LOW COMPLEXITY, TIME ACCURATE, MODEL ACCURATE ALGORITHMS IN COMPUTATIONAL FLUID DYNAMICS

by

Haiyun Zhao

Bachelor of Arts in Mathematics, University of Minnesota, 2014

Submitted to the Graduate Faculty of

the Kenneth P. Dietrich School of Arts and Sciences in partial

fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2019

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Haiyun Zhao

It was defended on

July 25, 2019

and approved by

Prof. William Layton, Dept. of Mathematics, University of Pittsburgh

Prof. Catalin Trenchea, Dept. of Mathematics, University of Pittsburgh

Prof. Michael Neilan, Dept. of Mathematics, University of Pittsburgh

Prof. Patrick Smolinski, Dept. of Mechanical Engineering and Materials Sciences, University of Pittsburgh

Dissertation Director: Prof. William Layton, Dept. of Mathematics, University of Pittsburgh

LOW COMPLEXITY, TIME ACCURATE, MODEL ACCURATE ALGORITHMS IN COMPUTATIONAL FLUID DYNAMICS

Haiyun Zhao, PhD

University of Pittsburgh, 2019

Computational fluid dynamics is an essential research area that is of crucial importance in comprehending of fluid flows in mechanical and hydrodynamic processes. Accurate, efficient and reliable simulation of flows occupies a central place in the development of computational science. In this work, we explore various numerical methods and utilize them to improve flow predication. Four research projects are conducted and show evidence in enhancement of accuracy, efficiency and reliability of prediction of fluid motion.

We first propose a low computationally complex, stable and adaptive method for time accurate approximation of the evolutionary stokes Darcy system and Navier-Stokes equations. The improved method post-processes the solutions of the Backward Euler scheme by adding no more than three lines to an existing program. Time accuracy is increased from first to second order and the overdamping of the Backward Euler method is removed. The second project is to develop an efficient method to describe magnetohydrodynamic flows at low magnetic Reynolds numbers. The decoupled method is based on the artificial compression and partitioned schemes. Computational efficiency is greatly improved because we only need to solve linear problems at each time step with systems decouple by physical processes. Last but not least, we introduce a way to correct the Baldwin-Lomax model for non-equilibrium turbulence, which is often considered impossible to simulate due to backscatter. The corrected Baldwin-Lomax model not only shows that effects of fluctuations on means are dissipative on time average but also can have bursts for which energy flow reverses. For each project, we present comprehensive error and stability analysis and provide different numerical exper-

iments to further support theoretical theories.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ALGORITHMS

PREFACE

Frist and foremost, I would like to express my deepest gratitude to my dissertation advisor, Professor William Layton, for the patient guidance, encouragement and advice he has provided throughout my time at University of Pittsburgh. His valuable suggestions and comments help me at various stages of my study. I feel extremely lucky to have a mentor who is always available for answering questions and who care so much about the well-being of his students.

I am grateful to the members of my committee, Professors, Catalin Trenchea, Michael Neilan and Patrick Smolinski for their inspiring discussions and kind willingness to provide help on my dissertation.

I would like to thank Professors Ivan Yotov, Yao Song, Hao Xu and Dr. Mike Sussman for their instructions on various mathematical topics.

A big thanks goes to all my fellow graduate students and visiting scholars at University of Pittsburgh, and especially to Victor DeCaria, Michael McLaughlin, Joseph Fiordilino, Yao Rong and Yi Li.

Finally, I would like to thank my parents for their endless love and constant support. Most of all, I thank my boyfriend, Xiyu, for being there for me during the ups and downs for the past four years.

1.0 INTRODUCTION

Accurate, reliable and efficient forecast of fluid flow is very important for scientific and engineering breakthroughs and advances. Meteorologists utilize numerical simulation of fluid flow to predict the weather and warn of natural disasters; oil and gas engineers can design and maintain optimal pipes network; doctors can prevent and cure arterial diseases by computational hemodynamics. In fact, the basic model of fluid prediction can be traced back to 1922, which was raised by L. Richardson for weather forecasting. In 1933, the earliest numerical solution for flow past a cylinder was developed by A. Thom [42]. In 1967, the first 3D model based on panels discretization was published by Douglas Aircraft. Nowadays, CFD is still one of the central scientific frontiers, and there are increasing number of researchers in this field seeking to obtain better understanding of fluid motion. However, fundamental barriers to accurate fluid predictions exist in many CFD applications. The thesis goal is to consider three fluid models from the basic Navier-Stokes equations, to equations of magnetohydrodynamic(MHD) flows which includes both NSE and Maxwell equation, to the more complex turbulence model where the velocity field is random, and develop low complexity, time accurate, model accurate methods that have the potential to break down the barriers in accuracy, reliability, and efficiency in fluid prediction, to establish mathematical foundations to provide theoretical support and to conduct numerical tests to confirm the obtained results.

In this dissertation, we first presents a low complexity, stable and time accurate method for the Stokes-Darcy system and the Navier-Stokes equations. The improved method requires a minimally intrusive modification to an existing program based on the fully implicit / backward Euler time discretization, does not add to the computational complexity, and is conceptually simple. The backward Euler approximation is simply post-processed with a linear time filter composed of no more than three steps. The time filter additionally removes the overdamping of Backward Euler while remaining unconditionally energy stable, proven herein. The second project is to develop a more computational efficient method to solve the time-dependent magnetohydrodynamic (MHD) flows at low magnetic Reynolds numbers. The constructed decoupling method is based on the artificial compression method (uncoupling the pressure and velocity) and partitioned method (uncoupling the velocity and electric potential). It allows us at each time step to solve linear problems, uncoupled by physical processes, per time step, which can greatly improve the computational efficiency. Last but not least, it is considered a challenge in many aspects to simulate complex turbulent flows not at statistical equilibrium. Because the most common approach, eddy viscosity model, fails to capture backscatter– intermittent energy flow from turbulent fluctuations back to the mean velocity. We present a way to correct the Baldwin-Lomax model to capture the non-equilibrium effects and prove the corrected models preserve important features of the true Reynolds stresses.

2.0 MATHEMATICAL PRELIMINARIES

This section introduces some widely used notations, inequalities and lemmas. The standard notations $H^k(\Omega)$, $H_0^k(\Omega)$, $W^{k,p}(\Omega)$ denote Sobolev spaces, and $L^p(\Omega)$ denotes L^p spaces, see [108]. The $H^k(\Omega)$ norm and $L^p(\Omega)$ $(p \neq 2)$ norm are denoted by $\|\cdot\|_k$ and $\|\cdot\|_{L^p}$, respectively. The $L^2(\Omega)$ norm is denoted by $\|\cdot\|$ and its corresponding inner products by $(\cdot,\cdot)$. Denote the dual space of $H_0^k(\Omega)$ by $H^{-k}(\Omega)$ and its norm by $\|\cdot\|_{-k}$. Furthermore, $\|\cdot\|_{\ell^p}$ is the ℓ^p-norm of vectors in R^d, see [117]. Constants C are different in different places throughout the paper, which do not depend on mesh size and time step but may depend on some known data such as $\Omega, \nu, \mathbf{f}, \cdots$, and so on. We introduce the following spaces and their norms:

$$L^p(0,T;L^q(\Omega)) := \{\mathbf{v}(\mathbf{x},t) : (\int_0^T \|\mathbf{v}(\cdot,t)\|_{L^q}^p dt)^{\frac{1}{p}} < \infty\},$$

$$L^\infty(0,T;L^q(\Omega)) := \{\mathbf{v}(\mathbf{x},t) : \sup_{0\le t\le T} \|\mathbf{v}(\cdot,t)\|_{L^q} < \infty\},$$

$$\|\mathbf{v}\|_{L^p(0,T;L^q)} = (\int_0^T \|\mathbf{v}(\cdot,t)\|_{L^q}^p dt)^{\frac{1}{p}}, \quad \|\mathbf{v}\|_{L^\infty(0,T;L^q)} = \sup_{0\le t\le T} \|\mathbf{v}(\cdot,t)\|_{L^q}.$$

Here, $1 \le p < \infty, 1 \le q \le \infty$. The velocity space X and pressure space Q are defined as follows.

$$X := H_0^1(\Omega)^d = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = 0\},$$

$$Q := L_0^2(\Omega)^d = \{q \in L^2(\Omega) : \int_\Omega q = 0\}.$$

The divergence free space V^2 of $X = W_0^{1,2}(\Omega)$ is given by

$$V := \{\mathbf{v} \in X : (\nabla\cdot\mathbf{v}, q) = 0 \quad \forall q \in Q)\}.$$

and the divergence free subspace of $W_0^{1,3}(\Omega)$ is similarly denoted V^3. Define

$$B(\mathbf{u},\mathbf{v}) := \mathbf{u}\cdot\nabla\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}, \quad b(\mathbf{u},\mathbf{v},\mathbf{w}) := (B(\mathbf{u},\mathbf{v}),\mathbf{w}), \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in X.$$

Then, we have

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}[b(\mathbf{u},\mathbf{v},\mathbf{w}) - b(\mathbf{u},\mathbf{w},\mathbf{v})], \qquad b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0,$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}), \quad \text{if } \ \mathbf{u}\in V.$$

Let Π_h be a set of triangulations of Ω with $\overline{\Omega} = \bigcup_{K\in\Pi_h} K$ ($h = \sup_{K\in\Pi_h} diam(K)$). It is uniformly regular when $h\to 0$. $X_h \subset X, Q_h \subset Q$ are finite element spaces that satisfy the discrete inf-sup condition: $\inf_{q\in Q_h}\sup_{\mathbf{v}\in X_h} \frac{(q,\nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\|\|q\|} \geq C > 0$. The subspace V_h of X_h is defined by

$$V_h := \{\mathbf{v}_h \in X_h : (\nabla\cdot\mathbf{v}_h, q_h) = 0 \quad \forall q_h\in Q_h\}.$$

Next, we present some inequalities and lemmas which will be used in later sections.

Lemma 1. (The discrete Gronwall's inequality) *Suppose that n and N are nonnegative integers, $n\leq N$. The real numbers $a_n, b_n, c_n, d_n, \triangle t$ are nonnegative and satisfy that*

$$a_N + \Delta t\sum_{n=0}^{N} b_n \leq \Delta t\sum_{n=0}^{N}(c_n a_n + d_n).$$

Then,

$$a_N + \Delta t\sum_{n=0}^{N} b_n \leq exp(\Delta t\sum_{n=0}^{N}\frac{c_n}{1-\Delta t c_n})(\Delta t\sum_{n=0}^{N} d_n),$$

provided that $\Delta t c_n < 1$ for each n.

Lemma 2. (Strong monotonicity and local Lipschitz-continuity) *There exist positive constants $\underline{C}$ and $\overline{C}$ such that for all $\mathbf{u}',\mathbf{u}'',\mathbf{v}\in (W^{1,3}(\Omega))^d$, we have*

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'', \nabla\times(\mathbf{u}'-\mathbf{u}''))\\ &\geq \underline{C}\|l^{\frac{2}{3}}(x)\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}^3,\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'', \nabla\times\mathbf{v})\\ &\leq \overline{C}\gamma\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3},\end{aligned} \tag{2.2}$$

where $l:\mathbf{x}\in\Omega\mapsto R$ is a non-negative function with $l\in L^\infty(\Omega)$, and $\gamma = \max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}'\|_{L^3}, \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}''\|_{L^3}\}$.

Remark 1. *Inequalities (A1) and (A2) can be found in [108, 116, 130]. Lemma 1 can be found in [138]. The proof of Lemma 2 is given in the Appendix.*

Lemma 3. *There exists $C > 0$ such that*

$$b(u, v, w) \leq C\|\nabla u\|\|\nabla v\|\|\nabla w\|, \qquad \forall\, u, v, w \in X$$
$$b(u, v, w) \leq C\|u\|\|v\|_2\|\nabla w\| \quad \forall u, w \in X, v \in X \cap H^2(\Omega).$$

Proof. See Lemma 2.1 on p. 12 of [105]. □

We use the following discrete Gronwall inequality found in [94, Lemma 5.1].

Lemma 4 (Discrete Gronwall Inequality)**.** *Let Δt, H, a_n, b_n, c_n, d_n (for integers $n \geq 0$) be non-negative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H, \quad \forall\, l \geq 0 \tag{2.3}$$

Suppose $\Delta t d_n < 1$ $\forall n$, then,

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \exp\Big(\Delta t \sum_{n=0}^{l} b_n \frac{d_n}{1 - \Delta t d_n}\Big)\Big(\Delta t \sum_{n=0}^{l} c_n + H\Big), \quad \forall\, l \geq 0 \tag{2.4}$$

Multiplying (6.1) by test functions $(v, q) \in (X, Q)$ and integrating by parts gives

$$(u_t, v) + b(u, u, v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) + (\nabla \cdot u, q) = (f, v), \quad (\nabla \cdot u, q) = 0. \tag{2.5}$$

To discretize the above system in space, we choose conforming finite element spaces for velocity $X^h \subset X$ and pressure $Q^h \subset Q$ satisfying the discrete inf-sup condition and the following approximation properties:

$$\begin{aligned}
&\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\|\|\nabla v_h\|} \geq \beta > 0,\\
&\inf_{v \in X^h} \|u - v\| \leq Ch^{k+1}\|u\|_{k+1}, \quad u \in H^{k+1}(\Omega)^d\\
&\inf_{v \in X^h} \|u - v\|_1 \leq Ch^{k+1}\|u\|_{k}, \quad u \in H^{k+1}(\Omega)^d\\
&\inf_{r \in Q^h} \|p - r\| \leq Ch^{s+1}\|p\|_{s+1}, \quad p \in H^{s+1}(\Omega)
\end{aligned} \tag{2.6}$$

h denotes the maximum triangle diameter. Examples of finite element spaces satisfying these conditions are the MINI [76] and Taylor-Hood [106] elements. The discretely divergence free subspace $V_h \in X_h$ is defined

$$V_h := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \quad \forall q_h \in Q_h\}.$$

The dual norms of X_h and V_h are

$$\|w\|_{X_h^*} := \sup_{v_h \in X_h} \frac{(w, v_h)}{\|\nabla v_h\|}, \|w\|_{V_h^*} := \sup_{v_h \in V_h} \frac{(w, v_h)}{\|\nabla v_h\|}.$$

The following Lemma from Galvin [87, p. 243] establishes the equivalence of these norms on V_h.

Lemma 5. *Suppose the discrete inf-sup condition holds, let $w \in V_h$, then there exists $C > 0$, independent of h, such that*

$$C\|w\|_{X_h^*} \leq \|w\|_{V_h^*} \leq \|w\|_{X_h^*}.$$

Lemma 5 is used to derive pressure error estimates with a technique shown in Fiordilino [86]. We will use the following, easily proven, algebraic identity.

Lemma 6. *The following identity holds.*

$$\left(\frac{3}{2}a - 2b + \frac{1}{2}c\right)\left(\frac{3}{2}a - b + \frac{1}{2}c\right) = \tag{2.7}$$
$$\left(\frac{a^2}{4} + \frac{(2a-b)^2}{4} + \frac{(a-b)^2}{4}\right) - \left(\frac{b^2}{4} + \frac{(2b-c)^2}{4} + \frac{(b-c)^2}{4}\right) + \frac{3}{4}(a - 2b + c)^2$$

3.0 LOW COMPLEXITY, TIME ACCURATE ALGORITHMS IN CFD-ADAPTIVE PARTITIONED METHODS FOR STOKES-DARCY SYSTEM

3.1 INTRODUCTION

The coupling of a fluid flowing between a porous media and a free flow region is a typical multi-physics and multi-domain problem, which plays an important role in many industrial and engineering applications and in transport between ground water and surface water. The Stokes-Darcy model is a fundamental model of this proem. Numerical methods for the coupled model have been extensively studied and tested, including finite element methods [74], spectral methods [70], discontinuous Galerkin methods [64], discontinuous finite volume methods [69], mortar element methods [46], boundary integral methods [68], hybrid discontinuous Galerkin methods [54], least square methods [57], optimization based methods[53], weak Galerkin methods [49], various domain decomposition methods [58], two grid methods [65], multigrid methods [44], and time partitioned methods [62].

Most recently research has focused on partitioned methods for the non-stationary Stokes-Darcy model. See [60],[61],[66],[67] for a first-order partitioned methods, [47],[62] for second order partitioned method and [48] for a third-order partitioned method. Current directions include methods using different timesteps in different subdomains [62] and higher order methods [47],[62], [48]. The aim of this research is to develop a fast partitioned method that provides time accurate approximations by time adaptive partitioned methods using time filters. Time filters are effective tools to offset the weakness of lower-order partitioned methods. Partitioning can be accomplished by using an IMEX method with explicit discretization of interface coupling terms. If this induces oscillations, time filters can also be used to damp non-physical oscillations here, as in GFD. Recently it was noted in [55] that time filters also can increase the time accuracy of simple, lower accuracy methods. This yields, at low cost, two approximations of different accuracy. Thus, it also gives a low cost error estimator for adapting the timestep to ensure time accuracy. This report develops, analyzes and tests adaptive algorithms, based on this idea, for the coupled, evolutionary Stokes-Darcy problem. The method herein is based on finite element discretization in space. The subdomain/subphysics terms are discretized in time by the usual (first order) fully implicit Backward Euler method. The coupling terms are discretized by the explicit second order's extrapolation method. These terms are skew symmetric (express conservation of material flowing from one subdomain into the other), must be treated explicitly to produce a partitioned method and represent the critical physical effect. For all these reasons we discretize by a second order extrapolation formula (rather than forward Euler). Adding 3 lines of code (time filtering the flow variables) increases the accuracy to $O(\Delta t^2)$ and gives (as noted above) an error estimator to adapt the time step.

The paper is organized as follows. Section 2 gives the coupled Stokes-Darcy model and the associated weak formulation. The BETF algorithm and the long time stability are given in Section 3. Section 4 is devoted to the error analysis of the fully discretized scheme. In Section 5, we introduce BE and BETF algorithm for variable time stepsize and construct adaptive algorithm with performing stepsize selections to control time accuracy and computational efficiency.We presented the numerical tests to illustrate the time accuracy of our numerical methods in Section 6. Final conclusions are given in Section 7.

3.1.1 PREVIOUS WORK ON TIME FILTERS

The first time filter called RA time filter was constructed by Robert[59] and analyzed by Asselin[45]. The combination of RA filter with leapfrog is used to control the leapfrog method's computational mode[73]. Williams [71][72] made an important modification to the RA filter, by proposing RAW time filter which increases the numerical accuracy for amplitude errors from the first order to the third-order accuracy. Li and Trenchea[63] proposed a higher-order Robert-Asselin (hoRA) type time filter which is non-intrusive, easily implementable and achieves third-order accuracy. The approach of using time filters, as herein, is quite recent but has already been shown to increase time accuracy in other flow problems. These include the fully coupled and nonlinear Navier-Stokes equations[52], slightly compressible flow problem [51] and shows promise for variable order, time adaptive method [50].

3.2 THE STOKES-DARCY MODEL AND WEAK FORMULATION

We consider the time-dependent Stokes-Darcy model consisting of Stokes equations and Darcy equations. The Stokes equations which describe the motion of free flow are given by: find the fluid velocity u: $\Omega_f \times [0,T] \to R^d$, the pressure p: $\Omega_f \times [0,T] \to R$ satisfying

$$\frac{\partial u}{\partial t} - \nu \triangle u + \nabla p = f_1 \qquad in\ \Omega_f, \tag{3.1}$$

$$\nabla \cdot u = 0 \qquad in\ \Omega_f. \tag{3.2}$$

In the porous medium region Ω_p, the Darcy equations which describe the behavior of the porous media flow is given by: find hydraulic head ϕ: $\Omega_p \times [0,T] \to R^d$ satisfying

$$S_0 \frac{\partial \phi}{\partial t} + \nabla \cdot u_p = f_2 \qquad in\ \Omega_p, \tag{3.3}$$

$$u_p = -K \nabla \phi \qquad in\ \Omega_p. \tag{3.4}$$

Combining the continuity equation (3.3) with $Darcy's$ law (3.4), we can get the following equation

$$S_0 \frac{\partial \phi}{\partial t} - \nabla \cdot (K \nabla \phi) = f_2 \qquad in\ \Omega_p, \tag{3.5}$$

Here, in Figure 1, $\Omega \in R^d (d = 2\ or\ 3)$ is a bounded domain and $\overline{\Omega} = \overline{\Omega_f} \cup \overline{\Omega_p}$. Ω_f and Ω_p are the fluid region and the porous medium region, respectively. And $\Gamma = \partial\Omega_f \cap \partial\Omega_p$ is the interface between the fluid and the porous media regions. Both Ω_f and Ω_p have Lipschitz continuous boundaries. Define $\Gamma_i = \partial\Omega_i \backslash \Gamma$ for $i = f, p$. Moreover, we denote by n_f and n_p the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, and τ_f the unit tangential vectors on the interface Γ. It is clear that $n_p = -n_f$ on Γ. Here, u and u_p denote the fluid velocity and the

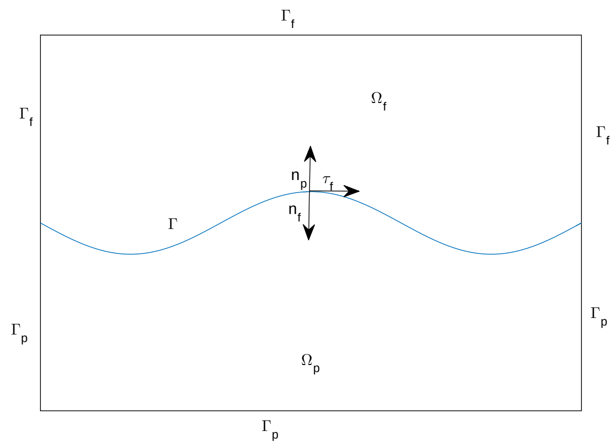


Figure 1: A global domain Ω consisting of the fluid region Ω_f and the porous media region Ω_p separated by the interface Γ.

specific discharge rate in the porous medium, p denotes the kinematic pressure, and f_1 and f_2 denote a general body force in Stokes equations and a source term in Darcy equations, ϕ denotes the hydraulic head, K is the hydraulic conductivity tensor, and S_0 is the soil compressibility. For simplicity, we assume that $K = \{K_{ii}\}_{d\times d}$ is a symmetric and positive definite matrix with the smallest eigenvalue $K_{min} > 0$. It is important to note that in many applications these are not $O(1)$ parameters.

We impose homogeneous Dirichlet boundary conditions and the initial condition:

$$u = 0 \qquad on\ \Gamma_f, \tag{3.6}$$

$$\phi = 0 \qquad on\ \Gamma_p, \tag{3.7}$$

$$u(x,0) = u^0 \qquad in\ \Omega_f, \tag{3.8}$$

$$\phi(x,0) = \phi^0 \qquad in\ \Omega_p. \tag{3.9}$$

The coupling conditions on the interface are the conservation of mass, the balance of normal forces and the Beavers-Joseph-Saffman condition:

$$u \cdot n_f + u_p \cdot n_p = 0 \qquad on\ \Gamma, \tag{3.10}$$

$$p - \nu n_f \cdot \frac{\partial u}{\partial n_f} = g\phi \qquad on\ \Gamma, \tag{3.11}$$

$$-\nu \tau_f \cdot \frac{\partial u}{\partial n_f} = \frac{\alpha \nu \sqrt{d}}{\sqrt{trace(\Pi)}} \tau_f \cdot u \qquad on\ \Gamma. \tag{3.12}$$

Here, d is the space dimension, g is the gravitational acceleration, α is a positive parameter depending on the properties of the medium and must be experimentally determined, and the permeability $\Pi = \frac{K\nu}{g}$. Equation (3.12) is the Beavers-Joseph-Saffman condition.

We introduce the following spaces

$$H_f = \{v \in (H^1(\Omega_f)^d : v = 0\ on\ \partial\Omega_f \backslash \Gamma\},$$
$$H_p = \{\psi \in H^1(\Omega_p) : \psi = 0\ on\ \partial\Omega_p \backslash \Gamma\},$$
$$Q = L_0^2(\Omega_f).$$

For the domain D, $(\cdot, \cdot)_D$ refers to the scalar inner product in D for $D = \Omega_f$ or Ω_p . In particular, we denote the $H^1(\Omega_{f/p})$ norm by $\| \cdot \|_{H_f/H_p}$, the $L^2(\Gamma)$ norm by $\| \cdot \|_\Gamma$ and the $L^2(\Omega_{f/p})$ norm by $\| \cdot \|_{f/p}$, and define the corresponding norms and the notation hereafter:

$$\|u\|_f = \|u\|_{L^2(\Omega_f)}, \qquad \|u\|_{H_f} = \|\nabla u\|_{L^2(\Omega_f)},$$

$$\|\phi\|_p = \|\phi\|_{L^2(\Omega_p)}, \qquad \|\phi\|_{H_p} = \|\nabla \phi\|_{L^2(\Omega_p)}.$$

With these notations, the weak formulation of the coupled Stokes-Darcy problem is given as follows: find $(u, \phi) \in H_f \times H_p$ and $p \in Q$ such that, $\forall t \in (0, T]$,

$$(u_t, v)_{\Omega_f} + gS_0(\phi_t, \psi)_{\Omega_p} + a_f(u, v) + a_p(\phi, \psi) + c_\Gamma(v, \phi) - c_\Gamma(u, \psi) + b(v, p)$$
$$= (f_1, v)_{\Omega_f} + g(f_2, \psi)_{\Omega_p} \quad \forall v \in H_f, \psi \in H_p, \tag{3.13}$$
$$b(u, q) = 0 \quad \forall q \in Q, \tag{3.14}$$

where

$$
\begin{aligned}
a_f(u,v) &= \nu(\nabla u, \nabla v)_{\Omega_f} + a_\Gamma(u,v),\\
a_p(\phi,\psi) &= g(K\nabla\phi, \nabla\psi)_{\Omega_p},\\
c_\Gamma(v,\psi) &= g\int_\Gamma \psi v\cdot n_f,\\
b(v,q) &= -(p, \nabla\cdot v)_{\Omega_f}.\\
a_\Gamma(u,v) &= \sum_{i=1}^{d-1}\int_\Gamma \frac{\alpha\sqrt{\nu g}}{\sqrt{trace(\Pi)}}(u\cdot\tau_i)(v\cdot\tau_i).
\end{aligned}
$$

For further investigation, we also recall the Poincaré, trace and Sobolev inequalities that are useful in the following analysis. There exist constants $C_d,\ C_s$, which depend only on the domain Ω_f, and $\widetilde{C}_d, \widetilde{C}_s$, which depend only on the domain Ω_p, such that, for all $v \in H_f$ and $\phi \in H_p$,

$$\|v\|_f \le C_d\|v\|_{H_f}, \qquad \|\phi\|_p \le \widetilde{C}_d\|\phi\|_{H_p}. \tag{3.15}$$

$$\|v\|_\Gamma \le C_s\|v\|_f^{\frac12}\|\nabla v\|_f^{\frac12}, \quad \|\phi\|_\Gamma \le \widetilde{C}_s\|\phi\|_p^{\frac12}\|\nabla\phi\|_p^{\frac12}. \tag{3.16}$$

Lemma 7. *[62] There exist constants $C_1 = C_s^2\widetilde{C}_s^2 \ge 0$ and $C_2 = C_d\widetilde{C}_d \ge 0$, such that for all $(v,\phi) \in H_f \times H_p$ and $\epsilon, \epsilon_1, \epsilon_2 > 0$, we have*

$$c_\Gamma(v,\phi) \le \epsilon\nu\|v\|_{H_f}^2 + \frac{g^2C_1C_2}{4\epsilon\nu}\|\phi\|_{H_p}^2, \tag{3.17}$$

$$c_\Gamma(v,\phi) \le \epsilon g\|K^{\frac12}\nabla\phi\|_p^2 + \frac{gC_1C_2}{4\epsilon K_{min}}\|u\|_{H_f}^2, \tag{3.18}$$

$$c_\Gamma(v,\phi) \le \epsilon_1\nu\|v\|_{H_f}^2 + \epsilon_2\|K^{\frac12}\nabla\phi\|_p^2 + \frac{g^4C_1^2C_d^2}{64\epsilon_1^2\epsilon_2\nu^2K_{min}}\|\phi\|_p^2, \tag{3.19}$$

$$c_\Gamma(v,\phi) \le \epsilon_1\|K^{\frac12}\nabla\phi\|_p^2 + \epsilon_2\nu\|v\|_{H_f}^2 + \frac{g^4C_1^2\widetilde{C}_d^2}{64\epsilon_1^2\epsilon_2\nu K_{min}^2}\|v\|_f^2, \tag{3.20}$$

$$c_\Gamma(v,\phi) \le \epsilon\nu\|v\|_{H_f}^2 + \frac{g^2C_1\widetilde{C}_IC_d}{4\epsilon\nu h}\|\phi\|_p^2, \tag{3.21}$$

$$c_\Gamma(v,\phi) \le \epsilon g\|K^{\frac12}\nabla\phi\|_p^2 + \frac{g^2C_1\widetilde{C}_dC_I}{4\epsilon gK_{min}h}\|u\|_f^2. \tag{3.22}$$

3.3 NUMERICAL ALGORITHMS

In this section, we propose the decoupled scheme for the coupled Stokes-Darcy model. We choose a uniform partition of $[0,T]$ with $t_m = m\Delta t,\ m = 0,1,...,N,$ where $\Delta t = \frac{T}{N}$, and (u^m, p^m, ϕ^m) denotes the discrete approximation in time by following schemes to $(u(t_m), p(t_m), \phi(t_m))$. These are presented below for constant time steps. Their variable Δt and adaptive versions are in later section.

3.3.1 BACKWARD EULER PLUS TIME FILTER (BETF)

- Given (u^0, p^0, ϕ^0) and (u^1, p^1, ϕ^1). Find $(\hat{u}^{m+1}, \hat{p}^{m+1}) \in (H_f, Q)$ with $m = 0, 1, ..., N-2$, such that for any $v \in H_f$, and $q \in Q$,

$$(\frac{\hat{u}^{m+1} - u^m}{\Delta t}, v)_{\Omega_f} - a_f(\hat{u}^{m+1}, v) + b(v, \hat{p}^{m+1}) = (f_1^{m+1}, v)_{\Omega_f} - c_\Gamma(v, 2\phi^m - \phi^{m-1}), \tag{3.23}$$

$$b(\hat{u}^{m+1}, q) = 0. \tag{3.24}$$

- Find $\hat{\phi}^{m+1} \in H_p$, with $m = 1, ..., N-2$, such that for any $\psi \in H_p$,

$$gS_0(\frac{\hat{\phi}^{m+1} - \phi^m}{\Delta t}, \psi)_{\Omega_p} + a_p(\hat{\phi}^{m+1}, \psi) = g(f_2^{m+1}, \psi)_{\Omega_p} + c_\Gamma(2u^m - u^{m-1}, \psi). \tag{3.25}$$

- Apply *time filter* to update the previous solutions $(\hat{u}^{m+1}, \hat{p}^{m+1}, \hat{\phi}^{m+1})$,

$$u^{m+1} = \hat{u}^{m+1} - \frac{1}{3}(\hat{u}^{m+1} - 2u^m + u^{m-1}), \tag{3.26}$$

$$p^{m+1} = \hat{p}^{m+1} - \frac{1}{3}(\hat{p}^{m+1} - 2p^m + p^{m-1}), \tag{3.27}$$

$$\phi^{m+1} = \hat{\phi}^{m+1} - \frac{1}{3}(\hat{\phi}^{m+1} - 2\phi^m + \phi^{m-1}). \tag{3.28}$$

3.3.2 EQUIVALENT BACKWARD EULER PLUS TIME FILTER

To analyze the algorithm we will eliminate the intermediate variables and reduce BETF to an equivalent 2 step method. This reduction is a repetition of the NSE case in [52] so we omit the routine algebraic details, yielding the following.

- Given (u^0, p^0, ϕ^0) and (u^1, p^1, ϕ^1), find $(u^{m+1}, p^{m+1}) \in (H_f, Q)$, with $m = 1, ..., N-1$, such that for any $v \in H_f$, and $q \in Q$,

$$\begin{aligned}
&(\frac{3u^{m+1} - 4u^m + u^{m-1}}{2\Delta t}, v)_{\Omega_f} - a_f(\frac{3}{2}u^{m+1} - u^m + \frac{1}{2}u^{m-1}, v) \\
&+ b(v, \frac{3}{2}p^{m+1} - p^m + \frac{1}{2}p^{m-1}) = (f_1^{m+1}, v)_{\Omega_f} - c_\Gamma(v, 2\phi^m - \phi^{m-1}), \qquad (3.29) \\
&b(\frac{3}{2}u^{m+1} - u^m + \frac{1}{2}u^{m-1}, q) = 0. \qquad (3.30)
\end{aligned}$$

- Given ϕ^0 and ϕ^1, find $\phi^{m+1} \in H_p$, with $m = 1, ..., N-1$, such that for any $\psi \in H_p$,

$$\begin{aligned}
&gS_0(\frac{3\phi^{m+1} - 4\phi^m + \phi^{m-1}}{2\Delta t}, \psi)_{\Omega_p} + a_p(\frac{3}{2}\phi^{m+1} - \phi^m + \frac{1}{2}\phi^{m-1}, \psi) \\
&= g(f_2^{m+1}, \psi)_{\Omega_p} + c_\Gamma(2u^m - u^{m-1}, \psi).
\end{aligned} \qquad (3.31)$$

Define the following difference operators:

$$A(u^{m+1}) = \frac{3}{2}u^{m+1} - 2u^m + \frac{1}{2}u^{m-1}, B(u^{m+1}) = \frac{3}{2}u^{m+1} - u^m + \frac{1}{2}u^{m-1}.$$

There are some important identities we need to use in the later section.

$$\begin{aligned}
&(A(u^{m+1}), u^{m+1})_{\Omega_f} = \frac{\|u^{m+1}\|_f^2 + \|2u^{m+1} - u^m\|_f^2}{4} \\
&- \frac{\|u^m\|_f^2 + \|2u^m - u^{m-1}\|_f^2}{4} + \frac{\|u^{m+1} - 2u^m + u^{m-1}\|_f^2}{4}, \\
&(B(u^{m+1}), u^{m+1})_{\Omega_f} = \frac{3\|u^{m+1}\|_f^2 + \|u^m\|_f^2}{4} - \frac{3\|u^m\|_f^2 + \|u^{m-1}\|_f^2}{4} \\
&+ \frac{\|u^{m+1} - u^m\|_f^2}{2} + \frac{\|u^{m+1} + u^{m-1}\|_f^2}{4}.
\end{aligned} \qquad (3.32)$$

3.3.3 TIME STABILITY OF THE DECOUPLED SCHEME

In this section, we prove the long time stability of BETF decoupled scheme for constant time step. Assume u^1 and u^0 is divergence free, i.e., $b(u^i, q) = 0, \forall q \in Q, i = 1, 0$. From (3.30), it is easy to know $b(u^i, q) = 0, \forall q \in Q, i \geq 0$. Long time stability holds under a condition (below) relating Δt to problem parameters. We make this condition explicit as parameters can vary from large to very small in different applications.

Theorem 1. *Define Energy*

$$
\begin{aligned}
E^{m+1} =& \frac{1}{4}(\|u^{m+1}\|_f^2 + \|2u^{m+1} - u^m\|_f^2) + \frac{\Delta t}{4}\big(3a_f(u^{m+1}, u^{m+1}) + a_f(u^m, u^m)\big) \\
&+ \frac{gS_0}{4}(\|\phi^{m+1}\|_p^2 + \|2\phi^{m+1} - \phi^m\|_p^2) + \frac{\Delta t}{4}\big(3a_p(\phi^{m+1}, \phi^{m+1}) + a_p(\phi^m, \phi^m)\big).
\end{aligned}
\tag{3.33}
$$

Suppose Δt satisfies the time step condition $\Delta t \leq min\{\frac{\nu K_{min}}{288C_1^2\widetilde{C}_d^2 g^2}, \frac{\nu^2 K_{min} S_0}{288C_1^2 C_d^2 g^2}\}$, then the decoupled scheme is stable uniformly in time and there holds

$$
\begin{aligned}
&E^N + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\leq& E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{3\nu\Delta t}{8}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) + \frac{3g\Delta t}{8}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2).
\end{aligned}
\tag{3.34}
$$

If there is no restriction on Δt, the decoupled scheme is stable in finite time and there holds

$$
\begin{aligned}
&E^N + \frac{1}{4}\sum_{m=1}^{N-1}\Big(\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + gS_0\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2\Big) \\
&+ \frac{13\nu\Delta t}{36}\sum_{m=1}^{N-1}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{9}\sum_{m=1}^{N-1}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{13g\Delta t}{36}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{9}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\leq& C(T)\Big\{E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{\nu\Delta t}{2}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) + \frac{g\Delta t}{2}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2)\Big\}.
\end{aligned}
\tag{3.35}
$$

with $C(T) = exp\big(\sum_{m=0}^{N} max\{\frac{CC_1^2C_d^2g^2\Delta t}{\nu^2K_{min}S_0}, \frac{CC_1^2\tilde{C}_d^2g^2\Delta t}{\nu K_{min}^2}\}\big)$.

Proof. In (3.29)-(3.31), we set $v = \Delta t u^{m+1}$, $q = \Delta t p^{m+1}$ and $\psi = \Delta t \phi^{m+1}$, and add them together

$$
\begin{aligned}
&(A(u^{m+1}), u^{m+1})_{\Omega_f} + gS_0(A(\phi^{m+1}), \phi^{m+1})_{\Omega_p} \\
&+ \Delta t a_f(B(u^{m+1}), u^{m+1}) + \Delta t a_p(B(\phi^{m+1}), \phi^{m+1}) \\
=&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1}))_{\Omega_p} \\
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m-1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}).
\end{aligned}
\tag{3.36}
$$

From (3.32) and (3.36), we get

$$
\begin{aligned}
&\frac{1}{4}(\|u^{m+1}\|_f^2 + \|2u^{m+1} - u^m\|_f^2) + \frac{\Delta t}{4}\big(3a_f(u^{m+1}, u^{m+1}) + a_f(u^m, u^m)\big) \\
&+ \frac{gS_0}{4}(\|\phi^{m+1}\|_p^2 + \|2\phi^{m+1} - \phi^m\|_p^2) + \frac{\Delta t}{4}\big(3a_p(\phi^{m+1}, \phi^{m+1}) + a_p(\phi^m, \phi^m)\big) \\
&- \frac{1}{4}(\|u^m\|_f^2 + \|2u^m - u^{m-1}\|_f^2) - \frac{\Delta t}{4}\big(3a_f(u^m, u^m) + a_f(u^{m-1}, u^{m-1})\big) \\
&- \frac{gS_0}{4}(\|\phi^m\|_p^2 + \|2\phi^m - \phi^{m-1}\|_p^2) - \frac{\Delta t}{4}\big(3a_p(\phi^m, \phi^m) + a_p(\phi^{m-1}, \phi^{m-1})\big) \\
&+ \frac{1}{4}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + \frac{gS_0}{4}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{\nu\Delta t}{2}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{4}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{2}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{4}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
=&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1})_{\Omega_p} \\
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m+1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}).
\end{aligned}
\tag{3.37}
$$

Then we can rearrange the equality

$$
\begin{aligned}
&E^{m+1} - E^m + \frac{1}{4}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + \frac{gS_0}{4}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{\nu\Delta t}{2}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{4}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{2}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{4}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
=&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1})_{\Omega_p} \\
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m+1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}).
\end{aligned}
\tag{3.38}
$$

Note that $u^{m+1} = \frac{1}{2}(u^{m+1} - u^m + u^{m+1} + u^{m-1} + u^m - u^{m-1})$ and $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, by using Young and Hölder inequalities, we have

$$
\begin{aligned}
&\Delta t(f_1^{m+1}, u^{m+1})_{\Omega_f} + g\Delta t(f_2^{m+1}, \phi^{m+1})_{\Omega_p} \\
\leq & \frac{3C_d^2 \Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2 \Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 + \frac{\nu \Delta t}{12} \parallel \nabla u^{m+1} \parallel_f^2 \\
&+ \frac{g\Delta t}{12} \parallel K^{\frac{1}{2}} \nabla \phi^{m+1} \parallel_p^2 \\
\leq & \frac{\nu \Delta t}{16} \Big(\|\nabla(u^{m+1} - u^m)\|_f^2 + \|\nabla(u^{m+1} + u^{m-1})\|_f^2 + \|\nabla(u^m - u^{m-1})\|_f^2 \Big) \\
&+ \frac{g\Delta t}{16} \Big(\|K^{\frac{1}{2}} \nabla(\phi^{m+1} - \phi^m)\|_p^2 \\
&+ \|K^{\frac{1}{2}} \nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 + \|K^{\frac{1}{2}} \nabla(\phi^m - \phi^{m-1})\|_p^2 \Big) \\
&+ \frac{3C_d^2 \Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2 \Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 .
\end{aligned}
\tag{3.39}
$$

Take $\epsilon_1 = \frac{1}{12}$, $\epsilon_2 = \frac{g}{32}$ in (3.19) and $\epsilon_1 = \frac{g}{12}$, $\epsilon_2 = \frac{1}{32}$ in (3.20), for the interface terms on the right hand side of (3.38),

$$
\begin{aligned}
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m-1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}) \\
=& - \Delta t c_\Gamma(u^{m+1}, \phi^{m+1} - 2\phi^m + \phi^{m-1}) + \Delta t c_\Gamma(u^{m+1} - 2u^m + u^{m-1}, \phi^{m+1}) \\
\leq & \frac{\nu \Delta t}{12} \|\nabla u^{m+1})\|_f^2 + \frac{g\Delta t}{32} \|K^{\frac{1}{2}}(\phi^{m+1} - 2\phi^m + \phi^{m-1})\|_p^2 \\
&+ \frac{g\Delta t}{12} \|K^{\frac{1}{2}} \nabla \phi^{m+1}\|_p^2 + \frac{\nu \Delta t}{32} \|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
&+ \frac{72 C_1^2 C_d^2 g^3 \Delta t}{\nu^2 K_{min}} \|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{72 C_1^2 \tilde{C}_d g^2 \Delta t}{\nu K_{min}^2} \|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
\leq & \frac{\nu \Delta t}{16} \Big(2\|\nabla(u^{m+1} - u^m)\|_f^2 + \|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ 2\|\nabla(u^m - u^{m-1})\|_f^2 \Big) \\
&+ \frac{g\Delta t}{16} \Big(2\|K^{\frac{1}{2}} \nabla(\phi^{m+1} - \phi^m)\|_p^2 + \|K^{\frac{1}{2}} \nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
&+ 2\|K^{\frac{1}{2}} \nabla(\phi^m - \phi^{m-1})\|_p^2 \Big) + \frac{72 C_1^2 C_d^2 g^3 \Delta t}{\nu^2 K_{min}} \|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{72 C_1^2 \tilde{C}_d g^2 \Delta t}{\nu K_{min}^2} \|u^{m+1} - 2u^m + u^{m-1}\|_f^2 .
\end{aligned}
\tag{3.40}
$$

Inserting (3.40) and (3.39) to (3.38), we arrive at

$$
\begin{aligned}
&E^{m+1} - E^m + (\frac{1}{4} - \frac{72C_1^2\tilde{C}_d g^2\Delta t}{\nu K_{min}^2})\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
&+ (\frac{gS_0}{4} - \frac{72C_1^2C_d^2g^3\Delta t}{\nu^2 K_{min}})\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{5\nu\Delta t}{16}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{5g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 \\
&+ \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\leq& \frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{3\nu\Delta t}{16}\|\nabla(u^m - u^{m-1})\|_f^2 + \frac{3g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\phi^m - \phi^{m-1})\|_p^2.
\end{aligned}
\tag{3.41}
$$

If Δt satisfies the time step condition $\Delta t \leq min\{\frac{\nu K_{min}}{288C_1^2\widetilde{C}_d^2 g^2}, \frac{\nu^2 K_{min} S_0}{288C_1^2C_d^2g^2}\}$, summing up (3.41) from $m = 1$ to $N - 1$ leads to

$$
\begin{aligned}
&E^N + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{8}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{8}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\leq& E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{3\nu\Delta t}{8}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) \\
&+ \frac{3g\Delta t}{8}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2).
\end{aligned}
\tag{3.42}
$$

Thus we complete the proof of the uniform in time stability. Next we are going to prove the unconditional, finite time stability of BETF scheme. Using $2u^m - u^{m-1} = -\frac{1}{2}(u^{m+1} - u^m) + \frac{3}{2}(u^m - u^{m-1}) + \frac{1}{2}(u^{m+1} + u^{m-1})$, $u^{m+1} = \frac{1}{2}(u^{m+1} - u^m + u^{m+1} + u^{m-1} + u^m - u^{m-1})$ and

$(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and taking $\epsilon_1 = \frac{1}{12}$, $\epsilon_2 = \frac{g}{54}$ in (3.19) and $\epsilon_1 = \frac{g}{12}$, $\epsilon_2 = \frac{1}{54}$ in (3.20), for the interface term on the right hand side of (3.38),

$$
\begin{aligned}
&- \Delta t c_\Gamma(u^{m+1}, 2\phi^m - \phi^{m-1}) + \Delta t c_\Gamma(2u^m - u^{m-1}, \phi^{m+1}) \\
\le& \frac{\nu \Delta t}{12}\|\nabla u^{m+1})\|_f^2 + \frac{g\Delta t}{54}\|K^{\frac{1}{2}}(2\phi^m - \phi^{m-1})\|_p^2 \\
&+ \frac{243 C_1^2 C_d^2 g^3 \Delta t}{2\nu^2 K_{min}}\|2\phi^m - \phi^{m-1}\|_p^2 \\
&+ \frac{g\Delta t}{12}\|K^{\frac{1}{2}}\nabla\phi^{m+1}\|_p^2 + \frac{\nu\Delta t}{54}\|2u^m - u^{m-1}\|_f^2 \\
&+ \frac{243 C_1^2 \tilde{C}_d g^2 \Delta t}{2\nu K_{min}^2}\|2u^m - u^{m-1}\|_f^2 \\
\le& \frac{\nu\Delta t}{144}\Big(11\|\nabla(u^{m+1} - u^m)\|_f^2 + 11\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ 27\|\nabla(u^m - u^{m-1})\|_f^2\Big) + \frac{g\Delta t}{144}\Big(11\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 \\
&+ 11\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 + 27\|K^{\frac{1}{2}}\nabla(\phi^m - \phi^{m-1})\|_p^2\Big) \\
&+ \frac{243 C_1^2 C_d^2 g^3 \Delta t}{2\nu^2 K_{min}}\|2\phi^m - \phi^{m-1}\|_p^2 + \frac{243 C_1^2 \tilde{C}_d g^2 \Delta t}{2\nu K_{min}^2}\|2u^m - u^{m-1}\|_f^2.
\end{aligned}
\tag{3.43}
$$

Inserting (3.43) and (3.39) to (3.38), we arrive at

$$
\begin{aligned}
&E^{m+1} - E^m + \frac{1}{4}\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 \\
&+ \frac{gS_0}{4}\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2 \\
&+ \frac{13\nu\Delta t}{36}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{9}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{13 g\Delta t}{36}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 \\
&+ \frac{g\Delta t}{9}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\le& \frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \frac{3g\widetilde{C}_d^2 \Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{\nu\Delta t}{4}\|\nabla(u^m - u^{m-1})\|_f^2 + \frac{g\Delta t}{4}\|K^{\frac{1}{2}}\nabla(\phi^m - \phi^{m-1})\|_p^2 \\
&+ \frac{243 C_1^2 C_d^2 g^3 \Delta t}{2\nu^2 K_{min}}\|2\phi^m - \phi^{m-1}\|_p^2 + \frac{243 C_1^2 \tilde{C}_d^2 g^2 \Delta t}{2\nu K_{min}^2}\|2u^m - u^{m-1}\|_f^2.
\end{aligned}
\tag{3.44}
$$

Summing up (3.41) from $m = 1$ to $N - 1$ leads to

$$
\begin{aligned}
&E^N + \frac{1}{4}\sum_{m=1}^{N-1}\Big(\|u^{m+1} - 2u^m + u^{m-1}\|_f^2 + gS_0\|\phi^{m+1} - 2\phi^m + \phi^{m-1}\|_p^2\Big) \\
&+ \frac{13\nu\Delta t}{36}\sum_{m=1}^{N-1}\|\nabla(u^{m+1} - u^m)\|_f^2 + \frac{\nu\Delta t}{9}\sum_{m=1}^{N-1}\|\nabla(u^{m+1} + u^{m-1})\|_f^2 \\
&+ \frac{13g\Delta t}{36}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} - \phi^m)\|_p^2 + \frac{g\Delta t}{9}\sum_{m=1}^{N-1}\|K^{\frac{1}{2}}\nabla(\phi^{m+1} + \phi^{m-1})\|_p^2 \\
\leq &E^1 + \sum_{m=1}^{N-1}\frac{3C_d^2\Delta t}{\nu} \parallel f_1^{m+1} \parallel_f^2 + \sum_{m=1}^{N-1}\frac{3g\widetilde{C}_d^2\Delta t}{K_{min}} \parallel f_2^{m+1} \parallel_p^2 \\
&+ \frac{\nu\Delta t}{2}(\|\nabla u^1\|_f^2 + \|\nabla u^0\|_f^2) + \frac{g\Delta t}{2}(\|K^{\frac{1}{2}}\nabla\phi^1\|_p^2 + \|K^{\frac{1}{2}}\nabla\phi^0\|_p^2) \\
&+ \frac{CC_1^2C_d^2g^2\Delta t}{\nu^2K_{min}S_0}\sum_{m=0}^{N-1}\frac{gS_0}{4}\|\phi^m\|_p^2 + \frac{CC_1^2\tilde{C}_d^2g^2\Delta t}{\nu K_{min}^2}\sum_{m=0}^{N-1}\frac{1}{4}\|u^m\|_f^2.
\end{aligned}
\tag{3.45}
$$

Applying the discrete Gronwall inequality, we get (3.35). □

3.4 ERROR ANALYSIS

This section gives an analysis of the error(constant time stepsize) of the fully discrete BETF algorithm, where spatial discretization is performed using finite element methods (FEMs). To discretize the Stokes-Darcy problem in space by finite element method, let h_i be a positive parameter and $\mathcal{T}_{h_i}$ be a regular partition of triangular or quadrilateral elements of Ω_i, $i = f, p$. We select continuous piecewise polynomials of degrees k, k, and $k-1$ for the finite element spaces $H_f^h \subset H_f, H_p^h \subset H_p, Q^h \subset Q$ which are conforming finite element spaces. We assume the fluid velocity space H_f^h and the pressure space Q^h satisfy the discrete inf-sup condition: there exists a positive constant γ, independent of h, such that, $\forall q \in Q^h, \exists\, v \in H_f^h, v \neq 0,$

$$
b(v, q) \geq \gamma\|v\|_{H_f}\|q\|_f. \tag{3.46}
$$

Furthermore, we will use the inverse inequalities: there exist constants $C_I, \widetilde{C}_I$, which depend on the angles in the finite element mesh, such that, for all $v \in H_f^h$ and $\phi \in H_p^h$,

$$\|\nabla v\|_f \le C_I h^{-1}\|v\|_f, \quad \|\nabla \phi\|_p \le \widetilde{C}_I h^{-1}\|\phi\|_p. \tag{3.47}$$

The fully discrete approximation of BETF algorithm is: Given (u_h^0, p_h^0, ϕ_h^0) and (u_h^1, p_h^1, ϕ_h^1), find $(u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}) \in (H_f^h, Q^h, H_p^h)$, with $m = 1, ..., N-1$, such that for any $v_h \in H_f^h$, $\psi_h \in H_p^h$ and $q_h \in Q^h$,

$$\begin{aligned}&(\frac{3u_h^{m+1} - 4u_h^m + u_h^{m-1}}{2\Delta t}, v_h)_{\Omega_f} - a_f(\frac{3}{2}u_h^{m+1} - u_h^m + \frac{1}{2}u_h^{m-1}, v_h)\\ &+ b(v_h, \frac{3}{2}p_h^{m+1} - p_h^m + \frac{1}{2}p_h^{m-1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, 2\phi_h^m - \phi_h^{m-1}),\end{aligned} \tag{3.48}$$

$$b(\frac{3}{2}u_h^{m+1} - u_h^m + \frac{1}{2}u_h^{m-1}, q_h) = 0, \tag{3.49}$$

$$\begin{aligned}&gS_0(\frac{3\phi_h^{m+1} - 4\phi_h^m + \phi_h^{m-1}}{2\Delta t}, \psi_h)_{\Omega_p} + a_p(\frac{3}{2}\phi_h^{m+1} - \phi_h^m + \frac{1}{2}\phi_h^{m-1}, \psi_h)\\ &= g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(2u_h^m - u_h^{m-1}, \psi_h).\end{aligned} \tag{3.50}$$

We assume that the solution of Stokes-Darcy problem satisfies the following regularity: $(u(t), \phi(t)) \in (H^{k+1}(\Omega_f)^d, H^{k+1}(\Omega_p))$ and $p(t) \in H^k(\Omega_f)$, define the linear projection operator $P : (u(t), \phi(t), p(t)) \in (H_f, H_p, Q) \to (\tilde{u}(t), \tilde{\phi}(t), \tilde{p}(t)) \in (H_f^h, H_p^h, Q^h), \forall t \in [0, T]$ by:

$$\begin{aligned}&a_f(u(t), v) + c_\Gamma(v, \phi(t)) + b(v, p(t)) + a_p(\phi(t), \psi) - c_\Gamma(u(t), \psi)\\ &= a_f(\tilde{u}(t), v) + c_\Gamma(v, \tilde{\phi}(t)) + b(v, \tilde{p}(t))\\ &+ a_p(\tilde{\phi}(t), \psi) - c_\Gamma(\tilde{u}(t), \psi) \quad \forall v(t) \in H_f^h, \psi(t) \in H_p^h,\end{aligned} \tag{3.51}$$

$$b(\tilde{u}(t), q) = 0 \quad \forall q(t) \in Q^h, \tag{3.52}$$

then we have the following error estimates:

$$\|\tilde{u}(t) - u(t)\|_f + h\|\tilde{u}(t) - u(t)\|_{H_f} \le Ch^{k+1}\|u(t)\|_{H^{k+1}(\Omega_f)^d}, \tag{3.53}$$

$$\|\tilde{\phi}(t) - \phi(t)\|_p + h\|\tilde{\phi}(t) - \phi(t)\|_{H_p} \le Ch^{k+1}\|\phi(t)\|_{H^{k+1}(\Omega_p)}, \tag{3.54}$$

$$\|\tilde{p}(t) - p(t)\|_f \le Ch^k\|p(t)\|_{H^k(\Omega_f)}. \tag{3.55}$$

For $\forall(v_h, \psi_h, q_h) \in (H_f^h, H_p^h, Q^h)$, the true solution $(u(t_{m+1}), p(t_{m+1}), \phi(t_{m+1}))$ satisfies:

$$\begin{aligned}
&(\frac{A(u(t_{m+1}))}{\Delta t}, v_h)_{\Omega_f} + a_f(B(u(t_{m+1})), v_h) + b(v_h, B(p(t_{m+1}))) \\
&= (\xi_f^{m+1}, v_h)_{\Omega_f} + (f_1^{m+1}, v_h)_{\Omega_f} - g\int_\Gamma \phi(t_{m+1}) v_h \cdot n_f \\
&+ a_f(B(u(t_{m+1})) - u(t_{m+1}), v_h) + b(v_h, B(p(t_{m+1})) - p(t_{m+1})),
\end{aligned} \tag{3.56}$$

$$- b(u(t_{m+1}), q_h) = 0, \tag{3.57}$$

$$\begin{aligned}
&gS_0(\frac{A(\phi(t_{m+1}))}{\Delta t}, \psi_h)_{\Omega_p} + a_p(B(\phi(t_{m+1})), \psi_h) = gS_0(\xi_p^{m+1}, \psi_h)_{\Omega_p} \\
&+ g(f_2^{m+1}, \psi_h)_{\Omega_p} + g\int_\Gamma \psi_h u(t_{m+1}) \cdot n_f + a_p(B(\phi(t_{m+1})) - \phi(t_{m+1}), \psi_h),
\end{aligned} \tag{3.58}$$

where, ξ_f^{m+1}, ξ_p^{m+1} are defined by

$$\xi_f^{m+1} := \tfrac{A(u(t_{m+1}))}{\Delta t} - u_t(t_{m+1}),\ \xi_p^{m+1} := \tfrac{B(\phi(t_{m+1}))}{2\Delta t} - \phi_t(t_{m+1}). \tag{3.59}$$

To derive the error estimates of BETF algorithm, we first give this method's consistency error.

Lemma 8. *The following inequalities hold:*

$$\|\tfrac{A(u(t_{m+1}))}{\Delta t}\|_f^2 \leq \tfrac{9}{2\triangle t}\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_f^2 dt, \tag{3.60}$$

$$\|\xi_f^{m+1}\|_f^2 \leq C\triangle t^3 \int_{t_m}^{t_{m+2}} \|u_{ttt}(t)\|_f^2 dt, \tag{3.61}$$

$$\|\widetilde{u}(t_{m+2}) - 2\widetilde{u}(t_{m+1}) + \widetilde{u}(t_m)\|_{H_f}^2 \leq C\triangle t^3 \int_{t_m}^{t_{m+2}} \|u_{tt}\|_{H_f}^2 dt, \tag{3.62}$$

$$\|B(u(t_{m+1})) - u(t_{m+1})\|_{H_f}^2 \leq C\triangle t^3 \int_{t_{m-1}}^{t_{m+1}} \|u_{tt}\|_{H_f}^2 dt, \tag{3.63}$$

$$\|\tfrac{A(\phi(t_{m+1}))}{\Delta t}\|_p^2 \leq \tfrac{9}{2\triangle t}\int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_p^2 dt, \tag{3.64}$$

$$\|\xi_p^{m+2}\|_p^2 \leq C\triangle t^3 \int_{t_m}^{t_{m+2}} \|\phi_{ttt}(t)\|_p^2 dt, \tag{3.65}$$

$$\|\widetilde{\phi}(t_{m+2}) - 2\widetilde{\phi}(t_{m+1}) - \widetilde{\phi}(t_m)\|_{H_p}^2 \leq C\triangle t^3 \int_{t_m}^{t_{m+2}} \|\phi_{tt}\|_{H_p}^2 dt, \tag{3.66}$$

$$\|B(\phi(t_{m+1})) - \phi(t_{m+1})\|_{H_p}^2 \leq C\triangle t^3 \int_{t_{m-1}}^{t_{m+1}} \|\phi_{tt}\|_{H_p}^2 dt. \tag{3.67}$$

Proof. The proof is similar to the Lemma 2 in [62]. □

Theorem 2. *For any $0 < t_N = T < \infty$, assume the solution satisfies the following regularity condition*

$$
\begin{aligned}
&u \in H^1(0,T;H^{k+1}(\Omega_f)) \cap H^2(0,T;H^1(\Omega_f)) \cap H^3(0,T;L^2(\Omega_f)),\\
&\phi \in H^1(0,T;H^{k+1}(\Omega_p)) \cap H^2(0,T;H^1(\Omega_p)) \cap H^3(0,T;L^2(\Omega_p)),\\
&p_{tt} \in L^2(0,T,L_0^2(\Omega_f)),
\end{aligned}
\tag{3.68}
$$

and Δt satisfies $\Delta t \le min\{\frac{\nu K_{min}}{CC_1^2\tilde{C}_d^2 g^2}, \frac{\nu^2 K_{min} S_0}{CC_1^2 C_d^2 g^2}\}$, then there exists a constant C independent of h and Δt, such that

$$
\begin{aligned}
&\|u(t_N) - u_h^N\|_f^2 + \|\phi(t_N) - \phi_h^N\|_p^2\\
&+ \sum_{m=1}^{N-1} \big(\frac{11}{8}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \frac{11}{8} g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2\big)\\
&+ \sum_{m=1}^{N-1} \big(\frac{3}{4} g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2\big)\\
\le & C\Delta t^4\Big(\int_0^T \|u_{ttt}\|_f^2 dt + \int_0^T \|\phi_{ttt}\|_p^2 dt\Big)\\
&+ C\Delta t^4\Big(\int_0^T \|\phi_{tt}\|_{H_p}^2 dt + \int_0^T \|u_{tt}\|_{H_f}^2 dt + \int_0^T \|p_{tt}\|_f^2 dt\Big)\\
&+ Ch^{2k+2}\Big(\int_0^T \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_0^T \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big)\\
&+ C\Delta t(\|\nabla\eta_u^1\|_f^2 + \|\nabla\eta_u^0\|_f^2) + C\Delta t(\|K^{\frac{1}{2}}\nabla(\eta_\phi^1)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^0)\|_p^2)\\
&+ (\|\eta_u^1\|^2 + \|2\eta_u^1 - \eta_u^0\|^2) + gS_0(\|\eta_\phi^1\|^2 + \|2\eta_\phi^1 - \eta_\phi^0\|^2).
\end{aligned}
\tag{3.69}
$$

Here and afterwards, we denote by C a generic positive constant which depends on the physical parameters ($\nu,\ g,\ S_0,\ K_{min}$), and it may has different values at different occasions.

Proof. Subtracting (3.56)-(3.58) from (3.48)-(3.50), we have the following error equations:

$$
\begin{aligned}
&(\frac{A(u(t_{m+1}) - u_h^{m+1})}{\Delta t}, v_h)_{\Omega_f} - a_f(B(u(t_{m+1}) - u_h^{m+1}), v_h) + b(v_h, B(p(t_{m+1}) - p_h^{m+1}))\\
&= (\xi_f^{m+1}, v_h)_{\Omega_f} - g\int_\Gamma (\phi(t_{m+1}) - 2\phi_h^m + \phi_h^{m-1}) v_h \cdot n_f\\
&+ a_f(B(u(t_{m+1})) - u(t_{m+1}), v_h) + b(v_h, B(p(t_{m+1})) - p(t_{m+1})),
\end{aligned}
\tag{3.70}
$$

$$
b(u(t_{m+1}) - B(u_h^{m+1}), q_h) = 0,
\tag{3.71}
$$

$$
\begin{aligned}
&gS_0(\frac{A(\phi(t_{m+1})-\phi_h^{m+1})}{\Delta t},\psi_h)_{\Omega_p}+a_p(B(\phi(t_{m+1})-\phi_h^{m+1}),\psi_h)\\
&=gS_0(\xi_p^{m+1},\psi_h)_{\Omega_p}+g\int_\Gamma\psi_h(u(t_{m+1})-2u_h^m+u_h^{m-1})\cdot n_f+a_p(B(\phi(t_{m+1}))-\phi^{m+1},\psi_h).
\end{aligned}
\tag{3.72}
$$

Let $\tilde{u}$, $\tilde{p}$ and $\tilde{\phi}$ be the projection of u, p and ϕ in H_f^h, Q^h and H_p^h. Denote the error as follows:

$$
u(t_{m+1})-u_h^{m+1}=u(t_{m+1})-\tilde{u}(t_{m+1})+\tilde{u}(t_{m+1})-u_h^{m+1}=\epsilon_u^{m+1}+\eta_u^{m+1}, \tag{3.73}
$$

$$
\phi(t_{m+1})-\phi_h^{m+1}=\phi(t_{m+1})-\tilde{\phi}(t_{m+1})+\tilde{\phi}(t_{m+1})-\phi_h^{m+1}=\epsilon_\phi^{m+1}+\eta_\phi^{m+1}, \tag{3.74}
$$

$$
p(t_{m+1})-p_h^{m+1}=p(t_{m+1})-\tilde{p}(t_{m+1})+\tilde{p}(t_{m+1})-p_h^{m+1}=\epsilon_p^{m+1}+\eta_p^{m+1}. \tag{3.75}
$$

Then we can rewrite (3.70)-(3.72):

$$
\begin{aligned}
&(\frac{A(\eta_u^{m+1})}{\Delta t},v_h)_{\Omega_f}+a_f(B(\eta_u^{m+1}),v_h)+b(v_h,B(\eta_p^{m+1}))=(\xi_f^{m+1},v_h)_{\Omega_f}\\
&-g\int_\Gamma(\phi(t_{m+1})-B(\phi(t_{m+1}))+B(\tilde{\phi}(t_{m+1}))\\
&-(2\tilde{\phi}(t_m)-\tilde{\phi}(t_{m-1})))v_h\cdot n_f-g\int_\Gamma(2\eta_\phi^m-\eta_\phi^{m-1})v_h\cdot n_f\\
&-a_f(B(\epsilon_u^{m+1}),v_h)-b(v_h,B(\epsilon_p^{m+1}))\\
&-g\int_\Gamma B(\epsilon_\phi^{m+1})v_h\cdot n_f+a_f(B(u(t_{m+1}))-u(t_{m+1}),v_h)\\
&+b(v_h,B(p(t_{m+1}))-p(t_{m+1}))-(\frac{A(\epsilon_u^{m+1})}{\Delta t},v_h)_{\Omega_f},
\end{aligned}
\tag{3.76}
$$

$$
b(u(t_{m+1})-B(u(t_{m+1}))+B(\eta_u^{m+1}),q_h)=-b(B(\epsilon_u^{m+1}),q_h), \tag{3.77}
$$

$$
\begin{aligned}
&gS_0(\frac{A(\eta_\phi^{m+1})}{\Delta t},\psi_h)_{\Omega_p}+a_p(B(\eta_\phi^{m+1}),\psi_h)\\
&=gS_0(\xi_p^{m+1},\psi_h)_{\Omega_p}+g\int_\Gamma\psi_h(2\eta_u^m-\eta_u^{m-1})\cdot n_f\\
&+g\int_\Gamma\psi_h(u(t_{m+1})\\
&-B(u(t_{m+1}))+B(\tilde{u}(t_{m+1}))-(2\tilde{u}(t_m)-\tilde{u}(t_{m-1})))\cdot n_f\\
&-gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t},\psi_h)_{\Omega_p}-a_p(B(\epsilon_\phi^{m+1}),\psi_h)\\
&+g\int_\Gamma\psi_h B(\epsilon_u^{m+1})\cdot n_f+a_p(B(\phi(t_{m+1}))-\phi(t_{m+1}),\psi_h).
\end{aligned}
\tag{3.78}
$$

Setting $v_h = \eta_u^{m+1}$, $q_h = \eta_p^{m+1}$ in (3.76) and (3.77) yields

$$
\begin{aligned}
&(\frac{A(\eta_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} + a_f(B(\eta_u^{m+1}), \eta_u^{m+1}) + b(\eta_u^{m+1}, B(\eta_p^{m+1})) = (\xi_f^{m+1}, \eta_u^{m+1})_{\Omega_f} \\
&- g\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m-1})))\eta_u^{m+1}\cdot n_f \\
&- g\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\eta_u^{m+1}\cdot n_f - (\frac{A(\epsilon_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} - a_f(B(\epsilon_u^{m+1}), \eta_u^{m+1}) \\
&- b(\eta_u^{m+1}, B(\epsilon_p^{m+1})) - g\int_\Gamma B(\epsilon_\phi^{m+1})\eta_u^{m+1}\cdot n_f + a_f(B(u(t_{m+1})) - u(t_{m+1}), \eta_u^{m+1}) \\
&+ b(\eta_u^{m+1}, B(p(t_{m+1})) - p(t_{m+1})),
\end{aligned}
\tag{3.79}
$$

$$
b(B(\eta_u^{m+1}), \eta_p^{m+1}) = -b(u(t_{m+1}) - B(u(t_{m+1})), \eta_p^{m+1}) - b(B(\epsilon_u^{m+1}), \eta_p^{m+1}). \tag{3.80}
$$

Choosing $\psi_h = \eta_\phi^{m+1}$ in (3.78) yields

$$
\begin{aligned}
&gS_0(\frac{A(\eta_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p} + a_p(B(\eta_\phi^{m+1}), \eta_\phi^{m+1}) = gS_0(\xi_p^{m+1}, \eta_\phi^{m+1})_{\Omega_p} \\
&+ g\int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f - gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p} \\
&+ g\int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1})))\cdot n_f \\
&- a_p(B(\epsilon_\phi^{m+1}), \eta_\phi^{m+1}) + g\int_\Gamma \eta_\phi^{m+1}B(\epsilon_u^{m+1})\cdot n_f + a_p(B(\phi(t_{m+1}) - \phi(t_{m+1}), \eta_\phi^{m+1}).
\end{aligned}
\tag{3.81}
$$

From (3.51), (3.52) and (3.77), we notice $b\big(B(\eta_u^{m+1}), q_h\big) = -b\big(u(t_{m+1}) - B(u(t_{m+1})), q_h\big) - b\big(B(\epsilon_u^{m+1}), q_h\big) = 0$, $-a_f\big(B(\epsilon_u^{m+1}), \eta_u^{m+1}\big) - b\big(\eta_u^{m+1}, B(\epsilon_p^{m+1})\big) + g\int_\Gamma B(\epsilon_\phi^{m+1})\eta_u^{m+1}\cdot n_f = 0$. Assuming $u_h^1 = \tilde{u}(t_1)$, $u_h^0 = \tilde{u}(t_0)$, and by (3.52) and $b\big(B(\eta_u^{m+1}), q_h\big) = 0$ we have $b(\eta_u^m, q_h) = 0\ \forall q_h \in Q^h$ for $m = 0, \ldots N$. From (3.32) and (3.36) and multiplying (3.79) by $4\Delta t$ gives

$$
\begin{aligned}
&(\|\eta_u^{m+1}\|_f^2 + \|2\eta_u^{m+1} - \eta_u^m\|_f^2) + 3\Delta t a_f(\eta_u^{m+1}, \eta_u^{m+1}) + \Delta t a_f(\eta_u^m, \eta_u^m) \\
&- (\|\eta_u^m\|_f^2 + \|2\eta_u^m - \eta_u^{m-1}\|_f^2) - 3\Delta t a_f(\eta_u^m, \eta_u^m) - \Delta t a_f(\eta_u^{m-1}, \eta_u^{m-1}) \\
&+ \|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 + 2\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
&- 4\Delta t b(\eta_u^{m+1}, B(p(t_{m+1})) - p(t_{m+1})) \\
=&4\Delta t(\xi_f^{m+1}, \eta_u^{m+1})_{\Omega_f} - 4g\Delta t\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\cdot\eta_u^{m+1}\cdot n_f \\
&- 4g\Delta t\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m-1})))\eta_u^{m+1}\cdot n_f \\
&- 4\Delta t\Big((\frac{A(\epsilon_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} - a_f(B(u(t_{m+1})) - u(t_{m+1}), \eta_u^{m+1})\Big).
\end{aligned}
\tag{3.82}
$$

Similarly, from (3.51), note that $a_p(B(\epsilon_\phi^{m+1}),\eta_\phi^{m+1}) + g\int_\Gamma \eta_\phi^{m+1} B(\epsilon_u^{m+1})\cdot n_f = 0$, from (3.32) and multiplying (3.81) by $4\Delta t$ yields

$$
\begin{aligned}
&gS_0(\|\eta_\phi^{m+1}\|_p^2 + \|2\eta_\phi^{m+1} - \eta_\phi^m\|_p^2) + 3\Delta t a_p(\eta_p^{m+1},\eta_p^{m+1}) + \Delta t a_p(\eta_p^m,\eta_p^m)\\
&- gS_0(\|\eta_\phi^m\|_p^2 + \|2\eta_\phi^m - \eta_\phi^{m-1}\|_p^2) - 3\Delta t a_p(\eta_p^m,\eta_p^m) - \Delta t a_p(\eta_p^{m-1},\eta_p^{m-1})\\
&+ gS_0\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + 2g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2\\
&+ g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2\\
=&4gS_0\Delta t(\xi_p^{m+1},\eta_\phi^{m+1})_{\Omega_p} + 4g\Delta t\int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f\\
&+ 4g\Delta t\int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1})))\cdot n_f\\
&- 4\Delta t\Big(gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t},\eta_\phi^{m+1})_{\Omega_p} - a_p(B(\phi^{m+1}) - \phi^{m+1},\eta_\phi^{m+1})\Big).
\end{aligned}
\tag{3.83}
$$

Let $F_{m+1} = (\|\eta_u^{m+1}\|_f^2 + \|2\eta_u^{m+1} - \eta_u^m\|_f^2) + 3\Delta t a_f(\eta_u^{m+1},\eta_u^{m+1}) + \Delta t a_f(\eta_u^m,\eta_u^m) + gS_0(\|\eta_\phi^{m+1}\|_p^2 + \|2\eta_\phi^{m+1} - \eta_\phi^m\|_p^2) + 3\Delta t a_p(\eta_p^{m+1},\eta_p^{m+1}) + \Delta t a_p(\eta_p^m,\eta_p^m)$, adding the above equalities (3.82)-(3.83) together, we get

$$
\begin{aligned}
&F_{m+1} - F_m + \|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 + 2\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2\\
&+ gS_0\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + 2g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2\\
&+ g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2\\
=&4\Delta t\Big((\xi_f^{m+1},\eta_u^{m+1})_{\Omega_f} - (\frac{A(\epsilon_u^{m+1})}{\Delta t},\eta_u^{m+1})_{\Omega_f}\\
&+ gS_0(\xi_p^{m+1},\eta_\phi^{m+1})_{\Omega_p} - gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t},\eta_\phi^{m+1})_{\Omega_p}\Big)\\
&- 4g\Delta t\Big(\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\cdot\eta_u^{m+1}\cdot n_f - \int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f\Big)\\
&- 4g\Delta t\Big(\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m-1})))\eta_u^{m+1}\cdot n_f\\
&- \int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1})))\cdot n_f\Big)\\
&+ 4\Delta t\Big(a_f(B(u(t_{m+1})) - u(t_{m+1}),\eta_u^{m+1}) + b(\eta_u^{m+1}, B(p(t_{m+1}) - p(t_{m+1})))\\
&+ a_p(B(\phi(t_{m+1})) - \phi(t_{m+1}),\eta_\phi^{m+1})\Big).
\end{aligned}
\tag{3.84}
$$

Based on Lemma 8 and $\|\frac{A(\epsilon_u^{m+1})}{\Delta t}\|^2 \leq \frac{9}{2\Delta t}\int_{t_{m-1}}^{t_{m+1}} \|u_t - \tilde{u}_t\|^2 dt$, the first term on the right hand side of (3.84) can be bounded by

$$
\begin{aligned}
&4\Delta t\Big((\xi_f^{m+1}, \eta_u^{m+1})_{\Omega_f} - (\frac{A(\epsilon_u^{m+1})}{\Delta t}, \eta_u^{m+1})_{\Omega_f} \\
&+ gS_0(\xi_p^{m+1}, \eta_\phi^{m+1})_{\Omega_p} - gS_0(\frac{A(\epsilon_\phi^{m+1})}{\Delta t}, \eta_\phi^{m+1})_{\Omega_p}\Big) \\
\leq & C\Delta t^4 \int_{t_{m-1}}^{t_{m+1}} \|u_{ttt}\|_f^2 dt + \frac{\nu\Delta t}{16}\|\nabla\eta_u^{m+1}\|_f^2 + C\Delta t\|\frac{A(\epsilon_u^{m+1})}{\Delta t}\|_f^2 \\
&+ C\Delta t^4 \int_{t_{m-1}}^{t_{m+1}} \|\phi_{ttt}\|_p^2 dt + \frac{g\Delta t}{16}\|K^{\frac{1}{2}}\nabla\eta_\phi^{m+1}\|_p^2 + C\Delta t\|\frac{A(\epsilon_\phi^{m+1})}{\Delta t}\|_p^2 \\
\leq & C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_{ttt}\|_f^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_{ttt}\|_p^2 dt\Big) \\
&+ Ch^{2k+2}\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big) \\
&+ \frac{\nu\Delta t}{16}\big(\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 + \|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2\big) \\
&+ \frac{g\Delta t}{16}\big(\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2\big).
\end{aligned}
\tag{3.85}
$$

Taking the same technique used in (3.40), we can bound the second term on the right hand side of (3.84)

$$
\begin{aligned}
&-4g\Delta t\Big(\int_\Gamma (2\eta_\phi^m - \eta_\phi^{m-1})\cdot\eta_u^{m+1}\cdot n_f - \int_\Gamma \eta_\phi^{m+1}(2\eta_u^m - \eta_u^{m-1})\cdot n_f\Big) \\
\leq & \frac{\nu\Delta t}{16}\Big(2\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 + 2\|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2\Big) \\
&+ \frac{g\Delta t}{16}\Big(2\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 \\
&+ 2\|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2\Big) \\
&+ \frac{CC_1^2C_d^2g^3\Delta t}{\nu^2K_{min}}\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 \\
&+ \frac{CC_1^2\tilde{C}_dg^2\Delta t}{\nu K_{min}^2}\|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2.
\end{aligned}
\tag{3.86}
$$

Taking $\epsilon = \frac{\Delta t}{12}$ in (3.17) and (3.18) and using $u^{m+1} = \frac{1}{2}(u^{m+1} - u^m + u^{m+1} + u^{m-1} + u^m - u^{m-1})$ and $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, the third term on the right hand side of (3.84) can be bounded by

$$
\begin{aligned}
&- 4g\Delta t\Big(\int_\Gamma (\phi(t_{m+1}) - B(\phi(t_{m+1})) + B(\tilde{\phi}(t_{m+1})) - (2\tilde{\phi}(t_m) - \tilde{\phi}(t_{m+1})))\eta_u^{m+1} \cdot n_f \\
&- \int_\Gamma \eta_\phi^{m+1}(u(t_{m+1}) - B(u(t_{m+1})) + B(\tilde{u}(t_{m+1})) - (2\tilde{u}(t_m) - \tilde{u}(t_{m-1}))) \cdot n_f\Big) \\
&= -4g\Delta t \int_\Gamma \Big((-\frac{1}{2}\phi(t_{m+1}) + \phi(t_m) - \frac{1}{2}\phi(t_{m-1})) + \frac{3}{2}(\tilde{\phi}(t_{m+1}) \\
&- 2\tilde{\phi}(t_m) + \tilde{\phi}(t_{m-1}))\Big)\eta_u^{m+1} \cdot n_f \\
&+ 4g\Delta t \int_\Gamma \eta_\phi^{m+1}\Big((-\frac{1}{2}u(t_{m+1}) + u(t_m) - \frac{1}{2}u(t_{m-1})) + \frac{3}{2}(\tilde{u}(t_{m+1}) \\
&- 2\tilde{u}(t_m) + \tilde{u}(t_{m-1}))\Big) \cdot n_f \\
&\le \frac{gC_3}{\nu}\Big(\|\nabla\phi(t_{m+1}) - 2\nabla\phi(t_m) + \nabla\phi(t_{m-1})\|_p^2 + \|\nabla\tilde{\phi}(t_{m+1}) - 2\nabla\tilde{\phi}(t_m) + \nabla\tilde{\phi}(t_{m-1})\|_p^2\Big) \\
&+ \frac{C_3}{K_{min}}\Big(\|\nabla u(t_{m+1}) - 2\nabla u(t_m) + \nabla u(t_{m-1})\|_f^2 + \|\nabla\tilde{u}(t_{m+1}) - 2\nabla\tilde{u}(t_m) + \nabla\tilde{u}(t_{m-1})\|_f^2\Big) \\
&+ \frac{\nu\Delta t}{16}\big(\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 \\
&+ \|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2\big) \\
&+ \frac{g\Delta t}{16}\big(\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2 \\
&+ \|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2\big),
\end{aligned}
\tag{3.87}
$$

where, the parameter $C_3 = 216C_1C_2g\Delta t$. From the definition of the bilinear forms a_f, a_p and b, the rest of the right hand side of the (3.84) can be bounded by the following inequality

$$
\begin{aligned}
&- 4\Delta t\Big(a_f(B(u(t_{m+1}) - u(t_{m+1}), \eta_u(t_{m+1})) \\
&+ b(\eta_u(t_{m+1}), B(p(t_{m+1})) - p(t_{m+1})) \\
&+ 4\Delta t a_p(B(\phi(t_{m+1})) - \phi(t_{m+1}), \eta_\phi(t_{m+1}))\Big) \\
&\le C\Delta t\Big(\|\nabla(B(u(t_{m+1})) - u(t_{m+1}))\|_f^2 + \|B(p(t_{m+1})) - p(t_{m+1})\|_f^2 \\
&+ \|K^{\frac{1}{2}}\nabla(B(\phi(t_{m+1})) - \phi(t_{m+1}))\|_p^2\Big) \\
&+ \frac{\nu\Delta t}{16}\big(\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 + \|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2\big) \\
&+ \frac{g\Delta t}{16}\big(\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2\big).
\end{aligned}
\tag{3.88}
$$

Combining all the above estimates (3.84)-(3.88) gives

$$
\begin{aligned}
&F_{m+1} - F_m + \|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 + \frac{27}{16}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 \\
&+ gS_0\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + \frac{27}{16}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 \\
&+ \frac{3}{4}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
\le& C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_{ttt}\|_f^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_{ttt}\|_p^2 dt\Big) \\
&+ Ch^{2k+2}\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big) \\
&+ \frac{CC_1^2C_d^2g^3\Delta t}{\nu^2 K_{min}}\|\eta_\phi^{m+1} - 2\eta_\phi^m + \eta_\phi^{m-1}\|_p^2 + \frac{CC_1^2\tilde{C}_d^2g^2\Delta t}{\nu K_{min}^2}\|\eta_u^{m+1} - 2\eta_u^m + \eta_u^{m-1}\|_f^2 \\
&+ C\Delta t\Big(\|\nabla\phi(t_{m+1}) - 2\nabla\phi(t_m) + \nabla\phi(t_{m-1})\|_p^2 + \|\nabla\tilde{\phi}(t_{m+1}) \\
&- 2\nabla\tilde{\phi}(t_m) + \nabla\tilde{\phi}(t_{m-1}))\|_p^2\Big) + C\Delta t\Big(\|\nabla u(t_{m+1}) \\
&- 2\nabla u(t_m) + \nabla u(t_{m-1})\|_f^2 + \|\nabla\tilde{u}(t_{m+1}) - 2\nabla\tilde{u}(t_m) + \nabla\tilde{u}(t_{m-1})\|_f^2\Big) \\
&+ C\Delta t\Big(\|\nabla(B(u(t_{m+1})) - u(t_{m+1}))\|_f^2 + \|B(p(t_{m+1})) - p(t_{m+1})\|_f^2 \\
&+ \|K^{\frac{1}{2}}\nabla(B(\phi(t_{m+1})) - \phi(t_{m+1}))\|_p^2\Big) + \frac{5\nu\Delta t}{16}\|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 \\
&+ \frac{5g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2.
\end{aligned}
\tag{3.89}
$$

Assuming $\Delta t \le min\{\frac{\nu K_{min}}{CC_1^2\tilde{C}_d^2g^2}, \frac{\nu^2 K_{min}S_0}{CC_1^2C_d^2g^2}\}$ and using (3.62)-(3.63), (3.66)-(3.67) yield

$$
\begin{aligned}
&F_{m+1} - F_m + \frac{27}{16}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \frac{27}{16}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2 \\
&+ \frac{3}{4}g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2 \\
\le& C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_{ttt}\|_f^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_{ttt}\|_p^2 dt\Big) \\
&+ Ch^{2k+2}\Big(\int_{t_{m-1}}^{t_{m+1}} \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big) \\
&+ C\Delta t^4\Big(\int_{t_{m-1}}^{t_{m+1}} \|\phi_{tt}\|_{H_p}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|u_{tt}\|_{H_f}^2 dt + \int_{t_{m-1}}^{t_{m+1}} \|p_{tt}\|_f^2 dt\Big) \\
&+ \frac{5\nu\Delta t}{16}\|\nabla(\eta_u^m - \eta_u^{m-1})\|_f^2 + \frac{5g\Delta t}{16}\|K^{\frac{1}{2}}\nabla(\eta_\phi^m - \eta_\phi^{m-1})\|_p^2.
\end{aligned}
\tag{3.90}
$$

Summing up from $m = 1$ to $m = N-1$ yields

$$
\begin{aligned}
&F_N + \sum_{m=1}^{N-1} (\frac{11}{8}\nu\Delta t\|\nabla(\eta_u^{m+1} - \eta_u^m)\|_f^2 + \frac{11}{8} g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} - \eta_\phi^m)\|_p^2) \\
&+ \sum_{m=1}^{N-1} (\frac{3}{4} g\Delta t\|K^{\frac{1}{2}}\nabla(\eta_\phi^{m+1} + \eta_\phi^{m-1})\|_p^2 + \frac{3}{4}\nu\Delta t\|\nabla(\eta_u^{m+1} + \eta_u^{m-1})\|_f^2) \\
\leq &F_1 + C\Delta t^4\Big(\int_0^T \|u_{ttt}\|_f^2 dt + \int_0^T \|\phi_{ttt}\|_p^2 dt\Big) \\
&+ C\Delta t^4\Big(\int_0^T \|\phi_{tt}\|_{H_p}^2 dt + \int_0^T \|u_{tt}\|_{H_f}^2 dt + \int_0^T \|p_{tt}\|_f^2 dt\Big) \\
&+ Ch^{2k+2}\Big(\int_0^T \|u_t\|_{H^{k+1}(\Omega_f)}^2 dt + \int_0^T \|\phi_t\|_{H^{k+1}(\Omega_p)}^2 dt\Big) \\
&+ \frac{5}{8}\nu\Delta t(\|\nabla\eta_u^1\|_f^2 + \|\nabla\eta_u^0\|_f^2) + \frac{5}{8} g\Delta t(\|K^{\frac{1}{2}}\nabla(\eta_\phi^1)\|_p^2 + \|K^{\frac{1}{2}}\nabla(\eta_\phi^0)\|_p^2).
\end{aligned}
\tag{3.91}
$$

From (3.53)-(3.55) and using triangle inequality on (3.73)-(3.75) yields (5.9). □

Remark 2. *: If there is no restriction on Δt, we can estimate the interface term in the same way as in 3.43, and we have the unconditional error analysis in the finite time interval.*

3.5 ADAPTIVE TIME FILTERED BE ALGORITHMS

By using Newton interpolation, the variable stepsize BDF methods of order p ($BDF-p$) can be written in [56]. Define the jth order backward divided difference by $\sigma^j u = u[t_{n+m}, t_{n+m-1}, ..., t_{n+m-j}]$, and the parameter in time filter by $\eta^{p+1} = \frac{\prod_{i=1}^{p}(t_{n+m}-t_{n+m-i})}{\sum_{j=1}^{p+1}(t_{n+m}-t_{n+m-j})^{-1}}$. Based on divided difference, let $k_m = t_{m+1} - t_m$, $\tau_m = \frac{k_{m+1}}{k_m}$. It is easy to state the variable stepsize Backward Euler plus time filter for Stokes-Darcy equations in terms of divided difference:

Given (u_h^0, p_h^0, ϕ_h^0) and (u_h^1, p_h^1, ϕ_h^1), find $(u_h^{m+1}, p_h^{m+1}, \phi^{m+1}) \in (H_f^h, Q^h, H_p^h)$, such that for $m = 0, 1, ..., N-1$:

BE for Stokes $\quad (\sigma^1 \hat{u}_h^{m+1}, v_h)_{\Omega_f} - a_f(\hat{u}_h^{m+1}, v_h) + b(v_h, \hat{p}^{m+1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, \phi_h^*),$

$\quad b(\hat{u}^{m+1}, q_h) = 0.$

BE for Darcy $\quad gS_0(\sigma^1 \hat{\phi}_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\hat{\phi}_h^{m+1}, \psi_h) = g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(u_h^*, \psi_h).$

Here, choosing $\phi_h^* = \phi_h^{m-1}$ and $u_h^* = u_h^{m-1}$ gives a standard variable time stepsize BE method for Stokes-Darcy equations and choosing $\phi_h^* = (1+\tau_{m-1})\phi_h^m - \tau_{m-1}\phi_h^{m-1}$, and $u_h^* = (1+\tau_{m-1})u_h^m - \tau_{m-1}u_h^{m-1}$ is explored herein. Implementing this change involves changing one line of code redefining ϕ_h^* and u_h^*. Apply time filter to update the previous solution

$$u_h^{m+1} = \hat{u}_h^{m+1} - \Big(\frac{k_{m+1}}{\frac{1}{k_{m+1}} + \frac{1}{k_{m+1}+k_m}}\Big)\sigma^2\hat{u}_h^{m+1}, \tag{3.92}$$

$$\phi_h^{m+1} = \hat{\phi}_h^{m+1} - \Big(\frac{k_{m+1}}{\frac{1}{k_{m+1}} + \frac{1}{k_{m+1}+k_m}}\Big)\sigma^2\hat{\phi}_h^{m+1}, \tag{3.93}$$

$$p_h^{m+1} = \hat{p}_h^{m+1} - \Big(\frac{k_{m+1}}{\frac{1}{k_{m+1}} + \frac{1}{k_{m+1}+k_m}}\Big)\sigma^2\hat{p}_h^{m+1}. \tag{3.94}$$

Using algebraic manipulation, the above three equality can be written in terms of stepsize ratio τ as follows:

$$u_h^{m+1} = \hat{u}_h^{m+1} - \frac{\tau_{m-1}(1+\tau_{m-1})}{1+2\tau_{m-1}}\Big(\frac{1}{1+\tau_{m-1}}\hat{u}_h^{m+1} - u_h^m + \frac{\tau_{m-1}}{1+\tau_{m-1}}u_h^{m-1}\Big), \tag{3.95}$$

$$p_h^{m+1} = \hat{p}_h^{m+1} - \frac{\tau_{m-1}(1+\tau_{m-1})}{1+2\tau_{m-1}}\Big(\frac{1}{1+\tau_{m-1}}\hat{p}_h^{m+1} - p_h^m + \frac{\tau_{m-1}}{1+\tau_{m-1}}p_h^{m-1}\Big), \tag{3.96}$$

$$\phi_h^{m+1} = \hat{\phi}_h^{m+1} - \frac{\tau_{m-1}(1+\tau_{m-1})}{1+2\tau_{m-1}}\Big(\frac{1}{1+\tau_{m-1}}\hat{\phi}_h^{m+1} - \phi_h^m + \frac{\tau_{m-1}}{1+\tau_{m-1}}\phi_h^{m-1}\Big). \tag{3.97}$$

The combination of BE, BEplustimefilter and general adaptive method lead to adaptive BE and adaptive BEplustimefilter algorithm. Since the time accuracy of BEplustimefilter is $O(k^2)$, we can use $\text{Est}_u = |u_h^{m+1} - \hat{u}_h^{m+1}|$ and $\text{Est}_\phi = |\phi_h^{m+1} - \hat{\phi}_h^{m+1}|$ as two estimate for the local error of velocity and hydraulic head in BE. In order to estimate the local error of velocity and hydraulic head for BEplustimefilter, it is easy to take $\text{Est}_u = \eta^3\sigma^3\hat{u}_h^{m+1}$, $\text{Est}_\phi = \eta^3\sigma^3\hat{\phi}_h^{m+1}$ as two estimate because σ^3 is the third order backward divided difference. We give a simple formula for stepsize selection which is the combination of some general adaptive methods. We denote γ and $\tilde{\gamma}$ two safety factors. The first safety factor γ is used to prevent the next step size becoming too big to decrease the chance that the next solution will be rejected. The effect of the second factor $\tilde{\gamma}$ is making the stepsize growing more slowly so that the recomputed solution is more likely to be accepted. We took $\gamma = 0.9$, and $\tilde{\gamma} = 0.6$.

Algorithm 1 (Adaptive BE)**.** *Let* $m = 1$. *Given* ε, $\tilde{\gamma}$, γ, $\{u_h^{m-1}, u_h^m\}$, $\{p_h^{m-1}, p_h^m\}$ *and* $\{\phi_h^{m-1}, \phi_h^m\}$, *compute* $\{u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}\}$ *by solving*

BE for Stokes

$$
\begin{aligned}
&(\sigma^1 u_h^{m+1}, v_h)_{\Omega_f} - a_f(u_h^{m+1}, v_h) + b(v_h, p^{m+1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, \phi_h^m),\\
&b(u_h^{m+1}, q_h) = 0.
\end{aligned}
$$

BE for Darcy

$$
gS_0(\sigma^1 \phi_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\phi_h^{m+1}, \psi_h) = g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(u_h^m, \psi_h).
$$

Choose

$$
Est_u = \eta^2 \sigma^2 \hat{u}_h^{m+1}, \quad Est_\phi = \eta^2 \sigma^2 \hat{\phi}_h^{m+1},
$$

if $\min\{|Est_u|, |Est_\phi|\} < \frac{\varepsilon}{4}$,

$$
\tau_m = \min\left\{2, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{2}}\right\}, \tag{3.98}
$$

$$
k_{m+1} = \gamma \cdot \tau_m \cdot k_m; \tag{3.99}
$$

if $\frac{\varepsilon}{4} \le \min\{|Est_u|, |Est_\phi|\} \le \varepsilon$,

$$
\tau_m = \min\left\{1, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{2}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{2}}\right\}, \tag{3.100}
$$

$$
k_{m+1} = \gamma \cdot \tau_m \cdot k_m. \tag{3.101}
$$

If none satisfy the tolerance, set

$$
k_m = \tilde{\gamma} \cdot \tau_{m-1} \cdot k_{m-1},
$$

and recompute the above steps.

Algorithm 2 (Adaptive BETF)**.** *Let* $m = 2$. *Given* ε, $\tilde{\gamma}$, γ, $\{u_h^{m-2}, u_h^{m-1}, u_h^m\}$, $\{p_h^{m-2}, p_h^{m-1}, p_h^m\}$, $\{\phi_h^{m-2}, \phi_h^{m-1}, \phi_h^m\}$, *compute* $\{u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}\}$ *by solving*

Modified BE for Stokes

$$
\begin{aligned}
&(\sigma^1 \hat{u}_h^{m+1}, v_h)_{\Omega_f} - a_f(\hat{u}_h^{m+1}, v_h) + b(v_h, \hat{p}^{m+1}) \\
&= (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, (1+\tau_{m-1})\phi_h^m - \tau_{m-1}\phi_h^{m-1}), \\
&b(\hat{u}^{m+1}, q_h) = 0.
\end{aligned}
$$

BE for Darcy

$$
\begin{aligned}
&gS_0(\sigma^1 \hat{\phi}_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\hat{\phi}_h^{m+1}, \psi_h) \\
&= g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma((1+\tau_{m-1})u^m - \tau_{m-1}u^{m-1}, \psi_h).
\end{aligned}
$$

Time filter for $\hat{u}_h^{m+1}, \hat{p}_h^{m+1}, \hat{\phi}_h^{m+1}$

$$
\begin{aligned}
u_h^{m+1} &= \hat{u}_h^{m+1} - \eta^2 \sigma^2 \hat{u}_h^{m+1}, \\
\phi_h^{m+1} &= \hat{\phi}_h^{m+1} - \eta^2 \sigma^2 \hat{\phi}_h^{m+1}, \\
p_h^{m+1} &= \hat{p}_h^{m+1} - \eta^2 \sigma^2 \hat{p}_h^{m+1}.
\end{aligned}
$$

Choose

$$
Est_u = \eta^3 \sigma^3 \hat{u}_h^{m+1}, \quad Est_\phi = \eta^3 \sigma^3 \hat{\phi}_h^{m+1},
$$

if $\min\{|Est_u|, |Est_\phi|\} < \frac{\varepsilon}{4}$,

$$
\tau_m = \min\left\{2, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{3}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{3}}\right\}, \tag{3.102}
$$

$$
k_{m+1} = \gamma \cdot \tau_m \cdot k_m; \tag{3.103}
$$

if $\frac{\varepsilon}{4} \leq \min\{|Est_u|, |Est_\phi|\} \leq \varepsilon$,

$$
\tau_m = \min\left\{1, \left(\frac{\varepsilon}{|Est_u|}\right)^{\frac{1}{3}}, \left(\frac{\varepsilon}{|Est_\phi|}\right)^{\frac{1}{3}}\right\}, \tag{3.104}
$$

$$
k_{m+1} = \gamma \cdot \tau_m \cdot k_m. \tag{3.105}
$$

If none satisfy the tolerance, set

$$
k_m = \tilde{\gamma} \cdot \tau_{m-1} \cdot k_{m-1},
$$

and recompute the above steps.

Extension to a Third Order Method

Recently [50] has shown how time filter can be extended from backward Euler to some higher order BDF methods. We also give a test of BDF2 discretization of subdomain terms with a third order extrapolation formula for the interface terms. Errors are estimated and accuracy is increased by an added time filter as presented next.

Algorithm 3 (BDF-2 plus time filter)**.** *Let* $\sigma^3 x^{m+1} = \sum_{i=0}^{3} C^i x^{m+1-i}$.
Given $\{u_h^{m-2}, u_h^{m-1}, u_h^m\}$, $\{p_h^{m-2}, p_h^{m-1}, p_h^m\}$ *and* $\{\phi_h^{m-2}, \phi_h^{m-1}, \phi_h^m\}$,
compute $\{u_h^{m+1}, p_h^{m+1}, \phi_h^{m+1}\}$ *by solving*

BDF-2 for Stokes

$$(\sigma^2 \hat{u}_h^{m+1}, v_h)_{\Omega_f} - a_f(\hat{u}_h^{m+1}, v_h) + b(v_h, \hat{p}^{m+1}) = (f_1^{m+1}, v_h)_{\Omega_f} - c_\Gamma(v_h, \phi_h^*),$$
$$b(\hat{u}^{m+1}, q_h) = 0.$$

BDF-2 for Darcy

$$gS_0(\sigma^2 \hat{\phi}_h^{m+1}, \psi_h)_{\Omega_p} + a_p(\hat{\phi}_h^{m+1}, \psi_h) = g(f_2^{m+1}, \psi_h)_{\Omega_p} + c_\Gamma(u_h^*, \psi_h).$$

Time filter for $\hat{u}_h^{m+1}, \hat{p}_h^{m+1}, \hat{\phi}_h^{m+1}$

$$u_h^{m+1} = \hat{u}_h^{m+1} - \eta^3 \sigma^3 \hat{u}_h^{m+1},$$
$$\phi_h^{m+1} = \hat{\phi}_h^{m+1} - \eta^3 \sigma^3 \hat{\phi}_h^{m+1},$$
$$p_h^{m+1} = \hat{p}_h^{m+1} - \eta^3 \sigma^3 \hat{p}_h^{m+1}.$$

where, $u_h^* = -\sum_{i=0}^{2} \frac{C^{i+1}}{C^0} u_h^{m-i}$, $\phi_h^* = -\sum_{i=0}^{2} \frac{C^{i+1}}{C^0} \phi_h^{m-i}$. We approximate u_h^{m+1} and ϕ_h^{m+1} by u_h^* and ϕ_h^* in the interface coupling terms due to $u_h^{m+1} = u_h^* + \frac{\sigma^{p+1} u_h^{m+1}}{C^0} = u_h^* + O(k^{p+1})$.

3.6 NUMERICAL EXPERIMENTS

In this section, three time filters are performed with constant time stepsize and variable time stepsize to show the validity and accuracy of the decoupled scheme. Furthermore, we implemented the codes using the software package FreeFEM++ .

Example 1: Consider the computational domain $\Omega_f = (0,1)\times(1,2), \Omega_p = (0,1)\times(0,1)$, and interface $\Gamma = (0,1)\times\{1\}$. We take the exact solution:

$$\phi(x,y,t) = [2-\pi sin(\pi x)][1-y-cos(\pi y)]cos(t),$$
$$u(x,y,t) = \Big([x^2(y-1)^2+y]cos(t), [-\frac{2}{3}x(y-1)^3+2-\pi sin(\pi x)]cos(t)\Big),$$
$$p(x,y,t) = [2-\pi sin(\pi x)]sin(\frac{\pi}{2}y)cos(t).$$

Table 1: The convergence performance for BE method at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	0.00173173		0.0746392		0.0181246	
1/16	0.000876655	0.9821	0.0371619	1.0061	0.00927994	0.9658
1/32	0.000441074	0.9910	0.0185575	1.0018	0.00469195	0.9839
1/48	0.000294664	0.9948	0.0123682	1.0007	0.00313929	0.9911
1/64	0.00022123	0.9964	0.00927512	1.0004	0.00235869	0.9938

Table 2: The convergence performance for BETF method at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	0.00917426		0.0228416		0.00820143	
1/16	0.00217503	2.0766	0.00600395	1.9277	0.00193804	2.0813
1/32	0.000519991	2.0645	0.00150016	2.0008	0.000463955	2.0625
1/48	0.000227587	2.0379	0.000666787	1.9998	0.000203143	2.0369
1/64	0.000127028	2.0270	0.000378827	1.9653	0.000113407	2.0263

Here, we set the parameters $\nu = 1, g = 1, z = 0, S_0 = 1, \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}} = 1$, and $K = kI$, where $k = 1$, and the initial conditions, boundary conditions, and the source terms follow from the exact solution. For BE and BETF, we use the well-known Taylor-Hood elements $(P2-P1)$

Table 3: Convergence for BDF2 at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	0.000202378		0.0113836		0.0015852	
1/16	4.79795e-005	2.0766	0.00276725	2.0404	0.000362859	2.1272
1/32	1.16465e-005	2.0425	0.0006805	2.0238	8.64945e-005	2.0687
1/48	5.12343e-006	2.0252	0.000300683	2.0144	3.78077e-005	2.0410
1/64	2.86697e-006	2.0181	0.00016863	2.0104	2.10875e-005	2.0294

Table 4: Convergence for BDF2 plus time filter at time $t_N = 1$ with $h = 1/120$.

Δt	$\frac{\|u(t_N)-u_h^N\|_f}{\|u(t_N)\|_f}$	rate	$\frac{\|p(t_N)-p_h^N\|_f}{\|p(t_N)\|_f}$	rate	$\frac{\|\phi(t_N)-\phi_h^N\|_p}{\|\phi(t_N)\|_p}$	rate
1/8	0.000583937		0.00165794		0.000485433	
1/16	7.92566e-05	2.8812	0.000199929	3.0518	6.59814e-05	2.8791
1/32	1.02708e-05	2.9480	2.57276e-05	2.9581	8.55089e-06	2.9479
1/48	3.07738e-06	2.9725	7.7075e-06	2.9728	2.56208e-06	2.9724
1/64	1.30534e-06	2.9811	3.27519e-06	2.9749	1.08676e-06	2.9811

Table 5: Convergence of global error for adaptive BE method with $h = 1/120$.

$\overline{\Delta t}$	$\left(\sum_{i=2}^{N} k_i \frac{\|u(t_i)-u_h^i\|_f^2}{\|u(t_i)\|_f^2}\right)^{\frac{1}{2}}$	rate	$\left(\sum_{i=2}^{N} k_i \frac{\|p(t_i)-p_h^i\|_f^2}{\|p(t_i)\|_f^2}\right)^{\frac{1}{2}}$	rate	$\left(\sum_{i=2}^{N} k_i \frac{\|\phi(t_i)-\phi_h^i\|_p^2}{\|\phi(t_i)\|_p^2}\right)^{\frac{1}{2}}$	rate
1/34	0.000236169		0.0119435		0.00234341	
1/114	7.42995e-005	0.9910	0.00377136	0.9528	0.00074303	0.9494
1/362	2.33819e-005	1.0006	0.00119749	0.9929	0.000233827	1.0006
1/1149	7.24488e-006	1.0144	0.000379229	0.9955	7.34297e-005	1.0028
1/1636	4.97812e-006	1.0619	0.000266619	0.9970	5.09029e-005	1.0369

Table 6: Convergence of global error for adaptive BETF method with $h = 1/120$.

$\overline{\Delta t}$	$\big(\sum_{i=2}^{N} k_i \frac{\|u(t_i)-u_h^i\|_f^2}{\|u(t_i)\|_f^2}\big)^{\frac{1}{2}}$	rate	$\big(\sum_{i=2}^{N} k_i \frac{\|p(t_i)-p_h^i\|_f^2}{\|p(t_i)\|_f^2}\big)^{\frac{1}{2}}$	rate	$\big(\sum_{i=2}^{N} k_i \frac{\|\phi(t_i)-\phi_h^i\|_p^2}{\|\phi(t_i)\|_p^2}\big)^{\frac{1}{2}}$	rate
1/13	0.0030781		0.0172693		0.00271955	
1/41	0.000331267	1.9407	0.00220551	1.7917	0.000292453	1.9414
1/119	4.00274e-005	1.9833	0.000546644	1.3091	3.5363e-005	1.9827
1/373	3.69748e-006	2.0849	0.000143857	1.1685	3.29478e-006	2.0774
1/553	1.66681e-006	2.0233	9.1707e-005	1.1433	1.52531e-006	1.9558

for fluid velocity and pressure and the continuous piecewise quadratic functions ($P2$) for hydraulic head. For BDF2 and BDF2 plus time filter, we use ($P3 - P2$) for fluid velocity and pressure and ($P3$) for hydraulic head. In Table 1-4, we use BE, BETF, BDF2 and BDF2 plus time filter with constant time stepsize to run simulations at final time $T = 1.0$ and set the mesh size $h = \frac{1}{120}$. The results show that the numerical convergence rate of BETF and BDF2 plus time filter are approximately second order and third order in time respectively for u, p and ϕ. In Figure 2-4, we present the log-log plots of the relative error for velocity, pressure and hydraulic head which clearly shows applying time filter leads to higher order. In Table 5-6, we allow the time stepsize to be variable, and final time and the same mesh from the constant stepsize test were used. Various tolerance were tested from 1e-1 to 1e-7. Since the time stepsize is variable, $\overline{\Delta t}$ in Table 5-6 is the average time stepsize. It can be seen the convergence order of adaptive BETF is also higher than adaptive BE. Figure 5-6 displays the log-log plots of the global error for velocity, pressure and hydraulic head. *It is observed that adaptive BETF reduced the total amount of work (number of steps taken), and it requires less time steps than adaptive BE for the smaller tolerance.*

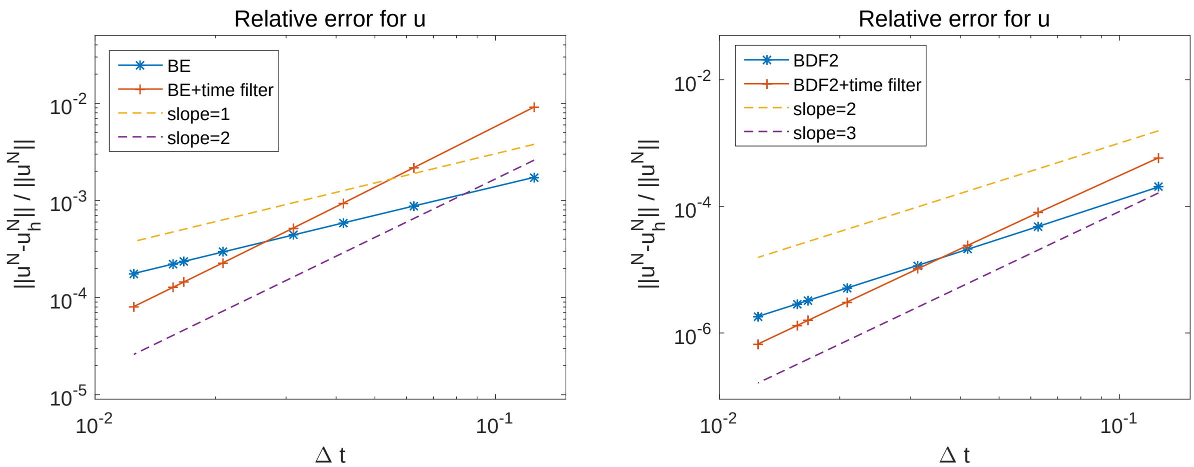


Figure 2: Relative errors of velocity u for BE, BE plus time filter, BDF2 and BDF2 plus time filter with constant time stepsize.

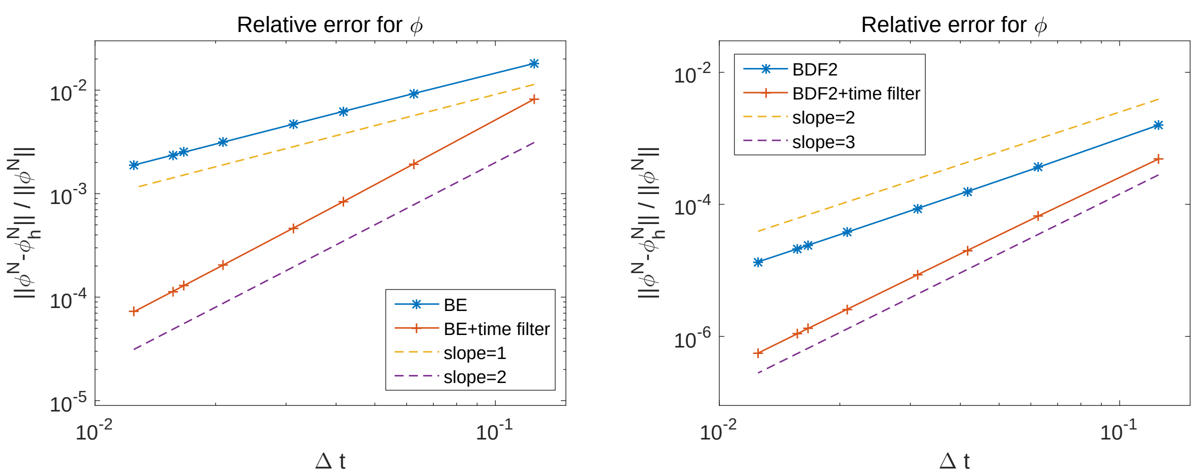


Figure 3: Relative errors of hydraulic head ϕ for BE, BE plus time filter, BDF2 and BDF2 plus time filter with constant time stepsize.

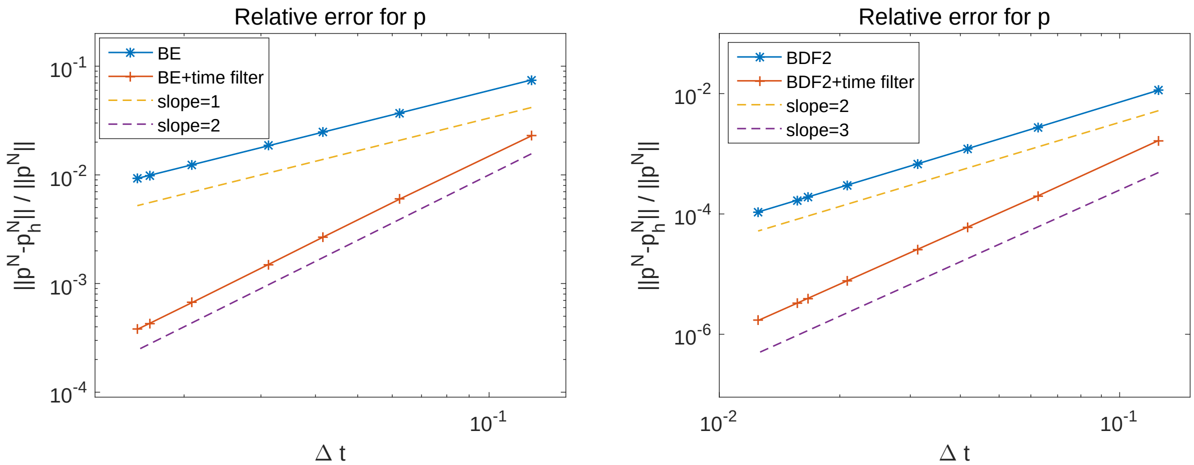


Figure 4: Relative errors of pressure p for BE, BE plus time filter, BDF2 and BDF2 plus time filter with constant time stepsize.

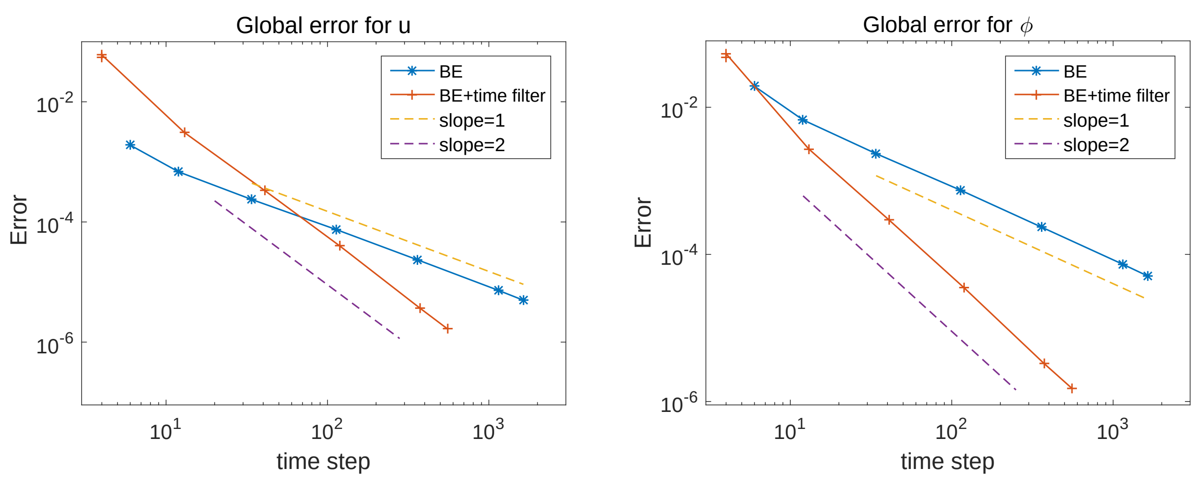


Figure 5: Global errors of velocity u (left) and hydraulic head ϕ (right) for adaptive BE and BE plus time filter.

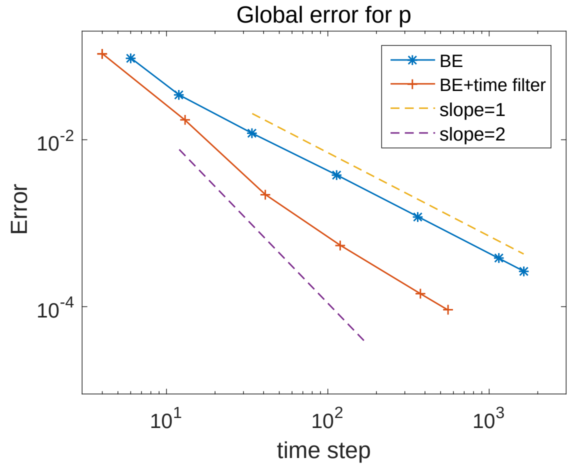


Figure 6: Global errors of pressure p for adaptive BE and BE plus time filter.

3.7 CONCLUSIONS

This paper develops, analyzes and tests a time-accurate partitioned method for the Stokes-Darcy equations. The method combines a time filter and Backward Euler scheme, is second order accurate and provide, at no extra complexity, an estimated the temporal error. This approach post-processes the solutions of Backward Euler scheme by adding three lines to original codes to increase the time accuracy from first order to second order. We prove long time stability and error estimates of Backward Euler plus time filter with constant time stepsize. Moreover, we extend the approach to variable time stepsize and construct adaptive algorithms. Numerical tests show convergence of our method and support the theoretical analysis.

4.0 LOW COMPLEXITY, TIME ACCURATE ALGORITHMS IN CFD-TIME FILTERED BACKWARD EULER FOR NAVIER-STOKES EQUATIONS

4.1 INTRODUCTION

The backward Euler time discretization is often used for complex, viscous flows due to its stability, rapid convergence to steady state solutions and simplicity to implement. However, it has poor time transient flow accuracy, [90], and can fail by overdamping a solution's dynamic behavior. For ODEs, adding a time filter to backward Euler, as in (1.3) below, yields two, embedded, A-stable approximations of first and second order accuracy, [93]. This report develops this idea into an adaptive time-step and adaptive order method for time accurate fluid flow simulation and gives an analysis of the resulting methods properties for constant time-steps. For constant time-steps, the resulting Algorithm 4 below involves adding only 1 extra line to a backward Euler code. The added filter step increases accuracy and adds negligible additional computational complexity, see Figure 7a and Figure 7b. Further, both time adaptivity and order adaptivity are easily implemented in a constant time step backward Euler code with $\mathcal{O}(20)$ added lines. Thus, algorithms herein have two main features. First, they can be implemented in a legacy code based on backward Euler without modifying the legacy components. Second, both time step and method order can easily be adapted due to the embedded structure of the method. The variable step, variable order step (VSVO) method is presented in Section 4.2 and tested in Section 4.6.2.

Even for constant time-steps and constant order, the method herein does not reduce to a standard / named method. Algorithm 4 with Option B is (for constant order and time-step) equivalent to a member of the known, 2 parameter family of second order, 2-step, A-stable one leg methods (OLMs), see Algorithm 7. Stability and velocity convergence of the (constant time step) general second order, two-step, A-stable method for the Navier-Stokes equations was proven already in [89], see equation (3.20) p. 185, and has been elaborated thereafter, e.g., [96]. Our *velocity* stability and error analysis, while necessary for completeness, parallels this previous work and is thus collected in appendix. On the other hand, Algorithm 4 with Option A does *not* fit within a general theory even for constant stepsize, and produces more accurate pressure approximations.

We begin by presenting the simplest, constant stepsize case to fix ideas. Consider the time dependent incompressible Navier-Stokes (NS) equations:

$$\begin{aligned}
&u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \text{ and } \nabla \cdot u = 0 \text{ in } \Omega, \\
&u = 0 \text{ on } \partial\Omega, \text{ and } \int_\Omega p\, dx = 0, \\
&u(x,0) = u_0(x) \text{ in } \Omega.
\end{aligned} \tag{4.1}$$

Here, $\Omega \subset R^d(d{=}2{,}3)$ is a bounded polyhedral domain; $u : \Omega \times [0,T] \to R^d$ is the fluid velocity; $p : \Omega \times (0,T] \to R$ is the fluid pressure. The body force $f(x,t)$ is known, and ν is the kinematic viscosity of the fluid.

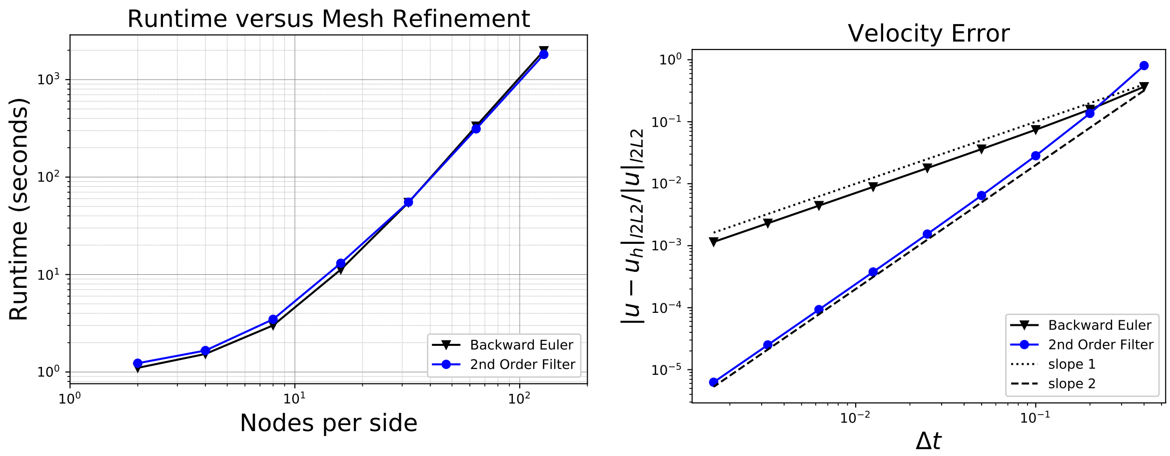
Suppressing the spacial discretization, the method calculates an intermediate velocity $\hat{u}^{n+1}$ using the backward Euler / fully implicit method. Time filters (requiring only two additional lines of code and not affecting the BE calculation) are applied to produce u^{n+1} and p^{n+1} follows:

Algorithm 4 (Constant $\triangle t$ BE plus time filter)**.** *With $u^* = \hat{u}^{n+1}$ (Implicit) or $u^* = 2u^n - u^{n-1}$ (Linearly-Implicit), Step 1: (Backward Euler)*

$$\begin{aligned}
&\frac{\hat{u}^{n+1} - u^n}{\Delta t} + u^* \cdot \nabla \hat{u}^{n+1} - \nu \Delta \hat{u}^{n+1} + \nabla \hat{p}^{n+1} = f(t^{n+1}), \\
&\nabla \cdot \hat{u}^{n+1} = 0,
\end{aligned} \tag{4.2}$$

Step 2: (Time Filter for velocity and pressure)

$$u^{n+1} = \hat{u}^{n+1} - \frac{1}{3}(\hat{u}^{n+1} - 2u^n + u^{n-1}) \tag{4.3}$$



(a) (b)

Figure 7: The time filter does not add to the computational complexity (Fig. 7a), yet increases the method to second order (Fig. 7b).

Option A: (No pressure filter)

$$p^{n+1} = \hat{p}^{n+1}.$$

Option B:

$$p^{n+1} = \hat{p}^{n+1} - \frac{1}{3}(\hat{p}^{n+1} - 2p^n + p^{n-1})$$

Algorithm 4A means Option A is used, and Algorithm 4B means Option B is used.

Its implementation in a backward Euler code does not require additional function evaluations or solves, only a minor increase in floating point operations. Figure 7a presents a runtime comparison with and without the filter step. It is apparent that the added computational complexity of Step 2 is negligible. However, adding the time filter step has a profound impact on solution quality, see Figure 7b.

Herein, we give a velocity stability and error analysis for constant timestep in appendix. Since (eliminating the intermediate step) the constant time-step method is equivalent to an A-stable, second order, two step method, its velocity analysis has only minor deviations from the analysis in [89] and [96]. We also give an analysis of the unfiltered pressure error, which does not have a parallel in [89] or [96]. The predicted (optimal) convergence rates are confirmed in numerical tests in Section 4.6. We prove the pressure approximation is stable and second order accurate provided only the velocity is filtered. The predicted second order pressure convergence, with or without filtering the pressure, is also confirmed in our tests, Figure 8. The rest of the paper is organized as follow. In Section 2, we give the full, self-adaptive VSVO algorithm for a general initial value problem. Section 3 introduces some important mathematical notations and preliminaries necessary and analyze the method for the Navier-Stokes equations. In Section 4, we prove unconditional, nonlinear energy stability. We analyze consistency error in Section 4.4.1.

4.1.1 RELATED WORK

Time filters are primarily used to stabilize leapfrog time discretizations of weather models; see [103], [77], [107]. In [93] it was shown that the time filter used herein increases accuracy to second order, preserves A-stability, anti-diffuses the backward Euler approximation and yields an error estimator useful for time adaptivity. The analysis in [93] is an application of classical numerical ODE theory and does not extend to the Navier-Stokes equations. For the constant time step case, our analysis is based on eliminating the intermediate approximation $\hat{u}^{n+1}$ and reducing the method to an equivalent two step, OLM (a twin of a linear multistep method). The velocity stability and convergence of the general A-stable OLM was analyzed for the NSE (semi-implicit, constant time step and without space discretization) in [89]. Thus, the constant time step, discrete *velocity* results herein follow from these results. There is considerable previous work on analysis of multistep time discretizations of various PDEs, e.g. Crouzeix and Raviart [82]. Baker, Dougalis, and Karakashian [78] gave a long-time error analysis of the BDF methods for the NSE under a small data condition. (We stress that the method herein is *not a BDF method*.) The analysis of the method in Girault and Raviart [89] was extended to include spacial discretizations in [96]. The work in [96] also shows how to choose those parameters to improve accuracy in higher Reynolds number flows - a significant contribution by itself. Other interesting extensions include the work of Gevici [88], Emmrich [84], [85], Jiang [95], Ravindran [102] and [98].

4.2 THE ADAPTIVE VSVO METHOD

Section 4.6.2 tests both the constant time step method and the method with adaptive step and adaptive order. This section will present the algorithmic details of adapting both the order and time step based on estimates of local truncation errors based on established methods [91]. The constant time step Algorithm 1.1 involves adding one (Option A) or two (Option B) lines to a backward Euler FEM code. The full self adaptive VSVO Algorithm 2.1 below adds $\mathcal{O}(20)$ lines. We first give the method for the initial value problem

$$y\,'(t) = f(t, y(t)),\ \text{for } t > 0 \text{ and } y(0) = y_0.$$

Denote the n^{th} time step size by Δt_n. Let $t^{n+1} = t^n + \Delta t_n$ and y^n an approximation to $y(t_n)$. The choice of filtering weights depend on $\omega_n = \Delta t_n/\Delta t_{n-1}$, Step 2 below. TOL is the user supplied tolerance on the allowable error per step.

Algorithm 5 (Variable Stepsize, Variable Order 1 and 2 (VSVO-12))**.**

Step 1 : Backward Euler

$$\frac{y_{(1)}^{n+1} - y^n}{\Delta t_n} = f(t_{n+1}, y_{(1)}^{n+1})$$

Step 2 : Time Filter

$$y_{(2)}^{n+1} = y_{(1)}^{n+1} - \frac{\omega_{n+1}}{2\omega_{n+1}+1}\left(y_{(1)}^{n+1} - (1+\omega_{n+1})y^n + \omega_{n+1}y^{n-1}\right)$$

Step 3 : Estimate error in $y_{(1)}^{n+1}$ and $y_{(2)}^{n+1}$.

$$EST_1 = y_{(2)}^{n+1} - y_{(1)}^{n+1}$$

$$EST_2 = \frac{\omega_n\omega_{n+1}(1+\omega_{n+1})}{1+2\omega_{n+1}+\omega_n\left(1+4\omega_{n+1}+3\omega_{n+1}^2\right)}\Bigg(y_{(2)}^{n+1} - \frac{(1+\omega_{n+1})(1+\omega_n(1+\omega_{n+1}))}{1+\omega_n}y^n + \omega_{n+1}(1+\omega_n(1+\omega_{n+1}))y^{n-1} - \frac{\omega_n^2\omega_{n+1}(1+\omega_{n+1})}{1+\omega_n}y^{n-2}\Bigg).$$

Step 4 : Check if tolerance is satisfied.

If $\|EST_1\| < TOL$ or $\|EST_2\| < TOL$, at least one approximation is acceptable. Go to Step 5a. Otherwise, the step is rejected. Go to Step 5b.

Step 5a : At least one approximation is accepted. Pick an order and stepsize to proceed.

If both approximations are acceptable, set

$$\Delta t_{(1)} = 0.9\Delta t_n \left(\frac{TOL}{\|EST_1\|}\right)^{\frac{1}{2}}, \qquad \Delta t_{(2)} = 0.9\Delta t^n \left(\frac{TOL}{\|EST_2\|}\right)^{\frac{1}{3}}.$$

Set

$$i = \arg max_{i\in\{1,2\}}\Delta t^{(i)}, \qquad \Delta t^{n+1} = \Delta t^{(i)}, \qquad t^{n+2} = t^{n+1} + \Delta t_{n+1}, \qquad y^{n+1} = y^{n+1}_{(i)}.$$

If only $y^{(1)}$ (resp. $y^{(2)}$) satisfies TOL, set $\Delta t_{n+1} = \Delta t^{(1)}$ (resp. $\Delta t^{(2)}$), and $y^{n+1} = y^{n+1}_{(1)}$ (resp. $y^{n+1}_{(2)}$). Proceed to Step 1 to calculate y^{n+2}.

Step 5b : Neither approximations satisfy TOL.

Set

$$\Delta t^{(1)} = 0.7\Delta t_n \left(\frac{TOL}{\|EST_1\|}\right)^{\frac{1}{2}}, \qquad \Delta t^{(2)} = 0.7\Delta t_n \left(\frac{TOL}{\|EST_2\|}\right)^{\frac{1}{3}}.$$

Set

$$i = \arg max_{i\in\{1,2\}}\Delta t^{(i)}, \qquad \Delta t_n = \Delta t^{(i)}, \qquad t^{n+1} = t^n + \Delta t_n$$

Return to Step 1 to try again.

For clarity, we have not mentioned several standard features such as setting a maximum and minimum timestep, the maximum or minimum stepsize ratio, etc.

The implementation above computes an estimation of the local errors in Step 3. EST_1 provides an estimation for the local error of the first order approximation $y^{(1)}_{n+1}$ since $y^{(2)}_{n+1}$ is a second order approximation. For a justification of EST_2, see appendix. The optimal next stepsizes for both approximations are predicted in a standard way in Steps 5a and 5b. The method order (first or second) is adapted by accepting whichever approximation satisfies the error tolerance criterion (Step 4) and yields the larger next time step by the choice of $i = \arg max\Delta t^{(i)}$.

Standard formulas, see e.g. [92], are used to pick the next stepsize. The numbers 0.9 in Step 5a and 0.7 in Step 5b are commonly used safety factors to make the next approximation more likely to be accepted.

One more line is needed for linearly implicit methods. For linearly implicit methods the point of linearization must also have $\mathcal{O}(\Delta t^2)$ accuracy. For example, with $u^* = u^n$

$$\frac{u^{n+1} - u^n}{\Delta t_n} + u^* \cdot \nabla u^{n+1} + \frac{1}{2}(\nabla \cdot u^*)u^{n+1} + \nabla p^{n+1} - \nu \Delta u^{n+1} = f^{n+1} \ \& \ \nabla \cdot u^{n+1} = 0 \quad (4.4)$$

is a common first order linearly implicit method. The required modification in the BE step to ensure second order accuracy after the filter is to shift the point of linearization from $u^* = u^n$ to

$$u^* = \left(1 + \frac{\Delta t_{n+1}}{\Delta t_n}\right) u^n - \frac{\Delta t_{n+1}}{\Delta t_n} u^{n-1} = (1 + \omega_n)\, u^n - \omega_n u^{n-1}.$$

Other simplifications. The algorithm can be simplified if only the time-step is adapted (not order adaptive). It can be further simplified using *extrapolation* where the second order approximation is adapted based on EST_1 (pessimistic for the second order approximation).

4.3 FULLY DISCRETE METHOD

Before starting to analyze stability and convergence, we state the fully discrete method.

Algorithm 6 (Fully Discrete Method)**.** *Given* $u_h^{n-1}, u_h^n \in X^h$ *(and if necessary, given* $p_h^{n-1}, p_h^n \in Q^h$*), find* $(\hat{u}_h^{n+1}, \hat{p}^{n+1}) \in (X^h, Q^h)$ *satisfying*

$$\left(\frac{\hat{u}_h^{n+1} - u_h^n}{\Delta t_n}, v_h\right) + b(\hat{u}_h^{n+1}, \hat{u}_h^{n+1}, v_h) + \nu(\nabla \hat{u}_h^{n+1}, \nabla v_h) - (\hat{p}_h^{n+1}, \nabla \cdot v_h) = (f(t^{n+1}), v_h), \quad (4.5)$$

$$(\nabla \cdot \hat{u}^{n+1}, q_h) = 0.$$

for all $(v_h, q_h) \in (X^h, Q^h)$*. Then compute*

$$u_h^{n+1} = \hat{u}_h^{n+1} - \frac{\omega_{n+1}}{2\omega_{n+1} + 1}\left(\hat{u}_h^{n+1} - (1 + \omega_{n+1})u_h^n + \omega_{n+1} u_h^{n-1}\right).$$

Option A: (No pressure filter)

$$p_h^{n+1} = \hat{p}_h^{n+1}.$$

Option B:

$$p_h^{n+1} = \hat{p}_h^{n+1} - \frac{\omega_{n+1}}{2\omega_{n+1}+1}\left(\hat{p}_h^{n+1} - (1+\omega_{n+1})p_h^n + \omega_{n+1}p_h^{n-1}\right).$$

The constant time-step stability and error analysis works with the following equivalent formulation of the method. We stress that what follows is not the preferred implementation since it only yields one approximation, while Algorithm 6 gives the embedded approximations $\hat{u}_h^{n+1}$ and u_h^{n+1} and an error estimator.

Algorithm 7 (Constant time-step, equivalent method)**.** *Assume the time-step is constant. Given (u_h^n, p_h^n) and (u_h^{n-1}, p_h^{n-1}), find (u_h^{n+1}, p_h^{n+1}) such that for all $(v_h, q_h) \in (X^h, Q^h)$,*
Option A

$$\begin{aligned}
&\left(\frac{\frac{3}{2}u_h^{n+1} - 2u_h^n + \frac{1}{2}u_h^{n-1}}{\Delta t}, v_h\right) + b\left(\frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}, \frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}, v_h\right) \qquad (4.6)\\
&+\nu\left(\nabla\left(\frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}\right), \nabla v_h\right) - \left(\mathbf{p_h^{n+1}}, \nabla\cdot\mathbf{v_h}\right) = \left(f^{n+1}, v_h\right),\\
&\left(\nabla\cdot\left(\frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}\right), q_h\right) = 0,
\end{aligned}$$

or Option B

$$\begin{aligned}
&\left(\frac{\frac{3}{2}u_h^{n+1} - 2u_h^n + \frac{1}{2}u_h^{n-1}}{\Delta t}, v_h\right) + b\left(\frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}, \frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}, v_h\right) \qquad (4.7)\\
&+\nu\left(\nabla\left(\frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}\right), \nabla v_h\right) - \left(\mathbf{\frac{3}{2}p_h^{n+1} - p_h^n + \frac{1}{2}p_h^{n-1}}, \nabla\cdot\mathbf{v_h}\right) = \left(f^{n+1}, v_h\right),\\
&\left(\nabla\cdot\left(\frac{3}{2}u_h^{n+1} - u_h^n + \frac{1}{2}u_h^{n-1}\right), q_h\right) = 0.
\end{aligned}$$

The pressure is highlighted in bold, and is the only difference between the two above equations. The time difference term of the above equivalent method is that of BDF2 but the remainder is different. This is *not* the standard BDF2 method.

Proposition 1. *Algorithm 6A (respectively B) is equivalent Algorithm 7A (respectively B).*

Proof. We will just prove the case for Option A since the other case is similar. Let (u_h^{n+1}, p_h^{n+1}) be the solution to Algorithm 6. By linearity of the time filter, $(u_h^{n+1}, p_h^{n+1}) \in (X^h, Q^h)$. We can write $\hat{u}_h^{n+1}$ in terms of u_h^{n+1},u_h^n, and u_h^{n-1} as $\hat{u}^{n+1} = \frac{3}{2}u^{n+1} - u^n + \frac{1}{2}u^{n-1}$. Substitute this into (4.5). Then (u_h^{n+1}, p_h^{n+1}) satisfies equation (4.6).

These steps can be reversed to show the converse. □

We next define the discrete kinetic energy, viscous and numerical dissipation terms that arise naturally from a G-stability analysis of Algorithm 7, *regardless* of whether Option A or B is used. The (constant time-step) discrete kinetic energy, discrete viscous energy dissipation rate and the numerical energy dissipation rate of Algorithm 7 are

$$\begin{aligned}
\text{discrete energy:} \quad & \mathcal{E}^n = \tfrac{1}{4}\left[\|u^n\|^2 + \|2u^n - u^{n-1}\|^2 + \|u^n - u^{n-1}\|^2\right], \\
\text{viscous dissipation:} \quad & \mathcal{D}^{n+1} = \Delta t \nu ||\nabla\left(\tfrac{3}{2}u^{n+1} - u^n + \tfrac{1}{2}u^{n-1}\right)||^2, \\
\text{numerical dissipation:} \quad & \mathcal{Z}^{n+1} = \tfrac{3}{4}\|u^{n+1} - 2u^n + u^{n-1}\|^2.
\end{aligned}$$

Remark 3. *As $\Delta t \to 0$, $\mathcal{E}^n$ is consistent with the kinetic energy $\frac{1}{2}\|u\|^2$ and $\mathcal{D}^n$ is consistent with the instantaneous viscous dissipation $\nu\|\nabla u\|^2$. The numerical dissipation $\mathcal{Z}^{n+1} \approx \frac{3}{4}\Delta t^4 \|u_{tt}(t^{n+1})\|^2$, is asymptotically smaller than the numerical dissipation of backward Euler, $\frac{1}{2}\Delta t^2 \|u_t(t^{n+1})\|^2$.*

The method's kinetic energy differs from that of BDF2, which is (e.g. [99])

$$\mathcal{E}^n_{BDF2} = \frac{1}{4}\left[\|u^n\|^2 + \|2u^n - u^{n-1}\|^2\right]$$

due to the term $\|u^n - u^{n-1}\|^2$ in $\mathcal{E}^n$ which is a dispersive penalization of a discrete acceleration.

Define the interpolation and difference operators as follows

Definition 1. *The interpolation operator I and difference operator D are*

$$I[w^{n+1}] = \frac{3}{2}w^{n+1} - w^n + \frac{1}{2}w^{n-1} \quad \text{and} \quad D[w^{n+1}] = \frac{3}{2}w^{n+1} - 2w^n + \frac{1}{2}w^{n-1}.$$

Formally, $I[w(t^{n+1})] = w(t^{n+1}) + \mathcal{O}(\Delta t^2)$, and $\frac{D[w(t^{n+1})]}{\Delta t} = w_t(t^{n+1}) + \mathcal{O}(\Delta t^2)$. This will be made more precise in the consistency error analysis in Section 4.4.1.

4.4 STABILITY AND ERROR ANALYSIS

We prove stability and error analysis of the *constant time-step* method. The velocity proofs parallel ones in [89] and [96] and are collected in appendix. The pressure analysis is presented in Section 5.

Theorem 3. *Assume the stepsize is constant. The following equality holds.*

$$\mathcal{E}^N + \sum_{n=1}^{N-1} \mathcal{D}^{n+1} + \sum_{n=1}^{N-1} \mathcal{Z}^{n+1} = \Delta t \sum_{n=1}^{N-1} (f, I[u_h^{n+1}]) + \mathcal{E}^1.$$

Proof. In Algorithm 7, set $v_h = I[u_h^{n+1}]$ and $q_h = p_h^{n+1}$ for Option A, or $q_h = I[p_h^{n+1}]$ for Option B, and add.

$$(D[u_h^{n+1}], I[u_h^{n+1}]) + \mathcal{D}^{n+1} = \Delta t (f, I[u_h^{n+1}]). \tag{4.8}$$

By Lemma 6 and Definition 1,

$$(D[u_h^{n+1}], I[u_h^{n+1}]) = \mathcal{E}^{n+1} - \mathcal{E}^n + \mathcal{Z}^{n+1}.$$

Thus, (4.8) can be written

$$\mathcal{E}^{n+1} - \mathcal{E}^n + \mathcal{D}^{n+1} + \mathcal{Z}^{n+1} = \Delta t (f(t^{n+1}), I[u_h^{n+1}]).$$

Summing over n from 1 to $N-1$ yields the result. □

This result is for the time stepping method applied to the Navier-Stokes equations. More generally, the constant time-step method of Algorithm 3.2 is G-Stable, a fact that follows from the equivalence of A and G-Stability [83]. We calculate the G matrix explicitly below.

Corollary 1. *Assume the time-step is constant. Backward Euler followed by the time filter is G-Stable with G matrix*

$$G = \begin{bmatrix} \frac{3}{2} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}.$$

Proof. Simply check that

$$[u^n, u^{n-1}] G \begin{bmatrix} u^n \\ u^{n-1} \end{bmatrix} = \frac{1}{4}\Big[|u^n|^2 + |2u^n - u^{n-1}|^2 + |u^n - u^{n-1}|^2 \Big].$$

□

4.4.1 CONSISTENCY ERROR

By manipulating (2.5), we derive the consistency error. The true solution to (2.5) satisfies

$$\begin{aligned}
&\left(\frac{D[u(t^{n+1})]}{\Delta t}, v_h\right) + b\left(I[u(t^{n+1})], I[u(t^{n+1})], v_h\right) \\
&+ \nu\left(\nabla I[u(t^{n+1})], \nabla v_h\right) - \left(p(t^{n+1}), \nabla \cdot v_h\right) \\
&= \left(\mathbf{f}^{n+1}, v_h\right) + \tau^{n+1}(u, p; v_h) \quad \forall v_h \in X_h.
\end{aligned} \tag{4.9}$$

If Option A is used (pressure is unfiltered),

$$\tau^{n+1}(u,p;v_h) = \tau_A^{n+1}(u,p;v_h) = \left(\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1}), v_h\right) \tag{4.10}$$

$$+b\left(I[u(t^{n+1})], I[u(t^{n+1})], v_h\right) - b(u(t^{n+1}), u(t^{n+1}), v_h) + \nu\left(\nabla(I[u(t^{n+1})] - u(t^{n+1})), \nabla v_h\right)$$

If Option B is used (pressure is filtered),

$$\tau^{n+1}(u,p;v_h) = \tau_A^{n+1}(u,p;v_h) - \left(I[p(t^{n+1})] - p(t^{n+1}), \nabla \cdot v_h\right) \tag{4.11}$$

Thus, filtering the pressure introduces a term that, while still second order, adds to the consistency error. We believe this is why Option A performs better in the numerical tests, Figure 8. Furthermore, Option B requires assuming additional regularity for convergence, see Theorem 11.

The terms in the consistency error are bounded in the following lemma.

Lemma 9 (Consistency)**.** *For u, p sufficiently smooth, we have*

$$\left\|\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\right\|^2 \leq \frac{6}{5}\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|u_{ttt}\|^2 dt,$$

$$\left\|I[u(t^{n+1})] - u(t^{n+1})\right\|^2 \leq \frac{4}{3}\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|u_{tt}\|^2 dt. \tag{4.12}$$

$$\left\|I[p(t^{n+1})] - p(t^{n+1})\right\|^2 \leq \frac{4}{3}\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|p_{tt}\|^2 dt. \tag{4.13}$$

Proof. See Appendix A. □

4.4.2 ERROR ESTIMATES FOR THE VELOCITY

Next, we analyze the convergence of Algorithm 7 and give an error estimate for the velocity. Let $t^n = n\Delta t$. Denote the errors $\mathbf{e}_u^n = u(t^n) - u_h^n$ and $e_p^n = p(t^n) - p_h^n$.

Theorem 4. *Assume that the true solution (u, p) satisfies the following regularity*

$$\begin{aligned} &u \in L^\infty(0,T;(H^{k+1}\Omega))^d), \quad u_t \in L^2(0,T;(H^{k+1}\Omega))^d), \quad u_{tt} \in L^2(0,T;(H^1\Omega))^d), \\ &u_{ttt} \in L^2(0,T;(L^2\Omega))^d), \qquad p \in L^2(0,T;(H^{s+1}(\Omega))^d). \end{aligned} \tag{4.14}$$

Additionally for Option B, assume $p_{tt} \in L^2(0,T;(L^2(\Omega))^d$. For (u_h^{n+1}, p_h^{n+1}) satisfying (4.6), we have the following estimate

$$\begin{aligned} &\|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \|e_u^N - e_u^{N-1}\|^2 + \sum_{n=1}^{N-1} 3\|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2 \\ &+ \nu\Delta t \sum_{n=1}^{N-1} \|\nabla I[e_u^{n+1}]\|^2 \le C\Big(h^{2k} + h^{2s+2} + \Delta t^4\Big) \end{aligned} \tag{4.15}$$

Proof. See appendix. □

4.5 PRESSURE STABILITY AND CONVERGENCE

4.5.1 STABILITY OF PRESSURE

We introduce the following discrete norms

$$|||\omega|||_{\infty,k} := \max_{0\le n\le T/\Delta t} \|\omega^n\|_k, \quad |||\omega|||_{2,k} := \left(\sum_{n=0}^{T/\Delta t-1} \Delta t\|\omega^n\|_k^2\right)^{1/2}. \tag{4.16}$$

In this section, we prove the pressure approximation is stable in $l^1(0,T;L^2(\Omega))$. We first give a corollary of Theorem 3 asserting the stability of the velocity approximation.

Corollary 2. *Suppose $f \in L^2(0,T;H^{-1}(\Omega)^d)$, then the velocity approximation satisfies*

$$\mathcal{E}^N + \frac{1}{2}\sum_{n=1}^{N-1}\mathcal{D}^{n+1} + \sum_{n=1}^{N-1}\mathcal{Z}^{n+1} \le \frac{1}{2\nu}|||f|||_{2,-1}^2 + \mathcal{E}^1.$$

Proof. Consider Theorem 3. Applying the Cauchy-Schwartz yields the inequality. □

We now prove the stability of the filtered pressure.

Theorem 5. *Suppose Corollary 2 holds, then the pressure approximation satisfies*

$$\begin{aligned} \beta\Delta t\sum_{n=1}^{N-1}\|p_h^{n+1}\| &\le C \qquad \text{for Option } A,\\ \beta\Delta t\sum_{n=1}^{N-1}\|I[p_h^{n+1}]\| &\le C \quad \text{for Option } B. \end{aligned} \tag{4.17}$$

Proof. We prove it for Option A, as the other case is similar. Isolating the discrete time derivative in (4.6), and restricting v_h to V_h yields

$$\begin{aligned} \left(\frac{D[u_h^{n+1}]}{\Delta t}, v_h\right) &= -b\left(I[u_h^{n+1}], I[u_h^{n+1}], v_h\right)\\ -\nu\left(\nabla I[u_h^{n+1}], \nabla v_h\right) &+ \left(f^{n+1}, v_h\right) \quad \forall v_h \in V_h. \end{aligned} \tag{4.18}$$

The terms on the right hand side of (4.18) can be bounded as follows,

$$\begin{aligned} &b\left(I[u_h^{n+1}], I[u_h^{n+1}], v_h\right) \le C\|\nabla I[u_h^{n+1}]\|\|\nabla I[u_h^{n+1}]\|\|\nabla v_h\|,\\ &-\nu\left(\nabla I[u_h^{n+1}], \nabla v_h\right) \le \nu\|\nabla I[u_h^{n+1}]\|\|\nabla v_h\|,\\ &\left(f^{n+1}, v_h\right) \le \|f^{n+1}\|_{-1}\|\nabla v_h\|. \end{aligned} \tag{4.19}$$

In equation (4.18), we can use the above estimates in (4.19), divide both sides by $\|\nabla v_h\|$, and take the supremum over $v_h \in V_h$. This gives

$$\left\|\frac{D[u_h^{n+1}]}{\Delta t}\right\|_{V_h^*} \le (C\|\nabla I[u_h^{n+1}]\| + \nu)\|\nabla I[u_h^{n+1}]\| + \|f^{n+1}\|_{-1}. \tag{4.20}$$

Lemma 5 implies

$$\left\|\frac{D[u_h^{n+1}]}{\Delta t}\right\|_{X_h^*} \le C\Big[(\|\nabla I[u_h^{n+1}]\| + 1)\|\nabla I[u_h^{n+1}]\| + \|f^{n+1}\|_{-1}\Big]. \tag{4.21}$$

Now consider Algorithm 7 again with $v_h \in X_h$. Isolating the pressure term in (4.6) and using the estimates from (4.19) yields

$$\begin{aligned} \left(p_h^{n+1}, \nabla\cdot v_h\right) &\le \left(\frac{D[u_h^{n+1}]}{\Delta t}, v_h\right)\\ +C(\|\nabla I[u_h^{n+1}]\| + 1)\|\nabla I[u_h^{n+1}]\|\|\nabla v_h\| &+ \|f^{n+1}\|_{-1}\|\nabla v_h\|. \end{aligned} \tag{4.22}$$

Divide both sides by $\|\nabla v_h\|$, take supremum over $v_h \in X_h$ and use the discrete inf-sup condition and the results in (4.22). Then,

$$\begin{aligned} &\beta\|p_h^{n+1}\| \\ &\leq C\Big[(\|\nabla I[u_h^{n+1}]\|+1)\|\nabla I[u_h^{n+1}]\|+\|f^{n+1}\|_{-1}\Big]. \end{aligned} \tag{4.23}$$

We then multiply by Δt, sum from $n=1$ to $n=N-1$, and apply Cauchy-Schwartz on the right hand,

$$\begin{aligned} &\beta\Delta t\sum_{n=1}^{N-1}\|p_h^{n+1}\| \\ &\leq C\Delta t\Big[(|||\nabla I[u_h^{n+1}]|||_{2,0}+1)|||\nabla I[u_h^{n+1}]|||_{2,0}+\|f^{n+1}\|_{2,-1}\Big]. \end{aligned} \tag{4.24}$$

Then using the result from velocity approximation, we get,

$$\begin{aligned} &\beta\Delta t\sum_{n=1}^{N-1}\|p_h^{n+1}\| \\ &\leq C\Big[(|||f|||_{2,-1}+1)|||f|||_{2,-1}+(\mathcal{E}^1+1)\mathcal{E}^1\Big]. \end{aligned} \tag{4.25}$$

□

4.5.2 ERROR ESTIMATES FOR THE PRESSURE

We now prove convergence of the pressure approximation in $l^1(0,T;L^2(\Omega))$. Denote the pressure error as $e_p^n = p(t^n) - p_h^n$.

Theorem 6. *Let u, p satisfy the equation (4.15). Let the assumption of regularity in Theorem 11 be satisfied. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\Delta t\beta\sum_{n=1}^{N-1}\|e_p^{n+1}\| \leq C\Big(h^k+h^{s+1}+\Delta t^2\Big) \qquad \text{for Option A}, \\ &\Delta t\beta\sum_{n=1}^{N-1}\|I[e_p^{n+1}]\| \leq C\Big(h^k+h^{s+1}+\Delta t^2\Big) \quad \text{for Option B}. \end{aligned} \tag{4.26}$$

Proof. Again, we only prove this for Option A since the other case requires only slight modification. Using the equations (A.3) and (A.4) yields

$$\begin{aligned} &\left(\frac{D[\phi(t^{n+1})]}{\Delta t}, v_h\right) = \left(\frac{D[\eta(t^{n+1})]}{\Delta t}, v_h\right) - b\left(I[e_u^{n+1}], I[u(t^{n+1})], v_h\right) \\ &- b\left(I[u_h^{n+1}], I[e_u^{n+1}], v_h\right) - \nu\left(\nabla I[e_u^{n+1}], \nabla v_h\right) \\ &+ \left(p(t^{n+1}) - \lambda_h^{n+1}, \nabla \cdot v_h\right) + \tau^{n+1}(u, p; v_h) \quad \forall v_h \in V_h. \end{aligned} \tag{4.27}$$

We bound the six individual terms on the right hand side of (4.27), term by term as follows:

$$\left(\frac{D[\eta(t^{n+1})]}{\Delta t}, v_h\right) \leq C\Delta t^{-\frac{1}{2}} \|\eta_t\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \|\nabla v_h\|, \tag{4.28}$$

$$-b\left(I[e_u^{n+1}], I[u(t^{n+1})], v_h\right) \leq C\|\nabla I[e_u^{n+1}]\| \|\nabla I[u(t^{n+1})]\| \|\nabla v_h\|, \tag{4.29}$$

$$-b\left(I[u_h^{n+1}], I[e_u^{n+1}], v_h\right) \leq C\|\nabla(I[u_h^{n+1}])\| \|\nabla I[e_u^{n+1}]\| \|\nabla v_h\|, \tag{4.30}$$

$$-\nu\left(\nabla I[e_u^{n+1}], \nabla v_h\right) \leq \nu\|\nabla I[e_u^{n+1}]\| \|\nabla v_h\|, \tag{4.31}$$

$$\left(p(t^{n+1}) - \lambda_h^{n+1}, \nabla \cdot v_h\right) \leq C\|p(t^{n+1}) - \lambda_h^{n+1}\| \|\nabla v_h\|, \tag{4.32}$$

$$\begin{aligned} \tau^{n+1}(u, p; v_h) \leq C\Delta t^{\frac{3}{2}} \Big(&\|u_{ttt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla u_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\ &+ \|\nabla u\|^2_{L^4(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla u_{tt}\|^2_{L^4(t^{n-1},t^{n+1};L^2(\Omega))}\Big) \|\nabla v_h\|. \end{aligned} \tag{4.33}$$

Considering equation (4.27) and Lemma 5 , using equations (4.28)-(4.33), dividing both sides by $\|\nabla v_h\|$ and taking a supremum over V_h gives

$$\begin{aligned} &\left\|\frac{D[\phi(t^{n+1})]}{\Delta t}\right\|_{X_h^*} \leq C\Big[\Delta t^{-\frac{1}{2}} \|\eta_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\| \\ &+ \|\nabla I[e_u^{n+1}]\|(\|\nabla I[u(t^{n+1})]\| + \|\nabla(I[u_h^{n+1}])\| + 1) \\ &+ \|p(t^{n+1}) - \lambda_h^{n+1}\| + \Delta t^{\frac{3}{2}} \Big(\|u_{ttt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla u_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\ &+ \|\nabla u\|^2_{L^4(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla u_{tt}\|^2_{L^4(t^{n-1},t^{n+1};L^2(\Omega))}\Big)\Big]. \end{aligned} \tag{4.34}$$

Separating the pressure error term $e_p^{n+1} = (p(t^{n+1}) - \lambda_h^{n+1}) - (p_h^{n+1} - \lambda_h^{n+1})$ and rearranging implies

$$\left(p_h^{n+1} - \lambda_h^{n+1}, \nabla \cdot v_h\right) = \left(\frac{D[\eta(t^{n+1})]}{\Delta t}, v_h\right) - \left(\frac{D[\phi(t^{n+1})]}{\Delta t}, v_h\right)$$
$$+\nu\left(\nabla I[e_u^{n+1}], \nabla v_h\right) - \left(e_p^{n+1}, \nabla \cdot v_h\right) - \left(p(t^{n+1}) - \lambda_h^{n+1}, v_h\right) + \tau^{n+1}(u, p; v_h) \quad \forall v_h \in X_h.$$

Consider the estimates in (4.28)-(4.34). Divide by $\|\nabla v_h\|$, take supremum over $v_h \in X_h$ and use discrete inf-sup condition to obtain,

$$\begin{aligned}
&\beta\|p_h^{n+1} - \lambda_h^{n+1}\| \leq C\Big[\Delta t^{-\frac{1}{2}}\|\eta_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&+ \|\nabla I[e_u^{n+1}]\|\Big(\|\nabla I[u(t^{n+1})]\| + \|\nabla(I[u_h^{n+1}])\| + 1\Big) \\
&+ \|p(t^{n+1}) - \lambda_h^{n+1}\| + \Delta t^{\frac{3}{2}}\Big(\|u_{ttt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla u_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\
&+ \|\nabla u\|^2_{L^4(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla u_{tt}\|^2_{L^4(t^{n-1},t^{n+1};L^2(\Omega))}\Big)\Big].
\end{aligned} \tag{4.35}$$

We multiply by Δt, sum from $n = 1$ to $n = N - 1$ and apply triangle inequality. This yields

$$\begin{aligned}
&\beta\Delta t\sum_{n=1}^{N-1}\|e_p^{n+1}\| \leq C\Big[\Delta t^{-\frac{1}{2}}\|\eta_t\|_{L^2(0,T;L^2(\Omega))} \\
&+ \||p(t^{n+1}) - \lambda_h^{n+1}\||_{2,0} + \||\nabla I[e_u^{n+1}]\||_{2,0} \\
&+ \Delta t^{\frac{5}{2}}\Big(\|u_{ttt}\|_{2,0} + \|\nabla u_{tt}\|_{2,0} + \||\nabla u\||^2_{4,0} + \|\nabla u_{tt}\|^2_{4,0}\Big)\Big].
\end{aligned} \tag{4.36}$$

Results from the equations ($A.19$) and ($A.22$) give the bounds for the first two terms. Using error estimates of the velocity on the third term and taking infimum over X_h and Q_h yield the result. □

4.6 NUMERICAL TESTS

We verify second order convergence for the new method through an exact solution in Section 4.6.1. Visualizations of the flow and benchmark quantities gives additional support to the increased accuracy of the new method in Section 4.6.3. The tests used P_2/P_1 and P_3/P_2 elements. All computations were performed with FEniCS [75].

4.6.1 TAYLOR-GREEN VORTEX

We apply the backward Euler and the backward Euler plus filter for the 2D Taylor-Green vortex. This test problem is historically used to assess accuracy and convergence rates in CFD [81]. The exact solution is given by

$$u = e^{-2\nu t}(\cos x \sin y, -\sin x \cos y) \text{ and } p = -\frac{1}{4}e^{-4\nu t}(\cos 2x + \cos 2y).$$

To test time accuracy, we solve using P_3/P_2 elements on a uniform mesh of 250×250 squares divided into 2 triangle per square. We take a series of time steps for which the total error is expected to be dominated by the temporal error. Since the true solution decays exponentially, we tabulate and display relative errors. Fig. 6.1 displays the relative errors for backward Euler, backward Euler plus filtering only the velocity (Algorithm 4A), and backward Euler plus filtering both the velocity and pressure (Algorithm 4B). Filtering the pressure does not affect the velocity solution, so the velocity error plot only shows two lines. The velocity error is $\mathcal{O}(\Delta t^2)$, as predicted, and significantly smaller than the backward Euler error. Thus, adding the filter step (1.3) reduces the velocity error substantially, Figure 8.1, at negligible cost, Figure 1.1. The pressure error is $\mathcal{O}(\Delta t^2)$ when either both u and p are filtered, or only u is filtered, which is consistent with our theoretical analysis. Filtering only u has smaller pressure error since the pressure filter introduces an extra consistency error term, see (4.11).

4.6.2 ADAPTIVE TEST

We test the time/order adaptive algorithm on a problem that showcases the superiority of the VSVO method over the constant stepsize, constant order method.

The Taylor-Green problem can be modified by replacing F with any differentiable function of t. With velocity and pressure defined as before, the required body force is

$$f(x, y, t) = (2\nu F(t) + F'(t))\langle \cos x \sin y, -\cos y \sin x\rangle.$$

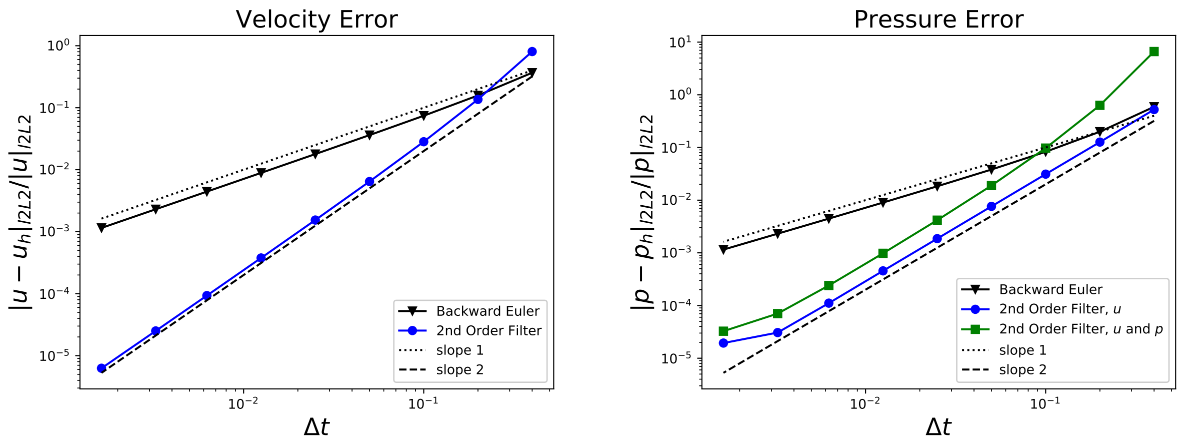


Figure 8: Convergence rates for the filtered quantities are second order as predicted. Filtering only the velocity produces the best pressure.

For $F(t)$, we construct a sharp transition function between 0 and 1. First, let

$$g(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \exp\left(-\frac{1}{(10t)^{10}}\right) & \text{if } t > 0 \end{cases}$$

This is a differentiable function, and $g(5) \equiv 1$ in double precision. Therefore, a differentiable (up to machine precision) function can be constructed with shifts and reflections of this function. This creates sections of flatness, and sections that rapidly change which require adaptivity to resolve efficiently. See Fig. 9 for the evolution of $\|u\|$ with time. All tests were initialized at rest spaced at a constant interval of $k = 0.1$, 100 nodes per side of the square using P_2/P_1 elements, and with final time of 45.

Figure 9 compares two numerical solutions. One is from Algorithm 4 (second order - nonadaptive), and the other is from Algorithm 5 (VSVO-12). With $TOL = 10^{-3}$, the VSVO-12 method takes 342 steps, which comprises 254 accepted steps, and 88 rejected steps. The constant stepsize method which took 535 steps does not accurately capture the energetic jumps.

Figure 10 shows the relative l^2L^2 velocity errors versus steps taken of VSVO-12 for seven different TOLs, starting at 10^{-1}, and dividing by ten down to 10^{-7}. This is compared with nonadaptive method (which has no rejected steps) sampled at several stepsizes. Both methods show second order convergence, but for smaller tolerances, VSVO-12 performs about 10^3 better than the nonadaptive method for the same amount of work.

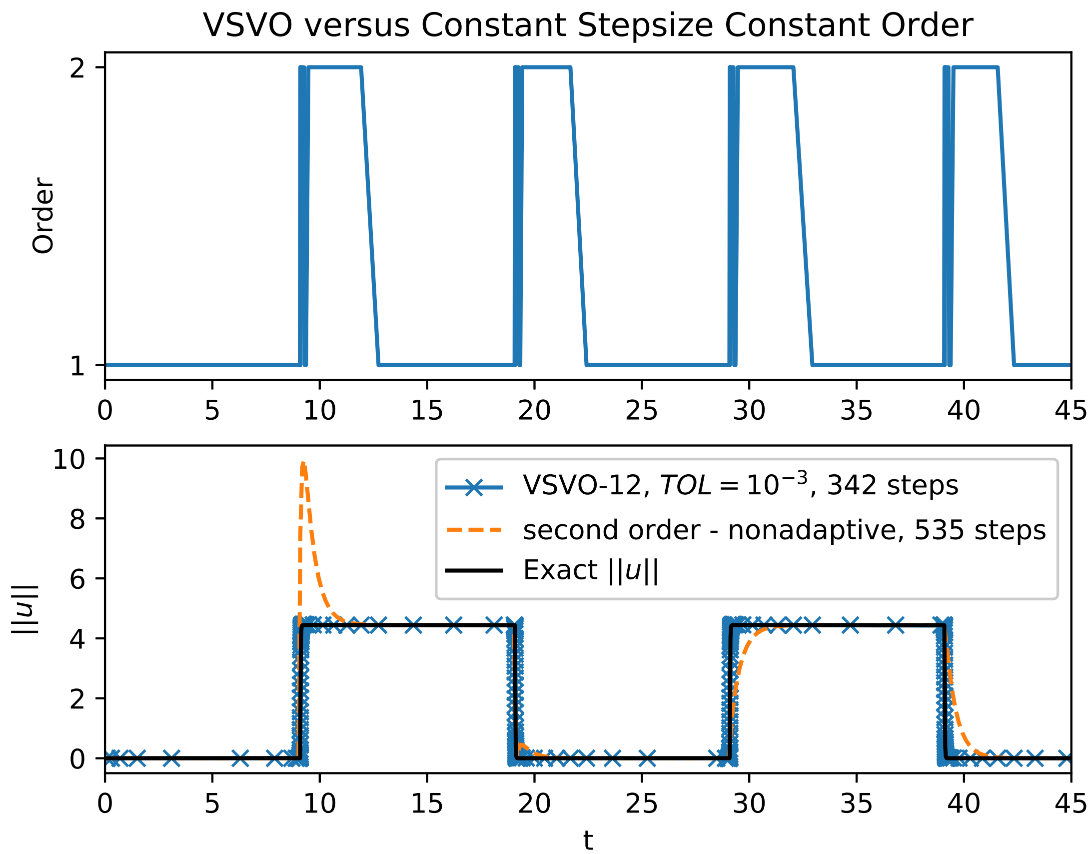


Figure 9: The nonadaptive second order method results in large overshoots and undershoots while requiring more work than the adaptive method.

4.6.3 FLOW AROUND A CYLINDER

We now use the benchmark problem of flow around a cylinder, originally proposed in [104], to test the improvement obtained using filters on flow quantities (drag, lift, and pressure drop) using values obtained via a DNS in [97] as a reference. This problem has also been used as a benchmark in [101],[100],[79],[80] and others. Let $\nu = 10^{-3}$, $f \equiv 0$, $T_{final} = 8$, and

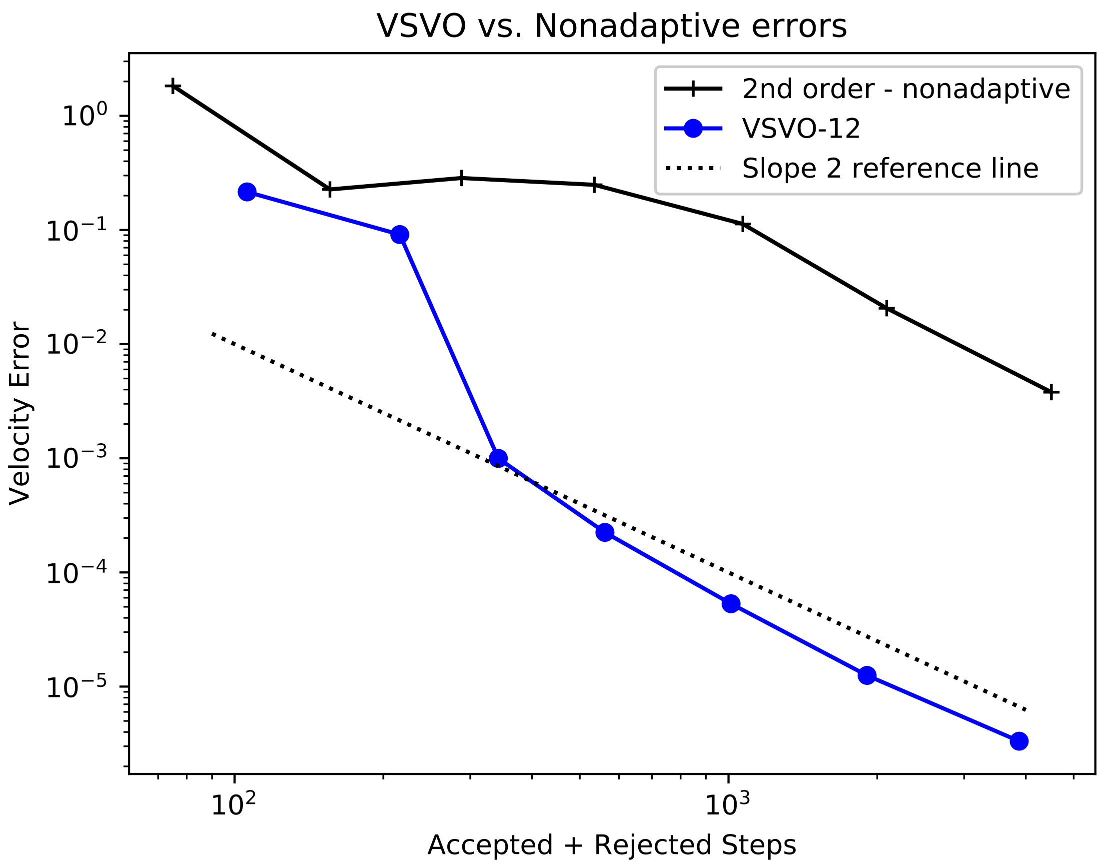


Figure 10: The VSVO-12 method performs three orders of magnitude better for the same amount of work compared to the nonadaptive 2nd order method for the test problem in Section 4.6.2. Each circle represents a different tolerance from $TOL = 10^{-1}$ to 10^{-7}.

i.e., a channel with a cylindrical cutout. A parabolic velocity of $u = 0.41^{-2}\sin(\pi t/8)(6y(0.41 - y), 0)$ is prescribed at the left and right boundaries. We used a spatial discretization with 479026 degrees of freedom with 1000 vertices on the boundary of the cylinder. The mesh used $P2/P1$ elements, and was obtained by adaptive refinement from solving the steady solution with $u = 0.41^{-2}(6y(0.41 - y), 0)$ as inflow and outflow boundary conditions.

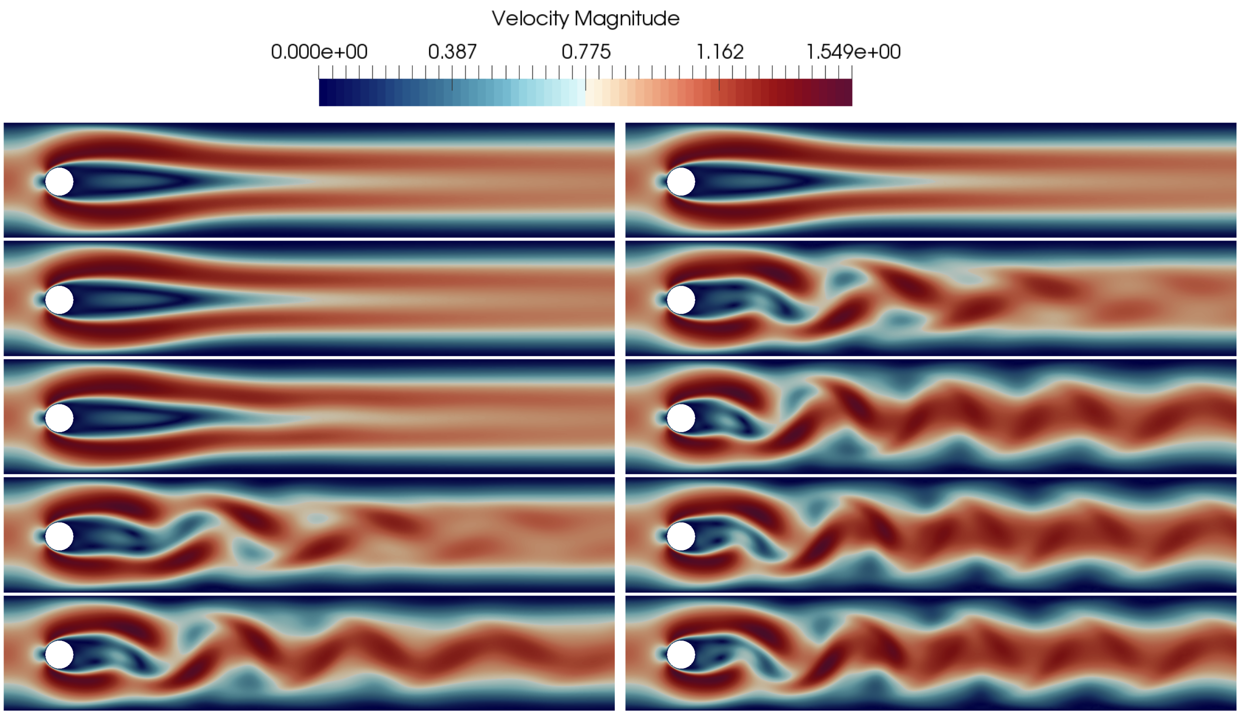
The correct behavior for this problem is that vortices shed off the cylinder as the in-let and outlet velocities increase. Fig. 11 shows snapshots of the flow at $t = 6$ for five successively halved Δt's. The Backward Euler approximation shows no vortex shedding for $\Delta t = 0.04, 0.02$, and 0.01. The filtered method of Algorithm 4 shows the qualitatively correct behavior from $\Delta t = 0.02$ on. Clearly, higher order and less dissipative methods are necessary to see dynamics for modestly large Δt.

It was demonstrated in [97] that the backward Euler time discretization greatly under predicts lift except for very small step sizes. Fig. 12 demonstrates that the time filter in Algorithm 4 corrects both the amplitude and phase error in the backward Euler approximation. Other quantities that were compared to reference values were the maximum drag $c_{d,\max}$, the time of max drag $t(c_{d,\max})$, time of maximum lift $t(c_{l,\max})$, and pressure drop across the cylinder at $t = 8$ are shown in Table 7.

The choice of whether or not to filter the pressure does not affect the velocity solution, the snapshots shown Figure 11 are the same for both choices. Table 7 shows that filtering u greatly improves the calculated flow quantities whether or not p is filtered.

4.7 CONCLUSIONS

This report presents a low computational and cognitive complexity, stable, time accurate and adaptive method for the Navier-Stokes equations. The improved method requires a minimally intrusive modification to an existing program based on the fully implicit / backward Euler time discretization, does not add to the computational complexity, and is conceptually simple. The backward Euler approximation is simply post-processed with a two-step, linear time filter. The time filter additionally removes the overdamping of Backward Euler while remaining unconditionally energy stable, proven herein. Even for constant stepsizes, the method does not reduce to a standard / named time stepping method but is related to a known 2-parameter family of A-stable, two step, second order methods. Numerical tests confirm the predicted convergence rates and the improved predictions of flow quantities such as drag and lift.



(a) Backward Euler

(b) Backward Euler Plus Filter

Figure 11: Flow snapshots at $t = 6$ with $\Delta t = 0.04$ (top), and Δt halving until $\Delta t = 0.0025$ (bottom). Backward Euler (left) destroys energy and suppresses oscillations, meaning that it can predict nearly steady state solutions when a time dependent one exists. The time filter (right) corrects this.

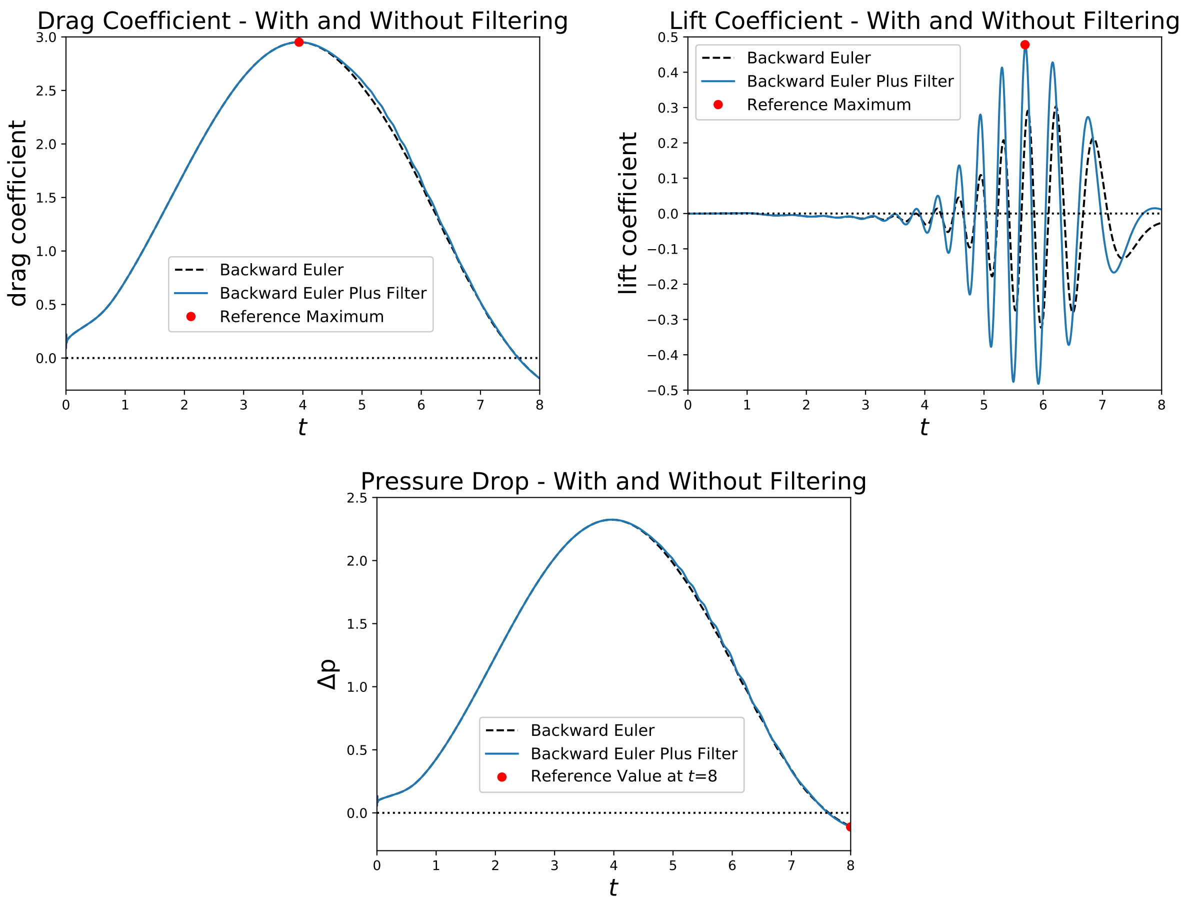


Figure 12: Lift of the Backward Euler solution and the filtered solution for $\Delta t = 0.0025$. The filtered solution correctly predicts both the time and magnitude of the maximum lift.

Table 7: Lift, drag, and pressure drop for cylinder problem.

Backward Euler

Δt	$t(c_{d,\max})$	$c_{d,\max}$	$t(c_{l,\max})$	$c_{l,\max}$	$\Delta p(8)$
0.04	3.92	2.95112558	0.88	0.00113655	-0.12675521
0.02	3.94	2.95064522	0.92	0.00117592	-0.12647232
0.01	3.93	2.95041574	7.17	0.02489640	-0.12433915
0.005	3.93	2.95031983	6.28	0.17588270	-0.10051423
0.0025	3.9325	2.95038901	6.215	0.30323034	-0.10699361

Backward Euler Plus Filter

0.04	3.92	2.95021463	7.56	0.00438111	-0.12628328
0.02	3.94	2.95026781	6.14	0.20559211	-0.11146505
0.01	3.93	2.95060684	5.81	0.40244197	-0.09943203
0.005	3.935	2.95082513	5.72	0.46074771	-0.11111586
0.0025	3.935	2.95089028	5.7	0.47414096	-0.11193754

Backward Euler Plus Filter u and p

0.04	3.92	2.95073993	7.52	0.00439864	-0.12642684
0.02	3.94	2.95039973	6.14	0.21101313	-0.11153593
0.01	3.93	2.95063962	5.81	0.40624697	-0.09945143
0.005	3.935	2.95083296	5.72	0.46192306	-0.11112049
0.0025	3.935	2.95089220	5.7	0.47444753	-0.11193859

Reference Values

—	3.93625	2.950921575	5.693125	0.47795	−0.1116

5.0 LOW COMPLEXITY ALGORITHMS IN CFD–ARTIFICAL COMPRESSION METHOD FOR MHD FLOWS

5.1 INTRODUCTION

We consider the time-dependent magnetohydrodynamic flows at low magnetic Reynolds numbers (denoted by R_m). The low-R_m MHD model (typical for terrestrial applications, see [12, 20, 23]) is given by: find fluid velocity $\mathbf{u} : \Omega \times [0,T] \to R^d$, pressure $p : \Omega \times [0,T] \to R$ and electric potential $\phi : \Omega \times [0,T] \to R$ satisfying

$$\begin{aligned}
&\frac{1}{N}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1}{M^2}\Delta\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{B}\times\nabla\phi + (\mathbf{u}\times\mathbf{B})\times\mathbf{B},\\
&\nabla\cdot\mathbf{u} = 0,\\
&\Delta\phi = \nabla\cdot(\mathbf{u}\times\mathbf{B}).\\
&\mathbf{u} = 0 \qquad \text{on } \partial\Omega\times[0,T],\\
&\phi = 0 \qquad \text{on } \partial\Omega\times[0,T],\\
&\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \forall\, x\in\Omega.
\end{aligned} \tag{5.1}$$

Here, body force $\mathbf{f}$, external magnetic field $\mathbf{B}$, and final time $T > 0$ are known. The domain $\Omega \subset R^d\,(d = 2\,\text{or}\,3)$ is a convex polygon or polyhedra. N is interaction parameter and M is Hartmann number, $\frac{N}{M^2} = \frac{1}{Re}$, where Re is the Reynolds number. Further, $\mathbf{u}_0(x) \in H_0^1(\Omega)^d$ and $\nabla\cdot\mathbf{u}_0 = 0$.

In this report, we give an anlysis of a classical artificial compression scheme from [28] adapted from the Navier-Stokes equations (NSE) to the model (5.1). Combined with two partitioned methods from [23], we gave two fully-decoupled methods. The schemes (Algorithm 8 and Algorithm 9) are based on (i) replacing $\nabla\cdot\mathbf{u}$ by $\varepsilon p_t + \nabla\cdot\mathbf{u} = 0$, (ii) time discretization by the implicit methods (Backward-Euler and BDF2) and (iii) treating the magnetic field terms explicitly to further uncouple the system into components. Theorem 11 below shows that for smooth solutions the error of Algorithm 8 is $\mathcal{O}(\Delta t + \varepsilon)$. Numerical tests in Section 5.5 also confirm that the error of Algorithm 8 and Algorithm 9 are $\mathcal{O}(\Delta t + \varepsilon)$ and $\mathcal{O}(\Delta t^2 + \varepsilon)$, respectively.

5.1.1 PREVIOUS WORK

MHD describes the behavior of the electrically conducting fluids in the presence of an external magnetic field. The study of MHD, initiated by Alfvén [1], has been widely developed in many fields of science including astrophysics, geophysics, engineering, and metallurgy. Applications include the studying of sunspots and solar flares, pumping and stirring of liquid metals, liquid metals cooling of nuclear reactors, forecasting of climate change, controlled thermonuclear fusion and sea water propulsion, see [2, 3, 4, 5, 6]. Most terrestrial applications, such as liquid metals, involve small magnetic Reynolds numbers, $R_m \ll 1$. In these cases, the magnetic field influences the conducting fluid via the Lorentz force, but the conducting fluid does not significantly perturb the magnetic field. Thus the magnetic field induced by the electrically conducting fluid motion is small and can be negligible compared with the imposed magnetic field. Neglecting the induced magnetic field, the general MHD flows can be simplified to the low-R_m MHD model considered herein.

In the recent years, there are many works on the MHD equations. For instance, references [18, 19, 13, 15, 16, 17] studied some effective iterative methods in finite element approximation for the steady MHD equations. For the time-dependent MHD equations, He [33] discussed an Euler semi-implicit scheme for the three-dimensional MHD equations. The decoupled fully discrete finite element schemes for the unsteady MHD equations were analyzed in [31, 32, 39]. Zhang, Su and Feng [14] analyzed a partitioned scheme based on Gauge-Uzawa finite element method for the 2D time-dependent MHD equations. The mathematical structure of the steady low-R_m MHD model was established by Peterson [12]. Numerical analysis of the evolutionary problem (5.1) was performed by Yuksel and Isik [25] (coupled implicit method), Yuksel and Ingram [20] (coupled Crank-Nicolson method), and

Rong, Hou and Zhang [35] (coupled spectral deferred correction method). Partitioned methods uncoupling the fluid velocity from the electric potential were analyzed in [23, 24, 36]. The Algorithm 9 and 8 herein continue this development uncoupling electric potential, pressure and individual components of the velocity.

5.1.2 THE SLIGHTLY COMPRESSIBLE MODEL

There are two forms of coupling in the above equation (5.1). One is the coupling between the fluid velocity $\mathbf{u}$ and the electric potential ϕ. The other is that the fluid velocity $\mathbf{u}$ and the pressure p are coupled by the incompressibility restriction $\nabla \cdot \mathbf{u} = 0$. Both couplings increase memory requirements, make the equations more difficult to solve numerically, and reduce computational efficiency. In the existing papers on numerical analysis of the MHD flows at low magnetic Reynolds numbers, most methods considered to solve the problem are monolithic methods in which the coupled problem is solved iteratively at each time step. Therefore, we study uncoupling methods for the time-dependent MHD flows at low magnetic Reynolds numbers. As to the coupling between $\mathbf{u}$ and p via $\nabla \cdot \mathbf{u} = 0$, the general idea to deal with this coupling is to relax the incompressibility constraint. There have been some such methods: the artificial compression method, penalty method, projection method, and pressure stabilization method (see [10, 9, 28, 30, 34, 37, 38, 13, 17, 29]). The artificial compression method (ACM), which was introduced by Chorin [9] and Temam [40], breaks the incompressibility restriction by adding a slightly compressible term εp_t ($\varepsilon > 0$ small) in $\nabla \cdot \mathbf{u} = 0$. This allows the pressure to be advanced in time explicitly. Using ACM, the slightly compressible model of (5.1) is given as follows.

$$\begin{aligned} &\frac{1}{N}(\mathbf{u}_t^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon) - \frac{1}{M^2}\Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} + \mathbf{B} \times \nabla \phi^\varepsilon + (\mathbf{u}^\varepsilon \times \mathbf{B}) \times \mathbf{B}, \\ &\varepsilon p_t^\varepsilon + \nabla \cdot \mathbf{u}^\varepsilon = 0, \qquad \Delta \phi^\varepsilon = \nabla \cdot (\mathbf{u}^\varepsilon \times \mathbf{B}), \end{aligned} \tag{5.2}$$

with the conditions

$$\begin{aligned} &\mathbf{u}^\varepsilon = 0 \qquad \text{on } \partial\Omega \times [0,T], \qquad \phi^\varepsilon = 0 \qquad \text{on } \partial\Omega \times [0,T], \\ &\mathbf{u}^\varepsilon(0) = \mathbf{u}_0, \quad p^\varepsilon(0) = p_0, \end{aligned} \tag{5.3}$$

where typically $\varepsilon = \mathcal{O}(\Delta t)$ or $\mathcal{O}(\Delta t^2)$. The function $p_0 \in L^2(\Omega)$ is arbitrarily chosen but independent of ε. The term $\frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon$ preserves skew-symmetry of the trilinear form. Since $\frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{u} = 0$ for the true solution $\mathbf{u}$ of (5.1) and $\varepsilon p_t = \mathcal{O}(\varepsilon)$, the consistency error of model (5.2)-(5.3) is clearly $\mathcal{O}(\varepsilon)$.

5.1.3 THE ARTIFICIAL COMPRESSION SCHEMES

Implicit-explicit (IMEX) methods have been widely used in to reduce the cost per time step in solving coupled systems of partial differential equations, see [8, 11, 31, 32, 33, 39]. In Algorithm 9 and 8 below, the coupling between the velocity $\mathbf{u}$ and electric potential ϕ are treated explicitly to uncouple the systems. The fluid velocity $\mathbf{u}$ and pressure p are further uncoupled using artificial compression method. The nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{u}$ is treated linearly implicitly to reduce complexity. Let $B(\mathbf{u}, \mathbf{v}) := \mathbf{u} \cdot \nabla \mathbf{v} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{v}$. Based on the IMEX partitioned schemes in [23], the following first order (Backward-Euler) and second order (BDF2) artificial compression schemes are introduced.

Algorithm 8 (Backward Euler). *Given* $\mathbf{u}^n, p^n, \phi^n$, *find* $\mathbf{u}^{n+1}, p^{n+1}, \phi^{n+1}$ *satisfying*

$$\begin{aligned} &\frac{1}{N}\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + B(\mathbf{u}^n, \mathbf{u}^{n+1})\right) - \frac{1}{M^2}\Delta \mathbf{u}^{n+1} + \nabla p^{n+1} \\ &\qquad\qquad\qquad\qquad = \mathbf{f}^{n+1} + \mathbf{B} \times \nabla \phi^n + (\mathbf{u}^{n+1} \times \mathbf{B}) \times \mathbf{B}, \\ &\varepsilon \frac{p^{n+1} - p^n}{\Delta t} + \nabla \cdot \mathbf{u}^{n+1} = 0, \\ &\Delta \phi^{n+1} = \nabla \cdot (\mathbf{u}^n \times \mathbf{B}). \end{aligned} \tag{5.4}$$

Algorithm 9 (BDF2). *Given* $\mathbf{u}^{n-1}, \mathbf{u}^n, p^n, \phi^{n-1}, \phi^n$, *find* $\mathbf{u}^{n+1}, p^{n+1}, \phi^{n+1}$ *satisfying*

$$\begin{aligned} &\frac{1}{N}\left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} + B(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1})\right) - \frac{1}{M^2}\Delta \mathbf{u}^{n+1} + \nabla p^{n+1} \\ &\qquad\qquad = \mathbf{f}^{n+1} + \mathbf{B} \times \nabla(2\phi^n - \phi^{n-1}) + (\mathbf{u}^{n+1} \times \mathbf{B}) \times \mathbf{B}, \\ &\varepsilon \frac{p^{n+1} - p^n}{\Delta t} + \nabla \cdot \mathbf{u}^{n+1} = 0, \\ &\Delta \phi^{n+1} = \nabla \cdot (\mathbf{u}^{n+1} \times \mathbf{B}). \end{aligned} \tag{5.5}$$

In both algorithms, the term ∇p^{n+1} can be eliminated by using $p^{n+1} = p^n - \frac{\Delta t}{\varepsilon}\nabla \cdot \mathbf{u}^{n+1}$. Thus, the calculation for Algorithm 8 (and similarly for Algorithm 9) proceeds as follows. Given $\mathbf{u}^n, p^n, \phi^n$, solve for $\mathbf{u}^{n+1}$:

$$\begin{aligned}\frac{1}{N}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + B(\mathbf{u}^n, \mathbf{u}^{n+1})) - \frac{1}{M^2}\Delta \mathbf{u}^{n+1} - \frac{\Delta t}{\varepsilon}\nabla\nabla\cdot\mathbf{u}^{n+1} - (\mathbf{u}^{n+1}\times\mathbf{B})\times\mathbf{B} \\ = \mathbf{f}^{n+1} + \mathbf{B}\times\nabla\phi^n - \nabla p^n.\end{aligned} \tag{5.6}$$

Perform an algebraic update of p^{n+1}:

$$p^{n+1} = p^n - \frac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^{n+1}. \tag{5.7}$$

Solve for ϕ^{n+1}:

$$\Delta\phi^{n+1} = \nabla\cdot(\mathbf{u}^n \times \mathbf{B}). \tag{5.8}$$

In Algorithm 8, the explicit treatment of the coupling terms $\mathbf{B}\times\nabla\phi$ and $\nabla\cdot(\mathbf{u}\times\mathbf{B})$ preserves unconditional stability, Section 5.3. In Algorithm 9, the coupling term $\nabla\cdot(\mathbf{u}\times\mathbf{B})$ is treated implicitly preserving stability (conditionally stable, Section 5.3) and higher accuracy. The derivation of a fully uncoupled, unconditionally, long time stable, second order method for (5.1) is an open problem. Since decoupling is through time discretization and can be applied to various space discretizations, we focus on analyzing the time discretization scheme for the slightly compressible model. For the numerical tests in Section 5.5, we use a standard finite element method for space discretizations.

5.2 ANALYSIS OF THE SLIGHTLY COMPRESSIBLE MODEL

In this section, we analyze the slightly compressible model (5.2)-(5.3). Firstly, we give a prior estimates for its solution $(\mathbf{u}^\varepsilon, p^\varepsilon, \phi^\varepsilon)$. Then, we show that $(\mathbf{u}^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ is an approximation to the true solution of the simplified MHD equations (5.1) when ε goes to 0.

The electric current density $\mathrm{J} := \sigma(-\nabla\phi + \mathbf{u}\times\mathbf{B})$ is an important electromagnetic quantity in MHD flows, see [21, 22]. Here the electrical conductivity σ is a constant. For convenient analysis, we define $\mathbf{j} := -\nabla\phi + \mathbf{u}\times\mathbf{B}$ and $\mathbf{j}^\varepsilon := -\nabla\phi^\varepsilon + \mathbf{u}^\varepsilon\times\mathbf{B}$.

Theorem 7. *Let* $(\mathbf{u}^\varepsilon, p^\varepsilon, \phi^\varepsilon)$ *be the solution of model* (5.2)-(5.3)*, then, with all bounds uniform in* ε*, we have*

$$\begin{aligned}
&\mathbf{u}^\varepsilon \in L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;H_0^1(\Omega)) \bigcap L^2(0,T;L^6(\Omega)),\\
&\sqrt{\varepsilon} p^\varepsilon \in L^\infty(0,T;L^2(\Omega)),\ \phi^\varepsilon \in L^2(0,T;H_0^1(\Omega)),\\
&\mathbf{j}^\varepsilon \in L^2(0,T;L^2(\Omega)),\\
&\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon \text{ and } (\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon \in L^2(0,T;L^1(\Omega)) \bigcap L^1(0,T;L^{\frac{3}{2}}(\Omega)).
\end{aligned} \tag{5.9}$$

Proof. Taking the inner product of the three equations in (5.2) with $\mathbf{u}^\varepsilon$, p^ε, and ϕ^ε, respectively, then summing up the three new equations, we obtain

$$\begin{aligned}
&\frac{1}{2N}\frac{d}{dt}\|\mathbf{u}^\varepsilon\|^2 + \frac{1}{M^2}\|\nabla \mathbf{u}^\varepsilon\|^2 + \|-\nabla\phi^\varepsilon + \mathbf{u}^\varepsilon \times \mathbf{B}\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|p^\varepsilon\|^2\\
&= (\mathbf{f}, \mathbf{u}^\varepsilon) \le \frac{1}{2M^2}\|\nabla \mathbf{u}^\varepsilon\|^2 + \frac{M^2}{2}\|\mathbf{f}\|_{-1}^2.
\end{aligned} \tag{5.10}$$

Thus, we have

$$\frac{1}{N}\frac{d}{dt}\|\mathbf{u}^\varepsilon\|^2 + \frac{1}{M^2}\|\nabla \mathbf{u}^\varepsilon\|^2 + \|-\nabla\phi^\varepsilon + \mathbf{u}^\varepsilon \times \mathbf{B}\|^2 + \varepsilon\frac{d}{dt}\|p^\varepsilon\|^2 \le M^2\|\mathbf{f}\|_{-1}^2. \tag{5.11}$$

Integration of (5.11) from 0 to t shows that

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{u}^\varepsilon(t)\|^2 + \frac{1}{M^2}\int_0^t \|\nabla \mathbf{u}^\varepsilon(s)\|^2 ds + \int_0^t \|-\nabla\phi^\varepsilon(s) + \mathbf{u}^\varepsilon(s) \times \mathbf{B}\|^2 ds + \varepsilon\|p^\varepsilon(t)\|^2\\
&\le M^2\|\mathbf{f}\|_{L^2(0,T;H^{-1})}^2 + \frac{1}{N}\|\mathbf{u}^\varepsilon(0)\|^2 + \varepsilon\|p^\varepsilon(0)\|^2, \quad 0 < t \le T.
\end{aligned} \tag{5.12}$$

Thus, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \frac{1}{N}\|\mathbf{u}^\varepsilon(t)\|^2 + \varepsilon\|p^\varepsilon(t)\|^2 \le c_1,\\
&c_1 = M^2\|\mathbf{f}\|_{L^2(0,T;H^{-1})}^2 + \frac{1}{N}\|\mathbf{u}_0\|^2 + \|p_0\|^2.
\end{aligned} \tag{5.13}$$

Here, since we are interested in small values of ε, we assume $\varepsilon \le 1$.

We also have

$$\int_0^T \|\nabla \mathbf{u}^\varepsilon(s)\|^2 ds \le M^2 c_1, \qquad \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds \le c_1. \tag{5.14}$$

Since

$$\nabla\phi^\varepsilon(s) = -\mathbf{j}^\varepsilon(s) + \mathbf{u}^\varepsilon(s) \times \mathbf{B}, \tag{5.15}$$

using the inequality $\|\mathbf{v_1} \times \mathbf{v_2}\| \leq 2\|\mathbf{v_1}\|_{L^\infty}\|\mathbf{v_2}\|$ and the Poincaré inequality, we can get

$$\begin{aligned}\int_0^T \|\nabla\phi^\varepsilon(s)\|^2 ds &\leq \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds + \int_0^T \|\mathbf{u}^\varepsilon(s) \times \mathbf{B}\|^2 ds \\ &\leq \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds + 4\|\mathbf{B}\|_{L^\infty}^2 \int_0^T \|\mathbf{u}^\varepsilon(s)\|^2 ds \\ &\leq \int_0^T \|\mathbf{j}^\varepsilon(s)\|^2 ds + C\|\mathbf{B}\|_{L^\infty}^2 \int_0^T \|\nabla\mathbf{u}^\varepsilon(s)\|^2 ds \\ &\leq (1 + CM^2\|\mathbf{B}\|_{L^\infty}^2)c_1.\end{aligned} \tag{5.16}$$

The remaining conditions follow from Hölder's inequality and the Sobolev embedding theorem. □

Next, we derive an error estimate for the sightly compressible model. Denote the error $\mathbf{e_u} = \mathbf{u} - \mathbf{u}^\varepsilon$, $e_\phi = \phi - \phi^\varepsilon$, $e_p = p - p^\varepsilon$, and $\mathbf{e_j} = \mathbf{j} - \mathbf{j}^\varepsilon$. Thus, we have $\mathbf{e_j} = -\nabla\mathbf{e}_\phi + \mathbf{e_u} \times \mathbf{B}$. The following theorem shows that $|\mathbf{u} - \mathbf{u}^\varepsilon|$ tends to zero in $L^\infty(0, T; L^2(\Omega))$ as $\varepsilon \to 0$. The order of convergence is at least $\mathcal{O}(\sqrt{\varepsilon})$.

Theorem 8. *Assume that the true solution* $\mathbf{u} \in L^2(0, T; H^2(\Omega))$ *and* $p_t \in L^2(0, T; L^2(\Omega))$, *then we have the following estimate*

$$\|\mathbf{e_u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{e_u}\|_{L^2(0,T;H_0^1(\Omega))} + \|\mathbf{e}_\phi\|_{L^2(0,T;H_0^1(\Omega))} + \|\sqrt{\varepsilon}e_p\|_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\varepsilon}. \tag{5.17}$$

Proof. Subtracting (5.2) from (5.1) , we obtain

$$\begin{aligned}&\frac{1}{N}\frac{\partial}{\partial t}\mathbf{e_u} + \frac{1}{N}B(\mathbf{e_u}, \mathbf{u}) + \frac{1}{N}B(\mathbf{u}^\varepsilon, \mathbf{e_u}) - \frac{1}{M^2}\Delta\mathbf{e_u} + \nabla e_p = \mathbf{B} \times \nabla e_\phi + (\mathbf{e_u} \times \mathbf{B}) \times \mathbf{B}, \\ &\varepsilon\frac{\partial}{\partial t}e_p + \nabla \cdot \mathbf{e_u} = \varepsilon p_t, \\ &\Delta e_\phi = \nabla \cdot (\mathbf{e_u} \times \mathbf{B}).\end{aligned} \tag{5.18}$$

Taking the inner product of the three equations in (5.18) with $\mathbf{e_u}$, e_p, and e_ϕ, respectively, then summing up the three new equations, we obtain

$$\frac{1}{2N}\frac{d}{dt}\|\mathbf{e_u}\|^2 + \frac{1}{N}b(\mathbf{e_u}, \mathbf{u}, \mathbf{e_u}) + \frac{1}{M^2}\|\nabla\mathbf{e_u}\|^2 + \| - \nabla e_\phi + \mathbf{e_u} \times \mathbf{B}\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|e_p\|^2 = (\varepsilon p_t, e_p). \tag{5.19}$$

Thus,

$$\begin{aligned}
&\frac{1}{2N}\frac{d}{dt}\|\mathbf{e_u}\|^2+\frac{1}{M^2}\|\nabla\mathbf{e_u}\|^2+\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2+\frac{\varepsilon}{2}\frac{d}{dt}\|e_p\|^2\\
&=(\varepsilon p_t,e_p)-\frac{1}{N}b(\mathbf{e_u},\mathbf{u},\mathbf{e_u})\\
&\le\varepsilon\|p_t\|\|e_p\|+\frac{C}{N}\|\mathbf{e_u}\|\|\mathbf{u}\|_2\|\nabla\mathbf{e_u}\|\\
&\le\frac{\varepsilon}{2}\|p_t\|^2+\frac{\varepsilon}{2}\|e_p\|^2+\frac{1}{2M^2}\|\nabla\mathbf{e_u}\|^2+\frac{CM^2}{2N^2}\|\mathbf{e_u}\|^2\|\mathbf{u}\|_2^2.
\end{aligned}\tag{5.20}$$

We have

$$\begin{aligned}
&\frac{1}{N}\frac{d}{dt}\|\mathbf{e_u}\|^2+\frac{1}{M^2}\|\nabla\mathbf{e_u}\|^2+\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2+\varepsilon\frac{d}{dt}\|e_p\|^2\\
&\le C\|\mathbf{u}\|_2^2\|\mathbf{e_u}\|^2+\varepsilon\|e_p\|^2+\varepsilon\|p_t\|^2.
\end{aligned}\tag{5.21}$$

Integrate (5.21) from 0 to t to obtain

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{e_u}(t)\|^2+\varepsilon\|e_p(t)\|^2+\frac{1}{M^2}\int_0^t\|\nabla\mathbf{e_u}\|^2ds+\int_0^t\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2ds\\
&\le\frac{1}{N}\|\mathbf{e_u}(0)\|^2+\varepsilon\|e_p(0)\|^2+C\int_0^t\|\mathbf{u}\|_2^2\|\mathbf{e_u}\|^2ds+\int_0^t\varepsilon\|e_p\|^2ds+\int_0^t\varepsilon\|p_t\|^2ds.
\end{aligned}\tag{5.22}$$

Using the Gronwall lemma, we get

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{e_u}\|^2+\varepsilon\|e_p\|^2+\frac{1}{M^2}\int_0^t\|\nabla\mathbf{e_u}\|^2ds+\int_0^t\|-\nabla e_\phi+\mathbf{e_u}\times\mathbf{B}\|^2ds\\
&\le C(\frac{1}{N}\|\mathbf{e_u}(0)\|^2+\varepsilon\|e_p(0)\|^2+\int_0^t\varepsilon\|p_t\|^2ds)\\
&\le C\varepsilon(\|e_p(0)\|^2+\int_0^t\|p_t\|^2ds),\ 0<t\le T.
\end{aligned}\tag{5.23}$$

Thus, we have, as $\varepsilon\to 0$,

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\mathbf{e_u}\|\le C\sqrt{\varepsilon}\to 0,\quad\int_0^T\|\nabla\mathbf{e_u}\|^2ds\le C\varepsilon\to 0,\\
&\int_0^T\|\mathbf{e_j}\|^2ds\le C\varepsilon\to 0,\quad\sup_{t\in[0,T]}\|\sqrt{\varepsilon}e_p\|\le C\sqrt{\varepsilon}\to 0.
\end{aligned}\tag{5.24}$$

Since

$$\nabla\mathbf{e}_\phi=-\mathbf{e_j}+\mathbf{e_u}\times\mathbf{B},\tag{5.25}$$

we can also deduce that

$$\int_0^T\|\nabla\mathbf{e}_\phi\|^2ds\le C\varepsilon\to 0,\quad as\ \varepsilon\to 0,\tag{5.26}$$

which completes the proof. □

5.3 STABILITY AND ERROR ANALYSIS

In this section, we analyze stability of Algorithm 8 and Algorithm 9, then give an á priori error estimate for Algorithm 8. Theorem 9 below shows that Algorithm 8 is unconditionally stable.

Theorem 9. *For $(\mathbf{u}^n, p^n, \phi^n)$ satisfying Algorithm 8, we have the following unconditional stability.*

$$\begin{aligned}
&\frac{1}{N}\|\mathbf{u}^m\|^2 + \frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2 + \varepsilon\|p^m\|^2 + \varepsilon\sum_{n=0}^{m-1}\|p^{n+1}-p^n\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2\\
&\quad + \Delta t\|\mathbf{u}^m\times\mathbf{B}\|^2 + \Delta t\|\nabla\phi^m\|^2 + \Delta t\sum_{n=0}^{m-1}(\|-\nabla\phi^n + \mathbf{u}^{n+1}\times\mathbf{B}\|^2\\
&+ \|-\nabla\phi^{n+1} + \mathbf{u}^n\times\mathbf{B}\|^2)\\
&\le M^2\Delta t\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2 + \frac{1}{N}\|\mathbf{u}^0\|^2 + \varepsilon\|p^0\|^2 + \Delta t\|\mathbf{u}^0\times\mathbf{B}\|^2 + \Delta t\|\nabla\phi^0\|^2.
\end{aligned} \tag{5.27}$$

Proof. Taking the inner product of the three equations in (5.4) with $\mathbf{u}^{n+1}$, p^{n+1}, and ϕ^{n+1}, respectively, and multiplying it by $2\Delta t$, we can obtain

$$\begin{aligned}
&\frac{1}{N}(\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2) + \frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2\\
&\quad - 2\Delta t(p^{n+1}, \nabla\cdot\mathbf{u}^{n+1}) + 2\Delta t(-\nabla\phi^n + \mathbf{u}^{n+1}\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B})\\
&= 2\Delta t(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}),\\
&\varepsilon(\|p^{n+1}\|^2 - \|p^n\|^2 + \|p^{n+1}-p^n\|^2) + 2\Delta t(p^{n+1}, \nabla\cdot\mathbf{u}^{n+1}) = 0,\\
&2\Delta t(-\nabla\phi^{n+1} + \mathbf{u}^n\times\mathbf{B}, -\nabla\phi^{n+1}) = 0.
\end{aligned} \tag{5.28}$$

Then summing up the three equations in (5.28), we have

$$\begin{aligned}
&\frac{1}{N}(\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2) + \frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2\\
&+ 2\Delta t(-\nabla\phi^n + \mathbf{u}^{n+1}\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B}) + 2\Delta t(-\nabla\phi^{n+1} + \mathbf{u}^n\times\mathbf{B}, -\nabla\phi^{n+1})\\
&+ \varepsilon(\|p^{n+1}\|^2 - \|p^n\|^2 + \|p^{n+1}-p^n\|^2) = 2\Delta t(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}).
\end{aligned} \tag{5.29}$$

By using the following identity

$$2(a+b,b)+2(c+d,c)=c^2-a^2+b^2-d^2+(a+b)^2+(c+d)^2, \tag{5.30}$$

we get

$$\begin{aligned}
&2\Delta t(-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B},\mathbf{u}^{n+1}\times\mathbf{B})+2\Delta t(-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B},-\nabla\phi^{n+1})\\
&=\Delta t\|\mathbf{u}^{n+1}\times\mathbf{B}\|^2-\Delta t\|\mathbf{u}^n\times\mathbf{B}\|^2+\Delta t\|\nabla\phi^{n+1}\|^2-\Delta t\|\nabla\phi^n\|^2\\
&\quad+\Delta t(\|-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B}\|^2).
\end{aligned} \tag{5.31}$$

Thus, (5.29) can be rewritten as

$$\begin{aligned}
&\frac{1}{N}(\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^n\|^2+\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2)+\varepsilon(\|p^{n+1}\|^2-\|p^n\|^2+\|p^{n+1}-p^n\|^2)\\
&\quad+\frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2+\Delta t\|\mathbf{u}^{n+1}\times\mathbf{B}\|^2-\Delta t\|\mathbf{u}^n\times\mathbf{B}\|^2+\Delta t\|\nabla\phi^{n+1}\|^2-\Delta t\|\nabla\phi^n\|^2\\
&\quad+\Delta t(\|-\nabla\phi^n+\mathbf{u}^{n+1}\times\mathbf{B}\|^2+\|-\nabla\phi^{n+1}+\mathbf{u}^n\times\mathbf{B}\|^2)\\
&=2\Delta t(\mathbf{f}^{n+1},\mathbf{u}^{n+1})\le M^2\Delta t\|\mathbf{f}^{n+1}\|_{-1}^2+\frac{\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2.
\end{aligned} \tag{5.32}$$

Finally, summing (5.32) from $n=0$ to $n=m-1$ completes the proof. □

The following theorem shows that Algorithm 9 is stable with a condition, relating the time step Δt with the problem data. Recall $\mathbf{j}^n:=-\nabla\phi^n+\mathbf{u}^n\times\mathbf{B}$.

Theorem 10. *For $(\mathbf{u}^n,p^n,\phi^n)$ satisfying Algorithm 9, if time step Δt satisfies*

$$\Delta t<\frac{1}{2N(1+C_p^2M^2\|\mathbf{B}\|_{L^\infty}^2)\|\mathbf{B}\|_{L^\infty}^2}, \tag{5.33}$$

we have the following stability.

$$\begin{aligned}
&\frac{1}{2N}\|\mathbf{u}^m\|^2+\frac{1}{2N}\|2\mathbf{u}^m-\mathbf{u}^{m-1}\|^2+\frac{\Delta t}{2M^2}\sum_{n=1}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2\\
&\quad+\Delta t\sum_{n=1}^{m-1}\|\mathbf{j}^{n+1}\|^2+\Delta t\sum_{n=1}^{m-1}\|2\mathbf{j}^n-\mathbf{j}^{n-1}\|^2\\
&\le\frac{1}{2N}\|\mathbf{u}^1\|^2+\frac{1}{2N}\|2\mathbf{u}^1-\mathbf{u}^0\|^2+2M^2\Delta t\sum_{n=1}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2.
\end{aligned} \tag{5.34}$$

Proof. Taking the inner product of the three equations in (5.5) with $\mathbf{u}^{n+1}$, p^{n+1}, and ϕ^{n+1}, respectively, and multiplying it by $2\Delta t$, we can obtain

$$\begin{aligned}
&\frac{1}{2N}(3\|\mathbf{u}^{n+1}\|^2 - 4\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n-1}\|^2) + \frac{1}{N}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 - \frac{1}{N}\|\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 \\
&\quad + \frac{1}{2N}\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 + \frac{2\Delta t}{M^2}\|\nabla \mathbf{u}^{n+1}\|^2 - 2\Delta t(p^{n+1}, \nabla\cdot\mathbf{u}^{n+1}) \\
&\quad + 2\Delta t(-\nabla(2\phi^n - \phi^{n-1}) + \mathbf{u}^{n+1}\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B}) \\
&= 2\Delta t(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}), \\
&\varepsilon(\|p^{n+1}\|^2 - \|p^n\|^2 + \|p^{n+1} - p^n\|^2) + 2\Delta t(p^{n+1}, \nabla\cdot\mathbf{u}^{n+1}) = 0, \\
&2\Delta t(-\nabla\phi^{n+1} + \mathbf{u}^{n+1}\times\mathbf{B}, -\nabla\phi^{n+1}) = 0,
\end{aligned} \tag{5.35}$$

where we use the identity

$$\frac{1}{2}(3a - 4b + c)a = \frac{1}{4}(3a^2 - 4b^2 + c^2) + \frac{1}{2}(a-b)^2 - \frac{1}{2}(b-c)^2 + \frac{1}{4}(a - 2b + c)^2.$$

The key issue is to deal with the term $2\Delta t(-\nabla(2\phi^n - \phi^{n-1}) + \mathbf{u}^{n+1}\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B})$. We have

$$\begin{aligned}
&2\Delta t(-\nabla(2\phi^n - \phi^{n-1}) + \mathbf{u}^{n+1}\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B}) \\
&= 2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1} + (\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B}) \\
&= 2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1}\times\mathbf{B}) + 2\Delta t((\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\times\mathbf{B}, \mathbf{u}^{n+1}\times\mathbf{B}) \\
&= 2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1}\times\mathbf{B}) \\
&\quad + \Delta t(\|\mathbf{u}^{n+1}\times\mathbf{B}\|^2 - \|(2\mathbf{u}^n - \mathbf{u}^{n-1})\times\mathbf{B}\|^2 + \|(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\times\mathbf{B}\|^2).
\end{aligned} \tag{5.36}$$

From the third equation in (5.5), we have

$$(2\mathbf{j}^n - \mathbf{j}^{n-1}, -\nabla\psi) = 0, \quad \forall\psi\in S. \tag{5.37}$$

Taking $\psi = \phi^{n+1}$ and adding it to the term $2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1}\times\mathbf{B})$ gives

$$\begin{aligned}
&2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{u}^{n+1}\times\mathbf{B}) = 2\Delta t(2\mathbf{j}^n - \mathbf{j}^{n-1}, \mathbf{j}^{n+1}) \\
&= \Delta t(\|\mathbf{j}^{n+1}\|^2 + \|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 - \|\mathbf{j}^{n+1} - 2\mathbf{j}^n + \mathbf{j}^{n-1}\|^2) \\
&= \Delta t(\|\mathbf{j}^{n+1}\|^2 + \|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 - \|(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\times\mathbf{B}\|^2 \\
&\quad + \|\nabla\phi^{n+1} - 2\nabla\phi^n + \nabla\phi^{n-1}\|^2),
\end{aligned} \tag{5.38}$$

where we use $\|\mathbf{j}^i\|^2 = \|\mathbf{u}^i \times \mathbf{B}\|^2 - \|\nabla\phi^i\|^2$, $\forall i \le n+1$. Combining (5.35), (5.36) and (5.38) yields

$$\begin{aligned}
&\frac{1}{2N}(3\|\mathbf{u}^{n+1}\|^2 - 4\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n-1}\|^2) + \frac{1}{N}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 - \frac{1}{N}\|\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 \\
&\quad + \frac{1}{2N}\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 + \frac{2\Delta t}{M^2}\|\nabla\mathbf{u}^{n+1}\|^2 \\
&\quad + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 + \Delta t\|\mathbf{j}^{n+1}\|^2 + \Delta t\|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 \\
&= 2\Delta t(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}) + \Delta t\|(2\mathbf{u}^n - \mathbf{u}^{n-1}) \times \mathbf{B}\|^2.
\end{aligned} \tag{5.39}$$

For an arbitrary $\delta > 0$, the term $\Delta t\|(2\mathbf{u}^n - \mathbf{u}^{n-1}) \times \mathbf{B}\|^2$ can be bounded by

$$\begin{aligned}
\Delta t\|(2\mathbf{u}^n - \mathbf{u}^{n-1}) \times \mathbf{B}\|^2 &= \Delta t\|(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) \times \mathbf{B}\|^2 + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 \\
&\quad - 2\Delta t((\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) \times \mathbf{B}, \mathbf{u}^{n+1} \times \mathbf{B}) \\
&\le \Delta t\|(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) \times \mathbf{B}\|^2 + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 \\
&\quad + \frac{\Delta t}{\delta^2}\|(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) \times \mathbf{B}\|^2 + \Delta t\delta^2\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 \\
&\le \Delta t(1 + \frac{1}{\delta^2})\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 \\
&\quad + \Delta t\|\mathbf{u}^{n+1} \times \mathbf{B}\|^2 + \Delta t C_p^2\delta^2\|\mathbf{B}\|_{L^\infty}^2\|\nabla\mathbf{u}^{n+1}\|^2,
\end{aligned} \tag{5.40}$$

where we use the Poincaré inequality $\|\mathbf{u}\| \le C_p\|\nabla\mathbf{u}\|$, $\forall \mathbf{u} \in X$. By taking $\delta = \frac{1}{C_p M\|\mathbf{B}\|_{L^\infty}}$, we can obtain

$$\begin{aligned}
&\frac{1}{2N}(3\|\mathbf{u}^{n+1}\|^2 - 4\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n-1}\|^2) + \frac{1}{N}\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 - \frac{1}{N}\|\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 \\
&\quad + (\frac{1}{2N} - \Delta t(1 + \frac{1}{\varepsilon^2})\|\mathbf{B}\|_{L^\infty}^2)\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2 \\
&\quad + \frac{\Delta t}{2M^2}\|\nabla\mathbf{u}^{n+1}\|^2 + \Delta t\|\mathbf{j}^{n+1}\|^2 + \Delta t\|2\mathbf{j}^n - \mathbf{j}^{n-1}\|^2 \\
&\le 2M^2\Delta t\|\mathbf{f}^{n+1}\|_{-1}^2.
\end{aligned} \tag{5.41}$$

Finally, under the condition (5.33), summing (5.41) from $n = 1$ to $n = m - 1$ completes the proof. □

Next, we analyze the convergency of Algorithm 8 and give an á priori error estimate for Algorithm 8. Since the error analysis of Algorithm 9 is similar to that of Algorithm 8 but considerably longer, we omit it. Denote $\mathbf{e}_{\mathbf{u}}^n = \mathbf{u}(t^n) - \mathbf{u}^n$, $e_p^n = p(t^n) - p^n$, and $e_\phi^n = \phi(t^n) - \phi^n$. The following theorem not only shows the convergence order of Algorithm 8 but can also indicates a way to choose the value of ε.

Theorem 11. *Assume that the true solution* $(\mathbf{u}, p, \phi)$ *satisfies the following regularity*

$$\begin{aligned} &\mathbf{u} \in L^\infty(0,T;H^2(\Omega)), \quad \mathbf{u}_t \in L^2(0,T;H^1(\Omega)), \quad \mathbf{u}_{tt} \in L^2(0,T;H^{-1}(\Omega)), \\ &p_t \in L^\infty(0,T;L^2(\Omega)), \quad p_{tt} \in L^2(0,T;L^2(\Omega)), \quad \phi_t \in L^2(0,T;H^1(\Omega)). \end{aligned} \tag{5.42}$$

For $(\mathbf{u}^n, p^n, \phi^n)$ *satisfying Algorithm 8, we have the following estimate*

$$\begin{aligned} &\frac{1}{2N}\|\mathbf{e}_\mathbf{u}^m\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|^2 + \Delta t\sum_{n=0}^{m-1}\|\nabla e_\phi^{n+1}\|^2 \\ &+ \Delta t\sum_{n=0}^{m-1}(\|-\nabla e_\phi^n + \mathbf{e}_\mathbf{u}^{n+1}\times\mathbf{B}\|^2 + \|-\nabla e_\phi^{n+1} + \mathbf{e}_\mathbf{u}^n\times\mathbf{B}\|^2) \le C(\Delta t^2 + \varepsilon^2), \end{aligned} \tag{5.43}$$

for sufficiently small Δt.

Proof. At time t^{n+1}, the true solution $(\mathbf{u}, p, \phi)$ satisfies

$$\begin{aligned} &\frac{1}{N}\Big(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} + B(\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}))\Big) - \frac{1}{M^2}\Delta\mathbf{u}(t^{n+1}) + \nabla p(t^{n+1}) \\ &- \mathbf{B}\times\nabla\phi(t^{n+1}) - (\mathbf{u}(t^{n+1})\times\mathbf{B})\times\mathbf{B} = \mathbf{f}^{n+1} + \frac{1}{N}\Big(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\Big), \\ &\varepsilon\frac{p(t^{n+1}) - p(t^n)}{\Delta t} + \nabla\cdot\mathbf{u}(t^{n+1}) = \frac{\varepsilon}{\Delta t}\int_{t^n}^{t^{n+1}} p_t dt, \\ &\Delta\phi(t^{n+1}) = \nabla\cdot(\mathbf{u}(t^{n+1})\times\mathbf{B}). \end{aligned} \tag{5.44}$$

Subtract (5.4) from (5.44) to obtain

$$\begin{aligned} &\frac{1}{N}\Big(\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t} + B(\mathbf{e}_\mathbf{u}^n, \mathbf{u}(t^{n+1})) + B(\mathbf{u}^n, \mathbf{e}_\mathbf{u}^{n+1}) + B(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}))\Big) \\ &\quad - \frac{1}{M^2}\Delta\mathbf{e}_\mathbf{u}^{n+1} + \nabla e_p^{n+1} - \mathbf{B}\times\nabla e_\phi^n - \mathbf{B}\times(\nabla\phi(t^{n+1}) - \nabla\phi(t^n)) - (\mathbf{e}_\mathbf{u}^{n+1}\times\mathbf{B})\times\mathbf{B} \\ &= \frac{1}{N}\Big(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\Big), \\ &\varepsilon\frac{e_p^{n+1} - e_p^n}{\Delta t} + \nabla\cdot\mathbf{e}_\mathbf{u}^{n+1} = \frac{\varepsilon}{\Delta t}\int_{t^n}^{t^{n+1}} p_t dt, \\ &\Delta e_\phi^{n+1} = \nabla\cdot(\mathbf{e}_\mathbf{u}^n\times\mathbf{B}) + \nabla\cdot((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\times\mathbf{B}). \end{aligned} \tag{5.45}$$

Taking the inner product of the three equations in (5.45) with $\mathbf{e}_{\mathbf{u}}^{n+1}$, e_p^{n+1}, and e_ϕ^{n+1}, respectively, then summing up the three new equations and multiplying it by $2\Delta t$, we obtain

$$\begin{aligned}
&\frac{1}{N}(\|\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{e}_{\mathbf{u}}^{n}\|^2 + \|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\|^2) + \varepsilon(\|e_p^{n+1}\|^2 - \|e_p^n\|^2 + \|e_p^{n+1} - e_p^n\|^2) \\
&\quad + \frac{2\Delta t}{M^2}\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t\|\mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 - \Delta t\|\mathbf{e}_{\mathbf{u}}^{n} \times \mathbf{B}\|^2 + \Delta t\|\nabla e_\phi^{n+1}\|^2 - \Delta t\|\nabla e_\phi^n\|^2 \\
&\quad + \Delta t(\| - \nabla e_\phi^n + \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 + \| - \nabla e_\phi^{n+1} + \mathbf{e}_{\mathbf{u}}^{n} \times \mathbf{B}\|^2) \\
&= \frac{2\Delta t}{N}(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) - \frac{2\Delta t}{N} b(\mathbf{e}_{\mathbf{u}}^{n}, \mathbf{u}(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) \\
&\quad - \frac{2\Delta t}{N} b(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) + 2\Delta t(\nabla\phi(t^{n+1}) - \nabla\phi(t^n), \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}) \\
&\quad + 2\Delta t((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \times \mathbf{B}, \nabla e_\phi^{n+1}) + 2\varepsilon(\int_{t^n}^{t^{n+1}} p_t dt, e_p^{n+1}),
\end{aligned} \tag{5.46}$$

where we use the identity (5.30) again. We have that

$$2\Delta t\|\nabla e_\phi^{n+1}\|^2 = 2\Delta t((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \times \mathbf{B}, \nabla e_\phi^{n+1}) + 2\Delta t(\nabla e_\phi^{n+1}, \mathbf{e}_{\mathbf{u}}^{n} \times \mathbf{B}) \tag{5.47}$$

Adding (5.46) to (5.47), we obtain

$$\begin{aligned}
&\frac{1}{N}(\|\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{e}_{\mathbf{u}}^{n}\|^2 + \|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^{n}\|^2) + \varepsilon(\|e_p^{n+1}\|^2 - \|e_p^n\|^2 + \|e_p^{n+1} - e_p^n\|^2) \\
&\quad + \frac{2\Delta t}{M^2}\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + 2\Delta t\|\nabla e_\phi^{n+1}\|^2 + \Delta t\|\mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 - \Delta t\|\mathbf{e}_{\mathbf{u}}^{n} \times \mathbf{B}\|^2 \\
&\quad + \Delta t\|\nabla e_\phi^{n+1}\|^2 - \Delta t\|\nabla e_\phi^n\|^2 + \Delta t(\| - \nabla e_\phi^n + \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 + \| - \nabla e_\phi^{n+1} + \mathbf{e}_{\mathbf{u}}^{n} \times \mathbf{B}\|^2) \\
&= \frac{2\Delta t}{N}(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) - \frac{2\Delta t}{N} b(\mathbf{e}_{\mathbf{u}}^{n}, \mathbf{u}(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) \\
&\quad - \frac{2\Delta t}{N} b(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) + 2\Delta t(\nabla\phi(t^{n+1}) - \nabla\phi(t^n), \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}) \\
&\quad + 4\Delta t((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \times \mathbf{B}, \nabla e_\phi^{n+1}) + 2\Delta t(\nabla e_\phi^{n+1}, \mathbf{e}_{\mathbf{u}}^{n} \times \mathbf{B}) + 2\varepsilon(\int_{t^n}^{t^{n+1}} p_t dt, e_p^{n+1}).
\end{aligned} \tag{5.48}$$

Next, we need to bound all the terms on the RHS of (5.48). For arbitrary $\sigma > 0, \sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$, we have the following estimates. The first term can be bounded as

$$\begin{aligned}
\frac{2\Delta t}{N}(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) \le & C\Delta t\|\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2 \\
& + \sigma\Delta t\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2.
\end{aligned} \tag{5.49}$$

The nonlinear terms can be bounded as

$$
\begin{aligned}
&-\frac{2\Delta t}{N} b(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) \leq C\Delta t \|\mathbf{e}_{\mathbf{u}}^n\| \|\mathbf{u}(t^{n+1})\|_2 \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\| \\
&\leq C\Delta t \|\mathbf{u}(t^{n+1})\|_2^2 \|\mathbf{e}_{\mathbf{u}}^n\|^2 + \sigma\Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2, \qquad (5.50) \\
&-\frac{2\Delta t}{N} b(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}), \mathbf{e}_{\mathbf{u}}^{n+1}) \leq C\Delta t \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\| \|\mathbf{u}(t^{n+1})\|_2 \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\| \\
&\leq C\Delta t \|\mathbf{u}(t^{n+1})\|_2^2 \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|^2 + \sigma\Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2. \qquad (5.51)
\end{aligned}
$$

We bound the term $2\Delta t(\nabla\phi(t^{n+1}) - \nabla\phi(t^n), \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B})$ as follows.

$$
\begin{aligned}
&2\Delta t(\nabla\phi(t^{n+1}) - \nabla\phi(t^n), \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}) \leq 4\Delta t \|\mathbf{e}_{\mathbf{u}}^{n+1}\| \|\mathbf{B}\|_{L^\infty} \|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\| \\
&\leq C\Delta t \|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\|^2 + \sigma\Delta t \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2.
\end{aligned} \qquad (5.52)
$$

The term $4\Delta t((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \times \mathbf{B}, \nabla e_\phi^{n+1})$ is bounded as

$$
\begin{aligned}
&4\Delta t((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \times \mathbf{B}, \nabla e_\phi^{n+1}) \leq 8\Delta t \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\| \|\mathbf{B}\|_{L^\infty} \|\nabla e_\phi^{n+1}\| \\
&\leq C\Delta t \|\mathbf{B}\|_{L^\infty}^2 \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|^2 + \sigma_1\Delta t \|\nabla e_\phi^{n+1}\|^2.
\end{aligned} \qquad (5.53)
$$

We also bound the term $2\Delta t(\nabla e_\phi^{n+1}, \mathbf{e}_{\mathbf{u}}^n \times \mathbf{B})$ by

$$
2\Delta t(\nabla e_\phi^{n+1}, \mathbf{e}_{\mathbf{u}}^n \times \mathbf{B}) \leq 4\Delta t \|\mathbf{e}_{\mathbf{u}}^n\| \|\mathbf{B}\|_{L^\infty} \|\nabla e_\phi^{n+1}\| \leq C\Delta t \|\mathbf{B}\|_{L^\infty}^2 \|\mathbf{e}_{\mathbf{u}}^n\|^2 + \sigma_1\Delta t \|\nabla e_\phi^{n+1}\|^2. \qquad (5.54)
$$

Lastly, we need to bound the remaining term $2\varepsilon(\int_{t^n}^{t^{n+1}} p_t dt, e_p^{n+1})$. It is known (see [28]) that if $p_t(t), p_{tt}(t) \in L^2(\Omega)/R$, there exists an unique $\varphi(t) \in H_0^1(\Omega)$, such that

$$
\nabla \cdot \varphi(t) = p_t(t), \quad \nabla \cdot \varphi_t(t) = p_{tt}(t) \qquad (5.55)
$$

and

$$
\|\varphi(t)\|_1 \leq C\|p_t(t)\|, \quad \|\varphi_t(t)\|_1 \leq C\|p_{tt}(t)\| \qquad for\ all\ t \in [0, T]. \qquad (5.56)
$$

From (5.45), we have, at time t^{n+1},

$$
\begin{aligned}
\nabla e_p^{n+1} = &-\frac{1}{N}\frac{\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n}{\Delta t} + \frac{1}{M^2}\Delta \mathbf{e}_{\mathbf{u}}^{n+1} - \frac{1}{N}B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1})) - \frac{1}{N}B(\mathbf{u}^n, \mathbf{e}_{\mathbf{u}}^{n+1}) \\
&- \frac{1}{N}B(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1})) + \mathbf{B} \times \nabla e_\phi^n + \mathbf{B} \times (\nabla\phi(t^{n+1}) - \nabla\phi(t^n)) \qquad (5.57) \\
&+ (\mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}) \times \mathbf{B} + \frac{1}{N}(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&2\varepsilon(\int_{t^n}^{t^{n+1}} p_t dt, e_p^{n+1}) = 2\varepsilon(\int_{t^n}^{t^{n+1}} \nabla\cdot\varphi(t) dt, e_p^{n+1}) = -2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \nabla e_p^{n+1}) \\
&= -2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}))) \\
&\quad - 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{M^2}\Delta\mathbf{e}_{\mathbf{u}}^{n+1}) - 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B}\times\nabla e_\phi^n) \\
&\quad - 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B}\times(\nabla\phi(t^{n+1}) - \nabla\phi(t^n))) \\
&\quad - 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, (\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B})\times\mathbf{B}) + 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\frac{\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^n}{\Delta t}) \\
&\quad + 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{u}^n, \mathbf{e}_{\mathbf{u}}^{n+1})) + 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1}))) \\
&\quad + 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{u}(t^{n+1}))).
\end{aligned}
\tag{5.58}
$$

Next, we need to bound all the terms on the RHS of (5.58). First, we have

$$
\begin{aligned}
&- 2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1}))) \\
&\le \frac{2\varepsilon}{N}\|\int_{t^n}^{t^{n+1}} \varphi dt\|_1 \|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1} \\
&\le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}} \|\varphi\|_1^2 dt)^{\frac{1}{2}} \|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1} \\
&\le C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|\varphi\|_1^2 dt + \Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2 \\
&\le C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2.
\end{aligned}
\tag{5.59}
$$

For the term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{M^2}\Delta\mathbf{e}_{\mathbf{u}}^{n+1})$, we have

$$
\begin{aligned}
-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{M^2}\Delta\mathbf{e}_{\mathbf{u}}^{n+1}) &= 2\varepsilon(\int_{t^n}^{t^{n+1}} \nabla\varphi dt, \frac{1}{M^2}\nabla\mathbf{e}_{\mathbf{u}}^{n+1}) \\
&\le \frac{2\varepsilon}{M^2}\|\int_{t^n}^{t^{n+1}} \nabla\varphi dt\|\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 \\
&\le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}} \|\nabla\varphi\|^2 dt)^{\frac{1}{2}}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 \\
&\le C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2.
\end{aligned}
\tag{5.60}
$$

For the term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times \nabla e_\phi^n)$, we have

$$\begin{aligned} &-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times \nabla e_\phi^n) \le 4\varepsilon\|\mathbf{B}\|_{L^\infty}\|\int_{t^n}^{t^{n+1}} \varphi dt\|\|\nabla e_\phi^n\| \\ &\le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}} \|\varphi\|^2 dt)^{\frac{1}{2}}\|\nabla e_\phi^n\| \le C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma_2 \Delta t \|\nabla e_\phi^n\|^2. \end{aligned} \tag{5.61}$$

The term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times (\nabla\phi(t^{n+1}) - \nabla\phi(t^n)))$ is bounded as

$$\begin{aligned} &-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{B} \times (\nabla\phi(t^{n+1}) - \nabla\phi(t^n))) \le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}} \|\varphi\|^2 dt)^{\frac{1}{2}}\|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\| \\ &\le C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \Delta t\|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\|^2. \end{aligned} \tag{5.62}$$

The term $-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, (\mathbf{e}_\mathbf{u}^{n+1} \times \mathbf{B}) \times \mathbf{B})$ is by

$$\begin{aligned} &-2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, (\mathbf{e}_\mathbf{u}^{n+1} \times \mathbf{B}) \times \mathbf{B}) \le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}} \|\varphi\|^2 dt)^{\frac{1}{2}}\|\mathbf{e}_\mathbf{u}^{n+1}\| \\ &\le C\varepsilon^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma\Delta t\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|^2. \end{aligned} \tag{5.63}$$

We bound the term $2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t})$ by

$$\begin{aligned} &2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}\frac{\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n}{\Delta t}) \\ &= \frac{2\varepsilon}{N\Delta t}[(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}) - (\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n)] - \frac{2\varepsilon}{N\Delta t}(\int_{t^n}^{t^{n+1}} \varphi dt - \int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n) \\ &\le \frac{2\varepsilon}{N\Delta t}[(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}) - (\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n)] + \frac{2\varepsilon\Delta t}{N}|(\varphi_t(\xi_n), \mathbf{e}_\mathbf{u}^n)| \\ &\le \frac{2\varepsilon}{N\Delta t}[(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}) - (\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n)] + \frac{2\varepsilon\Delta t}{N}\|\varphi_t(\xi_n)\|\|\mathbf{e}_\mathbf{u}^n\| \\ &\le \frac{2\varepsilon}{N\Delta t}[(\int_{t^n}^{t^{n+1}} \varphi dt, \mathbf{e}_\mathbf{u}^{n+1}) - (\int_{t^{n-1}}^{t^n} \varphi dt, \mathbf{e}_\mathbf{u}^n)] + C\varepsilon^2\Delta t\|p_{tt}(\xi_n)\|^2 + \sigma_3\Delta t\|\nabla\mathbf{e}_\mathbf{u}^n\|^2, \end{aligned} \tag{5.64}$$

where $\xi_n \in (t^{n-1}, t^{n+1})$. We bound the term $2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{u}^n, \mathbf{e}_\mathbf{u}^{n+1}))$ as

$$\begin{aligned} &2\varepsilon(\int_{t^n}^{t^{n+1}} \varphi dt, \frac{1}{N}B(\mathbf{u}^n, \mathbf{e}_\mathbf{u}^{n+1})) \le C\varepsilon\|\int_{t^n}^{t^{n+1}} \nabla\varphi dt\|\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|\|\nabla\mathbf{u}^n\| \\ &\le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}} \|\nabla\varphi\|^2 dt)^{\frac{1}{2}}\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|\|\nabla\mathbf{u}^n\| \le C\varepsilon^2\|\nabla\mathbf{u}^n\|^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt + \sigma\Delta t\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|^2. \end{aligned} \tag{5.65}$$

The term $2\varepsilon(\int_{t^n}^{t^{n+1}}\varphi dt, \frac{1}{N}B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1})))$ is bounded as

$$\begin{aligned}
&2\varepsilon(\int_{t^n}^{t^{n+1}}\varphi dt, \frac{1}{N}B(\mathbf{e}_{\mathbf{u}}^n, \mathbf{u}(t^{n+1}))) \le C\varepsilon\|\int_{t^n}^{t^{n+1}}\nabla\varphi dt\|\|\nabla\mathbf{e}_{\mathbf{u}}^n\|\|\nabla\mathbf{u}(t^{n+1})\| \\
&\le C\varepsilon^2\|\nabla\mathbf{u}(t^{n+1})\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt + \sigma_3\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^n\|^2.
\end{aligned} \tag{5.66}$$

For the last term, we have

$$\begin{aligned}
&2\varepsilon(\int_{t^n}^{t^{n+1}}\varphi dt, \frac{1}{N}B(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n), \mathbf{u}(t^{n+1}))) \\
&\le C\varepsilon\sqrt{\Delta t}(\int_{t^n}^{t^{n+1}}\|\nabla\varphi\|^2 dt)^{\frac{1}{2}}\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|\|\mathbf{u}(t^{n+1})\|_2 \\
&\le C\varepsilon^2\|\mathbf{u}(t^{n+1})\|_2^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt + \Delta t\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2.
\end{aligned} \tag{5.67}$$

Combining (5.59)-(5.67), we have

$$\begin{aligned}
&2\varepsilon(\int_{t^n}^{t^{n+1}}p_t dt, e_p^{n+1}) \le 3\sigma\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \sigma_2\Delta t\|\nabla e_\phi^n\|^2 + 2\sigma_3\Delta t\|\nabla\mathbf{e}_{\mathbf{u}}^n\|^2 \\
&+ \Delta t\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2 + \Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2 \\
&+ \Delta t\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|^2 + \frac{2\varepsilon}{N\Delta t}[(\int_{t^n}^{t^{n+1}}\varphi dt, \mathbf{e}_{\mathbf{u}}^{n+1}) - (\int_{t^{n-1}}^{t^n}\varphi dt, \mathbf{e}_{\mathbf{u}}^n)] \\
&+ C\varepsilon^2\Delta t\|p_{tt}(\xi_n)\|^2 + C\varepsilon^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt + C\varepsilon^2\|\nabla\mathbf{u}^n\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt.
\end{aligned} \tag{5.68}$$

Set $\sigma = \frac{1}{14M^2}, \sigma_1 = \frac{1}{4}, \sigma_2 = \frac{1}{2}, \sigma_3 = \frac{1}{4M^2}$. Combining (5.48)-(5.54) and (5.68), we have

$$\begin{aligned}
&\frac{1}{N}(\|\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{e}_{\mathbf{u}}^n\|^2 + \|\mathbf{e}_{\mathbf{u}}^{n+1}-\mathbf{e}_{\mathbf{u}}^n\|^2) + \varepsilon(\|e_p^{n+1}\|^2 - \|e_p^n\|^2 + \|e_p^{n+1}-e_p^n\|^2) \\
&+ \frac{\Delta t}{M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t\|\nabla e_\phi^{n+1}\|^2 + \Delta t\|\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2 - \Delta t\|\mathbf{e}_{\mathbf{u}}^n\times\mathbf{B}\|^2 \\
&+ \frac{\Delta t}{2M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 - \frac{\Delta t}{2M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|^2 + \frac{3}{2}\Delta t\|\nabla e_\phi^{n+1}\|^2 - \frac{3}{2}\Delta t\|\nabla e_\phi^n\|^2 \\
&+ \Delta t(\|-\nabla e_\phi^n + \mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{B}\|^2 + \|-\nabla e_\phi^{n+1} + \mathbf{e}_{\mathbf{u}}^n\times\mathbf{B}\|^2) \\
&\le C\Delta t(\|\mathbf{u}(t^{n+1})\|_2^2 + \|\mathbf{B}\|_{L^\infty}^2)\|\mathbf{e}_{\mathbf{u}}^n\|^2 + C\Delta t\|\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2 \\
&+ C\Delta t\|\nabla\phi(t^{n+1})-\nabla\phi(t^n)\|^2 + C\Delta t(1 + \|\mathbf{u}(t^{n+1})\|_2^2 + \|\mathbf{B}\|_{L^\infty}^2)\|\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)\|^2 \\
&+ \frac{2\varepsilon}{N\Delta t}[(\int_{t^n}^{t^{n+1}}\varphi dt, \mathbf{e}_{\mathbf{u}}^{n+1}) - (\int_{t^{n-1}}^{t^n}\varphi dt, \mathbf{e}_{\mathbf{u}}^n)] \\
&+ C\varepsilon^2\Delta t\|p_{tt}(\xi_n)\|^2 + C\varepsilon^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt + C\varepsilon^2\|\nabla\mathbf{u}^n\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt.
\end{aligned} \tag{5.69}$$

Summing (5.69) from $n = 0$ to $n = m - 1$, we obtain

$$
\begin{aligned}
&\frac{1}{N}\|\mathbf{e}_{\mathbf{u}}^m\|^2 + \frac{1}{N}\sum_{n=0}^{m-1}\|\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n\|^2 + \varepsilon\|e_p^m\|^2 + \sum_{n=0}^{m-1}\|e_p^{n+1} - e_p^n\|^2 + \frac{\Delta t}{M^2}\sum_{n=0}^{m-1}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 \\
&\quad + \Delta t\sum_{n=0}^{m-1}\|\nabla e_\phi^{n+1}\|^2 + \Delta t\|\mathbf{e}_{\mathbf{u}}^m \times \mathbf{B}\|^2 + \frac{3}{2}\Delta t\|\nabla e_\phi^m\|^2 + \frac{\Delta t}{2M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^m\|^2 \\
&\quad + \Delta t\sum_{n=0}^{m-1}(\| - \nabla e_\phi^n + \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 + \| - \nabla e_\phi^{n+1} + \mathbf{e}_{\mathbf{u}}^n \times \mathbf{B}\|^2) \\
&\le \frac{1}{N}\|\mathbf{e}_{\mathbf{u}}^0\|^2 + \varepsilon\|e_p^0\|^2 + \Delta t\|\mathbf{e}_{\mathbf{u}}^0 \times \mathbf{B}\|^2 + \frac{\Delta t}{2M^2}\|\nabla\mathbf{e}_{\mathbf{u}}^0\|^2 + \frac{3}{2}\Delta t\|\nabla e_\phi^0\|^2 + C\Delta t\sum_{n=0}^{m-1}\|\mathbf{e}_{\mathbf{u}}^n\|^2 \\
&\quad + C\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2 + C\Delta t\sum_{n=0}^{m-1}\|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\|^2 \\
&\quad + C\Delta t\sum_{n=0}^{m-1}\|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|^2 + \frac{2\varepsilon}{N\Delta t}(\int_{t^{m-1}}^{t^m}\varphi dt, \mathbf{e}_{\mathbf{u}}^m) + C\varepsilon^2\Delta t\sum_{n=0}^{m-1}\|p_{tt}(\xi_n)\|^2 \\
&\quad + C\varepsilon^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt + C\varepsilon^2\sum_{n=0}^{m-1}\|\nabla\mathbf{u}^{n+1}\|^2\int_{t^n}^{t^{n+1}}\|p_t\|^2 dt.
\end{aligned}
\tag{5.70}
$$

Next, we bound the terms in the right hand side of (5.70). First, we have

$$
\begin{aligned}
&C\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} - \mathbf{u}_t(t^{n+1})\|_{-1}^2 \le C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2 dt \\
&\le C\Delta t^2\|\mathbf{u}_{tt}\|_{2,-1}^2.
\end{aligned}
\tag{5.71}
$$

We also have

$$
C\Delta t\sum_{n=0}^{m-1}\|\nabla\phi(t^{n+1}) - \nabla\phi(t^n)\|^2 \le C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\nabla\phi_t\|^2 dt \le C\Delta t^2\|\phi_t\|_{2,1}^2, \tag{5.72}
$$

and

$$
C\Delta t\sum_{n=0}^{m-1}\|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|^2 \le C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{u}_t\|^2 dt \le C\Delta t^2\|\mathbf{u}_t\|_{2,0}^2. \tag{5.73}
$$

Moreover,

$$
\begin{aligned}
&\frac{2\varepsilon}{N\Delta t}(\int_{t^{m-1}}^{t^m}\varphi dt, \mathbf{e}_{\mathbf{u}}^m) = \frac{2\varepsilon}{N}(\varphi(\eta^m), \mathbf{e}_{\mathbf{u}}^m) \le \frac{2\varepsilon^2}{N}\|\varphi(\eta^m)\|^2 + \frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2 \\
&\le C\varepsilon^2\|p_t(\eta^m)\|^2 + \frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2 \le C\varepsilon^2\|p_t\|_{\infty,0}^2 + \frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2,
\end{aligned}
\tag{5.74}
$$

where $\eta^m \in (t^{m-1}, t^m)$. Lastly, it gives that

$$C\varepsilon^2 \sum_{n=0}^{m-1} \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt \leq C\varepsilon^2 \|p_t\|_{2,0}^2, \tag{5.75}$$

and

$$C\varepsilon^2 \sum_{n=0}^{m-1} \|\nabla \mathbf{u}^{n+1}\|^2 \int_{t^n}^{t^{n+1}} \|p_t\|^2 dt \leq C\varepsilon^2 \|p_t\|_{\infty,0}^2 \sum_{n=0}^{m-1} \Delta t \|\nabla \mathbf{u}^{n+1}\|^2 \leq C\varepsilon^2 \|p_t\|_{\infty,0}^2, \tag{5.76}$$

where we use the result in Theorem 9.

Combining (5.71)-(5.76) with (5.70), we can have

$$\begin{aligned} &\frac{1}{2N}\|\mathbf{e}_{\mathbf{u}}^m\|^2 + \frac{\Delta t}{M^2} \sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t \sum_{n=0}^{m-1} \|\nabla e_\phi^{n+1}\|^2 \\ &\quad + \Delta t \sum_{n=0}^{m-1} (\| - \nabla e_\phi^n + \mathbf{e}_{\mathbf{u}}^{n+1} \times \mathbf{B}\|^2 + \| - \nabla e_\phi^{n+1} + \mathbf{e}_{\mathbf{u}}^n \times \mathbf{B}\|^2) \\ &\leq C\Delta t \sum_{n=0}^{m-1} \|\mathbf{e}_{\mathbf{u}}^n\|^2 + C(\Delta t^2 + \varepsilon^2). \end{aligned} \tag{5.77}$$

Finally applying the discrete Gronwall lemma completes the proof. □

5.4 ANALYSIS OF NON-PHYSICAL ACOUSTIC WAVES

Artificial compression method can greatly speed up computations but can also introduce new physical flow behaviors associated with compressibility such as non-physical, fast, pressure oscillations (acoustic waves). These fast acoustic waves can yield restrictive time step conditions for explicit time discretization of the full system or pollute the pressure approximation. In this section, we analyze these non-physical acoustic waves through an acoustic waves equation for pressure. We rewrite the model (5.2) as follows.

$$\begin{aligned} &\frac{1}{N}(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{u}) - \frac{1}{N \cdot Re}\Delta \mathbf{u} + \nabla p = \mathbf{f} + \mathbf{j} \times \mathbf{B}, \\ &\varepsilon p_t + \nabla \cdot \mathbf{u} = 0, \\ &\nabla \cdot \mathbf{j} = 0. \end{aligned} \tag{5.78}$$

Taking the divergence of the first equation and $\partial/\partial t$ of the second equation in (5.78), we obtain

$$\begin{aligned}&\frac{1}{N}\nabla\cdot\mathbf{u}_t+\frac{1}{N}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u}-\frac{1}{\cdot Re}\Delta\mathbf{u})+\Delta p=\nabla\cdot\mathbf{f}+\nabla\cdot(\mathbf{j}\times\mathbf{B}),\\&\varepsilon p_{tt}+\nabla\cdot\mathbf{u}_t=0.\end{aligned}\tag{5.79}$$

Assume $\nabla\cdot\mathbf{f}=0$. Multiplying the first equation in (5.79) by N and eliminating $\nabla\cdot\mathbf{u}_t$ term, we can have

$$\varepsilon p_{tt}-N\Delta p=\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u}-\frac{1}{\cdot Re}\Delta\mathbf{u})-N\nabla\cdot(\mathbf{j}\times\mathbf{B}).\tag{5.80}$$

Since $\nabla\cdot(\mathbf{j}\times\mathbf{B})=(\nabla\times\mathbf{j})\cdot\mathbf{B}$ and $\nabla\times\mathbf{j}=\nabla\times(-\nabla\phi+\mathbf{u}\times\mathbf{B})=\nabla\times(\mathbf{u}\times\mathbf{B})$, we have $\nabla\times\mathbf{j}=-(\nabla\cdot\mathbf{u})|\mathbf{B}|$ and $\nabla\cdot(\mathbf{j}\times\mathbf{B})=-(\nabla\cdot\mathbf{u})|\mathbf{B}|^2$ in 2 dimension case. Thus equation (5.80) can be rewritten as

$$\varepsilon p_{tt}+\varepsilon N|\mathbf{B}|^2p_t-N\Delta p=\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u}-\frac{1}{\cdot Re}\Delta\mathbf{u}).\tag{5.81}$$

The pressure wave equation (5.81) shows that the non-physical oscillation in p will be damped by damping coefficient $N|\mathbf{B}|^2$. Thus, *the effect of magnetic field is to damp acoustic waves in slightly compressible flows*. The waves have energy input due to the right hand side terms. The speed of the non-physical acoustic wave is $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, which indicates that the wave speed goes up to ∞ as $\varepsilon\to 0$. We test the non-physical acoustic wave in Section 5.5.2. We compute and plot (figure 20 and figure 21) the pressure at origin $(0,0)$ on a time interval after initial transients pass to test if the wave speed increases as the time step $\Delta t\downarrow 0$.

Nonlinear Acoustics. We consider the effect of the nonlinear term in the right hand side of (5.81) on acoustic waves. Let the usual Lighthill sound source (see, [26] and [27] for justification) be denoted by

$$Q(\mathbf{u},\mathbf{u}):=\nabla\mathbf{u}:(\nabla\mathbf{u})^T=\sum_{i,j}\frac{\partial\mathbf{u}_i}{\partial\mathbf{x}_j}\frac{\partial\mathbf{u}_j}{\partial\mathbf{x}_i}\quad\text{for } d=2\text{ or }3.$$

Since we have the identity that

$$\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u})=Q(\mathbf{u},\mathbf{u})-\frac{1}{2}\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u})+\frac{1}{2}|\nabla\cdot\mathbf{u}|^2,$$

if the effect of slight compressibility on acoustic waves is negligible, we can obtain

$$\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{u} - \frac{1}{\cdot Re}\Delta \mathbf{u}) \simeq Q(\mathbf{u}, \mathbf{u}) \quad \text{if } \nabla \cdot \mathbf{u} \simeq 0.$$

Recall (5.2) that $\nabla \cdot \mathbf{u} = -\varepsilon p_t$. Then we have

$$\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{u} - \frac{1}{\cdot Re}\Delta \mathbf{u}) = Q(\mathbf{u}, \mathbf{u}) + \frac{\varepsilon}{2}\mathbf{u} \cdot \nabla p_t + \frac{\varepsilon^2}{2}|p_t|^2 + \frac{\varepsilon}{Re}\Delta p_t.$$

Thus (5.81) can be rewritten as

$$\varepsilon p_{tt} + \varepsilon N|\mathbf{B}|^2 p_t - N\Delta p = Q(\mathbf{u}, \mathbf{u}) + \frac{\varepsilon}{2}\mathbf{u} \cdot \nabla p_t + \frac{\varepsilon^2}{2}|p_t|^2 + \frac{\varepsilon}{Re}\Delta p_t. \tag{5.82}$$

$Q(\mathbf{u}, \mathbf{u})$ represents the physical sound source, i.e., the sound generated by the flow. The terms $\frac{\varepsilon}{2}\mathbf{u} \cdot \nabla p_t + \frac{\varepsilon^2}{2}|p_t|^2 + \frac{\varepsilon}{Re}\Delta p_t$ are the nonphysical sound sources and their values vanish as ε goes to 0. When $Re \gg 1$, the leading order term in nonphysical sound sources is $\frac{\varepsilon}{2}\mathbf{u} \cdot \nabla p_t$. In Section 5.5.2, we compute the relative size of the leading order term in non-physical sound sources to the Lighthill sound source

$$Ratio = \frac{\|\frac{\varepsilon}{2}\mathbf{u} \cdot \nabla p_t\|}{\|Q(\mathbf{u}, \mathbf{u})\|}.$$

The result is shown in figure 22 and figure 23. Looking at the vertical axis scales, we conclude that the nonphysical sound source is negligible compared to the physical one.

5.5 NUMERICAL TEST

In this section, we provide numerical experiments to test the convergence of Algorithm 8 and Algorithm 9, and the non-physical acoustic waves analyzed in Section 5.4. We utilize the P2-P1 Taylor-Hood mixed finite elements for fluid velocity and pressure and P2 finite element for electric potential. The software package $FreeFEM++$, see [142], are used for our simulation.

5.5.1 TESTING CONVERGENCE

Let the domain $\Omega = [0,1] \times [0,1]$, $Re = 1$, $N = 1$, $M = 1$ and $\mathbf{B} = (0,0,1)$. Consider the true solution $(\mathbf{u}, p, \phi)$ given as follows.

$$\mathbf{u}(x,y,t) = (2\pi\cos(2\pi x)\sin(2\pi y), -2\pi\sin(2\pi x)\cos(2\pi y), 0)e^{-5t},$$
$$p(x,y,t) = 0,$$
$$\phi(x,y,t) = (\cos(2\pi x)\cos(2\pi y) + x^2 - y^2)e^{-5t}.$$

The body force $\mathbf{f}$, boundary condition and initial condition are determined by the true solution.

We firstly compute the rate of convergence to confirm the effectiveness of our theoretical analysis for *Algorithm* 1. Set $\varepsilon = \Delta t$. Select $T = 1$, $h = \frac{1}{60}$ and then $\Delta t = \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{1}{50}, \frac{1}{60}$. We compute $\|\mathbf{e}_{\mathbf{u}}^m\|$, $(\sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$ and $(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$ to obtain the convergence rate.

Table 8: Errors and convergence rates of Algorithm 8.

Δt	$\|\mathbf{e}_{\mathbf{u}}^m\|$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$	Rate
1/20	6.0467e-2		2.2961e-1		2.5699e-1	
1/30	4.3838e-2	0.79	1.4885e-1	1.07	1.7247e-1	0.98
1/40	3.3862e-2	0.90	1.0901e-1	1.08	1.2983e-1	0.99
1/50	2.7299e-2	0.97	8.6329e-2	1.05	1.0411e-1	0.99
1/60	2.2684e-2	1.02	7.2225e-2	0.98	8.6917e-2	0.99

Table 8 confirms that the rate of convergence is first order in accord with the theoretical result of Theorem 11.

We also compute the rate of convergence for Algorithm 9. Since Algorithm 9 is second order accurate, we set $\varepsilon = \Delta t^2$. Select $T = 1$, $h = \Delta t$ and then $\Delta t = \frac{1}{20}, \frac{1}{40}, \frac{1}{60}, \frac{1}{80}, \frac{1}{100}$. We also compute $\|\mathbf{e}_{\mathbf{u}}^m\|$, $(\sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$ and $(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$.

Table 9 shows that the rate of convergence of Algorithm 9 is second order, as expected.

Table 9: Errors and convergence rates of Algorithm 9.

Δt	$\|\mathbf{e}_{\mathbf{u}}^m\|$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2)^{\frac{1}{2}}$	Rate	$(\Delta t \sum_{n=0}^{m-1} \|\nabla e_{\phi}^{n+1}\|^2)^{\frac{1}{2}}$	Rate
1/20	7.1314e-3		1.1478e-1		9.7290e-3	
1/40	1.7696e-3	2.01	3.6299e-2	1.66	2.9077e-3	1.72
1/60	7.6980e-4	2.05	1.7458e-2	1.81	1.3712e-3	1.85
1/80	4.2564e-4	2.06	1.0219e-2	1.86	7.9445e-4	1.90
1/100	2.6889e-4	2.06	6.6996e-3	1.89	5.1754e-4	1.92

5.5.2 TESTING NON-PHYSICAL ACOUSTIC WAVES

We explore the non-physical acoustic waves by the following 2D test problem (Flow between offset circles).

Let the domain $\Omega = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } (x - 0.5)^2 + y^2 \geq 0.1^2\}$, $\mathbf{B} = 1 \cdot \vec{k}$, the final time $T = 30$, and the body force $\mathbf{f} = (-4y(1 - x^2 - y^2), 4x(1 - x^2 - y^2))^{\mathrm{T}}$. Set $Re = 1000$ and $N = 1$, thus $M = \sqrt{N \cdot Re} = \sqrt{1000}$. For velocity boundary conditions, let $\mathbf{u} = 0$ on both circles. Similarly set $\varepsilon = \Delta t$ for Algorithm 8 ($\varepsilon = \Delta t^2$ for Algorithm 9). Choose time step $\Delta t = \frac{1}{25}, \frac{1}{50}, \frac{1}{100}$.

Firstly, we plot the pressure vs. t at $(0, 0)$. Figure 20 (figure 21) shows the results of Algorithm 8 (Algorithm 9) on a time interval $[25, 30]$ after initial transients pass. We find that the time evolution of the pressure at one point in space varies greatly as Δt changes. However, the wave's frequency has a clear pattern consistent with (5.81): As Δt decreases, the wave's frequency increases.

Figure 22 (Algorithm 8) and figure 23 (Algorithm 9) present the relative size of the leading order term in non-physical sound sources to the Lighthill sound source: $\frac{\|\frac{\varepsilon}{2}\mathbf{u}\cdot\nabla p_t\|}{\|Q(\mathbf{u},\mathbf{u})\|}$. It shows that $Q(\mathbf{u}, \mathbf{u})$ is the dominant forcing for oscillations in p and $\nabla \cdot \mathbf{u}$.

Lastly, in order to study how close the computing solution is to incompressible, we compute $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$. As shown in figure 25 (Algorithm 9), the relative size of $\nabla \cdot \mathbf{u}$ decreases as Δt decreases. However, in figure 24 (Algorithm 8), $\nabla \cdot \mathbf{u}$ fails to decrease

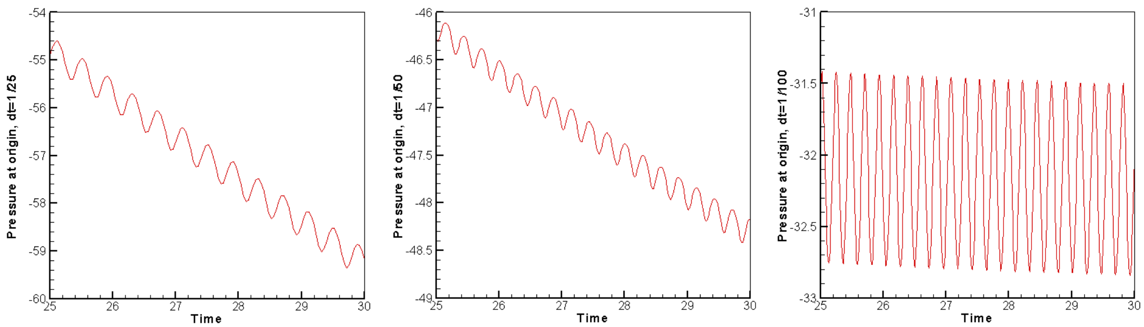


Figure 13: Pressure at (0,0) vs Time, Algorithm 8, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

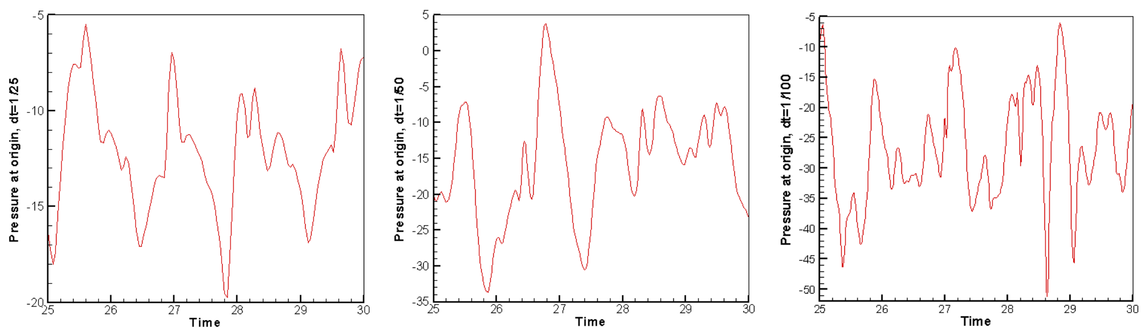


Figure 14: Pressure at (0,0) vs Time, Algorithm 9, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

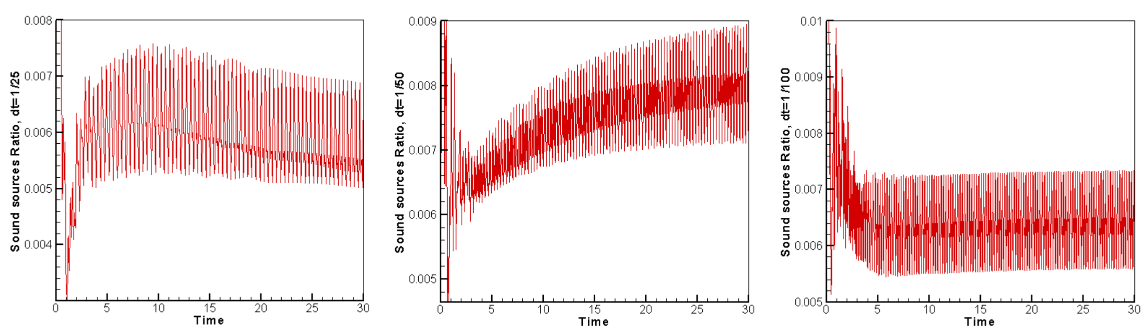


Figure 15: Non-physical acoustic source / Lighthill source, Algorithm 8, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

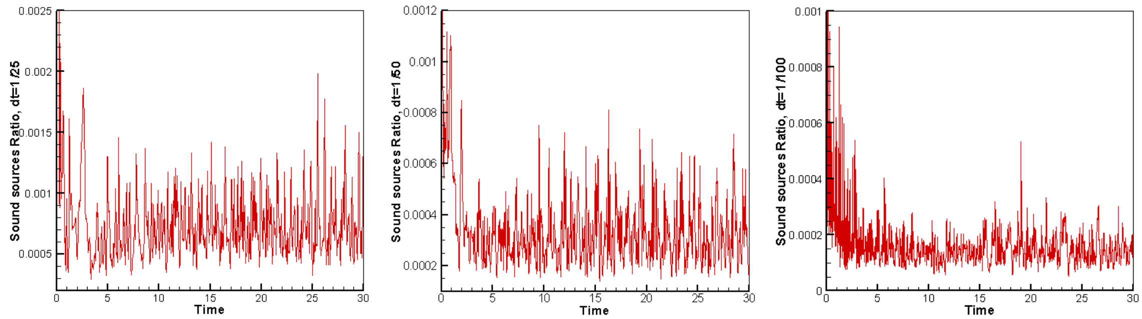


Figure 16: Non-physical acoustic source / Lighthill source, Algorithm 9, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

when Δt decreases, which implies that the selection of $\varepsilon = \Delta t$ or Δt^2 should influence the stability of the artificial compression method. For Algorithm 8, we could further stabilize this first order method. Thus, we consider to add a stabilization term $\gamma\nabla\nabla\cdot\mathbf{u}$ in Algorithm 8 ($\gamma = 10000$) and then recompute the relative size of $\nabla\cdot\mathbf{u}$. Figure 19 presents the computing results and shows that the relative size of $\nabla\cdot\mathbf{u}$ decreases as Δt decreases when we select a large γ in the stabilization term $\gamma\nabla\nabla\cdot\mathbf{u}$. Compared with figure 24, when we apply the stabilization term, the oscillation of the relative size of $\nabla\cdot\mathbf{u}$ becomes weaker, which shows the stabilization term $\gamma\nabla\nabla\cdot\mathbf{u}$ might be an effective way to dampen non-physical acoustic waves. Meanwhile, we find that as $\Delta t \downarrow 0$, $\|\nabla\cdot\mathbf{u}\|/\|\mathbf{u}\|$ appears to be more oscillating, which is also consistent with our analysis of non-physical acoustic waves since $\nabla\cdot\mathbf{u} = -\varepsilon p_t$.

5.6 CONCLUSIONS

In this paper, we construct two decoupled methods based on the artificial compression method (uncoupling the pressure and velocity) and partitioned method (uncoupling the velocity and electric potential) for magnetohydrodynamics flows at low magnetic Reynolds numbers. The methods we study allow us at each time step to solve linear problems, uncoupled by physical processes, per time step, which can greatly improve the computational

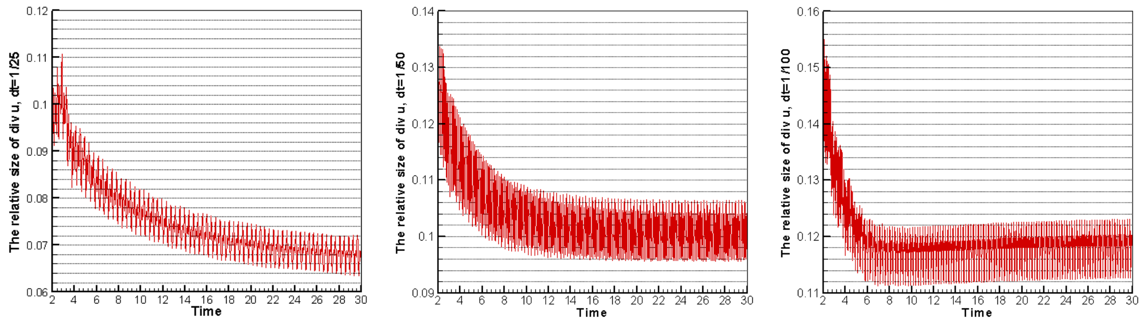


Figure 17: $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$, Algorithm 8, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

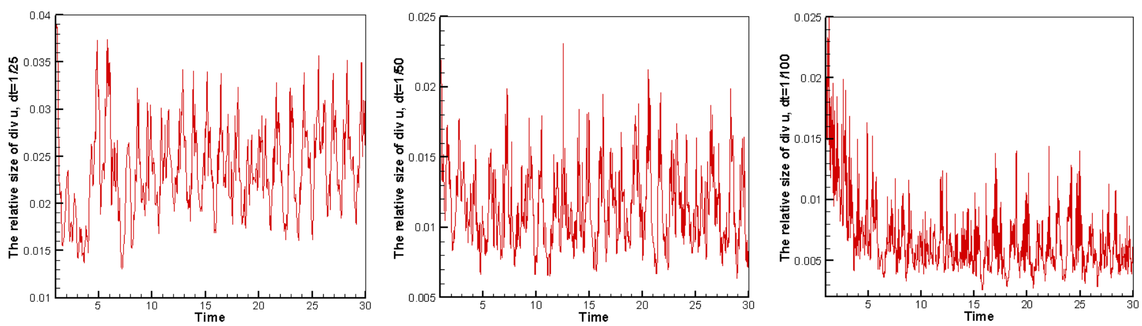


Figure 18: $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$, *Algorithm* 2, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

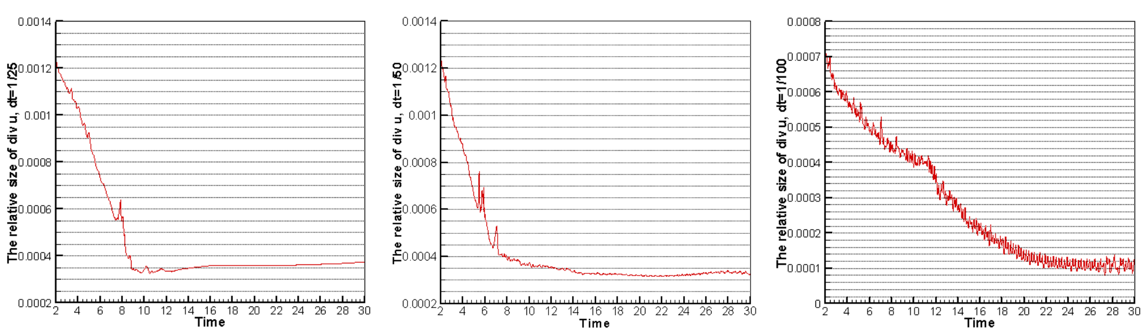


Figure 19: $\|\nabla \cdot \mathbf{u}\|/\|\mathbf{u}\|$, with $\gamma \nabla \nabla \cdot \mathbf{u}$, Algorithm 8, dt=1/25 (left), dt=1/50 (middle) and dt=1/100 (right).

efficiency. This paper gives the stability and error analysis, presents a brief analysis of the non-physical acoustic waves generated, and provides computational tests to support the theory.

6.0 MODEL ACCURATE ALGORITHMS IN CFD–EXTENSION OF BALDWIN-LOMAX MODEL TO NON-EQUILIBRIUM TURBULENCE

6.1 INTRODUCTION

The most common approach to the prediction of turbulent flow statistics is to add to the Navier-Stokes equations an eddy viscosity term, calibrate the term's coefficients, discretize the result and solve. A well-calibrated model and an effective numerical method have proven to predict reliably turbulent flows at statistical equilibrium. The question considered herein is how to extend such a model to non-equilibrium turbulence and how to adapt algorithms to the extended model. The first work on the approach herein was for the Smagorinsky model in [139]. Herein, we extend the Baldwin-Lomax model, shown below. In the modeling process, several choices must be made. We take a different path through these in developing the non-equilibrium extension than in [139]. Let us begin with the time dependent incompressible Navier-Stokes (NS) equations:

$$
\begin{aligned}
&\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \text{ and } \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\\
&\mathbf{u} = 0 \text{ on } \partial\Omega, \text{ and } \int_\Omega p\, d\mathbf{x} = 0, \qquad (6.1)\\
&\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.
\end{aligned}
$$

Here, $\Omega \subset R^d (d{=}2,3)$ is a bounded polyhedral domain; $\mathbf{u} : \Omega \times [0,T] \to R^d$ is the fluid velocity; $p : \Omega \times (0,T] \to R$ is the fluid pressure. The body force $\mathbf{f}$ is known. Re is the Reynolds number and $\nu = \frac{1}{Re}$.

There are many approaches to simulating turbulent flows, see [118, 119, 120, 121, 122]. One of the most commonly used is to model the ensemble-averaged Navier-Stokes equation by eddy viscosity, discretize then solve. In this approach, instantaneous variables are decomposed into the mean (ensemble averaging) and fluctuating components, then reintroduced into the governing equations to obtain the ensemble averaging equations. However, the system is not closed and contains Reynolds stress term that represents the effects of fluctuation. This Reynolds stress must be modeled in order to close the system.

Given the ensemble averaging of fluid velocity and pressure

$$\langle \mathbf{u} \rangle(\mathbf{x}, t) = \frac{1}{J}\sum_{j=1}^{J} \mathbf{u}(\mathbf{x}, t; \omega_j), \text{ and } \langle p \rangle(\mathbf{x}, t) = \frac{1}{J}\sum_{j=1}^{J} p(\mathbf{x}, t; \omega_j). \tag{6.2}$$

Decompose the pressure and velocity into mean and fluctuations:

$$\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}', \text{ and } p = \langle p \rangle + p', \tag{6.3}$$

where $\langle \mathbf{u} \rangle, \langle p \rangle$ are the mean and $\mathbf{u}', p'$ are the fluctuating components. Substituting the ensemble averaging variables into the NS equations (6.1) yields the ensemble averaging equations.

$$\begin{aligned} &\langle \mathbf{u} \rangle_t + \langle \mathbf{u} \rangle \cdot \nabla \langle \mathbf{u} \rangle - \nu \Delta \langle \mathbf{u} \rangle - \nabla \cdot \mathrm{R}(\mathbf{u}, \mathbf{u}) + \nabla \langle p \rangle = \mathbf{f}, \\ &\nabla \cdot \langle \mathbf{u} \rangle = 0, \end{aligned} \tag{6.4}$$

where the Reynolds stress $\mathrm{R}(\mathbf{u}, \mathbf{u}) := \langle \mathbf{u} \rangle \otimes \langle \mathbf{u} \rangle - \langle \mathbf{u} \otimes \mathbf{u} \rangle = -\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$, see, e.g., [121, 123, 130].

By the Boussinesq assumption and eddy viscosity hypothesis ([127, 128, 129, 132]), the Reynolds stress $\mathrm{R}(\mathbf{u}, \mathbf{u})$ is modeled by $(\nu_T(\langle \mathbf{u} \rangle)\nabla \langle \mathbf{u} \rangle)$. Note that the turbulence eddy viscosity $\nu_T(\langle \mathbf{u} \rangle) > 0$. This is the standard eddy viscosity (EV) model:

$$\begin{aligned} &\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} - \nu \Delta \mathbf{w} - \nabla \cdot (\nu_T(\mathbf{w})\nabla \mathbf{w}) + \nabla q = \mathbf{f}, \\ &\nabla \cdot \mathbf{w} = 0. \end{aligned} \tag{6.5}$$

The solution $(\mathbf{w}, q)$ of (6.5) is an approximation of the mean $(\langle \mathbf{u} \rangle, \langle p \rangle)$.

There have been many techniques to predict the turbulent eddy viscosity ν_T (e.g., [119, 120, 121, 122, 124, 134, 135, 136]). However, EV models have difficulties in simulating backscatter or complex turbulence not at statistical equilibrium, see, e.g., [131, 133, 137]. Since $\nu_T > 0$, the term $-\nabla \cdot (\nu_T(\mathbf{w})\nabla \mathbf{w})$ in (6.5) can only represent dissipative effects of the Reynolds stress. In order to precisely characterize backscatter, Jiang and Layton [139] presented two approaches to correcting EV models and obtained new models of turbulence not at statistical equilibrium, analyzed the corrected Smagorinsky model and gave algorithms for its discretization.

Like many EV models, the Baldwin-Lomax model (see, e.g., [119, 121, 141]) begins simply and then has evolved substantial complexity to model effects such as separation and wakes not well described by the basic model. We consider herein only its simplest form for which $\nu_T(\langle \mathbf{u} \rangle) = l(\mathbf{x})^2 |\nabla \times \langle \mathbf{u} \rangle|$. Here, $l(\mathbf{x})$ is a mixing length that depends on the distance to the wall. The model extension, analysis and algorithms herein are adapted to more intricate, algebraic $\nu_T(\langle \mathbf{u} \rangle)$ with only notational complexity. The corrected model studied herein is as follows.

$$\begin{aligned} &\mathbf{w}_t + \beta^2 \nabla \times (l^2(\mathbf{x}) \nabla \times \mathbf{w}_t) + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \times (l^2(\mathbf{x}) |\nabla \times \mathbf{w}| \nabla \times \mathbf{w}) - \nu \Delta \mathbf{w} + \nabla q = \mathbf{f}, \\ &\nabla \cdot \mathbf{w} = 0. \end{aligned} \tag{6.6}$$

Here, β is a positive model calibration parameter. The eddy viscosity term is expressed in rotational (curl - curl) form. The model's mixing length $l(\mathbf{x})$ has the property that $0 \leq l(\mathbf{x}) \to 0$ as $\mathbf{x} \to \partial\Omega$. The effect of the true Reynolds stress on the mean flow is, on time average, dissipative, see [129, 139, 140]. We prove in Section 6.2 that the time averaged effect of the terms that modeled the Reynolds stress is also dissipative.

$$\begin{aligned} &\liminf_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \mathbf{w}_t \cdot \nabla \times \mathbf{w} + l^2(\mathbf{x}) |\nabla \times \mathbf{w}|^3) d\mathbf{x} dt \\ &= \liminf_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega l^2(\mathbf{x}) |\nabla \times \mathbf{w}|^3 d\mathbf{x} dt \geq 0. \end{aligned}$$

Our numerical tests in Section 6.5 show that the term that models pointwise in time, statistical backscatter, $\beta^2 \nabla \times (l^2(\mathbf{x}) \nabla \times \mathbf{w}_t)$, does result in bursts, wherein

$$\int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \mathbf{w}_t \cdot \nabla \times \mathbf{w} + l^2(\mathbf{x}) |\nabla \times \mathbf{w}|^3) d\mathbf{x} < 0.$$

In other words, the eddy viscosity accounts for the persistent effect of the Reynolds stress while the new term accounts for statistical backscatter without artificial negative viscosities. We also present and analyze a numerical method for accurate solution of the new model

(6.6) in Section 6.3. The challenges that occur here are (i) to develop algorithms that are small extensions of the standard methods for the usual Baldwin-Lomax model, and (ii) to perform the analysis for coefficients $l(\mathbf{x}) \to 0$ at walls.

6.2 ANALYSIS OF THE CORRECTED EV MODEL

This section first recalls the important properties of standard EV models (6.5). Then it presents the derivation of the corrected scheme (6.6) based on (6.5) and shows that the new model (6.6) maintains the time-averaged dissipative effect of the Reynolds stress.

Taking the inner product of the first and second equation in ensemble averaging NS equations (6.4) with $\langle \mathbf{u} \rangle$ and $\langle p \rangle$ respectively, we can obtain the kinetic energy equation for the mean

$$\frac{1}{2}\frac{d}{dt}\|\langle \mathbf{u}\rangle\|^2 + \nu\|\nabla\langle \mathbf{u}\rangle\|^2 + \int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx = \int_\Omega \mathbf{f}\cdot\langle \mathbf{u}\rangle dx. \tag{6.7}$$

In (6.7), the right hand side term $\int_\Omega \mathbf{f}\cdot\langle \mathbf{u}\rangle dx$ is the energy input. The term $\frac{1}{2}\frac{d}{dt}\|\langle \mathbf{u}\rangle\|^2$ is the changing rate of the kinetic energy of the mean. The term $\nu\|\nabla\langle \mathbf{u}\rangle\|^2$ is the energy dissipation of the mean. The term $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx$ indicates the effect of fluctuations on the mean. Moreover, if the term $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx > 0$, the effect is dissipative but if $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx < 0$, fluctuations transfer energy back to the mean which is called backscatter.

There are two key properties of Reynolds stress (proven in [139]). Time averaged dissipativity:

$$\lim\inf_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dxdt = \lim\inf_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \nu\langle|\nabla\mathbf{u}'|^2\rangle dxdt \geq 0 \tag{6.8}$$

Second, the variance evolution equation:

$$\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx = \frac{1}{2}\frac{d}{dt}\int_\Omega\langle|\mathbf{u}'|^2\rangle dx + \int_\Omega \nu\langle|\nabla\mathbf{u}'|^2\rangle dx. \tag{6.9}$$

The inequality (6.8) is consistent with the Boussinesq assumption that the effects of turbulent fluctuations are dissipative on the mean in the time averaged case. The interpretation of (6.9) is as follows. For flows at statistical equilibrium $\frac{d}{dt}\int_\Omega\langle|\mathbf{u}'|^2\rangle dx = 0$, while the second term $\int_\Omega \nu\langle|\nabla \mathbf{u}'|^2\rangle dx$ is clearly dissipative. This term is modeled by an eddy viscosity acting on $\langle \mathbf{u}\rangle$ that dissipates energy pointwise in both space and time. Thus the (space averaged) pointwise in time deviation from dissipativity must arise from the first term on the RHS of (6.9). Therefore, the term $\frac{1}{2}\frac{d}{dt}\int_\Omega\langle|\mathbf{u}'|^2\rangle dx$ should be modeled in the corrected EV scheme to represent backscatter.

In addition, the proof of the Boussinesq hypothesis in [129] is by extracting information on fluctuations from the energy equality for realizations. It was done for strong solutions, the natural setting for turbulence theory since it can be phrased in terms of the Reynolds stresses. This report also gave, Section 2.1 p. 2356, the equation for evolution of space averaged variance and turbulence intensities as well as an extended survey of the long history on the problem. In [112] an extension was given for ensembles of both initial conditions and body forces. In [139] it was shown that (using a result of Duchon and Robert [110]) a similar proof holds for weak solutions and that the spacial localization of backscatter depends only on 4 quantities: the variance of u and ∇u, the skewness of u and the velocity-pressure covariance. The (highly nontrivial) connection between the formulation in terms of the Reynolds stresses and the above proof for weak solutions was recently made by Berselli and Lewandowski [109], which contains the most general formulation of the result. Dissipativity of fluctuations and convergence to statistical equilibrium was extended to space and time discretizations in [140] and to MHD turbulence (through the Elsässer variables) in [114]. Once it was pointed out that dissipativity, while violated at some instants in time, emerges for time averaged quantities, various forms of the Boussinesq hypothesis' proof are implicit in Reynolds transport theory. For example, space and time averaging the trace of the transport equations of the Reynolds stresses, Jovanovic [113] Section 5.1 p. 110, yields the dissipativity of fluctuations. For time, not statistical, averages dissipativity of fluctuations result appears already (well hidden) in equation (3.127) p. 75, in Chacón- Rebollo and Lewandowski [136]. This connection was developed further by Lewandowski [115] and in several new directions in [109].

The key point to describe the Reynolds stress term is how to model $\mathbf{u}'$ via $\langle \mathbf{u}\rangle$. By the Kolmogorov-Prandtl relation for the turbulent viscosity,

$$\nu_T(\langle \mathbf{u}\rangle) = c_l l\sqrt{k'}, \tag{6.10}$$

where $k' = \frac{1}{2}\langle|\mathbf{u}'|^2\rangle$, c_l is a proportionality constant and l is the mixing length. Consider the simplest case (the inner layer viscosity) of the Baldwin-Lomax model

$$\nu_T(\langle\mathbf{u}\rangle) = l^2|\nabla \times \langle\mathbf{u}\rangle|, \tag{6.11}$$

where l is the mixing length, such as $l(\mathbf{x}) = 0.41 \times \text{distance}\{x, \partial\Omega\}$. Combine (6.10) with (6.11) to obtain the fluctuation model

$$\text{action}(\mathbf{u}') \simeq \beta l \nabla \times \langle\mathbf{u}\rangle, \tag{6.12}$$

where $\beta > 0$ is a model calibration parameter. Then the Reynolds stresses term $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle\mathbf{u}\rangle dx$ becomes

$$\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle\mathbf{u}\rangle dx \simeq \frac{1}{2}\frac{d}{dt}\int_\Omega \beta^2 l^2 |\nabla \times \langle\mathbf{u}\rangle|^2 dx + \int_\Omega l^2 |\nabla \times \langle\mathbf{u}\rangle||\nabla \times \langle\mathbf{u}\rangle|^2 dx, \tag{6.13}$$

Notice the time derivative term comes from the fluctuation model (6.12) and the eddy viscosity term is based on (6.11). Then, (6.13) leads to the closure model

$$-\nabla \cdot \mathrm{R}(\mathbf{u},\mathbf{u}) \simeq \beta^2 \nabla \times (l^2 \nabla \times \langle\mathbf{u}\rangle_t) + \nabla \times (l^2 |\nabla \times \langle\mathbf{u}\rangle| \nabla \times \langle\mathbf{u}\rangle). \tag{6.14}$$

Combining (6.14) with (6.4) and calling the model's approximation to $\langle\mathbf{u}\rangle$ and $\langle p\rangle$, $\mathbf{w}$ and q yields model (6.6).

$$\mathbf{w}_t + \beta^2 \nabla \times (l^2(\mathbf{x}) \nabla \times \mathbf{w}_t) + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \times (l^2(\mathbf{x}) |\nabla \times \mathbf{w}| \nabla \times \mathbf{w}) - \nu \Delta \mathbf{w} + \nabla q = \mathbf{f},$$
$$\nabla \cdot \mathbf{w} = 0.$$

Remark 4. *Although existence theory for (6.6) is not the issue considered in this report, it is useful to note the model's mathematical structure. Compared with the NSE, this model contains two additional terms. The first is* $\beta^2\nabla\times(l^2\nabla\times\mathbf{w}_t)$ *which is similar to the commonly used dispersive regularization* $-\Delta\mathbf{w}_t$*. The second is* $\nabla\times(l^2|\nabla\times\mathbf{w}|\nabla\times\mathbf{w})$*. This second term is strongly monotone and locally Lipschitz being similar to a p-Laplacian term. For* $l>0$ *it is readily seen that Galerkin approximations in spaces of Stokes eigenfunctions belong to* $L^\infty(0,T;V^2)\cap L^3(0,T;V^3)$ *(and by a limit argument under mild conditions on* $l(x)$*) the solution* $\mathbf{w}$ *satisfies the regularity*

$$\mathbf{w}\in L^\infty(0,T;V^2)\cap L^3(0,T;V^3) \tag{6.15}$$

We therefore conjecture that (again under mild conditions on $l(x)$*) existence and uniqueness of strong solutions hold for the model.*

For our purposes herein, we shall assume the model has a solution in the following sense.

The weak formulation of (6.6), satisfied by sufficiently smooth solutions, is : Find $(\mathbf{w},q)\in(X,Q)$ satisfying

$$\begin{aligned}
&(\mathbf{w}_t,\mathbf{v})+(l^2(\mathbf{x})\nabla\times\mathbf{w}_t,\nabla\times\mathbf{v})+b(\mathbf{w},\mathbf{w},\mathbf{v})+(l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w},\nabla\times\mathbf{v})\\
&\quad+\nu(\nabla\mathbf{w},\nabla\mathbf{v})-(q,\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v}) \qquad \forall\mathbf{v}\in X, \\
&(\nabla\cdot\mathbf{w},r)=0 \qquad \forall r\in Q.
\end{aligned}\tag{6.16}$$

Definition 2. *A distributional solution* $\mathbf{w}$ *of model (6.6) is a strong solution if* $\mathbf{w}$ *has regularity (6.15),* $\mathbf{w}(x,t)\to\mathbf{w}_0(x)$ *in* $L^2(\Omega)$ *as* $t\to0$ *and if* $\mathbf{w}$ *satisfies the model's weak form ((2.1) above) for all* $\mathbf{v}\in L^\infty(0,T;V^2)\cap L^3(0,T;V^3)$*.*

Operationally, the above definition means one may set $\mathbf{v}=\mathbf{w}$ in (2.1). Our numerical tests in Section 6.5 include ones with different boundary conditions. For these the variational formulation and solution notion must be appropriately adapted. Next, we give a theoretical analysis of model (6.6) and show in Theorem 12 that this new model still maintains the property that the effects of turbulent fluctuations on the mean are dissipative in long time averaging sense.

Theorem 12. *Assume* $\mathbf{f} \in L^\infty(0,\infty; H^{-1}(\Omega))$*. For the strong solution* $\mathbf{w}$ *of model* (6.6)*, we have*

$$\mathbf{w},\ l\nabla \times \mathbf{w} \in L^\infty(0,\infty; L^2(\Omega)), \tag{6.17}$$

$$\liminf_{T\to\infty} [\frac{1}{T}\int_0^T \nu\|\nabla \mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T\int_\Omega l^2|\nabla\times\mathbf{w}|^3 d\mathbf{x}dt] = \liminf_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt, \tag{6.18}$$

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega [\beta^2\nabla\times(l^2\nabla\times\mathbf{w}_t)\cdot\mathbf{w} + l^2|\nabla\times\mathbf{w}|^3] d\mathbf{x}dt \geq 0. \tag{6.19}$$

Proof. Taking the inner product of model (6.6) with $\mathbf{w}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2) + \nu\|\nabla\mathbf{w}\|^2 + \int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}|^3 d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{w}d\mathbf{x}\\ &\leq \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2 + \frac{\nu}{2}\|\nabla\mathbf{w}\|^2.\end{aligned} \tag{6.20}$$

Rearrange the terms in inequality (6.20) to get

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2) + \nu\|\nabla\mathbf{w}\|^2 + 2\int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}|^3 d\mathbf{x} \leq \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2. \tag{6.21}$$

For the term $\nu\|\nabla\mathbf{w}\|^2$, we have

$$\begin{aligned}\nu\|\nabla\mathbf{w}\|^2 &\geq \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2}\|\nabla\mathbf{w}\|^2 = \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2}\|\nabla\times\mathbf{w}\|^2\\ &\geq \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2L_{max}}\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2,\end{aligned} \tag{6.22}$$

where we use the Poincaré inequality, and $L_{max} = \sup_{\mathbf{x}\in\Omega}|l(\mathbf{x})|^2$.

Since $2\int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}|^3 d\mathbf{x} \geq 0$, combining (6.22) and (6.21), we can obtain

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2) + \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2L_{max}}\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2 \leq \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2. \tag{6.23}$$

Let $g(t) := \|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2$, we have

$$g'(t) + \alpha g(t) \leq \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2 \leq \frac{1}{\nu}\|\mathbf{f}\|^2_{L^\infty(0,\infty;H^{-1}(\Omega))} < \infty, \tag{6.24}$$

where $\alpha = \max\{\frac{\nu}{2C_{PF}}, \frac{\nu}{2\beta^2 L_{max}}\} > 0$. By inequality (6.24), we can deduce (6.17). Then integrate (6.20) on $[0, T]$ and divide it by T to get

$$\begin{aligned}&\frac{1}{2T}(\|\mathbf{w}(T)\|^2 + \beta^2\|l\nabla\times\mathbf{w}(T)\|^2) + \frac{1}{T}\int_0^T \nu\|\nabla\mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T\int_\Omega l^2|\nabla\times\mathbf{w}|^3 d\mathbf{x}dt \\ &= \frac{1}{2T}(\|\mathbf{w}(0)\|^2 + \beta^2\|l\nabla\times\mathbf{w}(0)\|^2) + \frac{1}{T}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt.\end{aligned} \tag{6.25}$$

Using (6.17), we have

$$\begin{aligned}&\mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T \nu\|\nabla\mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T\int_\Omega l^2|\nabla\times\mathbf{w}|^3 d\mathbf{x}dt \\ &= \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt,\end{aligned} \tag{6.26}$$

which implies (6.18).

Next, consider the Reynolds stress term. Note that (6.25) can be rewritten as

$$\begin{aligned}&\frac{1}{2T}\|\mathbf{w}(T)\|^2 + \frac{\beta^2}{2T}\int_0^T \frac{d}{dt}\|l\nabla\times\mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T \nu\|\nabla\mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T\int_\Omega l^2|\nabla\times\mathbf{w}|^3 d\mathbf{x}dt \\ &= \frac{1}{2T}\|\mathbf{w}(0)\|^2 + \frac{1}{T}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt.\end{aligned} \tag{6.27}$$

Then by (6.26) and (6.27), we have

$$\begin{aligned}&\frac{\beta^2}{2T}\int_0^T \frac{d}{dt}\|l\nabla\times\mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T\int_\Omega l^2|\nabla\times\mathbf{w}|^3 d\mathbf{x}dt \\ &= \mathcal{O}(\frac{1}{T}) - \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt - \frac{1}{T}\int_0^T \nu\|\nabla\mathbf{w}\|^2 dt \\ &= \mathcal{O}(\frac{1}{T}) - \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T\int_\Omega l^2|\nabla\times\mathbf{w}|^3 d\mathbf{x}dt,\end{aligned} \tag{6.28}$$

which completes the proof of (6.19). □

6.3 THE FINITE ELEMENT APPROXIMATION

In this section, we present the finite element approximation for the corrected EV model (6.6) and analyze its stability and convergence. One deviation from the (now standard) numerical analysis of the NS equations is that the balance between the two model terms is not be accounted for. Alternatively speaking, our model has two new terms not in the NS equations. Their relative size, β, is important in the analysis .

The finite element approximation is : Find $(\mathbf{w}_h, q_h) \in (X_h, Q_h)$ satisfying

$$
\begin{aligned}
&(\mathbf{w}_{h,t}, \mathbf{v}_h) + \beta^2(l^2(\mathbf{x})\nabla \times \mathbf{w}_{h,t}, \nabla \times \mathbf{v}_h) + b(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{w}_h, \nabla \mathbf{v}_h) - (q_h, \nabla \cdot \mathbf{v}_h) \\
&\quad + (l^2(\mathbf{x})|\nabla \times \mathbf{w}_h|\nabla \times \mathbf{w}_h, \nabla \times \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in X_h, \\
&(\nabla \cdot \mathbf{w}_h, r_h) = 0 \qquad \forall r_h \in Q_h.
\end{aligned}
\tag{6.29}
$$

Theorem 13. *Method (6.29) is unconditionally energy stable. For all $0 < t \leq T$,*

$$
\begin{aligned}
&\|\mathbf{w}_h\|^2(t) + \beta^2 \|l(\mathbf{x})\nabla \times \mathbf{w}_h\|^2(t) + 2\int_0^t \int_\Omega |l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}_h|^3 d\mathbf{x} ds + \nu \int_0^t \|\nabla \mathbf{w}_h\|^2 ds \\
&\leq \frac{1}{\nu}\int_0^t \|\mathbf{f}\|_{-1}^2 ds + \|\mathbf{w}_h\|^2(0) + \|l(\mathbf{x})\nabla \times \mathbf{w}_h\|^2(0).
\end{aligned}
\tag{6.30}
$$

Proof. Set $\mathbf{v}_h = \mathbf{w}_h, r_h = q_h$ in (6.29) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{w}_h\|^2 + \frac{\beta^2}{2}\frac{d}{dt}\|l(\mathbf{x})\nabla \times \mathbf{w}_h\|^2 + \int_\Omega l^2(\mathbf{x})|\nabla \times \mathbf{w}_h|^3 d\mathbf{x} + \nu\|\nabla \mathbf{w}_h\|^2 \\
&= (\mathbf{f}, \mathbf{w}_h) \leq \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2 + \frac{\nu}{2}\|\nabla \mathbf{w}_h\|^2.
\end{aligned}
\tag{6.31}
$$

Multiplying (6.31) by 2 and integrating it from 0 to t complete the proof.

□

The next theorem shows the convergence result of method (6.29).

Theorem 14. *Assume* $\mathbf{w}$ *is a strong solution of model* (6.6)*. Suppose the following condition is satisfied:* $\nabla \mathbf{w} \in L^4(0,T;L^2(\Omega))$

$$\begin{aligned}
&\sup_{0\le t\le T}\|\mathbf{w}-\mathbf{w}_h\|^2(t) + \sup_{0\le t\le T}\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{w}_h)\|^2(t) + \nu\|\nabla(\mathbf{w}-\mathbf{w}_h)\|^2_{L^2(0,T;L^2)} \\
&\quad + \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{w}_h)\|^3_{L^3(0,T;L^3)} \\
&\le C[\inf_{\mathbf{v}_h\in\mathbf{X}_h, r_h\in Q_h}\{\sup_{0\le t\le T}\|\mathbf{w}-\mathbf{v}_h\|^2(t) + \sup_{0\le t\le T}\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)\|^2(t) \\
&\quad + \|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)_t\|^2_{L^2(0,T;L^2)} + \|\nabla(\mathbf{w}-\mathbf{v}_h)\|^2_{L^4(0,T;L^2)} + \|(\mathbf{w}-\mathbf{v}_h)_t\|^2_{L^2(0,T;L^2)} \\
&\quad + \|\nabla(\mathbf{w}-\mathbf{v}_h)\|^2_{L^2(0,T;L^2)} + \|q-r_h\|^2_{L^2(0,T;L^2)} \\
&\quad + \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)\|^{\frac{3}{2}}_{L^3(0,T;L^3)} + \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)\|^3_{L^3(0,T;L^3)}\} \\
&\quad + \|\mathbf{w}-\mathbf{w}_h\|^2(0) + \beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{w}_h)\|^2(0)].
\end{aligned} \tag{6.32}$$

Proof. Let $\widetilde{\mathbf{w}} : [0,T] \to V_h$ be arbitrary. Splitting the error $\mathbf{e} = \mathbf{w} - \mathbf{w}_h$ via $\mathbf{e} = \eta - \phi_h$, where $\eta = \mathbf{w} - \widetilde{\mathbf{w}}$, $\phi_h = \mathbf{w}_h - \widetilde{\mathbf{w}}$. Subtracting (6.29) from the weak formulation of (6.6), we obtain

$$\begin{aligned}
&(\mathbf{e}_t,\mathbf{v}_h) + \beta^2(l^2(\mathbf{x})\nabla\times\mathbf{e}_t,\nabla\times\mathbf{v}_h) + b(\mathbf{e},\mathbf{w},\mathbf{v}_h) + b(\mathbf{w}_h,\mathbf{e},\mathbf{v}_h) \\
&\quad + \nu(\nabla\mathbf{e},\nabla\mathbf{v}_h) - (q-q_h,\nabla\cdot\mathbf{v}_h) \\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w} - l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\mathbf{v}_h) \\
&\quad + (l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}} - l^2(\mathbf{x})|\nabla\times\mathbf{w}_h|\nabla\times\mathbf{w}_h,\nabla\times\mathbf{v}_h) \\
&= 0 \qquad \forall \mathbf{v}_h \in X_h, \\
&(\nabla\cdot\mathbf{e}, r_h) = 0 \qquad \forall r_h \in Q_h.
\end{aligned} \tag{6.33}$$

Setting $\mathbf{v}_h = \phi_h$ in (6.33), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \frac{\beta^2}{2}\frac{d}{dt}\|l(\mathbf{x})\nabla\times\phi_h\|^2 + \nu\|\nabla\phi_h\|^2 \\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{w}_h|\nabla\times\mathbf{w}_h - l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h) \\
&= (\eta_t,\phi_h) + (l^2(\mathbf{x})\nabla\times\eta_t,\nabla\times\phi_h) + \nu(\nabla\eta,\nabla\phi_h) - (q-r_h,\nabla\cdot\phi_h) \\
&\quad + b(\eta,\mathbf{w},\phi_h) - b(\phi_h,\mathbf{w},\phi_h) + b(\mathbf{w}_h,\eta,\phi_h) \\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w} - l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h),
\end{aligned} \tag{6.34}$$

where $r_h \in Q_h$ is arbitrary. Next, we need to bound the terms in (6.34). Using (2.1) in Lemma 2, we can obtain

$$(l^2(\mathbf{x})|\nabla \times \mathbf{w}_h|\nabla \times \mathbf{w}_h - l^2(\mathbf{x})|\nabla \times \widetilde{\mathbf{w}}|\nabla \times \widetilde{\mathbf{w}}, \nabla \times \phi_h) \geq \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \phi_h\|_{L^3}^3. \tag{6.35}$$

All the terms on the right hand side of (6.34) are bounded as follows. For the term (η_t, ϕ_h),

$$(\eta_t, \phi_h) \leq C\|\eta_t\|\|\nabla\phi_h\| \leq C\|\eta_t\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{6.36}$$

For the term $(l^2(\mathbf{x})\nabla \times \eta_t, \nabla \times \phi_h)$,

$$\begin{aligned}(l^2(\mathbf{x})\nabla \times \eta_t, \nabla \times \phi_h) &\leq \|l(\mathbf{x})\nabla \times \eta_t\|\|l(\mathbf{x})\nabla \times \phi_h\| \\ &\leq \frac{1}{2}\|l(\mathbf{x})\nabla \times \eta_t\|^2 + \frac{1}{2}\|l(\mathbf{x})\nabla \times \phi_h\|^2.\end{aligned} \tag{6.37}$$

The term $\nu(\nabla\eta, \nabla\phi_h)$ is bounded by

$$\nu(\nabla\eta, \nabla\phi_h) \leq C\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{6.38}$$

The term $-(q - r_h, \nabla \cdot \phi_h)$ is bounded by

$$-(q - r_h, \nabla \cdot \phi_h) \leq C\|q - r_h\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{6.39}$$

Using Lemma 3, we have the following estimate.

$$b(\eta, \mathbf{w}, \phi_h) \leq C\|\nabla\eta\|\|\nabla\mathbf{w}\|\|\nabla\phi_h\| \leq C\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{6.40}$$

$$b(\phi_h, \mathbf{w}, \phi_h) \leq C\|\phi_h\|^{\frac{1}{2}}\|\nabla\mathbf{w}\|\|\nabla\phi_h\|^{\frac{3}{2}} \leq C\|\nabla\mathbf{w}\|^4\|\phi_h\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{6.41}$$

$$\begin{aligned}b(\mathbf{w}_h, \eta, \phi_h) &\leq C\|\mathbf{w}_h\|^{\frac{1}{2}}\|\nabla\mathbf{w}_h\|^{\frac{1}{2}}\|\nabla\eta\|\|\nabla\phi_h\| \\ &\leq C\|\mathbf{w}_h\|\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2 \\ &\leq C\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2,\end{aligned} \tag{6.42}$$

where we use the stability bound $\sup_{0\le t\le T}\|\mathbf{w}_h\| \le C$ in Theorem 13. Lastly, by (2.2) in Lemma 2, we have

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w}-l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h)\\&\le\overline{C}\gamma\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3},\end{aligned}\tag{6.43}$$

where $\gamma=\max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3},\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\widetilde{\mathbf{w}}\|_{L^3}\}$. Since

$$\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\widetilde{\mathbf{w}}\|_{L^3}\le\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3},\tag{6.44}$$

there exists

$$\gamma=\max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3},\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\widetilde{\mathbf{w}}\|_{L^3}\}\le\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}.\tag{6.45}$$

Combining (6.45) with (6.43), we obtain

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w}-l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h)\\&\le\overline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}\\&\quad+\overline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^2\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}\\&\le C\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}^{\frac{3}{2}}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^{\frac{3}{2}}+C\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3+\frac{\underline{C}}{2}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}^3.\end{aligned}\tag{6.46}$$

Combining (6.35)-(6.42), (6.46) with (6.34) gives to

$$\begin{aligned}&\frac{d}{dt}(\|\phi_h\|^2+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2)+\nu\|\nabla\phi_h\|^2+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}^3\\&\le C\|\nabla\mathbf{w}\|^4\|\phi_h\|^2+\|l(\mathbf{x})\nabla\times\phi_h\|^2\\&\quad+\|l(\mathbf{x})\nabla\times\eta_t\|^2+C\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2+C\|\eta_t\|^2+C\|\nabla\eta\|^2+C\|q-r_h\|^2\\&\quad+C\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2+C\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}^{\frac{3}{2}}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^{\frac{3}{2}}+C\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3.\end{aligned}\tag{6.47}$$

Integrate (6.47) from 0 to t to obtain

$$\begin{aligned}&\|\phi_h\|^2(t)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t)+\int_0^t[\nu\|\nabla\phi_h\|^2+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}^3]ds\\&\le\|\phi_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0)\\&\quad+\int_0^t\max\{C\|\nabla\mathbf{w}\|^4,\frac{1}{\beta^2}\}(\|\phi_h\|^2+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2)ds\\&\quad+C\int_0^t[\|l(\mathbf{x})\nabla\times\eta_t\|^2+\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2+\|\eta_t\|^2+\|\nabla\eta\|^2+\|q-r_h\|^2\\&\quad+\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}^{\frac{3}{2}}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^{\frac{3}{2}}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3]ds.\end{aligned}\tag{6.48}$$

Since $\int_0^T \|\nabla \mathbf{w}\|^4 dt \le C$, using Gronwall's inequality, we have

$$\begin{aligned}
&\|\phi_h\|^2(t) + \beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t) + \int_0^t [\nu\|\nabla\phi_h\|^2 + \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}^3]ds \\
&\le C\{\|\phi_h\|^2(0) + \beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0) \\
&\quad + \int_0^t [\|l(\mathbf{x})\nabla\times\eta_t\|^2 + \|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2 + \|\eta_t\|^2 + \|\nabla\eta\|^2 + \|q-r_h\|^2 \\
&\quad + \|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2 + \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}^{\frac{3}{2}}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^{\frac{3}{2}} + \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3]ds\} \\
&\le C\{\|\phi_h\|^2(0) + \beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0) + \int_0^t \|l(\mathbf{x})\nabla\times\eta_t\|^2(s)ds \\
&\quad + (\int_0^t \|\nabla\mathbf{w}\|^4(s)ds)^{\frac{1}{2}}(\int_0^t \|\nabla\eta\|^4(s)ds)^{\frac{1}{2}} + \int_0^t \|\eta_t\|^2(s)ds + \int_0^t \|\nabla\eta\|^2(s)ds \\
&\quad + \int_0^t \|q-r_h\|^2(s)ds + (\int_0^t \|\nabla\mathbf{w}_h\|^2(s)ds)^{\frac{1}{2}}(\int_0^t \|\nabla\eta\|^4(s)ds)^{\frac{1}{2}} \\
&\quad + (\int_0^t \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|_{L^3}^3(s)ds)^{\frac{1}{2}}(\int_0^t \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3(s)ds)^{\frac{1}{2}} + \int_0^t \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3(s)ds\}.
\end{aligned} \tag{6.49}$$

With the stability bound $(\int_0^T \|\nabla\mathbf{w}_h\|^2(s)ds)^{\frac{1}{2}} \le C$, we can deduce

$$\begin{aligned}
&\|\phi_h\|^2(t) + \beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t) + \nu\int_0^t \|\nabla\phi_h\|^2(s)ds + \underline{C}\int_0^t \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3}^3(s)ds \\
&\le C\{\|\phi_h\|^2(0) + \beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0) + \int_0^t \|l(\mathbf{x})\nabla\times\eta_t\|^2(s)ds \\
&\quad + (\int_0^t \|\nabla\eta\|^4(s)ds)^{\frac{1}{2}} + \int_0^t \|\eta_t\|^2(s)ds + \int_0^t \|\nabla\eta\|^2(s)ds + \int_0^t \|q-r_h\|^2(s)ds \\
&\quad + (\int_0^t \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3(s)ds)^{\frac{1}{2}} + \int_0^t \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3}^3(s)ds\}.
\end{aligned} \tag{6.50}$$

Then we have

$$\begin{aligned}
&\sup_{0\le t\le T}\|\phi_h\|^2(t) + \sup_{0\le t\le T}\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t) + \nu\|\nabla\phi_h\|_{L^2(0,T;L^2)}^2 + \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|_{L^3(0,T;L^3)}^3 \\
&\le C\{\|\phi_h\|^2(0) + \beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0) + \|l(\mathbf{x})\nabla\times\eta_t\|_{L^2(0,T;L^2)}^2 \\
&\quad + \|\nabla\eta\|_{L^4(0,T;L^2)}^2 + \|\eta_t\|_{L^2(0,T;L^2)}^2 + \|\nabla\eta\|_{L^2(0,T;L^2)}^2 \\
&\quad + \|q-r_h\|_{L^2(0,T;L^2)}^2 + \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3(0,T;L^3)}^{\frac{3}{2}} + \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|_{L^3(0,T;L^3)}^3\}.
\end{aligned} \tag{6.51}$$

Adding (6.51) to

$$\sup_{0\leq t\leq T}\|\eta\|^2(t)+\sup_{0\leq t\leq T}\beta^2\|l(\mathbf{x})\nabla\times\eta\|^2(t)+\nu\|\nabla\eta\|^2_{L^2(0,T;L^2)}+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3(0,T;L^3)} \tag{6.52}$$

and then using the triangle inequality complete the proof.

□

6.4 TIME DISCRETIZATION

In this section, we discuss the time discretization scheme for the corrected EV model (6.6). Divide the time interval $[0,T]$ to m elements (t^n,t^{n+1}). Here, $\Delta t=\frac{T}{m}$, $t^n=n\Delta t$ for $n=0,1,2,\cdots,m$. We denote $\mathbf{w}^n=\mathbf{w}(t^n)$ and similarly for other variables.

Algorithm 10 (First order BE scheme)**.** *Given* $(\mathbf{w}^n,q^n)$*, find* $(\mathbf{w}^{n+1},q^{n+1})$ *satisfying*

$$\begin{aligned}
&(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\mathbf{v})+\beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\nabla\times\mathbf{v})+b(\mathbf{w}^{n+1},\mathbf{w}^{n+1},\mathbf{v})+\nu(\nabla\mathbf{w}^{n+1},\nabla\mathbf{v})\\
&\quad+(l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+1}|\nabla\times\mathbf{w}^{n+1},\nabla\times\mathbf{v})-(q^{n+1},\nabla\cdot\mathbf{v})=(\mathbf{f}^{n+1},\mathbf{v})\qquad\forall\mathbf{v}\in X,\\
&(\nabla\cdot\mathbf{w}^{n+1},r)=0\qquad\forall r\in Q.
\end{aligned} \tag{6.53}$$

An á priori bound on w^{n+1} is proven in the next theorem. The system (5.1) reduces to a finite dimensional nonlinear system with an á priori bound on any possible solution. Thus, existence of w^{n+1} then follows by a standard fixed point argument similar to the nonlinear system arising in the space and time discretized NSE case. When discretized in time but not in space, existence of w^{n+1} can also be proven by monotonicity techniques.

Theorem 15. *Algorithm 10 is unconditionally stable.*

$$\begin{aligned}
&\|\mathbf{w}^m\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{w}^{n+1}-\mathbf{w}^n\|^2+\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}^{n+1}-\mathbf{w}^n)\|^2\\
&\quad+2\Delta t\sum_{n=0}^{m-1}\int_\Omega|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}^{n+1}|^3d\mathbf{x}+\nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+1}\|^2\\
&\leq\frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|^2_{-1}+\|\mathbf{w}^0\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned} \tag{6.54}$$

Proof. Set $\mathbf{v} = \mathbf{w}^{n+1}, r = q^{n+1}$ in (10) to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{w}^{n+1}\|^2 - \|\mathbf{w}^n\|^2 + \|\mathbf{w}^{n+1} - \mathbf{w}^n\|^2) \\
&\quad + \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla \times \mathbf{w}^{n+1}\|^2 - \|l(\mathbf{x})\nabla \times \mathbf{w}^n\|^2 + \|l(\mathbf{x})\nabla \times \mathbf{w}^{n+1} - l(\mathbf{x})\nabla \times \mathbf{w}^n\|^2) \\
&\quad + \int_\Omega l^2(\mathbf{x})|\nabla \times \mathbf{w}^{n+1}|^3 d\mathbf{x} + \nu\|\nabla \mathbf{w}^{n+1}\|^2 \qquad (6.55) \\
&= (\mathbf{f}^{n+1}, \mathbf{w}^{n+1}) \\
&\le \frac{1}{2\nu}\|\mathbf{f}^{n+1}\|_{-1}^2 + \frac{\nu}{2}\|\nabla \mathbf{w}^{n+1}\|^2.
\end{aligned}$$

Multiplying (6.55) by $2\Delta t$ and summing it from $n = 0$ to $n = m - 1$ complete the proof.

□

Theorem 16. *Assume* $\mathbf{w}$ *is a strong solution of the model. Assume that the true solution* $\mathbf{w}$ *satisfies the following regularity.*

$$\nabla \mathbf{w} \in L^\infty(0, T; L^2), \quad \mathbf{w}_{tt} \in L^2(0, T; L^2), \quad l(\mathbf{x})\nabla \times \mathbf{w}_{tt} \in L^2(0, T; L^2). \qquad (6.56)$$

Then,

$$\|\mathbf{e}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^m\|^2 + \Delta t \sum_{n=0}^{m-1} \nu\|\nabla \mathbf{e}^{n+1}\|^2 + 2\Delta t \sum_{n=0}^{m-1} \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|_{L^3}^3 \le C\Delta t^2. \qquad (6.57)$$

Proof. At time t^{n+1}, the true solution $(\mathbf{w}, q)$ satisfies

$$\begin{aligned}
&(\frac{\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n)}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla \times \frac{\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n)}{\Delta t}, \nabla \times \mathbf{v}) + b(\mathbf{w}(t^{n+1}), \mathbf{w}(t^{n+1}), \mathbf{v}) \\
&\quad + (l^2(\mathbf{x})|\nabla \times \mathbf{w}(t^{n+1})|\nabla \times \mathbf{w}(t^{n+1}), \nabla \times \mathbf{v}) + \nu(\nabla \mathbf{w}(t^{n+1}), \nabla \mathbf{v}) - (p(t^{n+1}), \nabla \cdot \mathbf{v}) \\
&= (\mathbf{f}^{n+1}, \mathbf{v}) + (R^{n+1}, \mathbf{v}) + (l^2(\mathbf{x})\nabla \times R^{n+1}, \nabla \times \mathbf{v}) \qquad \forall \mathbf{v} \in X, \\
&(\nabla \cdot \mathbf{w}(t^{n+1}), r) = 0 \qquad \forall r \in Q,
\end{aligned} \qquad (6.58)$$

where $R^{n+1} = \frac{\mathbf{w}(t^{n+1})-\mathbf{w}(t^n)}{\Delta t} - \mathbf{w}_t(t^{n+1})$. Denote $\mathbf{e}^{n+1} = \mathbf{w}(t^{n+1}) - \mathbf{w}^{n+1}$. Subtracting (6.58) from (10), we have

$$
\begin{aligned}
&(\frac{\mathbf{e}^{n+1}-\mathbf{e}^n}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{e}^{n+1}-\mathbf{e}^n}{\Delta t}, \nabla\times\mathbf{v}) + b(\mathbf{e}^{n+1}, \mathbf{w}(t^{n+1}), \mathbf{v}) + b(\mathbf{w}^{n+1}, \mathbf{e}^{n+1}, \mathbf{v})\\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{w}(t^{n+1})|\nabla\times\mathbf{w}(t^{n+1}) - l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+1}|\nabla\times\mathbf{w}^{n+1}, \nabla\times\mathbf{v})\\
&\quad + \nu(\nabla\mathbf{e}^{n+1}, \nabla\mathbf{v}) - (p(t^{n+1}) - q^{n+1}, \nabla\cdot\mathbf{v})\\
&= (R^{n+1}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla\times R^{n+1}, \nabla\times\mathbf{v}) \qquad \forall\mathbf{v}\in X,\\
&(\nabla\cdot\mathbf{e}^{n+1}, r) = 0 \qquad \forall r\in Q.
\end{aligned}
\tag{6.59}
$$

Setting $\mathbf{v} = \mathbf{e}^{n+1}$ in (6.59), we can obtain

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n+1}-\mathbf{e}^n\|^2) + \nu\|\nabla\mathbf{e}^{n+1}\|^2\\
&\quad + \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^2 - \|l(\mathbf{x})\nabla\times\mathbf{e}^n\|^2 + \|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1} - l(\mathbf{x})\nabla\times\mathbf{e}^n\|^2)\\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{w}(t^{n+1})|\nabla\times\mathbf{w}(t^{n+1}) - l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+1}|\nabla\times\mathbf{w}^{n+1}, \nabla\times\mathbf{e}^{n+1})\\
&= (R^{n+1}, \mathbf{e}^{n+1}) + \beta^2(l^2(\mathbf{x})\nabla\times R^{n+1}, \nabla\times\mathbf{e}^{n+1}) - b(\mathbf{e}^{n+1}, \mathbf{w}(t^{n+1}), \mathbf{e}^{n+1}).
\end{aligned}
\tag{6.60}
$$

Next, we will bound the terms in (6.60). Using (2.1) in Lemma 2, we can obtain

$$
\begin{aligned}
&(l^2(\mathbf{x})|\nabla\times\mathbf{w}(t^{n+1})|\nabla\times\mathbf{w}(t^{n+1}) - l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+1}|\nabla\times\mathbf{w}^{n+1}, \nabla\times\mathbf{e}^{n+1})\\
&\geq \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^3_{L^3}.
\end{aligned}
\tag{6.61}
$$

For the term $(R^{n+1}, \mathbf{e}^{n+1})$,

$$
(R^{n+1}, \mathbf{e}^{n+1}) \leq C\|R^{n+1}\|\|\nabla\mathbf{e}^{n+1}\| \leq C\|R^{n+1}\|^2 + \frac{\nu}{4}\|\nabla\mathbf{e}^{n+1}\|^2. \tag{6.62}
$$

For the term $(l^2(\mathbf{x})\nabla\times R^{n+1}, \nabla\times\mathbf{e}^{n+1})$,

$$
\begin{aligned}
\beta^2(l^2(\mathbf{x})\nabla\times R^{n+1}, \nabla\times\mathbf{e}^{n+1}) &\leq \beta^2\|l(\mathbf{x})\nabla\times R^{n+1}\|\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|\\
&\leq \frac{\beta^2}{2}\|l(\mathbf{x})\nabla\times R^{n+1}\|^2 + \frac{\beta^2}{2}\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^2.
\end{aligned}
\tag{6.63}
$$

Lastly, the term $-b(\mathbf{e}^{n+1}, \mathbf{w}(t^{n+1}), \mathbf{e}^{n+1})$ is bounded by

$$
\begin{aligned}
-b(\mathbf{e}^{n+1}, \mathbf{w}(t^{n+1}), \mathbf{e}^{n+1}) &\leq C\|\mathbf{e}^{n+1}\|^{\frac{1}{2}}\|\nabla\mathbf{w}(t^{n+1})\|\|\nabla\mathbf{e}^{n+1}\|^{\frac{3}{2}}\\
&\leq C\|\mathbf{e}^{n+1}\|^2\|\nabla\mathbf{w}(t^{n+1})\|^4 + \frac{\nu}{4}\|\nabla\mathbf{e}^{n+1}\|^2.
\end{aligned}
\tag{6.64}
$$

Combining (6.61)-(6.64) with (6.60), we have

$$\frac{1}{2\Delta t}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|^2) + \frac{\nu}{2}\|\nabla \mathbf{e}^{n+1}\|^2 + \underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|_{L^3}^3$$
$$+ \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|^2 - \|l(\mathbf{x})\nabla \times \mathbf{e}^n\|^2 + \|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1} - l(\mathbf{x})\nabla \times \mathbf{e}^n\|^2)$$
$$\leq C\|\nabla \mathbf{w}(t^{n+1})\|^4\|\mathbf{e}^{n+1}\|^2 + \frac{\beta^2}{2}\|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|^2 + C\|R^{n+1}\|^2 + \frac{\beta^2}{2}\|l(\mathbf{x})\nabla \times R^{n+1}\|^2. \tag{6.65}$$

Multiplying (6.65) by $2\Delta t$ and summing it from $n = 0$ to $n = m - 1$, we obtain

$$\begin{aligned}
&\|\mathbf{e}^m\|^2 + \sum_{n=0}^{m-1}\|\mathbf{e}^{n+1} - \mathbf{e}^n\|^2 + \Delta t\sum_{n=0}^{m-1}\nu\|\nabla \mathbf{e}^{n+1}\|^2 + 2\Delta t\sum_{n=0}^{m-1}\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|_{L^3}^3\\
&\quad + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^m\|^2 + \Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1} - l(\mathbf{x})\nabla \times \mathbf{e}^n\|^2\\
&\leq \|\mathbf{e}^0\|^2 + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^0\|^2 + C\Delta t\sum_{n=0}^{m-1}\|\nabla \mathbf{w}(t^{n+1})\|^4\|\mathbf{e}^{n+1}\|^2\\
&\quad + \Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|^2 + C\Delta t\sum_{n=0}^{m-1}\|R^{n+1}\|^2 + \Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla \times R^{n+1}\|^2.
\end{aligned} \tag{6.66}$$

Since $\|\nabla \mathbf{w}\|_{L^\infty(0,T;L^2)} \leq C$, using Lemma 1, when $\Delta t < \frac{1}{\max\{C\|\nabla \mathbf{w}\|_{L^\infty(0,T;L^2)},1\}}$, we can obtain

$$\begin{aligned}
&\|\mathbf{e}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^m\|^2 + \Delta t\sum_{n=0}^{m-1}\nu\|\nabla \mathbf{e}^{n+1}\|^2 + 2\Delta t\sum_{n=0}^{m-1}\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|_{L^3}^3\\
&\leq C[\|\mathbf{e}^0\|^2 + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^0\|^2 + \Delta t\sum_{n=0}^{m-1}\|R^{n+1}\|^2 + \Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla \times R^{n+1}\|^2].
\end{aligned} \tag{6.67}$$

We also have

$$\begin{aligned}
\Delta t\sum_{n=0}^{m-1}\|R^{n+1}\|^2 &= \Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n)}{\Delta t} - \mathbf{w}_t(t^{n+1})\|^2\\
&\leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{w}_{tt}\|^2 ds = C\Delta t^2\|\mathbf{w}_{tt}\|_{L^2(0,T;L^2)},
\end{aligned} \tag{6.68}$$

and

$$
\begin{aligned}
\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times R^{n+1}\|^2 &= \Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times(\frac{\mathbf{w}(t^{n+1})-\mathbf{w}(t^n)}{\Delta t}-\mathbf{w}_t(t^{n+1}))\|^2 \\
&\leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}_{tt}\|^2 ds \\
&= C\Delta t^2\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}_{tt}\|_{L^2(0,T;L^2)}.
\end{aligned}
\tag{6.69}
$$

Finally, combining (6.68), (6.69), and (6.67) completes the proof.

□

Algorithm 11 (First order linearly implicit, Backward-Euler scheme)**.** *Given* $(\mathbf{w}^n, q^n)$*, find* $(\mathbf{w}^{n+1}, q^{n+1})$ *satisfying*

$$
\begin{aligned}
&(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\mathbf{v})+\beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\nabla\times\mathbf{v})+b(\mathbf{w}^n,\mathbf{w}^{n+1},\mathbf{v})+\nu(\nabla\mathbf{w}^{n+1},\nabla\mathbf{v}) \\
&\quad+(l^2(\mathbf{x})|\nabla\times\mathbf{w}^n|\nabla\times\mathbf{w}^{n+1},\nabla\times\mathbf{v})-(q^{n+1},\nabla\cdot\mathbf{v})=(\mathbf{f}^{n+1},\mathbf{v}) \qquad \forall\mathbf{v}\in X, \\
&(\nabla\cdot\mathbf{w}^{n+1},r)=0 \qquad \forall r\in Q.
\end{aligned}
\tag{6.70}
$$

Algorithm 11 leads to a linear problem for a continuous and coercive operator for w^{n+1}. Existence of w^{n+1} follows from the Lax-Milgram lemma.

Theorem 17. *Algorithm 11 is unconditionally stable.*

$$
\begin{aligned}
&\|\mathbf{w}^m\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{w}^{n+1}-\mathbf{w}^n\|^2+\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}^{n+1}-\mathbf{w}^n)\|^2 \\
&\quad+2\Delta t\sum_{n=0}^{m-1}\int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}^n|\cdot|\nabla\times\mathbf{w}^{n+1}|^2 d\mathbf{x}+\nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+1}\|^2 \\
&\leq\frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2+\|\mathbf{w}^0\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned}
\tag{6.71}
$$

Remark 5. *The proof of Theorem 17 is similar to that of Theorem 15. We omit it here.*

Algorithm 12 (Second order Crank-Nicolson scheme)**.** *Given* $(\mathbf{w}^n, q^n)$*, find* $(\mathbf{w}^{n+1}, q^{n+1})$ *satisfying*

$$
\begin{aligned}
&(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}, \nabla\times\mathbf{v}) + b(\mathbf{w}^{n+\frac{1}{2}}, \mathbf{w}^{n+\frac{1}{2}}, \mathbf{v}) + \nu(\nabla\mathbf{w}^{n+\frac{1}{2}}, \nabla\mathbf{v})\\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+\frac{1}{2}}|\nabla\times\mathbf{w}^{n+\frac{1}{2}}, \nabla\times\mathbf{v}) - (q^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) = (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}) \qquad \forall\mathbf{v}\in X,\\
&(\nabla\cdot\mathbf{w}^{n+1}, r) = 0 \qquad \forall r\in Q,
\end{aligned}
\tag{6.72}
$$

where we denote $\mathbf{w}^{n+\frac{1}{2}} = \frac{\mathbf{w}^n+\mathbf{w}^{n+1}}{2}$, and similarly for other variables.

Equations (6.72) in Algorithm 12 leads to a nonlinear system of equations for w^{n+1}. These equations can be rearranged into a nonlinear system for $w^{n+1/2}$ with the same structure as the nonlinear system occurring in Algorithm 10. By the same argument existence for $w^{n+1/2}$ follows. From this, by subtracting $w^n/2$, existence follows for w^{n+1}.

Theorem 18. *Algorithm 12 is unconditionally stable.*

$$
\begin{aligned}
&\|\mathbf{w}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2 + 2\Delta t\sum_{n=0}^{m-1}\int_\Omega |l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}^{n+\frac{1}{2}}|^3 d\mathbf{x} + \nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2\\
&\le \frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+\frac{1}{2}}\|_{-1}^2 + \|\mathbf{w}^0\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned}
\tag{6.73}
$$

Proof. Set $\mathbf{v} = \mathbf{w}^{n+\frac{1}{2}}$ in (6.72) to obtain

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{w}^{n+1}\|^2 - \|\mathbf{w}^n\|^2) + \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla\times\mathbf{w}^{n+1}\|^2 - \|l(\mathbf{x})\nabla\times\mathbf{w}^n\|^2)\\
&\quad + \int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+\frac{1}{2}}|^3 d\mathbf{x} + \nu\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2\\
&= (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{w}^{n+\frac{1}{2}}) \le \frac{1}{2\nu}\|\mathbf{f}^{n+\frac{1}{2}}\|_{-1}^2 + \frac{\nu}{2}\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2.
\end{aligned}
\tag{6.74}
$$

Multiplying (6.74) by $2\Delta t$ and summing it from $n=0$ to $n=m-1$ complete the proof.

□

Algorithm 13 (Second order Crank-Nicolson linearly extrapolation scheme)**.** *Given* $\mathbf{w}^{n-1}, \mathbf{w}^n, q^n$*, find* $(\mathbf{w}^{n+1}, q^{n+1})$ *satisfying*

$$
\begin{aligned}
&(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}, \nabla\times\mathbf{v}) + b(\varphi(\mathbf{w}^n), \mathbf{w}^{n+\frac{1}{2}}, \mathbf{v}) + \nu(\nabla\mathbf{w}^{n+\frac{1}{2}}, \nabla\mathbf{v})\\
&\quad + (l^2(\mathbf{x})|\nabla\times\varphi(\mathbf{w}^n)|\nabla\times\mathbf{w}^{n+\frac{1}{2}}, \nabla\times\mathbf{v}) - (q^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) = (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}) \qquad \forall\mathbf{v}\in X,\\
&(\nabla\cdot\mathbf{w}^{n+1}, r) = 0 \qquad \forall r\in Q,
\end{aligned}
\tag{6.75}
$$

where $\varphi(\mathbf{w}^n) = \frac{3}{2}\mathbf{w}^n - \frac{1}{2}\mathbf{w}^{n-1}$.

Algorithm 13 is linearly implicit. It leads to a system for w^{n+1} that is continuous and coercive. Existence of w^{n+1} then follows, as for the linearly implicit BE method, by the Lax-Milgram lemma.

Theorem 19. *Algorithm 13 is unconditionally stable.*

$$
\begin{aligned}
&\|\mathbf{w}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2 + 2\Delta t\sum_{n=0}^{m-1}\int_\Omega l^2(\mathbf{x})|\nabla\times\varphi(\mathbf{w}^n)||\nabla\times\mathbf{w}^{n+\frac{1}{2}}|^2 d\mathbf{x}\\
&\quad + \nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2\\
&\leq \frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+\frac{1}{2}}\|_{-1}^2 + \|\mathbf{w}^0\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned}
\tag{6.76}
$$

Remark 6. *The proof of Theorem* 19 *is similar to that of Theorem* 18*. We just omit it here.*

6.5 NUMERICAL TEST

Since we study the simplest realization of an algebraic model, our goal is simply to test if the model correction can exhibit backscatter in the form of a negative total dissipation at some times. We select a 2d test without a global rotational flow (due to the choice of the Baldwin-Lomax as the starting point). We use the first order scheme Algorithm 11 and the second order scheme Algorithm 13. For both methods, we compute the evolution of four components of dissipation in their approximation to the effect of the Reynolds stress on the mean kinetic energy. The P2-P1 Taylor-Hood mixed finite elements are utilized for discretization in space. FreeFem++ [142] is used for simulation.

Choose a rectangle domain $\Omega = [0,4] \times [0,1]$ with a square obstacle $\Omega = [0.5, 0.6] \times [0.45, 0.55]$ inside it. We compute the problem on a Delaunay-Vornoi generated triangular mesh with more mesh points near the obstacle area and less in other area, shown in Figure 20. The flow passes through this domain from left to right. For boundary conditions on the inflow boundary, we let $\mathbf{u}|_{x-\text{direction}} = 4y(1-y)$, $\mathbf{u}|_{y-\text{direction}} = 0$. On the right, out-flow, boundary, we impose the "do-nothing" outflow boundary condition, [126] p. 21 eqn. (2.37) and [111] p. 475. The no-slip condition $\mathbf{u} = 0$ is imposed on other boundaries. We take $\mathbf{f} = 0$, $T = 20$, $\Delta t = 0.01$, and $Re = 10,000$. Let $\bar{y}$ denote the distance of $\mathbf{x}$ to the nearest wall. The mixing length is chosen ([119] Chapter 3 e.g. eqn. (3.99) p. 76) to be

$$l(x) = \begin{cases} 0.41 \cdot \bar{y} & \text{when } 0 < \bar{y} < 0.2 \cdot Re^{-\frac{1}{2}} \\ 0.41 \cdot 0.2 \cdot Re^{-\frac{1}{2}} & \text{otherwise} \end{cases}$$

The degree of freedom is 17625, the shortest triangle edge is 0.0102605, and the longest is 0.097187. An alternatives is to pick $l(\mathbf{x}) = h$, the local meshwidth, in the spirit of large eddy simulation.

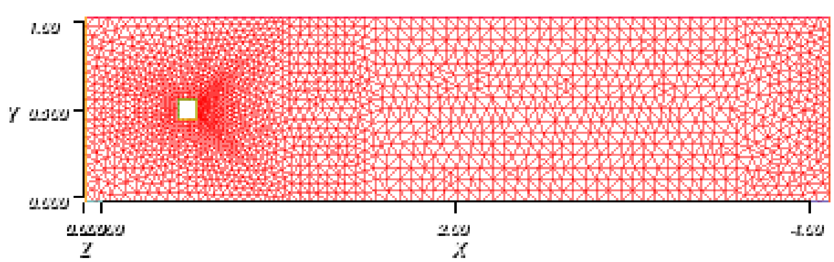


Figure 20: Mesh used in our computation.

For Algorithm 9, [130], since Backward-Euler scheme has substantial numerical dissipation, we select $\beta = 10$ and 100 to compute the following quantities, shown in Figure 21

($\beta = 10$), and Figure 22 ($\beta = 100$).

$$\text{Backscatter term (BST)} = \beta^2 \int_\Omega l^2(\mathbf{x}) \nabla \times \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} \cdot \nabla \times \mathbf{w}^{n+1} d\mathbf{x}$$

$$\text{Numerical dissipation (ND)} = \frac{\beta^2}{2\Delta t} \| l(\mathbf{x}) \nabla \times \mathbf{w}^{n+1} - l(\mathbf{x}) \nabla \times \mathbf{w}^n \|^2,$$

$$\text{Fluctuation dissipation (FD)} = \int_\Omega l^2(\mathbf{x}) |\nabla \times \mathbf{w}^n| |\nabla \times \mathbf{w}^{n+1}|^2 d\mathbf{x},$$

$$\text{Total Dissipation (TD)} = \int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} \cdot \nabla \times \mathbf{w}^{n+1} + l^2(\mathbf{x}) |\nabla \times \mathbf{w}^n| |\nabla \times \mathbf{w}^{n+1}|^2) d\mathbf{x}.$$

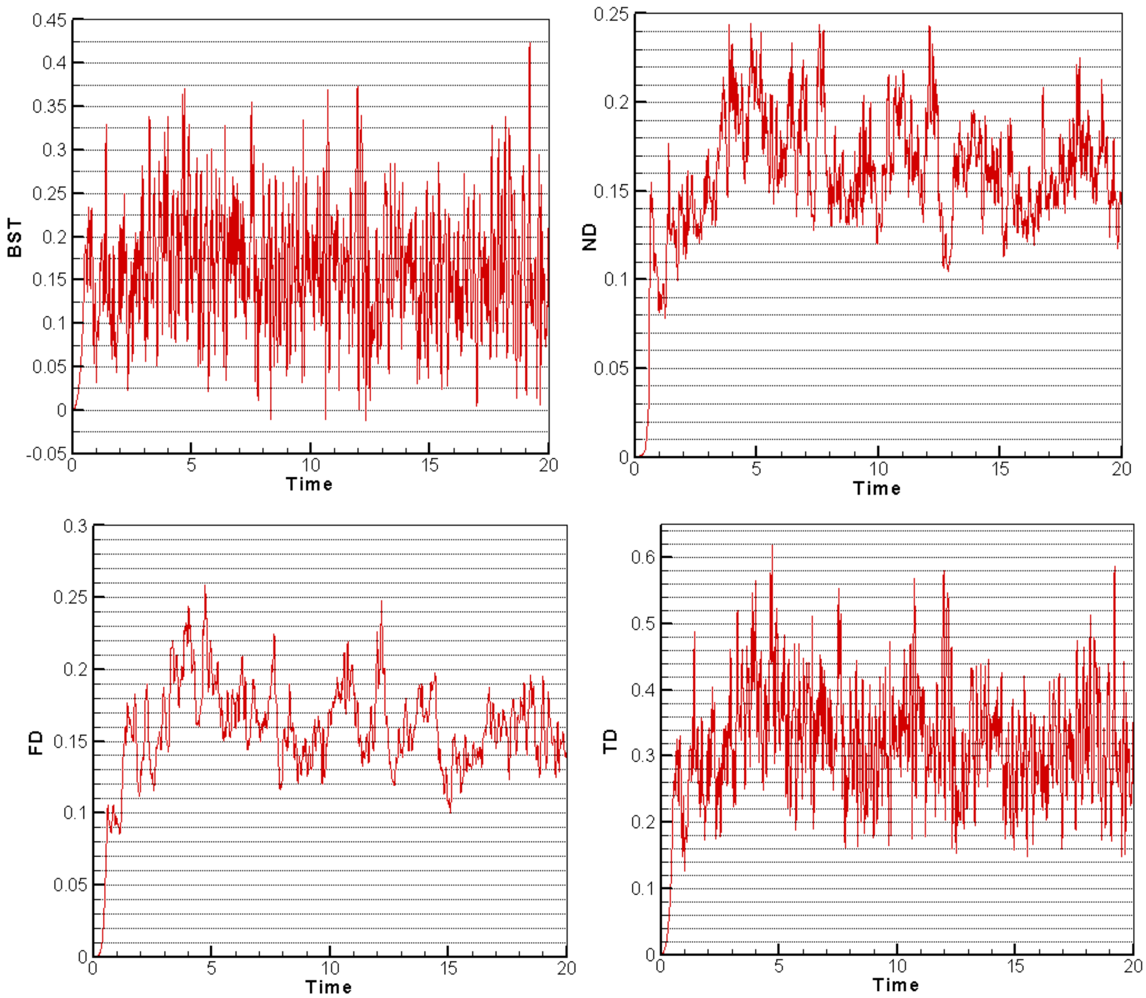


Figure 21: BST,ND,FD,TD vs Time, $\beta = 10$, Algorithm 11.

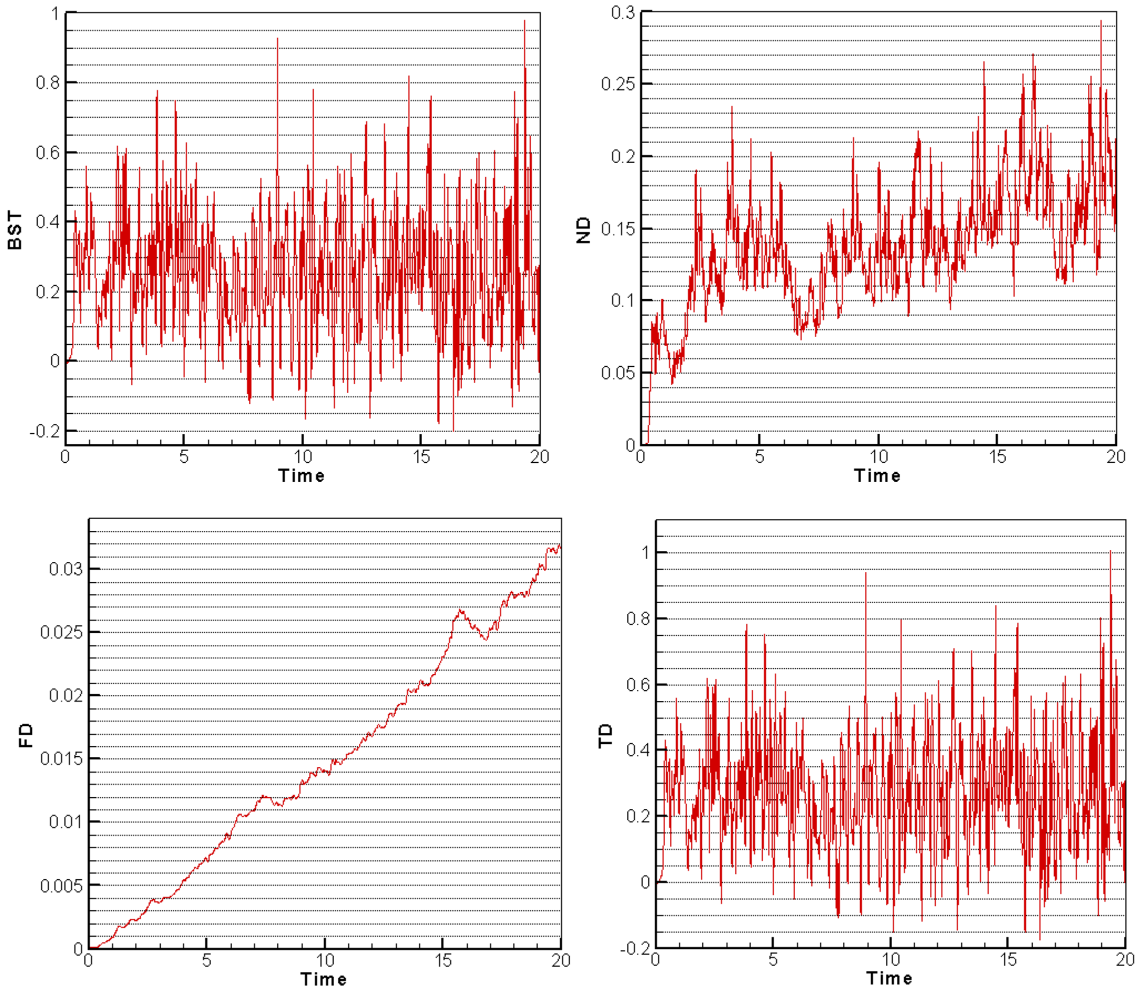


Figure 22: BST,ND,FD,TD vs Time, $\beta = 100$, Algorithm 11.

For Algorithm 13,[130] [143], we compute the following quantities, shown in Figure 23 ($\beta = 10$).

$$\begin{aligned}
\text{Backscatter term (BST)} &= \beta^2 \int_\Omega l^2(\mathbf{x}) \nabla \times \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} \cdot \nabla \times \mathbf{w}^{n+\frac{1}{2}} d\mathbf{x}, \\
\text{Fluctuation dissipation (FD)} &= \int_\Omega l^2(\mathbf{x}) |\nabla \times \varphi(\mathbf{w}^n)| |\nabla \times \mathbf{w}^{n+\frac{1}{2}}|^2 d\mathbf{x}, \\
\text{Total Dissipation (TD)} &= \int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} \cdot \nabla \times \mathbf{w}^{n+\frac{1}{2}} \\
&\quad + l^2(\mathbf{x}) |\nabla \times \varphi(\mathbf{w}^n)| |\nabla \times \mathbf{w}^{n+\frac{1}{2}}|^2) d\mathbf{x},
\end{aligned}$$

where $\varphi(\mathbf{w}^n) = \frac{3}{2}\mathbf{w}^n - \frac{1}{2}\mathbf{w}^{n-1}$ and $\mathbf{w}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{w}^{n+1} + \mathbf{w}^n)$.

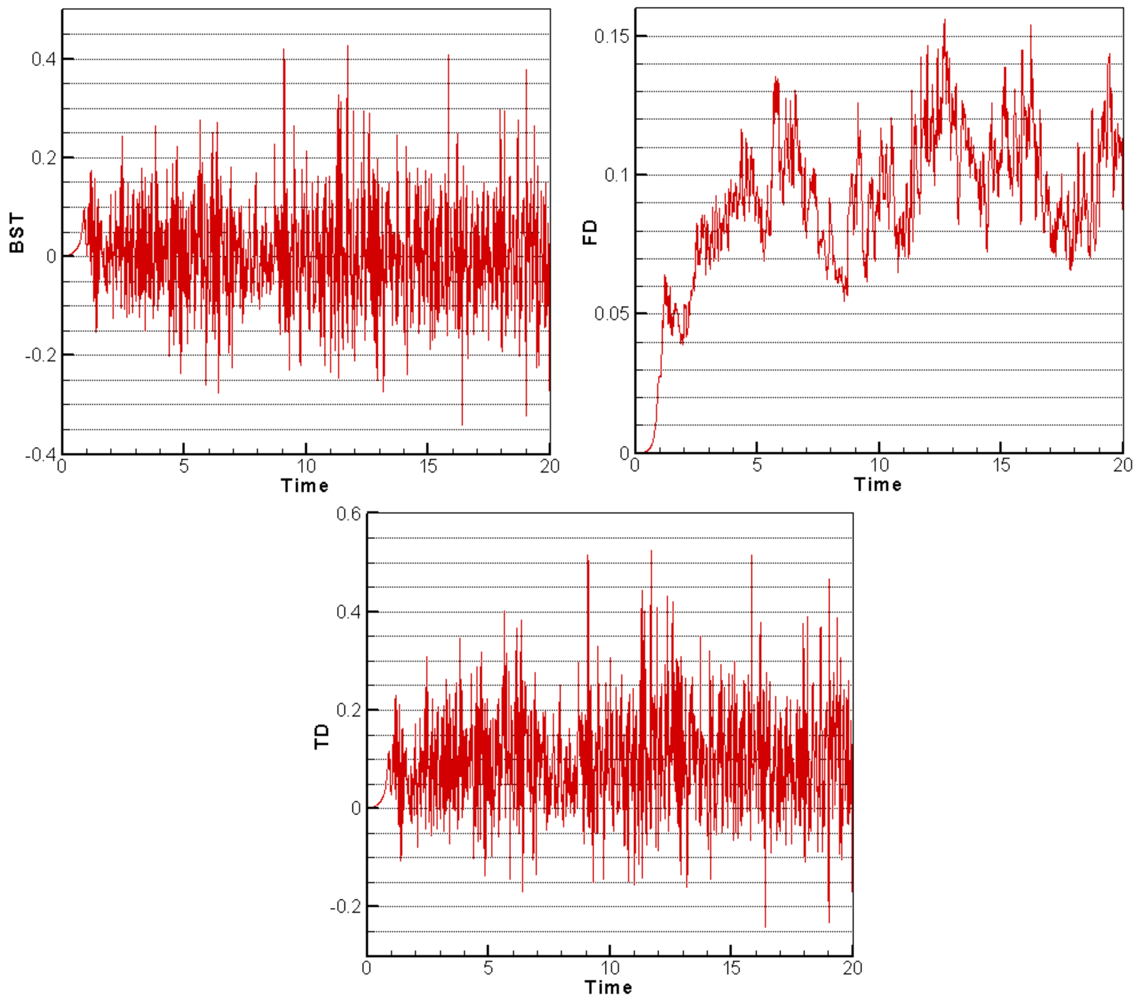


Figure 23: BST,FD,TD vs Time, $\beta = 10$, Algorithm 13.

In both Figure 22 and Figure 23, TD is negative at some times, indicating that backscatter occurs and energy is transferred from fluctuation to the mean at that moment. However in Figure 21, there is no appearance of negative values in TD. The results show that the numerical dissipation in Algorithm 11 (the backward Euler time discretization) is compared to the model dissipation and, for $\beta = 10$, dominates the term modeling the flow of energy from fluctuations back to the mean. Algorithm 13 does not contain any numerical dissipation. For Algorithm 13, $\beta = 10$, the effects of the added term does provide bursts of energy to the mean flow.

In order to test the convergence, we reduce both mesh size and time step, then recompute the backscatter term and dissipation terms. Since Algorithm 13 does not contain any numerical dissipation, we use Algorithm 13 to do this test.

Firstly, we double the mesh points of all edges. The degree of freedom is 69657, the shortest triangle edge is 0.00503187, and the longest is 0.0495673. We also halve the time step and take $\Delta t = 0.005$. Figure 24 shows the result.

We furthermore double the mesh points of all edges. The degree of freedom is 276438, the shortest triangle edge is 0.0024586, and the longest is 0.0270444. The time step $\Delta t = 0.0025$. Figure 25 shows the result.

Both Figure 24 and Figure 25 indicate the occurrence of backscatter at the same moments on successively refined meshes.

6.6 CONCLUSIONS

Complex turbulence not at statistical equilibrium is impossible to simulate using eddy viscosity models due to a backscatter. This research presents the way to correct the Baldwin-Lomax model for non-equilibrium effects and gives an analysis of the energy evolution in the corrected model. Furthermore, a finite element approximation of the corrected eddy model with first-order and second-order time discretization are also presented. A numerical test is given to support the theory.

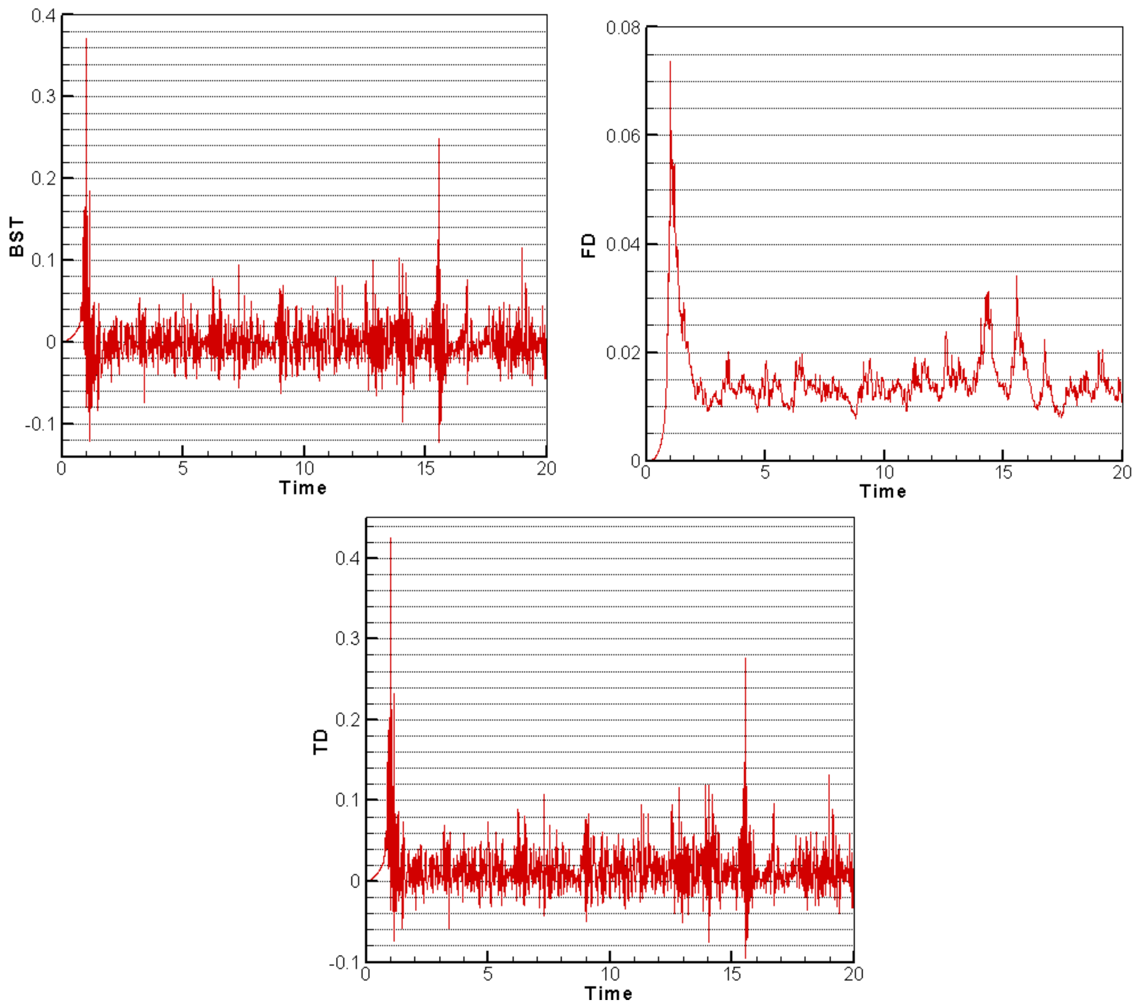


Figure 24: BST,FD,TD vs Time, $\beta = 10$, $\Delta t = 0.005$,Algorithm 13.

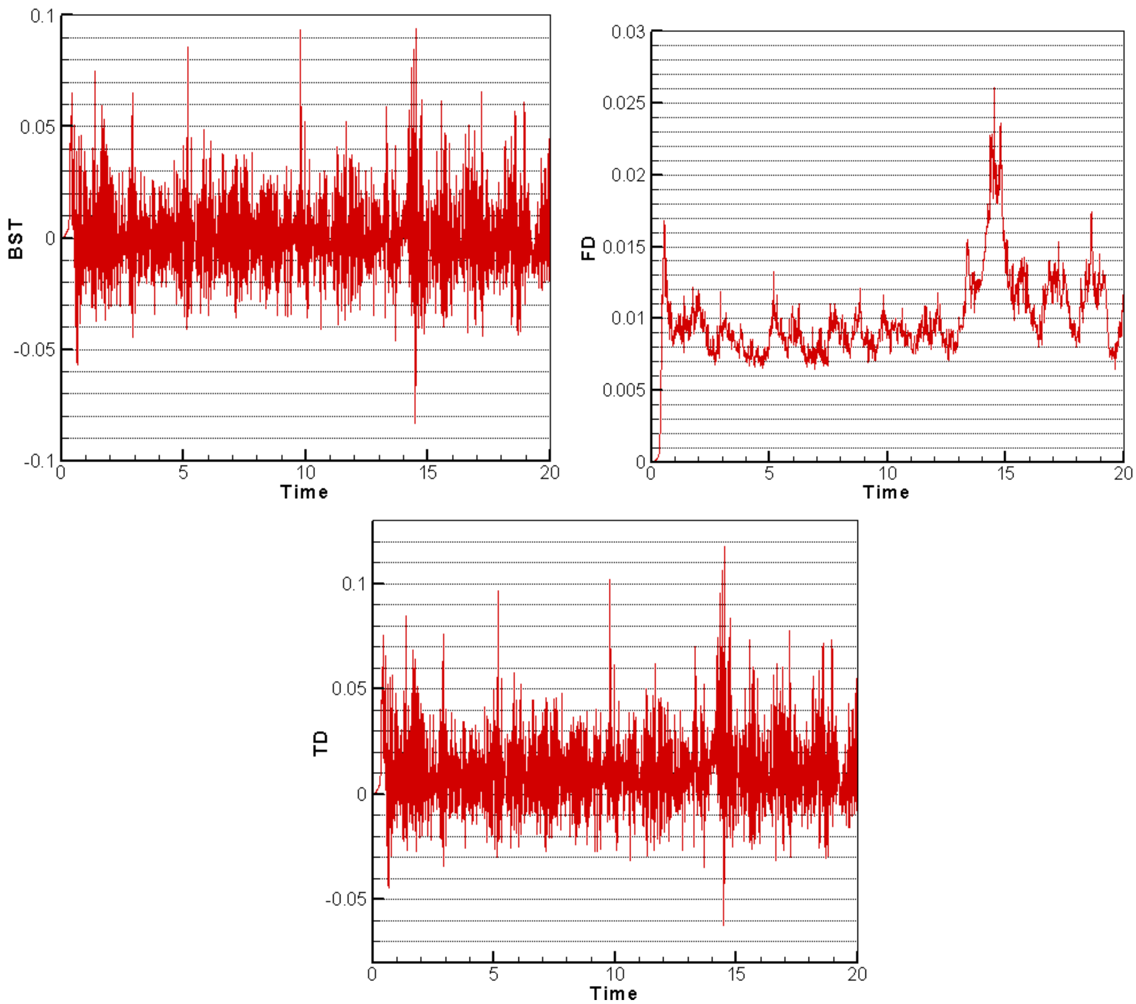


Figure 25: BST,FD,TD vs Time, $\beta = 10$, $\Delta t = 0.0025$,Algorithm 13.

7.0 CONCLUSIONS AND OPEN QUESTIONS

The major motivation of this thesis is to develop low complexity, time accurate, and model accurate methods to solve various problems in computational fluid dynamics. Rigorous mathematical foundation is provided through stability and error analysis to ensure accuracy and reliability of the methods. Numerical tests which are included for each project address the development of efficient methods and further prove the accuracy of fluid prediction.

In Chapter 2, we presented an BETF algorithm, construct an adaptive BETF algorithm and another time filter for pth order BDF method for Stokes-Darcy equation. Another second order method is derived by the combination of Backward Euler plus a time filter that is easily completed by adding three lines to the previous code and slightly modifying the matrix of right hand side based on the BE method. Both theoretical analysis and tests both indicate that adding the filter step to Backward Euler have the advantage of improving time accuracy and convergence order. The numerical tests performed verify that time adaptivity guarantees accuracy while decreasing storage required and overall complexity.

Accurate and stable time discretization is important for obtaining correct flow predictions. The backward Euler time discretization is a stable but inaccurate method. In Chapter 3, we have shown that for minimum extra programming effort, computational complexity, and storage, second order accuracy and unconditional stability can be obtained by adding a time filter. Due to the embedded and modular structure of the algorithm, both adaptive time-step and adaptive order are easily implemented in a code based on a backward Euler time discretization. Extension of the method and analysis to yet higher order time discretization is important as is exploring the effect of time filters on other methods possible for Step 1 of Algorithm 4. Analysis of the effect of time filters with moving and time dependent boundary conditions would also be a significant extension.

In Chapter 4, we construct two decoupled methods based on the artificial compression method and the partitioned method for the time-dependent magnetohydrodynamics flows at low magnetic Reynolds numbers. Theoretical analysis indicates that the error estimate of Algorithm 8 is $\|\mathbf{u}(t_n) - \mathbf{u}^n\| \le C(\Delta t + \varepsilon)\ \forall n \le T/\Delta t$. We also explore the non-physical acoustic waves that comes from the application of the artificial compression method and give a brief analysis for it. The numerical examples illustrate the correctness of our theoretical analysis. An open question is that the error estimate given in Theorem 8 shows $\|\mathbf{u}(t) - \mathbf{u}^\varepsilon(t)\| \le C\sqrt{\varepsilon}\ \forall t \in [0, T]$, which is not optimal in view of the error estimate in Theorem 11. The optimal error estimate for the slightly compressible model is necessary because it can indicate the relation between the coefficient ε and the time step Δt and thus suggest the optimal choice of ε according to different time-discretization schemes. The other open question is how to control the non-physical acoustic waves. Since the non-physical acoustic waves will increasingly influence the accuracy of computing solution as the time step goes to 0, it should be an interesting issue to study effective methods to solve this problem, such as adding stabilization terms (e.g. $\gamma\nabla\nabla\cdot\mathbf{u}$) or utilizing time filters.

Last but not least, we have considered the simplest form of the Baldwin-Lomax eddy viscosity model, making the simplest choices within the approach of [139] to adapt it to incorporate the effects of energy flow from fluctuations back to means (a form of statistical backscatter). For internal flow with no-slip boundary condition in both 2d and 3d, the effects of fluctuations on means are dissipative on time average but can have bursts (with time average zero) for which energy flow reverses. We have shown that the corrected Baldwin-Lomax model shares this property. We have given a stability analysis of two numerical methods for numerical approximation of the resulting model: one with substantial numerical dissipation and one without. Using this two methods, numerical tests confirm backscatter does occur and that the results obtained depend upon the numerical dissipation in the algorithms used and the single model calibration parameter β.

APPENDIX A

VELOCITY ERROR ANALYSIS

A.0.1 PROOF OF LEMMA 9

Proof. By Taylor's theorem with the integral remainder,

$$
\begin{aligned}
D[u(t^{n+1})] - \Delta t u_t(t^{n+1}) &= \frac{3}{2}u(t^{n+1}) - \Delta t u_t(t^{n+1}) \\
-2\left(u(t^{n+1}) - \Delta t u_t(t^{n+1}) + \frac{\Delta t^2}{2}u_{tt}(t^{n+1}))\right. &+ \left.\frac{1}{2}\int_{t^{n+1}}^{t^n} u_{ttt}(t)(t^n - t)^2 dt\right) \\
+\frac{1}{2}\left(u(t^{n+1}) - 2\Delta t u_t(t^{n+1}) + 2\Delta t^2 u_{tt}(t^{n+1}))\right. &+ \left.\frac{1}{2}\int_{t^{n+1}}^{t^{n-1}} u_{ttt}(t)(t^{n-1} - t)^2 dt\right) \\
&= -\int_{t^n}^{t^{n+1}} u_{ttt}(t^n - t)^2 dt - \frac{1}{4}\int_{t^{n-1}}^{t^{n+1}} u_{ttt}(t^{n-1} - t)^2 dt.
\end{aligned}
$$

These terms are first estimated by Cauchy-Schwarz.

$$
\left(\int_{t^n}^{t^{n+1}} u_{ttt}(t)(t^n - t)^2 dt\right)^2 \leq \int_{t^n}^{t^{n+1}} u_{ttt}^2 dt \int_{t^n}^{t^{n+1}} (t^n - t)^4 dt = \frac{\Delta t^5}{5}\int_{t^n}^{t^{n+1}} u_{ttt}^2 dt.
$$

$$
\frac{1}{16}\left(\int_{t^{n-1}}^{t^{n+1}} u_{ttt}(t)(t^{n-1} - t)^2 dt\right)^2 \leq \frac{1}{16}\int_{t^{n-1}}^{t^{n+1}} u_{ttt}^2 dt \int_{t^{n-1}}^{t^{n+1}} (t^{n-1} - t)^4 dt = \frac{2\Delta t^5}{5}\int_{t^{n-1}}^{t^{n+1}} u_{ttt}^2 dt.
$$

Thus,

$$\left(\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\right)^2 \leq \frac{6}{5}\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} u_{ttt}^2 dt.$$

Integrating with respect to x yields the first inequality. Next,

$$I[u(t^{n+1})] - u(t^{n+1}) = \frac{1}{2}u(t^{n+1}) - u(t^n) + \frac{1}{2}u(t^{n-1})$$
$$= \int_{t^n}^{t^{n+1}} u_{tt}(t)(t^{n+1} - t)dt + \int_{t^n}^{t^{n-1}} u_{tt}(t)(t^{n-1} - t)dt.$$

By similar steps,

$$\left(\int_{t^n}^{t^{n+1}} u_{tt}(t)(t^n - t)dt\right)^2 \leq \frac{\Delta t^3}{3} \int_{t^n}^{t^{n+1}} u_{tt}^2 dt.$$

$$\left(\int_{t^{n-1}}^{t^n} u_{tt}(t)(t^{n-1} - t)dt\right)^2 \leq \frac{\Delta t^3}{3} \int_{t^{n-1}}^{t^n} u_{tt}^2 dt.$$

Therefore,

$$\left(I[u(t^{n+1})] - u(t^{n+1})\right)^2 \leq \frac{4}{3}\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} u_{tt}^2 dt. \tag{A.1}$$

The last inequality can be proved using the same strategy. □

A.0.2 PROOF OF THEOREM 11

Proof. We prove this for Option A. A parallel proof exists for Option B. At $t^{n+1} = (n+1)\Delta t$, the true solution of (6.1) satisfies,

$$\begin{aligned}&\left(\frac{D[u(t^{n+1})]}{\Delta t}, v_h\right) + b\left(I[u(t^{n+1})], I[u(t^{n+1})], v_h\right)\\&+ \nu\left(\nabla I[u(t^{n+1})], \nabla v_h\right) - \left(p(t^{n+1}), \nabla \cdot v_h\right)\\&= \left(\mathbf{f}^{n+1}, v_h\right) + \tau^{n+1}(u, p; v_h) \quad \forall v_h \in X_h.\end{aligned} \tag{A.2}$$

Subtracting (4.6) from ($A.2$) yields

$$\begin{aligned}&\left(\frac{D[e_u^{n+1}]}{\Delta t}, v_h\right) + b\left(I[e_u^{n+1}], I[u(t^{n+1})], v_h\right)\\&+ b\left(I[u_h^{n+1}], I[e_u^{n+1}], v_h\right) + \nu\left(\nabla I[e_u^{n+1}], \nabla v_h\right)\\&- \left(e_p^{n+1}, \nabla \cdot v_h\right) = \tau^{n+1}(u, p; v_h).\end{aligned} \tag{A.3}$$

Decompose the error equation for velocity

$$u(t_{n+1}) - u_h^{n+1} = (u^{n+1} - \tilde{u}_h^{n+1}) + (\tilde{u}_h^{n+1} - u_h^{n+1}) = \eta^{n+1} + \phi_h^{n+1}. \tag{A.4}$$

where $\tilde{u}_h^{n+1}$ is the best approximation of $u(t^{n+1})$ in V_h.

Set $v_h = I[\phi_h^{n+1}]$. Using the identity (2.7) with $a = \phi_h^{n+1}$, $b = \phi_h^n$, $c = \phi_h^{n-1}$, (A.4), and applying $(\lambda_h, \nabla \cdot \phi_h) = 0$ for all $\lambda_h \in V^h$, equation ($A.3$) can be written

$$\begin{aligned}
&\frac{1}{4\Delta t}(\|\phi_h^{n+1}\|^2 + \|2\phi_h^{n+1} - \phi_h^n\|^2 + \|\phi_h^{n+1} - \phi_h^n\|^2) \\
&- \frac{1}{4\Delta t}(\|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 + \|\phi_h^n - \phi_h^{n-1}\|^2) \\
&+ \frac{3}{4\Delta t}\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \nu\|\nabla I[\phi_h^{n+1}]\|^2 \\
&= -\left(\frac{D[\eta^{n+1}]}{\Delta t}, I[\phi_h^{n+1}]\right) - b\left(I[\phi_h^{n+1}], I[u(t^{n+1})], I[\phi_h^{n+1}]\right) \\
&- b\left(I[u_h^{n+1}], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) - b\left(I[\eta^{n+1}], I[u(t^{n+1})], I[\phi_h^{n+1}]\right) \\
&+ \left(p(t^{n+1}) - \lambda_h^{n+1}, \nabla \cdot I[\phi_h^{n+1}]\right) - \nu\left(\nabla I[\eta^{n+1}], \nabla I[\phi_h^{n+1}]\right) \\
&+ \tau^{n+1}(u, p; I[\phi_h^{n+1}]).
\end{aligned} \tag{A.5}$$

The next step in the proof is to bound all the terms on the right hand side of ($A.5$) and absorb terms into the left hand side. For arbitrary $\varepsilon > 0$, the first term on the right hand side of (A.5) is bounded in the following way,

$$-\left(\frac{D[\eta^{n+1}]}{\Delta t}, I[\phi_h^{n+1}]\right) \leq \frac{1}{4\varepsilon}\left\|\frac{D[\eta^{n+1}]}{\Delta t}\right\|_{-1}^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2. \tag{A.6}$$

The first nonlinear term can be bounded as

$$\begin{aligned}
&- b\left(I[\phi_h^{n+1}], I[u(t^{n+1})], I[\phi_h^{n+1}]\right) \leq C\|I[\phi_h^{n+1}]\|\|I[u(t^{n+1})]\|_2\|\nabla I[\phi_h^{n+1}]\| \\
&\leq \frac{C^2}{4\varepsilon}\|I[\phi_h^{n+1}]\|^2\|I[u(t^{n+1})]\|_2^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2.
\end{aligned} \tag{A.7}$$

The second nonlinear term is estimated by rewriting it using (A.4) as follows

$$\begin{aligned}
&- b\left(I[u_h^{n+1}], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) = -b\left(I[u(t^{n+1})], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) \\
&+ b\left(I[\eta^{n+1}], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) + b\left(I[\phi_h^{n+1}], I[\eta^{n+1}], I[\phi_h^{n+1}]\right).
\end{aligned} \tag{A.8}$$

then find bounds for all terms on the right hand side of (A.8). We bound the third nonlinear term in (A.5) the same way as the first nonlinear term in (A.8).

$$
\begin{aligned}
&- b\left(I[u(t^{n+1})], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) \\
&\leq C\|\nabla I[u(t^{n+1})]\|\|\nabla I[\eta^{n+1}]\|\|\nabla I[\phi_h^{n+1}]\| \\
&\leq \frac{C^2}{4\varepsilon}\|u\|_{\infty,1}^2\|\nabla I[\eta^{n+1}]\|^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2,
\end{aligned} \tag{A.9}
$$

and

$$
b\left(I[\eta^{n+1}], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) \leq \frac{C^2}{4\varepsilon}\|\nabla I[\eta^{n+1}]\|^4 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2. \tag{A.10}
$$

Next, we have

$$
\begin{aligned}
&b\left(I[\phi_h^{n+1}], I[\eta^{n+1}], I[\phi_h^{n+1}]\right) \\
&\leq C\|I[\phi_h^{n+1}]\|^{\frac{1}{2}}\|\nabla I[\phi_h^{n+1}]\|^{\frac{1}{2}}\|\nabla I[\eta^{n+1}]\|\|\nabla I[\phi_h^{n+1}]\| \\
&\leq Ch^{\frac{-1}{2}}\|I[\phi_h^{n+1}]\|\|\nabla I[\eta^{n+1}]\|\|\nabla I[\phi_h^{n+1}]\| \\
&\leq Ch^{\frac{1}{2}}\|I[\phi_h^{n+1}]\|\|I[u(t^{n+1})]\|_2\|\nabla I[\phi_h^{n+1}]\| \\
&\leq \frac{C^2}{4\varepsilon}h\|I[\phi_h^{n+1}]\|^2\|I[u(t^{n+1})]\|_2^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2.
\end{aligned} \tag{A.11}
$$

The pressure can be bounded as follows

$$
\left(p(t^{n+1}) - \lambda_h^{n+1}, \nabla \cdot I[\phi_h^{n+1}]\right) \leq \frac{C^2}{4\varepsilon}\|p(t^{n+1}) - \lambda_h^{n+1}\|^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2. \tag{A.12}
$$

Then we can bound the term after the pressure,

$$
- \nu\left(\nabla I[\eta^{n+1}], \nabla(I[\phi_h^{n+1}])\right) \leq \frac{C^2}{4\varepsilon}\|\nabla I[\eta^{n+1}]\|^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2. \tag{A.13}
$$

Next we will bound all components of the consistency error $\tau^{n+1}(u, p; I[\phi_h^{n+1}])$.

$$
\begin{aligned}
&\left(\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1}), I[\phi_h^{n+1}]\right) \\
&\leq C\|\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\|\|\nabla I[\phi_h^{n+1}]\| \\
&\leq \frac{C^2}{4\varepsilon}\|\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\|^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2.
\end{aligned} \tag{A.14}
$$

$$
\begin{aligned}
&\nu\left(\nabla(I[u(t^{n+1})] - u(t^{n+1})), \nabla I[\phi_h^{n+1}]\right) \\
&\leq \frac{C^2}{4\varepsilon}\|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2 + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2.
\end{aligned} \tag{A.15}
$$

Setting $\varepsilon = \frac{\nu}{16}$, the nonlinear term in $\tau^{n+1}(u,p;I[\phi_h^{n+1}])$ is then estimated as follows,

$$
\begin{aligned}
& b\left(I[u(t^{n+1})], I[u(t^{n+1})], I[\phi_h^{n+1}]\right) - b(u(t^{n+1}), u(t^{n+1}), I[\phi_h^{n+1}]) \\
& = b\left(I[u(t^{n+1})] - u(t^{n+1}), I[u(t^{n+1})], I[\phi_h^{n+1}]\right) - b(u(t^{n+1}), I[u(t^{n+1})] - u(t^{n+1}), I[\phi_h^{n+1}]) \\
& \leq C\|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|\|\nabla I[\phi_h^{n+1}]\|\Big(\|\nabla I[u(t^{n+1})]\| + \|\nabla u(t^{n+1})\|\Big) \\
& \leq \frac{C^2}{4\varepsilon}\|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2\Big(\|\nabla I[u(t^{n+1})]\|^2 + \|\nabla u(t^{n+1})\|^2\Big) + \varepsilon\|\nabla I[\phi_h^{n+1}]\|^2.
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{4\Delta t}(\|\phi_h^{n+1}\|^2 + \|2\phi_h^{n+1} - \phi_h^n\|^2 + \|\phi_h^{n+1} - \phi_h^n\|^2) + \frac{\nu}{4}\|\nabla I[\phi_h^{n+1}]\|^2 \\
& - \frac{1}{4\Delta t}(\|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 + \|\phi_h^n - \phi_h^{n-1}\|^2) + \frac{3}{4\Delta t}\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
& \leq C\Big(\|\frac{D[\eta^{n+1}]}{\Delta t}\|_{-1}^2 + (1+h)\|I[\phi_h^{n+1}]\|^2\|I[u(t^{n+1})]\|_2^2 \\
& + \|u\|_{\infty,1}^2\|\nabla I[\eta^{n+1}]\|^2 + \|\nabla I[\eta^{n+1}]\|^4 + \|p(t^{n+1}) - \lambda_h^{n+1}\|^2 \\
& + \|\nabla I[\eta^{n+1}]\|^2 + \|\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\|^2 \\
& + \|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2 \\
& + \|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2(\|\nabla I[u(t^{n+1})]\|^2 + \|\nabla u(t^{n+1})\|^2)\Big).
\end{aligned}
\tag{A.16}
$$

Let $\kappa = C\nu\|u\|_{\infty,2}^2(1+h)$. Assume $\Delta t < \frac{1}{\kappa}$, summing from $n=1$ to $n=N-1$ and applying the discrete Gronwall lemma we obtain

$$
\begin{aligned}
& \|\phi_h^N\|^2 + \|2\phi_h^N - \phi_h^{N-1}\|^2 + \|\phi_h^N - \phi_h^{N-1}\|^2 \\
& + \sum_{n=1}^{N-1} 3\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \nu\Delta t\sum_{n=1}^{N-1}\|\nabla I[\phi_h^{n+1}]\|^2 \\
& \leq e^{\left(\frac{\Delta t\kappa(N-1)}{1-\Delta t\kappa}\right)}\Big(\|\phi_h^1\|^2 + \|2\phi_h^1 - \phi_h^0\|^2 + \|\phi_h^1 - \phi_h^0\|^2 + C\Delta t\sum_{n=1}^{N-1}\|\frac{D[\eta^{n+1}]}{\Delta t}\|_{-1}^2 \\
& + C\Delta t\nu(\|u\|_{\infty,1}^2 + 1)\sum_{n=1}^{N-1}\|\nabla I[\eta^{n+1}]\|^2 + C\Delta t\sum_{n=1}^{N-1}\|\nabla I[\eta^{n+1}]\|^4 \\
& + C\Delta t\sum_{n=1}^{N-1}\|p(t^{n+1}) - \lambda_h^{n+1}\|^2 + C\Delta t\sum_{n=1}^{N-1}\|\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\|^2 \\
& + C\Delta t\sum_{n=1}^{N-1}\|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2 \\
& + C\Delta t\sum_{n=1}^{N-1}\|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2(\|\nabla I[u(t^{n+1})]\|^2 + \|\nabla u(t^{n+1})\|^2)\Big).
\end{aligned}
\tag{A.17}
$$

The first three terms can be bounded as

$$
\begin{aligned}
&\|\phi_h^1\|^2 + \|2\phi_h^1 - \phi_h^0\|^2 + \|\phi_h^1 - \phi_h^0\|^2 \\
&\leq C\Big(\|u(t_1) - u_h^1\|^2 + \|(u(t_0) - u_h^0)\|^2\Big) + Ch^{2k+2}\||u\||^2_{\infty,k+1}.
\end{aligned} \tag{A.18}
$$

We bound the fourth term in (A.17) as follows

$$
\begin{aligned}
&\nu\Delta t \sum_{n=1}^{N-1} \|\frac{D[\eta^{n+1}]}{\Delta t}\|^2_{-1} = \nu\Delta t \sum_{n=1}^{N-1} \|\frac{\frac{3}{2}(\eta^{n+1} - \eta^n) - \frac{1}{2}(\eta^n - \eta^{n-1})}{\Delta t}\|^2_{-1} \\
&\leq C\sum_{n=0}^{N} \int_{t^{n-1}}^{t^{n+1}} \|\eta_t\|^2 ds \leq Ch^{2k+2}\|u_t\|^2_{2,k+1},
\end{aligned} \tag{A.19}
$$

and

$$
\begin{aligned}
&\Delta t(\nu\|u\|^2_{\infty,1} + \nu) \sum_{n=1}^{N-1} \|\nabla I[\eta^{n+1}]\|^2 \\
&\leq C\Delta t\nu(2\|u\|^2_{\infty,1} + 1) \max\left\{\frac{9}{4}, 4, \frac{1}{4}\right\} \sum_{n=1}^{N-1} 3\left(\|\nabla\eta^{n+1}\|^2 + \|\nabla\eta^n\|^2 + \|\nabla\eta^{n-1}\|^2\right) \\
&\leq C\Delta t \sum_{n=0}^{N} h^{2k}\|u^{n+1}\|^2_{k+1} = Ch^{2k}\||u\||^2_{2,k+1}.
\end{aligned} \tag{A.20}
$$

Similarly to (A.20), we also have

$$
\Delta t \sum_{n=1}^{N-1} \|\nabla I[\eta^{n+1}]\|^4 \leq C\Delta t \sum_{n=0}^{N} h^{4k}\|u^{t+1}\|^4_{k+1} = Ch^{4k}\||u\||^4_{4,k+1}. \tag{A.21}
$$

Observe that

$$
\nu\Delta t \sum_{n=1}^{N} \|p(t^{n+1}) - \lambda_h^{n+1}\|^2 \leq Ch^{2s+2}\||p\||^2_{2,s+1}. \tag{A.22}
$$

The terms from consistency error are bounded using Lemma 9.

$$
\nu\Delta t \sum_{n=1}^{N-1} \|\frac{D[u(t^{n+1})]}{\Delta t} - u_t(t^{n+1})\|^2 = C\Delta t^4 \sum_{n=0}^{N-1} \int_{t^{n-1}}^{t^{n+1}} \|u_{ttt}\|^2 dt = C\Delta t^4\|u_{ttt}\|^2_{2,0}. \tag{A.23}
$$

$$
\nu\Delta t \sum_{n=1}^{N-1} \|\nabla(I[u(t^{n+1})] - u(t^{n+1}))\|^2 \leq C\Delta t^4 \sum_{n=1}^{N-1} \int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{tt}\|^2 dt \leq C\Delta t^4\|\nabla u_{tt}\|^2_{2,0}. \tag{A.24}
$$

$$
\begin{aligned}
&\nu\Delta t\sum_{n=1}^{N-1}\|\nabla(I[u(t^{n+1})]-u(t^{n+1}))\|^2(\|\nabla I[u(t^{n+1})]\|^2+\|\nabla u(t^{n+1})\|^2)\\
&\leq C\Delta t\sum_{n=1}^{N-1}(\|\nabla I[u(t^{n+1})]\|^2+\|\nabla u(t^{n+1})\|^2)\Delta t^3\int_{t^n}^{t^{n+1}}\|\nabla u_{tt}\|^2dt\\
&\leq C\Delta t^4\sum_{n=1}^{N-1}(\int_{t^{n-1}}^{t^{n+1}}\|\nabla I[u(t^{n+1})]\|^4+\|\nabla u(t^{n+1})\|^4+\|\nabla u_{tt}\|^4dt)\\
&\leq C\Delta t^4(\||\nabla u|\|_{4,0}^4+\|\nabla u_{tt}\|_{4,0}^4).
\end{aligned}
\tag{A.25}
$$

Combining ($A.18$) - ($A.22$) gives

$$
\begin{aligned}
&\|\phi_h^N\|^2+\|2\phi_h^N-\phi_h^{N-1}\|^2+\|\phi_h^N-\phi_h^{N-1}\|^2+\sum_{n=1}^{N-1}3\|\phi_h^{n+1}-2\phi_h^n+\phi_h^{n-1}\|^2\\
&+\nu\Delta t\sum_{n=1}^{N-1}\|\nabla I[\phi_h^{n+1}]\|^2\\
&\leq C\Big(\|u(t_1)-u_h^1\|^2+\|(u(t_0)-u_h^0)\|^2+h^{2k+2}\||u|\|_{\infty,k+1}^2\\
&+h^{2k+2}\|u_t\|_{2,k+1}^2+h^{2k}\||u|\|_{2,k+1}^2+h^{4k}\||u|\|_{4,k+1}^4+h^{2s+2}\||p|\|_{2,s+1}^2\\
&+\Delta t^4(\|u_{ttt}\|_{2,0}^2+\|\nabla u_{tt}\|_{2,0}^2+\||\nabla u|\|_{4,0}^4+\|\nabla u_{tt}\|_{4,0}^4)\Big).
\end{aligned}
\tag{A.26}
$$

We add both sides of ($A.26$) with

$$
\begin{aligned}
&\|\eta^N\|^2+\|2\eta^N-\eta^{N-1}\|^2+\|\eta^N-\eta^{N-1}\|^2+\sum_{n=1}^{N-1}3\|\eta^{n+1}-2\eta^n+\eta^{n-1}\|^2\\
&+\nu\Delta t\sum_{n=1}^{N-1}\|\nabla(\frac{3}{2}\eta^{n+1}-\eta^n+\frac{1}{2}\eta^{n-1})\|^2.
\end{aligned}
\tag{A.27}
$$

and apply triangle inequality to get (4.15). □

APPENDIX B

SECOND ORDER ERROR ESTIMATOR

This section justifies the use of EST_2 as an error estimator for the second order approximation. A Taylor series calculation shows that the second order approximation $y^{n+1}_{(2)}$ in Algorithm 5 has the local truncation error (LTE) (for constant stepsize)

$$LTE = -\Delta t^3 \left(\frac{1}{3} y''' + \frac{1}{2} f_y y'' \right) + \mathcal{O}(\Delta t^4).$$

Consider the addition of a second time filter,

$$\begin{aligned}
&\text{Step 1} \quad : \quad \tfrac{y^{(1)}_{n+1} - y^n}{\Delta t} = f(t_{n+1}, y^{n+1}_{(1)}), \\
&\text{Step 2} \quad : \quad y^{n+1}_{(2)} = y^{n+1}_{(1)} - \tfrac{1}{3} \left\{ y^{n+1}_{(1)} - 2y^n + y^{n-1} \right\} \\
&\text{Step 3} \quad : \quad y_{n+1} = y^{n+1}_{(2)} - \tfrac{2}{11} \left\{ y^{n+1}_{(2)} - 3y^n + 3y^{n-1} - y^{n-2} \right\}
\end{aligned} \tag{B.1}$$

Another Taylor series calculation shows that the induced method has the LTE of

$$LTE = -\Delta t^3 \frac{1}{2} f_y y'' + \mathcal{O}(\Delta t^4),$$

Thus, y_{n+1} yields a more accurate (still second order) approximation, and

$$EST_2 = y^{n+1}_{(2)} - y_{n+1} = \frac{2}{11} \left\{ y^{(2)}_{n+1} - 3y^n + 3y^{n-1} - y^{n-2} \right\}$$

gives an estimate for the error of y_{n+1}. This is extended to variable stepsize using Newton interpolation, and written with stepsize ratios in Algorithm 5.

This is a nonstandard approach since one would normally use a higher order approximation to estimate the error. However, this is simple since it requires no additional function evaluations or Jacobians, and does not require solving a system of equations. Interestingly, (B.1) remains energy stable, and could be useful as a standalone constant stepsize method.

APPENDIX C

PROOF OF MONOTONICITY AND LIPSCHITZ CONTINUITY

Remark 7. *The proof of monotonicity and local Lipschitz continuity are similar to the analogous ones for the Smagorinsky model. Since the present work involves the rotational form, we include both for completeness. Our proofs are adapted from Lemma 8.5 in [125].*

Proof. (Proof of Lemma 2) First, we prove the strong monotonicity (2.1). Define an operator $\mathbf{F} : (L^3(\Omega))^d \to (L^{\frac{3}{2}}(\Omega))^d$ by

$$\mathbf{F}(\nabla \times \mathbf{u}) = l^2(\mathbf{x})|\nabla \times \mathbf{u}|\nabla \times \mathbf{u}, \tag{C.1}$$

where $\mathbf{u} \in (W^{1,3}(\Omega))^d$, $l : \mathbf{x} \in \Omega \mapsto R$ is a non-negative function and $l \in L^\infty(\Omega)$. We further define $\mathbf{u} = \tau\mathbf{u}' + (1-\tau)\mathbf{u}''$, $\tau \in [0,1]$. Then, we have

$$\mathbf{F}(\nabla \times \mathbf{u}') - \mathbf{F}(\nabla \times \mathbf{u}'') = \int_0^1 \frac{d}{d\tau}\mathbf{F}(\nabla \times \mathbf{u})d\tau. \tag{C.2}$$

Combine ($C.1$) and ($C.2$) to get

$$\begin{aligned}
&(l^2(\mathbf{x})|\nabla \times \mathbf{u}'|\nabla \times \mathbf{u}' - l^2(\mathbf{x})|\nabla \times \mathbf{u}''|\nabla \times \mathbf{u}'') \cdot (\nabla \times (\mathbf{u}' - \mathbf{u}''))\\
&= (\mathbf{F}(\nabla \times \mathbf{u}') - \mathbf{F}(\nabla \times \mathbf{u}'')) \cdot (\nabla \times (\mathbf{u}' - \mathbf{u}''))\\
&= \sum_{i=1}^{3}(\int_0^1 \frac{d}{d\tau}\mathbf{F}_i(\nabla \times \mathbf{u})d\tau)[(\frac{\partial \mathbf{u}'_k}{\partial x_j} - \frac{\partial \mathbf{u}'_j}{\partial x_k}) - (\frac{\partial \mathbf{u}''_k}{\partial x_j} - \frac{\partial \mathbf{u}''_j}{\partial x_k})].
\end{aligned} \tag{C.3}$$

Here, consider 3D case ($d = 3$). The notations $i, j, k = 1, 2, 3$, or $i, j, k = 2, 3, 1$, or $i, j, k = 3, 1, 2$. We have

$$\begin{aligned}&\frac{d}{d\tau}\mathbf{F}_i(\nabla\times\mathbf{u})\\&=(l^2(\mathbf{x})\frac{d}{d\tau}|\nabla\times\mathbf{u}|)(\frac{\partial\mathbf{u}_k}{\partial x_j}-\frac{\partial\mathbf{u}_j}{\partial x_k})+l^2(\mathbf{x})|\nabla\times\mathbf{u}|[(\frac{\partial\mathbf{u}'_k}{\partial x_j}-\frac{\partial\mathbf{u}'_j}{\partial x_k})-(\frac{\partial\mathbf{u}''_k}{\partial x_j}-\frac{\partial\mathbf{u}''_j}{\partial x_k})],\end{aligned}\tag{C.4}$$

and

$$\begin{aligned}\frac{d}{d\tau}|\nabla\times\mathbf{u}|&=\frac{d}{d\tau}(\sum_{l=1}^{3}[\tau(\frac{\partial\mathbf{u}'_n}{\partial x_m}-\frac{\partial\mathbf{u}'_m}{\partial x_n})+(1-\tau)(\frac{\partial\mathbf{u}''_n}{\partial x_m}-\frac{\partial\mathbf{u}''_m}{\partial x_n})]^2)^{\frac{1}{2}}\\&=\frac{1}{|\nabla\times\mathbf{u}|}\sum_{l=1}^{3}(\frac{\partial\mathbf{u}_n}{\partial x_m}-\frac{\partial\mathbf{u}_m}{\partial x_n})[(\frac{\partial\mathbf{u}'_n}{\partial x_m}-\frac{\partial\mathbf{u}'_m}{\partial x_n})-(\frac{\partial\mathbf{u}''_n}{\partial x_m}-\frac{\partial\mathbf{u}''_m}{\partial x_n})],\end{aligned}\tag{C.5}$$

where the notations $l, m, n = 1, 2, 3$, or $l, m, n = 2, 3, 1$, or $l, m, n = 3, 1, 2$.

By ($C.4$) and ($C.5$), we obtain

$$\begin{aligned}\frac{d}{d\tau}\mathbf{F}_i(\nabla\times\mathbf{u})=&\frac{l^2(\mathbf{x})}{|\nabla\times\mathbf{u}|}\sum_{l=1}^{3}(\frac{\partial\mathbf{u}_n}{\partial x_m}-\frac{\partial\mathbf{u}_m}{\partial x_n})[(\frac{\partial\mathbf{u}'_n}{\partial x_m}-\frac{\partial\mathbf{u}'_m}{\partial x_n})-(\frac{\partial\mathbf{u}''_n}{\partial x_m}-\frac{\partial\mathbf{u}''_m}{\partial x_n})](\frac{\partial\mathbf{u}_k}{\partial x_j}-\frac{\partial\mathbf{u}_j}{\partial x_k})\\&+l^2(\mathbf{x})|\nabla\times\mathbf{u}|[(\frac{\partial\mathbf{u}'_k}{\partial x_j}-\frac{\partial\mathbf{u}'_j}{\partial x_k})-(\frac{\partial\mathbf{u}''_k}{\partial x_j}-\frac{\partial\mathbf{u}''_j}{\partial x_k})].\end{aligned}\tag{C.6}$$

Combining ($C.6$) and ($C.3$), we have

$$(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'')\cdot(\nabla\times(\mathbf{u}'-\mathbf{u}''))=Q_1+Q_2.\tag{C.7}$$

Here,

$$\begin{aligned}Q_1=&\int_0^1\frac{l^2(\mathbf{x})}{|\nabla\times\mathbf{u}|}\sum_{i,l=1}^{3}(\frac{\partial\mathbf{u}_n}{\partial x_m}-\frac{\partial\mathbf{u}_m}{\partial x_n})(\frac{\partial\mathbf{u}_k}{\partial x_j}-\frac{\partial\mathbf{u}_j}{\partial x_k})\\&\cdot[(\frac{\partial\mathbf{u}'_n}{\partial x_m}-\frac{\partial\mathbf{u}'_m}{\partial x_n})-(\frac{\partial\mathbf{u}''_n}{\partial x_m}-\frac{\partial\mathbf{u}''_m}{\partial x_n})][(\frac{\partial\mathbf{u}'_k}{\partial x_j}-\frac{\partial\mathbf{u}'_j}{\partial x_k})-(\frac{\partial\mathbf{u}''_k}{\partial x_j}-\frac{\partial\mathbf{u}''_j}{\partial x_k})]d\tau\\&\geq 0.\end{aligned}\tag{C.8}$$

Furthermore, we have

$$
\begin{aligned}
Q_2 &= \int_0^1 l^2(\mathbf{x})|\nabla\times\mathbf{u}|\sum_{i=1}^{3}[(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 d\tau \\
&\geq C_d\int_0^1 l^2(\mathbf{x})\sum_{i=1}^{3}(\sum_{l=1}^{3}|\frac{\partial \mathbf{u}_n}{\partial x_m}-\frac{\partial \mathbf{u}_m}{\partial x_n}|)[(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 d\tau \\
&\geq C_d\sum_{i=1}^{3}\sum_{l=1}^{3} l^2(\mathbf{x})[\int_0^1 |\tau(\frac{\partial \mathbf{u}'_n}{\partial x_m}-\frac{\partial \mathbf{u}'_m}{\partial x_n})+(1-\tau)(\frac{\partial \mathbf{u}''_n}{\partial x_m}-\frac{\partial \mathbf{u}''_m}{\partial x_n})|d\tau] \\
&\qquad\qquad \cdot[(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 \\
&\geq C_d\sum_{i=1}^{3}\sum_{l=1}^{3} l^2(\mathbf{x})[\frac{1}{4}|(\frac{\partial \mathbf{u}'_n}{\partial x_m}-\frac{\partial \mathbf{u}'_m}{\partial x_n})-(\frac{\partial \mathbf{u}''_n}{\partial x_m}-\frac{\partial \mathbf{u}''_m}{\partial x_n})|][(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 \\
&\geq \underline{C}\sum_{i=1}^{3} l^2(\mathbf{x})|(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})|^3,
\end{aligned}
\tag{C.9}
$$

where we use those inequalities $\sqrt{a^2+b^2+c^2}\geq C_d(|a|+|b|+|c|)$ and $\int_0^1|\tau a+(1-\tau)b|d\tau \geq \frac{1}{4}|a-b|$, $\forall\, a,b,c\in R$.

By (C.7), (C.8) and (C.9), we have

$$
\begin{aligned}
&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}''),\nabla\times(\mathbf{u}'-\mathbf{u}'')) = \int_\Omega (Q_1+Q_2)d\mathbf{x} \\
&\geq \underline{C}\int_\Omega\sum_{i=1}^{3} l^2(\mathbf{x})|(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})|^3 d\mathbf{x} \\
&= \underline{C}\|l^{\frac{2}{3}}(x)\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}^3,
\end{aligned}
\tag{C.10}
$$

which completes the proof of strong monotonicity (2.1).

Next, we prove the local Lipschitz-continuity (2.2). Using triangle inequality, we have

$$
\begin{aligned}
&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'',\nabla\times\mathbf{v}) \\
&\leq (l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'',\nabla\times\mathbf{v}) \\
&\quad + (l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}''-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'',\nabla\times\mathbf{v}) \\
&\leq \|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'|\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3} \\
&\quad + \|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'|-l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}''|\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}''\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3}.
\end{aligned}
\tag{C.11}
$$

Since

$$\begin{aligned}\|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'|\|_{L^3} &= \int_\Omega (\sum_{i=1}^{3} |l^{\frac{2}{3}}(\mathbf{x})(\nabla\times\mathbf{u}')_i|^2)^{\frac{3}{2}} d\mathbf{x} \\ &\le \tilde{C}_d \int_\Omega (\sum_{i=1}^{3} |l^{\frac{2}{3}}(\mathbf{x})(\nabla\times\mathbf{u}')_i|^3) d\mathbf{x} \\ &= \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}'\|_{L^3},\end{aligned} \tag{C.12}$$

where we use the inequality $\|\mathbf{x}\|_{\ell^2} \le \tilde{C}_d \|\mathbf{x}\|_{\ell^3}$, $\forall\, \mathbf{x} \in R^d$. We also have

$$\begin{aligned}\|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'| - l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}''|\|_{L^3} &\le \|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}' - \nabla\times\mathbf{u}''|\|_{L^3} \\ &\le \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}.\end{aligned} \tag{C.13}$$

Finally, by (C.11), (C.12), and (C.13), we can obtain the local Lipschitz-continuity (2.2).

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'', \nabla\times\mathbf{v}) \\ &\le \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}'\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3} \\ &\quad + \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}''\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3} \\ &= \overline{C}\gamma \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3},\end{aligned} \tag{C.14}$$

where $\gamma = \max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}'\|_{L^3}, \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}''\|_{L^3}\}$.

□

BIBLIOGRAPHY

[1] Alfvén, H.: Existence of electromagnetic-hydrodynamic waves. Nature. 150, 405-406 (1942)

[2] Moffatt, H.K.: Field generation in electrically conducting fluids[M]. Cambridge University Press (1978)

[3] Barleon, L., Casal, V., Lenhart, L.: MHD flow in liquid-metal-cooled blankets. Fusion Engineering and Design. 14, 401-412 (1991)

[4] Davidson, P.A.: Magnetohydrodynamics in material processing. Annu. Rev. Fluid Mech. 31, 273-300 (1999)

[5] Lin, T.F., Gilbert, J.B., Kossowsky, R.: Sea-water magnetohydrodynamic propulsion for next-generation undersea vehicles. Pennsylvania State Univ State College Applied Research Lab (1990)

[6] Gerbeau, J.F., Bris, C.L., Lelièvre, T.: Mathmatical methods for the Magnetohydrodynamics of Liquid metals. Oxford University Press (2006)

[7] Adams, R.A.: Sobolev spaces. Academic press (2003)

[8] Ascher, U.M., Ruuth, S.J., Wetton, B.T.R.: Implicit-explicit methods for time-dependent partial differential equations[J]. SIAM Journal on Numerical Analysis. 32(3), 797-823 (1995)

[9] Chorin, A.J.: Numerical solution of the Navier-Stokes equations[J]. Mathematics of computation. 22(104), 745-762 (1968)

[10] Rannacher, R.: On Chorin's projection method for the incompressible Navier-Stokes equations. In The Navier-Stokes equations II theory and numerical methods. Springer Berlin Heidelberg. 167-183 (1992)

[11] Turek, S.: A comparative study of some time-stepping techniques for the incompressible Navier-Stokes equations: From fully implicit nonlinear schemes to semi-implicit projection methods[M]. IWR. (1995)

[12] Peterson, J.S.: On the finite element approximation of incompressible flows of an electrically conducting fluid[J]. Numerical Methods for Partial Differential Equations. 4(1), 57-68 (1988)

[13] Su, H., Feng, X., Huang, P.: Iterative methods in penalty finite element discretization for the steady MHD equations[J]. Computer Methods in Applied Mechanics and Engineering, 2016, 304: 521-545.

[14] Zhang, Q., Su, H., Feng, X.: A partitioned finite element scheme based on Gauge-Uzawa method for time-dependent MHD equations[J]. Numerical Algorithms, 2017: 1-19.

[15] Zhu, T., Su, H., Feng, X.: Some Uzawa-type finite element iterative methods for the steady incompressible magnetohydrodynamic equations[J]. Applied Mathematics and Computation, 2017, 302: 34-47.

[16] Wu, J., Liu, D., Feng, X., Huang, P.: An efficient two-step algorithm for the stationary incompressible magnetohydrodynamic equations[J]. Applied Mathematics and Computation, 2017, 302: 21-33.

[17] Su, H., Feng, X., Zhao, J.: Two-Level Penalty Newton Iterative Method for the 2D/3D Stationary Incompressible Magnetohydrodynamics Equations[J]. Journal of Scientific Computing, 2017, 70(3): 1144-1179.

[18] Dong, X., He, Y., Zhang, Y.: Convergence analysis of three finite element iterative methods for the 2D/3D stationary incompressible magnetohydrodynamics, Comput. Methods Appl. Mech. Engrg. 276 (2014) 287 C311.

[19] Dong, X., He, Y.: Two-level Newton iterative method for the 2D/3D stationary incompressible magnetohydrodynamics, J. Sci. Comput. 63 (2015) 426 C451.

[20] Yuksel, G., Ingram, R.: Numerical analysis of a finite element Crank-Nicolson discretization for MHD flows at small magnetic Reynolds numbers. International Journal of Numerical Analysis and Modeling. 10(1), 74-98 (2013)

[21] Roberts ,P.H.: An introduction to magnetohydrodynamics. Elsevier, USA (1967)

[22] Davidson ,P.A.: An Introduction to Magnetohydrodynamics. Cambridge University Press, United Kingdom (2001)

[23] Layton, W.J., Tran, H., Trenchea, C.: Numerical analysis of two partitioned methods for uncoupling evolutionary MHD flows. Numer. Meth. Part D.E. 30(4), 1083-1102 (2014)

[24] Layton W, Tran H, Trenchea C. Stability of partitioned methods for magnetohydrodynamics flows at small magnetic Reynolds number[J]. Recent advances in scientific computing and applications. 586(231) (2013)

[25] Yuksel, G., Isik, O.R.: Numerical analysis of Backward-Euler discretization for simplified magnetohydynamic flows. Applied Mathematical Modelling. 39, 1889-1898 (2015)

[26] Lighthill, M.J.: On sound generated aerodynamically. I. General theory[C]. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. The Royal Society, 1952, 211(1107): 564-587.

[27] Layton, W., Novotny, A.: The exact derivation of the Lighthill acoustic analogy for low Mach number flows. Advances in Math Fluid Mech. 247-279 (2009)

[28] Shen, J.: On error estimates of the penalty method for unsteady Navier CStokes equations[J]. SIAM Journal on Numerical Analysis. 32(2), 386-403 (1995)

[29] Fabrie, P., Galusinski, C.: The slightly compressible Navier-Stokes equations revisited[J]. Nonlinear Analysis: Theory, Methods and Applications. 46(8), 1165-1195 (2001)

[30] Shen, J.: On a new pseudocompressibility method for the incompressible Navier-Stokes equations[J]. Applied numerical mathematics. 21(1), 71-90 (1996)

[31] Zhang, G., He, Y.: Decoupled schemes for unsteady MHD equations. I. time discretization, Numer. Method Part. Diff. Equ. 33(3), 956-C973 (2017)

[32] Zhang, G., He, Y.: Decoupled schemes for unsteady MHD equations II: Finite element spatial discretization and numerical implementation. Comput. Math. Appl. 69, 1390-C1406 (2015)

[33] He, Y.: Unconditional convergence of the Euler semi-implicit scheme for the three-dimensional incompressible MHD equations. IMA Journal of Numerical Analysis. dru015, (2014)

[34] Shen, J.: Pseudo-compressibility methods for the unsteady incompressible Navier-Stokes equations. In Proceedings of the 1994 Beijing symposium on nonlinear evolution equations and infinite dynamical systems. 68-78 (1997)

[35] Rong, Y., Hou, Y., Zhang, Y.: Numerical analysis of a second order algorithm for simplified magnetohydrodynamic flows[J]. Advances in Computational Mathematics. 43:823-848 (2017)

[36] Rong, Y., Hou, Y.: A Partitioned Second-Order Method for Magnetohydrodynamic Flows at Small Magnetic Reynolds Numbers. Numerical Methods for Partial Differential Equations. 33(6), 1966-1986 (2017)

[37] Shen, J.: On error estimates of projection methods for Navier-Stokes equations: first-order schemes[J]. SIAM Journal on Numerical Analysis. 29(1), 57-77 (1992)

[38] Shen, J.: On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations[J]. Numerische Mathematik. 62(1), 49-73 (1992)

[39] Prohl, A.: Convergent finite element discretizations of the nonstationary incompressible magnetohydrodynamics system[J]. ESAIM: Mathematical Modelling and Numerical Analysis. 42(6), 1065-1087 (2008)

[40] Temam, R.: Sur l'approximation de la solution des quations de Navier-Stokes par la mthode des pas fractionnaires (I)[J]. Archive for Rational Mechanics and Analysis. 32(2), 135-153 (1969)

[41] Hecht, F., Pironneau, O.: FreeFem++. Webpage: http://www.freefem.org

[42] A, Thom.: The flow past circular cylinders at low speeds[J]. The Royal Society. 1364-5021

[43] Arbogast,T., Gomez, M.: A discretization and multigrid solver for a Darcy-Stokes system of three dimensional vuggy porous media. Computational Geosciences, 13(3):331–348, 2009.

[44] Arbogast,T., Gomez, M.: A discretization and multigrid solver for a Darcy-Stokes system of three dimensional vuggy porous media. Computational Geosciences, 13(3):331–348, 2009.

[45] Asselin, R.: Frequency filter for time integrations. Monthly Weather Review, 100(6):487, 1972.

[46] Bernardi, C., Rebollo, T., Hecht, F., Mghazli, Z.: Mortar finite element discretization of a model coupling Darcy and Stokes equations. ESIAM Mathematical Modelling and Numerical Analysis, 42(3):375–410, 2007.

[47] Chen, W., Gunzburger, M., Sun, D., Wang, X.: Efficient and long-time ac- curate second-order methods for Stokes-Darcy system. SIAM Journal on Numerical Analysis, 51(5):2563-2584, 2012.

[48] Chen, W., Gunzburger, M., Sun, D., Wang, X.: An efficient and long-time ac- curate third-order algorithm for the Stokes-Darcy system. Numerische Mathematik, 134(4):857–879, 2016.

[49] Chen, W., Gunzburger, M., Sun, D., Wang, X.: Weak Galerkin method for the coupled Darcy- Stokes flow. IMA Journal of Numerical Analysis, 36(2), 2014.

[50] DeCaria, V., Guzel, A., Layton, W., Li, Y.: A new embedded variable stepsize, variable order family of low computational complexity. arXiv:1810.06670 [math.NA], 2018.

[51] DeCaria, V., Layton, W., McLaughlin, M.: An analysis of the Robert-Asselin time filter for the correction of nonphysical acoustics in an artificial compression method. Nu- merical Methods for Partial Differential Equations, 12 2018.

[52] DeCaria, V., Layton, W., Zhao, H.: Analysis of a low complexity, time-accurate discretization of the Navier-Stokes equations. arXiv:1810.06705 [math.NA], 2018.

[53] V. J. Ervin, E. W. Jenkins, and Hyesuk Lee. Approximation of the Stokes-Darcy System by Optimization. Plenum Press, 2014.

[54] Gatica, N., Sequeira, F.: Analysis of the HDG method for the Stokes-Darcy coupling: HDG for Stokes-Darcy. Numerical Methods for Partial Differential Equations, 33(3), 2017.

[55] Guzel, A., Layton, W.: Time filters increase accuracy of the fully implicit method. Bit Numerical Mathematics, (3):1–15, 2018.

[56] Hay, A., Etienne, S., Pelletier, D., Garon,A.: Hp-adaptive time integration based on the BDF for viscous flows. Journal of Computational Physics, 291(C):151–176, 2015.

[57] Hessari, P.: Pseudospectral least squares method for Stokes-Darcy equations. SIAM Jour- nal on Numerical Analysis, 53(3):1195–1213, 2015.

[58] Jiang, B.: A parallel domain decomposition method for coupling of surface and groundwater flows. Computer Methods in Applied Mechanics and Engineering, 198(9):947–957, 2009.

[59] Kwizak, M., Robert, A.: A semi-implicit scheme for grid point atmospheric models of the primitive equations. Monthly Weather Review, 99(99), 1971.

[60] Layton, W., Tran, H., Xiong, X.: Long time stability of four methods for split- ting the evolutionary Stokes-Darcy problem into Stokes and Darcy subproblems. Journal of Computational and Applied Mathematics, 236(13):3198–3217, 2012.

[61] Layto, W., Trenchea, C:. Analysis of long time stability and errors of two parti- tioned methods for uncoupling evolutionary groundwater-surface water flows. SIAM Journal on Numerical Analysis, 51(51):248–272, 2013.

[62] Li, Y., Hou, Y.: A second-order partitioned method with different subdomain time steps for the evolutionary Stokes-Darcy system. Mathematical Methods in the Applied Sciences, 41(9), 2018.

[63] Li, Y., Trenchea, C:. A higher-order Robert-Asselin type time filter. Journal of Computational Physics, 259(2):23–32, 2014.

[64] Lipnikov, K., Vassilev, D., Yotov. I.: Discontinuous Galerkin and mimetic finite difference methods for coupled Stokes-Darcy flows on polygonal and polyhedral grids. Numerische Mathematik, 126(2):321–360, 2014.

[65] Mu, M., Xu, J.: A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow. SIAM Journal on Numerical Analysis, 45(5):1801–1813, 2007.

[66] Mu, M., Zhu., X.: Decoupled schemes for a non-stationary mixed Stokes-Darcy model. Mathematics of Computation, 79(270):707–731, 2010.

[67] Shan, L., Zheng, H., Layton, W.: A decoupling method with different subdomain time steps for the nonstationary Stokes-Darcy model. Numerical Methods for Partial Differential Equations, 29(2):549–583, 2013.

[68] Tlupova, S., Cortez, R.: Boundary integral solutions of coupled Stokes and Darcy flows. Journal of Computational Physics, 228(1):158–179, 2009.

[69] Wang, G., He, Y., Li, R.: Discontinuous finite volume methods for the stationary Stokes-Darcy problem. International Journal for Numerical Methods in Engineering, 107(5):395–418, 2016.

[70] Wang, W., Xu, C.: Spectral methods based on new formulations for coupled Stokes and Darcy equations. Journal of Computational Physics, 257(1):126–142, 2014.

[71] Williams, P.: A proposed modification to the Robert-Asselin time filter. Monthly Weather Review, 137(8):2538–2546, 2009.

[72] Williams, P.: The RAW filter: An improvement to the Robert-Asselin filter in semi-implicit integrations. Monthly Weather Review, 139(6):1996–2007, 2011.

[73] Williams, P.: Achieving seventh-order amplitude accuracy in leapfrog integrations. Monthly Weather Review, 141(9):3037–3051, 2013.

[74] Ervin, V. Approximation of coupled Stokes-Darcy flow in an axisymmetric domain. Computer Methods in Applied Mechanics and Engineering, 258(2):96–108, 2013.

[75] Alnæs, M., Blechta, J., Hake, j., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M., Wells, G.: The FEniCS project version 1.5. Archive of Numerical Software, 3(100), 2015.

[76] Arnold, D., Brezzi, F., Fortin, M.: A stable finite element for the Stokes equations. CALCOLO, 21(4):337–344, Dec 1984.

[77] Asselin, R.: Frequency filter for time integrations. Mon. Weather Rev, (100):487–490, 1972.

[78] Baker, G., Dougalis, A., Karakashian, O.: On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations. Mathematics of Computation, 39(160):339–375, 1982.

[79] Besier, M., Rannacher, R.: Goal-oriented spacetime adaptivity in the finite element Galerkin method for the computation of nonstationary incompressible flow. International Journal for Numerical Methods in Fluids, 70(9):1139–1166, January 2012.

[80] Charnyi, S., Heister, T., Olshanskii, M., Rebholz, L.: On conservation laws of Navier-Stokes Galerkin discretizations. Journal of Computational Physics, 337:289 – 308, 2017.

[81] Chorin, A.: The numerical solution of the Navier-Stokes equations for an incompressible fluid. Bull. Amer. Math. Soc., 73(6):928–931, 11 1967.

[82] Crouzeix, M., Raviart, P.: Approximation d equations d evolution lin eaires par des m ethodes multipas. Etude Numerique des Grands Systemes, 1976.

[83] Dahlquist, G.: G-stability is equivalent to A-stability. BIT Numerical Mathematics, 18(4):384–401, Dec 1978.

[84] Emmrich, E.: Error of the two-step BDF for the incompressible Navier-Stokes problem. ESAIM: M2AN, 38(5):757–764, 2004.

[85] Emmrich, E.: Stability and convergence of the two-step BDF for the incompressible Navier-Stokes Problem. 5:199–209, 01 2004.

[86] Fiordilino, J.: On pressure estimates for the Navier-Stokes equations. ArXiv e-prints, Mar. 2018.

[87] Galvin, K.: New subgrid artificial viscosity Galerkin methods for the Navier-Stokes equations. Computer Methods in Applied Mechanics and Engineering, 200(1):242 – 250, 2011.

[88] Geveci, T.: On the convergence of a time discretization scheme for the Navier-Stokes equations. Mathematics of Computation, 53(187):43–53, 1989.

[89] Girault, V., Raviart, P.: Finite Element Approximation of the Navier-Stokes Equations. Springer-Verlag Berlin Heidelberg, 1979.

[90] Gresho,P., Sani, R.: Incompressible Flow and the Finite Element Method. John Wiley and Sons, Inc., 1998.

[91] Griffiths, D., Higham, D.: Numerical methods for ordinary differential equations. Springer, 2010.

[92] Griffiths, D., Higham, D.: Numerical Methods for Ordinary Differential Equations. Springer-Verlag London Limited, 2010.

[93] Guzel,A., Layton, W.: Time filters increase accuracy of the fully implicit method. BIT Numerical Mathematics, 58(2):301–315, Jun 2018.

[94] Heywood, J., Rannacher, R.: Finite element approximation of the nonstationary Navier-Stokes prob-lem. part iv. error analysis for second-order time discretization. SIAM Journal on Numerical Analysis, 27(2):353384, 1990.

[95] Jiang, N.: A second-order ensemble method based on a blended backward differentiation formula timestep-ping scheme for time-dependent Navier-Stokes equations. Numerical Methods for Partial Differential Equa- tions, 33(1):34–61.

[96] Jiang, N., Mohebujjaman, N., Rebholz,L., Trenchea, C.: An optimally accurate discrete regulariza- tion for second order timestepping methods for Navier-Stokes equations. Computer Methods in Applied Mechanics and Engineering, 310:388 – 405, 2016.

[97] John, V.: Reference values for drag and lift of a two-dimensional time-dependent flow around a cylinder. International Journal for Numerical Methods in Fluids, 44(7):777–788, 2004.

[98] Layton, W., Mays, N., Neda, M., Trenchea, C.: Numerical analysis of modular regularization methods for the BDF2 time discretization of the Navier-Stokes equations. ESAIM: M2AN, 48(3):765–793, 2014.

[99] Layton, W., Trenchea, C.: Stability of two IMEX methods, CNLF and BDF2-AB2, for uncoupling systems of evolution equations. Applied Numerical Mathematics, 62:112–120, 2012.

[100] Liu, J., Liu, J., Pego, R.: Stable and accurate pressure approximation for unsteady incompressible viscous flow. Journal of Computational Physics, 229(9):3428 – 3453, 2010.

[101] Rang, J., Angermann, L.: New Rosenbrock W-Methods of order 3 for partial differential algebraic equa- tions of index 1. BIT Numerical Mathematics, 45(4):761–787, Dec 2005.

[102] Ravindran, S.: An extrapolated second order backward difference time-stepping scheme for the magneto- hydrodynamics system. Numerical Functional Analysis and Optimization, 37(8):990–1020, 2016.

[103] Robert, A.: An evaluation of the behaviour of planetary waves in an atmospheric model based on spherical harmonics. PhD thesis, McGill, 1965.

[104] Schafer, M., Turek, S.: Benchmark computations of laminar flow around a cylinder. In H. E.H., editor, Flow Simulation with High-Performance Computers II. Notes on Numerical Fluid Mechanics, volume 48, pages 547–566. Vieweg+Teubner Verlag, 1996.

[105] Temam, R.: Navier-Stokes Equations and Nonlinear Functional Analysis. Society for Industrial and Applied Mathematics, 1995.

[106] Verfurth, R.: Error estimates for a mixed finite element approximation of the Stokes equations. ESAIM: Mathematical Modelling and Numerical Analysis - Mod elisation Math ematique et Analyse Numerique, 18(2):175–182, 1984.

[107] Williams, P.: A proposed modification to the Robert-Asselin time filter. Monthly Weather Review, 137:2538–2546, 08 2009.

[108] Adams, R.A.: Sobolev spaces. Academic press, 2003.

[109] L.C. Berselli and R. Lewandowski, On the Reynolds time-averaged equations and the long-time behavior of Leray-Hopf weak solutions, with applications to ensemble averages, arXiv preprint arXiv:1801.08721, 2018

[110] J. Duchon and R. Robert, Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations, Nonlinearity,13, 249-255, 2000.

[111] P.M. Gresho and R.L.. Sani, Incompressible flow and the finite element method, Vol. 2: Isothermal laminar flows.

[112] N. Jiang and W. Layton, Numerical analysis of two ensemble eddy viscosity numerical regularizations of fluid motion, NMPDEs, 31 (2015) 630-651.

[113] J. Jovanovic,The statistical dynamics of turbulence, Springer, Berlin, 2004.

[114] A. Labovsky and W. Layton, Magnetohydrodynamic flows: Boussinesq conjecture, JMAA, Volume 434, 2016, 1665–1675.

[115] R. Lewandowski, Long-time turbulence model deduced from the Navier-Stokes equations, Chinese Annals of Mathematics, Series B, 36(2015) 883–894.

[116] R. Temam: Navier¡^{a}Stokes Equations: Theory and Numerical Analysis. North-Holland Publishing Company, New York, 1977.

[117] R.A. Horn and C.R. Johnson: Matrix Analysis. Cambridge University Press, Cambridge, England, 1990.

[118] H. Tennekes and J.L. Lumley: A first course in turbulence, MIT Press, Cambridge, 1972.

[119] D.C. Wilcox: Turbulence modeling for CFD[M]. La Canada, CA: DCW industries, 1998.

[120] J. Mathieu and J. Scott: An introduction to turbulent flow, Cambridge University Press, Cambridge, 2000.

[121] P.A. Durbin and B.A. Pettersson-Reif: Statistical Theory and Modeling for Turbulent Flows, Second Edition, Wiley, Chichester, 2011.

[122] W.K. George: Lectures in Turbulence for the 21st Century. Chalmers University of Technology, available at http://www.turbulence-online.com/, 2013.

[123] L.C. Berselli and F. Flandoli: On a stochastic approach to Eddy Viscosity models for turbulent flows[M]//Advances in Mathematical Fluid Mechanics. Springer Berlin Heidelberg, 2009: 55-81.

[124] B. Mohammadi and O. Pironneau: Analysis of the k-epsilon turbulence model[J]. France: Editions MASSON. 1993.

[125] V. John: Large eddy simulation of turbulent incompressible flows: analytical and numerical results for a class of LES models. Vol. 34. Springer Science and Business Media, 2012.

[126] V. John: Finite element methods for incompressible flow problems, Springer, Berlin, 2016.

[127] J. Boussinesq: Essai sur la th*é*orie des eaux courantes, M*é*moires pr*é*sent*é*s par divers savants *à* l'Acad*é*mie des Sciences, 23 (1877):1-680

[128] R.S. Rivlin: The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids. Quarterly of Applied Mathematics, 1957, 15(2): 212-215.

[129] W. Layton: The 1877 Boussinesq conjecture: Turbulent fluctuations are dissipative on the mean flow, available at www.mathematics.pitt.edu/research/technical-reports, 2014.

[130] W. Layton: Introduction to the numerical analysis of incompressible viscous flows. Vol. 6. SIAM, 2008.

[131] F.G. Schmitt: About Boussinesq's turbulent viscosity hypothesis: historical remarks and a direct evaluation of its validity. Comptes Rendus M*é*canique, Elsevier Masson, 335(9-10):617-627, 2007.

[132] B. de Saint-Venant: Note ¨¨|moire sur la dynamique des fluides[J]. Comptes Rendus, 1843, 17: 1240-1244.

[133] H. Frahnert and U. Dallmann: Examination of the eddy-viscosity concept regarding its physical justification[J]. New Results in Numerical and Experimental Fluid Mechanics III, 2002: 255-262.

[134] L.C. Berselli, T. Iliescu and W. Layton: Mathematics of large eddy simulation of turbulent flows[M]. Springer Science & Business Media, 2005.

[135] W. Layton and R. Lewandowski: Analysis of an eddy viscosity model for large eddy simulation of turbulent flows[J]. Journal of Mathematical Fluid Mechanics, 2002, 4(4): 374-399.

[136] T.C. Rebollo and R. Lewandowski: Mathematical and numerical foundations of turbulence models and applications[M]. Springer New York. 2014.

[137] V.P. Starr: Physics of negative viscosity phenomena[J]. 1968.

[138] J.G. Heywood and R. Rannacher: Finite-element approximations of the nonstationary Navier-Stokes problem. Part IV: Error estimates for second-order time discretization, SIAM J. Numer. Anal., 27 (1990), pp. 353-384.

[139] N. Jiang and W. Layton: Algorithms and models for turbulence not at statistical equilibrium[J]. Computers & Mathematics with Applications, 71(11): 2352-2372, 2016.

[140] N. Jiang, S. Kaya and W. Layton: Analysis of model variance for ensemble based turbulence modeling[J]. Computational Methods in Applied Mathematics, 2015, 15(2): 173-188.

[141] B. Baldwin and H. Lomax: Thin-layer approximation and algebraic model for separated turbulent flows[C]//16th Aerospace Sciences Meeting. 1978: 257.

[142] F. Hecht and O. Pironneau: FreeFem++. Webpage: http://www.freefem.org

[143] B.A. Baker: Galerkin approximation for the Navier-Stokes equations, Technical Report, 1976