MODELS AND MULTIPLICITIES: LOGICAL PICTURES IN HERTZ AND WITTGENSTEIN

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The most long-standing division amongst interpretations of Ludwig Wittgenstein's *Tractatus* is between *ontologically-oriented* and *logically-oriented* interpretations. On an ontologically-oriented interpretation, the *Tractatus* introduces unfamiliar entities—simple objects constituting logically independent states of affairs—in order to account for the sense of colloquial sentences. On a logically-oriented interpretation, in contrast, the sense of colloquial sentences is presupposed, and Tractarian simple objects do not play a special explanatory role.

I show that an unprecedented argument in favour of a logically-oriented interpretation emerges from an appreciation of the influence of Heinrich Hertz's *Principles of Mechanics*, particularly Hertz's notion of a *dynamical model*. According to Hertz, a dynamical model captures all of the essential content of a mechanical description. On this view a pendulum, a mass on a spring, and a vibrating string are all instantiations of the same mechanical system because they can all be represented by the same dynamical model. I show that understanding the central role of dynamical models in *Principles* provides crucial insights into Hertz's project as well as its influence on Wittgenstein. Just as *Principles* provides the analytic resources needed to bring out the essential content of mechanical descriptions, the *Tractatus* provides the analytic resources needed to bring out the essential content of colloquial sentences. Despite certain appearances to the contrary, neither Hertz nor Wittgenstein was arguing for the existence of unfamiliar ontological entities. Rather, they were aiming to display the significant content of ordinary descriptions—in classical mechanics and in natural language respectively.

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1.0 INTRODUCTION

> ‘A doubt which makes an impression on our mind cannot be removed by calling it metaphysical: every thoughtful mind as such has needs which scientific men are accustomed to denote as metaphysical.’
>
> Heinrich Hertz, *Principles of Mechanics*, p. 23

1.1 HERTZ AND WITTGENSTEIN

Although it is uncontroversial that Hertz was an important influence on Wittgenstein, his influence has mostly been treated quite peripherally. However, there are some striking pieces of textual evidence which indicate that Hertz's work may have had a more profound impact on Wittgenstein than is generally appreciated. Not only is Hertz's name one of the few to appear in the *Tractatus*,[1] Wittgenstein also considered using a quotation from Hertz's *Principles of Mechanics* as the motto for the *Philosophical Investigations*.[2] Besides this, Wittgenstein also made some rather grand attributions to Hertz's influence. One such attribution is a remark that Wittgenstein made during a talk to the Moral Sciences Club in 1939, saying that a passage from *Principles* seemed to him to sum up philosophy.[3] Another is a more autobiographical expression of a similar sentiment in *The Big Typescript*: ‘In the way I do

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[1] Cf. Kjærgaard (2002).

[2] Cf. Janik (2000) p. 149.

[3] Cf. McGuinness (2002a) ix and Kjærgaard (2002), p. 126.

philosophy, the whole task lies in arranging the expression in such a manner that convincing problems/insecurities disappear ((Hertz.))'[4] A further important piece of textual evidence is the following remark, in which Wittgenstein explicitly cited his influences (seemingly in chronological order):

> I think there is some truth in my idea that I am really only reproductive in my thinking. I think I have never *invented* a line of thinking but that it was always provided for me by someone else and I have done no more than passionately take it up for my work of clarification. This is how Boltzmann, Hertz, Schopenhauer, Frege, Russell, Kraus, Weininger, Spengler, and Saffra have influenced me. (Wittgenstein, 1984, p. 16e)

This remark indicates that Wittgenstein picked up at least one 'line of thinking' from Hertz. An overarching aim of this thesis will be to uncover such a line of thinking in the *Tractatus*.

Beginning with a wide-angle view, it is possible to identify a Hertzian influence that spans the breadth of Wittgenstein's philosophical career. Roughly, this influence concerns the idea of *dissolving* a philosophical problem—achieving a perspective from which the problem simply does not arise. The key passage where this idea is expressed by Hertz is probably one of the best known from the introduction to *Principles*, where Hertz discusses the confusion surrounding the term 'force' in classical mechanics:

> ...we have accumulated around the [term] 'force'... more relations than can be completely reconciled amongst themselves. We have an obscure feeling of this and want to have things cleared up. Our confused wish finds expression in the confused question as to the nature of force... But the answer which we want is not really an answer to this question. It is not by finding more and fresh relations and connections that it can be answered; but by removing the contradictions existing between those already known, and perhaps reducing their number. When these painful contradictions are removed, the question as to the nature of force will not have been answered; but our minds, no longer vexed, will cease to ask illegitimate questions. (Hertz, 1899, pp. 7-8)

Indeed, it is from this passage that Wittgenstein considered taking the motto for the *Investigations*. But the idea that philosophical problems should be dissolved rather than solved can also be recognised in the framing remarks of the *Tractatus*: in the preface, Wittgenstein says, 'The book deals with the problems of philosophy, and shows, I believe, that the reason why these problems are posed is that the logic of our language is misunderstood.' Then, on the last page of the book, we have the following:

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[4] Cf. ibid, p. 125

> The correct method in philosophy would really be the following: to say nothing except what can be said, i.e. sentences of natural science—i.e. something that has nothing to do with philosophy—and then, whenever someone else wanted to say something metaphysical, to demonstrate to him that he had failed to give a meaning to certain signs in his sentences. (6.53)[5]

The idea that philosophical problems are not genuine problems—and hence need to be dissolved rather than solved—is also evident at other important points in the text (cf. 4.003 in particular—'And it is not surprising that the deepest problems are in fact *not* problems at all'). Hence it is easy enough to see that, both early and late, Wittgenstein took a characteristically indirect approach to the kinds of philosophical problems he engaged with.

Promising as this may be, what has been said so far has remained at a high level of abstraction. What is wanted is a close study of what Hertz had in mind when writing the above passage in the introduction to *Principles*, particularly what he meant when he said: 'It is not by finding more and fresh relations and connections that [our confused question] can be answered; but by removing the contradictions existing between those already known, and perhaps reducing their number.' We will need to understand how exactly Hertz regarded himself as having achieved this with the concept of force through his reformulation of mechanics. We will then be in a position to explore why Wittgenstein was so drawn to Hertz in this regard, and saw his own work as following upon it.

1.2 REAPPROPRIATING HERTZ

Besides influencing Wittgenstein, *Principles* has had a significant influence on philosophy more broadly. Though Hertz's formulation of mechanics never "caught on",[6] his book was widely read by physicists at the time, and the philosophical introduction has had a lasting impact on philosophers of science ever since.[7] However, there are reasons to suspect that

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[5] Following convention, references to the *Tractatus* will be given by citing the line number. The translation used, with occasional modifications, is Pears and McGuinness (Wittgenstein, 1994).

[6] It is important to note that Hertz knew full well that *Principles* would make for a very poor textbook on mechanics, and had no ambitions for it to catch on in that sense. Indeed, Hertz had no qualms with the traditional presentation of mechanics for such purposes; cf. Hertz (1899), p. 40.

[7] See in particular the collection of papers in Baird et al. (1998).

some major aspects of *Principles* have been widely misunderstood.

Hertz is generally regarded as one amongst several of the major philosopher-scientists at the end of the nineteenth century who attempted to eliminate the notion of force from classical mechanics. The following passage from Max Jammer's *Concepts of Force* is representative of this amalgamation of nineteenth century attitudes towards the foundations of physics:

> Just as Maxwell conceived electromagnetic forces as due to the motion of concealed masses, or as Lord Kelvin reduced these effects to a mechanism of vortex atoms and Helmholtz to cyclical systems of concealed motion, so Hertz thought it necessary to account not only for electrodynamic forces, but also for gravitational forces, for all actions at a distance, and finally for all mechanical forces, by some mechanism of concealed masses and motions. But if such a approach is capable of gradually eliminating the mysterious forces from mechanics, declares Hertz, it should be possible entirely to prevent their entering into mechanics. (Jammer, 1999, p. 224)

The motivation for eliminating forces from mechanics is typically made out by appealing to the *unobservability* of forces, in contrast with the observable motions of masses. However, I will argue that such an interpretation of Hertz's motivation for writing *Principles* is not correct.

It is true enough that many of those working in physics at the end of the nineteenth century problematized the foundations of mechanics, and criticised the Newtonian definition of force.[8] Influenced in particular by the idea of the ether, various intricate mechanisms were suggested to account for the transmission of action-at-a-distance forces. More grandly, the 'energeticists' Wilhelm Ostwald and Georg Helm took energy to be primitive, and sought to show that force was a derived concept. Hertz himself, after critiquing both the traditional and energeticist formulations, took just space, time and mass as the primitive notions, and proceeded to recover the content of classical mechanics in supremely systematic fashion.

Perhaps surprisingly, however, Hertz begins his introduction to *Principles* not with a discussion of mechanics but with a general "picture theory" of representation:

> The most direct, and in a sense the most important, problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation. As a basis for the solution of this problem we always make use of our knowledge of events which have already occurred,

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[8] In this vein, Hertz himself cites Mach, Lodge, and Thomson and Tait (cf. Hertz (1899), p. 8).

> obtained by chance observation or by prearranged experiment. In endeavouring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves pictures [*Bilder*] or symbols of external objects; and the form which we give them is such that the necessary consequents of the pictures in thought are always the pictures of the necessary consequents in nature of the things pictured... We are thus enabled to be in advance of the facts, and to decide as to present affairs in accordance with the insight so obtained. The pictures which we here speak of are our conceptions of things. With the things themselves they are in conformity in *one* important respect, namely, in satisfying the above-mentioned requirement. For our purposes it is not necessary that they should be in conformity with the things in any other respect whatever. As a matter of fact, we do not know, nor have we any means of knowing, whether our conceptions of things are in conformity with them in any other than this *one* fundamental respect. (Hertz, 1899, p. 1)

Hertz's single fundamental requirement on pictures is linked to his emphasis on anticipating future events: the necessary consequents of a picture must give pictures of the consequents of what is represented. However, Hertz also makes the bold claim that we cannot know, even in principle, whether our conceptions of things are correct in any further sense. This epistemological modesty plays a fundamental role in shaping Hertz's formulation of mechanics. Hertz's notion of a picture (*Bild*) is not something visualizable; it is not some sort of imaginative aid for grasping an otherwise abstract idea. Indeed, insofar as such pictures do inevitably play a role in our theorizing, Hertz regards them as distracting from the *essential content* of a theory.[9] As I will be concerned to argue in chapter 2, Hertz's own formulation of mechanics is notably austere, just sufficient to provide abstract representations of the motions of mechanical systems in the form of *dynamical models*. A major theme of this thesis is that an adequate understanding of Hertz—not to mention an adequate understanding of Hertz's influence on Wittgenstein—requires appreciating the sense in which Hertzian pictures are *logical* pictures.

In understanding Hertz's motivations for writing *Principles*, it is important to note that Hertz was not challenging the *correctness* of the traditional formulation of mechanics, and had no expectations that his own formulation would replace it. Rather, what Hertz found unsatisfactory in the traditional formulation was the lack of logical perspicuity. In particular, Hertz went to some effort to impress upon his readers that there was a 'logical obscurity' in

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[9] Cf. Hertz (1893), p. 28: 'scientific accuracy requires of us that we should in no wise confuse the simple and homely figure, as it is presented to us by nature, with the gay garment which we use to clothe it.' Compare this with 4.002 in the *Tractatus*, quoted below.

the Newtonian conception of force, stemming from a subtle ambiguity in Newton's laws of motion:

> The force spoken of in [Newton's] definition and in the first two laws act upon a body in one definite direction. The sense of the third law is that forces always connect two bodies, and are directed from the first to the second as well as from the second to the first. It seems to me that the conception of force assumed and created in us by the third law on the one hand, and the first two laws on the other hand, are slightly different. This slight difference may be enough to produce the logical obscurity... (Hertz, 1899, p. 6)

The tension that Hertz believes he has identified between the conception of force operative in Newton's third law, and the slightly different conception operative in Newton's first two laws, will be analysed in detail below.[10] But it is worth noting immediately that Hertz does not express a concern that the notion of force is problematic *per se*, and in particular he does not express a concern that it is problematic because it is unobservable.

The final task that Hertz sets himself in his introduction is to provide an outline of his own formulation of mechanics. He first notes that it seems impossible to describe the behaviour of observable things in a lawlike way unless some reference is made to 'other, invisible things':

> We soon become aware that the totality of things visible and tangible do not form a universe conformable to law, in which the same results always follow from the same conditions. We become convinced that the manifold of the actual universe must be greater than the manifold of the universe which is directly revealed to us by our senses. (Hertz, 1899, p. 25)

Hence we are forced to postulate something beyond what is directly observable, and the notion of force in the traditional formulation of mechanics, or energy in the energeticist formulation, are paradigm examples of such invisible things. What is distinctive in Hertz's approach is that he instead postulates the existence of *hidden masses*. Hertz argues that there may be no need to postulate the existence of anything of a fundamentally different nature from the masses in motion that we *do* observe. Beginning with definitions of just time, space and mass, Hertz is then able to capture the core empirical content of mechanics in a single 'fundamental law':

> **Fundamental Law.** Every free system persists in its state of rest or of uniform motion in a straightest path. (§309)

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[10] See chapter 3 section 3.3.

The power of Hertz's fundamental law stems from the rich notion of a 'straightest path'. This is the straightest path in a system's *configuration space*—an abstract, high-dimensional space which incorporates information concerning the essential characteristics of the mechanical system at hand.[11] Hertz then offers the following explanation for how the notion of force will reemerge in this context:

> We soon find it convenient to introduce into our system the idea of force. However, it is not as something independent of us and apart from us that force now makes its appearance, but as a mathematical aid whose properties are entirely in our power. It cannot, therefore, in itself have anything mysterious to us. Thus according to our fundamental law, whenever two bodies belong to the same system, the motion of the one is determined by that of the other. The idea of force now comes in as follows. For assignable reasons we find it convenient to divide the determination of the one motion by the other into two steps. We thus say that the motion of the first body determines a force, and that this force then determines the motion of the second body. In this way force can with equal justice be regarded as being always a cause of motion, and at the same time a consequence of motion. Strictly speaking, it is a middle term conceived only between two motions. According to this conception the general properties of force must clearly follow as a necessary consequence of thought from the fundamental law; and if in possible experiences we see these properties confirmed, we can in no sense feel surprised, unless we are skeptical as to our fundamental law. Precisely the same is true of the idea of energy and of any other aids that may be introduced. (Hertz, 1899, p. 28)

In articulating clearly how the notion of force follows from the fundamental law, Hertz claims that *Principles* avoids the logical obscurity that is present in the customary representation of mechanics. Hence it should already be clear that it would be overly hasty to regard Hertz as engaged in the enterprise of simply *eliminating* the notion of force from mechanics. In spelling this out in further detail, however, I will be concerned to scrutinize the issue that Hertz *did* take with the traditional Newtonian formulation. These issues will be discussed in chapter 3.

1.3 HERTZ AND THE *TRACTATUS*

The first explicit reference to Hertz in the *Tractatus* is the following:

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[11] See below, chapter 2 section 2.3.2

> 4.04 In a sentence there must be exactly as many distinguishable parts as there are in the situation that it represents.
>
> The two must possess the same logical (mathematical) multiplicity. (Compare Hertz's *Mechanics* on dynamical models.)

One point of immediate note is that this remark refers to a series of passages set deep in the technical bulk of *Principles* (specifically §§418–428, about two thirds of the way through the book). Most commentary that discusses the connection between Hertz and Wittgenstein only considers Hertz's introduction, but the reference to dynamical models here not only suggests that Wittgenstein studied the entire text but also suggests that he drew inspiration from Hertz's mechanics taken as a connected whole.

The notion of multiplicity that Wittgenstein mentions at 4.04 is obscure. As a first approximation, the 'distinguishable parts' of a sentence are whatever grammatical components play some role in allowing it to express its sense. Wittgenstein also refers to the 'constituent parts' of a sentence at 4.024, and there is a brief ensuing discussion of the translation of individual words at 4.025 (indeed, at 4.026 words are referred to directly as 'simple signs'). 4.032 reminds us, however, that we should not think too crudely of distinct words as corresponding directly to the grammatical components of a sentence (as the separate roles played by the stem and ending of 'Ambulo' makes clear). But if we are to think of the true constituent parts of a sentence as Tractarian names, i.e. the simple signs in *elementary* sentences, then in fact there is a temptation to travel a great distance (via logical analysis) away from the overt grammatical components of colloquial language.[12] This, at any rate, might be regarded as the way to connect the Tractatarian notions of elementary sentences and names (as well as states of affairs and objects) with this section of the text, especially passages like the following:

> 4.0311 One name stands for one thing, another for another thing, and they are combined with one another. In this way the whole group—like a tableau vivant—presents a state of affairs.
> 4.0312 The possibility of sentences is based on the principle that objects have signs as their representatives.

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[12] Cf. 4.002, 'It is not humanly possible to gather immediately from it what the logic of language is. Language disguises thought. So much so, that from the outward form of the clothing it is impossible to infer the form of the thought beneath it'. It is noteworthy that Wittgenstein's language here is reminiscent of a well known remark from Hertz (1893), p. 28 (quoted above).

> My fundamental thought is that the 'logical constants' are not representatives; that there can be no representatives of the *logic* of facts.

In making progress with interpreting Wittgenstein's claim that sentence and situation must have the same multiplicity, it will clearly be useful to see what the comparison with Hertz's dynamical models comes down to. In this vein it is important to note that the passages which develop the notion of a dynamical model seem to tie together some of the central themes of *Principles*. In particular, Hertz states that the only knowledge that classical mechanics makes available is precisely the information contained in dynamical models:

> We can... in fact, have no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone, that the one set of systems are [dynamical] models of the other. (§427)[13]

For immediate purposes, this is particularly important because of its connection with Hertz's picture theory of representation:

> The relation of a dynamical model to the system of which it is regarded as the model, is precisely the same as the relation of the pictures which our minds forms of things to the things themselves... The agreement between mind and nature may therefore be likened to the agreement between two systems which are models of one another (§428)

Recall that Hertz's sole 'fundamental requirement' on a picture is that its necessary consequents must give pictures of the consequents of what it represents. Hence it is through the notion of a dynamical model that Hertz applies this fundamental requirement on pictures in general to the pictures provided by classical mechanics in particular. On Hertz's view, a pendulum, a mass on a spring, and a vibrating string are all instantiations of the *same* mechanical system because they are all represented by the same dynamical model. Dynamical models thus abstract away from the ontological constituents of what they represent. Although Hertz begins *Principles* by giving definitions of 'material systems', 'material points' and 'material particles' (*Massenteilchen*), he is not proposing an unfamiliar ontology in doing so—he does not think that empirical investigation will show all mechanical systems

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[13] From this point onwards, a section number without a further citation will be used to refer to passages from the main body of *Principles*. Note that I have made occasional modifications to the published translation.

to be *composed* of such entities. Rather, Hertz is building a perspicuous logical framework within which the propositions of mechanics can be cast.[14]

As I will be concerned to argue in the final chapter of this thesis, understanding the central role of dynamical models in *Principles* also points to a much deeper connection with Wittgenstein's project in the *Tractatus*. Interpretations of the *Tractatus* can be divided into two broad camps: *ontologically-oriented* interpretations and *logically-oriented* interpretations.[15] On an ontologically-oriented interpretation, the *Tractatus* introduces unfamiliar entities—simple objects constituting logically independent states of affairs—in order to account for the sense of colloquial sentences. On a logically-oriented interpretation, in contrast, the sense of colloquial sentences is presupposed, and Tractarian simple objects do not play a special explanatory role. As we will see, Wittgenstein's reference to dynamical models at 4.04 emerges as a critical piece of textual evidence in favour of a logically-oriented interpretation. Where Hertz's goal is to provide the analytic resources to display the essential content of mechanical descriptions, Wittgenstein's goal is to provide the analytic resources to display the essential content of descriptions *tout court*.

Recall that the upshot of doing this is supposed to be that certain misguided philosophical problems will lose their grip on us. (To borrow a Tractarian turn of phrase: *the solution is seen in the vanishing of the problem.*[16]) Here, again, the link with *Principles* is clear: Hertz's aim is to clarify the logic *of mechanics* so that 'our minds, no longer vexed, will cease to ask illegitimate questions'. As noted, although this theme of dissolving rather than solving philosophical problems is one that stays with Wittgenstein throughout his career, it is its presence in the *Tractatus* that will frame the overarching argument of this thesis.

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[14] Indeed, Hertz remains intentionally agnostic about the fundamental constituents of matter and thus side-steps the controversies of the period concerning the existence of atoms; cf. Lützen (2005), pp. 140-141.

[15] See below, chapter 4 section 4.1

[16] Cf. 6.521.

2.0 MECHANICS WITHOUT MECHANISMS

> ‘Even when one continued to speak of the fundamental concepts of theoretical physics as symbols, in order to avoid from the first any danger of ontological interpretation, there was a necessity of attributing to these very symbols themselves a theoretical meaning and therewith an “objective” content. Far from being merely arbitrary additions to what was given by direct observations they became essential factors with which alone an organization of the given, the fusion of the isolated details into the system of experience, was possible.
>
> The first great physicist actually to complete this turn of affairs and at the same time to grasp the full measure of its philosophical implications, was Heinrich Hertz, with whom began a new phase in the theory of physical methods.’
>
> Ernst Cassirer, *The Problem of Knowledge*, §V

2.1 AN UNTIMELY DEATH

On new year’s day of 1894, Hertz died just 36 years old. He had been heralded as one of the most promising scientists of his generation—‘predestined to open up to mankind many of the secrets which nature has hitherto concealed from us’, as Helmholtz put it (Hertz, 1899, vii). Hertz had dedicated the last few years of his life to a grand project in the foundations of physics, culminating in the posthumous publication of *Principles of Mechanics.* As he had prepared to send the manuscript to press, Hertz expressed trepidation about how it would be received, revealing to his parents that he had never shown it to another soul.[1] When *Principles* finally appeared it was received with high praise, but even as it was admired

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[1] Cf. Hertz’s letter to his parents, 19 November 1893 (Hertz, 1977, p. 343).

for its elegance and scope Hertz's contemporaries could not find in it the kinds of advances that they had hoped for. Indeed, there was a general sense of confusion regarding what *Principles* was supposed to have achieved. Hertz himself, of course, could not help. As Boltzmann lamented, at the same moment that Hertz's book was published 'his lips became for ever sealed to the thousand requests for clarification that are certainly not on the tip of my tongue alone' (Boltzmann, 1974, p. 90).

Nevertheless, *Principles* went on to have a remarkable impact on both physicists and philosophers. It has been regarded as marking 'the beginning of modern physics' (Mulligan, 2001, p. 151), a view defended emphatically by Cassirer and echoed more recently by van Fraassen.[2] Furthermore, almost all the leading physicists and scientifically-oriented philosophers of two generations read and reacted to *Principles*.[3] Crucially, however, almost all of these esteemed readers found Hertz's mechanics 'interesting and beautiful, but either baffling or unsuccessful, or both' (Preston, 2008a, p. 100). The sweeping influence of *Principles* makes the problem of finding a satisfactory interpretation of it all the more pressing, yet the difficulties in doing so remain as acute today as they did following Hertz's untimely death.

Hertz begins with three primitive notions—space, time, and mass—and proceeds to develop a sophisticated analytical framework in which to treat the mechanical properties of 'systems', defined as collections of material points with *connections* between them (equations relating their relative positions). Hertz then posits his single fundamental law: 'Every free system persists in its state of rest or of uniform motion in a straightest line' (§309). The grand claim of *Principles* is that the entire empirical content of classical mechanics is captured in this single statement. However, *Principles* does not merely treat mechanics more economically and systematically than previous formulations; Hertz also purports to demystify the notions of *force* and *energy*, deriving cleaned up versions of both from the spatial and temporal relations between masses. Hertz claims that by avoiding obscurities in Newton's laws of motion, certain confused questions which troubled his contemporaries simply won't arise.[4] To achieve all this, and to apply his framework to the full range of

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[2] Cf. Cassirer (1950) pp. 114 ff., and van Fraassen (2008) pp. 204 ff.

[3] Including Helmholtz, Mach, Boltzmann, Lorentz, FitzGerald, Einstein, Poincaré, Duhem, Carnap, Russell, and Wittgenstein Cf. Preston (2008a), p. 100 and Saunders (1998), p. 123.

[4] Cf. Hertz (1899) p. 8.

mechanical phenomena, Hertz introduces the notion of *hidden masses*:

> If we wish to obtain an picture of the universe which shall be well-rounded, complete, and conformable to law, we have to presuppose, behind the things which we see, other, invisible things—to imagine confederates concealed beyond the limits of our senses... We are free to assume that this hidden something is nought else than motion and mass again, motion and mass which differ from the visible ones not in themselves but in relation to us and to our usual means of perception. (Hertz (1899) p. 25)

However, it is here that we encounter the confused reaction of Hertz's readers. Helmholtz, in the introduction he wrote for *Principles*, remarked: 'Unfortunately [Hertz] has not given examples illustrating the manner in which he supposed such hypothetical mechanisms to act; to explain even the simplest cases of physical forces on these lines will clearly require much scientific insight and imaginative power' (Hertz, 1899, xx). Boltzmann went to considerable effort to try to construct the mechanisms that Hertz had apparently left out but without success, remarking: 'so long as even in the simplest cases no systems or only unduly complicated systems of hidden masses can be found that would solve the problem in the sense of Hertz's theory, the latter is only of purely academic interest' (Boltzmann, 1974, p. 90). And Mach was particularly pointed in drawing attention to the fact that such Hertzian mechanisms would oblige one 'to resort, even in simplest cases, to fantastic and even frequently questionable fictions' (Mach, 1960, p. 323). Modern commentators have been similarly unanimous in complaining about the difficulties of finding plausible Hertzian mechanisms. Lützen remarks, 'If Hertz had lived he would certainly have been hard pressed for a reaction to this problem' (Lützen, 2005, p. 278), or as Mulligan puts it, 'This criticism is quite valid and undoubtedly carried great weight with physicists in the decade after 1894' (Mulligan, 1998, p. 178).

The central goal of this chapter will be to resolve this persistent tension in interpreting Hertz's book. To begin, I will situate *Principles* in its historical context and identify the widespread tendency to regard Hertz's project as closely connected with the search for an ether mechanism. I will argue that this tendency has contributed to the confusion and dissatisfaction amongst Hertz's readers because it ties the value of his project to the prospects of finding such a mechanism. I will then turn to discuss Hertz's ideas concerning scientific representation; ideas that culminated in Hertz's "picture theory" of representation. With Hertz's austere account of representation in view, I will argue that it has been misleading to interpret *Principles* as closely connected with the quest for an ether mechanism, despite passages where Hertz seems to invite such an interpretation. More specifically, I will argue that a crucial role of Hertz's hypothesis of hidden masses has been widely overlooked. Rather than acting as an unwieldy proposal for the fundamental constituents of mechanical systems, Hertz's hypothesis *rules out* knowledge of such underlying entities.

2.2 THE QUEST FOR AN ETHER MECHANISM

The second half of the nineteenth century that encompassed Hertz's short career was characterized by fervent research in electromagnetism. The first volume of Maxwell's *Treatise* appeared in 1873, and Hertz's own groundbreaking observations of electric waves in 1888 established Maxwell's theory as canonical. For many physicists the most appealing aspect of that theory was the way in which it seemed to eschew instantaneous actions-at-a-distance in favour of the notion of waves propagating through a medium. Hertz's famous experiments were widely regarded as confirming this view of electromagnetism, and Kelvin introduced Hertz's collection of papers on the subject as a 'splendid consummation' of 'the nineteenth-century school of plenum, one ether for light, heat, electricity, magnetism' (Hertz, 1893, xv).[5] However, finding an ether mechanism which could account for electromagnetic phenomena remained a critical open problem.

In seeking an ether mechanism many of Hertz's contemporaries were inspired by the success of the kinetic theory of gases. That conception of a gas—a swarm of billiard-ball like atoms, colliding with each other according to ordinary Newtonian mechanics—had been extremely successful in both accounting for thermodynamical properties and leading to novel predictions. It was also admired for conveying a particularly satisfactory kind of understanding: the model *really represented* what a gas was like, at least approximately. Hence a widely held view was that it 'ought to be possible, at least in principle, to do the same thing for the ether: to find a mechanical model that reflected its true nature' (Hunt, 1991, pp. 76-77).

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[5] See also Mulligan (2001) p. 143: 'Hertz empirically confirmed Maxwell's electromagnetic waves; it was universally assumed that the ether was confirmed at the same time.'

Thus the historical context in which *Principles* appeared involved a plethora of increasingly intricate attempts to show how some kind of material ether, governed by ordinary mechanics, could account for electromagnetic effects. More generally, the promise of an ether mechanism that eschewed action-at-a-distance was a defining feature of theoretical physics around 1890, and the background against which Hertz turned to foundational work in mechanics. Indeed, an eloquent description of this situation is due to Hertz himself:

> More and more we feel that [the nature of the ether] is the all-important problem, and that the solution of it will not only reveal to us the nature of what used to be called imponderables, but also the nature of matter itself and of its most essential properties—weight and inertia. The quintessence of ancient systems of physical science is preserved for us in the assertion that all things have been fashioned out of fire and water. Just at present physics is more inclined to ask whether all things have not been fashioned out of the ether. (Hertz, 1896, pp. 326-327)

However, proposing a concrete ether mechanism was clearly not a *direct* goal of Hertz's book. In fact, before *Principles* was published Hertz had explicitly attempted to dispel such rumours concerning what it was he was working on:

> What you have heard about my works... is unfortunately without any foundation and I do not know how this opinion has been formed. I have not at all worked with the mechanics of the electric field, and I have not obtained anything concerning the motion of the ether. (Hertz to Emil Cohn, November 25 1891)[6]

Hertz's primary aim, as he himself emphasised, was to achieve a certain kind of *clarification* of classical mechanics as it stood. The letter to Cohn continues:

> This summer I have thought a great deal about the usual mechanics... In this area I would like to put something straight and arrange the concepts in such a way that one can see more clearly what are the definitions and what are the facts of experience, such as, for example, concepts of force and inertia. I am already convinced that it is possible to obtain great simplifications here. (ibid)

Thus most readers of *Principles*—both historical and contemporary—have regarded it as an attempt to lay the groundwork for some future ether mechanism, the details of which could be filled in later. But the inclination towards interpreting *Principles* this way has contributed to the dissatisfaction amongst Hertz's readers, for it ties the value of his project to the prospects

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[6] Translations of the letter are reproduced in Lützen (2005) p. 74 and Nordmann (1998) p. 160. For an extract of the original German text see Nordmann (1998) p. 169.

of filling in these details. We thus encounter a crucial unanswered question: how could Hertz's apparent attitude towards the difficulties of constructing such a mechanism have been so cavalier? Indeed, independently of the historical context, the content of *Principles* can also seem to invite this question itself.

2.3 AN OVERVIEW OF *PRINCIPLES*

Before proceeding, a note on terminology. In the opening paragraphs of *Principles* we find Hertz introducing 'material particles' and 'material points' in an interconnected series of definitions, leading up to the definition of a 'system'. The latter terms are literal translations from the German (*materieller Punkte* and *System* respectively), but translating Hertz's term *Massenteilchen* as 'material particle' is misleading. Hertz's *Massenteilchen* are, in an important sense, smaller—indeed, infinitely smaller—than his material points, and this conflicts with the ordinary understanding of 'particles' and 'points' in English. To avoid unhelpful associations, I will use '*Massenteilchen*' instead of 'material particle(s)' in what follows.[7]

2.3.1 Hertz's analytical framework

Principles is divided into two books: in the first, Hertz defines his terms and establishes an analytical (mathematical) framework; in the second, he explains how this framework is to be applied. The first book purports to be a priori 'in Kant's sense':

> The subject-matter of the first book is completely independent of experience. All the assertions made are a priori judgments in Kant's sense. They are based upon the laws of the internal intuition of, and upon the logical forms followed by, the person who makes the assertions; with his external experience they have no other connection than these intuitions and forms may have. (§1)[8]

The content of the first book is supposed to be compatible with any interactions with

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[7] In this I follow Lützen (2005), cf. p. 135

[8] For some discussion of Hertz's Kantian influences, see Hyder (2002) pp. 35-46, Lützen (2005) §10, and Leroux (2001) pp. 192-193

spatio-temporal objects whatsoever. Thus it is only in book two that we find the one proposition that Hertz regards as falsifiable: his fundamental law.

Following Kant, Hertz helps himself to ‘the space of Euclid’s geometry’ and ‘the time of our internal intuition’ (§2). In the case of mass, however, there is no associated Kantian form of intuition to appeal to, and Hertz’s avoidance of anything dependent on experience leads to a very minimal notion: the ‘mass’ contained in a given space is defined as the relative number of *Massenteilchen* in that space. Hence Hertz first defines *Massenteilchen* in order to give his definitions of mass, then proceeds to definitions of material points and, finally, systems.

Massenteilchen are represented completely by curves through space parametrized by time:

> **Definition 1.** A *Massenteilchen* is a characteristic by which we associate without ambiguity a given point in space at a given time with a given point in space at any other time. (§3)

Hertz also stipulates that any number of *Massenteilchen* can occupy the same location at the same time, allowing for the two definitions that follow:

> **Definition 2.** The number of *Massenteilchen* in any space, compared with the number of *Massenteilchen* in some chosen space at a fixed time, is called the mass contained in the first space.
>
> We may and shall consider the number of *Massenteilchen* in the space chosen for comparison to be infinitely great. The mass of the separate *Massenteilchen* will therefore, by the definition, be infinitely small. The mass in any given space may therefore have any rational or irrational value. (§4)

> **Definition 3.** A finite or infinitely small mass, conceived as being contained in an infinitely small space, is called a material point. (§5)

A material point may at first seem to be the familiar *point mass* by which standard presentations of mechanics routinely treat stars and atoms alike: a discrete object whose mass can be treated as situated at a point. However, according to Hertz’s definition of mass it must be possible for material points to contain infinite numbers of *Massenteilchen* if their mass values are to range over the real numbers. Hertz claims we can do this by ‘supposing the *Massenteilchen* to be of a higher order of infinitesimals than those material points which are regarded as being of infinitely small mass' (§5).[9] This relationship between the material points and the *Massenteilchen* is suggestive of the material points in continuum mechanics, which are integrated over to define the properties of continuous media. In fact, Hertz's introduction of *Massenteilchen* might have been intended, in part, as a way to preserve conservation of mass whilst allowing for continually varying mass-densities.[10]

The final definition in Hertz's first chapter is of a *system*:

> **Definition 4.** A number of material points considered simultaneously is called a system of material points, or briefly a system. The sum of the masses of the separate points is, by §4, the mass of the system. (§6)

Systems are simply collections of material points 'considered simultaneously'. For much of *Principles* Hertz concerns himself entirely with the mechanics of material systems (cf. §121), and shows that the connections of such a system can always be represented by 'equations of condition' of a canonical form (cf. §115 ff.). A great part of the ensuing work is in setting up the vocabulary to talk about the properties of such a system (its displacement, velocity, acceleration, and so on), and this vocabulary finds a natural home in the context of the *configuration space* associated with a system, to which we can now turn.[11]

2.3.2 Configuration Space

The basic idea of a system's configuration space is straightforward. A system of n material points has an associated configuration space with $3n$ dimensions—one dimension for each of the three coordinates of each of its points—so that every location in configuration space represents a conceivable position of the whole system. For example, the position of a system

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[9] Although Hertz is fudging the mathematical details here, we could flesh this out on Hertz's behalf using modern tools. For one suggestion along these lines see Lützen (2005) p. 139.

[10] For a discussion of attempts that were made to extend Hertz's framework to continuous systems, see Lützen (2005) p. 140 and p. 286. Note that, because Hertz's mechanics seems only directly applicable to discrete systems, commentators have not drawn on concepts in continuum mechanics in interpreting either Hertz's *Massenteilchen* or his material points. Although this may be a mistake, a full discussion of this issue is beyond the scope of this thesis.

[11] Hertz himself minimized his use of spatial language in this context, and in particular did not use the expression 'configuration space'. This is because Hertz was keen to play down any direct comparison between mathematical high-dimensional spaces and *physical* space. For a brief discussion of this point, see Lützen (2005) p. 110.

of three points can be given by specifying the nine coordinates in its associated configuration space.

When there are connections between the points there are corresponding limitations on which regions of configuration space are accessible. Specifically, each connection rules out the region that would correspond to "breaking" that connection. A rigid system in which no material point can move independently of any of the others has only six degrees of freedom; hence, no matter how many material points it has, such a system will always be located within a 6-dimensional subspace inside its configuration space. In general, the connections of a system always limit the accessible region of a $3n$-dimensional configuration space to a lower-dimensional subspace.[12]

Many of the key geometric properties of configuration space are given with its metrical properties, which Hertz derives by first defining the 'magnitude of the displacement of a system':

> The magnitude of the displacement of a system is the quadratic mean value of [i.e. the positive root of the arithmetic mean of the squares of] the magnitudes of the displacements of all its *Massenteilchen*. (§§28, 29)

Note here the reference to *Massenteilchen*.[13] If Hertz had calculated the displacements of the material points this would have resulted in configuration space having a standard Euclidean metric. In other words, the line element of configuration space would have taken the familiar Pythagorean form:

$$ds^2 = \sum_{i=1}^{3n} dx_i^2$$

However, calculating the displacements of the *Massenteilchen* instead of the material points "weights" the expression for the magnitude of the displacement of a system, so that the

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[12] In fact this is only true for *holonomous* connections (cf. §123). Hertz regarded it as important to incorporate non-holonomous connections within his framework, even though he could have regarded these as ultimately derivable from holonomous connections—cf. Lützen (2005) p. 193. In this section and the following I mainly limit my attention to holonomous systems; for some discussion of non-holonomous systems see Lützen (2005) §15.3.

[13] The need for the appearance of *Massenteilchen* in this definition is in fact the key reason why Hertz included them in his framework at all. For a detailed discussion of the development of the idea of *Massenteilchen* in the early drafts of *Principles* see Lützen (2005) pp. 146-158.

more massive points contribute more to the displacement.[14] Hertz thus has the raw material to develop a more exotic metric for configuration space, first moving to a definition of *infinitesimal* displacement of a system (cf. §54), and then to expressions for the lengths and curvatures of paths of systems in general (cf. §§104 ff.). This results in the line element of configuration space having the following form:

$$ds^2 = \sum_{i=1}^{3n} m_i dx_i^2$$

Weighting the expression for (infinitesimal) displacement thus links the metrical properties of configuration space to the particular mass distribution of the system at hand.

To appreciate the significance of this metric structure, it is helpful to approach it from a different direction.[15] If the velocity of the i-th material point is v_i, the total kinetic energy of a system is given by:

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2$$

From here, we could define the line element of configuration space as follows:

$$ds^2 = 2T dt^2 = \sum_{i=1}^{n} m_i v_i^2 dt^2$$

As $v_i = (dx_i^2 + dy_i^2 + dz_i^2)^{\frac{1}{2}} / dt$ this gives:

$$ds^2 = \sum_{i=1}^{n} m_i (dx_i^2 + dy_i^2 + dz_i^2)$$

Denoting the coordiantes of the μ-th point as $(x_{3\mu-2}, x_{3\mu-1}, x_{3\mu})$, and letting its mass be equal to $m_{3\mu-2} + m_{3\mu-1} + m_{3\mu}$, we can see that we have reccovered Hertz's expression for the line element:

$$ds^2 = \sum_{i=1}^{3n} m_i dx_i^2$$

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[14] For further discussion of the important role of *Massenteilchen* in defining the metric properties of configuration space, see Appendix A.

[15] Here I follow Lanczos (1962) p. 22.

Hence the total kinetic energy of the system can be written as $T = \frac{1}{2}m(\frac{ds}{dt})^2$, where m is the sum of the masses of the individual points. This means that the total kinetic energy of the system can be regarded as *the kinetic energy of a single point in configuration space.* Situating a mechanical problem within a configuration space of this structure thus carries over the mechanics of a single point to the mechanics of an arbitrary system.[16]

If a system has no connections at all between its points it moves in a *straight* path in its configuration space (which is indeed the *straightest* path available). Increasingly complex systems will have an increasing number of connections between their points. As each connection defines a $(3n - 1)$-dimensional hypersurface inside the system's configuration space, and as the path that a system traces out must lie on the intersection of the hypersurfaces determined by all of its connections, every additional connection causes the system's path to deviate further from the straight path that it would otherwise follow. Thus every new connection increases the curvature of the system's path. Hertz's fundamental law asserts that the motion of a free system (roughly, one that can be treated as isolated) always traces out a straightest path on this curved hypersurface, embedded within its $3n$-dimensional configuration space.[17]

The full elegance of Hertz's fundamental law as a kind of generalization of the principle of inertia is thus revealed. In Hertz's words: '[the fundamental law] asserts that if the connections of the system could be momentarily destroyed, its masses would become dispersed, moving in straight lines with uniform velocity, but that as this is impossible, they tend as nearly as possible to such a motion' (Hertz (1899) p. 28).

2.3.3 Hidden masses and cyclical coordinates

From what has been said so far it remains opaque how Hertz's fundamental law, on its own, could accommodate all the varied phenomena of mechanics. Of course, many canonical mechanical problems concern systems that are *not* free, such as systems acted on by forces. To

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[16] Cf. Lanczos (1962) p. 22: 'In this space one point is sufficient to represent the mechanical system, and hence we carry over the mechanics of a free particle to any mechanical system if we place that particle in a space of the proper number of dimensions and proper geometry.'

[17] A system can also be described in terms of its *general* coordinates (cf. §13)—see below, chapter 3 section 3.4

capture such systems within the scope of his fundamental law, Hertz allows a 'complete' free system to be decomposed into subsystems, and, in particular, to contain a *hidden* subsystem (cf. §429). Thus Hertz introduces the hidden masses that are particularly characteristic of his framework. This idea plays a fundamental role for Hertz: as already noted, it is what allows him to employ only space, time and mass as his primitive notions, and gives rise to one of the key advantages that he believes his own formulation of mechanics has over other formulations. For although Hertz thinks that the attempt to unify phenomena in a law-like way inevitably requires stipulating something that is not directly observable, he makes the case that this does not necessitate an appeal to a further primitive notion: 'We may admit that there is a hidden something at work, and yet deny that this something belongs to a special category.' (Hertz, 1899, p. 25).[18]

Hertz goes on, 'What we are accustomed to denote as force and as energy now become nothing more than an action of mass and motion, but not necessarily of mass and motion recognisable by our coarse senses.' (Hertz, 1899, p. 26). Here, Hertz appeals to Helmholtz's earlier work on cyclical systems. A cyclical coordinate is one whose effect on the properties of a system is due only to its change, not its absolute value. A system is then called cyclical if its energy can be approximated as a function of the rates of change of its cyclical coordinates (cf. §§546-549). As an intuitive example, consider the spinning ring of a gyroscope.[19] Each component part of the ring is immediately replaced by its neighbour as the gyroscope rotates. The positions of these components are thus paradigm cyclical coordinates: it is only their *rates of change* that affect the gyroscope's behaviour. Because of the conservation of angular momentum, a closed box with a spinning gyroscope fixed to the inside will resist certain changes in its motion, and hence such a setup could mimic the actions of an external force field.

The mathematical tools for describing hidden cyclical subsystems can thus be used to widen the scope of Hertz's fundamental law, accounting for motions which would ordinarily be explained by appealing to distant forces. In particular, Hertz treats a material system 'acted on by forces' as *coupled* to one or more other (hidden) material systems, such that

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[18] As Nordmann notes, Hertz's approach in this regard has an eminently respectable pedigree 'which can be traced back to Descartes and beyond' (Nordmann, 1998, p. 169).

[19] Here I follow Wilson (2007) pp. 12-13.

the systems have at least one coordinate in common (§450). He then defines a force as the effect that one such coupled system has upon the motion of another (§455), and goes on to show that defining force in this way aligns with the notion of force in customary approaches to mechanical problems to a remarkable degree.[20] However, Hertz's notion of force adds nothing beyond the application of the fundamental law to a system of connected material points: every *complete* system is itself free and moves on a straightest path in its own configuration space.

Thus, after deriving all the canonical treatments[21] of mechanical problems within his analytical framework, Hertz claims that *Principles* is 'capable of embracing the whole content of ordinary mechanics' (Hertz, 1899, xxii), and that 'no definite phenomena can at present be mentioned which would be inconsistent with the system' (Hertz, 1899, p. 36).

2.4 HERTZ AND ETHER MECHANISMS

At this point we can take a step back and consider the basis for the general inclination to regard Hertz as concerned with laying foundations for an ether mechanism. Hertz's *Massenteilchen* can seem to be fundamental particles of some kind, and he proposes that hidden cyclical subsystems can model the effects of distant forces. His project can thus seem to bear a close relationship with certain nineteenth century attempts to model the ether. A particularly noteworthy example is the "gyrostatic adynamic" ether mechanism proposed by Kelvin a few years before *Principles* was published.[22] In introducing this mechanism, Kelvin began by describing a network of spherical atoms arranged such that each lies at the centre of a tetrahedron of four others, linked to its four neighbours by rigid bars. The bars attach to the atoms in such a way that their end points can slide freely on the atoms' surfaces, thus allowing the whole structure to have a degree of flexibility. Furthermore, each bar is conceived as containing, along its length, two miniature gyroscopes:

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[20] I discuss Hertz's conception of force in further detail in the next chapter.

[21] Including those of Lagrange, Hamilton, d'Alembert, Gauss and Jacobi, as well as Galileo and Newton—cf. Hertz (1899) Book 2 chapter III.

[22] Cf. Schaffner (1972), pp. 194-203.

> Instead of a simple bar, let us take a bar of which the central part, for a third of its length for example, is composed of two rings in planes perpendicular to one another... Let the two rings be the exterior rings of gyroscopes, and let the axes of the interior rings be mounted perpendicularly to the line of the bar. (Schaffner, 1972, p. 195)[23]

Aligning the gyroscopes and setting them in motion gives the structure a kind of rotationally-dependent elasticity, differing from the behaviour of ordinary elastic solids due to the fact that the restoring forces depend on the rotations of the connecting bars away from their original orientations. Kelvin declared: 'This relation of the quasi-elastic forces with rotation, is just that which we require for the ether, and especially to explain the phenomena of electro-dynamics and magnetism' (Schaffner, 1972, p. 196). Kelvin then used this structure as the basis for a significantly more intricate mechanism, designed to produce no restoring forces other than restoring couples in the same axes as deforming rotations.

On the standard interpretation of *Principles*, Hertz was clarifying mechanics with the expectation that a mechanism like Kelvin's would prove to be a good representation (or at least a useful analogy) of the structure of the ether. Importantly, we can see this style of interpretation directly informing the attempts that were made to fill in what appeared as the gaps in Hertz's presentation.[24] These attempts aimed to give Hertz's mechanics some plausibility by showing that it was at least *possible* to construct "Hertzian mechanisms", crude and complicated as they might be.

Furthermore, there are certain passages in *Principles* which seem to suggest that Hertz was indeed hoping for precisely the kind of ether mechanism that many of his contemporaries were struggling to construct. The most overt such passage comes at the end of the introduction, where Hertz considers the merits of appealing to connections over distant forces, remarking: 'the balance of evidence will be entirely in favour of the [Hertzian formulation of mechanics] when a second approximation to the truth can be attained by tracing back the supposed actions-at-a-distance to motions in an all-pervading medium whose smallest parts are subjected to rigid connections' (Hertz, 1899, p. 41). Combining this with two other passages in which Hertz talks of 'seeking the ultimate connections in the world of atoms' (to be discussed below, section 2.6), it is hardly surprising that there exists an almost universal

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[23] For some discussion of Kelvin's model, see Schaffner (1972) pp. 68-75 and Stein (1981) p. 319.

[24] For brief surveys of these attempts, see Lützen (2005) pp. 274 ff. and Preston (2008b) pp. 59 ff.

inclination to read *Principles* as aiming to provide foundations for an ether mechanism. At any rate, commentators such as FitzGerald felt no hesitation in interpreting Hertz this way:

> Hertz sees in all actions the working of an underlying structure whose masses and motions are producing the effects on matter that we perceive, and what we call force and energy are due to the actions of these invisible structures, which he implicitly identifies with the ether. (Hertz and Mulligan, 1994, p. 371)

Moreover, as we have seen, many modern commentators continue to interpret *Principles* along the same lines:

> [Hertz's] overwhelming conviction of the importance of the aether, joined to his urge to reduce all physics to mechanics, eventually culminated in 1894 in the posthumous publication of his *Mechanics*. (Mulligan, 2001, p. 138)[25]

Such interpretations make Hertz's apparent attitude towards the difficulties of constructing a concrete ether mechanism seem remarkably cavalier. Indeed, it is against this backdrop that the problem of finding a plausible Hertzian mechanism seems acutely pressing. However, this way of reading *Principles* doesn't fully take into account a crucial aspect of Hertz's book: the picture theory of representation articulated in the introduction.

2.5 HERTZ'S PICTURE THEORY

Commentators who have engaged closely with the philosophical content of Hertz's introduction have recognized Hertz as a progenitor of the family of *structuralist* views developed

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[25] See also Saunders (1998) p. 126: 'my own view of the *Principles* is that Hertz intended to make a methodological proposal, and that he supposed that it would be given substance by a mechanical model of ether'; and Lützen (2005) p. 266: 'The sole aim of the book was to establish the theoretical foundation for a construction of such hidden systems or in other words for constructing a model of the ether'. Some commentators have even mistakenly claimed that *Principles* aimed to provide a *direct* model of the ether, cf. Hyder (2002) pp. 42-43: 'the gap in Hertz's picture of electromagnetism was occupied by the ether: How are we to imagine its polarisation?... To fill the gap would need a picture of these hidden material systems. Hertz's last book, *The Principles of Mechanics Presented in a New Form*, attempted to do just this.'

However, other commentators have resisted the suggestion that the goal of *Principles* was to lay the groundwork for an ether mechanism. In particular, Nordmann has pointed out that as Hertz's hidden masses are unobservable in principle, they are 'not subject to exploration even by physical undertakings of the future' (Nordmann, 1998, p. 160). Hence Nordmann suggests that Hertz's primary focus revolved 'around the conceptual problems of ordinary classical mechanics' (ibid). In a similar vein, D'Agostino has remarked: 'Since hidden quantities cannot be observed, they belong to a pure theoretical framework' (D'Agostino, 1993, p. 73). I pursue a similar line of interpretation in section 2.6 below.

by figures in the philosophy of science throughout the twentieth century. Roughly speaking, such views regard the representative content of a scientific theory as stemming from its structural features rather than from the objects that it posits. Ernst Cassirer was perhaps the earliest commentator to recognise the importance of Hertz's role in this regard. Far from seeing *Principles* as laying foundations for an ether mechanism, Cassirer regarded Hertz's project as a response to the problems that had emerged in such attempts:

> Every barely imaginable suggestion and combination had been exhausted in an effort to establish [the ether's] constitution until finally, after all endeavors had failed, a change in the whole intellectual orientation was effected and investigators began to submit to critical proof the assumption of its existence instead of continuing to examine into its nature. (Cassirer, 1950, p. 89)[26]

More recently, Leroux (2001) and van Fraassen (2008) have also emphasized Hertz's role in the movement away from the mechanistic approach encapsulated in the increasingly intricate nineteenth century attempts to find an ether mechanism. Van Fraassen even goes so far as to say, 'In Hertz's, and later Poincaré's, verdict we recognize a definite *goodbye* to the interrelation of matter and ether as a live topic in physics' (van Fraassen, 2008, p. 202).

In seeking to understand the lack of mechanisms in Hertz's book, we need to appreciate how Hertz's ideas concerning representation framed his project. Although the presentation of the 'picture theory' in the introduction to *Principles* has been relatively well-discussed in the literature,[27] it has not often been situated against the development of Hertz's earlier ideas.[28] Hertz discussed the role of pictures (*Bilder*) in scientific representation at least as early as his 1884 Kiel lectures[29]—a decade before *Principles* was published—and these ideas continued to develop throughout his work on electromagnetism.

The Kiel lectures are important for contextualizing Hertz's picture theory because it is here that Hertz introduced the distinction between the *essential* and *inessential* content of a scientific theory. Early in the lectures, Hertz discussed the desirability of gaining an picture of

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[26] Cf. also Cassirer (1950) pp. 103 ff.

[27] For example, Schaffner (1970), D'Agostino (1993), Majer (1998), and Lützen (2005) §§7-9.

[28] A notable exception is Lützen (2005), see in particular §8. See also van Fraassen (2008) §8, especially pp. 201 ff.

[29] The lectures have been published in German, "Die Constitution der Materie" (Hertz, 2013). Although much of this material has not yet been studied in proper detail, for some initial discussion see Hyder (2002) pp. 35-46 and Lützen (2005) pp. 97-101.

the workings of nature without thereby ascribing to the phenomena any superfluous features that attach to the picture via the imagination. An example where the imagination could be misleading would be attributing a colour to an atom simply because we can't imagine it otherwise. In such a case, Hertz argues, we simply have to regard colour as an *inessential* property, hence explicitly discount it as representing, or corresponding to, a property of the atom itself. Eight years later, having worked hard to distill the essential content out of Maxwell's sprawling *Treatise*, Hertz famously remarked:

> To the question, "What is Maxwell's theory?" I know of no shorter or more definite answer than the following:– Maxwell's theory is Maxwell's system of equations. Every theory which leads to the same system of equations, and therefore comprises the same possible phenomena, I would consider as being a form or special case of Maxwell's theory. (Hertz, 1893, p. 21)

This is particularly important for our purposes for the following reason. In drawing attention to the difficulty of finding plausible mechanisms within the framework of *Principles*, both Helmholtz and Mach claimed that, in their own cases, they would remain content with the analytical representation given by the relevant systems of equations.[30] But the fact that Helmholtz and Mach regarded themselves as thereby marking a contrast with Hertz is peculiar inasmuch as Hertz's concerns also lay precisely in the 'essential' content conveyed by the relevant equations, and had done so in a consistent and sustained way for a long time prior to his work on mechanics. In the context of his work in electromagnetism, Hertz makes this particularly clear:

> If we wish to lend more colour to the theory, there is nothing to prevent us from supplementing all this and aiding our powers of imagination by concrete representations of the various conceptions... But scientific accuracy requires of us that we should in no wise confuse the simple and homely figure, as it is presented to us by nature, with the gay garment which we use to clothe it. (Hertz, 1893, p. 28)

Here, the 'simple and homely figure' presented by nature is the system of relations determined by Maxwell's equations, to which a 'gay garment' can be added, if desired, from amongst the competing hypotheses *about the underlying workings of an ether.* More generally, Hertz's proposal is that when we think carefully about the picture of nature that a scientific theory

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[30] Cf. Hertz (1899) xix-xx and Mach (1960) p. 321.

conveys, we should attend to the *essential* features of that theory in its naked form. To do this, the theory should be reformulated so that 'its logical foundations [can] be easily recognised; all unessential ideas should be removed from it, and the relations of the essential ideas should be reduced to their simplest form' (Hertz, 1893, p. 195). This is exactly what Hertz took himself to have achieved in his theoretical work on electromagnetism before he turned to classical mechanics.

2.5.1 The picture theory in *Principles*

Hertz employed his picture theory in framing the entire purpose of *Principles*, and also in taking a stance from which to evaluate its success. With regard to the purpose of his book, Hertz was helpfully explicit in articulating his overall goal:

> The problem, whose solution the following investigation seeks, is this: to fill up the existing holes and specify a complete and definite presentation of the laws of mechanics, which is compatible with our present day knowledge, and in relation to the range of this knowledge is neither too narrow nor too broad. (Hertz, 1899, xxi)

To understand the motivation to formulate a 'complete and definite presentation of the laws of mechanics', we need to consider Hertz's dissatisfaction with the already existing presentations. The development of the desiderata of a satisfactory presentation, and the comparison of the extant formulations of mechanics with Hertz's own novel reformulation on this basis, is the main task of his introduction. Hertz thus compares three competing formulations of mechanics: the traditional Newtonian formulation; the more recent energetic formulation (which attempted to derive the notion of force from the notion of energy); and Hertz's own formulation.

This is the context in which Hertz presents his picture theory. However, before narrowing his focus to scientific theories (and formulations of mechanics in particular), Hertz discusses how such pictures function in representation quite generally, beginning with the following:

> The procedure which we use in order to draw deductions of the future from the past, and thereby obtain the striven for foresight, is this: we make for ourselves inner simulacra [*Scheinbilder*] or symbols of external objects, and indeed we make them in such a way that the necessary consequences of the pictures [*Bilder*] in thought are always again the pictures of the necessary consequences of the pictured objects... The pictures of which we speak are

> our conceptions of things; they have with the things *one* essential conformity, which lies in the fulfillment of the aforementioned requirement. (Hertz, 1899, p. 1)

Hertz goes on to specify three criteria by which to evaluate pictures: *permissibility* (*Zuläßigkeit*), *correctness* (*Richtigkeit*), and *appropriateness* (*Zweckmäßigkeit*). In brief: Hertz's notion of permissibility can be glossed as the demand of logical consistency. The second criterion—correctness—is stated more precisely in the form of Hertz's 'fundamental requirement' on pictures: 'the necessary consequences of the pictures in thought are always again the pictures of the necessary consequences of the pictured objects in nature'. Thus the necessary consequents of a correct picture give *successful predictions* of the relevant phenomena. (Importantly, Hertz emphasizes that respecting the fundamental requirement is the *only* 'essential conformity' between picture and what is pictured.) The final criterion—appropriateness—is more subtle than the other two. Hertz distinguishes two separate strands which speak to the appropriateness of a picture—its *distinctness* and its *simplicity*:

> Given two pictures of the same object, the more appropriate of them is the one which reflects more of the essential relations of the object than the other; the one which, we would say, is more distinct. Of two equally distinct pictures the more appropriate is the one which, besides the essential traits, contains the least number of unnecessary or empty relations, which is thus the simpler of the two. (Hertz, 1899, p. 2)

Hence it is here, in the criterion of appropriateness, that we find a development of Hertz's distinction between essential and inessential features of a picture. According to the account in *Principles*, one picture is more distinct than another if it captures more of the essential features of what it depicts. A picture can further improve its appropriateness by being stripped of any inessential features. Such a naked picture is thereby simpler.

Note that everything so far is meant to apply to pictures understood very broadly as 'our conceptions of things'. It is only after he has specified the three criteria of permissibility, correctness and appropriateness that Hertz turns to consider the pictures provided by scientific theories. The key difference in the case of a scientific picture is that it must be made clear which elements of the picture are operative in meeting the different criteria, for only in this way is the systematic improvement of pictures possible. Nevertheless, as we shall see, there are important ways in which Hertz's three criteria are intimately connected.

This becomes apparent if we examine how Hertz employed the criteria of the picture theory in criticizing the traditional formulation of mechanics, thereby indicating what he thought stood to be gained through his reformulation in *Principles*.

2.5.2 What *Principles* achieved

Hertz sets his three criteria to work in diagnosing what is problematic in 'the representation, differing in details but at root the same, in nearly every textbook which deals with the whole of mechanics, and in nearly every lecture course which disseminates the cumulative content of this science' (Hertz, 1899, p. 4). In an important series of passages, Hertz presents several reasons to doubt the logical perspicuity of the traditional formulation of mechanics. He begins with a critique of the notion of centrifugal force before turning to three 'general observations' as further evidence for his misgivings: the difficulty of expounding a rigorous and clear introduction to mechanics, the existence of disputes over the rigour of certain elementary theorems, and the pervasiveness of questions concerning the nature of *force*. Hertz summarizes the purpose of this extended polemic as follows:

> I have so severely questioned the permissibility of the picture under consideration in these remarks that it must appear that it was my aim to dispute and eventually to deny its permissibility. But my aim, and my opinion, do not go so far as this. Such logical uncertainties, which make us anxious about the reliability of the foundations of the subject, though they really exist, have clearly not prevented a single one of the countless successes which mechanics has won in its application to the facts. Thus they could not stem from contradictions between the essential characteristics of our picture, hence not from contradictions between those relations of mechanics which correspond to relations of things. Rather, they must be restricted to the inessential traits, to all those aspects which we ourselves have arbitrarily added to that essential content given by nature. (Hertz, 1899, p. 8)

What began, then, as a challenge to the permissibility of this picture is connected in the end to problems with its appropriateness; Hertz regarded the logical tension in the traditional formulation as stemming from inconsistencies in the *inessential* features of the picture. Here we have a further indication of the importance Hertz attached to clearly identifying essential features, and pruning down inessential features as far as possible. Crucially, what also comes into view at this point is what Hertz thought his novel reformulation of mechanics could achieve:

> Perhaps our objection is not at all with the contents of the outlined picture, but rather only with the form of their representation. We are certainly not too severe if we say that this representation has never attained complete scientific perfection; it yet lacks quite sufficiently sharp distinctions to distinguish what in the outlined picture arises from the laws of our thought, what from experience, and what from our own arbitrary choices... In this sense we grant, along with everyone, the permissibility of the contents of mechanics. But it is required by the dignity and importance of our subject that its logical purity is not only acknowledged with good will, but that a perfect representation would prove it (Hertz, 1899, pp. 8-9)

Hertz regarded his reformulation of mechanics as achieving two major things. The first was that it was clear which aspects of his picture were included for the sake of each of the three criteria. As already noted, Hertz believed the *correctness* of his picture came down to the scope and validity of the fundamental law alone.[31] As for *appropriateness* and *permissibility*, the evaluation of these are interconnected. In Hertz's presentation, the careful introduction of the primitive notions (space, time and mass) and the choice of a specific notational framework (the apparatus of differential geometry), along with his stringent axiomatic-deductive procedure, served to highlight how the framework logically cohered (how Hertz's propositions depended on one another), and where certain choices were being made (what alternative equivalent formulations of the fundamental law were possible, for example). The overall result of this leads to the second, and most important, achievement of the book: establishing the logical permissibility of mechanics beyond doubt. Indeed, Hertz strenuously emphasised that clarifying the logical structure of mechanics was his fundamental aim in writing *Principles*:

> I think that as far as logical permissibility is concerned [the picture of mechanics I have presented] will be found to satisfy the most rigid requirements, and I trust that others will be of the same opinion. This merit of the representation I consider to be of the greatest importance, indeed of unique importance. (Hertz, 1899, p. 33)

Thus we see that the sustained polemic challenging the clarity of the logical foundations of the traditional picture of mechanics was central in Hertz's motivations. To return to Hertz's preface, we have further clear confirmation of this fact:

> In the details I have not brought forward anything that is new and which could not be found

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[31] Hertz's evaluation of his success in this regard has been disputed; for some discussion see Lützen (2005) p. 132.

> in many books. What I hope is new, and to which alone I attach value, is the arrangement and presentation of the whole, and thus the logical, or, if one wants, the philosophical aspect of the matter. My work has accomplished its objective or failed insofar as it has gained something in this direction or not. (Hertz, 1899, xxiv)

2.6 'DESCENDING TO THE WORLD OF ATOMS'

We now need to address the passages in *Principles* where Hertz seemed to indicate that his aim was, after all, to lay the groundwork for an eventual ether theory in precisely the "mechanistic" sense of most of his contemporaries. As already noted, at the end of his introduction Hertz considers the plausibility of distant forces compared with rigid connections, seeming to make a direct appeal to developments in electromagnetism—and the concept of an ether—in support of his own formulation of mechanics:

> ...the balance of evidence will be entirely in favour of the [Hertzian formulation] when a second approximation to the truth can be attained by tracing back the supposed actions-at-a-distance to motions in an all-pervading medium whose smallest parts are subjected to rigid connections; a case which also seems to be nearly realised in the [sphere of electric and magnetic forces]. This is the field in which the decisive battle between these different fundamental assumptions of mechanics must be fought out. (Hertz, 1899, p. 41)

To make sense of these remarks we need to note that this passage occurs in the concluding paragraph of the introduction (pp. 40-41), a paragraph in which Hertz takes an entirely different stance from his discussion up until that point.[32] Earlier, Hertz had been concerned to bring out the difficulties the Newtonian picture faced with regard to its permissibility and its appropriateness, and had had no issue at all with its *correctness* (indeed, he remarked 'No one will deny that within the whole range of our experience up to the present the correctness is perfect', ibid p. 9). Here, at the conclusion of his introduction, Hertz turns this on its head:

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[32] Some commentators have noted this fact before, including Nordmann (1998) p. 163 and Lützen (2005) p. 118. (As Lützen puts it, 'the last two pages of the introduction read more as a second thought than as a conclusion'.) To my knowledge the only extended discussion of the new stance that Hertz adopts in these concluding passages is in Preston (2008b). However, my assessment of the significance of these passages differs from Preston's.

> We shall put the [Newtonian] and [Hertzian] pictures on an equality with respect to permissibility, by assuming that the first picture has been thrown into a form completely satisfactory from the logical point of view... We shall also put both pictures on an equality with respect to appropriateness, by assuming that the first picture has been rendered complete by suitable additions, and that the advantages of both in different directions are of equal value. We shall then have as our sole criterion the correctness of the pictures (Hertz, 1899, p. 40)

Thus the appeal to the concept of the ether that follows is in an extremely hypothetical context. Hertz is assuming that a project *analogous to his own in Principles* has been completed on behalf of the Newtonian picture, so that it can be regarded as on a level with the Hertzian picture in terms of its permissibility and appropriateness. For such a reformulation of the Newtonian picture to be successful, it would have to remove the obscurities concerning 'force' that Hertz took himself to have circumvented in *Principles.* Hence Hertz does not characterize the essential difference between these pictures in terms of a preference for distant forces over connections or vice versa here. Rather:

> ...if we try to express as briefly as possible the essential relations of the two representations, we come to this. The [Newtonian] picture assumes as the final constant elements in nature the relative accelerations of the masses with reference to each other: from these it incidentally deduces approximate, but only approximate, fixed relations between their positions. The [Hertzian] picture assumes as the strictly invariable elements of nature fixed relations between the positions: from these it deduces when the phenomena require it approximately, but only approximately, invariable relative accelerations between the masses. (Hertz, 1899, p. 41)

In the final analysis, Hertz claims that his own picture assumes exact relative displacements, whereas the Newtonain picture (if it can be reformulated in a logically perspicuous way) assumes exact relative accelerations.[33] Hertz points out it is likely that only one of these will seem plausible in the light of future accumulated data. Hence, in *this* context, Hertz notes that developments in electromagnetism speak in favour of exact relative displacements over exact relative accelerations, and hence (so the thought goes) future physics may indeed vindicate the Hertzian picture. For this situation to arise, the Newtonian picture would first have to be reformulated, and results from experimental physics would have to make significant strides forward. But Hertz's project in *Principles* is prior to all this:

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[33] This point is noted in Nordmann (1998) pp. 161-162.

> ...in order to arrive at such a decision it is first necessary to consider thoroughly the existing possibilities in all directions. To develop them in one special direction is the object of this treatise, an object which must necessarily be attained *even if we are still far from a possible decision*, and even if the decision should finally prove unfavourable to the picture here developed. (Hertz, 1899, p. 41, emphasis mine)

As noted, there are two other passages in *Principles* where Hertz refers to 'the world of atoms'. The first is earlier in the introduction, where Hertz responds to the worry that an appeal to connections already assumes the existence of forces. Hertz's interlocutor argues: surely it is precisely the presence of certain forces that maintains such fixed connections. To this Hertz replies, 'Your assertion is correct for the mode of thought of ordinary mechanics, but it is not correct independently of this mode of thought; it does not carry conviction to a mind which considers the facts without prejudice and as if for the first time' (Hertz, 1899, p. 34). His point is that there is no need to account for a fixed spatial relation between masses by appeal to forces if one is not already committed to the primacy of the latter. But Hertz's interlocutor pursues the matter, pointing out that all observed rigid connections in nature are only approximate, 'and the appearance of rigidity is only produced by the action of the elastic forces which continually annul the small deviations from the position of rest' (ibid). Hertz replies as follows:

> In seeking the actual rigid connections we shall perhaps have to descend to the world of atoms. But such considerations are out of place here; they do not affect the question whether it is logically permissible to treat of fixed connections as independent of forces and precedent to them. (Hertz, 1899, p. 34)

In the light of the previous discussion, we can see that Hertz's remarks here do not force the reading that his aim in *Principles* was to lay foundations for an ether mechanism. Note that this is compatible with Hertz's speculation that exact relative displacements may indeed be found at atomic length scales. Nevertheless, Hertz is unambiguous in stating that 'such considerations are out of place here'.[34]

The final passage in which Hertz refers to the 'world of atoms' occurs at the end of chapter II of Book 2:

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[34] Though it would take me too far afield to explore this here, it is clearly relevant that Hertz saw a clear separation between theoretical mechanics and experimental physics—cf. Hertz (1899) p. 27: 'To investigate in detail the connections of definite material systems is not the business of mechanics, but of experimental physics'.

> ...in all connections between sensible masses which physics discovers and mechanics uses, a sufficiently close investigation shows that they have only approximate validity, and therefore can only be derived connections. We are compelled to seek the ultimate connections in the world of atoms, and they are unknown to us. (§330)

This is, again, an accommodation of the fact that all observed rigid connections have so far turned out to be approximate. However, this section of the text (§§327-330) in fact highlights the way in which Hertz's project must be regarded as *separate* from an investigation into facts at atomic length scales. Here is how the passage just quoted continues:

> But even if [the ultimate connections in the world of atoms] were known to us we could not apply them to practical purposes, but should have to proceed as we now do. For the complete control over any problem always requires that the number of variables should be extremely small, whereas a return to the connections amongst the atoms would require the introduction of an immense number of variables. (§330)

Hertz points out that even if we *were* confident in our knowledge of phenomena in the atomic domain, it wouldn't change our approach to mechanical problems at larger length scales. For in the treatment of any problem (at any length scale), the free variables have to be kept to a workable number. Indeed, it is a key feature of Hertz's formulation of mechanics that he can explain clearly how his fundamental law can be applied to systems in *ignorance* of the microscopic details. Recall that a system's connections identify a lower-dimensional hypersurface within its $3n$-dimensional configuration space.[35] In general, one can apply the full apparatus of Hertz's mechanics as soon as one has identified equations of condition of the right form. Hertz makes clear that in applying the fundamental law it doesn't matter at all whether these equations represent underlying connections between the fundamental constituents of the system:

> If we know from experience that a system actually satisfies given equations of condition, then in applying the fundamental law it is quite indifferent whether these connections are original ones, i.e. whether they do not admit of a further physical explanation... or whether they are connections which may be represented as necessary consequences of other connections and of the fundamental law (§328)

Hertz argues that his own formulation of mechanics simply makes perspicuous the fact that every application of mechanics at ordinary length scales abstracts away from the un-

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[35] As noted in section 2.3, this is only strictly true for holonomous connections.

derlying microscopic details. This point is of fundamental importance in understanding the role of Hertz's hidden masses. As should now be emerging, their role in Hertz's framework is *not* to function as a proposal for the underlying microscopic constituents of systems. The most immediate role of the hypothesis of hidden masses is that it allows Hertz to accommodate the motion of unfree systems within his analytical framework. However, it also plays another crucially important role. Rather than being a proposal concerning the microscopic constituents of systems, the hypothesis of hidden masses *rules out* knowledge of the fundamental constituents of a system. This is because the only knowledge of a system that Hertz's mechanics delivers is the existence of a 'dynamical model' of that system:

> If we admit generally and without limitation that hypothetical masses (§301) can exist in nature in addition to those which can be directly determined by the balance, then it is impossible to carry our knowledge of the connections of natural systems further than is involved in specifying models of the actual systems. We can then, in fact, have no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone, that the one set of systems are models of the other. (§427)

It is important to appreciate how abstract such dynamical models are. Hertz calls two systems dynamical models of one another if it is possible to write down analytical representations of them which have: (i) the same number of coordinates, (ii) the same equations of condition, and (iii) the same expressions for the magnitude of a displacement (cf. §418). Thus, for instance, any symmetrical rigid system is a dynamical model of any other. The same applies to any system modeled as a simple harmonic oscillator—a mass on a spring, a pendulum, and a vibrating string are all dynamical models of one another. Indeed, 'An infinite number of systems, quite different physically, can be models of one and the same system. Any given system is a model of an infinite number of totally different systems' (§421). Thus it is built into Hertz's framework that the true composition of a material system is radically underdetermined.[36]

Note, here, the close relationship between Hertz's discussion of dynamical models and the picture theory of his introduction. When Hertz introduced the notion of an picture, he posited one fundamental requirement: the consequences of the picture in thought must

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[36] Among other places this point emerges in §536, where Hertz notes that it is 'permissible though arbitrary' to regard any material system whatsoever as composed of some number of coupled subsystems.

give rise to pictures of the consequences of the pictured objects. This requirement was an important *limitation* on how our pictures can represent things in the world: 'we do not know, and we have no way to learn, whether our conception of things conforms with them in any other way, except in this *one* fundamental respect alone' (Hertz, 1899, p. 1). With the hypothesis of hidden masses Hertz has shown how this requirement on pictures in general applies to the pictures provided by mechanics in particular. Hence it is in the discussion of dynamical models that Hertz makes his only explicit reference back to the general picture theory of his introduction:

> The relation of a dynamical model to the system of which it is regarded as the model, is precisely the same as the relation of the pictures which our mind forms of things to the things themselves... The agreement between mind and nature may therefore be likened to the agreement between two systems which are models of one another. (§428)

2.7 MECHANICS WITHOUT MECHANISMS

We began with the curious historical situation that followed the publication of Hertz's book. On the one hand, Hertz's contemporaries regarded *Principles* as a remarkably impressive work; on the other hand, they struggled to identify what it was that Hertz thought he had achieved in writing it. Formulating mechanics by eschewing actions-at-a-distance in favour of hidden masses and connections was all well and good, they thought, but without specifying how mechanisms of hidden masses could plausibly account for observed phenomena in concrete cases, the project was, as Boltzmann put it, doomed to be 'only of purely academic interest' or, at best, 'a programme for the distant future' (Boltzmann, 1974, p. 90).

Our task was thus to explain the absence of mechanisms in Hertz's book, and explain why Hertz seemed unperturbed by the difficulties of constructing such a mechanism. The path to the answer involved exploring the significance of Hertz's picture theory of representation, thereby reconstructing his rationale for distinguishing between the essential and inessential elements of a scientific theory. This brought out Hertz's commitment to distilling out the bare picture of mechanics, and separating this off from any inessential elements that attach to it via the imaginative aids we might employ in fleshing it out. Hence we saw that developing the kinds of mechanisms that Hertz's readers looked for would have been anathema to Hertz's intentions: in identifying the essential content of mechanics he intentionally avoided making any appeal to imaginative aids or concrete models.

We have seen that Hertz's own rhetoric and presentation can be particularly misleading on this issue, especially his introduction of *Massenteilchen* and hidden masses. The primary concern of this chapter has been to show that, rather than being speculative ontological posits, the introduction of such objects allowed Hertz to formulate suitably *abstract* descriptions of mechanical systems in the form of dynamical models. Hence the core value of Hertz's project is not tied to the prospects of finding a suitable ether mechanism. Indeed, in this vein it is important to bear in mind that Hertz never intended *Principles* to replace existing approaches of mechanical problems:

> In respect of [practical applications or the needs of mankind] it is scarcely possible that the usual representation of mechanics, which has been devised expressly for them, can ever be replaced by a more appropriate system. Our representation of mechanics bears towards the customary one somewhat the same relation that the systematic grammar of a language bears to a grammar devised for the purpose of enabling learners to become acquainted as quickly as possible with what they will require in daily life. The requirements of the two are very different, and they must differ widely in their arrangement if each is to be properly adapted to its purpose. (Hertz, 1899, p. 40)[37]

On the proposed interpretation, Hertz's formulation of mechanics provides only highly abstract descriptions of mechanical systems in the form of dynamical models. The sole criterion on the adequacy of a dynamical model is that it successfully models the system's evolution over time; hence the only ontological commitments that are relevant are the minimal commitments involved, for example, in recognising that both a mass on a spring and a pendulum are simple harmonic oscillators. This, I claim, is the core and lasting value of Hertz's project. At the same time, however, this conclusion needs to be tempered as an interpretation of Hertz's authorial intentions. Despite the substantial evidence canvassed above, it would be hard to deny that Hertz had a lingering sense that the image of mechanics presented in *Principles* was tied to the empirical claim that actions-at-a-distance could be accounted for, ultimately, in terms of contact actions. As we will see in the next chapter,

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[37] See also Lützen (2005) p. 263: 'Since [Hertz] could show that the usual principles of mechanics also hold in his picture of mechanics any analysis of a mechanical problem within the usual mechanics is, in a sense, also valid in his mechanics.'

this can be recognised in Hertz's distinction between Newton's third law of motion and his own action-reaction principle:

> [Newton's third law] is usually applied to actions-at-a-distance, i.e. to forces between bodies which have no common coordinates. But our mechanics does not recognise such actions. Thus in order to be able to adduce as a consequence of our proposition the fact that a planet attracts the sun with the same force that the sun attracts the planet, it is necessary that further data should be given as to the nature of the connection between the two bodies. (§469)

This is also linked to Hertz's discussion of the relative merits of appealing to exact displacements over exact accelerations. Indeed, Hertz was aware that *this* aspect of his project was speculative, and acknowledged that future experimental evidence might 'finally prove unfavourable to the picture here developed' (Hertz, 1899, p. 41). Thus, I do not maintain that Hertz regarded himself as prescinding entirely from substantive ontological commitments.

Nevertheless, as I will be concerned to argue in chapter 4, in order to explore Hertz's influence on Wittgenstein the austerely abstract interpretation of *Principles* is the appropriate interpretation to bear in mind. For although Hertz himself had a lingering sense of the empirical commitments that might yet distinguish between his own formulation of mechanics and the Newtonian formulation, Wittgenstein had no such interest in the future results of experimental physics. Before considering Hertz's influence on the *Tractatus*, however, let us first turn to a closer examination of Hertz's treatment of the notion of *force*.

3.0 A LOGICAL OBSCURITY

> '...science answers no *why*—it simply provides a shorthand description of the *how*... it therefore follows that if mass and force are to be used as scientific terms they must be symbols by aid of which we describe this *how*.'
>
> Karl Pearson, *The Grammar of Science*, p. 306

3.1 THE NATURE OF FORCE

Hertz finds a logical obscurity (*logische Trübung*) in the customary representation of mechanics – 'the representation, differing in details but at root the same, in nearly every textbook which deals with the whole of mechanics, and in nearly every lecture course which disseminates the cumulative content of this science' (Hertz, 1899, p. 4). In seeking to convey this obscurity to his readers, Hertz first examines the elementary problem of swinging a stone in a circle to illustrate the ease with which one encounters 'an undoubted hindrance to clear thinking' (Hertz, 1899, p. 5). He then offers three general observations as further evidence for the presence of this logical obscurity. The first is the dissatisfaction felt in introducing the basic concepts and definitions of mechanics, and the desire 'to move rapidly over the introductory material on to examples which speak for themselves' (Hertz, 1899, p. 7). The second is the existence of disagreements concerning the rigor of supposedly elementary theorems in mechanics, disagreements which 'in a logically complete science, such as pure mathematics... [are] utterly inconceivable' (ibid). The final observation is the concern felt in the physics community over the 'nature' (*Wesen*) of *force*:

> Weighty evidence seems to be furnished by the statements which one hears with wearisome frequency, that the nature of force is still a mystery, that one of the chief problems of physics is the investigation of the nature of force, and so on. In the same way electricians are continually attacked as to the nature of electricity. Now, why is it that people never in this way ask what is the nature of gold, or what is the nature of velocity? Is the nature of gold better known to us than that of electricity, or the nature of velocity better than that of force? Can we by our conceptions, by our words, completely represent the nature of any thing? Certainly not. (Hertz, 1899, pp. 7-8)

In this chapter, I seek an account of the logical obscurity that troubled Hertz. It turns out that this is no trivial task. FitzGerald suggested that Hertz had simply misunderstood Newton's third law:

> Hertz seems to consider that there is some outstanding confusion in applying the principle of equality of action and reaction, and appears to hold that by this principle the action on the body requires some reaction *in the body* whose acceleration is the effect of the force. He does not seem fully to appreciate that action and reaction are always on *different* bodies. From his consideration of this, and from a general review of our conception of force, he concludes that there is something mysterious about it, that its nature is a problem in physics, like the nature of electricity. (Hertz and Mulligan, 1994, p. 372)

FitzGerald's suggestion is that Hertz mistakenly thought that action-reaction force pairs act on one body, rather than on different bodies. Arnold Sommerfeld also took Hertz's troubles to relate to a misunderstanding of Newton's third law, and was thus similarly unconvinced by Hertz's critique.[1] But this is patently unsatisfactory inasmuch as it is implausible that Hertz had such an elementary misunderstanding of Newton's third law and that *this* would have led him to spend the last four years of his life reformulating mechanics.

In what follows I first present a close reading of Hertz's discussion of swinging a stone in a circle, aiming to identify an issue that does not merely demonstrate a misunderstanding on Hertz's part. I also draw attention to the fact that Hertz only intended to use this example to *gesture* at an underlying problem, and did not intend the example to directly reveal the logical obscurity in question. Thus in order to make further progress I turn to an exploration of the way in which the meaning of the term 'force' shifted during the historical development

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[1] Cf. Sommerfeld (1952) p. 60.

of mechanics. It is in this shift in meaning, I claim, that the source of the logical obscurity is to be found. I then unpack the notion of force that Hertz derives within his own framework, and show how Hertz's approach allowed him to circumvent the difficulties that lurked within the customary representation of mechanics.

3.2 SWINGING A STONE IN A CIRCLE

As noted, Hertz's first attempt to bring out the presence of the logical obscurity involves a consideration of swinging a stone in a circle. For reference, I quote this section of Hertz's introduction at some length here:

> Now, at first sight it may seem very far fetched that one could doubt the logical permissibility of [the Newtonian formulation of mechanics]. It may seem nearly impossible that one might come to find logical imperfections in a system which has been thought through over and over innumerable times by the best minds. But before one gives up the investigation altogether, one must ask whether *all* the *best* minds have really always felt satisfied with this system. In any case one must wonder at how easy it is from the start to link to the fundamental laws considerations which are completely ubiquitous in the ordinary parlance of mechanics and yet which place clear thought in unquestionable difficulty. Let us try to give an initial example. We swing a stone on a string around in a circle. We are thereby conscious of exerting a force on the stone. This force deflects the stone from continuing in a straight path, and if we alter this force, the mass of the stone, and the length of the string, we find that the motion of the stone is in fact in accordance with Newton's second law. Now, however, the third law demands an opposing force to the force which our hand exerts on the stone. To the question about this opposing force the familiar answer is: the stone imparts a force back onto the hand as a result of its centrifugal force, and this is indeed precisely equal and opposite to the force that we exert. Is, now, this mode of expression permissible? Is that which we now call centrifugal force something other than the inertia of the stone? May we, without destroying the clarity of our notions, take the effect of inertia twice into account, namely first as mass, and second as force? In our laws of motion force existed *before* motion and was the cause of motion. Can we pretend to ourselves that we have declared something about this new kind of force in our laws, or that by using the name "force" we have thereby conferred the properties of forces? All these questions are obviously to be answered in the negative, there is nothing for us but to explain it as follows: the description of centrifugal force as a force is improper; its name is, like the name 'living force', merely a historical artifact and its retention should be apologized for rather than justified. But what then of the demand of the third law, which requires a force that the inert stone exerts on the hand and which wants to be fulfilled by a real force, not by a mere name?
>
> I do not think that these difficulties are artificial or wantonly conjured up; they impose themselves upon us. Is not their origin to be traced back to the fundamental laws? The

> force of which we spoke in the definition and in the first two laws acts on a body in a single, particular direction. The sense of the third law is that forces always connect two bodies and are directed just as well from the first to the second as from the second to the first. The notion of force in this law, and the conception which the first two laws presume and suggest to us, seem to me to be slightly different, and this slight difference may be enough to produce the logical obscurity whose consequences are manifest in our example. (Hertz, 1899, pp. 5-6)

Note, first of all, that Hertz does not take the example of the swinging stone to itself expose the logical obscurity. Rather, Hertz is using the example to show that a completely standard way of talking about an ordinary mechanical problem leads immediately to confusion if subjected to scrutiny. Hertz's discussion can be paraphrased as follows.

When swinging a stone in a circle on the end of a string, we are aware of *exerting a force* with our hand. We can confirm that, in relation to the stone's circular motion, the magnitude of this force is consistent with Newton's second law. However, Newton's third law instructs us that an equal and opposite reaction force must act from the stone on our hand. The ordinary way to account for this reaction force would be to appeal to the *centrifugal* force, acting outward from the centre of the circle. But this is unsatisfactory for the following reason. The expression 'centrifugal force' is used to describe what appears as a force from the perspective of a rotating frame of reference. It is thus a (potentially misleading) way of describing how things appear from the perspective of an observer undergoing circular motion. To illustrate: an insect clinging to the stone would feel as if a force was pulling it off the surface even though no such force acts. (In the same way, an observer inside an accelerating vehicle feels as if a force is pulling them backwards.) It is for this reason that centrifugal force is called a *pseudo*-force.[2] Hence Hertz argues that its name 'is accepted as a historic tradition... we should rather apologise for its retention than endeavour to justify it.' However, acknowledging this simple point exposes why this way of talking about the swinging stone is unsatisfactory. It makes clear that, as things stand, this is an unsatisfactory way to account for the force *on our hand* when swinging the stone. This force does not arise because our hand is undergoing circular motion (it isn't), and it is certainly *not* a pseudo-force.

After making this brief critique of such 'usual modes of expression in mechanics', Hertz then claims that these difficulties can be traced back to Newton's laws. Specifically, Hertz

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[2] Sommerfeld calls centrifugal and similar inertial forces 'fictitious', cf. Sommerfeld (1952) p. 59.

argues that the conception of force as *a cause of motion*, as suggested by Newton's first two laws, does not fit entirely comfortably with the action-reaction forces required by Newton's third law. To anticipate a little of the later discussion: if we regard the conception of force in Newton's third law as a *symmetrical* and *synchronous* relation between two objects, then prima facie this seems somewhat different from the conception of force as a kind of directed power—acting from one body on another and *causing* motion in the latter (where the causal relation is understood as non-synchronous, compatible with the idea that forces propagate through space at a finite velocity, hence the force is 'present before' the motion that it causes). Hertz concludes by remarking, 'This slight difference may be enough to produce the logical obscurity of which the consequences are manifest in the above example' (Hertz, 1899, p. 6).

Hertz's conclusion is tentative; he is *gesturing* at what he takes to be a deep-rooted tension in the ordinary conception of mechanics, and appeals to 'general observations as evidence in support of the above-mentioned doubt.' He continues by appealing to the existence of similar doubts felt by others: 'It is not going too far to say that this representation has never attained scientific completeness... This is also the opinion of distinguished physicists who have thought over and discussed these questions' (Hertz, 1899, p. 8). In all of this Hertz is only *motivating* the need for a logically perspicuous reformulation of mechanics, and stops short of decisively demonstrating the need for it. Uncovering the deeper source of the logical obscurity that Hertz gestures at is evidently left to us.

3.3 'THE CUSTOMARY REPRESENTATION OF MECHANICS'

In order to identify Hertz's logical obscurity we need to appreciate Hertz's understanding of the domain in which it lies—the 'customary representation of mechanics':

> This is the path by which the great army of students travel and are inducted into the mysteries of mechanics. It closely follows the course of historical development and the sequence of discoveries. Its principal stages are distinguished by the names of Archimedes, Galileo, Newton, Lagrange. (Hertz, 1899, p. 4)

Newton, unsurprisingly, plays an especially important role in the development of this representation of mechanics. However, Hertz notes that Newton's laws were not sufficient for the full development of this image without the later addition of d'Alembert's principle:

> [Newton's] laws contain the seed of future developments; but they do not furnish any general expression for the influence of rigid spacial connections. Here d'Alembert's principle extends the general results of statics to the case of motion, and closes the series of independent fundamental statements which cannot be deduced from each other. From here on everything is deductive inference. (Hertz, 1899, p. 5)

Hertz's brief remarks gesture at a rich and complex history leading up to the textbooks and lecture courses of his time. It is possible to distinguish two traditions in the historical development of mechanics: the *vectorial* tradition, most recognisable in Newton's canonical laws of motion, and the *variational* tradition, attributable especially to Lagrange.[3] In the vectorial tradition, the primary objects are force vectors between point-masses. Complex systems can be regarded as arrays of such points, each pair of which has equal and opposite central forces acting along the line connecting them. The classical law of gravity acting between point-masses is the home territory of this tradition. By contrast, in the variational tradition the primary object is a scalar representation of some overall property of the system (normally, its total energy). Applications of extremal principles (which maximize or minimize some quantity) such as the principle of least action are the home territory of this tradition. The development of variational techniques was necessary in large part because of the inadequacy of the vectorial approach in dealing with problems which involve *constraints*, such as the motion of a bead constrained to slide along a rigid wire. The following section outlines these two traditions and traces the historical development from Newton's laws and d'Alembert's principle to the variational techniques of Lagrange and others. As will become apparent, the term 'force' underwent an important shift in meaning during this period.

3.3.1 The vectorial tradition

Newton's laws (in Motte's translation) read as follows:

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[3] Here I follow Lanczos (1962), cf. xvii

> **Law I:** Every body persists in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.
> **Law II:** The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.
> **Law III:** To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. (Newton, 1729, pp. 19-20)

Newton's use of the term 'body' (*corpus*) seems to be intended as an entirely generic and general term, encompassing any material object whatsoever. Thus in one fell swoop Newton's first law identifies the inertial motions of atoms, chairs and galaxies. However an immediate problem is that the first law is *false* on straightforward application to most such objects, even those which are more-or-less isolated. Far from persisting in a 'uniform motion', an object drifting in empty space might be spinning rapidly, with different parts undergoing various oscillations. Here is the point made by Karl Pearson:

> [a] body may not only be spinning about an axis, but may be, and as a general rule is, conceived as continually changing the axis about which it spins. The "state of rest or of uniform motion in a straight line" is thus *not* that which the physicist postulates to describe the motion of a body under the action of no forces. (Pearson, 1900, p. 322)

As Pearson notes, we can nevertheless apply the law correctly to a *representative point* associated with such a complex body, in particular its *centre of mass*. However, this is a result that must be *derived* (and, in particular, a result that requires an appeal to Newton's third law).[4] A more logically rigorous way to respond to this worry, then, would be take the first law to apply only to *point-masses*. As Pearson notes, this strategy would also make better sense of Newton's second and third law as well. For example, the 'equal and opposite' demand of the third law has a straightforward interpretation: the forces are given by a pair of vectors equal in magnitude and acting along the unique straight line that connects the two masses but in opposite directions. By contrast, for two extended bodies there is no such immediate way to determine at what point and in what direction the forces in question act:

> ...if the "change of motion" [in Newton's second law] is to be that of a body, not a [point-]particle, then we naturally ask which point of the body will have its motion changed in the direction of a straight line... [Regarding the third law,] the mutual action of two bodies is more complex than a reader just starting his study of mechanism would imagine, if he

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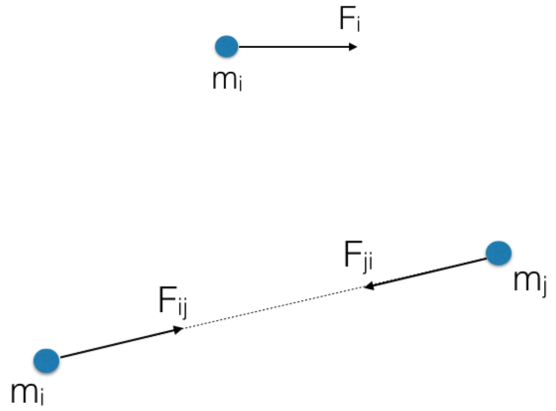
[4] For a typical (and typically informal) example of this kind of derivation, see Sommerfeld (1952) p. 25.

> naturally interpreted mutual action as corresponding to mutual acceleration in some one line. (Pearson, 1900, p. 324)

Although it would be anachronistic to regard Newton's own formulation of his laws as meant to apply only to point-masses, it is easy to see the appeal of a reconstruction of mechanics that begins only with the motions of punctiform bodies. The second and third laws can then be interpreted in a straightforward (and familiar) fashion (Figure 3.1).

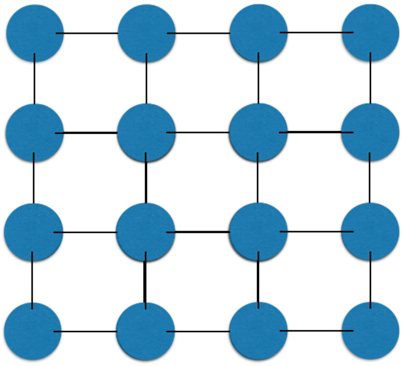
Figure 3.1: Vectorial interpretation of Newton's second and third laws



$$F_i = m_i a_i$$
$$F_{ij} = -F_{ji}$$

Extended bodies can be regarded as swarms or arrays of point-particles, held together by strong forces (Figure 3.2). However, this approach rapidly becomes extremely cumbersome, and this despite the fact that the details are largely redundant for most problems. (It's rarely important to know the forces between the constituent parts of a rigid body, for example, only that the body *as a whole* is rigid.) More generally, the problems that can be adequately dealt with using vectorial methods make up a relatively small portion of the broad domain of classical mechanics: 'For the solution of more involved problems, the geometrical methods of vectorial mechanics cease to be adequate and have to give way to more abstract analytical treatment' (Lanczos, 1962, p. 7).

Figure 3.2: Extended body

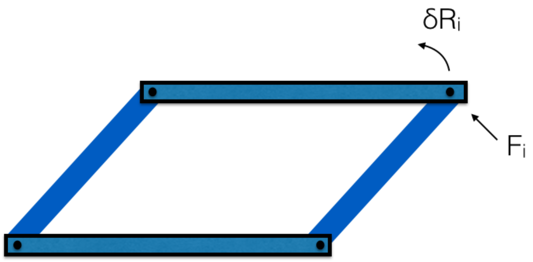


3.3.2 The variational tradition

As already noted, the incorporation of variational methods into the Newtonian framework was made possible by d'Alembert's principle.[5] To see this, we can first consider a simple formulation of the *principle of virtual work*.

Consider a system subject to given constraints, such as a connected mechanism of rigid bars. A *virtual displacement*, δR, of the system is an imagined slight change of its configuration consistent with the constraints, such as a small rotation without bending or breaking. The total *virtual work*, δw, is the sum of the impressed forces on the moving parts of the system, multiplied by the associated virtual displacements (Figure 3.3).

Figure 3.3: A virtual displacement



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[5] Variational methods evolved out of work in statics that well preceded Newton. For some discussion, see Sklar (2013) pp. 76-79.

$$\delta w = F_1.\delta R_1 + F_2.\delta R_2 + ... + F_n.\delta R_n$$

The principle of virtual work can be stated as follows: *a system will remain in equilibrium just in case the total virtual work of all the impressed forces vanishes*:

$$\delta w = \sum_{i=1}^{n} F_i.\delta R_i = 0$$

A particularly important development in mechanics was the extension of the principle of virtual work to apply to dynamics as well as kinematics. This can be achieved by defining the "force of inertia", $I = -ma$, which can be used to transform Newton's second law from $F = ma$ to $F + I = 0$. This in turn effectively generalises the notion of equilibrium so that it can be applied to systems in motion: adding such forces of inertia to the impressed forces allows an arbitrary mechanical system to be regarded as in equilibrium. Call the sum of the impressed force and the force of inertia the *effective force*, F^e. We now arrive at d'Alembert's principle[6]—a generalization of the principle of vitual work: *the total virtual work of the effective forces is zero for all virtual displacements*:[7]

$$\delta w = \sum_{i=1}^{n} F_i^e.\delta R_i = 0$$

It is relatively straightforward to use d'Alembert's principle to derive a true variational principle which requires the minimization of a scalar quantity. Given fixed endpoints, one simply integrates the virtual work with respect to time:

$$\int_{t_1}^{t_2} \delta w \; dt$$

If one requires that all virtual displacements are zero at t_1 and t_2, and defines the *Lagrangian function* as the difference between kinetic and potential energy, $L = T - V$, one can then derive the following identity:[8]

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[6] D'Alembert's work in this area developed an insight of James Bernoulli; see Sklar (2013) pp. 86-87 and pp. 96-99 for some discussion.

[7] Note that we assume that these displacements are *reversible*, see Lanczos (1962) pp. 86-87

[8] Cf. Lanczos (1962) pp. 112-113.

$$\int_{t_1}^{t_2} \delta w \; dt \;=\; \delta \int_{t_1}^{t_2} L \; dt$$

The vanishing of this quantity is Hamilton's principle. In words: given fixed endpoints, *the motion of an arbitrary mechanical system always occurs in such a way that the integral of the Lagrangian becomes stationary for arbitrary variations of the system's configuration.*

D'Alembert's principle paved the way to a battery of variational techniques, including the Lagrangian equations of motion and various formulations of the principle of least action. Such variational approaches to mechanical problems are remarkably powerful, allowing one to ascertain all the dynamical information of a system by minimizing a single scalar quantity.[9] However, we have now moved a significant distance away from Newton's original laws of motion. In particular, we have moved a significant difference from the Newtonian conception of force that we started with. When considering the Lagrangian for a system, we can derive a 'generalised' notion of force from the potential function, V.[10] However, this force will not in general be represented by a three-dimensional vector as in the vectorial tradition; it will typically be represented by a vector with as many components as the degrees of freedom of the mechanical system at hand. It is this shift in the meaning of the term 'force' that fed into Hertz's concerns with the customary representation of mechanics, or so I shall argue.

3.4 A LOGICAL OBSCURITY

There is an obvious tension between the notion of a 'force of inertia' ($I = -ma$) and the conception of force articulated in Newton's first two laws. In the first law a body continues at a constant velocity unless it is 'compelled to change that state by forces impressed thereon', and in the second law a force is responsible for 'the alteration of motion'. However, with the advent of the 'force of inertia' and the new equilibrium principle $F + I = 0$, forces are *always* balanced and systems are *always* in equilibrium. More specifically, the force of inertia does

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[9] It is worth noting, however, that d'Alembert's principle retains certain advantages over other variational principles. In particular, it can also accommodate forces which are *not* derivable from a scalar quantity.

[10] The class of forces that are derivable from a potential plays a prominent role in the treatment of a large class of mechanical problems; cf. Sklar (2013) pp. 99-100.

not compel a body to change its state of motion. From this perspective, the variational tradition appears to be abandoning many of the core tenets of the Newtonian conception of force.

A particularly clear way in which the vectorial and variational traditions diverge is on the question of whether forces can depend on velocity. In the vectorial tradition there was a clear rationale undergirding the expectation that all forces could ultimately be shown to be independent of velocity (depending only on relative distance). However, in the variational tradition constraint forces often clearly depend on velocity. The questionable nature of velocity-dependent forces is particularly manifest against the backdrop of certain nineteenth century attempts to treat mechanics as a unified fundamental theory.

In the period preceding Hertz's work on *Principles*, there was a widely held view that mechanics could be reduced to the actions of 'elementary forces' of a certain simple form. In particular, such elementary forces were expected to depend only on the relative distance between two objects and to act along a straight line connecting them (so-called 'central' forces). A key component of the attempt to reduce all of physics to mechanics was the idea that all phenomena could be reduced to the actions of such simple forces between simple objects. A clear illustration of the important role played by this idea is found in Helmholtz's seminal derivation of the 'conservation of force' (in modern terms: conservation of *energy*). In his opening remarks, Helmholtz made the following striking claim:

> The deduction of the propositions contained in the memoir may be based on either of two maxims; either on the maxim that it is not possible by any combination whatever of natural bodies to derive an unlimited amount of mechanical force, or on the assumption that all actions in nature can be ultimately referred to attractive or repulsive forces, *the intensity of which depends solely upon the distances between the points by which the forces are exerted.* That both these [maxims] are identical is shown at the commencement of the memoir itself. (Helmholtz, 1853, p. 114; emphasis added)

Thus Helmholtz attempted to prove that the conservation of energy was *equivalent* to the claim that all fundamental forces were central forces which depended only on relative distance. Although Helmholtz's proof is not satisfactory,[11] what is important for present purposes is that many physicists at this time were convinced that all forces could be reduced to

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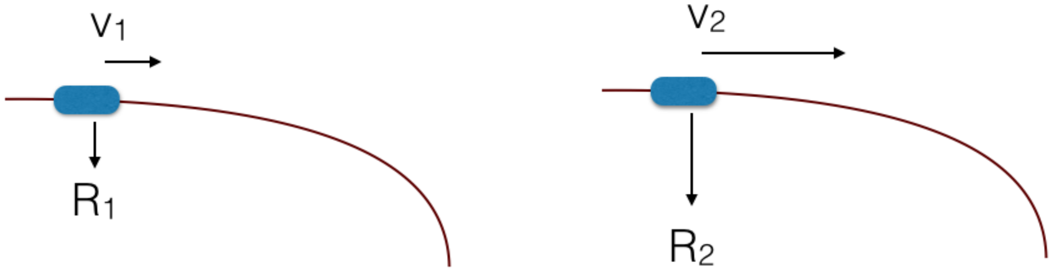
[11] Cf. Bevilacqua (1993) pp. 310 ff.

such elementary forces. Indeed, Helmholtz was conveying a widely shared expectation (albeit with particular enthusiasm) with such pronouncements as the following: '[The vocation of theoretical natural science] will be ended as soon as the reduction of natural phenomena to simple forces is complete, and the proof given that this is the only reduction to which the phenomena are capable' (Helmholtz, 1853, p. 118).

However, consider the simple constrained system of a bead sliding without friction along a rigid, curved wire.[12] As the bead travels along the wire, reaction (or *constraint*) forces will keep it on the trajectory determined by the wire's shape. As these constraint forces must perform no work, the resultant force on the bead that arises from the constraints must always act perpendicularly to the bead's instantaneous velocity. Thus *relative to the wire* the bead maintains a constant *generalized* velocity, hence moving in accordance with a *principle of generalized inertia* along this curved trajectory. In taking this approach we have unhesitatingly separated the forces acting on the bead into the external (impressed) forces and the internal (constraint) forces. However, the constraint forces *must* depend on the bead's velocity—if it slides faster, the reaction force bending its motion along the wire has to increase accordingly (Figure 3.4).

Figure 3.4: Beads on wires



The immediate response at this juncture might be to argue that constraint forces are not fundamental: at the microscopic scale, complex interactions are taking place, resulting in small fluctuations in the bead's velocity and even small distortions in the shape of the wire

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[12] As noted, this is precisely the kind of problem that is particularly poorly suited to the vectorial approach, and well suited to the variational.

itself. (The assumption that the wire is perfectly rigid must be some kind of idealisation after all.) The whole point of taking the variational approach to this problem was to avoid the need to delve into such complexity, and only consider the dominant aspects of the bead's motion. However, all of this only further highlights that the constraint forces ubiquitous in (and fundamental to) the variational approach are altogether quite different to vectorial forces. Although it is often assumed that, if needed, one could reduce constraint forces to underlying vectorial forces, no such "foundational" program has ever been successfully carried out. As Wilson (2013) notes, attempts to construct macroscopic bodies starting with atomic constituents runs into difficulties in specifying the special force laws which could make an array of point-masses cohere into an extended body, rigid or otherwise. Furthermore, going on to specify how such a body responds under impacts, for example, can be extremely difficult, requiring modeling assumptions that plainly have 'no relationship to any structure present in real-life materials' (Wilson, 2013, p. 68). Indeed, this fact shouldn't be particularly surprising—we have long since given up the expectation that interactions at atomic length scales are governed by classical physics.

The upshot of the discussion so far can be summarised as follows. Newton's first two laws suggest that a force is a directed quantity, *acting* on an object and *causing* it to accelerate. This is the intuitive idea of a force as a kind of push or pull. However, Newton's third law gestures at a somewhat different notion of force. According to this notion, forces exist *alongside* the accelerated motions of objects, and comprise *symmetric* and *synchronous* relations between them. This subtle ambiguity in the notion of force became more entrenched as more sophisticated mechanical techniques were developed. In particular, d'Alembert's principle required the positing of a conceptually thorny 'force of inertia'. The notion of *generalized* force that is then recoverable within the variational techniques of Lagrange and others is a high-dimensional vector that is tied to the degrees of freedom of the mechanical system under consideration. Hence it is very unclear what the relationship is between this notion of force and the original Newtonian notion of a kind of push or pull. Furthermore, there is a potential contradiction lurking in the question of the velocity (in)dependence of forces. The seemingly innocuous separation of constraint forces from impressed forces in the formulation of the principle of virtual work becomes problematic following d'Alembert's extension of the principle to encompass systems in motion. In such cases the required constraint forces are often clearly dependent on the system's velocity. However, such velocity dependence does not apply to any previous notion of force, and is explicitly *disallowed* in traditional derivations of the conservation of energy.

I claim that these difficulties, engendered by the subtle variations in the meaning of the term 'force', are in the background of Hertz's concerns regarding the customary representation of mechanics. From this point of view it is a testament to Hertz's insight that these difficulties can be linked to the ambiguous notion of force in Newton's original statement of the laws of motion—that the 'slight difference' between Newton's first two laws and the third law 'may be enough to produce the logical obscurity' (Hertz, 1899, p. 6).

3.5 HERTZ'S ALTERNATIVE APPROACH

In order to see how Hertz escapes such difficulties, let us now turn to consider the notion of force that Hertz derives within his own framework. Recall that, for a system composed of n material points, its associated configuration space will have $3n$ dimensions: one dimension for each of the three position coordinates of each of its points. The points themselves will in general have connections relating their relative positions. Hertz shows that such connections can always be described by writing down *equations of condition* of a canonical form (cf. §§125–128):

$$\sum_{\nu=1}^{3n} x_{\iota\nu} dx_{\nu} = 0$$

Here, the $x_{\iota\nu}$ are continuous functions of the coordinates x_{ν}. If there are m connections between the points, then there will be m equations of condition (so that the index ι runs from 1 to m). To recap: each connection lowers the number of the system's degrees of freedom by one, and rules out a region of the system's configuration space (i.e. the region which corresponds to "breaking" that connection). Thus each connection defines a hypersurface of $(3n-1)$ dimensions within the system's full $3n$-dimensional configuration space. The accessible region of configuration space will have a total of $3n-m$ dimensions: the intersections of the hypersurfaces determined by the connections. Hertz's fundamental law asserts that the motion of a free system always traces out a straightest path on this curved hypersurface embedded within its $3n$-dimensional configuration space.

A more abstract and more powerful characterization of a system can be achieved by describing it in terms of its *general* coordinates. These coordinates can be used to characterize a system directly in terms of its degrees of freedom. In particular, if general coordinates are used to incorporate *all* the system's connections, then the number of coordinates needed is equal to system's degrees of freedom.[13] In that case a system with r $(= 3n - m)$ degrees of freedom has r general coordinates, $p_1, ..., p_r$. Hertz introduces the notion of general coordinates as follows:

> We may also consider the system as determined by means of any r quantities $p_1..p_\rho..p_r$ whatever, as long as we agree to associate continuously a given value-system of these coordinates with a given position of the system, and conversely. The rectangular coordinates are therefore functions of these quantities, and conversely. The quantities p_ρ are called the general coordinates of the system. (§13)

In general, the $3n$ Cartesian coordinates of a system can be expressed as functions of r general coordinates:

$$x_1 = f_1(p_1, ..., p_r) \quad \ldots \quad z_{3n} = f_{3n}(p_1, ..., p_r)$$

The use of such coordinates is particularly important in Hertz's framework because of the primary role played by connections, i.e. conditions that give rise to *constraints*.[14] For a simple example of the usefulness of general coordinates in this context, consider a dumbbell system: two masses with a single rigid connection between them. If the locations of the two masses are given by the ordinary Cartesian coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then the representative point in configuration space can be given by using all six of these coordinates (Figure 3.5).

Note that if we removed the rigid connection, the six position coordinates would all be independent—any one of them could change without affecting the others. In that case the

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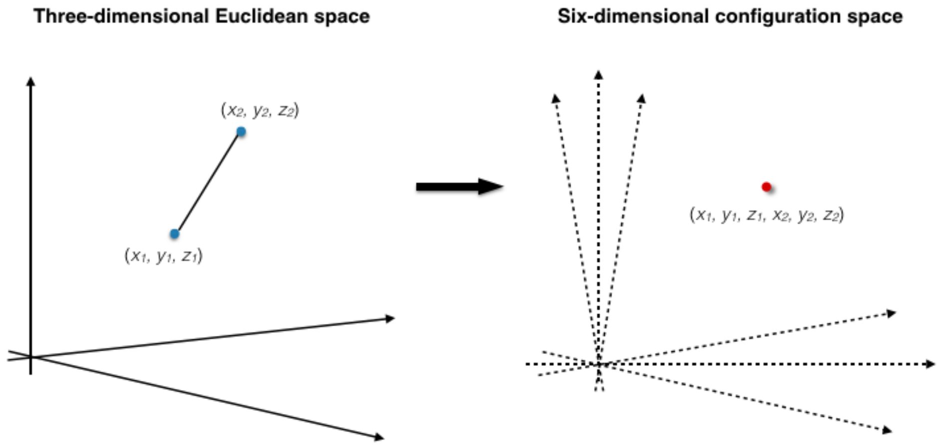
[13] If the general coordinates incorporate only a proper subset of the system's connections, then the number of the system's degrees of freedom is less than r.

[14] For Hertz's definition of constraint in terms of a system's connections, see §385.

Figure 3.5: Configuration space 1

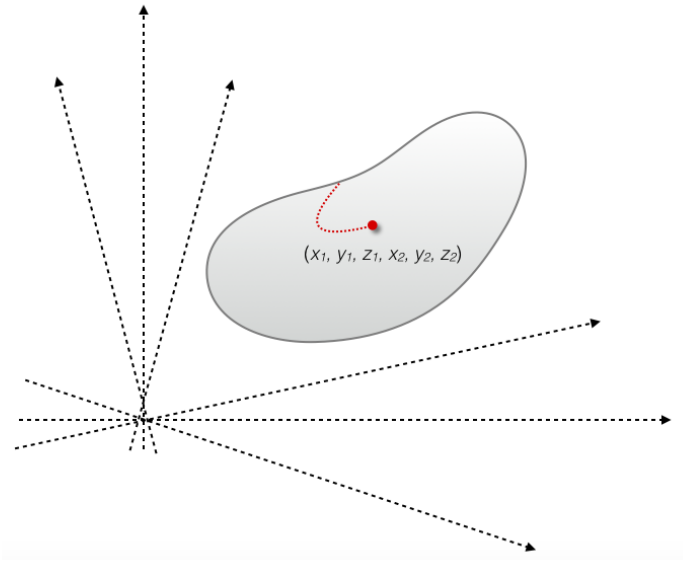


system would have six degrees of freedom: every point in the six-dimensional configuration space would then represent a possible configuration of the system. However, the presence of the rigid connection prevents the coordinates from being entirely independent of each other, and hence many points in configuration space represent impossible configurations (they would represent the rigid connection being broken). The connection thus determines a 'surface' of possible configurations, and motions of the system are represented by paths along this surface (Figure 3.6).

In fact, the existence of the connection results in this system having only *five* degrees of freedom: once we have specified any five of the coordinates, the remaining sixth coordinate is thereby determined. Rather than arbitrarily designate one coordinate as dependent on the other five, we can instead use three position coordinates to determine the system's center of mass, (X, Y, Z), and two angles to determine its orientation (θ, ϕ). Note that these five quantities *can* vary independently of each other (Figure 3.7).

Hence X, Y, Z, θ, and ϕ provide a set of specially adapted *general coordinates* for this system, mirroring its five degrees of freedom. The use of such coordinates leads to a particularly elegant characterization of a system's configuration space: the coordinates characterise a *curved* (Riemannian) space with *the same number of dimensions as the system's degrees*

Figure 3.6: Configuration space 2



of freedom. By using these coordinates we arrive immediately at the relevant "surface" of possible motions—the embedding space of impossible motions has disappeared from view (Figure 3.8). Characterizing a system in terms of its general coordinates gives rise to a more abstract characterization of a system's configuration space: the p_ρ are curvilinear coordinates of an r-dimensional Riemannian space.[15] Characterizing configuration space in terms of a system's general coordinates still preserves the elegance of the fundamental law—the motion of a free system still traces out a 'straightest path'.

3.5.1 'Systems acted on by forces'

Recall that, from Hertz's perspective, any apparently unfree system is *partial*, encompassed within a 'complete' system which is itself free (cf. §429). The cases of particular interest, unsurprisingly, are those in which the partial system is the only part of a complete system that is observable: 'When a part of a free system is considered an unfree system it is assumed that the rest of the system is more or less unknown, so that an immediate application of the fundamental law is impossible' (§430).

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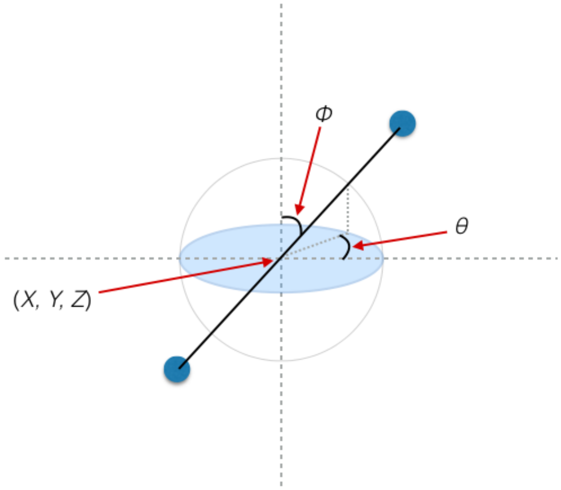
[15] The expression for the metric structure of this Riemannian space is reproduced in Appendix A.

Figure 3.7: General coordinates



Hertz considers two classes of partial systems: *guided systems* and *systems acted on by forces*. In the latter case, two (or more) material systems form a combined system by having at least one coordinate in common. Such systems are thereby 'coupled' to one another (cf. §450). It is here, then, that we find Hertz's reintroduction of the notion of force: 'By a *force* we understand the independently conceived effect which one of two coupled systems, as a consequence of the fundamental law, exerts upon the motion of the other' (§455).

Let the r coordinates p_ρ and the $\boldsymbol{r}$ coordinates $\boldsymbol{p_\rho}$ describe two coupled systems, A and B, respectively, and let us arrange the indices so that the common coordinates in both systems have the same index. Hence the fact that the systems are coupled is expressed via the condition that, for at least one value of ρ:

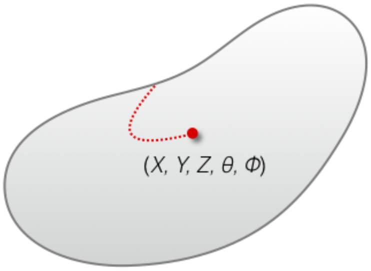
$$\boldsymbol{p_\rho} - p_\rho = 0$$

This immediately gives:

$$d\boldsymbol{p_\rho} - dp_\rho = 0$$

Each of the coupled systems has its own equations of condition. In general coordinates, these

Figure 3.8: Configuration space 3



are expressed as follows:[16]

$$\sum_{\rho=1}^{r} p_{\chi\rho} dp_\rho = 0$$

$$\sum_{\boldsymbol{\rho=1}}^{\boldsymbol{r}} \boldsymbol{p}_{\chi\rho} d\boldsymbol{p}_\rho = 0$$

We want to derive an expression for the effect that one coupled system has upon the other, and hence the force (as Hertz defines that term) that it exerts. Let us consider the force that the system B exerts on the system A.[17] For the *uncoupled* coordinates, the equations of motion of the system have the following form:[18]

$$mf_\rho + \sum_{\chi=1}^{k} p_{\chi\rho} P_\chi = 0$$

Here, f_ρ is a generalized acceleration vector, defined as the rate of change of the system's velocity (cf. §273). The $p_{\chi\rho}$ are the same factors that appear in the system's equations of condition, whilst the P_χ incorporate information about the curvature of the system's path.

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[16] Where before we had introduced $x_{\iota\nu}$ as continuous functions of the Cartesian coordinates, here the $p_{\chi\rho}$ are continuous functions of the general coordinates.

[17] We could, of course, equally well consider the effect that the system A has on the system B. As Hertz stresses, which of these effects should be called the 'force' and which the 'counterforce' is perfectly arbitrary, cf. §456.

[18] The reader interested in technical details is directed to Appendix B.

The only difference for the *coupled* coordinates turns out to be that we need to include an extra factor, P_ρ, in these equations (cf. §457):

$$mf_\rho + \sum_{\chi=1}^{k} p_{\chi\rho} P_\chi - P_\rho = 0$$

By setting $P_\rho = 0$ for all values of ρ that are not coupled coordinates, the above equations can be used as a more general expression for the equations of motion for the system (cf. §459). Thus the quantities P_ρ 'forms the analytical expression for the force which the system $\boldsymbol{p}_\rho$ exerts on the system p_ρ' (§460). Indeed: 'By this determination we place ourselves in agreement with the existing notation of mechanics; and the necessity for securing such an agreement sufficiently justifies us in choosing this particular determination out of several permissible ones' (§460).

3.6 HERTZ'S NOTION OF FORCE

Hertz's notion of force adds nothing beyond the application of the fundamental law to a system of connected material points. Even so, Hertz is able to show that his notion is in accord with the notion of force in the customary representation of mechanics to a remarkable degree.

Most immediately, Hertz's forces are vector quantities, both with regard to the systems that exert them and with regard to the systems that they are exerted on. All the expected compositional properties of forces are thus recovered: multiple forces can be added together to form a resultant force, and a single force can be analysed into components (cf. §§461 ff.). Hertz also points out that various *different* systems can exert the *same* force on a given system, and also that one given system can exert the same force on various different systems. Hence a force itself (considered simply as a certain vector quantity) can be considered on its own terms, independently of the partial system that is, so to speak, responsible for it (§§464 f.).

Hertz also derives sharpened up versions of Newton's three laws of motion, making explicit that the clearest and least ambiguous applications of the laws are to systems consisting of a *single point.* Regarding the first law, Hertz first observes that a free system with no connections 'persists in its condition of rest or uniform motion in a straight path' (§382). As we have already seen, this is just because in such cases the straight path in physical space is at the same time the straightest path in configuration space. Hertz then notes that Newton's first law follows as a corollary of this fact concerning connection-free systems in general, because it will obviously apply to a system consisting of just one material point:

> A free material point persists in its condition of rest or uniform motion in a straight path (Galileo's Law of Inertia or Newton's First Law). (§383)

Regarding the second law, Hertz similarly shows that if a system with no connections accelerates (i.e. by being coupled with another system, hence having a force exerted on it), then that acceleration 'takes place in the direction of the force which acts on the system, and its magnitude is equal to the magnitude of the force, divided by the mass of the system' (§494). What this motion will look like will of course depend entirely on the force in question (i.e. how this connection-free system is coupled to some other system). However, in the limiting case where the system we are considering consists of just one material point, its acceleration can always be given by an ordinary three-dimensional vector. Hence Hertz takes the case of a single point—again as the limiting case of a system without connections—as corresponding to a statement of Newton's second law:

> The acceleration of a single material point takes place in the direction of the force acting on it, and its magnitude is equal to the magnitude of the force, divided by the mass of the point (Newton's second law). (§495)

The case of the third law is more subtle: Hertz derives an action-reaction principle that has a close relationship with Newton's third law but is not equivalent to it (even for a single point). Earlier, we defined the vector P_ρ as the force which the system B exerts on the system A. This is a vector quantity with regard to system A because it gives the components of the force along the coordinates p_ρ. If we wish to regard this force as a vector quantity with regard to system B, it's components along $\boldsymbol{p}_{\boldsymbol{\rho}}$ can be denoted $\boldsymbol{P}'_{\boldsymbol{\rho}}$. Similarly, we can consider the "counterforce" that the system A exerts on the system B. The components of *this* force along the coordinates of system B are then denoted by $\boldsymbol{P}_{\boldsymbol{\rho}}$, and, as before, if we consider this force as a vector quantity with regard to system A, this is denoted by P'_ρ. Hertz then shows (§467) that:

$$P_\rho = -\boldsymbol{P_\rho}$$[19]

However, Hertz is careful to delineate this result from Newton's third law. The key difference lies in the fact that Newton's action-reaction principle, unlike Hertz's, is intended to apply to actions-at-a-distance:

> Newton's Law, as he intended it to be understood, contains our proposition completely; this is shown by the examples appended to his statement of the law. But Newton's Law contains more. At least it is usually applied to actions-at-a-distance, i.e. to forces between bodies which have no common coordinates. But our mechanics does not recognise such actions. (§469)

Hertz further observes that where Newton's action-reaction principle deviates from his own is precisely where it is open to criticism and doubt. Firstly, as we saw in section three, unless the two bodies are point-masses the line connecting them (along which the two forces are supposed to be equal and opposite) is open to ambiguity.[20] Secondly, as experimental results had been starting to confirm in Hertz's lifetime, the third law may simply be false when applied to electromagnetic interactions between distant bodies:

> ...the application of the principle of reaction to actions-at-a-distance commonly found in mechanics manifestly represents an experiential fact, concerning the correctness of which in all cases people are beginning to be doubtful. For instance, in Electromagnetics we are almost convinced that the mutual action between moving magnets is not in all cases strictly subject to the principle. (§470)

Most importantly for present purposes, Hertz's framework offers a thorough clarification of the relationship between vectorial and variational forces. In line with his re-statement of Newton's laws of motion, Hertz identifies the vectorial notion of force with the force on a *single* point. This is the class of Hertzian forces which are represented by ordinary three-dimensional vectors. He calls these forces 'Newtonian' or *elementary* forces. Thus Hertz

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[19] Or, equivalently: $P'_\rho = -\boldsymbol{P_\rho}'$. Hence $P_\rho = -P'_\rho$, $\boldsymbol{P_\rho} = -\boldsymbol{P_\rho}'$.

[20] In Hertz's words: 'when force and counter-force affect different bodies, it is not quite clear what is meant by opposite' (§470).

recovers a picture of ‘elementary mechanics’: Newtonian forces acting on point-masses. In contrast, Hertz calls the more general (higher-dimensional) force-vectors ‘Lagrangian’:

> As a rule, elementary mechanics means by forces only elementary forces. By way of distinction, the more general forms of forces hitherto considered by us are denoted as Lagrangian forces. Similarly we might denote the elementary forces as Galilean or Newtonian forces. (§476)

The fact that every system can be conceived as composed of material point in three-dimensional Euclidean space means that every Lagrangian force can be conceived as decomposable into elementary Newtonian forces (§479). In most cases, no useful such decomposition can be given, and in any case such a decomposition would be highly arbitrary. Nevertheless, from the perspective provided by Hertz’s framework, the relationship between general and ‘elementary’ forces is unproblematic.

Concerning the question of the velocity (in)dependence of forces, it is important to keep in mind that within Hertz’s framework, at base there is always nothing more than an application of the fundamental law to a free system of connected material points.[21] Because Hertz treats connections as fundamental he derives forces from constraints rather than the other way around. One result of this is that the distinction between ‘constraint forces’ and ‘impressed forces’ does not arise in a problematic way. Indeed, although the idea of an impressed force is more familiar, from Hertz’s perspective this notion is employed in situations in which we are particularly ignorant; situations in which we characterize the motion of the (partial) system in question with the limited information that we have.

In summary, Hertz’s notion of force is introduced as a way to succinctly capture the effect that one system can have on another (in particular, when the two are coupled together to form a combined system). In the general case, this is best represented by the high-dimensional ‘Lagrangian’ vector that is tied directly to the system’s degrees of freedom. Derivatively, we can recover the three-dimensional vectors in ordinary space that correspond to more familiar ‘Netwonian’ forces, either on the points making up a complex system, or on a single isolated point on its own. And from this perspective we can see that, if a partial system is all that is observable, a force is what appears to be the most immediate *cause* of

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[21] Furthermore, conservation of energy is derived as an immediate consequence of the fundamental law itself, cf. §§340 ff.

its acceleration. In an important sense, then, Hertz's approach reverses the direction of the historical development of mechanics.

As we will see, the way in which Hertz avoided the logical obscurity surrounding the Newtonian notion of force had a lasting impact on Wittgenstein's conception of the ambitions of philosophy.

4.0 MODELS AND MULTIPLICITIES

> ‘As I have often said, philosophy does not lead me to any renunciation, since I do not abstain from saying something, but rather abandon a certain combination of words as senseless. In another sense, however, philosophy requires a resignation, but one of feeling and not of intellect. And maybe that is what makes it so difficult for many. It can be difficult not to use an expression, just as it is difficult to hold back tears or an outburst of anger.’
>
> Ludwig Wittgenstein, *The Big Typescript*, §86

4.1 FUNDAMENTAL CONFUSIONS

In cases where a single term is used to perform subtly different functions, there may be competing demands on the term that cannot be met in a consistent way. If we fail to notice how this situation has arisen, we may misdiagnose what has gone wrong when these demands come into conflict. We may even attempt to introduce additional demands to those that are already present. However, the correct response would be to reduce the number of demands, or otherwise avoid the existing demands coming into conflict.

This kind of response is articulated clearly by Hertz in his diagnosis of the confusion surrounding the term ‘force’. In justifying the need for a reformulation of classical mechanics, Hertz diagnoses the persistent puzzlement over the *nature* of force in the following way:

> With the terms ‘velocity’ and ‘gold’ we connect a large number of relations to other terms; and between all these relations we find no contradictions which offend us. We are therefore

> satisfied and ask no further questions. But we have accumulated around the [term] 'force'... more relations than can be completely reconciled amongst themselves. We have an obscure feeling of this and want to have things cleared up. Our confused wish finds expression in the confused question as to the nature of force... But the answer which we want is not really an answer to this question. It is not by finding out more and fresh relations and connections that it can be answered; but by removing the contradictions existing between those already known, and thus perhaps by reducing their number. (Hertz, 1899, pp. 7-8)

Hertz is here claiming that the relations between 'force' and other terms have accumulated in such a way that they cannot be 'completely reconciled amongst themselves'. The discussion of the last chapter allows us to give a more precise characterization of this issue. The accumulation of conflicting relations around the term 'force' arises because it does not have a unitary meaning. In particular, the notion of an *impressed* force is not entirely consonant with the notion of a *constraint* force. The issue underlying Hertz's concern, then, is that the word 'force' is being used in mechanics in two subtly different ways.[1]

The source of confusion that Hertz's concern traces back to—the fact that a given term can be put to multiple uses; uses that might come into conflict, or even be strictly incompatible—generalises well beyond classical mechanics. In particular, such a source of confusion is a matter of central concern in the *Tractatus*:

> 3.323 In everyday language it very frequently happens that the same word has different modes of signification—and so belongs to different symbols—or that two words that have different modes of signification are employed in sentences in what is superficially the same way.
>
> Thus the word 'is' figures as the copula, as a sign for identity, and as an expression for existence; 'exist' figures as an intransitive verb like 'go', and 'identical' as an adjective; we speak of *something*, but also of *something's happening*.

For Wittgenstein, the fact that the varying uses of words can be masked by the superficial ways they occur in colloquial sentences is an acute philosophical danger. At 3.324 he writes: 'In this way the most fundamental confusions are easily produced (the whole of philosophy is full of them).'

As noted in the introduction to this thesis, the passage from Hertz's introduction quoted above resonated deeply with Wittgenstein, who even went so far as to suggest that it summed

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[1] Indeed, Hertz explicitly suggests that an ambiguity in the meaning of 'force' may be at the root of his concerns: 'The notion of force in [Newton's third law], and the conception which [Newton's] first two laws presume and suggest to us, seem to me to be slightly different, and this slight difference may be enough to produce the logical obscurity...' (Hertz, 1899, p. 6).

up philosophy.[2] In the *Tractatus*, Wittgenstein makes two specific references to *Principles*, one to Hertz's notion of dynamical models (at 4.04) and one to Hertz's notion of connections (at 6.361).[3] A central claim of the *Tractatus* is that ordinary sentences are analysable as truth-functions of elementary sentences, sentences which refer to simple objects. Interpretations of the *Tractatus*—particularly interpretations of the significance of this central claim—can be divided into two broad camps: *ontologically-oriented* interpretations and *logically-oriented* interpretations.[4]

On an ontologically-oriented interpretation, simple objects play a primary role: it is the existence of simple objects which imbues the names in elementary sentences with meaning, and the sense of colloquial sentences is then explained by appealing to the sense of elementary sentences. On a logically-oriented interpretation, however, the sense of colloquial sentences is not regarded as something that requires explanation, and simple objects do not play a special explanatory role. This central contrast can be summed up in the following way: according to an ontologically-oriented interpretation, the *Tractatus* takes a *bottom-up* approach to the analysis of colloquial sentences, whereas according to logically-oriented interpretation, the *Tractatus* takes a *top-down* approach.[5] To flesh this out in some more detail, it will be helpful to draw a comparison with the similarly contrasting approaches to analysis taken by Russell and by Frege.

During his 'logical atomism' period,[6] Russell works within an epistemological framework which privileges the notion of *acquaintance*. Russell describes this as a direct and unmediated relation between a subject and an object: 'I say that I am *acquainted* with an object when I have a direct cognitive relation to that object, i.e. when I am directly aware of the

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[2] Cf. McGuinness (2002a), ix and Kjærgaard (2002), pp. 125-126. See also Wittgenstein (2005), p. 310e: 'As I do philosophy, its entire task is to shape expression in such a way that certain worries disappear. ((Hertz.))'

[3] In what follows I will only be concerned with Wittgenstein's first reference to *Principles*. I intend to turn to Wittgenstein's second reference to *Principles* in future work.

[4] Proponents of an ontologically-oriented interpretation of the *Tractatus* include Black (1964), Griffin (1964), Hacker (1986), and Pears (1987). Proponents of a logically-oriented interpretation include Ishiguro (1969), Rhees (1970), Diamond (1988), Kremer (1997), McGuinness (2002b), Goldfarb (2011), and Ricketts (2014). For a similar division of interpretations of the *Tractatus* into two broad camps, see Kremer (1997) pp. 107-108.

[5] Note that my deployment of the expressions 'bottom-up' and 'top-down' is not exactly the same as that of other commentators, such as Kremer (1997) and Ricketts (2014), though it is closely related.

[6] I have in mind Russell's 1918 lectures, 'The philosophy of logical atomism' (Russell, 1919/2009), and the various earlier texts that feed into this work.

object itself' (Russell, 1910, p. 108). Examples of objects known by acquaintance include logical forms, universals, and sense-data. For Russell, it is impossible for an object to bear a logically proper name unless that object is known by acquaintance; hence an important achievement of analysis is to show that colloquial names are *not* names in this sense (rather, they are typically disguised definite descriptions). On Russell's view, the analysis of colloquial sentences terminates at the level of 'atomic' sentences, and the names that appear there will only be logically proper names. Acquaintance with objects imbues these names with meaning, and our ability to understand colloquial sentence stems, ultimately, from our knowledge of objects with which we are acquainted.[7] The *last* stage of analysis thus plays a primary role: it is acquaintance with objects which imbues the symbols in atomic sentences with meaning, and the meaningfulness of atomic sentences then 'flows upwards' to colloquial sentences. In this sense, then, Russell's approach to analysis is bottom-up

Characterised in this way, Russell's approach contrasts sharply with Frege's. Whereas Russell treats the simple symbols for items known by acquaintance as playing a primary role, and the meaningfulness of colloquial sentences as derived from them, Frege regards the constituents of a colloquial sentence as significant only via the examination of the sentence as a whole. As Thomas Ricketts has argued, this stems from the primacy Frege attributes to judgement.[8] The whole sentence—or, rather, the whole *proposition*: the sentence that expresses something that can be judged true or false—then becomes the primary unit of semantics. Indeed, Frege regards the privileging of whole judgements (hence whole propositions) as distinctive of his approach to logic:

> For in Aristotle, as in Boole, the logically primitive activity is the formation of concepts by abstraction... As opposed to this, I start from judgments and their contents, and not from concepts. (Frege, 1979, pp. 15-16)

On Frege's approach, it is the fact that our propositions express judgements that are true or false, and that we infer from certain propositions to others, that lends significance to the *parts* of our propositions—hence to notions like *concept* and *object*. Frege emphasises this approach to the analysis of sentences throughout his career. As early as 1880 he writes,

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[7] Cf. Russell (1910), p. 117: '*Every proposition which we can understand must be composed wholly of constituents with which we are acquainted*' (emphasis in original).

[8] Cf. Ricketts (1986), particularly pp. 313-321.

‘instead of putting a judgment together out of an individual as subject and already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgment’ (Frege, 1979, p. 253), and as late as 1919 he writes, ‘I do not begin with concepts and put them together to form a thought or a judgment; I come by the parts of a thought by analyzing a thought’ (Frege, 1979, p. 17).[9] Frege’s approach to analysis is thus top-down: the *first* stage of analysis plays a primary role. The significance of the symbols appearing in analysed sentences stems from the significance of the symbols in the original colloquial sentences, not the other way around. Indeed, the significance of colloquial sentences is not explained in terms of anything else; it is presupposed.

To sum up: on Russell’s bottom-up approach, the termination of analysis at the level of atomic sentences is what imbues colloquial sentences with the significance that they have. It is our acquaintance with objects that gives the symbols in atomic sentences meaning, and *in this way* our ability to understand colloquial sentences is accounted for. On Frege’s top-down approach, however, our ability to understand colloquial sentences is not a target explanandum. Rather, our evident ability to understand (and draw inferences from) colloquial sentences is itself what lends significance to the analysands of those sentences.

I can now flesh out my suggestion that an ontologically-oriented interpretation of the *Tractatus* carries with it a bottom-up approach to analysis, whereas a logically-oriented interpretation carries with it a top-down approach. Beginning with the former: on an ontologically-oriented interpretation, the argument for the existence of simple objects begins with a conception of naming that has clear similarities to Russell’s. On that conception, if an object did not exist then its name, absent a referent, would be meaningless. As the objects that we encounter in day-to-day life exist contingently, this implies that names of ordinary objects have their meanings contingently. In particular, the meaningfulness of a name depends on the component parts of the relevant object being arranged so as to constitute that object. Hence the sense of sentences in which such names occur seems to depend on the truth of further sentences—sentences describing the constitution of the named objects. Borrowing an example from the *Philosophical Investigations*, David Pears puts this

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[9] For a summary of some of the arguments and textual evidence in favour of this interpretation of Frege, see Goldfarb (2002) p. 190.

point in the following way: '[this] seems to make the sense of 'The broom is in the corner' depend on the truth of 'The brush is attached to the broomstick'.' (Pears, 1987, p. 77)[10]

Here a regress threatens: the component parts of an object that we pick out will typically have component parts themselves; thus the meanings of the names of these component parts will depend on the truth of yet further sentences. If this analysis into smaller and smaller components is not to go on forever, it must terminate on ontological *simples*—objects that have no internal structure at all. The meaningfulness of the names of such objects would then no longer depend on the truth of any further sentences. Having dived down to such depths, one could return to colloquial language reassured that it was firmly anchored in elementary sentences; sentences which referred to simple objects. The *Tractatus* is thus seen to expound a metaphysics of simple entities—one that bears a limited comparison with a Russellian metaphysics—and it is via its connection with the elements of this metaphysics that language finds its purchase on the world.

An ontologically-oriented interpretation thus carries with it a bottom-up conception of analysis: the *last* stage of analysis—elementary sentences consisting of names of simple objects—plays a primary role. Simple objects imbue the names in elementary sentences with meaning, and the sense of colloquial sentences is then accounted for by appealing to the sense of elementary sentences. This is the ontologically-oriented understanding of simple objects defended by Peter Hacker:

> The simple objects are, Wittgenstein thought, the final residue of analysis, the indecomposable elements that are the meanings of the unanalysable names that occur in elementary propositions. He *knew*, so he thought, that there must be such things. There must be unanalysable objects if language is to be related to the world... For only thus can the need for a firm anchor for language be met. (Hacker, 1986, pp. 65-66)

What alternative does a logically-oriented interpretation of the *Tractatus* provide? This will emerge more fully as the argument of this chapter progresses; here, I limit myself to a brief summary. On a logically-oriented interpretation, the notion of a simple object only finds its significance given the application and use of language, and the claim that such objects exist does not add anything beyond the claim that sentences have sense; that sentences

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[10] Cf. Wittgenstein (1953) §60.

represent or picture states of affairs. This is the kind of logically-oriented understanding of simple objects defended by Brian McGuinness:

> ...we must not think of the realm of reference as a mysterious, infinitely extended magazine of things, as if they were concrete objects, with which we might or might not be lucky enough in a full life to have acquaintance by, so to speak, coming across them in a street. There is already contained in language and thought the possibility of all objects that are possible. (McGuinness, 2002c, p. 91)

On this view, the central motivation to appeal to the logical apparatus of the *Tractatus*—including the notion of a simple object—is to display the logical relationships among colloquial sentences. A logically-oriented interpretation carries with it a top-down conception of analysis in the sense that the *first* stage of analysis—ordinary colloquial sentences and their manifest logical relationships—plays a primary role. In particular, the sense of elementary sentences stems from the sense of the original colloquial sentences, not the other way around.

Despite the fact that these interpretations present radically divergent ways of reading the *Tractatus*, it seems difficult to resolve this issue on purely textual grounds.[11] However, my central contention in what follows is that Wittgenstein's reference to Hertz's dynamical models at 4.04 is a striking piece of textual evidence in favour of a logically-oriented interpretation. More specifically, I will argue that Hertz's analysis of mechanical systems in *Principles* can be recognised as a precedent for a logically-oriented construal of Tractarian analysis, and that the recognition of Hertz's influence in this regard provides a distinctive conception of what the termination of Tractarian analysis might look like.

4.2 LOGICAL PICTURES

When Wittgenstein refers to *Principles* at 4.04 in the *Tractatus*, he indicates that Hertz's dynamical models are particularly helpful in bringing out what it means for a sentence to have the same 'logical (mathematical) multiplicity' as what it represents:

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[11] The purely textual defense of either interpretation can be bulwarked by appealing to sources besides the *Tractatus* itself—cf. Kremer (1997) p. 109.

> 4.04 In a sentence there must be exactly as many distinguishable parts as there are in the situation that it represents.
>
> The two must possess the same logical (mathematical) multiplicity. (Compare Hertz's *Mechanics* on dynamical models.)

The series of remarks in the 4.0s is the second and final time that the word 'picture' (*Bild*) features in the *Tractatus* in a sustained and prominent way. The first time is in the series of remarks in the 2.1s and 2.2s, where Wittgenstein articulates the Tractarian conception of picturing in general. The 4.0s then apply this conception of picturing to sentences. Wittgenstein's description of sentences as pictures is not figurative—he insists that a written sentence proves to be a picture 'even in the ordinary sense' (4.011). At 4.012 we are given an indication of how this might work: in a sentence of the form '*aRb*'—which says that *a* stands to *b* in the relation *R*—the propositional sign itself *looks like* two things of a certain kind, '*a*' and '*b*', related to each other by the fact that '*R*' stands between them.[12] But it is apparent that a sentence-sign like '*aRb*' is the exception rather than the rule: most declarative sentences don't look anything like the situations they assert to obtain. Although Wittgenstein repeatedly suggests that the operative conception of picturing in the *Tractatus* is, in certain central ways, a familiar one, it is difficult to see how ordinary sentences could possibly be pictures in a familiar sense. Our first task will be to address this puzzle.

In the 2.1s, we are told that the elements of a picture correspond to objects (2.13), and that what constitutes a picture is that its elements are related to one another in a determinate way (2.14). Following this, the notion of *pictorial form* is introduced:

> 2.15 The fact that the elements of a picture are related to one another in a determinate way represents that things are related to one another in the same way.
>
> Let us call this connection of its elements the structure of the picture, and let us call the possibility of this structure the pictorial form of the picture.

Pictorial form is 'the possibility that things are related to one another in the same way as the elements of the picture' (2.151). This is readily applicable to familiar pictures: the spatial relations in a spatial picture, or the relations among colours in a coloured picture, mirror the corresponding relations among the depicted objects. Hence a diagram showing a spatial arrangement of items could have been drawn with the items in different positions,

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[12] This example is also discussed earlier, at 3.1432.

for example, or a painting showing the colours of certain flowers could have been made using a different palette. These latent possibilities are a reflection of the possibilities that apply to the depicted objects themselves.[13] According to the *Tractatus*, it is pictorial form which imbues the correlations between the elements of a picture and the associated objects with the significance that they have (cf. 2.151-2.1514); a picture *is* a picture in virtue of the pictorial form that it shares with what it represents (2.16, 2.17).

To illustrate the notion of pictorial form further, imagine that I want to represent the order in which a truck, car, and taxi are parked outside my house by arranging a cup, book, and pen in a line on my desk.[14] Such a physical model is a particularly direct way of representing a spatial fact—the model can employ *exactly* the same kind of relations among its elements as the spatial arrangement of objects it depicts because, of course, the model is itself a spatial arrangement of objects. To simplify the example further, imagine that the only fact that I intend to represent is the mere linear ordering of the vehicles—i.e. which one is between the other two—rather than any further facts about their relative locations (which one is furthest to the left, how close together they are, and so on). The pictorial form that allows the items on my desk to represent the order in which the vehicles are parked can be seen in the possibility of arranging them accordingly once I have determined which item is correlated with which vehicle. Hence these two facts—the ordering of the items on my desk and the ordering of the vehicles outside—have the same pictorial form, and the one can represent the other.

So conceived, the same pictorial form can be easily manifested in more abstract representations. All that is required is that the possibilities of the relations among the elements of the representation is the same as the possibilities of the relations among the represented objects. For example, rather than arranging the items on my desk in a line I could instead place a coin on one of them to indicate that it is 'between' the other two. In this case it

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[13] Cf. 2.0131: 'A spatial object must be situated in infinite space. (A spatial point is an argument-place.) A speck in the visual field, though it need not be red, must have some colour: it is, so to speak, surrounded by colour-space. Notes must have some pitch, objects of the touch some degree of hardness, and so on.' See also 2.031-2.033.

[14] A more complicated example is the model of the car accident used in a Paris courtroom that Wittgenstein took as inspiration, cf. Wittgenstein (1998), p. 7. Note that it is reasonable to treat 'model' (*Modell*) and 'picture' (*Bild*) as synonyms in the context of the *Tractatus*; cf. 2.12 and 4.01. For some discussion of the connotations of '*Bild*' in this regard, see Rhees (1970), p. 4.

is the presence of the coin rather than the items' spatial arrangement that represents the ordering of the vehicles. That too would provide a picture of the vehicles by dint of having the same pictorial form; the same possibilities of relations among the elements of the model. (The coin could be placed on any of the items on my desk, thus mirroring the possibility that any of the vehicles is the one parked between the other two.) Beyond this, the same pictorial form could re-emerge in a great variety of abstract representations: in symbols on a piece of paper, in a series of sounds, in a certain pattern of wiggling of my eyebrows, and so on. All that is required is the possibility that the elements of the representation are related to one another in the same way as the depicted objects.

It might be objected that the items on my desk, let alone the elements of more abstract representations, manifestly do *not* have the same possibilities of relations as the vehicles they are representing. Many of the ways that I might manipulate the cup, book, and pen have nothing obvious corresponding to them concerning the parked vehicles outside: I might drop the pen on the floor or take the lid off and throw it away, for example. In response to this objection, note that in using the items on my desk as a model of the parked vehicles I must stipulate what features of the items are representationally significant. Until I have said otherwise, the colour or shape or size of the items does not represent anything at all. In the case where I arrange them in a line, the items on my desk *only* provide a model of the parked vehicles insofar as one of them is situated determinately between the other two, and dropping the pen on the floor is as representationally inert as the material the pen is made of. Of course, the way in which I am using these items as a model can be varied. Say I wanted to enhance my model by using a colour code to represent which of the vehicles (if any) has been left unlocked. The items I originally chose might suit the purpose already if, for instance, the pen is red whilst the book and cup both happen to be blue, and if only the taxi is unlocked. In that case I can simply declare that red means unlocked, blue means locked, and that the pen (suitably placed) corresponds to the taxi. However, enhancing the model in this way clearly carries with it the possibility of using another red item (say, a book with a red cover) to represent the possibility that one of the other vehicles is also unlocked. The possibilities of the relations among the elements in my model now includes a binary colour scheme as well as a linear ordering, and *in this way* my model stands to represent the locked or unlocked states of the vehicles as well as the order in which they are parked. Hence the pictorial form shared by the model and the situations it stands to depict is now different to what it was before.

Let this stand as a proposed interpretation of 2.12–2.17. To recap: in a picture, the elements of the picture correspond to objects (2.13) and the fact that the pictorial elements are related to one another in a determinate way represents that the depicted objects are related to one another in a determinate way (2.15). Pictorial form is the possibility that the elements of the picture are related to one another in the same way as the things that those elements depict (2.151). Pictorial form is therefore identical in a picture and the fact depicted (2.16), and it is through its pictorial form that a picture *is* a picture (2.161, 2.17). The notion of pictorial form thus plays a particularly central role: '*That* is how a picture is attached to reality; it reaches right out to it' (2.1511).

Let us turn to consider how this conception of picturing can be applied to sentences. The claim at 4.01 that a sentence is a picture is justified at 4.02 with the seemingly unrelated observation that 'we can understand the sense of a sentence-sign without its having been explained to us'. Wittgenstein elaborates on this by noting that, although the meaning of an unfamiliar word needs to be explained, the meaning of an unfamiliar *sentence* is readily comprehensible (4.026); a sentence is typically understood by anyone who understands its constituents (4.024). This leads naturally into 4.03: 'A sentence must use old expressions to communicate a new sense.' But what has all this got to do with the idea that a sentence is a picture?

Wittgenstein's emphasis on the construction of unfamiliar sentences from familiar words becomes intelligible with the central role of pictorial form in view. Roughly, it is the possibilities of the relations among the elements of a sentence (the words) which can mirror the possibilities of the relations among the elements of the represented situation. Here, then, is how a sentence is a picture. Returning to the example of the vehicles parked outside my house, I can *describe* the relative positions of the vehicles just as I can *describe* the ordering of the items on my desk. By dint of the fact that they can represent the linear ordering of three individuals, those sentences then have *the same pictorial form* as the fact that the vehicles in the street (or the items on my desk) are arranged in a particular order. Thus I can describe the arrangement of the vehicles by saying: "the car is parked between the truck and the taxi".[15] In this sentence, the pictorial form that it shares with the situation it depicts can be seen in the possibility of rearranging the sentential elements—the words—as required. Hence permuting the words 'car' and 'truck' to form a new sentence is like switching around the cup and the book on my desk.[16]

Pictorial form thus provides a central bridge between the 2.1s and the 4.0s. If a sentence is to be a picture of reality (4.01), then what it must have in common with reality, in order to be able to depict it in the way it does, is its pictorial form (2.17). This is how sentences prove to be pictures 'even in the ordinary sense'. Note, however, that it is specifically a *logical* picture that is mentioned at 4.03:

> 4.03 A sentence must use old expressions to communicate a new sense.
>
> A sentence communicates a situation to us, and so it must be *essentially* connected with the situation.
>
> And the connection is precisely that it is its logical picture
>
> A sentence states something only in so far as it is a picture.

The notion of a logical picture is introduced in the concluding remarks of the 2.1s:

> 2.18 What any picture, of whatever form, must have in common with reality, in order to be able to depict it—correctly or incorrectly—in any way at all, is logical form, i.e. the form of reality.
> 2.181 A picture whose pictorial form is logical form is called a logical picture.
> 2.182 Every picture is *at the same time* a logical one. (On the other hand, not every picture is, for example, a spatial one.)
> 2.19 Logical pictures can depict the world.

Logical form is common to all pictures which can depict reality 'correctly or incorrectly'. Given any two possible situations, the obtaining or non-obtaining of one will either guarantee the obtaining or non-obtaining of the other, or vice versa, or else they will be independent. Thus: the correctness or incorrectness of one picture will imply the correctness or incorrectness of a second picture, or vice versa, or else *they* will be independent. Because every picture (that can depict reality correctly or incorrectly) sits in its logical relationships with other

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[15] Here, obviously enough, the words 'truck', 'car' and 'taxi' each goes proxy for the relevant vehicle, while 'x is parked between y and z' conveys their linear ordering.

[16] I will not consider issues concerning truth-functionally complex sentences here. For some discussion, see Ricketts (1996) §IV.

such pictures, every picture is a logical picture. One reason why the notion of logical picture is mentioned at 4.03 is because the sentences of natural language can adopt a multitude of pictorial forms. I can describe a spatial arrangement of objects ("the book is between the coffee cup and the pen") or a pattern of colours ("the flowers are pink, blue, and crimson"), and many other situations besides. In all cases, however, these sentences are still logical pictures of what they represent: 'every picture is *at the same time* a logical one'.

In the discussion of picturing so far, there has been no need to depart from straightforward and familiar examples, examples in which the depicted objects are just ordinary things like pens, flowers, and taxis. However, immediately preceding the discussion of picturing in the 2.1s is the grand metaphysics of the 2.0s: the *states of affairs* that are configurations of *simple objects*; objects which themselves make up 'the substance of the world' (2.021). Although many of these remarks about configurations of objects in states of affairs are straightforwardly incompatible with ordinary facts concerning, say, the relative positions of items on my desk, immediately following the discussion of picturing in the 2.1s, 2.201 declares: 'A picture depicts reality by representing a possibility of existence and non-existence of states of affairs.'[17] Similarly, immediately following the discussion of the pictorial nature of sentences in the 4.0s, 4.1 declares: 'Sentences represent the existence and non-existence of states of affairs.' Corresponding to the configuration of objects in states of affairs is the configuration of names in elementary sentences (4.21, 4.22). We are thus faced with the question: how is it that an ordinary picture is a representation of the obtaining and non-obtaining or logically independent states of affairs (cf. 2.201)? Or in the context of language: how is it that a colloquial sentence is a truth-function of logically independent elementary sentences (cf. 4.4, 5)?

The answer to this question (or part of an answer) will require us to return, eventually, to the notion of multiplicity and the reference to Hertz's dynamical models at 4.04. But the starting place is the idea that both the colloquial sentence and the corresponding fully analysed sentence are *projections* of the situation they represent.

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[17] See also 2.11.

4.3 PROJECTION

The notion of projection is introduced and discussed in the remarks following 3.1, and then occurs once again at 4.0141:

> 3.11 We use the perceptible sign of a sentence (whether written or spoken, etc.) as a projection of a possible situation.
>
> Thinking the sense of the sentence is the method of projection.[18]

> 4.0141 In the fact that there is a general rule by which the musician is able to read the symphony from the score, and that there is a rule by which one could reconstruct the symphony from the line on a gramophone record and from this again—by means of the first rule—construct the score, herein lies the internal similarity between these things which at first sight seem to be entirely different. And the rule is the law of projection which projects the symphony into the language of the musical score. It is the rule of translation of this language into the language of the gramophone record.

4.0141 is embedded in the discussion of the essentially pictorial nature of sentences. The following remark at 4.015 asserts that 'all imagery, all our pictorial modes of expression, is contained in the logic of depiction', and at 4.016 Wittgenstein claims that alphabets developed out of hieroglyphs 'without losing what was essential to picturing'. Recall that a picture *is* a picture in virtue of the pictorial form that it shares with what it represents.[19] This is what imbues the correlations between the elements of a picture and the associated objects with the significance that they have.[20] Wittgenstein describes such correlations as 'the feelers of the picture's elements, with which the picture touches reality' (2.1515), a metaphor that he couples with a simile of a measuring ruler:

> 2.1512 [A picture] is laid against reality like a measure.
> 2.15121 Only the end-points of the graduating lines actually *touch* the object that is to be measured.

Here the notion of projection comes through clearly: both the feelers and the graduating lines act as *lines of projection*, connecting the picture to the situation depicted. Hence projection can be recognised as intrinsic to the Tractarian conception of picturing from the

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[18] The translation of this particular sentence is the subject of some discussion; see Rhees (1970) p. 39, Winch (1987) pp. 13-14, and McGuinness (2002b) p. 91.

[19] Cf. 2.16, 2.17.

[20] Cf. 2.151-2.1514.

get-go. Following the introduction of projection at 3.11, 3.12 tells us that a sentence is 'a propositional sign in its projective relation to the world'. Then at 3.13 we have the following:

> 3.13 A sentence includes all that the projection includes, but not what is projected.
>
> Therefore, though what is projected is not itself included, its possibility is.
>
> A sentence, therefore, does not actually contain its sense, but does contain the possibility of expressing it.
>
> ('The content of a sentence' means the content of a sentence that has sense.)
>
> A sentence contains the form, but not the content of its sense.

Recall that the arrangement of items on my desk had the same pictorial form as the parked vehicles by dint of the possibility of rearranging them in the relevant ways. The possibility of rearranging the items on my desk is identical to the possibility of rearranging the parked vehicles; it is the possibility common to *any* linear ordering of three things. Here is a sense, then, in which a model 'contains' the possibility of the situation it represents while not containing the situation itself.[21] On this reading, to say that a representation contains 'the form, but not the content' of what it represents is simply to say that they share the same pictorial form.

Exactly the same is true of sentences—sentences that also share the pictorial form of what they represent (else they could not represent it). In the case where I describe the arrangement of the parked vehicles, the possibility of rearranging the words in the relevant body of sentences is, again, just the possibility of linearly ordering three individuals. Thus at 3.14 we have an anticipation of the idea emphasised in the run up to 4.03—that sentences are, after all, particular arrangements of words: 'What constitutes a propositional sign is that in it its elements (the words) stand in a determinate relation to one another.' Indeed, Wittgenstein draws the comparison between the arrangement of words in a sentence and the arrangement of the elements of a physical model closer still:

> 3.1431 The essence of a propositional sign is very clearly seen if we imagine one composed of spatial objects (such as tables, chairs, and books) instead of written signs.
>
> Then the spatial arrangement of these things will express the sense of the proposition.
>
> 3.1432 Instead of, 'The complex sign "aRb" says that a stands to b in the relation R', we ought to put, '*That* "a" stands to "b" in a certain relation says *that* aRb.'

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[21] The latter notion might bring to mind one of Russell's memorable remarks to Frege: 'I believe that in spite of all its snowfields Mont Blanc is itself a component part of what is actually asserted in the proposition 'Mont Blanc is more than 4,000 meters high'.' 12th December 1904; reprinted in Frege (1982) p. 169.

At the heart of the Tractarian picture theory is the claim that the use of a sentence-sign as a projection is like the use of any picture or model. Just as we can think the sense of a sentence (3.11), so too we can 'think the sense of' an arrangement of physical objects. Indeed, in all of the various ways that we can represent states of affairs, we use arrangements of perceptible things as projections of possible situations (cf. 4.015).

As noted, 4.0141 is the only place outside of the 3.1s where the term 'projection' reoccurs. Here, we are offered another comparison between linguistic and non-linguistic representation, and the notion of projection is also extended in the following way. Given two representations of the same fact (such as a gramophone record and the corresponding score), it is not only possible to project from either of them to what they represent (to play the record, say, or to read the score), but also possible to project from one representation to the other (to write a score by listening to the record, or make a record by playing from the score). There are thus 'laws of projection': *general rules* which carry us from one representation to another, such as the 'rule of translation' from the language of musical notation to the language of gramophone records.[22] Again, this comparison between linguistic and non-linguistic representation is not metaphorical: 'A gramophone record, the musical idea, the written notes, and the sound-waves, all stand to one another in the same internal relation of depicting that holds between language and the world' (4.014).

We can now connect the notion of projection with the notion of multiplicity. At 4.03, Wittgenstein claims that a sentence must be '*essentially* connected' with the situation it depicts, and that this essential connection 'is precisely that it is its logical picture.' The notion of multiplicity is then introduced at 4.04 as a feature of this essential connection—sentence and situation *must* have exactly as many distinguishable parts as one another; they must have the same multiplicity. When Wittgenstein introduces the notion of projection at 3.11 he is considering the ordinary colloquial sentence ('whether spoken or written'), but at 3.2 he turns to a consideration of the completely analysed sentence. According to 4.03, both a colloquial and a completely analysed sentence are *only* sentences by dint of being logical

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[22] Both the simple idea of projection and the potential complexities involved are evident in this example. Although we may be perfectly confident that the written score can indeed be reconstructed from the gramophone record, actually carrying out such a reconstruction could prove very difficult in practice, especially in the absence of a record player!

pictures of what they represent. The discussion of projection as applied to the gramophone record and the musical score at 4.0141 can thus be carried over and applied to the colloquial sentence and the fully analysed sentence. Just as there is a general rule by which we can translate between the record and the score, so there is a general rule by which we can translate between the colloquial sentence and the fully analysed sentence. Herein lies the 'internal similarity between these things which at first sight seem to be entirely different'.[23]

We have thus seen the drawing together of the notions of *logical picture*, *projection* and *multiplicity*. For a sentence to represent a situation is for it to be a logical picture of that situation; for a sentence to be a logical picture is for there to be a method of projection that employs its essential connection with that situation; and if a sentence can be projected in this way then it must have the same multiplicity as the situation it depicts. Importantly, this is as true for the fully analysed sentence as it is for the colloquial sentence: both are logical pictures of the situation they represent, both can be used as a projection of that situation, and *all three* (the colloquial sentence, the fully analysed sentence, and the situation itself) must have the same multiplicity.

4.4 DYNAMICAL MODELS

Why, then, does Wittgenstein refer his readers to Hertz's dynamical models at 4.04? A dynamical model captures the *number* and *type* of the degrees of freedom of the target phenomenon, reflecting these in the *dimensionality* and *geometry* of configuration space. In this way a dynamical model provides a complete specification of the target system: it captures all and only the mechanical features of that system. (The underlying ontological details are not reflected in a dynamical model precisely because those details are irrelevant for the task of characterising a system's mechanical behaviour.) As a result, dynamical models make perspicuous the significant content of ordinary mechanical descriptions. It is

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[23] Cf. 3.343: 'Definitions are rules for translating from one language into another. Any correct sign-language must be translatable into any other in accordance with such rules: it is *this* that they all have in common.' We will see how this idea connects up with the central Tractarian notion of the *general sentence-form*, below.

not that such ordinary descriptions *fail* to represent a system's degrees of freedom; rather, the ordinary description may fail to make these features of the system explicit, and may imbue other, inessential, features with unwarranted significance. As an illustrative case, recall the example of the dumbbell system from the previous chapter.[24] In that example it is evident that the use of ordinary Cartesian coordinates obscures—but does not falsify—the true number of degrees of freedom of the target system. In contrast, a description using specially adapted general coordinates can present the degrees of freedom of the system explicitly.

Let us examine the distinctive way that Hertz allows for the construction of dynamical models by appealing to subsystems of hidden masses. In ordinary mechanical descriptions, forces do work on a system by accelerating the observed masses, thereby converting potential energy into kinetic energy. In the framework of *Principles*, Hertz models potential energy as the kinetic energy of hidden masses instead, as one might imagine tapping into the kinetic energy stored in a system of hidden flywheels. However, Hertz's procedure can then appear perplexing in the way explored in chapter 1. In the absence of plausible mechanisms that would actually produce the observed motions, Hertz's approach can seem remarkably speculative. But such an interpretation misses the central role of dynamical models in *Principles* and the connection with Hertz's picture theory. As we have seen, it is precisely through the hypothesis of hidden masses that Hertz limits what can be learnt about a mechanical system to what is conveyed by a dynamical model:

> If we admit generally and without limitation that hypothetical masses (§301) can exist in nature in addition to those which can be directly determined by the balance, then it is impossible to carry our knowledge of the connections of natural systems further than is involved in specifying models of the actual systems. We can then, in fact, have no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone—that the one set of systems are models of the other. (§427)

The significance of Hertz's hidden masses can be made clear by examining the method of *Lagrange multipliers*. Lagrange multipliers provide a powerful method for solving constrained variational problems by transforming to a dynamically equivalent 'free' problem. A generic

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[24] See above—chapter 3, section 3.4.

variational problem involves determining the extreme value of some function, $F(p_1, ..., p_r)$, by finding where its variation vanishes:

$$\sum_{\rho=1}^{r} \frac{\partial F}{\partial p_\rho} \delta p_\rho = 0 \tag{4.1}$$

In an unconstrained problem the variables p_ρ are independent of one another, but in a constrained problem there will be at least one *constraint equation*:

$$f(p_1, .., p_r) = 0 \tag{4.2}$$

This prevents the variables p_ρ from varying independently. In fact, we could re-arrange our constraint equation to make the final variable explicitly dependent on the others:

$$p_r = f'(p_1, ..., p_{r-1}) \tag{4.3}$$

We can then use this to eliminate p_r from (4.1) and solve for the remaining independent variables. More generally: we can re-arrange k constraint equations so as to eliminate k variables, then solve for the $r - k$ independent variables that we are left with. However, proceeding in this manner can be extremely cumbersome. Moreover, if the constraint equations are symmetric then dividing the variables into those that are dependent and those that are independent will be entirely arbitrary. It is here, then, that the method of Lagrange multipliers shows its strength: rather than reducing the number of variables in this way, we can introduce *additional* variables which allow us to work with a dynamically equivalent 'free' problem instead.

To do this, we take the variation of our constraint equation, (4.2):

$$\sum_{\rho=1}^{r} \frac{\partial f}{\partial p_\rho} \delta p_\rho = 0 \tag{4.4}$$

Because it is equal to zero, we can multiply the left hand side of this equation by an arbitrary factor, λ, and then add it to the left hand side of (4.1). In other words, solving our original variational problem is equivalent to solving the following variational problem:

$$\sum_{\rho=1}^{r} \frac{\partial F}{\partial p_\rho} \delta p_\rho + \lambda \sum_{\rho=1}^{r} \frac{\partial f}{\partial p_\rho} \delta p_\rho = 0 \tag{4.5}$$

So far we have merely added zero to the original problem. However, we can now use the hitherto undetermined value of λ to eliminate the coefficients of p_r by specifying that λ satisfies the following equation:

$$\frac{\partial F}{\partial p_r} \delta p_r + \lambda \frac{\partial f}{\partial p_r} \delta p_r = 0 \tag{4.6}$$

Fixing the value of λ in this way renders our problem independent of p_r. We can then proceed *as if* our original r variables are independent. More generally: given k constraint equations we can introduce k multipliers and fix their values so as to eliminate the coefficients of the final k variables. Again, we are then free to proceed *as if* the original r variables are independent. Thus, rather than using the constraint equations to solve for the $r-k$ quantities $p_1, ..., p_{r-k}$, we introduce additional factors and solve for the $r+k$ quantities $p_1, ..., p_r$ and $\lambda_1, ..., \lambda_k$. The advantage of proceeding in this way is that we can now apply the methods appropriate to an unconstrained problem to our modified function:

$$F^{\star} = F + \lambda_1 f_1 + ... + \lambda_k f_k \tag{4.7}$$

Let us witness this technique as applied to Hamilton's principle. As we saw in chapter 3, this principle requires the vanishing of a system's Lagrangian:

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{4.8}$$

Using Lagrange multipliers, if the system is subject to constraints then instead of solving (4.8) we solve:

$$\delta \int_{t_1}^{t_2} L \, dt + \delta \int_{t_1}^{t_2} (\lambda_1 \delta f_1 + ... + \lambda_k \delta f_k) \, dt = 0 \tag{4.9}$$

Importantly, we can regard this as applying Hamilton's principle directly to a modified Lagrangian, $L^\star = L + \lambda_1 f_1 + ... + \lambda_k f_k$:

$$\delta \int_{t_1}^{t_2} L^\star \, dt = 0 \tag{4.10}$$

We can regard $L^\star$ as describing a system that is *dynamically equivalent* to the original system. As before, the advantage of considering $L^\star$ instead of L is that we can regard the former as 'free': we can apply the techniques appropriate to an unconstrained problem.

For Hertz, Lagrange multipliers provide a uniform and simple way to express the equations of motion of *any* mechanical system. In the context of *Principles*, the problem of determining a system's equations of motion reduces to the problem of minimizing the curvature of the system's path through configuration space. Hertz thus derives necessary and sufficient conditions for the curvature of a system's path to take a minimum value (§§158,159) and then derives a general expression for a system's equations of motion (cf. §371).[25] Employing a generalized acceleration vector, f_ρ (§277), the equations of motion for a system with no connections is very simple:

$$mf_\rho = 0 \tag{4.11}$$

However, a system with connections has k 'equations of constraint':

$$\sum_{\rho=1}^{r} p_{\chi\rho} p'_\rho = 0 \tag{4.12}$$

As before, we could proceed by re-arranging these constraint equations so as to express some of the variables as dependent on the others, and then solve for the remaining $r - k$ independent variables that we are left with. Instead of doing this, Hertz introduces undetermined Lagrange multipliers, P_χ, and incorporates these into the system's equations of motions as follows:

$$mf_\rho + \sum_{\chi=1}^{k} p_{\chi\rho} P_\chi = 0 \tag{4.13}$$

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[25] For a more detailed discussion of Hertz's derivation of a system's equations of motion, see Appendix B.

Hertz can then solve for the $r + k$ variables p_ρ and P_χ while *treating them as independent*. The only difference for a 'system acted on by forces' (a coupled system) is that we simply apply this trick a second time: we introduce r additional factors P_ρ (where $P_\rho = 0$ for any uncoupled coordinates) and solve for a total of $2r + k$ variables:

$$mf_\rho + \sum_{\chi=1}^{k} p_{\chi\rho} P_\chi + P_\rho = 0 \tag{4.14}$$

In terms of the core impact on Hertz's formulation of mechanics, this is what the introduction of hidden masses amounts to: the hypothesis of hidden masses leads to a simple and uniform method for writing down the equations of motion for any mechanical system. On Hertz's picture theory, correctly modelling a system's motion over time is the ultimate achievement of a physical description, and this is precisely what is achieved by capturing a system's degrees of freedom in a dynamical model. Once one is freed from the task of seeking concrete mechanisms of hidden masses, the power of Hertz's approach becomes clear. In fact, it becomes a great advantage of Hertz's framework that phenomena can be modelled very *simply* by employing dynamical models in this way:

> In order to determine beforehand the course of the natural motion of a material system, it is sufficient to have a model of that system. The model may be much simpler than the system whose motion it represents. (§425)

Hertz's framework thus provides a uniform method for displaying the degrees of freedom of mechanical systems; a uniform method for displaying the essential content of ordinary mechanical descriptions. With this in view, Wittgenstein's reference to dynamical models at 4.04 becomes intelligible. The 'multiplicity' that is shared by the ordinary mechanical description, the Hertzian description, and the phenomenon itself is the number and type of the system's degrees of freedom.[26] This multiplicity is present at least implicitly, perhaps obscurely, in the ordinary mechanical description, but displayed perspicuously by a dynamical model. A system's degrees of freedom is thus the mechanical analogue for the multiplicity that is common between the colloquial sentence, fully analysed sentence, and depicted situation in the *Tractatus*.

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[26] Note that, here and elsewhere, I am intentionally trading on the ambiguity of the term 'system', which can be used to refer both to the symbolic representation and the real world phenomenon.

4.5 THE MULTIPLICITY OF A SENTENCE

Although it is possible to identify the multiplicity of a mechanical system concretely via its degrees of freedom, it is harder to identify the multiplicity of a sentence in a similarly concrete way.[27] However, we can make progress on this front by turning to the remarks following 4.04. At 4.041 Wittgenstein claims that the multiplicity of a sentence 'cannot itself be the subject of depiction', rather one 'cannot get away from it when depicting.'[28] Wittgenstein then elaborates on this remark through a discussion of the way in which certain variants of the generality notation would fail to be adequate because they *lack* the necessary multiplicity:

> 4.0411 If, for example, we wanted to express what we now write as '$\forall x(fx)$'[29] by putting an affix in front of 'fx'—for instance by writing '$Gen.fx$'—it would not be adequate: we should not know what was being generalised. If we wanted to signalize it with an affix '$_g$'—for instance by writing '$f(x_g)$'—that would not be adequate either: we should not know the scope of the generality-sign.
>
> If we were to try to do it by introducing a mark into the argument places—for instance by writing '$(G, G).F(G, G)$'—it would not be adequate: we should not be able to establish the identity of the variables. And so on.
>
> All these modes of signifying are inadequate because they lack the necessary mathematical multiplicity.

The problems that Wittgenstein identifies in these variant notations are reasonably straightforward. Taking the second variant as an example, if we attempted to notate generality using an affix—such as by replacing '$\forall x(fx)$' with '$f(x_g)$'—this would be inadequate because 'we should not know the scope of the generality-sign'. Consider the following two propositional functions:

1. fx
2. $fx \supset p$

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[27] Furthermore, in the context of the *Tractatus* we have no 'external' perspective—no perspective outside of logic and language from which to reflect on logic and language. This is a peculiar and central problem at the heart of the *Tractatus* that has no analogue in *Principles*. Here I just mention it in passing.

[28] It would distract from my main theme to turn to a discussion of why the multiplicity of a sentence is something that can be shown, and not said. For one proposal for how to interpret 4.041 in this regard however, see Kremer (1992).

[29] Note that I have substituted the more familiar notation, '$\forall x(fx)$', for the notation used in the *Tractatus*, '$(x).fx$'

Using the standard notation, we can form two sentences with the universal quantifier as follows:

1*. $(\forall x)(fx)$

2*. $(\forall x)(fx \supset p)$

From 1* we can form a third sentence:

3*. $(\forall x)(fx) \supset p$

Note that the scope of the quantifier tracks the difference between sentences 2* and 3*. However, using the variant notation our original sentences would be translated as follows:

1⋆. $f(x_g)$

2⋆. $f(x_g) \supset p$

We can now no longer form a third sentence in the way we did before: the attempt to do so simply reproduces $f(x_g) \supset p$. Hence, as Wittgenstein remarks, the variant notation fails to indicate the scope of the generality-sign. The other two variant notations face similar problems: we can't replace '$\forall x(fx)$' with '$Gen.fx$' because we need to be able to identify the bound variable, and we can't replace '$\forall x(fx)$' with '$(G,G).F(G,G)$' because we need to be able to distinguish different variables (x, y, z, etc.), particularly when one quantifier appears within the scope of another.

The three variants suggest that the multiplicity of the generality notation can be recognised in the scope of the generalization, the identification of bound variables, and the distinction of different variables. Given that range of examples, however, it seems that we might as well regard *any* significant feature of the notation as falling under the heading 'multiplicity'. In the case of an adequate generality notation, 4.0411 helps to specify what these significant features are. As Michael Kremer has argued, the three variants help us to see that *any* adequate generality notation must be able to identify bound variables, determine a quantifier's scope, and allow for one quantifier to occur within the scope of another (cf. Kremer (1992), pp. 411-412). The upshot seems to be this: as Wittgenstein is using the term, 'multiplicity' encompasses all the features of a notation that are necessary for it to do the work it purports to do.

Here, however, we should keep in mind Wittgenstein's distinction between the *essential* and merely *accidental* features of a sentence (or symbol):

> 3.34 A sentence possesses essential and accidental features.
>
> Accidental features are those that result from the particular way in which the propositional sign is produced. Essential features are those without which the sentence could not express its sense.
>
> 3.341 So what is essential in a sentence is what all sentences that can express the same sense have in common.
>
> And similarly, in general, what is essential in a symbol is what all symbols that can serve the same purpose have in common.

Equivalent (and hence equally adequate) notations have different features, and some of the features that are needed when using one notation are not needed in another.[30] But such features are *accidental*—they result from 'the particular way the propositional signs are produced'. What is essential, by contrast, is what all adequate notations *have in common*. This makes the identification of such features a difficult task, and on this point Wittgenstein's reference to Hertz is particularly helpful. Within the limited scope of *Principles*, what all adequate notations have in common are the resources to represent the number and type of a mechanical system's degrees of freedom. Within the much broader scope of the *Tractatus*, however, what all adequate notations have in common are the resources to represent any situation *at all*. In aiming to identify the essential features of sentences *tout court*, our task becomes, in the words of 4.5: 'to give the sentences of *any* sign language *whatsoever* in such a way that every possible sense can be expressed by a symbol satisfying the description, and every symbol satisfying the description can give a sense'. Here, then, we have arrived at the central Tractarian notion of the *general sentence-form*.

The general sentence-form represents an arbitrary truth-function of an arbitrary number of independent elementary sentences, employing iterated applications of joint-negation to capture the familiar logical operations (conjunction, disjunction, etc.).[31] According to the *Tractatus*, every sentence with sense can be written as a truth-function of elementary

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[30] Compare, for example, the use of parentheses in Russellian notation with the absence of parentheses in Polish notation.

[31] The extent of the logical resources that the general sentence-form makes available should not be underestimated. For discussions of how far these resources extend, see for example Floyd (2002) and Ricketts (2014).

sentences; thus every sentence with sense is an instance of the general sentence-form. The resources for analysis that the general sentence-form makes available include truth-operations applied to elementary sentences and to sentences that are themselves truth-functions of elementary sentences. At the level of independent elementary sentences the resources for analysis also include the interlocking forms of elementary sentences and forms of names of objects.[32]

The construction of sentences from elementary sentences can be illustrated by returning to one of the variant notations canvassed in 4.0411. In particular, note that replacing '$\forall x(fx)$' with '$Gen.(x)$' runs into trouble with a sentence such as the following:

$$\forall(x)(\forall(y)Rxy \supset \forall(y)Ryx)$$

It seems that the variant notation simply lacks sufficient resources to reproduce this sentence.[33] But the resources that the general sentence-form has available fare better. To see how to construct this sentence from elementary sentences, let us first treat Rab as an elementary sentence.[34] From here, our next step is to replace the name 'b' with a variable 'y' to form the propositional function Ray. The familiar quantifier notation provides a shorthand for the logical sum (conjunction) of the values of this function, $\forall(y)Ray$, and in a similar way we can form the sentence $\forall(y)Rya$ from the elementary sentence Rba. From these two sentences we now form the conditional $\forall(y)Ray \supset \forall(y)Rya$.[35] Note that it is important that the same name, 'a', occurs in the antecedent and the consequent (derived from the fact that both were formed by starting with elementary sentences which contained 'a'). At this point, we can replace 'a' with the variable 'x' to arrive at a new propositional function: $\forall(y)Rxy \supset \forall(y)Ryx$. Forming the logical sum of the values of this function, we arrive back

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[32] The form of a particular object is its possibilities of being related to other objects in states of affairs (2.0141), and the form of a particular state of affairs is the possibility that objects be related in that way (2.031-2.033). The form of a name then mirrors the form of the object it names, and the form of an elementary sentence mirrors the form of the state of affairs it asserts to obtain.

[33] Here I follow Kremer (1992) p. 415.

[34] It is a commonplace in the literature to treat a sentence such as 'Rab' as a candidate elementary sentence while putting to one side the various controversial issues concerning what a genuine example of an elementary sentence might be. Though nothing here turns on such issues, I follow Ricketts (2014) in regarding the asymmetric relation exemplified by Rab as in fact constructed from more basic elementary sentences—cf. Ricketts (2014) pp. 273-274.

[35] To see how to construct a conditional sentence using successive applications of joint-negation, see Ricketts (2013) p. 127.

at our original sentence: $\forall(x)(\forall(y)Rxy \supset \forall(y)Ryx)$.

The discussion of the multiplicity of generality notation in 4.0411 provides a central example of the essential features of sentences, in particular the essential features of sentences capable of expressing generality. Following from this, and taking a cue from Wittgenstein's reference to Hertz, I propose that multiplicity in the *Tractatus* encompasses the essential features of *any* sentence. These are precisely the features which the complete analysis of a sentence makes explicit. At the same time, such features must already be present, if tacit, in the colloquial sentence. (If they were not, the colloquial sentence would not be able to express its sense.) This has a clear parallel in *Principles*: a mechanical system's degrees of freedom must be tacit in an ordinary mechanical description of that system, otherwise it would not *be* a description of that system. A dynamical model simply stands to make a system's degrees of freedom explicit.

In Hertz's context, the notion of degrees of freedom is the relevant *multiplicity*; the number and type of a system's degrees of freedom constitute the essential features of that system. This is what is common in the ordinary description, the fully analysed description, and the target phenomenon. What corresponds to degrees of freedom in the *Tractatus*? Writing a sentence as a truth-function of elementary sentences shows which truth-possibilities of elementary sentences the sentence agrees and disagrees with (4.4). Implication relations between colloquial sentences are then analysed as follows: if the truth-possibilities of elementary sentences with which a given sentence agrees include within them the truth-possibilities with which another sentence agrees, then the first sentence follows from (is implied by) the second. Wittgenstein describes this case by saying that the sense of the second sentence is *contained* in the sense of the first (5.122). This is made vivid by truth-table notation: the fact that the truth of 'p' is implied by the truth of '$p.q$' is shown by the fact that the truth-possibilities with which 'p' expresses agreement (the first and third rows of the truth-table, below) include the truth-possibility with which '$p.q$' expresses agreement (the first row):

p	q	p	$p.q$
T	T	T	T
F	T	F	F
T	F	T	F
F	F	F	F

Other logical relationships can be accommodated in a similar fashion—if the truth-possibilities with which one sentence expresses agreement are at the same time the truth-possibilities with which a second sentence expresses disagreement, then the truth of either sentence implies the falsity of the other (their senses *exclude* each other), and so on. Tractarian analysis thus employs the logical resources made available by the general sentence-form to capture the logical relationships among colloquial sentences in terms of sense inclusion and exclusion; in terms of agreement and disagreement with truth-possibilities of elementary sentences. In this way, the complete analysis of colloquial sentences makes their logical relationships explicit.

According to the *Tractatus*, such logical relationships can also be recognised as holding between the situations that sentences depict. The opening of the *Tractatus* famously declares: 'The world is the totality of facts, not things', and 'The facts in logical space are the world' (1.1, 1.13). On this view, situations—just like sentences—occupy locations in logical space. Thus, just as sentences carry with them their logical relationships with other sentences, facts carry with them their logical relationships with other facts. Here we have found a central commonality between the colloquial sentence, the fully analysed sentence, and the depicted situation: all three are, so to speak, linked to the same node in logical space. Here too we find the significance of the claim that sentences—and pictures more generally—represent a possibility of the existence and non-existence of states of affairs (2.11, 2.201, 4.1). If a sentence is true then the situation it depicts obtains; in that case a number of elementary sentences will also be true, and so a number of states of affairs will also obtain. Here is a sense, then, in which sentence and situation have exactly as many distinguishable parts as one another—a sense in which the colloquial sentence, fully analysed sentence, and depicted situation have the same multiplicity.

The comparison of Tractarian analysis with Hertzian analysis—and the understanding of the notion of multiplicity that this comparison makes available—substantiates a suggestive metaphor for the relationship between the colloquial sentence and the fully analysed sentence articulated by Cora Diamond:

> On the one hand there is the fully analysed sentence, which would lay out clearly in front of us what function of what expressions a sentence really is. To get what is going to be on the other hand we have to think of lifting up an ordinary sentence, and noticing, attached to it, like little wires, all the sentences which entail that it is true or that it is not. The ordinary sentence, together with *all* its little wires, is the same sentence as the fully analysed one. So we can understand the ordinary sentence even though we do not know how to carry out its full analysis. (Diamond, 1988, p. 19)

To make a parallel remark concerning *Principles*, we can say that the ordinary mechanical description is indeed the same description as the corresponding (fully analysed) dynamical model. And, of course, we can note that physicists can (and still do) understand ordinary mechanical descriptions in perfect ignorance of dynamical models. The achievement of Hertzian analysis is to display the multiplicity of ordinary mechanical descriptions, thus making perspicuous the essential features of those ordinary descriptions. Similarly, the achievement of Tractarian analysis is to display the multiplicity of ordinary colloquial sentences, thus making perspicuous the essential features of those sentences. Just as in the mechanical case, however, there is of course no need to wait upon such analysis in order to get on with using ordinary sentences.

4.6 MODELS AND MULTIPLICITIES

A central feature of a logically-oriented interpretation of the *Tractatus*—a feature which can be illustrated through the comparison with *Principles*—can be glossed as follows: Wittgenstein's motivation to introduce names of simple objects is to provide a uniform method for displaying the logical relations among colloquial sentences in terms of sense inclusion and exclusion, so that all sentences with sense can be recognised as instances of the general sentence-form. The parallel feature of *Principles* is the following: Hertz's motivation to introduce hidden masses and *Massenteilchen* is to provide a uniform method for displaying the degrees of freedom of mechanical systems in dynamical models, so that all mechanical phenomena can be recognised as falling under the fundamental law.

'Multiplicity' encompasses all the *essential* features of a representation—all the features that allow it to do the representational work it purports to do. This can be spelt out in the context of *Principles* as follows. On Hertz's austere account of the essential content of classical mechanics, the only information that a mechanical description conveys is what is contained in a dynamical model.[36] In the context of *Principles*, multiplicity encompasses all the essential features of descriptions in classical mechanics. In the context of the *Tractatus*, however, multiplicity encompasses all the essential features of descriptions *tout court*. In both cases, the essential features are typically tacit in ordinary or colloquial descriptions, but displayed explicitly when those descriptions are written in their fully analysed form.

On an ontolgoically-oriented interpretation, simple objects imbue the names in elementary sentences with meaning, and elementary sentences ground the meaningfulness of colloquial sentences. It is thus the simple objects occurring in states of affairs, independently of language and thought, which gives significance to the forms of names and the forms of elementary sentences. On a logically-oriented interpretation, by contrast, analysis uncovers whatever forms of elementary sentences and forms of names are needed in order to capture the manifest logical relationships among colloquial sentences. On this view, it is the sense of colloquial sentences which accounts for the sense of elementary sentences (and hence the meaningfulness of the names of simple objects), not the other way around. Indeed, elementary sentences and the names of simple objects do not have any significance apart from the analysis of colloquial sentences.[37]

This ontologically-oriented and logically-oriented distinction can also be used to characterize contrasting interpretations of *Principles*. On an ontologically-oriented interpretation, Hertz's *Massenteilchen* would be an unfamiliar kind of fundamental particle, and the hypothesis of hidden masses would be a bold ontological gambit. On such an interpretation,

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[36] As Hertz articulates the point at §427: '[We can], in fact, have no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone—that the one set of systems are models of the other.'

[37] Cf. Ricketts (2014) p. 275: 'This view of Tractarian objects explains why Wittgenstein avoids offering any informative characterization of what kinds of objects there are or might be. Given the holism of Tractarian analysis, there is no such characterization in advance of analysis. There is only the rewriting of sentences to make explicit what truth-functions of which elementary sentences they are.'

if it turns out that such entities don't actually exist, then so much the worse for Hertz's grand reformulation of mechanics. On a logically-oriented interpretation however—the kind of interpretation defended in this thesis—the entire motivation to talk in terms of material points and *Massenteilchen* is to display the essential content of mechanical descriptions in the form of dynamical models. Given a particular mechanical phenomenon, we can analyse it into a connected system of material points, introducing hidden masses to correctly capture its degrees of freedom as needed. The relative masses of these material points (both hidden and visible) is what then determines the relative numbers of *Massenteilchen* occupying those locations at those times. On a logically-oriented view, the material points and *Massenteilchen* do not have significance apart from the descriptions in which they occur—they are introduced to allow for a uniform analysis of mechanical phenomena, not as a speculative ontological posit.

As I have been concerned to show earlier in this thesis, there are strong and mutually supporting lines of evidence in favour of this kind of logically-oriented interpretation of Hertz's project. This evidence emerges at three important places in particular: in Hertz's original motivation to introduce *Massenteilchen*, in the role of the hypothesis of hidden masses, and in the overarching significance of Hertz's picture theory of representation.

Hertz's primary motivation to introduce *Massenteilchen* is to derive the appropriate equation for the displacement of a system (i.e. the appropriate metric structure for configuration space).[38] If *Massenteilchen* are interpreted as a strange kind of fundamental particle, this motivation appears wholly inadequate. On the other hand, if *Massenteilchen* are interpreted as an analytical device that allows for a perspicuous description of mechanical systems, then such a motivation is just what one might expect. Regarding Hertz's hidden masses, we have seen that their fundamental role is to allow for a uniform analysis of mechanical phenomena. This role is evident both in Hertz's strategy for accommodating forces by introducing additional cyclic coordinates (i.e. appealing to connected sub-systems of hidden masses) and also in Hertz's justification of the adequacy of dynamical models as *complete* descriptions of mechanical systems. Hertz makes clear that the hypothesis of hidden masses rules out knowledge of fundamental ontological structure, and that what we can actually learn about

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[38] See above, chapter 2 section 2.3.2

a mechanical system is precisely the information conveyed by a dynamical model.[39] This brings us to the picture theory of representation that frames *Principles*. On Hertz's austere account of representation, the ultimate achievement of a mechanical description is to correctly model the target system's motion over time, hence to capture that system's degrees of freedom in a dynamical model. It is no accident, then, that it is precisely in the discussion of dynamical models that Hertz refers back to the picture theory of his introduction.[40]

It is also worth recalling that Hertz makes clear that he is engaged in a task of *clarification*; that his aim in *Principles* is to distill the 'essential content' of classical mechanics from its customary representation.[41] In this vein, it is helpful to recall Hertz's earlier theoretical achievement of distilling the essential content of the theory of electromagnetism from Maxwell's sprawling *Treatise*. On a logically-oriented interpretation of *Principles*, the Hertzian slogan—'Maxwell's theory is Maxwell's system of equations'—takes on a special significance, showing that Hertz's austere conception of the essential content of a scientific theory significantly predated his work on classical mechanics. This is what developed into the picture theory of *Principles*.[42] Given Hertz's account of representation, the task of clarifying classical mechanics requires explicitly abstracting away from the imaginative devices and constructable models that might be employed in ordinary treatments of mechanical problems.[43] Hertz's succinct statement of his ambitions at the end of his preface is thus entirely consonant with a logically-oriented interpretation of his work:

> In the details I have not brought forward anything that is new and which could not be found in many books. What I hope is new, and to which alone I attach value, is the arrangement

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[39] Cf. §427: 'it is impossible to carry our knowledge of the connections of natural systems further than is involved in specifying models of the actual systems. We can then, in fact, have no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone—that the one set of systems are models of the other.'

[40] Cf. §428: 'The relation of a dynamical model to the system of which it is regarded as the model, is precisely the same as the relation of the pictures which our mind forms of things to the things themselves'.

[41] Recall Hertz's letter to Cohn in November of 1891: 'I would like to put something straight and arrange the concepts in such a way that one can see more clearly what are the definitions and what are the facts of experience, such as, for example, concepts of force and inertia. I am already convinced that it is possible to obtain great simplifications here.' (cf. Lützen (2005) p. 74)

[42] Recall that in order to clarify electromagnetism, Hertz wrote that the theory 'should be so constructed as to allow its logical foundations to be easily recognised; all unessential ideas should be removed from it, and the relations of the essential ideas should be reduced to their simplest form' (Hertz, 1893, p. 195).

[43] Here we might recall one final famous Hertzian remark, from *Electric Waves*: 'scientific accuracy requires of us that we should in no wise confuse the simple and homely figure, as it is presented to us by nature, with the gay garment which we use to clothe it' (Hertz, 1893, p. 28).

> and presentation of the whole, and thus the logical, or, if one wants, the philosophical aspect of the matter. My work has accomplished its objective or failed insofar as it has gained something in this direction or not. (Hertz, 1899, xxiv)

Both Hertz and Wittgenstein introduce unfamiliar entities, whether in the form of *Massenteilchen* and hidden masses or in the form of simple objects. An overarching concern of this thesis has been the seemingly speculative nature of such a procedure. In Hertz's case, the uniformity and simplicity that can be attained in the description of mechanical systems (and in particular, the fact that all systems can be seen to fall under the fundamental law) serves as the major motivation for approaching mechanical problems in a novel and unfamiliar way:

> ...we are bound to answer the question how a new, unusual, and comprehensive mode of expression justifies itself, and what advantages we expect from using it. In answering this question we specify as the first advantage that it enables us to render the most general and comprehensive statements with great simplicity and brevity. In fact, propositions relating to whole systems do not require more words or more ideas than are usually employed in referring to a single point. (Hertz, 1899, pp. 30-31)

In Wittgenstein's case, the uniformity and simplicity attained in accommodating logical relationships among colloquial sentences in terms of sense containment and exclusion (and in particular, the fact that all sentences can be seen to be instances of the general sentence-form) serves as a parallel motivation for the introduction of simple objects. This is, of course, tied to a logically-oriented interpretation of the *Tractatus*. As I hope to have brought out, such an interpretation rescues the *Tractatus* from seeming disappointingly speculative, or even particularly implausible. More specifically, I have argued that Wittgenstein's reference to Hertz's dynamical models at 4.04 should be recognised as a critical piece of textual evidence in favour of a logically-oriented interpretation of the *Tractatus*. I have also argued that Hertz provides a distinctive conception of the termination of logically-oriented analysis; a conception on which analysis does *not* lead to a complete specification of ontological constitution.

There is one further aspect of the comparison between *Principles* and the *Tractatus* that should be noted here. Recall that one of Hertz's overarching goals is to alleviate confusions that trace back to the conflicting demands on the term 'force'. Hertz disentangles these demands by providing a notational framework in which all forces can be treated on the model of constraint forces, but it is important to note that Hertz has not thereby answered the question: *what is the nature of force?* In the famous passage from Hertz's introduction which so resonated with Wittgenstein, Hertz claims that 'the answer which we want is not really an answer to this question'. Here is how that passage concludes:

> When these painful contradictions are removed, the question as to the nature of force will not have been answered; but our minds, no longer vexed, will cease to ask illegitimate questions. (Hertz, 1899, p. 8)

Hertz's suggestion, then, is that once a certain perspective is achieved, certain confused questions will no longer seem pressing. It would be uncontentious to claim that this idea played an important role in Wittgenstein's later conception of the ambitions of philosophy. What is less widely appreciated, however, is the extent of this Hertzian influence already in the *Tractatus*.[44] The central concern of this final chapter has been to interpret Wittgenstein's reference to Hertz's dynamical models at 4.04 and thereby uncover the parallels between Wittgenstein's analysis of sentences and Hertz's analysis of mechanical systems. All this, however, only bulwarks the claim that Wittgenstein's conception of philosophical problems was deeply influenced by Hertz already in the *Tractatus*: Wittgenstein took inspiration both from the way in which Hertz provided an analytical framework for classical mechanics *and* from Hertz's conception of what providing such an analytical framework achieved.

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[44] Michael Kremer is one of the relatively few commentators who recognises Hertz's influence here: 'It is true that all of the passages [where Wittgenstein refers to Hertz's introduction] are drawn from his middle to late works. But Wittgenstein encountered Hertz's ideas even before he became a student of Russell's, and it is my contention that the conception of philosophical problems and their solution that he found in Hertz was crucial to his approach to philosophy from the beginning.' (Kremer, 2012, p. 16)

APPENDIX A

MASSENTEILCHEN AND THE GEOMETRY OF CONFIGURATION SPACE

In *Principles*, the geometry of configuration space is specially adapted to the mechanical system under consideration (see above, chapter 2 section 2.3.2). More specifically, the determination of distances and angles—the *metric structure* of configuration space—is linked to the particular mass-distribution of the system. In order to derive this metric structure, Hertz introduces the notion of *Massenteilchen*.

In ordinary Euclidean space, the distance between two points is determined by the familiar Pythagorean metric:

$$ds^2 = dx^2 + dy^2 + dz^2$$

To transpose the structure of ordinary Euclidean geometry into configuration space, we would simply have to generalise the Pythagorean metric to a higher number of dimensions. A system with n material points has a configuration space with $3n$ dimensions: $x_1, x_2, x_3, ..., x_{3n-2}, x_{3n-1}, x_{3n}$. Hence a generalised Pythagorean metric has the form:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + ... + dx_{3n-2}^2 + dx_{3n-1}^2 + dx_{3n}^2$$

Or more compactly:

$$ds^2 = \sum_{i=1}^{3n} dx_i^2$$

However, this Euclidean structure is insufficient to do the work that Hertz requires. Thus Hertz derives a more exotic metric structure, beginning by first defining the 'magnitude of the displacement of a system' as follows:

> The magnitude of the displacement of a system is the quadratic mean value of the magnitudes of the displacements of all its *Massenteilchen*. (Hertz, 1899, §29)

Note here the reference to *Massenteilchen*. If this definition had referred to material points, this would have resulted in configuration space having a Pythagorean metric. However, calculating the displacements of the *Massenteilchen* instead of the material points *weights* the expression for the magnitude of the displacement of a system, so that the more massive points contribute more to the displacement.[1] Using this definition, Hertz derives a metric for configuration space of the following form (cf. Hertz (1899) §55):

$$ds^2 = \frac{1}{m} \sum_{i=1}^{3n} m_i dx_i^2$$

Here, m is the total mass of the system (i.e. the sum of the masses of the material points) and the m_i are defined so that the mass of the μ-th material point is proportional to $m_{3\mu-2} + m_{3\mu-1} + m_{3\mu}$.

If instead of using $3n$ Cartesian coordinates $(x_1, ..., x_{3n})$, we use r general coordinates $(p_1, ..., p_r)$,[2] then the metric structure of configuration space takes the following form (Hertz, 1899, §57):

$$ds^2 = \sum_{\rho=1}^{r} \sum_{\sigma=1}^{r} a_{\rho\sigma} dp_\rho dp_\sigma$$

Here, the $a_{\rho\sigma}$ are defined as follows:

$$a_{\rho\sigma} = \frac{1}{m} \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_\rho} \frac{\partial x_\nu}{\partial p_\sigma}$$

It is worth emphasising the importance of this metric structure. Hertz is only in a position to describe all mechanical systems with a single fundamental law because the geometrical

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[1] The demonstration that this was indeed the primary motivation for Hertz to introduce *Massenteilchen* is due to Lützen. For his detailed discussion of the development of the idea of *Massenteilchen* in the early drafts of *Principles*, see Lützen (2005) pp. 146-158.

[2] Note that r is the number of the degrees of freedom of the system.

structure of configuration space itself incorporates the necessary information regarding the spatial distribution of the masses making up a system. The key result is that the total kinetic energy of the system can be represented by the kinetic energy of a *single point* in configuration space.[3] It is in this way that the mechanics of a single point is carried over to the mechanics of an arbitrary system:

> In this space one point is sufficient to represent the mechanical system, and hence we carry over the mechanics of a free particle to any mechanical system if we place that particle in a space of the proper number of dimensions and proper geometry. (Lanczos, 1962, p. 22)

In the context of this thesis, the following should be emphasised. First, the *Massenteilchen* are the true simple objects of Hertz's framework. In *Principles*, it is the relative number of *Massenteilchen* that defines the notion of mass.[4] Thus the *parts* (so to speak) of a material point—the *Massenteilchen* that occupy that spatio-temporal location—have a clear (indeed, a paramount) logical role. The *Massenteilchen* themselves have no parts. However, this point can be (and has been) misconstrued if *Massenteilchen* are regarded as a strange kind of fundamental particle. But to interpret them as such would make Hertz's framework both speculative and implausible.[5] Just as with Hertz's notion of a material point, *Massenteilchen* are defined entirely by their logical role—it is in the construction of configuration space, and thus the construction of dynamical models, that they find their significance.

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[3] See above, chapter 2 section 2.3.2.

[4] Cf. Hertz (1899) §4: 'The number of *Massenteilchen* in any space, compared with the number of *Massenteilchen* in some chosen space at a fixed time, is called the mass contained in the first space.'

[5] Hence, as we saw in chapter 2, commentators who have regarded *Principles* as closely connected with the search for an ether mechanism have been severely disappointed.

APPENDIX B

HERTZ'S EQUATIONS OF MOTION

When using rectangular coordinates, the condition that a system's path has a minimum curvature is equivalent to the condition that the following quantity takes a minimum value (cf. §§106, 155):

$$\frac{1}{2}\sum_{\nu=1}^{3n}\frac{m_\nu}{m}(x''_\nu)^2 \tag{B.1}$$

Here, m is the total mass of the system and x''_ν are the second partial differentials of the coordinates with respect to path length (cf. §100). When using r general coordinates, p_ρ, the expression for the curvature of the system's path takes a significantly more complex form (cf.§108):

$$\sum_{\rho=1}^{r}\sum_{\sigma=1}^{r}\left(a_{\rho\sigma}p''_\rho p''_\sigma+\sum_{\tau=1}^{r}\left(2\frac{\partial a_{\rho\sigma}}{\partial p_\tau}-\frac{\partial a_{\rho\tau}}{\partial p_\sigma}\right)p'_\rho p'_\tau p'_\sigma+\sum_{\lambda=1}^{r}\sum_{\mu=1}^{r}a_{\rho\sigma\lambda\mu}p'_\rho p'_\sigma p'_\lambda p'_\mu\right) \tag{B.2}$$

Here, $a_{\rho\sigma}$ and $a_{\rho\sigma\lambda\mu}$ are notational short-hands, defined as follows:

$$a_{\rho\sigma}=\frac{1}{m}\sum_{\nu=1}^{3n}m_\nu\frac{\partial x_\nu}{\partial p_\rho}\frac{\partial x_\nu}{\partial p_\sigma} \tag{B.3}$$

$$a_{\rho\sigma\lambda\mu}=\sum_{\nu=1}^{3n}m_\nu\frac{\partial a_{\nu\sigma}}{\partial p_\lambda}\frac{\partial a_{\nu\rho}}{\partial p_\mu} \tag{B.4}$$

Despite their differences, (B.1) and (B.2) are simply two expressions for the curvature of a path through configuration space; i.e. the quantity that we wish to minimize.

As we have seen, if $r < 3n$ then the general coordinates themselves incorporate at least some of the system's connections.[1] Any further connections can be represented by k 'equations of condition' of the following form (§130):[2]

$$\sum_{\rho=1}^{r} p_{\chi\rho} p'_{\rho} = 0 \tag{B.5}$$

Each one of these equations acts as a constraint equation, exactly analogous to $f(p_1, ..., p_r) = 0$. Thus Hertz multiplies the k equations of condition (B.5) by Lagrange multipliers Π_χ and then adds the left hand side of these equations to the partial differentials of (B.2). In this way, Hertz derives necessary and sufficient conditions for a path to be a straightest path (§§158-159):

$$\sum_{\sigma=1}^{r} a_{\rho\sigma} p''_{\sigma} + \sum_{\sigma=1}^{r}\sum_{\tau=1}^{r} \left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho} p'_\sigma p'_\tau \right) + \sum_{\chi=1}^{k} p_{\chi\rho} \Pi_\chi = 0 \tag{B.6}$$

These can be used to write down the differential equations of motion for a system, i.e. a set of differential equations—with time as the independent variable and the coordinates of the system as the dependent variables—which are sufficient to determine the motion of the system (when supplemented by initial conditions). Introducing P_χ as an abbreviation for $mv^2\, \Pi_\chi$, the differential equations of motion that Hertz derives are the following (§371):

$$m \left(\sum_{\sigma=1}^{r} a_{\rho\sigma} \ddot{p}_\sigma + \sum_{\sigma=1}^{r}\sum_{\tau=1}^{r} \frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho} \dot{p}_\sigma \dot{p}_\tau \right) + \sum_{\chi=1}^{k} p_{\chi\rho} P_\chi = 0 \tag{B.7}$$

We can employ the acceleration vector to write these equations in a much more condensed form. This is because the component of a system's acceleration in the direction of each

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[1] If the general coordinates incorporate *all* the system's connections, then r is equal to the system's degrees of freedom; if the general coordinates incorporate a proper subset of the system's connections, then the number of the system's degrees of freedom is less than r. See above, chapter 3 section 3.5.

[2] Note that the number of degrees of freedom of a system is independent of the choice of coordinates (cf. §136).

coordinate is (§277):

$$f_\rho = \sum_{\sigma=1}^{r} a_{\rho\sigma}\ddot{p}_\sigma + \sum_{\sigma=1}^{r}\sum_{\tau=1}^{r}\left(\frac{\partial a_{\rho\sigma}}{\partial p_\tau} - \frac{1}{2}\frac{\partial a_{\sigma\tau}}{\partial p_\rho}\right)\dot{p}_\sigma\dot{p}_\tau \tag{B.8}$$

Thus a system's equations of motion can be written very compactly as follows (§372):

$$mf_\rho + \sum_{\chi=1}^{k} p_{\chi\rho}P_\chi = 0 \tag{B.9}$$

As we have seen, the only difference for a *coupled* (unfree) system is that we need to introduce an additional factor, P_ρ:

$$mf_\rho + \sum_{\chi=1}^{k} p_{\chi\rho}P_\chi + P_\rho = 0 \tag{B.10}$$

Setting P_ρ to zero for the uncoupled coordinates, we then have a general expression for the equations of motion for any mechanical system. *This* is the overarching motivation for Hertz's introduction of hidden masses.

BIBLIOGRAPHY

Davis Baird, RI Hughes, and Alfred Nordmann. *Heinrich Hertz: Classical physicist, modern philosopher*, volume 198. Springer, 1998.

Fabio Bevilacqua. Helmholtz's "Ueber die Erhaltung der Kraft". In *Hermann von Helmholtz and the Foundations of Nineteenth-Century Science*. Univ of California Press, 1993.

Max Black. *A companion to Wittgenstein's Tractatus*. Cornell University Press, 1964.

Ludwig Boltzmann. *Theoretical physics and philosophical problems: selected writings*. Reidel, 1974.

Ernst Cassirer. *The problem of knowledge: Philosophy, science, and history since Hegel*. Yale University Press, 1950.

Salvo D'Agostino. Hertz's researches and their place in nineteenth century theoretical physics. *Centaurus*, 36(1):46–77, 1993.

Cora Diamond. Throwing away the ladder. *Philosophy*, 63(243):5–27, 1988.

Juliet Floyd. Number and ascriptions of number in Wittgenstein's Tractatus. In *From Frege to Wittgenstein: Perspectives on Early Analytic Philosophy*, chapter 13, pages 308–352. Oxford University Press, 2002.

Gottlob Frege. *Posthumous Writings*. Basil Blackwell, Oxford, 1979.

Gottlob Frege. *Philosophical and mathematical correspondence*. University of Chicago Press, 1982.

Warren Goldfarb. Wittgenstein's understanding of Frege. In *From Frege to Wittgenstein: Perspectives on early analytic philosophy*, chapter 8, pages 185–200. Oxford University Press, 2002.

Warren Goldfarb. Das Überwinden. In *Beyond the Tractatus wars: the new Wittgenstein debate*, chapter 1, pages 6–21. Routledge, 2011.

James Griffin. *Wittgenstein's logical atomism*. Oxford, Clarendon Press, 1964.

PMS Hacker. *Insight and illusion: Themes in the philosophy of Wittgenstein.* Oxford, Clarendon Press, 1986.

Hermann von Helmholtz. On the conservation of force; a physical memoir. *Scientific memoires, selected from the transactions of foreign academies of science, and from foreign journals*, pages 114–162, 1853.

Heinrich Hertz. *Electric waves.* Macmillan and Co., 1893.

Heinrich Hertz. *Miscellaneous papers, translated from first German edition (1895).* Macmillan and Co., 1896.

Heinrich Hertz. *The principles of mechanics presented in a new form.* Macmillan and Co., 1899.

Heinrich Hertz. *Memoirs, letters, diaries (L. Brinner, M. Hertz, C. Susskind trs. Originally arranged by J. Hertz. prepared by M. Hertz and C. Susskind).* San Francisco Press, San Francisco, CA, 1977.

Heinrich Hertz. *Die Constitution der Materie: eine Vorlesung über die Grundlagen der Physik aus dem Jahre 1884.* Springer-Verlag, 2013.

Heinrich Hertz and Joseph Francis Mulligan. *Heinrich Rudolf Hertz (1857-1894): A Collection of Articles and Addresses.* Garland Pub., 1994.

Bruce Hunt. *The Maxwellians.* Cornell University Press, 1991.

David Hyder. Kantian metaphysics and Hertzian mechanics. *Vienna Circle Institute Yearbook (10): The Vienna Circle and Logical Empiricism*, 10:35–48, 2002.

Hide Ishiguro. Use and reference of names. *Studies in the Philosophy of Wittgenstein*, 20, 1969.

Max Jammer. *Concepts of force: a study in the foundations of dynamics.* Courier Corporation, 1999.

Allan Janik. *Wittgenstein's Vienna revisited.* Transaction Publishers, 2000.

Peter C Kjærgaard. Hertz and Wittgenstein's philosophy of science. *Journal for general philosophy of science*, 33(1):121–149, 2002.

Michael Kremer. The multiplicity of general propositions. *Nous*, 26(4):409–426, 1992.

Michael Kremer. Contextualism and holism in the early Wittgenstein: From Prototractatus to Tractatus. *Philosophical Topics*, 25(2):87–120, 1997.

Michael Kremer. Russell's merit. In Zalabardo, José L, editor, *Wittgenstein's early philosophy.* Oxford University Press, 2012.

Cornelius Lanczos. *The variational principles of mechanics.* University of Toronto Press, 1962.

Jean Leroux. “Picture theories” as forerunners of the semantic approach to scientific theories. *International Studies in the Philosophy of Science*, 15(2):189–197, 2001.

Jesper Lützen. *Mechanistic images in geometric form: Heinrich Hertz's ‘Principles of Mechanics’.* OUP Oxford, 2005.

Ernst Mach. *The science of mechanics: A critical and historical accounts of its development.* The Open Court Publishing Company, 6th edition, 1960.

Ulrich Majer. Heinrich Hertz's picture-conception of theories: Its elaboration by Hilbert, Weyl, and Ramsey. In *Heinrich Hertz: Classical physicist, Modern Philosopher*, pages 225–242. Springer, 1998.

Brian McGuinness. *Approaches to Wittgenstein.* Routledge, 2002a.

Brian McGuinness. The supposed realism of the Tractatus. In *Approaches to Wittgenstein*, pages 82–94. Routledge, London, 2002b.

Brian McGuinness. Language and reality. In *Approaches to Wittgenstein*, pages 95–102. Routledge, London, 2002c.

Joseph F Mulligan. The reception of Heinrich Hertz's Principles of Mechanics by his contemporaries. In *Heinrich Hertz: Classical Physicist, Modern Philosopher*, pages 173–181. Springer, 1998.

Joseph F Mulligan. The aether and Heinrich Hertz's The Principles of Mechanics Presented in a New Form. *Physics in Perspective*, 3(2):136–164, 2001.

Isaac Newton. *The mathematical principles of natural philosophy*, volume 1. London, 1729.

Alfred Nordmann. “Everything could be different”: The Principles of Mechanics and the limits of physics. In *Heinrich Hertz: Classical physicist, modern philosopher*, pages 155–171. Springer, 1998.

David Pears. *The false prison: A study of the development of Wittgenstein's philosophy*, volume 1. Oxford, Clarendon Press, 1987.

Karl Pearson. *The grammar of science.* Adam and Charles Black, 2nd edition, 1900.

John Preston. Mach and Hertz's mechanics. *Studies in History and Philosophy of Science Part A*, 39(1):91–101, 2008a.

John Preston. Hertz, Wittgenstein and philosophical method. *Philosophical Investigations*, 31(1):48–67, 2008b.

Rush Rhees. *Discussions of Wittgenstein.* Schocken Books Inc, New York, 1970.

Thomas G Ricketts. Objectivity and objecthood: Frege's metaphysics of judgment. In *Frege synthesized*, pages 65–95. Springer, 1986.

Thomas G Ricketts. Pictures, logic, and the limits of sense in Wittgenstein's Tractatus. In Hans Sluga and David G Stern, editors, *The Cambridge Companion to Wittgenstein*. Cambridge University Press, 1996.

Thomas G Ricketts. Logical segmentation and generality in Wittgenstein's Tractatus. In Peter Sullivan and Michael Potter, editors, *Wittgenstein's Tractatus: history and interpretation*. OUP Oxford, 2013.

Thomas G Ricketts. Analysis, independence, simplicity, and the general sentence-form. *Philosophical Topics*, 42(2):263–288, 2014.

Bertrand Russell. Knowledge by acquaintance and knowledge by description. In *Proceedings of the Aristotelian Society*, volume 11, pages 108–128, 1910.

Bertrand Russell. *The philosophy of logical atomism*. Routledge, 1919/2009.

Simon Saunders. Hertz's principles. In *Heinrich Hertz: Classical Physicist, Modern Philosopher*, pages 123–154. Springer, 1998.

Kenneth F Schaffner. Outlines of a logic of comparative theory evaluation with special attention to pre-and post-relativistic electrodynamics. In *Minnesota Studies in the Philosophy of Science*, volume V, pages 311–355. 1970.

Kenneth F Schaffner. *Nineteenth-century aether theories*. Pergamon Press, 1972.

Lawrence Sklar. *Philosophy and the foundations of dynamics*. Cambridge University Press, 2013.

Arnold Sommerfeld. *Lectures on theoretical physics*. Academic press, 1952.

Howard Stein. 'Subtler forms of matter' in the period following Maxwell. In *Conceptions of Ether*, pages 309–340. Cambridge University Press, 1981.

Bas C van Fraassen. *Scientific representation: Paradoxes of perspective*. Clarendon press, Oxford, 2008.

Mark Wilson. Duhem before breakfast. *PhilSci Archive - http://philsci-archive.pitt.edu/3373/*, 2007.

Mark Wilson. What is 'classical mechanics' anyway? In *The Oxford handbook of philosophy of physics*, pages 43–106. Oxford University Press, 2013.

Peter Winch. *Trying to make sense*. Blackwell Oxford, 1987.

Ludwig Wittgenstein. *Philosophical investigations*. Oxford, Blackwell, 1953.

Ludwig Wittgenstein. *Culture and value.* University of Chicago Press, 1984.

Ludwig Wittgenstein. *Tractatus logico-philosophicus.* Routledge, 1994.

Ludwig Wittgenstein. *Notebooks, 1914-1916.* Blackwell, 2nd edition, 1998.

Ludwig Wittgenstein. *The big typescript TS 213 (German-English Scholars' Ed.).* Blackwell, 2005.