# Non-singular three dimensional arbitrary shape acoustic cloaks composed of homogeneous parts

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Acoustic metamaterials are artificial materials whose properties can be controlled at will. Acoustic cloaking is an important application. It can make some space acoustically invisible. In this paper, a model of transformation acoustics is proposed for a general tetrahedron. It contains three parts, each with homogeneous properties. Since most cloaks can be approximated by polyhedrons, they can be divided into a series of tetrahedrons. Thus, most cloaks can be built with homogeneous parts. Helmholtz equations are solved for the space with two polyhedral cloaks with COMSOL Multiphysics finite element software. The results show that the cloaks work well in hiding the space acoustically. In the models, all properties of each part are non-singular. Since all the parameters affect the properties of each part and some also affect the performance of cloaking, a balance can be found between performance and properties. It provides an easier and more realizable way to fabricate acoustic cloaks.

## 1. Introduction

Acoustic metamaterials are artificial materials whose properties can be designed at will. Thus, the propagation of waves inside the acoustic metamaterials can be controlled. With acoustic metamaterials, many applications previously only in imaginary can be realized in future. One application on the way is acoustic cloaking. It can make some space acoustically invisible.

The cloak that is first proposed with specific properties is in electromagnetic field. Pendry, et al [1] derived the properties of a spherical cloak in EM field based on transformation optics. Later, Cummer, et al [2] presented the 2D annular acoustic cloaks by comparing the 2D time harmonic acoustic equations with 2D Maxwell equations. Chen and Chan [3] found that spherical acoustic cloaks could be designed with transformation acoustics as well. The properties of the perfect annular cloak or spherical cloak are singular at the inner boundary. To overcome this difficulty, reduced cloak [4] and carpet cloak [5] are proposed as a solution with sacrifice of some cloaking performance.

Some models have been manufactured and tested. Popa, et al [6] designed and experimentally demonstrated a 2D ground cloak. After that, Zigoneanu, et al [7] fabricated an omnidirectional 3D ground cloak. Zhang, et al [8] fabricated a 2D annular cloak based on a transmission line approach.

Besides inertial cloaks as mentioned above, pentamode materials can also be used to design cloaks. Norris [9] found that either density or stiffness or both can be anisotropic. Chen, et al [10] proposed a latticed pentamode acoustic cloak.

In theory, cloaks with more complicated boundaries have been studied. Several EM cloaks of more complicated shapes have been presented based on transformation optics.[11-14] The authors [15] proposed a method for designing acoustic cloaks of arbitrary shapes. The properties of the cloaks are anisotropic and inhomogeneous.

Li, et al [15] proposed a near-perfect EM cloak consisting of homogeneous parts. Wang, et al [16] extended it to a three-dimensional model and derived diamond-shaped EM cloaks. In acoustics, Li, et al [17] developed acoustic cloaks with homogeneous parts in 2D and 3D. In their 3D model, all the vertices are on the axes, and the cloaked space is mapped to an area.

In this paper, a general tetrahedron is used in transformation acoustics. The cloaked space is mapped to a tetrahedron. In particular cases, the cloaked space is mapped to an area or a line. The cloak in each section contains three parts, each with homogeneous properties. Since most 3D cloaks can be approximated by a polyhedron, which can be divided into a series of tetrahedrons, they can be built with homogeneous parts. Helmholtz equations are solved for the space with the two polyhedral cloaks with COMSOL Multiphysics finite element software. The results show that the cloaks work well in hiding the space acoustically. All the dimension parameters affect the properties of each part, and some also affect the performance of cloaking. Two important factors are investigated at last.

## 2. Method

Most cloaks can be divided into small sections, each of which is the difference of two tetrahedrons, as shown in Fig. 1. The Tetrahedron $OA_1B_1C_1$ is the cloaked space. $A_0$, $B_0$ and $C_0$ are virtual points which are used to derive properties. Without generality, the edge $OA_2$ is put on the x-axis, and the face $OA_2B_2$ is put on the xy plane. This model is used to derive the properties of a section from an arbitrary cloak.

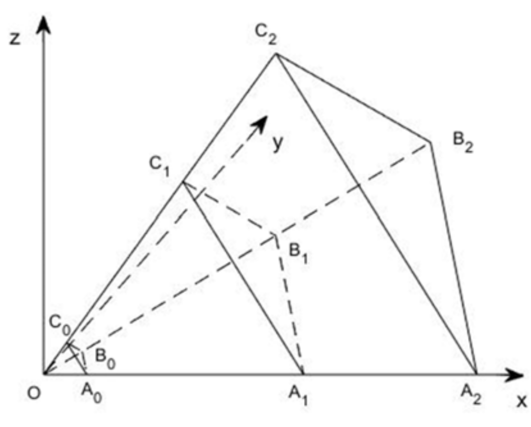


FIG.1 A section in an acoustic cloak with arbitrary shape

The section will be divided into three parts, each with homogeneous properties. A three-step method is performed, as shown in Fig.2. In the first step, Tetrahedron $OA_0B_2C_2$ is expanded to $OA_1B_2C_2$ and $A_0A_2B_2C_2$ is compressed to $A_1A_2B_2C_2$ with linear transform along $OA_2$. In the second step, the $OA_1C_2B_0$ is expanded to $OA_1C_2B_1$ and $B_0A_1C_2B_2$ is compressed to $B_1A_1C_2B_2$ with linear transform along $OB_2$. In the third step, $C_0A_1B_1C_2$ is compressed to $C_1A_1B_1C_2$ along $OC_2$.

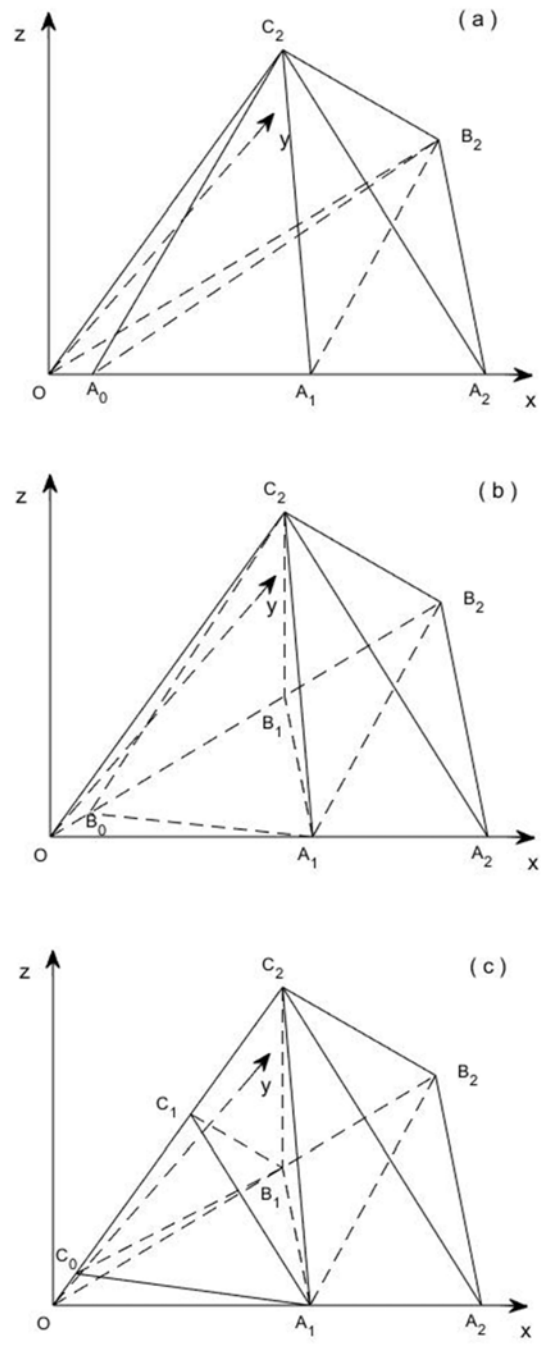


FIG.2 Schematic diagram of the three-step mapping (a) first step (b) second step (c) third step

Set $OA_0 = \lambda_1 OA_2$, $OA_1 = \lambda_2 OA_2$, $OB_0 = \mu_1 OB_2$, $OB_1 = \mu_2 OB_2$, $OC_0 = \xi_1 OC_2$, $OC_1 = \xi_2 OC_2$. $\lambda_i$, $\mu_i$ and $\xi_i$ (i=1,2) are all constants.

The points in tetrahedron $A_0A_2B_2C_2$ after transformation with respect to the points in tetrahedron $A_1A_2B_2C_2$ before transformation are

$$\begin{bmatrix} x_{III1} \\ y_{III1} \\ z_{III1} \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}_1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} c_4 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

where,

$$c_1 = \frac{1-\lambda_2}{1-\lambda_1}$$

$$c_2 = \frac{\lambda_2-\lambda_1}{1-\lambda_1}\frac{x_{B_2}-x_{A_2}}{y_{B_2}}$$

$$c_3 = -\frac{\lambda_2-\lambda_1}{1-\lambda_1}\frac{x_{B_2}y_{C_2}-x_{C_2}y_{B_2}+x_{A_2}\left(y_{B_2}-y_{C_2}\right)}{y_{B_2}z_{C_2}}$$

$$c_4 = \frac{\lambda_2-\lambda_1}{1-\lambda_1}x_{A_2}$$

The points in Tetrahedron $OA_0B_2C_2$ after transformation with respect to the points in Tetrahedron $OA_1B_2C_2$ before transformation are

$$\begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} = \underbrace{\begin{bmatrix} c_5 & c_6 & c_7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}_2} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2)$$

where,

$$c_5 = \frac{\lambda_2}{\lambda_1}$$

$$c_6 = \frac{\lambda_1-\lambda_2}{\lambda_1}\frac{x_{B_2}}{y_{B_2}}$$

$$c_7 = -\frac{\lambda_1-\lambda_2}{\lambda_1}\frac{x_{B_2}y_{C_2}-x_{C_2}y_{B_2}}{y_{B_2}z_{C_2}}$$

The second step continues after the first step. The points in Tetrahedron $B_1A_1C_2B_2$ after the second transformation with respect to those in Tetrahedron $B_1A_1C_2B_2$ before second transformation are

$$\begin{bmatrix} x_{III2} \\ y_{III2} \\ z_{III2} \end{bmatrix} = \underbrace{\begin{bmatrix} c_8 & c_9 & c_{10} \\ c_{12} & c_{13} & c_{14} \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}_3} \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} + \begin{bmatrix} c_{11} \\ c_{15} \\ 0 \end{bmatrix} \quad (3)$$

where,

$$c_8 = \frac{\mu_1-\mu_2}{1-\mu_1}\frac{x_{B_2}}{\lambda_2 x_{A_2}}+1$$

$$c_9 = \frac{\mu_1-\mu_2}{1-\mu_1}\frac{x_{B_2}}{y_{B_2}}\left(1-\frac{x_{B_2}}{\lambda_2 x_{A_2}}\right)$$

$$c_{10} = \frac{\mu_2-\mu_1}{1-\mu_1}\left[\frac{x_{B_2}}{z_{C_2}}\left(\frac{x_{C_2}}{\lambda_2 x_{A_2}}-1\right)+\frac{x_{B_2}y_{C_2}}{y_{B_2}z_{C_2}}\left(1-\frac{x_{B_2}}{\lambda_2 x_{A_2}}\right)\right]$$

$$c_{11} = \frac{\mu_2-\mu_1}{1-\mu_1}x_{B_2}$$

$$c_{12} = \frac{\mu_1-\mu_2}{1-\mu_1}\frac{y_{B_2}}{\lambda_2 x_{A_2}}$$

$$c_{13} = \frac{\mu_1-\mu_2}{1-\mu_1}\left(1-\frac{x_{B_2}}{\lambda_2 x_{A_2}}\right)+1$$

$$c_{14} = \frac{\mu_2-\mu_1}{1-\mu_1}\left[\frac{y_{B_2}}{z_{C_2}}\left(\frac{x_{C_2}}{\lambda_2 x_{A_2}}-1\right)+\frac{y_{C_2}}{z_{C_2}}\left(1-\frac{x_{B_2}}{\lambda_2 x_{A_2}}\right)\right]$$

$$c_{15} = \frac{\mu_2-\mu_1}{1-\mu_1}y_{B_2}$$

The points in Tetrahedron $OA_1C_2B_1$ after the second transformation with respect to those in Tetrahedron $OA_1C_2B_0$ before the second transformation are

$$\begin{bmatrix} x_{II} \\ y_{II} \\ z_{II} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & c_{16} & c_{17} \\ 0 & c_{18} & c_{19} \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}_4} \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} \quad (4)$$

where,

$$c_{16} = -\frac{\mu_1-\mu_2}{\mu_1}\frac{x_{B_2}}{y_{B_2}}$$

$$c_{17}=\frac{\mu_1-\mu_2}{\mu_1}\frac{x_{B_2}y_{C_2}}{y_{B_2}z_{C_2}}$$

$$c_{18}=\frac{\mu_2}{\mu_1}$$

$$c_{19}=\frac{\mu_1-\mu_2}{\mu_1}\frac{y_{C_2}}{z_{C_2}}$$

The third step continues after the two steps. The points in Tetrahedron $C_0A_1B_1C_2$ after the third transformation with respect to those in Tetrahedron $C_1A_1B_1C_2$ before the third transformation are

$$\begin{bmatrix} x_{III3} \\ y_{III3} \\ z_{III3} \end{bmatrix} = \underbrace{\begin{bmatrix} c_{20} & c_{21} & c_{22} \\ c_{24} & c_{25} & c_{26} \\ c_{28} & c_{29} & c_{30} \end{bmatrix}}_{\boldsymbol{A}_5} \begin{bmatrix} x_{II} \\ y_{II} \\ z_{II} \end{bmatrix} + \begin{bmatrix} c_{23} \\ c_{27} \\ c_{31} \end{bmatrix} \tag{5}$$

where,

$$c_{20}=1-\frac{\xi_2-\xi_1}{1-\xi_1}\frac{x_{C_2}}{\lambda_2 x_{A_2}}$$

$$c_{21}=\frac{\xi_2-\xi_1}{1-\xi_1}\frac{x_{C_2}}{\mu_2 y_{B_2}}\left(\frac{\mu_2 x_{B_2}}{\lambda_2 x_{A_2}}-1\right)$$

$$c_{22}=\frac{\xi_2-\xi_1}{1-\xi_1}\frac{x_{C_2}}{z_{C_2}}\left[\frac{x_{C_2}}{\lambda_2 x_{A_2}}-\frac{y_{C_2}}{\mu_2 y_{B_2}}\left(\frac{\mu_2 x_{B_2}}{\lambda_2 x_{A_2}}-1\right)-1\right]$$

$$c_{23}=\frac{\xi_2-\xi_1}{1-\xi_1}x_{C_2}$$

$$c_{24}=-\frac{\xi_2-\xi_1}{1-\xi_1}\frac{y_{C_2}}{\lambda_2 x_{A_2}}$$

$$c_{25}=\frac{\xi_2-\xi_1}{1-\xi_1}\frac{y_{C_2}}{\mu_2 y_{B_2}}\left(\frac{\mu_2 x_{B_2}}{\lambda_2 x_{A_2}}-1\right)+1$$

$$c_{26}=\frac{\xi_2-\xi_1}{1-\xi_1}\frac{y_{C_2}}{z_{C_2}}\left[\frac{x_{C_2}}{\lambda_2 x_{A_2}}-\frac{y_{C_2}}{\mu_2 y_{B_2}}\left(\frac{\mu_2 x_{B_2}}{\lambda_2 x_{A_2}}-1\right)-1\right]$$

$$c_{27}=\frac{\xi_2-\xi_1}{1-\xi_1}y_{C_2}$$

$$c_{28}=-\frac{\xi_2-\xi_1}{1-\xi_1}\frac{z_{C_2}}{\lambda_2 x_{A_2}}$$

$$c_{29}=\frac{\xi_2-\xi_1}{1-\xi_1}\frac{z_{C_2}}{\mu_2 y_{B_2}}\left(\frac{\mu_2 x_{B_2}}{\lambda_2 x_{A_2}}-1\right)$$

$$c_{30}=\frac{\xi_2-\xi_1}{1-\xi_1}\left(\frac{x_{C_2}}{\lambda_2 x_{A_2}}-\frac{y_{C_2}}{\mu_2 y_{B_2}}\left(\frac{\mu_2 x_{B_2}}{\lambda_2 x_{A_2}}-1\right)-1\right)+1$$

$$c_{31}=\frac{\xi_2-\xi_1}{1-\xi_1}z_{C_2}$$

With the transformation relations, the properties of each part are

$$\begin{gathered} \overline{\overline{\rho}} = \rho_0 \det(\boldsymbol{J})\left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1} \\ \kappa = \kappa_0 \det(\boldsymbol{J}) \end{gathered} \tag{6}$$

where $\boldsymbol{J}$ is the Jacobian matrix, which is defined as

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x_{III}}{\partial x} & \frac{\partial x_{III}}{\partial y} & \frac{\partial x_{III}}{\partial z} \\ \frac{\partial y_{III}}{\partial x} & \frac{\partial y_{III}}{\partial y} & \frac{\partial y_{III}}{\partial z} \\ \frac{\partial z_{III}}{\partial x} & \frac{\partial z_{III}}{\partial y} & \frac{\partial z_{III}}{\partial z} \end{bmatrix} \tag{7}$$

For Tetrahedron $A_1A_2B_2C_2$, the Jacobian matrix is

$$\boldsymbol{J}_1 = \boldsymbol{A}_1 \tag{8}$$

For Tetrahedron $B_1A_1C_2B_2$, the Jacobian matrix is

$$\boldsymbol{J}_2 = \boldsymbol{A}_3\boldsymbol{A}_2 \tag{9}$$

For Tetrahedron $C_1A_1B_1C_2$, the Jacobian matrix is

$$\boldsymbol{J}_3 = \boldsymbol{A}_5\boldsymbol{A}_4\boldsymbol{A}_2 \tag{10}$$

Since $\mathbf{A}_i$ (i=1,2,3,4,5) are free of x and y, all the Jacobian matrices are constant. With Eq. (6) ,the properties in each part are homogeneous.

## 3. Simulation

Two models are developed with this method. When $OA_2$, $OB_2$ and $OC_2$ are perpendicular to each other, it is an octohedral cloak, as shown in Fig. 3. Let $\lambda_1 = \mu_1 = \xi_1 = 0.05$, and $\lambda_2 = \mu_2 = \xi_2 = 0.5$. The cloaked space is an octohedral which is mapped to a smaller one in virtual space.

Helmholtz equations for a space with the octohedral cloak were solved with COMSOL Multiphysics finite element analysis software. The results are shown in Fig. 3. It can be seen that the cloak reduces the effect of the cloaked area. From the outside of the cloak, it seems the wave propagates in a homogeneous medium.

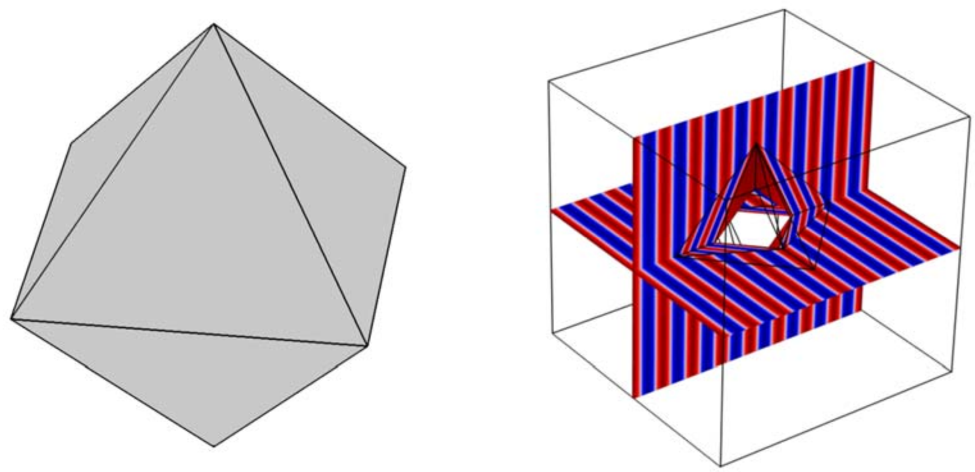

FIG. 3 An octohedral cloak and its cloaking performance

More complicated polyhedral cloak can also be designed with this method. Actually, most 3D cloaks can be approximated by polyhedrons. Thus, most cloaks can be built with this method.

A polyhedron is shown in Fig. 4. It contains 32 faces, which is an approximation of a sphere when $|OA_2|=|OB_2|=|OC_2|$ for every section. Let $\lambda_1 = \mu_1 = \xi_1 = 0.05$, and $\lambda_2 = \mu_2 = \xi_2 = 0.5$. The properties of each section can be calculated.

Helmholtz equations for the space with the polyhedral cloak were conducted with COMSOL Multiphysics finite element analysis software. The results are shown in Fig. 4. It can be seen from the figure that the cloak works well in reducing the effect of the cloaked area.

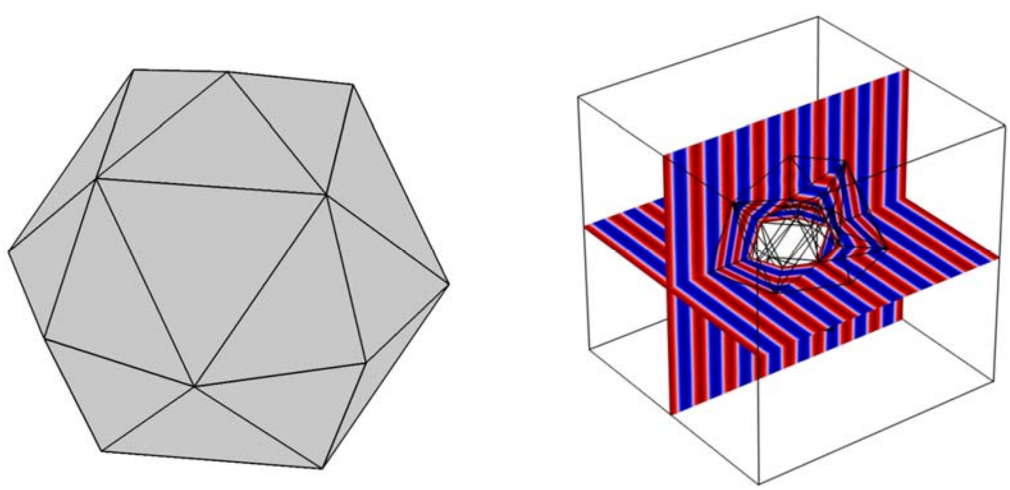

FIG. 4 A polyhedral cloak and its cloaking performance

The properties of the two models are different. That means the properties of each section can be affected by the dimension parameters. Although all the parameters can change independently, a simple model is used. Let $|OA_2|=|OB_2|=|OC_2|$, $\lambda_2 = \mu_2 = \xi_2 = 0.5$, $\lambda_1 = \mu_1 = \xi_1 = \eta$, and $\angle A_2OB_2=\angle B_2OC_2=\angle C_2OA_2=\theta$. Then, analyze the effect of θ and η separately. When η=0.05, the principal velocities of each part vary with θ, as shown in Fig. 5. When θ=5 , the principal velocities of each part vary with η, as shown in Fig. 6.

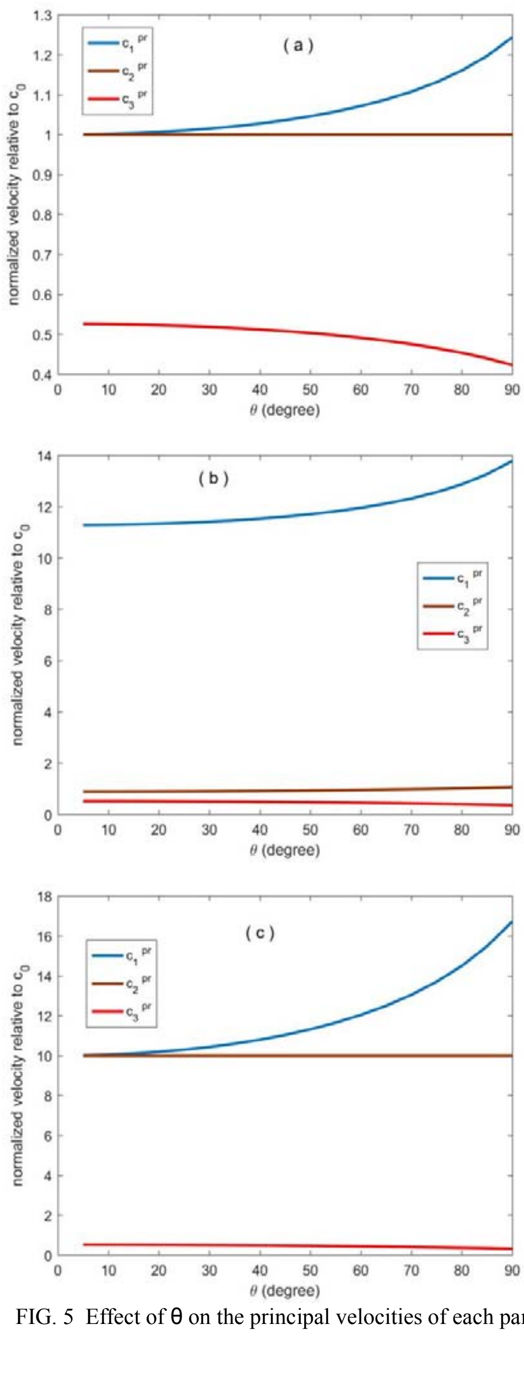


FIG. 5 Effect of θ on the principal velocities of each part

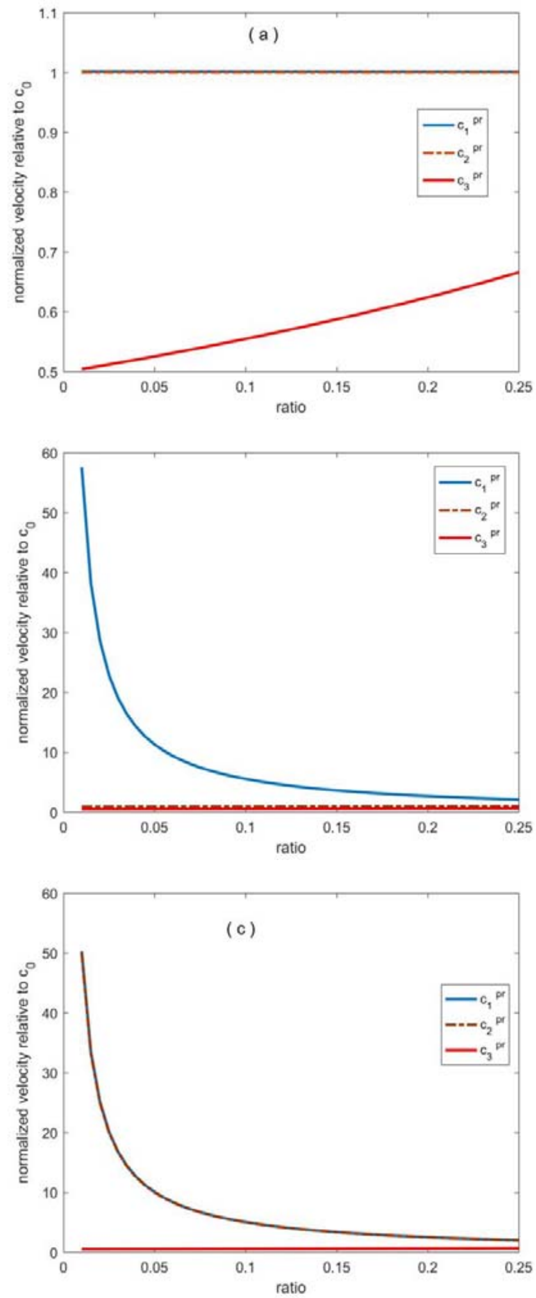


FIG. 6 Effect of η on the principal velocities of each part

It can be seen from the figures that the two factors affect the properties evidently. In Fig. 5, the deviation of three principal velocities within each part decreases with the decrease of θ. Particularly for part I and II, two principal velocities approaches to be the same. In Fig. 6, the deviation of three principal velocities within each part decreases with the increase of η. η is the size of tetrahedron to which the cloaked space is mapped. Increase of η means decrease of cloaking performance.

## 4. Conclusion

Most cloaks can be approximated by polyhedrons, which can be divided into a series of tetrahedrons. A method of transformation acoustics is proposed for a general tetrahedron. It contains three parts, each with homogeneous properties. Thus, most cloaks can be built with homogeneous parts.

Two polyhedral cloaks are designed with this method as examples. One is an octohedron and another has 32 faces, which is an approximation of a sphere. Helmholtz equations are solved for the space with the cloaks. The results show that the cloaks work well in reducing reflections and shadows. It means that acoustic cloaks with arbitrary shapes can be designed with this method and work well.

All the dimension parameters affect the properties of each part. Certain parameters also affect the performance of the cloaks. A balance can be made between cloaking performance and cloak properties. It provides an easier way to design and fabricate acoustic cloaks.

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