_{y\in\mathbb{R}^n}\int_{x\in B_R(y)} \frac{|f(y)-f(x)|^p}{|y-x|^{n+sp}} dx\,dy + \int_{y\in\mathbb{R}^n}\int_{|x-y|>R} \frac{|f(y)-f(x)|^p}{|y-x|^{n+sp}}\,dx\,dy \\
&=: I_1 + I_2.
\end{aligned}$$

I_1 is already what we want. As for I_2 we have

$$\begin{aligned}
I_2 &= \int_{y\in\mathbb{R}^n}\int_{|x-y|>R} \frac{|f(y)-f(x)|^p}{|y-x|^{n+sp}} dx\,dy \\
&\lesssim \int_{y\in\mathbb{R}^n}\int_{|x-y|>R} \frac{|f(y)|^p}{|y-x|^{n+sp}}\,dx\,dy + \int_{y\in\mathbb{R}^n}\int_{|x-y|>R} \frac{|f(x)|^p}{|y-x|^{n+sp}}\,dx\,dy \\
&= \int_{y\in\mathbb{R}^n}\int_{|x-y|>R} \frac{|f(y)|^p}{|y-x|^{n+sp}} dx\,dy + \int_{x\in\mathbb{R}^n}\int_{|y-x|>R} \frac{|f(x)|^p}{|y-x|^{n+sp}} dy\,dx \\
&= C(n,s,p,R)\left(\int_{y\in\mathbb{R}^n} |f(y)|^p\,dy + \int_{x\in\mathbb{R}^n} |f(x)|^p\,dx\right) = C\|f\|^p_{L^p(\mathbb{R}^n)}
\end{aligned}$$

which completes the proof. □

Proposition 3.5.7. *$C_c^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.*

Proof. Using Propositions 3.5.4 and 3.5.5, we may assume $f \in W^{s,p}(\mathbb{R}^n)$ with compact support in B_1. Let $a > 0$ be such that f has support in B_{1-a}. Let f_t be the standard mollifications of f in $L^p(\mathbb{R}^n)$, with $f_t \to f$ as $t \to 0$ and $f_t \in C_c^\infty(B_{1-a})$. Note that smooth compactly supported functions are in $W^{s,p}(\mathbb{R}^n)$ by Proposition 3.5.2. For convenience, we adopt the notation

$$\varphi(h,x,y) = \frac{|h(x)-h(y)|^p}{|x-y|^{n+sp}}.$$

We compute

$$\begin{aligned}
[f-f_t]^p_{W^{s,p}} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(f-f_t,x,y)\,dxdy \\
&\le 2\,(I_1+I_2+I_3)
\end{aligned} \tag{3.23}$$

where

$$I_1 = \int_{B_1}\int_{\mathbb{R}^n\setminus B_1} \varphi(f-f_t,x,y)\,dx\,dy,$$

$$I_2 = \int_{B_1} \int_{B_\delta(y)} \varphi(f - f_t, x, y)\, dx\, dy,$$
$$I_3 = \int_{B_1} \int_{B_1 \setminus B_\delta(y)} \varphi(f - f_t, x, y)\, dx\, dy.$$

I_1 and I_3 we estimate as:

$$I_1 \leq \frac{C}{a^{sp}} \left\| f - f_t \right\|^p_{L^p(B_1)}, \qquad I_3 \leq C\,(n, s, p) \cdot \frac{1}{\delta^{n+sp}} \left\| f - f_t \right\|^p_{L^p(B_1)}.$$

I_2 is more subtle:

$$I_2 \leq 2^p\,(I_4 + I_5)$$

where

$$I_4 = \int_{B_1} \int_{B_\delta(y)} \varphi(f, x, y)\, dx\, dy \qquad I_5 = \int_{B_1} \int_{B_\delta(y)} \varphi(f_t, x, y)\, dxdy.$$

By absolute continuity of the Lebesgue Integral, $I_4 < \varepsilon$ for δ sufficiently small. Then we are able to estimate

$$\begin{aligned}
I_5 &= \int_{B_1} \int_{B_\delta(y)} |x - y|^{-(n+sp)} \left| \int_{B_t} (f(x - z) - f(y - z)) \eta_t(z)\, dz \right|^p\, dx\, dy \\
&\leq \int_{B_1} \int_{B_\delta(y)} \int_{B_t} \varphi(f, x - z, y - z)\, \eta_t(z)\, dz\, dx\, dy \\
&= \int_{z \in B_t} \int_{y \in B_1 - z} \int_{x \in B_\delta(y)} \varphi(f, x, y)\, \eta_t(z)\, dx\, dy\, dz \\
&= I_4 < \varepsilon
\end{aligned}$$

independent of t, provided $t < a/2$. Thus we can let δ be such that $I_2 < \varepsilon/3$. Then we can take t small enough that $I_1 < \varepsilon/3$ and $I_3 < \varepsilon/3$. Returning to (3.23), we have

$$[f - f_t]^p_{W^{s,p}(\mathbb{R}^n)} < \varepsilon.$$

□

As for Sobolev Spaces, Slobodetskiĭ spaces enjoy extension from Lipschitz domains Ω to $\mathbb{R}^n$.

Theorem 3.5.8. *For a domain* $\Omega \subseteq \mathbb{R}^n$ *with Lipschitz boundary,* $0 < s < 1$*, and* $1 \leq p < \infty$*, there exists a bounded extension operator* $\mathrm{ext}_\Omega : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^n)$.

This theorem is proven in [12].

Our reason for using Slobodetskiĭ spaces is the following characterization of traces of Sobolev functions, due to Gagliardo [17].

Theorem 3.5.9 (Traces and Extensions)**.** *$W^{1-\frac{1}{p},p}(\mathbb{R}^n)$ is the trace space of $W^{1,p}(\mathbb{R}^{n+1}_+)$. More precisely,*

1. *For $u \in C_c^\infty(\mathbb{R}^{n+1}_+)$ we have*
$$[\operatorname{tr} u]_{W^{1-\frac{1}{p},p}(\mathbb{R}^n)} \lesssim [u]_{W^{1,p}(\mathbb{R}^{n+1}_+)}.$$
2. *For $v \in W^{1-\frac{1}{p},p}(\mathbb{R}^n)$ define the extension operator* ext *by the formula*
$$\operatorname{ext} v(x', t) = \eta_t * v(x'), \quad (x', t) \in \mathbb{R}^{n+1}_+,$$
where we fix a smooth family of convolution kernels η_t supported in B_t as in Definition 3.2.20. Then we have
$$[\operatorname{ext} v]_{W^{1,p}(\mathbb{R}^{n+1}_+)} \lesssim [v]_{W^{1-1/p,p}(\mathbb{R}^n)}.$$
3. *For the extension operator of part 2, we will have for $\gamma > 1 - \frac{1}{p}$*
$$[\operatorname{ext} v]_{W^{1,p}(B_R \times [0,1])} \leq C(\gamma) R^n [v]_{C^{0,\gamma}}. \tag{3.24}$$

Proof. 1. We omit the proof of Claim 1. It will not be used in the sequel. See [25] for a readable proof or [17] for the original paper of Gagliardo.

2. This claim is easier and can be proven directly. Observe,

$$\begin{aligned}
\frac{\partial}{\partial t}[\operatorname{ext} v(x', t)] &= \frac{\partial}{\partial t}\left[\int_{\mathbb{R}^n} \eta_t(x' - y') v(y')\, dy'\right] \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial t}\left[\frac{1}{t^n}\eta\left(\frac{x' - y'}{t}\right)\right] v(y')\, dy' \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial t}\left[\frac{1}{t^n}\eta\left(\frac{x' - y'}{t}\right)\right] (v(y') - v(x'))\, dy' \\
&= -\int_{\mathbb{R}^n} \left(\frac{n}{t^{n+1}}\eta\left(\frac{x' - y'}{t}\right) + \frac{1}{t^{n+2}}\nabla\eta\left(\frac{x' - y'}{t}\right) \cdot (x' - y')\right)(v(y') - v(x'))\, dy' \\
\left|\frac{\partial}{\partial t}[\operatorname{ext} v(x', t)]\right| &\lesssim \int_{B_t(x')} \frac{|v(y') - v(x')|}{t^{n+1}}\, dy'.
\end{aligned}$$

And by a similar argument (or simply see (3.4)), we have

$$\left|\frac{\partial}{\partial x_i}[\operatorname{ext} v(x',t)]\right| \lesssim \int_{B_t(x')} \frac{|v(y') - v(x')|}{t^{n+1}}\, dy'.$$

Thus, the total gradient of $\operatorname{ext} v$ in $\mathbb{R}^{n+1}$ enjoys the bound

$$|\nabla \operatorname{ext} v(x',t)| \lesssim \int_{B_t(x')} \frac{|v(y') - v(x')|}{t^{n+1}}\, dy'.$$

Raising to the p and applying Jensen's Inequality, find

$$|\nabla \operatorname{ext} v(x',t)|^p \lesssim \int_{B_t(x')} \frac{|v(y') - v(x')|^p}{t^{n+p}}\, dy'. \tag{3.25}$$

Obtaining the desired estimate is a routine integration:

$$\begin{aligned}
[\operatorname{ext} v]^p_{W^{1,p}(\mathbb{R}^{n+1})} &= \int_{\mathbb{R}^{n+1}} |\nabla \operatorname{ext} v(x',t)|^p\, dx'\, dt \\
&\lesssim \int_{x'\in\mathbb{R}^n} \int_{t\in\mathbb{R}} \int_{y'\in B_t(x')} \frac{|v(y') - v(x')|^p}{t^{n+p}}\, dy'\, dt\, dx' \\
&\lesssim \int_{x'\in\mathbb{R}^n} \int_{y'\in\mathbb{R}^n} \int_{t=|y'-x'|}^{t\to\infty} \frac{|v(y') - v(x')|^p}{t^{n+p}}\, dy'\, dx' \\
&\approx \int_{x'\in\mathbb{R}^n} \int_{y'\in\mathbb{R}^n} \frac{|v(y') - v(x')|^p}{|y' - x'|^{n+p-1}} dy'\, dx' \\
&= [v]^p_{W^{1-\frac{1}{p},p}(\mathbb{R}^n)}.
\end{aligned}$$

3. Beginning with (3.25),

$$\begin{aligned}
[\operatorname{ext} v]^p_{W^{1,p}(B_R\times[0,1]} &\lesssim \int_{x'\in B_R} \int_{t=0}^{t=1} \int_{y'\in B_t(x')} \frac{|v(y') - v(x')|^p}{t^{n+p}}\, dy'\, dt\, dx' \\
&\leq [v]^p_{C^{0,\gamma}} \int_{x'\in B_R} \int_{t=0}^{t=1} \frac{t^{\gamma p+n-1}}{t^{n+p}}\, dt\, dx' \\
&\approx R^n [v]^p_{C^{0,\gamma}}.
\end{aligned}$$

□

Theorem 3.5.10 (Sobolev Slobodetskiĭ Inclusion)**.** *Let $B_R \subseteq \mathbb{R}^n$ be a ball of radius R and $f \in W^{1,p}(B_R)$, and $s \in (0,1)$. Then $f \in W^{s,p}(B_R)$ and we have*

$$[f]_{W^{s,p}(B_R)} \lesssim R^{1-s}[Df]_{L^p(B_R)}. \tag{3.26}$$

That is, we have a continuous inclusion $W^{1,p}(B_R) \subseteq W^{s,p}(B_R)$.

Proof. For $\xi, x \in B_R$, let $L_{\xi,x} = \{y \in B_R : \xi \in \overline{xy}\}$. Reasoning as usual with Jensen's inequality,

$$|f(x) - f(y)|^p \leq |x-y|^{p-1} \int_{\xi \in \overline{xy}} |\nabla f(\xi)|^p \, d\xi$$

and so we can estimate

$$\begin{aligned}
[f]^p_{W^{1-\frac{1}{p},p}(B_R)} &= \int_{B_R} \int_{B_R} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \\
&\leq \int_{B_R} \int_{B_R} \frac{1}{|x-y|^{n+sp}} \left(\int_{\xi \in \overline{xy}} |Df(\xi)| \, d\xi \right)^p \, dx \, dy \\
&\leq \int_{B_R} \int_{B_R} \int_{\xi \in \overline{xy}} \frac{|Df(\xi)|^p}{|x-y|^{n-p(1-s)+1}} \, d\xi \, dx \, dy \\
&= \int_{\xi \in B_R} \int_{x \in B_R} \int_{y \in L_{\xi,x}} \frac{|Df(\xi)|^p}{|x-y|^{n-p(1-s)+1}} \, dy \, dx \, d\xi \\
&\lesssim \int_{\xi \in B_R} \int_{x \in B_R} \left(\frac{|Df(\xi)|^p}{|x-\xi|^{n-p(1-s)}} \right) dx \, d\xi \\
&\approx R^{p(1-s)} \int_{\xi \in B_R} |Df(\xi)|^p \\
&= R^{p(1-s)} [Df]^p_{L^p(B_R)}.
\end{aligned}$$

□

4.0 MANIFOLDS

4.1 FUNDAMENTAL NOTIONS

Definition 4.1.1. An n-dimensional *manifold* is a pair $M, \mathcal{A}$ of a paracompact Hausdorff space M and an atlas $\mathcal{A}$ for M. An *atlas* is a collection $\mathcal{A} = (\varphi_\alpha : \tilde{U}_\alpha \to U_\alpha)_{\alpha \in \mathcal{J}}$ of homeomorphisms φ_α called *coordinate systems*, from open domains $\tilde{U}_\alpha \to \mathbb{R}^n$ onto open subsets $U_\alpha \subseteq M$ with the property that the sets U_α cover M, and that the *transition maps*

$$_\beta\varphi_\alpha = \varphi_\beta^{-1} \circ \varphi_\alpha : \tilde{U}_{\beta\alpha} \to \tilde{U}_{\alpha\beta}$$

are smooth diffeomorphisms, where $\tilde{U}_{\alpha\beta} = \varphi_\beta^{-1}[U_\alpha]$. The sets U_α are called *coordinate patches* of M. Given a coordinate system $\varphi : \tilde{U} \to U$, we can write $\varphi^{-1} = (x^1, \ldots, x^n)$. The component functions $x^i : U \to \mathbb{R}^n$ are called *coordinate functions* of the coordinate system φ.

Given a manifold $M, \mathcal{A}$, a set X, a function $f : M \to X$, and a coordinate system φ we define the *representation* of f in the coordinate system φ to be the function $f_\varphi : \tilde{U} \to X$ given by

$$f_\varphi = f \circ \varphi.$$

Definition 4.1.2. If $(M, \mathcal{A})$ is a manifold and $f : M \to \mathbb{R}$ is a continuous function, we say f is *smooth* if its representation in any coordinate system is smooth.

Definition 4.1.3. Let $(M, \mathcal{A})$ and $(N, \mathcal{A}')$ be two manifolds, $f : M \to N$ a map, $\mathcal{A} \ni \varphi : \tilde{U} \to U$, $\mathcal{A}' \ni \psi : \tilde{V} \to V$ coordinate systems on M and N. Define the *representation* of f with respect to the coordinate systems φ and ψ, denoted $_\psi f_\varphi : U \cap f^{-1}[V] \to f[U] \cap V$ via the formula

$$_\psi f_\varphi = \psi^{-1} \circ f \circ \varphi.$$

Definition 4.1.4. We say that a mapping $f : M \to N$ between manifolds is *smooth* if its representation is smooth for all coordinate systems φ and ψ on M and N respectively. If $p \in M$, we say that f is *smooth near* p if there exist coordinate systems $\varphi : \tilde{U} \to U \ni p$ and $\psi : \tilde{V} \to V \ni f(p)$ such that the coordinate representation of f is smooth.

Remark 4.1.5. $f : M \to N$ is smooth if and only if it is smooth near all $p \in M$.

Remark 4.1.6. The spaces $\mathbb{R}^n$ have a natural n-dimensional manifold structure with atlases given by the identity mappings. While there are other manifold structures for $\mathbb{R}^n$, we always use this one without further comment.

Remark 4.1.7. Given a manifold $(M, \mathcal{A})$, the Hausdorff Maximal Principle implies there is a unique maximal atlas $\overline{\mathcal{A}}$ containing $\mathcal{A}$ which we will call the *maximal atlas* for M. A function is smooth on $(M, \mathcal{A})$ if and only if it is smooth in the maximal atlas. For each $p \in M$, there is a coordinate system in the maximal atlas $\varphi : B^n \to U$ from the unit ball with $\varphi(0) = p$. This is called a *coordinate system centered at* p.

Definition 4.1.8 (Nice Atlas)**.** A finite atlas $\mathcal{A} = \{\varphi_i : B_i \to M\}_{i=1}^N$ for M will be called *nice* if the coordinate charts φ_i extend to smooth maps $\tilde{\varphi}_i : B_i' \to M$ on some larger sets $\mathbb{R}^n \supseteq B_i' \Supset B_i$.

Remark 4.1.9. This assumption is made so that $\varphi_i^{-1} \circ \varphi_j$ are smooth maps with derivatives of all orders bounded. This is only going to be an issue in Section 4.4 where we define function spaces on manifolds.

Definition 4.1.10 (Tangent Space, Tangent Vector)**.** Fix $p \in M$ and now let X be the set of smooth curves $\gamma : I \to M$ where $I \subset \mathbb{R}$ is an interval containing 0 and $\gamma(0) = p$. Define the *tangent space* to M at p, denoted T_pM, to be the quotient $X/\sim$ under the equivalence relation $\sim$ identifying γ_1 and γ_2 if their coordinate representations have equal derivative at 0 in any coordinate system near p. Given such a curve $\gamma \in X$, let γ_p' denote the equivalence class $[\gamma]$. Define scalar multiplication and addition by

$$c\gamma_p' = [c(\varphi^{-1} \circ \gamma)]$$
$$\gamma_1' + \gamma_2' = [\varphi^{-1} \circ \gamma_1 + \varphi^{-1} \circ \gamma_2].$$

where φ is any coordinate system *centered* at p (remember this means $\varphi(0) = p$). Call γ'_p the *tangent vector* to γ at p. The resulting vector space of all such tangent vectors is called the *tangent space* to M at p, and is denoted T_pM.

Definition 4.1.11 (Cotangent Space, Differential)**.** Fix a point p in a manifold M. Let X be the set of smooth maps $f : U \to \mathbb{R}$ defined on a neighborhood of p. Define the *cotangent space* to M at p, denoted T^*_pM, to be the quotient $X/\sim$ under the equivalence relation $\sim$ identifying maps f and g if the coordinate representation of $f - g$ has derivative zero at $\varphi(p)$ in any coordinate system φ near p. Given such a smooth map $f \in X$, let df_p denote the equivalence class $[f]$ in $X/\sim$. Call this the *differential* of f at p. In an obvious way, the set of all such differentials form a vector space, which we call the *cotangent space* to M at p, denoted T^*_pM.

Remark 4.1.12. Notice that T^*_pM can in a natural way be identified as the dual space of T_pM. Indeed, if $df_p \in T^*_pM$ and $\gamma'_p \in T_pM$, we can define

$$(df_p, \gamma'_p) = \frac{d}{dt}\bigg|_{t=0} f \circ \gamma(t)$$

where f and γ are *any* representatives of df_p and γ'_p.

Definition 4.1.13. Fix a point p in a manifold M and a coordinate system $\varphi : \tilde{U} \to U$ near p. We define the *standard basis* for T^*_pM to be the set $dx^1_p, \ldots, dx^n_p$, where $x^1, \ldots, x^n$ are the coordinate functions of φ. Also, if we define $\frac{\partial}{\partial x_i}\big|_p$ to be the tangent vector to the curves $t \mapsto \varphi^{-1}(p) + te_i$, this is a basis for T_pM which is dual to dx^i_p, since we evidently have

$$\left(dx^i_p, \frac{\partial}{\partial x_j}\bigg|_p\right) = \delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

4.2 DIFFERENTIAL FORMS

Fix $k \in \{1, \ldots, m\}$. For each $p \in M$, we can associate the vector space $\bigwedge^k_p M = \bigwedge^k(T^*_pM)$. As a set, we define the *exterior k-bundle* to be the (disjoint) union of these spaces,

$$\bigwedge^k M = \bigsqcup_{p \in M} \bigwedge^k_p M.$$

Let $\pi : \bigwedge^k M \to M$ denote the projection map sending forms in $\bigwedge_p^k M$ to $p \in M$. A differential k-form is a map $\omega : M \to \bigwedge^k M$ with $\pi \circ \omega =$ identity. Given $p \in M$ and a coordinate chart $\varphi : \mathbb{B}^m \to U \subseteq M$ about p, recall that we have a basis for $T_q^* M$ at any $q \in U$, namely $dx_1, \ldots, dx_m$. Consequently, if we use the notation

$$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

whenever $I = \{i_1 < i_2 < \ldots < i_k\}$ is a subset of $\{1, \ldots, n\}$, then $(dx_I)_q$ forms a basis for $\bigwedge_q^k M$ for all $q \in U$, see part 3 of Theorem 2.3.10. Consequently, any differential form ω has *coordinate representation* $\omega_\varphi = (a_I)_I$ so that $\omega = \sum_I a_I dx_I$ on U. These coordinate representations of k-forms with vectors in $\mathbb{R}^{\binom{n}{k}}$ give $\bigwedge^k M$ the structure of an $n + \binom{n}{k}$-dimensional manifold. This allows us to define a *smooth* differential k-form. This is a differential k-form ω which is smooth as a map from M to $\bigwedge^k M$. Equivalently, ω is smooth if the coordinate representations $(a_I)_I$ are smooth functions on any coordinate chart. We refer to the space of smooth differential k-forms by $C^\infty(\bigwedge^k M)$.

There is a rich algebraic structure associated with differential forms. They are a module over $C^\infty(M)$. So if $f \in C^\infty(M)$ and $\omega \in C^\infty(\bigwedge^k M)$ then $f\omega \in C^\infty(\bigwedge^k M)$. $C^\infty(\bigwedge^k M)$ also has a multiplicative structure $\wedge$ making this space an *algebra*. Notice that the operator d can be viewed as sending a smooth function f to the smooth differential 1-form df,

$$d : C^\infty(\textstyle\bigwedge^0 M) \to C^\infty(\textstyle\bigwedge^1 M).$$

It turns out there is a *unique* way to extend this operator to all of $C^\infty(\bigwedge M)$ to have the property that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \tag{4.1}$$

where $\omega \in C^\infty(\bigwedge^k M)$ and $\eta \in C^\infty(\bigwedge^\ell M)$. It is by the formula

$$d(a_I\, dx_I) = \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I \quad \text{(summation over } I \text{ and } j)$$

This construction is called the *exterior derivative* on M, still denoted d. It satisfies $d(d\omega) = 0$.

There is yet more algebraic structure. Given a smooth mapping $f : M \to N$ between manifolds, there is an induced homomorphism—both of the $C^\infty(M)$-module structure and the algebra structure:

$$f^* : C^\infty(\textstyle\bigwedge N) \to C^\infty(\textstyle\bigwedge M)$$

called the pullback operator. It is defined pointwise by

$$f^*\omega|_p = (Df)^*(\omega|_{f(p)})$$

where $(Df)^* : \bigwedge T^*_{f(p)}N \to \bigwedge T^*_p M$ is the pullback operator associated with the linear map $Df : T_p M \to T_{f(p)}N$ (Definition 2.3.11). Pullback respects the algebraic structure $\wedge$ and d for $C^\infty(\bigwedge M)$ already discussed. Namely, we have

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta \tag{4.2}$$

$$f^*(d\omega) = d(f^*\omega). \tag{4.3}$$

The first of these is a trivial consequence of how the pullback associated with a linear transformation is defined. To verify the second, it suffices to check it for functions $\omega \in C^\infty(\bigwedge^0 M) = C^\infty(M)$. Then, since any k-form can be expressed as $a_I dx_I$ locally, (4.2) implies (4.3).

Notice that the fibers of $\bigwedge^n M$ are one-dimensional by part 3 of Theorem 2.3.10, and so for each $p \in M$, $\bigwedge^n_p(M\backslash\{0\})$ consists of two connected components. This does not, however, imply that $\Omega M := \bigcup_{p\in M}(\bigwedge^n_p M\backslash\{0_p\})$ consists of two connected components. If ΩM does have two connected components, it is said to be an *orientable* manifold. Equivalently, M is orientable if there exists a non-vanishing n-form ω on M.

Definition 4.2.1. By *oriented manifold*, we mean an orientable manifold M *together* with a choice of one of the two connected components of ΩM. This is often done by specifying a non-vanishing differential form $\omega \in C^\infty(\bigwedge^n M)$.

Definition 4.2.2. A *manifold with boundary* is a paracompact Hausdorff space M together with an *interior* atlas $\mathcal{A}^o$ and a *boundary* atlas $\mathcal{A}^\partial$, satisfying the following properties: The boundary atlas is a collection of homeomorphisms $\mathcal{A}^\partial = \{\varphi_i^\partial : \mathbb{R}^n \supset B^n_+ \to M\}$ from the upper-half unit ball B^n_+ into M such that their restrictions to $B^{n-1} \subset \mathbb{R}^{n-1}$ form an atlas for $\bigcup \varphi_i(B^{n-1})$, making this subset of M an $(n-1)$-dimensional manifold, which we refer to as the *boundary* of M, denoted ∂M. The interior atlas $\mathcal{A}^o = \{\varphi_i^o : B^n \to M\}$ is a collection of homeomorphisms into M. All transition maps $\varphi_i^{-1} \circ \varphi_j$ are smooth on the interior of their domain, where φ_i and φ_j can come from $\mathcal{A}^o$ or $\mathcal{A}^\partial$.

Notice that, if a manifold M with boundary is orientable, then so is ∂M. Indeed, fix a finite *positively oriented* covering $\mathcal{B} = \mathcal{B}^o \cup \mathcal{B}^\partial$ of M by boundary and interior coordinate charts, and fix a partition of unity $\{\psi_i\}$ subordinate to this covering. Then $-\sum_{\mathcal{B}^\partial} \psi_i dx_1^{(i)} \wedge \ldots \wedge dx_{n-1}^{(i)}$ is an orientation of ∂M, where $x_1^{(i)}, \ldots x_n^{(i)}$ are the coordinate functions in the i^{th} coordinate chart in $\mathcal{B}^\partial$. This orientation (note the minus sign) is the *standard orientation* of ∂M.

4.3 STOKES' THEOREM

Definition 4.3.1. Given an oriented manifold M, a finite *positively oriented* coordinate covering $\mathcal{B} = \{\varphi_i : U_i \to M\}$, a parition of unity $\Psi = \{\psi_i\}$ subordinate to $\mathcal{B}$, and an n-form ω on M, we define the *integral*,

$$\int_M \omega = \sum_i \int_{U_i} \varphi_i^*(\psi_i \omega) \tag{4.4}$$

where we define the integral of an n-form $\eta = a(x)dx_1 \wedge \ldots \wedge dx_n$ on $U \subseteq \mathbb{R}^n$ by

$$\int_U \eta = \int_U a(x)\, dx \tag{4.5}$$

Lemma 4.3.2. *Let $\eta = a(x)dx_1 \wedge \ldots \wedge dx_n$ with a integrable with compact support inside an open set $U \subseteq \mathbb{R}^n$, and let $g : V \to U$ be a positively oriented diffeomorphism from an open set $V \subseteq \mathbb{R}^n$. Then*

$$\int_U \eta = \int_V g^*\eta \tag{4.6}$$

Proof. Notice that

$$g^*\eta = a \circ g(x)\, dg_1 \wedge \ldots \wedge dg_n = a \circ g(x) \mathcal{J} g(x)\, dx_1 \wedge \ldots \wedge dx_n$$

Therefore (since $\mathcal{J} g$ is positive),

$$\int_U \eta = \int_U a(x)\, dx = \int_V a \circ g(x)\, \mathcal{J} g\, dx = \int_V g^*\eta.$$

□

Proposition 4.3.3. *The quantity (4.4) is independent of the choice of coordinate covering $\mathcal{B}$ and partition of unity Ψ.*

Proof. Let $\mathscr{B}_1 = \{\varphi_{i,(1)}\}$ and $\mathscr{B}_2 = \{\varphi_{j,(2)}\}$ be two choices of coordinate coverings of M, and let $\Psi_1 = \{\psi_i^{(1)}\}$ and $\Psi_2 = \{\psi_i^{(2)}\}$ be partitions of unity. Denote by $\int_M^{(1)} \omega$ and $\int_M^{(2)} \omega$ the corresponding integrals given in (4.4). For notational convenience define $W_{ij} = U_i \cap \varphi_{i,(1)}^{-1} \circ \varphi_{j,(2)}(V_j)$ and $W^{ij} = \varphi_{j,(2)}^{-1} \circ \varphi_{i,(1)}(U_i) \cap V_j$. The point is, $\varphi_{j,(2)} \circ (\varphi_{i,(1)})^{-1}$ is a (positively oriented!) diffeomorphism $W^{ij} \to W_{ij}$. Then we can compute

$$\begin{aligned}
\int_M^{(1)} \omega &= \sum_i \int_{U_i} \varphi_{i,(1)}^*(\psi_i^{(1)}\omega) \\
&= \sum_i \sum_j \int_{W_{ij}} (\varphi_{i,(1)})^*(\psi_j^{(2)}\psi_i^{(1)}\omega) \\
&= \sum_j \sum_i \int_{W^{ij}} (\varphi_{j,(2)} \circ \varphi_{i,(1)}^{-1})^* \varphi_{i,(1)}^*(\psi_j^{(2)}\psi_i^{(1)}\omega) \\
&= \sum_j \sum_i \int_{W^{ij}} \varphi_{j,(2)}^*(\psi_j^{(2)}\psi_i^{(1)}\omega) \\
&= \sum_j \int_{V_j} \varphi_{j,(2)}^*(\psi_j^{(2)}\omega) \\
&= \int_M^{(2)} \omega.
\end{aligned}$$

□

Theorem 4.3.4 (Stokes' Theorem)**.** *For an oriented manifold M with boundary and $\omega \in C^\infty(\bigwedge^{n-1} M)$*

$$\int_M d\omega = \int_{\partial M} \omega \tag{4.7}$$

where ∂M is given standard orientation, and on the right side, "ω" is understood to be $\iota^*\omega$, where $\iota : \partial M \to M$ is the embedding map.

Proof. Fix a finite coordinate cover with interior $\mathscr{B}^o = \{\varphi_{i,o}\}$ and boundary cover $\mathscr{B}^\partial = \{\varphi_{j,\partial}\}$. We can write $\varphi_{i,o}^*(\psi_i\omega) = a_i^{\hat{k}} dx_{\hat{k}}$ where $\hat{k} = \{1, \ldots, n\}\backslash\{k\}$, and summation over $k \in \{1, \ldots, n\}$ is implied.

$$\begin{aligned}
\int_M d\omega &= \sum_i \int_M d(\psi_{i,o}\omega) + \sum_j \int_M d(\psi_{i,\partial}\omega) \\
&= \sum_i \int_{B_i} \varphi_{i,o}^*(\psi_{i,o}\omega) + \sum_j \int_{B_{j,+}} \varphi_{j,\partial}^*(\psi_{j,\partial}\omega) \\
&= \sum_i \sum_{k=1}^n \int_{B_i} \frac{\partial a_{i,o}^{\hat{k}}}{\partial x_k} + \sum_j \sum_{k=1}^{n-1} \int_{B_{j,+}} \frac{\partial a_{j,\partial}^{\hat{k}}}{\partial x_k} + \sum_j \int_{B_{j,+}} \frac{\partial a_{j,\partial}^{\hat{n}}}{\partial x_n}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_j \int_{B_{j,+}} \frac{\partial a^{\hat{n}}_{j,\partial}}{\partial x_n} \\
&= \sum_j \int_{\partial B_{j,+}} -a^{\hat{n}}_{j,\partial} dx_{\hat{n}} \\
&= \sum_j \sum_{k=1}^{n} \int_{\partial B_{j,+}} -a^{\hat{k}}_{j,\partial} dx_{\hat{k}} \\
&= -\sum_j \int_{\partial B_{j,+}} \varphi_{j,\partial}\big|^*_{\partial B_{j,+}} (\psi_{j,\partial}\omega) \\
&= \int_{\partial M} \omega
\end{aligned}
$$

where in the last step, we used the orientation convention that the restrictions $\varphi_{j,\partial}\big|_{\partial B_{j,+}}$ are a negatively oriented coordinate covering of ∂M. □

4.4 FUNCTION SPACES AND SMOOTH APPROXIMATION ON MANIFOLDS

In Chapter 3, we introduced various function spaces on Euclidean spaces. Each of these spaces permitted approximation in norm by smooth functions—with the notable exception of Hölder functions, which required a somewhat weaker statement. We show that with suitable definitions, all of this holds for the corresponding function spaces of differential forms on a manifold M.

Definition 4.4.1. We will denote the space of continuous functions on M by $C(M)$ and give it the norm $\|f\|_\infty = \sup_{p\in M} |f(p)|$.

Definition 4.4.2. Given a finite covering $\mathcal{B} = \{\varphi_i : B_1 \to M\}$ of M by coordinate charts and a partition of unity $\Psi = \{\psi_i\}$ on M subordinate to the cover $\mathcal{B}$, and given $f \in C^k(M)$, define

$$
\|f\|_{C^k(M);\mathcal{B},\Psi} = \max_{1\leq i\leq N} \|f \circ \varphi_i\|_{C^k}.
$$

While it feels unsatisfactory to define the norm as with respect to some choice of covering, it turns out that we at least always get equivalent norms regardless of the choice, provided the coverings are nice (see Definition 4.1.8):

Proposition 4.4.3. *Given two nice finite coverings $\mathcal{B}_1$ and $\mathcal{B}_2$ of M by coordinate charts, we have*

$$\|f\|_{C^k(M);\mathcal{B}_1} \approx \|f\|_{C^k(M);\mathcal{B}_2}.$$

Proof. Fix a coordinate chart $\varphi_i^{(1)} : B_i^{(1)} \to M$ in $\mathcal{B}_1$ and $\varphi_j^{(2)} : B_j^{(2)} \to M$ in $\mathcal{B}_2$. Pick a multi-index α. Now compute

$$\begin{aligned} D^\alpha(f \circ \varphi_i) &= D^\alpha[f \circ \varphi_j \circ (\varphi_j^{-1} \circ \varphi_i)] \\ &= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta(f \circ \varphi_j) D^{\alpha-\beta}(\varphi_j^{-1} \circ \varphi_i) \\ &\le C_{ij} \|f \circ \varphi_j\|. \end{aligned}$$

Since the coverings are nice, the constants C_{ij} which depend on the derivatives of $\varphi_j^{-1} \circ \varphi_i$ are finite. Thus taking supremum over $x \in B_i^{(1)}$, and then taking maximum over i and j, we find

$$[f]_{C^k(M);\mathcal{B}_1} \le \|f\|_{C^k(M);\mathcal{B}_2}.$$

□

Remark 4.4.4. In the sequel, when we write $\|f\|_{C^k(M)}$, we make the implicit assumption that a "preferred covering" $\mathcal{B}$ has been chosen and fixed for M.

Definition 4.4.5. Given a nice covering of coordinate charts $\mathcal{B} = \{\varphi_i : B_i \to M\}_{i=1}^N$ for a manifold M, a partition of unity $\Psi = \{\psi_i\}_{i=1}^N$, $1 \le p < \infty$ and measurable $f : M \to \mathbb{R}$, define

$$\|f\|_{L^p;\mathcal{B};\Psi} = \left(\sum_{i=1}^N \int_{B_i} |f \circ \varphi_i|^p \psi_i \circ \varphi_i \right)^{1/p}.$$

Proposition 4.4.6. *For any choices $\mathcal{B}_1, \Psi_1$ and $\mathcal{B}_2, \Psi_2$, the corresponding L^p-norms on M are equivalent:*

$$\|f\|_{L^p;\mathcal{B}_1,\Psi_1} \approx \|f\|_{L^p;\mathcal{B}_2,\Psi_2}.$$

Proof.

$$
\begin{aligned}
\|f\|^p_{L^p;\Psi_1} &= \sum_{i=1}^{N} \int_{B_i} |f \circ \varphi_i^{(1)}|^p \, \psi_i^{(1)} \circ \varphi_i^{(1)} && \text{definition of } L^p\text{-norm} \\
&= \sum_{i=1}^{N_1} \int_{B_i} \sum_{j=1}^{N_2} |f \circ \varphi_i^{(1)}|^p \, \psi_j^{(2)} \circ \varphi_i^{(1)} \, \psi_i^{(1)} \circ \varphi_i^{(1)} && \psi_j^{(2)} \text{ is a partition of unity.} \\
&= \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \int_{B_i} |f \circ \varphi_i^{(1)}|^p \, \psi_j^{(2)} \circ \varphi_i^{(1)} \, \psi_i^{(1)} \circ \varphi_i^{(1)} && \text{switching orders of summation} \\
&= \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \int_{B_j} |f \circ \varphi_j^{(2)}|^p \, \psi_j^{(2)} \circ \varphi_j^{(2)} \, \psi_i^{(1)} \circ \varphi_j^{(2)} \, \mathcal{J}_{ij} && \text{change of variables } \varphi_i^{(1)\,-1} \circ \varphi_j^{(2)} \\
&\le C \sum_{j=1}^{N_2} \int_{B_j} \sum_{i=1}^{N_1} |f \circ \varphi_j^{(2)}|^p \psi_j^{(2)} \circ \varphi_j^{(2)} \, \psi_i^{(1)} \circ \varphi_j^{(2)} && \text{letting } C = \sup_{i,j,x} \mathcal{J}_{ij}(x) \\
&= C \sum_{j=1}^{N_2} \int_{B_j} |f \circ \varphi_j^{(2)}|^p \, \psi_j^{(2)} \circ \varphi_j^{(2)} && \psi_i^{(1)} \text{ is a partition of unity} \\
&= C\|f\|^p_{L^p;\mathcal{B}_2,\Psi_2}
\end{aligned}
$$

This proves that the norms are equivalent up to a constant which depends on the Jacobians of the change-of-coordinate maps, which are bounded since the coordinate coverings are assumed to be nice. In fact, a close look at the proof shows the constants do *not* depend on the partition of unity Ψ. □

Definition 4.4.7 (Standard Mollification with respect to a Partition of Unity)**.** Given a manifold M, a fixed finite covering by coordinate charts $\mathcal{B} = \{\varphi_i : B_i \to M\}$ and smooth partition of unity $\Psi = \{\psi_i\}$, we define the *standard mollification* (with respect to $\mathcal{B}$ and Ψ)

$$
f_t = \sum_i ((\eta_t * (f \circ \varphi_i)) \circ \varphi_i^{-1}) \, \psi_i. \tag{4.8}
$$

Remark 4.4.8. Notice that $\eta_t * (f \circ \varphi_i)$ is only defined inside of B_i, but this poses no issue for the definition since ψ_i is supported inside $\varphi_i(B_i)$. We simply declare a summand in the expression to be zero outside $\varphi_i(B_i)$.

Proposition 4.4.9. *$C^\infty(M)$ is dense in $L^p(M)$.*

Proof. Given $f \in L^p(M)$ we will show that the standard mollifications f_t with respect to some arbitrary coordinate covering $(\mathcal{B}, \Psi)$, converge to f in L^p. We compute

$$\begin{aligned}
\|f_t - f\|_{L^p}^p &= \sum_{j=1}^N \int_{B_j} |(f_t - f)\circ\varphi_j|^p\,\psi_j\circ\varphi_j \\
&= \sum_{j=1}^N \int_{B_j} \left|\left(\sum_{i=1}^N (\eta_t * (f\circ\varphi_i))\circ(\varphi_i^{-1}\circ\varphi_j)\,\psi_i\circ\varphi_j\right) - f\circ\varphi_j\right|^p \psi_j\circ\varphi_j \\
&= \sum_{j=1}^N \int_{B_j} \left|\sum_{i=1}^N \Big((\eta_t * (f\circ\varphi_i))\circ(\varphi_i^{-1}\circ\varphi_j) - f\circ\varphi_j\Big)\,\psi_i\circ\varphi_j\right|^p \psi_j\circ\varphi_j \\
&\leq \sum_{j=1}^N \int_{B_j} \sum_{i=1}^N \left|(\eta_t * (f\circ\varphi_i))\circ(\varphi_i^{-1}\circ\varphi_j) - f\circ\varphi_j\right|^p \psi_i\circ\varphi_j\,\psi_j\circ\varphi_j \\
&= \sum_{j=1}^N \sum_{i=1}^N \int_{B_j} \left|(\eta_t * (f\circ\varphi_i))\circ(\varphi_i^{-1}\circ\varphi_j) - f\circ\varphi_j\right|^p \psi_i\circ\varphi_j\;\psi_j\circ\varphi_j \\
&= \sum_{j=1}^N \sum_{i=1}^N \int_{B_i} |(\eta_t * (f\circ\varphi_i)) - f\circ\varphi_i|^p\;\psi_i\circ\varphi_i\,\psi_j\circ\varphi_i\,\mathcal{J}_{ij} \\
&\leq C\sum_{i=1}^N \int_{B_i} |(\eta_t * (f\circ\varphi_i)) - f\circ\varphi_i|^p\;\psi_i\circ\varphi_i \\
&\xrightarrow{t\to 0} 0
\end{aligned}$$

where in the last step we applied Theorem 3.2.21 to show that the smooth approximations $\eta_t * (f\circ\varphi_i)$ converge to $f\circ\varphi_i$. The constant C is the maximum of the Jacobian determinants $\mathcal{J}_{ij} = \det D(\varphi_i^{-1}\circ\varphi_j)$ which appeared in the change-of-variables in the sixth step. □

Definition 4.4.10. Given a nice covering of coordinate charts $\mathcal{B}$ for a manifold M, partition of unity Ψ, and $f \in L^p(M)$ define

$$\begin{aligned}
[f]_{W^{k,p}(M);\mathcal{B},\Psi} &= \left(\sum_{i=1}^N \int_{B_i} |\nabla^k(f\circ\varphi_i)|^p\psi_i\circ\varphi_i\right)^{1/p}, \\
\|f\|_{W^{k,p}(M);\mathcal{B},\Psi} &= \left(\|f\|_{L^p(M)}^p + \sum_{\ell=1}^k [f]^p_{W^{\ell,p}(M),\mathcal{B},\Psi}\right)^{1/p}.
\end{aligned}$$

whenever the weak derivatives $\nabla^k(f\circ\varphi_i)$ up to order k exist and are in $L^p(B_i)$ for each φ_i. If this is the case, we shall say that $f \in W^{k,p}(M;\mathcal{B},\Psi)$.

Proposition 4.4.11. *Given two nice coverings and partitions of unity* $(\mathcal{B}_1, \Psi_1)$ *and* $\mathcal{B}_2, \Psi_2$ *of* M *and* $f : M \to \mathbb{R}$ *the weak derivatives* $\nabla^k(f \circ \varphi_i^{(1)})$ *exist if and only if those of* $\nabla^k(f \circ \varphi_j^{(2)})$ *exist, and we have*

$$[f]_{W^{k,p};\mathcal{B}_1,\Psi_1} \approx [f]_{W^{k,p};\mathcal{B}_2,\Psi_2}.$$

Proposition 4.4.12. *For* $f \in W^{k,p}(M)$ *the standard mollifications* f_t *given in Definition 4.4.7 converge to* f *in* $W^{k,p}(M)$*.*

Definition 4.4.13. Let $0 < s < 1$, $1 \leq p < \infty$, $\mathcal{B}$ a nice coordinate covering of M, and Ψ a partition of unity, and $f \in L^p(M)$. We define the Slobodetskiĭ semi-norm on M,

$$[f]_{W^{s,p}(M,\mathcal{B},\Psi)} = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_i \circ \psi_i(y) \varphi_j \circ \psi_j(x) \frac{|\varphi_i \circ f(y) - \varphi_j \circ f(x)|^p}{|y - x|^{n+sp}} \, dx \, dy \right)^{1/p}. \tag{4.9}$$

Proposition 4.4.14. $W^{s,p}(M, \mathcal{B}_1, \Psi_1)$ *and* $W^{s,p}(M, \mathcal{B}_2, \Psi_2)$ *are equivalent norms for two nice coordinate covers and partitions of unity.*

Proposition 4.4.15. *Let* $f \in W^{s,p}(M)$*. Then the standard mollifications* f_t *given by Definition 4.8 converge to* f *in* $W^{s,p}(M)$*.*

Theorem 4.4.16. *Given a compact Riemannian manifold without boundary* M *and, nice coordinate cover* $\mathcal{B}$*, and partition of unity* Ψ*, define an extension operator for* $u \in W^{1-\frac{1}{p},p}(M, \mathcal{B}, \Psi)$ *with the formula*

$$\begin{aligned} \operatorname{ext} &: W^{1-\frac{1}{p},p}(M) \to W^{1,p}(M \times [0,1]) \\ \operatorname{ext} u(x,t) &= u_t(x) \quad x \in M\,,\ t \in [0,1] \end{aligned} \tag{4.10}$$

where $u_t(x)$ *is the standard mollification of* u *on* M *with respect to the coordinates* $(\mathcal{B}, \Psi)$*. (See Definition 4.4.7). Then we have*

1. $\operatorname{ext} : W^{1-\frac{1}{p},p}(M) \to W^{1,p}(M \times [0,1])$ *is a bounded linear operator.*
2. $\operatorname{tr} : W^{1,p}(M \times [0,1]) \to W^{1-\frac{1}{p},p}(M)$ *is bounded and a left inverse of* ext*, which is to say that* $u = \operatorname{tr}(\operatorname{ext} u)$ *for all* $u \in W^{1-\frac{1}{p},p}(M)$*.*
3. *For* $\gamma > 1 - \frac{1}{p}$ *and a coordinate ball* $B_R \subset M$ *we have*

$$[\operatorname{ext} v]_{W^{1,p}(B_R \times [0,1])} \leq C(\gamma) R^n [v]_{C^{0,\gamma}}. \tag{4.11}$$

We also define the cut-off *extension operator*

$$\begin{aligned} \operatorname{ext}_{\text{mod}} &: W^{1-\frac{1}{p},p}(M) \to W^{1,p}(M \times [0,1]) \\ \operatorname{ext}_{\text{mod}} u(x,t) &= u_t(x)\chi(t) \quad x \in M, t \in [0,1] \end{aligned} \tag{4.12}$$

where $[0,1/2] \prec \chi \prec [0,1)$ *is a fixed smooth cut-off.* $\operatorname{ext}_{\text{mod}}$ *has the following properties:*

1. $\operatorname{ext}_{\text{mod}} : W^{1-\frac{1}{p},p}(M) \to W^{1,p}(M \times [0,1])$ *is a bounded linear operator.*
2. $\operatorname{ext}_{\text{mod}} u$ *is supported away from* $M \times \{1\}$.
3. $\operatorname{tr} : W^{1,p}(M \times [0,1]) \to W^{1-\frac{1}{p},p}(M)$ *is bounded and a left inverse of* $\operatorname{ext}_{\text{mod}}$, *which is to say that* $u = \operatorname{tr}(\operatorname{ext}_{\text{mod}} u)$ *for all* $u \in W^{1-\frac{1}{p},p}(M)$.
4. *The restriction of* $\operatorname{ext}_{\text{mod}}$ *to* $W^{1,p}(M) \subseteq W^{1-\frac{1}{p},p}(M)$ *(recall Theorem 3.5.10) is bounded as an operator*
$$\operatorname{ext}_{\text{mod}} : W^{1,p}(M) \to W^{1,p}(M \times [0,1]). \tag{4.13}$$

We will not provide the proofs of Proposition 4.4.14, 4.4.15, or Theorem 4.4.16. Gagliardo extension was explained already for Euclidean spaces in Theorem 3.5.9, and the computations needed to extend these results to manifolds has been outlined in earlier results of this section. We merely remark that the boundedness of (4.13) is a consequence of Property 1 and of the boundedness of the inclusion $W^{1,p}(M) \subseteq W^{s,p}(M)$ proven in Theorem (3.5.10).

Finally, with all these spaces $C^m(M)$, $L^p(M)$, $W^{m,p}(M)$, $W^{s,p}(M)$, we also define the corresponding spaces of differential forms $C^m(\bigwedge^k M)$, $L^p(\bigwedge^k M)$, $W^{m,p}(\bigwedge^k M)$, $W^{s,p}(\bigwedge^k M)$ to be those spaces for which the coefficient functions with respect to a fixed atlas belong to the spaces C^m, L^p, $W^{m,p}$, $W^{s,p}$. As usual, the norms depend on the choice of a finite atlas for M, but all such norms are equivalent.

With these definitions, we state and prove one last simple but fundamental lemma on the relationships between smooth mollification and pullback.

Lemma 4.4.17. *For* $f \in C^{0,\gamma}(\mathbb{R}^n, \mathbb{R}^N)$, $t > 0$, *and* $\kappa \in L^\infty(\bigwedge^k \mathbb{R}^N)$

$$\|f_t^* \kappa\|_{L^\infty} \lesssim \|\kappa\|_{L^\infty} [f]^k_{C^{0,\gamma}} t^{-k(1-\gamma)}. \tag{4.14}$$

Proof. Denote points in $\mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and z. Denote points in $\mathbb{R}^N$ with $y = (y_1, \ldots, y_m)$. Write $\kappa = \sum_\alpha \psi_\alpha dy_\alpha$ where the summation is over multi-indices $\alpha = \{\alpha_1 < \ldots < \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\psi_\alpha \in C^0(\mathbb{R}^N)$. Now

$$f_t^*\kappa = \sum_\alpha \psi_\alpha \circ f_t \, df^t_{\alpha_1} \wedge \ldots \wedge df^t_{\alpha_k}$$

and we have

$$df^t_{\alpha_j} = \sum_{i=1}^n \frac{\partial f^t_{\alpha_j}}{\partial x_i} dx_i.$$

Thus

$$\|f_t^*\kappa\|_{L^\infty} \lesssim \|\kappa\|_{L^\infty} [f_t]^k_{C^1}. \tag{4.15}$$

So we have only to estimate the C^1 semi-norm of f_t. Fix $\beta \in \{1, \ldots, n\}$ and $\alpha \in \{1, \ldots, m\}$ and $x \in \mathbb{R}^n$ and estimate, using (3.4)

$$\begin{aligned}\left|\frac{\partial f^t_\alpha}{\partial x_\beta}\right| &\leq t^{-m-1} \int_{\mathbb{R}^n} |f_\alpha(x+z) - f_\alpha(x)| \left|\frac{\partial \eta}{\partial x_\beta}(t^{-1}z)\right| dz \\ &\leq \|\nabla\eta\|_{L^1} [f]_{C^{0,\gamma}} t^{\gamma-1}. \end{aligned} \tag{4.16}$$

The estimate (4.14) follows from (4.15) and (4.16). □

4.5 RIEMANNIAN MANIFOLDS

So far in this thesis we have not considered metric structures on manifolds. To study quantities like distance and curvature on a manifold, we need to introduce additional structure. The most elegant and efficient way to do this turns out to be to introduce an *inner product* on the tangent bundle.

Definition 4.5.1. A *Riemannian manifold* is a manifold M together with a smoothly varying inner product g on TM.

That is, at each point $p \in M$, g_p is an inner product on T_pM; and this association is smooth in the sense that $g(X, Y)(p)$ will be a smooth function for any smooth vector fields X and Y on M.

For $X \in T_pM$, there is a unique $\xi \in T_p^*M$ such that

$$\xi(Y) = g_p(X, Y) \quad \forall Y \in T_pM.$$

Recall, the Riesz Representation Theorem 2.2.11 says that this association is an isomorphism with an inverse

$$\mathrm{rep} : T_p^*M \to T_pM$$
$$\xi \mapsto X$$

Using this isomorphism, we can extend the defintion of g from T_pM to T_p^*M with the obvious formula

$$g(\xi, \eta) = (\mathrm{rep}(\xi), \eta) \quad \xi, \eta \in T_p^*M$$

Definition 4.5.2. Given an oriented Riemannian manifold (M, g, ω) with metric g and orientation ω, we define the volume form vol to be the unique n-form on M with the property that, for each $p \in M$

$$\mathrm{vol}_p = \xi_1 \wedge \ldots \wedge \xi_n$$

where $\xi_1, \ldots, \xi_n \in T_p^*M$ form an orthonormal basis with respect to g (extended to T_p^*M as above) with positive orientation with respect to ω.

Given positively oriented orthonormal bases $\xi_1, \ldots \xi_n$ and $\xi_1', \ldots \xi_n'$ for T_p^*M, we have $\xi_1 \wedge \ldots \wedge \xi_n = \xi_1' \wedge \ldots \wedge \xi_n'$, so that vol_p is independent of the choice of $\xi_1, \ldots, \xi_n$. One can also show that vol_p is smooth. Indeed, in a coordinate chart, let $\eta_1, \ldots, \eta_n$ be a smooth basis of one-forms and then construct an orthonormal basis using the Gram-Schmidt orthonormalization algorithm. This algorithm will preserve smoothness in the outputs $\xi_1, \ldots, \xi_n$, since the Riemannian metric is smooth.

To summarize what we did above, an inner product g on T_pM allows us to naturally identify T_pM and T_p^*M. This identification then allowed us to extend g to T_p^*M. We can take this further. By the isomorphism (2.9), our natural isomorphism between T_pM and T_p^*M extends to

$$(\textstyle\bigwedge^k(T_p^*M))^* \cong \textstyle\bigwedge^k(T_p^{**}M) \cong \textstyle\bigwedge^k(T_p^*M)$$

If we call this map $\mathrm{rep} : \bigwedge^k(T_p^*M) \to (\bigwedge^k(T_p^*M))^*$, then we can now extend the inner product g from T_pM to all of $\bigwedge_p^k M$ by the formula

$$g(\xi, \eta) = (\mathrm{rep}(\xi), \eta) \quad \xi, \eta \in \textstyle\bigwedge^k(T_p^*M)$$

Thus we have shown that the inner product structure of a Riemannian manifold extends naturally to the exterior bundle. Now we can define a Hilbert space $L^2(\bigwedge^k M)$ with inner product

$$(\omega, \varphi) = \int_M g(\omega, \varphi)\, d\,\mathrm{vol} \tag{4.17}$$

where the set $L^2(\bigwedge^k M)$ consists of those differential forms ω for which $(\omega, \omega) < \infty$.

This is sometimes presented with slightly different language, using the so-called *Hodge star* operator:

Definition 4.5.3. Let (M, g, ω) be an *oriented* Riemannian manifold, $p \in M$, and $\xi \in \bigwedge_p^k M$. Define for the moment the square bracket $[\,\cdot\,,\,\cdot\,] : \bigwedge_p^{n-k} M \times \bigwedge_p^k M \to \mathbb{R}$ by the formula

$$\eta \wedge \xi = [\eta, \xi]\, \mathrm{vol}_p$$

We define the *Hodge star* $*\xi$ to be the unique form in $\bigwedge_p^{n-k} M$ satisfying

$$[\eta, \xi] = g(\eta, *\xi) \quad \forall \eta \in \textstyle\bigwedge_p^k M$$

Proposition 4.5.4. *For* $0 \le k \le n$*, the Hodge star is an isomorphism*

$$* : \textstyle\bigwedge_p^k M \to \bigwedge_p^{n-k} M$$

Proof. $*$ is clearly linear, so it suffices to establish that the $\ker(*)$ is trivial. Suppose $*\xi = 0$. That means, $\eta \wedge \xi = 0$ for all $(n-k)$-forms η. But in local coordinates, we can write $\xi = \sum_I a_I dx_I$. If $\xi \neq 0$, then it would be easy to find η satisfying $\xi \wedge \eta = dx_1 \wedge \ldots dx_n$. $\square$

Remark 4.5.5. We can define

$$(\omega, \varphi) = \int_M \omega \wedge *\varphi \tag{4.18}$$

and this definition is equivalent to (4.17).

It turns out that the Hodge $*$ operator is convenient for expressing the L^2-adjoint of the exterior derivative operator d.

Definition 4.5.6. Define $\delta = (-1)^{n(k+1)+1} * d*$ on $C^\infty(\bigwedge^k M)$.

Proposition 4.5.7. *δ is the formal adjoint of d on $C^\infty(\bigwedge^k M)$ with respect to the L^2 inner product. That is, if $\omega \in C^\infty(\bigwedge^k M)$ then we have*

$$(\delta\omega, \varphi) = (\omega, d\varphi) \quad \forall \omega, \varphi \in C^\infty(\textstyle\bigwedge^{k-1} M) \tag{4.19}$$

Definition 4.5.8. Given a Riemannian manifold (M, g), define the *intrinsic* metric on M,

$$d(p, q) = \inf_{\gamma : p \rightsquigarrow q} \mathrm{length}_g(\gamma) \tag{4.20}$$

where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to M$ with startpoint $\gamma(0) = p$ and endpoint $\gamma(1) = q$; and $\mathrm{length}_g(\gamma)$ is defined in the natural way,

$$\mathrm{length}_g(\gamma) := \int_0^1 |\gamma'(t)|\, dt := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt.$$

Remark 4.5.9. It will be particularly useful to note that if $f \in W^{1,p}(\bigwedge^k M)$, then its $W^{1,p}$-norm can be computed as

$$\|f\|_{W^{1,p}(\bigwedge^k M)} \approx \|f\|_{L^p(\bigwedge^k M)} + \|df\|_{L^p(\bigwedge^k M)} + \|\delta f\|_{L^p(\bigwedge^k M)}. \tag{4.21}$$

Indeed, in normal coordinates centered at $x_0 \in M$ we can write $f = \sum_I f_I dx_I$ and then, since the vector fields dx_j are an orthonormal basis for $T^*_{x_0}M$, we have

$$df|_{x_0} = \sum_I \sum_{j \notin I} \frac{\partial f_I}{\partial x_j} \quad \delta f|_{x_0} = \sum_I \sum_{j \in I} \frac{\partial f_I}{\partial x_j}.$$

and so the differential forms df and δf "contain all partial derivatives of f near x_0." which gives (4.21).

5.0 CLASSICAL HODGE THEORY

Throughout this chapter, we fix a compact oriented riemannian manifold M without boundary, of dimension n.

5.1 THE LAPLACE-BELTRAMI OPERATOR AND POISSON EQUATION

Definition 5.1.1. The Laplace-Beltrami Operator Δ is defined, for $\omega \in C^\infty(\bigwedge^k M)$,

$$\Delta\omega = d\delta\omega + \delta d\omega. \tag{5.1}$$

We will say that ω is *harmonic* if it satisfies $\Delta\omega = 0$ on M. we will denote the space of harmonic forms on M with $H^k(M)$ or simply H^k if the manifold is understood. If $f \in C^\infty(\bigwedge^k M)$, we will say that $f \in H^\perp$ if $\int_M f \wedge *\omega = 0$ for all harmonic $\omega \in H^k$.

Remark 5.1.2. Observe that, if $\omega \in H^k(M)$, then we must have $d\omega = 0$ and $\delta\omega = 0$. Indeed, the definition of δ gives

$$0 = \int_M \Delta\omega \wedge *\omega = \int_M d\omega \wedge *d\omega + \delta\omega \wedge *\delta\omega = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

Definition 5.1.3. We will say that ω is *weakly harmonic* if $(d\omega, d\omega) + (\delta\omega, \delta\omega) = 0$. We will denote the space of weakly harmonic k-forms $\overline{H}^k(M)$.

Consider the following problem: given a smooth differential form $f \in C^\infty(\bigwedge^k M)$, we ask if there exists a smooth differential form $\omega \in C^\infty(\bigwedge^k M)$ with the property that

$$\Delta\omega = f \tag{5.2}$$

It turns out that the natural spaces in which to study the Poisson Equation are the Sobolev spaces. Thus, if $\varphi \in L^2(\bigwedge^k M)$, we can say that $\omega \in W^{2,2}(\bigwedge^k M)$ is a solution to the Poisson Equation (5.2) if $\Delta\omega = f$ in L^2 since ω has weak second derivatives. However, if we multiply (5.2) on both sides by a test form $\varphi \in C^\infty(\bigwedge^k M)$ and integrate over M:

$$\int_M \Delta\omega \wedge *\varphi = \int_M f \wedge *\varphi \quad \forall\varphi \in C^\infty(\textstyle\bigwedge^k M).$$

Then, since $\Delta = d\delta + \delta d$, we can use integration by parts (4.19) and make the following our definition of a *weak solution*:

Definition 5.1.4. Given $f \in L^2(\bigwedge^k M)$, we say that $\omega \in W^{1,2}(\bigwedge^k M)$ is a *weak solution* of the Poisson Equation (5.2) if there holds

$$\int_M d\omega \wedge *d\varphi + \delta\omega \wedge *\delta\varphi = \int_M f \wedge *\varphi \quad \forall\varphi \in W^{1,2}(\textstyle\bigwedge^k M). \tag{5.3}$$

5.2 ELLIPTIC REGULARITY

We now prove that solutions to the Poisson Equation, if they exist, are regular in some sense. Throughout this section and the next section, we will use angle brackets as follows: for $\omega, \varphi \in W^{1,2}(\bigwedge^k M)$, we define

$$\langle\omega, \varphi\rangle = \int_M d\omega \wedge *d\varphi + \delta\omega \wedge *\delta\varphi. \tag{5.4}$$

Lemma 5.2.1. *Suppose $\omega, \varphi \in W^{1,2}(\bigwedge^k \mathbb{R}^n)$ are smooth k-forms with compact support. Using the notation $\nabla\omega$ for the vector of all partial derivatives of all components of ω, we have*

$$\int_{\mathbb{R}^n} (\nabla\omega, \nabla\varphi) = \langle\omega, \varphi\rangle. \tag{5.5}$$

In particular,

$$\|\nabla\omega\|_{L^2} \approx \|d\omega\|_{L^2} + \|\delta\omega\|_{L^2}. \tag{5.6}$$

Proof. Both sides of (5.5) are bilinear functionals of ω and φ, and so it suffices to check that it is true for the components. Moreover, approximating ω and φ with $C_c^\infty(\bigwedge^k \mathbb{R}^n)$ forms, it suffices to prove the equality for smooth forms ω and φ, and by bilinearity we can assume they can be written as $\omega = a\,dx_I$ and $\varphi = b\,dx_J$ with $\#(I) = \#(J)$, where a and b are smooth functions on $\mathbb{R}^n$. We then consider three distinct cases.

Case 1: $\#(I \cap J) < k - 1$

Then both sides of (5.5) are zero.

Case 2: $I \backslash J = \{q\}$, $J \backslash I = \{r\}$

$$\begin{aligned}\langle \omega, \varphi \rangle &= \int_{\mathbb{R}^n} \pm \left(\frac{\partial a}{\partial x_r} \frac{\partial b}{\partial x_q} - \frac{\partial a}{\partial x_q} \frac{\partial b}{\partial x_r} \right) = \int_{\mathbb{R}^n} \pm \left(\frac{\partial}{\partial x_r} \left[a \frac{\partial b}{\partial x_q} \right] - \frac{\partial}{\partial x_q} \left[a \frac{\partial b}{\partial x_r} \right] \right) \\ &= 0 = \int_{\mathbb{R}^n} (\nabla \omega, \nabla \varphi)\end{aligned}$$

Case 3: $I = J$

$$\begin{aligned}\langle \omega, \varphi \rangle &= \sum_{i \notin I} \int_{\mathbb{R}^n} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_i} + \sum_{i \in I} \int_{\mathbb{R}^n} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_i} = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_i} \\ &= \int_{\mathbb{R}^n} (\nabla \omega, \nabla \varphi)\end{aligned}$$

which completes the proof in all cases. □

The equivalence (5.6) can be extended from Euclidean space to $W^{1,2}$ forms on Riemannian manifolds. The argument somewhat tedious: one takes smooth approximation of the form, uses the compactness of M to construct a finite collection of coordinate patches for which the standard tangent vector basis $\frac{\partial}{\partial x_i}$ is "nearly orthogonal" and a partition of unity subordinate to those patches; then one invokes Lemma 5.2.1 on the patches. This process introduces an error $C\|\omega\|_{L^2(M)}$ (from the derivatives of the partition of unity and the error implicit in the phrase "nearly orthogonal"). The reader may consult [23, Section 4], which also proves it in the L^p setting, $p \neq 2$. We state the result as follows:

Theorem 5.2.2 (Gaffney Inequality)**.** *For* $\omega \in W^{1,2}(\bigwedge^k M)$ *we have*

$$\|\omega\|_{W^{1,2}(M)} \lesssim \|\omega\|_{L^2(M)} + \|d\omega\|_{L^2(M)} + \|\delta\omega\|_{L^2(M)}.$$

Proposition 5.2.3 (Poincaré Inequality). *For $\omega \in W^{1,2}(\bigwedge^k M) \cap (\tilde{H}^k(M))^\perp$ there holds*

$$\|\omega\|_{L^2(M)} \lesssim \|d\omega\|_{L^2(M)} + \|\delta\omega\|_{L^2(M)}. \tag{5.7}$$

Also, on balls $B_\rho(x_0) \subseteq M$,

$$\int_{B_\rho(x_0)} |\omega - \omega_{x_0,\rho}|^2 \lesssim \rho^2 \int_{B_\rho(x_0)} |d\omega|^2 + |\delta\omega|^2. \tag{5.8}$$

Proof. We first prove (5.7). Suppose it were false. Then we could find a sequence $\omega_n \in (\tilde{H}^k(M))^\perp$ with $\|\omega_n\|_{L^2} = 1$ and $\|d\omega_n\|_{L^2} + \|\delta\omega_n\|_{L^2} \to 0$. By the Gaffney Inequality, $\|\nabla\omega_n\|_{L^2}$ is bounded. By the Rellich-Kondrachov Theorem 3.4.12 there is a convergent subsequence which we may also call ω_n. Now for all $\varphi \in C^\infty(\bigwedge^k M)$ we have

$$0 \leftarrow \int_M d\omega_n \wedge *d\varphi + \delta\omega_n \wedge *\delta\varphi = \int_M \omega_n \wedge *\Delta\varphi \to (\omega, \Delta\varphi) = (\nabla\omega, \nabla\varphi).$$

Remember, this is true for smooth φ. But taking a sequence φ_k approaching ω in $W^{1,2}$-norm, we see that in fact,

$$(\nabla\omega, \nabla\omega) = 0$$

which is to say, $\omega \in \tilde{H}^k(M)$. Since we assumed $\omega \in (\tilde{H}^k(M))^\perp$, we must have $\omega = 0$ contradicting $\|\omega\|_{L^2} = 1$. This contradiction shows that the inequality (5.7) must hold for some constant. ◻

Remark 5.2.4. In light of Remark 4.5.9, Poincaré's Inequality shows that

$$\|\omega\|_{L^2(M)} \lesssim [\omega]_{W^{1,2}(M)}$$

for $\omega \in W^{1,2}(\bigwedge^k M) \cap (\overline{H}^k(M))^\perp$.

Theorem 5.2.5 (Global Caccioppoli Inequality). *Suppose $f \in L^2(\bigwedge^k M)$ and $\omega \in W^{1,2}(\bigwedge^k M)$ is a weak solution to the Poisson equation $\Delta\omega = f$. Then we have*

$$\|\omega\|_{W^{1,2}} \lesssim \|\omega\|_{L^2} + \|f\|_{L^2}. \tag{5.9}$$

In fact, if we additionally assume that $\omega \in (H^k(M))^\perp$,

$$\|\omega\|_{W^{1,2}} \lesssim \|f\|_{L^2}. \tag{5.10}$$

Proof. We first use the Gaffney Inequality, and then we use ω as a test-function, i.e., we invoke (5.1.4) with $\varphi = \omega$:

$$\begin{aligned}
\|\omega\|_{W^{1,2}}^2 &\lesssim \|\omega\|_{L^2}^2 + \|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 \\
&= \|\omega\|_{L^2}^2 + \int_M d\omega \wedge *d\omega + \delta\omega \wedge *\delta\omega \\
&= \|\omega\|_{L^2}^2 + \int_M f \wedge *\omega \\
&\lesssim \|f\|_{L^2}^2 + \|\omega\|_{L^2}^2.
\end{aligned}$$

To prove the second claim, we of course need Poincaré. Repeating the computation, now assuming additionally that $\omega \in (H^k(M))^\perp$, and temporarily dispensing with the $\lesssim$ notation for clarity,

$$\begin{aligned}
\|\omega\|_{W^{1,2}}^2 &\leq C_1(\|\omega\|_{L^2}^2 + \|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2) \\
&\leq C_2(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2) \\
&= C_2 \int_M f \wedge *\omega \\
&\leq C_2 \|f\|_{L^2} \|\omega\|_{L^2} \\
&\leq C_\varepsilon \|f\|_{L^2}^2 + \varepsilon C_2 \|\omega\|_{L^2}^2 \\
&\leq C_\varepsilon \|f\|_{L^2}^2 + \varepsilon C_2 \|\omega\|_{W^{1,2}}^2.
\end{aligned}$$

If we take ε in Young's Inequality small enough that $\varepsilon C_2 = 1/2$, then we can subtract $\frac{1}{2}\|\omega\|_{W^{1,2}}^2$ from both sides to find the estimate (5.10). $\square$

Proposition 5.2.6 (Local Caccioppoli Inequality)**.** *Suppose $f \in L^2(\bigwedge^k M)$, $\omega \in W^{1,2}(\bigwedge^k M)$ is a weak solution to the Poisson equation $\Delta\omega = f$. Then there exists R_0 such that, for all $x_0 \in M$, there is a coordinate patch $\varphi : U \to M$ centered at x_0 with $B_{R_0}(x_0) \subseteq \varphi(U)$), satisfying*

$$\int_{B_\rho} |Du|^2 \lesssim \frac{1}{(R-\rho)^2} \int_{B_R} |u - u_{x_0,R}|^2 \tag{5.11}$$

for all $0 < \rho < R < R_0$, where u is the coordinate representation of ω in the coordinate system φ.

The proof of Proposition 5.2.6 is much longer than that of the global Caccioppoli inequality because it requires the use of a cut-off function $B_\rho(x_0) \prec \eta \prec B_R(x_0)$. It is, however, standard, and a proof in the language of differential forms (as we are using it here) is in [33]. The technique of cut-off functions and test functions to prove Caccioppoli-type inequalities is explained in detail in the reference [14, Chapter 6].

Theorem 5.2.7. *The space of harmonic k-forms on M is finite-dimensional.*

Proof. Let (ω_n) be any L^2-bounded sequence of harmonic forms. By the global Caccioppoli inequality (5.9), it is also $W^{1,2}$-bounded. Now by the Rellich-Kondrachov compactness theorem 3.4.12 there is a convergent subsequence in L^2-norm. Thus the space of harmonic forms equipped with L^2-norm is a locally compact Hilbert space, hence finite-dimensional. $\square$

Corollary 5.2.8. *$W^{\ell,2}(\bigwedge^k M)$ can be decomposed orthogonally (with respect to the usual L^2 inner product)*

$$W^{\ell,2}(\textstyle\bigwedge^k M) = H^k(M) \oplus ((H^k(M))^\perp \cap W^{\ell,2}(\textstyle\bigwedge^k M))$$

Proof. Since $H^k(M) \subseteq L^2(\bigwedge^k M)$ is finite-dimensional, it is closed. So we have $L^2 = H^k \oplus (H^k)^\perp$ by Proposition 2.2.10. The corollary follows. $\square$

On many occasions throughout our work we will make reference to forms which belong to a space $X \cap H^\perp(\bigwedge^k M)$ where X may be $C^{k,\gamma}$, $W^{s,p}$, $W^{k,p}$, etc. This will be meant as short-hand for

$$X \cap H^\perp(\textstyle\bigwedge^k M) = \left\{\omega \in X(\textstyle\bigwedge^k M) \ : \ \int_M \omega \wedge *h = 0 \forall h \in H^k(M)\right\} \tag{5.12}$$

The following two theorems on higher-order regularity are proved in [33, Chapter 4]

Theorem 5.2.9 (Second Order Regularity)**.** *Suppose $f \in L^2(\bigwedge^k M)$ and $\omega \in W^{1,2}(\bigwedge^k M)$ is a weak solution to the Poisson equation (5.2). Then in fact, $\omega \in W^{2,2}(\bigwedge^k M)$ with the inequality*

$$\|\omega\|_{W^{2,2}(\bigwedge^k M)} \lesssim \|\omega\|_{L^2(\bigwedge^k M)} + \|f\|_{L^2(\bigwedge^k M)}.$$

In fact, this estimate is local, in the sense that if $\Delta\omega = f$ only on an open set $\Omega \subseteq M$, there still holds for $U \Subset \Omega$

$$\|\omega\|_{W^{2,2}(\bigwedge^k U)} \lesssim \|\omega\|_{L^2(\bigwedge^k \Omega)} + \|f\|_{L^2(\bigwedge^k \Omega)}.$$

Theorem 5.2.10 (Higher-Order Regularity)**.** *Suppose $f \in W^{\ell,2}(\bigwedge^k M)$ and $\omega \in W^{1,2}(\bigwedge^k M)$ is a weak solution to the Poisson equation $\Delta\omega = f$. Then in fact $\omega \in W^{\ell+2,2}(\bigwedge^k M)$ with the inequality*

$$\|\omega\|_{W^{\ell+2,2}(\bigwedge^k M)} \lesssim \|\omega\|_{L^2(\bigwedge^k M)} + \|f\|_{W^{\ell,2}(\bigwedge^k M)} \tag{5.13}$$

and this estimate is local: for $U \Subset \Omega \subseteq M$, and if ω is only a weak solution on Ω we have the local estimate

$$\|\omega\|_{W^{\ell+2,2}(\bigwedge^k U)} \lesssim \|\omega\|_{L^2(\bigwedge^k \Omega)} + \|f\|_{W^{\ell,2}(\bigwedge^k \Omega)}.$$

Corollary 5.2.11. *Weakly harmonic forms $\omega \in \overline{H}^k(M)$ have weak derivatives of every order, with estimate*

$$\|\omega\|_{W^{k,2}(\bigwedge^k M)} \leq C_{k,M}\|\omega\|_{L^2(\bigwedge^k M)}$$

Finally, as a consequence of the previous corollary and the Sobolev Embedding Theorem 3.4.14, we obtain

Corollary 5.2.12. *Harmonic forms are smooth:*

$$\tilde{H}^k(M) = H^k(M).$$

Remark 5.2.13. We will no longer use $\overline{H}^k(M)$ to refer to the space of weakly harmonic k-forms on M since they coincide with the classically harmonic k-forms $H^k(M)$.

This concludes our brief review of the classical *a priori* regularity estimates for the Poisson equation $\Delta\omega = f$. In this setting, *a priori* refers to the fact that we have not even shown whether such solutions exist—merely that *if* they exist, they satisfy certain estimates. In the next section, we fully answer the question of *whether* a Poisson equation $\Delta\omega = f$ can be solved.

5.3 HODGE DECOMPOSITION

Theorem 5.3.1. *Let $f \in L^2(\bigwedge^k M)$. The Poisson equation $\Delta\omega = f$ is solvable for $\omega \in W^{2,2}(\bigwedge^k M)$ if and only if f is* orthogonal *to $H^k(M)$ with respect to the L^2 inner product. In this case, ω satisfies*

$$\|\omega\|_{W^{2,2}(\bigwedge^k M)} \lesssim \|f\|_{L^2(\bigwedge^k M)} + \|\omega\|_{L^2(\bigwedge^k M)}. \tag{5.14}$$

If we take the unique solution $\omega \in H^\perp$, then it enjoys the estimate

$$\|\omega\|_{W^{2,2}(\bigwedge^k M)} \lesssim \|f\|_{L^2(\bigwedge^k M)}. \tag{5.15}$$

Proof. Observe that $\mathcal{H} = W^{1,2}(\bigwedge^k M) \cap (H^k)^\perp$ is a Hilbert space when equipped with the inner product

$$\langle \omega, \eta \rangle = \int_M d\omega \wedge *d\eta + \delta\omega \wedge *\delta\eta. \tag{5.16}$$

Indeed, by the Pioncaré inequality (Theorem 5.2.3), $\|\omega\|_{L^2}^2 \leq \langle \omega, \omega \rangle$ so $\langle \omega, \omega \rangle = 0$ only if $\omega = 0$. Now observe that $\mathcal{H} \ni \varphi \mapsto \int_M f \wedge *\varphi$ is a bounded linear functional on $\mathcal{H}$. By the Riesz Representation Theorem 2.2.11, there is a representative ω such that

$$\langle \omega, \varphi \rangle = \int_M f \wedge *\varphi \quad \forall \varphi \in \mathcal{H}.$$

Now if $\varphi \in W^{1,2}(\bigwedge^k M)$, we can decompose it $\varphi = \varphi^H + \varphi^\perp$, and we still have

$$\langle \omega, \varphi \rangle = \langle \omega, \varphi^H \rangle + \langle \omega, \varphi^\perp \rangle = 0 + \langle \omega, \varphi^\perp \rangle = \int_M f \wedge *\varphi^\perp = \int_M f \wedge *\varphi$$

so $\Delta\omega = f$ in the weak sense. By Theorem 5.2.9, $\omega \in W^{2,2}(\bigwedge^k M)$. □

Corollary 5.3.2 (The Classical Hodge Decomposition)**.** *Let $f \in W^{\ell,2}(\bigwedge^k M)$. The Poisson equation $\Delta\omega = f$ is solvable for $\omega \in W^{\ell+2,2}(\bigwedge^k M)$ if and only if f is orthogonal to $H^k(M)$ with respect to the L^2 inner product. In this case, the solution ω satisfies*

$$\|\omega\|_{W^{\ell+2,2}(\bigwedge^k M)} \lesssim \|\omega\|_{L^2(\bigwedge^k M)} + \|f\|_{W^{\ell,2}(\bigwedge^k M)}. \tag{5.17}$$

Moreover, if we take ω to be the unique solution in $H^\perp$ then we have

$$\|\omega\|_{W^{\ell+2,2}(\bigwedge^k M)} \lesssim \|f\|_{W^{\ell,2}(\bigwedge^k M)}. \tag{5.18}$$

Proof. The previous theorem gives the existence of a solution $\omega \in W^{2,2}(\bigwedge^k M)$. Applying Theorem 5.2.10, we have $\omega \in W^{\ell+2,2}$ with the claimed estimate. □

Corollary 5.3.3. *If $f \in C^\infty(\bigwedge^k M)$ is smooth, then there exists a unique solution $\omega \in C^\infty(\bigwedge^k M) \cap H^\perp$ satisfying $\Delta\omega = f$ in the classical sense. All solutions are given by $\omega + h$ where $h \in H^k(M)$, and they are all smooth.*

Proof. By Corollary 5.3.2 there is a unique weak solution $\omega \in W^{\ell,2}(\bigwedge^k M) \cap H^\perp$ for all $\ell \in \mathbb{N}$. The Sobolev Embedding Theorem 3.4.14 implies $\omega \in C^m(\bigwedge^k M)$ for all $m \in \mathbb{N}$. □

Proposition 5.3.4. *Given a domain $\Omega \subsetneq M$ diffeomorphic to a Euclidean ball, and $f \in W^{\ell,2}(\bigwedge^k \Omega)$, there exists unique $\omega \in W^{\ell+2,2}(\bigwedge^k B)$ such that $\Delta\omega = f$ in B and $\omega = 0$ at the boundary of B.*

Proof. Thanks to the Gaffney Inequality (Theorem 5.2.2) and Poincaré inequality (Theorem 5.2.3) we have $\|\omega\|_{L^2}^2 \lesssim \|\nabla\omega\|_{L^2} \lesssim \langle\omega, \omega\rangle$ where we have defined $\langle\omega, \varphi\rangle$ in the same way as in the proof of Hodge Decomposition above, (5.16), and so this is an inner product. Once again, Riesz Representation establishes the existence of a solution ω satisfying the proposition. Theorem 5.2.10 shows that $\omega \in W^{\ell+2,2}(\bigwedge^k B)$. □

Remark 5.3.5. We have been referring to the solution ω to the Poisson equation $\Delta\omega = f$ as a "Hodge Decomposition" of f. Historically, Hodge Decomposition refers to the fact that a differential form f can be *decomposed* $f = h + d\alpha + \delta\beta$ where h is harmonic, $\delta\alpha = 0$, and $d\beta = 0$. This is an immediate consequence of the fact that the Poisson equation $\Delta\omega = f$ has a solution. Indeed, once we find a solution $\omega \in H^\perp$, we can take $\alpha = \delta\omega$ and $\beta = d\omega$, and then $h = f - d\alpha - \delta\beta$.

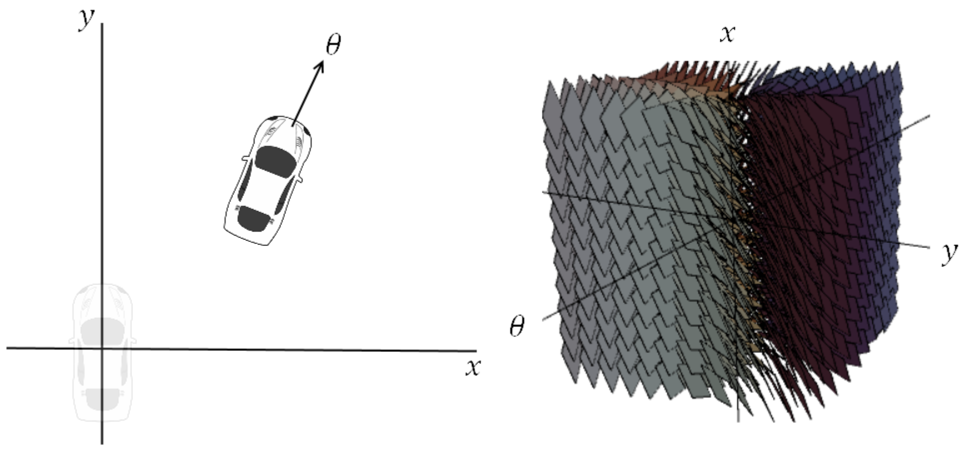
6.0 SUB-RIEMANNIAN MANIFOLDS

6.1 EXAMPLES AND MOTIVATION

Consider the deceptively simple problem of a driver who wishes parallel park their car at position (x, y) in a parking spot at $(0, 0)$. Throughout the process, the state of the car is tracked with three coordinates: x and y are the car's position; $\theta \in \mathbb{S}^1$ denotes the (say, clockwise) angle made by the car with the y-axis (Figure 6.1a).

Our problem can be stated as, *find the shortest legal path from, say,* $(x, y, \theta) = (1, 0, 0)$ *to* $(x, y, \theta) = (0, 0, 0)$. This problem captures everything relevant in the definition of a *sub-riemannian manifold*: we have a manifold representing some state space $\mathbb{M}$, which for us is $(x, y, \theta) \in \mathbb{M} = \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$; there is a notion of *path-length*: for this problem, we might say that path-length is the length of the driver's path through $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$, which is to say, $\mathbb{M}$ has a metric, making it a Riemannian manifold; and finally, there is a notion of *permissible paths*: the path $t \mapsto (1 - t, 0, 0)$ would be the "shortest" path from $(1, 0, 0)$ to $(0, 0, 0)$, but it is not permissible. It corresponds to the car sliding into the parking spot without any use of the steering wheel, which we assume to not be possible!

Let's take time to elaborate on this notion of permissible paths, as this is the distinguishing feature which sets apart sub-Riemannian manifolds from ordinary Riemannian manifolds. At any moment, the car is constrained to two types of motion: forward-and-backward in the direction it is currently pointing, or rotation. (Let us ignore that cars cannot *just* rotate without forward motion. This admittedly would complicate the model.) If the car is at $p = (x, y, \theta)$, then the permissible directions of travel form a two-dimensional subspace of $T_p\mathbb{M}$ spanned by the vectors $\Theta = \frac{\partial}{\partial \theta}$ and $X = \sin(\theta)\frac{\partial}{\partial x} + \cos(\theta)\frac{\partial}{\partial y}$. Let's refer to this subspace by $H_p\mathbb{M} \subset T_p\mathbb{M}$. See Figure 6.1a, 6.1b for a visualization.



(a) The Parallel Parking Problem (b) The Horizontal Distribution on $\mathbb{M}$

Figure 6.1: The horizontal distribution of the Heisenberg group

The fact that the parallel parking problem is solvable—the fact that the car can move from $(1, 0, 0)$ to $(0, 0, 0)$—is a consequence of the fact that the vector fields X and Θ generate $T_p\mathbb{M}$ under Lie brackets. Indeed,

$$Y := [\Theta, X] = \cos\theta \frac{\partial}{\partial x} - \sin\theta \frac{\partial}{\partial y}$$

so that $\{\Theta, X, [\Theta, X]\}$ span $T_p\mathbb{M}$ at every point $p \in \mathbb{M}$. Visually, if one walks along the vector fields $\Theta, X, -\Theta, -X$ in that order, one moves in the direction of Y (Figure 6.2a, 6.2b).

With this notion of permissible paths, we let the distance between two points p and q in $\mathbb{M}$ be the length of the shortest permissible path between them. The resulting metric space is an example of a sub-Riemannian manifold, which we briefly formalize in the next section.

6.2 DEFINITIONS AND BASIC NOTIONS

Definition 6.2.1. A Sub-Riemannian Manifold is a triple $(\mathbb{M}, H\mathbb{M}, g)$ consisting of a connected n-dimensional manifold $\mathbb{M}$; a *horizontal distribution* $H\mathbb{M}$ which is a smoothly varying subspace

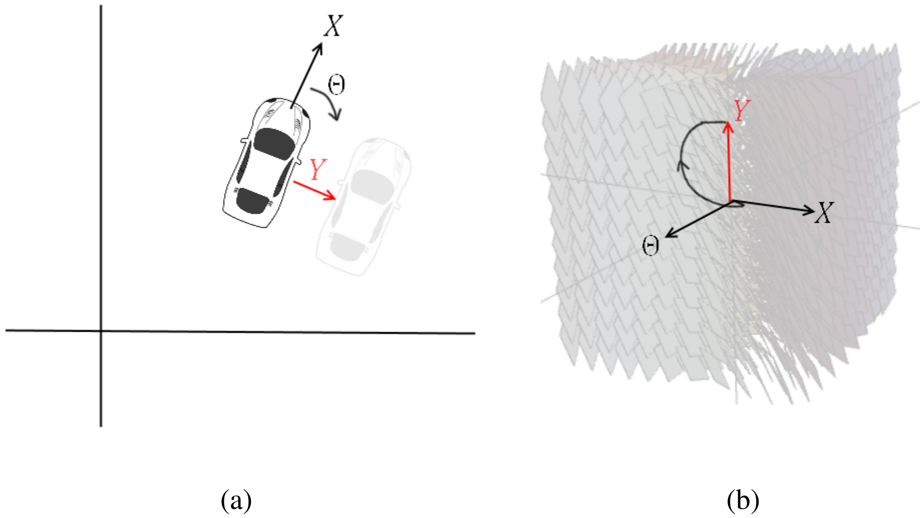


Figure 6.2: Lie bracket generation of the tangent space of the Heisenberg group

$H_p\mathbb{M} \subseteq T_p\mathbb{M}$ of constant dimension $k \leq n$, which *generates* $T_p\mathbb{M}$ as a Lie Algebra at each $p \in \mathbb{M}$; and a smoothly varying inner product g on $H\mathbb{M}$.

We clarify what we mean by generating the Lie Algebra at a point:

Definition 6.2.2. A horizontal distribution $H\mathbb{M} \subseteq T\mathbb{M}$ *generates the Lie algebra* at p if, on a neighborhood of p, we can find vector fields $X_1, \ldots, X_k$ in $H\mathbb{M}$ such that they and their iterated Lie brackets $X_i, [X_i, X_j], [[X_i, X_j], X_k], \ldots$ span $T_p\mathbb{M}$.

Remark 6.2.3. $H\mathbb{M}$ is lie bracket generating if and only if *any* vector fields $X_1, \ldots, X_k$ which pointwise span $T_q\mathbb{M}$ on a neighborhood about p generate the Lie algebra at p.

Theorem 6.2.4 (Chow). *Any points p, q in a sub-Riemannian manifold can be joined by a horizontal curve $\gamma : [0, 1] \to \mathbb{M}$.*

It seems that virtually any physical system has sub-Riemannian structure. Physical systems obey laws, and these laws can be viewed as restrictions on the possible paths of evolution of the system. For example, the space of all anatomical configurations of a cat is a manifold $\mathbb{M}$. The fact that cats, however stubborn, must obey the law of conservation of angular momentum when they fall is a sub-Riemannian structure $H\mathbb{M}$. The fact that a cat can right itself when dropped from an upside-down orientation means that $H\mathbb{M}$ generates the tangent space $T\mathbb{M}$. The fact that cats can do this quickly enough to land on their feet means they are adept navigators of the sub-Riemannian manifold $\mathbb{M}$.

Whatever their applications, we will study sub-Riemannian manifolds for their rich metric structure. We're going to avoid a comprehensive introduction to sub-Riemannian geometry, opting in this thesis to focus on the Heisenberg Group, to be introduced in the next section. For a proof of Chow's Theorem and a solid reference on these general objects, the reader may consult the collection [5], especially [4] and [19] therein. [9] is also an excellent introduction.

Definition 6.2.5 (Carnot-Caratheodory Metric)**.** Given two points p and q in a sub-Riemannian manifold $(\mathbb{M}, H\mathbb{M}, g)$, we define the Carnot-Caratheodory distance between them

$$\inf_{\gamma: p \rightsquigarrow q} \operatorname{length} \gamma \tag{6.1}$$

where the infimum is taken over all Lipschitz paths $\gamma : [0,1] \to \mathbb{M}$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma'(t) \in H_{\gamma(t)}\mathbb{M}$ almost everywhere, and where length γ is measured

$$\operatorname{length} \gamma = \int_0^1 \sqrt{g(\gamma'(s), \gamma'(s))}\, ds.$$

In the next section, we will outline the properties that make sub-Riemannian manifolds so enticing to an analytic geometer or geometric analyst as it were.

6.3 THE HEISENBERG GROUP

On $\mathbb{R}^{2n+1} = \{(x_1, y_1, \ldots, x_n, y_n, t) : x_i, y_i, t \in \mathbb{R}\}$ define the group law $*$ by the formula

$$\begin{aligned}(x_1, y_1, \ldots, x_n, y_n, t) &* (x_1', y_1', \ldots, x_n', y_n', t') \\ &= \left(x_1 + x_1', y_1 + y_1', \ldots, x_n + x_n', y_n + y_n', t + t' - 2\sum_{i=1}^n x_i y_i' - y_i x_i' \right).\end{aligned} \tag{6.2}$$

This group law makes $(\mathbb{R}^{2n+1}, *)$ a Lie group, called the Heisenberg Group, denoted $\mathbb{H}_n$. Denote the corresponding Lie algebra of left-invariant vector fields with the symbol $\mathfrak{h}_n$. Fix the basis

$\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ for $\mathfrak{h}_n$ consisting of the left-invariant vector fields in $\mathbb{H}_n$ satisfying $X_i|_0 = \frac{\partial}{\partial x_i}$, $Y_i|_0 = \frac{\partial}{\partial y_i}$, $T|_0 = \frac{\partial}{\partial t}$. Define the *horizontal distribution* to be the sub-bundle

$$H_p\mathbb{H}_n = \operatorname{span}\{X_1, Y_1, \ldots, X_n, Y_n\} \subseteq T_p\mathbb{H}_n$$

Endow $H\mathbb{H}_n$ with the metric $\mathbf{g}$ for which $X_1, Y_1, \ldots, X_n, Y_n$ form an orthonormal basis. Since $[X_i, Y_i] = -4T$, Chow's Theorem 6.2.4 implies that any two points in $\mathbb{H}_n$ can be connected by horizontal curves, and so $(\mathbb{H}_n, H\mathbb{H}_n, \mathbf{g})$ is a sub-Riemannian manifold with Carnot-Caratheodory metric given by the formula (6.2.5).

Define the *contact form*

$$\alpha = dt + 2\sum_{i=1}^{n}(x_i dy_i - y_i dx_i)\,. \tag{6.3}$$

One can check that $H\mathbb{H}_n = \ker\alpha$. The following two lemmas are useful for understanding the Carnot Caratheodory metric and motivating the questions asked about the Heisenberg Group.

Lemma 6.3.1. *(Properties of the Carnot-Caratheodory metric)*

1. *The Carnot-Caratheodory distance from the origin O to a point $(0, t)$ on the t-axis is $4\sqrt{\pi}\sqrt{t}$.*
2. *For fixed $t \in \mathbb{R}$ and $x \in \mathbb{R}^{2n}$, the map $\mathbb{R} \ni s \mapsto (s\,x, t) \in \mathbb{H}_n$ is horizontal, hence 1-Lipschitz.*
3. *For a constant $C > 0$, for all $p, q \in \mathbb{R}^{2n+1}$, $d_{cc}(p, q) \leq C|p - q|^{1/2} + |p - q|$.*

Proof. See the formula given for geodesics in [22] to prove the first claim. For the second claim, one can directly check that $f^*\alpha = 0$. For the last claim, first assume $p = O$ is the origin and let $q = (q_x, q_t)$. Now use the first two claims to obtain a piecewise curve from O to $(0, q_t)$ to q_x, q_t of total length $4\sqrt{\pi}\sqrt{t} + |q_x|$. For the general case, use left-invariance of the d_{cc}-metric: $d_{\mathrm{cc}}(p, q) = d_{\mathrm{cc}}(0, p^{-1} * q)$. □

Lemma 6.3.2. *Suppose $f : \mathbb{R}^k \to \mathbb{H}_n$ is $C^{0,\gamma}$ for some $\gamma > 1/2$. Then f is horizontal at any point where f is differentiable.*

Proof. We prove the contrapositive: suppose f is differentiable but not horizontal at a certain point $x \in \mathbb{R}^k$. We will show that f is not $C^{0,\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$. It suffices to prove it for $k = 1$, $x = 0$, and $f(0) = 0$ with $f^*\alpha|_0 \neq 0$. This means we can write

$$f'(0) = a_1\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial y_1} + \ldots + a_n\frac{\partial}{\partial x_n} + b_n\frac{\partial}{\partial y_n} + c\frac{\partial}{\partial t}$$

where $c \neq 0$. Let $\mathbf{x} = (a_1, b_1, \ldots, a_n, b_n)$ and we can write (by the definition of differentiability)

$$f(s) = (s\mathbf{x}, cs) + s\,\varphi(s)$$

where $|\varphi(s)| \to 0$ as $s \to 0$. Let O denote the origin, $P = (0, cs)$, $Q = (s\mathbf{x}, cs)$, and $R = f(s)$. Then, using Lemma 6.3.1,

$$d_{\mathrm{cc}}(f(0), f(s)) = d_{\mathrm{cc}}(0, R) \geq d_{\mathrm{cc}}(0, P) - d_{\mathrm{cc}}(P, Q) - d_{\mathrm{cc}}(Q, R)$$

$$\geq \left(4\sqrt{\pi c} - \sqrt{C\,|\varphi(s)|}\right)\sqrt{s} - |\mathbf{x}|\,s.$$

For any $M > 0$ and any $\varepsilon > 0$ this expresion is greater than $Ms^{\frac{1}{2}+\varepsilon}$ for s close to 0. □

The main point is that a mapping f into the Heisenberg group must be horizontal in order to be $> \frac{1}{2}$-Hölder continuous.

Now let us take a brief look ahead. Proposition 7.2.4 (and its corollary) will show that smooth horizontal maps are low-rank. Thereafter, we will show that smoothness is not needed: maps which are $> \frac{1}{2}$-Hölder continuous are *approximately horizontal* in the sense of Hölder pullbacks as defined by Theorem 8.2.1. This is stated precisely in Lemma 9.1.1.

This concludes our introduction to the metric structure of the Heisenberg group. A more thorough introduction can be found in [9]. The Heisenberg Group is also one of the principal examples in the collection [5]. $\mathbb{H}_1$ is even related to the "parallel parking" example of the previous section, as a kind of local approximation. This is made precise in [4, Section 5.5]. It arises in numerous other contexts such as complex geometry, algorithm analysis, and image processing.

7.0 HORIZONTAL SUBMANIFOLDS IN THE HEISENBERG GROUP

The titular objects of this chapter are the primary subject of study for the rest of this thesis.

7.1 SPHERE EMBEDDING

Before proving what manifolds *cannot* be embedded in the Heisenberg Group, it is important to take note of what embeddings *are* possible. The following construction (Theorem 7.1.3) of horizontal embeddings from $\mathbb{S}^n$ into $\mathbb{H}_n$ is adapted directly from [11, Chapter 3].

We have already seen in Section 6.3 that for any Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^{2n}$ there is a horizontal lift $\Gamma : [0,1] \to \mathbb{H}_n$. Indeed, if

$$\gamma(s) = (x_1(s), y_1(s), \ldots, x_n(s), y_n(s))$$

then we can take $\Gamma(s) = (\gamma(s), t(s))$ with

$$t(s) = t_0 + 2\sum_{i=1}^{n} \int_0^s y(\sigma)x'(\sigma) - x(\sigma)y'(\sigma)\, d\sigma.$$

If we introduce the notation

$$\beta = \frac{1}{2}\sum_{i=1}^{n} x_i dy_i - y_i dx_i$$

then we can write the contact form (Definition 6.3) α as $\alpha = dt - 4\beta$ and the horizontality condition reads

$$t(s) = t_0 + 4\int_0^s \gamma^*\beta.$$

With this formula it is easy to characterize those closed curves $\gamma : \mathbb{S}^1 \to \mathbb{R}^{2n}$ which lift to a closed curve $\Gamma : \mathbb{S}^1 \to \mathbb{H}_n$:

Lemma 7.1.1. *A Lipschitz curve $\gamma : \mathbb{S}^1 \to \mathbb{H}_n$ has a horizontal lift $\Gamma = (\gamma, t)$ if and only if*

$$\int_{\mathbb{S}^1} \gamma^* \beta = 0.$$

To generalize this, let us characterize all those maps $f : M \to \mathbb{R}^{2n}$ which have a horizontal lift $F = (f, t)$. First, if f has a horizontal lift, then that lift will satisfy

$$F^* \alpha = dt - 4 f^* \beta = 0$$

which implies that $f^* \beta = \frac{1}{4} dt$ is an exact form. Conversely, if $f^* \beta$ is exact, then we can define t by the formula

$$dt = 4 f^* \beta \tag{7.1}$$

and then $F = (f, t)$ is a horizontal lift of f. So we have proven

Lemma 7.1.2. *A smooth map $f : M \to \mathbb{H}_n$ has a horizontal lift $F = (f, t)$ if and only if $f^* \beta$ is an exact one-form on M.*

With Lemma 1, it is straightforward to find smooth horizontal embeddings $f : \mathbb{S}^1 \to \mathbb{H}_1$. Take any smooth curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ which encloses zero *signed* area, e.g.

$$\gamma(s) = (\sin(s), \cos(s) \sin(s)).$$

This map satisfies the assumption of Lemma 7.1.1, so it has a horizontal lift $\Gamma = (\gamma, t) : \mathbb{S}^1 \to \mathbb{H}_1$, where $t(s) = 2\cos(s) - \frac{2}{3}\cos^3(s)$. Since the only self-intersection of γ is $\gamma(0) = \gamma(\pi) = (0, 0)$, and since $t(0) \neq t(\pi)$, we conclude that Γ is a horizontal embedding $\Gamma : \mathbb{S}^1 \hookrightarrow \mathbb{H}_1$.

To generalize this examaple, notice that the "figure 8" map γ can be viewed as the restriction of $(x, y) \mapsto (y, xy)$ to $\mathbb{S}^1$, and the lift has the simple formula

$$(x, y) \mapsto \left(y, xy, 2x - \frac{2}{3} x^3 \right).$$

To produce a horizontal embedding $\mathbb{S}^n \hookrightarrow \mathbb{H}_n$, we view $\mathbb{S}^n$ as the set $\{|x|^2 = 1\} \subset \mathbb{R}^{n+1}$ labeling coorinates $x = (x_0, x_1, \ldots, x_n)$. Define the map $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{2n}$

$$\tilde{f}(x_0, x_1, \ldots, x_n) = (x_1, x_0 x_1, \ldots, x_n, x_0 x_n)$$

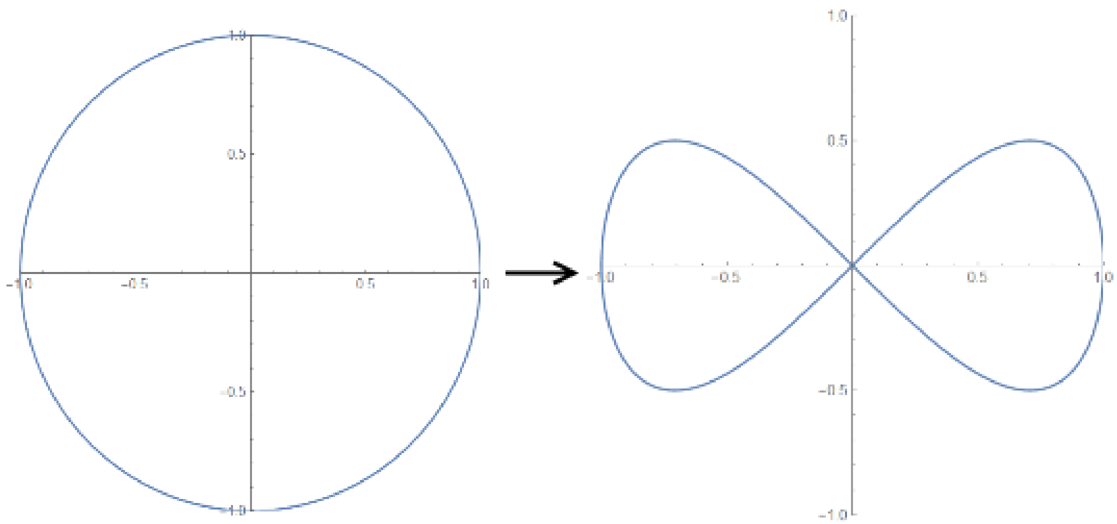


Figure 7.1: Circle Embedding Projection

and $f = \tilde{f}\big|_{\mathbb{S}^n}$. Denoting $N = (1, 0, \ldots, 0)$ and $S = (-1, 0, \ldots, 0) \in \mathbb{S}^n$, the only self-intersection of f is $f(N) = f(S) = 0$. Now we have $f = \tilde{f} \circ \iota$ where $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is the standard embedding.

We can then compute

$$f^*\beta = (\tilde{f} \circ \iota)^*\beta = \iota^* \tilde{f}^*\beta = \iota^* \left(\frac{1}{2} \sum_{i=1}^{n} \tilde{f}_{x_i} d\tilde{f}_{y_i} - \tilde{f}_{y_i} d\tilde{f}_{x_i} \right) = \iota^* \left(\frac{1}{2} \sum_{i=1}^{n} x_i(x_0 dx_i + x_i dx_0) - x_0 x_i dx_i \right)$$
$$= \iota^* \left(\frac{1}{2} \sum_{i=1}^{n} x_i^2 dx_0 \right) = \frac{1}{2}(1 - x_0^2) dx_0 = d\left(\frac{1}{2} x_0 + \frac{1}{6} x_0^3 \right).$$

Now by (7.1) we can take

$$t(x_0, x_1, \ldots, x_n) = 2x_0 + \frac{2}{3} x_0^3$$

and then $F = (f, t) : \mathbb{S}^n \to \mathbb{R}^{2n+1}$ is a horizontal lift of f. Since $\tilde{f}$ is full-rank on $\mathbb{S}^n$ except at N and S, and anyway f is full-rank at those points, f is an immersion. Hence F is an immersion. Finally, since f is one-to-one except for $f(N) = f(S)$, and anyway $F(N) \neq F(S)$, we have shown that

Theorem 7.1.3. *The map*

$$F : (x_0, x_1, \dots, x_n) \mapsto (x_1, x_0x_1, \dots x_n, x_0x_n, 2x_0 + \frac{2}{3}x_0^3) \tag{7.2}$$

is a smooth horizontal embedding $F : \mathbb{S}^n \hookrightarrow \mathbb{H}_n$.

7.2 RANK-OBSTRUCTION TO HORIZONTALITY

So far we have been able to show that there are at least some horizontal immersions and embeddings from n-dimensional manifolds into $\mathbb{H}_n$. But since the horizontal distribution of $\mathbb{H}_n$ is $2n$-dimensional, it makes sense to ask about the existence of immersions and embeddings from manifolds of dimension $n + 1 \leq k \leq 2n$. Our answer to the question is the following

Theorem 7.2.1. *If* $f : M \to \mathbb{H}_n$ *is any smooth horizontal map into* $\mathbb{H}_n$ *from a manifold, we have* $\operatorname{rank}(Df) \leq n$ *everywhere.*

Now we prove it.

Lemma 7.2.2 (Lefschetz)**.** *Denoting coordinates* $(x_1, y_1, \dots, x_n, y_n)$ *in* $\mathbb{R}^{2n}$*, the mapping*

$$L = \omega \wedge (\cdot) : \bigwedge^{n-1} \mathbb{R}^{2n} \to \bigwedge^{n+1} \mathbb{R}^{2n} \tag{7.3}$$

is a linear isomorphism, where ω *is the standard symplectic form* $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Proof. The dimensions of $\bigwedge^{n-1} \mathbb{R}^{2n}$ and $\bigwedge^{n+1} \mathbb{R}^{2n}$ are equal, so we will simply demonstrate surjectivity by providing the explicit inverse formula. After relabeling the axes, any basis vector for $\bigwedge^{n+1} \mathbb{R}^{2n}$ can be written in the form

$$\kappa = dx_1 \wedge dy_1 \wedge \dots \wedge dx_\ell \wedge dy_\ell \wedge dx_{\ell+1} \wedge \dots \wedge dx_{\ell+r}$$

where $2\ell + r = n + 1$. Let us find explicitly a form τ such that $\omega \wedge \tau = \kappa$. This will suffice to establish the inverse formula.

We use multi-index subscripts as follows:

$$dP_j = dx_j \wedge dy_j, \quad dx_J = dx_{j_1} \wedge \dots \wedge dx_{j_k}, \quad dP_J = dP_{j_1} \wedge \dots \wedge dP_{j_k}$$

where J is a multi-index $J = \langle j_1, \ldots, j_k \rangle$ with $j_i \in \{1, \ldots, n\}$ listed in increasing order. We define $|J| := k$ to be the number of entries. Fix $I = \langle 1, 2, \ldots, \ell \rangle$ and $I' = \langle \ell + 1, \ldots, \ell + r \rangle$ so that $\kappa = dP_I \wedge dx_{I'}$. Let $\mathcal{I}_q^s$ be the set of all multi-indices $J = \langle i_1, \ldots, i_s \rangle$ with precisely s entries $i_1 < \ldots < i_s$ with $i_1, \ldots, i_q \in \{1, \ldots, \ell\}$ and $i_{q+1}, \ldots, i_s \in \{\ell + r + 1, \ldots, n\}$. Now define

$$\tau_1 = \sum_{J \in \mathcal{I}_{\ell-1}^{\ell-1}} dP_J \wedge dx_{I'}$$

and notice that

$$\omega \wedge \tau_1 = \ell\kappa + \sum_{J \in \mathcal{I}_{\ell-1}^{\ell}} dP_J \wedge dx_{I'}$$

The reader may take this observation as inspiration for the following tactic: define

$$\tau_j = \sum_{J \in \mathcal{I}_{\ell-j}^{\ell-1}} dP_J \wedge dp_{I'} \quad j = 1, \ldots, \ell$$

$$E_j = \sum_{J \in \mathcal{I}_{\ell-j}^{\ell}} dP_J \wedge dx_{I'}$$

and we have the identities

$$\begin{aligned} \omega \wedge \tau_1 &= \ell\,\kappa + E_1 \\ \omega \wedge \tau_j &= (\ell - j + 1)E_{j-1} + jE_j \quad j = 2, \ldots, \ell - 1 \\ \omega \wedge \tau_\ell &= E_{\ell-1}. \end{aligned} \tag{7.4}$$

Crucially, the third of these is where we used the fact that $2l + r = n + 1$. The idea now is to obtain a telescoping sum with κ as the final sum. So we take

$$\tau = \sum_{j=1}^{\ell} \frac{(-1)^{j-1}\,(\ell - j)!\,(j-1)!}{\ell!} \tau_j$$

and the identities (7.4) imply $\omega \wedge \tau = \kappa$. ◻

Remark 7.2.3. For an instructive example with which to walk through this proof, consider the case $n = 4$, $\ell = 2$, and $r = 1$. Now we have

$$\kappa = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \in \bigwedge^5 \mathbb{R}^8$$

We then defined

$$\begin{aligned}
\tau_1 &= dx_1 \wedge dy_1 \wedge dx_3 + dx_2 \wedge dy_2 \wedge dx_3 \\
\tau_2 &= dx_3 \wedge dx_4 \wedge dy_4 \\
E_1 &= dx_1 \wedge dy_1 \wedge dx_3 \wedge dx_4 \wedge dy_4 + dx_2 \wedge dy_2 \wedge dx_3 \wedge dx_4 \wedge dy_4
\end{aligned}$$

and the identities (7.4) are readily verified, as well as

$$\omega \wedge \tau = \kappa$$

where we have set

$$\tau = \frac{1}{2}\tau_1 - \frac{1}{2}\tau_2.$$

Note that the symplectic form ω can, in a natural way, be identified with $d\alpha$, the exterior derivative of the contact form on $\mathbb{R}^{2n+1} = \mathbb{H}_n$. With this identification in mind we prove

Proposition 7.2.4. *In $\mathbb{R}^{2n+1}$ denote coordinates $(x_1, y_1, \ldots, x_n, y_n, t)$. Let $\kappa \in C^\infty\left(\bigwedge^k \mathbb{R}^{2n+1}\right)$ and α the contact form on $\mathbb{H}_{2n}$. Then there exist smooth forms $\sigma \in C^\infty\left(\bigwedge^{k-1} \mathbb{R}^{2n+1}\right)$ and $\tau \in C^\infty\left(\bigwedge^{k-2} \mathbb{R}^{2n+1}\right)$ such that*

$$\kappa = \alpha \wedge \sigma + d\alpha \wedge \tau.$$

That is, the differential ideal generated by α contains the k-forms, $k > n$.

Proof. It suffices to prove it when $k = n + 1$. Relabel $(x_1, y_1, \ldots, x_n, y_n, t)$ as $(p_1, p_2, \ldots, p_{2n}, t)$ and write

$$\kappa =: \left(\sum_{\substack{|I|=k \\ 2n+1 \notin I}} \psi_I dp_I \right) + dt \wedge \left(\sum_{\substack{|J|=k-1 \\ 2n+1 \notin J}} \varphi_J dp_J \right) =: \beta + dt \wedge \sigma.$$

Observe that $\kappa - \alpha \wedge \sigma$ is a sum of forms not involving dt. That is,

$$\kappa - \alpha \wedge \sigma =: \tilde{\kappa} =: \sum_{\substack{|I|=k \\ 2n+1 \notin I}} \eta_I dp_I.$$

That is, we can treat $\tilde{\kappa}$ as a smooth map $\tilde{\kappa} : \mathbb{R}^{2n+1} \to \bigwedge^{n+1} \mathbb{R}^{2n}$. With this identification, take $\tau = L^{-1}\tilde{\kappa}$ (pointwise), where L is the operator (7.3) and we have

$$d\alpha \wedge \tau = \omega \wedge \tau = LL^{-1}\tilde{\kappa} = \tilde{\kappa}$$

and therefore

$$\kappa = \alpha \wedge \sigma + d\alpha \wedge \tau.$$

□

We now prove Theorem 7.2.1 with this result.

Proof of Theorem 7.2.1. $f^*\alpha = 0$ everywhere by definition. Now if $k > n$, then for any k-forms κ on $\mathbb{R}^{2n+1}$ use Proposition 7.2.4 to write $\kappa = \alpha \wedge \sigma + d\alpha \wedge \tau$ and we then have

$$f^*\kappa = f^*(\alpha \wedge \sigma + d\alpha \wedge \tau) = f^*\alpha \wedge f^*\sigma + d(f^*\alpha) \wedge f^*\tau = 0.$$

Consequently $\mathrm{rank}(f) \leq k$. □

7.3 GROMOV'S QUESTION

We have now seen that $f : \mathbb{R}^k \to \mathbb{H}_n$ cannot simultaneously be (1) smooth, (2) high rank ($\mathrm{rank}\, f > n$), and (3) horizontal. we have also seen that horizontality of f is equivalent to $f \in \mathrm{Lip}(\mathbb{R}^k, \mathbb{H}_n)$. Recall that, conversely, if f fails to be horizontal, then f is $C^{1/2*}$. Hence, the property of a mapping f into $\mathbb{H}_n$ being more than 1/2-Hölder continuous can be viewed as a generalized horizontality – that is, a notion of horizontality that applies to non-smooth maps. Gromov asked ([19]) whether there are any "interesting" maps in this class.

Question 7.3.1 (Gromov)**.** *For $k > n$, does there exist an* embedding $f : \mathbb{R}^k \to \mathbb{H}_n$ *of class $C^{0,\gamma}$ for any $\gamma > 1/2$?*

Demanding that f be an embedding eliminates trivialities like a constant map. Of course, if f is smooth, then this assumption would mean that f was full rank. Since that's impossible, any example of such an embedding cannot be differentiable on any open set. In fact, for the case of mappings $f : \mathbb{R}^2 \to \mathbb{H}_1$, LeDonne and Züst [24] showed f cannot be of essentially bounded variation, and, perhaps more suggestively, almost every vertical line that intersects the image of f intersects the surface on a cantor set!

Gromov proved that any counter-example embedding must be $\gamma \leq 2/3$ Hölder continuous. The existence or non-existence of a counter-example with $1/2 < \gamma \leq 2/3$ remains an open question. In the following chapters we explore the question. Along the way we will prove Gromov's $2/3$ result in a satisfyingly simple way.

8.0 HÖLDER CONTINUOUS MAPS HAVE JACOBIANS

In this chapter we prove that, in a distributional sense, we can define the pullback of a differential form by a Hölder continuous function. In Section 8.3 we show that it is possible to solve the Poisson equation $\Delta\omega = f^*\kappa$ even when the right-hand side is one of these distributional pullbacks, and we provide regularity estimates for ω sufficient for the applications in Chapter 9.

8.1 SLOBODETSKIĬ MAPPINGS HAVE JACOBIANS

In [7], Brezis and Nguyen characterized the Sobolev Spaces in which one has a reasonable notion of a Jacobian. While it is "clear" that $f \in W^{1,N}(\mathbb{R}^N, \mathbb{R}^N)$ has an integrable Jacobian, the following theorem expands that class to an even broader class of functions which have Jacobians. In fact, in that paper, Brezis and Nguyen also showed that this space $W^{1-\frac{1}{N},N}$ is the largest space among Sobolev spaces $W^{s,p}$ in which one has a Jacobian ([7, Theorems 3 and 4]). For our purposes we state the theorem without the optimality claim, and provide a marginally simpler proof.

Theorem 8.1.1 (Brezis, Nguyen 2011)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded domain, and $f \in W^{1-\frac{1}{N},N}(\Omega, \mathbb{R}^N)$. Then, for any $\psi \in C^1_c(\Omega)$ and any choice of smooth approximation $f_t \to f$ in $W^{1-\frac{1}{N},N}$-norm, the limit*

$$\langle f, \psi\rangle := \lim_{t\to 0} \int_\Omega \det(Df_t)\,\psi \tag{8.1}$$

exists and is independent of the choice of smooth approximation.

Before we give the proof, take note of the following simple identity: if $\xi_1, \ldots, \xi_k$ and $\eta_1, \ldots, \eta_k$

are 1-forms in $\bigwedge^1 \mathbb{R}^n$, then we have

$$(\xi_1 \wedge \ldots \wedge \xi_k) - (\eta_1 \wedge \ldots \wedge \eta_k) = \sum_{i=1}^{k} \eta_1 \wedge \ldots \wedge \eta_{i-1} \wedge (\xi_i - \eta_i) \wedge \xi_{i+1} \wedge \ldots \wedge \xi_k. \tag{8.2}$$

With this in mind let us proceed.

Proof. We present Brezis and Nguyen's proof in the language of differential forms. Let g and h be any two smooth functions $g, h : \Omega \to \mathbb{R}^N$. We will show that if $\|g - h\|_{W^{1-\frac{1}{N},N}}$ is small, then the difference $\int_\Omega \det(Dg)\psi - \int_\Omega \det(Dh)\psi$ is also small. This will effectively prove the theorem, by showing that the right side of (8.1) is Cauchy as $t \to 0$.

In what follows, let $\tilde{g}$ and $\tilde{h}$ be extensions of g and h to $\mathbb{R}^N$ as in Theorem 3.5.8. Then let G and H be extensions of $\tilde{g}$ and $\tilde{h}$ to $\mathbb{R}^{N+1}$, as in Theorem 3.5.9. Also take an extension $\Psi \in C_c^1(\Omega \times [0,1])$ of ψ. It suffices to take $\Psi(x,t) = \psi(x)\eta(t)$ where η is a fixed smooth function $[0,1/2] \prec \eta \prec [0,1)$. The point is, we want to have $\|\Psi\|_{C^1} \lesssim \|\psi\|_{C^1}$, and Ψ vanishes on $(\Omega \times \{1\}) \cup (\partial\Omega \times [0,1])$. Now observe,

$$\begin{aligned}
\Big| \int_\Omega \det(Dg)\psi - \int_\Omega \det(Dh)\psi \Big| &= \Big| \int_\Omega \psi(dg_1 \wedge \ldots \wedge dg_N - dh_1 \wedge \ldots \wedge dh_N) \Big| \\
&\leq \Big| \int_{\Omega \times [0,1]} d\Psi \wedge (dG_1 \wedge \ldots \wedge dG_N - dH_1 \wedge \ldots \wedge dH_N) \Big| && \text{Stokes' Theorem} \\
&\leq \Big| \int_{\Omega \times [0,1]} \sum_{i=1}^{N} d\Psi \wedge dH_1 \wedge \ldots \wedge dH_{i-1} \wedge d(G_i - H_i) \wedge dG_{i+1} \wedge \ldots \wedge dG_N \Big| && \text{due to (8.2)} \\
&\lesssim \Big| \int_{\Omega \times [0,1]} |\nabla\Psi| |\nabla G - \nabla H| (|\nabla G| + |\nabla H|)^{N-1} \Big| \\
&\leq \|\nabla\Psi\|_{L^\infty(\Omega\times[0,1])} \|\nabla G - \nabla H\|_{L^N(\Omega\times[0,1])} \Big(\|\nabla G\|^{N-1}_{L^N(\Omega\times[0,1])} + \|\nabla H\|^{N-1}_{L^N(\Omega\times[0,1])} \Big) && \text{Hölder's Inequality} \\
&\leq \|\psi\|_{C^1} \|g - h\|_{W^{1-\frac{1}{N},N}(\Omega)} \Big(\|g\|^{N-1}_{W^{1-\frac{1}{N},N}(\Omega)} + \|h\|^{N-1}_{W^{1-\frac{1}{N},N}(\Omega)} \Big). && \text{Theorem 3.5.9}
\end{aligned} \tag{8.3}$$

With this estimate in mind, let $\tilde{f}$ be an extension of f to $\mathbb{R}^N$ and then f_t be smooth approximations of $\tilde{f}$ in $W^{1-\frac{1}{N},N}(\mathbb{R}^N, \mathbb{R}^N)$ as in Proposition 3.5.7. Since $\|f_{t_1} - f_{t_2}\|_{W^{1-\frac{1}{N},N}} \to 0$ as $t_1, t_2 \to 0$, and also since $\|f_t\|_{W^{1-\frac{1}{N},N}} \lesssim \|f\|_{W^{1-\frac{1}{N},N}}$, we have by (8.3)

$$\Big| \int_\Omega \det(Df_{t_1})\psi - \int_\Omega \det(Df_{t_2})\psi \Big| \lesssim \|\psi\|_{C^1} \|f\|^{N-1}_{W^{1-\frac{1}{N},N}} \|f_{t_1} - f_{t_2}\|_{W^{1-\frac{1}{N},N}} \to 0$$

as $t_1, t_2 \to 0$, so the limit (8.1) exists. $\square$

The importance of this result for us cannot be overstated. While $\det(Df)$ in this context does not have to exist in the classical sense at any points in Ω, this theorem tells us that we can nonetheless make sense of the quantity $\int_\Omega \det(Df)\psi$. We say that $\det(Df)$ is a *distribution*—not a function, but a member of a broader class of mathematical objects which can be integrated (or "tested") against smooth functions.

8.2 HÖLDER PULLBACKS

We have just seen that mappings in certain Slobodetskiĭ spaces have a distributional Jacobian. We have also seen (see Proposition 3.5.2) that Hölder spaces embed in these Slobodetskiĭ spaces. Since pullbacks can be viewed as a generalized Jacobian determinant, it would seem to follow that Hölder mappings can, in some way, pull back a differential form. We make this intuition precise.

Theorem 8.2.1. *Let $0 \leq k \leq n$ be integers, M a compact oriented n-dimensional Riemannian manifold without boundary, $f \in W^{1-\frac{1}{k+1},k+1}(M,\mathbb{R}^N)$, and $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^N)$ a smooth k-form on $\mathbb{R}^N$. Then the limit*

$$|\langle f^*\kappa, \tau\rangle| = \lim_{t\to 0} \int_M f_t^*\kappa \wedge *\tau. \tag{8.4}$$

exists for $\tau \in L^\infty \cap W^{1-\frac{1}{k+1},k+1}(\bigwedge^k M)$, independent of the choice of smooth approximations $f_t \to f$ in $W^{1-\frac{1}{k+1},k+1}(M,\mathbb{R}^N)$. Moreover, $f^*\kappa$, thus defined, can be viewed as a bounded linear functional on $L^\infty \cap W^{1-\frac{1}{k+1},k+1}(\bigwedge^k M)$ with the bound

$$|\langle f^*\kappa, \tau\rangle| \lesssim \|\kappa\|_{C^1}(\|f\|^k_{W^{1-\frac{1}{k+1},k+1}} + \|f\|^{k+1}_{W^{1-\frac{1}{k+1},k+1}})(\|\tau\|_{W^{1-\frac{1}{k+1},k+1}} + \|\tau\|_{L^\infty}). \tag{8.5}$$

Proof. Let $f, g \in C^\infty(M,\mathbb{R}^N)$, $\tau \in C^{0,\gamma}(\bigwedge^k N)$. Using Theorem 4.4.16 (4.10), extend f and g to F and G in $W^{1,k+1}(M\times[0,1],\mathbb{R}^N)$; using (4.12) extend $*\tau$ to $T \in W^{1,k+1}(\bigwedge^{n-k}(M\times[0,1]))$ with T supported away from $M\times\{1\}$. Observe, therefore, by Stokes' Theorem,

$$\begin{aligned}\left|\int_M f^*\kappa\wedge *\tau - g^*\kappa\wedge *\tau\right| &= \left|\int_M F^*\kappa\wedge T - G^*\kappa\wedge T\right| \\ &= \left|\int_{M\times[0,1]} (F^*(d\kappa) - G^*(d\kappa))\wedge T + (-1)^k(F^*\kappa - G^*\kappa)\wedge dT\right|. \end{aligned}\tag{8.6}$$

The differences-of-pullbacks can be estimated with a similar telescoping argument as that used in the previous section, viz. (8.2): we can write $\kappa = \psi\, dy_1 \wedge \ldots \wedge dy_k$ (actually κ is a finite sum $\kappa = \sum_I \psi_I dy_I$ of such forms, but we can apply the following argument to each of them) and then

$$\begin{aligned} |F^*\kappa - G^*\kappa| &= |\psi \circ F\, dF_1 \wedge \ldots \wedge dF_k - \psi \circ G\, dG_1 \wedge \ldots \wedge dG_k| \\ &\leq |(\psi \circ F - \psi \circ G)\, dF_1 \wedge \ldots \wedge dF_k| \\ &\quad + \sum_{i=1}^{l} |\psi \circ G\, dG_1 \wedge \ldots \wedge dG_{i-1} \wedge d\,(F_i - G_i) \wedge dF_{i+1} \wedge \ldots \wedge dF_k| \\ &\lesssim \|\kappa\|_{C^{0,1}} |F - G||DF|^k + \|\kappa\|_\infty |DF - DG|(|DF| + |DG|)^{k-1}. \end{aligned} \tag{8.7}$$

In the last line we used $\psi(F(x)) - \psi(G(x)) \leq \|\psi\|_{C^{0,1}} |F(x) - G(x)|$. Similarly

$$|F^*(d\kappa) - G^*(d\kappa)| \lesssim \|\kappa\|_{C^{1,1}} |F - G||DF|^{k+1} + \|\kappa\|_{C^1} |DF - DG|(|DF| + |DG|)^k. \tag{8.8}$$

With these observations, (8.6) becomes (writing $s = 1 - \frac{1}{k+1}$ for more compact notation)

$$\begin{aligned} \left| \int_M f^*\kappa \wedge *\tau - g^*\kappa \wedge *\tau \right| &\leq \|\kappa\|_{C^{1,1}} \|T\|_\infty \int_{M\times[0,1]} |F - G||DF|^{k+1} + |DF - DG|(|DF| + |DG|)^k \\ &\quad + \|\kappa\|_{C^1} \int_{M\times[0,1]} (|F - G||DF|^k + |DF - DG|(|DF| + |DG|)^{k-1})|DT| \end{aligned}$$

$$\begin{aligned} \leq \|\kappa\|_{C^{1,1}} (\|\tau\|_{W^{s,k+1}} + \|\tau\|_{L^\infty}) \|f - g\|_{W^{s,k+1}} \Bigg(&\|f\|_{W^{s,k+1}}^{k+1} + \|f\|_{W^{s,k+1}}^k + (\|f\|_{W^{s,k+1}} + \|g\|_{W^{s,k+1}})^k \\ &+ (\|f\|_{W^{s,k+1}} + \|g\|_{W^{s,k+1}})^{k-1} \Bigg). \end{aligned} \tag{8.9}$$

In the last step we used the boundedness of the extensions, per Theorem 4.4.16

$$\|DF\|_{L^{k+1}} \lesssim \|f\|_{W^{1-\frac{1}{k+1},k+1}}, \qquad \|DG\|_{L^{k+1}} \lesssim \|g\|_{W^{1-\frac{1}{k+1},k+1}}, \qquad \|DT\|_{L^{k+1}} \lesssim \|\tau\|_{W^{1-\frac{1}{k+1},k+1}}.$$

Now suppose $f \in W^{1-\frac{1}{k+1},k+1}(M, \mathbb{R}^N)$ and let f_t be any sequence of smooth approximations in $W^{1-\frac{1}{k+1},k+1}$ (as is shown to be possible by Proposition 4.4.15). (8.9) implies the limit (8.4) exists, as it is Cauchy.

Taking $g = 0$ and repeating the estimate—this time without any need for the telescoping trick—will establish (8.5). Observe, for smooth f, and once again using capital letters F and T to denote extensions to $M \times [0, 1]$,

$$
\begin{aligned}
\left|\int_M f^*\kappa \wedge *\tau\right| &= \int_{M\times[0,1]} F^*(d\kappa)\wedge *T + F^*\kappa \wedge *dT \\
&\lesssim \|\kappa\|_{C^1} \int_{M\times[0,1]} |DF|^{k+1}|T| + |DF|^k|DT| \\
&\lesssim \|\kappa\|_{C^1}\|DF\|^{k+1}_{L^{k+1}(M\times[0,1])}\|T\|_{L^\infty(M\times[0,1])} + \|DF\|^k_{L^{k+1}(M\times[0,1])}\|DT\|_{L^{k+1}(M\times[0,1])} \\
&\lesssim \|\kappa\|_{C^1}\left(\|f\|^{k+1}_{W^{1-\frac{1}{k+1}}(M)} + \|f\|^k_{W^{1-\frac{1}{k+1}}(M)}\right)\left(\|\tau\|_{W^{1-\frac{1}{k+1}}(M)} + \|\tau\|_{L^\infty(M)}\right).
\end{aligned}
$$

and then for general $f \in W^{1-\frac{1}{k+1}}(M,\mathbb{R}^N)$ not smooth, (8.5) follows by smooth approximation. □

Definition 8.2.2. When the assumptions of Theorem 8.2.1 hold, and $\tau \in L^\infty \cap W^{1-\frac{1}{k+1},k+1}(\bigwedge^k M)$, we will use the notation

$$\int_M f^*\kappa \wedge *\tau := \lim_{t\to 0}\int_M f_t^*\kappa \wedge *\tau \tag{8.10}$$

instead of writing $\langle f^*\kappa, \tau\rangle$, when no confusion can arise.

Recall (see Proposition 3.5.2) that $C^{0,\gamma}$ spaces embed in $W^{s,p}$ when $\gamma > s$. We thus obtain immediately as a corollary,

Corollary 8.2.3. *If $f \in C^{0,\gamma}(M,\mathbb{R}^N)$ with $\gamma > 1 - \frac{1}{k+1}$, then $f^*\kappa$ is well-defined in the distributional sense* (8.4).

We conclude this section with a theorem to demonstrate that the pullback operator f^*, just defined, is non-trivial at least if f is "topologically non-trivial."

Theorem 8.2.4. *Let $k \le n$ be integers, $\gamma > 1 - \frac{1}{k+1}$, and $f \in C^{0,\gamma}(\mathbb{S}^k,\mathbb{R}^N)$ be a topological embedding. Then there exists $\omega \in C^\infty(\bigwedge^k \mathbb{R}^N)$ with the property that $d\omega = 0$ on a neighborhood of $f(\mathbb{S}^k)$, with (see Remark 8.2.2)*

$$\int_{\mathbb{S}^k} f^*\omega = 1.$$

Proof. Fix N throughout, and we proceed by induction on k. For $k = 0$ we have $\mathbb{S}^0 = \{\pm 1\}$ and we can take any *function* ω with $\omega \equiv 1$ on a neighborhood of $f(+1)$ and $\omega \equiv 0$ on a neighborhood of $f(-1)$. For the inductive step, we suppose the result holds for $k-1$ and $f \in C^{0,\gamma}(\mathbb{S}^k,\mathbb{R}^N)$ is an embedding. Write S^+ for the closed northern hemisphere and S_- for the closed southern hemisphere of $\mathbb{S}^k$, so that $\mathbb{S}^k = S^+ \cup S_-$ and $\mathbb{S}^{k-1} = E := S^+ \cap S_- \subset \mathbb{S}^k$. Let $\tilde{S}^+ := f(S^+)$, $\tilde{S}_- := f(S_-)$, $\tilde{E} := f(E)$, and

$$\tilde{E}_\delta = \{y \in \mathbb{R}^N \ : \ \mathrm{dist}(y, f(E)) < \delta\}.$$

By the induction hypothesis applied to $f|_E$, for some $\delta > 0$ we can find $\eta \in C^\infty(\bigwedge^{m-1} \mathbb{R}^N)$ with $d\eta \equiv 0$ on $\tilde{E}_\delta$ and $\int_E f^*\eta = 1$. Let $\{\psi^+, \psi_-, \psi_E\}$ be a partition of unity subordinate to the cover of $\mathbb{R}^N$, $\{\mathbb{R}^N \backslash \tilde{S}_-, \mathbb{R}^N \backslash \tilde{S}^+, \tilde{E}_\delta\}$. Notice that we can write

$$d\eta = \psi^+ \, d\eta + \psi_- \, d\eta = \omega + \sigma,$$

where $\omega := \psi^+ \, d\eta \in C^\infty(\bigwedge^m \mathbb{R}^n)$ and $\sigma := \psi_- \, d\eta \in C^\infty(\bigwedge^m \mathbb{R}^n)$. We claim this ω is the form we sought. First observe, since $d\eta = \omega + \sigma$, we have $d^2\eta = 0 = d\omega + d\sigma$ so that $d\omega = -d\sigma$, and that therefore $\mathrm{spt}\,(d\omega) = \mathrm{spt}\,(d\sigma)$. But since ω is supported away from $\tilde{S}_-$ and σ is supported away from $\tilde{S}^+$, they must both be supported away from $f(\mathbb{S}^k)$.

Evidently σ is also supported away from $f_t(S^+)$ for sufficiently small $t > 0$. So we complete the induction:

$$\int_{\mathbb{S}^k} f^*\omega \xleftarrow{0 \leftarrow t} \int_{\mathbb{S}^k} f_t^*\omega = \int_{S^+} f_t^*\omega = \int_{S^+} f_t^*(d\eta - \sigma) = \int_{S^+} d(f_t^*\eta) - 0 = \int_E f_t^*\eta \xrightarrow{t \to 0} 1.$$

□

8.3 THE POISSON EQUATION WITH HÖLDER PULLBACKS

In Section 9.3 we will need to solve the Poisson equation $\Delta\omega = f^*\kappa$ where f is a $C^{0,\gamma}$ map and κ is a smooth differential form. Doing so will allow us to develop a parametric version of the Hopf invariant for such Hölder maps for appropriately large $\gamma \in (0, 1]$. The main result of this section is Theorem 8.3.1, which we state here as motivation for what follows.

Recall, in the following statement, that $H^\perp$ is the space of differential forms which are L^2-orthogonal to the harmonic forms, see (5.12).

Theorem 8.3.1. *Let M be a compact n-dimensional Riemannian manifold without boundary, $f \in C^{0,\gamma}(M,\mathbb{R}^N)$, $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^N)$, $1 - \frac{1}{k+1} < \gamma < 1$, so that $\sigma = 1 - k(1-\gamma)$ satisfies $0 < \sigma < 1$. Then there exists a unique $\omega \in C^{1,\sigma}(\bigwedge^k M) \cap H^\perp$ satisfying the* weak *Poisson equation*

$$\Delta\omega = f^*\kappa. \tag{8.11}$$

More precisely, for all $\varphi \in C^\infty(\bigwedge^k M)$ we have

$$\int_M d\omega \wedge *d\varphi + \delta\omega \wedge *\delta\varphi = \int_M f^*\kappa \wedge *\varphi, \tag{8.12}$$

where the right-hand side is understood through the Definition 8.2.2. Moreover, we have the estimate

$$\|\omega\|_{C^{1,\sigma}(\bigwedge^k M)} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}. \tag{8.13}$$

This type of estimate—bounding the $C^{k,\gamma}$-norm of a solution to an elliptic PDE—is often called a Schauder-type estimate. The remaining sub-sections of this section will provide a proof. Before continuing on to those, the reader should note that this section on Schauder estimates is *not* used for the proof of Gromov's non-Embedding Theorem 9.2.2 in Section 9.2.

8.3.1 Some Lemmas Needed for Schauder Estimates

The lemmas in this sub-section are essentially in [18, Chapter 5], but we restate and prove them for differential forms for convenience.

Lemma 8.3.2. *For real numbers $A, B, \alpha, \beta, R_0 > 0$ with $\alpha > \beta$, there exists $c = c(A, \alpha, \beta)$ with the following property: If $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is non-negative non-decreasing with*

$$\phi(\rho) \leq A\left(\frac{\rho}{R}\right)^\alpha \phi(R) + BR^\beta, \quad 0 < \rho < R < R_0, \tag{8.14}$$

then in fact

$$\frac{1}{\rho^\beta}\phi(\rho) \leq \frac{c}{R^\beta}\phi(R) + cB, \quad 0 < \rho < R < R_0. \tag{8.15}$$

Proof. Select $0 < \tau < 1$ so that

$$A\tau^{\alpha-\beta} < \frac{1}{2}. \tag{8.16}$$

Then use $\rho = \tau R$ in (8.14) to obtain

$$\phi(\tau R) \le A\tau^{\alpha}\phi(R) + BR^{\beta}.$$

Iterate, and obtain inductively

$$\phi\left(\tau^k R\right) \le A^k\tau^{\alpha k}\phi(R) + BR^{\beta}\tau^{(k-1)\beta}\sum_{j=0}^{k-1}\left(A\tau^{\alpha-\beta}\right)^j.$$

Let $\tau^{k+1}R \le \rho \le \tau^k R$; sum the series on the right using (8.16) ; divide by ρ^{β}. Obtain

$$\frac{1}{\rho^{\beta}}\phi(\rho) \le \frac{1}{\rho^{\beta}}\phi\left(\tau^k R\right) \le \frac{1}{\rho^{\beta}}\left(A^k\tau^{\alpha k}\phi(R) + 2BR^{\beta}\tau^{(k-1)\beta}\right).$$

Using $\tau^{k+1}R \le \rho$ obtain

$$\begin{aligned}\frac{1}{\rho^{\beta}}\phi(\rho) &\le \frac{A^k\tau^{(\alpha-\beta)k}}{\tau^{\beta}}\cdot\frac{1}{R^{\beta}}\phi(R) + \frac{2}{\tau^{2\beta}}B \\ &= \frac{\left(1/2^k\right)}{\tau^{\beta}}\cdot\frac{1}{R^{\beta}}\phi(R) + \frac{2}{\tau^{2\beta}}B.\end{aligned}$$

So the lemma is proven with $c = 2/\tau^{2\beta}$. □

Remark 8.3.3. In what follows we will need to refer to the integral of a differential form $\omega \in L^1_{\mathrm{loc}}(\bigwedge^k M)$ on a small open set $B \subseteq M$. This cannot be done in a coordinate-free way, but rather we must specify a coordinate patch $\varphi : \tilde{U} \to U$ with $B \subseteq U$. Then we make for our definition

$$\int_{B,\varphi}\omega := \int_{\varphi^{-1}(B)}\omega_{\varphi} \in \mathbb{R}^{\binom{n}{k}}$$

where ω_{φ} is the coordinate representation of ω in the coordinate system $\varphi : \tilde{U} \to U$. This identification of differential forms with their coordinate representations in $\mathbb{R}^{\binom{n}{k}}$ is called *local trivialization* of the fiber bundle $\bigwedge^k M$.

Lemma 8.3.4. *Let M be a Riemannian manifold with a fixed, finite atlas $\mathcal{A} = \{\varphi_i : \tilde{U}_i \to U_i\}$. Let $\Omega \subseteq M$ be an open set and $\omega \in C^\infty(\bigwedge^k \Omega)$ a harmonic differential form. Let $x_0 \in \Omega$ be fixed. Let R_0 be the Lebesgue number of the covering*[1]*, and let $0 < \rho < R < R_0$. Then for some coordinate patch $\varphi \in \mathcal{A}$ we have*

$$\int_{B_\rho(x_0)} |\omega|^2 \leq C_{M,\mathcal{A}} \left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\omega|^2, \tag{8.17}$$

$$\int_{B_\rho(x_0)} |\omega - \omega_{x_0,\rho}|^2 \leq C_{M,\mathcal{A}} \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\omega - \omega_{x_0,R}|^2. \tag{8.18}$$

Proof. We can do the computations in normal coordinates centered at x_0. Let u be the coordinate representation of ω in these coordinates. To prove (8.17), clearly we can assume $\rho < R/2$. Define $v(x) = u(Rx)$ on B_1. By Corollary 5.2.11, $\|v\|_{W^{k,2}(B_{1/2})} \lesssim C_k \|v\|_{L^2(B_1)}$ for all $k \in \mathbb{N}$. Take $k > n/2$ so that, by the Sobolev Embedding Theorem 3.4.14, v is continuous with $\|v\|_{C^0} \lesssim \|v\|_{W^{k,2}(B_1)} \lesssim C_k \|v\|_{L^2(B_1)}$. Then we can estimate,

$$\int_{B_\rho} |u|^2 = R^n \int_{B_{\rho/R}} |v|^2 \lesssim \rho^n \|v\|^2_{C^0(B_{\rho/R})} \lesssim \rho^n \|v\|^2_{L^2(B_1)} = \rho^n \int_{B_1} |v|^2 = \left(\frac{\rho}{R}\right)^n \int_{B_R} |u|^2.$$

This proves (8.17).

For the second inequality, continue to assume $\rho < R/2$ without loss of generality. Find,

$$\begin{aligned}
\int_{B_\rho} |\omega - \omega_{x_0,\rho}|^2 &\lesssim \rho^2 \int_{B_\rho} |d\omega|^2 + |\delta\omega|^2 && \text{Proposition 5.2.3} \\
&\lesssim \rho^2 \left(\frac{\rho}{R}\right)^n \int_{B_{R/2}} |d\omega|^2 + |\delta\omega|^2 && \text{by (8.17), since } d\omega \text{ and } \delta\omega \text{ are harmonic} \\
&\lesssim \rho^2 \left(\frac{\rho}{R}\right)^n \frac{1}{R^2} \int_{B_R} |\omega - \omega_{x_0,R}|^2 && \text{Caccioppoli inequality 5.2.6,}
\end{aligned}$$

which proves (8.18). ∎

[1]Given a compact metric space X and covering $\{U_i\}$, there exists a number $R_0 > 0$ such that, for all $x \in X$ and $0 < R < R_0$, $B_{R_0}(x)$ is contained inside one of the open sets U_i. This number R_0 is called the Lebesgue number of the covering $\{U_i\}$

8.3.2 A Priori Estimates

Proposition 8.3.5. *Let k be an integer, $0 < k < n$, $f \in C^\infty(M, \mathbb{R}^m) \cap H^\perp$ with $1 - \frac{1}{2(k+1)}$, and $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^m)$ (See Definition 5.1.1). Let $\omega \in C^\infty(\bigwedge^k M)$ be a solution to the Poisson equation guaranteed by Classical Hodge Decomposition (Corollary 5.3.2),*

$$\Delta\omega = f^*\kappa.$$

Then we have the estimate

$$\|\omega\|_{W^{1,2}(\bigwedge^k M)} \lesssim \|\omega\|_{L^2(\bigwedge^k M)} + [f]^k_{C^{0,\gamma}(\bigwedge^k M)} + [f]^{k+1}_{C^{0,\gamma}(\bigwedge^k M)}. \tag{8.19}$$

Or, if $\omega \in H^\perp$, then

$$\|\omega\|_{W^{1,2}(\bigwedge^k M)} \lesssim [f]^k_{C^{0,\gamma}(\bigwedge^k M)} + [f]^{k+1}_{C^{0,\gamma}(\bigwedge^k M)}. \tag{8.20}$$

Recall, $H^\perp$ is the L^2-orthogonal compliment to the (finite-dimensional) space of harmonic forms.

Proof. First assume that $\omega \in H^\perp$. We will treat the general case below. As in the last chapter, the capital symbols F, Ω refer to the Gagliardo extensions of f and ω to $M \times [0, 1]$. As before, F uses the extension (4.10) while Ω uses the cut-off extension (4.12). Observe.

$$\begin{aligned}
[\omega]^2_{W^{1,2}} &\approx \int_M d\omega \wedge *d\omega + \delta\omega \wedge *\delta\omega && \text{Gaffney's Inequality 5.2.2} \\
&= \int_M \Delta\omega \wedge *\omega && \text{Integration by Parts} \\
&= \int_M f^*\kappa \wedge *\omega && \text{because } \Delta\omega = f^*\kappa \\
&= \int_{M\times[0,1]} d(F^*\kappa \wedge \Omega) && \text{Stokes' Theorem 4.3.4} \\
&= \int_{M\times[0,1]} F^*d\kappa \wedge \Omega + (-1)^k F^*\kappa \wedge d\Omega && \text{Product rule (4.1)} \\
&\lesssim \|\kappa\|^{k+1}_{C^1} \int_{M\times[0,1]} |\nabla F|^{k+1}|\Omega| + \|\kappa\|^k_{C^1} \int_{M\times[0,1]} |\nabla F|^k|\nabla\Omega| \\
&\leq \|\kappa\|^{k+1}_{C^1}\|\nabla F\|^{k+1}_{L^{2(k+1)}}\|\Omega\|_{L^2} + \|\kappa\|^k_{C^1}\|\nabla F\|^k_{L^{2k}}\|\nabla\Omega\|_{L^2} && \text{Hölder's Inequality} \\
&\leq C(\kappa)\|\omega\|_{W^{1,2}}\left([f]^{k+1}_{W^{1-\frac{1}{2(k+1)},2(k+1)}} + [f]^k_{W^{1-\frac{1}{2k},2k}}\right) && \text{Theorem 4.4.16, (4.13)} \\
&\lesssim C(\kappa)\|\omega\|_{W^{1,2}}\left([f]^{k+1}_{C^{0,1-\frac{1}{2(k+1)}+\varepsilon}} + [f]^k_{C^{0,1-\frac{1}{2k}+\varepsilon}}\right) && \text{Proposition 3.5.2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim C(\kappa)[\omega]_{W^{1,2}}\left([f]^{k+1}_{C^{0,1-\frac{1}{2(k+1)}+\varepsilon}} + [f]^{k}_{C^{0,1-\frac{1}{2k}+\varepsilon}}\right) && \text{Remark following Proposition 5.2.3}\\
&\lesssim C(\kappa)[\omega]_{W^{1,2}}\left([f]^{k}_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}\right). && \text{Proposition 3.1.11}
\end{aligned}
\tag{8.21}
$$

Dividing by $[\omega]_{W^{1,2}}$ on both sides gives the desired inequality $[\omega]^2_{W^{1,2}} \lesssim C([f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}})$ for the case where $\omega \in H^{\perp}$. Then, for general ω we can orthogonally decompose $\omega = \omega^H + \omega^{\perp}$ and estimate

$$
\begin{aligned}
\|\omega\|^2_{W^{1,2}(\bigwedge^k M)} &= \|\omega\|^2_{L^2(\bigwedge^k M)} + [\omega]^2_{W^{1,2}(\bigwedge^k M)}\\
&\lesssim \|\omega\|_{L^2(\bigwedge^k M)} + [\omega^H]^2_{W^{1,2}(\bigwedge^k M)} + [\omega^{\perp}]^2_{W^{1,2}}\\
&\lesssim \|\omega\|^2_{L^2} + \|\omega^H\|^2_{L^2} + [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}} && \text{by (8.21) and Theorem 5.2.5}\\
&\lesssim \|\omega\|^2_{L^2} + [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}.
\end{aligned}
$$

□

Definition 8.3.6 (Dual Campanato Space $\mathcal{L}^{2,\sigma-1}$)**.** For $\eta \in C^{\infty}(\bigwedge^k M)$ and $0 < \sigma < 1$ we define the quantity $\|\eta\|_{\mathcal{L}^{2,\sigma-1}}$ to be the infimum over those numbers α for which

$$
\int_{B_R(x)} \eta \wedge *v \leq \alpha R^{\frac{n}{2}+\sigma}[v]_{W^{1,2}(M)}
$$

for all radii $R > 0$, $x \in M$, and $v \in W^{1,2}_0(\bigwedge^k B_R(x))$.

Remark 8.3.7. We will call this the "dual Campanato norm", but the reader should be advised that this is not standard terminology.

Theorem 8.3.8 (Schauder a priori estimates)**.** *Let M be an n-dimensional compact oriented Riemannian manifold without boundary and $\eta \in C^{\infty}(\bigwedge^k M)$. If $\omega \in C^{\infty}(\bigwedge^k M)$ satisfies*

$$
\Delta\omega = \eta, \tag{8.22}
$$

then for any $0 < \sigma < 1$,

$$
[\nabla\omega]_{C^{0,\sigma}} \lesssim \|\eta\|_{\mathcal{L}^{2,\sigma-1}} + \|\nabla\omega\|_{L^2}.
$$

Proof. The claim follows in a very similar manner to the proof of Schauder estimates, e.g. [18, Chapter 5], but the right-hand side requires some more care. Using a test-form $\alpha \in C^1(\bigwedge^k M)$ we have

$$\int_M d\omega \wedge *d\alpha + \delta\omega \wedge *\delta\alpha = \int_M \eta \wedge *\alpha.$$

We proceed with Schauder-type estimates as in [18]. Fix an atlas for M; Let R_0 be the Lebesgue number of the atlas. We will ultimately show that (see Remark 8.3.3)

$$\frac{1}{\rho^{n+2\sigma}} \int_{B_\rho(x_0)} |\nabla\omega - (\nabla\omega)_{x,\rho}|^2 \lesssim \|\eta\|_{\mathcal{L}^{2,\sigma-1}(M)} + [\omega]_{W^{1,2}(M)}. \tag{8.23}$$

From this estimate and Campanato's Theorem (Theorem 3.3.2) it will follow that $\nabla\omega \in C^{0,\sigma}$. So we set out to prove (8.23).

Fix a ball $B_R(x)$ with $R < R_0$, so that the ball is inside one of the patches of the atlas. We will not distinguish in our notation between forms and their representation in this coordinate system. No confusion should arise from this. On B_R we can write $\omega = u + v$ where (See Proposition 5.3.4)

$$\left\{\begin{array}{ll} \Delta u = 0 & \text{in } B_R \\ u = \omega & \text{on } \partial B_R \end{array}\right\}, \quad \left\{\begin{array}{ll} \Delta v = \eta & \text{in } B_R \\ v = 0 & \text{on } \partial B_R \end{array}\right\}. \tag{8.24}$$

We then have, for $0 < \rho < R$,

$$\int_{B_\rho(x)} |d\omega - (d\omega)_{x,\rho}|^2 \lesssim \int_{B_\rho(x)} |du - (du)_\rho|^2 + \int_{B_\rho(x)} |dv - (dv)_\rho|^2. \tag{8.25}$$

Since u is harmonic, so is du (since $\Delta du = d\Delta u = 0$), and we can estimate, by (8.18),

$$\int_{B_\rho(x)} |du - (du)_\rho|^2 \leq C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x)} |du - (du)_{x_0,R}|^2 \tag{8.26}$$

whence follows

$$\int_{B_\rho(x)} |d\omega - (d\omega)_\rho|^2 \leq C_1\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x)} |d\omega - (d\omega)_{x,R}|^2 + C_2 \int_{B_R(x)} |dv|^2. \tag{8.27}$$

Precisely the same reasoning with δ in place of d gives

$$\int_{B_\rho(x)} |\delta\omega - (\delta\omega)_\rho|^2 \leq C_1\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x)} |\delta\omega - (\delta\omega)_{x,R}|^2 + C_2 \int_{B_R(x)} |\delta v|^2. \tag{8.28}$$

Next we estimate $\int_{B_R(x)} |dv|^2 + |\delta v|^2$. Observe, by (8.24)

$$\int_{B_R(x)} dv \wedge *dv + \delta v \wedge *\delta v = \int_{B_R(x)} \eta \wedge *v \leq R^{\frac{n}{2}+\sigma} \|\eta\|_{\mathcal{L}^{2,\sigma-1}} \|Dv\|_{L^2(B_R(x))}$$
$$\approx R^{\frac{n}{2}+\sigma} \|\eta\|_{\mathcal{L}^{2,\sigma-1}} (\|dv\|_{L^2(B_R(x))} + \|\delta v\|_{L^2(B_R(x))}),$$

where in the last step we used Remark 4.5.9. Divide by $(\|dv\|_{L^2(B_R(x))} + \|\delta v\|_{L^2(B_R(x))})$ and square both sides to find,

$$\int_{B_R} |dv|^2 + |\delta v|^2 \lesssim R^{n+2\sigma} \|\eta\|^2_{\mathcal{L}^{2,\sigma-1}}.$$

So (8.27) and (8.28) become

$$\int_{B_\rho} |\nabla\omega - (\nabla\omega)_\rho|^2 \leq C_3 \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |\nabla\omega - (\nabla\omega)_R|^2 + C_4 R^{n+2\sigma} \|\eta\|^2_{\mathcal{L}^{2,\sigma-1}}.$$

Using Lemma 8.3.2 with $\phi(\rho) = \int_{B_\rho} |\nabla\omega - (\nabla\omega)_\rho|^2$, this improves to

$$\frac{1}{\rho^{n+2\sigma}} \int_{B_\rho} |\nabla\omega - (\nabla\omega)_\rho|^2 \leq \frac{C_3}{R^{n+2\sigma}} \int_{B_R} |\nabla\omega - (\nabla\omega)_R|^2 + C_4 \|\eta\|^2_{\mathcal{L}^{2,\sigma-1}}.$$

Recall, this is for arbitrary coordinate balls $B_\rho(x) \subset B_R(x) \subset M$ with $R < R_0$. So we have,

$$\frac{1}{\rho^{n+2\sigma}} \int_{B_\rho} |\nabla\omega - (\nabla\omega)_\rho|^2 \leq \frac{C_3}{R_0^{n+2\sigma}} \|\nabla\omega\|_{L^2} + C_4 \|\eta\|^2_{\mathcal{L}^{2,\sigma-1}}$$

which is to say,

$$[\nabla\omega]_{\mathcal{L}^{2,n+2\sigma}(M)} \lesssim [\omega]_{W^{1,2}(M)} + \|\eta\|_{\mathcal{L}^{2,\sigma-1}(M)}.$$

Since the Campanato norm is equivalent to the Hölder norm (Theorem 3.3.2),

$$[\nabla\omega]_{C^{0,\sigma}(M)} \lesssim [\omega]_{W^{1,2}(M)} + \|\eta\|_{\mathcal{L}^{2,\sigma-1}(M)}.$$

$\square$

8.3.3 One last lemma to characterize Hölder pullbacks

The following lemma connects Theorem 8.3.8 to Theorem 8.3.1.

Lemma 8.3.9 (Hölder pullbacks belong to dual Campanato spaces)**.** *Let $f \in C^{0,\gamma}(M, \mathbb{R}^N)$, $1 - \frac{1}{k+1} < \gamma < 1$, $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^N)$, $\sigma = 1 - k(1-\gamma)$, (note that $0 < \sigma \leq 1$), and $R > 0$ be sufficiently small such that $B_R(x)$ is diffeomorphic to a Euclidean ball for all $x \in M$. Then, for all $x \in M$ and $v \in W_0^{1,2}(\bigwedge^k B_R(x))$, we have*

$$\int_{B_R(x)} f^*\kappa \wedge *v \lesssim R^{\frac{n}{2}+\sigma} \|\nabla v\|_{L^2} \left([f]^k_{C^{0,\gamma}(M,\mathbb{R}^N)} + [f]^{k+1}_{C^{0,\gamma}(M,\mathbb{R}^N)} \right). \tag{8.29}$$

That is, $f^*\kappa$ is in the dual Campanato space $\mathcal{L}^{2,\sigma-1}(\bigwedge^k M)$ with the estimate

$$[f^*\kappa]_{\mathcal{L}^{2,\sigma-1}(\bigwedge^k M)} \lesssim [f]^k_{C^{0,\gamma}(M,\mathbb{R}^N)} + [f]^{k+1}_{C^{0,\gamma}(M,\mathbb{R}^N)}. \tag{8.30}$$

Proof. Using Theorem 4.4.16 (4.10), let $F \in W^{1,k+1}(M \times [0,1])$ be the extension of f to $M \times [0,1]$. Let $V(x,t) = \mathrm{ext}_{\mathrm{mod}}\, v(x,t)$ be the cut-off extension (4.12) of v from M to $M \times [0,1]$ supported away from $M \times \{1\}$. Then, applying Stokes' Theorem

$$\begin{aligned}\int_M f^*\kappa \wedge *v &= \int_{\mathrm{Cyl}_R} F^*d\kappa \wedge V + F^*\kappa \wedge dV \\ &\lesssim \int_{M\times[0,1]} |DF|^{k+1}|V| + |DF|^k|\nabla V|. \end{aligned} \tag{8.31}$$

We estimate these separately. Throughout what follows, let $\varepsilon = \gamma - (1 - \frac{1}{k+1})$. Recall the estimates

$$\begin{aligned}|\nabla F(x,t)|^{k+1} &\lesssim [f]^{k+1}_{C^{0,\gamma}} \frac{1}{t^{1-(k+1)\varepsilon}}, \\ |V(x,t)| &\lesssim \int_{y\in B_t(x)} \frac{|v(y)|}{t^n}, \\ |\nabla F(x,t)|^k &\lesssim [f]^k_{C^{0,\gamma}} \frac{1}{t^{1-\frac{1}{k+1}+k\varepsilon}}, \\ |\nabla V(x,t)| &\lesssim \int_{y\in B_t(x)} \frac{|\nabla v(y)|}{t^n}.\end{aligned}$$

Using these in (8.31) we find that

$$\int_M f^*\kappa \wedge *v \lesssim \int_{M\times[0,1]} |\nabla F|^{k+1}|V| + |\nabla F|^k|\nabla V|$$

$$
\begin{aligned}
&\lesssim \int_{x\in M}\int_{t=0}^{t=1}\int_{y\in B_t(x)} \frac{[f]^{k+1}_{C^{0,\gamma}}|v(y)|}{t^{n+1-(k+1)\varepsilon}} + \frac{[f]^{k}_{C^{0,\gamma}}|\nabla v(y)|}{t^{n+1-\frac{1}{k+1}+k\varepsilon}} \\
&= \int_{y\in B_R}\int_{t=0}^{t=1}\int_{x\in B_t(y)} \frac{[f]^{k+1}_{C^{0,\gamma}}|v(y)|}{t^{n+1-(k+1)\varepsilon}} + \frac{[f]^{k}_{C^{0,\gamma}}|\nabla v(y)|}{t^{n+1-\frac{1}{k+1}+k\varepsilon}}.
\end{aligned}
$$

Using the fact that v is supported inside B_R, it is routine to estimate this by

$$
\int_M f^*\kappa \wedge *v \lesssim R^{(k+1)\varepsilon}[f]^{k+1}_{C^{0,\gamma}}\int_{B_R}|v| + R^{\frac{1}{k+1}+k\varepsilon}[f]^{k}_{C^{0,\gamma}}\int_{B_R}|\nabla v|.
$$

Now use the Poincaré Inequality (3.22) on the first integral on the right:

$$
\int_M f^*\kappa \wedge *v \lesssim R^{1+(k+1)\varepsilon}[f]^{k+1}_{C^{0,\gamma}}\int_{B_R}|\nabla v| + R^{\frac{1}{k+1}+k\varepsilon}[f]^{k}_{C^{0,\gamma}}\int_{B_R}|\nabla v|,
$$

and then estimate the integrals on the right with Jensen's Inequality (3.3):

$$
\int_M f^*\kappa \wedge *v \lesssim R^{\frac{n}{2}+1+(k+1)\varepsilon}[f]^{k+1}_{C^{0,\gamma}}\|\nabla v\|_{L^2} + R^{\frac{n}{2}+\frac{1}{k+1}+k\varepsilon}[f]^{k}_{C^{0,\gamma}}\|\nabla v\|_{L^2}.
$$

Since M is compact, we can assume without loss of generality that $R < 1$. Then we obtain

$$
\begin{aligned}
\int_M f^*\kappa \wedge *v &\lesssim R^{\frac{n}{2}+\frac{1}{k+1}+k\varepsilon}([f]^{k+1}_{C^{0,\gamma}} + [f]^{k}_{C^{0,\gamma}})\|\nabla v\|_{L^2} \\
&= R^{\frac{n}{2}+\sigma}([f]^{k+1}_{C^{0,\gamma}} + [f]^{k}_{C^{0,\gamma}})\|\nabla v\|_{L^2}.
\end{aligned}
$$

□

8.3.4 At last, the Poisson Equation for Hölder Pullbacks

We re-state and prove the Theorem 8.3.1 at the beginning of this section.

Theorem 8.3.10. *Let M be a compact n-dimensional Riemannian manifold without boundary, $f \in C^{0,\gamma}(M, \mathbb{R}^N)$, $\kappa \in C^\infty(\bigwedge^k \mathbb{R}^N)$, $1 - \frac{1}{k+1} < \gamma < 1$, so that $\sigma = 1 - k(1-\gamma)$ satisfies $0 < \sigma < 1$. Then there exists a unique $\omega \in C^{1,\sigma}(\bigwedge^k M) \cap H^\perp$ satisfying the* weak *Poisson equation*

$$\Delta\omega = f^*\kappa. \tag{8.32}$$

More precisely, for all $\varphi \in C^\infty(\bigwedge^k M)$ we have

$$\int_M d\omega \wedge *d\varphi + \delta\omega \wedge *\delta\varphi = \int_M f^*\kappa \wedge *\varphi, \tag{8.33}$$

where the right-hand side is understood through the Definition 8.2.2. Moreover, we have the estimate

$$\|\omega\|_{C^{1,\sigma}(\bigwedge^k M)} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}. \tag{8.34}$$

Proof. First suppose f is smooth so that a smooth solution ω exists to $\Delta\omega = f^*\kappa$. Also, assume for now that $\omega \in H^\perp$. Beginning with Lemma 8.3.9, (8.30), we know that

$$[f^*\kappa]_{\mathcal{L}^{2,\sigma-1}} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}.$$

So we may apply Theorem 8.3.8 with $\eta = f^*\kappa$ to find that

$$\|\nabla\omega\|_{C^{0,\sigma}} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}} + [\nabla\omega]_{L^2}.$$

Finally, applying Proposition 8.3.5, conclude

$$\|\nabla\omega\|_{C^{0,\sigma}} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}.$$

This proves the estimate (8.34) for the case where f is smooth. Now we dispense with that assumption. If $f \in C^{0,\gamma}(M, \mathbb{R}^N)$, then the smooth mollifications f_t remain bounded $[f_t]_{C^{0,\gamma}} \le [f]_{C^{0,\gamma}}$. Obtain smooth $\omega_t \in C^\infty(\bigwedge^k M)$ with $\Delta\omega_t = f_t^*\kappa$ and $\|\omega_t\|_{C^{1,\sigma}(\bigwedge^k M)} \lesssim [f_t]^k_{C^{0,\gamma}} + [f_t]_{C^{0,\gamma}} \le [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}$. By Proposition 3.1.11, ω_t has subsequence which is converging to a limit $\omega \in C^{1,\sigma'}$ for any $\sigma' < \sigma$. But by the second part of Proposition 3.1.11, $\omega \in C^{1,\sigma}$ with the same bound $\|\nabla\omega\|_{C^{0,\sigma}} \lesssim [f]^k_{C^{0,\gamma}} + [f]^{k+1}_{C^{0,\gamma}}$. Notice then that for any test-form $\varphi \in C^\infty(\bigwedge^k M)$, we have

$$\int_M d\omega \wedge *d\varphi + \delta\omega \wedge *\delta\varphi \xleftarrow{0\leftarrow t} \int_M d\omega_t \wedge *d\varphi + \delta\omega_t \wedge *\delta\varphi = \int_M \Delta\omega_t \wedge *\varphi$$
$$= \int_M f_t^*\kappa \wedge *\varphi \xrightarrow{t\to 0} \int_M f^*\kappa \wedge *\varphi$$

proving (8.33). The uniqueness of ω in $H^\perp$ is clear. □

Remark 8.3.11. In fact we proved slightly more: if f_t is $C^{0,\gamma}$-bounded and converging uniformly to f (say, if f_t are mollifications of f), then the solution ω is the $C^{1,\sigma'}$-limit of the solutions to $\Delta\omega_t = f_t^*\kappa$, for any $\sigma' < \sigma$.

9.0 HÖLDER MAPPINGS INTO THE HEISENBERG GROUP

Now armed with weapons which allow us to do analysis with pullbacks $f^*\kappa$ even when f is merely Hölder continuous, we can obliterate certain questions about the Heisenberg Group. We also describe how the techniques could be extended to other sub-riemannian manifolds. At the end of this chapter, the question remains: what about $C^{0,\frac{1}{2}+}(\mathbb{R}^2, \mathbb{H}_1)$ maps? Or, for that matter, $C^{0,\frac{1}{2}+}(\mathbb{R}^k, \mathbb{H}_n)$ for $k > n$? While definitive answers as to whether any interesting maps of these types exist remain elusive, the ultimate and penultimate sections offer some partial answers.

9.1 HÖLDER MAPS INTO $\mathbb{H}_N$ ARE WEAKLY LOW RANK

Recall what we have discussed so far about rank, horizontality, and Hölder continuity. In section 7.2, we showed that a smooth horizontal map $f : \mathbb{R}^d \to \mathbb{H}_n$ must have rank everywhere $\operatorname{rank}(Df) \leq n$. Thus, by Section 7.2, if $f : \mathbb{R}^d \to \mathbb{H}_n$ is a smooth and $f \in C^{0,\gamma}(\mathbb{R}^d, \mathbb{H}_n)$ with $\gamma > 1/2$, then $\operatorname{rank}(Df) \leq n$. Gromov's question about Hölder mapping into the Heisenberg group, posed in Section 7.3, asks what can be said if $f \in C^{0,\gamma}(\mathbb{R}^d, \mathbb{H}_n)$ if we dispense with the assumption that f is smooth. In this section and the next section we prove Theorem 9.2.2, a slightly stronger version of Gromov's non-embedding result 9.2.1.

Lemma 9.1.1. *For all (bounded) $f \in C^{0,\gamma}(\mathbb{R}^d, \mathbb{H}_n)$ and $t > 0$ we have*

$$\|f_t^*\alpha\|_{L^\infty} \leq C_{\|f\|_{C^0}} [f]^2_{C^{0,\gamma}} t^{2\gamma - 1}$$

where α is the contact form and f_t is the standard mollification of f.

Proof. In this proof, and *only* in this proof, we will temporarily change our notation to denote points in the Heisenberg Group with coordinates $p = (p_{x_1}, p_{y_1}, \ldots, p_{x_n}, p_{y_n}, p_T)$ with T instead of t to avoid confusion with the symbol t we have been using for mollifications f_t.

We define a kind of local linear approximation to the contact form,

$$\begin{aligned} \varphi(p, q) &= (p^{-1} * q)_T \\ &= q_T - p_T - 2\sum_{i=1}^{n} p_{x_i}(q_{y_i} - p_{y_i}) - p_{y_i}(q_{x_i} - p_{x_i}). \end{aligned} \tag{9.1}$$

Recall (see Section 6.3) that we can use the Koranyi metric as an equivalent metric to the Carnot-Caratheodory metric on $\mathbb{H}_n$.

$$d_K(p, q) = |\pi p - \pi q| + |\varphi(p, q)|^{1/2}$$

where π is the projection onto $\mathbb{R}^{2n}$ and $|\cdot|$ denotes Euclidean distance (or absolute value). The fact that $f \in C^{0,\gamma}(\mathbb{R}^d, \mathbb{H}_n)$ tells us that for $\xi, \zeta \in \mathbb{R}^d$,

$$d_K(f(\xi), f(\zeta)) = |\pi f(\zeta) - \pi f(\xi)| + |\varphi(f(\xi), f(\zeta))|^{1/2} \leq C[f]_{C^{0,\gamma}}|\xi - \zeta|^{\gamma}. \tag{9.2}$$

Fix $\xi \in \mathbb{R}^d$ and $i \in \{1, \ldots, k\}$ and estimate

$$\begin{aligned} f_t^*\alpha\left[\frac{\partial}{\partial \xi_i}\right] &= \left(df_T^t + 2\sum_{i=1}^{n} f_{x_i}^t df_{y_i}^t - f_{y_i}^t df_{x_i}^t\right)\left[\frac{\partial}{\partial \xi_i}\right] \\ &= \left|\frac{\partial f_T^t}{\partial \xi_i} + 2\sum_{i=1}^{n} f_{x_i}^t \frac{\partial f_{y_i}^t}{\partial \xi_i} - f_{y_i}^t \frac{\partial f_{x_i}^t}{\partial \xi_i}\right| \\ &= |I_1 + I_2 + I_3| \end{aligned} \tag{9.3}$$

where

$$I_1 := t^{-n-1}\int_{\mathbb{R}^k} \varphi(f(\xi), f(\xi + \zeta))\frac{\partial \eta}{\partial \xi_i}(t^{-1}\zeta)d\zeta,$$

$$I_2 := 2t^{-n-1}\sum_{i=1}^{n}\left(\int_{\mathbb{R}^k}[f_{x_i}(\xi + \zeta) - f_{x_i}(\xi)]\,\eta(t^{-1}\zeta)d\zeta\right)\left(\int_{\mathbb{R}^k}\left[f_{y_i}(\xi + \zeta) - f_{y_i}(\xi)\right]\frac{\partial \eta}{\partial \xi_i}(t^{-1}\zeta)\,d\zeta\right),$$

$$I_3 := 2t^{-n-1}\sum_{i=1}^{n}\left(\int_{\mathbb{R}^k}\left[f_{y_i}(\xi + \zeta) - f_{y_i}(\xi)\right]\eta(t^{-1}\zeta)d\zeta\right)\left(\int_{\mathbb{R}^k}[f_{x_i}(\xi + \zeta) - f_{x_i}(\xi)]\frac{\partial \eta}{\partial \xi_i}(t^{-1}\zeta)\,d\zeta\right).$$

Here we have repeatedly used the convolution derivative formula (3.4). Using (9.2) we have

$$|I_1| \lesssim C\|\nabla\eta\|_{L^1}[f]_{C^{0,\gamma}}^2 t^{2\gamma}.$$

Moreover, by Section 6.3, $[f]_{C^{0,\gamma}(\mathbb{R}^d,\mathbb{R}^{2n+1})} \leq C(\|f\|_{C^0})[f]_{C^{0,\gamma}(\mathbb{R}^d,\mathbb{H}_n)}$, and so we have also

$$|I_2|, |I_3| \leq 2n\|\nabla\eta\|_{L^1}[f]^2_{C^{0,\gamma}} t^{2\gamma}.$$

Put these estimates into (9.3) to obtain the final estimate

$$\|f_t^*\alpha\|_{C^0} \leq C(\|f\|_{C^0})[f]^2_{C^{0,\gamma}} t^{2\gamma-1}.$$

□

Remark 9.1.2. Compare this result to Lemma 4.4.17.

Proposition 9.1.3. *Let M be an $(n+1)$-dimensional manifold, $1 - \frac{1}{n+1} < \gamma$, $f \in C^{0,\gamma}(M, \mathbb{H}_n)$, and $\kappa \in C^\infty(\bigwedge^{n+1}\mathbb{R}^{2n+1})$. Then*

$$\int_M f^*\kappa = 0. \tag{9.4}$$

Proof. By Proposition 7.2.4, we can find smooth forms $\sigma \in C^\infty(\bigwedge^{n-1}\mathbb{R}^{2n+1})$ and $\tau \in C^\infty(\bigwedge^{n}\mathbb{R}^{2n+1})$ with $\kappa = \alpha \wedge \tau + d\alpha \wedge \sigma$. Then we can compute

$$\begin{aligned}
\int_M f^*\kappa \xleftarrow{0\leftarrow t} \int_M f_t^*\kappa &= \int_M f_t^*(d\alpha \wedge \sigma + \alpha \wedge \tau) \\
&= \int_M d\left(f_t^*\alpha\right) \wedge f_t^*\sigma + f_t^*\alpha \wedge f_t^*\tau \\
&= \int_M f_t^*\alpha \wedge f_t^*(d\sigma + \tau) \\
&\lesssim \mathrm{vol}(M)\|f_t^*\alpha\|_{L^\infty}\|f_t^*(d\sigma+\tau)\|_{L^\infty} \\
&\lesssim \left([f]^2_{C^{0,\gamma}} t^{2\gamma-1}\right)\left([f]^n_{C^{0,\gamma}} t^{-n(1-\gamma)}\right) \\
&= [f]^{n+2}_{C^{0,\gamma}} t^{(n+2)\gamma-(n+1)} \\
&\xrightarrow{t\to 0} 0.
\end{aligned}$$

□

9.2 GROMOV'S NON-HÖLDER EMBEDDING THEOREM

We prove that $\mathbb{R}^k$ does not γ-Hölder-embed into $\mathbb{H}_n$ for large γ and k.

Theorem 9.2.1 (Gromov)**.** *There does not exist a topological embedding $f \in C^{0,\gamma}(\mathbb{R}^k, \mathbb{H}_n)$ for $k > n$ and $\gamma > 1 - \frac{1}{n+2}$.*

In fact, we will prove somewhat more:

Theorem 9.2.2. *Suppose $n < k$, $1/2 < \gamma \leq 1$ and $1 - \frac{1}{n+1} < \theta \leq 1$, and*

$$2\gamma - 1 > n(1 - \theta).$$

Then there does not exist a map $f : \overline{\mathbb{B}}^k \to \mathbb{R}^{2n+1}$ with $f \in C^{0,\gamma}(\overline{\mathbb{B}}^k, \mathbb{H}_n) \cap C^{0,\theta}(\overline{\mathbb{B}}^k, \mathbb{R}^{2n+1})$ and *with the property that $f|_{\mathbb{S}^{k-1}}$ is a topological embedding.*

Proof. We may assume $k = n + 1$ since higher dimensional balls $\mathbb{B}^k$ contain embedded copies of $\mathbb{B}^{n+1}$. We will refer to the boundary as $\partial\mathbb{B}^{n+1} = \mathbb{S}^n$. Suppose the theorem were false. Then let f be a counter-example to the theorem. By Theorem 8.2.4 there exists a smooth form $\omega \in C^\infty(\bigwedge^n \mathbb{R}^{2n+1})$ such that $\int_{\mathbb{S}^n} f^*\omega = 1$. By Proposition 7.2.4, $d\omega = \alpha \wedge \sigma + d\alpha \wedge \tau$ where σ and τ are smooth forms. So, using Theorem 8.2.1 as well as Lemmas 4.4.17 and 9.1.1,we have

$$\begin{aligned}
1 = \int_{\mathbb{S}^n} f^*\omega &\xleftarrow{0 \leftarrow t} \int_{\mathbb{S}^n} f_t^*\omega \\
&= \int_{\mathbb{B}^{n+1}} f_t^*(\alpha \wedge \sigma + d\alpha \wedge \tau) \\
&\leq \left| \int_{\mathbb{B}^{n+1}} f_t^*(\alpha \wedge (\sigma + d\tau)) \right| + \left| \int_{\mathbb{S}^n} f_t^*(\alpha \wedge \tau) \right| \\
&\lesssim \|f_t^*\alpha\|_\infty (\|f_t^*\sigma\|_\infty + \|f_t^*\tau\|_\infty + \|f_t^*(d\tau)\|_\infty) \\
&\lesssim \varepsilon^{2\gamma-1}(\varepsilon^{n(\theta-1)} + \varepsilon^{(n-1)(\theta-1)} + \varepsilon^{n(\theta-1)}) \\
&\leq \varepsilon^{2\gamma-1-n(1-\theta)} \\
&\xrightarrow{t \to 0} 0.
\end{aligned}$$

This proves that $1 \leq 0$. This contradiction completes the proof. ◻

Remark 9.2.3. Notice that Gromov's theorem follows from this one, with $\gamma = \theta = 1 - \frac{1}{n+2}$.

Remark 9.2.4. With $\theta = 1$, this result implies a different result of Hajłasz, Balogh, and Wildrick [3] for Lipschitz maps.

9.3 THE GENERALIZED HOPF INVARIANT

The following ideas have been explored for Lipschitz maps in [21]. Our tools so-far developed allow for a near-effortless extension of these past results to Hölder mappings. We begin with a parametric Hopf invariant.

Definition 9.3.1 (Hölder Hopf Invariant)**.** Let $\phi : \mathbb{S}^{2n} \to \mathbb{H}_{2n}$ be a smooth horizontal embedding (see Theorem 7.1.3), vol be the volume form on $\mathbb{S}^{2n}$, κ be a $2n$-form on $\mathbb{H}_{2n}$ with $\phi^*\kappa = \text{vol}$, $1 - \frac{1}{2n(2n+1)} < \gamma < 1$, and $f \in C^{0,\gamma}(\mathbb{S}^{4n-1}, \mathbb{H}_{2n})$. Define the Hopf invariant of f,

$$\mathcal{H}_\kappa(f) := \int_{\mathbb{S}^{4n-1}} f^*\kappa \wedge \delta\omega \tag{9.5}$$

where ω is a solution in $C^{0,\sigma}(\mathbb{S}^{4n-1})$ to $\Delta\omega = f^*\kappa$ as in Theorem 8.3.1, where $\sigma = 1 - (2n)(1-\gamma)$.

Remark 9.3.2. Theorem 8.3.1 guarantees the existence of a solution $\omega \in C^{0,\sigma}$ to the Poisson equation $\Delta\omega = f^*\kappa$ for any $\sigma = 1 - k(1-\gamma)$. In this case, $k = 2n$ and so $\sigma > 1 - \frac{1}{2n+1}$. Thus $\delta\omega \in C^{0,\sigma} \subseteq L^\infty \cap W^{1-\frac{1}{2n+1},2n+1}$, and so Theorem 8.2.1 shows that the expression (9.5) is well-defined in the sense of distributional pullback equation (8.4).

Proposition 9.3.3. *Let κ be as in the definition of the Hopf invariant, $1 - \frac{1}{2n(2n+1)} < \gamma' < \gamma \le 1$ and suppose $f \in C^{0,\gamma}(\mathbb{S}^{4n-1}, \mathbb{R}^{4n+1})$, and suppose $f_t \to f$ uniformly and the f_t are bounded in $C^{0,\gamma}$-norm. (For example f_t can be standard mollifications of f.) Then $\mathcal{H}_\kappa(f_t) \to \mathcal{H}_\kappa(f)$.*

Proof. Let ω_t be $C^{1,\sigma}$ solutions to $\Delta\omega_t = f_t^*\kappa$ with $\omega_t \to \omega$ uniformly and in $C^{0,\sigma'}$. This is possible by Theorem 8.3.1 and Remark 8.3.11. Then we can estimate

$$\begin{aligned}
|\mathcal{H}_\kappa(f) - \mathcal{H}_\kappa(f_t)| &= \left|\int_M f^*\kappa \wedge \delta\omega - \int_M f_t^*\kappa \wedge \delta\omega_t\right| \\
&\le \left|\int_M f^*\kappa \wedge \delta\omega - \int_M f_t^*\kappa \wedge \delta\omega\right| + \left|\int_M f_t^*\kappa \wedge (\delta\omega - \delta\omega_t)\right| \\
&=: I_1 + I_2.
\end{aligned}$$

We have $I_1 \to 0$ by the definition of Hölder pullbacks (8.10). Also, Theorem 8.2.1, estimate (8.5) gives

$$I_2 \lesssim (\|f\|_{C^{0,\gamma'}}^{2n} + \|f\|_{C^{0,\gamma'}}^{2n+1})\|\delta(\omega - \omega_t)\|_{C^{0,\gamma'}} \le \|f\|_{C^{0,\gamma'}}^{2n}\|\omega - \omega_t\|_{C^{1,\gamma'}} \xrightarrow{t\to 0} 0$$

which completes the proof. □

Our interest in the Hopf Invariant—and indeed the reason it is called an *invariant*—is that two maps f and g into the Heisenberg group will have the same Hopf Invariant if there is a suitable homotopy between them. We make the following definition.

Definition 9.3.4. Given two maps $f, g : X \to Y$ between metric spaces X and Y, we will say that f and g are $C^{0,\gamma}$-homotopic if there exists a map $H \in C^{0,\gamma}(X \times [0,1], Y)$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Remark 9.3.5. In particular, f and g must themselves be $C^{0,\gamma}$ in order for them to be $C^{0,\gamma}$-homotopic.

Proposition 9.3.6. *Suppose $\gamma > 1 - \frac{1}{2n(2n+1)}$ and let $f, g \in C^{0,\gamma}(\mathbb{S}^{4n-1}, \mathbb{H}_{2n})$ be maps which are $C^{0,\gamma}$-homotopic. Then $\mathcal{H}_\kappa(f) = \mathcal{H}_\kappa(g)$.*

Proof. Let $H \in C^{0,\gamma}(\mathbb{S}^{4n-1} \times [0,1], \mathbb{H}_{2n})$ be the homotopy. By reparametrization we may assume that $H(\cdot, t) = f(\cdot)$ for all t near 0 and likewise $H(\cdot, t) = g(\cdot)$ for t near 1. We can also extend H to $M = \mathbb{S}^{4n-1} \times (\mathbb{R}/2\mathbb{Z}) \cong \mathbb{S}^{4n-1} \times \mathbb{S}^1$. The restriction of H to $X = \mathbb{S}^{4n-1} \times [0,1]$ is still a homotopy between f and g.

Let H_t denote the smooth approximations in $C^{0,\gamma'}(M, \mathbb{R}^{4n+1})$ to H (see Corollary 3.2.22); modify them so that $H_t(\cdot, s) = f_t(\cdot)$ for s near 0 and similarly $H_t(\cdot, s) = g_t(\cdot)$ when s near 1.

For $t > 0$ let $\tilde{\omega}_t$ denote the unique solution in $C^\infty(\bigwedge^{2n} M) \cap H^\perp$ to $\Delta \tilde{\omega}_t = H_t^* \kappa$ on M given by Theorem 8.3.1. Let $\omega_{t,0}$ and $\omega_{t,1}$ denote the solutions on $\mathbb{S}^{4n-1}$ to $\Delta \omega_{t,0} = f_t^* \kappa$ and $\Delta \omega_{t,1} = g_t^* \kappa$, also given by Theorem 8.3.1 . Let $\iota_s : \mathbb{S}^{4n-1} \to M$ denote the canonical embedding $\theta \mapsto (\theta, s)$. Note $\iota_0^* H_t^* \kappa = f_t^* \kappa$ and $\iota_1^* H_t^* \kappa = g_t^* \kappa$ for small t. We need the nontrivial facts that

$$\int_{\mathbb{S}^{4n-1}} f_t^* \kappa \wedge \iota_0^* \delta \tilde{\omega}_t \to \mathcal{H}_\kappa(f) \tag{A}$$

$$\int_{\mathbb{S}^{4n-1}} g_t^* \kappa \wedge \iota_1^* \delta \tilde{\omega}_t \to \mathcal{H}_\kappa(g) \tag{B}$$

$$\|\delta d \tilde{\omega}_t\|_{C^{0,\gamma'}} \to 0. \tag{C}$$

These are not obvious because we do not (necessarily) have $\omega_{t,0} = \iota_0^* \tilde{\omega}_t$ or $\omega_{t,1} = \iota_1^* \tilde{\omega}_t$. We assume these facts for now, and prove them in a moment. First observe that (A) and (B) allow us to apply Stokes' Theorem:

$$\mathcal{H}_\kappa(f) - \mathcal{H}_\kappa(g) \xleftarrow{0 \leftarrow t} \int_{\mathbb{S}^{4n-1}} f_t^* \kappa \wedge \iota_0^* \delta \tilde{\omega}_t - \int_{\mathbb{S}^{4n-1}} g_t^* \kappa \wedge \iota_1^* \delta \tilde{\omega}_t$$

$$
\begin{aligned}
&= \int_{\partial X} H_t^* \kappa \wedge \delta \tilde{\omega}_t \\
&= \int_X H_t^*(d\kappa) \wedge \delta \tilde{\omega}_t + \int_X H_t^* \kappa \wedge d\delta \tilde{\omega}_t \\
&= \int_X H_t^*(d\kappa) \wedge \delta \tilde{\omega}_t + \int_X H_t^* \kappa \wedge \Delta \tilde{\omega}_t - \int_X H_t^* \kappa \wedge \delta d \tilde{\omega}_t \\
&= \int_X H_t^*(d\kappa) \wedge \delta \tilde{\omega}_t + \int_X H_t^*(\kappa \wedge \kappa) + \int_X H_t^* \kappa \wedge \delta d \tilde{\omega}_t \\
&\xrightarrow{t \to 0} 0
\end{aligned}
$$

where we have estimated the first two integrals in the last line using Proposition 9.1.1 and C^0-boundedness of $\delta\tilde{\omega}_t$; and we have estimated the third using estimate (C) and Theorem 8.2.1. □

Proof of facts (A), (B), (C). We first prove (A). The proof of (C) will fall out of the estimates for free. (B) is similar to (A). Begin with the definition:

$$
\begin{aligned}
\mathcal{H}_\kappa(f_t) &= \int_{\mathbb{S}^{4n-1}} f_t^* \kappa \wedge \delta \omega_{t,0} \\
&= \int_{\mathbb{S}^{4n-1}} \iota_0^*(H_t^* \kappa \wedge \delta \tilde{\omega}_t) - \delta \omega_{t,0} \wedge d(\delta \omega_{t,0} - \iota_0^* \delta \tilde{\omega}_t) + \delta d \omega_{t,0} \wedge (\delta \omega_{t,0} - \iota_0^* \delta \tilde{\omega}_t)
\end{aligned} \tag{9.6}
$$

and, because of Proposition 9.3.3 (continuity of the $\mathcal{H}_\kappa$), it suffices to show that the second and third terms' integrals approach zero as $t \to 0$. Observe, by Proposition 9.1.1,

$$
d\delta d \tilde{\omega}_t = d(\Delta \tilde{\omega}_t) = dH_t^* \kappa = H_t^*(d\kappa) \xrightarrow{C^0} 0
$$

$$
d\delta d \omega_{t,0} = d(\Delta \tilde{\omega}_{t,0}) = d f_t^* \kappa = f_t^*(d\kappa) \xrightarrow{C^0} 0
$$

Therefore,

$$
\|\delta d \tilde{\omega}_t\|_{L^2} = (\delta d \tilde{\omega}_t, \delta d \tilde{\omega}_t)^{1/2} = (d\tilde{\omega}_t, d\delta d \tilde{\omega}_t)^{1/2}
$$

$$
\leq C \|d\tilde{\omega}_t\|_{C^0}^{1/2} \|d\delta d \tilde{\omega}_t\|_{C^0}^{1/2} \to 0 \tag{9.7}
$$

since $\|d\tilde{\omega}_t\|_{C^0}$ is bounded and $\|d\delta d\tilde{\omega}_t\|_{C^0}$ approaches zero. Similarly,

$$
\|\delta d \omega_{t,0}\|_{L^2} \to 0. \tag{9.8}
$$

Moreover, since

$$
d\delta \omega_{t,0} + \delta d \omega_{t,0} = f_t^* \kappa = \iota_0^* H_t^* \kappa = d \iota_0^* \delta \tilde{\omega}_t + \iota_0^* \delta d \tilde{\omega}_t
$$

we have

$$d(\delta\omega_{f_t} - \iota_0^*\delta\tilde{\omega}_t) = \iota_0^*\delta d\tilde{\omega}_t - \delta d\omega_{t,0}.$$

Conclude from (9.7) and (9.8) that

$$\|d(\delta\omega_{t,0} - \iota_0^*\delta\tilde{\omega}_t)\|_{L^2} \to 0. \tag{9.9}$$

Taking the limit in (9.6) as $t \to 0$ and using estimates (9.9) and (9.8) we obtain (A), since $f_t = H_t \circ \iota_0$. Obtain (B) similarly. Since $\delta d\tilde{\omega}_t$ is bounded in C^1 and approaching zero in L^2, we have $\|\delta d\tilde{\omega}_t\|_{C^{0,\gamma'}} \to 0$, which is (C). □

Theorem 9.3.7. *There is a smooth horizontal map $f : \mathbb{S}^{4n-1} \to \mathbb{H}_{2n}$ which cannot be contracted to a point via a homotopy $H \in C^{0,\gamma}(\mathbb{S}^{4n-1} \times [0,1], \mathbb{H}_{2n})$.*

Proof. Let $h : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ be the Hopf fibration (see [6, §18]). Let $\phi : \mathbb{S}^{2n} \to \mathbb{H}_{2n}$ be any smooth horizontal embedding such as (7.2). Consider the map $f = \phi \circ h$. Since $\mathcal{H}_\kappa(f) \neq 0$ (again, [6, §18]), f cannot be $C^{0,\gamma}$-null-homotopic by Proposition 9.3.6. □

Remark 9.3.8. This last result means that the Hölder homotopy group $\pi^\gamma_{4n-1}(\mathbb{H}_{2n})$ is non-trivial. It is similar to Gromov's non-embedding Theorem 9.2.2 in that we show that there are certain maps f from a sphere into the Heisenberg group which cannot be extended to a Hölder-continuous map on the ball (for sufficiently large Hölder exponent γ).

9.4 ALMOST-EVERYWHERE HORIZONTAL SURFACES

Thus far, we have seen that smooth horizontal mappings $F : \mathbb{R}^k \to \mathbb{H}_n$ cannot be full rank for $k > n$. In particular, there cannot exist a smooth function $g : \mathbb{R}^2 \to \mathbb{R}$ such that the graph $F : \mathbb{R}^2 \to \mathbb{H}_1$ given by $F(x) = (x, g(x))$ is horizontal. Indeed, since $F^*\alpha = 0$, f would need to satisfy $\frac{\partial g}{\partial x} = 2y$ and $\frac{\partial g}{\partial y} = -2x$ which is already impossible by Clairaut's Theorem. However, even if f were merely *Lipschitz* (hence Clairaut's Theorem would not apply), we could still argue that

$$g(1,1) = g(0,0) + \int_0^1 \frac{\partial g}{\partial x}(t,0)\,dt + \int_0^1 \frac{\partial g}{\partial y}(1,t)\,dt = g(0,0) - 2,$$

$$g(1,1) = g(0,0) + \int_0^1 \frac{\partial g}{\partial y}(0,t)\,dt + \int_0^1 \frac{\partial g}{\partial x}(t,1)\,dt = g(0,0) + 2$$

which is another contradiction. So this argument shows that there is not even a Lipschitz function $g : \mathbb{R}^2 \to \mathbb{R}$ (or indeed even an absolutely continuous-on-lines function, i.e. a Sobolev function) with this horizontal graph property.

Rather than admit defeat, let us return to the question of finding a function $g : \mathbb{R}^2 \to \mathbb{R}$ with the property that

$$\begin{aligned} \frac{\partial g}{\partial x} &= 2y \\ \frac{\partial g}{\partial y} &= -2x \end{aligned} \tag{9.10}$$

In 1917 Lusin [26] proved that for every measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable a.e. and such that $g'(x) = f(x)$ for almost all $x \in \mathbb{R}$. A first important step toward a generalization of Lusin's theorem to higher dimensions was obtained by Alberti [2] who proved that any measurable function on $\mathbb{R}^n$ coincides with the gradient of a C^1 function up to a set of an arbitrarily small measure. Using methods of Alberti, Moonens and Pfeffer [27] established the complete higher dimensional version of the Lusin theorem, and then Francos [16] extended the theorem to higher order derivatives. Francos proved that if f_α, $|\alpha| = m$ are measurable functions in an open set $\Omega \subset \mathbb{R}^n$, then there is a function $g \in C^{m-1}(\Omega)$ that is m times differentiable a.e. and such that for all $|\alpha| = m$, $D^\alpha g = f_\alpha$ a.e. It is easy to see that in general one cannot require that $g \in C^{m-1,1}_{loc}$, i.e. one cannot assume that the derivatives of order $m-1$ are Lipschitz continuous. We just saw this with (9.10). Nonetheless Francos' Theorem shows that (9.10) can be solved *almost everywhere*.

We will now improve somewhat upon this result. We will construct a function g with *any* modulus of continuity of derivatives of order $m-1$ which is worse than that of a Lipschitz function.

Theorem 9.4.1. *Let $\Omega \subset \mathbb{R}^n$ be open, $m \geq 1$ an integer, and let $f = (f_\alpha)_{|\alpha|=m}$, be a collection of measurable functions $f_\alpha : \Omega \to \mathbb{R}$, $|\alpha| = m$. Let $\sigma > 0$ and let $\mu : [0,\infty) \to [0,\infty)$ be a continuous function with $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. Then there is a function $g \in C^{m-1}(\mathbb{R}^n)$ that is m-times differentiable a.e., and such that*

(i) $D^\alpha g = f_\alpha$ a.e. on Ω for all $|\alpha| = m$;

(ii) $\|D^\gamma g\|_{L^\infty(\mathbb{R}^n)} < \sigma$ *for all* $0 \le |\gamma| \le m-1$*;*

(iii)

$$|D^\gamma g(x) - D^\gamma g(y)| \le \sigma |x-y|$$

for all $x, y \in \mathbb{R}^n$ *and all* $0 \le |\gamma| \le m-2$*;*

(iv)

$$|D^\gamma g(x) - D^\gamma g(y)| \le \frac{|x-y|}{\mu(|x-y|)}$$

for all $x, y \in \mathbb{R}^n$ *and all* $|\gamma| = m-1$.

In particular, we can take g *such that the derivatives* $D^\gamma g$*,* $|\gamma| = m-1$*, are* λ*-Hölder continuous simultaneously for all* $\lambda \in (0,1)$.

Here $\mu(t) = O(t)$ as $t \to \infty$ means that $\mu(t) \le Ct$ for all $t \ge t_0$.

Corollary 9.4.2. *There exists an almost-everywhere differentiable function* $g : \mathbb{R}^{2n} \to \mathbb{R}$ *with any modulus of continuity less than that of a Lipschitz function, such that the graph* $F(x) = (x, g(x))$ *is almost-everywhere horizontal.*

Indeed we can simply designate the derivatives $\frac{\partial g}{\partial x_i} = 2y_i$ and $\frac{\partial g}{\partial y_i} = -2x_i$ in Theorem 9.4.1.

Remark 9.4.3. This does not mean we found $> 1/2$ Hölder continuous map into $\mathbb{H}_n$. That is because while F is almost Lipschitz as a mapping into $\mathbb{R}^{2n+1}$, that by no means implies that F is almost Lipschitz as a mapping into $\mathbb{H}_n$. For example, if F is γ-Hölder continuous as a mapping into $\mathbb{R}^{2n+1}$, we can only guarantee that F is $\gamma/2$-Hölder continuous as a mapping into $\mathbb{H}_n$. In fact there is no reason to expect that this construction does any better than $1/2$-Hölder continuity in the best case. we will consider a more hopeful approach in the next section.

Parts (1)-(4) of the next lemma are due to Francos [16, Theorem 2.4] which is a generalization of an earlier result of Alberti [2, Theorem 1]. Estimates (5) and (6) are new.

Lemma 9.4.4. *Let* $\Omega \subset \mathbb{R}^n$ *be open with* $|\Omega| < \infty$*. Let* $m \ge 1$ *be an integer, and let* $f = (f_\alpha)_{|\alpha|=m}$*, be a collection of measurable functions* $f_\alpha : \Omega \to \mathbb{R}$*,* $|\alpha| = m$*. Let* $\mu : [0,\infty) \to [0,\infty)$ *be a continuous function with* $\mu(0) = 0$ *and* $\mu(t) = O(t)$ *as* $t \to \infty$*. Let* $\varepsilon, \sigma > 0$*. Then there is a function* $g \in C_c^m(\Omega)$ *and a compact set* $K \subset \Omega$ *such that*

(1) $|\Omega \setminus K| < \varepsilon$*;*

(2) $D^\alpha g(x) = f_\alpha(x)$ *for all* $x \in K$ *and* $|\alpha| = m$*;*

(3)

$$\|D^\alpha g\|_p \le C(n,m)(\varepsilon/|\Omega|)^{\frac{1}{p}-m}\|f\|_p$$

for all $|\alpha| = m$ *and* $1 \le p \le \infty$*;*

(4) $\|D^\gamma g\|_\infty < \sigma$ *for all* $0 \le |\gamma| < m$*;*

(5)

$$|D^\gamma g(x) - D^\gamma g(y)| \le \sigma|x-y|$$

for all $x, y \in \mathbb{R}^n$ *and all* $0 \le |\gamma| \le m-2$*;*

(6)

$$|D^\gamma g(x) - D^\gamma g(y)| \le \frac{|x-y|}{\mu(|x-y|)}$$

for all $x, y \in \mathbb{R}^n$ *and all* $|\gamma| = m-1$.

Proof. For the proof of existence of $g \in C_c^m(\Omega)$ with properties (1)-(4), see [16, Theorem 2.4]. We need to prove that g can be modified in such a way that (5) and (6) are also satisfied.

Let $K' \subset \Omega$ be a compact set such that $|\Omega \setminus K'| < \varepsilon/2$ and $f|_{K'}$ is bounded. Let $\tilde{f} = f\chi_{K'}$, where $\chi_{K'}$ is the characteristic function of K'. Clearly $\|\tilde{f}\|_\infty < \infty$. By continuity of μ we can find $\delta > 0$ such that

$$\mu(t) \le \frac{\varepsilon^m}{\sqrt{n}C(n,m)|\Omega|^m\|\tilde{f}\|_\infty} \quad \text{for all } 0 \le t \le \delta.$$

Here $C(n,m)$ is the constant from the inequality at (3). In particular if $0 < |x-y| \le \delta$, then

$$\sqrt{n}C(n,m)\varepsilon^{-m}|\Omega|^m\|\tilde{f}\|_\infty \le \frac{1}{\mu(|x-y|)}\,.$$

Let $M = \sup\{\mu(t)/t : \ t \ge \delta\}$, then M is finite because $\mu(t) = O(t)$ as $t \to \infty$. Applying (1)-(4) to $\tilde{f}$ we can find $g \in C_c^m(\mathbb{R}^n)$ and a compact set $K'' \subset \Omega$ such that

(1’) $|\Omega \setminus K''| < \varepsilon/2$;

(2’) $D^\alpha g(x) = f_\alpha(x)$ for all $x \in K' \cap K''$ and $|\alpha| = m$;

(3’)

$$\|D^\alpha g\|_p \le C(n,m)(\varepsilon/|\Omega|)^{\frac{1}{p}-m}\|\tilde{f}\|_p$$

for all $|\alpha| = m$ and $1 \le p \le \infty$;

(4’)
$$\|D^\gamma g\|_\infty < \min\left\{\frac{\sigma}{\sqrt{n}}, \frac{1}{2M},\right\}$$

for all $0 \le |\gamma| < m$.

Let $K = K' \cap K''$, then $|\Omega \setminus K| < \varepsilon$ and it is easy to see that the function g has the properties (1)-(4) from the statement of the lemma. It remains to prove properties (5) and (6).

If $0 \le |\gamma| \le m-2$, then (4’) yields

$$|D^\gamma g(x) - D^\gamma g(y)| \le \|\nabla D^\gamma g\|_\infty |x-y| \le \sigma|x-y|.$$

Now let $|\gamma| = m-1$. If $|x-y| \ge \delta$, then

$$|D^\gamma g(x) - D^\gamma g(y)| \le 2\|D^\gamma g\|_\infty \le \frac{1}{M} \le \frac{|x-y|}{\mu(|x-y|)}\,.$$

If $0 < |x-y| < \delta$, then (3’) with $p = \infty$ yields

$$\begin{aligned}|D^\gamma g(x) - D^\gamma g(y)| &\le \|\nabla D^\gamma g\|_\infty |x-y| \\ &\le \sqrt{n}C(n,m)\varepsilon^{-m}|\Omega|^m\|\tilde{f}\|_\infty |x-y| \le \frac{|x-y|}{\mu(|x-y|)}\,.\end{aligned}$$

The proof is complete. □

Now we can complete the proof of Theorem 9.4.1. We follow the argument used in [16] and [27], and the only main modification is that we are using the improved estimates from Lemma 9.4.4.

Proof. Let $U_1 = \Omega \cap B(0,1)$ and let $V_1 \subset\subset U_1$ be open with $|U_1 \setminus V_1| < 1/4$. Using Lemma 9.4.4, we can find a compact set $K_1 \subset V_1$ with $|V_1 \setminus K_1| < 1/4$ and a function $g_1 \in C^m_c(V_1)$ such that

(a) $D^\alpha g_1(x) = f_\alpha(x)$ for all $|\alpha| = m$ and $x \in K_1$;

(b) $|D^\gamma g_1(x)| < 2^{-1}\sigma \min\{\mathrm{dist}^2\left(x, U_1^c\right), 1\}$, for all $x \in \mathbb{R}^n$ and $|\gamma| < m$;

(c)
$$|D^\gamma g_1(x) - D^\gamma g_1(y)| \le 2^{-1}\sigma|x-y|$$

for all $x, y \in \mathbb{R}^n$ and all $0 \le |\gamma| \le m-2$;

(d)

$$|D^{\gamma}g_1(x) - D^{\gamma}g_1(y)| \le 2^{-1}\frac{|x-y|}{\mu(|x-y|)}$$

for all $x, y \in \mathbb{R}^n$ and all $|\gamma| = m - 1$.

We now proceed with an inductive definition. Suppose that the sets $K_1, \ldots, K_{k-1}$, and the functions $g_1, \ldots, g_{k-1}$, are defined for some $k \ge 2$. Let $U_k = \Omega \cap B(0,k) \setminus (K_1 \cup \ldots \cup K_{k-1})$ and let $V_k \Subset U_k$ be open with $|U_k \setminus V_k| < 2^{-k-1}$. Using Lemma 9.4.4, we find a compact set $K_k \subset V_k$ with $|V_k \setminus K_k| < 2^{-k-1}$and a function $g_k \in C_c^m(V_k)$ such that

(a') $D^{\alpha}g_k(x) = f_{\alpha}(x) - \sum_{j=1}^{k-1} D^{\alpha}g_j(x)$, for all $|\alpha| = m$ and $x \in K_k$;

(b') $|D^{\gamma}g_k(x)| < 2^{-k}\sigma \min\left\{\operatorname{dist}^2\left(x, U_k^c\right), 1\right\}$, $x \in \mathbb{R}^n$, $|\gamma| < m$;

(c')

$$|D^{\gamma}g_k(x) - D^{\gamma}g_k(y)| < 2^{-k}\sigma|x-y|$$

for all $x, y \in \mathbb{R}^n$ and all $0 \le |\gamma| \le m-2$;

(d')

$$|D^{\gamma}g_k(x) - D^{\gamma}g_k(y)| \le 2^{-k}\frac{|x-y|}{\mu(|x-y|)}$$

for all $x, y \in \mathbb{R}^n$ and all $|\gamma| = m - 1$.

We now take $g = \sum_{k=1}^{\infty} g_k$. We will prove that g satisfies the claim of the theorem. First, to see that $g \in C^{m-1}(\mathbb{R}^n)$, we observe that, by (b'),

$$\sum_{k=1}^{\infty} \|D^{\gamma}g_k\|_{L^{\infty}(\mathbb{R}^n)} < \sigma$$

for all $|\gamma| \le m-1$, which implies C^{m-1} differentiability, and this proves (ii). Properties (iii) and (iv) now follow immediately from (c') and (d'). Let $C = \bigcup_{k=1}^{\infty} K_k$, then we have $|\Omega \setminus C| = 0$ and it remains to prove that g is m-times differentiable at all points of C, and that $D^{\alpha}g = f_{\alpha}$, $|\alpha| = m$ on C. Fix $x \in C$, then $x \in K_k$ for some k. We write $g = p + q$ where $p = \sum_{j=1}^{k} g_j$ and $q = \sum_{j=k+1}^{\infty} g_j$. Now by (a'), we have $D^{\alpha}p(x) = f_{\alpha}(x)$ for $|\alpha| = m$, so it remains show show that q is m-times differentiable at x and that $D^{\alpha}q(x) = 0$ for $|\alpha| = m$. Fix $|\gamma| = m-1$ and consider, for $0 \ne h \in \mathbb{R}^n$, the difference quotient $|D^\gamma q(x+h) - D^\gamma q(x)|\,|h|^{-1}$. We actually have $D^\gamma q(x) = 0$, because $x \in K_k$ and $\operatorname{supp} D^\gamma g_j \cap K_k = \emptyset$ for $j > k$. Hence

$$\frac{|D^\gamma q(x+h) - D^\gamma q(x)|}{|h|} \le \frac{1}{|h|} \sum_{j=k+1}^{\infty} |D^\gamma g_j(x+h)|\,.$$

If $D^\gamma g_j(x+h) \neq 0$, $j \ge k+1$, then $x + h \in U_j$. In this case, since also $x \in K_k \subset U_j^c$, we must have $\operatorname{dist}(x+h, U_j^c) \le \operatorname{dist}(x+h, x) = |h|$. Hence by (b') we have $|D^\gamma g_j(x+h)| \le 2^{-j}\sigma|h|^2$. Thus

$$\frac{|D^\gamma q(x+h) - D^\gamma q(x)|}{|h|} \le \frac{1}{|h|} \sum_{j=k+1}^{\infty} 2^{-j}\sigma|h|^2 \le \sigma|h| \to 0 \quad \text{as } h \to 0$$

which proves that the derivative $DD^\gamma q(x) = 0$ equals zero for any $|\gamma| = m-1$. This also completes the proof of (i).

In particular, if we define

$$\mu(t) = \begin{cases} 0 & t = 0 \\ |\log t|^{-1} & 0 < t \le e^{-1} \\ et & t > e^{-1}, \end{cases}$$

then evidently μ satisfies the hypotheses of Theorem 9.4.1, and for every $\lambda \in (0,1)$ there is number $C_\lambda > 0$ such that

$$\mu(t) > C_\lambda t^{1-\lambda}, \quad t > 0.$$

In that case derivatives $D^\gamma g$, $|\gamma| = m-1$ satisfy

$$|D^\gamma g(x) - D^\gamma g(y)| \le C_\lambda^{-1}|x-y|^\lambda \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda \in (0,1).$$

The proof is complete. □

9.5 NUMERICAL SURFACES IN THE HEISENBERG GROUP

Approaching the end of our journey, there remains a gap between $1/2$ and $2/3$. In Section 9.2, we demonstrated that horizontal embeddings $f : \mathbb{S}^1 \to \mathbb{H}_1$) (which are plentiful) cannot be extended to $\overline{f} \in C^{0,\gamma}(\mathbb{B}^2, \mathbb{H}_1)$. Since any *smooth* extension $\overline{f}$ to $\mathbb{B}^2$ must be rank 2 on some open set, and since such a map must therefore be no more than $1/2$-Hölder continuous (see Theorem 7.2.1), there remains a natural question as to whether f can be extended in any way to $\mathbb{B}^2$ so as to be $C^{0,\gamma'}$ for any value of $\gamma' \in (1/2, 2/3]$, if we dispense with the assumption of smoothness. This is Gromov's Hölder equivalence question. See the introductory chapter of [19], wherein Gromov states, "It is not at all clear what is the precise geometrical (infinitesimal) significance of f being C^α with respect to C-C metrics *without* assuming f is smooth." Gromov goes on to conjecture that $\gamma > 1/2$ Hölder extensions do not exist. There is, however, reason to take seriously the possibility that this conjecture is false.

Conjecture 9.5.1. *Suppose* $1/2 < \gamma \leq 2/3$ *and* f *is a* γ*-Hölder mapping from* $\mathbb{S}^1$ *to* $\mathbb{H}_1$*. Then* f *can be extended to a* γ*-Hölder map on* $\mathbb{B}^2$.

We make this conjecture based on the following numerical evidence for the existence of a maps $f : \mathbb{B}^2 \to \mathbb{H}_1$ which are horizontal embeddings on the boundary.

First, we can of course replace $\mathbb{B}^2$ with $B = [0,1]^2$ and $\mathbb{S}^1$ with ∂B. Fix an explicit smooth or piecewise smooth horizontal map on ∂B. Choose any four points $p_1, p_2, p_3, p_4 \in \mathbb{H}$ (in the unit ball, say), and let f be the map which sends the corners of $B = [0,1]^2$ to those points, and interpolates between them with Carnot-Caratheodory geodesics, so that f is now defined on ∂B and is piecewise smooth horizontal—hence Lipschitz as a map into $\mathbb{H}_1$.

We then extend f to the interior of B as follows. Fix $p \in B$. Now let $B = B_0 \supset B_1 \supset B_2 \supset \ldots$ be the sequence of dyadic squares converging to p. We now define f on each of the B_k inductively. Suppose f has been defined on B_k. Denote the corners of B_k by P_1, P_2, P_3, P_4 (listed clockwise, say) and the midpoints M_1, M_2, M_3, M_4, where $M_i = \text{midpoint}(\overline{P_i P_{i+1}})$ (with cyclic notation). Use lower-case letters for the images, so that $f(P_i) = p_i$ and $f(M_i) = m_i$. Let C be the barycenter of B_k, and define c to be the barycenter of m_1, m_2, m_3, m_4. Now define f on the line segments $\overline{M_i C}$ to be the geodesic from m_i to c. Now f is defined on the boundary of B_{k+1}, completing the inductive definition.

At this time, it is not at all clear that f is even well-defined at p. Remarkably, in numerical testing, the diameters of the images $f(\partial B_0), f(\partial B_1), f(\partial B_2)\dots$ shrink at very nearly the rate

$$\operatorname{diam}\left(f(\partial B_k)\right) \leq \frac{C}{2^{k\gamma}}$$

with γ well above $1/2$ in all experiments performed. By experiment, we mean, a choice of four initial points $p_1, p_2, p_3, p_4 \in \mathbb{H}_1$ and test-point $p \in [0,1]^2$. If this were uniformly true for *all points* $p \in [0,1]^2$ for some $\gamma > 1/2$, and for all starting configurations p_1, p_2, p_3, p_4 it would imply the conjecture for that value of γ.

Consider the plot of $-\log_2(\operatorname{diam}(f(B_i)))$ against the iteration i (figure 9.1). For this initial configuration (p_1, p_2, p_3, p_4) and test-point p, the line of best fit was found to have a slope of 0.69, indicating that f would seem to be 0.69-Hölder continuous near this point p. Essentially the same behavior is observed regardless of the choice of point p.

Of course, checking any finite number of points for Hölder continuity does not make a convincing case, as it remains entirely plausible that f is not $> 1/2$-Hölder continuous—or indeed, not even continuous!—on a set of singularities of measure zero. In principle, these singularities, if they exist, would not be found by this kind of numerical experimentation. Nonetheless, the consistent clustering of γ-values close to $2/3$ and consistently above $1/2$ is tantalizing and warrants further investigation. We conclude with the picture of a surface generated numerically according to the above construction, whose Hölder regularity remains contested.

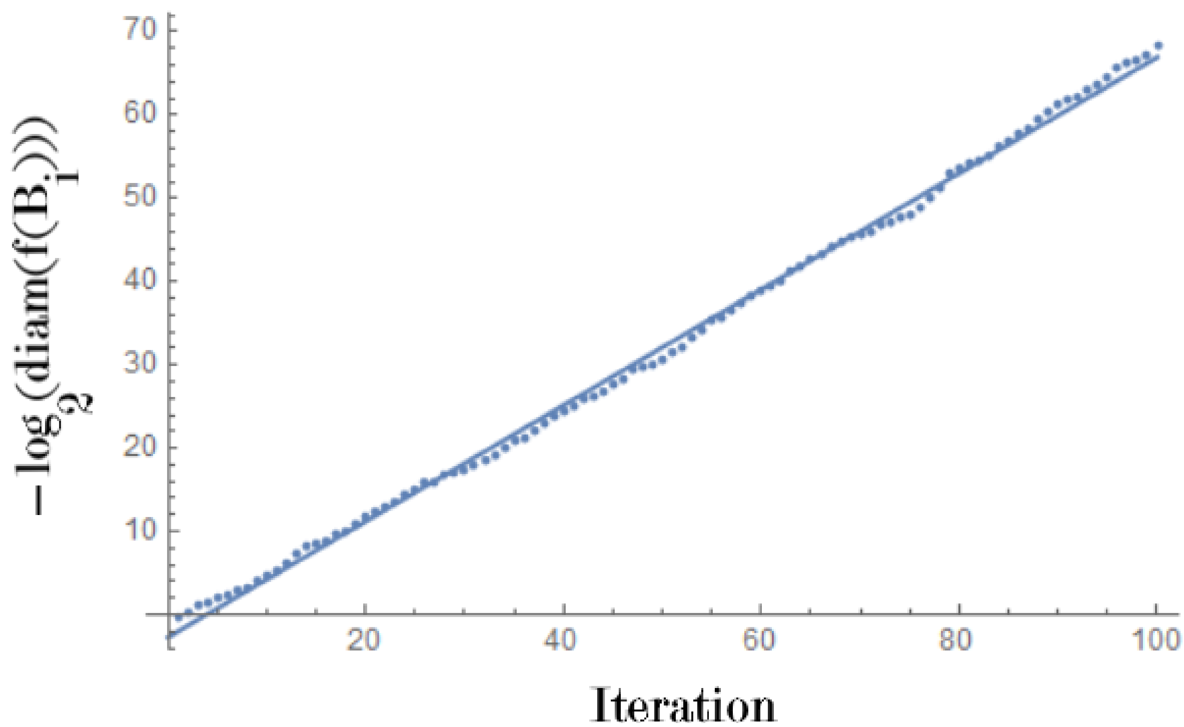


Figure 9.1: Diameter Versus Iteration. Diameters shrink at approximately the rate that should be expected of a $C^{0,2/3}$ function.

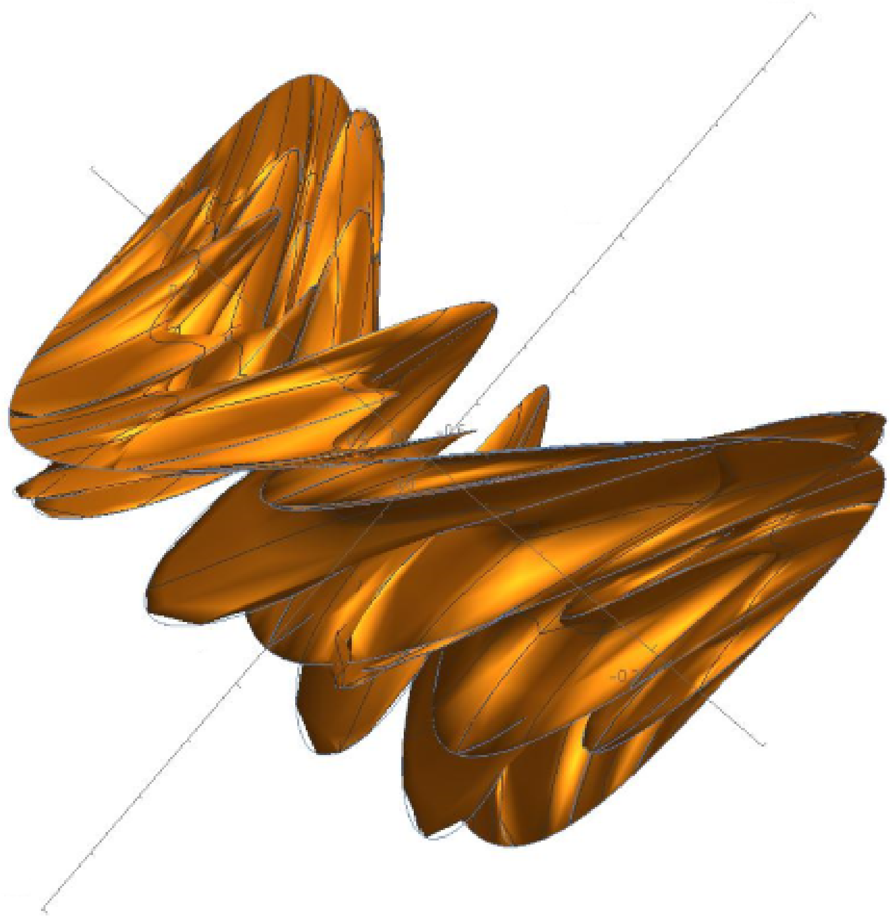


Figure 9.2: A Hölder continuous surface in the Heisenberg group? This image is the result of using three iterations of the dyadic geodesic bisection procedure, and then rendering a smooth interpolation of the resulting geodesic squares.

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