GLOBAL EXISTENCE AND REGULARITY FOR THE ACTIVE LIQUID CRYSTAL SYSTEM

by

Rongfang Zhang

B.S., Xiamen University, 2009

M.S., Xiamen University, 2012

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This dissertation was presented

by

Rongfang Zhang

It was defended on

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and approved by

Prof. Dehua Wang, Department of Mathematics

Prof. Ming Chen, Department of Mathematics

Prof. Gautam Iyer, Department of Mathematical Sciences, Carnegie Mellon University

Prof. Huiqiang Jiang, Department of Mathematics

Dissertation Director: Prof. Dehua Wang, Department of Mathematics

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Rongfang Zhang, PhD

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We study the hydrodynamics of active liquid crystals in the Beris-Edwards hydrodynamic framework with the Landau-de Gennes Q-tensor order parameter to describe liquid crystalline ordering. For the incompressible case, the existence of global weak solutions in two and three spatial dimensions is established. The higher regularity of the weak solutions and the weak-strong uniqueness are also obtained by using the Littlewood-Paley decomposition in two space dimensions. For the inhomogeneous incompressible case, Faedo-Galerkin's method is adopted to construct the solutions for the initial-boundary value problem. Two levels of approximations are used and the weak convergence is obtained through compactness estimates by new techniques due to the active terms. The existence of global weak solutions in dimension two and three is established in a bounded domain. For the compressible case where the concentration of the active particles in the system is not constant, we prove the existence of global weak solutions for this active system in three space dimensions by the three-level approximations and weak convergence argument. New techniques and estimates are developed to overcome the difficulties caused by the active terms.

Keywords: Active hydrodynamics, active liquid crystals, Navier-Stokes equations, incompressible flow, inhomogeneous incompressible flow, compressible flow, weak solutions, strong solutions, global well-posedness, regularity, weak-strong uniqueness, weak convergence.

TABLE OF CONTENTS

PREFACE

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1.0 INTRODUCTION

Liquid crystals are classical examples of mesophases that are intermediate between solids and liquids [23].They often combine physical properties of both liquids and solids, and in general liquid crystals can be divided into thermotropic, lyotropic, and metallotropic phases, according to their different optical properties. Nematic liquid crystals are one of the most common liquid crystalline phases; nematics are complex liquids with a certain degree of long-range orientational order. That is, the constituent molecules are typically rod-like or elongated, and these elongated molecules flow about freely as in a conventional liquid but, whilst flowing, they tend to align along certain distinguished directions [23, 63].

There are several competing mathematical theories for nematic liquid crystals in the literature, such as the Doi-Onsager theory proposed by Doi [12] in 1986 and Onsager [48] in 1949, the Oseen-Frank theory proposed by Oseen [49] in 1933 and Frank [21] in 1958, the Ericksen-Leslie theory proposed by Ericksen [14] in 1961 and Leslie [35] in 1968, and the Landau-de Gennes theory proposed by Gennes [23] in 1995. The first one is a molecular kinetic theory, and the remaining three are continuum macroscopic theories. These theories can be derived or related to each other, under some assumptions. For instance, Kuzzu-Doi [33] and E-Zhang [13] formally derived the Ericksen-Leslie equation from the Doi-Onsager equations by taking small Deborah number limit. Wang-Zhang-Zhang [66] justified this formal derivation before the first singular time of the Ericksen-Leslie equations. Wang-Zhang-Zhang [67] presented a rigorous derivation of the Ericksen-Leslie equations from the Beris-Edwards model in the Landau-de Gennes framework. Ball-Majumdar [3] and Ball-Zarnescu [4] studied the differences and the overlap between the Oseen-Frank theory and the Landau-de Gennes theory. See [36, 37, 40, 41] for further discussions.

Active hydrodynamics describe fluids with active constituent particles that have collective motion and are constantly maintained out of equilibrium by internal energy sources, rather than by the external forces applied to the system [30, 44]. In particular, when the particles have elongated shapes, usually the collective motion induces the particles to demonstrate orientational ordering at high concentration. Thus, there are natural analogies with nematic liquid crystals.

Active hydrodynamics have wide applications and have attracted much attention in recent decades. For example, many biophysical systems are classified as active nematics, including microtubule bundles [58], cytoskeletal filaments [32], actin filaments [8], dense suspensions of microswimmers [68], bacteria [10], catalytic motors [52], and even nonliving analogues such as monolayers of vibrated granular rods [42]. For more information and discussions, see [5, 12, 29, 30, 32, 53, 54, 56] and the references therein. Active nematic systems are distinguished from their well-studied passive counterparts since the constituent particles are active; that is, it is the energy consumed and dissipated by the active particles that drives the system out of equilibrium, rather than the external force applied at the boundary of the system, like a shear flow. Consequently, active dynamics are truly striking, and many novel effects have been observed in active systems, like the occurrence of giant density fluctuations [44, 46, 55], the spontaneous laminar flow [27, 43, 64], unconventional rheological properties [20, 28, 60], low Reynolds number turbulence [30, 68], and very different spatial and temporal patterns compared to passive systems [9, 24, 44, 45, 57] arising from the interaction of the orientational order and the flow.

Whilst active liquid crystals are popular in the theoretical physics community, a rigorous mathematical description of active nematics is relatively new. There are phenomenological models for active liquid crystals in [54], and a common approach is to add phenomenological active terms to the hydrodynamic theories for nematic liquid crystals. In this thesis, we use the Landau-de Gennes Q-tensor description that is one of the most comprehensive descriptions, which can describe primary and secondary directions of nematic alignment along with variations in the degree of nematic order [37]. More precisely, the Landau-de Gennes Q-tensor order parameter is a d-dimensional symmetric and traceless matrix for the d-dimensional case; the isotropic phase is defined by $Q = 0$. We remark that two-dimensional Q-tensors have been used to successfully model severely confined three-dimensional nematic systems that are effectively invariant in the third dimension.

Regarding the related mathematical contributions in the passive Q-tensor liquid crystal framework, Paicu-Zarnescu [50, 51] proved the existence of global weak solutions to the coupled incompressible Navier-Stokes and Q-tensor system for $d = 2, 3$, as well as the existence of global regular solutions with sufficiently regular initial data for $d = 2$. Wilkinson [69] obtained the existence and regularity of weak solutions on the d–dimensional torus over a certain singular potential. In [19], Feireisl-Rocca-Schimperna-Zarnescu derived the global-in-time weak solutions in the Q-tensor framework, with arbitrary physically relevant initial data in case of a singular bulk potential proposed in Ball-Majumdar [3]. Wang-Xu-Yu [65] established the existence and long-time dynamics of globally defined weak solutions for the coupled compressible Navier-Stokes and Q-tensor system. See [11] and the references therein for more results and discussions.

In Chapter 2, we study the incompressible active system to establish the existence of weak solutions in two and three spatial dimensions, along with the existence of regular solutions and the uniqueness of weak-strong solutions in the two-dimensional case, motivated by the work of Paicu-Zarnescu [50, 51] for the passive system. Since we are dealing with the active system, we need to conquer some new difficulties. Firstly, by using the general energy method, we obtain *a priori* estimates for the system (2.0.2), based on some crucial cancellations. Those cancellations turn out to be very important in the proof of the existence of weak solutions in $\mathbb{R}^d, d = 2, , 3$, higher regularity and the uniqueness of weak-strong solutions in $\mathbb{R}^2$. However, due to the appearance of the active term $\kappa Q_{\alpha\beta}$, we can only obtain an energy inequality, instead of the perfect Lyapunov functional for the smooth solutions of the system. Here we mention that the symmetry and traceless properties of the Q-tensor play a key role in the validity of the cancellations (see also Appendix A). Also the property of the Q-tensor (A.0.1) is very important in order to derive the H^1-estimate for the Q-tensor in Proposition 2.1.2, since the bulk potential in the Landau-de Gennes energy density (the terms independent of ∇Q) is not always positive. Additionally, in order to obtain the weak solutions, we need to add the extra terms $-\varepsilon\partial_\alpha Q_{\beta\gamma}(u \cdot \nabla Q_{\beta\gamma})|u \cdot \nabla Q|$ and $\varepsilon\nabla \cdot (\nabla u|\nabla u|^2)$ to the system to control some non-vanishing terms in the energy estimates for the approximate system (2.2.6), due to the nonlinearity of the terms: $J_n(R_\varepsilon u^n \nabla Q^{(n)})$

and $J_n(R_\varepsilon\Omega^n Q^{(n)} - Q^{(n)}R_\varepsilon\Omega^n)$ (see §2.2 for notations). In §2.1–§2.2, the cancellations for these terms work very well. However, in §2.3, when we seek the regular solutions in $\mathbb{R}^2$, the cancellation for the terms, $\lambda|Q|D$ and $\lambda\nabla\cdot(|Q|H)$, does not hold perfectly as before. We use the Littlewood-Paley decomposition to reduce these terms to two new terms that can be controlled by the previous cancellation idea (see Appendix B for the details). We also need to pay more attention to the higher order terms of Q in the elastic stress tensor of the system.

In Chapter 3, we consider the inhomogeneous version of the active hydrodynamic model studied in Chapter 2, in particular, the fluid flow is governed by the inhomogeneous incompressible Navier-Stokes equations, the motion of the order parameter Q is represented by a parabolic type equation, and both with the extra nonlinear coupling terms as forcing terms. We establish the existence of global weak solutions in dimension two and three for this inhomogeneous coupled system (3.0.2)−(3.0.5) with the initial-boundary conditions (3.0.6)-(3.0.9). In our system, the highly nonlinear terms cause more difficulties mathematically. Similar to Chapter 2, we can deal with these by using the symmetric and traceless properties of the Q-tensor and the cancellation rules we mentioned in Chapter 2. The Friedrichs' scheme was used to construct the solutions in Chapter 2 for the incompressible active liquid crystal system in the whole space. In this chapter, we adopt the Faedo-Galerkin's method to construct the solutions for the initial-boundary value problem in a bounded domain. Two levels of approximations are used and the weak convergence is obtained through compactness estimates. One level of the approximations is related to lifting the density above zero to avoid the vacuum. The other is the approximation from the finite dimensional space to the infinite one. In order to obtain the necessary compactness results, the force term in the inhomogeneous Navier-Stokes equations is required to be in H_x^{-1} as in Lions [38]. However, we can only obtain $L_t^\infty H_x^1 \cap L_t^2 H_x^2$ regularity for Q from the Q-tensor equation. By using the structure of the system, we are able to overcome this difficulty from similar cancellations we mentioned in the *a priori* estimates. Finally with all the compactness estimates we can pass to the limit in the approximate solutions to establish the existence of global weak solutions to the system (3.0.2)−(3.0.5) with the initial-boundary conditions (3.0.6)-(3.0.9).

In Chapter 4, we prove the existence of the compressible active liquid crystal system with varying concentration of the active particles by using three-level approximations [61], including the Faedo-Galerkin approximation, artificial viscosity, artificial pressure, as well as the weak convergence argument, in the spirit of [17, 18]. This approach was used to construct weak solutions to the compressible Beris-Edwards in [65]. As discussed above, new techniques are needed to overcome the difficulties arising from the concentration equation and its coupling with both the fluid and Q-tensor equations. Firstly, by using the Faedo-Galerkin approximation, for any fixed u_n in the finite-dimensional space $C(0,T;X_n)$ (see (4.2.17)), we obtain a unique solution $(\rho[u_n], c[u_n], Q[u_n])$ of the initial-boundary value problem (4.2.1)–(4.2.2) and (4.2.4). In Lemma 4.2.2, system (4.2.13) has complicated interaction terms which cause difficulties in the proof of both the uniqueness of the solution $(c[u_n], Q[u_n])$ and the traceless property of $Q[u_n]$. The tracelessness of the Q-tensor is an important property that guarantees the validity of the cancellation rules in order to treat the highly nonlinear terms caused by the appearance of the concentration and the Q-tensor. In the proof of the tracelessness, we make the L^2-estimate of $\operatorname{tr} Q$ from an energy inequality, which implies the tracelessness of Q when combined Gronwall's inequality and the tracelessness of the initial condition of Q. As we mentioned before, the highly nonlinear interaction terms cause difficulties in this procedure. Here we can see that the concentration equation has the maximum principle, which provides the L^∞–bound for the concentration. Moreover, we show that the solutions to system (4.2.13) are in $L^\infty(0,T;H^1(\mathcal{O})) \cap L^2(0,T;H^2(\mathcal{O}))$, which provides sufficient regularity for the interaction terms so that they stay bounded, and we can prove the uniqueness of the solution and the traceless property of $Q[u_n]$. Then, substituting $(\rho[u_n], c[u_n], Q[u_n])$ into the variational problem of the momentum equation, we can construct a contraction map and obtain a local solution (ρ_n, c_n, u_n, Q_n) of the approximation system (4.2.1)–(4.2.4) on the time interval $[0, T_n]$. Moreover, the global existence of solutions follows from the uniform energy estimate of the approximation system. In order to pass to the limit for solutions (ρ_n, c_n, u_n, Q_n) as $n \to \infty$ to obtain a solution $(\rho_{\varepsilon,\delta}, c_{\varepsilon,\delta}, u_{\varepsilon,\delta}, Q_{\varepsilon,\delta})$ for the approximation system in the infinite space, we need enough integrability of the solutions. First, the maximum principle satisfied by the concentration equation provides sufficient integrability for the concentration c_n. This, together with the regularity of (u_n, Q_n) obtained in the uniform energy estimate, gives enough compactness for the nonlinear interaction terms of c_n and Q_n in our system. It actually also plays a crucial role when the artificial viscosity and the artificial pressure tend to zero. With the above results and the artificial pressure and viscosity in the approximation system, we have enough regularity and integrability of the density. These integrability and compactness results allow us to pass to the limit as $n \to \infty$ and obtain a solution for the approximation system in the infinite space. Secondly, we let the artificial viscosity ε tend to zero to recover the original continuity equation. Here we employ the convergence of the effective viscous flux sequence to deal with the lack of regularity of the density sequence to retrieve the compactness results of the solutions. Lastly, we pass to the limit of the vanishing artificial pressure sequence to obtain a finite-energy weak solution of the original problem, including the vacuum case. Again, we do not have enough integrability for the density. Similarly to the vanishing artificial viscosity procedure, the convergence of the effective viscous flux sequence gives us the much needed higher regularity for the density. We remark that, although the weak convergence argument is similar to that in [17], owing to the extra terms and difficulties, we will provide most of the details for the sake of completeness and convenience to the reader.

In summary, the rest of the dissertation is organized as follows. In Chapter 2, we establish the existence of weak solutions for the incompressible active liquid crystal system in $\mathbb{R}^d, d = 2, 3$. Moreover, by using the Littlewood-Paley decomposition, we achieve the higher regularity of the corresponding weak solution and the uniqueness of the weak and strong solution in $\mathbb{R}^2$ with suitable initial data. In Chapter 3, we prove the existence of solutions to the approximation problems by using the Faedo-Galerkin's method. In Chapter 4, we apply the Faedo-Galerkin's method with three levels of approximations to prove the existence of the solutions to the compressible active liquid crystal system with non-constant concentration of the active particles in the fluid in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$. In Appendix A, we provide some important preliminary estimates that we use extensively in this thesis. In Appendix B, we provide the detailed estimates for inequality (2.3.6).

2.0 INCOMPRESSIBLE ACTIVE LIQUID CRYSTALS

In this chapter, we consider the following hydrodynamic partial differential equations that model spatio-temporal pattern formation in incompressible active nematic systems [25, 29]:

$$\begin{cases} \partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H, \\ (\partial_t + u_\beta\partial_\beta)u_\alpha + \partial_\alpha P - \mu\Delta u_\alpha = \partial_\beta\tau_{\alpha\beta} + \partial_\beta\sigma_{\alpha\beta}, \\ \nabla\cdot u = 0, \end{cases} \tag{2.0.1}$$

where $u \in \mathbb{R}^d$, $d = 2$ or 3, is the flow velocity; P is the pressure; Q is the nematic tensor order parameter that is a traceless and symmetric $d\times d$ matrix; $\mu > 0$ denotes the viscosity coefficient; $\Gamma^{-1} > 0$ is the rotational viscosity; $\lambda \in \mathbb{R}$ stands for the nematic alignment parameter; $D = \frac{1}{2}(\nabla u + \nabla u^\top)$ and $\Omega = \frac{1}{2}(\nabla u - \nabla u^\top)$ are the symmetric and antisymmetric part of the strain tensor with $(\nabla u)_{\alpha\beta} = \partial_\beta u_\alpha$. Hereafter, we use the Einstein summation convention, *i.e.*, the repeated indices are summed over, and $\alpha, \beta = 1, 2, \ldots, d$. The time variable is $t \geq 0$, the space variable is $x = (x_1, \ldots, x_d)$, and $\partial_\beta = \frac{\partial}{\partial x_\beta}$. Moreover, the molecular tensor

$$H = H[Q] = K\Delta Q - \frac{k}{2}(c - c_*)Q + b\big(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d\big) - cQ\,\mathrm{tr}(Q^2)$$

describes the relaxational dynamics of the nematic phase and can be obtained from the Landau-de Gennes free energy, *i.e.*, $H_{\alpha\beta} = -\frac{\delta\mathcal{F}}{\delta Q_{\alpha\beta}}$, where

$$\mathcal{F} = \int \Big(\frac{k}{4}(c - c^*)\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|\mathrm{tr}(Q^2)|^2 + \frac{K}{2}|\nabla Q|^2\Big)dA,$$

with K the elastic constant for the one-constant elastic energy density, c the concentration of active units and c^* the critical concentration for the isotropic-nematic transition, and $k > 0$

and $b \in \mathbb{R}$ are material-dependent constants. We note that the analysis of this paper holds for all real b, but b is usually taken to be positive in the literature. In what follows, we set $K = k = 1$ for simplicity of notation. The stress tensor $\sigma_{\alpha\beta}$ is

$$\sigma_{\alpha\beta} = \sigma^r_{\alpha\beta} + \sigma^a_{\alpha\beta},$$

with

$$\sigma^r_{\alpha\beta} = -\lambda|Q|H_{\alpha\beta} + Q_{\alpha\delta}H_{\delta\beta} - H_{\alpha\delta}Q_{\delta\beta}, \qquad \sigma^a_{\alpha\beta} = \sigma_* c^2 Q_{\alpha\beta},$$

where $\sigma^r_{\alpha\beta}$ is the elastic stress tensor due to the nematic elasticity, and $\sigma^a_{\alpha\beta}$ is the active contribution which describes contractile or extensile stresses exerted by the active particles in the direction of the director field ($\sigma_* > 0$ for the contractile case, and $\sigma_* < 0$ for the extensile case). The symmetric additional stress tensor is denoted by

$$\tau_{\alpha\beta} := -(\nabla Q \odot \nabla Q)_{\alpha\beta} = -\partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta}.$$

In the rest of this paper, we consider the case when $c > 0$ is a constant. We set

$$a = \frac{1}{2}(c - c_*), \quad \kappa = \sigma_* c^2.$$

Then system (2.0.1) becomes

$$\begin{cases} \partial_t Q + (u \cdot \nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H, \\ \partial_t u + (u \cdot \nabla)u + \nabla P - \mu\Delta u = -\nabla \cdot (\nabla Q \odot \nabla Q) - \lambda\nabla \cdot (|Q|H) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \nabla \cdot (Q\Delta Q - \Delta Q Q) + \kappa\nabla \cdot Q, \\ \nabla \cdot u = 0, \end{cases} \tag{2.0.2}$$

with

$$H = \Delta Q - aQ + b\big(Q^2 - \frac{\operatorname{tr}(Q^2)}{d}\mathrm{I}_d\big) - cQ\operatorname{tr}(Q^2),$$

the constants $c > 0$, $\Gamma > 0$, $\mu > 0$, $a, b, \lambda, \kappa \in \mathbb{R}$, and $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$.

2.1 THE DISSIPATION PRINCIPLE AND *A PRIORI* ESTIMATES

In this section, by using the energy method, we derive the dissipation principle in Proposition 2.1.1 and the *a priori* estimates in Proposition 2.1.2 for system (2.0.2).

For the sake of convenience, we first introduce some notations. We denote $H^k(\mathcal{O})$, with $k \geq 1$ integer, as the Sobolev space that consists of all functions v in $L^2(\mathcal{O})$ such that $D^\nu v$ is in $L^2(\mathcal{O})$ for every multi-index $\nu = (\nu_1, \cdots, \nu_d)$, $0 \leq |\nu| \leq k$, where $D^\nu := \partial_1^{\nu_1} \cdots \partial_d^{\nu_d}$ is the distributional derivative. The space $H^k(\mathcal{O})$ is equipped with norm $\|\cdot\|_{H^k}$ defined by

$$\|v\|_{H^k}^2 := \sum_{0 \leq |\nu| \leq k} \|D^\nu v\|_{L^2}^2.$$

The space $H^{-k}(\mathcal{O})$, with $k \geq 1$ integer, is defined as the dual spaces of $H_0^k(\mathcal{O})$, equipped with the norm:

$$\|v\|_{H^{-k}} := \sup_{\varphi \in H_0^k, \|\varphi\|_{H_0^k} \leq 1} |(v, \varphi)|,$$

where $(\cdot, \cdot)$ stands for the inner product in L^2. For example, if a and b are vector functions, then

$$(a, b) = \int_{\mathcal{O}} a(x) \cdot b(x)\, dx,$$

and if A and B are matrices, then

$$(A, B) = \int_{\mathcal{O}} A : B\, dx$$

with $A : B = \mathrm{tr}(AB)$. We denote by $S_0^d \subset \mathbb{M}^{d \times d}$ the space of symmetric traceless Q-tensors in d-dimension, that is,

$$S_0^d := \left\{ Q \in \mathbb{M}^{d \times d} : \; Q_{\alpha\beta} = Q_{\beta\alpha}, \; \mathrm{tr}(Q) = 0, \; \alpha, \beta = 1, \cdots, d \right\}.$$

We define the norm of a matrix by using the Frobenius norm denoted by

$$|Q| := \sqrt{\mathrm{tr}(Q^2)} = \sqrt{Q_{\alpha\beta} Q_{\alpha\beta}}.$$

With respect to this norm, we can define the Sobolev spaces for the Q-tensors, for example,

$$H^1(\mathbb{R}^d, S_0^d) := \left\{ Q : \mathcal{O} \to S_0^d : \; \int_{\mathcal{O}} \left(|Q(x)|^2 + |\nabla Q(x)|^2 \right) dx < \infty \right\},$$

where $|Q|^2 := tr(Q^2)$ and $|\nabla Q|^2 := \partial_\delta Q_{\alpha\beta}\partial_\delta Q_{\alpha\beta}$. We also denote $|\Delta Q|^2 := \Delta Q_{\alpha\beta}\Delta Q_{\alpha\beta}$.

Let us denote the Landau-de Gennes free energy for the nematic liquid crystals [23] by

$$\mathbf{F}(Q) := \int_{\mathcal{O}} \left(\frac{1}{2}|\nabla Q|^2 + \frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4\right) dx. \tag{2.1.1}$$

Moreover, by adding the kinetic energy to $\mathbf{F}(Q)$, we denote the energy of system (2.0.2) by

$$E(t) := \mathbf{F}(Q) + \frac{1}{2}\int_{\mathcal{O}} |u|^2 dx. \tag{2.1.2}$$

In this chapter, $\mathcal{O} = \mathbb{R}^d$ for $d = 2, 3$.

Proposition 2.1.1. *Let (Q, u) be a smooth solution of system (2.0.2) such that*

$$Q \in L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d)) \tag{2.1.3}$$

and

$$u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d)) \tag{2.1.4}$$

for $d = 2, 3$. Then, for any given $T > 0$, we have

$$\frac{d}{dt}E(t) + \frac{\mu}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \Gamma \int_{\mathbb{R}^d} \mathrm{tr}(H^2) dx \le C(\kappa, \mu)\int_{\mathbb{R}^d} |Q|^2 dx \quad \textit{for any } t \in (0, T). \tag{2.1.5}$$

Proof. We take the summation of the first equation in (2.0.2) multiplied by $-H$ and the second equation in (2.0.2) multiplied by u, take the trace, and then integrate by parts over $\mathbb{R}^d$ to find

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} \big(\frac{1}{2}|\nabla Q|^2 + \frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 + \frac{1}{2}|u|^2\big) dx + \mu\|\nabla u\|_{L^2}^2 + \Gamma\int_{\mathbb{R}^d} \mathrm{tr}(H^2) dx \\
&= (u\cdot\nabla Q, \Delta Q) - (u\cdot\nabla Q, aQ - b(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d) + cQ|Q|^2) - (\Omega Q - Q\Omega, \Delta Q) \\
&\quad + (\Omega Q - Q\Omega, aQ - b(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d) + cQ|Q|^2) - \lambda(|Q|D, H) \\
&\quad - (\nabla\cdot(\nabla Q \odot \nabla Q), u) + \lambda(|Q|H, \nabla u) + (\nabla\cdot(Q\Delta Q - \Delta Q Q), u) - \kappa(Q, \nabla u) \\
&= \sum_{i=1}^{8} \mathcal{I}_i - \kappa(Q, \nabla u) \\
&\le \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + C(\kappa, \mu)\|Q\|_{L^2}^2,
\end{aligned}$$

where, in the last inequality, besides Cauchy's inequality, we have used that $\mathcal{I}_2 = 0$ (since $\nabla \cdot u = 0$), $\mathcal{I}_3 + \mathcal{I}_8 = 0$ (by Lemma A.0.5), $\mathcal{I}_1 + \mathcal{I}_6 = 0$, $\mathcal{I}_4 = 0$, and $\mathcal{I}_5 + \mathcal{I}_7 = 0$, as shown below:

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_6 &= (u \cdot \nabla Q, \Delta Q) - (\nabla \cdot (\nabla Q \odot \nabla Q), u) \\
&= \int u_\alpha \partial_\alpha Q_{\delta\gamma} \Delta Q_{\delta\gamma} dx - \int \partial_\alpha \partial_\beta Q_{\delta\gamma} \partial_\beta Q_{\delta\gamma} u_\alpha dx - \int \partial_\alpha Q_{\delta\gamma} \partial_\beta \partial_\beta Q_{\delta\gamma} u_\alpha dx \\
&= - \int \partial_\alpha \partial_\beta Q_{\delta\gamma} \partial_\beta Q_{\delta\gamma} u_\alpha dx \\
&= 0,
\end{aligned}
$$

and, by the fact that Q is symmetric and Ω is skew symmetric,

$$
\begin{aligned}
\mathcal{I}_4 &= (\Omega Q - Q\Omega, aQ - b\big(Q^2 - \frac{\mathrm{tr}(Q^2)}{d} \mathrm{I}_d\big) + cQ|Q|^2) \\
&= -(\Omega Q + Q\Omega, aQ - b\big(Q^2 - \frac{\mathrm{tr}(Q^2)}{d} \mathrm{I}_d\big) + cQ|Q|^2) \\
&\quad + 2(\Omega Q, aQ - b\big(Q^2 - \frac{\mathrm{tr}(Q^2)}{d} \mathrm{I}_d\big) + cQ|Q|^2) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_5 + \mathcal{I}_7 &= \lambda(|Q|H, \nabla u) - \lambda(|Q|D, H) = \lambda(|Q|H, \nabla u) - \lambda(|Q|H, D) \\
&= \lambda(|Q|H, \nabla u - D) = \lambda(|Q|H, \Omega) = 0.
\end{aligned}
$$

□

Remark 2.1.1. For the passive system considered in [50], a perfect Lyapunov functional is available. However, for the active system as analyzed here, only an energy inequality (2.1.5) is obtained above, which is not a Lyapunov functional in general.

Based on Proposition 2.1.1 and Gronwall's inequality (Lemma A.0.9), we have the following *a priori* estimates.

Proposition 2.1.2. *Let (Q, u) be a smooth solution of system (2.0.2) in $\mathbb{R}^d, d = 2, 3$, with smooth initial data $(\bar{Q}(x), \bar{u}(x))$. If $(\bar{Q}, \bar{u}) \in H^1 \times L^2$, then, for any $t > 0$,*

$$\|Q(t,\cdot)\|_{H^1} \le C_1 e^{C_2 t}\big(\|\bar{Q}\|_{H^1}^2 + \|\bar{u}\|_{L^2}^2\big), \tag{2.1.6}$$

and

$$\frac{1}{2}\|u(t,\cdot)\|_{L^2}^2 + \frac{\mu}{4}\int_0^t \|\nabla u(s,\cdot)\|_{L^2}^2 ds \le C_3\big(\|\bar{Q}\|_{H^1}^2 + \|\bar{u}\|_{L^2}^2\big)e^{C_2 t} + C_4, \tag{2.1.7}$$

where constants $C_i, 1 \le i \le 4$, depend on $(a, b, c, \kappa, \mu, \lambda, \Gamma, \bar{Q}, \bar{u})$.

Proof. From the energy estimate in Proposition 2.1.1 and the symmetric property of Q, we have

$$\begin{aligned}
&\frac{d}{dt}E(t) + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \Gamma\|\Delta Q\|_{L^2}^2 + a^2\Gamma\|Q\|_{L^2}^2 + c^2\Gamma\|Q\|_{L^6}^6 \\
&\quad + b^2\Gamma\int_{\mathbb{R}^d} \operatorname{tr}\big((Q^2 - \frac{\operatorname{tr}(Q^2)}{d}\mathrm{I}_d)^2\big)dx \\
&\le C(\kappa,\mu)\|Q\|_{L^2}^2 + 2a\Gamma(\Delta Q, Q) - 2ac\Gamma\|Q\|_{L^4}^4 - 2b\Gamma(\Delta Q, Q^2) \\
&\quad + 2bc\Gamma(Q\operatorname{tr}(Q^2), Q^2) + 2ab\Gamma(Q, Q^2) + 2c\Gamma(\Delta Q, Q\operatorname{tr}(Q^2)) \\
&= C(\kappa,\mu)\|Q\|_{L^2}^2 - 2a\Gamma\|\nabla Q\|_{L^2}^2 - 2ac\Gamma\|Q\|_{L^4}^4 + \sum_{i=1}^4 \mathcal{I}_i.
\end{aligned} \tag{2.1.8}$$

We now derive the estimates for $\mathcal{I}_i$, $1 \le i \le 4$. First, we have

$$\mathcal{I}_1 = -2b\Gamma(\Delta Q, Q^2) \le \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + C(b^2,\Gamma)\|Q\|_{L^4}^4. \tag{2.1.9}$$

From (A.0.1), we have the following estimates for $\mathcal{I}_2$ and $\mathcal{I}_3$ by choosing an appropriate $\varepsilon > 0$:

$$\begin{aligned}
\mathcal{I}_2 &= 2bc\Gamma(Q\operatorname{tr}(Q^2), Q^2) = 2bc\Gamma\int_{\mathbb{R}^d}\operatorname{tr}(Q^3)|Q|^2 dx \\
&\le 2|b|c\Gamma\int_{\mathbb{R}^d}\big(\frac{\varepsilon}{4}|Q|^4 + \frac{1}{\varepsilon}|Q|^2\big)|Q|^2 dx \\
&= \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 + C(b^2,\Gamma)\|Q\|_{L^4}^4,
\end{aligned} \tag{2.1.10}$$

and

$$\mathcal{I}_3 = 2ab\Gamma(Q, Q^2) = 2ab\Gamma\int_{\mathbb{R}^d}\operatorname{tr}(Q^3)dx \le C(a,b,\Gamma)\big(\|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big). \tag{2.1.11}$$

Moreover, we observe that

$$
\begin{aligned}
\mathcal{I}_4 &= 2c\Gamma(\Delta Q, Q\,\mathrm{tr}(Q^2)) = 2c\Gamma\int_{\mathbb{R}^d}\partial_{\gamma\gamma}Q_{\alpha\beta}Q_{\alpha\beta}\mathrm{tr}(Q^2)dx \\
&= -2c\Gamma\int_{\mathbb{R}^d}\partial_\gamma Q_{\alpha\beta}\partial_\gamma Q_{\alpha\beta}\mathrm{tr}(Q^2)dx - 2c\Gamma\int_{\mathbb{R}^d}\partial_\gamma Q_{\alpha\beta}Q_{\alpha\beta}\partial_\gamma\mathrm{tr}(Q^2)dx \\
&= -2c\Gamma\int_{\mathbb{R}^d}|\nabla Q|^2|Q|^2dx - c\Gamma\int_{\mathbb{R}^d}|\nabla\mathrm{tr}(Q^2)|^2dx \le 0.
\end{aligned}
\tag{2.1.12}
$$

Combining (2.1.8) with (2.1.9)−(2.1.12), we have

$$
\begin{aligned}
&\frac{d}{dt}E(t) + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 + b^2\Gamma\int_{\mathbb{R}^d}\mathrm{tr}((Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d)^2)dx \\
&\le C(a^2, b^2, c, \kappa, \mu, \Gamma)\big(\|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big).
\end{aligned}
\tag{2.1.13}
$$

Here we should clarify that, since the potential terms, *i.e.*, the Q–terms without derivatives in $E(t)$ do not always sum to a positive quantity, (2.1.13) does not always yield the desired estimates for Q directly. However, we can deal with this issue as follows.

Case I: $a > 0$ and sufficiently large. By the property of Q in (A.0.1), we have

$$
\begin{aligned}
\frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 &\ge \frac{a}{2}|Q|^2 - \frac{b}{3}\left(\frac{\varepsilon}{4}|\mathrm{tr}(Q^2)|^2 + \frac{1}{\varepsilon}\mathrm{tr}(Q^2)\right) + \frac{c}{4}|Q|^4 \\
&= \left(\frac{a}{2} - \frac{b}{3\varepsilon}\right)|Q|^2 + \left(\frac{c}{4} - \frac{b}{12}\varepsilon\right)|Q|^4.
\end{aligned}
\tag{2.1.14}
$$

When $\varepsilon > 0$ is sufficiently small and $a > 0$ is sufficiently large, both $\frac{a}{2} - \frac{b}{3\varepsilon}$ and $\frac{c}{4} - \frac{b}{12}\varepsilon$ can be positive. As a result, we can obtain the H^1–estimates of Q directly from (2.1.13) by Gronwall's inequality (Lemma A.0.9).

Case II: For all other a. In this case, the sum of the Q–terms without derivatives in $E(t)$ may be negative. Thus, we have to deal with the L^2–estimates of Q separately to obtain the H^1–estimates for Q. In fact, we multiply the first equation in (2.0.2) by Q, take the trace, and integrate over $\mathbb{R}^d$ by parts to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|Q\|_{L^2}^2 &= -\Gamma\|\nabla Q\|_{L^2}^2 - a\Gamma\|Q\|_{L^2}^2 - c\Gamma\|Q\|_{L^4}^4 + b\Gamma\int_{\mathbb{R}^d}\mathrm{tr}(Q^3)dx + \lambda(|Q|D, Q) \\
&\le -\Gamma\|\nabla Q\|_{L^2}^2 - a\Gamma\|Q\|_{L^2}^2 - c\Gamma\|Q\|_{L^4}^4 + C(a^2, b^2, \Gamma)\big(\|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big) \\
&\quad + \varepsilon|\lambda|\|\nabla u\|_{L^2}^2 + C(\varepsilon)|\lambda|\|Q\|_{L^4}^4 \\
&\le -\Gamma\|\nabla Q\|_{L^2}^2 + \varepsilon|\lambda|\|\nabla u\|_{L^2}^2 + \bar{C}\big(\|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big),
\end{aligned}
\tag{2.1.15}
$$

where $\varepsilon > 0$ will be decided later, and $\bar{C} = \bar{C}(a, b, c, \lambda, \Gamma, \varepsilon)$. Motivated by Case I, we notice that, for any $Q \in S_0^d$, there exists a positive, sufficiently large constant $M = M(a, b, c)$ such that

$$0 \le \frac{M}{2}|Q|^2 + \frac{c}{8}|Q|^4 \le (M + \frac{a}{2})|Q|^2 - \frac{b}{3}\text{tr}(Q^3) + \frac{c}{4}|Q|^4. \tag{2.1.16}$$

Multiplying (2.1.15) by $2M$, adding it to (2.1.13), and letting $\varepsilon = \frac{\mu}{8|\lambda|M}$, we have

$$\begin{aligned} &\frac{d}{dt}\big(E(t) + M\|Q\|_{L^2}^2\big) + \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 \\ &\le C\big(\|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big), \end{aligned} \tag{2.1.17}$$

where $C = C(a, b, c, \kappa, \mu, \lambda, \Gamma, M)$.

Then the desired estimates (2.1.6)–(2.1.7) for (Q, u) follow from Gronwall's inequality (Lemma A.0.9). □

2.2 WEAK SOLUTIONS

In this section, we prove the existence of weak solutions for system (2.0.2) with suitable initial data for $d = 2, 3$.

Definition 2.2.1. (Q, u) is called a weak solution of system (2.0.2) with the initial data:

$$Q(0, x) = \bar{Q}(x) \in H^1(\mathbb{R}^d), \quad u(0, x) = \bar{u}(x) \in L^2(\mathbb{R}^d), \quad \nabla \cdot \bar{u}(x) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^d), \tag{2.2.1}$$

if (Q, u) satisfies the following:

(i) $Q \in L^\infty_{\text{loc}}(\mathbb{R}_+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^2)$ and $u \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1)$;

(ii) For every compactly supported $\varphi \in C^\infty([0,\infty)\times\mathbb{R}^d; S_0^d)$ and $\psi \in C^\infty([0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$ with $\nabla\cdot\psi = 0$,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d}\big(Q:\partial_t\varphi+\Gamma\Delta Q:\varphi+Q:(u\cdot\nabla_x\varphi)-(Q\Omega-\Omega Q-\lambda|Q|D):\varphi\big)dxdt\\&=\Gamma\int_0^\infty\int_{\mathbb{R}^d}\Big(aQ-b\big(Q^2-\frac{\operatorname{tr}(Q^2)}{d}\mathrm{I}_d\big)+cQ\operatorname{tr}(Q^2)\Big):\varphi\,dxdt-\int_{\mathbb{R}^d}\bar{Q}(x):\varphi(0,x)\,dx,\end{aligned} \tag{2.2.2}$$

and

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d}\big(-u\cdot\partial_t\psi-u\cdot(u\cdot\nabla_x\psi)+\mu\nabla u:\nabla\psi^\top\big)dxdt-\int_{\mathbb{R}^d}\bar{u}(x)\cdot\psi(0,x)\,dx\\&=\int_0^\infty\int_{\mathbb{R}^d}\Big(\nabla Q\odot\nabla Q+\lambda|Q|H-(Q\Delta Q-\Delta QQ)-\kappa Q\Big):\nabla\psi\,dxdt.\end{aligned} \tag{2.2.3}$$

Theorem 2.2.1. *There exists a weak solution (Q,u) of system (2.0.2) subject to the initial conditions (2.2.1), for $d=2,3$, satisfying*

$$Q\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^1)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^2),\quad u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1). \tag{2.2.4}$$

Before proving Theorem 2.2.1, we introduce some useful notations:

(i) R_ε is the convolution operator with kernel $\varepsilon^{-d}\chi(\varepsilon^{-1}\cdot)$, where $\chi\in C_0^\infty$ is a radial positive function such that

$$\int_{\mathbb{R}^d}\chi(y)\mathrm{d}y=1.$$

(ii) The mollifying operator J_n, $n=1,2,\cdots$, is defined by

$$\mathcal{F}(J_nf)(\xi):=1_{[2^{-n},2^n]}(|\xi|)\mathcal{F}(f)(\xi),$$

where $\mathcal{F}$ is the Fourier transform.

(iii) $\mathcal{P}$ is the Leray projector onto divergence-free vector fields, *i.e.*,

$$\mathcal{P}:L^2(\Omega)\to H=\left\{\mathbf{w}\in(L^2(\Omega))^d:\nabla\cdot\mathbf{w}=0\right\},$$

which can be explicitly described in the Fourier domain by the tensor as

$$\mathcal{F}(\mathcal{P}(\mathbf{w}))(\xi)=\Big(\mathrm{I}_d-\frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathcal{F}(\mathbf{w})(\xi).$$

Next we prove Theorem 2.2.1 in the following three subsections. In §2.2.1, we construct regularized approximate solutions $(Q_\varepsilon^{(n)}, u_\varepsilon^n)$ to the approximation system (2.2.5). In §2.2.2, similarly to Proposition 2.1.2, we obtain some *a priori* bounds in (2.2.7), which allow us to obtain the convergence result (2.2.10) for $(Q_\varepsilon^{(n)}, u_\varepsilon^n)$ by the Aubin-Lions compactness lemma. Moreover, we can pass to the limit as n goes to infinity to achieve the weak solution $(Q_\varepsilon, u_\varepsilon)$ of the modified system (2.2.11). In §2.2.3, by studying the $\varepsilon \to 0$ limit of the modified system (2.2.11), we obtain the weak solution to system (2.0.2). However, we cannot use the previous *a priori* bounds in (2.2.7), since those bounds are not uniform with respect to ε. Instead, we need to repeat a similar procedure in order to obtain the uniform bounds in (2.2.14)–(2.2.15).

2.2.1 Regularized Approximation System

Let us consider the following approximation system for the active hydrodynamic system (2.0.2), followed by the classical Friedrichs' scheme, for any fixed $\varepsilon > 0$ and $n > 0$ (from now on, solution $(Q_\varepsilon^{(n)}, u_\varepsilon^n)$ is denoted by $(Q^{(n)}, u^n)$ for simplicity of notation when no confusion arises):

$$
\begin{cases}
\partial_t Q^{(n)} + J_n((\mathcal{P}J_nR_\varepsilon u^n \cdot \nabla)J_nQ^{(n)}) - J_n(\mathcal{P}J_n(R_\varepsilon\Omega^n)J_nQ^{(n)}) \\
\qquad\qquad + J_n(J_nQ^{(n)}\mathcal{P}J_n(R_\varepsilon\Omega^n)) - \lambda J_n(|J_nQ^{(n)}|\mathcal{P}J_n(R_\varepsilon D^n)) = \Gamma J_nH^{(n)}, \\
\partial_t u^n + \mathcal{P}J_n((\mathcal{P}J_nR_\varepsilon u^n \cdot \nabla)\mathcal{P}J_nu^n) - \mu\Delta\mathcal{P}J_nu^n \\
\qquad = -\varepsilon\mathcal{P}J_nR_\varepsilon(\partial_\alpha J_nQ^{(n)}_{\beta\gamma}(R_\varepsilon J_nu^n\cdot\nabla J_nQ^{(n)}_{\beta\gamma})|R_\varepsilon J_nu^n\cdot\nabla J_nQ^{(n)}|) \\
\qquad\quad + \varepsilon\mathcal{P}\nabla\cdot J_nR_\varepsilon(\nabla J_nR_\varepsilon u^n|\nabla J_nR_\varepsilon u^n|^2) \\
\qquad\quad - \mathcal{P}\nabla\cdot J_nR_\varepsilon(\nabla J_nQ^{(n)} \odot \nabla J_nQ^{(n)}) - \lambda\mathcal{P}\nabla\cdot J_nR_\varepsilon(|J_nQ^{(n)}|J_nH^{(n)}) \\
\qquad\quad + \mathcal{P}\nabla\cdot J_nR_\varepsilon(J_nQ^{(n)}\Delta J_nQ^{(n)} - \Delta J_nQ^{(n)}J_nQ^{(n)}) + \kappa\mathcal{P}\nabla\cdot J_nR_\varepsilon Q^{(n)}, \\
(Q^{(n)}, u^n)|_{t=0} = (J_nR_\varepsilon\bar{Q}, J_nR_\varepsilon\bar{u}),
\end{cases}
\tag{2.2.5}
$$

where

$$
H^{(n)} = \Delta Q^{(n)} - aQ^{(n)} + b\big((J_nQ^{(n)})^2 - \frac{\mathrm{tr}((J_nQ^{(n)})^2)}{d}\mathrm{I}_d\big) - cJ_nQ^{(n)}\mathrm{tr}((J_nQ^{(n)})^2).
$$

This approximate system can be regarded as a system of ordinary differential equations in L^2. By checking the conditions of the Cauchy-Lipschitz theorem [62], we know that it admits a unique maximal solution $(Q^{(n)}, u^n) \in C^1([0, T_n); L^2(\mathbb{R}^d; \mathbb{R}^{d\times d}) \times L^2(\mathbb{R}^d, \mathbb{R}^d))$ on some time interval $[0, T_n)$. By simple calculation, $(\mathcal{P}J_n)^2 = \mathcal{P}J_n$ and $J_n^2 = J_n$, so that pair $(J_nQ^{(n)}, \mathcal{P}J_nu^n)$ is also a solution of (2.2.5). By uniqueness, $(J_nQ^{(n)}, \mathcal{P}J_nu^n) = (Q^{(n)}, u^n)$. Therefore, $(Q^{(n)}, u^n)$ also satisfies the following system:

$$
\begin{cases}
\partial_t Q^{(n)} + J_n\big(R_\varepsilon u^n \nabla Q^{(n)}\big) - J_n(R_\varepsilon \Omega^n Q^{(n)} - Q^{(n)} R_\varepsilon \Omega^n) - \lambda J_n(|Q^{(n)}| R_\varepsilon D^n) = \Gamma J_n \bar{H}^{(n)}, \\
\partial_t u^n + \mathcal{P}J_n(R_\varepsilon u^n \nabla u^n) - \mu \Delta u^n \\
\qquad = -\varepsilon \mathcal{P} J_n R_\varepsilon\big(\partial_\alpha Q^{(n)}_{\beta\gamma}(R_\varepsilon u^n \cdot \nabla Q^{(n)}_{\beta\gamma}) |R_\varepsilon u^n \cdot \nabla Q^{(n)}|\big) \\
\qquad\quad + \varepsilon \mathcal{P} \nabla \cdot J_n R_\varepsilon (\nabla R_\varepsilon u^n |\nabla R_\varepsilon u^n|^2) \\
\qquad\quad - \mathcal{P}\nabla \cdot J_n R_\varepsilon(\nabla Q^{(n)} \odot \nabla Q^{(n)}) - \lambda \mathcal{P} \nabla \cdot J_n R_\varepsilon(|Q^{(n)}| J_n \bar{H}^{(n)}) \\
\qquad\quad + \mathcal{P}\nabla \cdot J_n R_\varepsilon\big(Q^{(n)} \Delta Q^{(n)} - \Delta Q^{(n)} Q^{(n)}\big) + \kappa \mathcal{P} \nabla \cdot (J_n R_\varepsilon Q^{(n)}), \\
(Q^{(n)}, u^n)|_{t=0} = (J_n R_\varepsilon \bar{Q}, J_n R_\varepsilon \bar{u}),
\end{cases}
\tag{2.2.6}
$$

where $(Q^{(n)}, u^n) \in C^1([0, T_n); \cap_{k=1}^\infty H^k)$ and

$$
\bar{H}^n = \Delta Q^{(n)} - aQ^{(n)} + b\big((Q^{(n)})^2 - \frac{\mathrm{tr}((Q^{(n)})^2)}{d} \mathrm{I}_d\big) - cQ^{(n)} \mathrm{tr}((Q^{(n)})^2).
$$

Remark 2.2.1. It is easy to see that, if $Q^{(n)}$ is a solution to system (2.2.5), so is $(Q^{(n)})^\top$. Hence, $Q^{(n)} = (Q^{(n)})^\top$ *a.e.*, in $[0, T_n] \times \mathbb{R}^d$, by the uniqueness of the solution. Moreover, from now on, we will work with the solution of the system (2.2.6) with this symmetry property, instead of the solution to the system (2.2.5).

2.2.2 Compactness and Convergence as $n \to \infty$ for System (2.2.6)

First, we need to derive some *a priori* estimates for the system (2.2.6) in the following proposition.

Proposition 2.2.1. *The solution $(Q^{(n)}, u^n)$ of the system (2.2.6) satisfies the following estimates, which are independent of n, for any $T < \infty$:*

$$\sup_n \|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3(0,T;L^3)} + \sup_n \|\nabla R_\varepsilon u^n\|_{L^4(0,T;L^4)} \le C,$$
$$\sup_n \|Q^{(n)}\|_{L^2(0,T;H^2)\cap L^\infty(0,T;H^1\cap L^4)} + \sup_n \|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2(0,T;L^2)} \le C, \tag{2.2.7}$$
$$\sup_n \|u^n\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \le C,$$

provided the initial data $(\bar{Q}, \bar{u}) \in H^1 \times L^2$. Moreover, if $\bar{Q} \in S_0^d$, then $Q^{(n)} \in S_0^d$.

Proof. Similarly to the proof of Proposition 2.1.1, we sum up the first equation in (2.2.6) multiplied by $-(\Delta Q^{(n)} - aQ^{(n)} + b(Q^{(n)})^2 - cQ^{(n)}\mathrm{tr}((Q^{(n)})^2))$ and the second equation multiplied by u^n, take the trace, and integrate by parts over $\mathbb{R}^d$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla Q^{(n)}|^2 + \frac{a}{2}|Q^{(n)}|^2 - \frac{b}{3}\mathrm{tr}((Q^{(n)})^3) + \frac{c}{4}|Q^{(n)}|^4 + \frac{1}{2}|u^n|^2\right)dx \\
&\quad + \mu\|\nabla u^n\|_{L^2}^2 + \Gamma\|\Delta Q^{(n)}\|_{L^2}^2 + a^2\Gamma\|Q^{(n)}\|_{L^2}^2 + 2ac\Gamma\|Q^{(n)}\|_{L^4}^4 \\
&\quad + b^2\Gamma\|J_n(Q^{(n)})^2\|_{L^2}^2 + c^2\Gamma\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + 2a\Gamma\|\nabla Q^{(n)}\|_{L^2}^2 \\
&\quad + \varepsilon\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3}^3 + \varepsilon\|\nabla R_\varepsilon u^n\|_{L^4}^4 \\
&= (R_\varepsilon u^n \nabla Q^{(n)}, \Delta Q^{(n)}) + (J_n(R_\varepsilon u^n \nabla Q^{(n)}), bJ_n(Q^{(n)})^2 - cJ_n(Q^{(n)}|Q^{(n)}|^2)) \\
&\quad - (R_\varepsilon \Omega^n Q^{(n)} - Q^{(n)} R_\varepsilon \Omega^n, \Delta Q^{(n)}) + 2c\Gamma(\Delta Q^{(n)}, Q^{(n)}|Q^{(n)}|^2) \\
&\quad + (J_n(R_\varepsilon \Omega^n Q^{(n)} - Q^{(n)} R_\varepsilon \Omega^n), -bJ_n(Q^{(n)})^2 + cJ_n(Q^{(n)}|Q^{(n)}|^2)) \\
&\quad - \lambda(|Q^{(n)}|R_\varepsilon D^n, J_n(\Delta Q^{(n)} - aQ^{(n)} + b(Q^{(n)})^2 - cQ^{(n)}|Q^{(n)}|^2)) \\
&\quad + 2\Gamma(-b\Delta Q^{(n)} + abQ^{(n)} + bcJ_n(Q^{(n)}|Q^{(n)}|^2), J_n(Q^{(n)})^2) \\
&\quad - (\nabla\cdot(\nabla Q^{(n)} \odot \nabla Q^{(n)}), R_\varepsilon u^n) + \lambda(|Q^{(n)}|J_n\bar{H}^{(n)}, \nabla R_\varepsilon u^n) \\
&\quad + (\nabla\cdot(Q^{(n)}\Delta Q^{(n)} - \Delta Q^{(n)} Q^{(n)}), R_\varepsilon u^n) - \kappa(Q^{(n)}, \nabla R_\varepsilon u^n) \\
&\quad + \frac{b\Gamma}{d}(\mathrm{tr}((Q^{(n)})^2)\mathrm{I}_d, J_n(\Delta Q^{(n)} - aQ^{(n)} + b(Q^{(n)})^2 - cQ^{(n)}|Q^{(n)}|^2)) \\
&= \sum_{i=1}^{12}\mathcal{I}_i.
\end{aligned} \tag{2.2.8}$$

By using cancellations analogous to §2.1, we have

$$\mathcal{I}_1 + \mathcal{I}_8 = 0, \quad \mathcal{I}_3 + \mathcal{I}_{10} = 0, \quad \mathcal{I}_6 + \mathcal{I}_9 = 0, \quad \mathcal{I}_4 \le 0.$$

The remaining terms can be estimated as follows:

$$
\begin{aligned}
\mathcal{I}_2 &= \big(J_n(R_\varepsilon u^n \nabla Q^{(n)}), bJ_n(Q^{(n)})^2 - cJ_n(Q^{(n)}|Q^{(n)}|^2)\big) \\
&\leq (\frac{2}{\Gamma} + \frac{1}{4C(b^2)})\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^2}^2 + \frac{c^2\Gamma}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + C(b^2)\|Q^{(n)}\|_{L^4}^4 \\
&\leq \frac{\varepsilon}{2}\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3}^3 + C(b^2, \varepsilon, \Gamma)\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^1} \\
&\quad + \frac{c^2\Gamma}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + C(b^2)\|Q^{(n)}\|_{L^4}^4 \\
&\leq \frac{\varepsilon}{2}\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3}^3 + C(b^2, \varepsilon, \Gamma)\left(\|u^n\|_{L^2}^2 + \|\nabla Q^{(n)}\|_{L^2}^2\right) \\
&\quad + \frac{c^2\Gamma}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + C(b^2)\|Q^{(n)}\|_{L^4}^4,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_5 &= \big(J_n(R_\varepsilon \Omega^n Q^{(n)} - Q^{(n)} R_\varepsilon \Omega^n), -bJ_n(Q^{(n)})^2 + cJ_n(Q^{(n)}|Q^{(n)}|^2)\big) \\
&\leq \big(\frac{4}{\Gamma} + \frac{1}{2C(b^2)}\big)\|R_\varepsilon \Omega^n Q^{(n)}\|_{L^2}^2 + \frac{\Gamma c^2}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + C(b^2)\|Q^{(n)}\|_{L^4}^4 \\
&\leq \frac{\varepsilon}{2}\|\nabla R_\varepsilon u^n\|_{L^4}^4 + C(b^2, \Gamma, \varepsilon)\|Q^{(n)}\|_{L^4}^4 + \frac{\Gamma c^2}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_7 &= 2\Gamma\big(-b\Delta Q^{(n)} + abQ^{(n)} + bcJ_n(Q^{(n)}|Q^{(n)}|^2), J_n(Q^{(n)})^2\big) \\
&\leq \frac{\Gamma}{4}\|\Delta Q^{(n)}\|_{L^2}^2 + \frac{c^2\Gamma}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + C(a^2, b^2, \Gamma)\big(\|Q^{(n)}\|_{L^2}^2 + \|Q^{(n)}\|_{L^4}^4\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{11} &= -\kappa(Q^{(n)}, \nabla R_\varepsilon u^n) \leq \frac{\mu}{4}\|\nabla R_\varepsilon u^n\|_{L^2}^2 + C(\kappa^2, \mu)\|Q^{(n)}\|_{L^2}^2 \\
&\leq \frac{\mu}{4}\|\nabla u^n\|_{L^2}^2 + C(\kappa^2, \mu)\|Q^{(n)}\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{12} &= \frac{b\Gamma}{d}(\mathrm{tr}((Q^{(n)})^2)\mathrm{I}_d, J_n(\Delta Q^{(n)} - aQ^{(n)} + b(Q^{(n)})^2 - cQ^{(n)}|Q^{(n)}|^2)) \\
&\leq \frac{\Gamma}{4}\|\Delta Q^{(n)}\|_{L^2}^2 + \frac{c^2\Gamma}{8}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + C\|Q^{(n)}\|_{L^2}^2 + C\|Q^{(n)}\|_{L^4}^4.
\end{aligned}
$$

Substituting all the above estimates into (2.2.8), we have

$$
\begin{aligned}
&\frac{d}{dt}E^n(t) + \frac{3\mu}{4}\|\nabla u^n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q^{(n)}\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 \\
&\quad + \frac{\varepsilon}{2}\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3}^3 + \frac{\varepsilon}{2}\|\nabla R_\varepsilon u^n\|_{L^4}^4 \\
&\leq C\left(\|Q^{(n)}\|_{L^2}^2 + \|Q^{(n)}\|_{L^4}^4 + \|u^n\|_{L^2}^2 + \|\nabla Q^{(n)}\|_{L^2}^2\right),
\end{aligned}
\tag{2.2.9}
$$

where C depends on $a, b, c, \kappa, \Gamma, \mu$, and ε, and

$$E^n(t) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla Q^{(n)}|^2 + \frac{a}{2}|Q^{(n)}|^2 - \frac{b}{3}\mathrm{tr}((Q^{(n)})^3) + \frac{c}{4}|Q^{(n)}|^4 + \frac{1}{2}|u^n|^2 \right) dx.$$

Again, by the same reasoning as in Proposition 2.1.2, $E^n(t)$ may be negative. We also need to estimate the L^2–norm of the Q-tensor separately in order to obtain the desired H^1–estimates for Q. Multiplying the first equation in system (2.2.6) by $Q^{(n)}$, taking the trace, integrating over $\mathbb{R}^d$ by parts, multiplying the result by $2M$ ($M > 0$ is sufficiently large), and adding it to (2.2.9), we have

$$\begin{aligned}
&\frac{d}{dt}(E^n(t) + M\|Q^{(n)}\|_{L^2}^2) + \frac{\mu}{2}\|\nabla u^n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q^{(n)}\|_{L^2}^2 \\
&\quad + \frac{c^2\Gamma}{2}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2 + \frac{\varepsilon}{2}\|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3}^3 + \frac{\varepsilon}{2}\|\nabla R_\varepsilon u^n\|_{L^4}^4 \\
&\quad \le C(\|Q^{(n)}\|_{L^2}^2 + \|Q^{(n)}\|_{L^4}^4 + \|u^n\|_{L^2}^2 + \|\nabla Q^{(n)}\|_{L^2}^2),
\end{aligned}$$

where $C = C(a, b, c, \kappa, \lambda, \Gamma, \varepsilon, \mu, M)$ is a constant, independent of n.

From the above estimate, along with Gronwall's inequality (Lemma A.0.9), we can conclude the *a priori* bounds in (2.2.7), which are independent of n, for any $T < \infty$.

In order to prove that $Q^{(n)} \in S_0^d$, besides the symmetry property mentioned in Remark 2.2.1, it remains to show $\mathrm{tr}(Q^{(n)}) = 0$. We take the trace on both sides of the first equation in system (2.2.6) and use $Q^{(n)} = (Q^{(n)})^\top$, $(\Omega^n)^\top = -\Omega^n$, and $\mathrm{tr}(D^n) = \mathrm{div}(u^n) = 0$ to obtain the following initial value problem:

$$\begin{cases}
\partial_t \mathrm{tr}(Q^{(n)}) + J_n\big(R_\varepsilon u^n \cdot \nabla \mathrm{tr}(Q^{(n)})\big) = \Gamma J_n(\Delta \mathrm{tr}(Q^{(n)}) - a\,\mathrm{tr}(Q^{(n)}) - c\,\mathrm{tr}(Q^{(n)})\mathrm{tr}(Q^{(n)})^2), \\
\mathrm{tr}(Q^{(n)})|_{t=0} = J_n R_\varepsilon \mathrm{tr}(\bar{Q}) = 0.
\end{cases}$$

Multiplying the above equation by $\mathrm{tr}(Q^{(n)})$, integrating by parts over $\mathbb{R}^d$, and using $J_n Q^{(n)} = Q^{(n)}$ and the uniform bounds of $Q^{(n)}$ in (2.2.7), we have

$$\begin{aligned}
&\frac{d}{dt}\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2 + \Gamma\|\nabla \mathrm{tr}(Q^{(n)})\|_{L^2}^2 = -a\Gamma\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2 - c\Gamma\int_{\mathbb{R}^d} |\mathrm{tr}(Q^{(n)})|^2|Q^{(n)}|^2 dx \\
&\le -a\Gamma\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2 + C\|Q^{(n)}\|_{L^6}^2\|\mathrm{tr}(Q^{(n)})\|_{L^6}\|\mathrm{tr}(Q^{(n)})\|_{L^2} \\
&\le -a\Gamma\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2 + C\|Q^{(n)}\|_{H^1}^2\|\nabla \mathrm{tr}(Q^{(n)})\|_{L^2}^{\frac{d}{3}}\|\mathrm{tr}(Q^{(n)})\|_{L^2}^{2-\frac{d}{3}} \\
&\le \frac{\Gamma}{2}\|\nabla \mathrm{tr}(Q^{(n)})\|_{L^2}^2 + C\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2,
\end{aligned}$$

thus,

$$\frac{d}{dt}\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2 \leq C\|\mathrm{tr}(Q^{(n)})\|_{L^2}^2,$$

where we have used the Sobolev imbedding, the Gagliardo-Nirenberg interpolation inequality in Lemma A.0.7 and the Cauchy inequality. Hence, we conclude that $\mathrm{tr}(Q^{(n)}) = 0$ by the initial condition. □

Then we can conclude from the uniform estimates in (2.2.7) that $T_n = \infty$. In addition, by using system (2.2.6) and the above estimates, we can compute the bounds for $\partial_t(Q^{(n)}, u^n)$ in some $L^1(0,T;H^{-N})$ for large enough N. Then, by the classical Aubin-Lions compactness lemma (Lemma A.0.6), we conclude that, subject to a subsequence,

$$\begin{aligned}
&Q^{(n)} \rightharpoonup Q \quad \text{in } L^2(0,T;H^2), \qquad Q^{(n)} \to Q \quad \text{in } L^2(0,T;H^{2-\delta}_{\mathrm{loc}}) \ \text{ for any } \delta \in (0, 2+N),\\
&Q^{(n)}(t) \rightharpoonup Q(t) \qquad \text{in } H^1 \ \text{ for any } t \in \mathbb{R}_+,\\
&Q^{(n)} \rightharpoonup Q \text{ in } L^p(0,T;H^1),\ Q^{(n)} \to Q \text{ in } L^p(0,T;H^{1-\delta}_{\mathrm{loc}}) \ \text{ for any } \delta \in (0,1+N),\ p \in [2,\infty],\\
&u^n \rightharpoonup u \quad \text{in } L^2(0,T;H^1), \qquad u^n \to u \quad \text{in } L^2(0,T;H^{1-\delta}_{\mathrm{loc}}) \ \text{ for any } \delta \in (0,1+N),\\
&u^n(t) \rightharpoonup u(t) \quad \text{in } L^2 \quad \text{for any } \ t \in \mathbb{R}_+.
\end{aligned} \tag{2.2.10}$$

As a result, we can pass to the limit as n goes to infinity to obtain a weak solution $(Q_\varepsilon, u_\varepsilon)$ of the following modified system (for simplicity, we denote $(Q_\varepsilon, u_\varepsilon)$ by (Q, u) when no confusion arises):

$$\begin{cases}
\partial_t Q + R_\varepsilon u \cdot \nabla Q + Q R_\varepsilon \Omega - R_\varepsilon \Omega Q - \lambda |Q| R_\varepsilon D = \Gamma H,\\
\partial_t u + \mathcal{P}(R_\varepsilon u \cdot \nabla u) - \mu \Delta u\\
\quad = -\varepsilon \mathcal{P} R_\varepsilon (\partial_\alpha Q_{\beta\gamma}(R_\varepsilon u \cdot \nabla Q_{\beta\gamma})|R_\varepsilon u \cdot \nabla Q|) + \varepsilon \mathcal{P}\nabla \cdot R_\varepsilon(\nabla R_\varepsilon u |\nabla R_\varepsilon u|^2)\\
\qquad - \mathcal{P}\nabla \cdot R_\varepsilon(\nabla Q \odot \nabla Q) - \lambda \nabla \cdot \mathcal{P} R_\varepsilon(|Q|H) + \mathcal{P}\nabla \cdot R_\varepsilon(Q\Delta Q - \Delta Q Q)\\
\qquad + \kappa \mathcal{P}\nabla \cdot R_\varepsilon Q,\\
(Q,u)|_{t=0} = (R_\varepsilon \bar{Q}, R_\varepsilon \bar{u}),
\end{cases} \tag{2.2.11}$$

such that

$$Q_\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^2), \quad u_\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1). \tag{2.2.12}$$

Remark 2.2.2. This modified system is obtained by mollifying the coefficients of the Q-tensor equation and the forcing terms of the velocity equation, and by adding the extra terms given by $-\varepsilon\partial_\alpha Q_{\beta\gamma}(u\cdot\nabla Q_{\beta\gamma})|u\cdot\nabla Q|$ and $\varepsilon\nabla\cdot(\nabla u|\nabla u|^2)$ to the velocity equation. These two terms are needed to estimate some *bad* terms which do not disappear in the energy estimates. Moreover, we would like to mention that the above procedure for obtaining the solution to system (2.2.11) follows from the classical Friedrichs' scheme. We also point out that the solutions to (2.2.11) are smooth, because we can bootstrap the regularity improvement provided by the linear heat equation to obtain the smooth regularity of Q, and bootstrap the regularity improvement provided by the linear advection equation to obtain the smooth regularity of u.

2.2.3 Compactness and Convergence as $\varepsilon \to 0$ for System (2.2.11)

In this subsection, we show that, by passing to the $\varepsilon \to 0$ limit in system (2.2.11), we can obtain a weak solution of the system (2.0.2). In order to do so, we need to achieve some uniform bounds for solution $(Q_\varepsilon, u_\varepsilon)$. Although, as a limit of $(Q_\varepsilon^{(n)}, u_\varepsilon^n)$, $(Q_\varepsilon, u_\varepsilon)$ still satisfies the *a priori* bounds in (2.2.7), we cannot apply these bounds in this step because they are not uniform with respect to ε. Therefore, we need to find new *a priori* bounds for the system (2.2.11). For simplicity of notation, we denote $(Q_\varepsilon, u_\varepsilon)$ by (Q, u) when no confusion arises below.

Applying the same procedure as in §2.1, that is, multiplying the first equation in (2.2.11) by $-H + 2MQ$ (with M a sufficiently large positive constant), taking the trace, integrating by parts over $\mathbb{R}^d$, and adding this to the second equation in (2.2.11) multiplied by u and integrated by parts over $\mathbb{R}^d$, with the analogous cancellations as in §2.1, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla Q|^2 + (\frac{a}{2}+M)|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 + \frac{1}{2}|u|^2\right)dx \\
&\quad + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 + \varepsilon\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^3 + \varepsilon\|\nabla R_\varepsilon u\|_{L^4}^4 \\
&\quad \le C(\|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^4}^4),
\end{aligned} \tag{2.2.13}$$

where $C = C(a, b, c, \kappa, \mu, \lambda, \Gamma, M)$ is independent of ε. Then, by Gronwall's inequality (Lemma A.0.9), we have the following *a priori* bounds, independent of ε, such that, for any $T < \infty$,

$$\sup_{\varepsilon} \|Q_\varepsilon\|_{L^2(0,T;H^2)\cap L^\infty(0,T;H^1\cap L^4)\cap L^6(0,T;L^6)} + \sup_{\varepsilon} \|u_\varepsilon\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \leq C. \tag{2.2.14}$$

Moreover, for any $\varepsilon > 0$, we have

$$\varepsilon\|R_\varepsilon u \cdot \nabla Q\|^3_{L^3(0,T;L^3)} + \varepsilon\|\nabla R_\varepsilon u\|^4_{L^4(0,T;L^4)} \leq C, \tag{2.2.15}$$

with constant C independent of ε. In addition, since $(Q_\varepsilon, u_\varepsilon)$ satisfies system (2.2.11), along with (2.2.14)–(2.2.15), we can obtain bounds for $\partial_t(Q_\varepsilon, u_\varepsilon)$ in $L^1(0,T;H^{-2})$. In order to do this, we need to estimate the H^{-2}-norm of each of the other terms in system (2.2.11), aside from $(\partial_t Q_\varepsilon, \partial_t u_\varepsilon)$. Since the estimates are similar to each other, we just show some tricky ones in the following:

$$\begin{aligned}\|\lambda|Q|R_\varepsilon D\|_{H^{-2}} &= \sup_{\varphi\in S_0^d, \|\varphi\|_{H_0^2}\leq 1} (\lambda|Q|R_\varepsilon D, \varphi) \leq C\|\nabla u\|_{L^2}\|Q\|_{L^2}\|\varphi\|_{L^\infty} \\ &\leq C\|\nabla u\|_{L^2}\|Q\|_{L^2}\|\varphi\|_{H_0^2} \leq C\|\nabla u\|_{L^2}\|Q\|_{L^2},\end{aligned}$$

$$\begin{aligned}\|\varepsilon\mathcal{P}\nabla\cdot R_\varepsilon(\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2)\|_{H^{-2}} &= \sup_{\|\psi\|_{H_0^2}\leq 1, \operatorname{div}(\psi)=0} (\varepsilon\mathcal{P}\nabla\cdot R_\varepsilon(\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2), \psi) \\ &= \sup_{\|\psi\|_{H_0^2}\leq 1, \operatorname{div}(\psi)=0} (\varepsilon\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2, R_\varepsilon\nabla\psi) \leq C\varepsilon\|\nabla R_\varepsilon u\|_{L^2}\|\nabla R_\varepsilon u\|^2_{L^4} \\ &\leq C\varepsilon\|\nabla u\|_{L^2}\|\nabla R_\varepsilon u\|^2_{L^4},\end{aligned}$$

$$\begin{aligned}\|\lambda\nabla\cdot\mathcal{P}R_\varepsilon(|Q|H)\|_{H^{-2}} &= \sup_{\|\psi\|_{H_0^2}\leq 1, \operatorname{div}(\psi)=0} (\lambda\nabla\cdot\mathcal{P}R_\varepsilon(|Q|H), \psi) \\ &= \sup_{\|\psi\|_{H_0^2}\leq 1, \operatorname{div}(\psi)=0} \Big(\lambda|Q|\big(\Delta Q - aQ + b(Q^2 - \frac{\operatorname{tr}(Q^2)}{d}\mathrm{I}_d) - cQ\operatorname{tr}(Q^2)\big), R_\varepsilon\nabla\psi\Big) \\ &\leq C\|Q\|_{L^2}\big(\|\Delta Q\|_{L^2} + \|Q\|_{L^2} + \|Q\|^2_{L^4} + \|Q\|^3_{L^6}\big),\end{aligned}$$

and

$$\|\kappa\mathcal{P}\nabla\cdot R_\varepsilon Q\|_{H^{-2}} = \sup_{\|\psi\|_{H_0^2}\leq 1, \operatorname{div}(\psi)=0} (\kappa\mathcal{P}\nabla\cdot R_\varepsilon Q, \psi) \leq C\|Q\|_{L^2}.$$

By the Sobolev interpolation inequality (Lemma A.0.7), one has

$$\|\nabla Q\|_{L^3} \le C\|D^2Q\|_{L^2}^{\frac{1}{2}}\|Q\|_{L^6}^{\frac{1}{2}}. \tag{2.2.16}$$

From the above inequality, we have

$$\begin{aligned}
&\|\varepsilon\mathcal{P}R_\varepsilon(\partial_\alpha Q_{\beta\gamma}(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})|R_\varepsilon u\cdot\nabla Q|)\|_{H^{-2}}\\
&=\sup_{\|\psi\|_{H_0^2}\le 1,\mathrm{div}(\psi)=0}(\varepsilon\mathcal{P}R_\varepsilon(\partial_\alpha Q_{\beta\gamma}(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})|R_\varepsilon u\cdot\nabla Q|),\psi)\\
&=\sup_{\|\psi\|_{H_0^2}\le 1,\mathrm{div}(\psi)=0}(\varepsilon\partial_\alpha Q_{\beta\gamma}(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})|R_\varepsilon u\cdot\nabla Q|,R_\varepsilon\psi)\\
&\le C\varepsilon\|\nabla Q\|_{L^3}\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^2\|\psi\|_{L^\infty}\\
&\le C\varepsilon\|D^2Q\|_{L^2}^{\frac{1}{2}}\|Q\|_{L^6}^{\frac{1}{2}}\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^2\|\psi\|_{L^\infty}\\
&\le C\varepsilon\big(\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^3+\|D^2Q\|_{L^2}^2+\|Q\|_{L^6}^6\big).
\end{aligned}$$

From all the above estimates, together with the uniform bounds (2.2.14)–(2.2.15), we can conclude that $(\partial_t u_\varepsilon,\partial_t Q_\varepsilon)\in L^1(0,T;H^{-2})$. Then, by the classical Aubin-Lions compactness lemma (Lemma A.0.6), we know that there exist $Q\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^1)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^2)$ and $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1)$ such that, subject to a subsequence, we have

$$\begin{aligned}
&Q_\varepsilon\rightharpoonup Q\quad\text{in } L^2(0,T;H^2),\quad Q_\varepsilon\to Q\quad\text{in } L^2(0,T;H^{2-\delta}_{\mathrm{loc}})\quad\text{for any }\delta\in(0,4),\\
&Q_\varepsilon(t)\rightharpoonup Q(t)\quad\text{in } H^1\quad\text{for any } t\in\mathbb{R}_+,\\
&Q_\varepsilon\rightharpoonup Q\quad\text{in } L^p(0,T;H^1),\quad Q_\varepsilon\to Q\quad\text{in } L^p(0,T;H^{1-\delta}_{\mathrm{loc}})\quad\text{for any }\delta\in(0,3),p\in[2,\infty),\\
&u_\varepsilon\rightharpoonup u\quad\text{in } L^2(0,T;H^1)\quad\text{and}\quad u_\varepsilon\to u\quad\text{in } L^2(0,T;H^{1-\delta}_{\mathrm{loc}})\quad\text{for any }\delta\in(0,3),\\
&u_\varepsilon(t)\rightharpoonup u(t)\quad\text{in } L^2\quad\text{for any } t\in\mathbb{R}_+.
\end{aligned}\tag{2.2.17}$$

With the above result, we can pass to the limit in the weak solution $(Q_\varepsilon,u_\varepsilon)$ of system (2.2.11), as $\varepsilon\to 0$, to obtain a weak solution (Q,u) of system (2.0.2) satisfying (2.2.2)–(2.2.3). In the following, we focus on some terms that are not so easy to deal with.

First, we observe that

$$\begin{aligned}
&\partial_\beta(Q_\varepsilon)_{\gamma\delta}R_\varepsilon(\partial_\beta\psi_\alpha)-\partial_\beta Q_{\gamma\delta}\partial_\beta\psi_\alpha\\
&=\partial_\beta(Q_\varepsilon)_{\gamma\delta}\big(R_\varepsilon(\partial_\beta\psi_\alpha)-\partial_\beta\psi_\alpha\big)+\big(\partial_\beta(Q_\varepsilon)_{\gamma\delta}-\partial_\beta Q_{\gamma\delta}\big)\partial_\beta\psi_\alpha\\
&=\mathcal{I}^1_\varepsilon+\mathcal{I}^2_\varepsilon
\end{aligned}\tag{2.2.18}$$

converges to zero strongly in $L^2(0,T;L^2)$. This is owing to the fact that $\mathcal{I}^1_\varepsilon$ converges to zero strongly in $L^2(0,T;L^2)$, since $R_\varepsilon(\partial_\beta\psi_\alpha)-\partial_\beta\psi_\alpha$ converges strongly to zero in any $L^p(0,T;L^q)$ (ψ is compactly supported and smooth) and Q_ε is uniformly bounded in $L^\infty(0,T;H^1)$, and the fact that $\mathcal{I}^2_\varepsilon$ converges to zero strongly in $L^2(0,T;L^2)$, since Q_ε converges to Q strongly in $L^2(0,T;H^{2-\delta}_{\text{loc}})$ and ψ is compactly supported and smooth.
Combining the above facts with the weak convergence of Q_ε in $L^2(0,T;H^2)$ in (2.2.17) yields

$$\begin{aligned}
\int_0^\infty\int_{\mathbb{R}^d}R_\varepsilon\big(\partial_\alpha(Q_\varepsilon)_{\gamma\delta}\partial_\beta(Q_\varepsilon)_{\gamma\delta}\big)\partial_\beta\psi_\alpha dxdt&=\int_0^\infty\int_{\mathbb{R}^d}\partial_\alpha(Q_\varepsilon)_{\gamma\delta}\partial_\beta(Q_\varepsilon)_{\gamma\delta}R_\varepsilon\partial_\beta\psi_\alpha dxdt\\
&\to\int_0^\infty\int_{\mathbb{R}^d}\partial_\alpha Q_{\gamma\delta}\partial_\beta Q_{\gamma\delta}\partial_\beta\psi_\alpha dxdt.
\end{aligned}$$

Moreover, from the strong convergence of Q_ε in $L^p(0,T;H^{1-\delta}_{\text{loc}})$ for $p\in[1,\infty)$ in (2.2.17) and the uniform bound of Q_ε in $L^6(0,T;L^6)$ in (2.2.14), we have

$$|Q_\varepsilon|^3Q_\varepsilon\to|Q|^3Q,\quad |Q_\varepsilon|Q_\varepsilon^2\to|Q|Q^2\qquad\text{in }\;L^1(0,T;L^1_{\text{loc}}).$$

As a result, we have the following convergence:

$$\begin{aligned}
&\lambda\int_0^\infty\int_{\mathbb{R}^d}R_\varepsilon(|Q_\varepsilon|H^\varepsilon):\nabla\psi\,dxdt=\lambda\int_0^\infty\int_{\mathbb{R}^d}|Q_\varepsilon|H^\varepsilon:R_\varepsilon\nabla\psi\,dxdt\\
&=\lambda\int_0^\infty\int_{\mathbb{R}^d}|Q_\varepsilon|\Big(\Delta Q_\varepsilon-aQ_\varepsilon+b\big(Q_\varepsilon^2+\frac{\text{tr}((Q_\varepsilon)^2)}{d}\text{I}_d\big)-cQ_\varepsilon\text{tr}((Q_\varepsilon)^2)\Big):R_\varepsilon\nabla\psi\,dxdt\\
&\to\lambda\int_0^\infty\int_{\mathbb{R}^d}|Q|H:\nabla\psi\,dxdt.
\end{aligned}\tag{2.2.19}$$

Finally, from the uniform bounds of $\varepsilon\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^3$ in (2.2.15), along with (2.2.16), we have

$$\begin{aligned}
&\varepsilon\int_0^\infty\int_{\mathbb{R}^d} R_\varepsilon\big(|R_\varepsilon u\cdot\nabla Q|(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})\partial_\alpha Q_{\beta\gamma}\big)\psi_\alpha\,dxdt\\
&=\varepsilon\int_0^T\int_{\mathbb{R}^d}\big(|R_\varepsilon u\cdot\nabla Q|(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})\partial_\alpha Q_{\beta\gamma}\big)R_\varepsilon\psi_\alpha\,dxdt\\
&\le\varepsilon\int_0^T\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^2\|\nabla Q\|_{L^3}\|R_\varepsilon\psi\|_{L^\infty}dt\\
&\le C\varepsilon\int_0^T\|R_\varepsilon u\cdot\nabla Q\|_{L^3}^2\|D^2Q\|_{L^2}^{\frac12}\|Q\|_{L^6}^{\frac12}dt\\
&\le C\varepsilon\|R_\varepsilon u\cdot\nabla Q\|_{L^3(0,T;L^3)}^2\|Q\|_{L^6(0,T;L^6)}^{\frac12}\|Q\|_{L^2(0,T;H^2)}^{\frac12}\\
&\to 0.
\end{aligned}\tag{2.2.20}$$

Similarly to the above estimate, by using the uniform bound of $\varepsilon\|\nabla R_\varepsilon u\|_{L^4}^4$ in (2.2.15), we also conclude that $\varepsilon\int R_\varepsilon\big(\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2\big):\nabla\psi\,dxdt\to 0$ as $\varepsilon\to 0$.

Then the proof of Theorem 2.2.1 is completed.

We remark that, in the proof above, because of the active terms in our system (2.0.2), in order to obtain the convergence of the approximate solutions, we need to establish a higher integrability of Q in time and use the Sobolev interpolation inequalities to achieve the uniform H^{-2}–estimates for the extra terms added to the approximate system, which is different from the passive case treated in [50].

2.3 HIGHER REGULARITY IN TWO DIMENSIONS

In this section, we prove that, in the two-dimensional case, system (2.0.2) has solutions with higher regularity, provided with sufficiently regular initial data. The result is stated in Theorem 2.3.1. We mention that we use the Littlewood-Paley decomposition to help us improve the regularity of the solution. Of course, we can also obtain the higher regularity by differentiating the equations $k\ge 1$ times in system (2.0.2). However, this requires the initial data $(\bar{Q},\bar{u})$ to be at least in $H^2\times H^1$, rather than in $H^{s+1}\times H^s$ for $s>0$ in the Littlewood-Paley method.

Remark 2.3.1. For any traceless, symmetric, 2×2 matrix Q,

$$Q^2 - \frac{\mathrm{tr}(Q^2)}{2}\mathrm{I}_2 = 0,$$

which means that H reduces to a simpler form

$$H = \Delta Q - aQ - c\,Q\,\mathrm{tr}(Q^2) \qquad \text{for } x \in \mathbb{R}^2.$$

Theorem 2.3.1. *Let $s > 0$ and $(\bar{Q}, \bar{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$. There exists a global solution $(Q(t,x), u(t,x))$ to system (2.0.2) with the initial conditions:*

$$Q(0,x) = \bar{Q}(x), \qquad u(0,x) = \bar{u}(x) \tag{2.3.1}$$

such that

$$Q \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)),$$

$$u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{s}(\mathbb{R}^2)),$$

and

$$\|\nabla Q(t,\cdot)\|^2_{H^s} + \|u(t,\cdot)\|^2_{H^s} \leq Ce^{e^{Ct}}, \tag{2.3.2}$$

where constant C depends on $\bar{Q}, \bar{u}, a, b, c,$ and Γ.

To prove this theorem, we restrict ourselves to system (2.0.2) in two spatial dimensions and give the *a priori* estimates for the smooth solutions of this system. Of course, the same estimates independent of ε can also be obtained, if we use the modified system (2.2.11), whose solutions are smooth as we mentioned in Remark 2.2.2. Moreover, we use the Littlewood-Paley decomposition to obtain the *a priori* estimates of the solution. Firstly, we apply Δ_q (see Appendix A for the notations), with $q \in \mathbb{N}$, to the equations in system (2.0.2) to obtain the estimates of high frequencies of the solution. Secondly, by applying S_0 to this system, we obtain the estimates of low frequencies. Finally, we achieve the high regularity of the solution by combining the high and low frequencies together and by using Gronwall's inequalities (Lemma A.0.9).

Proof. We begin with the estimates of high frequencies. Applying Δ_q to the first equation in (2.0.2), using Bony's paraproduct decomposition, multiplying the equation by $-\Delta\Delta_q Q$, taking the trace, and integrating by parts over $\mathbb{R}^2$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta_q Q\|_{L^2}^2 + \Gamma\|\Delta\Delta_q Q\|_{L^2}^2 + (\Delta_q\Omega S_{q-1}Q - S_{q-1}Q\Delta_q\Omega, \Delta\Delta_q Q)\\
&= (\Delta_q(u\nabla Q), \Delta\Delta_q Q) - \sum_{|q'-q|\le 5}([\Delta_q; S_{q'-1}Q_{\gamma\beta}]\Delta_{q'}\Omega_{\alpha\gamma}, \Delta\Delta_q Q_{\alpha\beta})\\
&\quad - \sum_{|q'-q|\le 5}((S_{q'-1}Q_{\gamma\beta} - S_{q-1}Q_{\gamma\beta})\Delta_q\Delta_{q'}\Omega_{\alpha\gamma}, \Delta\Delta_q Q_{\alpha\beta})\\
&\quad - \sum_{q'>q-5}(\Delta_q(S_{q'+2}\Omega_{\alpha\gamma}\Delta_{q'}Q_{\gamma\beta}), \Delta\Delta_q Q_{\alpha\beta})\\
&\quad + \sum_{|q'-q|\le 5}([\Delta_q; S_{q'-1}Q_{\alpha\gamma}]\Delta_{q'}\Omega_{\gamma\beta}, \Delta\Delta_q Q_{\alpha\beta})\\
&\quad + \sum_{|q'-q|\le 5}((S_{q'-1}Q_{\alpha\gamma} - S_{q-1}Q_{\alpha\gamma})\Delta_q\Delta_{q'}\Omega_{\gamma\beta}, \Delta\Delta_q Q_{\alpha\beta})\\
&\quad + \sum_{q'>q-5}(\Delta_q(S_{q'+2}\Omega_{\gamma\beta}\Delta_{q'}Q_{\alpha\gamma}), \Delta\Delta_q Q_{\alpha\beta})\\
&\quad - \lambda(\Delta_q(|Q|D), \Delta\Delta_q Q) + \Gamma a(\Delta_q Q, \Delta\Delta_q Q) + \Gamma c(\Delta_q(Q\mathrm{tr}(Q^2)), \Delta\Delta_q Q)\\
&= \sum_{i=1}^{10}\mathcal{I}_i.
\end{aligned} \tag{2.3.3}$$

Similarly, we can apply Δ_q to the second equation in (2.0.2), use Bony's paraproduct decomposition again, multiply the equation by $\Delta_q u$, and integrate by parts over $\mathbb{R}^2$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 + \mu\|\nabla\Delta_q u\|_{L^2}^2 + (S_{q-1}Q\Delta_q\Delta Q - \Delta_q\Delta Q S_{q-1}Q, \Delta_q\nabla u)\\
&= -(\Delta_q(u\cdot\nabla u), \Delta_q u) + (\Delta_q(\partial_\alpha Q_{\gamma\delta}\partial_\beta Q_{\gamma\delta}), \Delta_q\partial_\beta u_\alpha)\\
&\quad - \sum_{|q'-q|\le 5}([\Delta_q; S_{q'-1}Q_{\alpha\gamma}]\Delta_{q'}\Delta Q_{\gamma\beta}, \Delta_q\partial_\beta u_\alpha)\\
&\quad - \sum_{|q'-q|\le 5}((S_{q'-1}Q_{\alpha\gamma} - S_{q-1}Q_{\alpha\gamma})\Delta_q\Delta_{q'}\Delta Q_{\gamma\beta}, \Delta_q\partial_\beta u_\alpha)\\
&\quad - \sum_{q'>q-5}(\Delta_q(S_{q'+2}\Delta Q_{\gamma\beta}\Delta_{q'}Q_{\alpha\gamma}), \Delta_q\partial_\beta u_\alpha)\\
&\quad + \sum_{|q'-q|\le 5}([\Delta_q; S_{q'-1}Q_{\gamma\beta}]\Delta_{q'}\Delta Q_{\alpha\gamma}, \Delta_q\partial_\beta u_\alpha)
\end{aligned} \tag{2.3.4}$$

$$
\begin{aligned}
&+\sum_{|q'-q|\le 5}((S_{q'-1}Q_{\gamma\beta}-S_{q-1}Q_{\gamma\beta})\Delta_q\Delta_{q'}\Delta Q_{\alpha\gamma},\Delta_q\partial_\beta u_\alpha)\\
&+\sum_{q'>q-5}(\Delta_q(S_{q'+2}\Delta Q_{\alpha\gamma}\Delta_{q'}Q_{\gamma\beta}),\Delta_q\partial_\beta u_\alpha)+\lambda(\Delta_q(|Q|\Delta Q),\nabla\Delta_q u)\\
&-a\lambda(\Delta_q(|Q|Q),\nabla\Delta_q u)-c\lambda(\Delta_q(|Q|Q\,\mathrm{tr}(Q^2)),\nabla\Delta_q u)-\kappa(\Delta_q Q,\nabla\Delta_q u)\\
=&\sum_{j=1}^{12}\mathcal{J}_j.
\end{aligned}
$$

Adding up (2.3.3)–(2.3.4) and using Lemma A.0.5, we have

$$
\frac{1}{2}\frac{d}{dt}\big(\|\nabla\Delta_q Q\|_{L^2}^2+\|\Delta_q u\|_{L^2}^2\big)+\mu\|\nabla\Delta_q u\|_{L^2}^2+\Gamma\|\Delta\Delta_q Q\|_{L^2}^2=\sum_{i=1}^{10}\mathcal{I}_i+\sum_{j=1}^{12}\mathcal{J}_j. \tag{2.3.5}
$$

Set

$$
\varphi(t):=\|\nabla Q\|_{H^s}^2+\|u\|_{H^s}^2,\quad \varphi_1(t):=\|S_0\nabla Q\|_{L^2}^2+\|S_0 u\|_{L^2}^2,\quad \varphi_2(t):=\varphi(t)-\varphi_1(t),
$$

with φ_1 and φ_2 representing the low-frequency part and high-frequency part of φ, respectively. Then (2.3.5) leads to the following estimate (see Appendix B for the details):

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\varphi_2(t)+\sum_{q\in\mathbb{N}}2^{2qs}\big(\mu\|\Delta_q\nabla u\|_{L^2}^2+\Gamma\|\Delta_q\Delta Q\|_{L^2}^2\big)\\
\le& C\big(1+\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\|\nabla Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2+\|Q\|_{L^2}^2+\|Q\|_{L^4}^4+\|Q\|_{L^6}^6\big)\varphi(t)\\
&+\frac{\Gamma}{4}\|\Delta Q\|_{H^s}^2+\frac{\mu}{4}\|\nabla u\|_{H^s}^2.
\end{aligned}
\tag{2.3.6}
$$

Next, we estimate the low frequencies. Applying S_0 to the first equation in (2.0.2), multiplying by $-S_0\Delta Q$, taking the trace, and integrating by parts over $\mathbb{R}^2$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla S_0 Q\|_{L^2}^2+\Gamma\|S_0\Delta Q\|_{L^2}^2\\
&=(S_0(u\cdot\nabla Q),S_0\Delta Q)+(S_0(Q\Omega-\Omega Q),S_0\Delta Q)\\
&\quad-\lambda(S_0(|Q|\Omega),S_0\Delta Q)+a\Gamma(S_0Q,S_0\Delta Q)+c\Gamma(S_0(Q\mathrm{tr}(Q^2)),S_0\Delta Q)\\
&\le C\|u\|_{L^4}\|\nabla Q\|_{L^4}\|S_0\Delta Q\|_{L^2}+\|S_0(Q\Omega-\Omega Q)\|_{L^2}\|S_0\Delta Q\|_{L^2}\\
&\quad+C\|S_0(|Q|\Omega)\|_{L^2}\|S_0\Delta Q\|_{L^2}+C\|S_0\nabla Q\|_{L^2}^2+C\|Q\|_{L^6}^3\|S_0\Delta Q\|_{L^2}\\
&\le C\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla Q\|_{L^2}^{\frac12}\|\Delta Q\|_{L^2}^{\frac12}\|S_0\Delta Q\|_{L^2}+\|S_0(Q\Omega-\Omega Q)\|_{L^1}\|S_0\Delta Q\|_{L^2}
\end{aligned}
\tag{2.3.7}
$$

$$
\begin{aligned}
&+ C\|S_0(|Q|\Omega)\|_{L^1}\|S_0\Delta Q\|_{L^2} + C\|S_0\nabla Q\|_{L^2}^2 + C\|Q\|_{L^6}^3\|S_0\Delta Q\|_{L^2} \\
&\le C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla Q\|_{L^2}^{\frac{1}{2}}\|\Delta Q\|_{L^2}^{\frac{1}{2}}\|S_0\Delta Q\|_{L^2} + C\|Q\|_{L^2}\|\nabla u\|_{L^2}\|S_0\Delta Q\|_{L^2} \\
&+ C\|S_0\nabla Q\|_{L^2}^2 + C\|Q\|_{L^6}^3\|S_0\Delta Q\|_{L^2} \\
&\le \frac{\Gamma}{4}\|S_0\Delta Q\|_{L^2}^2 + C\|\nabla Q\|_{H^s}^2 + C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C\|\nabla Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2 \\
&+ C\|Q\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C\|Q\|_{L^6}^6.
\end{aligned}
$$

Then we apply S_0 to the second equation in (2.0.2), multiply by $S_0 u$, and integrate by parts over $\mathbb{R}^2$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|S_0 u\|_{L^2}^2 + \mu\|S_0\nabla u\|_{L^2}^2 \\
&= -(S_0(u\cdot\nabla u), S_0 u) + (S_0(\nabla Q\odot\nabla Q), S_0\nabla u) - (S_0\nabla\cdot(Q\Delta Q - \Delta Q Q), S_0 u) \\
&\quad - \lambda(S_0\nabla\cdot(|Q|(\Delta Q - aQ - cQ\operatorname{tr}(Q^2))), S_0 u) + \kappa(S_0\nabla\cdot Q, S_0 u) \\
&\le \|S_0(u\cdot\nabla u)\|_{L^2}\|S_0 u\|_{L^2} + \|S_0(\nabla Q\odot\nabla Q)\|_{L^2}\|S_0\nabla u\|_{L^2} \\
&\quad + |\lambda|\|S_0(|Q|(\Delta Q - aQ - cQ\operatorname{tr}(Q^2)))\|_{L^2}\|S_0\nabla u\|_{L^2} \\
&\quad + \|S_0(Q\Delta Q - \Delta Q Q)\|_{L^2}\|S_0\nabla u\|_{L^2} + |\kappa|\|S_0\nabla\cdot Q\|_{L^2}\|S_0 u\|_{L^2} \\
&\le C\|S_0(u\cdot\nabla u)\|_{L^1}\|S_0 u\|_{L^2} + C\|S_0(\nabla Q\odot\nabla Q)\|_{L^1}\|S_0\nabla u\|_{L^2} \qquad (2.3.8) \\
&\quad + C\|S_0(|Q|(\Delta Q - aQ - cQ\operatorname{tr}(Q^2)))\|_{L^1}\|S_0\nabla u\|_{L^2} \\
&\quad + C\|S_0(Q\Delta Q - \Delta Q Q)\|_{L^1}\|S_0\nabla u\|_{L^2} + C\|S_0\nabla\cdot Q\|_{L^2}\|S_0 u\|_{L^2} \\
&\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|S_0 u\|_{L^2} + C\|\nabla Q\|_{L^2}^2\|S_0\nabla u\|_{L^2} \\
&\quad + C\big(\|Q\|_{L^2}\|\Delta Q\|_{L^2} + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big)\|S_0\nabla u\|_{L^2} + C\|S_0\nabla Q\|_{L^2}\|S_0 u\|_{L^2} \\
&\le \frac{\mu}{4}\|S_0\nabla u\|_{L^2}^2 + C\|u\|_{H^s}^2 + C\|\nabla Q\|_{H^s}^2 + C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C\|\nabla Q\|_{L^2}^4 \\
&\quad + C\big(\|Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2 + \|Q\|_{L^2}^4 + \|Q\|_{L^4}^8\big).
\end{aligned}
$$

We add (2.3.7)–(2.3.8) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\varphi_1(t) + \frac{3\mu}{4}\|S_0\nabla u\|_{L^2}^2 + \frac{3\Gamma}{4}\|S_0\Delta Q\|_{L^2}^2 \\
&\le C\varphi(t) + C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C\|\nabla Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2 + C\|Q\|_{L^2}^2\|\nabla u\|_{L^2}^2 \qquad (2.3.9) \\
&\quad + C\|Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2 + C\big(\|Q\|_{L^2}^4 + \|Q\|_{L^4}^8 + \|Q\|_{L^6}^6 + \|\nabla Q\|_{L^2}^4\big).
\end{aligned}
$$

Finally, we derive the estimates of the high norms. Adding (2.3.6) and (2.3.9), we have

$$\frac{d}{\mathrm{d}t}\varphi(t)+\frac{\mu}{2}\|\nabla u\|_{H^s}^2+\frac{\Gamma}{2}\|\Delta Q\|_{H^s}^2\le A(t)\varphi(t)+B(t), \tag{2.3.10}$$

with

$$\begin{aligned}
A(t)=&C\big(1+\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\|\nabla Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2+\|Q\|_{L^2}^2+\|Q\|_{L^4}^4+\|Q\|_{L^6}^6\big),\\
B(t)=&C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+C\|\nabla Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2+C\|Q\|_{L^2}^2\|\nabla u\|_{L^2}^2\\
&+C\|Q\|_{L^2}^2\|\Delta Q\|_{L^2}^2+C\big(\|Q\|_{L^2}^4+\|Q\|_{L^4}^8+\|Q\|_{L^6}^6+\|\nabla Q\|_{L^2}^4\big).
\end{aligned}$$

From the *a priori* estimates in Proposition 2.1.2, we know that both $A(t)$ and $B(t)$ belong to $L^1(0,T)$ and increase exponentially in time. Then, by Gronwall's inequality (Lemma A.0.9), we can conclude that $\varphi(t)$ increases double exponentially in time as stated in (2.3.2). □

2.4 WEAK-STRONG UNIQUENESS IN DIMENSION TWO

In this section, we prove that a global weak solution and a strong one must coincide, provided that they have the same initial data $(\bar{Q},\bar{u})\in H^{s+1}(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ for $s>0$. The result is stated in the following theorem.

Theorem 2.4.1. *Let $(\bar{Q},\bar{u})\in H^{s+1}(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$, with $s>0$, be the initial data. By Theorem 2.2.1, there exists a weak solution (Q_1,u_1) of system (2.0.2) such that*

$$Q_1\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^1)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^2),\qquad u_1\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1). \tag{2.4.1}$$

Theorem 2.3.1 gives the existence of a strong solution (Q_2,u_2) such that

$$Q_2\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{s+1})\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^{s+2}),\qquad u_2\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^s)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^{s+1}). \tag{2.4.2}$$

Then $(Q_1,u_1)=(Q_2,u_2)$.

Proof. Denote $\delta Q := Q_1 - Q_2$ and $\delta u := u_1 - u_2$. Then $(\delta Q, \delta u)$ satisfies the following system:

$$\begin{cases}
\partial_t \delta Q + \delta u \cdot \nabla \delta Q - \delta\Omega \delta Q + \delta Q \delta\Omega + \delta u \cdot \nabla Q_2 + u_2 \cdot \nabla \delta Q + Q_2 \delta\Omega + \delta Q \Omega_2 \\
-\delta\Omega Q_2 - \Omega_2 \delta Q \\
\qquad = \lambda |Q_1| \delta D + \lambda(|Q_1| - |Q_2|) D_2 \\
\qquad\quad + \Gamma\big(\Delta \delta Q - a\delta Q - c(\delta Q \mathrm{tr}(Q_1^2) + Q_2 \mathrm{tr}(Q_1 \delta Q + \delta Q Q_2))\big), \\
\partial_t \delta u + \mathcal{P}(\delta u \cdot \nabla \delta u) \\
\qquad = \mu \Delta \delta u - \mathcal{P}(\nabla \cdot (\nabla \delta Q \odot \nabla \delta Q)) \\
\qquad\quad + \mathcal{P}(\nabla \cdot (\delta Q \Delta \delta Q - \Delta \delta Q \delta Q)) - \mathcal{P}(u_2 \cdot \nabla \delta u + \delta u \cdot \nabla u_2) \\
\qquad\quad + \mathcal{P}(\nabla \cdot (\delta Q \Delta Q_2 + Q_2 \Delta \delta Q - \Delta \delta Q Q_2 - \Delta Q_2 \delta Q)) \\
\qquad\quad - \mathcal{P}(\nabla \cdot (\nabla \delta Q \odot \nabla Q_2 + \nabla Q_2 \odot \nabla \delta Q)) - \lambda \mathcal{P}(\nabla \cdot (|Q_1|(\Delta \delta Q - a\delta Q))) \\
\qquad\quad - \lambda \mathcal{P}(\nabla \cdot (|Q_1| - |Q_2|)(\Delta Q_2 - aQ_2)) + \lambda c \mathcal{P}\big(\nabla \cdot (|Q_1| \delta Q \mathrm{tr}(Q_1^2))\big) \\
\qquad\quad + \lambda c \mathcal{P}\big(\nabla \cdot (|Q_1| Q_2 \mathrm{tr}(Q_1 \delta Q + \delta Q Q_2))\big) \\
\qquad\quad + \lambda c \mathcal{P}\big(\nabla \cdot ((|Q_1| - |Q_2|) Q_2 \mathrm{tr}(Q_2^2))\big) + \kappa \mathcal{P}(\nabla \cdot \delta Q).
\end{cases} \tag{2.4.3}$$

Similarly to the proof of Proposition 2.1.1, we multiply the first equation in (2.4.3) by $-\Delta \delta Q + \delta Q$, take the trace, and integrate by parts over $\mathbb{R}^2$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\nabla \delta Q\|_{L^2}^2 + \|\delta Q\|_{L^2}^2\big) - (\delta u \cdot \nabla \delta Q, \Delta \delta Q) - (\delta Q \delta\Omega - \delta\Omega \delta Q, \Delta \delta Q) \\
&\quad - (\delta u \cdot \nabla Q_2 + u_2 \cdot \nabla \delta Q + \delta Q \Omega_2 - \Omega_2 \delta Q, \Delta \delta Q) + (\delta u \cdot \nabla Q_2, \delta Q) \\
&\quad - (Q_2 \delta\Omega - \delta\Omega Q_2, \Delta \delta Q) + (Q_2 \delta\Omega - \delta\Omega Q_2, \delta Q) \\
&= -\Gamma \|\Delta \delta Q\|_{L^2}^2 - \Gamma \|\nabla \delta Q\|_{L^2}^2 - a\Gamma \|\nabla \delta Q\|_{L^2}^2 - a\Gamma \|\delta Q\|_{L^2}^2 \\
&\quad + \lambda(|Q_1| \delta D + (|Q_1| - |Q_2|) D_2, -\Delta \delta Q + \delta Q) + c\Gamma(\delta Q \mathrm{tr}(Q_1^2), \Delta \delta Q) \\
&\quad + c\Gamma(Q_2 \mathrm{tr}(Q_1 \delta Q + \delta Q Q_2), \Delta \delta Q) - c\Gamma(\delta Q |Q_1|^2, \delta Q) \\
&\quad - c\Gamma(Q_2 \mathrm{tr}(Q_1 \delta Q + \delta Q Q_2), \delta Q).
\end{aligned} \tag{2.4.4}$$

Multiplying the second equation in (2.4.3) by δu and integrating by parts over $\mathbb{R}^2$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2+\mu\|\nabla\delta u\|_{L^2}^2\\
&=-(\nabla\cdot(\nabla\delta Q\odot\nabla\delta Q),\delta u)-(\delta Q\Delta\delta Q-\Delta\delta Q\delta Q,\nabla\delta u^\top)\\
&\quad-(u_2\cdot\nabla\delta u+\delta u\cdot\nabla u_2,\delta u)-(Q_2\Delta\delta Q-\Delta\delta QQ_2,\nabla\delta u^\top)\\
&\quad-(\delta Q\Delta Q_2-\Delta Q_2\delta Q,\nabla\delta u^\top)+(\nabla\delta Q\odot\nabla Q_2+\nabla Q_2\odot\nabla\delta Q,\nabla\delta u^\top)\\
&\quad+\lambda(|Q_1|\Delta\delta Q,\nabla\delta u)-a\lambda(|Q_1|\delta Q_1,\nabla\delta u)+\lambda((|Q_1|-|Q_2|)(\Delta Q_2-aQ_2),\nabla\delta u)\\
&\quad-c\lambda(|Q_1|\delta Q\operatorname{tr}(Q_1^2),\nabla\delta u)-c\lambda\big(|Q_1|Q_2\operatorname{tr}(Q_1\delta Q+\delta QQ_2),\nabla\delta u\big)\\
&\quad-c\lambda((|Q_1|-|Q_2|)Q_2\operatorname{tr}(Q_2^2),\nabla\delta u)-\kappa(\delta Q,\nabla\delta u).
\end{aligned}\tag{2.4.5}$$

Adding (2.4.4) and (2.4.5) together, and performing analogous cancellations to those in the proof of Proposition 2.1.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\nabla\delta Q\|_{L^2}^2+\|\delta Q\|_{L^2}^2+\|\delta u\|_{L^2}^2\big)+\mu\|\nabla\delta u\|_{L^2}^2+\Gamma\big(\|\Delta\delta Q\|_{L^2}^2+\|\nabla\delta Q\|_{L^2}^2\big)\\
&=(\delta u\cdot\nabla Q_2+u_2\cdot\nabla\delta Q,\Delta\delta Q)+(\delta Q\Omega_2-\Omega_2\delta Q,\Delta\delta Q)-(\delta u\cdot\nabla Q_2,\delta Q)\\
&\quad+\lambda(|Q_1|\delta D,\delta Q)-\lambda(|Q_1|-|Q_2|)D_2,\Delta\delta Q-\delta Q)+c\Gamma(\delta Q\operatorname{tr}(Q_1^2),\Delta\delta Q)\\
&\quad+c\Gamma(Q_2\operatorname{tr}(Q_1\delta Q+\delta QQ_2),\Delta\delta Q)-c\Gamma(\delta Q|Q_1|^2,\delta Q)-c\Gamma(Q_2\operatorname{tr}(Q_1\delta Q+\delta QQ_2),\delta Q)\\
&\quad+(\delta u\cdot\nabla\delta u,u_2)-(\delta Q\Delta Q_2-\Delta Q_2\delta Q,\nabla\delta u^\top)+(\nabla\delta Q\odot\nabla Q_2+\nabla Q_2\odot\nabla\delta Q,\nabla\delta u^\top)\\
&\quad-a\lambda(|Q_1|\delta Q,\nabla\delta u)+\lambda((|Q_1|-|Q_2|)(\Delta Q_2-aQ_2),\nabla\delta u)-c\lambda(|Q_1|\delta Q\operatorname{tr}(Q_1^2),\nabla\delta u)\\
&\quad-c\lambda(|Q_1|Q_2\operatorname{tr}(Q_1\delta Q+\delta QQ_2),\nabla\delta u)-c\lambda((|Q_1|-|Q_2|)Q_2\operatorname{tr}(Q_2^2),\nabla\delta u)\\
&\quad-\kappa(\delta Q,\nabla\delta u)-a\Gamma\big(\|\nabla\delta Q\|_{L^2}^2+\|\delta Q\|_{L^2}^2\big)\\
&\le\|\Delta\delta Q\|_{L^2}\big(\|\delta u\|_{L^2}\|\nabla Q_2\|_{L^\infty}+\|\nabla\delta Q\|_{L^2}\|u_2\|_{L^\infty}+2\|\delta Q\|_{L^{\frac{2}{s}}}\|\Omega_2\|_{L^{\frac{2}{1-s}}}\big)\\
&\quad+\|\delta Q\|_{L^2}\|\delta u\|_{L^2}\|\nabla Q_2\|_{L^\infty}+|\lambda|\|\delta Q\|_{L^2}\|Q_1\|_{L^\infty}\|\nabla\delta u\|_{L^2}\\
&\quad+|\lambda|\big(\|\Delta\delta Q\|_{L^2}+\|\delta Q\|_{L^2}\big)\|\delta Q\|_{L^{\frac{2}{s}}}\|D_2\|_{L^{\frac{2}{1-s}}}+c\Gamma\|\Delta\delta Q\|_{L^2}\|\delta Q\|_{L^4}\|Q_1\|_{L^8}^2\\
&\quad+c\Gamma\|\Delta\delta Q\|_{L^2}\|Q_2\|_{L^\infty}\big(\|Q_1\|_{L^4}+\|Q_2\|_{L^4}\big)\|\delta Q\|_{L^4}+c\Gamma\|\delta Q\|_{L^4}^2\|Q_1\|_{L^2}\|Q_1\|_{L^\infty}\\
&\quad+c\Gamma\|\delta Q\|_{L^4}^2\|Q_2\|_{L^2}\big(\|Q_1\|_{L^\infty}+\|Q_2\|_{L^\infty}\big)+\|u_2\|_{L^\infty}\|\delta u\|_{L^2}\|\nabla\delta u\|_{L^2}\\
&\quad+2\|\delta Q\|_{L^{\frac{2}{s}}}\|\Delta Q_2\|_{L^{\frac{2}{1-s}}}\|\nabla\delta u\|_{L^2}+2\|\nabla\delta Q\|_{L^2}\|\nabla\delta u\|_{L^2}\|\nabla Q_2\|_{L^\infty}\\
&\quad+|a\lambda|\|\nabla\delta u\|_{L^2}\|\delta Q\|_{L^2}\|Q_1\|_{L^\infty}+C\|\nabla\delta u\|_{L^2}\big(\|\delta Q\|_{L^{\frac{2}{s}}}\|\Delta Q_2\|_{L^{\frac{2}{1-s}}}+\|\delta Q\|_{L^2}\|Q_2\|_{L^\infty}\big)
\end{aligned}$$

$$
\begin{aligned}
&+ c|\lambda|\|\nabla\delta u\|_{L^2}\|\delta Q\|_{L^4}\big(\|Q_1\|_{L^\infty}(\|Q_1\|_{L^8}^2+\|Q_2\|_{L^8}^2)+\|Q_2\|_{L^8}^2\|Q_2\|_{L^\infty}\big)\\
&+ |\kappa|\|\nabla\delta u\|_{L^2}\|\delta Q\|_{L^2} - a\Gamma\big(\|\nabla\delta Q\|_{L^2}^2+\|\delta Q\|_{L^2}^2\big)\\
\le\ & \frac{\Gamma}{2}\|\Delta\delta Q\|_{L^2}^2+\frac{\mu}{2}\|\nabla\delta u\|_{L^2}^2 + C\big(\|\nabla Q_2\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2\big)\|\delta u\|_{L^2}^2\\
&+ C\big(1+\|Q_2\|_{L^\infty}^2+\|Q_1\|_{L^\infty}^2\big)\|\delta Q\|_{L^2}^2 + C(1+\|u_2\|_{L^\infty}^2+\|\nabla Q_2\|_{L^\infty}^2)\|\nabla\delta Q\|_{L^2}^2\\
&+ C\Gamma\big(\|Q_1\|_{L^8}^4+\|Q_2\|_{L^\infty}^2(\|Q_1\|_{L^4}^2+\|Q_2\|_{L^4}^2)+\|Q_1\|_{L^8}^4\|Q_1\|_{L^\infty}^2+\|Q_2\|_{L^8}^4\|Q_2\|_{L^\infty}^2\\
&\qquad + \|Q_1\|_{L^\infty}^2(\|Q_1\|_{L^8}^4+\|Q_2\|_{L^8}^4)+(\|Q_1\|_{L^\infty}+\|Q_2\|_{L^\infty})(\|Q_1\|_{L^2}+\|Q_2\|_{L^2})\big)\|\delta Q\|_{L^4}^2\\
&+ C\big(\|\nabla u_2\|_{L^{\frac{2}{1-s}}}^2+\|\Delta Q_2\|_{L^{\frac{2}{1-s}}}^2\big)\|\delta Q\|_{L^{\frac{2}{s}}}^2.
\end{aligned}
$$

Since we restrict ourselves to the two-dimensional case, $\|\delta Q\|_{L^4}^2$ and $\|\delta Q\|_{L^{\frac{2}{s}}}^2$ can be controlled by $\|\delta Q\|_{L^2}^2+\|\nabla\delta Q\|_{L^2}^2$. Moreover, we know from the imbedding theorem that

$$
\|\nabla u_2\|_{L^{\frac{2}{1-s}}}\le C\|u_2\|_{H^{s+1}},\quad \|\Delta Q_2\|_{L^{\frac{2}{1-s}}}\le C\|Q_2\|_{H^{s+2}}.
$$

In addition, from the conditions of Q_1, u_1, Q_2, and u_2, *i.e.*, (2.4.1)–(2.4.2), we know that all the coefficients of $\|\delta u\|_{L^2}^2$, $\|\delta Q\|_{L^2}^2$, $\|\nabla\delta Q\|_{L^2}^2$, $\|\delta Q\|_{L^4}^2$, and $\|\delta Q\|_{L^{\frac{2}{s}}}^2$ are integrable with respect to time. Therefore, we can use Gronwall's inequality (Lemma A.0.9) to conclude the uniqueness of the solution. □

3.0 INHOMOGENEOUS INCOMPRESSIBLE ACTIVE LIQUID CRYSTALS

In this Chapter, we consider the following hydrodynamic equations of inhomogeneous incompressible flows of active nematic liquid crystals [25, 26, 29] in a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ for $d = 2, 3$:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P - \mu \Delta u = \nabla \cdot \tau + \nabla \cdot \sigma, \\ \partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q - \lambda |Q| D = \Gamma H[Q], \\ \nabla \cdot u = 0, \end{cases} \tag{3.0.1}$$

where ρ is the density of the fluid, and as in Chapter 2, $u \in \mathbb{R}^d$ represents the flow velocity, Q is the nematic tensor order parameter, P stands for the pressure, $\mu > 0$ denotes the viscosity coefficient, $\Gamma^{-1} > 0$ is the rotational viscosity, $\lambda \in \mathbb{R}$ is the nematic alignment parameter, $D = \frac{1}{2}(\nabla u + \nabla u^\top)$ and $\Omega = \frac{1}{2}(\nabla u - \nabla u^\top)$ are the symmetric and antisymmetric part of the strain tensor with $(\nabla u)_{\alpha\beta} = \partial_\beta u_\alpha$. Hereafter, we use the Einstein summation convention, i.e. we sum over the repeated indices. Moreover, the molecular tensor

$$H = H[Q] = K\Delta Q - \frac{k}{2}(c - c_*)Q + b\big(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d\big) - cQ\,\mathrm{tr}(Q^2),$$

describes the relaxation dynamics of the nematic phase and can be derived from the Landau-de Gennes free energy, i.e., $H_{\alpha\beta} = -\delta\mathcal{F}/\delta Q_{\alpha\beta}$, where

$$\mathcal{F} = \int \Big(\frac{k}{4}(c - c^*)\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|\mathrm{tr}(Q^2)|^2 + \frac{K}{2}|\nabla Q|^2\Big) dA,$$

with K the elastic constant for the one-constant elastic energy density, c the concentration of active units and c^* the critical concentration for the isotropic-nematic transition, and $k > 0$

and $b \in \mathbb{R}$ the material-dependent constants. We note that the analysis of this paper holds for all real b, although b is usually taken to be positive in the literature. In what follows, we set $K = k = 1$ for simplicity of notation. The stress tensor $\sigma = (\sigma^{ij})$ is

$$\sigma^{ij} = \sigma_r^{ij} + \sigma_a^{ij},$$

with

$$\sigma_r^{ij} = -\lambda|Q|H^{ij}[Q] + Q^{ik}H^{kj}[Q] - H^{ik}[Q]Q^{kj}, \quad \sigma_a^{ij} = \sigma_* c^2 Q^{ij},$$

where σ_r^{ij} is the elastic stress tensor due to the nematic elasticity and σ_a^{ij} is the active contribution which describes contractile or extensile stresses exerted by the active particles in the direction of the director field ($\sigma_* > 0$ for the contractile case and $\sigma_* < 0$ for the extensile case). The symmetric additional stress tensor is denoted by:

$$\tau^{ij} = -\partial_j Q^{kl} \partial_i Q^{kl} = -(\nabla Q \odot \nabla Q)^{ij}.$$

In the rest of this chapter, we consider the case where $c > 0$ is a constant. We set,

$$a = \frac{1}{2}(c - c_*), \quad \kappa = \sigma_* c^2.$$

Then system (3.0.1) becomes:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{3.0.2}$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P = \mu \Delta u - \nabla \cdot (\nabla Q \odot \nabla Q) - \lambda \nabla \cdot (|Q|H[Q]) + \nabla \cdot (Q\Delta Q - \Delta Q Q) + \kappa \nabla \cdot Q, \tag{3.0.3}$$

$$\partial_t Q + (u \cdot \nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H[Q], \tag{3.0.4}$$

$$\nabla \cdot u = 0, \tag{3.0.5}$$

with

$$H = H[Q] = \Delta Q - aQ + b\big(Q^2 - \frac{\operatorname{tr}(Q^2)}{d}\mathrm{I}_d\big) - cQ\operatorname{tr}(Q^2),$$

and the constants $c > 0, \Gamma > 0, \mu > 0, a, b, \lambda, \kappa \in \mathbb{R}$, $(x,t) \in \mathcal{O} \times \mathbb{R}^+$. The system is subject to the following initial conditions:

$$\rho|_{t=0} = \bar{\rho}(x) \in L^\infty(\mathcal{O}), \quad \bar{\rho} \geq 0, \tag{3.0.6}$$

$$\rho u|_{t=0} = \bar{m}(x) \in L^2(\mathcal{O}), \quad \bar{m} = 0 \text{ where } \bar{\rho} = 0, \quad \frac{|\bar{m}|^2}{\bar{\rho}} \in L^1(\mathcal{O}), \tag{3.0.7}$$

$$Q|_{t=0} = \bar{Q}(x) \in H^1(\mathcal{O}), \text{ and } \bar{Q} \in S_0^d \ a.e. \text{ in } \mathcal{O}, \tag{3.0.8}$$

and the following boundary conditions:

$$u(x,t) = 0, \quad Q(x,t) = \bar{Q}(x), \quad \text{for} \quad (x,t) \in \partial\mathcal{O} \times (0,\infty), \tag{3.0.9}$$

where $S_0^d := \{Q \in \mathbb{M}^{d\times d} : Q^{ij} = Q^{ij}, \ \mathrm{tr}(Q) = 0, \ i,j = 1,\cdots,d\}$ is the space of Q-tensors in d-dimension.

3.1 THE *A PRIORI* ESTIMATES

In this section, we derive the *a priori* estimates for the system (3.0.2)−(3.0.5). Here we continue to use the notations in Chapter 2, with $\mathcal{O}$ a bounded domain in $\mathbb{R}^d$ for $d = 2,3$. Moreover, we introduce the following function spaces:

$$V_1 = \{v \in L^2(\mathcal{O}) : \operatorname{div} u = 0, \text{ in } \mathcal{D}'\},$$
$$V_2 = \{v \in H_0^1(\mathcal{O}) : \operatorname{div} u = 0\}.$$

We still denote the Landau-de Gennes free energy for the nematic liquid crystals [23] by

$$\mathbf{F}(Q) := \int_{\mathcal{O}} \left(\frac{1}{2}|\nabla Q|^2 + \frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 \right) dx. \tag{3.1.1}$$

Moreover, the energy of the system (3.0.2)-(3.0.5) can be represented as

$$E(t) := \mathbf{F}(Q) + \int_{\mathcal{O}} \frac{1}{2}\rho|u|^2 \mathrm{d}x, \tag{3.1.2}$$

by adding the kinetic energy to the Landau-de Gennes free energy.

Since the bulk potential in the Landau-de Gennes energy density is not always positive as we mentioned in Chapter 2, we can redefine the energy of the system in the following way. By (A.0.1) we know

$$\mathrm{tr}(Q^3) \le \frac{\varepsilon}{4}|\mathrm{tr}(Q^2)|^2 + \frac{1}{\varepsilon}\mathrm{tr}(Q^2) \qquad \text{for any } \varepsilon > 0, \text{ and } d\times d \text{ matrix } Q. \tag{3.1.3}$$

Then there exists a sufficiently large constant $M = M(a, b, c) > 0$, such that

$$\begin{aligned}
&(M + \frac{a}{2})|Q|^2 - \frac{b}{3}\text{tr}(Q^3) + \frac{c}{4}|Q|^4 \\
&\geq (M + \frac{a}{2})|Q|^2 - \frac{b}{3}\left(\frac{3\varepsilon}{8}\text{tr}^2(Q^2) + \frac{1}{\varepsilon}\text{tr}(Q^2)\right) + \frac{c}{4}|Q|^4 \\
&= (M + \frac{a}{2} - \frac{b}{3\varepsilon})|Q|^2 + (\frac{c}{4} - \frac{b}{8}\varepsilon)|Q|^4 \\
&\geq \frac{M}{2}|Q|^2 + \frac{c}{8}|Q|^4 \geq 0.
\end{aligned} \tag{3.1.4}$$

We define the positive energy of the system by

$$E^M(t) := \int_{\mathcal{O}} \left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|\nabla Q|^2 + (\frac{a}{2} + M)|Q|^2 - \frac{b}{3}\text{tr}(Q^3) + \frac{c}{4}|Q|^4\right) dx. \tag{3.1.5}$$

Proposition 3.1.1. *Let (ρ, u, Q) be a smooth solution of the problem (3.0.2)-(3.0.5), with smooth initial data $(\bar{\rho}(x), \bar{m}(x), \bar{Q}(x))$. If $(\bar{\rho}(x), \bar{m}(x), \bar{Q}) \in L^\infty \times L^2 \times H^1$, the following energy inequality holds for any $T > 0$,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E^M(t) + \frac{1}{2}\int_0^t \int_{\mathcal{O}} (\mu|\nabla u|^2 + \Gamma|\Delta Q|^2 + c^2\Gamma|Q|^6)\mathrm{d}x\mathrm{d}s \\
&\leq Ce^{Ct}\int_{\mathcal{O}} (\frac{|\bar{m}|^2}{\bar{\rho}} + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla\bar{Q}|^2)\mathrm{d}x, \quad \textit{for a.e. } t \in [0, T].
\end{aligned} \tag{3.1.6}$$

Moreover, we have

$$0 \leq \rho(x, t) \leq \|\bar{\rho}\|_{L^\infty(\mathcal{O})}, \; \textit{for any}\; t \in (0, T), \tag{3.1.7}$$

$$\|\sqrt{\rho}u\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} + \mu\|\nabla u\|^2_{L^2(0,T;L^2(\mathcal{O}))} \leq C, \tag{3.1.8}$$

$$\|Q\|_{L^\infty(0,T;H^1(\mathcal{O})\cap L^4(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))\cap L^6(0,T;L^6(\mathcal{O}))} \leq C. \tag{3.1.9}$$

Hereafter C is a constant that depends on the material coefficients $a, b, c, \kappa, \mu, \lambda, \Gamma$ and the initial data.

Proof. First, since ρ satisfies the transport-type equations, we obtain (3.1.7). By using the density equation (3.0.2) and the boundary equation (3.0.9), we know

$$((\rho u)_t, u) + (\rho u \otimes u, \nabla u) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{O}} \rho |u|^2 \mathrm{d}x. \tag{3.1.10}$$

Then we take the summation of the equation (3.0.3) multiplied by u and the equation (3.0.4) multiplied by $-H + 2MQ$ (M is a sufficiently large constant as in (3.1.4)), take the trace, and integrate by parts over $\mathcal{O}$ to get

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}E^M(t) + \mu\|\nabla u\|_{L^2}^2 + \Gamma\|\Delta Q\|_{L^2}^2 + (a^2 + 2Ma)\Gamma\|Q\|_{L^2}^2 + c^2\Gamma\|Q\|_{L^6}^6 + 2(a+M)\Gamma\|\nabla Q\|_{L^2}^2 \\
&\quad + 2(a+M)c\Gamma\|Q\|_{L^4}^4 + b^2\Gamma\int_{\mathcal{O}} \mathrm{tr}(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d)^2 \mathrm{d}x \\
&= (u\cdot\nabla Q, \Delta Q) - (u\cdot\nabla Q, aQ - b(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}Id) + cQ|Q|^2) - (\Omega Q - Q\Omega, \Delta Q) \\
&\quad + (\Omega Q - Q\Omega, aQ - b(Q^2 - \frac{\mathrm{tr}(Q^2)}{d}Id) + cQ|Q|^2) - \lambda(|Q|D, H) \\
&\quad - (\nabla\cdot(\nabla Q\odot\nabla Q), u) + \lambda(|Q|H, \nabla u) + (\nabla\cdot(Q\Delta Q - \Delta Q Q), u) - \kappa(Q, \nabla u) \\
&\quad - 2b\Gamma\left(\Delta Q, Q^2\right) + 2bc\Gamma\left(Q\mathrm{tr}(Q^2), Q^2\right) + 2(a+M)b\Gamma\left(Q, Q^2\right) + 2c\Gamma\left(\Delta Q, Q\mathrm{tr}(Q^2)\right) \\
&\quad + 2\lambda M(|Q|D, Q) \\
&= \sum_{i=1}^{14} \mathcal{I}_i.
\end{aligned}$$

We now derive the estimates for the terms $\mathcal{I}_i$, $1 \le i \le 14$, one by one. First, $\mathcal{I}_2 = 0$ (by $\mathrm{div}\, u = 0$) and $\mathcal{I}_3 + \mathcal{I}_8 = 0$ (by Lemma A.0.5). In the following, we shall show that $\mathcal{I}_1 + \mathcal{I}_6 = 0$, $\mathcal{I}_5 + \mathcal{I}_7 = 0$, $\mathcal{I}_4 = 0$, $\mathcal{I}_{13} \le 0$, and the estimates for all other terms. A direct computation shows that

$$\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_6 &= (u\cdot\nabla Q, \Delta Q) - (\nabla\cdot(\nabla Q\odot\nabla Q), u) \\
&= \int_{\mathcal{O}} u^i\partial_i Q^{jk}\Delta Q^{jk}\mathrm{d}x - \int_{\mathcal{O}} \partial_i\partial_j Q^{kl}\partial_j Q^{kl}u^i + \partial_i Q^{kl}\partial_j\partial_j Q^{kl}u^i\mathrm{d}x \\
&= -\int_{\mathcal{O}} \partial_i\partial_j Q^{kl}\partial_j Q^{kl}u^i\mathrm{d}x = \frac{1}{2}\int_{\mathcal{O}} |\nabla Q|^2\mathrm{div}u\mathrm{d}x = 0.
\end{aligned}$$

By the fact that Q is symmetric and Ω is skew-smmetric we have,

$$\mathcal{I}_4 = (\Omega Q - Q\Omega, aQ - b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{Id}] + cQ|Q|^2)$$

$$
\begin{aligned}
&= -(\Omega Q + Q\Omega, aQ - b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{Id}] + cQ|Q|^2) \\
&\quad + 2(\Omega Q, aQ - b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{Id}] + cQ|Q|^2) \\
&= 0, \\
\mathcal{I}_5 + \mathcal{I}_7 &= \lambda(|Q|H, \nabla u) - \lambda(|Q|D, H) = \lambda(|Q|H, \nabla u) - \lambda(|Q|H, D) \\
&= \lambda(|Q|H, \nabla u - D) = \lambda(|Q|H, \Omega) = 0.
\end{aligned}
$$

Moreover, by Young's inequality, we get

$$
\begin{aligned}
\mathcal{I}_9 &= -\kappa(Q, \nabla u) \le \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + C\|Q\|_{L^2}^2, \\
\mathcal{I}_{10} &= -2b\Gamma(\Delta Q, Q^2) \le \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + C\|Q\|_{L^4}^4, \\
\mathcal{I}_{14} &= 2\lambda M(|Q|D, Q) \le \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + C\|Q\|_{L^4}^4.
\end{aligned}
$$

From (A.0.1), we have the following estimates for $\mathcal{I}_{11}, \mathcal{I}_{12}$, of course, by choosing the appropriate $\varepsilon > 0$,

$$
\begin{aligned}
\mathcal{I}_{11} &= 2bc\Gamma\left(Q\mathrm{tr}(Q^2), Q^2\right) = 2bc\Gamma\int_{\mathcal{O}} \mathrm{tr}(Q)^3|Q|^2\mathrm{d}x \\
&\le 2|b|c\Gamma\int_{\mathcal{O}}(\frac{3\varepsilon}{8}|Q|^4 + \frac{1}{\varepsilon}|Q|^2)|Q|^2\mathrm{d}x \\
&= \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 + C\|Q\|_{L^4}^4,
\end{aligned}
$$

$$
\mathcal{I}_{12} = 2(a+M)b\Gamma\left(Q, Q^2\right) = 2(a+M)b\Gamma\int_{\mathcal{O}} \mathrm{tr}(Q)^3\mathrm{d}x \le C(\|Q\|_{L^2}^2 + \|Q\|_{L^4}^4).
$$

Finally,

$$
\begin{aligned}
\mathcal{I}_{13} &= 2c\Gamma\left(\Delta Q, Q\mathrm{tr}(Q^2)\right) = 2c\Gamma\int_{\mathcal{O}} \partial_{kk}Q^{ij}Q^{ij}\mathrm{tr}(Q^2)\mathrm{d}x \\
&= -2c\Gamma\int_{\mathcal{O}} \partial_k Q^{ij}\partial_k Q^{ij}\mathrm{tr}(Q^2)\mathrm{d}x - 2c\Gamma\int_{\mathcal{O}} \partial_k Q^{ij}Q^{ij}\partial_k\mathrm{tr}(Q^2)\mathrm{d}x \\
&= -2c\Gamma\int_{\mathcal{O}} |\nabla Q|^2|Q|^2\mathrm{d}x - c\Gamma\int_{\mathcal{O}} |\nabla\mathrm{tr}(Q^2)|^2\mathrm{d}x \le 0.
\end{aligned}
$$

With all the above estimates, we have

$$
\begin{aligned}
&\frac{d}{\mathrm{d}t}E^M(t) + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 \\
&\le C(\|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4),
\end{aligned} \tag{3.1.11}
$$

where $C = C(a, b, c, \kappa, \mu, \lambda, \Gamma, M)$. Then the desired estimates (3.1.6), (3.1.8) and (3.1.9) follow from (3.1.11) and the Gronwall's inequality.

□

Next, let us introduce the definition of the global weak solutions.

Definition 3.1.1. (ρ, u, Q) is a global weak solution to the system (3.0.2)-(3.0.5) with the initial and boundary conditions (3.0.6)-(3.0.9), if for any $T > 0$, the following conditions are satisfied:

$$\begin{cases} \rho \geq 0, \rho \in L^\infty((0,T) \times \mathcal{O}), \rho \in C([0,T]; L^p(\mathcal{O})),\ 1 \leq p < \infty, \\ u \in L^2(0,T; V_2(\mathcal{O})),\ \text{and}\ \rho|u|^2 \in L^\infty(0,T; L^1(\mathcal{O})), \\ Q \in L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O})),\ \text{and}\ Q \in S_0^d\ a.e.\ \text{in}\ [0,T] \times \mathcal{O}; \end{cases} \tag{3.1.12}$$

moreover, for any $\zeta \in C^1([0,T] \times \mathcal{O})$ with $\zeta(T, \cdot) = 0$,

$$-\int_0^T \int_{\mathcal{O}} (\rho \partial_t \zeta + \rho u \cdot \nabla_x \zeta) \mathrm{d}x\mathrm{d}t = \int_{\mathcal{O}} \bar{\rho} \zeta(0, x) \mathrm{d}x; \tag{3.1.13}$$

for any $\psi \in C^1([0,T] \times \mathcal{O})$ with $\mathrm{div}_x \psi = 0$ and $\psi(T, \cdot) = 0$,

$$\begin{aligned} &\int_0^T \int_{\mathcal{O}} (-\rho u \cdot \partial_t \psi - (\rho u \otimes u) : \nabla_x \psi + \mu \nabla_x u : \nabla_x \psi) \mathrm{d}x\mathrm{d}t - \int_{\mathcal{O}} \bar{m}(x) \cdot \psi(0,x) \mathrm{d}x \\ &= \int_0^T \int_{\mathcal{O}} (\nabla Q \odot \nabla Q + \lambda |Q| H[Q] - Q\Delta Q + \Delta Q Q - \kappa Q) : \nabla_x \psi \mathrm{d}x\mathrm{d}t; \end{aligned} \tag{3.1.14}$$

for any $\varphi \in C^1([0,T] \times \mathcal{O})$ with $\varphi \in S_0^d$ and $\varphi(T, \cdot) = 0$,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \Big(Q : \partial_t \varphi + \Gamma \Delta Q : \varphi + Q : (u \cdot \nabla_x \varphi) - (Q\Omega - \Omega Q - \lambda |Q| D) : \varphi \Big) dxdt \\ &= \Gamma \int_0^T \int_{\mathbb{R}^d} \Big(aQ - b\big(Q^2 - \frac{\mathrm{tr}(Q^2)}{d} \mathrm{I}_d\big) + cQ\mathrm{tr}(Q^2) \Big) : \varphi dxdt \\ &\quad - \int_{\mathbb{R}^d} \bar{Q}(x) : \varphi(0,x) dx; \end{aligned} \tag{3.1.15}$$

and finally, the following energy inequality holds:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} E^M(t) + \frac{1}{2} \int_0^t \int_{\mathcal{O}} (\mu |\nabla u|^2 + \Gamma |\Delta Q|^2 + c^2 \Gamma |Q|^6) \mathrm{d}x\mathrm{d}s \\ &\leq Ce^{Ct} \int_{\mathcal{O}} (\frac{|\bar{m}|^2}{\bar{\rho}} + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla \bar{Q}|^2) \mathrm{d}x, \quad \text{for } a.e.\ t \in [0,T]. \end{aligned} \tag{3.1.16}$$

Our main result states as follows:

Theorem 3.1.1. *The problem (3.0.2)–(3.0.5) admits a weak solution* (ρ, u, Q) *in the sense of Definition 3.1.1 under the assumptions* (3.0.6)-(3.0.9) *on the initial and boundary conditions.*

We will prove Theorem 3.1.1 using the Faedo-Galerkin approximation for constructing the solution.

3.2 THE APPROXIMATION SCHEME

In this section, we present the approximation system, and prove the existence of the approximate solutions.

3.2.1 The approximation scheme

Let $\{\psi_n\} \in C_0^\infty(\mathcal{O})$ be an orthonormal basis of V_1. Now, we define a sequence of finite dimensional spaces

$$X_n = \text{span}\{\psi_1, \psi_2, \cdots, \psi_n\}, \quad n = 1, 2, \cdots, \tag{3.2.1}$$

and let

$$Y_n = C([0,T]; X_n).$$

Because the presence of vacuum is allowed, we will choose our approximate initial density to be bounded away from zero as follows. We set

$$\tilde{\rho} = \begin{cases} \bar{\rho}, & \text{in } \mathcal{O}, \\ 1, & \mathbb{R}^d \setminus \mathcal{O}, \end{cases}$$

and let

$$(\bar{\rho})_\varepsilon = (\tilde{\rho} * \eta_\varepsilon)|_{\mathcal{O}},$$

where η is the standard mollifier in $\mathbb{R}^d$. Then the initial density for the approximation system is

$$\rho|_{t=0} = \overline{\rho^\varepsilon} = (\bar{\rho})_\varepsilon + \varepsilon. \tag{3.2.2}$$

It is easy to see that $\varepsilon \le \overline{\rho^\varepsilon} \le C$, for some universal constant C independent of ε. Moreover, we have $\overline{\rho^\varepsilon} \in C^\infty$ and $\overline{\rho^\varepsilon} \to \bar{\rho}$ in $L^p(\mathcal{O})$ for all $1 \le p < \infty$, as $\varepsilon \to 0$.

Next, we consider the initial data for the velocity u. First let us introduce the following lemma on the Hodge-de Rham type decompostion due to Lions [38].

Lemma 3.2.1. *Let $N \ge 2$, $\rho \in L^\infty(\mathbb{R}^d)$ such that $0 < \underline{\varrho} \le \rho$ a.e. on $\mathbb{R}^d$ for some $\underline{\varrho} \in (0,\infty)$. Then there exists two bounded operators P_ρ, Q_ρ on $L^2(\mathbb{R}^d)$ such that for all $m \in L^2(\mathbb{R}^d)$, $(m_p, m_q) = (P_\rho m, Q_\rho m)$ is the unique solution in $L^2(\mathbb{R}^d)$ of*

$$m = m_p + m_q, \quad (-\Delta)^{-\frac{1}{2}}\mathrm{div}(\rho^{-1} m_p) = 0, \quad (-\Delta)^{-\frac{1}{2}}\mathrm{rot}(m_q) = 0.$$

Furthermore, if $\rho_n \in L^\infty(\mathbb{R}^d)$, $\underline{\varrho} \le \rho_n \le \overline{\varrho}$ a.e. on $\mathbb{R}^d$ for some $0 < \underline{\varrho} \le \overline{\varrho} < \infty$ and ρ_n converges a.e. to ρ, then $(P_{\rho_n} m_n, Q_{\rho_n} m_n)$ converges weakly in $L^2(\mathbb{R}^d)$ to $(P_\rho m, Q_\rho m)$ whenever m_n converges weakly to m.

We set

$$(\sqrt{\bar{\rho}})_\varepsilon = (\sqrt{\tilde{\rho}} * \eta_\varepsilon)|_{\mathcal{O}},$$

and define

$$\overline{m^\varepsilon} = \frac{\bar{m}}{\sqrt{\bar{\rho}}}(\sqrt{\bar{\rho}})_\varepsilon.$$

Obviously,

$$\overline{m^\varepsilon} \to \bar{m} \quad \text{in } L^2(\mathcal{O}), \quad \text{and} \quad \frac{\overline{m^\varepsilon}}{\sqrt{\overline{\rho^\varepsilon}}} \to \frac{\bar{m}}{\sqrt{\bar{\rho}}} \quad \text{in } L^2(\mathcal{O}).$$

By Lemma 3.2.1, we have

$$\overline{m^\varepsilon} = \overline{\rho^\varepsilon} u_{0,\varepsilon} + Q_{\overline{\rho^\varepsilon}} \overline{m^\varepsilon},$$

where $u_{0,\varepsilon} \in L^2(\mathcal{O})$ and $\operatorname{div} u_{0,\varepsilon} = 0$, and $Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon} \in L^2(\mathcal{O})$, $\nabla \times Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon} = 0$ in $\mathcal{D}'$. Now we impose the initial data for the velocity as

$$u|_{t=0} = u_{0,n} = P_n u_{0,\varepsilon}, \tag{3.2.3}$$

with P_n the orthogonal projection in V_1 onto X_n.

Lastly, we impose the initial data for the Q-tensor as

$$Q|_{t=0} = \bar{Q}. \tag{3.2.4}$$

With the initial data defined above, our approximate solution can be stated as:

Definition 3.2.1. (ρ_n, u_n, Q_n) is an approximate solution if the following equations are satisfied in the weak sense of Definition 3.1.1,

$$(\rho_n)_t + \nabla \cdot (\rho_n u_n) = 0, \tag{3.2.5}$$

$$\begin{aligned}(\rho_n u_n)_t + \nabla \cdot (\rho_n u_n \otimes u_n) + \nabla p_n - \mu \Delta u_n &= -\nabla \cdot (\nabla Q_n \odot \nabla Q_n) - \lambda \nabla \cdot (|Q_n| H[Q_n]) \\ &\quad + \nabla \cdot (Q_n \Delta Q_n - \Delta Q_n Q_n) + \kappa \nabla \cdot Q_n,\end{aligned} \tag{3.2.6}$$

$$(Q_n)_t + (u_n \cdot \nabla) Q_n + Q_n \Omega_n - \Omega_n Q_n - \lambda |Q_n| D_n = \Gamma H[Q_n], \tag{3.2.7}$$

$$\nabla \cdot u_n = 0, \tag{3.2.8}$$

with the initial conditions (3.2.2)-(3.2.4), boundary conditions (3.0.9), and the test function space in (3.1.14) replaced by the restriction on X_n. Moreover,

$$\rho_n \in C([0,T] \times \mathcal{O}), \quad u_n \in Y_n, \quad Q_n \in L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O})).$$

Under the above definition, we have the following existence result of approximate solutions.

Theorem 3.2.1. *For any $T > 0$, the problem (3.2.5)-(3.2.8) admits a global weak solution (ρ_n, u_n, Q_n) with the initial conditions (3.2.2)-(3.2.4) and the boundary conditions (3.0.9).*

We remark that the approximate solution (ρ_n, u_n, Q_n) depends also on ε.

3.2.2 The Neumann problem for the density and the Q-tensor

In order to prove Theorem 3.2.1, let us first introduce the following existence results for the density and Q-tensor equations, which are classical and may be found in [17].

Lemma 3.2.2. *For each $u \in C([0,T]; C_0^\infty(\bar{\mathcal{O}}, \mathbb{R}^d))$ with $\operatorname{div} u = 0$, there exists a mapping $S = S[u]$,*

$$S : C([0,T]; C^\infty(\bar{\mathcal{O}}, \mathbb{R}^d)) \to C([0,T]; C^\infty(\bar{\mathcal{O}})),$$

satisfying the following properties:

(i) $\rho = S[u]$ is the unique classical solution of the initial-value problem

$$\begin{cases} \rho_t + \operatorname{div}(u\rho) = 0, \\ \rho|_{t=0} = \bar{\rho} \in C^\infty(\bar{\mathcal{O}}), \end{cases} \tag{3.2.9}$$

(ii) $\rho \in C^1([0,T]; C^\infty(\mathcal{O}))$, and $\varepsilon \le S[u](t,x) \le C$, for all $t \in [0,T]$;

(iii)

$$\|S[u_1] - S[u_2]\|_{C([0,T];L^2(\mathcal{O}))} \le TC(T)\|u_1 - u_2\|_{C([0,T];C_0^1(\mathcal{O}))}, \tag{3.2.10}$$

for any u_1, u_2 in the set

$$\mathcal{N}_N := \{v | v \in C([0,T]; C_0^\infty(\bar{\mathcal{O}}, \mathbb{R}^d)), \operatorname{div} v = 0, \text{ and } \|v\|_{C([0,T];C_0^1(\bar{\mathcal{O}}))} \le N\}, \tag{3.2.11}$$

for some suitable constant $N > 0$.

Proof. Since $\operatorname{div} u = 0$, (3.2.9) is just a transport equation. We integrate along the characteristic line, *i.e.* solve the following problem:

$$\frac{\mathrm{d}X}{\mathrm{d}s} = u(X,s), \quad X(t;(x,t)) = x, \quad \text{for } (x,t) \in \mathcal{O} \times [0,T]. \tag{3.2.12}$$

By the standard theory of the transport equation, and the fact that $u \in C([0,T]; C_0^\infty(\mathcal{O}))$, there exists a unique smooth solution X to (3.2.12) and the solution satisfies

$$\rho(t,x) = \overline{\rho^\varepsilon}(X(0;(x,t))), \quad \text{for all } t \in [0,T],\ x \in \mathcal{O}. \tag{3.2.13}$$

It is easy to see that $\varepsilon \le \rho \le C$. Since $\overline{\rho^\varepsilon}$ is smooth, we know that ρ is bounded in $C([0,T]; C^\infty(\mathcal{O}))$ and it is unique.

In order to show the continuity of S, let ρ_1, ρ_2 represent the solutions to (3.2.9) corresponding to u_1, $u_2 \in \mathcal{N}_N$. And we denote by $\tilde{\rho} = \rho_1 - \rho_2$, it satisfies,

$$\begin{cases} \tilde{\rho}_t + u_2 \cdot \nabla \tilde{\rho} + (u_1 - u_2) \cdot \nabla \rho_1 = 0, \\ \tilde{\rho}|_{t=0} = 0. \end{cases}$$

Multiplying $\tilde{\rho}$ on the both sides of the above equation, and integrating by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{\rho}\|_{L^2}^2 = -2 \int_{\mathcal{O}} (u_1 - u_2) \cdot \nabla \rho_1 \tilde{\rho} \mathrm{d}x \leq C \|u_1 - u_2\|_{L^2}^2 + C \|\tilde{\rho}\|_{L^2}^2.$$

Then by the Gronwall's inequality, we get

$$\|\tilde{\rho}(t, \cdot)\|_{L^2}^2 \leq C \mathrm{e}^{Ct} \int_0^t \|(u_1 - u_2)(s, \cdot)\|_{L^2}^2 \mathrm{d}s,$$

which implies (3.2.10). □

Lemma 3.2.3. *For each $u \in C([0,T]; C_0^2(\bar{\mathcal{O}}, \mathbb{R}^d))$ with $u|_{\partial \mathcal{O}} = 0$, and $\bar{Q}(x)$ satisfied (3.0.8), there exists a unique solution to the following initial-boundary value problem,*

$$\begin{cases} \partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q - \lambda |Q| D = \Gamma H, \\ Q|_{t=0} = Q|_{\partial \mathcal{O}} = \bar{Q}(x). \end{cases} \tag{3.2.14}$$

with $Q \in L^\infty([0,T]; H^1(\mathcal{O})) \cap L^2([0,T]; H^2(\mathcal{O}))$. Moreover, the above mapping $u \mapsto Q[u]$ is continuous from $\mathcal{N}_N$ to $L^\infty([0,T]; H^1(\mathcal{O})) \cap L^2([0,T]; H^2(\mathcal{O}))$, and $Q[u] \in S_0^d$ a.e. in $[0,T] \times \mathcal{O}$.

Proof. The existence of the solution Q can be achieved by the standard parabolic theory [34]. Next, we show that Q belongs to $L^\infty(0,T;H^1(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))$.

Assume $u\in\mathcal{N}_N$. First, similar to Proposition 3.1.1, we multiply the first equation in (3.2.14) by $-H[Q]+2MQ$ (M is a sufficient large constant as in (3.1.4)), take the trace, integrate by parts over $\mathcal{O}$, and then obtain,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E^M(t)+\Gamma\|\Delta Q\|_{L^2}^2+a^2\Gamma\|Q\|_{L^2}^2+c^2\Gamma\|Q\|_{L^6}^6+b^2\Gamma\int_{\mathcal{O}}\mathrm{tr}(Q^2-\frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d)^2\mathrm{d}x\\
&=(u\cdot\nabla Q,\Delta Q)-(u\cdot\nabla Q,aQ-b[Q^2-\frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d]+cQ|Q|^2)-(\Omega Q-Q\Omega,\Delta Q)\\
&\quad+(\Omega Q-Q\Omega,aQ-b[Q^2-\frac{\mathrm{tr}(Q^2)}{d}\mathrm{Id}]+cQ|Q|^2)-\lambda(|Q|D,H[Q])+2a\Gamma\left(\Delta Q,Q\right)\\
&\quad-2b\Gamma\left(\Delta Q,Q^2\right)+2bc\Gamma\left(Q\mathrm{tr}(Q^2),Q^2\right)+2(a+M)b\Gamma\left(Q,Q^2\right)+2c\Gamma\left(\Delta Q,Q\mathrm{tr}(Q^2)\right)\\
&\quad-2M\Gamma\|\nabla Q\|_{L^2}^2-2aM\Gamma\|Q\|_{L^2}^2+2\lambda M(|Q|D,Q)-2(a+M)c\Gamma\|Q\|_{L^4}^4\\
&\le\frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2+\frac{c^2\Gamma}{2}\|Q\|_{L^6}^6+C(N,M)(\|\nabla Q\|_{L^2}^2+\|Q\|_{L^2}^2+\|Q\|_{L^4}^4).
\end{aligned}
$$

Then by Gronwall's inequality, we have

$$\|Q\|_{L^\infty(0,T;H^1(\mathcal{O}))}+\|Q\|_{L^2(0,T;H^2(\mathcal{O}))}\le C,$$

where $C>0$ depends on $M,N,a,b,c,\lambda,\Gamma,T,\|\bar{Q}\|_{H^1(\mathcal{O})}$.

For the uniqueness, we denote by $\tilde{Q}=Q_1-Q_2$, with Q_1 and Q_2 two solutions to the problem (3.2.14). Then it holds,

$$
\begin{cases}
\tilde{Q}_t+u\cdot\nabla\tilde{Q}-\Omega\tilde{Q}+\tilde{Q}\Omega-\lambda(|Q_1|-|Q_2|)D-\Gamma\Delta\tilde{Q}+a\Gamma\tilde{Q}\\
=\Gamma(b[\tilde{Q}(Q_1+Q_2)-\frac{\mathrm{tr}(\tilde{Q}(Q_1+Q_2))}{d}\mathrm{I}_d]-c\tilde{Q}\mathrm{tr}Q_1^2-cQ_2\mathrm{tr}(\tilde{Q}(Q_1+Q_2))),\\
\tilde{Q}|_{t=0}=\tilde{Q}|_{\partial\mathcal{O}}=0.
\end{cases}
\tag{3.2.15}
$$

Multiply equation (3.2.15) by $\tilde{Q}$, take the trace, integrate by parts over $\mathcal{O}$, we obtain,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{Q}\|_{L^2}^2+\Gamma\|\nabla\tilde{Q}\|_{L^2}^2\\
&=-a\Gamma\|\tilde{Q}\|_{L^2}^2-(u\cdot\nabla\tilde{Q},\tilde{Q})+(\Omega\tilde{Q}-\tilde{Q}\Omega,\tilde{Q})+\lambda((|Q_1|-|Q_2|)D,\tilde{Q})\\
&\quad+\Gamma(b[\tilde{Q}(Q_1+Q_2)-\frac{\mathrm{tr}(\tilde{Q}(Q_1+Q_2))\mathrm{I}_d}{d}]-c\tilde{Q}\mathrm{tr}Q_1^2-cQ_2\mathrm{tr}(\tilde{Q}(Q_1+Q_2)),\tilde{Q})\\
&\le C\|\tilde{Q}\|_{L^2}^2+N\|\nabla\tilde{Q}\|_{L^2}\|\tilde{Q}\|_{L^2}+C\|\tilde{Q}\|_{L^2}\|\tilde{Q}\|_{L^6}(\|Q_1+Q_2\|_{L^3}+\|Q_1\|_{L^6}^2+\|Q_2\|_{L^6}^2)
\end{aligned}
$$

$$\leq \frac{\Gamma}{2}\|\nabla \tilde{Q}\|_{L^2}^2 + C\|\tilde{Q}\|_{L^2}^2,$$

where C depends on $a, b, c, \lambda, \Gamma, N, \|Q_1\|_{L^\infty(0,T;H^1(\mathcal{O}))}, \|Q_2\|_{L^\infty(0,T;H^1(\mathcal{O}))}$, and in the last step, we used Sobolev embedding inequality, Poincar$\acute{e}$ inequality and Young's inequality. Then, by applying Gronwall's inequality to the above iequality, we obtain the desired uniqueness result.

In the following, we shall show that the map $u \mapsto Q[u]$ is continuous. Let $\{u_n\}$ be a bounded sequence in $\mathcal{N}_N$, with

$$\lim_{n\to\infty} \|u_n - u\|_{C(0,T;C_0^2(\bar{\mathcal{O}}))} = 0, \tag{3.2.16}$$

for some $u \in C(0,T;C_0^2(\bar{\mathcal{O}}))$. Set $Q_n = Q_n[u_n], Q = Q[u]$, and $\tilde{Q}_n = Q_n - Q$. Taking the difference of the equations of Q_n and Q, multiplying the resulting equation by $-\Delta\tilde{Q}_n$, taking the trace, integrating by parts over $\mathcal{O}$, we obtain,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tilde{Q}_n\|_{L^2}^2 + \Gamma\|\Delta\tilde{Q}_n\|_{L^2}^2 \\
&= (u_n\cdot\nabla\tilde{Q}_n + (u_n-u)\cdot\nabla Q, \Delta\tilde{Q}_n) - (\Omega_n\tilde{Q}_n + (\Omega_n-\Omega)Q, \Delta\tilde{Q}_n) \\
&\quad + (\tilde{Q}_n\Omega_n + Q(\Omega_n-\Omega), \Delta\tilde{Q}_n) - \lambda(|Q_n|(D_n-D) + (|Q_n|-|Q|)D, \Delta\tilde{Q}_n) \\
&\quad + a\Gamma(\tilde{Q}_n, \Delta\tilde{Q}_n) - b\Gamma(\tilde{Q}_n(Q_n+Q) - \frac{\mathrm{tr}(\tilde{Q}_n(Q_n+Q))}{d}\mathrm{I}_d, \Delta\tilde{Q}_n) \\
&\quad + c\Gamma(|Q_n|^2\tilde{Q}_n + Q\mathrm{tr}(\tilde{Q}_n(Q_n+Q)), \Delta\tilde{Q}_n) \\
&= \sum_{i=1}^{7}\mathcal{I}_i.
\end{aligned}$$

In the following, we shall estimate the terms on the right hand side of the above equation one by one, by using the Sobolev embedding inequality, Poincar$\acute{e}$ inequality and Young's inequality:

$$\begin{aligned}
\mathcal{I}_1 &= (u_n\cdot\nabla\tilde{Q}_n + (u_n-u)\cdot\nabla Q, \Delta\tilde{Q}_n) \\
&\leq (\|u_n\|_{L^\infty}\|\nabla\tilde{Q}_n\|_{L^2} + \|u_n-u\|_{L^\infty}\|\nabla Q\|_{L^2})\|\Delta\tilde{Q}_n\|_{L^2} \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2 + C\|u_n-u\|_{L^\infty}^2. \\
\mathcal{I}_2 &= -(\Omega_n\tilde{Q}_n + (\Omega_n-\Omega)Q, \Delta\tilde{Q}_n)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\Omega_n\|_{L^\infty}\|\tilde{Q}_n\|_{L^2}\|\Delta\tilde{Q}_n\|_{L^2} + \|\nabla u_n - \nabla u\|_{L^\infty}\|Q\|_{L^2}\|\Delta\tilde{Q}_n\|_{L^2} \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2 \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2. \\
\mathcal{I}_3 &= (\tilde{Q}_n\Omega_n + Q(\Omega_n - \Omega), \Delta\tilde{Q}_n) \\
&\leq \|\tilde{Q}_n\|_{L^2}\|\Omega_n\|_{L^\infty}\|\Delta\tilde{Q}_n\|_{L^2} + \|Q\|_{L^2}\|\Omega_n - \Omega\|_{L^\infty}\|\Delta\tilde{Q}_n\|_{L^2} \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2 \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2. \\
\mathcal{I}_4 &= -\lambda(|Q_n|(D_n - D) + (|Q_n| - |Q|)D, \Delta\tilde{Q}_n) \\
&\leq (\|Q_n\|_{L^2}\|\nabla u_n - \nabla u\|_{L^\infty} + \|\tilde{Q}_n\|_{L^2}\|\nabla u\|_{L^\infty})\|\Delta\tilde{Q}_n\|_{L^2} \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2 \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2. \\
\mathcal{I}_5 &= a\Gamma(\tilde{Q}_n, \Delta\tilde{Q}_n) = -a\Gamma\|\nabla\tilde{Q}_n\|_{L^2}^2. \\
\mathcal{I}_6 &= -b\Gamma(\tilde{Q}_n(Q_n + Q) + \frac{\mathrm{tr}(\tilde{Q}_n(Q_n + Q))}{3}\mathrm{I}_3, \Delta\tilde{Q}_n) \\
&\leq C\|\tilde{Q}_n\|_{L^3}\|Q_n + Q\|_{L^6}\|\Delta\tilde{Q}_n\|_{L^2} \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2. \\
\mathcal{I}_7 &= c\Gamma(|Q_n|^2\tilde{Q}_n + Q\mathrm{tr}(\tilde{Q}_n(Q_n + Q)), \Delta\tilde{Q}_n) \\
&\leq C\Gamma(\|Q_n\|_{L^6}^2 + \|Q_n\|_{L^6}\|Q\|_{L^6} + \|Q\|_{L^6}^2)\|\tilde{Q}_n\|_{L^6}\|\Delta\tilde{Q}_n\|_{L^2} \\
&\leq \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2.
\end{aligned}
$$

Combining all the above estimates, we get,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tilde{Q}_n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta\tilde{Q}_n\|_{L^2}^2 \leq C\|u_n - u\|_{C_0^2(\mathcal{O})}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2.
$$

Then by Gronwall's inequality, we have

$$
\|\nabla\tilde{Q}_n(t,\cdot)\|_{L^2}^2 + \Gamma\int_0^t \|\Delta\tilde{Q}_n(s,\cdot)\|_{L^2}^2 \mathrm{d}s \leq CTe^{Ct}\|u_n - u\|_{C([0,T];C_0^2(\mathcal{O}))}^2. \tag{3.2.17}
$$

As $n \to \infty$, we conclude that

$$\lim_{n\to\infty}(\|\tilde{Q}_n\|_{L^\infty(0,T;H^1(\mathcal{O}))} + \|\tilde{Q}_n\|_{L^2(0,T;H^2(\mathcal{O}))}) = 0. \tag{3.2.18}$$

Finally, we shall show $Q \in S_0^d$, that is, $Q^T = Q$, and $\mathrm{tr}Q = 0$ *a.e.* in $[0,T] \times \mathcal{O}$. It is obvious to see that if Q is a solution to the problem (3.2.14), so is Q^T. Then by the uniqueness of the solution we proved earlier, we know that $Q^T = Q$. The only thing left is to show that $\mathrm{tr}Q = 0$. Let us take the trace on both side of the first equation in (3.2.14) to get

$$\partial_t(\mathrm{tr}Q) + u \cdot \nabla \mathrm{tr}Q = \Gamma(\Delta \mathrm{tr}Q - a\mathrm{tr}Q - c\mathrm{tr}Q\mathrm{tr}(Q^2)), \tag{3.2.19}$$

with $\mathrm{tr}Q|_{t=0} = \mathrm{tr}Q|_{\partial\mathcal{O}} = 0$. Here we use the fact that $Q^T = Q$, $\Omega^T = -\Omega$ and $\mathrm{tr}D = \mathrm{div}\, u = 0$. Then we multiply the equation (3.2.19) by $\mathrm{tr}Q$, and integrate by parts over $\mathcal{O}$ to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{tr}Q\|_{L^2}^2 + \Gamma\|\nabla\mathrm{tr}Q\|_{L^2}^2 = -a\Gamma\|\mathrm{tr}Q\|_{L^2}^2 - c\Gamma\int_{\mathcal{O}} |\mathrm{tr}Q|^2\mathrm{tr}(Q^2)\mathrm{d}x \\
&\le -a\Gamma\|\mathrm{tr}Q\|_{L^2}^2 + C\|Q\|_{L^6}^2\|\mathrm{tr}Q\|_{L^6}\|\mathrm{tr}Q\|_{L^2} \\
&\le -a\Gamma\|\mathrm{tr}Q\|_{L^2}^2 + C\|\nabla\mathrm{tr}Q\|_{L^2}\|\mathrm{tr}Q\|_{L^2} \\
&\le \frac{\Gamma}{2}\|\nabla\mathrm{tr}Q\|_{L^2}^2 + C\|\mathrm{tr}Q\|_{L^2}^2.
\end{aligned}$$

Applying the Gronwall's inequality again, we complete the proof. □

3.2.3 Existence of the solution to the approximation scheme

In this subsection, we finish the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. Let $\bar{u}_n \in Y_n$, we first consider the following two problems:

$$\begin{cases} (\rho_n)_t + \nabla\cdot(\bar{u}_n\rho_n) = 0, \\ \rho_n|_{t=0} = \overline{\rho^\varepsilon}, \end{cases} \tag{3.2.20}$$

$$\begin{cases} (Q_n)_t + (\bar{u}_n\cdot\nabla)Q_n + Q_n\bar{\Omega}_n - \bar{\Omega}_nQ_n - \lambda|Q_n|\bar{D}_n = \Gamma H[Q_n], \\ Q_n|_{t=0} = Q_n|_{\partial\mathcal{O}} = \bar{Q}, \end{cases} \tag{3.2.21}$$

with $\bar{\Omega}_n = (\nabla\bar{u}_n - \nabla\bar{u}_n^\top)/2$ and $\bar{D}_n = (\nabla\bar{u}_n + \nabla\bar{u}_n^\top)/2$. By Lemma 3.2.2, we have a unique classical solution $\rho_n = S[\bar{u}_n]$ to the initial-value problem (3.2.20), and from Lemma 3.2.3 the problem (3.2.21) has a unique solution $Q_n = Q[\bar{u}_n]$.

Then we consider the Navier-Stokes equations and look for the solution $u_n \in Y_n$ to the following variational approximation problem:

$$\begin{aligned}
&\int_{\mathcal{O}} \rho(t,x)u_n(t,x)\cdot\psi(x)\mathrm{d}x + \int_0^t\int_{\mathcal{O}} \operatorname{div}(\rho u_n \otimes \nabla u_n)\cdot\psi \mathrm{d}x\mathrm{d}s + \mu\int_0^t\int_{\mathcal{O}} \nabla u_n : \nabla\psi \mathrm{d}x\mathrm{d}s \\
&= \int_0^t\int_{\mathcal{O}} (\nabla Q\odot\nabla Q + \lambda|Q|H[Q] - Q\Delta Q + \Delta Q Q - \kappa Q) : \nabla\psi\mathrm{d}x\mathrm{d}s + \int_{\mathcal{O}} \overline{\rho^\varepsilon} u_{0,n}\cdot\psi(x)\mathrm{d}x,
\end{aligned} \tag{3.2.22}$$

for any $t \in [0,T]$, and $\psi \in X_n$.

Next, we introduce a family of operators as in [17],

$$\mathcal{M}[\rho] : X_n \mapsto X_n^*, \quad \mathcal{M}[\rho]v(w) = \int_{\mathcal{O}} \rho v\cdot w\mathrm{d}x, \tag{3.2.23}$$

for any $v, w \in X_n$. As stated in [17], we know that this operator is invertible provided ρ is strictly positive in $\mathcal{O}$, and the functional

$$\rho \mapsto \mathcal{M}^{-1}[\rho] \tag{3.2.24}$$

maps from $N_\eta = \{\rho \in L^1(\mathcal{O}) | \inf_{x\in\mathcal{O}} \rho \geq \eta > 0\}$ into $\mathcal{L}(X_n^*, X_n)$ with the following property,

$$\|\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2]\|_{\mathcal{L}(X_n^*,X_n)} \leq C(n,\eta)\|\rho_1 - \rho_2\|_{L^1(\mathcal{O})}. \tag{3.2.25}$$

From the above together with Theorems 3.2.2 and 3.2.3, where we take $\rho_n = S[u_n]$, and $Q_n = Q[u_n]$, we can rewrite the variational problem (3.2.22) as:

$$u_n(t) = \mathcal{M}^{-1}[\rho_n]\left(q^* + \int_0^t \mathcal{N}[\rho_n(s), u_n(s), Q_n(s)]\mathrm{d}s\right), \tag{3.2.26}$$

with

$$(q^*, \psi) = \int_{\mathcal{O}} \overline{\rho^\varepsilon} u_{0,n}\cdot\psi\mathrm{d}x, \tag{3.2.27}$$

$$
\begin{aligned}
(\mathcal{N}[\rho_n,u_n,Q_n],\psi) = &-\int_{\mathcal{O}} \mathrm{div}(\rho_n u_n\otimes u_n)\cdot\psi \mathrm{d}x - \mu\int_{\mathcal{O}} \nabla u_n : \nabla\psi \mathrm{d}x \\
&+ \int_{\mathcal{O}} (\nabla Q_n \odot \nabla Q_n + \lambda|Q_n|H[Q_n] - Q_n\Delta Q_n + \Delta Q_n Q_n - \kappa Q_n) : \nabla\psi \mathrm{d}x,
\end{aligned} \tag{3.2.28}
$$

for any $t\in[0,T]$, and $\psi\in X_n$. Therefore, combining (3.2.10), (3.2.17) and (3.2.25), we achieve a local solution (ρ_n,u_n,Q_n) to the problem (3.2.5), (3.2.7), (3.2.22), with initial data (3.2.2)-(3.2.4) and boundary data (3.0.9) on a short time interval $[0,T_n], T_n\le T$, by using the standard fixed point theorem on $C([0,T];X_n)$. In order to extend the existence time T_n to T for any $n=1,2,\cdots$, we need to prove that u_n stays bounded in X_n for the whole interval $[0,T_n]$. Hence, in the following, we will prove an energy inequality in the same way as Proposition 3.1.1. Taking the sum of (3.2.6) multiplied by u_n and (3.2.7) multiplied by $-H[Q_n]+MQ_n$ (M is a sufficiently large constant as in (3.1.4)), taking the trace, and integrating by parts over $\mathcal{O}$, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_n^M(t) + \mu\|\nabla u_n\|_{L^2}^2 + \Gamma\|\Delta Q_n\|_{L^2}^2 + (a^2+2aM)\Gamma\|Q_n\|_{L^2}^2 + c^2\Gamma\|Q_n\|_{L^6}^6 \\
&\quad + 2M\Gamma(\|\nabla Q_n\|_{L^2}^2 + c\|Q_n\|_{L^4}^4) + b^2\Gamma\int_{\mathcal{O}} \mathrm{tr}(Q_n^2 - \frac{\mathrm{tr}(Q_n^2)}{d}Id)^2 \mathrm{d}x \\
&= -\kappa(Q_n,\nabla u_n) + 2a\Gamma\left(\Delta Q_n, Q_n\right) - 2ac\Gamma\|Q_n\|_{L^4}^4 - 2b\Gamma\left(\Delta Q_n, Q_n^2\right) \\
&\quad + 2bc\Gamma\left(Q_n\mathrm{tr}(Q_n^2), Q_n^2\right) + 2(a+M)b\Gamma\left(Q_n, Q_n^2\right) + 2c\Gamma\left(\Delta Q_n, Q_n\mathrm{tr}(Q_n^2)\right) \\
&\quad + 2\lambda M(|Q_n|D_n, Q_n)) \\
&\le \frac{\mu}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q_n\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q_n\|_{L^6}^6 + C(\|Q_n\|_{L^2}^2 + \|Q_n\|_{L^4}^4),
\end{aligned}
$$

which gives,

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_n^M(t) + \frac{\mu}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q_n\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q_n\|_{L^6}^6 \le C(\|Q_n\|_{L^2}^2 + \|Q_n\|_{L^4}^4), \tag{3.2.29}
$$

where C is independent of n and ε, and

$$
E_n^M(t) = \int_{\mathcal{O}} \left(\frac{1}{2}\rho_n|u_n|^2 + \frac{1}{2}|\nabla Q_n|^2 + (\frac{a}{2}+M)|Q_n|^2 - \frac{b}{3}\mathrm{tr}(Q_n^3) + \frac{c}{4}|Q_n|^4\right) dx. \tag{3.2.30}
$$

Then from Gronwall's inequality, the above inequality yields, for $a.e.\ t \in [0, T_n]$,

$$\begin{aligned}
&\int_{\mathcal{O}} (\rho_n |u_n|^2 + |\nabla Q_n|^2 + M|Q_n|^2 + \frac{c}{4}|Q_n|^4) dx + \int_0^t \int_{\mathcal{O}} (\mu |\nabla u_n|^2 + \Gamma |\Delta Q_n|^2 + c^2 \Gamma |Q_n|^6) \mathrm{d}x \mathrm{d}s \\
&\leq C e^{Ct} \int_{\mathcal{O}} (\overline{\rho^{\varepsilon}} |u_{0,n}|^2 + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla \bar{Q}|^2) \mathrm{d}x \\
&\leq C e^{Ct} \int_{\mathcal{O}} (\overline{\rho^{\varepsilon}} |u_{0,\varepsilon}|^2 + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla \bar{Q}|^2) \mathrm{d}x \leq C,
\end{aligned} \tag{3.2.31}$$

where C is independent of n. Then we obtain the following uniform estimates:

$$\mu \int_0^{T_n} \|\nabla u_n\|_{L^2}^2 \mathrm{d}t \leq C, \tag{3.2.32}$$

$$\sup_{t \in [0, T_n]} \int_{\mathcal{O}} \rho_n |u_n|^2 \mathrm{d}x \leq C. \tag{3.2.33}$$

Since X_n is a finite dimensional space, we can deduce from Lemma 3.2.2 that there exists a constant $C = C(n, \bar{\rho}, \bar{m}, \bar{Q}, a, b, c, \mathcal{O})$, such that

$$0 < \varepsilon \leq \rho_n(t, x) \leq C, \quad \forall\, t \in (0, T_n),\ x \in \mathcal{O}, \tag{3.2.34}$$

which, combined with (3.2.33) and the fact that L^∞ and L^2 norms are equivalent on X_n, yields

$$\sup_{t \in [0, T_n]} (\|u_n(t, \cdot)\|_{L^\infty(\mathcal{O})} + \|\nabla u_n(t, \cdot)\|_{L^\infty(\mathcal{O})}) \leq C(n, E_\delta(0), M, \mathcal{O}). \tag{3.2.35}$$

Then we can extend the existence time interval $[0, T_n)$ of the solution (u_n, ρ_n, Q_n) to $[0, T]$, for any $T \in [0, \infty)$. □

3.3 CONVERGENCE OF THE APPROXIMATE PROBLEM

In this section, we shall complete the proof of the main Theorem 3.1.1, by taking the limit of the approximation system as $n \to \infty$ and $\varepsilon \to 0$.

3.3.1 The existence of the first level approximate solutions as $n \to \infty$

First we keep ε as a fixed constant, then ρ_n is bounded away from zero uniformly in n. Thus from the energy inequality (3.2.31), we obtain the following estimates of the solution (ρ_n, u_n, Q_n) which are independent of n,

$$0 < \varepsilon \leq \rho_n(t,x) \leq C, \quad \forall t \in (0,T), x \in \mathcal{O}, \tag{3.3.1}$$

$$\|\sqrt{\rho_n} u_n\|_{L^\infty(0,T;L^2(\mathcal{O}))} \leq C, \tag{3.3.2}$$

$$\|u_n\|_{L^\infty(0,T;L^2(\mathcal{O}))\cap L^2(0,T;H_0^1(\mathcal{O}))} \leq C, \tag{3.3.3}$$

$$\|Q_n\|_{L^\infty(0,T;H^1(\mathcal{O})\cap L^4(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))\cap L^6(0,T;L^6(\mathcal{O}))} \leq C. \tag{3.3.4}$$

The above estimates give the following convergence results for any fixed $\varepsilon > 0$ as $n \to \infty$:

$$u_n \rightharpoonup u \text{ in } L^2(0,T;H_0^1(\mathcal{O})), \tag{3.3.5}$$

$$Q_n \rightharpoonup Q \text{ in } L^2(0,T;H^2(\mathcal{O})), \tag{3.3.6}$$

$$\sqrt{\rho_n} u_n \rightharpoonup^* \sqrt{\rho} u \text{ in } L^\infty(0,T;L^2(\mathcal{O})). \tag{3.3.7}$$

However the above convergence is not strong enough to pass every term in the system to a limit. We need to prove more compactness results as follows.

First, we apply Theorem A.0.1 to obtain some compactness for ρ_n and u_n. The only condition we need to verify is that for any $\psi \in \mathcal{D}$ with $\operatorname{div} \psi = 0$, the following holds

$$|(\partial_t(\rho_n u_n), \psi)| \leq C\|\psi\|_{L^2(0,T;W^{1,3}(\mathcal{O}))}. \tag{3.3.8}$$

To this end, we show that other terms aside from $\partial_t(\rho_n u_n)$ in equation (3.2.6) belong to $W^{-1,\frac{3}{2}}(\mathcal{O})$. In fact, by using the Sobolev embedding inequality and the Poincaré inequality, we have

$$\begin{aligned}
&\|\rho_n u_n \otimes u_n\|_{L_t^2 L_x^{\frac{3}{2}}} \leq C\Big(\int_0^T \|\sqrt{\rho_n} u_n\|_{L^2}^2 \|u_n\|_{L^6}^2 \mathrm{d}t\Big)^{\frac{1}{2}} \leq C\Big(\int_0^T \|\sqrt{\rho_n} u_n\|_{L^2}^2 \|u_n\|_{H^1} \mathrm{d}t\Big)^{\frac{1}{2}} \\
&\leq C\|\sqrt{\rho_n} u_n\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 H_x^1}, \\
&\|\lambda |Q_n| H[Q_n]\|_{L_t^2 L_x^{\frac{3}{2}}} = \|\lambda |Q_n| \big(\Delta Q_n - aQ_n + b[Q_n^2 - \frac{\operatorname{tr}(Q_n^2)}{d}\mathrm{I}_d] - cQ_n|Q_n|^2\big)\|_{L_t^2 L_x^{\frac{3}{2}}} \\
&\leq C\Big(\int_0^T \|Q_n\|_{L^6}^2 \|\Delta Q_n\|_{L^2}^2 + \|Q_n\|_{L^3}^4 + \|Q_n\|_{L^3}^2 \|Q_n\|_{L^6}^4 + \|Q_n\|_{L^6}^8 \mathrm{d}t\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C\big(\|Q_n\|_{L_t^\infty H_x^1} + \|Q_n\|_{L_t^\infty H_x^1}^2 + \|Q_n\|_{L_t^\infty H_x^1}^3\big)\|Q_n\|_{L_t^2 H_x^2},$$

$$\|\nabla Q_n \odot \nabla Q_n + Q_n \Delta Q_n - \Delta Q_n Q_n + \kappa Q_n\|_{L_t^2 L_x^{\frac{3}{2}}}$$

$$\leq C\Big(\int_0^T \|\nabla Q_n\|_{L^2}^2 \|\nabla Q_n\|_{L^6}^2 + \|Q_n\|_{L^6}^2 \|\Delta Q_n\|_{L^2}^2 + \|Q_n\|_{L^2}^2 |\mathcal{O}|^{\frac{1}{3}} \mathrm{d}t\Big)^{\frac{1}{2}}$$

$$\leq C\big(1 + \|Q_n\|_{L_t^\infty H_x^1}\big)\|Q_n\|_{L_t^2 H_x^2},$$

which imply that the first order derivatives of all the above terms are in $L^2(0,T;W^{-1,\frac{3}{2}}(\mathcal{O}))$, and so is $\partial_t(\rho_n u_n)$. Then by Theorem A.0.1, we have

$$\rho_n \to \rho \text{ in } C([0,T];L^q(\mathcal{O})), \tag{3.3.9}$$

$$\sqrt{\rho_n} u_n \to \rho u \text{ in } L^p(0,T;L^r(\mathcal{O})), \tag{3.3.10}$$

$$\rho_n u_n \to \rho u \text{ in } L^p(0,T;L^r(\mathcal{O})), \tag{3.3.11}$$

for any $1 \leq q < \infty$, $2 < p < \infty$, and $1 \leq r < \frac{2dp}{dp-4}$.

Moreover, similarly to the above statement, from the estimates (3.3.3) and (3.3.4) along with the fact that $\partial_t Q_n$ satisfies (3.2.7), we know that $\partial_t Q_n \in L^2(0,T;L^{\frac{3}{2}}(\mathcal{O}))$, and

$$\|(u_n \cdot \nabla)Q_n\|_{L_t^2 L_x^{\frac{3}{2}}} \leq C(\int_0^T \|u_n\|_{L^6}^2 \|\nabla Q_n\|_{L^2}^2 \mathrm{d}t)^{\frac{1}{2}} \leq C\|u_n\|_{L_t^2 (H_0^1)_x} \|\nabla Q_n\|_{L_t^\infty L_x^2},$$

$$\|\lambda |Q_n| D_n\|_{L_t^2 L_x^{\frac{3}{2}}} \leq C(\int_0^T \|Q_n\|_{L^6}^2 \|\nabla u_n\|_{L^2}^2 \mathrm{d}t)^{\frac{1}{2}} \leq C\|Q_n\|_{L_t^\infty H_x^1} \|\nabla u_n\|_{L_{t,x}^2},$$

$$\|\Gamma H[Q_n]\|_{L_t^2 L_x^{\frac{3}{2}}} = \Gamma\|\Delta Q_n - aQ_n + b[Q_n^2 - \frac{\mathrm{tr}(Q_n^2)}{d}\mathrm{I}_d] - cQ_n|Q_n|^2\|_{L_t^2 L_x^{\frac{3}{2}}}$$

$$\leq C(\int_0^T (\|\Delta Q_n\|_{L_x^2}^2 |\mathcal{O}|^{\frac{1}{3}} + \|Q_n\|_{L_x^2}^2 |\mathcal{O}|^{\frac{1}{3}} + \|Q_n\|_{L_x^3}^4 + \|Q_n\|_{L_x^6}^4 \|Q_n\|_{L_x^3}^2)\mathrm{d}t)^{\frac{1}{2}}$$

$$\leq C(\|\Delta Q_n\|_{L_{t,x}^2} + \|Q_n\|_{L_{t,x}^2} + \|Q_n\|_{L_t^2 H_x^2}(\|Q_n\|_{L_t^\infty H_x^1} + \|Q_n\|_{L_t^\infty H_x^1}^2)).$$

Then by the Aubin-Lions Lemma (Lemma A.0.6), we have

$$Q_n \rightharpoonup Q \text{ in } L^2(0,T;H^2(\mathcal{O})), \quad Q_n \to Q \text{ in } L^2(0,T;H^1(\mathcal{O})). \tag{3.3.12}$$

Obviously, (3.2.3) implies the strong convergence of $u_{0,n}$ to $u_{0,\varepsilon}$ in $L^2((0,T)\times\mathcal{O})$. Thus we can pass to limit as n goes to infinity in system (3.2.5)-(3.2.7) in $\mathcal{D}'((0,T)\times\mathcal{O})$, i.e.

(ρ, u, Q) is a weak solution to this system. Regarding the limit of the energy, from the above convergence results, we know that

$$
\begin{aligned}
&E^M(t) \leq \liminf_{n\to\infty} E_n^M(t),\\
&\int_0^t \int_{\mathcal{O}} (\mu|\nabla u|^2 + \Gamma|\Delta Q|^2 + c^2\Gamma|Q|^6)\mathrm{d}x\mathrm{d}s\\
&\leq \liminf_{n\to\infty} \int_0^t \int_{\mathcal{O}} (\mu|\nabla u_n|^2 + \Gamma|\Delta Q_n|^2 + c^2\Gamma|Q_n|^6)\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

For the initial data in the energy inequality, by the definition of $u_{0,n}$ we have

$$
\int_{\mathcal{O}} \overline{\rho^\varepsilon}|u_{0,n}|^2\mathrm{d}x \to \int_{\mathcal{O}} \overline{\rho^\varepsilon}|u_{0,\varepsilon}|^2\mathrm{d}x.
$$

In summary, we obtain the following results:

Proposition 3.3.1. *For any $T > 0$, there is a solution $(\rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$ which satisfies the system* (3.0.2)-(3.0.5) *in the weak sense of Definition 3.1.1, with the initial data*

$$
\rho|_{t=0} = \overline{\rho^\varepsilon}, \quad u|_{t=0} = u_{0,\varepsilon}, \quad Q|_{t=0} = \bar{Q}, \tag{3.3.13}
$$

and the boundary condition (3.0.9). *In addition, the solution satisfies the following energy inequality:*

$$
\begin{aligned}
&E^M(t) + \frac{1}{2}\int_0^T \int_{\mathcal{O}} (\mu|\nabla u|^2 + \Gamma|\Delta Q|^2 + c^2\Gamma|Q|^6)\mathrm{d}x\mathrm{d}t\\
&\leq Ce^{CT}\int_{\mathcal{O}} (\overline{\rho^\varepsilon}|u_{0,\varepsilon}|^2 + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla\bar{Q}|^2)\mathrm{d}x.
\end{aligned} \tag{3.3.14}
$$

3.3.2 Pass to the limit as $\varepsilon \to 0$

The difficulty here is the degeneracy of the lower bound of ρ, but we can resolve this using Theorem A.0.1. First let us consider the initial kinetic energy. Since we have $\nabla \times Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon} = 0$ in $\mathcal{D}'$, we know

$$
\begin{aligned}
&\int_{\mathcal{O}} \overline{\rho^\varepsilon}|u_{0,\varepsilon}|^2 \mathrm{d}x = \int_{\mathcal{O}} \frac{1}{\overline{\rho^\varepsilon}}|\overline{m^\varepsilon} - Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon}|^2 \mathrm{d}x \\
&= \int_{\mathcal{O}} \frac{1}{\overline{\rho^\varepsilon}}(|\overline{m^\varepsilon}|^2 + |Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon}|^2 - 2\overline{m^\varepsilon}Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon})\mathrm{d}x \\
&= \int_{\mathcal{O}} \frac{1}{\overline{\rho^\varepsilon}}(|\overline{m^\varepsilon}|^2 + |Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon}|^2 - 2(\overline{\rho^\varepsilon}u_{0,\varepsilon} + Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon})Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon})\mathrm{d}x \\
&= \int_{\mathcal{O}} \frac{1}{\overline{\rho^\varepsilon}}(|\overline{m^\varepsilon}|^2 - |Q_{\overline{\rho^\varepsilon}}\overline{m^\varepsilon}|^2)\mathrm{d}x.
\end{aligned}
$$

For each ε, the solution $(\rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$ satisfies the following energy inequality from (3.3.14),

$$
\begin{aligned}
&2E_\varepsilon^M(t) + \int_0^T \int_{\mathcal{O}} \mu|\nabla u_\varepsilon|^2 + \Gamma|\Delta Q_\varepsilon|^2 + c^2\Gamma|Q_\varepsilon|^6 \mathrm{d}x\mathrm{d}t \\
&\leq Ce^{CT}\int_{\mathcal{O}} (\overline{\rho^\varepsilon}|u_{0,\varepsilon}|^2 + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla\bar{Q}|^2)\mathrm{d}x \\
&\leq Ce^{CT}\int_{\mathcal{O}} (\frac{|\overline{m^\varepsilon}|^2}{\overline{\rho^\varepsilon}} + |\bar{Q}|^2 + |\bar{Q}|^4 + |\nabla\bar{Q}|^2)\mathrm{d}x,
\end{aligned}
$$

with

$$
E_\varepsilon^M(t) := \int_{\mathcal{O}} \left(\frac{1}{2}\rho_\varepsilon|u_\varepsilon|^2 + \frac{1}{2}|\nabla Q_\varepsilon|^2 + (\frac{a}{2} + M)|Q_\varepsilon|^2 - \frac{b}{3}\mathrm{tr}(Q_\varepsilon^3) + \frac{c}{4}|Q_\varepsilon|^4\right) dx.
$$

Since $\overline{m^\varepsilon}(\overline{\rho^\varepsilon})^{-\frac{1}{2}} \to \bar{m}\bar{\rho}^{-\frac{1}{2}}$ in $L^2(\mathcal{O})$ as $\varepsilon \to 0$, we have the following uniform bounds with respect to ε,

$$\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{O}))} \leq C, \tag{3.3.15}$$

$$\|u_\varepsilon\|_{L^2(0,T;H_0^1(\mathcal{O}))} \leq C, \tag{3.3.16}$$

$$\|Q_\varepsilon\|_{L^\infty(0,T;H^1(\mathcal{O})\cap L^4(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))\cap L^6(0,T;L^6(\mathcal{O}))} \leq C. \tag{3.3.17}$$

Then we obtain

$$u_\varepsilon \rightharpoonup u \text{ in } L^2(0,T;H_0^1(\mathcal{O})), \tag{3.3.18}$$

$$Q_\varepsilon \rightharpoonup Q \text{ in } L^2(0,T;H^2(\mathcal{O})), \tag{3.3.19}$$

$$\sqrt{\rho_\varepsilon}u_\varepsilon \rightharpoonup^* \sqrt{\rho}u \text{ in } L^\infty(0,T;L^2(\mathcal{O})). \tag{3.3.20}$$

In order to pass to the limit as $\varepsilon \to 0$ in the approximate system, the above convergence results are not strong enough. Then similarly to Subsection 3.3.1, we need to get some compactness results. By the same way as (3.3.8), we also can verify the condition

$$|(\partial_t(\rho_\varepsilon u_\varepsilon),\psi)| \le C\|\psi\|_{L^2(0,T;W^{1,3}(\mathcal{O}))}. \tag{3.3.21}$$

Then Theorem A.0.1 yields

$$\begin{aligned}
&\rho_\varepsilon \to \rho \text{ in } C([0,T];L^q(\mathcal{O})),\\
&\sqrt{\rho_\varepsilon}u_\varepsilon \to \rho u \text{ in } L^p(0,T;L^r(\mathcal{O})),\\
&\rho_\varepsilon u_\varepsilon \to \rho u \text{ in } L^p(0,T;L^r(\mathcal{O})),
\end{aligned}$$

for any $1 \le q < \infty$, $2 < p < \infty$, and $1 \le r < \frac{2dp}{dp-4}$.

Moreover, similarly to (3.3.12), we also have

$$Q_\varepsilon \rightharpoonup Q \text{ in } L^2(0,T;H^2(\mathcal{O})), \quad Q_\varepsilon \to Q \text{ in } L^2(0,T;H^1(\mathcal{O})). \tag{3.3.22}$$

The above boundedness and convergence results imply that

$$\begin{aligned}
&\operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) \to \operatorname{div}(\rho u \otimes u) \text{ in } \mathcal{D}',\\
&\int_0^T\int_{\mathcal{O}} \nabla\cdot(\nabla Q_\varepsilon \odot \nabla Q_\varepsilon + \lambda|Q_\varepsilon|H[Q_\varepsilon] - Q_\varepsilon\Delta Q_\varepsilon + \Delta Q_\varepsilon Q_\varepsilon - \kappa Q_\varepsilon) : \nabla_x\psi \mathrm{d}x\mathrm{d}t\\
&\to \int_0^T\int_{\mathcal{O}} \nabla\cdot(\nabla Q \odot \nabla Q + \lambda|Q|H[Q] - Q\Delta Q + \Delta Q Q - \kappa Q) : \nabla_x\psi \mathrm{d}x\mathrm{d}t,\\
&\int_0^T\int_{\mathcal{O}} (u_\varepsilon\cdot\nabla_x Q_\varepsilon + Q_\varepsilon\Omega_\varepsilon - \Omega_\varepsilon Q_\varepsilon - \lambda|Q_\varepsilon|D_\varepsilon) : \varphi \mathrm{d}x\mathrm{d}t\\
&\to \int_0^T\int_{\mathcal{O}} (u\cdot\nabla_x Q + Q\Omega - \Omega Q - \lambda|Q|D) : \varphi \mathrm{d}x\mathrm{d}t,\\
&\int_0^T\int_{\mathcal{O}} H[Q_\varepsilon] : \varphi \mathrm{d}x\mathrm{d}t \to \int_0^T\int_{\mathcal{O}} H[Q] : \varphi \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

for any $\psi \in \mathcal{D}$ with $\operatorname{div}\psi = 0$, and $\varphi \in \mathcal{D}$ with $\varphi \in S_0^d$. For the initial data in the Navier-Stokes equations, by the decomposition and the convergence of $\overline{m^\varepsilon}$, we have

$$\int_{\mathcal{O}} \overline{\rho^\varepsilon}u_{0,\varepsilon}\cdot\psi \mathrm{d}x = \int_{\mathcal{O}} \overline{m^\varepsilon}\cdot\psi \mathrm{d}x \to \int_{\mathcal{O}} \bar{m}\cdot\psi \mathrm{d}x, \tag{3.3.23}$$

for any $\psi \in \mathcal{O}$ with $\operatorname{div} \psi = 0$. Then we can pass to the limit to obtain the weak solution (ρ, u, Q) of the problem (3.0.2)-(3.0.5) with the initial condition (3.0.6)-(3.0.8) and the boundary condition (3.0.9).

4.0 COMPRESSIBLE ACTIVE LIQUID CRYSTALS

In this chapter, we build on the work in Chapter 2 to analyze the following system for compressible flows of active nematic liquid crystals [25,29] in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$:

$$\begin{cases} \partial_t c + (u \cdot \nabla) c = D_0 \Delta c, \\ \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u = \nabla \cdot \tau + \nabla \cdot \sigma, \\ \partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c], \end{cases} \tag{4.0.1}$$

where c is the concentration of active particles, ρ is the density of the fluid, $u \in \mathbb{R}^3$ is the flow velocity, the nematic tensor order parameter Q is a traceless and symmetric 3×3 matrix, $P = \kappa \rho^\gamma$ denotes the pressure with adiabatic constant $\gamma > 1$, $D_0 > 0$ is the diffusion constant, $\mu > 0$ and $\nu > 0$ are the viscosity coefficients, $\Gamma^{-1} > 0$ is the rotational viscosity, and $\Omega = \frac{1}{2}\big(\nabla u - \nabla u^\top\big)$ is the antisymmetric part of the strain tensor. Moreover, the tensor:

$$H[Q, c] := K \Delta Q - \frac{k}{2}(c - c_*) Q + b\Big(Q^2 - \frac{\operatorname{tr}(Q^2)}{3} \mathrm{I}_3\Big) - c_* Q \operatorname{tr}(Q^2)$$

describes the relaxational dynamics of the nematic phase, which can be obtained from the Landau-de Gennes free energy, *i.e.*, $H_{\alpha\beta} = -\frac{\delta \mathcal{F}}{\delta Q_{\alpha\beta}}$ with

$$\mathcal{F} = \int \Big(\frac{k}{4}(c - c_*) \operatorname{tr}(Q^2) - \frac{b}{3} \operatorname{tr}(Q^3) + \frac{c_*}{4} |\operatorname{tr}(Q^2)|^2 + \frac{K}{2} |\nabla Q|^2\Big) dA,$$

where K is the elastic constant for the one-constant elastic energy density, c_* is the critical concentration for the isotropic-nematic transition, and $k > 0$ and $b \in \mathbb{R}$ are material-dependent constants. Without loss of generality, we take $K = k = 1$ in this paper. The stress tensor $\sigma = (\sigma^{ij})$ has two contributions:

$$\sigma^{ij} = \sigma_r^{ij} + \sigma_a^{ij},$$

with

$$\begin{aligned} \sigma_r^{ij} &= Q^{ik} H^{kj}[Q,c] - H^{ik}[Q,c] Q^{kj}, \\ \sigma_a^{ij} &= \sigma_* c^2 Q^{ij}, \end{aligned}$$

where σ_r^{ij} is the stress due to the nematic elasticity, and σ_a^{ij} is the active contribution which describes contractile ($\sigma_* > 0$) or extensile ($\sigma_* < 0$) stresses exerted by the active particles along the director field. The symmetric additional stress tensor is denoted by

$$\tau^{ij} = \mathrm{F}(Q)\delta_{ij} - \partial_j Q^{kl} \partial_i Q^{kl} = \mathrm{F}(Q)\delta_{ij} - (\nabla Q \odot \nabla Q)^{ij}$$

with

$$\mathrm{F}(Q) = \frac{1}{2}|\nabla Q|^2 + \frac{1}{2}\mathrm{tr}(Q^2) + \frac{c_*}{4}\mathrm{tr}^2(Q^2).$$

Here and elsewhere, we use the Einstein summation convention, *i.e.*, we sum over the repeated indices.

We rewrite system (4.0.1) as

$$\partial_t c + (u \cdot \nabla) c = D_0 \Delta c, \tag{4.0.2}$$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{4.0.3}$$

$$\begin{aligned} \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma = \mu \Delta u + (\nu + \mu)\nabla \mathrm{div}\, u + \nabla \cdot \big(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q\big) \\ + \nabla \cdot (Q\Delta Q - \Delta Q Q) + \sigma_* \nabla \cdot (c^2 Q), \end{aligned} \tag{4.0.4}$$

$$\partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q,c], \tag{4.0.5}$$

with

$$H[Q,c] = \Delta Q - \frac{c - c_*}{2} Q + b\Big(Q^2 - \frac{\mathrm{tr}(Q^2)}{3} I_3\Big) - c_* Q \mathrm{tr}(Q^2),$$

and $\Gamma > 0, D_0 > 0, \mu > 0, \nu > 0, c_* > 0, b, \sigma_* \in \mathbb{R}$, $\gamma > \frac{3}{2}$, $(x,t) \in \mathcal{O} \times \mathbb{R}^+$; subject to the following initial conditions:

$$(c, \rho, \rho u, Q)|_{t=0} = (c_0, \rho_0, m_0, Q_0)(x) \qquad \text{for } x \in \mathcal{O} \subset \mathbb{R}^3, \tag{4.0.6}$$

with

$$\begin{aligned} &c_0 \in H^1(\mathcal{O}), \qquad 0 < \underline{c} \le c_0 \le \overline{c} < \infty, \\ &Q_0 \in H^1(\mathcal{O}), \qquad Q_0 \in S_0^3 \quad a.e. \text{ in } \mathcal{O}, \end{aligned}$$

and the following boundary conditions on $\partial\mathcal{O}$ with unit outward normal $\vec{n}$:

$$\nabla c \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \qquad u|_{\partial\mathcal{O}} = 0, \qquad \nabla Q \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \tag{4.0.7}$$

satisfying the following compatibility conditions:

$$\rho_0 \in L^\gamma(\mathcal{O}), \quad \rho_0 \ge 0; \qquad m_0 \in L^1(\mathcal{O}), \quad m_0 = 0 \text{ if } \rho_0 = 0; \qquad \frac{|m_0|^2}{\rho_0} \in L^1(\mathcal{O}). \tag{4.0.8}$$

The hydrodynamic equations in (4.0.1), or (4.0.2)–(4.0.5), are from [30] with some differences, primarily for technical reasons. Namely, in the concentration equation, the diffusion constants are assumed to be the same in all directions, and the active current is assumed to be zero, which is equivalent to setting $\alpha_1 = 0$ in equations (15a)–(15c) in [30]. Furthermore, the flow-aligning parameter λ in [30] is assumed to be zero; this is not a severe restriction, but just implies that we are in the flow-tumbling regime. Finally, we have also neglected one of the terms in the passive "nematic" stress which does not feature for the two-dimensional systems but can play a role for the three-dimensional systems. Despite these simplifications compared to the successful model presented in [30], our work is a first step in the rigorous analysis of initial-boundary value problems for compressible active nematics in two and three space dimensions, and the mathematical approach developed here is different from the previous approaches in Chapter 2 and Chapter 3 and [50].

In the simplified system above, the fluid flow is dictated by the compressible Navier-Stokes equations; the particle concentration in the fluid and the evolution of the order parameter Q are governed by the parabolic-type equations, with extra nonlinear coupling terms as forcing terms. The term, $\mathrm{F}(Q)$, is added to close the energy in our compressible system. Since our system reduces to the compressible Navier-Stokes system in the absence of the concentration c and the Q-tensor, the best result we could expect can not be better than those in [16–18], in which the existence of finite-energy weak solutions of the compressible Navier-Stokes system (allowing initial vacuum) was proved for $\gamma > \frac{3}{2}$.

In this paper, our aim is to prove the existence of global weak solutions of this compressible coupled system (4.0.2)−(4.0.8) in three space dimensions. In our system, owing to the varying concentration $c = c(x,t)$, we multiply the Q-tensor equation by $-\big(\Delta Q - Q - c_* Q\mathrm{tr}(Q^2)\big)$, rather than $-H[Q,c]$, to avoid dealing with the interaction terms of the concentration and the Q-tensor and obtain the dissipation and *a priori* estimates for the system. Moreover, the cubic term of the Q-tensor does not appear in the energy with this strategy, so that a positive total energy for this system can be obtained, unlike in Chapter 2 and Chapter 3, where a specific positive energy for the system is re-defined by using the property of the Q-tensor, *i.e.*, (3.1.5) in Chapter 3. Furthermore, the highly nonlinear terms in this system cause new mathematical difficulties compared to the incompressible liquid crystal system in Chapter 2. However, since the maximum principle holds for the concentration equation (4.0.2) for c (*i.e.*, c is bounded if the initial condition (4.2.5) is satisfied; see Lemma 4.2.2), the highly nonlinear terms can be dealt with via using some cancellation rules as in Lemma A.0.5 and (4.1.6) in Proposition 4.1.1. We remark that the symmetry and tracelessness of the Q-tensor play a key role in the cancellations which are crucial for the proof of the existence of weak solutions. For example, in order to obtain the essential compactness results, the force term in the compressible Navier-Stokes equations should belong to $H^{-1}(\mathcal{O})$ due to Lions [38]. However, the regularity of Q obtained from the Q-tensor equation is $L_t^\infty H_x^1 \cap L_t^2 H_x^2$, which is not enough to achieve this condition. Owing to the cancellations, all the higher order nonlinear terms together vanish, so we do not need to deal with them.

In this part of the thesis, we apply the Faedo-Galerkin's method [61] with three levels of approximations to prove the existence of the solutions of the initial-boundary value problem (4.0.2)–(4.0.8) in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$. The first level of approximation concerns the artificial pressure due to the possibility of vanishing density and lower integrability of the density. Here we lift the density above zero to avoid the vacuum and add the artificial pressure to increase the integrability of the density. The second level corresponds to the artificial viscosity, which changes the continuity equation from the hyperbolic to parabolic type which ensures higher regularity. The last level is the approximation from the finite-dimensional to infinite-dimensional space. By the weak convergence argument, we obtain the global existence of finite-energy weak solutions defined as follows:

Definition 4.0.1. For any $T > 0$, (c, ρ, u, Q) is a finite-energy weak solution of problem (4.0.2)−(4.0.8) if the following conditions are satisfied:

(i) $c > 0$, $c \in L^\infty(0,T;L^2(\mathcal{O})) \cap L^2(0,T;H^1(\mathcal{O}))$; $\rho \geq 0$, $\rho \in L^\infty(0,T;L^\gamma(\mathcal{O}))$; $u \in L^2(0,T;H_0^1(\mathcal{O}))$, $Q \in L^\infty(0,T;H^1(\mathcal{O})) \cap L^2(0,T;H^2(\mathcal{O}))$, and $Q \in S_0^3$ *a.e.* in $\mathcal{O}_T = [0,T] \times \mathcal{O}$.

(ii) Equations (4.0.2)−(4.0.5) are valid in $\mathcal{D}'(\mathcal{O}_T)$. Moreover, (4.0.3) is valid in $\mathcal{D}'(0,T;\mathbb{R}^3)$, if (ρ, u) are extended to be zero on $\mathbb{R}^3 \setminus \mathcal{O}$.

(iii) Energy $E(t)$ is locally integrable on $(0,T)$ and satisfies the energy inequality:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \frac{D_0}{2}\|\nabla c\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + (\nu+\mu)\|\operatorname{div} u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c_*^2\Gamma}{2}\|Q\|_{L^6}^6 \\ &\leq C\big(\|u\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big) \qquad \text{in } \mathcal{D}'(0,T),\end{aligned}$$

where

$$E(t) := \int_{\mathcal{O}} \Big(\frac{1}{2}|c|^2 + \frac{1}{2}\rho|u|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|Q|^2 + \frac{1}{2}|\nabla Q|^2 + \frac{c_*}{4}|Q|^4\Big)\mathrm{d}x.$$

(iv) Equation (4.0.3) is satisfied in the sense of renormalized solutions; that is, for any function $g \in C^1(\mathbb{R})$ with the property:

$$g'(z) \equiv 0 \qquad \text{for all } z \geq M \text{ for a sufficiently large constant } M, \tag{4.0.9}$$

then

$$\partial_t g(\rho) + \operatorname{div}(g(\rho)u) + \big(g'(\rho)\rho - g(\rho)\big)\operatorname{div} u = 0 \qquad \text{in } \mathcal{D}'(0,T). \tag{4.0.10}$$

Our main result reads:

Theorem 4.0.1. *Let $\gamma > \frac{3}{2}$ and $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain of the class $C^{2+\tau}$ for $\tau > 0$. Assume that the initial data function $(c_0, \rho_0, m_0, Q_0)(x)$ satisfies the compatibility conditions (4.0.8). Then, for any $T > 0$, problem (4.0.2)−(4.0.7) admits a finite-energy weak solution $(c, \rho, u, Q)(t, x)$ on $\mathcal{O}_T$.*

4.1 THE DISSIPATION PRINCIPLE AND *A PRIORI* ESTIMATES

In this section, we derive the dissipation principle and *a priori* estimates in Proposition 4.1.1 for system (4.0.2)−(4.0.5).

Proposition 4.1.1. *Let (c, ρ, u, Q) be a smooth solution of problem (4.0.3)–(4.0.8). Then there exists $C > 0$ depending only on $(D_0, b, c_*, \sigma_*, \mu, \nu, \Gamma)$ and the initial data such that, for a given T,*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \frac{D_0}{2}\|\nabla c\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + (\nu+\mu)\|\mathrm{div}\, u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c_*^2\Gamma}{2}\|Q\|_{L^6}^6 \\ &\le C\big(\|u\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big) \qquad \text{for all } t\in(0,T).\end{aligned} \tag{4.1.1}$$

In addition, if $(c_0, \rho_0, m_0, Q_0)(x) \in L^\infty \times L^\gamma \times L^2 \times H^1$, then

$$\|c\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} + \|c\|_{L^\infty(0,T;L^2(\mathcal{O}))\cap L^2(0,T;H^1(\mathcal{O}))} \le C, \tag{4.1.2}$$

$$\|\rho\|_{L^\infty(0,T;L^\gamma(\mathcal{O}))} \le C, \tag{4.1.3}$$

$$\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(\mathcal{O}))}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2(0,T;L^2(\mathcal{O}))}^2 + (\nu+\mu)\|\mathrm{div}\, u\|_{L^2(0,T;L^2(\mathcal{O}))}^2 \le C, \tag{4.1.4}$$

$$\|Q\|_{L^\infty(0,T;H^1(\mathcal{O})\cap L^4(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))\cap L^6(0,T;L^6(\mathcal{O}))} \le C. \tag{4.1.5}$$

Proof. The L^∞–bound for the concentration $c(t,x)$ follows from the maximum principle and its initial condition $c_0 \in L^\infty$. Using the continuity equation (4.0.3) and the boundary equation (4.0.7), we have

$$\begin{aligned}&((\rho u)_t, u) - (\rho u\otimes u, \nabla u)\\
&= \int_{\mathcal{O}} \rho_t|u|^2\,\mathrm{d}x + \int_{\mathcal{O}} \rho\partial_t u^i u^i\,\mathrm{d}x - \int_{\mathcal{O}} \rho u^i u^j \partial_j u^i\,\mathrm{d}x\\
&= \int_{\mathcal{O}} \rho_t|u|^2\,\mathrm{d}x + \frac12\int_{\mathcal{O}} \rho\partial_t |u|^2\,\mathrm{d}x + \frac12\int_{\mathcal{O}} |u|^2\mathrm{div}\,(\rho u)\,\mathrm{d}x\\
&= \int_{\mathcal{O}} \rho_t|u|^2\,\mathrm{d}x + \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \rho|u|^2\,\mathrm{d}x - \frac12\int_{\mathcal{O}} \rho_t|u|^2\,\mathrm{d}x + \frac12\int_{\mathcal{O}} |u|^2\mathrm{div}\,(\rho u)\,\mathrm{d}x\\
&= \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \rho|u|^2\,\mathrm{d}x + \frac12\int_{\mathcal{O}} |u|^2\big(\rho_t + \mathrm{div}\,(\rho u)\big)\,\mathrm{d}x\\
&= \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \rho|u|^2\,\mathrm{d}x,\end{aligned}$$

and

$$
\begin{aligned}
(\rho^\gamma, \operatorname{div} u) &= -\int_{\mathcal{O}} \rho^{\gamma-1}(\rho_t + \nabla\rho\cdot u)\,\mathrm{d}x \\
&= -\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}\rho^\gamma\,\mathrm{d}x - \frac{1}{\gamma}\int_{\mathcal{O}}\partial_i\rho^\gamma u^i\,\mathrm{d}x = -\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}\rho^\gamma\,\mathrm{d}x + \frac{1}{\gamma}\int_{\mathcal{O}}\rho^\gamma \operatorname{div} u\,\mathrm{d}x,
\end{aligned}
$$

which gives

$$
(\rho^\gamma, \operatorname{div} u) = \frac{1}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}\rho^\gamma\,\mathrm{d}x.
$$

We take the inner product of equation (4.0.2) with c, equation (4.0.4) with u, and equation (4.0.5) with $-\big(\Delta Q - Q - c_* Q\operatorname{tr}(Q^2)\big)$ respectively, sum them up, and then integrate by parts over $\mathcal{O}$ to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}\Big(\frac{1}{2}|(c,\sqrt{\rho}u)|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|(Q,\nabla Q)|^2 + \frac{c_*}{4}|Q|^4\Big)\mathrm{d}x + D_0\|\nabla c\|_{L^2}^2 \\
&\quad + \mu\|\nabla u\|_{L^2}^2 + (\nu+\mu)\|\operatorname{div} u\|_{L^2}^2 + \Gamma\|\Delta Q\|_{L^2}^2 + \Gamma\|\nabla Q\|_{L^2}^2 + c_*\Gamma\|Q\|_{L^4}^4 + c_*^2\Gamma\|Q\|_{L^6}^6 \\
&= -(u\cdot\nabla c, c) - (\nabla\cdot(\nabla Q\odot\nabla Q), u) - (\mathrm{F}(Q)\mathrm{I}_3, \nabla u) + (\nabla\cdot(Q\Delta Q - \Delta Q Q), u) \\
&\quad - \sigma_*(c^2 Q, \nabla u) + (u\cdot\nabla Q, \Delta Q) - (u\cdot\nabla Q, Q + c_* Q|Q|^2) - (\Omega Q - Q\Omega, \Delta Q) \\
&\quad + \big(\Omega Q - Q\Omega, Q + c_* Q|Q|^2\big) + \Gamma(\frac{c-c_*}{2}Q, \Delta Q - Q - c_* Q\operatorname{tr}(Q^2)) \\
&\quad - b\Gamma(Q^2, \Delta Q) + b\Gamma(Q^2, Q + c_* Q\operatorname{tr}(Q^2)) + 2c_*\Gamma(Q|Q|^2, \Delta Q) \\
&= \sum_{i=1}^{13}\mathcal{I}_i.
\end{aligned}
$$

First, by Lemma A.0.5, we have

$$
\mathcal{I}_4 + \mathcal{I}_8 = 0.
$$

By simple calculation,

$$
\mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_6 + \mathcal{I}_7 = 0, \qquad \mathcal{I}_9 = 0,
$$

as shown below:

$$
\begin{aligned}
&\mathcal{I}_2+\mathcal{I}_6+\mathcal{I}_7\\
&=-(\nabla\cdot(\nabla Q\odot\nabla Q),u)+(u\cdot\nabla Q,\Delta Q)-(u\cdot\nabla Q,Q+c_*Q|Q|^2)\\
&=-\int_{\mathcal{O}}\partial_j\big(\partial_iQ^{kl}\partial_jQ^{kl}\big)u^i\,\mathrm{d}x+\int_{\mathcal{O}}u^i\partial_iQ^{kl}\Delta Q^{kl}\,\mathrm{d}x-\int_{\mathcal{O}}u^i\partial_iQ^{kl}\big(Q^{kl}+c_*Q^{kl}|Q|^2\big)\,\mathrm{d}x\\
&=-\int_{\mathcal{O}}\big(\partial_i\partial_jQ^{kl}\partial_jQ^{kl}u^i+\partial_iQ^{kl}\partial_j\partial_jQ^{kl}u^i\big)\,\mathrm{d}x+\int_{\mathcal{O}}u^i\partial_iQ^{kl}\Delta Q^{kl}\,\mathrm{d}x\\
&\quad-\int_{\mathcal{O}}u^i\partial_i\big(\frac{1}{2}|Q|^2+\frac{c_*}{4}\mathrm{tr}^2(Q^2)\big)\,\mathrm{d}x\\
&=-\int_{\mathcal{O}}\partial_i\partial_jQ^{kl}\partial_jQ^{kl}u^i\,\mathrm{d}x+\int_{\mathcal{O}}\big(\frac{1}{2}|Q|^2+\frac{c_*}{4}\mathrm{tr}^2(Q^2)\big)\mathrm{div}\,u\,\mathrm{d}x\\
&=\int_{\mathcal{O}}\big(\frac{1}{2}|\nabla Q|^2+\frac{1}{2}|Q|^2+\frac{c_*}{4}\mathrm{tr}^2(Q^2)\big)\mathrm{div}\,u\,\mathrm{d}x=-\mathcal{I}_3,
\end{aligned}
\tag{4.1.6}
$$

and

$$
\mathcal{I}_9=(\Omega Q-Q\Omega,Q+c_*Q|Q|^2)=-(\Omega Q+Q\Omega,Q+c_*Q|Q|^2)+2(\Omega Q,Q+c_*Q|Q|^2)=0,
$$

where we have used the fact that Q is symmetric and Ω is skew-symmetric.

Moreover, by the Young inequality, we have

$$
\begin{aligned}
|\mathcal{I}_1|&=|(u\cdot\nabla c,c)|\le C\|u\|_{L^2}\|\nabla c\|_{L^2}\|c\|_{L^\infty}\le\frac{D_0}{4}\|\nabla c\|_{L^2}^2+C\|u\|_{L^2}^2,\\
|\mathcal{I}_5|&=|\sigma_*(c^2Q,\nabla u)|\le C\|c\|_{L^\infty}^2\|\nabla u\|_{L^2}\|Q\|_{L^2}\le\frac{\mu}{4}\|\nabla u\|_{L^2}^2+C\|Q\|_{L^2}^2,\\
|\mathcal{I}_{10}|&=\Gamma\big|\big(\frac{c-c_*}{2}Q,\Delta Q-Q-c_*Q\,\mathrm{tr}(Q^2)\big)\big|\le\frac{\Gamma}{8}\|\Delta Q\|_{L^2}^2+C\|Q\|_{L^2}^2+C\|Q\|_{L^4}^4,\\
|\mathcal{I}_{11}|&=|-b\Gamma(Q^2,\Delta Q)|\le\frac{\Gamma}{8}\|\Delta Q\|_{L^2}^2+C\|Q\|_{L^4}^4,\\
|\mathcal{I}_{12}|&=|b\Gamma(Q^2,Q+c_*Q\,\mathrm{tr}(Q^2))|=\big|b\Gamma\int_{\mathcal{O}}\big(\mathrm{tr}(Q)\big)^3\,\mathrm{d}x+bc_*\Gamma\int_{\mathcal{O}}\big(\mathrm{tr}(Q)\big)^3|Q|^2\,\mathrm{d}x\big|\\
&\le\frac{c^2\Gamma}{2}\|Q\|_{L^6}^6+C\|Q\|_{L^2}^2+C\|Q\|_{L^4}^4,\\
\mathcal{I}_{13}&=2c_*\Gamma(Q|Q|^2,\Delta Q)=2c_*\Gamma\int_{\mathcal{O}}\partial_{kk}Q^{ij}Q^{ij}\mathrm{tr}(Q^2)\,\mathrm{d}x\\
&=-2c_*\Gamma\int_{\mathcal{O}}\partial_kQ^{ij}\partial_kQ^{ij}\mathrm{tr}(Q^2)\,\mathrm{d}x-2c_*\Gamma\int_{\mathcal{O}}\partial_kQ^{ij}Q^{ij}\partial_k\mathrm{tr}(Q^2)\,\mathrm{d}x\\
&=-2c_*\Gamma\int_{\mathcal{O}}|\nabla Q|^2|Q|^2\,\mathrm{d}x-c_*\Gamma\int_{\mathcal{O}}|\nabla\mathrm{tr}(Q^2)|^2\,\mathrm{d}x\le 0.
\end{aligned}
$$

Combining all the above estimates, we obtain the desired result. $\square$

Corollary 4.1.1. *For any smooth solution* (c, ρ, u, Q) *of problem* (4.0.2)–(4.0.8),

$$Q \in L^{10}(\mathcal{O}_T) \cap L^\infty(0, T; H^1(\mathcal{O})) \cap L^2(0, T; H^2(\mathcal{O})), \quad \nabla Q \in L^{\frac{10}{3}}(\mathcal{O}_T).$$

Proof. From estimate (4.1.5) and the Gagliardo-Nirenberg inequality in Lemma A.0.7, we have

$$\begin{aligned}
&\|Q\|_{L^{10}(\mathcal{O})} \le C\|Q\|_{L^6(\mathcal{O})}^{\frac{4}{5}}\|\Delta Q\|_{L^2(\mathcal{O})}^{\frac{1}{5}} + C\|Q\|_{L^6(\mathcal{O})},\\
&\|\nabla Q\|_{L^{\frac{10}{3}}(\mathcal{O})} \le C\|\nabla Q\|_{L^2(\mathcal{O})}^{\frac{2}{5}}\|\Delta Q\|_{L^2(\mathcal{O})}^{\frac{3}{5}} + C\|\nabla Q\|_{L^2(\mathcal{O})}.
\end{aligned}$$

Then the proof is complete. □

4.2 THE FAEDO-GALERKIN APPROXIMATION

In this section, we use the Faedo-Galerkin method to construct a solution (c_n, ρ_n, u_n, Q_n) of the initial-boundary value problem (4.2.1)–(4.2.4) below, which is an approximation system to (4.0.2)–(4.0.5). Since this initial-boundary value problem involves coupling terms with the concentration, velocity, and Q-tensor in the force, such as the high nonlinear terms $H[Q, c], F(Q)$, and c^2Q, and the terms with high order derivative $Q\Delta Q - \Delta Q Q$, we need to handle all these couplings carefully in order to construct the approximation successfully by using the Faedo-Galerkin method. First, we show that there is a unique solution $(\rho[u_n], c[u_n], Q[u_n])$ to the initial-boundary problem (4.2.1)–(4.2.2) and (4.2.4) for any u_n in the finite-dimensional space $C(0, T; X_n)$. Then, substituting $(\rho[u_n], c[u_n], Q[u_n])$ into the variational problem of the momentum equation, we can obtain a local solution (ρ_n, c_n, u_n, Q_n) of the approximation system (4.2.1)–(4.2.4) on the time interval $[0, T_n]$ by using the contraction map theorem. Thanks to the cancellation rule we mentioned previously and the fact that the concentration equation obeys the maximum principle, we can overcome the difficulties caused by the complicated coupling terms of the system both in the proof of extending the local solution to a global one by the uniform energy estimates of the system with respect to n and in the proof of the existence of the first level approximation solution as $n \to \infty$.

For fixed $\delta > 0$ and $\varepsilon > 0$, we solve the following approximation problem:

$$\partial_t c + (u \cdot \nabla) c = D_0 \Delta c, \tag{4.2.1}$$

$$\partial_t \rho + \nabla \cdot (\rho u) = \varepsilon \Delta \rho, \tag{4.2.2}$$

$$\begin{aligned} &\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla\big(\rho^\gamma + \delta \nabla \rho^\beta\big) + \varepsilon(\nabla\rho \cdot \nabla) u \\ &\quad = \mu \Delta u + (\nu + \mu) \nabla \operatorname{div} u + \nabla \cdot (\mathrm{F}(Q) \mathrm{I}_3 - \nabla Q \odot \nabla Q) \\ &\qquad + \nabla \cdot (Q \Delta Q - \Delta Q) + \sigma_* \nabla \cdot (c^2 Q), \end{aligned} \tag{4.2.3}$$

$$\partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c], \tag{4.2.4}$$

complemented with the modified initial conditions:

$$c|_{t=0} = c_0 \in H^1(\mathcal{O}), \qquad 0 < \underline{c} \le c_0(x) \le \bar{c}, \tag{4.2.5}$$

$$\rho|_{t=0} = \rho_0 \in C^3(\bar{\mathcal{O}}), \qquad 0 < \underline{\varrho} \le \rho_0(x) \le \bar{\varrho}, \tag{4.2.6}$$

$$(\rho u)|_{t=0} = m_0(x) \in C^2(\bar{\mathcal{O}}), \tag{4.2.7}$$

$$Q|_{t=0} = Q_0(x) \in H^1(\mathcal{O}), \qquad Q_0 \in S_0^3 \ \ a.e. \text{ in } \mathcal{O}, \tag{4.2.8}$$

and the boundary conditions on $\partial\mathcal{O}$ with unit outward normal $\vec{n}$:

$$\nabla c \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \qquad \nabla \rho \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \tag{4.2.9}$$

$$u|_{\partial\mathcal{O}} = 0, \qquad \frac{\partial Q}{\partial \vec{n}}|_{\partial\mathcal{O}} = 0, \tag{4.2.10}$$

where $\underline{c}$, $\bar{c}$, $\underline{\varrho}$, and $\bar{\varrho}$ are positive constants.

Remark 4.2.1. We now remark on the extra terms in the approximation system. The vanishing viscosity $\varepsilon\Delta\rho$ converts equation (4.0.3) from the hyperbolic to parabolic type and provides the higher regularity of ρ. The term, $\delta\nabla\rho^\beta$, is added to obtain the higher integrability of ρ for some constant $\beta > 0$. The extra term $\varepsilon\nabla\rho \cdot \nabla u$ is needed to cancel some *bad* terms that do not vanish in the energy estimates.

4.2.1 The Neumann problem for the density and Q-tensor

We first state the following existence results, which can be found in [17].

Lemma 4.2.1. *Assume that $u \in C([0,T]; C^2(\bar{\mathcal{O}}, \mathbb{R}^3))$ with $u|_{\partial\mathcal{O}} = 0$. Then there exists the following mapping $S = S[u]$:*

$$S : C([0,T]; C^2(\bar{\mathcal{O}}, \mathbb{R}^3)) \to C([0,T]; C^3(\bar{\mathcal{O}}))$$

such that

(i) $\rho = S[u]$ *is the unique classical solution of* (4.2.2), (4.2.6)–(4.2.7), *and* (4.2.9),

(ii) For all $t \geq 0$,

$$\underline{\varrho} \exp\Big(- \int_0^t \|\operatorname{div} u(s)\|_{L^\infty(\mathcal{O})} \mathrm{d}s \Big) \leq S[u](t,x) \leq \bar{\varrho} \exp\Big(\int_0^t \|\operatorname{div} u(s)\|_{L^\infty(\mathcal{O})} \mathrm{d}s \Big),$$

(iii) For any u_1 and u_2 in the set:

$$\mathcal{N}_N = \{ v \,:\, v \in C([0,T]; C_0^2(\mathcal{O}, \mathbb{R}^3)),\ \|v\|_{C([0,T];C_0^2(\bar{\mathcal{O}}, \mathbb{R}^3))} \leq N \} \tag{4.2.11}$$

with some suitable constant $N > 0$,

$$\|S[u_1] - S[u_2]\|_{C([0,T];H^1(\mathcal{O}))} \leq TC(N,T) \|u_1 - u_2\|_{C([0,T];H_0^1(\mathcal{O}))}. \tag{4.2.12}$$

Lemma 4.2.2. *For each $u \in C([0,T]; C_0^2(\bar{\mathcal{O}}, \mathbb{R}^3))$ with $u|_{\partial\mathcal{O}} = 0$, there exists a unique solution of the following initial-boundary value problem:*

$$\begin{cases} \partial_t c + (u \cdot \nabla) c = D_0 \Delta c, \\ \partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c], \\ Q|_{t=0} = Q_0(x) \in H^1(\mathcal{O}), \qquad \nabla Q \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \\ c|_{t=0} = c_0(x) \in H^1(\mathcal{O}), \qquad \nabla c \cdot \vec{n}|_{\partial\mathcal{O}} = 0 \end{cases} \tag{4.2.13}$$

with $0 < \underline{c} \leq c_0 \leq \bar{c} < \infty$ such that $Q \in L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O}))$, $c \in L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O}))$, and $0 < \underline{c} \leq c \leq \bar{c} < \infty$. Moreover, the above mapping $u \mapsto (c[u], Q[u])$ is continuous from $\mathcal{N}_N$ to $\big(L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O}))\big) \times \big(L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O}))\big)$, and $Q[u] \in S_0^3$ a.e. in $\mathcal{O}_T$.

Proof. We divide the proof into five steps.

1. The existence of a solution (c, Q) can be achieved by the standard parabolic theory [34]. The boundedness of c is guaranteed by the fact that the maximum principle is valid for the first equation in system (4.2.13) and the initial condition of c. Next, we show that (c, Q) belongs to $\big(L^\infty(0,T;H^1(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))\big)\times\big(L^\infty(0,T;H^1(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O}))\big)$.

2. Assume that $u \in \mathcal{N}_N$. First, let us take the sum of the first equation in (4.2.13) multiplied by $c-\Delta c$ and the second equation in (4.2.13) multiplied by $-\big(\Delta Q - Q - c_* Q\mathrm{tr}(Q^2)\big)$, take the trace, and integrate by parts over $\mathcal{O}$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}\big(|(c,Q,\nabla c,\nabla Q)|^2+\frac{c_*}{2}|Q|^4\big)\mathrm{d}x\\
&\quad + D_0\|(\nabla c,\Delta c)\|_{L^2}^2+\Gamma\|(\nabla Q,\Delta Q)\|_{L^2}^2+c_*\Gamma\|Q\|_{L^4}^4+c_*^2\Gamma\|Q\|_{L^6}^6\\
&=(u\cdot\nabla c,\Delta c)-(u\cdot\nabla c,c)+(u\cdot\nabla Q,\Delta Q)-(u\cdot\nabla Q,Q+c_*Q|Q|^2)-(\Omega Q-Q\Omega,\Delta Q)\\
&\quad+\big(\Omega Q-Q\Omega,Q+c_*Q|Q|^2\big)+\frac{1}{2}\Gamma((c-c_*)Q,\Delta Q-Q-c_*Q\,\mathrm{tr}(Q^2))\\
&\quad-b\Gamma(Q^2,\Delta Q-Q-c_*Q\,\mathrm{tr}(Q^2))+2c_*\Gamma(Q|Q|^2,\Delta Q)\\
&\le\frac{D_0}{2}\|\Delta c\|_{L^2}^2+\frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2+\frac{c_*^2\Gamma}{2}\|Q\|_{L^6}^6+C\big(\|(c,Q,\nabla c,\nabla Q)\|_{L^2}^2+\|Q\|_{L^4}^4\big).
\end{aligned}
$$

By the Gronwall inequality, we obtain

$$
\|(c,Q)\|_{L^\infty(0,T;H^1(\mathcal{O}))}+\|(c,Q)\|_{L^2(0,T;H^2(\mathcal{O}))}\le C,
$$

where $C>0$ depends on $N, b, c_*, \Gamma, T, \|Q_0\|_{H^1(\mathcal{O})}$, and $\|c_0\|_{H^1(\mathcal{O})}$.

3. For the uniqueness, we denote $(\tilde{c},\tilde{Q})=(c_1-c_2,Q_1-Q_2)$ for any two solutions (c_1,Q_1) and (c_2,Q_2) of (4.2.13). Then $(\tilde{c},\tilde{Q})$ satisfies

$$
\begin{cases}
\partial_t\tilde{c}+u\cdot\nabla\tilde{c}=D_0\Delta\tilde{c},\\
\partial_t\tilde{Q}+u\cdot\nabla\tilde{Q}-\Omega\tilde{Q}+\tilde{Q}\Omega-\Gamma\Delta\tilde{Q}+\frac{1}{2}\Gamma\tilde{c}Q_1+\frac{1}{2}\Gamma\tilde{Q}(c_2-c_*)\\
\quad=\Gamma\Big(b\big(\tilde{Q}(Q_1+Q_2)-\frac{1}{3}\mathrm{tr}(\tilde{Q}(Q_1+Q_2))\mathrm{I}_3\big)-c_*\tilde{Q}\,\mathrm{tr}\,Q_1^2-c_*Q_2\,\mathrm{tr}(\tilde{Q}(Q_1+Q_2))\Big)
\end{cases}
\tag{4.2.14}
$$

with $(\tilde{c},\tilde{Q})|_{t=0}=(0,0)$, $\nabla\tilde{Q}\cdot\vec{n}|_{\partial\mathcal{O}}=0$, and $\nabla\tilde{c}\cdot\vec{n}|_{\partial\mathcal{O}}=0$. We sum up the first equation in (4.2.14) multiplied by $\tilde{c}$ and the second equation in (4.2.14) multiplied by $\tilde{Q}$, take the trace, and integrate by parts over $\mathcal{O}$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\tilde{c},\tilde{Q})\|_{L^2}^2+D_0\|\nabla\tilde{c}\|_{L^2}^2+\Gamma\|\nabla\tilde{Q}\|_{L^2}^2\\
&=-(u\cdot\nabla\tilde{c},\tilde{c})-(u\cdot\nabla\tilde{Q},\tilde{Q})+(\Omega\tilde{Q}-\tilde{Q}\Omega,\tilde{Q})-\frac{\Gamma}{2}(\tilde{c}Q_1,\tilde{Q})-\frac{\Gamma}{2}((c_2-c_*)\tilde{Q},\tilde{Q})\\
&\quad+\Gamma(b(\tilde{Q}(Q_1+Q_2)-\frac{1}{3}\mathrm{tr}(\tilde{Q}(Q_1+Q_2))\mathrm{I}_3)-c_*\tilde{Q}\,\mathrm{tr}\,Q_1^2-c_*Q_2\,\mathrm{tr}(\tilde{Q}(Q_1+Q_2)),\,\tilde{Q})\\
&\le\frac{D_0}{2}\|\nabla\tilde{c}\|_{L^2}^2+C\|\tilde{Q}\|_{L^2}\|\tilde{Q}\|_{L^6}(\|Q_1+Q_2\|_{L^3}+\|(Q_1,Q_2)\|_{L^6}^2)\\
&\quad+C\big(\|(\tilde{c},\tilde{Q})\|_{L^2}^2+\|\nabla\tilde{Q}\|_{L^2}\|\tilde{Q}\|_{L^2}\big)\\
&\le\frac{D_0}{2}\|\nabla\tilde{c}\|_{L^2}^2+\frac{\Gamma}{2}\|\nabla\tilde{Q}\|_{L^2}^2+C\|(\tilde{c},\tilde{Q})\|_{L^2}^2,
\end{aligned}
$$

where C depends on b,c_*,Γ,N, and $\|Q_0\|_{H^1}$, and we have also used Sobolev's embedding inequality, Poincaré inequality, and Young's inequality in the last step. Therefore, applying the Gronwall inequality, we obtain the desired uniqueness result.

4. Now we show that the map: $u\mapsto(c[u],Q[u])$ is continuous. Let $\{u_n\}$ be a bounded sequence in $\mathcal{N}_N$ with

$$\lim_{n\to\infty}\|u_n-u\|_{C(0,T;C_0^2(\bar{\mathcal{O}}))}=0$$

for some $u\in C(0,T;C_0^2(\bar{\mathcal{O}}))$. Denote $(c_n,Q_n)=(c[u_n],Q[u_n])$, $(c,Q)=(c[u],Q[u])$, and $(\tilde{c}_n,\tilde{Q}_n)=(c_n-c,Q_n-Q)$. Taking the difference of the equations satisfied by (c_n,Q_n) and (c,Q), multiplying the resulting equations by $(-\Delta\tilde{c}_n,-\Delta\tilde{Q}_n)$, taking the trace, and integrating by parts over $\mathcal{O}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\tilde{c}_n,\nabla\tilde{Q}_n)\|_{L^2}^2+D_0\|\Delta\tilde{c}_n\|_{L^2}^2+\Gamma\|\Delta\tilde{Q}_n\|_{L^2}^2\\
&=(u_n\cdot\nabla\tilde{c}_n+(u_n-u)\cdot\nabla c,\Delta\tilde{c}_n)+(u_n\cdot\nabla\tilde{Q}_n+(u_n-u)\cdot\nabla Q,\Delta\tilde{Q}_n)\\
&\quad-(\Omega_n\tilde{Q}_n+(\Omega_n-\Omega)Q,\Delta\tilde{Q}_n)+(\tilde{Q}_n\Omega_n+Q(\Omega_n-\Omega),\Delta\tilde{Q}_n)\\
&\quad+\frac{\Gamma}{2}(\tilde{c}_nQ_n+(c-c_*)\tilde{Q}_n,\Delta\tilde{Q}_n)-\frac{1}{3}b\Gamma(3\tilde{Q}_n(Q_n+Q)+\mathrm{tr}(\tilde{Q}_n(Q_n+Q))\mathrm{I}_3,\Delta\tilde{Q}_n)\\
&\quad+c_*\Gamma(|Q_n|^2\tilde{Q}_n+Q(|Q_n|^2-|Q|^2),\Delta\tilde{Q}_n)\\
&=\sum_{i=1}^{7}\mathcal{I}_i.
\end{aligned}
$$

In the following, we estimate all the terms on the right-hand side of the above equation:

$$
\begin{aligned}
|\mathcal{I}_1| &= |(u_n \cdot \nabla \tilde{c}_n + (u_n - u) \cdot \nabla c, \Delta \tilde{c}_n)| \\
&\le C\big(\|u_n\|_{L^\infty} \|\nabla \tilde{c}_n\|_{L^2} + \|u_n - u\|_{L^\infty} \|\nabla c\|_{L^2}\big) \|\Delta \tilde{c}_n\|_{L^2} \\
&\le \frac{D_0}{2} \|\Delta \tilde{c}_n\|_{L^2}^2 + C\|\nabla \tilde{c}_n\|_{L^2}^2 + C\|u_n - u\|_{L^\infty}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_2| &= |(u_n \cdot \nabla \tilde{Q}_n + (u_n - u) \cdot \nabla Q, \Delta \tilde{Q}_n)| \\
&\le C\big(\|u_n\|_{L^\infty} \|\nabla \tilde{Q}_n\|_{L^2} + \|u_n - u\|_{L^\infty} \|\nabla Q\|_{L^2}\big) \|\Delta \tilde{Q}_n\|_{L^2} \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\nabla \tilde{Q}_n\|_{L^2}^2 + C\|u_n - u\|_{L^\infty}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_3| &= |(\Omega_n \tilde{Q}_n + (\Omega_n - \Omega) Q, \Delta \tilde{Q}_n)| \\
&\le C\|\Omega_n\|_{L^\infty} \|\tilde{Q}_n\|_{L^2} \|\Delta \tilde{Q}_n\|_{L^2} + \|\nabla u_n - \nabla u\|_{L^\infty} \|Q\|_{L^2} \|\Delta \tilde{Q}_n\|_{L^2} \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2 \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\nabla \tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_4| &= |(\tilde{Q}_n \Omega_n + Q(\Omega_n - \Omega), \Delta \tilde{Q}_n)| \\
&\le C\|\tilde{Q}_n\|_{L^2} \|\Omega_n\|_{L^\infty} \|\Delta \tilde{Q}_n\|_{L^2} + \|Q\|_{L^2} \|\Omega_n - \Omega\|_{L^\infty} \|\Delta \tilde{Q}_n\|_{L^2} \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2 \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\nabla \tilde{Q}_n\|_{L^2}^2 + C\|\nabla u_n - \nabla u\|_{L^\infty}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_5| &= \frac{\Gamma}{2} |(\tilde{c}_n Q_n + (c - c_*) \tilde{Q}_n, \Delta \tilde{Q}_n)| \le C\big(\|\tilde{c}_n\|_{L^4} \|Q_n\|_{L^4} + \|c - c_*\|_{L^\infty} \|\tilde{Q}_n\|_{L^2}\big) \|\Delta \tilde{Q}_n\|_{L^2} \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\nabla \tilde{c}_n\|_{L^2}^2 + C\|\tilde{Q}_n\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_6| &= \frac{1}{3} b\Gamma |(3\tilde{Q}_n(Q_n + Q) + \mathrm{tr}(\tilde{Q}_n(Q_n + Q))\mathrm{I}_3, \Delta \tilde{Q}_n)| \le C\|\tilde{Q}_n\|_{L^3} \|Q_n + Q\|_{L^6} \|\Delta \tilde{Q}_n\|_{L^2} \\
&\le \frac{\Gamma}{12} \|\Delta \tilde{Q}_n\|_{L^2}^2 + C\|\nabla \tilde{Q}_n\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{I}_7| &= c_*\Gamma|(|Q_n|^2\tilde{Q}_n + Q(|Q_n|^2 - |Q|^2), \Delta\tilde{Q}_n)| \\
&\le C\Gamma\big(\|Q_n\|_{L^6}^2 + \|Q_n\|_{L^6}\|Q\|_{L^6} + \|Q\|_{L^6}^2\big)\|\tilde{Q}_n\|_{L^6}\|\Delta\tilde{Q}_n\|_{L^2} \\
&\le \frac{\Gamma}{12}\|\Delta\tilde{Q}_n\|_{L^2}^2 + C\|\nabla\tilde{Q}_n\|_{L^2}^2.
\end{aligned}
$$

Combining all the above estimates, we conclude

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\tilde{c}_n, \nabla\tilde{Q}_n)\|_{L^2}^2 + \frac{D_0}{2}\|\Delta\tilde{c}_n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta\tilde{Q}_n\|_{L^2}^2 \\
&\le C\|u_n - u\|_{C_0^2(\mathcal{O})}^2 + C\|(\nabla\tilde{c}_n, \nabla\tilde{Q}_n)\|_{L^2}^2.
\end{aligned}
$$

Then, by the Gronwall inequality, we have

$$
\begin{aligned}
&\|(\nabla\tilde{c}_n(t,\cdot), \nabla\tilde{Q}_n(t,\cdot))\|_{L^2}^2 + \Gamma\int_0^t \|(\Delta\tilde{c}_n(s,\cdot), \Delta\tilde{Q}_n(s,\cdot))\|_{L^2}^2\,\mathrm{d}s \\
&\le CTe^{CT}\|u_n - u\|_{L^\infty(0,T;C_0^2(\mathcal{O}))}^2.
\end{aligned} \tag{4.2.15}
$$

As $n \to \infty$, we conclude

$$
\lim_{n\to\infty}\big(\|(\tilde{c}_n, \tilde{Q}_n)\|_{L^\infty(0,T;H^1)} + \|(\tilde{c}_n, \tilde{Q}_n)\|_{L^2(0,T;H^2)}\big) = 0.
$$

5. Finally, we show that $Q \in S_0^3$, *i.e.*, $Q^\top = Q$ and $\operatorname{tr} Q = 0$ *a.e.* in $\mathcal{O}_T$. It is clear that, if Q is a solution of problem (4.2.13), so is $Q^\top$. Then, by the uniqueness of the solution, we know that $Q^\top = Q$.

Then the only thing left is to show that $\operatorname{tr} Q = 0$. We take the trace on both sides of the second equation in (4.2.13) to obtain

$$
\partial_t(\operatorname{tr} Q) + u\cdot\nabla\operatorname{tr} Q = \Gamma\Big(\Delta\operatorname{tr} Q - \frac{1}{2}(c - c_*)\operatorname{tr} Q - c_*\operatorname{tr} Q\operatorname{tr} Q^2\Big), \tag{4.2.16}
$$

with $\operatorname{tr} Q|_{t=0} = \operatorname{tr} Q_0 = 0$ and $\nabla\operatorname{tr} Q\cdot\vec{n}|_{\partial\mathcal{O}} = 0$, where we have used the fact that $Q^\top = Q$, $\Omega^\top = -\Omega$, and $\operatorname{tr}(Q\Omega) = \operatorname{tr}(\Omega Q)$. Then we multiply equation (4.2.16) by $\operatorname{tr} Q$ and integrate by parts over $\mathcal{O}$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\operatorname{tr} Q\|_{L^2}^2 + \Gamma\|\nabla\operatorname{tr} Q\|_{L^2}^2 \\
&= -\frac{1}{2}\Gamma((c - c_*)\operatorname{tr} Q, \operatorname{tr} Q) - c_*\Gamma\int_{\mathcal{O}}|\operatorname{tr} Q|^2\operatorname{tr} Q^2\mathrm{d}x - \int_{\mathcal{O}} u\cdot\nabla(\operatorname{tr} Q)\operatorname{tr} Q\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\mathrm{tr}\, Q\|_{L^2}^2 + C\|Q\|_{L^6}^2\|\mathrm{tr}\, Q\|_{L^6}\|\mathrm{tr}\, Q\|_{L^2} + C\|\nabla \mathrm{tr}\, Q\|_{L^2}\|\mathrm{tr}\, Q\|_{L^2}\\
&\le C\|\mathrm{tr}\, Q\|_{L^2}^2 + C\|\nabla\mathrm{tr}\, Q\|_{L^2}\|\mathrm{tr}\, Q\|_{L^2} + C\|\nabla \mathrm{tr}\, Q\|_{L^2}\|\mathrm{tr}\, Q\|_{L^2}\\
&\le \frac{\Gamma}{2}\|\nabla\mathrm{tr}\, Q\|_{L^2}^2 + C\|\mathrm{tr}\, Q\|_{L^2}^2.
\end{aligned}
$$

Applying the Gronwall inequality again, we complete the proof. □

4.2.2 The Faedo-Galerkin approximation scheme

In this section, we proceed to solve (4.2.3) by the Faedo-Galerkin approximation scheme. Let $\{\psi_n\}$ be a family of smooth eigenfunctions of the Laplacian operator:

$$
\begin{aligned}
&-\Delta\psi_n = \lambda_n\psi_n \qquad \text{on } \mathcal{O},\\
&\psi_n|_{\partial\mathcal{O}} = 0,
\end{aligned}
$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots$ are eigenvalues. We know that the eigenfunctions $\{\psi_n\}_{n=1}^{\infty}$ form an orthogonal basis of $H_0^1(\mathcal{O})$.

Now, consider a sequence of finite-dimensional spaces:

$$
X_n = \mathrm{span}\{\psi_1, \psi_2, \cdots, \psi_n\}, \qquad n = 1, 2, \cdots, \tag{4.2.17}
$$

and look for solutions $u_n \in C(0, T; X_n)$ to the following variational approximation problem:

$$
\begin{aligned}
&\int_{\mathcal{O}} \rho(t,x)u_n(t,x)\cdot\psi(x)\,\mathrm{d}x - \int_{\mathcal{O}} m_0(x)\cdot\psi(x)\,\mathrm{d}x\\
&= \int_0^t\int_{\mathcal{O}} \big(\mu\Delta u_n + (\mu+\nu)\nabla\mathrm{div}\, u_n - \mathrm{div}\,(\rho u_n\otimes u_n) - \nabla(\rho^\gamma + \delta\rho^\beta) - \varepsilon\nabla\rho\cdot\nabla u_n\big)\cdot\psi\,\mathrm{d}x\mathrm{d}s\\
&\quad + \int_0^t\int_{\mathcal{O}} \nabla\cdot\big(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q + Q\Delta Q - \Delta Q Q + \sigma_* c^2 Q\big)\cdot\psi\,\mathrm{d}x\mathrm{d}s
\end{aligned}
\tag{4.2.18}
$$

for any $t \in [0, T]$, and $\psi \in X_n$.

Next we introduce a family of operators, as in [17]:

$$
\mathcal{M}[\rho] : X_n \mapsto X_n^*, \qquad (\mathcal{M}[\rho]v, w) = \int_{\mathcal{O}} \rho v\cdot w\,\mathrm{d}x \quad \text{for any } v, w \in X_n.
$$

The map:

$$\rho \mapsto \mathcal{M}^{-1}[\rho]$$

from $N_\eta = \{\rho \in L^1(\mathcal{O}) \ : \ \inf_{x\in\Omega} \rho \geq \eta > 0\}$ into $\mathcal{L}(X_n^*, X_n)$ has the following property:

$$\|\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2]\|_{\mathcal{L}(X_n^*, X_n)} \leq C(n,\eta)\|\rho_1 - \rho_2\|_{L^1(\mathcal{O})}. \tag{4.2.19}$$

Using Theorems 4.2.1–4.2.2 with $\rho_n = S[u_n]$, $c_n = c[u_n]$, and $Q_n = Q[u_n]$, we can rewrite the variational problem (4.2.18) as

$$u_n(t) = \mathcal{M}^{-1}[\rho_n]\left(m^* + \int_0^t \mathcal{N}[c_n(s), \rho_n(s), u_n(s), Q_n(s)]\mathrm{d}s\right)$$

with

$$\begin{aligned}
&(m^*, \psi) = \int_{\mathcal{O}} m_0 \cdot \psi \, \mathrm{d}x, \\
&(\mathcal{N}[c_n, \rho_n, u_n, Q_n], \psi) \\
&= \int_{\mathcal{O}} \big(\mu\Delta u_n + (\mu+\nu)\nabla \mathrm{div}\, u_n - \mathrm{div}\,(\rho_n u_n \otimes u_n) - \nabla(\rho_n^\gamma + \delta\rho_n^\beta)\big)\cdot \psi \, \mathrm{d}x \\
&\quad - \varepsilon \int_{\mathcal{O}} (\nabla\rho_n \cdot \nabla) u_n \cdot \psi \, \mathrm{d}x + \int_{\mathcal{O}} \nabla \cdot \big(-\nabla Q_n \odot \nabla Q_n + \mathrm{F}(Q_n)\mathrm{I}_3\big)\cdot \psi \, \mathrm{d}x \\
&\quad + \int_{\mathcal{O}} \nabla \cdot \big(Q_n\Delta Q_n - \Delta Q_n Q_n + \sigma_* c_n^2 Q_n\big)\cdot\psi \, \mathrm{d}x
\end{aligned}$$

for any $t \in [0,T]$ and $\psi \in X_n$. Therefore, combining (4.2.12), (4.2.15), and (4.2.19), we achieve a local solution (c_n, ρ_n, u_n, Q_n) of problem (4.2.1)–(4.2.2), (4.2.4), and (4.2.18), with initial-boundary data (4.2.6)–(4.2.10) on a short time interval $[0, T_n], T_n \leq T$, by using the standard fixed point theorem on $C(0,T;X_n)$. In order to extend the existence time T_n to T for any $n = 1, 2, \cdots$, we need to prove that u_n stays bounded in X_n for the whole interval $[0, T_n]$. Hence, in the following, we establish an energy inequality as in Proposition 4.1.1.

Differentiate (4.2.18) with respect to t and take $\psi = u_n$ as a test function to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \big(\frac{1}{2}\rho_n|u_n|^2 + \frac{\rho_n^\gamma}{\gamma-1} + \frac{\delta\rho_n^\beta}{\beta-1}\big)\,\mathrm{d}x + \mu\|\nabla u_n\|_{L^2}^2 + (\nu+\mu)\|\mathrm{div}\, u_n\|_{L^2}^2 \\
&\qquad + \varepsilon \int_{\mathcal{O}} (\gamma\rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2 \, \mathrm{d}x \\
&= \int_{\mathcal{O}} \nabla\cdot\big(-\nabla Q_n \odot \nabla Q_n + \mathrm{F}(Q_n)\mathrm{I}_3 + Q_n\Delta Q_n - \Delta Q_n Q_n + \sigma_* c_n^2 Q_n\big)\cdot u_n \, \mathrm{d}x,
\end{aligned} \tag{4.2.20}$$

where we have used the following equalities as in Proposition 4.1.1:

$$\begin{aligned}
&\int_{\mathcal{O}} \partial_t(\rho_n u_n)\cdot u_n \,\mathrm{d}x + \int_{\mathcal{O}} \mathrm{div}\,(\rho_n u_n \otimes u_n)\cdot u_n \,\mathrm{d}x \\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \rho_n |u_n|^2 \,\mathrm{d}x - \varepsilon \int_{\mathcal{O}} (\nabla\rho_n\cdot\nabla)u_n \cdot u_n \,\mathrm{d}x, \\
&\int_{\mathcal{O}} \nabla\rho_n^\gamma \cdot u_n \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \frac{\rho_n^\gamma}{\gamma-1}\,\mathrm{d}x + \varepsilon\gamma\int_{\mathcal{O}} \rho_n^{\gamma-2}|\nabla\rho_n|^2 \,\mathrm{d}x, \\
&\int_{\mathcal{O}} \delta\nabla\rho_n^\beta \cdot u_n \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \frac{\delta\rho_n^\beta}{\beta-1}\,\mathrm{d}x + \varepsilon\delta\beta\int_{\mathcal{O}} \rho_n^{\beta-2}|\nabla\rho_n|^2 \,\mathrm{d}x.
\end{aligned}$$

Then we take the inner product of (4.2.1) with c_n, (4.2.4) with $-\big(\Delta Q_n - Q_n - c_* Q_n \mathrm{tr}(Q_n^2)\big)$, add the resulting equations to (4.2.20) and integrate by parts over $\mathcal{O}$ to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_\delta^n(t) + D_0\|\nabla c_n\|_{L^2}^2 + \mu\|\nabla u_n\|_{L^2}^2 + (\mu+\nu)\|\mathrm{div}\,u_n\|_{L^2}^2 + \Gamma\|(\nabla Q_n, \Delta Q_n)\|_{L^2}^2 \\
&\quad + c_*\Gamma\|Q_n\|_{L^4}^4 + c_*^2\Gamma\|Q_n\|_{L^6}^6 + \varepsilon\int_{\mathcal{O}} (\gamma\rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2 \,\mathrm{d}x \\
&= \int_{\mathcal{O}} \nabla\cdot\big(-\nabla Q_n \odot \nabla Q_n + \mathrm{F}(Q_n)\mathrm{I}_3 + Q_n\Delta Q_n - \Delta Q_n Q_n + \sigma_* c_n^2 Q_n\big)\cdot u_n \,\mathrm{d}x \\
&\quad + (u_n\cdot\nabla Q_n, \Delta Q_n) - (u_n\cdot\nabla Q_n, Q_n + c_* Q_n|Q_n|^2) - (\Omega_n Q_n - Q_n\Omega_n, \Delta Q_n) \\
&\quad + \big(\Omega_n Q_n - Q_n\Omega_n, Q_n + c_* Q_n|Q_n|^2\big) + \frac{1}{2}\Gamma((c_n - c_*)Q_n, \Delta Q_n - Q_n - c_* Q_n\mathrm{tr}(Q_n^2)) \\
&\quad - b\Gamma(Q_n^2, \Delta Q_n - Q_n - c_* Q_n\mathrm{tr}(Q_n^2)) + 2c_*\Gamma(Q_n|Q_n|^2, \Delta Q_n) - (u_n\cdot\nabla c_n, c_n) \\
&\le \frac{D_0}{2}\|\nabla c_n\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q_n\|_{L^2}^2 + \frac{c_*\Gamma}{2}\|Q_n\|_{L^6}^6 + C\big(\|(u_n, Q_n)\|_{L^2}^2 + \|Q_n\|_{L^4}^4\big)
\end{aligned}$$

where C is independent of n and ε, and

$$E_\delta^n(t) = \int_{\mathcal{O}} \Big(\frac{1}{2}|c_n|^2 + \frac{1}{2}\rho_n|u_n|^2 + \frac{\rho_n^\gamma}{\gamma-1} + \frac{\delta\rho_n^\beta}{\beta-1} + \mathrm{F}(Q_n)\Big)\,\mathrm{d}x.$$

This implies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_\delta^n(t) + \frac{D_0}{2}\|\nabla c_n\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u_n\|_{L^2}^2 + (\mu+\nu)\|\mathrm{div}\,u_n\|_{L^2}^2 + \Gamma\|\nabla Q_n\|_{L^2}^2 \\
&\quad + \frac{\Gamma}{2}\|\Delta Q_n\|_{L^2}^2 + \frac{c_*^2\Gamma}{2}\|Q_n\|_{L^6}^6 + \varepsilon\int_{\mathcal{O}} \big(\gamma\rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2}\big)|\nabla\rho_n|^2 \mathrm{d}x \\
&\le C\big(\|(u_n, Q_n)\|_{L^2}^2 + \|Q_n\|_{L^4}^4\big),
\end{aligned} \tag{4.2.21}$$

The above inequality yields

$$\mu\int_0^{T_n} \|\nabla u_n\|_{L^2}^2 \,\mathrm{d}t \le C, \tag{4.2.22}$$

$$\sup_{t\in[0,T_n]}\int_{\mathcal{O}}\rho_n|u_n|^2\,\mathrm{d}x\le C, \tag{4.2.23}$$

where C is a constant independent of n. Since X_n is a finite-dimension space, we can deduce from Lemma 4.2.1 that there exists a constant $C=C(n,c_0,\rho_0,m_0,Q_0,b,\mathcal{O})$ such that

$$0<C\le\rho_n(t,x)\le\frac{1}{C}\qquad\text{for all } t\in(0,T_n)\text{ and } x\in\mathcal{O}, \tag{4.2.24}$$

which, combined with (4.2.23) and the fact that the L^∞ and L^2 norms are equivalent on X_n, yields

$$\sup_{t\in[0,T_n]}\|(u_n,\nabla u_n)(t,\cdot)\|_{L^\infty(\mathcal{O})}\le C(n,E_\delta^n(0),N,\mathcal{O}).$$

Then we can extend the existence time-interval $[0,T_n]$ of (c_n,ρ_n,u_n,Q_n) to $[0,T]$.

We summarize the results in this subsection in the following lemma.

Lemma 4.2.3. *Let $\beta\ge 4$. Then there exists a solution (c_n,ρ_n,u_n,Q_n) of problem (4.2.1)–(4.2.2), (4.2.4), and (4.2.18) with the corresponding initial-boundary data (4.2.5)–(4.2.10). Moreover, the following estimates hold*:

$$0<\underline{c}\le c_n(t,x)\le\bar{c},\quad \|c_n\|_{L^\infty(0,T;L^2(\mathcal{O}))\cap L^2(0,T;H^1(\mathcal{O}))}\le C, \tag{4.2.25}$$

$$\sup_{t\in[0,T]}\|\rho_n(t,\cdot)\|_{L^\gamma(\mathcal{O})}^\gamma\le C, \tag{4.2.26}$$

$$\delta\sup_{t\in[0,T]}\|\rho_n(t,\cdot)\|_{L^\beta(\mathcal{O})}^\beta\le C, \tag{4.2.27}$$

$$\varepsilon\|\nabla\rho_n^{\frac{\gamma}{2}}\|_{L^2(0,T;L^2(\mathcal{O}))}^2+\varepsilon\delta\|\nabla\rho_n^{\frac{\beta}{2}}\|_{L^2(0,T;L^2(\mathcal{O}))}^2\le C, \tag{4.2.28}$$

$$\sup_{t\in[0,T]}\|\sqrt{\rho_n}(t,\cdot)u_n(t,\cdot)\|_{L^2(\mathcal{O})}^2\le C, \tag{4.2.29}$$

$$\|u_n\|_{L^2(0,T;H_0^1(\mathcal{O}))}\le C, \tag{4.2.30}$$

$$\|\rho_n\|_{L^{\beta+1}(\mathcal{O}_T)}\le C, \tag{4.2.31}$$

$$\varepsilon\|\nabla\rho_n\|_{L^2(0,T;L^2(\mathcal{O}))}^2\le C, \tag{4.2.32}$$

$$\|Q_n\|_{L^\infty(0,T;H^1(\mathcal{O})\cap L^4(\mathcal{O}))\cap L^2(0,T;H^2(\mathcal{O})\cap L^6(\mathcal{O}))}\le C, \tag{4.2.33}$$

$$\|Q_n\|_{L^{10}(\mathcal{O}_T)}\le C, \tag{4.2.34}$$

$$\|\nabla Q_n\|_{L^{\frac{10}{3}}(\mathcal{O}_T)}\le C, \tag{4.2.35}$$

where C is a constant independent of n and ε.

Proof. Estimates (4.2.25)–(4.2.30) and (4.2.33) follow from the energy estimate (4.2.21). Moreover, we can use similar methods to obtain (4.2.34)–(4.2.35) as in Corollary 4.1.1. We only need to show (4.2.31)–(4.2.32).

From (4.2.28), $\rho_n^{\frac{\beta}{2}} \in L_t^2 H_x^1$. This, together with the embedding: $H^1(\mathcal{O}) \subset L^6(\mathcal{O})$, yields

$$\|\rho_n^{\beta}\|_{L_t^1 L_x^3} \leq C \tag{4.2.36}$$

with C independent of n. Combining (4.2.27), (4.2.36), and the interpolation (pp. 623, [15]):

$$\|\rho_n^{\beta}\|_{L_x^2} \leq C \|\rho_n^{\beta}\|_{L_x^1}^{\frac{1}{4}} \|\rho_n^{\beta}\|_{L_x^3}^{\frac{3}{4}},$$

we have

$$\|\rho_n^{\beta}\|_{L_t^{\frac{4}{3}} L_x^2}^{\frac{4}{3}} \leq C \int_0^T \|\rho_n^{\beta}\|_{L_x^1}^{\frac{1}{3}} \|\rho_n^{\beta}\|_{L_x^3} \mathrm{d}t \leq C \|\rho_n^{\beta}\|_{L_t^\infty L_x^1}^{\frac{1}{3}} \|\rho_n^{\beta}\|_{L_t^1 L_x^3} \leq C,$$

which implies that $\rho_n \in L^{\frac{4}{3}\beta}(0,T; L^{2\beta}(\mathcal{O}))$. Moreover, this, together with the following interpolation:

$$\|\rho_n\|_{L_x^{\frac{4}{3}\beta}} \leq C \|\rho_n\|_{L_x^{\beta}}^{\frac{1}{2}} \|\rho_n\|_{L_x^{2\beta}}^{\frac{1}{2}},$$

gives

$$\|\rho_n\|_{L_{t,x}^{\frac{4}{3}\beta}}^{\frac{4\beta}{3}} \leq C \int_0^T \|\rho_n\|_{L_x^{\beta}}^{\frac{2}{3}\beta} \|\rho_n\|_{L_x^{2\beta}}^{\frac{2}{3}\beta} \,\mathrm{d}t \leq C(\delta),$$

i.e., $\rho_n \in L^{\frac{4}{3}\beta}(\mathcal{O}_T)$. Then, if $\beta \geq 3$ ($\Leftrightarrow \frac{4}{3}\beta \geq \beta + 1$), we have

$$\|\rho_n\|_{L^{\beta+1}(\mathcal{O}_T)} \leq C(\delta).$$

Regarding (4.2.32), we multiply (4.2.2) by ρ_n and integrate by parts over $\mathcal{O}$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_n\|_{L_x^2}^2 + \varepsilon \|\nabla \rho_n\|_{L_x^2}^2 = -\frac{1}{2}\int_{\mathcal{O}} \operatorname{div} u_n \, \rho_n^2 \,\mathrm{d}x \leq C \|\operatorname{div} u_n\|_{L_x^2} \|\rho_n\|_{L_x^4}^2,$$

which yields

$$\varepsilon \|\nabla \rho_n\|_{L^2(\mathcal{O}_T)}^2 \leq \frac{1}{2}\big(\|\rho_0\|_{L_x^2}^2 + \sqrt{T}\|\rho_n\|_{L_t^\infty L_x^4}^2 \|\operatorname{div} u_n\|_{L_{t,x}^2}\big).$$

Therefore, (4.2.32) follows from (4.2.27) and (4.2.30), provided $\beta \geq 4$. $\square$

4.2.3 The existence of the first level approximate solutions

In this subsection, we obtain a solution (c, ρ, u, Q) of problem (4.2.1)−(4.2.10), by letting $n \to \infty$. We do not distinguish between the sequence convergence and the subsequence convergence for the sake of convenience. Assume that $\beta > 4$ and $\gamma > \frac{3}{2}$. It follows from [17] that, as $n \to \infty$,

$$\rho_n \to \rho \qquad \text{in } L^4(\mathcal{O}_T), \tag{4.2.37}$$

$$\rho_n^\gamma \to \rho^\gamma, \ \ \rho_n^\beta \to \rho^\beta \qquad \text{in } L^1(\mathcal{O}_T) \text{ if } \beta > \gamma, \tag{4.2.38}$$

$$u_n \rightharpoonup u \qquad \text{in } L^2(0, T; H_0^1(\mathcal{O})). \tag{4.2.39}$$

Moreover, we can infer that $\partial_t c_n \in L^1(0, T; H^{-1}(\mathcal{O}))$ from (4.2.25), (4.2.30), and the fact that $\partial_t c_n$ satisfies equation (4.2.1). Applying the Aubin-Lions lemma (Lemma A.0.6), we have

$$c_n \rightharpoonup c \quad \text{in } L^2(0, T; H^1(\mathcal{O})), \qquad c_n \to c \quad \text{in } L^2(0, T; L^2(\mathcal{O})). \tag{4.2.40}$$

Thus, we know that the limit function c is a weak solution to (4.2.1). Similarly, estimates (4.2.25), (4.2.30), and (4.2.33), along with the fact that $\partial_t Q_n$ satisfies (4.2.4), yield that $\partial_t Q_n \in L^2(0, T; L^{\frac{3}{2}}(\mathcal{O}))$. In fact,

$$\begin{aligned}
&\|(u_n \cdot \nabla) Q_n\|_{L_t^2 L_x^{\frac{3}{2}}} \leq C\Big(\int_0^T \|u_n\|_{L_x^6}^2 \|\nabla Q_n\|_{L_x^2}^2 \mathrm{d}t\Big)^{\frac{1}{2}} \leq C \|u_n\|_{L_t^2 (H_0^1)_x} \|\nabla Q_n\|_{L_t^\infty L_x^2} \leq C, \\
&\|Q_n \Omega_n - \Omega_n Q_n\|_{L_t^2 L_x^{\frac{3}{2}}} \leq C\Big(\int_0^T \|Q_n\|_{L_x^6}^2 \|\nabla u_n\|_{L_x^2}^2 \mathrm{d}t\Big)^{\frac{1}{2}} \leq C \|Q_n\|_{L_t^\infty H_x^1} \|\nabla u_n\|_{L^2(\mathcal{O}_T)} \leq C, \\
&\|\Gamma H[Q_n, c_n]\|_{L_t^2 L_x^{\frac{3}{2}}} = \Gamma \|\Delta Q_n - \frac{c_n - c_*}{2} Q_n + b[Q_n^2 - \frac{\mathrm{tr}(Q_n^2)}{3} \mathrm{I}_d] - c_* Q_n |Q_n|^2\|_{L_t^2 L_x^{\frac{3}{2}}} \\
&\leq C\Big(\|(Q_n, \Delta Q_n)\|_{L^2(\mathcal{O}_T)} + \big(\|c_n\|_{L_t^2 H_x^1} + \|Q_n\|_{L_t^2 H_x^2}\big)\big(\|Q_n\|_{L_t^\infty H_x^1} + \|Q_n\|_{L_t^\infty H_x^1}^2\big)\Big) \leq C.
\end{aligned}$$

Then, by the Aubin-Lions lemma, we have

$$Q_n \rightharpoonup Q \ \ \text{in } L^2(0, T; H^2(\mathcal{O})), \qquad Q_n \to Q \ \ \text{in } L^2(0, T; H^1(\mathcal{O})). \tag{4.2.41}$$

This ensures that we can pass to the limit in equation (4.2.4) in $\mathcal{D}'(\mathcal{O}_T)$ as $n \to \infty$, *i.e.*, Q is a weak solution of (4.2.4).

Furthermore, we also see that $\rho_n u_n$ is bounded in $L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}))$ with $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$ (since $\gamma > \frac{3}{2}$), by using (4.2.26) and (4.2.29)–(4.2.30). In fact,

$$\|\rho_n u_n\|_{L_x^{\frac{2\gamma}{\gamma+1}}} \le C\|\rho_n\|_{L_x^\gamma}^{\frac{1}{2}}\|\sqrt{\rho_n}u_n\|_{L_x^2}. \tag{4.2.42}$$

This, together with (4.2.37) and (4.2.39), yields

$$\rho_n u_n \overset{*}{\rightharpoonup} \rho u \quad \text{in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})).$$

Then we can conclude that ρ is a weak solution of (4.2.2) and we can pass to the limit in (4.2.2) as $n \to \infty$.

In the following, we show that the limit function u satisfies equation (4.2.18), by using Corollary A.0.1. Then we need to establish convergence results for the terms: $\rho_n u_n \otimes u_n$ and $\nabla\rho_n \cdot \nabla u_n$, which require more estimates for the density. From Lemma 2.4 in [17], we know that (4.2.2) holds in the following strong sense:

Lemma 4.2.4. *There exist $r > 1$ and $q > 2$ such that*

$$\begin{aligned} &\partial_t\rho_n, \Delta\rho_n && \textit{are bounded in } L^r(\mathcal{O}_T), \\ &\nabla\rho_n && \textit{is bounded in } L^q(0,T;L^2(\mathcal{O})), \end{aligned}$$

independently with respect to n. Consequently, the limit function ρ belongs to the same class and satisfies equation (4.2.2) almost everywhere on $\mathcal{O}_T$ and the boundary conditions (4.2.9) in the sense of traces.

To continue the proof, we first show that $\int_\mathcal{O} \rho_n u_n \cdot \psi \,\mathrm{d}x$ is equi-continuous in t, for any fixed test function $\psi \in X_n$ in (4.2.18). Using Lemmas 4.2.3–4.2.4, we see that, for any $0 < \xi < 1$,

$$\begin{aligned} &\int_t^{t+\xi}\int_\mathcal{O} \big(\mu\Delta u_n + (\mu+\nu)\nabla \mathrm{div}\, u_n\big)\cdot\psi \,\mathrm{d}x\mathrm{d}s \le C\int_t^{t+\xi} \|\nabla u_n\|_{L_x^2}\|\nabla\psi\|_{L_x^2}\,\mathrm{d}s \le C\sqrt{\xi}, \\ &\int_t^{t+\xi}\int_\mathcal{O} \mathrm{div}\,(\rho_n u_n\otimes u_n)\cdot\psi \,\mathrm{d}x\mathrm{d}s \le \int_t^{t+\xi} \|\sqrt{\rho_n}u_n\|_{L_x^2}^2\|\nabla\psi\|_{L_x^\infty}\,\mathrm{d}s \le C\xi, \\ &\int_t^{t+\xi}\int_\mathcal{O} \nabla(\rho_n^\gamma + \delta\rho_n^\beta)\cdot\psi \,\mathrm{d}x\mathrm{d}s \le \int_t^{t+\xi} \big(\|\rho_n\|_{L_x^\gamma}^\gamma + \delta\|\rho_n\|_{L_x^\beta}^\beta\big)\|\mathrm{div}\,\psi\|_{L_x^\infty}\,\mathrm{d}s \le C\xi, \end{aligned}$$

$$\varepsilon\int_t^{t+\xi}\int_{\mathcal{O}}\nabla\rho_n\cdot\nabla u_n\cdot\psi\,\mathrm{d}x\mathrm{d}s\le C\varepsilon\|\nabla\rho_n\|_{L^q_tL^2_x}\|\nabla u_n\|_{L^2_tL^2_x}\xi^{\frac{1}{2}-\frac{1}{q}}\le C\varepsilon\xi^{\frac{1}{2}-\frac{1}{q}}\quad\text{for } q>2,$$

$$\begin{aligned}
&\int_t^{t+\xi}\int_{\mathcal{O}}\nabla\cdot\mathrm{F}(Q_n)\mathrm{I}_3\cdot\psi\,\mathrm{d}x\mathrm{d}s=-\frac{1}{2}\int_0^{t+\xi}\int_{\mathcal{O}}\big(|(Q_n,\nabla Q_n)|^2+\frac{c_*}{2}|Q_n|^4\big)\mathrm{div}\,\psi\,\mathrm{d}x\mathrm{d}s\\
&\quad\le C\int_t^{t+\xi}\big(\|(Q_n,\nabla Q_n)\|^2_{L^2_x}+\|Q_n\|^4_{L^4_x}\big)\|\mathrm{div}\,\psi\|_{L^\infty_x}\,\mathrm{d}s\le C\xi,\\
&\int_t^{t+\xi}\int_{\mathcal{O}}\nabla\cdot\big(-\nabla Q_n\odot\nabla Q_n+Q_n\Delta Q_n-\Delta Q_nQ_n+\sigma_*c_n^2Q_n\big)\cdot\psi\,\mathrm{d}x\mathrm{d}s\\
&\quad\le C\int_t^{t+\xi}\big((\|\nabla Q_n\|^2_{L^2_x}+\|Q_n\|_{L^2_x}\|\Delta Q_n\|_{L^2_x})\|\nabla\psi\|_{L^\infty_x}+\|c_n\|^2_{L^\infty}\|Q_n\|_{L^2_x}\|\nabla\psi\|_{L^2_x}\big)\,\mathrm{d}s\\
&\quad\le C\big(\xi+\sqrt{\xi}\big).
\end{aligned}$$

Together with the fact that ρ_nu_n is uniformly bounded in $L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O}))$ with respect to n and X_n is dense in $L^{\frac{\gamma-1}{2\gamma}}(\mathcal{O})$, we conclude by Corollary A.0.1 that

$$\rho_nu_n\to\rho u\qquad\text{in } C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\mathcal{O}))\quad\text{as}\quad n\to\infty.\tag{4.2.43}$$

Since $\frac{2\gamma}{\gamma+1}>\frac{6}{5}$ $(\gamma>\frac{3}{2})$, by Proposition A.0.2, (4.2.43) yields

$$\rho_nu_n\to\rho u\qquad\text{in } C([0,T];H^{-1}(\mathcal{O})).$$

This, combined with (4.2.39), yields

$$\rho_nu_n\otimes u_n\to\rho u\otimes u\qquad\text{in } \mathcal{D}^{'}(\mathcal{O}_T).\tag{4.2.44}$$

Next, let us elaborate on the convergence result for the term: $\nabla u_n\cdot\nabla\rho_n$. We multiply (4.2.2) by ρ_n and integrate by parts to obtain

$$\|\rho_n(t,\cdot)\|^2_{L^2_x}+2\varepsilon\int_0^t\|\nabla\rho_n(t,\cdot)\|^2_{L^2_x}\,\mathrm{d}s=-\int_0^t\int_{\mathcal{O}}\mathrm{div}\,u_n\,|\rho_n|^2\,\mathrm{d}x\mathrm{d}s+\|\rho_0\|^2_{L^2_x}.\tag{4.2.45}$$

By Lemma 4.2.4, we know that the limit function ρ also satisfies (4.2.2). Applying the same argument to ρ as above, we have

$$\|\rho(t,\cdot)\|^2_{L^2_x}+2\varepsilon\int_0^t\|\nabla\rho(t,\cdot)\|^2_{L^2_x}\,\mathrm{d}s=-\int_0^t\int_{\mathcal{O}}\mathrm{div}\,u\,|\rho|^2\,\mathrm{d}x\mathrm{d}s+\|\rho_0\|^2_{L^2_x}.\tag{4.2.46}$$

Differentiating (4.2.45) with respect to t, we use (4.2.27), (4.2.30), and Lemma 4.2.4 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_n(t,\cdot)\|_{L^2_x}^2 = -2\varepsilon\|\nabla\rho_n(t,\cdot)\|_{L^2_x}^2 - \int_{\mathcal{O}} \operatorname{div} u_n\, |\rho_n|^2 \,\mathrm{d}x \in L^q(0,T), \quad 1<q<2,$$

which implies that $\|\rho_n(t,\cdot)\|_{L^2_x}^2$ is equi-continuous. Then we conclude that $\|\rho_n(t,\cdot)\|_{L^2_x}^2$ converges in $C([0,T])$ by the Arzela–Ascoli theorem. Moreover, from (4.2.37), (4.2.39), (4.2.45)–(4.2.46), and Lemma 4.2.4, we have

$$\begin{aligned}
&\|\rho_n(t,\cdot)\|_{L^2_x} \to \|\rho(t,\cdot)\|_{L^2_x} \qquad \text{for any } t,\\
&\|\nabla\rho_n\|_{L^2(\mathcal{O}_T)} \to \|\nabla\rho\|_{L^2(\mathcal{O}_T)}.
\end{aligned}$$

Since $\nabla\rho_n \rightharpoonup \nabla\rho$, it yields

$$\nabla\rho_n \to \nabla\rho \qquad \text{in } L^2(\mathcal{O}_T).$$

Then we have

$$\nabla\rho_n\cdot\nabla u_n \to \nabla\rho\cdot\nabla u \qquad \text{in } \mathcal{D}'(\mathcal{O}_T).$$

In addition, by (4.2.40)–(4.2.41), we have

$$\begin{aligned}
&\int_0^t\int_{\mathcal{O}} \nabla\cdot\big(\mathrm{F}(Q_n)\mathrm{I}_3 - \nabla Q_n\odot\nabla Q_n + Q_n\Delta Q_n - \Delta Q_n Q_n + \sigma_* c_n^2 Q_n\big)\cdot\psi\,\mathrm{d}x\mathrm{d}s\\
&\to \int_0^t\int_{\mathcal{O}} \nabla\cdot\big(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q + Q\Delta Q - \Delta Q Q + \sigma_* c^2 Q\big)\cdot\psi\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Then we can pass to the limit in equation (4.2.18) as $n\to\infty$. We deduce that the limit function (c,ρ,u,Q) is a weak solution of problem (4.2.1)−(4.2.10). Finally, let us summarize the results in this section in the following:

Proposition 4.2.1. *Suppose $\beta > \max\{4,\gamma\}$. Then there exists a weak solution (c,ρ,u,Q) of problem (4.2.1)−(4.2.10) with the same estimates as in Lemma 4.2.3 and $Q\in S_0^3$ a.e. in $\mathcal{O}_T$. Moreover, the energy inequality (4.2.21) and estimates (4.2.25)−(4.2.35) hold for (c,ρ,u,Q). Finally, we can find $r>1$ such that $\rho_t, \Delta\rho\in L^r(\mathcal{O}_T)$, and equation (4.2.2) is satisfied a.e. in $\mathcal{O}_T$.*

4.3 THE VANISHING ARTIFICIAL VISCOSITY LIMIT

In this section, we let $\varepsilon \to 0$ in (4.2.1)−(4.2.4). We denote $(c_\varepsilon, \rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$ the solution of problem (4.2.1)−(4.2.10), which we have obtained in Proposition 4.2.1. However, unlike the previous step, we do not have the higher integrability of the density as in Lemma 4.2.4. The boundedness of ρ_ε in $L^\infty(0,T; L^\gamma(\mathcal{O}) \cap L^\beta(\mathcal{O}))$ can guarantee only that ρ_ε^β converges to a Radon measure as $\varepsilon \to 0$, which is not easy to deal with. Thus, it is essential to obtain the strong compactness of ρ_ε in $L^1(\mathcal{O}_T)$. In order to do this, we need to use the uniform estimates (4.2.25)−(4.2.35) fully for $(c_\varepsilon, \rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$, since the system includes some highly nonlinear coupling terms, such as the last three terms in (4.3.5) when we prove the higher integrability of the density and the convergence of the last four terms in (4.3.29) to (4.3.30) in the proof of the convergence of the effective viscous flux. Moreover, we introduce the useful operator $\mathcal{B}$ related to the equation: $\operatorname{div} v = f$. See [6, 7, 22] for the construction and the proof of the following properties of the operator $\mathcal{B}$: For the problem

$$\operatorname{div} v = f, \qquad v|_{\partial\mathcal{O}} = 0, \tag{4.3.1}$$

there exists a linear operator $\mathcal{B} = [B_1, B_2, B_3]$ with the following properties:

(i) $\mathcal{B} : \{f \in L^p(\mathcal{O}) \,:\, \int_{\mathcal{O}} f \,\mathrm{d}x = 0\} \mapsto \left(W_0^{1,p}(\mathcal{O})\right)^3$ is a bounded linear operator such that, for any $1 < p < \infty$,

$$\|\mathcal{B}[f]\|_{W_0^{1,p}(\mathcal{O})} \leq C(p)\|f\|_{L^p(\mathcal{O})}; \tag{4.3.2}$$

(ii) $v = \mathcal{B}[f]$ is a solution of problem (4.3.1);

(iii) If there is a vector function $\mathbf{g} \in \left(L^r(\mathcal{O})\right)^3$ with $\mathbf{g} \cdot \vec{n}|_{\partial\mathcal{O}} = 0$, then

$$\|\mathcal{B}[\operatorname{div} \mathbf{g}]\|_{L^r(\mathcal{O})} \leq C(p)\|\mathbf{g}\|_{L^r(\mathcal{O})}, \tag{4.3.3}$$

where $r \in (1, \infty)$ is arbitrary.

4.3.1 Estimates of the density independent of ϵ

We take the quantities:

$$\psi(t)\mathcal{B}[\rho_\varepsilon - \bar{m}], \tag{4.3.4}$$

with $\psi \in \mathcal{D}(0,T)$, $0 \le \psi \le 1$, and $\bar{m} = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \rho_0(x)\mathrm{d}x$, as test functions for (4.2.3). Note that $\bar{m}$ is a constant such that this test function is well defined. We have the following result.

Lemma 4.3.1. *Assume that $(c_\varepsilon, \rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$ is the solution of problem (4.2.1)–(4.2.10) constructed in Proposition 4.2.1. Then*

$$\|\rho_\varepsilon\|^{\gamma+1}_{L^{\gamma+1}(\mathcal{O}_T)} + \delta\|\rho_\varepsilon\|^{\beta+1}_{L^{\beta+1}(\mathcal{O}_T)} \le C,$$

with C independent of ε.

Proof. The proof is similar to Lemma 3.1 in [17]. Let us apply the test function (4.3.4) to (4.2.3). Then, by a direct calculation, we have

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}} \psi\big(\rho_\varepsilon^{\gamma+1} + \delta\rho_\varepsilon^{\beta+1}\big)\,\mathrm{d}x\mathrm{d}t \\
&= \bar{m}\int_0^T \psi\int_{\mathcal{O}} (\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\,\mathrm{d}x\mathrm{d}t + (\mu+\nu)\int_0^T\psi\int_{\mathcal{O}}(\rho_\varepsilon - \bar{m})\mathrm{div}\, u_\varepsilon\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi_t\int_{\mathcal{O}} \rho_\varepsilon u_\varepsilon\cdot\mathcal{B}[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t + \mu\int_0^T\psi\int_{\mathcal{O}}\partial_j u_\varepsilon^i \partial_j B_i[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\psi\int_{\mathcal{O}}\rho_\varepsilon u_\varepsilon^i u_\varepsilon^j \partial_j B_i[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t - \varepsilon\int_0^T\psi\int_{\mathcal{O}}\rho_\varepsilon u_\varepsilon\cdot\mathcal{B}(\Delta\rho_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\psi\int_{\mathcal{O}}\rho_\varepsilon u_\varepsilon\cdot\mathcal{B}[\mathrm{div}\,(\rho_\varepsilon u_\varepsilon)]\,\mathrm{d}x\mathrm{d}t + \varepsilon\int_0^T\psi\int_{\mathcal{O}}\partial_j u_\varepsilon^i\partial_j\rho_\varepsilon B_i[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\psi\int_{\mathcal{O}}\nabla\cdot(\nabla Q_\varepsilon\otimes\nabla Q_\varepsilon - \mathrm{F}(Q_\varepsilon)\mathrm{I}_3)\cdot\mathcal{B}[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sigma_*\int_0^T\psi\int_{\mathcal{O}} c_\varepsilon^2 Q_\varepsilon : \nabla\mathcal{B}[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\psi\int_{\mathcal{O}}\nabla\cdot(Q_\varepsilon\Delta Q_\varepsilon - \Delta Q_\varepsilon Q_\varepsilon)\cdot\mathcal{B}[\rho_\varepsilon - \bar{m}]\,\mathrm{d}x\mathrm{d}t \\
&= \sum_{i=1}^{11}\mathcal{I}_i.
\end{aligned} \tag{4.3.5}$$

Next, we estimate the terms on the right-hand side of the above equality by using the boundedness of solution $(c_\varepsilon, \rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$ obtained in Proposition 4.2.1, in which the universal constant $C>0$ is independent of ε:

$$
\begin{aligned}
|\mathcal{I}_1| &= \Big|\bar m\int_0^T\psi\int_{\mathcal{O}}(\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta)\,\mathrm{d}x\mathrm{d}t\Big| \le C\|\psi\|_{L^\infty(0,T)}\big(\|\rho_\varepsilon\|^\gamma_{L^\infty_tL^\gamma_x}+\delta\|\rho_\varepsilon\|^\beta_{L^\infty_tL^\beta_x}\big)\le C\|\psi\|_{L^\infty(0,T)},\\
|\mathcal{I}_2| &= (\mu+\nu)\Big|\int_0^T\psi\int_{\mathcal{O}}(\rho_\varepsilon-\bar m)\mathrm{div}\,u_\varepsilon\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le C\sqrt{T}\|\psi\|_{L^\infty(0,T)}\big(\|\rho_\varepsilon\|_{L^\infty_tL^2_x}+\bar m|\mathcal{O}|^{\frac12}\big)\|\mathrm{div}\,u_\varepsilon\|_{L^2_{t,x}}\le C\|\psi\|_{L^\infty(0,T)},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_3| &= \Big|\int_0^T\psi_t\int_{\mathcal{O}}\rho_\varepsilon u_\varepsilon\cdot\mathcal{B}[\rho_\varepsilon-\bar m]\,\mathrm{d}x\mathrm{d}t\Big|\le C\int_0^T|\psi_t|\|\sqrt{\rho_\varepsilon}\|_{L^4_x}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2_x}\|\mathcal{B}[\rho_\varepsilon-\bar m]\|_{L^4_x}\,\mathrm{d}t\\
&\le C\|\rho_\varepsilon\|^{\frac12}_{L^\infty_tL^2_x}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty_tL^2_x}\|\rho_\varepsilon-\bar m\|_{L^\infty_tL^2_x}\int_0^T|\psi_t|\mathrm{d}t\le C\|\psi_t\|_{L^1(0,T)},\\
|\mathcal{I}_4| &= \mu\Big|\int_0^T\psi\int_{\mathcal{O}}\partial_ju^i_\varepsilon\partial_jB_i[\rho_\varepsilon-\bar m]\,\mathrm{d}x\mathrm{d}t\Big|\le\mu\int_0^T|\psi|\|\nabla u_\varepsilon\|_{L^2_x}\|\nabla\mathcal{B}[\rho_\varepsilon-\bar m]\|_{L^2_x}\,\mathrm{d}t\\
&\le\mu\int_0^T|\psi|\|\nabla u_\varepsilon\|_{L^2_x}\|\rho_\varepsilon-\bar m\|_{L^2_x}\,\mathrm{d}t\le C\|\psi\|_{L^\infty(0,T)},\\
|\mathcal{I}_5| &= \Big|\int_0^T\psi\int_{\mathcal{O}}\rho_\varepsilon u^i_\varepsilon u^j_\varepsilon\partial_jB_i[\rho_\varepsilon-\bar m]\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le C\int_0^T|\psi|\|\rho_\varepsilon\|_{L^3_x}\|u_\varepsilon\|^2_{L^6_x}\|\nabla\mathcal{B}[\rho_\varepsilon-\bar m]\|_{L^3_x}\,\mathrm{d}t\\
&\le C\int_0^T|\psi|\|\rho_\varepsilon\|_{L^3_x}\|u_\varepsilon\|^2_{L^6_x}\|\rho_\varepsilon-\bar m\|_{L^3_x}\,\mathrm{d}t\le C\|\psi\|_{L^\infty(0,T)},\\
|\mathcal{I}_6| &= \varepsilon\Big|\int_0^T\psi\int_{\mathcal{O}}\rho_\varepsilon u_\varepsilon\cdot\mathcal{B}[\Delta\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t\Big|\le\varepsilon\int_0^T|\psi|\|\rho_\varepsilon\|_{L^3_x}\|u_\varepsilon\|_{L^6_x}\|\mathcal{B}[\Delta\rho_\varepsilon]\|_{L^2_x}\,\mathrm{d}t\\
&\le C\varepsilon\int_0^T|\psi|\|\rho_\varepsilon\|_{L^3_x}\|u_\varepsilon\|_{L^6_x}\|\nabla\rho_\varepsilon\|_{L^2_x}\,\mathrm{d}t\le C\|\psi\|_{L^\infty(0,T)}\qquad\text{for }\varepsilon<1,\\
|\mathcal{I}_7| &= \Big|\int_0^T\psi\int_{\mathcal{O}}\rho_\varepsilon u_\varepsilon\cdot\mathcal{B}[\mathrm{div}\,(\rho_\varepsilon u_\varepsilon)]\,\mathrm{d}x\mathrm{d}t\Big|\le\int_0^T|\psi|\|\rho_\varepsilon\|_{L^3_x}\|u_\varepsilon\|_{L^6_x}\|\mathcal{B}[\mathrm{div}\,(\rho_\varepsilon u_\varepsilon)]\|_{L^2_x}\,\mathrm{d}t\\
&\le\int_0^T|\psi|\|\rho_\varepsilon\|_{L^3_x}\|u_\varepsilon\|_{L^6_x}\|\rho_\varepsilon u_\varepsilon\|_{L^2_x}\,\mathrm{d}t\le C\int_0^T|\psi|\|\rho_\varepsilon\|^2_{L^3_x}\|u_\varepsilon\|^2_{L^6_x}\,\mathrm{d}t\le C\|\psi\|_{L^\infty(0,T)}.
\end{aligned}
$$

Since $\beta>4$, by using the Sobolev embedding (Lemma A.0.7 in Appendix A), we have

$$
\begin{aligned}
\|B[\rho_\varepsilon-\bar m]\|_{L^\infty_x}&\le C_1\|\nabla B[\rho_\varepsilon-\bar m]\|^{\frac{3}{\beta}}_{L^\beta_x}\|B[\rho_\varepsilon-\bar m]\|^{1-\frac{3}{\beta}}_{L^\beta_x}+C_2\|B[\rho_\varepsilon-\bar m]\|_{L^\beta_x}\\
&\le C\|\rho_\varepsilon-\bar m\|_{L^\beta_x}.
\end{aligned}\tag{4.3.6}
$$

Then we have

$$
\begin{aligned}
|\mathcal{I}_8| &= \varepsilon\Big|\int_0^T \psi \int_{\mathcal{O}} \nabla u_\varepsilon \cdot \mathcal{B}[\rho_\varepsilon - \bar{m}]\nabla\rho_\varepsilon \,\mathrm{d}x\mathrm{d}t\Big| \\
&\le \varepsilon \int_0^T |\psi| \|\nabla u_\varepsilon\|_{L_x^2} \|\nabla\rho_\varepsilon\|_{L_x^2} \|\mathcal{B}[\rho_\varepsilon - \bar{m}]\|_{L_x^\infty} \,\mathrm{d}t \\
&\le \varepsilon \int_0^T |\psi| \|\nabla u_\varepsilon\|_{L_x^2} \|\nabla\rho_\varepsilon\|_{L_x^2} \|\rho_\varepsilon - \bar{m}\|_{L_x^\beta} \,\mathrm{d}t \le C\|\psi\|_{L^\infty(0,T)},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_9| &= \Big|\int_0^T \psi \int_{\mathcal{O}} \nabla\cdot(\nabla Q_\varepsilon \otimes \nabla Q_\varepsilon - \mathrm{F}(Q_\varepsilon)\mathrm{I}_3) \cdot \mathcal{B}[\rho_\varepsilon - \bar{m}] \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le C\int_0^T |\psi| \Big(\|\nabla Q_\varepsilon\|^2_{L_x^{\frac{10}{3}}} \|\nabla\mathcal{B}[\rho_\varepsilon - \bar{m}]\|_{L_x^{\frac{5}{2}}} + \Big|\int_{\mathcal{O}} \mathrm{F}(Q_\varepsilon)\mathrm{div}\,\mathcal{B}[\rho_\varepsilon - \bar{m}] \,\mathrm{d}x\Big|\Big) \,\mathrm{d}t \\
&\le C\int_0^T |\psi| \Big(\|\nabla Q_\varepsilon\|^2_{L_x^{\frac{10}{3}}} \|\nabla\mathcal{B}[\rho_\varepsilon - \bar{m}]\|_{L_x^{\frac{5}{2}}} + \Big|\int_{\mathcal{O}} \mathrm{F}(Q_\varepsilon)(\rho_\varepsilon - \bar{m}) \mathrm{d}x\Big|\Big) \,\mathrm{d}t \\
&\le C\int_0^T |\psi| \big(\|\nabla Q_\varepsilon\|^2_{L_x^{\frac{10}{3}}} \|\rho_\varepsilon - \bar{m}\|_{L_x^{\frac{5}{2}}} + (\|Q_\varepsilon\|^2_{L_x^5} + \|Q_\varepsilon\|^4_{L_x^{10}}) \|\rho_\varepsilon - \bar{m}\|_{L_x^{\frac{5}{3}}} \big) \mathrm{d}t \\
&\le C\|\psi\|_{L^\infty(0,T)},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_{10}| &= \sigma_* \Big|\int_0^T \psi \int_{\mathcal{O}} c_\varepsilon^2 Q_\varepsilon \cdot \nabla\mathcal{B}[\rho_\varepsilon - \bar{m}] \,\mathrm{d}x\mathrm{d}t\Big| \\
&\le C\|c_\varepsilon\|^2_{L_{t,x}^\infty} \int_0^T |\psi| \|Q_\varepsilon\|_{L_x^2} \|\nabla\mathcal{B}[\rho_\varepsilon - \bar{m}]\|_{L_x^2} \,\mathrm{d}t \le C\|\psi\|_{L^\infty(0,T)},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_{11}| &= \Big|\int_0^T \psi \int_{\mathcal{O}} \nabla\cdot(Q_\varepsilon\Delta Q_\varepsilon - \Delta Q_\varepsilon Q_\varepsilon) \cdot \mathcal{B}[\rho_\varepsilon - \bar{m}] \,\mathrm{d}x\mathrm{d}t\Big| \\
&\le C\int_0^T ||\psi| \|Q_\varepsilon\|_{L_x^4} \|\Delta Q_\varepsilon\|_{L_x^2} \|\nabla\mathcal{B}[\rho_\varepsilon - \bar{m}]\|_{L_x^4} \,\mathrm{d}t \\
&\le C\int_0^T |\psi| \|Q_\varepsilon\|_{L_x^4} \|\Delta Q_\varepsilon\|_{L_x^2} \|\rho_\varepsilon - \bar{m}\|_{L_x^4} \,\mathrm{d}t \le C\|\psi\|_{L^\infty(0,T)}.
\end{aligned}
$$

Combining all the above estimates together, we have

$$
\int_0^T \int_{\mathcal{O}} \psi\big(\rho_\varepsilon^{\gamma+1} + \delta\rho_\varepsilon^{\beta+1}\big) \,\mathrm{d}x\mathrm{d}t \le C\big(\|\psi\|_{L^\infty(0,T)} + \|\psi_t\|_{L^1(0,T)}\big). \tag{4.3.7}
$$

Then we can take $\psi = \psi^k \in \mathcal{D}(0,T)$ in (4.3.7), with $\|\psi^k\|_{L^\infty(0,T)} + \|\psi_t^k\|_{L^1(0,T)} \le C$ uniformly bounded independent of k, and $\phi^k(t) \to 1$ for $t \in (0,T)$ as $k \to \infty$, and then let $k \to \infty$ in the resulting (4.3.7) to conclude our desired result. □

Remark 4.3.1. Lemma 4.3.1 implies that ρ_ε has higher integrability, which provides the weak convergence result for the pressure, *i.e.*, $P_\varepsilon = \rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta \rightharpoonup p$ in $L^{\frac{\beta+1}{\beta}}(\mathcal{O}_T)$.

4.3.2 Limit passage of $\varepsilon \to 0$

In this subsection, we fix parameter δ, and pass to the limit $\varepsilon \to 0$ in equations (4.2.1)–(4.2.4). To begin with, similarly to (4.2.40)–(4.2.41) in §4.2.3, we have

$$c_\varepsilon \rightharpoonup c \quad \text{in } L^2(0,T;H^1(\mathcal{O})), \qquad c_\varepsilon \to c \quad \text{in } L^2(0,T;L^2(\mathcal{O})), \tag{4.3.8}$$

$$Q_\varepsilon \rightharpoonup Q \quad \text{in } L^2(0,T;H^2(\mathcal{O})), \qquad Q_\varepsilon \to Q \quad \text{in } L^2(0,T;H^1(\mathcal{O})). \tag{4.3.9}$$

From the boundedness of ρ_ε in $L^{\beta+1}(\mathcal{O}_T)$, $\sqrt{\varepsilon}\nabla\rho_\varepsilon$ in $L^2(\mathcal{O}_T)$, and u_ε in $L^2(0,T;H_0^1(\mathcal{O}))$, we know that

$$\rho_\varepsilon \rightharpoonup \rho \qquad \text{in } L^{\beta+1}(\mathcal{O}_T), \tag{4.3.10}$$

$$u_\varepsilon \rightharpoonup u \qquad \text{in } L^2(0,T;H_0^1(\mathcal{O})), \tag{4.3.11}$$

$$\varepsilon\nabla\rho_\varepsilon \cdot \nabla u_\varepsilon \to 0 \qquad \text{in } L^1(\mathcal{O}_T), \tag{4.3.12}$$

$$\varepsilon\Delta\rho_\varepsilon \to 0 \qquad \text{in } L^2(0,T;H^{-1}(\mathcal{O})). \tag{4.3.13}$$

Moreover, we can also obtain the following convergence results as in §4.2.3:

$$\rho_\varepsilon \to \rho \qquad \text{in } C([0,T];L^\beta_{\text{weak}}(\mathcal{O})), \tag{4.3.14}$$

$$\rho_\varepsilon \to \rho \qquad \text{in } C([0,T];L^\gamma_{\text{weak}}(\mathcal{O})), \tag{4.3.15}$$

$$\rho_\varepsilon u_\varepsilon \to \rho u \qquad \text{in } C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\mathcal{O})), \tag{4.3.16}$$

$$\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \rho u \otimes u \qquad \text{in } \mathcal{D}'(\mathcal{O}_T). \tag{4.3.17}$$

Finally, we conclude that the limit vector function (c,ρ,u,Q) satisfies the following equations in $\mathcal{D}'(\mathcal{O}_T)$:

$$\partial_t c + u\cdot\nabla c = D_0 c, \tag{4.3.18}$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 0, \tag{4.3.19}$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla p = \mu\Delta u + (\nu+\mu)\nabla\operatorname{div} u + \nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q) \tag{4.3.20}$$

$$+ \nabla\cdot(Q\Delta Q - \Delta Q Q) + \sigma_*\nabla\cdot(c^2 Q), \tag{4.3.21}$$

$$\partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q = \Gamma H[Q,c], \tag{4.3.22}$$

along with the initial–boundary conditions (4.2.6)−(4.2.10), with

$$P_\varepsilon = \rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta \rightharpoonup p \qquad \text{in } L^{\frac{\beta+1}{\beta}}(\mathcal{O}_T) \tag{4.3.23}$$

for $\beta > \max\{4, \gamma\}$.

In the next step, we show that $p = \rho^\gamma + \delta\rho^\beta$, which is equivalent to the strong convergence of ρ_ε in $L^1(\mathcal{O}_T)$.

4.3.3 The effective viscous flux

The quantity, $\mathfrak{E}_\varepsilon := \rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta - (\nu+2\mu)\operatorname{div} u_\varepsilon$, is usually called the effective viscous flux, and its corresponding weak convergence limit is $\mathfrak{E} := p - (\nu+2\mu)\operatorname{div} u$. The properties of $\mathfrak{E}_\varepsilon$ (*cf.* [31,39,59]) play an important role in our problem. We introduce the following operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\mathcal{A}_j[v] = \partial_j \Delta^{-1} v$$

with the Fourier transform:

$$\mathcal{F}(\mathcal{A}_j)(\xi) = -\frac{i\xi_j}{|\xi|^2},$$

and enjoying the following properties:

$$\operatorname{div} \mathcal{A}[v] = v, \qquad \Delta\mathcal{A}_i[v] = \partial_i v, \tag{4.3.24}$$

$$\|\mathcal{A}_i[v]\|_{W^{1,p}(\mathcal{O})} \le C(p,\mathcal{O})\|v\|_{L^p(\mathbb{R}^3)} \qquad \text{for } 1 < p < \infty, \tag{4.3.25}$$

$$\|\mathcal{A}_i[v]\|_{L^q(\mathcal{O})} \le C(p,q,\mathcal{O})\|v\|_{L^p(\mathbb{R}^3)} \qquad \text{if } p < \infty \text{ and } \frac{1}{q} \ge \frac{1}{p} - \frac{1}{3}, \tag{4.3.26}$$

$$\|\mathcal{A}_i[v]\|_{L^\infty(\mathcal{O})} \le C(p,\mathcal{O})\|v\|_{L^p(\mathbb{R}^3)} \qquad \text{for all } p > 3. \tag{4.3.27}$$

Lemma 4.3.2. *Let $(c_\varepsilon, \rho_\varepsilon, u_\varepsilon, Q_\varepsilon)$ be a sequence of solutions constructed in Proposition 4.2.1, and let (c, ρ, u, Q, p) be the limits satisfying (4.3.18)−(4.3.23). Then*

$$\lim_{\varepsilon\to 0^+} \int_0^T \psi \int_{\mathcal{O}} \phi\, \mathfrak{E}_\varepsilon \rho_\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_0^T \psi \int_{\mathcal{O}} \phi\, \mathfrak{E}\rho \,\mathrm{d}x\mathrm{d}t$$

for any $\psi \in \mathcal{D}(0,T)$ and $\phi \in \mathcal{D}(\mathcal{O})$.

Proof. We prove this lemma based on the div-curl lemma of compensated compactness. By Proposition 4.2.1, we know that $(\rho_\varepsilon, u_\varepsilon)$ satisfies (4.2.2) *a.e.* on $\mathcal{O}_T$ with boundary condition (4.2.9). If $(\rho_\varepsilon, u_\varepsilon)$ are extended to be zero outside $\mathcal{O}$, then it satisfies

$$\partial_t \rho_\varepsilon + \nabla\cdot(\rho_\varepsilon u_\varepsilon) = \varepsilon \mathrm{div}\,(1_{\mathcal{O}}\nabla\rho_\varepsilon) \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3),$$

with $1_{\mathcal{O}}$ the characteristic function on $\mathcal{O}$. Consider the following test function to (4.2.3) as

$$\varphi_\varepsilon(t,x) = \psi(t)\phi(x)\mathcal{A}[\rho_\varepsilon], \tag{4.3.28}$$

with $\psi \in \mathcal{D}(0,T)$ and $\phi \in \mathcal{D}(\mathcal{O})$. Similarly to (4.3.5), we apply the test function (4.3.28) to (4.2.3). By a direct calculation, we have

$$\begin{aligned}
&\int_0^T \psi \int_{\mathcal{O}} \phi\big(\rho_\varepsilon^{\gamma+1} + \delta\rho_\varepsilon^{\beta+1} - (\nu+2\mu)\mathrm{div}\,u_\varepsilon\big)\rho_\varepsilon \,\mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \psi \int_{\mathcal{O}} (\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\partial_i\phi\,\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t + (\mu+\nu)\int_0^T \psi \int_{\mathcal{O}} \mathrm{div}\,u_\varepsilon\,\partial_i\phi\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \mu\int_0^T \psi \int_{\mathcal{O}} \partial_j u_\varepsilon^i \partial_j\phi\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t - \mu\int_0^T \psi \int_{\mathcal{O}} u_\varepsilon^i\partial_j\phi\partial_j\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \mu\int_0^T \psi \int_{\mathcal{O}} u_\varepsilon^i\partial_i\phi\rho_\varepsilon\,\mathrm{d}x\mathrm{d}t - \varepsilon\int_0^T \psi \int_{\mathcal{O}} \phi\rho_\varepsilon u_\varepsilon^i\mathcal{A}_i\,\mathrm{div}\,(1_\Omega\nabla\rho_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \psi \int_{\mathcal{O}} \phi\rho_\varepsilon u_\varepsilon^i\big(\mathcal{A}_i[\mathrm{div}(\rho_\varepsilon u_\varepsilon)] - u_\varepsilon^j\partial_j\mathcal{A}_i[\rho_\varepsilon]\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi_t \int_{\mathcal{O}} \phi\rho_\varepsilon u_\varepsilon^i\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t - \int_0^T \psi \int_{\mathcal{O}} \rho_\varepsilon u_\varepsilon^i u_\varepsilon^j\partial_j\phi\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon\int_0^T \psi \int_{\mathcal{O}} \partial_j u_\varepsilon^i\partial_j\rho_\varepsilon\phi\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t + \sigma_*\int_0^T \psi \int_{\mathcal{O}} c_\varepsilon^2 Q_\varepsilon^{ij}\big(\partial_j\phi\mathcal{A}_i[\rho_\varepsilon] + \phi\partial_j\mathcal{A}_i[\rho_\varepsilon]\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \big(\partial_i Q_\varepsilon^{kl}\partial_j Q_\varepsilon^{kl} - \mathrm{F}(Q_\varepsilon)\delta_{ij}\big)\partial_j\phi\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \big(\partial_i Q_\varepsilon^{kl}\partial_j Q_\varepsilon^{kl} - \mathrm{F}(Q_\varepsilon)\delta_{ij}\big)\phi\partial_j\mathcal{A}_i[\rho_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \psi \int_{\mathcal{O}} \big(Q_\varepsilon^{ik}\Delta Q_\varepsilon^{kj} - \Delta Q_\varepsilon^{ik} Q_\varepsilon^{kj}\big)\big(\partial_j\phi\mathcal{A}_i[\rho_\varepsilon] + \phi\partial_j\mathcal{A}_i[\rho_\varepsilon]\big)\,\mathrm{d}x\mathrm{d}t \\
&= \sum_{i=1}^{14}\mathcal{I}_i.
\end{aligned} \tag{4.3.29}$$

Remark 4.3.2. The function ρ_ε extended by zero outside $\mathcal{O}$ admits the time derivative as

$$\partial_t \rho_\varepsilon = \begin{cases} \varepsilon \Delta \rho_\varepsilon - \operatorname{div}(\rho_\varepsilon u_\varepsilon) & \text{in } \mathcal{O}, \\ 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{O}. \end{cases}$$

Since $u_\varepsilon|_{\partial\mathcal{O}} = 0$, we have

$$\operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \qquad \text{on } \mathbb{R}^3 \setminus \mathcal{O}.$$

Moreover, since $\nabla \rho_\varepsilon \cdot \vec{n}|_{\partial\mathcal{O}} = 0$ and $\Delta \rho_\varepsilon \in L^r(\mathcal{O}_T)$ for some $r > 1$, we have

$$\operatorname{div}(1_{\mathcal{O}} \nabla \rho_\varepsilon) = \begin{cases} \Delta \rho_\varepsilon & \text{in } \mathcal{O}, \\ 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{O}. \end{cases}$$

Next, we do the zero extension to the limit function ρ to $\mathbb{R}^3$, and repeat the same procedure above. This step is guaranteed by the following results from [17]:

Lemma 4.3.3. *Assume that $(\rho, u) \in L^2(\mathcal{O}_T) \times L^2(0,T; H_0^1(\mathcal{O}))$ is a solution of (4.3.19) in $\mathcal{D}'(\mathcal{O}_T)$. Then, extending (ρ, u) to be zero in $\mathbb{R}^3 \setminus \mathcal{O}$, equation (4.3.19) still holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.*

Here, let us apply the test function $\varphi = \psi(t)\phi(x)\mathcal{A}[\rho]$ to (4.3.21). By a similar calculation as before, we have

$$\begin{aligned}
&\int_0^T \psi \int_{\mathcal{O}} \phi\big(p - (\nu + 2\mu)\operatorname{div} u\big)\rho \, \mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \psi \int_{\mathcal{O}} p \partial_i \phi \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t + (\mu + \nu) \int_0^T \psi \int_{\mathcal{O}} \operatorname{div} u \partial_i \phi \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t \\
&\quad + \mu \int_0^T \psi \int_{\mathcal{O}} \partial_j u^i \partial_j \phi \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t - \mu \int_0^T \psi \int_{\mathcal{O}} u^i \partial_j \phi \partial_j \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t \\
&\quad + \mu \int_0^T \psi \int_{\mathcal{O}} u^i \partial_i \phi \rho \, \mathrm{d}x\mathrm{d}t + \int_0^T \psi \int_{\mathcal{O}} \phi \rho u^i \big(\mathcal{A}_i[\operatorname{div}(\rho u)] - u^j \partial_j \mathcal{A}_i[\rho]\big) \, \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi_t \int_{\mathcal{O}} \phi \rho u^i \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t - \int_0^T \psi \int_{\mathcal{O}} \rho u^i u^j \partial_j \phi \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t \\
&\quad + \sigma_* \int_0^T \psi \int_{\mathcal{O}} c^2 Q^{ij} \big(\partial_j \phi \mathcal{A}_i[\rho] + \phi \partial_j \mathcal{A}_i[\rho]\big) \, \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \big(\partial_i Q^{kl} \partial_j Q^{kl} - \mathrm{F}(Q)\delta_{ij}\big) \partial_j \phi \mathcal{A}_i[\rho] \, \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \psi \int_{\mathcal{O}} \big(\partial_i Q^{kl} \partial_j Q^{kl} - \mathrm{F}(Q)\delta_{ij}\big) \phi \partial_j \mathcal{A}_i[\rho] \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \psi \int_{\mathcal{O}} \big(Q^{ik}\Delta Q^{kj} - \Delta Q^{ik} Q^{kj}\big)\big(\partial_j \phi \mathcal{A}_i[\rho] + \phi \partial_j \mathcal{A}_i[\rho]\big) \,\mathrm{d}x\mathrm{d}t \\
= &\sum_{j=1}^{12} \mathcal{J}_i. \qquad (4.3.30)
\end{aligned}
$$

Next, in order to prove Lemma 4.3.2, we need to show that the right-hand side of (4.3.29) converges to the right-hand side of (4.3.30). First, similarly to (4.2.43), by the uniform bound of ρ_ε in $L^\infty(0,T;L^\beta(\mathcal{O}))$, we have

$$
\rho_\varepsilon \to \rho \qquad \text{in } C([0,T];L^\beta_{\text{weak}}(\mathcal{O})) \ \text{ as } \varepsilon \to 0.
$$

Then, for any $\phi \in (L^\beta(\mathcal{O}))^*$, we extend ϕ by zero outside of $\mathcal{O}$ to obtain

$$
\begin{aligned}
&(\partial_j \mathcal{A}_i[\rho_\varepsilon], \phi)_{L^\beta(\mathcal{O})\times(L^\beta(\mathcal{O}))^*} = \int_{\mathbb{R}^3} \partial_j \mathcal{A}_i[\rho_\varepsilon]\phi \,\mathrm{d}x = \int_{\mathbb{R}^3} \frac{\hat{\rho}_\varepsilon \xi_i \xi_j}{|\xi|^2} \hat{\phi} \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^3} \rho_\varepsilon \partial_j \mathcal{A}_i[\phi] \,\mathrm{d}x = (\rho_\varepsilon, \partial_j \mathcal{A}_i[\phi])_{L^\beta(\mathcal{O})\times(L^\beta(\mathcal{O}))^*} \\
&\longrightarrow (\rho, \partial_j \mathcal{A}_i[\phi])_{L^\beta(\mathcal{O})\times(L^\beta(\mathcal{O}))^*} = (\partial_j \mathcal{A}_i[\rho], \phi)_{L^\beta(\mathcal{O})\times(L^\beta(\mathcal{O}))^*} \qquad \text{as } \varepsilon \to 0,
\end{aligned}
$$

from which we infer

$$
\mathcal{A}[\rho_\varepsilon] \to \mathcal{A}[\rho] \qquad \text{in } C([0,T], W^{1,\beta}_{\text{weak}}(\mathcal{O})).
$$

This, combining with Proposition A.0.2 ($\beta > \frac{6}{5}$) and the compact imbedding $W^{1,\beta}(\mathcal{O}) \Subset C(\bar{\mathcal{O}})$, gives

$$
\nabla \mathcal{A}[\rho_\varepsilon] \to \nabla \mathcal{A}[\rho] \qquad \text{in } C([0,T];H^{-1}(\mathcal{O})), \tag{4.3.31}
$$

$$
\mathcal{A}[\rho_\varepsilon] \to \mathcal{A}[\rho] \qquad \text{in } C(\overline{\mathcal{O}_T}). \tag{4.3.32}
$$

From (4.3.11) and (4.3.32), we see that, as $\varepsilon \to 0$,

$$
\mathcal{I}_2 \to \mathcal{J}_2, \qquad \mathcal{I}_3 \to \mathcal{J}_3.
$$

By (4.3.16) and (4.3.32), we find that, as $\varepsilon \to 0$,

$$
\mathcal{I}_8 \to \mathcal{J}_7.
$$

From (4.3.11) and (4.3.16), we have

$$\|\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon\|_{L_x^{\frac{6\gamma}{3+4\gamma}}} \le C\|\rho_\varepsilon u_\varepsilon\|_{L_x^{\frac{2\gamma}{\gamma+1}}}\|u_\varepsilon\|_{L_x^6},$$

which, combined with (4.3.17), gives

$$\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \rightharpoonup \rho u \otimes u \qquad \text{in } L^2(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\mathcal{O})).$$

Thus, together with (4.3.32), we see that, if $\beta > \frac{6\gamma}{2\gamma-3}$,

$$\mathcal{I}_9 \to \mathcal{J}_8 \qquad \text{as } \varepsilon \to 0.$$

Combining (4.3.11) with (4.3.31), we obtain that, as $\varepsilon \to 0$,

$$\mathcal{I}_4 \to \mathcal{J}_4.$$

In the same way as above, since $\beta > 4$, we know

$$\mathcal{I}_5 \to \mathcal{J}_5.$$

It follows from the boundedness of ρ_ε in $L^\beta(\mathcal{O})$, $\rho_\varepsilon u_\varepsilon$ in $L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})$, and (4.3.25) that

$$\begin{aligned}
&\|\rho_\varepsilon \partial_j \mathcal{A}_i[\rho_\varepsilon u_\varepsilon^j] - \rho_\varepsilon u_\varepsilon^j \partial_j \mathcal{A}_i[\rho_\varepsilon]\|_{L_x^\alpha} \\
&\le \|\rho_\varepsilon\|_{L_x^\beta}\|\partial_j \mathcal{A}_i[\rho_\varepsilon u_\varepsilon^j]\|_{L_x^{\frac{2\gamma}{\gamma+1}}} + \|\rho_\varepsilon u_\varepsilon\|_{L_x^{\frac{2\gamma}{\gamma+1}}}\|\partial_j \mathcal{A}_i[\rho_\varepsilon]\|_{L_x^\beta} \\
&\le \|\rho_\varepsilon\|_{L_x^\beta}\|\rho_\varepsilon u_\varepsilon\|_{L_x^{\frac{2\gamma}{\gamma+1}}} \le C,
\end{aligned}$$

where $\frac{1}{\alpha} = \frac{\gamma+1}{2\gamma} + \frac{1}{\beta} < \frac{5}{6}$, if $\beta > \frac{6\gamma}{2\gamma-3}$. Then we conclude

$$\rho_\varepsilon \partial_j \mathcal{A}_i[\rho_\varepsilon u_\varepsilon^j] - \rho_\varepsilon u_\varepsilon^j \partial_j \mathcal{A}_i[\rho_\varepsilon] \in L^\infty(0,T;L^\alpha(\mathcal{O})).$$

From Lemma 3.4 in [17] and the compact embedding of $L^\alpha(\mathcal{O})$ in $H^{-1}(\mathcal{O})$, we infer that

$$\rho_\varepsilon \partial_j \mathcal{A}_i[\rho_\varepsilon u_\varepsilon^j] - \rho_\varepsilon u_\varepsilon^j \partial_j \mathcal{A}_i[\rho_\varepsilon] \to \rho \partial_j \mathcal{A}_i[\rho u^j] - \rho u^j \partial_j \mathcal{A}_i[\rho] \qquad \text{in } H^{-1}(\mathcal{O}).$$

Then, after applying Lebesgue convergence theorem, we obtain

$$\rho_\varepsilon \partial_j \mathcal{A}_i[\rho_\varepsilon u_\varepsilon^j] - \rho_\varepsilon u_\varepsilon^j \partial_j \mathcal{A}_i[\rho_\varepsilon] \to \rho \partial_j \mathcal{A}_i[\rho u^j] - \rho u^j \partial_j \mathcal{A}_i[\rho] \qquad \text{in } L^2(0,T;H^{-1}(\mathcal{O})),$$

which, combined with (4.3.11), yields

$$\mathcal{I}_7 \to \mathcal{J}_6 \qquad \text{as } \varepsilon \to 0.$$

Moreover, we have

$$\begin{aligned}
|\mathcal{I}_6| &= \varepsilon \Big| \int_0^T \psi \int_{\mathcal{O}} \phi \rho_\varepsilon u_\varepsilon^i \mathcal{A}_i[\mathrm{div}(1_\Omega \nabla \rho_\varepsilon)] \,\mathrm{d}x\mathrm{d}t \Big| \\
&\le C\varepsilon \int_0^T \|\rho_\varepsilon\|_{L^3_x} \|u_\varepsilon\|_{L^6_x} \|\mathcal{A}_i[\mathrm{div}(1_\Omega \nabla \rho_\varepsilon)]\|_{L^2_x} \mathrm{d}t \\
&\le C\varepsilon \int_0^T \|\rho_\varepsilon\|_{L^3_x} \|u_\varepsilon\|_{L^6_x} \|\nabla \rho_\varepsilon\|_{L^2_x} \mathrm{d}t \\
&\le C\varepsilon \|\rho_\varepsilon\|_{L^\infty_t L^3_x} \|u_\varepsilon\|_{L^2_t L^6_x} \|\nabla \rho_\varepsilon\|_{L^2_t L^2_x} \le C\sqrt{\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Since $\beta > 3$, we have

$$\begin{aligned}
|\mathcal{I}_{10}| &= \varepsilon \Big| \int_0^T \psi \int_{\mathcal{O}} \partial_j u_\varepsilon^i \partial_j \rho_\varepsilon \phi \mathcal{A}_i[\rho_\varepsilon] \,\mathrm{d}x\mathrm{d}t \Big| \\
&\le C\varepsilon \int_0^T \|\nabla u_\varepsilon\|_{L^2_x} \|\nabla \rho_\varepsilon\|_{L^2_x} \|\mathcal{A}_i[\rho_\varepsilon]\|_{L^\infty_x} \,\mathrm{d}t \\
&\le C\varepsilon \int_0^T \|\nabla u_\varepsilon\|_{L^2_x} \|\nabla \rho_\varepsilon\|_{L^2_x} \|\rho_\varepsilon\|_{L^\beta_x} \,\mathrm{d}t \\
&\le C\varepsilon \|\nabla u_\varepsilon\|_{L^2_t L^2_x} \|\nabla \rho_\varepsilon\|_{L^2_t L^2_x} \|\rho_\varepsilon\|_{L^\infty_t L^\beta_x} \le C\sqrt{\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

From (4.3.8)–(4.3.9) and (4.3.31)–(4.3.32), we have

$$\mathcal{I}_{11} = \sigma_* \int_0^T \psi \int_{\mathcal{O}} c_\varepsilon^2 Q_\varepsilon^{ij} \big(\partial_j \phi \mathcal{A}_i[\rho_\varepsilon] + \phi \partial_j \mathcal{A}_i[\rho_\varepsilon]\big) \,\mathrm{d}x\mathrm{d}t \to \mathcal{J}_9.$$

From (4.3.9), (4.3.32), and the boundedness of Q_ε in $L^2(0,T;H^2(\mathcal{O}))$, we have

$$\mathcal{I}_{12} = \int_0^T \psi \int_{\mathcal{O}} \big(\partial_i Q_\varepsilon^{kl} \partial_j Q_\varepsilon^{kl} - \mathrm{F}(Q_\varepsilon)\delta_{ij}\big) \partial_j \phi \mathcal{A}_i[\rho_\varepsilon] \,\mathrm{d}x\mathrm{d}t \to \mathcal{J}_{10},$$

$$\mathcal{I}_{13} = \int_0^T \psi \int_{\mathcal{O}} \big(\partial_i Q_\varepsilon^{kl} \partial_j Q_\varepsilon^{kl} - \mathrm{F}(Q_\varepsilon)\delta_{ij}\big) \phi \partial_j \mathcal{A}_i[\rho_\varepsilon] \,\mathrm{d}x\mathrm{d}t \to \mathcal{J}_{11},$$

and

$$\mathcal{I}_{14} = -\int_0^T \psi \int_{\mathcal{O}} \big(Q_\varepsilon^{ik} \Delta Q_\varepsilon^{kj} - \Delta Q_\varepsilon^{ik} Q_\varepsilon^{kj}\big) \big(\partial_j \phi \mathcal{A}_i[\rho_\varepsilon] + \phi \partial_j \mathcal{A}_i[\rho_\varepsilon]\big) \,\mathrm{d}x\mathrm{d}t \to \mathcal{J}_{12}.$$

□

Next, following the same argument as in Subsection 3.5 in [16], we obtain the strong convergence of the density sequence ρ_ε in $L^1(\mathcal{O}_T)$, *i.e.* $p = \rho^\gamma + \delta\rho^\beta$.

4.3.4 Strong convergence of density

In this subsection, we prove that

$$p = \rho^{\gamma} + \delta\rho^{\beta}, \tag{4.3.33}$$

which is the strong convergence of the sequence ρ_ε in $L^1(\mathcal{O}_T)$.

From (4.3.10) and (4.3.11), we know that the limit functions $\rho \in L^{\beta+1}(\mathcal{O}_T)$ $(\beta > 4)$, $u \in L^2(0,T;H_0^1)$. Moreover, since (ρ, u) satisfies (4.3.19) in distribution sense, by Lemma A.0.8, we know that (ρ, u) is a renormalized solution, *i.e.* (ρ, u) satisfies (A.0.3). We choose $g(z) = z\log(z)$ to get

$$\partial_t(\rho\log\rho) + \mathrm{div}(\rho\log\rho u) + \rho\mathrm{div}\, u = 0, \quad \text{in} \quad \mathcal{D}^{'}(\mathcal{O}_T). \tag{4.3.34}$$

Then by integrating (4.3.34), we obtain,

$$\int_0^T\int_{\mathcal{O}} \rho\mathrm{div}\, u\mathrm{d}x\mathrm{d}t = \int_{\mathcal{O}} \rho_0\log\rho_0\mathrm{d}x - \int_{\mathcal{O}} \rho(T)\log\rho(T)\mathrm{d}x. \tag{4.3.35}$$

Further, since ρ_ε solves (4.2.2) *a.e.* on $\mathcal{O}_T$, we multiply $g^{'}(\rho_\varepsilon)$ on both sides of (4.2.2) to get

$$\partial_t g(\rho_\varepsilon) + \mathrm{div}(g(\rho_\varepsilon)u_\varepsilon) + (g^{'}(\rho_\varepsilon)\rho_\varepsilon - g(\rho_\varepsilon))\mathrm{div}\, u_\varepsilon = \varepsilon\Delta g(\rho_\varepsilon) - \varepsilon g^{''}(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2.$$

From the convexity of $g(z) = z\log z$, we have

$$(\rho_\varepsilon\log\rho_\varepsilon)_t + \mathrm{div}(\rho_\varepsilon u_\varepsilon\log\rho_\varepsilon) + \rho_\varepsilon\mathrm{div}\, u_\varepsilon - \varepsilon\Delta g(\rho_\varepsilon) = -\varepsilon\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} \le 0,$$

which implies

$$\int_0^T\int_{\mathcal{O}} \rho_\varepsilon\mathrm{div}\, u_\varepsilon\mathrm{d}x\mathrm{d}t \le \int_{\mathcal{O}} \rho_0\log\rho_0\mathrm{d}x - \int_{\mathcal{O}} \rho_\varepsilon(T)\log\rho_\varepsilon(T)\mathrm{d}x. \tag{4.3.36}$$

Take two nondecreasing sequences $\psi_n(t) \in \mathcal{D}(0,T), \phi_n(x) \in \mathcal{D}(\mathcal{O})$ with

$$\begin{cases} \psi_n \to 1, \psi_n \ge 0, \quad \text{and} \quad \psi_n(t) = 1, \quad \text{for} \quad t \ge \frac{1}{n}, \quad \text{or} \quad t \le T - \frac{1}{n}; \\ \phi_n \to 1, \phi_n \ge 0, \quad \phi_n(x) = 1, \quad \text{for} \quad x \in \{y \in \mathcal{O} | \mathrm{dist}(x, \partial\mathcal{O}) \ge \frac{1}{n}\}. \end{cases}$$

From Lemma 4.3.2, along with (4.3.35) and (4.3.36), we get for all $n = 1, 2, \cdots$,

$$\limsup_{\varepsilon\to 0^+} \int_0^T \psi_n \int_{\mathcal{O}} \phi_n(\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\rho_\varepsilon \mathrm{d}x\mathrm{d}t \le \int_0^T \int_{\mathcal{O}} p\rho \mathrm{d}x\mathrm{d}t. \tag{4.3.37}$$

In order to get (4.3.33), we use Minty's trick and the fact that the function $p(z) = z^\gamma + \delta z^\beta$ is monotone to obtain the inequality

$$\int_0^T \psi_n \int_{\mathcal{O}} \phi_n(p(\rho_\varepsilon) - p(v))(\rho_\varepsilon - v)\mathrm{d}x\mathrm{d}t \ge 0.$$

Let us apply $\limsup_{\varepsilon\to 0}$ to both sides of the above inequality to obtain

$$\begin{aligned}
0 &\le \limsup_{\varepsilon\to 0} \int_0^T \psi_n \int_{\mathcal{O}} \phi_n(p(\rho_\varepsilon) - p(v))(\rho_\varepsilon - v)\mathrm{d}x\mathrm{d}t \\
&= \limsup_{\varepsilon\to 0} \left\{ \int_0^T \psi_n \int_{\mathcal{O}} \phi_n p(\rho_\varepsilon)\rho_\varepsilon \mathrm{d}x\mathrm{d}t - \int_0^T \psi_n \int_{\mathcal{O}} \phi_n(p(v)\rho_\varepsilon + p(\rho_\varepsilon)v - p(v)v)\mathrm{d}x\mathrm{d}t \right\} \\
&\le \int_0^T \psi_n \int_{\mathcal{O}} \phi_n p\rho \mathrm{d}x\mathrm{d}t - \int_0^T \psi_n \int_{\mathcal{O}} \phi_n(p(v)\rho + pv - p(v)v)\mathrm{d}x\mathrm{d}t \\
&\to \int_0^T \int_{\mathcal{O}} p\rho \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathcal{O}} (p(v)\rho + pv - p(v)v)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathcal{O}} (p - p(v))(\rho - v)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

with the fact that $\rho_\varepsilon \rightharpoonup \rho$, $p(\rho_\varepsilon) \rightharpoonup p$, and (4.3.37). Then take $v = \rho + \alpha\varphi$ with $\alpha \to 0$, and $\varphi \in \mathcal{D}(\mathcal{O})$, we have

$$p = \rho^\gamma + \delta\rho^\beta, \quad a.e. \ in \quad \mathcal{O}_T.$$

Now we have finished the proof of the vanishing artificial viscosity limit for our approximation system. Let us summarize the results in this section in the next proposition.

Proposition 4.3.1. *Let $\beta > \max\{\frac{6\gamma}{2\gamma-3}, \gamma, 4\}$. Then, for any given $T > 0$ and $\delta > 0$, there exists a finite-energy weak solution (c, ρ, u, Q) of the problem:*

$$\partial_t c + u \cdot \nabla c = D_0 \Delta c, \tag{4.3.38}$$

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{4.3.39}$$

$$\begin{aligned}
\partial_t(\rho u) + \nabla\cdot(\rho u \otimes u) + \nabla(\rho^\gamma + \delta\rho^\beta) &= \mu\Delta u + (\nu+\mu)\nabla \mathrm{div}\, u + \nabla\cdot(Q\Delta Q - \Delta Q Q) \\
&\quad + \nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q) + \sigma_* \nabla\cdot(c^2 Q),
\end{aligned} \tag{4.3.40}$$

$$\partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c], \tag{4.3.41}$$

with initial-boundary conditions (4.2.5)−(4.2.10)*. Moreover, equation* (4.3.39) *holds in the sense of renormalized solutions on* $\mathcal{D}'((0,T) \times \mathbb{R}^3)$*, provided that* (ρ, u) *are extended to be zero on* $\mathbb{R}^3 \setminus \mathcal{O}$*. In addition, the following estimates are valid:*

$$0 < \underline{c} \le c(x,t) \le \bar{c}, \quad \|c\|_{L^\infty(0,T;L^2(\mathcal{O})) \cap L^2(0,T;H^1(\mathcal{O}))} \le C, \tag{4.3.42}$$

$$\sup_{t \in [0,T]} \|\rho(t,\cdot)\|^\gamma_{L^\gamma(\mathcal{O})} \le C, \tag{4.3.43}$$

$$\delta \sup_{t \in [0,T]} \|\rho(t,\cdot)\|^\beta_{L^\beta(\mathcal{O})} \le C, \tag{4.3.44}$$

$$\sup_{t \in [0,T]} \|\sqrt{\rho}(t,\cdot) u(t,\cdot)\|^2_{L^2(\mathcal{O})} \le C, \tag{4.3.45}$$

$$\|u\|_{L^2(0,T;H^1_0(\mathcal{O}))} \le C, \tag{4.3.46}$$

$$\|Q\|_{L^\infty(0,T;H^1(\mathcal{O}) \cap L^4(\mathcal{O})) \cap L^2(0,T;H^2(\mathcal{O}) \cap L^6(\mathcal{O}))} \le C, \tag{4.3.47}$$

$$\|Q\|_{L^{10}(\mathcal{O}_T)} \le C, \tag{4.3.48}$$

$$\|\nabla Q\|_{L^{\frac{10}{3}}(\mathcal{O}_T)} \le C, \tag{4.3.49}$$

where C *is a constant independent of* ε*.*

Remark 4.3.3. The initial conditions (4.2.6)−(4.2.8) are satisfied in the weak sense, since

$$\rho_\varepsilon \to \rho \ \text{ in } C([0,T]; L^\beta_{\text{weak}}(\mathcal{O})), \qquad \rho_\varepsilon u_\varepsilon \to \rho u \ \text{ in } C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\mathcal{O}))$$

from (4.3.10)–(4.3.11).

4.4 PASSING TO THE LIMIT IN THE ARTIFICIAL PRESSURE

We denote by $(c_\delta, \rho_\delta, u_\delta, Q_\delta)$ the approximate solutions constructed in Proposition 4.3.1. In this section, we let $\delta \to 0$ in (4.3.38)$-$(4.3.41) to obtain the solution of the original problem (4.0.2)$-$(4.0.5).

In order for the solution (ρ_δ, u_δ) to satisfy the initial condition (4.2.6)–(4.2.7) in Proposition 4.3.1, we first modify the general initial data (ρ_0, m_0) to satisfy the compatibility condition (4.0.8). As in [17], it is easy to find a sequence $\tilde\rho_\delta \in C_0^3(\mathcal{O})$ with the property:

$$0 \le \tilde\rho_\delta(x) \le \frac{1}{2}\delta^{-\frac{1}{\beta}}, \qquad \|\tilde\rho_\delta - \rho_0\|_{L^2(\mathcal{O})} < \delta.$$

Take $\rho_{0,\delta} = \tilde\rho_\delta + \delta$. From (4.2.6)–(4.2.7), we have

$$0 < \delta \le \rho_{0,\delta}(x) \le \delta^{-\frac{1}{\beta}}, \qquad \nabla\rho_{0,\delta}\cdot\vec{n}|_{\partial\mathcal{O}} = 0, \tag{4.4.1}$$

with

$$\rho_{0,\delta} \to \rho_0 \qquad \text{in } L^\gamma(\mathcal{O}) \text{ as } \delta \to 0. \tag{4.4.2}$$

Set

$$\tilde q_\delta(x) = \begin{cases} m_0(x)\sqrt{\frac{\rho_{0,\delta}}{\rho_0}} & \text{if } \rho_0(x) > 0, \\ 0 & \text{if } \rho_0(x) = 0. \end{cases}$$

Then it follows from (4.0.8) that $\frac{|\tilde q_\delta|^2}{\rho_{0,\delta}}$ is uniformly bounded in $L^1(\mathcal{O})$. It is also direct to find $h_\delta \in C^2(\bar{\mathcal{O}})$ such that

$$\Big\|\frac{\tilde q_\delta}{\sqrt{\rho_{0,\delta}}} - h_\delta\Big\|_{L^2(\mathcal{O})} < \delta.$$

Taking $m_{0,\delta} = h_\delta\sqrt{\rho_{0,\delta}}$, we can check that

$$\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} \qquad \text{is uniformly bounded in } L^1(\mathcal{O}) \text{ with respect to } \delta > 0, \tag{4.4.3}$$

and

$$m_{0,\delta} \to m_0 \qquad \text{in } L^1(\mathcal{O}) \text{ as } \delta \to 0. \tag{4.4.4}$$

From now on, we deal with the sequence of approximate solutions $(c_\delta, \rho_\delta, u_\delta, Q_\delta)$ of problem (4.3.38)$-$(4.3.41) with the initial data $(c_0, \rho_{0,\delta}, m_{0,\delta}, Q_0)$. The existence of such a solution is provided by Proposition 4.3.1. We notice that all the corresponding estimates in Proposition 4.3.1 are independent of δ, by virtue of (4.4.1) and (4.4.3).

4.4.1 On the integrability of the density

Now we provide some pressure estimates independent of $\delta > 0$. From (4.3.43), $\rho_\delta \in L^\infty(0,T;L^\gamma(\Omega))$ uniformly in δ yields that $\rho_\delta^\gamma \rightharpoonup \mu$ by measure, as $\delta \to 0$. Hence, we need the higher integrability of ρ_δ. The technique is similar to that in §4.3.1. In addition, we need to make estimates carefully for the highly nonlinear terms like the last three terms in (4.4.7)–(4.4.8) by using the estimates of the solution obtained in Proposition 4.3.1.

Since the continuity equation (4.3.39) is satisfied in the sense of renormalized solutions in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, we can apply the standard mollifying operator on both sides of (4.3.39) to obtain

$$\partial_t S_m[g(\rho_\delta)] + \mathrm{div}(S_m[g(\rho_\delta)u_\delta]) + S_m[(g'(\rho_\delta)\rho_\delta - g(\rho_\delta))\mathrm{div}\, u_\delta] = r_m,$$

with

$$r_m \to 0 \qquad \text{in } L^2(0,T;L^2(\mathbb{R}^3)) \text{ as } m\to\infty.$$

As in §4.3.1, we use operator $\mathcal{B}$ to construct the test function as

$$\phi(t,x) = \psi(t)\mathcal{B}\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\,\mathrm{d}y\big], \tag{4.4.5}$$

with

$$\psi \in \mathcal{D}(0,T), \qquad \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\,\mathrm{d}y := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} S_m[g(\rho_\delta)]\,\mathrm{d}y.$$

Since $\phi|_{\partial\mathcal{O}} = 0$, $\phi \in L^\infty(0,T;H_0^1(\mathcal{O}))$, and $\partial_t\phi \in L^2(0,T;H_0^1(\mathcal{O}))$, we know that ϕ can be used as a test function for (4.3.40). We approximate the function: $g(z) = z^\theta, 0<\theta\le 1$, by a sequence of functions $\{z^\theta\chi_n(z)\}_{n=1}^\infty$, where $\{\chi_n(z)\}_{n=1}^\infty$ are cutoff functions with

$$\chi_n(z) = \begin{cases} 1 & \text{if } z\in[0,n], \\ 0 & \text{if } z > 2n. \end{cases}$$

Then, from (4.3.42)–(4.3.49), we employ the same techniques as in Lemma 4.3.1 to obtain

Lemma 4.4.1. *If $\gamma > \frac{3}{2}$, there exists a constant $\theta \in (0, \min\{\frac{1}{4}, \frac{2\gamma}{3} - 1\})$ depending only on γ, and $C > 0$ independent of δ such that*

$$\int_0^T \int_{\mathcal{O}} \big(\rho_\delta^{\gamma+\theta} + \delta\rho_\delta^{\beta+\theta}\big)\,\mathrm{d}x\mathrm{d}t \le C. \tag{4.4.6}$$

Proof. Applying the test function (4.4.5) with the choice of $\psi = \psi(t) \ge 0$ to (4.3.40), we have

$$\begin{aligned}
&\int_0^T \psi(t) \int_{\mathcal{O}} \big(\rho_\delta^\gamma + \delta\rho_\delta^\beta\big) S_m[g(\rho_\delta)]\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \psi(t) \int_{\mathcal{O}} \big(\rho_\delta^\gamma + \delta\rho_\delta^\beta\big)\mathrm{d}x \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\,\mathrm{d}y\mathrm{d}t + (\mu+\nu)\int_0^T \psi \int_{\mathcal{O}} \operatorname{div} u_\delta S_m[g(\rho_\delta)]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi_t \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \mu\int_0^T \psi \int_{\mathcal{O}} \partial_j u_\delta^i \partial_j B_i\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta^i u_\delta^j \partial_j B_i\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[r_m - \fint_{\mathcal{O}} r_m\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[S_m[(g'(\rho_\delta)\rho_\delta - g(\rho_\delta))\operatorname{div} u_\delta] \\
&\qquad\qquad\qquad - \fint_{\mathcal{O}} S_m[(g'(\rho_\delta)\rho_\delta - g(\rho_\delta))\operatorname{div} u_\delta]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[\operatorname{div}(S_m[g(\rho_\delta)]u_\delta)\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \big(\nabla Q_\delta \otimes \nabla Q_\delta - \mathrm{F}(Q_\delta)\mathrm{I}_3\big) : \nabla\mathcal{B}\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sigma_* \int_0^T \psi \int_{\mathcal{O}} c_\delta^2 Q_\delta : \nabla\mathcal{B}\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \nabla\cdot\big(Q_\delta\Delta Q_\delta - \Delta Q_\delta Q_\delta\big)\cdot\mathcal{B}\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&= \sum_{i=1}^{11} \mathcal{I}_i.
\end{aligned} \tag{4.4.7}$$

By the estimates in Proposition 4.3.1, we can pass to the limit in (4.4.7) as $m \to \infty$ to obtain

$$\int_0^T \psi(t) \int_{\mathcal{O}} \big(\rho_\delta^\gamma + \delta\rho_\delta^\beta\big) g(\rho_\delta)\,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^T \psi(t) \int_{\mathcal{O}} \big(\rho_\delta^\gamma + \delta\rho_\delta^\beta\big)\,\mathrm{d}x \fint_{\mathcal{O}} g(\rho_\delta)\,\mathrm{d}y\mathrm{d}t + (\mu+\nu)\int_0^T \psi \int_{\mathcal{O}} \mathrm{div}\, u_\delta\, g(\rho_\delta)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi_t \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \mu \int_0^T \psi \int_{\mathcal{O}} \partial_j u_\delta^i \partial_j B_i\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta^i u_\delta^j \partial_j B_i\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[(g'(\rho_\delta)\rho_\delta - g(\rho_\delta))\mathrm{div}\, u_\delta - \fint_{\mathcal{O}} (g'(\rho_\delta)\rho_\delta - g(\rho_\delta))\mathrm{div}\, u_\delta \mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[\mathrm{div}\,(g(\rho_\delta)u_\delta)\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \big(\nabla Q_\delta \otimes \nabla Q_\delta - \mathrm{F}(Q_\delta)\mathrm{I}_3\big) : \nabla\mathcal{B}\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sigma_* \int_0^T \psi \int_{\mathcal{O}} c_\delta^2 Q_\delta : \nabla\mathcal{B}\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \psi \int_{\mathcal{O}} \nabla\cdot\big(Q_\delta\Delta Q_\delta - \Delta Q_\delta Q_\delta\big)\cdot\mathcal{B}\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&= \sum_{j=1}^{10} \mathcal{J}_j.
\end{aligned} \tag{4.4.8}
$$

For example, we have the following estimates: Since $\rho_\delta \in L^{\beta+1}(\mathcal{O}_T)$ and $g'(z) = 0$ for sufficiently large z, we have

$$
\begin{aligned}
&\Big|\int_0^T \psi \int_{\mathcal{O}} \rho_\delta^\beta \big(S_m[g(\rho_\delta)] - g(\rho_\delta)\big)\,\mathrm{d}x\mathrm{d}t\Big| \\
&\le C\|\rho_\delta\|_{L_{t,x}^{\beta+1}}^\beta \|S_m[g(\rho_\delta)] - g(\rho_\delta)\|_{L_{t,x}^{\beta+1}} \\
&\le C(\delta)\|S_m[g(\rho_\delta)] - g(\rho_\delta)\|_{L_{t,x}^{\beta+1}} \to 0 \qquad \text{as } m\to\infty.
\end{aligned}
$$

By the property of $\mathcal{B}$ in (4.3.2), we have

$$
\begin{aligned}
\big\|\partial_j B_i\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\big\|_{L_t^\infty L_x^3} &\le C\|S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}x\|_{L_t^\infty L_x^3} \\
&\le C\|g(\rho_\delta)\|_{L_t^\infty L_x^3} \le C(\delta),
\end{aligned}
$$

where $C(\delta)$ is independent of m. This shows that

$$
\partial_j B_i\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big] \overset{*}{\rightharpoonup} G \qquad \text{weak-star in } L_t^\infty L_x^3.
$$

On the other hand, since

$$
\begin{aligned}
&\big\|\partial_j B_i\big[S_m[g(\rho_\delta)] - g(\rho_\delta) - \fint_{\mathcal{O}} (S_m[g(\rho_\delta)] - g(\rho_\delta))\,\mathrm{d}x\big]\big\|_{L^2_t L^2_x} \\
&\le C\|S_m[g(\rho_\delta)] - g(\rho_\delta) - \fint_{\mathcal{O}} \big(S_m[g(\rho_\delta)] - g(\rho_\delta)\big)\,\mathrm{d}x\|_{L^2_t L^2_x} \\
&\le C\|S_m[g(\rho_\delta)] - g(\rho_\delta)\|_{L^2_t L^2_x} \to 0,
\end{aligned}
$$

we have

$$
\partial_j B_i\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big] \overset{*}{\rightharpoonup} \partial_j B_i\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big] \quad \text{weak-star in } L^\infty_t L^3_x.
$$

Then

$$
\begin{aligned}
\mathcal{I}_5 &= \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u^i_\delta u^j_\delta \partial_j B_i\big[S_m[g(\rho_\delta)] - \fint_{\mathcal{O}} S_m[g(\rho_\delta)]\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \\
&\to \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u^i_\delta u^j_\delta \partial_j B_i\big[g(\rho_\delta) - \fint_{\mathcal{O}} g(\rho_\delta)\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t = \mathcal{J}_5.
\end{aligned}
$$

By (4.3.3), we have

$$
\begin{aligned}
&\mathcal{B}\big[\mathrm{div}\,(S_m[g(\rho_\delta)]u_\delta)\big] \to \mathcal{B}\big[\mathrm{div}\,(g(\rho_\delta)u_\delta)\big] && \text{strongly in } L^{\frac{3}{2}}_t L^2_x, \\
&\mathcal{B}\big[\mathrm{div}\,(S_m[g(\rho_\delta)]u_\delta)\big] \rightharpoonup G && \text{weakly in } L^2(Q_T).
\end{aligned}
$$

Thus, we obtain

$$
\mathcal{B}\big[\mathrm{div}\,(S_m[g(\rho_\delta)]u_\delta)\big] \rightharpoonup \mathcal{B}\big[\mathrm{div}\,(g(\rho_\delta)u_\delta)\big] \qquad \text{weakly in } L^2(Q_T).
$$

Moreover, since $\psi\rho_\delta u_\delta \in L^2(Q_T)$, we have

$$
\mathcal{I}_8 = \int_0^T \int_{\mathcal{O}} \psi\rho_\delta u_\delta \cdot \mathcal{B}\big[\mathrm{div}\,(S_m[g(\rho_\delta)]u_\delta)\big]\mathrm{d}x\mathrm{d}t \to \int_0^T \int_{\mathcal{O}} \psi\rho_\delta u_\delta \cdot \mathcal{B}\big[\mathrm{div}\,(g(\rho_\delta)u_\delta)\big]\mathrm{d}x\mathrm{d}t = \mathcal{J}_7
$$

as $m \to \infty$. Then, as we mentioned before, we can use a sequence of functions $\{z^\theta \chi_n(z)\}$ to approximate $g(z) = z^\theta$ to obtain

$$
\int_0^T \int_{\mathcal{O}} \psi\big(\rho_\delta^{\gamma+\theta} + \delta\rho_\delta^{\beta+\theta}\big)\,\mathrm{d}x\mathrm{d}t = \sum_{i=1}^{10} \mathcal{J}_i, \tag{4.4.9}
$$

where $g(z)$ is substituted by z^θ in every $\mathcal{J}_i$, $i = 1, 2, \cdots, 10$.

Next, we estimate the terms on the right-hand side of (4.4.9) to obtain the condition for θ, for which the universal constant C is independent of δ:

$$
\begin{aligned}
|\mathcal{J}_1| &= \Big| \int_0^T \psi \int_{\mathcal{O}} (\rho_\delta^\gamma + \delta \rho_\delta^\beta)\,\mathrm{d}x \fint_{\mathcal{O}} \rho_\delta^\theta \,\mathrm{d}x\mathrm{d}t \Big| \\
&\le C\|\psi\|_{L^\infty(0,T)} \big(\|\rho_\delta\|^\gamma_{L^\infty_t L^\gamma_x} + \delta \|\rho_\delta\|^\beta_{L^\infty_t L^\beta_x} \big) \|\rho_\delta\|^\theta_{L^\infty_t L^\theta_x} \le C\|\psi\|_{L^\infty(0,T)}. \\
|\mathcal{J}_2| &= (\mu+\nu) \Big| \int_0^T \psi \int_{\mathcal{O}} \rho_\delta^\theta \mathrm{div}\, u_\delta \,\mathrm{d}x\mathrm{d}t \Big| \le C\|\nabla u_\delta\|_{L^2(Q_T)} \|\rho_\delta^\theta\|_{L^\infty_t L^2_x} \\
&\le C\|\psi\|_{L^\infty(0,T)} \qquad \text{if } \theta \le \tfrac{\gamma}{2}. \\
|\mathcal{J}_3| &= \Big| \int_0^T \psi_t \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \Big| \\
&\le C \int_0^T |\psi_t| \|\sqrt{\rho_\delta}\|_{L^{2\gamma}_x} \|\sqrt{\rho_\delta} u_\delta\|_{L^2_x} \big\| \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big] \big\|_{L_x^{\frac{2\gamma}{\gamma-1}}} \mathrm{d}t \\
&\le C \int_0^T |\psi_t| \|\rho_\delta\|^{\frac12}_{L^\gamma_x} \|\sqrt{\rho_\delta} u_\delta\|_{L^2_x} \big\| \rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y \big\|_{L_x^{\frac{2\gamma}{\gamma-1}}} \mathrm{d}t \\
&\le CT \|\rho_\delta\|^{\frac12}_{L^\infty_t L^\gamma_x} \|\sqrt{\rho_\delta} u_\delta\|_{L^\infty_t L^2_x} \|\rho_\delta\|^\theta_{L^\infty_t L_x^{\frac{2\gamma}{\gamma-1}\theta}} \|\psi\|_{L^1(0,T)} \\
&\le C(T)\|\psi\|_{L^1(0,T)} \qquad \text{if } \theta \le \tfrac{\gamma-1}{2}. \\
|\mathcal{J}_4| &= \mu \Big| \int_0^T \psi \int_{\mathcal{O}} \partial_j u_\delta^i \partial_j B_i\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big]\mathrm{d}x\mathrm{d}t \Big| \\
&\le \mu \int_0^T |\psi| \|\nabla u_\delta\|_{L^2_x} \big\| \nabla \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big] \big\|_{L^2_x} \mathrm{d}t \\
&\le \mu \int_0^T |\psi| \|\nabla u_\delta\|_{L^2_x} \|\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\|_{L^2_x} \mathrm{d}t \le C\|\psi\|_{L^\infty(0,T)} \qquad \text{if } \theta \le \tfrac{\gamma}{2}.
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{J}_5| &= \Big| \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta^i u_\delta^j \partial_j B_i\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big]\mathrm{d}x\mathrm{d}t \Big| \\
&\le C \int_0^T |\psi| \|\rho_\delta\|_{L^\gamma_x} \|u_\delta\|^2_{L^6_x} \big\| \nabla \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big] \big\|_{L_x^{\frac{3\gamma}{2\gamma-3}}} \mathrm{d}t \\
&\le C \int_0^T |\psi| \|\rho_\delta\|_{L^\gamma_x} \|u_\delta\|^2_{L^6_x} \|\rho_\delta^\theta\|_{L_x^{\frac{3\gamma}{2\gamma-3}}} \mathrm{d}t \le C\|\psi\|_{L^\infty(0,T)} \qquad \text{if } \theta \le \tfrac{2\gamma}{3} - 1.
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{J}_6| &= (1-\theta) \Big| \int_0^T \psi \int_{\mathcal{O}} \rho_\delta u_\delta \cdot \mathcal{B}\big[\rho_\delta^\theta \mathrm{div}\, u_\delta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{div}\, u_\delta \mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t \Big| \\
&\le C \int_0^T |\psi| \|\rho_\delta\|_{L^\gamma_x} \|u_\delta\|_{L^6_x} \big\| \mathcal{B}\big[\rho_\delta^\theta \mathrm{div}\, u_\delta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{div}\, u_\delta \mathrm{d}y\big] \big\|_{L_x^{\frac{6\gamma}{5\gamma-6}}} \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_0^T|\psi|\|u_\delta\|_{L^6_x}\big\|\mathcal{B}\big[\rho_\delta^\theta \mathrm{div}\, u_\delta-\fint_{\mathcal{O}}\rho_\delta^\theta \mathrm{div}\, u_\delta \mathrm{d}y\big]\big\|_{W^{1,p}_x}\mathrm{d}t\\
&\le C\int_0^T|\psi|\|u_\delta\|_{L^6_x}\|\rho_\delta^\theta \mathrm{div}\, u_\delta\|_{L^p_x}\mathrm{d}t\\
&\le C\int_0^T|\psi|\|u_\delta\|_{L^6_x}\|\nabla u_\delta\|_{L^2_x}\|\rho_\delta^\theta\|_{L^q_x}\mathrm{d}t\\
&\le C\int_0^T|\psi|\|u_\delta\|^2_{H^1_x}\|\rho_\delta\|_{L^\gamma_x}\mathrm{d}t\le C\|\psi\|_{L^\infty(0,T)},
\end{aligned}
$$

where $p=\frac{6\gamma}{7\gamma-6}$ if $\gamma<6$, $p=\frac{3}{2}$ if $\gamma\ge 6$; $q=\frac{3\gamma}{2\gamma-3}$ if $\gamma<6$, $p=6$ if $\gamma\ge 6$; $\theta\le\frac{2}{3}\gamma-1$ and $\theta\le 1$.

$$
\begin{aligned}
|\mathcal{J}_7|&=\Big|\int_0^T\psi\int_{\mathcal{O}}\rho_\delta u_\delta\cdot\mathcal{B}\big[\mathrm{div}(\rho_\delta^\theta u_\delta)\big]\,\mathrm{d}x\mathrm{d}t\Big|\le\int_0^T|\psi|\|\rho_\delta\|_{L^\gamma_x}\|u_\delta\|_{L^6_x}\big\|\mathcal{B}[\mathrm{div}(\rho_\delta^\theta u_\delta)]\big\|_{L_x^{\frac{6\gamma}{5\gamma-6}}}\mathrm{d}t\\
&\le\int_0^T|\psi|\|u_\delta\|_{L^6_x}\|\rho_\delta^\theta u_\delta\|_{L_x^{\frac{6\gamma}{5\gamma-6}}}\mathrm{d}t\le\int_0^T|\psi|\|u_\delta\|^2_{L^6_x}\|\rho_\delta^\theta\|_{L_x^{\frac{3\gamma}{2\gamma-3}}}\mathrm{d}t\\
&\le C\|u_\delta\|^2_{L^2_tH^1_x}\|\rho_\delta\|^\theta_{L^\infty_tL^\gamma_x}\|\psi\|_{L^\infty(0,T)}\\
&\le C\|\psi\|_{L^\infty(0,T)}\qquad\text{if }\theta\le\tfrac{2\gamma-3}{3}.
\end{aligned}
$$

Similarly to (4.3.6), we have

$$
\begin{aligned}
&\big\|B[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\,\mathrm{d}y]\big\|_{L^\infty_x}\\
&\le C_1\big\|\nabla B\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\,\mathrm{d}y]\big\|^{\frac{3}{\gamma}}_{L^\gamma_x}\big\|B\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\,\mathrm{d}y]\big\|^{1-\frac{3}{\gamma}}_{L^\gamma_x}+C_2\big\|B\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\,\mathrm{d}y\big]\big\|_{L^\gamma_x}\\
&\le C\|\rho_\delta^\theta\|_{L^\gamma_x}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|\mathcal{J}_8|&=\Big|\int_0^T\psi\int_{\mathcal{O}}\big(\nabla Q_\delta\otimes\nabla Q_\delta-\mathrm{F}(Q_\delta)\mathrm{I}_3\big):\nabla\mathcal{B}\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y\big]\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le C\int_0^T|\psi|\Big(\|\nabla Q_\delta\|^2_{L_x^{\frac{10}{3}}}\big\|\nabla\mathcal{B}\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y\big]\big\|_{L_x^{\frac{5}{2}}}+\Big|\int_{\mathcal{O}}\mathrm{F}(Q_\delta)\mathrm{div}\mathcal{B}\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y\big]\mathrm{d}x\Big|\Big)\mathrm{d}t\\
&\le C\int_0^T|\psi|\Big(\|\nabla Q_\delta\|^2_{L_x^{\frac{10}{3}}}\big\|\nabla\mathcal{B}\big[\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y\big]\big\|_{L_x^{\frac{5}{2}}}+\Big|\int_{\mathcal{O}}\mathrm{F}(Q_\delta)(\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y)\mathrm{d}x\Big|\Big)\mathrm{d}t\\
&\le C\int_0^T|\psi|\Big(\|\nabla Q_\delta\|^2_{L_x^{\frac{10}{3}}}\big\|\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y\big\|_{L_x^{\frac{5}{2}}}+\big(\|Q_\delta\|^2_{L^5_x}+\|Q_\delta\|^4_{L^{10}_x}\big)\big\|\rho_\delta^\theta-\fint_{\mathcal{O}}\rho_\delta^\theta\mathrm{d}y\big\|_{L_x^{\frac{5}{3}}}\Big)\mathrm{d}t\\
&\le C\|\psi\|_{L^\infty(0,T)}\qquad\text{if }\theta\le\tfrac{2}{5}.
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_9| &= \sigma_* \Big| \int_0^T \psi \int_{\mathcal{O}} c_\delta^2 Q_\delta \cdot \nabla \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big] \mathrm{d}x\mathrm{d}t \Big| \\
&\le C \|c_\delta\|_{L^\infty_{t,x}}^2 \int_0^T |\psi| \|Q_\delta\|_{L^2_x} \big\| \nabla \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \mathrm{d}y\big] \big\|_{L^2_x} \mathrm{d}t \\
&\le C \int_0^T |\psi| \|Q_\delta\|_{L^2_x} \|\rho_\delta^\theta\|_{L^2_x} \mathrm{d}t \le C \|\psi\|_{L^\infty(0,T)} \qquad \text{if } \theta \le \tfrac{\gamma}{2}.
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_{10}| &= \Big| \int_0^T \psi \int_{\mathcal{O}} \nabla \cdot \big(Q_\delta \Delta Q_\delta - \Delta Q_\delta Q_\delta\big) \cdot \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \, \mathrm{d}y\big] \, \mathrm{d}x\mathrm{d}t \Big| \\
&\le \int_0^T |\psi| \|Q_\delta\|_{L^4_x} \|\Delta Q_\delta\|_{L^2_x} \big\| \nabla \mathcal{B}\big[\rho_\delta^\theta - \fint_{\mathcal{O}} \rho_\delta^\theta \, \mathrm{d}y\big] \big\|_{L^4_x} \mathrm{d}t \\
&\le \int_0^T |\psi| \|Q_\delta\|_{L^4_x} \|\Delta Q_\delta\|_{L^2_x} \|\rho_\delta^\theta\|_{L^4_x} \mathrm{d}t \le C \|\psi\|_{L^\infty(0,T)} \qquad \text{if } \theta \le \tfrac{1}{4}.
\end{aligned}
$$

Combining the above estimates with (4.4.9), we have

$$
\int_0^T \int_{\mathcal{O}} \psi \big(\rho_\delta^{\gamma+\theta} + \delta \rho_\delta^{\beta+\theta}\big) \, \mathrm{d}x\mathrm{d}t \le C \big(\|\psi\|_{L^\infty(0,T)} + \|\psi_t\|_{L^1(0,T)} \big). \tag{4.4.10}
$$

Then we can again take $\psi = \psi^k \in \mathcal{D}(0,T)$ in (4.4.10), with $\|\psi^k\|_{L^\infty(0,T)} + \|\psi_t^k\|_{L^1(0,T)} \le C$ uniformly bounded independent of k, and $\phi^k(t) \to 1$ for $t \in (0,T)$ as $k \to \infty$, and then let $k \to \infty$ in the resulting (4.4.10) to conclude our desired result. □

4.4.2 The limit passage and the effective viscous flux

We infer from the uniform estimates (4.3.42)−(4.3.49) in Proposition 4.3.1 and Lemma 4.4.1 that, as $\delta \to 0$,

$$
\begin{aligned}
&c_\delta \rightharpoonup c && \text{in } L^2(0,T;H^1(\mathcal{O})),\\
&\rho_\delta \to \rho && \text{in } C([0,T];L^\gamma_{\text{weak}}(\mathcal{O})),\\
&u_\delta \rightharpoonup u && \text{in } L^2(0,T;H^1_0(\mathcal{O})),\\
&\rho_\delta u_\delta \to \rho u && \text{in } C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\mathcal{O})),\\
&Q_\delta \rightharpoonup Q && \text{in } L^2(0,T;H^2(\mathcal{O})),\\
&Q_\delta \to Q && \text{in } L^2(0,T;H^1(\mathcal{O})),\\
&\rho_\delta^\gamma \to \overline{\rho^\gamma} && \text{in } L^{\frac{\gamma+\theta}{\gamma}}(\mathcal{O}_T).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\rho_\delta u_\delta \otimes u_\delta \to \rho u \otimes u \qquad \text{in } \mathcal{D}'(\mathcal{O}_T),\\
&\mathrm{F}(Q_\delta)\mathrm{I}_3 - \nabla Q_\delta \odot \nabla Q_\delta + (Q_\delta \Delta Q_\delta - \Delta Q_\delta Q_\delta) + \sigma_* c_\delta^2 Q_\delta\\
&\qquad \to \mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q + (Q\Delta Q - \Delta Q Q) + \sigma_* c^2 Q,\\
&\delta \rho_\delta^\beta \to 0 \qquad \text{in } L^1(\mathcal{O}_T).
\end{aligned}
$$

Then the limit functions (c,ρ,u,Q) satisfy

$$c_t + u\cdot\nabla c = D_0 \Delta c, \tag{4.4.11}$$

$$\partial_t \rho + \nabla\cdot(\rho u) = 0, \tag{4.4.12}$$

$$(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla\overline{\rho^\gamma} = \mu\Delta u + (\nu+\mu)\nabla \mathrm{div}\, u + \nabla\cdot(Q\Delta Q - \Delta Q Q) \tag{4.4.13}$$

$$+\nabla\cdot\big(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q\big) + \sigma_*\nabla\cdot(c^2 Q), \tag{4.4.14}$$

$$\partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q = \Gamma H[Q,c] \tag{4.4.15}$$

in $\mathcal{D}'(\mathcal{O}_T)$, with the initial-boundary conditions (4.0.6)−(4.0.8), due to (4.4.2) and (4.4.4).

Next, in order to prove that (c,ρ,u,Q) is a weak solution of (4.0.2)−(4.0.8), we only need to show that $\overline{\rho^\gamma} = \rho^\gamma$ *a.e.* in $\mathcal{O}_T$, or equivalently, the strong convergence of ρ_δ in

$L^1(\mathcal{O})$. Moreover, we need to show that (ρ, u) is a renormalized solution of (4.4.12). From Lemma 4.4.1, the best estimate is $\rho \in L^{\gamma+\frac{2}{3}\gamma-1}(\mathcal{O}_T)$. Then $\gamma > \frac{3}{2}$ is not enough to guarantee that ρ is square integrable, and that (ρ, u) is a renormalized solution by Lemma A.0.8. In order to deal with this difficulty, we introduce the cut-off function $T_k(z) = kT(\frac{z}{k})$ for $z \in \mathbb{R}$, $k = 1, 2, \cdots$, where T is a smooth and concave function satisfying

$$T(z) = \begin{cases} z & \text{if } z \leq 1, \\ 2 & \text{if } z \geq 3. \end{cases}$$

Since (ρ_δ, u_δ) is a renormalized solution of (4.3.39), taking $g(z) = T_k(z)$, we obtain

$$\partial_t\big(T_k(\rho_\delta)\big) + \operatorname{div}\big(T_k(\rho_\delta)u_\delta\big) + \big(T_k'(\rho_\delta) - T_k(\rho_\delta)\big)\operatorname{div} u_\delta = 0 \qquad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3), \quad (4.4.16)$$

where $T_k(\rho_\delta) \in L^\infty(\mathcal{O}_T)$ for any fixed k. Then

$$T_k(\rho_\delta) \overset{*}{\rightharpoonup} \overline{T_k(\rho)} \qquad \text{in } L^\infty(\mathcal{O}_T).$$

Moreover, since $\partial_t T_k(\rho_\delta)$ satisfies (4.4.16), as before, we have

$$T_k(\rho_\delta) \to \overline{T_k(\rho)} \qquad \text{in } C([0,T]; L^p_{\text{weak}}(\mathcal{O})), \quad \forall\, 1 \leq p < \infty.$$

Letting $\delta \to 0$, it yields

$$\partial_t \overline{T_k(\rho)} + \operatorname{div}\big(\overline{T_k(\rho)}u\big) + \overline{(T_k'(\rho) - T_k(\rho))\operatorname{div} u} = 0 \qquad \text{in } \mathcal{D}'(\mathcal{O}_T),$$

where

$$\big(T_k'(\rho_\delta) - T_k(\rho_\delta)\big)\operatorname{div} u_\delta \rightharpoonup \overline{\big(T_k'(\rho) - T_k(\rho)\big)\operatorname{div} u} \qquad \text{in } L^2(\mathcal{O}_T).$$

4.4.3 The effective viscous flux

Similarly to Lemma 4.3.2, we define the effective viscous flux as $\tilde{\mathfrak{E}}_\delta := \rho_\delta^\gamma - (\nu + 2\mu)\mathrm{div}\, u_\delta$, and its correspondingly weak convergence limit $\tilde{\mathfrak{E}} := \overline{\rho^\gamma} - (\nu + 2\mu)\mathrm{div}\, u$. Then we have

Lemma 4.4.2. *Assume (ρ_δ, u_δ) is a family of the approximate solutions constructed in Proposition 4.3.1. Then*

$$\lim_{\delta\to 0^+} \int_0^T \psi\Big(\int_{\mathcal{O}} \phi \tilde{\mathfrak{E}}_\delta T_k(\rho_\delta)\,\mathrm{d}x\Big)\mathrm{d}t = \int_0^T \psi\Big(\int_{\mathcal{O}} \phi \tilde{\mathfrak{E}} \overline{T_k(\rho)}\,\mathrm{d}x\Big)\mathrm{d}t$$

for any $\psi \in \mathcal{D}(0,T)$ and $\phi \in \mathcal{D}(\mathcal{O})$.

4.4.4 Renormalized solutions

The following lemma implies that $T_k(\rho) - \overline{T_k(\rho)} \in L^2(\mathcal{O}_T)$, which helps us establish that the limit function (ρ, u) is a renormalized solution.

Lemma 4.4.3 (The amplitude of oscillations)**.** *There exists a constant C, independent of k, such that*

$$\limsup_{\delta\to 0} \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\gamma+1}(\mathcal{O}_T)} \le C. \tag{4.4.17}$$

The proof of this lemma is the same as that for Lemma 4.3 of Subsection 4.4 in [17].

Remark 4.4.1. By the concavity of the norm, we know from Lemma 4.4.3 that

$$\|\overline{T_k(\rho)} - T_k(\rho)\|_{L^{\gamma+1}(\mathcal{O}_T)} \le C. \tag{4.4.18}$$

From the proof of Lemma 4.4.3, we have

$$0 \le \limsup_{\delta\to 0} \|T_k(\rho_\delta) - T_k(\rho)\|^{\gamma+1}_{L^{\gamma+1}(\mathcal{O}_T)} \le \lim_{\delta\to 0} \int_0^T \int_{\mathcal{O}_T} \big(\rho_\delta^\gamma T_k(\rho_\delta) - \overline{\rho^\gamma}\,\overline{T_k(\rho)}\big)\mathrm{d}x\mathrm{d}t. \tag{4.4.19}$$

Based on the uniform estimate of the amplitude of oscillations in Lemma 4.4.3, we see that the limit function (ρ, u) satisfies (4.4.12) in the renormalized sense.

Lemma 4.4.4. *The limit function (ρ, u) is a renormalized solution to (4.4.12); that is,*

$$\partial_t g(\rho) + \mathrm{div}(g(\rho)u) + \big(g'(\rho)\rho - g(\rho)\big)\mathrm{div}\, u = 0 \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{4.4.20}$$

for any $g \in C^1(\mathbb{R})$ with the property that $g'(z) \equiv 0$ when $z \ge M$ for sufficiently large constant M, provided (ρ, u) are prolonged zero outside $\mathcal{O}$.

The detailed proof can be found in Subsection 4.5 of [17] for Lemma 4.4 there.

4.4.5 Strong convergence of density ρ_δ

We now give an outline of the proof for the strong convergence of density ρ_δ, *i.e.*, $\overline{\rho^\gamma} = \rho^\gamma$, where $\rho_\delta^\gamma \to \overline{\rho^\gamma}$ in $L^{\frac{\gamma+\theta}{\gamma}}(\mathcal{O}_T)$ for the completeness of the proof of Theorem 4.0.1.

Introduce a family of function in $C^1(\mathbb{R}^+) \cap C[0,\infty)$:

$$L_k(z) = \begin{cases} z \log z & \text{for } 0 \leq z < k, \\ z \log k + z \int_k^z \frac{T_k(s)}{s^2} \mathrm{d}s & \text{for } z \geq k. \end{cases}$$

By the construction of L_k, we know that L_k is a linear function for large z. In particular, we see that, for $z \geq 3k$,

$$L_k(z) = \beta_k z - 2k$$

with

$$\beta_k = \log k + \int_k^{3k} \frac{T_k(s)}{s^2} \mathrm{d}s + \frac{2}{3}.$$

Then, if $g_k(z) := L_k(z) - \beta_k z$, we obtain that $g_k(z) \in C^1(\mathbb{R}^+) \cap C[0,\infty)$, $g_k'(z) = 0$ for z is sufficiently large, and

$$g_k'(z) z - g_k(z) = T_k(z).$$

By Proposition 4.3.1 and Lemma 4.4.4, we know that (ρ_δ, u_δ) and (ρ, u) are renormalized solutions to (4.4.12). Then we substitute function g by g_k in the definition of renormalized solutions and take the difference of these two equations to obtain

$$\partial_t \big(L_k(\rho_\delta) - L_k(\rho) \big) + \mathrm{div} \big(L_k(\rho_\delta) u_\delta - L_k(\rho) u \big) + T_k(\rho_\delta) \mathrm{div} u_\delta - T_k(\rho) \mathrm{div}\, u = 0 \tag{4.4.21}$$

in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.

Since L_k is linear when z is large, $L_k(\rho_\delta)$ is uniformly bounded with respect to δ in $L^\infty(0,T; L^\gamma(\mathcal{O}))$ so that

$$L_k(\rho_\delta) \overset{*}{\rightharpoonup} \overline{L_k(\rho)} \qquad \text{in } L_t^\infty L_x^\gamma \text{ as } \delta \to 0.$$

Moreover, since $L_k(\rho_\delta)$ is a renormalized solution, similarly as before, we have

$$L_k(\rho_\delta) \to \overline{L_k(\rho)} \qquad \text{in } C([0,T], L^\gamma_{\text{weak}}(\mathcal{O})) \cap C([0,T]; H^{-1}(\mathcal{O})) \text{ as } \delta \to 0. \tag{4.4.22}$$

Now, using function $\phi(x) \in \mathcal{D}(\mathcal{O})$ to test (4.4.21) and then integrate over $(0,t)$, we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \big(L_k(\rho_\delta) - L_k(\rho)\big)(t,x)\phi(x)\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \big(L_k(\rho_{0,\delta}) - L_k(\rho_0)\big)\phi\,\mathrm{d}x + \int_0^t\int_{\mathcal{O}} \big(L_k(\rho_\delta)u_\delta - L_k(\rho)u\big)\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t\int_{\mathcal{O}} \big(T_k(\rho_\delta)\mathrm{div}u_\delta - T_k(\rho)\mathrm{div}\,u\big)\phi\,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Sending $\delta \to 0$, we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \big(\overline{L_k(\rho)} - L_k(\rho)\big)(t,x)\phi(x)\,\mathrm{d}x \\
&= \int_0^t\int_{\mathcal{O}} \big(\overline{L_k(\rho)} - L_k(\rho)\big)u\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}s - \lim_{\delta\to 0}\int_0^t\int_{\mathcal{O}} \big(T_k(\rho_\delta)\mathrm{div}u_\delta - T_k(\rho)\mathrm{div}\,u\big)\phi\,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Taking $\phi = \phi_m$ with $\phi_m(z) \to 1_{\mathcal{O}}(z)$ in the above equation and sending $m \to \infty$, we have

$$
\int_{\mathcal{O}} \big(\overline{L_k(\rho)} - L_k(\rho)\big)(t)\,\mathrm{d}x = \int_0^t\int_{\mathcal{O}} T_k(\rho)\,\mathrm{div}\,u\,\mathrm{d}x\mathrm{d}s - \lim_{\delta\to 0}\int_0^t\int_{\mathcal{O}} T_k(\rho_\delta)\mathrm{div}u_\delta\,\mathrm{d}x\mathrm{d}s. \tag{4.4.23}
$$

Then, using Lemmas 4.4.2–4.4.3, we find that the right-hand side of the above equation is non-positive, which yields

$$
\lim_{k\to\infty}\int_{\mathcal{O}} \big(\overline{L_k(\rho)} - L_k(\rho)\big)(t)\,\mathrm{d}x \le 0 \qquad \text{for } t \in [0,T]. \tag{4.4.24}
$$

Moreover, by the definition of L_k and the absolution continuity of $\rho\log\rho \in L^1(\mathcal{O}_T)$, we have

$$
\begin{aligned}
\|L_k(\rho) - \rho\log\rho\|_{L^1(\mathcal{O}_T)} &\le \int\int_{\{\rho\ge k\}} |L_k(\rho) - \rho\log\rho|\,\mathrm{d}x\mathrm{d}t \\
&\le C\int\int_{\{\rho\ge k\}} |\rho\log\rho|\,\mathrm{d}x\mathrm{d}t \to 0 \qquad \text{as } k\to\infty.
\end{aligned} \tag{4.4.25}
$$

Similarly, we have

$$
\begin{aligned}
\|L_k(\rho_\delta) - \rho_\delta\log\rho_\delta\|_{L^1(\mathcal{O}_T)} &\le \int\int_{\{\rho_\delta\ge k\}} |L_k(\rho_\delta) - \rho_\delta\log\rho_\delta|\,\mathrm{d}x\mathrm{d}t \\
&\le \int\int_{\{\rho_\delta\ge k\}} \frac{\log k + \int_k^{\rho_\delta}\frac{T_k(s)}{s^2}\mathrm{d}s + \log\rho_\delta}{\rho_\delta^{\gamma-1}}\rho_\delta^\gamma\,\mathrm{d}x\mathrm{d}t \\
&\le C(\varepsilon)k^{1+\varepsilon-\gamma}\int\int_{\{\rho_\delta\ge k\}} \rho_\delta^\gamma\,\mathrm{d}x\mathrm{d}t \to 0 \qquad \text{as } k\to\infty.
\end{aligned}
$$

This, together with the lower-semicontinuity of the norm, we have

$$\|\overline{L_k(\rho)} - \overline{\rho\log\rho}\|_{L^1(\mathcal{O}_T)} \le \liminf_{\delta\to 0} \|L_k(\rho_\delta) - \rho_\delta \log\rho_\delta\|_{L^1(\mathcal{O}_T)} \to 0, \quad \text{as } k\to\infty. \tag{4.4.26}$$

Finally, combining (4.4.24)−(4.4.26), we have

$$\int_{\mathcal{O}} \big(\overline{\rho\log\rho} - \rho\log\rho\big)(t)\mathrm{d}x \le 0.$$

Moreover, since $\overline{\rho\log\rho} \ge \rho\log\rho$, we see that $\overline{\rho\log\rho} = \rho\log\rho$ for *a.e.* $(t,x)\in\mathcal{O}_T$. In addition, by the restrict concavity of function $z\log z$, we have

$$\rho_\delta \to \rho \qquad \text{in } L^p(\mathcal{O}_T) \text{ for any } p\in[1,\gamma+\theta).$$

Then we conclude

$$\overline{\rho^\gamma} = \rho^\gamma, \quad a.e.$$

Therefore, we complete the proof of Theorem 4.0.1.

APPENDIX A

SOME BASIC THEORIES AND LEMMAS

In this appendix, we review some important theories and lemmas that are used extensively in this thesis.

First, let us introduce the Littlewood-Paley theory; see [2] for details.

Proposition A.0.1 (Dyadic Partition of Unity)**.** *Let $\mathcal{C}$ be annulus $\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exist radial functions χ and φ, valued in interval $[0,1]$, belonging to $\mathcal{D}(B(0,\frac{4}{3}))$ and $\mathcal{D}(\mathcal{C})$, respectively, such that*

$$\begin{aligned}
&\chi(\xi) + \sum_{j\geq 1} \varphi(2^{-j}\xi) = 1 \qquad \textit{for any } \xi \in \mathbb{R}^d, \\
&\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \qquad \textit{for any } \xi \in \mathbb{R}^d \setminus \{0\}, \\
&|j - j^{'}| \geq 2 \implies \mathrm{supp}(\varphi(2^{-j}\cdot)) \cap \mathrm{supp}(\varphi(2^{-j^{'}}\cdot)) = \emptyset, \\
&j \geq 1 \implies \mathrm{supp}(\chi) \cap \mathrm{supp}(\varphi(2^{-j}\cdot)) = \emptyset.
\end{aligned}$$

Definition A.0.1. The homogeneous dyadic blocks Δ_j and the homogeneous low-frequency cut-off operators S_j are defined for all $j \in \mathbb{Z}$ by

$$\begin{aligned}
\Delta_j u &= \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\mathcal{F}u) = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy, \\
S_j u &= \mathcal{F}^{-1}(\chi(2^{-j}\xi)\mathcal{F}u) = 2^{jd}\int_{\mathbb{R}^d} \tilde{h}(2^j y)u(x-y)dy,
\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^d$, and

$$h := \mathcal{F}^{-1}\varphi, \quad \tilde{h} := \mathcal{F}^{-1}\chi.$$

We define the Sobolev norm of space H^s as

$$\|u\|_{H^s} := \big(\|S_0 u\|_{L^2}^2 + \sum_{j\in\mathbb{N}} 2^{2qs}\|\Delta_j u\|_{L^2}^2\big)^{\frac{1}{2}}.$$

Next, we recall the Bony's paraproduct decomposition for two appropriately smooth functions u and v:

$$uv = T_u v + T_v u + R(u,v) = T_u v + T_v^{'} u,$$

where

$$\begin{aligned} &T_u v := \sum_j S_{j-1}u\Delta_j v, \quad R(u,v) := \sum_{|j-j'|\le 1} \Delta_j u \Delta_{j'} v,\\ &T_v^{'} u = T_v u + R(u,v) = \sum_j S_{j+2} v\Delta_j u. \end{aligned}$$

Then we have

$$\begin{aligned} \Delta_j(uv) &= \Delta_j T_u v + \Delta_j T_v^{'} u\\ &= \Delta_j \sum_{j'} S_{j'-1}u\Delta_{j'}v + \Delta_j \sum_{j'} S_{j'+2}v\Delta_{j'}u\\ &= \sum_{|j-j'|\le 5} \Delta_j(S_{j'-1}u\Delta_{j'}v) + \sum_{j'>j-5} \Delta_j(S_{j'+2}v\Delta_{j'}u)\\ &= S_{j-1}u\Delta_j v + \sum_{|j-j'|\le 5} [\Delta_j, S_{j'-1}u]\Delta_{j'}v\\ &\quad + \sum_{|j-j'|\le 5} (S_{j'-1}u - S_{j-1}u)\Delta_j\Delta_{j'}v + \sum_{j'>j-5} \Delta_j(S_{j'+2}v\Delta_{j'}u). \end{aligned}$$

In order to obtain the uniqueness and higher regularity of the weak-strong solutions in §2.3–§2.4, we now recall the following useful inequalities that we use extensively. These inequalities follows from the Bernstein-type lemma and the commutator estimates in Chapter 2 of [2] by the construction of Δ_j.

Bernstein-type inequalities:

$$\|\nabla S_j u\|_{L^p} \leq C2^j \|u\|_{L^p} \qquad \text{for any } 1 \leq p \leq \infty,$$
$$C2^j \|\Delta_j u\|_{L^p} \leq \|\nabla \Delta_j u\|_{L^p} \leq C2^j \|\Delta_j u\|_{L^p} \qquad \text{for any } 1 \leq p \leq \infty,$$
$$\|\Delta_j u\|_{L^q} \leq C2^{d(\frac{1}{p}-\frac{1}{q})j} \|\Delta_j u\|_{L^p} \qquad \text{with } \ 1 \leq p \leq q,$$
$$\|S_j u\|_{L^q} \leq C2^{d(\frac{1}{p}-\frac{1}{q})j} \|S_j u\|_{L^p} \qquad \text{with } \ 1 \leq p \leq q.$$

Commutator estimates:

$$\|[\Delta_j, u]v\|_{L^r} \leq C2^{-j} \|\nabla u\|_{L^p} \|v\|_{L^q} \qquad \text{with } \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

where C is independent of p, q, and r.

In order to deal with the high nonlinear interaction terms in the system, we need to introduce the following important cancellation rules:

Lemma A.0.5. *Let Q and Q' be two $d \times d$ symmetric matrices, and let $\Omega = \frac{1}{2}(\nabla u - \nabla u^\top)$ be the vorticity with $(\nabla u)_{\alpha\beta} = \partial_\beta u_\alpha$. Then*

$$(\Omega Q' - Q'\Omega, \Delta Q) - (\nabla \cdot (Q'\Delta Q - \Delta Q Q'), u) = 0.$$

Proof. For any two $d \times d$ matrices A and B, we know that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Then we have

$$\begin{aligned}
&(\Omega Q' - Q'\Omega, \Delta Q) - (\nabla \cdot (Q'\Delta Q - \Delta Q Q'), u) \\
&= (\Omega Q' - Q'\Omega, \Delta Q) + (Q'\Delta Q - \Delta Q Q', \nabla u^\top) \\
&= (\Omega Q', \Delta Q) - (Q'\Omega, \Delta Q) + (Q'\Delta Q - \Delta Q Q', \nabla u^\top) \\
&= (Q'\Delta Q, \Omega) - (\Delta Q Q', \Omega) + (Q'\Delta Q - \Delta Q Q', \nabla u^\top) \\
&= (Q'\Delta Q - \Delta Q Q', \Omega + \nabla u^\top) \\
&= (Q'\Delta Q - \Delta Q Q', D) = 0,
\end{aligned}$$

where, in the last equality, we use the fact that Q, Q', and D are symmetric. □

Remark A.0.2. For any $d \times d$ matrix Q, we have

$$\mathrm{tr}(Q^3) \leq \frac{\varepsilon}{4}|\mathrm{tr}(Q^2)|^2 + \frac{1}{\varepsilon}\mathrm{tr}(Q^2) \qquad \text{for any } \varepsilon > 0. \tag{A.0.1}$$

Proof. Let x, y, and z be the eigenvalues of Q. Then we have

$$\mathrm{tr}(Q) = x + y + z, \quad \mathrm{tr}(Q^2) = x^2 + y^2 + z^2, \quad \mathrm{tr}(Q^3) = x^3 + y^3 + z^3.$$

Since it is obvious that $\mathrm{tr}(Q^4) \leq |\mathrm{tr}(Q^2)|^2$, we only need to show that, for any $\varepsilon > 0$, the following inequality is true:

$$\mathrm{tr}(Q^3) \leq \frac{3\varepsilon}{8}\mathrm{tr}(Q^4) + \frac{1}{\varepsilon}\mathrm{tr}(Q^2).$$

By Young's inequality with ε, we have

$$x^3 \leq \frac{x^2}{\varepsilon} + \frac{\varepsilon}{4}x^4, \quad y^3 \leq \frac{y^2}{\varepsilon} + \frac{\varepsilon}{4}y^4, \quad z^3 \leq \frac{z^2}{\varepsilon} + \frac{\varepsilon}{4}z^4.$$

Then we obtain the desired inequality. □

Lemma A.0.6 (Aubin-Lions lemma [1]). *Let X_0, X, and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$, let X_0 be compactly embedded in X, and let X be continuously embedded in X_1. For $1 \leq p, q \leq \infty$, let*

$$W = \left\{u \in L^p(0,T;X_0) \,:\, \dot{u} \in L^q(0,T;X_1)\right\}.$$

Then

(i) If $p < \infty$, then the embedding of W into $L^p(0,T;X)$ is compact.

(ii) If $p = \infty$ and $q > 1$, then the embedding of W into $C(0,T;X)$ is compact.

Lemma A.0.7 (Gagliardo-Nirenberg interpolation inequality [47]). *Let $1 \le q, r \le \infty$, and the function $u : \mathcal{O} \to \mathbb{R}$ defined on a bounded Lipschitz domain $\mathcal{O} \subseteq \mathbb{R}^d$. For $0 \le j < m$, the following inequalities hold*

$$\|D^j u\|_{L^p} \le C_1 \|D^m u\|_{L^r}^a \|u\|_{L^q}^{1-a} + C_2 \|u\|_{L^s}, \tag{A.0.2}$$

where

$$\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{d}) + (1-a)\frac{1}{q}, \quad \frac{j}{m} \le a \le 1,$$

and $s > 0$ is arbitrary, C_1 and C_2 depend on $\mathcal{O}, m, d$.

Next, we introduce a sufficient condition for a solution (ρ, u) to be a renormalized solution.

Lemma A.0.8. *Let $\mathcal{O} \subseteq \mathbb{R}^3$ be a bounded domain. Let $\rho \in L^2(\mathcal{O}_T)$ and $u \in L^2(0, T; H^1(\mathcal{O}))$ such that*

$$\partial_t \rho + \nabla \cdot (\rho u) = 0 \qquad in\ \mathcal{D}'(\mathcal{O}_T).$$

Then

$$\partial_t g(\rho) + \operatorname{div}(g(\rho)u) + \big(g'(\rho)\rho - g(\rho)\big)\operatorname{div} u = 0 \qquad in\ \mathcal{D}'(\mathcal{O}_T) \tag{A.0.3}$$

for any $g \in C^1(\mathbb{R})$ with the property that $g'(z) \equiv 0$ when $z \ge M$ for sufficiently large constant M, i.e., (ρ, u) is a renormalized solution.

Theorem A.0.1. *(Theorem 2.4 in [38]) Assume*

$$\begin{aligned}
0 \le \rho_n \le C \quad a.e.\ on\ \mathcal{O} \times (0,T), \\
\operatorname{div} u_n = 0 \quad a.e.\ on\ \ \mathcal{O} \times (0,T), \quad \|u_n\|_{L^2(0,T;H^1(\mathcal{O}))} \le C, \\
\frac{\partial \rho_n}{\partial t} + \operatorname{div}(\rho_n u_n) = 0, \quad in \quad \mathcal{D}'(\mathbb{R}^d \times (0,T)), \\
\rho_{0,n} \to \bar{\rho} \quad in\ L^1(\mathcal{O}), \quad u_n \rightharpoonup u \quad in\ L^2(0,T; H^1_0(O)),
\end{aligned}$$

where $\bar{\rho}$ satisfies $0 \le \bar{\rho} \le C$ a.e., and C denotes various positive constants independent of n. Then *(i)* ρ_n *converges in* $C([0,T];L^p(\mathcal{O}))$ *for all* $1 \le p < \infty$ *to the unique solution* ρ*, bounded on* $\mathcal{O} \times (0,T)$*, of*

$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^d \times (0,T)), \\ \rho \in C([0,T]; L^1(\mathcal{O})), \rho|_{t=0} = \bar{\rho} \quad \text{a.e. in } \mathcal{O}. \end{cases} \tag{A.0.4}$$

(ii) If we assume in addition that $\rho_n|u_n|^2$ *is bounded in* $L^\infty(0,T;L^1(\mathcal{O}))$ *and that we have for some* $q \in (1,\infty)$*,* $m \ge 1$*,*

$$|(\frac{\partial}{\partial t}(\rho_n u_n), \psi)| \le C\|\psi\|_{L^q(0,T;W^{m,q}(\mathcal{O}))} \tag{A.0.5}$$

for all $\psi \in L^q(0,T;W^{m,q}(\mathcal{O}))$*, such that,* $\mathrm{div}\,\psi = 0$ *on* $\mathbb{R}^d \times (0,T)$*. Then,* $\sqrt{\rho_n}u_n$ *converges to* $\sqrt{\rho}u$ *in* $L^p(0,T;L^r(\mathcal{O}))$ *for* $2 < p < \infty$*,* $1 \le r < \frac{2dp}{dp-4}$*, and* u_n *converges to* u *in* $L^\theta(0,T;L^{\frac{d\theta}{d-2}}(\mathcal{O}))$ *for* $1 \le \theta < 2$ *on the set* $\{\rho > 0\}$ *(if* $d = 2$*,* $\frac{d\theta}{d-2}$ *is replaced by an arbitrary* r *in* $[0,\infty)$*).*

Definition A.0.2. The metric space $C([0,T];X^*_{\text{weak}})$ contains all the functions $v : [0,T] \mapsto X^*$ which are continuous with respect to the weak topology. We say

$$v_n \to v \qquad \text{in } C([0,T]; X^*_{\text{weak}}),$$

if $(v_n(t), \phi) \to (v(t), \phi)$ uniformly with respect to $t \in [0,T]$ for any $\phi \in X$.

In the following corollary, we introduce a sufficient condition for a family of functions to converge in $C([0,T];X^*_{\text{weak}})$ (see Corollary 2.1 in [18]).

Corollary A.0.1. *Let* X *be a separable Banach space. Assume that* $v_n : [0,T] \to X^*$*,* $n = 1, 2, \cdots$*, is a sequence of measurable functions such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|v_n(t)\|_{X^*} \le M, \qquad \text{uniformly in } n = 1, 2, \cdots.$$

Moreover, let the family of functions:

$$(v_n, \phi) : t \mapsto (v_n(t), \phi), \qquad t \in [0,T], n = 1, 2, \cdots,$$

be equi-continuous for any fixed ϕ *belonging to a dense subset in* X*.*

Then $v_n \in C([0,T];X^*_{\text{weak}})$ *for any* $n = 1, 2, \cdots$*, and there exist* $v \in C([0,T];X^*_{\text{weak}})$ *and a subsequence (still denoted)* v_n *such that*

$$v_n \to v \qquad \text{in } C([0,T]; X^*_{\text{weak}}) \quad \text{as } n \to \infty.$$

Proposition A.0.2. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^3$. If $v_n \in L^\infty(0,T;L^p(\mathcal{O}))$ for $p > \frac{6}{5}$, and $v_n \to v$ in $C([0,T], L^p_{\text{weak}}(\mathcal{O}))$, then $v_n \to v$ in $C([0,T], H^{-1}(\mathcal{O}))$.*

Lemma A.0.9 (Gronwall's inequality)**.** *Let $\alpha \geq 0$ and $\beta \geq 0$ be integrable functions on $[0,T]$. If a differentiable function Y satisfies the differential inequality:*

$$Y'(t) \leq \alpha(t)Y(t) + \beta(t) \qquad \text{for } t \in [0,T],$$

then

$$Y(t) \leq Y(0)\exp\Big(\int_0^t \alpha(s)ds\Big) + \int_0^t \beta(s)\exp\Big(\int_s^t \alpha(\tau)d\tau\Big)\,ds \qquad \text{for any } t \in [0,T].$$

APPENDIX B

THE ESTIMATES OF INEQUALITY (2.3.6)

Before proving inequality (2.3.6), let us first introduce the following Lemma.

Lemma B.0.10. *Let $u \in H^s \cap L^p$ with $p \geq 1$ and $s > 0$. Then, for any $k \geq 2$ and $q \in \mathbb{N}$,*

$$\|\Delta_q u^k\|_{L^p} \leq C 2^{-qs} a_{q,k}(t) \|u\|^{k-1}_{L^{p(k-1)}} \|\nabla u\|_{H^s}, \tag{B.0.1}$$

where $\{a_{q,k}(t)\}_{q\in\mathbb{N}}$ is a sequence in l^2.

Proof. We prove this lemma by induction.

Firstly, for $k = 2$, by using the Bony's paraproduct decomposition, we have

$$\begin{aligned}
\Delta_q(u^2) = & S_{q-1}u\Delta_q u + \sum_{|q-q'|\leq 5} [\Delta_q, S_{q'-1}u]\Delta_{q'} u \\
& + \sum_{|q-q'|\leq 5} (S_{q'-1}u - S_{q-1}u)\Delta_q\Delta_{q'} u + \sum_{q'>q-5} \Delta_q(S_{q'+2}u\Delta_{q'} u) \\
= & \sum_{1\leq i\leq 4} \mathcal{I}_i.
\end{aligned}$$

Let us calculate the right side term by term as follows,

$$\begin{aligned}
\|\mathcal{I}_1\|_{L^p} = \|S_{q-1}u\Delta_q u\|_{L^p} &\leq C\|u\|_{L^p}\|\Delta_q u\|_{L^\infty} \leq C\|u\|_{L^p}\|\Delta_q \nabla u\|_{L^2} \\
&\leq C 2^{-qs}\bar{a}_q^{(1)}(t)\|u\|_{L^p}\|\nabla u\|_{H^s},
\end{aligned}$$

where $\{\bar{a}_q^{(1)}(t)\}_{q\in\mathbb{N}}$ is a sequence in l^2,

$$
\begin{aligned}
\|\mathcal{I}_2\|_{L^p} &= \|\sum_{|q-q'|\le 5}[\Delta_q, S_{q'-1}u]\Delta_{q'}u\|_{L^p} \le C\sum_{|q-q'|\le 5}2^{-q}\|\nabla S_{q'-1}u\|_{L^p}\|\Delta_{q'}u\|_{L^\infty}\\
&\le C\sum_{|q-q'|\le 5}2^{q'-q}\|u\|_{L^p}\|\Delta_{q'}\nabla u\|_{L^2}\\
&\le C2^{-qs}\sum_{|q-q'|\le 5}2^{(q'-q)(1-s)}\bar{a}_{q'}^{(1)}(t)\|u\|_{L^p}\|\nabla u\|_{H^s}\\
&\le C2^{-qs}\bar{a}_q^{(2)}(t)\|u\|_{L^p}\|\nabla u\|_{H^s},
\end{aligned}
$$

where $\{\bar{a}_q^{(2)}(t)\}_{q\in\mathbb{N}} = \{\sum_{|q-q'|\le 5}2^{(q'-q)(1-s)}\bar{a}_{q'}^{(1)}(t)\}_{q\in\mathbb{N}}$ is a sequence in l^2,

$$
\begin{aligned}
\|\mathcal{I}_3\|_{L^p} &= \|\sum_{|q-q'|\le 5}(S_{q'-1}u - S_{q-1}u)\Delta_q\Delta_{q'}u\|_{L^p}\\
&\le C\sum_{|q-q'|\le 5}\|S_{q'-1}u - S_{q-1}u\|_{L^p}\|\Delta_q\Delta_{q'}u\|_{L^\infty}\\
&\le C\sum_{|q-q'|\le 5}\|u\|_{L^p}\|\Delta_{q'}\nabla u\|_{L^2}\\
&\le C2^{-qs}\sum_{|q-q'|\le 5}2^{(q-q')s}\bar{a}_{q'}^{(1)}(t)\|u\|_{L^p}\|\nabla u\|_{H^s}\\
&\le C2^{-qs}\bar{a}_q^{(3)}(t)\|u\|_{L^p}\|\nabla u\|_{H^s},
\end{aligned}
$$

with $\{\bar{a}_q^{(3)}(t)\}_{q\in\mathbb{N}} = \{\sum_{|q-q'|\le 5}2^{(q-q')s}\bar{a}_{q'}^{(1)}(t)\}_{q\in\mathbb{N}} \in l^2$,

$$
\begin{aligned}
\|\mathcal{I}_4\|_{L^p} &= \|\sum_{q'>q-5}\Delta_q(S_{q'+2}u\Delta_{q'}u)\|_{L^p} \le C\sum_{q'>q-5}\|S_{q'+2}u\|_{L^p}\|\Delta_{q'}u\|_{L^\infty}\\
&\le C\sum_{q'>q-5}\|u\|_{L^p}\|\Delta_{q'}\nabla u\|_{L^2} \le C2^{-qs}\sum_{q'>q-5}2^{(q-q')s}\|u\|_{L^p}\bar{a}_{q'}^{(1)}(t)\|\nabla u\|_{H^s}\\
&\le C2^{-qs}\bar{a}_q^{(4)}(t)\|u\|_{L^p}\|\nabla u\|_{H^s},
\end{aligned}
$$

with $\{\bar{a}_q^{(4)}(t)\}_{q\in\mathbb{N}} = \{\sum_{q'>q-5}2^{(q-q')s}\bar{a}_{q'}^{(1)}(t)\}_{q\in\mathbb{N}} \in l^2$.

Then taking $a_{q,2}(t) = \max_{1\le k\le 4}\{\bar{a}_q^{(k)}(t)\}$, we know that the result holds for $k=2$.

Next, we first assume the statement is valid for k and then check the validity for the case $k+1$. Similarly to the previous steps, we have

$$\begin{aligned}\Delta_q(u^{k+1}) = \Delta_q(u^k u) &= S_{q-1}u^k\Delta_q u + \sum_{|q-q'|\le 5}[\Delta_q, S_{q'-1}u^k]\Delta_{q'}u \\ &+ \sum_{|q-q'|\le 5}(S_{q'-1}u^k - S_{q-1}u^k)\Delta_q\Delta_{q'}u + \sum_{q'>q-5}\Delta_q(S_{q'+2}u\Delta_{q'}u^k) \\ &= \sum_{1\le i\le 4}\mathcal{J}_i,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{J}_1\|_{L^p} &= \|S_{q-1}u^k\Delta_q u\|_{L^p} \le C\|u^k\|_{L^p}\|\Delta_q u\|_{L^\infty} \le C\|u\|_{L^{pk}}^k\|\Delta_q\nabla u\|_{L^2} \\ &\le C2^{-qs}\bar{a}_q^{(1)}(t)\|u\|_{L^{pk}}^k\|\nabla u\|_{H^s},\end{aligned}$$

$$\begin{aligned}\|\mathcal{J}_2\|_{L^p} &= \|\sum_{|q-q'|\le 5}[\Delta_q, S_{q'-1}u^k]\Delta_{q'}u\|_{L^p} \le C\sum_{|q-q'|\le 5}2^{-q}\|\nabla S_{q'-1}u^k\|_{L^p}\|\Delta_{q'}u\|_{L^\infty} \\ &\le C\sum_{|q-q'|\le 5}2^{q'-q}\|u^k\|_{L^p}\|\Delta_{q'}\nabla u\|_{L^2} \\ &\le C2^{-qs}\sum_{|q-q'|\le 5}2^{(q'-q)(1-s)}\bar{a}_{q'}^{(1)}(t)\|u\|_{L^{pk}}^k\|\nabla u\|_{H^s} \\ &\le C2^{-qs}\bar{a}_q^{(2)}(t)\|u\|_{L^{pk}}^k\|\nabla u\|_{H^s},\end{aligned}$$

$$\begin{aligned}\|\mathcal{J}_3\|_{L^p} &= \|\sum_{|q-q'|\le 5}(S_{q'-1}u^k - S_{q-1}u^k)\Delta_q\Delta_{q'}u\|_{L^p} \\ &\le C\sum_{|q-q'|\le 5}\|S_{q'-1}u^k - S_{q-1}u^k\|_{L^p}\|\Delta_q u\|_{L^\infty} \\ &\le C\sum_{|q-q'|\le 5}\|u^k\|_{L^p}\|\Delta_q\nabla u\|_{L^2} \\ &\le C2^{-qs}\bar{a}_q^{(1)}(t)\|u\|_{L^{pk}}^k\|\nabla u\|_{H^s},\end{aligned}$$

and from the induction assumption, one has

$$\|\mathcal{J}_4\|_{L^p} = \|\sum_{q'>q-5}\Delta_q(S_{q'+2}u\Delta_{q'}u^k)\|_{L^p} \le C\sum_{q'>q-5}\|u\|_{L^{kp}}\|\Delta_{q'}u^k\|_{L^{\frac{kp}{k-1}}}$$

$$
\begin{aligned}
&\leq C \sum_{q'>q-5} \|u\|_{L^{kp}} 2^{-q's} a_{q',k}(t) \|u\|_{L^{kp}}^{k-1} \|\nabla u\|_{H^s} \\
&\leq C 2^{-qs} \sum_{q'>q-5} 2^{(q-q')s} a_{q',k}(t) \|u\|_{L^{pk}}^{k} \|\nabla u\|_{H^s} \\
&\leq C 2^{-qs} \bar{a}_{q,k+1}(t) \|u\|_{L^{pk}}^{k} \|\nabla u\|_{H^s},
\end{aligned}
$$

with $\{\bar{a}_{q,k+1}(t)\}_{q\in\mathbb{N}} = \{\sum_{q'>q-5} 2^{(q-q')s} a_{q',k}(t)\}_{q\in\mathbb{N}} \in l^2$.

Then taking $a_{q,k+1}(t) = \max\{\bar{a}_q^{(1)}(t), \bar{a}_q^{(2)}(t), \bar{a}_{q,k+1}(t)\}$, we know that the statement is true for the case $k+1$. By induction, we complete the proof. □

Then, in order to derive (2.3.6), we estimate the terms on the right-hand side of (2.3.5) one by one. For the detailed estimates of the following terms, we refer the readers to the appendix in [51]:

$$
\begin{aligned}
|\mathcal{I}_1| \leq C 2^{-2qs} b_q(t) \big(& \|u\|_{L^2}^{\frac12} \|\nabla u\|_{L^2}^{\frac12} \|\nabla Q\|_{H^s}^{\frac12} \|\Delta Q\|_{H^s}^{\frac32} \\
& + \|\nabla Q\|_{L^2}^{\frac12} \|\Delta Q\|_{L^2}^{\frac12} \|u\|_{H^s}^{\frac12} \|\nabla u\|_{H^s}^{\frac12} \|\Delta Q\|_{H^s} \big), \\
|\mathcal{I}_2| + |\mathcal{I}_3| + |\mathcal{I}_5| + |\mathcal{I}_6| &\leq C 2^{-2qs} b_q(t) \|\nabla Q\|_{L^2}^{\frac12} \|\Delta Q\|_{L^2}^{\frac12} \|u\|_{H^s}^{\frac12} \|\nabla u\|_{H^s}^{\frac12} \|\Delta Q\|_{H^s}, \\
|\mathcal{I}_4| + |\mathcal{I}_7| &\leq C 2^{-2qs} b_q(t) \|u\|_{L^2}^{\frac12} \|\nabla u\|_{L^2}^{\frac12} \|\nabla Q\|_{H^s}^{\frac12} \|\Delta Q\|_{H^s}^{\frac32}, \\
|\mathcal{J}_1| &\leq C 2^{-2qs} b_q(t) \|u\|_{L^2}^{\frac12} \|\nabla u\|_{L^2}^{\frac12} \|u\|_{H^s}^{\frac12} \|\nabla u\|_{H^s}^{\frac32}, \\
|\mathcal{J}_2| &\leq C 2^{-2qs} b_q(t) \|\nabla Q\|_{L^2}^{\frac12} \|\Delta Q\|_{L^2}^{\frac12} \|Q\|_{H^s}^{\frac12} \|\nabla Q\|_{H^s}^{\frac12} \|\nabla u\|_{H^s}, \\
|\mathcal{J}_3| + |\mathcal{J}_4| + |\mathcal{J}_6| + |\mathcal{J}_7| &\leq C 2^{-2qs} b_q(t) \|\nabla Q\|_{L^2}^{\frac12} \|\Delta Q\|_{L^2}^{\frac12} \|u\|_{H^s}^{\frac12} \|\nabla u\|_{H^s}^{\frac12} \|\Delta Q\|_{H^s}, \\
|\mathcal{J}_5| + |\mathcal{J}_8| &\leq C 2^{-2qs} b_q(t) \|\nabla Q\|_{L^2}^{\frac12} \|\Delta Q\|_{L^2}^{\frac12} \|\nabla Q\|_{H^s}^{\frac12} \|\Delta Q\|_{H^s}^{\frac12} \|\nabla u\|_{H^s},
\end{aligned}
$$

where $\{b_q(t)\}_{q\in\mathbb{N}}$ is a sequence in l^1.

Next we will give the details of the estimates for the remaining terms:

$$
\begin{aligned}
|\mathcal{I}_9| &= |\Gamma a(\Delta_q Q, \Delta\Delta_q Q)| = \Gamma a \|\Delta_q \nabla Q\|_{L^2}^2 \leq C 2^{-2qs} b_q^{(1)}(t) \|\nabla Q\|_{H^s}^2, \\
|\mathcal{J}_{12}| &= |\kappa(\Delta_q Q, \Delta_q \nabla u)| = |\kappa(\Delta_q \nabla Q, \Delta_q u)| \\
&\leq C \|\Delta_q \nabla Q\|_{L^2} \|\Delta_q u\|_{L^2} \leq C 2^{-2qs} b_q^{(2)}(t) \|\nabla Q\|_{H^s} \|u\|_{H^s},
\end{aligned}
$$

for some $\{b_q^{(i)}(t)\}_{q\in\mathbb{N}} \in l^1$ with $i = 1, 2$. Applying Lemma B.0.10, we have the following estimates:

$$
\begin{aligned}
|\mathcal{I}_{10}| &= |\Gamma c(\Delta_q(Q\mathrm{tr}(Q^2)), \Delta\Delta_q Q)| \leq C\|\Delta_q(Q\mathrm{tr}(Q^2))\|_{L^2}\|\Delta\Delta_q Q\|_{L^2} \\
&\leq C2^{-qs}a_{q,3}(t)\|Q\|_{L^4}^2\|\nabla Q\|_{H^s}2^{-qs}\bar{a}_q^{(1)}(t)\|\Delta Q\|_{H^s} \\
&\leq C2^{-2qs}b_q^{(3)}(t)\|Q\|_{L^4}^2\|\nabla Q\|_{H^s}\|\Delta Q\|_{H^s}, \\
|\mathcal{J}_{10}| &= |-a\lambda\big(\Delta_q(|Q|Q), \nabla\Delta_q u\big)| \\
&\leq C\|\Delta_q(|Q|Q)\|_{L^2}\|\nabla\Delta_q u\|_{L^2} \\
&\leq C2^{-qs}a_{q,2}(t)\|Q\|_{L^2}\|\nabla Q\|_{H^s}2^{-qs}\bar{a}_q^{(1)}(t)\|\nabla u\|_{H^s} \\
&\leq C(s)2^{-2qs}b_q^{(4)}(t)\|Q\|_{L^2}\|\nabla Q\|_{H^s}\|\nabla u\|_{H^s}, \\
|\mathcal{J}_{11}| &= |-c\lambda(\Delta_q(|Q|Q\mathrm{tr}(Q)^2), \nabla\Delta_q u)| \\
&\leq C\|\Delta_q(|Q|Q\mathrm{tr}(Q^2))\|_{L^2}\|\nabla\Delta_q u\|_{L^2} \\
&\leq C2^{-qs}a_{q,4}(t)\|Q\|_{L^6}^3\|\nabla Q\|_{H^s}2^{-qs}\bar{a}_q^{(1)}(t)\|\nabla u\|_{H^s} \\
&\leq C2^{-2qs}b_q^{(5)}(t)\|Q\|_{L^6}^3\|\nabla Q\|_{H^s}\|\nabla u\|_{H^s},
\end{aligned}
$$

for some $\{b_q^{(i)}(t)\}_{q\in\mathbb{N}} \in l^1$ with $i = 3, 4, 5$.

Finally, for the remaining two terms $\mathcal{I}_8$ and $\mathcal{J}_9$ it is not easy to obtain the bounds directly. The key idea is to combine them together and identify the cancellation factors between them for the estimates. Applying the Bony's decomposition, we have

$$
\begin{aligned}
\mathcal{I}_8 &= -\lambda(\Delta_q(|Q|D), \Delta\Delta_q Q) \\
&= -\lambda(S_{q-1}(|Q|)\Delta_q D, \Delta\Delta_q Q) - \lambda\sum_{|q'-q|\leq 5}([\Delta_q; S_{q'}(|Q|)]\Delta_{q'}D, \Delta\Delta_q Q) \\
&\quad - \lambda\sum_{|q'-q|\leq 5}((S_{q'-1}|Q| - S_{q-1}|Q|)\Delta_q\Delta_{q'}D, \Delta\Delta_q Q) \\
&\quad - \lambda\sum_{q'>q-5}(\Delta_q(S_{q'+2}D\Delta_{q'}|Q|), \Delta\Delta_q Q) \\
&= \lambda\sum_{i=1}^{4}\mathcal{I}_{8,i}, \\
\mathcal{J}_9 &= \lambda(\Delta_q(|Q|\Delta Q), \nabla\Delta_q u) = \lambda(S_{q-1}|Q|\Delta_q\Delta Q, \nabla\Delta_q u) + \lambda([\Delta_q; S_{q'}|Q|]\Delta_{q'}\Delta Q, \nabla\Delta_q u)
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\sum_{|q'-q|\le 5}((S_{q'-1}|Q|-S_{q-1}|Q|)\Delta_q\Delta_{q'}\Delta Q,\nabla\Delta_q u)\\
&+\lambda\sum_{q'>q-5}(\Delta_q(S_{q'+2}\Delta Q\Delta_{q'}|Q|),\nabla\Delta_q u)\\
=&\ \lambda\sum_{i=1}^{4}\mathcal{I}_{9,i}.
\end{aligned}
$$

Since Q is symmetric and $\Omega=\nabla u-D$ is skew-symmetric, we find that, for $1\le i\le 3$,

$$
\mathcal{I}_{8,i}+\mathcal{J}_{9,i}=0.
$$

Then

$$
\mathcal{I}_8+\mathcal{J}_9=\mathcal{I}_{8,4}+\mathcal{J}_{9,4}.
$$

In addition, we have

$$
\begin{aligned}
|\mathcal{I}_{8,4}|=&\Big|\sum_{q'>q-5}(\Delta_q(S_{q'+2}D\Delta_{q'}|Q|),\Delta\Delta_q Q)\Big|\\
\le&\ C\sum_{q'>q-5}\|S_{q'+2}\nabla u\|_{L^4}\|\Delta_{q'}|Q|\|_{L^4}\|\Delta\Delta_q Q\|_{L^2}\\
\le&\ C\sum_{q'>q-5}2^{q'+2}\|u\|_{L^4}2^{-q'}\|\Delta_{q'}\nabla Q\|_{L^4}\|\Delta\Delta_q Q\|_{L^2}\\
\le&\ C\sum_{q'>q-5}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta_{q'}\nabla Q\|_{L^2}^{\frac{1}{2}}\|\Delta_{q'}\Delta Q\|_{L^2}^{\frac{1}{2}}\|\Delta_q\Delta Q\|_{H^s}\\
\le&\ C2^{-2qs}\sum_{q'>q-5}2^{(q-q')s}\bar{a}_{q'}^{(1)}(t)\bar{a}_q^{(1)}(t)\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla Q\|_{H^s}^{\frac{1}{2}}\|\Delta Q\|_{H^s}^{\frac{3}{2}}\\
\le&\ C2^{-2qs}b_q^{(6)}(t)\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla Q\|_{H^s}^{\frac{1}{2}}\|\Delta Q\|_{H^s}^{\frac{3}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{I}_{9,4}|=&\Big|\sum_{q'>q-5}(\Delta_q(S_{q'+2}\Delta Q\Delta_{q'}|Q|),\nabla\Delta_q u)\Big|\\
\le&\ C\sum_{q'>q-5}\|S_{q'+2}\Delta Q\|_{L^4}\|\Delta_{q'}Q\|_{L^4}\|\Delta_q\nabla u\|_{L^2}\\
\le&\ C\sum_{q'>q-5}2^{q'}\|\nabla Q\|_{L^4}2^{-q'}\|\Delta_{q'}\nabla Q\|_{L^4}\|\Delta_q\nabla u\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{q'>q-5} \|\nabla Q\|_{L^2}^{\frac{1}{2}} \|\Delta Q\|_{L^2}^{\frac{1}{2}} \|\Delta_{q'} \nabla Q\|_{L^2}^{\frac{1}{2}} \|\Delta_{q'} \Delta Q\|_{L^2}^{\frac{1}{2}} \|\Delta_q \nabla u\|_{L^2} \\
&\leq C \sum_{q'>q-5} \|\nabla Q\|_{L^2}^{\frac{1}{2}} \|\Delta Q\|_{L^2}^{\frac{1}{2}} 2^{-q's} \bar{a}_{q'}^{(1)}(t) \|\nabla Q\|_{H^s}^{\frac{1}{2}} \|\Delta_{q'} \Delta Q\|_{H^s}^{\frac{1}{2}} 2^{-qs} \bar{a}_q^{(1)}(t) \|\nabla u\|_{H^s} \\
&\leq C 2^{-2qs} b_q^{(6)}(t) \|\nabla Q\|_{L^2}^{\frac{1}{2}} \|\Delta Q\|_{L^2}^{\frac{1}{2}} \|\nabla Q\|_{H^s}^{\frac{1}{2}} \|\Delta Q\|_{H^s}^{\frac{1}{2}} \|\nabla u\|_{H^s},
\end{aligned}
$$

with $\{b_q^{(6)}(t)\}_{q\in\mathbb{N}} = \{\sum_{q'>q-5} 2^{(q-q')s} \bar{a}_{q'}^{(1)}(t) \bar{a}_q^{(1)}(t)\}_{q\in\mathbb{N}} \in l^1$.

Multiplying all the above estimates by 2^{2qs}, adding them together, taking the sum in q for $q \in \mathbb{N}$, noticing that $\{b_q(t)\}_{q\in\mathbb{N}}, \{b_q^{(i)}(t)\}_{q\in\mathbb{N}} \in l^1$ with $1 \leq i \leq 6$, and using the Cauchy inequality with suitable ε, *i.e.*, $ab \leq \varepsilon a^2 + \frac{C}{\varepsilon} b^2$, we obtain the desired estimate (2.3.6).

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